UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

ASYMPTOTIC DESCRIPTION OF DYNAMICS IN PLASMA AND WATER-WAVES TYPE MODELS

TESIS PARA OPTAR AL GRADO DE
DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

MARÍA EUGENIA MARTÍNEZ MARTINI

PROFESOR GUÍA:
CLAUDIO ANTONIO MUÑOZ CERÓN

MIEMBROS DE LA COMISIÓN:
HANNE VAN DEN BOSCH
CARLOS KENIG
MICHAL KOWALCZYK
FRÉDÉRIC ROUSSET
CATHERINE SULEM-BIERSTONE.

Esta tesis ha sido parcialmente financiada por Fondecyt 1191412 y 1150202, Conicyt PFCHA/Doctorado Nacional/ 2019-21192076, ECOS-Sud C18E06, CMM ANID PIA AFB170001, CMM ANID BASAL ACE210010 y CMM ANID BASAL FB210005

SANTIAGO DE CHILE
2021

RESUMEN DE LA MEMORIA PARA OPTAR
AL TÍTULO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA,
MENCIÓN MODELACIÓN MATEMÁTICA
POR: MARÍA EUGENIA MARTÍNEZ MARTINI
FECHA: 2021
PROF. GUÍA: CLAUDIO ANTONIO MUÑOZ CERÓN

ASYMPTOTIC DESCRIPTION OF DYNAMICS IN PLASMA AND WATER-WAVES TYPE MODELS

This thesis is devoted to the study of the asymptotic dynamics in several fluid models of key interest. These are on the one hand related to the classical Schrödinger equation and on the other hand can be derived from the Zakharov Water Waves model, in the Craig-Sulem-Zakharov formulation. Precisely, the models to be considered here are the following: the Nonlinear Schrödinger equation (NLS), the Hartree equation, the Zakharov and Klein-Gordon-Zakharov systems, the Zakharov-Rubenchik/Benney-Roskes system, and finally, the Zakharov Water Waves problem. All these models are analyzed in one dimension, which is interesting because of the lack of suitable dispersive estimates in the nonlinear setting.

Additionally, in terms of key questions, this thesis is divided in three main parts: a first part where local virial estimates will provide spatial decay on compact sets, a second part where time-expanding virial estimates permit to prove extensive decay properties for completely untreated physical systems, and finally, a constructive part related to the existence of nonlinear solitary waves in variable bottom water waves.

First of all, Chapter 2 is devoted to the study of the Nonlinear Schrödinger equation (with and without potential), and the Hartree equation. We consider the decay property as a form of nonlinear scattering in one dimension. We use virial identities to prove decay in compact intervals in space for both equations under oddness (and sometimes smallness) condition on the initial data.

Zakharov and Klein-Gordon-Zakharov systems are considered in Chapter 3, where we prove two types of decay: one in compact intervals around the origin, and another one for the energy norm in compact intervals along curves outside the light cone. No oddness conditions are imposed for the second result.

Chapter 4 analyses decay properties of the Zakharov-Rubenchik/Benney-Roskes system using different virial techniques. This time, we want to show strong local decay in extensive regions of space, which is somehow unknown in the previous models. Taking advantage of the underlying characteristic curves for the equation under work, a new virial method is introduced to prove a decay result in growing compact intervals around these curves. We also prove decay of the energy norm along curves outside the light cone.

The main part of this thesis is Chapter 5, which is concerned with the solitary wave problem for the Zakharov water waves equation under variable domain. Indeed, we assume a slightly changing, nonflat bottom. Adapting and extending the techniques introduced by Martel, and recently by Ming, Rousset and Tzvetkov, we prove the existence of a soliton-like solution for the nonflat bottom problem before it encounters the strong interaction regime. The collision problem remains the main open question to be considered in the future.

Finally, Chapter 6 is devoted to the conclusion of this work, as well as new ideas and forthcoming projects. Some of them have been remained elusive for us during these years, but we plan to attack them in the near future.

RESUMEN DE LA MEMORIA PARA OPTAR
AL TÍTULO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA,
MENCIÓN MODELACIÓN MATEMÁTICA
POR: MARÍA EUGENIA MARTÍNEZ MARTINI
FECHA: 2021
PROF. GUÍA: CLAUDIO ANTONIO MUÑOZ CERÓN

ASYMPTOTIC DESCRIPTION OF DYNAMICS IN PLASMA AND WATER-WAVES TYPE MODELS

Esta tesis está dedicada al estudio de la dinámica asintótica en varios modelos de fluidos de interés clave. Estos están por un lado relacionados con la ecuación clásica de Schrödinger y por otro lado pueden derivarse del modelo de Zakharov Water Waves, en la formulación de Craig-Sulem-Zakharov. Precisamente, los modelos a considerar aquí son los siguientes: la ecuación no lineal de Schrödinger (NLS) y Hartree, los sistemas de Zakharov y Klein-Gordon-Zakharov, el sistema de Zakharov-Rubenchik/Benney-Roskes, y finalmente, el problema de Water Waves de Zakharov. Todos estos modelos se analizan en una dimensión, muy interesante debido a la falta de estimaciones nolineales dispersivas adecuadas.

Además, en términos de preguntas clave, esta tesis se divide en tres partes principales: una primera parte donde las estimaciones viriales locales proporcionarán decaimiento en conjuntos de espacio compactos, una segunda donde las estimaciones viriales que se expanden en el tiempo permiten demostrar propiedades de dispersión para sistemas físicos que no han sido tratados, y finalmente, una parte constructiva relacionada con la existencia de ondas solitarias no lineales en Water Waves de fondo variables.

En primer lugar, el Capítulo 2 está dedicado al estudio de las ecuaciones NLS (con y sin potencial) y Hartree. Consideramos la propiedad de decaimiento como una forma de dispersión no lineal. Usamos identidades viriales para probar dispersión en intervalos compactos en espacio en condiciones de imparidad (y a veces, pequeñez) para el dato inicial.

Los sistemas de Zakharov y Klein-Gordon-Zakharov se consideran en el Capítulo 3, donde probamos dos tipos de decaimiento: uno en intervalos compactos alrededor del origen y otro para la norma de energía en intervalos compactos a lo largo de curvas fuera del cono de luz. No se imponen condiciones de imparidad para el segundo resultado.

El Capítulo 4 analiza las propiedades de decaimiento del sistema Zakharov-Rubenchik / Benney-Roskes utilizando diferentes técnicas viriales. Esta vez, queremos mostrar un fuerte decaimiento local en extensas regiones del espacio, desconocidas para los modelos anteriores. Aprovechando las curvas características subyacentes para la ecuación en estudio, se introduce un nuevo método virial para probar un resultado de dispersión en intervalos compactos crecientes alrededor de estas curvas. También demostramos la dispersión de la norma de enería a lo largo de las curvas fuera del cono de luz.

La parte principal de esta tesis es el Capítulo 5, que trata el problema de onda solitaria para la ecuación ondas de agua de Zakharov, bajo un dominio variable. Asumimos un fondo no plano ligeramente cambiante. Adaptando y ampliando las técnicas introducidas por Martel, y por Ming, Rousset y Tzvetkov, probamos la existencia de una solución tipo solitón para el problema de fondo no plano antes de que encuentre el régimen de interacción fuerte. El problema de las colisiones sigue siendo la principal pregunta abierta a considerar en el futuro.

Finalmente, el Capítulo 6 está dedicado a la conclusión de este trabajo, así como a nuevas ideas y proyectos futuros, algunos esquivos durante estos años.

Agradecimientos

Esta tesis se considera, a todas luces, un producto de mi persona. Sin embargo, para ser honesta, creo que fue un trabajo en conjunto, no sólo por la ayuda matemática que recibí en forma de discusión e ideas, sino también por el apoyo emocional que me dio la gente de mi entorno a lo largo de los últimos cinco años. Tomo el doctorado como un proyecto personal, no sólo académico, y no estaría escribiendo estos agradecimientos si no fuera por quienes me acompañaron y me dieron afecto en el proceso.

En primer lugar, me gustaría agradecer a mi familia. Mamá y papá, gracias por apoyarme a ciegas en cualquiera de mis proyectos, sin cuestionar mis decisiones ni siquiera cuando tal vez quisieran. Por enseñarme que no hay carreras de chicas ni de chicos, por escucharme con paciencia cuando se me ocurre hablarles de matemáticas, por aprender lo que son las ecuaciones dispersivas. Su cariño es invaluable para mí.

También quiero agradecer a mis amigos, Rodrigo, Eve y Christopher, por ser mi familia en Chile, el almuerzo de los domingos, los cafés después del almuerzo y una inacabable fuente de apoyo y consuelo. Llevo la oficina 416 en mi corazón.

A Nata, que fue mi amiga, una hermana mayor prestada que no duda en ofrecerme su ayuda cuando la necesito, que me saca de apuros a último momento. Gracias por ser un salvavidas adentro del DIM y una compatriota con quien puedo compartir argentinismos sin tener que explicarlos. Aprovecho también para agradecer a las y los funcionarios del DIM, en particular a Silvia Mariano, Karen Hernández y a Cookie, por su trabajo y esfuerzo para hacer del departamento un agradable lugar de trabajo.

A mis amigos, chilenos o argentinos, matemáticos o no, por ser mi espacio de descargo y de recreación. A mis amigas de Río Cuarto, que monopolizan mis pausas virtuales y mis vacaciones; a mis amigos matemáticos, por escuchar mis miedos y compartir mis almuerzos.

A Rodrigo, por acompañarme en la cuarentena no tan autoimpuesta justo antes de la defensa, por su cariño y su comprensión en el último tramo del doctorado.

I would like to thank Prof. Frédéric Rousset and Juan Soler, for their kindness and hospitality during my visit to their universities. Also, I should like to express my gratitude to Prof. Hanne Van Den Bosch, Prof. Carlos Kenig, Prof. Michal Kowalczyk, Prof. Frédéric Rousset and Prof. Catherine Sulem for having accepted to be part of the commission of this thesis. Thank you for your time and dedication reading this work.

Finalmente, quiero agradecer enormemente a mi director de tesis, Prof. Claudio Muñoz, por tener siempre un trato amable y preocuparse por el bienestar emocional y físico de sus alumnos, por guiarme en mi carrera y no sólo en mi doctorado. Gracias, Claudio, por no limitarte a ser sólo director de tesis, sino también consejero y mentor.

Contents

Part I

Introduction

Chapter 1

Introduction

Contents

1.1 Preliminaries

Physics, as every science, became an important source of questions and, eventually, (although, not always) of answers. As a consequence, the joint growth of both physics and mathematics, the last one being the main tool to model the observable world, became more complex and more synergetic. Such evolution of mathematical methods for application to problems in physics is commonly referred to as mathematical physics. The study of Partial Differential Equations is, perhaps, one of the theories most closely associated with this concept.

In physics, dispersion relations constitute the characterization of the plane wave motion in a medium. They represent an important part of mathematical physics and cover various types of classical and quantum scattering phenomena, describing the behaviour and interaction between solutions to partial differential equations. Hence, the study of dispersive equations becomes essential to understand physical events. In this context, the Schrödinger equation is probably one of the most relevant equations of the area. Not only it constitutes the basis for quantum mechanics, it is also a good dispersive model, as it is usually simpler in terms of techniques than others in the area, such as the wave equation or the Korteweg-de Vries equation.

In fluid dynamics, dispersion of water waves usually means that waves of different wavelengths travel at different phase speeds, that is, *frequency dispersion*. In fluids with a free surface, water waves are travelling waves dealing with surface tension and gravity forces, that coerce its elevation into the resting state. Such interaction between restoring forces and surface elevation shape what is considered a dispersive medium, and different mediums imply specific mathematical models. For instance, if the water depth is large compared to the wave length of the water waves (deep water), envelope solitons described by Schrödinger equation may occur. Another deep water model for the description of gravitational waves is the Zakharov-Rubenchik/Benney-Roskes system. In this context, an important model that has exhibited an increasing interest in the bibliography over the last years is the Zakharov/Craig-Sulem formulation. It arises when considering a non-vanishing shoreline and assuming that the flow is at rest at infinity, which essentially means to be away from the coast. We will give a more detailed derivation of the Zakharov/Craig-Sulem model in Chapter 5.

Other examples of dispersive fluid models that we will consider in this work are the Zakharov systems, which describe long-wavelength small-amplitude Langmuir oscillations in a ionized plasma, and Zakharov-Rubenchik/Benney-Roskes, mainly treated in Chapter 3.

This thesis was made with important collaborations and research visits. Part of this work was done while I was visiting Université Paris-Saclay (Paris) and Universidad de Granada FisyMat (Granada). I would like to acknowledge professors Juan Soler and Frédéric Rousset for their great support and their help making these travels possible.

Before moving to the description of the models considered in this thesis, we shortly recall some important notions for this work. We concentrate ourselves in the concept of dispersion and decay.

1.1.1 Notion of dispersion and decay

Dispersion occurs when pure plane waves of different wavelengths have different propagation velocities, so that a wave packet of mixed wavelengths tends to spread out in space; they *scatter*. A *linear* PDE is said to be dispersive if plane wave type solutions present such dispersion.

For instance, let us consider the linear Schrödinger equation

$$\mathrm{i}u_t + \Delta u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^{\mathrm{d}}. \tag{1.1}$$

Then, wave plane solutions of amplitude A of the form $u(t,x) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)}$ satisfy that the frequency ω is given by the square of the wave-number k, namely $\omega(k) = |k|^2$. Moreover, using Fourier transform we obtain an explicit convolution representation from which we also have the dispersive inequality:

$$\|u(t)\|_{L^\infty(\mathbb{R}^{\mathrm{d}})} \lesssim_{\mathrm{d}} |t|^{-\mathrm{d}/2}\|u(0)\|_{L^1(\mathbb{R}^{\mathrm{d}})}. \tag{1.2}$$

Such decay estimate holds for any solution whose initial data $u(0)$ belongs to $L^1(\mathbb{R}^{\mathrm{d}})$.

Now, the non-linear case is another matter in terms of what to expect from solutions and their asymptotic behaviour. The main issue in this regard is the existence of non-decaying solutions such as *solitons*, that is, travelling non-dissipative waves that maintain their shape while they propagate at a constant velocity. Moreover, the interest produced by the dichotomy between the existence of dispersion and solitary waves for non-linear models eventually hinted what is called the "soliton resolution conjecture". This conjecture essentially means that solutions with generic initial data should eventually resolve into a finite number of solitons, moving at different speeds, plus a radiative term which goes to zero. Hence, we can no longer count on decaying solutions without any further condition. The balance between non-linearity and dispersion enables the existence of solitary waves and enriches the study of *large-time behaviour* of solutions.

In this thesis, when dealing with non-linear dispersive models, we will say that a form of *scattering or dispersion* is present if a solution decays to zero in some sense as time tends to ∞. In particular, we refer to *scattering to a free solution* when a global solution behaves asymptotically like solutions to the linear equation. More precisely, for the nonlinear Schödinger equation (NLS):

Definition 1.1. *A global solution u of NLS equation is said to* scatter in the space X to a free solution *as $t \to \pm\infty$ if there exists $u_\pm \in X$ such that*

$$\lim_{t\to\pm\infty} \|\mathrm{e}^{\mathrm{i}t\Delta}u_\pm - u(t,\cdot)\|_X = 0,$$

where $\mathrm{e}^{\mathrm{i}t\Delta}$ *is the semi-group associated to the Scrödinger equation.*

The smaller the dimension, the less probable that scattering occur, and modifications are needed. This thesis is concerned with decay properties for the following **one dimensional** Schrödinger type models: NLS and Hartree equations, Zakharov, Klein-Gordon Zakharov (KG Zakharov), and Zakharov-Rubenchik/Benney Roskes systems (ZR/BR). When proving decay results, the main goal is to avoid encounters with solitary waves, multi-solitions and other non-decaying solutions. To do so, we shall make use of appropriate virial identities, constructed essentially to ensure soliton-free regimes. In addition, in Section 5, we shall study the existence for soliton-type solutions to the Zakharov Water Waves system (ZWW). We will construct a solution to the ZWW that behaves asymptotically (as time tends to $-\infty$) like a travelling wave. To that end, we shall also study the already known solitary waves for ZWW (in this case, in the flat-bottom regime).

1.2 Introducing Schrödinger Models

Schrödinger equation was first introduced by Erwin Schrödinger in 1925, as he decided to find a proper 3-dimensional wave equation for the electron. It is the fundamental law of non-relativistic quantum mechanics, a physical theory that deals with those phenomena that occur at microscopic scales of the order of Planck's constant. From a mathematical point of view, the Schrödinger equation is a delicate problem, and it has a jumble of properties of parabolic and hyperbolic equations.

Schrödinger's nonlinear equation has received great attention in mathematics over the past 50 years, in part because of its many applications, such as in nonlinear optics or in deep water models. Variants and generalizations appeared over the years, extending its study and giving rise to new models.

We are concerned with the following models: NLS and Hartree equations, Zakharov, Klein-Gordon Zakharov, and Zakharov-Rubenchik/Benney Roskes systems. As mentioned before, we are interested in their scattering properties and long-time behaviour, which ultimately imply the need to understand their non-decaying solutions, as well. Throughout this section, we shall introduce the precise models, along with some basic properties, and a summarized study on their non-decaying solutions.

1.2.1 Nonlinear Schrödinger equation

In Chapter 2, we will consider the **non-linear Schrödinger equation (NLS)**

$$\mathrm{i}u_t + u_{xx} = \mu V(x)u + f\left(|u|^2\right)u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1.3}$$

where the potential $V:\mathbb{R}\to\mathbb{R}$ is a *Schwartz even function* and $f:\mathbb{R}\to\mathbb{R}$ is a function such that for $1<p<5$,

$$|f(s)| \lesssim s^{\frac{p-1}{2}},$$

and satisfies that $f\circ s^2$ is locally Lipschitz continuous.

In the particular case,

$$f(s) = \sigma s^{\frac{p-1}{2}}, \quad 1<p<5$$

the **semilinear Schrödinger equation** is recovered. We will say the equation is focusing when $\sigma = 1$. Otherwise ($\sigma = -1$), we will be in the defocusing case. For $p = 3$, the model is integrable.

The equation (1.3) is Hamiltonian, and it is characterized by having at least the following conservation laws, defined as Mass, Energy and Momentum, respectively:

$$M(u(t)) := \int_{\mathbb{R}} |u(t)|^2 \mathrm{d}x = M(u(0)), \tag{1.4}$$

$$E(u(t)) := \int_{\mathbb{R}} |\nabla u(t)|^2 \mathrm{d}x - \sigma \tfrac{2}{p+1} |u(t)|^{p+1} \mathrm{d}x = E(u(0)), \tag{1.5}$$

$$P(u(t)) := \mathrm{Im} \int_{\mathbb{R}} u(t) \overline{u}_x(t) \mathrm{d}x = P(u(0)). \tag{1.6}$$

It is well-known that this one-dimensional semilinear Schödinger equation is globally well-posed for initial data in $H^1(\mathbb{R})$ when $1 < p < 5$, and blow up may occur if $p \geqslant 5$, as was proved by Glassey, [25], Merle-Raphaël [43] and other subsequent works.

Solitons, multi-solitons and breathers: For a better understanding of decay and scattering for NLS (1.3), we need to study non-decaying solutions and how to rule them out from our results. Equation (1.3) presents solitons or solitary waves of the form

$$u(t,x) = \mathrm{e}^{\mathrm{i}ct} Q_c(x)$$

where $Q_c > 0$ is a stationary solution to the ODE $Q_c'' + |Q_c|^{(p-1)} Q - cQ_c = 0$, $Q_c \in H^1$. Early on, Zakharov and Shabat proved that this solution is even in space and small in H^1 provided $c \gg 1$, (see [81]). Moreover, because of the many symmetries that the equation presents, for any $v, x_0, \gamma \in \mathbb{R}$,

$$u(t,x) = Q_c(x - x_0 - vt) \mathrm{e}^{\mathrm{i}((1/2)vx - (1/4)v^2 t + ct + \gamma)}$$

is also a solution. Such solutions are stable, as stated, for instance, by Cazenave-Lions [8] and Weinstein [74].

In addition, there are also multi-solitons solutions that are *not* even, which means that even though parity conditions are a good way to rule out solitary waves, they are not completely effective. Indeed, Martel, Merle and Tsai [35] proved the existence (and H^1 stability) of solutions that decompose like the sum of solitary waves with non-zero speed plus radiation (later improved by Vihn [52]), and these solutions could very well be odd.

Finally, equation (1.3) can present breathers, that is periodic in time solutions but with non-trivial period. Such is the case for the (scattering) **critical** one-dimensional NLS ($p = 3$), that possesses explicit breather solutions, such as the Satsuma-Yajima breather [66],

$$B_{SY}(t,x) := \frac{4\sqrt{2}\mathrm{e}^{\mathrm{i}t}(\cosh(3x) + 3\mathrm{e}^{8\mathrm{i}t}\cosh(x))}{\cosh(4x) + 4\cosh(2x) + 3\cosh(8t)},$$

or the Peregrine breather [63],

$$B_P(t,x) := \mathrm{e}^{\mathrm{i}t} \left(1 - \frac{4(1 + 2\mathrm{i}t)}{1 + 4t^2 + 2x^2} \right).$$

Both breathers satisfy that are even and arbitrarily small.

1.2.2 Hartree equation

The Hartree equation is a non-local model that can be written in one dimension of space as

$$\mathrm{i}u_t + u_{xx} = \sigma\left(|x|^{-\alpha} * |u|^2\right)u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{1.7}$$

where $0 < \alpha < 1$ and $\sigma = \pm 1$ determines the focusing ($\sigma = -1$) or defocusing ($\sigma = 1$) nature of the model.

This equation was first derived by Douglas Hartree in 1927, as he sets himself the goal to first calculate the solutions to Schrödinger's equation for individual electrons. The solution to the original Schrödinger equation is a wave function which describes all of the electrons. He assumed that the nucleus together with the electrons formed a spherically symmetric field and found an equation for each electron. Then, the wavefunction of a system could be computed as a combination of wavefunctions of individual particles, solving Hartree's equations for each electron.

The Hartree equation (1.7) is locally well-posed, extended to global well posedness for small initial data. In addition, this equation conserves the Mass, Momentum and Energy, defined (respectively) below:

$$\begin{aligned}
M(u(t)) &:= \int_{\mathbb{R}} |u(t)|^2 \mathrm{d}x = M_0,\\
E(u(t)) &:= \frac{1}{2}\int_{\mathbb{R}} |\nabla u(t)|^2 \mathrm{d}x + \frac{\sigma}{4}\int_{\mathbb{R}} \left(|x|^{-a} * |u|^2\right)|u|^2 \mathrm{d}x = E_0,\\
P(u(t)) &:= \operatorname{Im}\int_{\mathbb{R}} u(t)\overline{u}_x(t)\mathrm{d}x = P_0.
\end{aligned}$$

Non-decaying solutions for Hartree: Regarding the existence of solitary waves, the situation is fairly similar to NLS. Focusing Hartree equation (1.7) ($\sigma = 1$) admits solitary waves solutions (or solitons) of the form

$$u(t,x) = \mathrm{e}^{\mathrm{i}ct}Q_c(x) \in H^1$$

where the ground state $Q_c : \mathbb{R} \to \mathbb{R}$ is an H^1-solution of the Choquard equation $\Delta Q + \left(\frac{1}{|x|^a} * |Q|^2\right)Q - cQ = 0$, $c \in \mathbb{R}$. These solutions are, up to translation and inversion of the sign, positive and radially symmetric functions, see, for instance, the works of Cingolani-Secchi-Squassina [11] and Ruiz [64] for the proofs of such properties, or VanSchaftingen for a beautiful review of the Choquard equation [46]. Moreover, solitary waves for the focusing Hartree equation are stable, as was proven by Cazenave and Lions in [8]. It is not expected for the *defocusing* Hartree equation ($\sigma = -1$) to present soliton-like solutions.

1.2.3 Zakharov and Klein-Gordon Zakharov system

In Chapter 3, we will deal with the Cauchy problem for the one dimensional Zakharov system

$$\begin{aligned}
&\mathrm{i}u_t + \Delta u = nu, && (t,x)\in\mathbb{R}\times\mathbb{R},\\
&\alpha^{-2}n_{tt} - \Delta n = \Delta|u|^2, && (t,x)\in\mathbb{R}\times\mathbb{R},\\
&(u,n)(t=0) = (u_0,n_0),\ n_t(t=0) = n_1,
\end{aligned} \tag{1.8}$$

where $u(t,x):\mathbb{R}\times\mathbb{R}\to\mathbb{C}$, $n(t,x):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\alpha>0$.

Chapter 3 also studies the Cauchy problem for Klein-Gordon Zakharov system in one dimension

$$\begin{aligned}
&c^{-2}u_{tt}-\Delta u+c^2u=-nu, && (t,x)\in\mathbb{R}\times\mathbb{R},\\
&\alpha^{-2}n_{tt}-\Delta n=\Delta|u|^2 && (t,x)\in\mathbb{R}\times\mathbb{R},\\
&(u,n)\,(t=0)=(u_0,n_0),\ \ (u_t,n_t)\,(t=0)=(u_1,n_1), &&
\end{aligned}\tag{1.9}$$

where $u(t,x):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $n(t,x):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $\alpha>0,\ c>0$.

Plasma is one of the most abundant form of ordinary matter in the universe, second only to dark matter and dark energy, greatly studied in astrophysics, and can be artificially generated by heating a neutral gas or subjecting it to a strong electromagnetic field. The Zakharov systems, first derived by Zakharov in 1921 [79], model long-wavelength small-amplitude Langmuir waves (rapid oscillations of the electron density in conducting media) in a ionized plasma. They describe nonlinear interactions between high-frequency, electromagnetic waves and low-frequency, acoustic type waves. Here, the unknowns represent the mean mode of the ionic fluctuations of density in the plasma n and the changing amplitude of electric field u, which varies slowly compared to the unperturbed plasma frequency. Constants α and c are the ion sound speed and the plasma frequency, respectively.

Zakharov systems can be derived from the two-fluid Euler–Maxwell system by considering a plasma as two interpenetrating fluids, an electron fluid and an ion fluid. See the work of Sulem-Sulem [71] and of Texier [72], where the authors expose a very detailed derivation of the model.

An appealing aspect of these equations are the limiting cases. From the Klein-Gordon-Zakharov system (1.9) in the high frequency case ($c\gg 1$), one could recover Zakharov (1.8) as a limiting equation. Indeed, if one considers $\tilde{u}=\mathrm{e}^{\mathrm{i}c^2t}u$ in (1.9), then it follows that

$$\begin{aligned}
&c^{-2}\tilde{u}_{tt}-2\mathrm{i}\tilde{u}_t-\Delta\tilde{u}=-n\tilde{u},\\
&\alpha^{-2}n_{tt}-\Delta n=\Delta|\tilde{u}|^2.
\end{aligned}$$

Thus, formally, taking $c\to\infty$, the Zakharov system (1.8) is obtained.

Another interesting limiting case accurs in the subsonic regime ($\alpha<c$), in which density perturbations are changing slowly. This would imply that taking $\alpha\to\infty$, the Langmuir waves follow the cubic semilinear Schrodinger equation. Both subsonic and high frequency limits were extensively studied in the work of Masmoudi and Nakanishi [36]-[39].

System (1.8) in one dimension is globally well-posed for initial data in $H^1(\mathbb{R})\times L^2(\mathbb{R})\times\hat{H}^{-1}(\mathbb{R})$, where

$$w\in\hat{H}^s \text{ if there exists } v:\mathbb{R}^{\mathrm{d}}\to\mathbb{R}^{\mathrm{d}} \text{ such that } w=\nabla\cdot v \text{ and } \|w\|_{\hat{H}^s}=\|v\|_{H^{s+1}}.$$

Some works in this regard are Sulem-Sulem [70] for local well-posedness in $\mathrm{d}=1,2,3$, extended to global well posedness by Ozawa-Tsutsumi [60], Colliander [13] and Pecher [62].

For KGZ (1.9), well-posedness results hold in the energy space $H^1 \times L^2 \times L^2 \times \hat{H}^{-1}$. See, for instance, Ozawa-Tsutaya-Tsutsumi [57]-[58], Otha-Todorova [56], Masmoudi-Nakanishi in [39]-[40].

The Zakharov system (1.8) preserves the mass $\|u(t)\|_{L^2(\mathbb{R})} = \|u(0)\|_{L^2(\mathbb{R})}$ and both systems preserve the energy:

- Energy associated to Zakharov (1.8)

$$H_S(t) := \int_{\mathbb{R}} |\nabla u(t,x)|^2 + \frac{1}{2}\Big(|n(t,x)|^2 + \frac{1}{\alpha^2}|D^{-1}n_t(t,x)|^2\Big) + n(t,x)|u(t,x)|^2 \mathrm{d}x,$$

- Energy associated to KGZ (1.9)

$$\begin{aligned} H_{KG}(t) = \int_{\mathbb{R}} & c^2|u(t,x)|^2 + |\nabla u(t,x)|^2 + \frac{1}{c^2}|u_t(t,x)|^2 + \frac{1}{2}|n(t,x)|^2 \\ & + \frac{1}{2}\big|\alpha D^{-1}n_t(t,x)\big|^2 + n(t,x)|u(t,x)|^2 \mathrm{d}x = H_{KG}(0), \end{aligned}$$

where $D = \sqrt{-\Delta}$. We point out that the energy for both systems could very well be negative. Nevertheless, an interesting property of the one-dimensional Zakharov systems (1.8) and (1.9) is the fact that, even though not conserved, it is possible to find uniform bounds for the energy norm of global solutions. Indeed, using Gagliardo-Nirenberg inequality:

$$\|u\|_{L^4}^4 \leqslant \|u_x\|_{L^2}\|u\|_{L^2}^3$$

one can prove the existence of constants depending on initial such that:

- For a global solution of Zakharov system (1.8):

$$\int_{\mathbb{R}} \left(|u_x(t,x)|^2 + |u(t,x)|^2 + |D^{-1}n_t(t,x)|^2 + |n(t,x)|^2\right) \mathrm{d}x \leqslant K_S, \tag{1.10}$$

- For a global solution of KGZ system (1.9):

$$\int_{\mathbb{R}} |u_t(t,x)|^2 + |u_x(t,x)|^2 + |u(t,x)|^2 + |n(t,x)|^2 + |D^{-1}n_t(t,x)|^2 \mathrm{d}x \leqslant K_{KG}. \tag{1.11}$$

  A more detailed proof of both bounds can be found in Chapter 3, Lemmas 3.6 and 3.13.

Non-decaying solution for Zakharov sistem: Solitary waves for the Zakharov system are solutions of the form

$$u(t,x) = \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\frac{c}{2}(x-ct)}u_{\omega,c}(x-ct),\ n(t,x) = n_{\omega,c}(x-ct), \tag{1.12}$$

where $u_{\omega,c}$ and $n_{\omega,c}$ are even functions (the explicit formula can be found in Chapter 3) and c, ω are real numbers satisfying $4\omega + c^2 \geqslant 0$ and $1 - c^2 > 0$. Under such conditions, the travelling wave turns out to be orbitally stable, as proven by Wu in [77]. Moreover, for solitons (u, n) described by (1.12), it is easy to see that u, n are both even in space, which inspires parity conditions for our results.

In addition to soliton-like solutions, there also exist solutions that blow up both in finite and infinite time. Merle [41, 42] adapts the Glassey technics [25] and uses virial identies to prove that negative energy solutions blow up. Indeed, he considered the perturbed virial quantity:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{4}\int_{\mathbb{R}}|x|^2|u|^2+\int_0^t\frac{1}{\alpha^2}\left(x\cdot D^{-1}n_t\right)n\right)=\mathrm{d}H_s-(\mathrm{d}-2)\int_{\mathbb{R}}|\nabla u|^2-\frac{1}{\alpha^2}(\mathrm{d}-1)\int_{\mathbb{R}}|D^{-1}n_t|^2.$$

Consequently, for solutions to (1.8) in dimension $\mathrm{d}=2,3$ and such that $H_s<0$, blow-up is obtained.

Non-decaying solutions for KGZ: Similarly to the Zakharov equation, travelling waves for the KGZ system (1.9) can be constructed as solutions described by

$$\begin{aligned}u(t,x)&=\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\frac{c}{2}(x-ct)}u_{\omega,c}(x-ct),\\ n(t,x)&=n_{\omega,c}(x-ct),\end{aligned}$$

with even functions $u_{\omega,c}$, $n_{\omega,c}$, described more precisely in Chapter 3. Then, solitary waves exist when the real constants ω and c satisfy $1-c^2-\omega^2>0$. Moreover, Chen [10] proved they are orbitally stable.

1.2.4 Zakharov-Rubenchik/Benney-Roskes system

Chapter 4 deals with the decay properties for solutions of the initial value problem (IVP) associated with the Zakharov-Rubenchik/Benney-Roskes (ZR/BR) system in one space dimension

$$\begin{array}{ll}\mathrm{i}\partial_t\psi+\omega\partial_x^2\psi=\gamma\big(\eta-\frac{1}{2}\alpha\rho+q|\psi|^2\big)\psi, & (t,x)\in\mathbb{R}\times\mathbb{R}\\ \theta\partial_t\rho+\partial_x\big(\eta-\alpha\rho\big)=-\gamma\partial_x(|\psi|^2), & (t,x)\in\mathbb{R}\times\mathbb{R}\\ \theta\partial_t\eta+\partial_x\big(\beta\rho-\alpha\eta\big)=\frac{1}{2}\alpha\gamma\partial_x(|\psi|^2), & (t,x)\in\mathbb{R}\times\mathbb{R}\\ (\psi,\rho,\eta)\,(t=0,x)=(\psi_0,\rho_0,\eta_0)\,, & \end{array}\tag{1.13}$$

where

$$\omega>0,\quad \beta>0,\quad \gamma>0,\quad \beta-\alpha^2>0,\quad 0<\theta<1,\quad \text{and}\quad q:=\gamma+\frac{\alpha(\alpha\gamma-1)}{2(\beta-\alpha^2)},$$

are all real parameters.

Model (1.13) corresponds to the one-dimensional case of the most general system derived by Zakharov and Rubenchik [80] to describe the interaction of spectrally narrow high-frequency wave packets of small amplitude with low-frequency acoustic type oscillations. The unknown $\psi(t,x)$, $\rho(t,x)$ and $\eta(t,x)$ stand for:

- $\psi:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$, the amplitude of the carrying (high frequency) waves,
- $\rho,\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, low-frequency oscillations.

This system was also independently found by Benney and Roskes [5] in the context of gravity waves. System (1.13) has also been derived in several other physical situations, such as for example, in the study of Alfvén waves (transverse oscillations of the magnetic fields) in the Magneto-Hydrodynamics equations (see for instance [9, 61]).

A characteristic that makes this model so rich is the many limiting cases it contains. For instance, in the supersonic limit, the classical (scalar) Zakharov system (1.8) is recovered. In the subsonic limit it is possible to obtain (formally) the Davey-Stewartson, a generalization in dimensions d $= 2, 3$ of Schrödinger equations, but the rigorous proof remains still open. In the one dimensional case, we can also consider the adiabiatic limit, that is, to take $\theta \to 0$ in (1.13), from where we can formally see that $\rho(t,x)$ and $\eta(t,x)$ satisfy now the following relations

$$\rho = -\frac{\gamma\alpha}{2(\beta-\alpha^2)}|\psi|^2, \quad \eta = -\gamma\frac{\beta-\alpha^2/2}{\beta-\alpha^2}|\psi|^2.$$

Then, we infer that the complex amplitude ψ solves the cubic nonlinear Schrödinger equation

$$\mathrm{i}\partial_t\psi + \omega\partial_x^2\psi = -\frac{\gamma\alpha}{3(\beta-\alpha^2)}|\psi|^2\psi.$$

A rigorous justification of such limit was introduced by Oliveira in [54].

On the other hand, the ZR/BR system (1.13) is Hamiltonian and preserves the Mass, Energy and Momentum:

$$\begin{aligned}
M\big(\psi(t),\rho(t),\eta(t)\big) &:= \int_{\mathbb{R}} |\psi(t,x)|^2 \mathrm{d}x = M(\psi_0,\rho_0,\eta_0),\\
E\big(\psi(t),\rho(t),\eta(t)\big) &:= \int_{\mathbb{R}} \Big(\omega|\psi_x|^2 + \tfrac{\gamma q}{2}|\psi|^4 + \tfrac{\beta}{2}\rho^2 + \tfrac{1}{2}\eta^2 + \tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2 - \alpha\rho\eta\Big)\mathrm{d}x\\
&= E(\psi_0,\rho_0,\eta_0),\\
P\big(\psi(t),\rho(t),\eta(t)\big) &:= \mathrm{Im}\int_{\mathbb{R}} \psi\overline{\psi}_x - \theta\int_{\mathbb{R}} \rho(t,x)\eta(t,x)\mathrm{d}x = P(\psi_0,\rho_0,\eta_0).x
\end{aligned}$$

Similar to the Zakharov systems, introduced in the previous subsection, even though the energy could very well be negative, one can use it along with the mass to find a uniform bound on the energy norm. Indeed, thanks to Gagliardo-Nirenberg inequality, one finds that

$$\|\psi(t)\|_{H^1}^2 + \|\rho(t)\|_{L^2}^2 + \|\eta(t)\|_{L^2}^2 \leqslant C, \qquad \forall t \in \mathbb{R}.$$

The precise statement (and proof) can be found in Chapter 4, Lemma 4.5.

System (1.13) is globally well-posed for the one-dimensional case in the energy space $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$. The first approach in this regard is found in [53], where Oliveira proved local and global well-posedness in the space $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$, later improved by Linares and Matheus [33] to well-posedness for initial data in the energy space $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

Existence of solitary waves:

Regarding the existence of solitary waves, in the case $\beta - \alpha^2 > 0$, $\gamma > 0$ and $\theta < 1$, Oliveira has proved in [53] the existence and the orbital stability of solitary waves of the form

$$\big(\psi,\rho,\eta\big)(t,x) := \big(\mathrm{e}^{\mathrm{i}\lambda t}\mathrm{e}^{\mathrm{i}cx/2\omega}R(x-ct), a(c)|R(x-ct)|^2, b(c)|R(x-ct)|^2\big), \tag{1.14}$$

where $\lambda \in \mathbb{R}$, $c \geqslant 0$ and $R(\cdot)$ is an positive, even and exponentially decaying complex-valued function, while $a(c)$ and $b(c)$ are given by the following formulas

$$a(c) := -\frac{\gamma(\beta - \frac{\alpha}{2}(c\theta + \alpha))}{\beta - (c\theta + \alpha)^2}, \quad b(c) := -\frac{\gamma(c\theta + \frac{1}{2}\alpha)}{\beta - (c\theta + \alpha)^2}.$$

under the conditions

$$a(c) - \frac{\alpha}{2}b(c) + q < 0 \quad \text{and} \quad \frac{c^2}{4\omega} - \lambda < 0. \tag{1.15}$$

In particular, we point out that condition (1.15) implies the existence of standing solitary waves only for $\alpha \neq 0$. In other words, for the specific case $\alpha = 0$, it is not necessary to avoid standing waves in decay results, since they are not expected to happen.

1.2.5 The virial method

Virial, a word derived from the latin *vis* meaning "force" or "energy", was first used in the context of physics by Clausius in the so called Virial Theorem. The original statement of Clausius' theorem reads: "The mean vis viva of the system is equal to its virial", which, in other words, means that the average kinetic energy is equal to $\frac{1}{2}$ the average potential energy.

In mathematics, virial identities are related to conservation laws. The idea is to study the properties of a conserved quantity, perturbed with an appropriate weight. In dispersive equations, the virial method was introduced by Glassey [25] in 1977, as he presented a blow-up result for the focusing NLS equation. Although the result holds for a more general nonlinearity, let us explain the idea with the semilinear Schrödinger equation. Taking advantage of the conserved momentum (1.6), Glassey considered a weight $|x| = r$ and computed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Im}\int_{\mathbb{R}^{\mathrm{d}}} r\overline{u}u_r \mathrm{d}x\right) = \left[\mathrm{d}\left(\frac{p+1}{2} - 1\right) - 2\right]\int_{\mathbb{R}^{\mathrm{d}}} |\nabla u|^2 \mathrm{d}x - \mathrm{d}\left(\frac{p+1}{2} - 1\right) E_0,$$

where E_0 stands for the conserved energy, defined in (1.5), and d is the space dimension. The $\|\cdot\|_{H^1}$ norm is bounded by below by $\mathrm{Im}\int_{\mathbb{R}^{\mathrm{d}}} r\overline{u}u_r \mathrm{d}x$, which turns out to be increasing assuming $E_0 < 0$ and $p > 1 + \frac{4}{\mathrm{d}}$. Therefore, as a consequence, solutions with initial data such that $E_0 < 0$ blow up. Needless to say, the monotonicity of the modified momentum plays a crucial role in the proof.

Since then, virial methods have been generalized to meet different needs and adapt to various equations. They are used not only to prove blow-up, but also stability or inestability results, and even scattering, almost every time taking advantage of monotonic quantities. In a few words, the idea is the following: if you intend on proving blow-up, you want the norm to be larger than something increasing, whereas if what you need is decay, than you would look for the norm to be smaller than something decreasing. In this thesis, we use virial methods to prove decay properties for the Scrödinger models previously mentioned.

We proceed to give an idea of the method. To that end, let us consider again our typical example, the NLS equation. For an appropriate (bounded) weight $\omega \in C^\infty$ and making use of the conserved momentum, we prove that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Im}\int_{\mathbb{R}} \omega u \overline{u}_x \mathrm{d}x\right) \geqslant 0.$$

Then, integrating in time, we can deduce decay properties for the H^1-norm localized in the support of the weight, under certain conditions on the solution (mainly, small initial data for the focusing case). More precisely,

$$\int_0^{\infty} \|u(t)\|^2_{H^1_\omega} \mathrm{d}t \leqslant C.$$

Consequently, the results depend on the conserved quantity considered and on the weight used, since it defines the region in which the decay property holds. In Subsection 1.4, we present the main theorems proved in this thesis.

1.3 Zakharov Water Waves

1.3.1 Derivation of the model

Zakharov water waves arises as a free surface model for an irrotational and incompressible fluid under the influence of gravity. Such fluid is considered in a domain with rigid bottom:

$$\Omega_t = \left\{(x,z) \in \mathbb{R}^2 \text{ such that } -ha_\varepsilon(x) \leqslant z \leqslant \eta(t,x)\right\},$$

where $h > 0$, $a_\varepsilon : \mathbb{R} \to \mathbb{R}$ (to be properly defined later) and $\eta : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is the (unknown) free surface elevation.

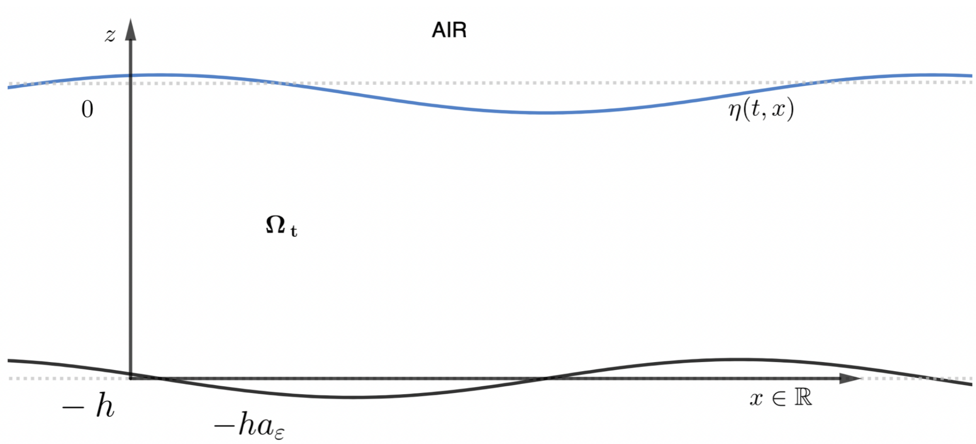


Figure 1.1: Description of the domain.

The following assumptions are made on the fluid and on the flow:
(H1) The fluid is homogeneous and inviscid.

(H2) The fluid is incompressible.
(H3) The flow is irrotational.
(H4) The surface and the bottom can be parametrized as graphs above the still water level.
(H5) The fluid particles do not cross the bottom.
(H6) The fluid particles do not cross the surface.
(H7) There is surface tension.
(H8) The fluid is at rest at infinity.
(H9) The water depth is always bounded from below by a nonnegative constant.

We denote $\mathbf{u}$ the **velocity of the fluid**. By (H3), there exists a scalar function Φ such that inside the fluid domain Ω_t,

$$\mathbf{u} = (\partial_x \Phi, \partial_z \Phi) = \nabla_{x,z}\Phi \quad \text{ in } \Omega.$$

Assumptions (H1) – (H2) imply that the velocity of the fluid $\mathbf{u}$ follows the the *free surface Euler equation*, which, interms of the velocity potential Φ, turn into the *free surface Bernoulli equations*:

$$\begin{aligned} &\partial_t \Phi + \frac{1}{2}|\nabla_{x,z}\Phi|^2 + gz = -\frac{1}{\rho}(P - P_{atm}) && \text{in } \Omega_t. \\ &\Delta_{x,z}\Phi = 0 \qquad (\nabla \cdot u = 0) && \text{in } \Omega_t. \end{aligned}$$

where $P(t,x,z)$ is the pressure at time t at the point (x,z) and P_{atm} is the constant atmospheric pressure. If $\partial_{\mathbf{n}}$ is the upwards normal derivative, the fact that the fluid does not cross the bottom nor the surface ((H5) – (H6)) imply the following boundary conditions

$$\begin{aligned} &\partial_{\mathbf{n}}\Phi = 0, && \text{on } \{z = -ha_\varepsilon\}. \\ &\partial_t \eta + \sqrt{1 + |\partial_x \eta|^2}\ \partial_{\mathbf{n}}\Phi = 0 && \text{on } \{z = \eta(t,x)\}. \end{aligned} \tag{1.16}$$

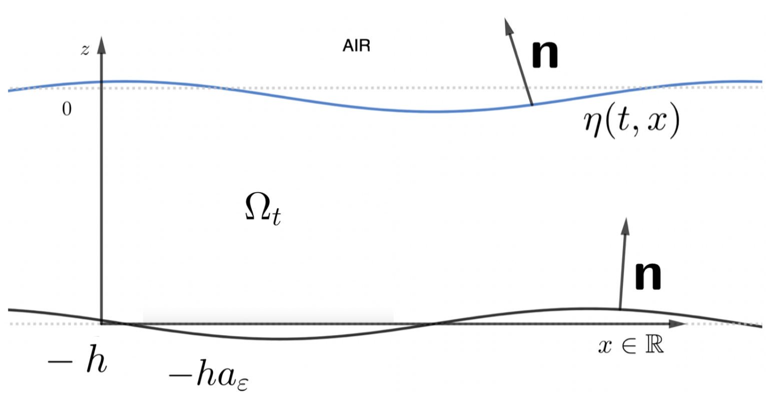


Figure 1.2: The unitary normal vector pointing upwards.

Finally, since we are considering surface tension, over the surface we obtain the condition:

$$\frac{1}{\rho}(P - P_{atm}) = -\beta \nabla \cdot \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) \qquad \text{on } \{z = \eta(t,x)\},$$

where β is the surface tension coefficient.

As noted by Zakharov, if Φ denotes the velocity potential, then η along with the trace of the velocity potential at the surface $\varphi = \Phi|_{z=\eta}$ fully determines the flow. Later, Craig, Sulem and Sulem [16] introduced a new formulation invoking the Dirichlet-Neumann operator

$$G[\eta, a] : \varphi \mapsto \sqrt{1 + |\nabla\eta|^2}\, \partial_{\mathbf{n}}\Phi\big|_{z=\eta}.$$

Taking into account the pressure over the surface, the system that models the fluid reads

$$\begin{cases} \partial_t \eta = G[\eta, a]\varphi \\ \partial_t\varphi = -\frac{1}{2}\left|\partial_x\varphi\right|^2 + \frac{1}{2}\dfrac{\left(G[\eta, a]\varphi + \partial_x\varphi\cdot\partial_x\eta\right)^2}{1 + \left|\partial_x\eta\right|^2} - g\eta + \beta\partial_x\cdot\left(\dfrac{\partial_x\eta}{\sqrt{1+\left|\partial_x\eta\right|^2}}\right) \end{cases} \tag{1.17}$$

where g is the gravitational constant and β is the tension surface coefficient. The velocity potential can be recovered as the solution to the elliptic equation

$$\begin{cases} \Delta_{X,z}\Phi = 0 & (t, X, z) \in [0,\infty) \times \Omega_t \\ \Phi\big|_{z=\eta} = \varphi & \\ \partial_n\Phi\big|_{z=-H+a} = 0. & \end{cases} \tag{1.18}$$

System (1.17) has a Hamiltonian structure [78] in the variable (η, φ). Indeed, we can re-write (1.17) as

$$\partial_t \begin{pmatrix} \eta \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \partial_\eta \mathcal{H} \\ \partial_\varphi \mathcal{H} \end{pmatrix}$$

where the Hamiltonian $\mathcal{H}$ is the total energy given by

$$\mathcal{H}(\eta, \varphi) = \frac{1}{2}\int_{\mathbb{R}^2} \varphi G[\eta, a]\varphi + g\eta^2 + 2\beta\left(\sqrt{1 + |\nabla\eta|^2} - 1\right) \mathrm{d}x\mathrm{d}z. \tag{1.19}$$

Regarding the well-posedness for the Zakharov water waves problem in the presence of surface tension, the 3-dimensional Zakharov water-waves problem is globally well-posed for small initial data under rather restrictive smothness conditions, as studied by Germain, Masmoudi and Shatah [19, 20]. For the 2-dimensional problem, local well-posedness for the Cauchy problem in the space $H^{s+1/2} \times H^s$, $s > 5/2$ is proved by Alazard, Burq and Zuily [1].

Existence and long-time behavior of solitary waves:

The study of solitary waves for equation (1.17) was mainly devoted to the flat-bottom case ($a_\varepsilon = 1$). Indeed, existence of solitary waves [3] of speed $c \sim \sqrt{gh}$ was shown when the parameters g, β and h satisfy the condition

$$\frac{gh}{c^2} = 1 + \lambda^2, \quad \frac{\beta}{hc^2} > \frac{1}{\lambda},$$

for $\varepsilon > 0$ sufficiently small. These travelling waves are solutions Q_c of the form

$$Q_c(x - ct) = (\eta_c(x - ct), \varphi_c(x - ct)) = \left(h\eta_\lambda(h^{-1}(x - ct)), ch\varphi_\lambda(h^{-1}(x - ct))\right)$$

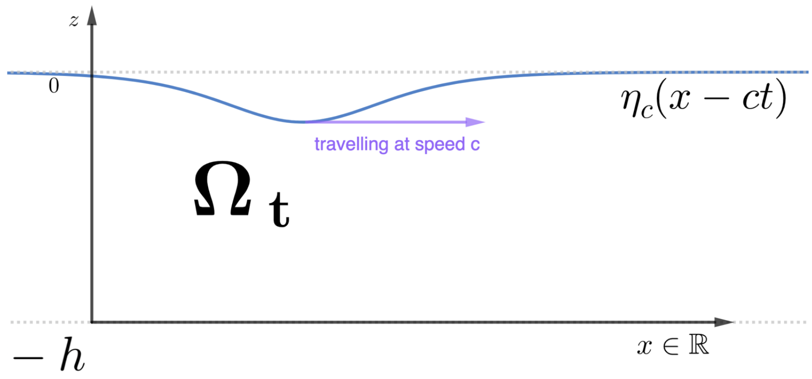


Figure 1.3: AK Solitary wave.

with

$$\eta_\lambda(x) = \lambda^2 \Theta_1(\lambda x, \lambda) \quad \varphi_\lambda(x) = \lambda \Theta_2(\lambda x, \lambda)$$

These profiles Θ_1 and Θ_2 satisfy an exponential decay. As noted in [3, 65], the profiles $\Theta_1(x,\lambda)$ and $\Theta_2(x,\lambda)$ have smooth expansions in λ. Then, we are entitled to study the particular case $\lambda = 0$, for which we get

$$\Theta_1(x,0) = \cosh^{-2}\left(\frac{x}{2(\beta/(hc^2) - 1/3)^{1/2}}\right).$$

Thus, the KdV solitary wave is recovered.

A rather characteristic property of the one-dimensional Zakharov water waves model is the fact that the surface of the fluid is invariant by translation. As a result, usual Lyapounov stability cannot be expected. Nevertheless, orbital stability of the solitary waves holds, as was proven by Mielke [44]. On the other hand, when considering the 2-dimensional case, the situation changes, and solitons turn out to be unstable under transverse perturbations [65]. There also exist multi-solitons solutions, that is solutions that are time asymptotic to a sum of decoupling solitary waves, as constructed by Ming, Rousset and Tzvetkov in [45].

1.4 Main results

The results proved in this thesis are part of the following articles:

1. M. E. Martínez, *Decay of small odd solutions for long range Schrödinger and Hartree equations in one dimension*, published in Nonlinearity, 2020. (Chapter 2).

2. M. E. Martínez, *On the decay problem for the Zakharov and Klein–Gordon–Zakharov systems in one dimension*, published in Journal of Evolutions Equations, 2021. (Chapter 3).

3. M. E. Martínez and J. M. Palacios, *On long-time behavior of solutions of the Zakharov-Rubenchik/Benney-Roskes system* , accepted in Nonlinearity, available in arXiv, 2021. (Chapter 4).

4. M. E. Martínez, *Existence of solitary waves in the Water Waves Zakharov system with slowly varying bottom*, preprint 2021. (Chapter 5).

Now we briefly describe these results.

1.4.1 Asymptotic dynamics of small solutions for NLS and Hartree equations

Local decay results for NLS:

A great deal of literature proves that for the subcritical (in the sense of GWP and scattering) semilinear NLS equation ($3 < p < 5$), scattering to a free solution exists (see, for instance, Ginibre and Velo [24], Tsutsumi [73] and Nakanishi-Ozawa [50], to name a few). Nevertheless, Strauss [69] and Barab [4] showed that one cannot expect the same scattering for the critical ($p = 3$) and super critical case ($p < 3$). Instead, modified scattering is believed to occur.

The first results on *modified scattering* for d dimensional NLS under small initial conditions, were introduced by Ozawa [55] and by Ginibre and Ozawa [22]. Moreover, it was shown that solutions u of such equations present the decay

$$\|u(t)\|_{L^\infty} \lesssim (1+|t|)^{-\mathrm{d}/2}, \tag{1.20}$$

when the initial data is sufficiently small in weighted Sobolev spaces (see also Hayashi-Naumkin [18], and Kato-Pusateri [30], for instance).

Modified scattering also holds for the NLS case with potential V, although it seems necessary to assume spectral conditions on the functional $-\frac{1}{2}\Delta + V$, see [17, 51, 21].

In this thesis, we focus on the decay problem for the super-critical case ($p < 3$), with and without potential (although the result is for a more general nonlinearity, including the critical $p = 1$ and subcritical $3 < p < 5$ cases). We present the first result, that deals with decay for the Schrödinger equation (1.3) without potential in compact intervals:

Theorem 1.2 (Theorem (2.1), Chapter 2)**.** *Suppose $u(t) \in H^1(\mathbb{R})$ is a global odd solution of the equation* (1.3) *for $1 < p < 5$ and $\mu = 0$ such that, for some $\varepsilon > 0$ small,*

$$\|u(t=0)\|_{H^1(\mathbb{R})} \leqslant \varepsilon. \tag{1.21}$$

Then,

$$\lim_{t\to\infty}\Big(\|u(t)\|_{L^2(I)} + \|u(t)\|_{L^\infty(I)}\Big) = 0, \tag{1.22}$$

for any bounded interval $I \subset \mathbb{R}$. Moreover, if the equation is defocusing, the smallness condition (1.21) *is not needed.*

Similarly, the following theorem deals with the decay in compact intervals for NLS with potential:

Theorem 1.3 (Theorem 2.3, Chapter 2)**.** *Assume $V \neq 0$ even as in* (1.3). *Under the assumptions of Theorem 2.1, suppose additionally that V satisfies*

$$\int_{\mathbb{R}} (|V(x)| + |V'(x)|) \cosh(2x) \mathrm{d}x < +\infty. \tag{1.23}$$

Then there exists $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$, (1.22) *holds for any bounded interval $I \subset \mathbb{R}$.*

The proofs of Theorems 2.1 and 2.3 combine the use of a virial identity, derived from the conserved momentum, and spectral characteristics of the Schrödinger operator. Indeed, spectral properties are used to justify the monotonicity of for the adapted momentum, as commented in Subsection 1.2.5. To do so, the oddnes condition comes into play to rule out solitary waves and breathers. On the other hand, the fact that we are considering fixed intervals, allows us to forget about not only travelling waves, but also solutions that can be written as a sum of solitary waves with different speeds (and sufficiently away from each other) plus radiation. This is due to the fact that, since such solitons are moving at a speed different to zero, then for any given fixed interval, one can wait sufficiently long and, eventually, all solitons move away from our space interval.

Local decay for Hartree:

Similar to the critical and super-critical NLS equation ($1 < p \leqslant 3$), scattering to a free solution does not exist for Hartree equation, as was stated recently by Murphy and Nakanishi [49].

Nevertheless, modified scattering holds for Hartree equation for dimension $\mathrm{d} \geqslant 2$ and the solution presents the expected dispersive estimation (1.20). The first result in this regard shows decay for Hartree equation with Coulomb potential

$$g(u) = \sigma \left(|x|^{-1} * |u|^2\right) u, \quad \mathrm{d} \geqslant 2,$$

and small initial condition [55, 22]. See also [18] or [30] . We point out that,as general as the results mentioned above are, they do deal with the decay problem for the one-dimensional case.

It is our goal to prove decay properties for the Hartree equation (both focusing and defocusing) in the one-dimensional case. When the dimension is larger than two ($\mathrm{d} \geqslant 2$), there seems to be an extensive study of the long-time behavior of solutions. Nevertheless, it has come to our attention that the case $\mathrm{d} = 1$ is still fairly open. A first step in that direction is the following result:

Theorem 1.4 (Theorem 2.4, Chapter 2. Defocusing Hartree equation)**.** *Suppose that $u \in H^1(\mathbb{R})$ is a global odd solution of equation* (2.1) *with* (2.5) *and $\sigma = 1$. Then*

$$\lim_{t\to\infty} (\|u(t)\|_{L^2(I)} + \|u(t)\|_{L^\infty(I)}) = 0, \tag{1.24}$$

for any bounded interval $I \subset \mathbb{R}$.

The arguments that we used in Chapter 2 are based on the previous work of [31] and [32], where the decay problem for the Klein-Gordon equation is considered. Because of the the dynamics of the Schrödinger model, in the virial method some uncontrollable H^2 terms arise, which prevents us from proving decay in the energy norm.

1.4.2 Zakharov systems

Decay for Zakharov system:

It is known that for dimension d = 2, 3, there exist solutions to the Zakharov equation (1.8) that decay to zero in the energy space $H^1 \times L^2 \times \hat{H}^{-1}$. Indeed, Ozawa and Tsutsumi [60], Ginibre and Velo [23], and Shimomura [68], proved existence and uniqueness of asymptotically free solutions of (1.8). Scattering theory for Zakharov system as we know it is presented by Guo and Nakanishi [27], where they prove that all (energy) small, radially symmetric solutions for the Zakharov system in d = 3 scatter. By using generalized Strichartz estimates for the Schrödinger equation, Guo, Lee, Nakanishi and Wang in [26] were able to improve [27] by showing scattering of small solutions without the radial assumption.

In Chapter 3, we will be interested in the decay problem for the one dimensional Zakharov system. We study the decay in very specific regions: compact intervals around curves that exist outside the light cone. We present the first result, dealing with decay in compact intervals:

Theorem 1.5 (Theorem 3.1). *Assume $E_s < \infty$. Let $(u, n, v) \in C(\mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ be a solution of* (3.4) *such that u is odd and satisfies, for some $\varepsilon > 0$ small,*

$$\sup_{t \geqslant 0} \|u(t)\|_{H^1(\mathbb{R})} < \varepsilon.$$

Then, for every compact interval $I \subset \mathbb{R}$,

$$\lim_{t\to\infty} \|u(t)\|_{L^\infty(I)} + \|u(t)\|_{L^2(I)} + \|n(t)\|_{L^2(I)} + \|D^{-1}n_t(t)\|_{L^2(I)} = 0. \tag{1.25}$$

The proof of Theorem 3.1 is based on the Schrödinger case, and follows the idea of Theorem 2.1. In the same spirit, we construct a virial identity from the conserved momentum for the Zakharov system.

Our second result deals with decay in far field regions along curves.

Theorem 1.6 (Theorem 3.2, Chapter 3). *Assume (u, n) is a global solution of* (1.8)*. Then, for any pair of constants c_1, c_2, the following holds:*

1. *If $(u, n, v) \in C(\mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}))$, then, for any $\mu \in C^1(\mathbb{R})$ satisfying $\mu(t) \gtrsim t\log(t)^{1+\delta}$, $\delta > 0$, and setting $\Omega_\mu(t) := \{x \in \mathbb{R} : c_1\mu(t) \leqslant |x| \leqslant c_2\mu(t)\}$, the following limit holds*
$$\lim_{t\to\infty} \|u(t)\|_{L^2(\Omega_\mu(t))} = 0. \tag{1.26}$$

2. *If* $(u, n, D^{-1}n_t) \in C\left(\mathbb{R}^+, H^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})\right)$ *and there exists* $f(t) \in C^1(\mathbb{R})$ *a non-decreasing function such that*
$$\|u(t)\|_{H^2(\mathbb{R})} \lesssim f(t), \tag{1.27}$$
then, for any $\mu \in C^1(\mathbb{R})$ *satisfying* $\mu(t) \gtrsim t\log(t)^{1+\delta}f(t)$, $\delta > 0$,
$$\lim_{t\to\infty} \|u(t)\|_{H^1(\Omega_\mu(t))} + \|n(t)\|_{L^2(\Omega_\mu(t))} + \|D^{-1}n_t(t)\|_{L^2(\Omega_\mu(t))} = 0. \tag{1.28}$$

The proof of Theorem 3.2 follows an argument introduced by Muñoz, Ponce and Saut in [48], where they deal with the long time behaviour of intermediate long wave equation. Hypothesis (1.27) it is not due to the method, but on the model itself. It works mainly controlling H^2-terms that appear in the dynamics of the H^1-norm of u. However, it allows us to obtain decay of the $\|\cdot\|_{H^1}$-norm, which was not present in results established in [48] nor in Theorem 2.1 for NLS.

An intuitive sketch of the region where the decay is described can be seen in Figure 1.4.

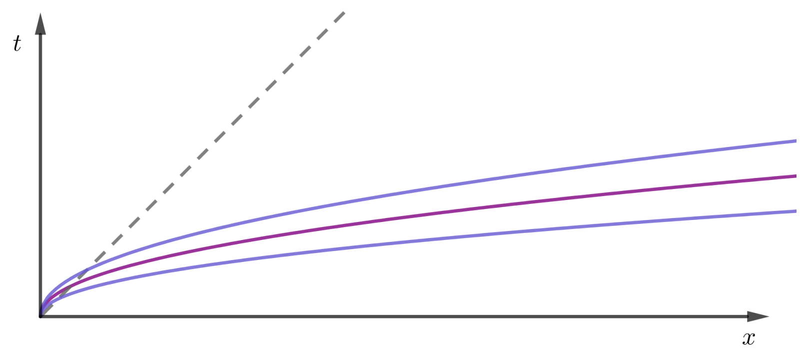


Figure 1.4: Sketch of Ω_μ

Decay for Zakharov and KGZ.

Following the idea in [27], Guo, Nakanishi and Wang [28] proved scattering in the energy space for radially symmetric solutions with small energy for the system (1.9) in three dimensions, as well. In [29], they continue the study of global dynamics of radial solutions in three dimensions and find a dichotomy between scattering and blow-up. More specifically, relying on virial identities, they show that if the initial data is radially symmetric and its energy is below the energy of the ground state then the solution to (3.2) can either (for both i $= 1, 2$): scatter or blow up in finite time.

As we did for (1.8), we prove decay of solutions to (1.9) in two different ways: over compact intervals of time and over far field regions along curves. Our result for compact intervals is the following:

Theorem 1.7 (Theorem 3.4, Chapter 3)**.** *Let*
$$(u, u_t, n, D^{-1}n_t) \in C\left(\mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})\right)$$

be a solution of (1.9) *such that* u *is odd and satisfies*

$$\sup_{t\geqslant 0}\|u(t)\|_{H^1(\mathbb{R})}\leqslant\varepsilon \quad \textit{and} \quad \sup_{t\geqslant 0}\|u_t(t)\|_{L^2(\mathbb{R})}\leqslant C \tag{1.29}$$

for some $C>0$ *and* $\varepsilon>0$ *small. Then, for every compact interval* $I\subset\mathbb{R}$,

$$\lim_{t\to\infty}\|u(t)\|_{H^1(I)}+\|u_t(t)\|_{L^2(I)}+\|n(t)\|_{L^2(I)}+\|D^{-1}n_t(t)\|_{L^2(I)}=0. \tag{1.30}$$

The last theorem is devoted to the decay of the solutions to (3.15) in regions along curves outside the light cone:

Theorem 1.8 (Theorem 3.5, Chapter 3)**.** *Let*

$$(u,u_t,D^{-1}n_t,v)\in C\left(\mathbb{R}^+,H^1(\mathbb{R})\times L^2(\mathbb{R})\times L^2(\mathbb{R})\times L^2(\mathbb{R})\right)$$

be a global solution to (1.9) *such that*

$$\sup_{t\geqslant 0}\|u(t)\|_{H^1(\mathbb{R})}\leqslant\varepsilon \tag{1.31}$$

for some $0<\varepsilon\leqslant 1$. *Then, for any pair of constants* c_1,c_2*, and* $\mu\in C^1(\mathbb{R})$ *satisfying* $\mu(t)\gtrsim t\log(t)^{1+\delta}$, $\delta>0$, *and setting and setting* $\Omega_\mu(t):=\{x\in\mathbb{R}:\ c_1\mu(t)\leqslant|x|\leqslant c_2\mu(t)\}$, *th efollowing limit holds,*

$$\lim_{t\to\infty}\|u(t)\|_{H^1(\Omega_\mu(t))}+\|u_t(t)\|_{L^2\Omega_\mu(t)}+\|n(t)\|_{L^2(\Omega_\mu(t))}+\|D^{-1}n_t(t)\|_{L^2(\Omega_\mu(t))}=0. \tag{1.32}$$

Notice that for the KGZ system, no hypothesis regarding a controlled growth of the H^2-norm was needed.

1.4.3 Zakharov-Rubenchik/Benney-Roskes

In Chapter 4, we deal with the decay problem for the Zakharov-Rubenchik/Benney-Roskes (ZR/BR) system (1.13). The main idea that inspired the first result is the fact that one can recover transport equations from the acoustic type functions η,ρ. Indeed notice that we can make the change of variables

$$\mu_1=\sqrt{\beta}\rho+\eta,\quad \mu_2=-\sqrt{\beta}\rho+\eta.$$

Then, if (ψ,η,ρ) a solution to (1.13), we get that μ_1 and μ_2 solve the **transport equations**

$$\partial_t\mu_1+(\sqrt{\beta}-\alpha)\partial_x\mu_1=\gamma\left(-\sqrt{\beta}+\frac{\alpha}{2}\right)\partial_x(|\psi|^2),$$
$$\partial_t\mu_2-(\sqrt{\beta}+\alpha)\partial_x\mu_1=\gamma\left(\sqrt{\beta}+\frac{\alpha}{2}\right)\partial_x(|\psi|^2).$$

Take, for instance, μ_1. Thus, *formally*, integrating by parts

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\mu_1(t,x+(\sqrt{\beta}-\alpha)t)\mathrm{d}x=\gamma\left(-\sqrt{\beta}+\frac{\alpha}{2}\right)\int_{\mathbb{R}}\partial_x(|\psi|^2)(t,x+(\sqrt{\beta}-\alpha)t)\mathrm{d}x=0.$$

Although this quantity might not be well-defined for solutions in the energy space, this "conservation law" inspired the following result:

Theorem 1.9 (Theorem 4.1, Chapter 4). *Let $\upsilon_\pm := \pm\theta^{-1}\big(\sqrt{\beta}\pm\alpha\big)$ fixed. Consider $(\psi,\rho,\eta)\in C(\mathbb{R},H^1\times L^2\times L^2)$ to be any solution to system (4.1) emanating from an initial data $(\psi_0,\rho_0,\eta_0)\in H^1\times L^2\times L^2$. Then, for any $c\in\mathbb{R}_+$, the following inequality holds*

$$\int_0^{+\infty}\frac{1}{\mu_*(t)}\int_{\Omega_\pm(t)}|\psi(t,x)|^2\mathrm{d}x\mathrm{d}t<+\infty,$$

where $\Omega_\pm(t):=\{x\in\mathbb{R}:\,-c\lambda(t)\leqslant x-\upsilon_\pm t\leqslant c\lambda(t)\}$, $\kappa:=10^{100}$ and

$$\lambda(t):=t^{2/3}\log\log^{-2/3}(\kappa+t)\quad\textit{and}\quad\mu_*(t):=t\log(\kappa+t)\log\log(\kappa+t).$$

Furthermore, we have the following scenarios:

1. *If $\pm\alpha<0$, then, the following inequality holds*

$$\int_0^{+\infty}\frac{1}{\mu_*(t)}\int_{\Omega_\pm(t)}\big(|\psi_x(t,x)|^2+|\psi(t,x)|^2+\rho^2(t,x)+\eta^2(t,x)\big)\mathrm{d}x\mathrm{d}t<+\infty.$$

   In particular, we have that

$$\liminf_{t\to+\infty}\int_{\Omega_\pm(t)}\big(|\psi_x(t,x)|^2+|\psi(t,x)|^2+\rho^2(t,x)+\eta^2(t,x)\big)\mathrm{d}x=0.$$

2. *If $\alpha=0$, then, the following inequality holds*

$$\int_0^{+\infty}\frac{1}{\mu_*(t)}\int_{\Omega_0(t)}\Big(|\psi_x(t,x)|^2+|\psi(t,x)|^4+\eta^2(t,x)+\rho^2(t,x)\Big)\mathrm{d}x\mathrm{d}t<+\infty,$$

   where λ and μ_* defined as above and $\Omega_0(t):=\{x\in\mathbb{R}:\,c\lambda(t)\leqslant|x|\leqslant C\lambda(t)\}$. In particular, the following is satisfied

$$\liminf_{t\to+\infty}\int_{\Omega_0(t)}\Big(|\psi_x(t,x)|^2+|\psi(t,x)|^4+\eta^2(t,x)+\rho^2(t,x)\Big)\mathrm{d}x=0.$$

Remark 1.1. In the previous statement, the condition $\pm\alpha<0$ must be understood according to the sets $\Omega_\pm$. In other words, if $+\alpha<0$, then both results for Ω_+ hold, while if $-\alpha<0$, then both results for Ω_- hold. Notice that if $\alpha<0$, the result for Ω_- is not necessarily true.

This result follows the idea from the decay results in compact intervals for NLS, Hartree and Zakharov systems, but differs in the sense that it does not involve spectral properties. In addition, we point out the lack of parity conditions, caused by the fact that parity is not conserved by the flow. Also, we were able to approach compacts sets centered at the origin because from (1.15) we already know that standing waves do not occur if $\alpha=0$

Our second main result states that, in the so-called far-field region, solutions (in the energy space) must decay to zero.

Theorem 1.10 (Theoreom 4.2, Chapter 4). *Let $(\psi,\rho,\eta)\in C(\mathbb{R},H^1\times L^2\times L^2)$ be any solution to system (4.1) emanating from an initial data $(\psi_0,\rho_0,\eta_0)\in H^1\times L^2\times L^2$. Then, for any pair of constants $c_1,c_2>0$ the following properties holds:*

1. *Consider any non-negative function $\zeta \in C^1(\mathbb{R})$ satisfying that, there exists $\delta > 0$ such that, for all $t > 0$ it holds*

$$\zeta(t) \gtrsim t\log(\kappa + t)^{1+\delta} \quad \text{and} \quad \zeta'(t) \gtrsim \log(\kappa + t)^{\delta+1}.$$

Then, setting $\Omega_\zeta(t) := \{x \in \mathbb{R} : c_1\zeta(t) \leqslant |x| \leqslant c_2\zeta(t)\}$, the following limit holds

$$\lim_{t\to\infty} \|\psi(t)\|_{L^2(\Omega_\zeta(t))} = 0. \tag{1.33}$$

2. *Assume additionally that $(\psi, \rho, \eta) \in C(\mathbb{R}, H^2 \times H^1 \times H^1)$ is a solution emanating from an initial data $(\psi_0, \rho_0, \eta_0) \in H^2 \times H^1 \times H^1$. Then, for any non-negative $\zeta \in C^1(\mathbb{R})$ satisfying that, there exists $\delta > 0$ such that, for all $t > 0$,*

$$\zeta(t) \gtrsim t^{2+\delta} \quad \text{and} \quad \zeta'(t) \gtrsim t^{1+\delta},$$

the following decay for the local energy norm holds

$$\lim_{t\to\infty} \left(\|\psi(t)\|_{H^1(\Omega_\zeta(t))} + \|\rho(t)\|_{L^2(\Omega_\zeta(t))} + \|\eta(t)\|_{L^2(\Omega_\zeta(t))}\right) = 0.$$

The condition 1.33 is expected to happen at least for smooth initial data. Indeed, a polinomial bound for the growth of the H^s-norm of ψ was stated in [33]. More specifically, they proved that, for smooth initial data, solutions to system (4.1) satisfies the following property:

$$\|\psi\|_{H^s(\mathbb{R})} \lesssim 1 + |t|^{(s-1)^+}.$$

1.4.4 The solitary wave for Zakharov water waves

Chapther 5 is devoted to the study of the solitary wave for the water waves Zakharov system (1.17). Specifically, we analyse the behavior of the solitary wave solution of the flat-bottom problem (that is, when $a_\varepsilon = 0$), given by Amick and Kirchgässner (AK) [3] when the bottom actually presents a (slight) change at some point. We consider a *a slightly changing bottom* described by $a_\varepsilon = a(\varepsilon\cdot) \in C_b^2(\mathbb{R})$, where $\varepsilon > 0$ and a is assumed to satisfy the following conditions:

$$\begin{gathered}\text{There exist } K > 0,\ 0 < \kappa < 1 \text{ and } \gamma_1, \gamma_2 > 0 \text{ such that:}\\ 1 - \kappa < a(r) < 1, \quad \forall r \in \mathbb{R},\\ 1 - a(r) \leqslant K\mathrm{e}^{\gamma_2 r}, \quad \forall r \leqslant 0,\\ \lim_{r\to-\infty} a(r) = 1,\ \lim_{r\to\infty} a(r) = 1 - \kappa,\\ |a'(r)| < K\mathrm{e}^{-\gamma_1|r|}, \quad \forall r \in \mathbb{R},\\ a' \text{ does not change sign.}\end{gathered} \tag{1.34}$$

However simple the sketch of the approach might be (Figure 1.5), in reality the situation is fairly different. The description of the bottom has a (nonlinear) non-local interaction

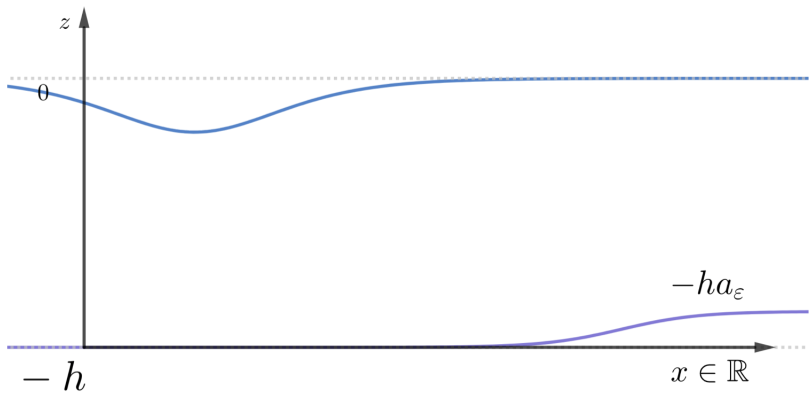


Figure 1.5: A solitary wave in nonflat bottom.

with the flow (see the boundary condition on the elliptic equation (1.16)). Consequently, one cannot assume the existence of the AK solitary wave. Instead, our goal is to prove the existence of a solution

$$\begin{pmatrix} \eta \\ \varphi \end{pmatrix} \to \begin{pmatrix} \eta_c \\ \varphi_c \end{pmatrix} (x + A - ct) \quad \text{as} \quad t \to -\infty. \tag{1.35}$$

where $A \in \mathbb{R}$ is a large (safe) distance away from the changing point $A \gg 1$, and $Q_c = \big(\eta_c, \varphi_c\big)$ is a solitary wave of the flat-bottom problem (the Amick-Kirchgässner solitaary wave). It will be convenient to define

$$\mathbf{R}(t,x) = \mathbf{Q}_c(x - ct + A).$$

The precise result reads:

Theorem 1.11 (Theorem 5.2, Chapter 5)**.** *Let us fix $s \geqslant 0$. Suppose that the speed $c > 0$ satisfy* (5.10) *with a parameter λ. Then, there exists λ^* such that for every $\lambda \in (0, \lambda^*)$, and $A > 0$ sufficiently large (depending on ε), there exists a solution $\boldsymbol{U} = (\eta, \varphi)^t$ to* (5.6) *defined in the time interval $(-\infty, 0]$, that satisfies*

$$\boldsymbol{U} - \boldsymbol{R} \in \mathcal{C}_b((-\infty, 0], H^s(\mathbb{R}) \times H^s(\mathbb{R})),$$

and

$$\lim_{t\to\infty} \| \boldsymbol{U}(t) - \boldsymbol{R}(t) \|_{H^s(\mathbb{R})} = 0.$$

The proof of Theorem 5.2 is based on the argumnts used by Ming, Rousset and Tzvetkov to prove multisolitons-like solutions for the Zakharov water waves problem with flat bottom [45]. The most important step for such goal is the construction of an approximate solution $\mathbf{U}_{ap}$, approximate in the sense that $\partial_t \mathbf{U}_{ap} = \mathcal{F}(\mathbf{U}_{ap}) + \mathbf{r}_{ap} \to \mathcal{F}(\mathbf{U}_{ap})$ exponentially fast. To that end, we construct:

$$\mathbf{U}_{ap}(t,x) = \mathbf{R}(t,x) + \sum_{j=1}^{N} \rho^j \mathbf{V}_j(t,x),$$

where $\mathbf{V}_j(t,x)$ (to be defined) are solutions to linear problems (linearization of (1.17) about the solitary wave) with exponentially decaying source terms. The decay of both the error

$\mathbf{r}_{ap}$ produced by the approximate solution and the solutions V_j of the linearized problem are described in the following theorem:

Theorem 1.12 (Theorem 5.13, Chapter 5)**.** *For every $N \in \mathbb{N}$, there exists*

$$\boldsymbol{U}_{ap} = \boldsymbol{Q}_c + \boldsymbol{V} = \boldsymbol{Q}_c + \sum_{j=1}^{N} \rho^j \, \boldsymbol{V}_j(t,x),$$

where $\boldsymbol{V}_j \in C^\infty(\mathbb{R}, H^\infty(\mathbb{R}))$ such that

$$|\, \boldsymbol{V}_j|_{E^s} \leqslant A^{(2j-1)/4} C_{s,j}(\delta_0) \mathrm{e}^{-j\delta_0 c|t|} \quad \forall t \leqslant 0. \tag{1.36}$$

In addition, $\boldsymbol{U}_{ap}$ is an approximate solution of (5.6) *in the sense that the remainder $\boldsymbol{r}_{ap}$ defined as*

$$\partial_t \boldsymbol{U}_{ap} - \mathcal{F}(\boldsymbol{U}_{ap}) = \boldsymbol{r}_{ap}$$

satisfies the exponential decay

$$|\boldsymbol{r}_{ap}|_{E^s} \leqslant A^{(2N+1)/4} C_{N,s}(\delta_0) \rho^{N+1} \mathrm{e}^{-(N+1)\delta_0 c|t|} \quad \forall t \leqslant 0.$$

This result provides, as far as we understand, the first construction of a solitary wave like solution in the case of non flat bottom. Now the collision problem becomes key to understand, since the bottom strongly interacts with this solitary wave after some large positive time.

Bibliography

[1] T. Alazard, N. Burq, C. Zuily, *On the water-wave equations with surface tension* Duke Math. J. Volume 158, Number 3, pp. 413–499 (2011).

[2] T. Alazard, N. Burq, C. Zuily, *On the Cauchy problem for gravity water waves*, Invent. math. 198, pp. 71–163 (2014).

[3] C.J. Amick and K. Kirchgassner, *A theory of solitary water-waves in the presence of surface tension*, Arch. Ration. Mech. Analysis 105, pp. 1–49 (1989).

[4] J. E. Barab *Nonexistence of asymptotically free solutions for a nonlinear Schroödinger equation*, Journal of Mathematical Physics 25, 3270 (1984); doi: 10.1063/1.526074

[5] D. J. Benney, G. J. Roskes, *Wave instability*, Studies in Applied Math., 48 (1969), pp 455-472.

[6] L. Bergé, *Wave collapse in physics: principles and applications to light and plasma waves*, Phys. Rep., Vol. 303, pp. 259–370 (1998).

[7] T. Cazenave. *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, New York University, New York, 2003.

[8] T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. 85 (1982), no. 4, 549–561.

[9] S. Champeaux, D. Laveder, T. Passot, E L. Sulem, *Remarks on the parallel propagation of small-amplitude dispersive Alfvén waves*, Nonlinear Processes in Geophysics 6 (1999), 169

[10] L. Chen, *Orbital stability of solitary waves for the Klein-Gordon-Zakharov equations*, Acta Math. Appl. Sin., Vol. 15, No. 1, pp. 54–64 (1999).

[11] S. Cingolani, S. Secchi and M. Squassina, *Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities.*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 140(5) (2017), pp. 973-1009.

[12] J. C. Cordero, *Supersonic limit for the Zakharov-Rubenchik system*, J. Differential Equations 261 (2016), no. 9, 5260-5288.

[13] J. Colliander, *The initial value problem for the zakharov system*, Thesis, University of Illinois at Urbana-Champaign (1997).

[14] W. Craig, *An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits*, Commun. Partial Differ. Equ. 10(8), pp. 787–1003 (1985).

[15] W. Craig, P. Guyenne, J. Hammack, D. Henderson and C. Sulem *Solitary water wave interactions*, Physycs of Fluids Vol. 18, No. 5, id. 057106-057106-25 (2006)

[16] W. Craig, C. Sulem, P.-L. Sulem, *Nonlinear modulation of gravity waves: A rigorous approach*, Nonlinearity 5, pp. 497–522 (1992).

[17] S. Cuccagna, N. Visciglia and V. Georgiev, *Decay and scattering of small solutions of pure power NLS in $\mathbb{R}$ with $p > 3$ and with a potential*, Comm. Pure Appl. Math, 67 (2014), pp. 957–981.

[18] N. Hayashi and P. I. Naumkin, *Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations*, Amer. J. Math. 120 (1998), pp. 369–389.

[19] P. Germain, N. Masmoudi, J. Shatah *Global solutions for the gravity water waves equation in dimension 3*, C. R. Acad. Sci. Paris, Ser. I 347 pp. 897–902 (2012).

[20] P Germain, N Masmoudi, J Shatah *Global existence for capillary water waves* Communications on Pure and Applied Mathematics 68 (4), pp. 625–687 (2015)

[21] P. Germain, F. Pusateri and F. Rousset, *The nonlinear Schrödinger equation with a potential*, Annales de l'Institut Henri Poincaré C, Analyse non linéaire 35 (2018), pp. 1477–1530.

[22] J. Ginibre and T. Ozawa, *Long Range Scattering for Non-Linear Schrodinger and Hartree Equations in Space Dimension $n > 2$*, Commun. Math. Phys. 151 (1993), pp. 619–645.

[23] J. Ginibre and G. Velo, *Scattering theory for the Zakharov system*, Hokkaido Mathematical Journal 35, no. 4, pp. 865–892 (2006).

[24] J. Ginibre, and G. Velo, *On a class of nonlinear Schrödinger equations. I: The Cauchy problem*, J. Funct. Anal. 32 (1979), 1–32.

[25] R. Glassey, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. 18 (1977), no. 9, pp. 1794–1797.

[26] Z. Guo, S. Lee, K. Nakanishi and C. Wang, *Generalized Strichartz Estimates and Scattering for 3D Zakharov System*, Commun. Math. Phys. 331, pp. 239–259 (2014). doi:10.1007/s00220-014-2006-0

[27] Z. Guo and K. Nakanishi, *Small Energy Scattering for Zakharov*, Int. Math. Res. Not. IMRN, Vol. 2014, No. 9, pp. 2327–2342 (2013). doi:10.1093/imrn/rns296

[28] Z. Guo, K. Nakanishi and S. Wang, *Small energy scattering for the Klein-Gordon-*

Zakharov system with radial symmetry, Math. Res. Lett., Vol. 21, No. 04, pp. 733–755 (2014).

[29] Z. Guo, K. Nakanishi and S. Wang, *Global Dynamics below the Ground State Energy for the Klein-Gordon-Zakharov System in the 3D Radial Case*, Communications in Partial Differential Equations, Vol. 39, pp. 1158–1184 (2014).

[30] J. Kato and F. Pusateri, *A new proof of long range scattering for critical nonlinear Schrödinger equations*, Differential Integral Equations 24 (2011), pp. 923–940.

[31] M. Kowalczyk, Y. Martel and C. Muñoz, *Kink dynamics in the ϕ^4 model: asymptotic stability for odd perturbations in the energy space*, J. Amer. Math. Soc. 30 (2017), 769–798.

[32] M. Kowalczyk, Y. Martel and C. Muñoz, *Nonexistence of small, odd breathers for a class of nonlinear wave equations*, Lett. Math. Phys. 107 (2017), pp. 921–931.

[33] F. Linares, C. Matheus, *Well posedness for the 1D Zakharov-Rubenchik system*, Adv. Differential Equations 14 (2009), no. 3-4, 261-288.

[34] H. Luong, N. Mauser, J-C Saut, *On the Cauchy problem for the Zakharov-Rubenchik/Benney-Roskes system*, Commun. Pure Appl. Anal. 17 (2018), no. 4, 1573-1594.

[35] Y. Martel, F. Merle and T.-P. Tsai, *Stability in H^1 of the sum of K solitary waves for some nonlinear Schrödinger equations*, Duke Math. J. 133 (2006), 405–466.

[36] N. Masmoudi and K. Nakanishi, *Nonrelativistic limit from Maxwell-Klein-Gordon and Maxwell-Dirac to Poisson-Schrödinger*, Int. Math. Res. Not. 13, pp. 697–734 (2003).

[37] N. Masmoudi and K. Nakanishi, *From the Klein-Gordon-Zakharov system to the nonlinear Schrödinger equation*, J. Hyperbolic Differ. Equ. 2 (4), pp. 975–1008 (2005).

[38] N. Masmoudi and K. Nakanishi, *From the Klein-Gordon-Zakharov system to a singular nonlinear Schrödinger system*, Annales de Annales de l'I. H. P. (C) Non Linear Analysis Vol. 27, No. 4, pp. 1073–1096 (2010).

[39] N. Masmoudi and K. Nakanishi, *Energy convergence for singular limits of Zakharov type systems* Invent. math., Vol. 172, pp. 535–583 (2008). doi: 10.1007/s00222-008-0110-5

[40] N. Masmoudi and K. Nakanishi, *Uniqueness of Solutions for Zakharov Systems*, Funkcialaj Ekvacioj Vol. 52, No. 2, pp. 233–253 (2009).

[41] F. Merle, *Blow-up Results of Viriel Type for Zakharov Equations*, Commun. Math. Phys. 175, pp. 433–455 (1996).

[42] F. Merle, *Lower Bounds for the Blowup Rate of Solutions of the Zakharov Equation in Dimension Two*, Commun. Pure Appl. Anal., Vol. XLIX, pp. 765–794 (1996).

[43] F. Merle and P. Raphaël, *The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation*, Ann. of Math. (2) **161** (2005), no. 1, 157–222.

[44] A. Mielke, *On the energetic stability of solitary water waves*, Phil. Trans. R. Soc. Lond. A 360, pp. 2337–2358 (2002).

[45] M. Ming, F. Rousset and N. Tzvetkov *Multi-solitons and Related Solutions for the Water-waves System* SIAM J. Math. Anal., 47 (1), pp. 897–954 (2013).

[46] V. Moroz and J. Van Schaftingen, *A guide to the Choquard equation*, J. J. Fixed Point Theory Appl. 19 (2017), pp. 773–813.

[47] C. Munõz, *Soliton dynamics for the generalized KdV equations in a slowly mediun*, Analysis and PDE, Vol. 4, No. 4, pp. 573–638 (2011).

[48] C. Muñoz, G. Ponce and J.-C. Saut, *On the long time behavoir of solutions to the Intermediate Long Wave equation*, arXiv:1910.03897.

[49] J. Murphy, and K. Nakanishi, *Failure of scattering to solitary waves for the long-range nonlinear Schrödinger equations*, preprint arXiv: 1906.01802.

[50] K. Nakanishi, and T. Ozawa, *Remarks on scattering for nonlinear Schrödinger equations*, NoDEA 2002 Vol. 9, Issue 1, pp. 45–68.

[51] I. P. Naumkin, *Sharp asymptotic behavior of solutions for cubic nonlinear Schrödinger equations with a potential*, J. Math. Phys. 57 (2016), pp. 051501.

[52] T. V. Nguyen, *Existence of multi-solitary waves with logarithmic relative distances for the NLS equation*, Comptes Rendus Mathematique, 357 (2019), pp. 13–58.

[53] F. Oliveira, *Stability of the solitons for the one-dimensional Zakharov-Rubenchik equation*, Phys. D 175 (2003), no. 3-4, 220-240.

[54] F. Oliveira, *Adiabatic limit of the Zakharov-Rubenchik equation*, Rep. Math. Phys. 61 (2008), no. 1, 13-27.

[55] T. Ozawa, *Long range scattering for nonlinear Schrödinger equations in one space dimension*, Communications in Mathematical Physics (1991) DOI: 10.1007/BF02101876.

[56] M. Ohta and G. Todorova, *Strong instability of standing waves for nonlinear Klein-Gordon equation and Klein-Gordon-Zakharov system*, SIAM Journal on Mathematical Analysis Vol. 38, No. 6, pp. 1912–1931 (2006).

[57] T. Ozawa, K.Tsutaya and Y. Tsutsumi, *Normal form and global solutions for the Klein-Gordon-Zakharov equations* Annales de l'I. H. P. (C), Vol. 12, No. 4, pp. 459–503 (1995).

[58] T. Ozawa, K.Tsutaya and Y. Tsutsumi, *Well-posedness in energy space for the Cauchy problem of the Klein-Gordon-Zakharov equations with different propagation speeds in three space dimensions*, Math. Ann. 313, pp. 127–140 (1999).

[59] T. Ozawa and Y. Tsutsumi, *The nonlinear Schrödinger limit and the initial layer of the Zakharov equations*, Proc. Japan Acad. Ser. A Math. Sci. 67, no. 4, pp. 113–116 (1991).

[60] T. Ozawa and Y. Tsutsumi, *Existence and Smoothing Effect of Solutions for the Zakharov Equations*, Publ RIMS, Kyoto Univ. 28, pp. 329–361 (1992).

[61] T. Passot, P.-L. Sulem, C. Sulem, *Generalization of acoustic fronts by focusing wave-packets*, Physica D 94 (1996), 168-187.

[62] H. Pecher, *Global Well-Posedness below Energy Space for the 1-Dimensional Zakharov System*, Int. Math. Res. Not. IMRN, No. 19, pp. 1027–1056 (2001).

[63] D. H. Peregrine, *Water waves, nonlinear Schrödinger equations and their solutions*, J. Austral. Math. Soc. Ser. B 25, pp. 16–43 (1983).

[64] D. Ruiz, *On the Schrödinger-Poisson-Slater System: Behavior of Minimizers, Radial and Nonradial Cases*, Arch Rational Mech Anal 198 (2010), pp. 349–368.

[65] F. Rousset and N. Tzvetkov, *Transverse instability of the line solitary water-waves*, Invent. Math. 184, pp. 257–388 (2011).

[66] J. Satsuma, and N. Yajima, *Initial Value Problems of One-Dimensional Self-Modulation of Nonlinear Waves in Dispersive Media*, Supplement of the Progress of Theoretical Physics, No. 55, 1974, 284–306.

[67] G. Schneider, E. Wayne, *The long wave limit for the water wave problem I. The case of zero surface tension*, Commun. Pure. Appl. Math. 53(12), pp. 1475–1535 (2000).

[68] A. Shimomura, *Scattering theory for the Zakharov equation in three espace dimensions*, Commun. Contemp. Math. 6, No. 6, pp. 881–899 (2004). doi:10.1142/S0219199704001574

[69] W. Strauss, *Nonlinear Scattering Theory*, Scattering Theory in Mathematical Physics, edited by J. A. Lavita and J-P. Marchand (Reidel, Dordrecht, Holland, 1974), pp. 53-78.

[70] C. Sulem and P. L. Sulem, *Quelques résulatats de régularité pour les équation de la turbulence de Langmuir*. C. R. Acad. Sci. Paris 289, pp. 173–176 (1979).

[71] C. Sulem, P.-L. Sulem, *The Nonlinear Schrödinger Equation. Self-Focusing and Wave Collapse*, Applied Mathematical Sciences Series Volume 139, Springer-Verlag New York (1999).

[72] B. Texier, *Derivation of the Zakharov Equations* Archive for Rational Mechanics and Analysis 184, pp. 121–183 (2008).

[73] Y. Tsutsumi, *Scattering problem for nonlinear Schroödinger equations*, Annales de l'IHP, section A, tome 43, No. 3 (1985), p. 321-347.

[74] M. I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution*

equations, Comm. Pure Appl. Math. **39**, (1986) 51–68.

[75] S. Wu, *Almost global wellposedness of the 2–D full water wave problem*, Invent math 177, pp. 45–135 (2009).

[76] S. Wu, *Global wellposedness of the 3–D full water wave problem*, Invent math 184, pp. 125—220 (2011).

[77] Y. Wu, *Orbital stability of solitary waves of Zakharov system*, J. Math. Phys. Vol. 35, No. 5, pp. 2413–2422 (1994). doi:10.1063/1.530512

[78] V. E. Zakharov, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. Applied Mech. Tech. Phys. 9, pp. 190–194 (1968).

[79] V. E. Zakharov, *Collapse of Langmuir Waves*, Zh. Eksp. Teor. Fiz. 62, pp. 1745–1759 (1972).

[80] V. E. Zakharov and A. M. Rubenchik, *Nonlinear interaction of high-frequency and low-frequency waves*, Prikl. Mat. Techn. Phys., 5 (1972), pp 84-98.

[81] V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, JETP, 34 (1), pp. 62–69.

Part II

Two Schrödinger models

Chapter 2

Decay of small odd solutions for long range Schrödinger and Hartree equations in one dimension

Abstract. We consider the long time asymptotics of (not necessarily small) odd solutions to the nonlinear Schrödinger equation with semi-linear and nonlocal Hartree nonlinearities, in one dimension of space. We assume data in the energy space $H^1(\mathbb{R})$ only, and we prove decay to zero in compact regions of space as time tends to infinity. We give three different results where decay holds: semilinear NLS, NLS with a suitable potential, and defocusing Hartree. The proof is based on the use of suitable virial identities, in the spirit of nonlinear Klein-Gordon models [30], and covers scattering sub, critical and supercritical (long range) nonlinearities. No spectral assumptions on the NLS with potential are needed.

This chapter has been published as: M. E. Martínez, *Decay of small odd solutions for long range Schrödinger and Hartree equations in one dimension*, Nonlinearity, 2020.

Contents

2.1 Introduction

In this section of chapter II our goal is to study the long time behavoir of *small odd* global solutions of the one-dimensional nonlinear Schrödinger (NLS) and Hartree equations

$$\mathrm{i}u_t + u_{xx} = g(u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{2.1}$$

In the Schrödinger case (see Ginibre-Velo [22], Cazenave-Weissler [8] and Cazenave [6]), we shall assume that the nonlinearity takes the form

$$g(u) = \mu V(x)u + f\left(|u|^2\right)u, \tag{2.2}$$

where the potential $V : \mathbb{R} \to \mathbb{R}$ is a *Schwartz even function* and $f : \mathbb{R} \to \mathbb{R}$ is a function such that for $1 < p < 5$ (L^2 subcritical case),

$$|f(s)| \lesssim s^{\frac{p-1}{2}}, \tag{2.3}$$

and that satisfies that $f \circ s^2$ is locally Lipschitz continuous. In this context, we denote $F(s) = \int_0^s f(v)\mathrm{d}v$, for all $s > 0$, and

$$G(u) = \frac{\mu}{2}\int_{\mathbb{R}} V(x)|u|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} F(|u|^2)\mathrm{d}x.$$

In the Hartree case, we have

$$g(u) = \sigma\left(W * |u|^2\right)u, \quad G(u) = \frac{\sigma}{4}\int_{\mathbb{R}}\left(W * |u|^2\right)|u|^2\mathrm{d}x, \tag{2.4}$$

where $\sigma = \pm 1$ and the potential W is given by

$$W(x) = \frac{1}{|x|^a}, \quad \text{with} \quad 0 < a < 1. \tag{2.5}$$

The equation (2.1) is Hamiltonian, and it is characterized by having at least the following conservation laws:

- Mass:
$$M(u(t)) := \int_{\mathbb{R}} |u(t)|^2\mathrm{d}x = M(u(0)), \tag{2.6}$$
- Energy:
$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}} |\nabla u(t)|^2\mathrm{d}x + G(u(t)) = E(u(0)), \tag{2.7}$$
- Momentum:
$$P(u(t)) := \mathrm{Im}\int_{\mathbb{R}} u(t)\overline{u}_x(t)\mathrm{d}x = P(u(0)). \tag{2.8}$$

The NLS equation (2.1)-(2.2) with nonlinearity $f(s) = \pm s^{\frac{p-1}{2}}$ is commonly known as the semilinear Schrödinger equation [6]. In particular, if $f(s) = -s^{\frac{p-1}{2}}$, we say that the equation is focusing, while the defocusing case takes place when $f(s) = s^{\frac{p-1}{2}}$. It is well-known that this one-dimensional semilinear Schödinger equation is globally well-posed for initial data in $H^1(\mathbb{R})$ when $1 < p < 5$, and blow up may occur if $p \geqslant 5$, see e.g. [23, 35] and subsequent works.

On the other hand, the Hartree equation (2.1) with (2.4) is also locally well-posed in $H^1(\mathbb{R})$, and globally well-posed for small data, see [6, Corollary 6.1.5] for instance. This comes from the fact that the potential W in (2.5) is an even function that satisfies the following properties:

- $W \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$,
- The function $(W * |u|^2)|u|^2$ is integrable. For the case (2.5), one has the estimate

$$\int_{\mathbb{R}} \left(|x|^{-a} * |u|^2\right) |u|^2 \mathrm{d}x < \infty,$$

(we prove this using the Hardy-Littlewood-Sobolev inequality [32, Theorem 4.3, p. 106] with $p = r = \frac{2}{2-a}$).

This means that we are in the case of [6, Example 3.2.11] and [6, Corollary 4.3.3], which implies the local well-posedness of the Hartree equation.

In this paper we are interested in the asymptotic behavoir of small solutions to (2.1), both in the NLS case (with and without potential), and in the nonlocal Hartree case, at least in the defocusing case. The literature on this subject is huge; we present now a (far from complete) account of the most relevant results.

It is known that for subcritical (in the sense of GWP and scattering) semilinear NLS equation ($f(s) = \pm s^{\frac{p-1}{2}}$, $3 < p < 5$), scattering to a free solution exists (see, for instance, Ginibre and Velo [22], Tsutsumi [54] and Nakanishi-Ozawa [40]). Nevertheless, in Strauss [51] and Barab [3] it was proven that one cannot expect the same scattering for the critical ($p = 3$) and super critical case ($p < 3$), and modified scattering is believed to occur. This was generalized recently by Murphy and Nakanishi [38] for the semilinear NLS equation with potential and Hartree-type nonlinearities as (2.5).

Precisely, *modified scattering* for d dimensional critical NLS equation with nonlinearities

$$g(u) = \sigma |u|^{p-1} u, \quad p = 1 + \frac{2}{\mathrm{d}}, \quad \mathrm{d} = 1, 2, 3;$$

and the Hartree equation with Coulomb potential

$$g(u) = \sigma \left(|x|^{-1} * |u|^2\right) u, \quad \mathrm{d} \geqslant 2,$$

and small initial condition, was first proved by Ozawa [43] and by Ginibre and Ozawa [21]. Moreover, it was shown that solutions u of such equations present the decay

$$\|u(t)\|_{L^\infty} \lesssim (1 + |t|)^{-\mathrm{d}/2}, \tag{2.9}$$

when the initial data is sufficiently small in weighted Sobolev spaces (see also Hayashi-Naumkin [16], and Kato-Pusateri [27], for instance). Through a thorough analysis of the solution profile, a simplified proof of scattering in the critical defocusing NLS and Hartree equations has been exhibited in [27].

Similar recent results hold for the NLS case *with a potential*, as was shown by Cuccagna, Visciglia and Georgiev [14] for $p > 3$, and Naumkin [41] and Germain-Pusateri-Rousset [20] for the critical case $p = 3$ (see also [19]). Nakanishi [39] considered 3D NLS with a potential having a single negative eigenvalue, and proved asymptotics for large time. Indeed, assuming that the potential V is such that $-\frac{1}{2}\Delta + V$ does not have negative eigenvalues nor resonances at zero, they were able to prove the decay (1.20) for solutions of subcritical ($p > 3$) and critical ($p = 3$) NLS equation in one dimension. However different the methods to prove this decay are from each other, it is not clear to us if they still hold by assuming less restrictive spectral conditions.

Finally, following idea introduced in [29], about considering odd data only, Delort [17] proved modified scattering for small (smaller than a parameter ϵ) odd solutions u to (2.9) with data in $H^{0,1} \cap H^N$, N large, and showed (among other things) the precise decomposition for large time

$$u(t,x) = \frac{\epsilon}{\sqrt{t}} A_\epsilon\left(\frac{x}{t}\right) \exp\left[-\mathrm{i}\frac{x^2}{2t} + \mathrm{i}\epsilon^2 \log t \left|A_\epsilon\left(\frac{x}{t}\right)\right|^2\right] + r(t,x),$$

where the continuous function A_ϵ is bounded in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\theta \in (0, \frac{1}{4})$ and

$$\|r(t,\cdot)\|_{L^\infty} = O(\epsilon t^{-\frac{3}{4}+\theta}), \quad \|A_\epsilon(x)\langle tx\rangle^{-2}\|_{L^\infty} = O(\epsilon t^{-\frac{1}{4}+\theta}),$$

and

$$\|r(t,\cdot)\|_{L^2} = O(\epsilon t^{-\frac{1}{4}+\theta}), \quad \|A_\epsilon(x)\langle tx\rangle^{-2}\|_{L^2} = O(\epsilon t^{-\frac{5}{8}+\frac{\theta}{2}}).$$

Notice that all positive decay/scattering results above mentioned cannot deal with the one dimensional NLS (for $p < 3$) and Hartree equations. This is in part explained by the lack of precise nonlinear estimates in the case of long range nonlinearities.

Our main goal in this paper is to extend in some sense the recently mentioned results [41, 20, 27, 17] and show decay of small solutions to the above equations, regardless the (supercritical with respect to scattering) power of the nonlinearity. In particular, we consider nonlinearities NLS with $1 < p < 5$ and Hartree long range supercritical in one dimension.

Our first result covers the NLS case without potential ($1 < p < 5$).

Theorem 2.1. *Suppose $u(t) \in H^1(\mathbb{R})$ is a global odd solution of the equation* (2.1)-(2.2) *and $\mu = 0$ such that, for some $\varepsilon > 0$ small,*

$$\|u(t=0)\|_{H^1(\mathbb{R})} \leqslant \varepsilon. \tag{2.10}$$

Then,

$$\lim_{t\to\infty} \Big(\|u(t)\|_{L^2(I)} + \|u(t)\|_{L^\infty(I)} \Big) = 0, \tag{2.11}$$

for any bounded interval $I \subset \mathbb{R}$. Moreover, if the equation is defocusing, the smallness condition (2.10) *is not needed.*

Remark 2.1. NLS (2.1) preserves the oddness of the initial data along the flow.

Remark 2.2. As far as we could understand, Theorem 2.1 is the first decay result for small data NLS in the long range supercritical nonlinearities $1 < p < 3$. Although we do not give a precise description of a possible limiting profile as in the previous literature, our results show dispersion after all.

Remark 2.3. Theorem 2.1 is sharp. Indeed, it is not true for $u(t) \in H^1$ even. A simple counterexample in this case is the non decaying soliton itself:

$$u(t,x) = Q_c(x)\mathrm{e}^{\mathrm{i}ct}, \qquad 0 < c \ll 1, \tag{2.12}$$

and $Q_c > 0$ solving $Q_c'' - cQ_c + Q_c^p = 0$, $Q_c \in H^1$. Note that this solution is even in space and small in H^1 provided $c \ll 1$. Also, the Satsuma-Yajima breather solution (see [46] and [1, eqn. (1.16)]) is an arbitrarily small non decaying even solution to NLS (2.2) in the integrable [57] case $p = 3$.

Remark 2.4. For an interval $I = I(t)$ growing in time, Theorem 2.1 is also sharp. Indeed, see the works [33, 42] for the construction of odd solutions composed of two solitary waves with non zero speeds for finite time. These asymptotic 2-soliton solutions can be arbitrarily small in the energy space, but they separate each other as time evolves, leaving any compact region in space for sufficiently large time. In this sense, these solutions do not contradict Theorem 2.1.

Remark 2.5. From the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} x|u(t,x)|^2 \mathrm{d}x = -2\,\mathrm{Im}\int_{\mathbb{R}} u(t)\overline{u}_x(t)\mathrm{d}x = -2P(u(t)),$$

valid if $xu(t=0) \in L^2$, we can see that nontrivial, non decaying periodic-in-time solutions (i.e. breathers) of NLS may exist only if their momentum vanishes. See [37] for more details on these properties of breather solutions.

Remark 2.6. Sometimes, instead of assuming odd data, the additional assumption $\|xu(t = 0)\|_{L^2} \ll 1$ is considered. This condition works with even data, and rules out the existence of small solitary waves as in (2.12), since solitary waves with small H^1-norm satisfy $\|xQ_c\|_{L^2} \gg 1$.

Remark 2.7. Note that (2.11) does not contain the $\dot{H}^1$ norm of the solution. This is a standard open issue in the field, see e.g. [17] for similar results. In our case, the lack of control on the decay of this semi-norm is due to the emergence of uncontrolled H^2 terms in the dynamics of the energy norm.

Remark 2.8. In the defocusing case, we expect better results. For instance, we can prove that $\liminf_{t\to+\infty} \|u_x(t)\|_{L^2(|x|\lesssim|t|\log^{-1}|t|)} = 0$, but a better decay property is out of reach for the moment.

The proof of Theorem 2.1 is based on the introduction of a virial identity adapted to the NLS dynamics. Following the ideas presented in [30, 29], which considered the nonlinear Klein-Gordon case, we use here a functional adapted to the momentum (2.8). Once this virial identity is established, decay is proved in a standard form.

Compared with the available results for Klein-Gordon [30], where H^1 decay is proven, the main novelty here is that we avoid the lack of H^1 decay in time for NLS (Remark 2.7) by proving time decay in L^∞_x instead; also, we consider the cases of NLS with a nontrivial potential and with Hartree nonlinearities (see below), both of important physical interest, and not treated in [30].

Using inverse scattering techniques, Deift and Zhou [15] described the asymptotic behavior of solutions of the defocusing, nearly integrable quintic perturbation of cubic NLS

$$\mathrm{i}u_t + u_{xx} = |u|^2 u + \epsilon |u|^4 u, \quad \epsilon > 0. \tag{2.13}$$

Using the techniques of this paper, we are able to give a partially complementary result to the one stated in [15]:

Corollary 2.2. *Let $\epsilon \neq 0$, and let $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ be a global small odd solution of* (2.13). *Then* (2.11) *is satisfied.*

The proof of this result immediately follows from Theorem 2.1.

Our second result deals with NLS (2.1) with nonzero potential in (2.2). In this case, we also provide time decay results in the case μV small and spatially decaying fast enough, complementing [14, 17, 41, 20].

Theorem 2.3 (NLS with potential)**.** *Assume $V \neq 0$ even as in* (2.2). *Under the assumptions of Theorem 2.1, suppose additionally that V satisfies*

$$\int_{\mathbb{R}} (|V(x)| + |V'(x)|) \cosh(2x) \mathrm{d}x < +\infty. \tag{2.14}$$

Then there exists $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$, (2.11) *holds for any bounded interval $I \subset \mathbb{R}$.*

Remark 2.9. Note that Theorem 2.3 does not require that the operator $-\partial_x^2 \pm \mu V$ satisfies specific spectral properties as in [14, 41, 20]; only the decay hypothesis (2.14) is needed. In particular, no *non resonance* condition is needed for having (2.11). This fact reveals that the non resonance condition is essentially linked to the evenness of the involved data.

Remark 2.10. We can ask for V decaying slower than in (2.14), but proofs are probably more complicated; we hope to consider this problem elsewhere.

Finally, we deal with the Hartree case.

Theorem 2.4 (Defocusing Hartree equation)**.** *Suppose that $u \in H^1(\mathbb{R})$ is a global odd solution of equation* (2.1) *with* (2.5) *and $\sigma = 1$. Then*

$$\lim_{t\to\infty} (\|u(t)\|_{L^2(I)} + \|u(t)\|_{L^\infty(I)}) = 0, \tag{2.15}$$

for any bounded interval $I \subset \mathbb{R}$.

Remark 2.11. Theorem 2.4 proves the non-existence of odd standing waves solutions for the equation (2.1) with defocusing Hartree type non-linearities.

Remark 2.12. Theorem 2.4 does not include the focusing case, which is an open problem of independent interest. In that sense, the scattering problem for the d $\geqslant 2$ generalized Hartree equation was recently treated in Arora-Roudenko [2].

Remark 2.13. Focusing Hartree equation (2.1) with (2.5) ($\sigma = -1$) admits solitary waves solutions (or solitons)

$$u(t,x) = \mathrm{e}^{\mathrm{i}ct} Q_c(x) \in H^1$$

where $Q_c : \mathbb{R} \to \mathbb{R}$ is an H^1-solution of the Choquard equation

$$\Delta Q + \left(\frac{1}{|x|^a} * |Q|^p \right) Q - \lambda Q = 0, \quad c \in \mathbb{R}. \tag{2.16}$$

These solutions are, up to translation and inversion of the sign, positive and radially symmetric functions [9, 36]. Moreover, solitary waves for the focusing Hartree equation are stable, as was proven by Cazenave and Lions in [7]. See also Ruiz [45] for more details on solitary waves for Hartree.

Remark 2.14 (NLS around solitary waves)**.** Solitary waves in mass subcritical NLS exist and they are stable. The first results on stability were provided by Cazenave and Lions in [7], where orbital stability of solitary waves for the NLS equation (2.1)-(2.2) without potential was proven (see also [56, 25]). Stability of several NLS solitons well-decoupled was proved in [34], and in [26] for the integrable case. The asymptotic stability for the same equation was studied by Buslaev and Perelman in [4] in the supercritical regime; this result was later generalized by Cuccagna in [10, 11, 13] for dimensions d $\geqslant 3$, and under special spectral conditions on the linearized operator around the solitary wave. The one dimensional case, under similar spectral assumptions and even data perturbations of the standing wave, was studied by Buslaev and Sulem [5]. For the NLS equation with potential (2.1)-(2.2), results for asymptotic stability of ground states (also, under spectral conditions) were provided by Soffer and Weinstein in [49, 50], see also [48, 18, 52, 53], and [44] for the case of multi-solitons in general dimensions. We believe that some of the ideas in this paper can be generalized to the case of asymptotic stability for solitary waves, but with harder proofs. See e.g. the recent paper by Cuccagna and Maeda [12], and the NLKG paper by Kowalczyk, Martel and Muñoz [31].

Notation

To simplify the notation we will denote $u_1 = \text{Re } u$, $u_2 = \text{Im } u$. Let $\alpha(x) \geqslant 0$ be a weight. We also denote by

$$\begin{aligned} \|u(t)\|^2_{L^2_\alpha(\mathbb{R})} &:= \int_{\mathbb{R}} \alpha(x)|u(t,x)|^2 \mathrm{d}x, \\ \|u(t)\|^2_{H^1_\alpha(\mathbb{R})} &:= \int_{\mathbb{R}} \alpha(x) \left(|u_x(t,x)|^2 + |u(t,x)|^2\right) \mathrm{d}x, \end{aligned} \tag{2.17}$$

the weighted L^2-norm and H^1-norm with weight α.

Organization of this paper

This paper is written as follows. In Section 2.2 we prove Theorem 2.1, NLS without potential. Section 2.3 is devoted to the proof of Theorem 2.3, namely NLS with potential. Finally, Section 2.4 deals with the Hartree case (Theorem 2.4).

2.2 Schrödinger equation without potential

In this Section we prove Theorem 2.1. Consider the equation (2.1) with (2.2) and $V \equiv 0$. That is,

$$\mathrm{i}u_t + u_{xx} = f\left(|u|^2\right) u, \qquad u \in H^1 \text{ odd.} \tag{2.18}$$

As claimed in the introduction, the proof here follows the ideas in [30], with some minor differences.

2.2.1 A virial identity

We shall introduce a standard virial identity adapted to (2.18). Let $\varphi \in C^\infty(\mathbb{R})$ be bounded and to be chosen later, $u(t) \in H^1(\mathbb{R})$ a solution of equation (2.18) and define

$$I(u(t)) := \text{Im} \int_{\mathbb{R}} \varphi(x) u(t,x) \overline{u}_x(t,x) \mathrm{d}x. \tag{2.19}$$

Then we have the following:

Lemma 2.5. *For $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ one has $I(u(t))$ well-defined and bounded in time. Moreover, we have the virial identity*

$$-\frac{\mathrm{d}}{\mathrm{d}t} I(t) = 2\int_{\mathbb{R}} \varphi_x |u_x|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi_{xxx}|u|^2 \mathrm{d}x - \int_{\mathbb{R}} \varphi_x \left[F\left(|u|^2\right) - f(|u|^2)|u|^2\right] \mathrm{d}x. \tag{2.20}$$

Proof. Let $u(t) \in H^1(\mathbb{R})$ such that it satisfies equation (2.18). Then, we integrate by parts

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) &= \operatorname{Im}\int_{\mathbb{R}} \varphi u_t \overline{u}_x \mathrm{d}x + \operatorname{Im}\int_{\mathbb{R}} \varphi u \overline{u}_{xt} \mathrm{d}x \\ &= \operatorname{Im}\int_{\mathbb{R}} \varphi u_t \overline{u}_x \mathrm{d}x - \operatorname{Im}\int_{\mathbb{R}} (\varphi u)_x \overline{u}_t \mathrm{d}x.\end{aligned}$$

Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) &= -\operatorname{Im}\int_{\mathbb{R}} \mathrm{i}\varphi\,(\mathrm{i}u_t)\,\overline{u}_x \mathrm{d}x - \operatorname{Im}\int_{\mathbb{R}} \mathrm{i}\,(\varphi u)_x\,\overline{\mathrm{i}u_t} \mathrm{d}x \\ &= -\operatorname{Re}\int_{\mathbb{R}} \varphi \overline{\mathrm{i}u_t} u_x \mathrm{d}x - \operatorname{Re}\int_{\mathbb{R}} (\varphi u)_x\,\overline{\mathrm{i}u_t} \mathrm{d}x.\end{aligned}$$

Computing the derivative on the second term above

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) &= -2\operatorname{Re}\int_{\mathbb{R}} \varphi \overline{\mathrm{i}u_t} u_x \mathrm{d}x - \operatorname{Re}\int_{\mathbb{R}} \varphi_x u \overline{\mathrm{i}u_t} \mathrm{d}x \\ &= -2\operatorname{Re}\int_{\mathbb{R}} \varphi (\mathrm{i}u_t) \overline{u}_x \mathrm{d}x - \operatorname{Re}\int_{\mathbb{R}} \varphi_x\,(\mathrm{i}u_t)\,\overline{u} \mathrm{d}x.\end{aligned} \tag{2.21}$$

Thus, using (2.18), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) = {}& 2\operatorname{Re}\int_{\mathbb{R}} \varphi u_{xx} \overline{u}_x \mathrm{d}x + \operatorname{Re}\int_{\mathbb{R}} \varphi_x u_{xx} \overline{u} \mathrm{d}x \\ & - 2\operatorname{Re}\int_{\mathbb{R}} \varphi f\left(|u|^2\right) u \overline{u}_x \mathrm{d}x - \operatorname{Re}\int_{\mathbb{R}} \varphi_x f\left(|u|^2\right) u \overline{u} \mathrm{d}x.\end{aligned}$$

We notice that $2\operatorname{Re}\left(u_x \overline{u}\right) = 2\operatorname{Re}\left(u \overline{u}_x\right) = \left(|u|^2\right)_x$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) = {}& \int_{\mathbb{R}} \varphi \left(|u_x|^2\right)_x \mathrm{d}x + \operatorname{Re}\int_{\mathbb{R}} \varphi_x u_{xx} \overline{u} \mathrm{d}x \\ & - \int_{\mathbb{R}} \varphi f\left(|u|^2\right) \left(|u|^2\right)_x \mathrm{d}x - \int_{\mathbb{R}} \varphi_x f\left(|u|^2\right) |u|^2 \mathrm{d}x.\end{aligned}$$

Recall the definition of $F(s) = \int_0^s f(v)\mathrm{d}v$, which implies that $\left(F(s)\right)_x = f(s)s_x$. Furthermore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) = {}& \int_{\mathbb{R}} \varphi \left(|u_x|^2\right)_x \mathrm{d}x + \operatorname{Re}\int_{\mathbb{R}} \varphi_x \overline{u} u_{xx} \mathrm{d}x \\ & - \int_{\mathbb{R}} \varphi \left(F\left(|u|^2\right)\right)_x \mathrm{d}x - \int_{\mathbb{R}} \varphi_x f\left(|u|^2\right) |u|^2 \mathrm{d}x.\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) &= -2\int_{\mathbb{R}} \varphi_x |u_x|^2 \mathrm{d}x - \operatorname{Re}\int_{\mathbb{R}} \varphi_{xx} \overline{u} u_x \mathrm{d}x + \int_{\mathbb{R}} \varphi_x \left[F\left(|u|^2\right) - f\left(|u|^2\right)|u|^2\right] \mathrm{d}x \\ &= -2\int_{\mathbb{R}} \varphi_x |u_x|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi_{xx} \left(|u|^2\right)_x \mathrm{d}x + \int_{\mathbb{R}} \varphi_x \left[F\left(|u|^2\right) - f\left(|u|^2\right)|u|^2\right] \mathrm{d}x.\end{aligned}$$

We integrate by parts again on the second term to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) = -2\int_{\mathbb{R}} \varphi_x |u_x|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varphi_{xxx} |u|^2 \mathrm{d}x + \int_{\mathbb{R}} \varphi_x \left[F\left(|u|^2\right) - f\left(|u|^2\right)|u|^2\right] \mathrm{d}x.$$

■

2.2.2 Analysis of a bilinear form

With the identity (2.20) in mind, we define the bilinear form

$$B(w) = 2\int_{\mathbb{R}} \varphi_x w_x^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi_{xxx} w^2 \mathrm{d}x, \quad w = u_{\mathrm{i}}, \quad \mathrm{i} = 1, 2. \tag{2.22}$$

Here, $u = u_1 + \mathrm{i}u_2$, with u_1, u_2 real-valued.

Let $\lambda \in (1, \infty)$. As we explained before, our intention is to prove some estimation of B using the weighted H_α^1-norm introduced in (2.17). To obtain this, we will consider $\varphi(x) = \lambda \tanh\left(\frac{x}{\lambda}\right)$ on the virial identity (2.20) and define the auxiliar function $\alpha(x) = \sqrt{\varphi_x(x)}$. Now, we estimate each term of the bilinear form B:

$$\begin{aligned}
\int_{\mathbb{R}} (\alpha w)_x^2 \mathrm{d}x &= \int_{\mathbb{R}} \alpha^2 (w_x)^2 \mathrm{d}x + 2\int_{\mathbb{R}} \alpha\alpha_x w w_x \mathrm{d}x + \int_{\mathbb{R}} (\alpha_x)^2 w^2 \mathrm{d}x \\
&= \int_{\mathbb{R}} \varphi_x (w_x)^2 \mathrm{d}x + \int_{\mathbb{R}} \alpha\alpha_x \left(w^2\right)_x \mathrm{d}x + \int_{\mathbb{R}} (\alpha_x)^2 w^2 \mathrm{d}x \\
&= \int_{\mathbb{R}} \varphi_x (w_x)^2 \mathrm{d}x - \int_{\mathbb{R}} \alpha\alpha_{xx} w^2 \mathrm{d}x,
\end{aligned}$$

using integration by parts in the last equality. Thus

$$\int_{\mathbb{R}} \varphi_x (w_x)^2 \mathrm{d}x = \int_{\mathbb{R}} (\alpha w)_x^2 \mathrm{d}x + \int_{\mathbb{R}} \frac{\alpha_{xx}}{\alpha} (\alpha w)^2 \mathrm{d}x. \tag{2.23}$$

Furthermore, noticing that $\varphi_{xxx} = (\alpha^2)_{xx} = 2\left(\alpha\alpha_{xx} + \alpha_x^2\right)$, we get

$$\int_{\mathbb{R}} \varphi_{xxx} w^2 \mathrm{d}x = 2\int_{\mathbb{R}} \left(\frac{\alpha_{xx}}{\alpha} + \frac{\alpha_x^2}{\alpha^2}\right) (\alpha w)^2 \mathrm{d}x. \tag{2.24}$$

Hence, from (2.23) and (2.24),

$$B(w) = 2\int_{\mathbb{R}} (\alpha w)_x^2 \mathrm{d}x - \int_{\mathbb{R}} \left(\frac{\alpha_x^2}{\alpha^2} - \frac{\alpha_{xx}}{\alpha}\right) (\alpha w)^2 \mathrm{d}x.$$

Since $\alpha(x) = \operatorname{sech}\left(\frac{x}{\lambda}\right)$, then

$$\begin{aligned}
\alpha_x(x) &= -\frac{1}{\lambda} \operatorname{sech}\left(\frac{x}{\lambda}\right) \tanh\left(\frac{x}{\lambda}\right) \\
\alpha_{xx}(x) &= \frac{1}{\lambda^2} \left(\operatorname{sech}\left(\frac{x}{\lambda}\right) \tanh\left(\frac{x}{\lambda}\right) - \operatorname{sech}^3\left(\frac{x}{\lambda}\right)\right)
\end{aligned}$$

which implies that

$$B(w) = 2\int_{\mathbb{R}} (\alpha w)_x^2 \mathrm{d}x - \frac{1}{\lambda^2}\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) (\alpha w)^2 \mathrm{d}x.$$

In order to prove Theorem 2.1 we need to prove that the bilineal part of (2.20) is coercive in some way. To be more precise, we would like the following

$$B(w) \geqslant \int_{\mathbb{R}} (\alpha w)_x^2 \mathrm{d}x. \tag{2.25}$$

We introduce the auxiliar function $v = \alpha w$. Then we can set

$$\mathcal{B}(v) = 2\int_{\mathbb{R}} v_x^2 \mathrm{d}x - \frac{1}{\lambda^2}\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) v^2 \mathrm{d}x$$

so that

$$\mathcal{B}(v) = B(w).$$

This way, coercivity of the operator $\mathcal{B}$ implies (2.25). We recall now

Proposition 2.6 (See [30]). *Let $v \in H^1(\mathbb{R})$ be odd, $\lambda > 0$. Then*

$$\mathcal{B}(v) \geqslant \frac{3}{2}\int_{\mathbb{R}} v_x^2 \mathrm{d}x. \tag{2.26}$$

Sketch of proof. We write

$$\mathcal{B}(v) = \frac{3}{2}\int_{\mathbb{R}} v_x^2 \mathrm{d}x + \frac{1}{2}\left(\int_{\mathbb{R}} v_x^2 \mathrm{d}x - \frac{2}{\lambda^2}\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) v^2 \mathrm{d}x\right).$$

Notice that

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{2}{\lambda^2}\operatorname{sech}^2\left(\frac{x}{\lambda}\right)$$

has only one negative eigenvalue corresponding to an even eigenfunction. This comes from the fact that (see [24, Exercise 12]) the index of the operator

$$-\frac{h^2}{\nu}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma \operatorname{sech}^2\left(\frac{x}{a}\right)$$

is equal to the largest integer N such that

$$N < \frac{1}{2}\sqrt{8\gamma\nu a^2 h^{-2} + 1} - \frac{1}{2}.$$

Since v is odd,

$$\int_{\mathbb{R}} v_x^2 \mathrm{d}x - \frac{2}{\lambda^2}\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) v^2 \mathrm{d}x \geqslant 0, \tag{2.27}$$

and then (2.26) holds. ■

2.2.3 Estimates of the terms on (2.20)

Lemma 2.7. *Let $u \in H^1(\mathbb{R})$ be odd. Then for some $C > 0$, $u = u_1 + \mathrm{i}u_2$,*

$$\|u\|_{H^1_\alpha(\mathbb{R})}^2 \leqslant C\left(B(u_1) + B(u_2)\right). \tag{2.28}$$

Proof. We take $\lambda = 100$. First, notice that from (2.27), we have

$$\int_{\mathbb{R}} (\alpha w)_x^2 \geqslant \frac{2}{100^2}\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{100}\right)(\alpha w)^2 \mathrm{d}x.$$

Using Proposition 2.6, this implies that

$$B(w) \geqslant \frac{3}{2}\int_{\mathbb{R}} (\alpha w)_x \, \mathrm{d}x \gtrsim \int_{\mathbb{R}} \operatorname{sech}^4\left(\frac{x}{100}\right) u^2 \mathrm{d}x \gtrsim \int_{\mathbb{R}} \operatorname{sech}(x) w^2 \mathrm{d}x. \tag{2.29}$$

Thus,

$$\int_{\mathbb{R}} \operatorname{sech}(x) u_{\mathrm{i}}^2 \mathrm{d}x \lesssim B(u_{\mathrm{i}}), \quad \mathrm{i} = 1, 2. \tag{2.30}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}} (\alpha w)_x^2 \, \mathrm{d}x &\gtrsim \int_{\mathbb{R}} \alpha^2 (\alpha w)_x^2 \, \mathrm{d}x \\ &= \int_{\mathbb{R}} \alpha^4 w_x^2 \mathrm{d}x + \int_{\mathbb{R}} \alpha^3 \alpha_x \left(w^2\right)_x \mathrm{d}x + \int_{\mathbb{R}} \alpha^2 \alpha_x^2 w^2 \mathrm{d}x.\end{aligned}$$

We integrate by parts,

$$\begin{aligned}\int_{\mathbb{R}} (\alpha w)_x^2 \, \mathrm{d}x &\gtrsim \int_{\mathbb{R}} \alpha^4 w_x^2 \mathrm{d}x - \int_{\mathbb{R}} \left(\alpha^3 \alpha_x\right)_x w^2 \mathrm{d}x + \int_{\mathbb{R}} \alpha^2 \alpha_x^2 w^2 \mathrm{d}x \\ &= \int_{\mathbb{R}} \alpha^4 w_x^2 \mathrm{d}x - \int_{\mathbb{R}} \left(2\alpha^2 \alpha_x^2 + \alpha^3 \alpha_{xx}\right) w^2 \mathrm{d}x.\end{aligned}$$

Then, from the definition of α,

$$\int_{\mathbb{R}} (\alpha w)_x^2 \, \mathrm{d}x \gtrsim \int_{\mathbb{R}} \operatorname{sech}(x) w_x^2 \mathrm{d}x - \int_{\mathbb{R}} \operatorname{sech}^4\left(\frac{x}{100}\right) w^2 \mathrm{d}x.$$

In other words,

$$\int_{\mathbb{R}} \operatorname{sech}(x) w_x^2 \mathrm{d}x \lesssim \int_{\mathbb{R}} (\alpha w)_x^2 \, \mathrm{d}x + \int_{\mathbb{R}} \operatorname{sech}^4\left(\frac{x}{100}\right) w^2 \mathrm{d}x.$$

Then, from (2.29), we have that

$$\int_{\mathbb{R}} \operatorname{sech}(x) w_x^2 \mathrm{d}x \lesssim \int_{\mathbb{R}} (\alpha w)_x^2 \, \mathrm{d}x. \tag{2.31}$$

Hence, using Proposition 2.6,

$$\int_{\mathbb{R}} \operatorname{sech}(x) u_{\mathrm{i}x}^2 \mathrm{d}x \lesssim B(u_{\mathrm{i}}), \quad \mathrm{i} = 1, 2. \tag{2.32}$$

Finally, from (2.30) and (2.32), we get

$$\|u(t)\|_{H^1_\alpha(\mathbb{R})}^2 \lesssim B(u_1) + B(u_2).$$

■

Lemma 2.8. *There exists $\varepsilon > 0$ such that:*
If u is an odd solution of (2.18) satisfying (2.10), then

$$-\frac{\mathrm{d}}{\mathrm{d}t} I(u(t)) \geqslant C \|u(t)\|_{H^1_\alpha(\mathbb{R})}^2. \tag{2.33}$$

where $C > 0$.

Proof. Recall from (2.20) and the analysis of the previous section that

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) &= 2\int_{\mathbb{R}}\varphi_x|u_x|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}}\varphi_{xxx}|u|^2\mathrm{d}x - \int_{\mathbb{R}}\varphi_x\left[F\left(|u|^2\right) - f(|u|^2)|u|^2\right]\mathrm{d}x\\
&= B(u_1) + B(u_2) - \int_{\mathbb{R}}\varphi_x\left[F\left(|u|^2\right) - f(|u|^2)|u|^2\right]\mathrm{d}x.
\end{aligned}$$

Consequently, in order to complete the proof, we need to control the remaining terms of (2.20), since the terms involving the bilinear form B have already been estimated by Lemma 2.7.

Note that

$$\left|F\left(|u|^2\right) - f(|u|^2)|u|^2\right| \lesssim |u|^{p+1}.$$

Since u is odd,

$$\begin{aligned}
\int_{\mathbb{R}}\operatorname{sech}^2\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x &= 2\int_0^{\infty}\operatorname{sech}^2\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x\\
&= 2\int_0^{\infty}\operatorname{sech}^{-(p-1)}\left(\frac{x}{\lambda}\right)\operatorname{sech}^{p+1}\left(\frac{x}{\lambda}\right)|u|^{p+1}\\
&\simeq \int_0^{\infty}\mathrm{e}^{(p-1)x/\lambda}\operatorname{sech}^{p+1}\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x.
\end{aligned}$$

With a slight abuse of notation, set $v(t,x) := \operatorname{sech}\left(\frac{x}{\lambda}\right)u(t,x)$. Note that $v(t,0) = 0$ and vanishes at infinity $\forall t \in \mathbb{R}$. Then, integrating by parts,

$$\begin{aligned}
\int_0^{\infty}\mathrm{e}^{(p-1)x/\lambda}|v|^{p+1}\mathrm{d}x &= -\frac{\lambda}{p-1}\int_0^{\infty}\mathrm{e}^{(p-1)x/\lambda}\left(|v|^{p+1}\right)_x\mathrm{d}x\\
&= -\frac{\lambda(p+1)}{p-1}\mathrm{Re}\int_0^{\infty}\mathrm{e}^{(p-1)x/\lambda}|v|^{p-1}\bar{v}v_x\mathrm{d}x.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^{\infty}\mathrm{e}^{(p-1)x/\lambda}|v|^{p+1}\mathrm{d}x &= -\frac{\lambda(p+1)}{p-1}\mathrm{Re}\int_0^{\infty}\mathrm{e}^{(p-1)x/2\lambda}|v|^{\frac{p-1}{2}}\bar{v}v_x\left(\mathrm{e}^{(p-1)x/2\lambda}|v|^{\frac{p-1}{2}}\right)\mathrm{d}x\\
&\lesssim \|u\|_{L^\infty(\mathbb{R})}^{(p-1)/2}\mathrm{Re}\int_0^{\infty}\mathrm{e}^{(p-1)x/2\lambda}|v|^{\frac{p-1}{2}}\bar{v}v_x\mathrm{d}x\\
&\lesssim \|u\|_{L^\infty(\mathbb{R})}^{(p-1)/2}\int_0^{\infty}\mathrm{e}^{(p-1)x/2\lambda}|v|^{\frac{p-1}{2}}|v||v_x|\mathrm{d}x\\
&= \|u\|_{L^\infty(\mathbb{R})}^{(p-1)/2}\int_0^{\infty}\mathrm{e}^{(p-1)x/2\lambda}|v|^{\frac{p+1}{2}}|v_x|\mathrm{d}x.
\end{aligned}$$

By Young's inequality,

$$\begin{aligned}
\int_0^{\infty}\mathrm{e}^{(p-1)x/\lambda}|v|^{p+1}\mathrm{d}x &\lesssim \|u\|_{L^\infty(\mathbb{R})}^{p-1}\int_0^{\infty}|v_x|^2\mathrm{d}x + \int_0^{\infty}\mathrm{e}^{(p-1)x/\lambda}|v|^{p+1}\mathrm{d}x\\
&\simeq \|u\|_{L^\infty(\mathbb{R})}^{p-1}\int_0^{\infty}|v_x|^2\mathrm{d}x + \int_0^{\infty}\operatorname{sech}^{p-1}\left(\frac{x}{\lambda}\right)\operatorname{sech}^{p+1}\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x\\
&= \|u\|_{L^\infty(\mathbb{R})}^{p-1}\int_0^{\infty}|v_x|^2\mathrm{d}x + \int_0^{\infty}\operatorname{sech}^2\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x,
\end{aligned}$$

which actually means that

$$\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x \lesssim \|u\|_{L^\infty(\mathbb{R})}^{p-1}\int_0^\infty |\left(\alpha u\right)_x|^2\mathrm{d}x.$$

By Sobolev's embedding,

$$\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x \lesssim \|u\|_{H^1(\mathbb{R})}^{p-1}\int_0^\infty |\left(\alpha u\right)_x|^2\mathrm{d}x.$$

Now, it is a fact that for every $0 < \varepsilon < 1$, there exists $\delta(\varepsilon)$ such that $\|u(0)\|_{H^1(\mathbb{R})} \leqslant \delta(\epsilon)$ implies that $\sup\limits_{t\in\mathbb{R}} \|u\|_{H^1(\mathbb{R})} < \varepsilon$ (see [6, Corollary 6.1.4] or the conservation of energy and mass (2.7)-(2.6)). This way, from Proposition 2.6, we get

$$\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x \lesssim \varepsilon^{p-1}\left(B(u_1)+B(u_2)\right).$$

So, choosing ε sufficiently small, (2.33) is proved. ■

Remark 2.15 (Defocusing case)**.** Note that in the semilinear defocusing case $f(|u|^2) = |u|^{p-1}$,

$$F\left(|u|^2\right) - f(|u|^2)|u|^2 = \left(\frac{2}{p+1}-1\right)|u|^{p+1}.$$

Since $p > 1$, $\frac{2}{p+1}|u|^{p+1} - 1 < 0$ which means that the remaining term on (2.20) involving the nonlinearity is positive:

$$-\int_{\mathbb{R}} \varphi_x\left[F\left(|u|^2\right) - f(|u|^2)|u|^2\right]\mathrm{d}x \geqslant 0$$

and then Lemma 2.7 is enough to conclude Lemma 2.8.

With this estimation, we can now prove the key to get Theorem 2.1.

Proposition 2.9. *There exists a constant $C > 0$ such that*

$$\int_0^\infty \|u(t)\|_{H_\alpha^1(\mathbb{R})}^2\mathrm{d}t \leqslant C\varepsilon^2. \tag{2.34}$$

Proof. Let $\tau > 0$. We integrate (2.33) over $[0,\tau]$

$$\int_0^\tau \|u(t)\|_{H_\alpha^1(\mathbb{R})}^2\mathrm{d}t \leqslant C\left(I(u(0)) - I(u(\tau))\right) \leqslant CI(u(0))$$

From Hölder inequality and (2.10) we get that

$$I(u(0)) \leqslant \|u(0)\|_{L^2(\mathbb{R})}\|u_x(0)\|_{L^2(\mathbb{R})} \leqslant \varepsilon^2.$$

This last fact implies that

$$\int_0^\tau \|u(t)\|_{H_\alpha^1(\mathbb{R})}^2\mathrm{d}t \leqslant C\varepsilon^2.$$

Now, taking $\tau \to \infty$, we conclude the proof. ■

2.2.4 End of proof of Theorem 2.1

Now Theorem 2.1 is ready to be proved:

Step 1: The L^2 norm tends to zero: Let $\varphi \in C^\infty(\mathbb{R})$ be bounded. Then we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\varphi|u(t)|^2\mathrm{d}x\right) &= \mathrm{Re}\int_{\mathbb{R}}\varphi\overline{u}u_t\mathrm{d}x = -\mathrm{Re}\int \mathbb{R}R\mathrm{i}\varphi\overline{u}\left(\mathrm{i}u_t\right)\mathrm{d}x \\ &= \mathrm{Im}\int_{\mathbb{R}}\varphi\overline{u}\left(\mathrm{i}u_t\right)\mathrm{d}x.\end{aligned}$$

Hence, using equation (2.18) and integrating by parts

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\varphi|u(t)|^2\mathrm{d}x\right) &= -\mathrm{Im}\int_{\mathbb{R}}\varphi\overline{u}u_{xx}\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\varphi f(|u^2|)u\overline{u}\mathrm{d}x \\ &= \mathrm{Im}\int_{\mathbb{R}}\varphi\overline{u}_x u_x\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\varphi_x\overline{u}u_x\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\varphi f(|u^2|)|u|^2\mathrm{d}x \\ &= \mathrm{Im}\int_{\mathbb{R}}\varphi|u_x|^2\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\varphi_x\overline{u}u_x\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\varphi f(|u^2|)|u|^2\mathrm{d}x.\end{aligned}$$

Since the integrals on the first and third term are real, we get the following identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\varphi|u(t)|^2\right) = \mathrm{Im}\int_{\mathbb{R}}\varphi_x\overline{u}u_x\mathrm{d}x. \tag{2.35}$$

Thus

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\varphi|u(t)|^2\mathrm{d}x\right)\right| &\leqslant \int_{\mathbb{R}}|\varphi_x||\overline{u}(t)||u_x(t)|\mathrm{d}x \\ &\lesssim \int_{\mathbb{R}}|\varphi_x||\overline{u}(t)|^2\mathrm{d}x + \int_{\mathbb{R}}|\varphi_x||u_x(t)|^2\mathrm{d}x.\end{aligned}$$

We take $\varphi(x) = \mathrm{sech}(x)$ and get

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2_\alpha(\mathbb{R})}\right| &= \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}}\mathrm{sech}(x)|u(t,x)|^2\mathrm{d}x\right)\right| \\ &\lesssim \int_{\mathbb{R}}\mathrm{sech}(x)|\overline{u}(t,x)|^2\mathrm{d}x + \int_{\mathbb{R}}\mathrm{sech}(x)|u_x(t,x)|^2\mathrm{d}x = \|u(t)\|^2_{H^1_\alpha(\mathbb{R})}.\end{aligned}$$

From (2.34), there exists a sequence $t_n \in \mathbb{R}$, $t_n \to \infty$ such that $\|u(t_n)\|^2_{L^2_w(\mathbb{R})} \to 0$. Consider $t \in \mathbb{R}$, integrate over $[t, t_n]$, and take $t_n \to \infty$. Then

$$\|u(t)\|^2_{L^2_\alpha(\mathbb{R})} \lesssim \int_t^\infty \|u(s)\|^2_{H^1_\alpha(\mathbb{R})}\mathrm{d}s.$$

In consequence

$$\lim_{t\to\infty}\|u(t)\|_{L^2_w(\mathbb{R})} = 0. \tag{2.36}$$

Step 2: The L^∞ norm tends to zero: We state the following:

Claim 2.10. *For every interval I there exists $\tilde{x}(t) \in I$ such that, as t tends to infinity,*

$$|u(t, \tilde{x}(t))|^2 \to 0.$$

Proof. Let $I \in \mathbb{R}$ be an interval. By contradiction, Suppose that there exists $\varepsilon_0 > 0$ such that $\forall n > 0$, $\exists t_n > n$

$$|u(t_n, x)|^2 > \varepsilon_0 \quad \forall x \in I.$$

Integrating over I, we get

$$\int_I |u(t_n, x)|^2 \mathrm{d}x > |I|\varepsilon_0,$$

which contradicts (2.36). ■

Let $x \in I$. By Fundamental Theorem of calculus and Hölder's inequality

$$\begin{aligned} |u(t,x)|^2 - |u(t,\tilde{x}(t))|^2 &= \int_{\tilde{x}(t)}^{x} \left(|u|^2\right)_x \mathrm{d}x \leqslant 2\int_{\tilde{x}(t)}^{x} |u||u_x|\mathrm{d}x \\ &\leqslant 2\|u(t)\|_{L^2(I)}\|u_x(t)\|_{L^2(I)}. \end{aligned}$$

Then we get

$$|u(t,x)|^2 \lesssim |u(t,\tilde{x}(t))|^2 + 2\|u(t)\|_{L^2(I)}\|u_x(t)\|_{L^2(I)}, \quad \forall x \in I. \tag{2.37}$$

Now, since (2.10) holds for $\varepsilon > 0$ as small as needed,

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1(\mathbb{R})} < \infty.$$

Also, this smallness condition is not needed if the nonlinearity is defocusing. Hence, taking $t \to \infty$ in (3.38), from Claim 2.10 and (2.36), we get that

$$|u(t,x)|^2 \to 0, \quad \forall x \in I.$$

Which implies (2.11). The proof of Theorem 2.1 is complete.

2.3 NLS with potential

This section is devoted to the proof of Theorem 2.3. We consider now the NLS equation with a nontrivial potential V:

$$\mathrm{i}u_t + u_{xx} = \mu V(x)u + f\left(|u|^2\right)u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{2.38}$$

As done in the previous section, we introduce a virial identity that will be used to estimate the H^1_α-norm of a solution of equation (2.38). However, because of the potential term V, new estimates must be proved in order to get Theorem 2.3.

2.3.1 Virial Identity

Suppose again $\varphi \in C^\infty(\mathbb{R})$ bounded and recall from Subsection 2.2.1 the definition

$$I(u(t)) = \operatorname{Im} \int_{\mathbb{R}} \varphi(x) u(t,x) \overline{u}_x(t,x) \mathrm{d}x.$$

Following the proof of Lemma 2.5, we have now

Lemma 2.11. *Let $u(t) \in H^1(\mathbb{R})$ be a bounded in time solution of equation (2.38). Then*

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t} I(t) = & \, 2\int_{\mathbb{R}} \varphi_x |u_x|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi_{xxx} |u|^2 \mathrm{d}x - \mu \int_{\mathbb{R}} \varphi V_x |u|^2 \mathrm{d}x \\ & - \int_{\mathbb{R}} \varphi_x \left[F\left(|u|^2\right) - f(|u|^2)|u|^2 \right] \mathrm{d}x. \end{aligned} \tag{2.39}$$

Sketch of proof. From the proof of Lemma 2.5 (equation (2.21)) we know that

$$\frac{\mathrm{d}}{\mathrm{d}t} I(u(t)) = -2\operatorname{Re} \int_{\mathbb{R}} \varphi(\mathrm{i}u_t)\overline{u}_x \mathrm{d}x - \operatorname{Re} \int_{\mathbb{R}} \varphi_x \left(\mathrm{i}u_t\right) \overline{u} \mathrm{d}x.$$

We use (2.38) to obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I(u(t)) = & \, 2\operatorname{Re} \int_{\mathbb{R}} \varphi u_{xx} \overline{u}_x \mathrm{d}x + \operatorname{Re} \int_{\mathbb{R}} \varphi_x u_{xx} \overline{u} \mathrm{d}x - 2\mu \operatorname{Re} \int_{\mathbb{R}} \varphi V u \overline{u}_x \mathrm{d}x \\ & - \mu \operatorname{Re} \int_{\mathbb{R}} \varphi_x V u \overline{u} \mathrm{d}x - 2\operatorname{Re} \int_{\mathbb{R}} \varphi f\left(|u|^2\right) u \overline{u}_x \mathrm{d}x - \operatorname{Re} \int_{\mathbb{R}} \varphi_x f\left(|u|^2\right) u \overline{u} \mathrm{d}x. \end{aligned}$$

From the last equation, we are only interested in the terms involving the potential V, since the rest of them were analyzed in the proof of Lemma 2.5. Then we compute

$$\begin{aligned} 2\operatorname{Re} \int_{\mathbb{R}} \varphi V u \overline{u}_x \mathrm{d}x + \operatorname{Re} \int_{\mathbb{R}} \varphi_x V u \overline{u} \mathrm{d}x &= \int_{\mathbb{R}} \varphi V \left(|u|^2\right)_x \mathrm{d}x + \int_{\mathbb{R}} \varphi_x V |u|^2 \mathrm{d}x \\ &= -\int_{\mathbb{R}} \varphi V_x |u|^2 \mathrm{d}x. \end{aligned}$$

Combining this with Lemma 2.5, we conclude (2.39). ■

2.3.2 Analysis of a modified bilinear form

In the following analysis, we will see more clearly the difference between the cases with and without potential. In this occasion, we define a new bilinear form ($u = u_1 + \mathrm{i}u_2$, $u_{\mathrm{i}} \in \mathbb{R}$)

$$B(w) = 2\int_{\mathbb{R}} \varphi_x w_x^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi_{xxx} w^2 \mathrm{d}x - \mu \int_{\mathbb{R}} \varphi V_x w^2 \mathrm{d}x, \quad w = u_{\mathrm{i}}, \quad \mathrm{i} = 1,2.$$

Consider $\lambda \in (1,\infty)$, $\varphi(x) = \lambda \tanh\left(\frac{x}{\lambda}\right)$ and $\alpha(x) = \sqrt{\varphi_x(x)}$. Since $\alpha^2 = \varphi_x$, we can write

$$\int_{\mathbb{R}} \varphi V_x w^2 \mathrm{d}x = \int_{\mathbb{R}} V_x \frac{\varphi}{\varphi_x} \left(\alpha w\right)^2 \mathrm{d}x. \tag{2.40}$$

Thus, from (2.23), (2.40) and (2.24),

$$B(w) = 2\int_{\mathbb{R}} (\alpha w)_x^2 \,\mathrm{d}x - \int_{\mathbb{R}} \left(\frac{\alpha_x^2}{\alpha^2} - \frac{\alpha_{xx}}{\alpha}\right) (\alpha w)^2 \,\mathrm{d}x - \mu \int_{\mathbb{R}} V_x \frac{\varphi}{\varphi_x} (\alpha w)^2 \,\mathrm{d}x.$$

Then, from computations of subsection 2.2.2 we have that

$$B(w) = 2\int_{\mathbb{R}} (\alpha w)_x^2 \,\mathrm{d}x - \frac{1}{\lambda^2} \int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) (\alpha w)^2 \,\mathrm{d}x - \mu \int_{\mathbb{R}} V_x \frac{\varphi}{\varphi_x} (\alpha w)^2 \,\mathrm{d}x.$$

We set

$$\mathcal{B}(v) = 2\int_{\mathbb{R}} v_x^2 \mathrm{d}x - \frac{1}{\lambda^2} \int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) v^2 \mathrm{d}x - \mu \int_{\mathbb{R}} V_x \frac{\varphi}{\varphi_x} v^2 \mathrm{d}x,$$

where $v = \alpha w$. Then

$$\mathcal{B}(v) = B(w).$$

Now we prove a modified version of Proposition 2.6.

Proposition 2.12. *Let $v \in H^1(\mathbb{R})$ be odd. Then, for $\lambda > 0$ sufficiently small,*

$$\mathcal{B}(v) \geqslant \frac{1}{2} \int_{\mathbb{R}} v_x^2 \mathrm{d}x. \tag{2.41}$$

Proof. We introduce

$$\mathcal{L}(v) = \int_{\mathbb{R}} v_x^2 \mathrm{d}x - \frac{1}{\lambda^2} \int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) v^2 \mathrm{d}x$$

and

$$\mathcal{K}(v) = \int_{\mathbb{R}} v_x^2 \mathrm{d}x + \mu \int_{\mathbb{R}} V_0 v^2 \mathrm{d}x,$$

where $V_0 = -V_x \frac{\varphi}{\varphi_x}$. Then,

$$\mathcal{B}(v) = \mathcal{L}(v) + \mathcal{K}(v).$$

Arguing as in the proof of Proposition 2.6, we write

$$\mathcal{L}(v) = \frac{1}{2} \int_{\mathbb{R}} v_x^2 \mathrm{d}x + \frac{1}{2} \left(\int_{\mathbb{R}} v_x^2 \mathrm{d}x - \frac{2}{\lambda^2} \int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) v^2 \mathrm{d}x \right).$$

Since v is odd,

$$\int_{\mathbb{R}} v_x^2 \mathrm{d}x - \frac{2}{\lambda^2} \int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) v^2 \mathrm{d}x \geqslant 0,$$

because the index N of such an operator is the integer that satisfies $N < \frac{1}{2}\sqrt{17} - \frac{1}{2} < 2$. Hence, we get that

$$\mathcal{L}(v) \geqslant \frac{1}{2} \int_{\mathbb{R}} v_x^2 \mathrm{d}x. \tag{2.42}$$

Then, in order to get the (2.41) it will be sufficient to demonstrate that $\mathcal{K}(v) \geqslant 0$.

To prove the positiveness of $\mathcal{K}$, we make use of the following result by Simon [47, Theorem 2.5] (see also [28] for improved results):

Lemma 2.13. *Let V_0 be a non-identically zero potential that obeys*

$$\int_{\mathbb{R}} \left(1+x^2\right) |V_0(x)| \mathrm{d}x < \infty.$$

Then

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu V_0$$

has a unique negative eigenvalue for all positive μ sufficiently small if and only if

$$\int_{\mathbb{R}} V_0(x) \mathrm{d}x \leqslant 0. \tag{2.43}$$

Moreover, since V_0 is even, such an eigenvalue is associated to an even eigenfunction.

Remark 2.16. We remark that in the case $\int_{\mathbb{R}} V_0 > 0$ there is no negative eigenvalue $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu V_0$, $\mu > 0$ sufficiently small.

Notice that, from the definition of φ and (2.14), we have

$$\int_{\mathbb{R}} V_0 \mathrm{d}x = -\int_{\mathbb{R}} V_x \frac{\varphi}{\varphi_x} \mathrm{d}x = -\lambda \int_{\mathbb{R}} V_x \sinh\left(\frac{x}{\lambda}\right) \cosh\left(\frac{x}{\lambda}\right) \mathrm{d}x$$

We integrate by parts and get

$$\int_{\mathbb{R}} V_0 \mathrm{d}x = \int_{\mathbb{R}} \cosh\left(\frac{2x}{\lambda}\right) V \mathrm{d}x.$$

Since $\lambda > 1$, (2.14) tells us that V_0 integrates in space. Besides, since V is a Schwartz function,

$$\int_{\mathbb{R}} \left(1+x^2\right) \left|V_x \frac{\varphi}{\varphi_x}\right| \mathrm{d}x \leqslant \int_{\mathbb{R}} \left(1+x^2\right) |V_x| \cosh\left(\frac{2x}{\lambda}\right) \mathrm{d}x < \infty.$$

Then, Lemma 2.13 implies that there exists $\mu_0 > 0$ such that

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu V_0$$

has a unique negative eigenvalue for all $\mu < \mu_0$ and $\lambda > 1$. Since the corresponding eigenfunction is even, we have $\mathcal{K}(v) \geqslant 0$ for v odd. ■

The conclusion that we obtain from Proposition 2.41 is that for $\mathrm{i} = 1, 2$,

$$B(u_{\mathrm{i}}) \geqslant \frac{1}{2} \int_{\mathbb{R}} (\alpha u_{\mathrm{i}})_x^2 \mathrm{d}x.$$

This property of the bilinear form B will allow us to get an estimation of the operator $\frac{\mathrm{d}}{\mathrm{d}t} I(u(t))$ that will lead us to conclude the proof of Theorem 2.3.

2.3.3 Estimates of the terms on (3.25)

Lemma 2.14. *Let u be an odd solution of (2.38). Then,*

$$\|u\|_{H^1_\alpha(\mathbb{R})}^2 \leqslant C\left(B(u_1)+B(u_2)\right). \tag{2.44}$$

for some $C>0$.

Proof. Direct from Lemma 2.7. ∎

Lemma 2.15. *There exists $\varepsilon>0$ such that for every odd solution u of (2.38) satisfying*

$$\|u(t)\|_{H^1(\mathbb{R})} \leqslant \varepsilon \quad \forall t\in\mathbb{R}, \tag{2.45}$$

then

$$-\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) \geqslant C\|u(t)\|_{H^1_\alpha(\mathbb{R})}^2. \tag{2.46}$$

where $C>0$.

Proof. The virial identity we have is

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}I(t) &= 2\int_{\mathbb{R}}\varphi_x|u_x|^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}}\varphi_{xxx}|u|^2\mathrm{d}x-\mu\int_{\mathbb{R}}\varphi V_x|u|^2\mathrm{d}x\\
&\quad-\int_{\mathbb{R}}\varphi_x\left[F\left(|u|^2\right)-f(|u|^2)|u|^2\right]\mathrm{d}x\\
&= B(u_1)+B(u_2)-\int_{\mathbb{R}}\varphi_x\left[F\left(|u|^2\right)-f(|u|^2)|u|^2\right]\mathrm{d}x.
\end{aligned}$$

As we already have an estimation for $B(u_1)+B(u_2)$ given by Lemma 2.14, we need to check that the remaining terms can be controled. Replicating the proof of Lemma 2.8, we get that

$$\begin{aligned}
\int_{\mathbb{R}}\varphi_x\left[F(|u|^2)-f(|u|^2)|u|^2\right]\mathrm{d}x &\lesssim \int_{\mathbb{R}}\operatorname{sech}^2\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x\\
&\lesssim \|u\|_{H^1(\mathbb{R})}^{p-1}\int_0^\infty\left|\left(\operatorname{sech}\left(\frac{x}{\lambda}\right)u_1\right)_x\right|^2\mathrm{d}x\\
&\quad+\|u\|_{H^1(\mathbb{R})}^{p-1}\int_0^\infty\left|\left(\operatorname{sech}\left(\frac{x}{\lambda}\right)u_2\right)_x\right|^2\mathrm{d}x.
\end{aligned}$$

Thus, Proposition 2.12 implies that

$$\int_{\mathbb{R}}\varphi_x\left[F(|u|^2)-f(|u|^2)|u|^2\right]\mathrm{d}x \lesssim \|u\|_{H^1(\mathbb{R})}^{p-1}\big(B(u_1)+B(u_2)\big).$$

Now, since $\|u\|_{H^1(\mathbb{R})}$ is small enough, we conclude. (In the defocusing case, this condition is not needed.) ∎

We can modify the proof of Proposition 2.34, using Lemma 2.15 instead of Lemma 2.8 to obtain the following:

Proposition 2.16. *There exists a constant $C>0$ such that*

$$\int_0^\infty \|u(t)\|_{H^1_\alpha(\mathbb{R})}^2\mathrm{d}t \leqslant C\epsilon^2. \tag{2.47}$$

2.3.4 Proof main result

Step 1: The L^2 norm tends to zero:

Let $\varphi \in C^\infty(\mathbb{R})$. Since $\operatorname{Im}\int_{\mathbb{R}} \varphi V|u|^2 \mathrm{d}x = 0$, computing as in Subsection 2.2.4, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}} \varphi |u(t)|^2\right) = \operatorname{Im}\int_{\mathbb{R}} \varphi_x \overline{u} u_x \mathrm{d}x \tag{2.48}$$

This identity implies that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}} \varphi |u(t)|^2 \mathrm{d}x\right)\right| \lesssim \int_{\mathbb{R}} |\varphi_x||\overline{u}(t)|^2 \mathrm{d}x + \int_{\mathbb{R}} |\varphi_x||u_x(t)|^2 \mathrm{d}x.$$

Taking $\varphi(x) = \operatorname{sech}(x)$ we obtain

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2_\alpha(\mathbb{R})}\right| &= \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} \operatorname{sech}(x)|u(t)|^2\right)\right| \\ &\lesssim \int_{\mathbb{R}} \operatorname{sech}(x)|\overline{u}(t,x)|^2 \mathrm{d}x + \int_{\mathbb{R}} \operatorname{sech}(x)|u_x(t,x)|^2 \mathrm{d}x = \|u(t)\|^2_{H^1_\alpha(\mathbb{R})}.\end{aligned}$$

From (2.47), there exists a sequence $t_n \in \mathbb{R}$, $t_n \to \infty$ such that $\|u(t_n)\|^2_{L^2_\alpha(\mathbb{R})} \to 0$. Consider $t \in \mathbb{R}$, integrate over $[t_n, t]$, and take $t_n \to \infty$. Then

$$\|u(t)\|^2_{L^2\alpha(\mathbb{R})} \lesssim \int_t^\infty \|u(t)\|^2_{H^1_\alpha(\mathbb{R})} \mathrm{d}t.$$

Passing to the limit

$$\lim_{t\to\infty} \|u(t_n)\|_{L^2_\alpha(\mathbb{R})} = 0. \tag{2.49}$$

Step 2: The L^∞ norm tends to zero:

One uses the same arguments as in Subsection 2.2.4. We skip the proof.

2.4 The Hartree Equation. Proof of Theorem 2.4

Our goal in this section is to extend Theorem 2.1 to the Hartree equation,

$$\mathrm{i}u_t + u_{xx} = \sigma\left(|x|^{-a} * |u|^2\right) u, \quad (t,x) \in \mathbb{R}\times\mathbb{R} \tag{2.50}$$

where $\sigma = \pm 1$ and $0 < a < 1$. We start out with a virial identity.

2.4.1 Virial Identity

As before (see (2.19)), let us consider $\varphi \in C^\infty(\mathbb{R})$ bounded and let

$$J(u(t)) := \operatorname{Im} \int_{\mathbb{R}} \varphi(x) u(t,x) \overline{u}_x(t,x) \mathrm{d}x, \tag{2.51}$$

then we state the following result.

Lemma 2.17. *Let $u \in H^1(\mathbb{R})$ be a solution of (2.50), then*

$$-\mathit{Im} J(u(t)) = 2\int_{\mathbb{R}} \varphi_x |u_x|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi_{xxx}|u|^2 \mathrm{d}x + \sigma a \int_{\mathbb{R}} \varphi \left(\frac{x}{|x|^{a+2}} * |u|^2 \right) |u|^2 \mathrm{d}x. \tag{2.52}$$

Proof. Recall (2.21) from the proof of Lemma 3.25,

$$\frac{\mathrm{d}}{\mathrm{d}t} J(u(t)) = -2\operatorname{Re} \int_{\mathbb{R}} \varphi (\mathrm{i}u_t) \overline{u}_x \mathrm{d}x - \operatorname{Re} \int_{\mathbb{R}} \varphi_x \overline{u} (\mathrm{i}u_t) \mathrm{d}x.$$

We use (2.50) to get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} J(u(t)) &= 2\operatorname{Re} \int_{\mathbb{R}} \varphi u_{xx} \overline{u}_x \mathrm{d}x + \operatorname{Re} \int_{\mathbb{R}} \varphi_x \overline{u} u_{xx} \mathrm{d}x \\
&\quad - \sigma 2\operatorname{Re} \int_{\mathbb{R}} \varphi \left(|x|^{-a} * |u|^2 \right) u \overline{u}_x \mathrm{d}x - \sigma \operatorname{Re} \int_{\mathbb{R}} \varphi_x \left(|x|^{-a} * |u|^2 \right) u \overline{u} \mathrm{d}x \\
&= \int_{\mathbb{R}} \varphi \left(|u_x|^2 \right)_x \mathrm{d}x + \operatorname{Re} \int_{\mathbb{R}} \varphi_x \overline{u} u_{xx} \mathrm{d}x \\
&\quad - \sigma \int_{\mathbb{R}} \varphi \left(|x|^{-a} * |u|^2 \right) \left(|u|^2 \right)_x \mathrm{d}x - \sigma \int_{\mathbb{R}} \varphi_x \left(|x|^{-a} * |u|^2 \right) |u|^2 \mathrm{d}x.
\end{aligned}$$

We integrate by parts once on the last term and twice on the second term to obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} J(u(t)) &= -2 \int_{\mathbb{R}} \varphi_x |u_x|^2 \mathrm{d}x - \operatorname{Re} \int_{\mathbb{R}} \varphi_{xx} \overline{u} u_x \mathrm{d}x + \sigma \int_{\mathbb{R}} \varphi \left(|x|^{-a} * |u|^2 \right)_x |u|^2 \mathrm{d}x \\
&= -2 \int_{\mathbb{R}} \varphi_x |u_x|^2 \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}} \varphi_{xx} \left(|u|^2 \right)_x \mathrm{d}x + \sigma \int_{\mathbb{R}} \varphi \left(|x|^{-a} * |u|^2 \right)_x |u|^2 \mathrm{d}x \\
&= -2 \int_{\mathbb{R}} \varphi_x |u_x|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}} \varphi_{xxx} |u|^2 \mathrm{d}x + \sigma \int_{\mathbb{R}} \varphi \left(|x|^{-a} * |u|^2 \right)_x |u|^2 \mathrm{d}x.
\end{aligned}$$

Computing the derivative on the last term,

$$\frac{\mathrm{d}}{\mathrm{d}t} J(u(t)) = -2 \int_{\mathbb{R}} \varphi_x |u_x|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}} \varphi_{xxx} |u|^2 \mathrm{d}x - \sigma a \int_{\mathbb{R}} \varphi \left(\frac{x}{|x|^{a+2}} * |u|^2 \right)_x |u|^2 \mathrm{d}x.$$

■

Let us analyze the RHS of (2.52). Notice that if φ is a non-decreasing weight function, the integral on the last term in (2.52) is positive:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) \frac{x-y}{|x-y|^{a+2}} |u(y)|^2 |u(x)|^2 \mathrm{d}y \mathrm{d}x \geqslant 0. \tag{2.53}$$

Indeed, we compute (all the computations below are justified by choosing suitably compactly supported functions, and taking the standard limit procedure)

$$\begin{aligned}\int_{\mathbb{R}} \varphi \left(|x|^{-a} * |u|^2\right)_x |u|^2 \mathrm{d}x =& -a \int_{\mathbb{R}} \varphi \left(\frac{x}{|x|^{a+2}} * |u|^2\right) |u|^2 \mathrm{d}x \\ =& -a \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) \frac{x-y}{|x-y|^{a+2}} |u(y)|^2 |u(x)|^2 \mathrm{d}y \mathrm{d}x.\end{aligned}$$

We have that

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) \frac{x-y}{|x-y|^{a+2}} |u(y)|^2 |u(x)|^2 \mathrm{d}y \mathrm{d}x =& \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\varphi(x) - \varphi(y)\right) \frac{x-y}{|x-y|^{a+2}} |u(y)|^2 |u(x)|^2 \mathrm{d}y \mathrm{d}x \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(y) \frac{x-y}{|x-y|^{a+2}} |u(y)|^2 |u(x)|^2 \mathrm{d}y \mathrm{d}x.\end{aligned}$$

After a change of variables on the second integral, we get

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) \frac{x-y}{|x-y|^{a+2}} |u(y)|^2 |u(x)|^2 \mathrm{d}y \mathrm{d}x =& \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\varphi(x) - \varphi(y)\right) \frac{x-y}{|x-y|^{a+2}} |u(y)|^2 |u(x)|^2 \mathrm{d}y \mathrm{d}x \\ &- \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) \frac{x-y}{|x-y|^{a+2}} |u(y)|^2 |u(x)|^2 \mathrm{d}y \mathrm{d}x.\end{aligned}$$

Then, we obtain that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) \frac{x-y}{|x-y|^{a+2}} |u(y)|^2 |u(x)|^2 \mathrm{d}y \mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\varphi(x) - \varphi(y)\right) \frac{x-y}{|x-y|^{a+2}} |u(y)|^2 |u(x)|^2 \mathrm{d}y \mathrm{d}x.$$

If φ is non-decreasing, then $\left(\varphi(x) - \varphi(y)\right)(x-y) \geqslant 0$. Moreover,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left(\varphi(x) - \varphi(y)\right) \frac{x-y}{|x-y|^{a+2}} |u(y)|^2 |u(x)|^2 \mathrm{d}y \mathrm{d}x \geqslant 0.$$

This implies that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) \frac{x-y}{|x-y|^{a+2}} |u(y)|^2 |u(x)|^2 \mathrm{d}y \mathrm{d}x \geqslant 0,$$

as claimed.

2.4.2 Proof of Theorem 2.4

Assume $\sigma = 1$ in (2.50) and let $u = u_1 + \mathrm{i}u_2 \in H^1(\mathbb{R})$ be an odd solution of this equation. As done in Section 2.2, we define the bilinear form

$$B(u_{\mathrm{i}}) = 2 \int_{\mathbb{R}} \varphi_x u_{\mathrm{i}x}^2 \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}} \varphi_{xxx} u_{\mathrm{i}}^2 \mathrm{d}x, \quad \mathrm{i} = 1, 2.$$

This means that we can re-write the virial identity (2.51) as

$$-\frac{\mathrm{d}}{\mathrm{d}t} J(u(t)) = B(u_1) + B(u_2) - \sigma \int_{\mathbb{R}} \varphi \left(|x|^{-a} * |u|^2\right)_x |u|^2 \mathrm{d}x. \tag{2.54}$$

Now, as usual, take $\lambda > 1$, $\varphi = \lambda \tanh\left(\frac{x}{\lambda}\right)$ and $\alpha = \sqrt{\varphi_x}$. From (3.28) and (3.29) and reasoning as before, we have that

$$B(u_{\mathrm{i}}) = 2\int_{\mathbb{R}} (\alpha u_{\mathrm{i}})_x^2 \,\mathrm{d}x - \frac{1}{\lambda^2}\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right)(\alpha u_{\mathrm{i}})^2 \,\mathrm{d}x, \quad \mathrm{i} = 1, 2.$$

Thus, Proposition 2.6 implies that

$$B(u_{\mathrm{i}}) \geqslant \frac{3}{2}\int_{\mathbb{R}} (\alpha u_{\mathrm{i}})_x^2 \,\mathrm{d}x, \quad \text{for } \mathrm{i} = 1, 2. \tag{2.55}$$

Moreover, if we consider

$$\|u(t)\|_{H^1_\alpha(\mathbb{R})} = \int_{\mathbb{R}} \operatorname{sech}(x) u^2(t,x)\mathrm{d}x + \int_{\mathbb{R}} \operatorname{sech}(x) u_x^2(t,x)\mathrm{d}x,$$

then, from Proposition 2.7 we obtain

$$\|u\|^2_{H^1_\alpha(\mathbb{R})} \lesssim B(u_1) + B(u_2). \tag{2.56}$$

Since

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x)\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x \geqslant 0,$$

it follows that

$$-\frac{\mathrm{d}}{\mathrm{d}t}J(u(t)) \geqslant \|u\|^2_{H^1_\alpha(\mathbb{R})}.$$

Replicating the proof of Proposition 2.9, we use the last inequality to obtain

$$\int_0^\infty \|u(t)\|^2_{H^1_\alpha(\mathbb{R})}\mathrm{d}t \leqslant C\varepsilon^2. \tag{2.57}$$

Step 1: The L^2 norm tends to zero: Let $\phi \in C^\infty(\mathbb{R})$ bounded. Then we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}} \phi|u(t)|^2\mathrm{d}x\right) &= \mathrm{Re}\int_{\mathbb{R}} \phi\overline{u}u_t\mathrm{d}x \\
&= -\mathrm{Re}\int_{\mathbb{R}} \mathrm{i}\phi\overline{u}\left(\mathrm{i}u_t\right)\mathrm{d}x \\
&= \mathrm{Im}\int_{\mathbb{R}} \phi\overline{u}\left(\mathrm{i}u_t\right)\mathrm{d}x.
\end{aligned}$$

Hence, using equation (2.50) with $\sigma = 1$ and integrating by parts

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}} \phi|u(t)|^2\mathrm{d}x\right) &= -\mathrm{Im}\int_{\mathbb{R}} \phi\overline{u}u_{xx}\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}} \phi\left(|x|^{-a} * |u|^2\right)u\overline{u}\mathrm{d}x \\
&= \mathrm{Im}\int_{\mathbb{R}} \phi\overline{u}_x u_x\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}} \phi\left(|x|^{-a} * |u|^2\right)|u|^2\mathrm{d}x \\
&= \mathrm{Im}\int_{\mathbb{R}} \phi|u_x|^2\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}} \phi_x\overline{u}u_x\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}} \phi\left(|x|^{-a} * |u|^2\right)|u|^2\mathrm{d}x.
\end{aligned}$$

Since the only integral that can have an imaginary part is the second one, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\phi|u(t)|^2\right) = \operatorname{Im}\int_{\mathbb{R}}\phi_x\overline{u}u_x\mathrm{d}x \tag{2.58}$$

Thus

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\phi|u(t)|^2\mathrm{d}x\right)\right| &\leqslant \int_{\mathbb{R}}|\phi_x||\overline{u}(t)||u_x(t)|\mathrm{d}x\\ &\lesssim \int_{\mathbb{R}}|\phi_x||\overline{u}(t)|^2\mathrm{d}x + \int_{\mathbb{R}}|\phi_x||u_x(t)|^2\mathrm{d}x.\end{aligned}$$

We take $\phi(x) = \operatorname{sech}(x)$ and get

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2_\alpha(\mathbb{R})}\right| &= \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}}\operatorname{sech}(x)|u(t)|^2\right)\right|\\ &\lesssim \int_{\mathbb{R}}\operatorname{sech}(x)|\overline{u}(t,x)|^2\mathrm{d}x + \int_{\mathbb{R}}\operatorname{sech}(x)|u_x(t,x)|^2\mathrm{d}x = \|u(t)\|^2_{H^1_\alpha(\mathbb{R})}.\end{aligned}$$

From (2.57), there exists a sequence $t_n \in \mathbb{R}$, $t_n \to \infty$ such that $\|u(t_n)\|^2_{L^2_\alpha(\mathbb{R})} \to 0$. Consider $t \in \mathbb{R}$, integrate over $[t, t_n]$, and take $t_n \to \infty$. Then

$$\|u(t)\|^2_{L^2_\alpha(\mathbb{R})} \lesssim \int_t^\infty \|u(s)\|^2_{H^1_\alpha(\mathbb{R})}\mathrm{d}s.$$

In consequence

$$\lim_{t\to\infty}\|u(t)\|_{L^2_\alpha(\mathbb{R})} = 0. \tag{2.59}$$

The rest of the proof is exactly the same as in the proofs of Theorems 2.1 and 2.3.

Bibliography

[1] M. A. Alejo, L. Fanelli, and C. Muñoz, *Stability and instability of breathers in the $U(1)$ Sasa-Satsuma and Nonlinear Schrödinger models*, preprint arXiv:1901.10381 (2019).

[2] A. K. Arora and S. Roudenko, *Global behavior of solutions to the focusing generalized Hartree equation*, preprint arXiv:1904.05339v1 (2019).

[3] J. E. Barab *Nonexistence of asymptotically free solutions for a nonlinear Schroödinger equation*, Journal of Mathematical Physics 25, 3270 (1984); doi: 10.1063/1.526074.

[4] V. S. Buslaev and G. Perelman, *Nonlinear scattering: the states that are closed to a soliton*, Journal of Mathematical Sciences, Vol. 77, no. 3 (1995), pp. 3161–3169.

[5] V. S. Buslaev and C. Sulem, *On asymptotic stability of solitary waves for the nonlinear Schrödinger equations*, Annales de l'Institut Henri Poincaré (C), Vol. 20, Issue 3 (2003), pp. 419–475.

[6] T. Cazenave. *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, New York University, New York, 2003.

[7] T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. 85 (1982), no. 4, pp. 549–561.

[8] T. Cazenave and F. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Anal. 14 (1990), no. 10, pp. 807–836.

[9] S. Cingolani, S. Secchi and M. Squassina, *Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities.*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 140(5) (2017), pp. 973–1009.

[10] S. Cuccagna, *Stabilization of solutions to nonlinear Schrödinger equations*, Comm. Pure Appl. Math., 54 (2001), pp. 1110–1145.

[11] S. Cuccagna, *The Hamiltonian Structure of the Nonlinear Schrödinger Equation and the Asymptotic Stability of its Ground States*, Commun. Math. Phys. 305 (2011), pp. 279–331.

[12] S. Cuccagna, and M. Maeda *On stability of small solitons of the 1–D NLS with a trapping delta potential*, arXiv:1904.11869.

[13] S. Cuccagna and T. Mizumachi, *On Asymptotic Stability in Energy Space of Ground States for Nonlinear Schroödinger Equations*, Commun. Math. Phys. 284 (2008), pp. 51–77.

[14] S. Cuccagna, N. Visciglia and V. Georgiev, *Decay and scattering of small solutions of pure power NLS in $\mathbb{R}$ with $p > 3$ and with a potential*, Comm. Pure Appl. Math, 67 (2014), pp. 957–981.

[15] P. Deift and X. Zhou, *Perturbation theory for the infinite-dimensional integrable system on the line. A case study*, Acta Math, Vol. 188, Number 2 (2002), pp. 163–262.

[16] N. Hayashi and P. I. Naumkin, *Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations*, Amer. J. Math. 120 (1998), pp. 369–389.

[17] J.-M. Delort, *Modified scattering for odd solutions of cubic nonlinear Schrödinger equations with potential in dimension one*, preprint $\langle \mathrm{hal} - 01396705 \rangle$ (2016).

[18] Z. Gang, and I. M. Sigal, *Relaxation of solitons in nonlinear Schrödinger equations with potential*, Adv. Math. 216 (2007) 443–490.

[19] P. Germain, F. Pusateri and F. Rousset, *Asymptotic stability of solitons for mKdV*, Adv. Math. 299 (2016), pp. 272–330.

[20] P. Germain, F. Pusateri and F. Rousset, *The nonlinear Schrödinger equation with a potential*, Annales de l'Institut Henri Poincaré (C), 35 (2018), pp. 1477–1530.

[21] J. Ginibre and T. Ozawa, *Long Range Scattering for Non-Linear Schrodinger and Hartree Equations in Space Dimension $n > 2$*, Commun. Math. Phys. 151 (1993), pp. 619–645.

[22] J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equations. I: The Cauchy problem*, J. Funct. Anal. 32 (1979), pp. 1–32.

[23] R. Glassey, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. 18 (1977), no. 9, pp. 1794–1797.

[24] I. I. Gol'dman, V. D. Krivchenkov, B. T. Geilikman, E. Marquit, E. Lepa, *Problems in quantum mechanics*, Authorised revised ed. Edited by B. T. Geilikman; translated from the Russian by E. Marquit and E. Lepa, Pergamon Press, 1961.

[25] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I.* J. Funct. Anal. 74 (1987), no. 1, pp. 160–197.

[26] T. Kapitula, *On the stability of N-solitons in integrable systems*, Nonlinearity, 20 (2007) pp. 879–907.

[27] J. Kato and F. Pusateri, *A new proof of long range scattering for critical nonlinear Schrödinger equations*, Differential Integral Equations 24 (2011), pp. 923–940.

[28] M. Klaus, *On the Bound State of Schrödinger Operators in One Dimension*, Annals of

Physics 108 (1977), pp. 288–300.

[29] M. Kowalczyk, Y. Martel and C. Muñoz, *Kink dynamics in the ϕ^4 model: asymptotic stability for odd perturbations in the energy space*, J. Amer. Math. Soc. 30 (2017), pp. 769–798.

[30] M. Kowalczyk, Y. Martel and C. Muñoz, *Nonexistence of small, odd breathers for a class of nonlinear wave equations*, Lett. Math. Phys. 107 (2017), pp. 921–931.

[31] M. Kowalczyk, Y. Martel and C. Muñoz, *Soliton dynamic for the 1D NLKG equation with symmetry and in the absence of internal modes*, preprint: arXiv:1903.12460.

[32] E. Lieb and M. Loss, *Analysis*, 2nd. Ed., Graduate Studies in Mathematics, Vol. 14, Amer. Math. Soc., 2001.

[33] Y. Martel and F. Merle, *Multi-solitary waves for nonlinear Schrödinger equations*, Annales de l'Institut Henri Poincaré (C) 23 (2006), pp. 849–864.

[34] Y. Martel, F. Merle and T.-P. Tsai, *Stability in H^1 of the sum of K solitary waves for some nonlinear Schrödinger equations*, Duke Math. J. 133 (2006), pp. 405–466.

[35] F. Merle and P. Raphaël, *The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation*, Ann. of Math. (2) **161** (2005), no. 1, pp. 157–222.

[36] V. Moroz and J. Van Schaftingen, *A guide to the Choquard equation*, J. Fixed Point Theory Appl. 19 (2017), pp. 773–813.

[37] C. Muñoz and G. Ponce, *Breathers and the dynamics of solutions to the KdV type equations*, Comm. Math. Phys. April 2019, Volume 367, Issue 2, pp 581–598.

[38] J. Murphy and K. Nakanishi, *Failure of scattering to solitary waves for the long-range nonlinear Schrödinger equations*, preprint arXiv:1906.01802.

[39] K. Nakanishi, *Global dynamics below excited solitons for the nonlinear Schrödinger equation with a potential*, J. Math. Soc. Japan 69 (2017), no. 4, pp. 1353–1401.

[40] K. Nakanishi and T. Ozawa, *Remarks on scattering for nonlinear Schrödinger equations*, NoDEA 2002 Vol. 9, Issue 1, pp. 45–68.

[41] I. P. Naumkin, *Sharp asymptotic behavior of solutions for cubic nonlinear Schrödinger equations with a potential*, J. Math. Phys. 57(5):051501 (2016).

[42] T. V. Nguyen, *Existence of multi-solitary waves with logarithmic relative distances for the NLS equation*, Comptes Rendus Mathematique, 357 (2019), pp. 13–58.

[43] T. Ozawa, *Long range scattering for nonlinear Schrödinger equations in one space dimension*, Comm. Math. Phys. 139 (1991), no. 3, pp. 479–493.

[44] I. Rodnianski, W. Schlag and A. Soffer, *Asymptotic stability of N-soliton states of NLS*

(2003), preprint arXiv:0309114.

[45] D. Ruiz, *On the Schrödinger-Poisson-Slater System: Behavior of Minimizers, Radial and Nonradial Cases*, Arch. Rational. Mech. Anal. 198 (2010), pp. 349–368.

[46] J. Satsuma and N. Yajima, *Initial Value Problems of One-Dimensional Self-Modulation of Nonlinear Waves in Dispersive Media*, Supplement of the Progress of Theoretical Physics, No. 55, 1974, pp. 284–306.

[47] B. Simon. *The Bound State of Weakly Coupled Schrödinger Operators in One and Two Dimensions*, Annals of Physics 97 (1976), pp. 279–288.

[48] A. Soffer and M. I. Weinstein, *Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations*, Invent. Math., March 1999, Volume 136, Issue 1, pp. 9–74.

[49] A. Soffer and M. I. Weinstein, *Multichannel Nonlinear Scattering for Nonintegrable Equations* Commun. Math. Phys. 133, (1990), pp. 119–146.

[50] A. Soffer and M. I. Weinstein, *Multichannel Nonlinear Scattering for Nonintegrable Equations II. The Case of Anisotropic Potentials and Data*, Journal of Differential Equations, Vol. 98, Issue 2 (1992), pp. 376–390.

[51] W. Strauss, *Nonlinear Scattering Theory*, Scattering Theory in Mathematical Physics, edited by J. A. Lavita and J-P. Marchand (Reidel, Dordrecht, Holland, 1974), pp. 53–78.

[52] T.-P. Tsai, and H.-T. Yau, *Asymptotic dynamics of nonlinear Schrödinger equations: resonance-dominated and dispersion-dominated solutions*, Comm. Pure Appl. Math. 55 (2002), no. 2, 153–216.

[53] T.-P. Tsai, and H.-T. Yau, *Relaxation of Excited States in Nonlinear Schrödinger Equations*, Int. Math. Res. Not. 2002, no. 31, 1629–1673.

[54] Y. Tsutsumi, *Scattering problem for nonlinear Schrödinger equations*, Annales de l'Institut Henri Poincaré, section A, tome 43, no 3 (1985), pp. 321–347.

[55] Y. Tsutsumi, *L^2-solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcial. Ekvac. 30 (1987), no. 1, pp. 115–125.

[56] M. I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39**, (1986) pp. 51–68.

[57] V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, JETP, 34 (1), pp. 62–69.

Chapter 3

On the decay problem for the Zakharov and Klein-Gordon Zakharov systems in one dimension

Abstract. We are interested in the long time asymptotic behaviour of solutions to the scalar Zakharov system

$$\begin{aligned} \mathrm{i}u_t + \Delta u &= nu, \\ n_{tt} - \Delta n &= \Delta |u|^2 \end{aligned}$$

and the Klein-Gordon Zakharov system

$$\begin{aligned} u_{tt} - \Delta u + u &= -nu, \\ n_{tt} - \Delta n &= \Delta |u|^2 \end{aligned}$$

in one dimension of space. For these two systems, we give two results proving decay of solutions for initial data in the energy space. The first result deals with decay over compact intervals assuming smallness and parity conditions (u odd). The second result proves decay in far field regions along curves for solutions whose growth can be dominated by an increasing C^1 function. No smallness condition is needed to prove this last result for the Zakharov system. We argue relying on the use of suitable virial identities appropiate for the equations and follow the technics of [20, 23] and [32].

This chapter has been published as M. E. Martínez, *On the decay problem for the Zakharov and Klein–Gordon–Zakharov systems in one dimension*, Journal of Evolutions Equations, 2021.

Contents

3.1 Introduction

In this work, we are concerned with the one dimensional Zakharov system

$$\begin{aligned} &\mathrm{i}u_t + \Delta u = nu, && (t,x) \in \mathbb{R}\times\mathbb{R}, \\ &\alpha^{-2}n_{tt} - \Delta n = \Delta|u|^2, && (t,x) \in \mathbb{R}\times\mathbb{R}, \end{aligned} \tag{3.1}$$

with initial data

$$u(t=0,x) = u_0(x), \quad n(t=0,x) = n_0(x), \quad n_t(t=0,x) = n_1(x).$$

where $u(t,x) : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$, $n(t,x) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\alpha > 0$.

We are also interested in the Klein-Gordon Zakharov system in one dimension

$$\begin{aligned} &c^{-2}u_{tt} - \Delta u + c^2 u = -nu, && (t,x) \in \mathbb{R}\times\mathbb{R}, \\ &\alpha^{-2}n_{tt} - \Delta n = \Delta|u|^2 && (t,x) \in \mathbb{R}\times\mathbb{R}, \end{aligned} \tag{3.2}$$

with initial data

$$u(t=0,x) = u_0(x), \quad u_t(t=0,x) = u_1(x)$$
$$n(t=0,x) = n_0(x), \quad n_t(t=0,x) = n_1(x).$$

where $u(t,x) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $n(t,x) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $\alpha > 0,\ c > 0$.

The Zakharov systems are simplified models for the description of long-wavelength small-amplitude Langmuir oscillations in a ionized plasma [49]. Langmuir waves are rapid oscillations of the electron density; electrons and ions oscillate out of phase. Zakharov equations model the nonlinear interactions between the mean mode of the ionic fluctuations of density in the plasma n and the changing amplitude of electric field u, which varies slowly compared to the unperturbed plasma frequency. The constant α is the ion sound speed and c is the plasma frequency.

In the *subsonic limit* ($\alpha \to \infty$), in which density perturbations are changing slowly, the term n_{tt} of the wave equation in (3.1) is negligible. This would imply that the Langmuir waves follow the cubic NLS equation

$$\mathrm{i}u_t + \Delta u + |u|^2 u = 0.$$

If one considers $\tilde{u} = \mathrm{e}^{\mathrm{i}c^2t}$ in (3.2), then it follows that

$$\begin{aligned} c^{-2}\tilde{u}_{tt} - 2\mathrm{i}\tilde{u}_t - \Delta\tilde{u} &= -n\tilde{u}, \\ \alpha^{-2}n_{tt} - \Delta n &= \Delta|\tilde{u}|^2. \end{aligned}$$

Thus, formally, in the *high-frequency limit* (that is, taking $c \to \infty$) of the Klein-Gordon-Zakharov (3.2), the Zakharov system is recovered.

These high-frequency and subsonic limits were extensively studied in [1, 38, 46] and [26]-[29]. See, also, [45, 42, 4] for more details on the physical derivation.

The Zakharov system (3.1) preserves the mass $\|u(t)\|_{L^2(\mathbb{R})} = \|u(0)\|_{L^2(\mathbb{R})}$ and the energy

$$H_S(t) := \int_{\mathbb{R}} |\nabla u(t,x)|^2 + \frac{1}{2}\Big(|n(t,x)|^2 + \frac{1}{\alpha^2}|D^{-1}n_t(t,x)|^2\Big) + n(t,x)|u(t,x)|^2 \mathrm{d}x,$$

where $D = \sqrt{-\Delta}$. The Klein-Gordon-Zakharov system (3.2) preserves the following energy, as well:

$$\begin{aligned} H_{KG}(t) = \int_{\mathbb{R}} & c^2|u(t,x)|^2 + |\nabla u(t,x)|^2 + \frac{1}{c^2}|u_t(t,x)|^2 + \frac{1}{2}|n(t,x)|^2 \\ &+ \frac{1}{2}\big||\alpha D|^{-1}n_t(t,x)\big|^2 + n(t,x)|u(t,x)|^2 \mathrm{d}x = H_{KG}(0). \end{aligned}$$

The system (3.1) in one dimension is globally well-posed for initial data in $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times \hat{H}^{-1}(\mathbb{R})$, where

$$w \in \hat{H}^s \text{ if there exists } v : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{\mathrm{d}} \text{ such that } w = \nabla \cdot v \text{ and } \|w\|_{\hat{H}^s} = \|v\|_{H^{s+1}}.$$

The first approach in this regard was presented by Sulem and Sulem in [44], where they stated local well-posedness of (3.1) for dimensions $\mathrm{d} = 1, 2, 3$ and initial data

$$(u_0, n_0, n_1) \in H^m(\mathbb{R}^{\mathrm{d}}) \times H^{m-1}(\mathbb{R}^{\mathrm{d}}) \times \left(H^{m-2} \cap \hat{H}^{-1}\right)(\mathbb{R}^{\mathrm{d}}), \quad m \geqslant 3.$$

Using the Brezis-Gallouët inequality

$$\|u\|_{L^\infty} \lesssim 1 + \|u\|_{H^1}(\ln(1 + \|\Delta u\|_{L^2})), \tag{3.3}$$

valid if $u \in H^2(\mathbb{R}^2)$, Added and Added in [1] improved [44] to global well-posedness for small initial data in the 2-dimensional case. The local well posedness result ($\mathrm{d} = 1, 2, 3$) was refined by Ozawa and Tsutsumi [39], for data $(u_0, n_0, n_1) \in H^2 \times H^1 \times L^2$, and by Colliander [7] for $(u_0, n_0, n_1) \in H^1 \times L^2 \times \hat{H}^{-1}$. In fact, in [7], using a priori estimates on the H^1-norm of u, Colliander shows global well-posedness for small data in the one-dimensional case. Finally, on [40], Pecher proves that the Zakharov system is globally well posed for rough data $(u_0, n_0, n_1) \in H^s \times L^2 \times \hat{H}^{l-1}$, $1 > s > 9/10$ for dimension $\mathrm{d} = 1$ without any smallness condition. More results on local and global well-posedness for other dimensions and more general nonlinearities are stated in [5, 8, 10, 31], and on the torus in [3].

Regarding well-posedness for system (3.2), the first result was presented in [36], where the authors followed a method based on the theory of normal forms to prove that (3.2) in dimensions d $= 1, 2, 3$ with $c = \alpha = 1$, admits a unique global solutions for small initial data with rather restrictive regularity conditions. Also, they give a completeness result for the 3-dimensional case, as they show the existence of global solutions that tend to behave asymptotically (when $t \to \infty$) as the free solutions.

Using Sobolev invariant spaces, Tsutaya [47], improved the regularity conditions of the global existence result in [36] for dimension d $= 3$. The low-frequency case ($0 < \alpha < 1$) in three dimensions was addressed in [37], where the authors rely on the different propagation speed (normalized $c = 1$ in the Klein Gordon equation, while assumed $0 < \alpha < 1$ in the wave equation) to prove local and then global well-posedness for small initial data in the energy space:

$$(u_0, u_1, n_0, n_1) \in H^1 \times L^2 \times L^2 \times \hat{H}^{-1}.$$

Following the idea stated in [37], Otha and Todorova [35] extended the well-posedness result for all $\alpha > 0$ and d < 3. The high-frequency, subsonic case in dimension d $= 3$ was later treated by Masmoudi and Nakanishi in [30]-[31], where they presented local well-posedness in the energy space under the assumption $\alpha < c$.

In this paper, we are interested in the decay of solutions to systems (3.1) and (3.2).

It is known that for dimension d $= 2, 3$, there exist solutions to (3.1) that decay to zero in the energy space $H^1 \times L^2 \times \hat{H}^{-1}$. Indeed, Ozawa and Tsutsumi [39], Ginibre and Velo [11], and Shimomura [43], proved existence and uniqueness of asymptotically free solutions of (3.1) by solving the system with final data given at $t \to \infty$, instead of the initial value problem; that is, for u_+ and n_+ free solutions of the Schrödinger and wave equations respectively,

$$\|u(t) - u_+(t)\|_{H^1} + \|\nabla n(t) - \nabla n_+\|_{L^2} + \|\partial_t n(t) - \partial_t n_+(t)\|_{L^2} \to 0, \text{ as } t \to \infty.$$

Completeness results were also obtained for the 3-dimensional Klein-Gordon-Zakharov (3.2). In fact, Ozawa, Tsutaya and Tsutsumi [36] proved the existence of global solutions that behave asymptotically as free solutions in space

$$\sum_{j=0,1} \|\partial_t^j (u(t) - u_+(t))\|_{H^{52-j}} + \sum_{j=0,1} \|\partial_t^j (n(t) - n_+(t))\|_{H^{51-j}} \to 0 \quad \text{as} \quad t \to \infty.$$

In [13], Guo and Nakanishi prove that radially symmetric solutions for the Zakharov system in d $= 3$ with small energy do scatter. By using generalized Strichartz estimates for the Schrödinger equation, Guo, Lee, Nakanishi and Wang in [12] were able to improve [13] by showing scattering of small solutions without the radial assumption.

Following the idea in [13], Guo, Nakanishi and Wang [14] proved scattering in the energy space for radially symmetric solutions with small energy for the system (3.2) in three dimensions, as well. In [15], they continue the study of global dynamics of radial solutions in three dimensions and find a dichotomy between scattering and blow-up. More specifically, relying on virial identities, they show that if the initial data is radially symmetric and its energy is below the energy of the ground state then the solution to (3.2) can either (for both i $= 1, 2$):

- scatter when $J_{\mathrm{i}}(u_0) \geqslant 0$, or

- blow up in finite time when $J_{\mathrm{i}}(u_0) < 0$,

where J_{i} are scaling derivative of the static Klein-Gordon energy:

$$J_0(v) = \int |u|^2 + |\nabla u|^2 - |u|^4 \mathrm{d}x \quad \text{and} \quad J_2(v) = \int |\nabla u|^2 - \frac{3}{4}|u|^4 \mathrm{d}x.$$

The behavior of radially symmetric solutions of (3.1) was also studied in [24]. Using virial identities, Merle in [24] showed blow up at either finite or infinity time for radially symmetric solutions to (3.1) that satisfy $E_s < 0$ for $\mathrm{d} = 2, 3$. In [25], Merle improved the results in [24] by presenting lower estimates on the blow-up of the Zakharov system in the 2-dimensional case.

Notice that all positive decay/scattering results above mentioned do not deal with the case $\mathrm{d} = 1$.

From now on, we consider the one-dimensional case. In the following subsections, we introduced a reduction of order for the Zakharov and Klein-Gordon-Zakharov systems and present the main results of this work.

3.1.1 Main results for Zakharov system

In order to simplify the computations, from now on we consider system (3.1) with $\alpha = 1$, although the analysis still works for $\alpha \neq 1$. With the purpose of reducing (3.1) into a first order system, we introduce the real function v such that:

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} = nu, \\ &n_t + v_x = 0, \\ &v_t + \left(n + |u|^2\right)_x = 0, \end{aligned} \tag{3.4}$$

and

$$u(t = 0, x) = u_0(x), \;\; n(t = 0, x) = n_0(x), \;\; v(t = 0, x) = v_0.$$

Such supposition is possible because we study the Zakharov system (3.1) in the Hamiltonian case, meaning that we assume that there exists $v_0 \in L^2(\mathbb{R})$ such that $-\nabla \cdot v_0 = n_t(0)$; property that is preserved by the flow. This way, to consider $(u, n, n_t) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times \dot{H}^{-1}(\mathbb{R})$ solution of (3.1) is equivalent to study $(u, n, v) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$ solution to (3.4).

The system (3.4) preserves:

- Mass:

$$M_s(t) := \int_{\mathbb{R}} |u(t, x)|^2 \mathrm{d}x = M_s(0), \tag{3.5}$$

- Energy:

$$E_s(t) := \int_{\mathbb{R}} |u_x(t, x)|^2 + \frac{1}{2}\Big(|n(t, x)|^2 + |v(t, x)|^2\Big) + n(t, x)|u(t, x)|^2 \mathrm{d}x = E_s(0), \tag{3.6}$$

- Momentum:

$$P_s(t) := \mathrm{Im} \int_{\mathbb{R}} u(t,x)\overline{u}_x(t,x)\mathrm{d}x - \int_{\mathbb{R}} v(t,x)n(t,x)\ \mathrm{d}x = P_s(0). \tag{3.7}$$

In the present work, we show decay for solutions of (3.4) in one dimension in two different ways. On one hand, we prove decay on any compact interval for solutions to (3.4) under parity assumptions (u odd). On the other hand, we are able to show decay, without any oddness condition but with sufficient regularity ($\|u(t)\|_{H^2} \in L^\infty$), in regions along curves outside the "light cone".

Theorem 3.1. *Assume $E_s < \infty$. Let $(u,n,v) \in C\left(\mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})\right)$ be a solution of* (3.4) *such that u is odd and satisfies, for some $\varepsilon > 0$ small,*

$$\sup_{t\geqslant 0} \|u(t)\|_{H^1(\mathbb{R})} < \varepsilon. \tag{3.8}$$

Then, for every compact interval $I \subset \mathbb{R}$,

$$\lim_{t\to\infty} \|u(t)\|_{L^\infty(I)} + \|u(t)\|_{L^2(I)} + \|n(t)\|_{L^2(I)} + \|v(t)\|_{L^2(I)} = 0. \tag{3.9}$$

Remark 3.1. Asking for u to be odd implies necessarily for n to be even. This property is preserved by the flow.

Remark 3.2. The fact that u is odd allows to rule out solitary waves. The first result regarding solitary waves was stated by Wu in [48], where he proves existence and orbital stability of solutions

$$u(t,x) = \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}q(x-ct)}u_{\omega,c}(x-ct), \tag{3.10}$$

and

$$n(t,x) = n_{\omega,c}(x-ct), \tag{3.11}$$

for

$$u_{\omega,c}(x) = \sqrt{\frac{(4\omega+c^2)(1-c^2)}{2}}\,\mathrm{sech}\left(\frac{\sqrt{4\omega+c^2}}{2}x\right),$$

$$n_{\omega,c}(x) = \left(2\omega + \frac{c^2}{2}\right)\mathrm{sech}^2\left(\frac{\sqrt{4\omega+c^2}}{2}x\right), \quad q = \frac{c}{2},$$

satisfying

$$4\omega + c^2 \geqslant 0 \ \text{ and } \ 1 - c^2 > 0.$$

Angulo and Banquet [2] studied existence of periodic travelling wave forms such as (3.10)-(3.11), in this case for $u_{\omega,c}$ and $n_{\omega,c}$ being periodic functions, and prove their orbital stability as well. See also [9, 34, 17, 18, 50] for other results on solitary waves for generalized Zakharov systems.

Remark 3.3. We do not prove decay in the energy space $H^1 \times L^2 \times \dot{H}^{-1}(\mathbb{R})$. This is because uncontrolled H^2-terms emerge when considering semi-norm $\dot{H}^1$ for the solution u of the Schrödinger equation. We show L^∞ decay instead.

Remark 3.4. The result also holds for a generalized Zakharov system when adding a potential term $|u|^p u$ in the Schrödinger equation. See [23] for the details on how to treat the new non-linear term.

Remark 3.5. In [41, 22], the authors study the asymptotic behaviour of the Zakharov-Rubenchik system. They prove that solutions blow up if the energy is negative and give instability results for the solitary wave in the case d = 3. Such system is of special interest since in the supersonic limit it has been proven that it converges to (3.1).

The proof of Theorem 3.1 is based on the use of suitable virial identities. The argument follows from [20, 21], where the authors deal with the Klein Gordon case. The idea is to argue as in [23], where a functional adapted to the momentum for the nonlinear Schödinger equation was considered. Unfortunately, the identity used for the NLS equation, which allows us to conclude, it is not appropriate in this case. Instead, as in [24, 25], we need to work with a virial identity that comes from the quantity

$$P(t) := \mathrm{Im} \int_{\mathbb{R}} u(t,x)\overline{u}_x(t,x)\mathrm{d}x - \int_{\mathbb{R}} v(t,x)n(t,x)\ \mathrm{d}x.$$

Such virial has an uncontrolled term that we manage by adding the condition (3.8).

Our second result deals with decay in far field regions along curves.

Theorem 3.2. *Assume $E_s < \infty$ and $M_s < \infty$. Let (u,n,v) satisfy (3.4).*

1. *If $(u,n,v) \in C\left(\mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})\right)$, then, for any $\mu \in C^1(\mathbb{R})$ satisfying $\mu(t) \gtrsim t\log(t)^{1+\delta}$, $\delta > 0$,*
$$\lim_{t\to\infty} \|u(t)\|_{L^2(|x|\sim\mu(t))} = 0. \tag{3.12}$$

2. *If $(u,n,v) \in C\left(\mathbb{R}^+, H^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})\right)$ and there exists $f(t) \in C^1(\mathbb{R})$ a non-decreasing function such that*
$$\|u(t)\|_{H^2(\mathbb{R})} \lesssim f(t), \tag{3.13}$$
then, for any $\mu \in C^1(\mathbb{R})$ satisfying $\mu(t) \gtrsim t\log(t)^{1+\delta} f(t)$, $\delta > 0$,
$$\lim_{t\to\infty} \|u(t)\|_{H^1(|x|\sim\mu(t))} + \|n(t)\|_{L^2(|x|\sim\mu(t))} + \|v(t)\|_{L^2(|x|\sim\mu(t))} = 0. \tag{3.14}$$

A direct consequence of the proof of Theorem 3.2 is the following result for the NLS equation:

Corollary 3.3. *Let $u(t) \in H^1(\mathbb{R})$ be a solution of the non-linear Schrödinger equation*

$$\mathrm{i}u_t + u_{xx} \pm |u|^{p-1}u = 0,$$

where $1 < p < 5$, with initial data $u(t=0,x) = u_0$ satisfying $\|u(t=0)\|_{H^1(\mathbb{R})} < \infty$. Then,

$$\lim_{t\to\infty} \|u(t)\|_{L^2(|x|\sim\mu(t))} = 0.$$

The proof of Theorem 3.2 follows an argument recently introduced by Muñoz, Ponce and Saut in [32], where they deal with the long time behaviour of intermediate long wave equation. This method proves to be independent of the integrability of the equation and does not need size restriction. However, when dealing with the Zakharov system, because of the presence of uncontrolled H^2-terms in the dynamics of the H^1-norm of u, we need the additional condition (3.13). Note that such condition allows as to obtain decay of the $\|\cdot\|_{H^1}$-norm, which was not present in results established in [32].

3.1.2 Main results for Klein-Gordon-Zakharov system

We will consider system (3.2) with $\alpha = c = 1$, although the computations still hold for different values of $\alpha, c \in \mathbb{R}$. As we did for the Zakharov system (3.1), we reduce (3.2) by introducing a real function v satisfying $-\nabla \cdot v = n_t$ for all $t \geqslant 0$. That is, we get a new first order system,

$$\begin{aligned} &u_{tt} - u_{xx} + u = -nu, \\ &n_t + v_x = 0, \\ &v_t + \left(n + |u|^2\right)_x = 0, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} u(t=0,x) = u_0(x), \quad u_t(t=0,x) = u_1(x) \\ n(t=0,x) = n_0(x), \quad v(t=0,x) = v_0(x). \end{aligned}$$

The system (3.15) preserves:

- Energy:

$$\begin{aligned} E_{KG}(t) := \int_{\mathbb{R}} &|u(t,x)|^2 + |u_x(t,x)|^2 + |u_t(t,x)|^2 \\ &+ \frac{1}{2}\Big(|n(t,x)|^2 + |v(t,x)|^2\Big) + n(t,x)|u(t,x)|^2 \mathrm{d}x = E_{KG}(0), \end{aligned} \tag{3.16}$$

- Momentum:

$$P_{KG}(t) := \int_{\mathbb{R}} u_t(t,x)u_x(t,x)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} v(t,x)n(t,x)\ \mathrm{d}x = P_{KG}(0). \tag{3.17}$$

As we did for (3.4), we prove decay of solutions to (3.15) in two different ways: over compact intervals of time and over far field regions along curves. Our result for compact intervals is the following:

Theorem 3.4. *Assume $E_{KG} < \infty$. Let $(u, u_t, n, v) \in C\left(\mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})\right)$ be a solution of (3.4) such that u is odd and satisfies*

$$\sup_{t\geqslant 0} \|u(t)\|_{H^1(\mathbb{R})} \leqslant \varepsilon \quad \textit{and} \quad \sup_{t\geqslant 0} \|u_t(t)\|_{L^2(\mathbb{R})} \leqslant C \tag{3.18}$$

for some $C > 0$ and $\varepsilon > 0$ small. Then, for every compact interval $I \subset \mathbb{R}$,

$$\lim_{t\to\infty} \|u(t)\|_{H^1(I)} + \|u_t(t)\|_{L^2(I)} + \|n(t)\|_{L^2(I)} + \|v(t)\|_{L^2(I)} = 0. \tag{3.19}$$

Remark 3.6. The oddness condition rules out solitary waves. Indeed, solitary waves of (3.15) exist and they are orbitally stable. They were first introduced by Chen in [6], where he stated that solitons of the form

$$\begin{aligned} u(t,x) &= \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}q(x-ct)}u_{\omega,c}(x-ct), \\ n(t,x) &= n_{\omega,c}(x-ct), \end{aligned}$$

with

$$\begin{aligned} u_{\omega,c}(x) &= \sqrt{2(1-c^2-\omega^2)}\,\mathrm{sech}\left(\frac{\sqrt{1-c^2-\omega^2}}{1-c^2}x\right), \\ n_{\omega,c}(x) &= -2\frac{(1-c^2-\omega^2)}{1-c^2}\,\mathrm{sech}^2\left(\frac{\sqrt{1-c^2-\omega^2}}{1-c^2}x\right), \quad q=\frac{\omega c}{1-c^2}. \end{aligned}$$

exists when the real constants ω and c satisfy $1-c^2-\omega^2>0$. There also exist solitary waves $(u_{\omega,c},(u_t)_{\omega,c},n_{\omega,c},v_{\omega,c})$ of (3.15) of the form

$$\begin{aligned} u_{\omega,c}(x) = \sqrt{2(1-c^2-\omega^2)}\,\mathrm{sech}\left(\frac{\sqrt{1-c^2-\omega^2}}{1-c^2}x\right)\mathrm{e}^{\mathrm{i}\frac{\omega c}{1-c^2}x}, &\quad (u_t)_{\omega,c} = \left(\mathrm{i}\omega + c\frac{\partial}{\partial x}\right)u_{\omega,c}(x), \\ n_{\omega,c}(x) &= -2\frac{(1-c^2-\omega^2)}{1-c^2}\,\mathrm{sech}^2\left(\frac{\sqrt{1-c^2-\omega^2}}{1-c^2}x\right), \\ v_{\omega,c}(x) &= 2c\frac{(1-c^2-\omega^2)}{1-c^2}\,\mathrm{sech}^2\left(\frac{\sqrt{1-c^2-\omega^2}}{1-c^2}x\right), \end{aligned}$$

with $1-2c^2-2\omega^2<0$ and are orbitally stable [6].

Remark 3.7. The result holds when considering cubic nonlinear KGZ system, that is, when adding an additional term $|u|^2u$ in the Klein Gordon equation. We do not address this case in the proof, but it follows naturally from the analysis of the non-linear term in [20].

The proof of this results follows more closely the idea in [20]. Indeed, we construct a virial identity that comes from the momentum:

$$P_{KG} = \int_{\mathbb{R}} u_t(t,x)u_x(t,x)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} v(t,x)n(t,x)\mathrm{d}x.$$

But, since there are uncontrolled terms involving u and u_t in the identity from the potential, we need to consider u_t uniformly bounded. Notice that we obtain now decay in the whole energy norm (that is, even for the H^1-norm).

The last theorem is devoted to the decay of the solutions to (3.15) in regions along curves outside the light cone:

Theorem 3.5. *Assume $E_{KG}<0$. If $(u,u_t,n,v)\in C\left(\mathbb{R}^+, H^1(\mathbb{R})\times L^2(\mathbb{R})\times L^2(\mathbb{R})\times L^2(\mathbb{R})\right)$ is a solution to* (3.15) *such that*

$$\sup_{t\geqslant 0}\|u(t)\|_{H^1(\mathbb{R})}\leqslant\varepsilon \tag{3.20}$$

for some $0<\varepsilon\leqslant 1$, then, for any $\mu\in C^1(\mathbb{R})$ satisfying $\mu(t)\gtrsim t\log(t)^{1+\delta}$, $\delta>0$,

$$\lim_{t\to\infty}\|u(t)\|_{H^1(|x|\sim\mu(t))}+\|u_t(t)\|_{L^2(|x|\sim\mu(t))}+\|n(t)\|_{L^2(|x|\sim\mu(t))}+\|v(t)\|_{L^2(|x|\sim\mu(t))}=0. \tag{3.21}$$

Notation

We introduce

$$\begin{aligned}\|u(t)\|^2_{L^2_\omega(\mathbb{R})} &:= \int_{\mathbb{R}} \omega(x)|u(t,x)|^2 \mathrm{d}x,\\ \|u(t)\|^2_{H^1_\omega(\mathbb{R})} &:= \int_{\mathbb{R}} \omega(x)\left(|u_x(t,x)|^2 + |u(t,x)|^2\right)\mathrm{d}x,\end{aligned} \tag{3.22}$$

as the weighted L^2-norm and H^1-norm.

This paper is organized as follows. In Section 3.2 we prove Theorem 3.1; the virial argument is given in Subsection 3.2.1. Section 3.3 is devoted to the proof of Theorem 3.2. Sections 3.4 and 3.5 contain the KGZ system results, Theorems 3.4 and 3.5, respectively.

3.2 Decay on compact intervals for Zakharov

This section is devoted to the proof of Theorem 3.1. Before we begin with the virial analysis, we give the following result, which states boundness of the energy norm for every solution to (3.4) with finite energy, It will be useful also in Section 3.3.

Lemma 3.6. *Let $(u,n,v) \in C\left(\mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})\right)$ be a solution of (3.4) such that $E_s < \infty$ and $M_s < \infty$. Then, there exists $K_s > 0$ (K_s depending only on the initial data) such that*

$$\int_{\mathbb{R}} \left(|u_x(t,x)|^2 + |u(t,x)|^2 + |v(t,x)|^2 + |n(t,x)|^2\right)\mathrm{d}x \leqslant K_s. \tag{3.23}$$

Proof. We have that

$$\begin{aligned}\int_{\mathbb{R}} |u_x|^2 + \frac{1}{2}\left(|v|^2 + |n|^2\right)\mathrm{d}x &= \int_{\mathbb{R}} |u_x|^2 + \frac{1}{2}\left(|v|^2 + |n|^2\right) + 2n|u|^2\mathrm{d}x - 2\int_{\mathbb{R}} n|u|^2\mathrm{d}x\\ &\leqslant \int_{\mathbb{R}} |u_x|^2 + \frac{1}{2}\left(|v|^2 + |n|^2\right) + 2n|u|^2\mathrm{d}x + 2\int_{\mathbb{R}} |n||u|^2\mathrm{d}x.\end{aligned}$$

Using Young inequality for products, for $\epsilon > 0$ we get

$$\begin{aligned}&\int_{\mathbb{R}} |u_x|^2 + \frac{1}{2}\left(|v|^2 + |n|^2\right)\mathrm{d}x\\ &\leqslant \int_{\mathbb{R}} |u_x|^2 + \frac{1}{2}\left(|v|^2 + |n|^2\right) + 2n|u|^2\mathrm{d}x + \frac{1}{\epsilon}\int_{\mathbb{R}} |n|^2\mathrm{d}x + \epsilon\int_{\mathbb{R}} |u|^4\mathrm{d}x.\end{aligned} \tag{3.24}$$

Now, Gagliardo-Nirenberg inequality [19, 33], implies that

$$\int_{\mathbb{R}} |u|^4\mathrm{d}x \leqslant C_{GN}\|u_x\|_{L^2(\mathbb{R})}\|u\|^3_{L^2(\mathbb{R})} \leqslant \frac{C_{GN}}{2}\|u_x\|^2_{L^2(\mathbb{R})} + \frac{C_{GN}}{2}\|u\|^6_{L^2(\mathbb{R})},$$

where

$$C_{GN} = \frac{\sqrt{3}}{3},$$

and Q is the solution to $Q''+Q^3-Q=0$. Then, going back to (3.24) and taking, for instance, $\epsilon=2$, we obtain

$$\int_{\mathbb{R}} |\nabla u|^2 + \frac{1}{2}\left(|v|^2+|n|^2\right)\mathrm{d}x \leqslant 2E_s(0) + \frac{\sqrt{3}}{6}M_s(0)^3$$

Which means that there exists a constant K_s depending on $M_s(0)$ and $E_s(0)$ such that (3.23) holds. ∎

3.2.1 Virial argument

Step 1: Virial identity.
We now introduce suitable virial identities that allow us to work out our argument. Let $\varphi \in C^\infty(\mathbb{R})$ be a bounded real function. Define

$$I(t) = \mathrm{Im}\int_{\mathbb{R}} \varphi(x)u(t,x)\overline{u}_x(t,x)\mathrm{d}x - \int_{\mathbb{R}} \varphi(x)v(t,x)n(t,x)\mathrm{d}x.$$

Then, we have the following:

Lemma 3.7 (Virial identity). *Let (u,n,v) be a solution to (3.4). Then,*

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) = 2\int_{\mathbb{R}} \varphi'(x)\left|u_x(t,x)\right|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi'''(x)|u(t,x)|^2\mathrm{d}x + \int_{\mathbb{R}} \varphi'(x)n(t,x)|u(t,x)|^2 \\ + \frac{1}{2}\int_{\mathbb{R}} \varphi'(x)|n(t,x)|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varphi'|v(t,x)|^2\mathrm{d}x.\end{aligned} \tag{3.25}$$

Proof. We compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) =& \mathrm{Im}\int_{\mathbb{R}} \varphi(x)u_t(t,x)\overline{u}_x(t,x)\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}} \varphi(x)u(t,x)\overline{u}_{tx}(t,x)\mathrm{d}x \\ &- \int_{\mathbb{R}} \varphi(x)v_t(t,x)n(t,x)\mathrm{d}x - \int_{\mathbb{R}} \varphi(x)v(t,x)n_t(t,x)\mathrm{d}x.\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}I(t) &= 2\mathrm{Im}\int_{\mathbb{R}} \varphi(x)u_t(t,x)\overline{u}_x(t,x)\mathrm{d}x - \mathrm{Im}\int_{\mathbb{R}} \varphi'(x)u(t,x)\overline{u}_t(t,x)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}} \varphi(x)v_t(t,x)n(t,x)\mathrm{d}x - \int_{\mathbb{R}} \varphi(x)v(t,x)n_t(t,x)\mathrm{d}x \\ &= -2\mathrm{Re}\int_{\mathbb{R}} \varphi(x)\mathrm{i}u_t(t,x)\overline{u}_x(t,x)\mathrm{d}x - \mathrm{Re}\int_{\mathbb{R}} \varphi'(x)u(t,x)\overline{\mathrm{i}u_t}(t,x)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}} \varphi(x)v_t(t,x)n(t,x)\mathrm{d}x - \int_{\mathbb{R}} \varphi v(t,x)n_t(t,x)\mathrm{d}x.\end{aligned}$$

Now, since (u,n,v) is a solution of the system (3.4),

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}I(t) = & \int_{\mathbb{R}} \varphi(x)\left(|u_x(t,x)|^2\right)_x \mathrm{d}x - \int_{\mathbb{R}} \varphi(x)n(t,x)\left(|u(t,x)|^2\right)_x \mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}} \varphi'(x)u(t,x)\overline{u}_{xx}(t,x)\mathrm{d}x \\
& - \int_{\mathbb{R}} \varphi'(x)|u(t,x)|^2 n(t,x)\mathrm{d}x + \int_{\mathbb{R}} \varphi(x)\left(|u(t,x)|^2 + n(t,x)\right)_x n(t,x)\mathrm{d}x \\
& + \frac{1}{2}\int_{\mathbb{R}} \varphi(x)\left(|v(t,x)|^2\right)_x \mathrm{d}x.
\end{aligned}
$$

We integrate by parts once more and get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}I(t) = & -2\int_{\mathbb{R}} \varphi'(x)\left|u_x(t,x)\right|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi''(x)\left(|u(t,x)|^2\right)_x \mathrm{d}x - \int_{\mathbb{R}} \varphi'(x)n(t,x)|u(t,x)|^2 \\
& - \int_{\mathbb{R}} \varphi'(x)|n(t,x)|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varphi(x)\left(|n(t,x)|^2\right)_x \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi'(x)|v(t,x)|^2\mathrm{d}x \\
= & -2\int_{\mathbb{R}} \varphi'(x)\left|u_x(t,x)\right|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varphi'''(x)|u(t,x)|^2\mathrm{d}x - \int_{\mathbb{R}} \varphi'(x)n(t,x)|u(t,x)|^2 \\
& - \frac{1}{2}\int_{\mathbb{R}} \varphi'(x)|n(t,x)|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi'(x)|v(t,x)|^2\mathrm{d}x.
\end{aligned}
$$

Then, the identity follows. ■

Step 2: Estimations of the terms on the virial.
In order to find a more compact expression of (3.25), we define the bilinear form

$$B(u) = 2\int_{\mathbb{R}} \varphi'\left|u_x\right|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi'''|u|^2\mathrm{d}x. \tag{3.26}$$

Then identity (3.25) turns into

$$-\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) = B(u) + \int_{\mathbb{R}} \varphi' n|u|^2 + \frac{1}{2}\int_{\mathbb{R}} \varphi'|n|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varphi'|v|^2\mathrm{d}x. \tag{3.27}$$

Following the argument in [23], for $\lambda > 0$, let us take $\varphi(x) = \lambda\tanh\left(\frac{x}{\lambda}\right)$ and $\omega(x) = \sqrt{\varphi'(x)}$. Notice that if $u = u_1 + \mathrm{i}u_2$, where u_1, u_2 are real functions, then $B(u) = B(u_1) + B(u_2)$. We take $\eta = u_{\mathrm{i}}$, $\mathrm{i} = 1,2$, and find estimations for $B(\eta)$. From now on, we are going to assume u odd, which implies that η is also odd. Note that, by integration by parts,

$$
\begin{aligned}
\int_{\mathbb{R}} (\omega\eta)_x^2\,\mathrm{d}x &= \int_{\mathbb{R}} \varphi'\left(\eta_x\right)^2\mathrm{d}x + \int_{\mathbb{R}} \omega\omega'\left(\eta^2\right)_x\mathrm{d}x + \int_{\mathbb{R}} \left(\omega'\right)^2\eta^2\mathrm{d}x \\
&= \int_{\mathbb{R}} \varphi'\left(\eta_x\right)^2\mathrm{d}x - \int_{\mathbb{R}} \omega\omega''\eta^2\mathrm{d}x,
\end{aligned}
$$

It follows that

$$\int_{\mathbb{R}} \varphi'\left(\eta_x\right)^2\mathrm{d}x = \int_{\mathbb{R}} (\omega\eta)_x^2\,\mathrm{d}x + \int_{\mathbb{R}} \frac{\omega''}{\omega}(\omega\eta)^2\,\mathrm{d}x. \tag{3.28}$$

On the other hand, we can re-write $\varphi''' = (\omega^2)'' = 2\left(\omega\omega'' + (\omega')^2\right)$, and obtain

$$\int_{\mathbb{R}} \varphi''' \eta^2 \mathrm{d}x = 2\int_{\mathbb{R}} \left(\frac{\omega''}{\omega} + \frac{(\omega')^2}{\omega^2}\right) (\omega\eta)^2 \,\mathrm{d}x. \tag{3.29}$$

Thus, from (3.28) and (3.29),

$$B(\eta) = 2\int_{\mathbb{R}} (\omega\eta)_x^2 \,\mathrm{d}x - \int_{\mathbb{R}} \left(\frac{(\omega')^2}{\omega^2} - \frac{\omega''}{\omega}\right) (\omega\eta)^2 \,\mathrm{d}x.$$

Since $\omega(x) = \operatorname{sech}\left(\frac{x}{\lambda}\right)$, then

$$B(\eta) = 2\int_{\mathbb{R}} (\omega\eta)_x^2 \,\mathrm{d}x - \frac{1}{\lambda^2}\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) (\omega\eta)^2 \,\mathrm{d}x.$$

Now, introducing a new variable $\zeta = \omega\eta$, we set

$$\mathcal{B}(\zeta) = 2\int_{\mathbb{R}} \zeta_x^2 \mathrm{d}x - \frac{1}{\lambda^2}\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) \zeta^2 \mathrm{d}x$$

so that

$$\mathcal{B}(\zeta) = B(\eta).$$

At this point, we would like to prove that the bilinear $\mathcal{B}$ is coercive, which would imply

$$B(\eta) = \mathcal{B}(\zeta) \gtrsim \|\zeta_x\|_{L^2(\mathbb{R})}^2 = \|\left(\omega\eta\right)_x\|_{L^2(\mathbb{R})}^2$$

That way, we could have an estimation for the bilinear part on (3.27), $B(u)$, using the weighted norm $\|\cdot\|_{H^1_\omega}$.

Lemma 3.8 (See [23]). *Let $\zeta \in H^1(\mathbb{R})$ be odd. Then,*

$$\mathcal{B}(\zeta) \geqslant \frac{3}{2}\int_{\mathbb{R}} \zeta_x^2 \mathrm{d}x.$$

We refer to [23, Proposition 2.2] for the details of the proof.

Finally, to conclude the analysis of the linear term $B(u)$, we need to bound this term by $\|u\|_{H^1_\omega(\mathbb{R})}$.

Lemma 3.9 (See [23]). *Let $u \in H^1(\mathbb{R})$ be odd, $u = u_1 + \mathrm{i}u_2$. Then there exists a positive constant $c_0 < 1$ such that*

$$B(u) \geqslant c_0\|u\|_{H^1_\omega(\mathbb{R})}^2 \tag{3.30}$$

We omit the proof, see [23, Lemma 2.3].

Step 3: Conclusion of the argument.
The key ingredient of the virial argument is the subsequent proposition:

Proposition 3.10. *Let (u, n, v) be a solution of* (3.4) *such that u is odd and satisfies* (3.8)*, for $\varepsilon > 0$ sufficiently small. Then, there exists $C > 0$ such that*

$$\int_0^\infty \|u(t)\|^2_{H^1_w(\mathbb{R})} + \frac{1}{2}\|v(t)\|^2_{L^2_w(\mathbb{R})} + \frac{1}{2}\|n(t)\|^2_{L^2_w(\mathbb{R})}\mathrm{d}t \leqslant C. \tag{3.31}$$

In particular,

$$\int_0^\infty \|u_x(t)\|^2_{L^2_w(\mathbb{R})} + \frac{1}{2}\|v(t)\|^2_{L^2_w(\mathbb{R})} + \frac{1}{2}\|n(t)\|^2_{L^2_w(\mathbb{R})}\mathrm{d}t \leqslant C. \tag{3.32}$$

Proof. From (3.27) and (3.30), we have that

$$-\frac{\mathrm{d}}{\mathrm{d}t}I(t) \geqslant c_0\|u(t)\|^2_{H^1_\omega(\mathbb{R})} + \frac{1}{2}\int_{\mathbb{R}} \varphi'\left(v^2 + n^2\right)\mathrm{d}x + \int_{\mathbb{R}} \varphi' n|u|^2\mathrm{d}x. \tag{3.33}$$

Now, following the idea of the proof of Lemma 3.6, by Young inequality, for some $\epsilon > 0$ we get

$$\int_{\mathbb{R}} \varphi'|n||u|^2\mathrm{d}x \leqslant \frac{1}{2\epsilon}\int_{\mathbb{R}} \varphi' n^2\mathrm{d}x + \frac{\epsilon}{2}\int_{\mathbb{R}} \varphi'|u|^4\mathrm{d}x. \tag{3.34}$$

At this point, we need to absorb the negative terms using the weighted-norm (3.22). Since u is odd,

$$\begin{aligned}
\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right)|u|^4\mathrm{d}x &= 2\int_0^\infty \operatorname{sech}^2\left(\frac{x}{\lambda}\right)|u|^4\mathrm{d}x \\
&= 2\int_0^\infty \operatorname{sech}^{-2}\left(\frac{x}{\lambda}\right)\operatorname{sech}^4\left(\frac{x}{\lambda}\right)|u|^4 \\
&\simeq \int_0^\infty \mathrm{e}^{2x/\lambda}\operatorname{sech}^4\left(\frac{x}{\lambda}\right)|u|^4\mathrm{d}x.
\end{aligned}$$

With a slight abuse of notation, set $\zeta(t, x) := \operatorname{sech}\left(\frac{x}{\lambda}\right)u(t, x)$. Note that $\zeta(t, 0) = 0$ and vanishes at infinity $\forall t \in \mathbb{R}$. Then, integrating by parts,

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{2x/\lambda}|\zeta|^4\mathrm{d}x &= -\frac{\lambda}{2}\int_0^\infty \mathrm{e}^{2x/\lambda}\left(|\zeta|^4\right)_x\mathrm{d}x \\
&= -2\lambda\mathrm{Re}\int_0^\infty \mathrm{e}^{2x/\lambda}|\zeta|^2\bar{\zeta}\zeta_x\mathrm{d}x.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{2x/\lambda}|\zeta|^4\mathrm{d}x &= -2\lambda\mathrm{Re}\int_0^\infty \mathrm{e}^{x/\lambda}|\zeta|\bar{\zeta}\zeta_x\left(\mathrm{e}^{x/\lambda}|\zeta|\right)\mathrm{d}x \\
&\lesssim \|u\|_{L^\infty(\mathbb{R})}\mathrm{Re}\int_0^\infty \mathrm{e}^{x/\lambda}|\zeta|\bar{\zeta}\zeta_x\mathrm{d}x \\
&\lesssim \|u\|_{L^\infty(\mathbb{R})}\int_0^\infty \mathrm{e}^{x/\lambda}|\zeta|^2|\zeta_x|\mathrm{d}x.
\end{aligned}$$

By Young's inequality,

$$\begin{aligned}\int_0^\infty \mathrm{e}^{2x/\lambda}|\zeta|^4 \mathrm{d}x &\lesssim \|u\|_{L^\infty(\mathbb{R})}\int_0^\infty |\zeta_x|^2\mathrm{d}x + \|u\|_{L^\infty(\mathbb{R})}\int_0^\infty \mathrm{e}^{2x/\lambda}|\zeta|^4\mathrm{d}x\\
&\simeq \|u\|_{L^\infty(\mathbb{R})}\int_0^\infty |\zeta_x|^2\mathrm{d}x + \|u\|_{L^\infty(\mathbb{R})}\int_0^\infty \operatorname{sech}^{-2}\left(\frac{x}{\lambda}\right)\operatorname{sech}^4\left(\frac{x}{\lambda}\right)|u|^4\mathrm{d}x\\
&= \|u\|_{L^\infty(\mathbb{R})}\int_0^\infty |\zeta_x|^2\mathrm{d}x + \|u\|_{L^\infty(\mathbb{R})}\int_0^\infty \operatorname{sech}^{2}\left(\frac{x}{\lambda}\right)|u|^4\mathrm{d}x.\end{aligned}$$

By Sobolev's embedding and (3.8) with $0 < \varepsilon < 1$, this actually means that

$$\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right)|u|^4\mathrm{d}x \lesssim \varepsilon\int_0^\infty |\left(\omega u\right)_x|^2\mathrm{d}x.$$

From Lemma 3.8 and Lemma 3.9, we obtain

$$\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right)|u|^4\mathrm{d}x \lesssim c_0\varepsilon\|u\|^2_{H^1_\omega}.$$

Then, going back to (3.33), taking $\epsilon = 2$ and ε sufficiently small, one gets

$$-\frac{\mathrm{d}}{\mathrm{d}t}I \gtrsim \|u(t)\|^2_{H^1_\omega(\mathbb{R})} + \|n(t)\|^2_{L^2_\omega(\mathbb{R})} + \|v(t)\|^2_{L^2_\omega(\mathbb{R})}.$$

Now, we integrate in time over $[0,\tau]$ for $\tau > 0$,

$$\int_0^\tau \|u(t)\|^2_{H^1_\omega(\mathbb{R})} + \|n(t)\|^2_{L^2_\omega(\mathbb{R})} + \|v(t)\|^2_{L^2_\omega(\mathbb{R})}\mathrm{d}t \lesssim |I(\tau)| + |I(0)|.$$

Thanks to Lemma 3.6, one obtains that

$$|I(t)| \leqslant \|u(t)\|^2_{H^1(\mathbb{R})} + \|n(t)\|^2_{L^2(\mathbb{R})} + \|v(t)\|^2_{L^2(\mathbb{R})} \leqslant K_s, \quad \forall t \geqslant 0.$$

Finally, taking $\tau \to \infty$, we conclude. ■

3.2.2 Proof of Theorem 3.1:

Now, we proceed to conclude the proof of Theorem 3.1.

Let $\phi \in C^\infty(\mathbb{R})$. Using equation (3.4), we can compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\phi\left(|u|^2+|v|^2+|n|^2\right)\mathrm{d}x = -\,2\mathrm{Im}\int_{\mathbb{R}}\phi' u\overline{u}_x\mathrm{d}t - 2\int_{\mathbb{R}}\phi v\left(|u|^2+n\right)_x\mathrm{d}x - 2\int_{\mathbb{R}}\phi n v_x\mathrm{d}x.$$

By integration by parts, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\phi\left(|u|^2+|v|^2+|n|^2\right)\mathrm{d}x = -2\mathrm{Im}\int_{\mathbb{R}}\phi' u\overline{u}_x\mathrm{d}t + 2\int_{\mathbb{R}}\phi' vn\ \mathrm{d}x - 2\int_{\mathbb{R}}\phi v\left(|u|^2\right)_x\mathrm{d}x.$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\phi\left(|u|^2+|v|^2+|n|^2\right)\mathrm{d}x=-2\mathrm{Im}\int_{\mathbb{R}}\phi' u\overline{u}_x\mathrm{d}x+2\int_{\mathbb{R}}\phi' vn\ \mathrm{d}x-4\mathrm{Re}\int_{\mathbb{R}}\phi vu\overline{u}_x\mathrm{d}x. \quad (3.35)$$

From Hölder inequality and (3.8), taking $\phi(x)=\mathrm{sech}(x)$, we can conclude that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|^2_{L^2_w(\mathbb{R})}+\frac{1}{2}\|v(t)\|^2_{L^2_w(\mathbb{R})}+\frac{1}{2}\|n(t)\|^2_{L^2_w(\mathbb{R})}\right)\\&\leqslant 2\int_{\mathbb{R}}\mathrm{sech}(x)\left(|u|^2+|u_x|^2+|v|^2+|n|^2\right)\mathrm{d}x+2\varepsilon\int_{\mathbb{R}}\mathrm{sech}(x)\left(|v|^2+|u_x|^2\right)\mathrm{d}x\\&\lesssim \|u(t)\|^2_{H^1_w(\mathbb{R})}+\frac{1}{2}\|n(t)\|^2_{L^2_w(\mathbb{R})}+\frac{1}{2}\|v(t)\|^2_{L^2_w(\mathbb{R})}.\end{aligned} \quad (3.36)$$

By (3.32) we have that there exists a sequence $\{t_n\}\subset\mathbb{R}$, $t_n\to\infty$ such that

$$\|u(t_n)\|^2_{L^2_w(\mathbb{R})}+\frac{1}{2}\|v(t_n)\|^2_{L^2_w(\mathbb{R})}+\frac{1}{2}\|n(t_n)\|^2_{L^2_w(\mathbb{R})}\to 0.$$

We integrate (3.36) over $[t,t_n]$, for some $t\in\mathbb{R}$ and take $t_n\to\infty$.

$$\begin{aligned}&\|u(t)\|^2_{L^2_w(\mathbb{R})}+\frac{1}{2}\|v(t)\|^2_{L^2_w(\mathbb{R})}+\frac{1}{2}\|n(t)\|^2_{L^2_w(\mathbb{R})}\\&\lesssim\int_t^{\infty}\|u(s)\|^2_{H^1_w(\mathbb{R})}+\frac{1}{2}\|n(s)\|^2_{L^2_w(\mathbb{R})}+\frac{1}{2}\|v(s)\|^2_{L^2_w(\mathbb{R})}\mathrm{d}s.\end{aligned}$$

Finally, taking $t\to\infty$, in view of (3.32), we obtain

$$\lim_{t\to\infty}\|u(t)\|^2_{L^2_w(\mathbb{R})}+\|v(t)\|^2_{L^2_w(\mathbb{R})}+\|n(t)\|^2_{L^2_w(\mathbb{R})}=0. \quad (3.37)$$

To show decay of the L^{∞}-norm, we use the following claim, proven in [23]:

Claim 3.11. *For every interval I there exists $\tilde{x}(t)\in I$ such that, as t tends to infinity,*

$$|u(t,\tilde{x}(t))|^2\to 0.$$

Now, if $x\in I$, by Fundamental Theorem of calculus and Hölder's inequality

$$\begin{aligned}|u(t,x)|^2-|u(t,\tilde{x}(t))|^2&=\int_{\tilde{x}(t)}^{x}\left(|u|^2\right)_x\mathrm{d}x\leqslant 2\int_{\tilde{x}(t)}^{x}|u||u_x|\mathrm{d}x\\&\leqslant 2\|u(t)\|_{L^2(I)}\|u_x(t)\|_{L^2(I)}.\end{aligned}$$

Then,

$$|u(t,x)|^2\lesssim|u(t,\tilde{x}(t))|^2+2\|u(t)\|_{L^2(I)}\|u_x(t)\|_{L^2(I)},\quad \forall x\in I. \quad (3.38)$$

Using Lemma 3.6, we get

$$\sup_{t\in\mathbb{R}}\|u(t)\|_{H^1(\mathbb{R})}<\infty.$$

Hence, taking $t\to\infty$ in (3.38), from Claim 3.11 and (3.37), we get that

$$|u(t,x)|^2\to 0,\quad \forall x\in I.$$

3.3 Decay in regions along curves for Zakharov

From now on, let us assume $\lambda, \mu \in C^1(\mathbb{R})$ are functions depending on time. For $\varphi \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, define

$$K(t) = \frac{1}{2}\int_{\mathbb{R}} \varphi\left(\frac{x+\mu(t)}{\lambda(t)}\right)|u(t,x)|^2 \mathrm{d}x,$$

and

$$J(t) = \int_{\mathbb{R}} \varphi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|u_x(t,x)|^2 + \frac{1}{2}|v(t,x)|^2 + \frac{1}{2}|n(t,x)|^2 + n(t,x)|u(t,x)|^2 + |u(t,x)|^2\right)\mathrm{d}x.$$

As we did in the previous section, we obtain a virial identity from which we are going to construct the argument.

Lemma 3.12. *Let $(u,n,v) \in H^1 \times L^2 \times L^2$ a solution to (3.4). Then,*

1.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}K(t) =& \frac{1}{\lambda(t)} Im \int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\overline{u}(t,x)u_x(t,x)\mathrm{d}x + \frac{\mu'(t)}{2\lambda(t)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)|u(t,x)|^2\mathrm{d}x \\ &- \frac{\lambda'(t)}{2\lambda(t)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\frac{x+\mu(t)}{\lambda(t)}\right)|u(t,x)|^2\mathrm{d}x.\end{aligned} \tag{3.39}$$

2.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}J(t) =& \frac{1}{\lambda(t)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\left(n+|u|^2\right)v\right)(t,x)\mathrm{d}x \\ &+ \frac{2}{\lambda(t)} Im \int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\overline{u}_x u_{xx} + n\overline{u}u_x + \overline{u}u_x\right)(t,x)\mathrm{d}x \\ &+ \frac{\mu'(t)}{2\lambda(t)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(2|u_x|^2 + 2n|u|^2 + |u|^2 + |v|^2 + |n|^2\right)(t,x)\mathrm{d}x \\ &- \frac{\lambda'(t)}{2\lambda(t)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(2|u_x|^2 + 2n|u|^2 + |u|^2 + |v|^2 + |n|^2\right)(t,x)\mathrm{d}x.\end{aligned} \tag{3.40}$$

Proof. The proof follows from (3.35). ■

Let us consider $\varphi \in C^2(\mathbb{R})$ a decreasing bounded funtion such that $\varphi(s) = 1$ for $s \leqslant -1$ and $\varphi(s) = 0$ for $s \geqslant 0$. Thus, we get that $\mathrm{supp}(\varphi') \subset [-1,0]$ and

$$\varphi'(s) \leqslant 0, \quad \varphi'(s)s \geqslant 0 \quad \forall s \in \mathbb{R}.$$

Now, we want to take λ such that λ^{-1} integrates finite in time over an interval such as $[T,\infty]$, $T > 0$. With this idea in mind, define, for $\delta > 0$ and $t \geqslant 2$, $\lambda(s) = t\log^{1+\delta}(t)$ and $\mu(t) = \lambda(t)$.

In fact, we are considering $\mu = \lambda$ but in the following we will see that is possible to take $\mu(t) \gtrsim \lambda(t)$ and the computations would still work. Then, for $t \geqslant 2$ we have

$$\frac{\lambda'(t)}{\lambda(t)} = \frac{\mu'(t)}{\lambda(t)} \geqslant \frac{1}{t} + \frac{1+\delta}{t\log(t)}.$$

So now, whereas $\lambda^{-1}(t)$ is integrable in $[2,\infty]$, $\frac{\lambda'}{\lambda}$ is not.

3.3.1 First part of the proof of Theorem 3.2

In this subsection, we prove (3.12). The idea is to take adventage of the non-integrability of $\frac{\lambda'}{\lambda}$ (and of $\frac{\mu'}{\lambda}$, as well) by using the virial identity from $K(t)$. Let us rearrange the terms in (3.39):

$$\begin{aligned}\frac{\lambda'(t)}{2\lambda(t)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\frac{x+\mu(t)}{\lambda(t)}\right)|u(t,x)|^2\mathrm{d}x - \frac{\mu'(t)}{2\lambda(t)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)|u(t,x)|^2\mathrm{d}x \\ = -\frac{\mathrm{d}}{\mathrm{d}t}K(t) + \frac{1}{\lambda(t)}\mathrm{Im}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\overline{u}(t,x)u_x(t,x)\mathrm{d}x\end{aligned} \tag{3.41}$$

Notice that, because of our election of φ, each term on the LHS is positive. Now, our aim is to control the RHS so that we get integrability on that part of the equation. Indeed, computing the integral of the RHS over $[2,\infty]$, from Lemma 3.6 we have that

$$\begin{aligned}\int_2^\infty \frac{1}{\lambda(\tau)}\mathrm{Im}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(t)}{\lambda(\tau)}\right)\overline{u}(\tau,x)u_x(\tau,x)\mathrm{d}x\mathrm{d}\tau &\leqslant \int_2^\infty \frac{1}{\lambda(\tau)}\|u(\tau)\|_{L^2(\mathbb{R})}\|u_x(\tau)\|_{L^2(\mathbb{R})}\mathrm{d}\tau \\ &\lesssim \int_2^\infty \frac{1}{\lambda(\tau)}\mathrm{d}\tau < \infty.\end{aligned}$$

Thus, we integrate equation (3.41) in time and obtain:

$$\int_2^\infty \frac{\lambda'(\tau)}{2\lambda(\tau)}\int_{\mathbb{R}}\left[\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right) - \varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\right]|u(\tau,x)|^2\mathrm{d}x\mathrm{d}\tau < \infty. \tag{3.42}$$

This implies the existence of a sequence $\{t_n\} \subset \mathbb{R}$, $t_n \to \infty$, such that

$$\int_{\mathbb{R}}\left[\varphi'\left(\frac{x+\mu(t_n)}{\lambda(t_n)}\right)\left(\frac{x+\mu(t_n)}{\lambda(t_n)}\right) - \varphi'\left(\frac{x+\mu(t_n)}{\lambda(t)}\right)\right]|u(t_n,x)|^2\mathrm{d}x \to 0. \tag{3.43}$$

Now, take $\phi \in C_0^1(\mathbb{R})$ such that $\phi(s) \in [0,1]$ for all $s \in \mathbb{R}$, $\mathrm{supp}(\phi) = [-3/4, -1, 4]$, satisfying

$$\phi(s) \lesssim |\varphi'(s)| \text{ and } |\phi'(s)| \lesssim |\varphi'(s)| \text{ for all } s \in \mathbb{R}.$$

Then, if we consider ϕ instead φ in (3.39), we obtain

$$\begin{aligned}&\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)|u(t,x)|^2\mathrm{d}x\right| \\ &\lesssim \frac{1}{\lambda(t)} + \frac{\mu'(t)}{2\lambda(t)}\int_{\mathbb{R}}\left|\phi'\left(\frac{x+\mu(t)}{\lambda(t)}\right) - \phi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\frac{x+\mu(t)}{\lambda(t)}\right)\right||u(t,x)|^2\mathrm{d}x \\ &\lesssim \frac{1}{\lambda(t)} + \frac{\mu'(t)}{2\lambda(t)}\int_{\mathbb{R}}\left|\phi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\right||u(t,x)|^2\mathrm{d}x,\end{aligned}$$

Integrating over $[t, t_n]$ and taking $t_n \to \infty$, one gets from (3.43) that

$$\left|\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)|u(t,x)|^2 \mathrm{d}x\right| \lesssim \int_t^\infty \frac{1}{\lambda(\tau)}\mathrm{d}\tau + \int_t^\infty \frac{\mu'(\tau)}{2\lambda(\tau)}\int_{\mathbb{R}}\left|\phi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\right||u(\tau,x)|^2 \mathrm{d}x\mathrm{d}\tau.$$

Thus, because (3.42) holds, we can take $t \to \infty$ and conclude that

$$\lim_{t\to\infty}\left|\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)|u(t,x)|^2 \mathrm{d}x\right| \lesssim 0$$

Finally, the region of the convergence comes from the fact that $-\frac{3}{4} \leqslant \frac{x+\mu(t)}{\lambda(t)} - \frac{1}{4}$ is equivalent to $x \sim \mu(t)$.

3.3.2 Second part of the proof of Theorem 3.2

This section deals with the proof of (3.14). The idea of the proof is the same as before, but since our aim is to show decay of the solution (u, n, v), we need to consider the virial identity (3.40). As before, we re-write the terms in (3.40) and get

$$\begin{aligned}
&\frac{\lambda'(t)}{2\lambda(t)}\int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\frac{x+\mu(t)}{\lambda(t)}\right)\Big(2|u_x|^2 + 2n|u|^2 + |u|^2 + |v|^2 + |n|^2\Big)(t,x)\mathrm{d}x \\
&-\frac{\mu'(t)}{2\lambda(t)}\int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\Big(2|u_x|^2 + 2n|u|^2 + |u|^2 + |v|^2 + |n|^2\Big)(t,x)\mathrm{d}x \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}J(t) + \frac{1}{\lambda(t)}\mathrm{Im}\int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\Big(2\overline{u}_x u_{xx} + \overline{u}u_x\Big)(t,x)\mathrm{d}x \\
&+\frac{1}{\lambda(t)}\int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\Big(2\mathrm{Im}\; n\overline{u}u_x + \left(n+|u|^2\right)v\Big)(t,x)\mathrm{d}x.
\end{aligned} \tag{3.44}$$

Just as before, we need to take λ such that λ^{-1} integrates finite in time over an interval $[T, \infty]$, $T > 0$. Then, for $\delta > 0$ and $t \geqslant 2$, we define $\lambda(s) = t\log^{1+\delta}(t)f(t)$ and $\mu(t) = \lambda(t)$ (although the computations will still work for $\mu(t) \gtrsim \lambda(t)$). Then, for $t \geqslant 2$ we have

$$\frac{\lambda'(t)}{\lambda(t)} = \frac{\mu'(t)}{\lambda(t)} \geqslant \frac{1}{t} + \frac{1+\delta}{t\log(t)}.$$

Notice that, if $\|u(t)\|_{H^2(\mathbb{R})} \leqslant C$ for all $t \geqslant 0$ for some $C > 0$, then we take $f(t) = C$ and get that $\lambda(t)^{-1} = \frac{1}{t\log^{1+\delta}(t)f(t)} \approx \frac{1}{t\log^{1+\delta}(t)}$. Furthermore, in the case $\|u(t)\|_{H^2(\mathbb{R})}$ increasing infinitely, there would exist $T > 0$ and $C > 0$ such that $\frac{1}{f(t)} \leqslant C$ for $t \geqslant T$. Then, either way, we get that there exists $T \geqslant 2$

$$\lambda(t)^{-1} \lesssim \frac{1}{t\log^{1+\delta}(t)}, \quad \text{for } t > T.$$

Thus, we get that λ^{-1} integrates finite over $[T, \infty)$, while $\frac{\lambda'}{\lambda}$ does not.

We estimate each term on the RHS of (3.40) after integrating in time. From Lemma 3.6 we have that

$$\begin{aligned}\int_T^\infty \frac{2}{\lambda(\tau)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\overline{u}_x(\tau,x)u_{xx}(\tau,x)\mathrm{d}x\mathrm{d}\tau &\lesssim \int_T^\infty \frac{1}{\lambda(\tau)}\|u_x(\tau)\|_{L^2(\mathbb{R})}\|u_{xx}(\tau)\|_{L^2(\mathbb{R})}\mathrm{d}\tau\\ &\lesssim \int_T^\infty \frac{f(\tau)}{\lambda(\tau)}\mathrm{d}\tau<\infty.\end{aligned}$$

The same way, we get

$$\int_T^\infty \frac{2}{\lambda(\tau)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\overline{u}(\tau,x)u_x(\tau,x)\mathrm{d}x\mathrm{d}\tau<\infty.$$

Also, from Lemma 3.6 and Sobolev embeddings,

$$\begin{aligned}&\int_T^\infty \frac{2}{\lambda(\tau)}\mathrm{Im}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)n(\tau,x)\overline{u}(\tau,x)u_x(\tau,x)\mathrm{d}x\mathrm{d}t\\ &\lesssim \|u(t)\|_{L^\infty(\mathbb{R})}\int_T^\infty \frac{2}{\lambda(\tau)}\int_{\mathbb{R}}|n(\tau,x)|^2+|u_x(\tau,x)|^2\mathrm{d}x\mathrm{d}t\lesssim \int_T^\infty \frac{2}{\lambda(\tau)}\mathrm{d}t\leqslant\infty,\end{aligned}$$

$$\int_T^\infty \frac{1}{\lambda(\tau)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)n(\tau,x)v(\tau,x)\mathrm{d}x\mathrm{d}t\lesssim\int_T^\infty \frac{1}{\lambda(\tau)}\int_{\mathbb{R}}|n(\tau,x)|^2+|v(\tau,x)|^2\mathrm{d}x\mathrm{d}t<\infty$$

and

$$\int_T^\infty \frac{1}{\lambda(\tau)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)|u(\tau,x)|^2v(\tau,x)\mathrm{d}x\mathrm{d}t\lesssim\|u(t)\|_{L^\infty(\mathbb{R})}\int_T^\infty \frac{1}{\lambda(\tau)}\int_{\mathbb{R}}|u(\tau,x)|^2+|v(\tau,x)|^2\mathrm{d}x\mathrm{d}t<\infty.$$

Consequently, integrating (3.44) over $[T,\infty]$, one obtains

$$\begin{aligned}&\int_T^\infty \frac{\lambda'(\tau)}{2\lambda(\tau)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\Big(2|u_x|^2+2n|u|^2+|u|^2+|v|^2+|n|^2\Big)(\tau,x)\mathrm{d}x\\ &\quad-\frac{\mu'(\tau)}{2\lambda(\tau)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\Big(2|u_x|^2+2n|u|^2+|u|^2+|v|^2+|n|^2\Big)(\tau,x)\mathrm{d}x\ \mathrm{d}\tau<\infty.\end{aligned}\tag{3.45}$$

Furthermore,

$$\int_T^\infty \frac{\mu'(\tau)}{2\lambda(\tau)}\int_{\mathbb{R}}\left|\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\right|\Big(2|u_x|^2+|u|^2+|v|^2+|n|^2\Big)(\tau,x)\mathrm{d}x\mathrm{d}t<\infty.\tag{3.46}$$

Indeed, from (3.45) and Young inequality for products,

$$\begin{aligned}\infty>&\int_T^\infty \frac{\lambda'(\tau)}{2\lambda(\tau)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\Big(2|u_x|^2+2n|u|^2+|u|^2+|v|^2+|n|^2\Big)(\tau,x)\mathrm{d}x\\ &-\frac{\mu'(\tau)}{2\lambda(\tau)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\Big(2|u_x|^2+2n|u|^2+|u|^2+|v|^2+|n|^2\Big)(\tau,x)\mathrm{d}x\ \mathrm{d}t\\ \geqslant&\int_T^\infty \frac{\lambda'(\tau)}{2\lambda(\tau)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\Big(2|u_x|^2+|u|^2+|v|^2+\frac{1}{2}|n|^2-2|u|^4\Big)(\tau,x)\mathrm{d}x\\ &-\frac{\mu'(\tau)}{2\lambda(\tau)}\int_{\mathbb{R}}\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\Big(2|u_x|^2+|u|^2+|v|^2+\frac{1}{2}|n|^2-2|u|^4\Big)(\tau,x)\mathrm{d}x\ \mathrm{d}t.\end{aligned}$$

By Sobolev embedding and (3.42), we have that

$$\begin{aligned}
&\int_T^\infty \frac{\mu'(\tau)}{\lambda(\tau)} \int_{\mathbb{R}} \left(\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right) - \varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\right)|u(\tau,x)|^4 \mathrm{d}x\mathrm{d}t \\
&\lesssim \int_T^\infty \frac{\mu'(\tau)}{\lambda(\tau)} \int_{\mathbb{R}} \left(\varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right) - \varphi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\right)|u(\tau,x)|^2 \mathrm{d}x\mathrm{d}t < \infty.
\end{aligned}$$

Then, (3.46) holds. Thus, there exists a sequence $\{t_n\}$, $t_n \to \infty$ satisfying

$$\int_{\mathbb{R}} \left|\varphi'\left(\frac{x+\mu(t_n)}{\lambda(t_n)}\right)\right| \Big(|u_x|^2 + |u|^2 + |v|^2 + |n|^2\Big)(t_n,x)\mathrm{d}x \to 0. \tag{3.47}$$

Furthermore, using Sobolev embedding and Lemma 3.6, we have from (3.47) that

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t_n)}{\lambda(t_n)}\right) n(t_n,x)|u(t_n,x)|^2 \mathrm{d}x\right| \\
&\lesssim \int_{\mathbb{R}} \left|\varphi'\left(\frac{x+\mu(t_n)}{\lambda(t_n)}\right)\right| |n(t_n,x)|^2 \mathrm{d}x + \int_{\mathbb{R}} \left|\varphi'\left(\frac{x+\mu(t_n)}{\lambda(t_n)}\right)\right| |u(t_n,x)|^2 \mathrm{d}x \to 0.
\end{aligned}$$

Then,

$$\left|\int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t_n)}{\lambda(t_n)}\right) \Big(|u_x|^2 + |u|^2 + |v|^2 + |n|^2 + n|u|^2\Big)(t_n,x)\mathrm{d}x\right| \to 0. \tag{3.48}$$

We argue as before and consider $\phi \in C_0^1(\mathbb{R})$ such that $\phi(s) \in [0,1]$ for all $s \in \mathbb{R}$, $\mathrm{supp}(\phi) = [-3/4, -1, 4]$,

$$\phi(s) \lesssim |\varphi'(s)| \text{ and } |\phi'(s)| \lesssim |\varphi'(s)| \text{ for all } s \in \mathbb{R}.$$

One gets,

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\Big(|u_x|^2 + \frac{1}{2}|u|^2 + \frac{1}{2}|n|^2 + \frac{1}{2}|v|^2 + n|u|^2\Big)(t,x)\mathrm{d}x\right| \\
&\lesssim \frac{2}{\lambda(t)}\int_{\mathbb{R}} \left|\phi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\right| \left|\Big(\overline{u}_x u_{xx} + n\overline{u}u_x\Big)(t,x)\right| \mathrm{d}x \\
&\quad + \frac{1}{\lambda(t)}\int_{\mathbb{R}} \left|\phi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\right| \left|\Big(\overline{u}u_x + \left(n + |u|^2\right)v\Big)(t,x)\right| \mathrm{d}x \\
&\quad + \frac{\mu'(t)}{2\lambda(t)}\int_{\mathbb{R}} \left|\phi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\right| \left|\Big(2|u_x|^2 + 2n|u|^2 + |u|^2 + |v|^2 + |n|^2\Big)(t,x)\right| \mathrm{d}x \\
&\lesssim \frac{1}{\lambda(t)} + \frac{\mu'(t)}{2\lambda(t)}\int_{\mathbb{R}} \left|\phi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\right| \left|\Big(2|u_x|^2 + 2|n|^2 + |u|^4 + |u|^2 + |v|^2\Big)(t,x)\right| \mathrm{d}x
\end{aligned}$$

Integrate over $[t, t_n]$, $t \geqslant T$ and take $t_n \to \infty$. Thanks to (3.48), we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\Big(|u_x|^2 + \frac{1}{2}|u|^2 + \frac{1}{2}|n|^2 + \frac{1}{2}|v|^2 + n|u|^2\Big)(t,x)\mathrm{d}x\right| \\
&\lesssim \int_t^\infty \frac{1}{\lambda(\tau)}\mathrm{d}t + \int_t^\infty \frac{\mu'(\tau)}{2\lambda(\tau)}\int_{\mathbb{R}} \left|\phi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\right| \left|\Big(2|u_x|^2 + |n|^2 + |u|^4 + |u|^2 + |v|^2\Big)(\tau,x)\right| \mathrm{d}x\mathrm{d}t
\end{aligned}$$

Now, as in subsection 3.3.1, taking $t \to \infty$, we obtain

$$\lim_{t\to\infty}\left|\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\Big(|u_x|^2 + \frac{1}{2}|u|^2 + \frac{1}{2}|n|^2 + \frac{1}{2}|v|^2 + n|u|^2\Big)(t,x)\mathrm{d}x\right| \leqslant 0$$

Note that by Hölder inequality and Lemma 3.6,

$$\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(n|u|^2\right)(t,x)\mathrm{d}x \lesssim \|n(t)\|_{L^2(\mathbb{R})}\left(\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)|u(t,x)|^2\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\lesssim \left(\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)|u(t,x)|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

Thanks to (3.12), we conclude the proof.

3.4 Decay on compact intervals for Klein-Gordon-Zakharov

Before we present the proof of Theorems 3.4 and 3.5, we give an estimation of the energy norm of a solution for (3.15), that will be useful in the following.

Lemma 3.13. *Let $(u,u_t,n,v) \in C\left(\mathbb{R}^+, H^1(\mathbb{R})\times L^2(\mathbb{R})\times L^2(\mathbb{R})\times L^2(\mathbb{R})\right)$ be a solution of (3.4) such that $E_{KG}<\infty$ and it satisfies (3.18) for some $C>0$ and $\varepsilon>0$ not necessarily small. Then, there exists $K_{KG}>0$ such that*

$$\int_{\mathbb{R}} |u_t(t,x)|^2+|u_x(t,x)|^2+|u(t,x)|^2+|n(t,x)|^2+|v(t,x)|^2\mathrm{d}x \leqslant K_{KG}.$$

Proof. We write

$$\frac{1}{2}\int_{\mathbb{R}} |u_t|^2+|u_x|^2+|u|^2+|n|^2+|v|^2\mathrm{d}x$$
$$\leqslant \int_{\mathbb{R}} |u_t|^2+\frac{1}{2}|u_x|^2+|u|^2+\frac{1}{2}|n|^2+|v|^2+2n|u|^2\mathrm{d}x - 2\int_{\mathbb{R}} n|u|^2\mathrm{d}x.$$

The first integral in the RHS can be bounded by the energy. Thus, we need to control the remaining term. By Young inequality and Gagliardo-Nirenberg inequality [19, 33], for $\epsilon>0$ we have that

$$2\int_{\mathbb{R}} nu^2\mathrm{d}x \leqslant \frac{1}{\epsilon}\int_{\mathbb{R}} n^2\mathrm{d}x+\epsilon\int_{\mathbb{R}} u^4\mathrm{d}x \leqslant \frac{1}{\epsilon}\int_{\mathbb{R}} n^2\mathrm{d}x+\epsilon C_{GN}\|u_x\|_{L^2(\mathbb{R})}\|u\|^3_{L^2(\mathbb{R})},$$

where

$$C_{GN}=\frac{\sqrt{3}}{3}$$

and Q is the solution to $Q''+Q^3-Q=0$. Then, taking $\epsilon=2$, we get

$$2\int_{\mathbb{R}} nu^2\mathrm{d}x \leqslant \frac{1}{2}\int_{\mathbb{R}} n^2\mathrm{d}x+2\frac{\sqrt{3}}{3}\|u_x\|_{L^2(\mathbb{R})}\|u\|^3_{L^2(\mathbb{R})} \leqslant \frac{1}{2}\int_{\mathbb{R}} n^2\mathrm{d}x+\frac{\sqrt{3}}{3}\int_{\mathbb{R}} u_x^2\mathrm{d}x+\frac{\sqrt{3}}{3}\|u\|^6_{L^2(\mathbb{R})}.$$

Finally, this means that

$$\frac{1}{2}\int_{\mathbb{R}} |u_t|^2+|u_x|^2+|u|^2+|n|^2+|v|^2\mathrm{d}x \leqslant 2E_0+\frac{\sqrt{3}}{3}\|u\|^6_{L^2(\mathbb{R})}.$$

Thanks to (3.18), we conclude. ∎

3.4.1 Virial argument

During this section, we are going to consider (u, n, v) a solution such that u is odd and satisfies (3.18), for some $C > 0$ and ε small. As in Section 3.2, let $\varphi \in C^\infty(\mathbb{R})$ a bounded real function and define

$$I(t) = 2\int_{\mathbb{R}} \varphi(x)u_x(t,x)u_t(t,x)\mathrm{d}x - \int_{\mathbb{R}} \varphi(x)v(t,x)n(t,x)\mathrm{d}x + \int_{\mathbb{R}} \varphi'(x)u(t,x)u_t(t,x)\mathrm{d}x.$$

We get the following virial identity:

Lemma 3.14 (Virial Identity)**.** *Let (u, u_t, n, v) be a solution to* (3.15)*. Then,*

$$-\frac{\mathrm{d}}{\mathrm{d}t}I(t) = 2\int_{\mathbb{R}} \varphi' u_x^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi''' u^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varphi' n^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varphi' v^2\mathrm{d}x + \int_{\mathbb{R}} \varphi' u^2 n\mathrm{d}x. \quad (3.49)$$

Proof. Using equation (3.15), we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}I(t) =& 2\int_{\mathbb{R}} \varphi u_{xt}u_t\mathrm{d}x + 2\int_{\mathbb{R}} \varphi u_x u_{xx}\mathrm{d}x - 2\int_{\mathbb{R}} \varphi u_x u\mathrm{d}x - 2\int_{\mathbb{R}} \varphi n u_x u\mathrm{d}x + \int_{\mathbb{R}} \varphi\left(n+u^2\right)_x n\mathrm{d}x \\
&+ \int_{\mathbb{R}} \varphi v v_x\mathrm{d}x + 2\int_{\mathbb{R}} \varphi' u_t u_t\mathrm{d}x + \int_{\mathbb{R}} \varphi' u u_{xx}\mathrm{d}x - \int_{\mathbb{R}} \varphi' u u\mathrm{d}x - \int_{\mathbb{R}} \varphi' u u n\mathrm{d}x \\
=& \int_{\mathbb{R}} \varphi(u_t^2)_x\mathrm{d}x + \int_{\mathbb{R}} \varphi(u_x^2)_x\mathrm{d}x - \int_{\mathbb{R}} \varphi(u^2)_x\mathrm{d}x - \int_{\mathbb{R}} \varphi n(u^2)_x\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varphi(n^2)_x\mathrm{d}x \\
&+ \int_{\mathbb{R}} \varphi\left(u^2\right)_x n\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varphi(v^2)_x\mathrm{d}x + \int_{\mathbb{R}} \varphi' u_t^2\mathrm{d}x + \int_{\mathbb{R}} \varphi' u u_{xx}\mathrm{d}x - \int_{\mathbb{R}} \varphi' u^2\mathrm{d}x - \int_{\mathbb{R}} \varphi' u^2 n\mathrm{d}x.
\end{aligned}$$

We integrate by parts

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}I(t) =& -2\int_{\mathbb{R}} \varphi' u_x^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi' n^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi' v^2\mathrm{d}x - \int_{\mathbb{R}} \varphi'' u u_x\mathrm{d}x - \int_{\mathbb{R}} \varphi' u^2 n\mathrm{d}x \\
=& -2\int_{\mathbb{R}} \varphi' u_x^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varphi''' u^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi' n^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi' v^2\mathrm{d}x - \int_{\mathbb{R}} \varphi' u^2 n\mathrm{d}x.
\end{aligned}$$

■

Notice that now we have a very similiar virial identity to the one obtained in Subsection 3.2.1. In fact, the RHS is the same. Then, we are entitled to use the estimations for the bilinear part of (3.25). Indeed, we can write

$$-\frac{\mathrm{d}}{\mathrm{d}t}I(t) = B(u) + \frac{1}{2}\int_{\mathbb{R}} \varphi' n^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varphi' v^2\mathrm{d}x + \int_{\mathbb{R}} \varphi' u^2 n\mathrm{d}x,$$

where B is defined in (3.26). Just as before, for $\lambda > 0$, consider $\varphi(x) = \lambda\tanh(x/\lambda)$ and $\omega(x) = \sqrt{\varphi'(x)}$. Finally, using the arguments in Subsection 3.2.1 and by estimation (3.30), we have that

$$-\frac{\mathrm{d}}{\mathrm{d}t}I(t) \gtrsim \|u(t)\|_{H^1_\omega(\mathbb{R})}^2 + \frac{1}{2}\int_{\mathbb{R}} \varphi' n^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varphi' v^2\mathrm{d}x + \int_{\mathbb{R}} \varphi' u^2 n\mathrm{d}x, \quad (3.50)$$

where $\|\cdot\|_{H^1_\omega(\mathbb{R})}$ is the weighted-norm introduced in (3.22).

In order to conclude the argument, we present the following proposition:

Proposition 3.15. *Let (u, n, v) be a solution of (3.15). Then, there exists $C > 0$ such that*

$$\int_0^\infty \|u_t(t)\|_{L^2_\omega(\mathbb{R})} + \|u(t)\|_{H^1_\omega(\mathbb{R})} + \|n(t)\|_{L^2_\omega(\mathbb{R})} + \|v(t)\|_{L^2_\omega(\mathbb{R})}\mathrm{d}t \leqslant C.$$

Proof. Thanks to (3.50) and (3.18), the proof follow as in Proposition 3.10. ∎

3.4.2 Conclusion of the proof

Let ϕ be a $C^\infty(\mathbb{R})$ function to be defined later. Then, since (u, n, v) is a solution to equation (3.15),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{R}} \phi(x)\left(|u|^2 + |u_t|^2 + |u_x|^2 + |n|^2 + |v|^2\right)(t,x)\mathrm{d}x \\
&= \int_{\mathbb{R}} \phi(x)u_t(t,x)u_{xx}(t,x)\mathrm{d}x - \int_R \phi(x)n(t,x)u_t(t,x)u(t,x)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}} \phi(x)u_x(t,x)u_{xt}(t,x)\mathrm{d}x - \int_{\mathbb{R}} \phi(x)n(t,x)v_x(t,x)\mathrm{d}x - \int_{\mathbb{R}} \phi(x)v(t,x)\left(n + |u|^2\right)_x(t,x)\mathrm{d}x.
\end{aligned}$$

After integration by parts, one gets

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{R}} \phi(x)\left(|u|^2 + |u_t|^2 + |u_x|^2 + |n|^2 + |v|^2\right)(t,x)\mathrm{d}x \\
&= -\int_{\mathbb{R}} \phi'(x)u_t(t,x)u_x(t,x)\mathrm{d}x - \int_{\mathbb{R}} \phi(x)n(t,x)u_t(t,x)u(t,x)\mathrm{d}x + \int_{\mathbb{R}} \phi'(x)v(t,x)n(t,x)\mathrm{d}x \\
&\quad + 2\int_{\mathbb{R}} \phi(x)v(t,x)u(t,x)u_x(t,x)\mathrm{d}x.
\end{aligned} \tag{3.51}$$

Thus, if we take $\phi(x) = \mathrm{sech}(x)$, we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{R}} \mathrm{sech}(x)\left(|u|^2 + |u_t|^2 + |u_x|^2 + |n|^2 + |v|^2\right)(t,x)\mathrm{d}x \\
&\quad\lesssim \|u(t)\|^2_{H^1_\omega(\mathbb{R})} + \|u_t(t)\|^2_{L^2_\omega(\mathbb{R})} + \|v(t)\|^2_{L^2_\omega(\mathbb{R})} + \|n(t)\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Proposition 3.15 implies the existence of a sequence $\{t_n\} \subset \mathbb{R}$, $t_n \to \infty$, such that

$$\|u(t_n)\|^2_{H^1_\omega(\mathbb{R})} + \|u_t(t_n)\|^2_{L^2_\omega(\mathbb{R})} + \|v(t_n)\|^2_{L^2_\omega(\mathbb{R})} + \|n(t_n)\|^2_{L^2(\mathbb{R})} \to 0.$$

Then, we integrate over $[t, t_n]$, take $t_n \to \infty$ and obtain

$$\begin{aligned}
&\|u(t)\|^2_{H^1_\omega(\mathbb{R})} + \|u_t(t)\|^2_{L^2_\omega(\mathbb{R})} + \|v(t)\|^2_{L^2_\omega(\mathbb{R})} + \|n(t)\|^2_{L^2(\mathbb{R})} \\
&\lesssim \int_t^\infty \|u(\tau)\|^2_{H^1_\omega(\mathbb{R})} + \|u_t(\tau)\|^2_{L^2_\omega(\mathbb{R})} + \|v(\tau)\|^2_{L^2_\omega(\mathbb{R})} + \|n(\tau)\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau.
\end{aligned}$$

Thanks to Proposition 3.15, the RHS of the last equation is finite. Consequently, we can take $t \to \infty$ and conclude the proof.

3.5 Decay in regions along curves for Klein-Gordon-Zakharov

To construct the virial identity, consider $\varphi \in C^2(\mathbb{R})$ a bounded real function and $\lambda, \mu \in C^1(\mathbb{R})$ functions depending on time. We define

$$J(t) = \frac{1}{2}\int_{\mathbb{R}} \varphi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t,x)\mathrm{d}x.$$

Lemma 3.16. *Let (u,n,v) be a solution to* (3.15). *Then,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}J(t) = {}& \frac{1}{\lambda(t)}\int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\frac{1}{2}vn + \frac{1}{2}v|u|^2 - u_t u_x\right)(t,x)\mathrm{d}x \\ &+ \frac{\mu'(t)}{2\lambda(t)}\int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t,x)\mathrm{d}x \\ &- \frac{\lambda'(t)}{2\lambda(t)}\int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t,x)\mathrm{d}x.\end{aligned}$$

We skip the proof of Lemma 3.16, since it follows from (3.51)

3.5.1 Proof Theorem 3.5

As we did in Section 3.3, we consider $\varphi \in C^2(\mathbb{R})$ a decreasing function satisfying $\varphi(s) = 1$ for $s \leqslant -1$ and $\varphi(s) = 0$ for $s \geqslant 0$. It follows that $\mathrm{supp}(\varphi') \subset [-1,0)$ and

$$\varphi'(s) \leqslant 0, \quad \varphi'(s)s \geqslant 0 \quad \forall s \in \mathbb{R}.$$

Also, for $\delta > 0$, we take $\lambda(t) = t\log^{1+\delta}(t)$ and $\mu(t) = \lambda(t)$. Just as before, we are going to take adventage of the fact that λ^{-1} is integrable in time over the time interval $[T,\infty)$, for some $T \geqslant 2$, while $\frac{\lambda'}{\lambda}$ is not.

We re-arrange the virial identity (3.16) as:

$$\begin{aligned}&\frac{\lambda'(t)}{2\lambda(t)}\int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t,x)\mathrm{d}x \\ &- \frac{\mu'(t)}{2\lambda(t)}\int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t,x)\mathrm{d}x \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}J(t) + \frac{1}{\lambda(t)}\int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\frac{1}{2}vn + \frac{1}{2}v|u|^2 - u_t u_x\right)(t,x)\mathrm{d}x.\end{aligned} \tag{3.52}$$

We have that, by Gagliardo-Nirenberg inequality and Lemma 3.13

$$\begin{aligned}&\int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\frac{1}{2}vn + \frac{1}{2}v|u|^2 - u_t u_x\right)(t,x)\mathrm{d}x \\ &\lesssim \|v(t)\|_{L^2(\mathbb{R})}^2 + \|n(t)\|_{L^2(\mathbb{R})}^2 + \|u_x(t)\|_{H^1(\mathbb{R})}^2 + \|u_t(t)\|_{L^2(\mathbb{R})}^2 + \|u(t)\|_{L^2(\mathbb{R})}^6 \leqslant K,\end{aligned}$$

where $K > 0$, is a constant depending on the energy and the L^2-norm of u. Thus, we integrate equation (3.52) in time over $[2, \infty)$ and get that

$$\int_2^{\infty} + \frac{\lambda'(t)}{2\lambda(t)} \int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t,x)\mathrm{d}x$$
$$- \frac{\mu'(t)}{2\lambda(t)} \int_{\mathbb{R}} \varphi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t,x)\mathrm{d}x\mathrm{d}t < \infty.$$

Note that by Sobolev embeddings and (3.20),

$$\begin{aligned} -\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|n||u|^2\right)(t,x)\mathrm{d}x &\geqslant -\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\frac{1}{3}|n(t,x)|^2 + \frac{3}{4}|u(t,x)|^4\right)\mathrm{d}x \\ &\geqslant -\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(\frac{1}{3}|n(t,x)|^2 + \frac{3}{4}|u(t,x)|^2\right)\mathrm{d}x. \end{aligned} \tag{3.53}$$

Then, we argue as in Subsection 3.3.2 and obtain that there exists a sequence of time $\{t_n\}$, $t_n \to \infty$, such that,

$$\int_{\mathbb{R}} \left|\varphi'\left(\frac{x+\mu(t_n)}{\lambda(t_n)}\right)\right|\left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t_n,x)\mathrm{d}x \to 0. \tag{3.54}$$

As in Section 3.3, we consider $\phi \in C_0^1(\mathbb{R})$ such that $\phi(s) \in [0,1]$ for all $s \in \mathbb{R}$, $\mathrm{supp}(\phi) = [-3/4, -1, 4]$,

$$\phi(s) \lesssim |\varphi'(s)| \text{ and } |\phi'(s)| \lesssim |\varphi'(s)| \text{ for all } s \in \mathbb{R}.$$

Following the computations for φ, one gets,

$$\begin{aligned} &\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|n|^2 + \frac{1}{2}|v|^2 + n|u|^2\right)(t,x)\mathrm{d}x\right| \\ &\quad\lesssim \frac{1}{\lambda(t)}\int_{\mathbb{R}} \left|\phi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\right|\left|\left(\frac{1}{2}vn + \frac{1}{2}v|u|^2 - u_t u_x\right)(t,x)\right|\mathrm{d}x \\ &\quad+ \frac{\mu'(t)}{2\lambda(t)}\int_{\mathbb{R}} \left|\phi'\left(\frac{x+\mu(t)}{\lambda(t)}\right)\right|\left|\left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|n|^2 + n|u|^2\right)(t,x)\right|\mathrm{d}x. \end{aligned}$$

Integrate over $[t, t_n]$, $t \geqslant T$ and take $t_n \to \infty$. Thanks to (3.54), we obtain

$$\begin{aligned} &\left|\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|n|^2 + \frac{1}{2}|v|^2 + n|u|^2\right)(t,x)\mathrm{d}x\right| \\ &\lesssim \int_t^{\infty} \frac{1}{\lambda(\tau)}\mathrm{d}\tau + \int_t^{\infty} \frac{\mu'(\tau)}{2\lambda(\tau)}\int_{\mathbb{R}} \left|\phi'\left(\frac{x+\mu(\tau)}{\lambda(\tau)}\right)\right|\left|\left(|u_x|^2 + |u|^2 + |u_t|^2 + |n|^2 + |v|^2 + n|u|^2\right)(\tau,x)\right|\mathrm{d}x\mathrm{d}\tau \end{aligned}$$

Now, taking $t \to \infty$, we obtain

$$\lim_{t\to\infty}\left|\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|n|^2 + \frac{1}{2}|v|^2 + n|u|^2\right)(t,x)\mathrm{d}x\right| \leqslant 0$$

Consequently, taking into account (3.53), one gets

$$\begin{aligned} &\lim_{t\to\infty}\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|u_x|^2 + \frac{1}{4}|u|^2 + |u_t|^2 + \frac{1}{6}|n|^2 + \frac{1}{2}|v|^2\right)(t,x)\mathrm{d}x \\ &\leqslant \lim_{t\to\infty}\int_{\mathbb{R}} \phi\left(\frac{x+\mu(t)}{\lambda(t)}\right)\left(|u_x|^2 + |u|^2 + |u_t|^2 + \frac{1}{2}|n|^2 + \frac{1}{2}|v|^2 + n|u|^2\right)(t,x)\mathrm{d}x = 0. \end{aligned}$$

Then, (3.21) follows.

Bibliography

[1] H. Added and S. Added, *Existence globale de solutions fortes pour les é quations de la turbulence de Langmuir en dimension 2*, C. R. Acad. Sci. Paris 299, pp. 551–554 (1984).

[2] J. Angulo, C. Banquet, *Orbital stability for the periodic Zakharov system*, Nonlinearity 24, pp. 2913–2932 (2011).

[3] J. Bourgain, *On the Cauchy and invariant problem for the periodic Zakharov system*, Duke Math. J. (C), Vol. 76, No. 1, pp. 175–202 (1994).

[4] L. Bergé, *Wave collapse in physics: principles and applications to light and plasma waves*, Phys. Rep., Vol. 303, pp. 259–370 (1998).

[5] J. Bourgain and J. Colliander, *On Wellposedness of the Zakharov System*, Int. Math. Res. Not. IMRN, No. 11, pp. 515–546 (1996).

[6] L. Chen, *Orbital stability of solitary waves for the Klein-Gordon-Zakharov equations*, Acta Math. Appl. Sin., Vol. 15, No. 1, pp. 54–64 (1999).

[7] J. Colliander, *The initial value problem for the zakharov system*, Thesis, University of Illinois at Urbana-Champaign (1997).

[8] J. Colliander, *Wellposedness for Zakharov Systems with Generalized Nonlinearity*, Journal of Differential Equations 148, pp. 351–363 (1998).

[9] S. Fang, C. Guo, B. Guo, *Exact traveling wave solutions of modified Zakharov equations for plasmas with quantum correction*, Acta Math. Sin. 32B(3), pp. 1073–1082 (2012).

[10] J. Ginibre, Y. Tsutsumi and G. Velo, *On the Cauchy Problem for the Zakharov System*, J. Funct. Anal. 151, pp. 384–436 (1997).

[11] J. Ginibre and G. Velo, *Scattering theory for the Zakharov system*, Hokkaido Mathematical Journal 35, no. 4, pp. 865–892 (2006).

[12] Z. Guo, S. Lee, K. Nakanishi and C. Wang, *Generalized Strichartz Estimates and Scattering for 3D Zakharov System*, Commun. Math. Phys. 331, pp. 239–259 (2014). doi:10.1007/s00220-014-2006-0

[13] Z. Guo and K. Nakanishi, *Small Energy Scattering for Zakharov*, Int. Math. Res. Not.

IMRN, Vol. 2014, No. 9, pp. 2327–2342 (2013). doi:10.1093/imrn/rns296

[14] Z. Guo, K. Nakanishi and S. Wang, *Small energy scattering for the Klein-Gordon-Zakharov system with radial symmetry*, Math. Res. Lett., Vol. 21, No. 04, pp. 733–755 (2014).

[15] Z. Guo, K. Nakanishi and S. Wang, *Global Dynamics below the Ground State Energy for the Klein-Gordon-Zakharov System in the 3D Radial Case*, Communications in Partial Differential Equations, Vol. 39, pp. 1158–1184 (2014).

[16] Z. Guo, K. Nakanishi and S. Wang, *Global Dynamics below the Ground State Energy for the Klein-Gordon-Zakharov System in the 3D Radial Case*, Communications in Partial Differential Equations, Vol. 39, pp. 1158–1184, (2014). doi: 10.1080/03605302.2013.836715

[17] Y. Hui *Orbital stability of solitary waves for the generalized Zakharov system* J. Partial Diff. Eqns Vol. 20, No 3, pp. 252–264 (2007)

[18] Y. Hui, *Orbital stability of solitary waves for the generalized Zakharov system*, Adv. Math. 35, pp. 635–637 (2006).

[19] R. Killip and M. Visan, *Nonlinear Schrödinger Equations at Critical Regularity* Clay Mathematics Proceedings, Vol. 10 (2009).

[20] M. Kowalczyk, Y. Martel and C. Muñoz, *Kink dynamics in the ϕ^4 model: asymptotic stability for odd perturbations in the energy space*, J. Amer. Math. Soc. 30 (2017), 769–798.

[21] M. Kowalczyk, Y. Martel and C. Muñoz, *Nonexistence of small, odd breathers for a class of nonlinear wave equations*, Lett. Math. Phys. 107 (2017), pp. 921–931.

[22] H. Luong, N. J. Mauser, J.-C. Saut, *On the Cauchy problem for the Zakharov-Rubenchik/ Benney-Roskes system*, Commun. Pure Appl. Anal., Vol. 17, No. 4, pp. 1573–1594 (2018). doi:10.3934/cpaa.2018075

[23] M. E. Martínez, *Decay of small odd solutions for long range Schrödinger and Hartree equations in one dimension*, Nonlinearity, Vol. 33, No. 3., pp. 1156–1182 (2020).

[24] F. Merle, *Blow-up Results of Viriel Type for Zakharov Equations*, Commun. Math. Phys. 175, pp. 433–455 (1996).

[25] F. Merle, *Lower Bounds for the Blowup Rate of Solutions of the Zakharov Equation in Dimension Two*, Commun. Pure Appl. Anal., Vol. XLIX, pp. 765–794 (1996).

[26] N. Masmoudi and K. Nakanishi, *From nonlinear Klein-Gordon equation to a system of coupled nonlinear Schrödinger equations*, Math. Ann. 324 (2), pp. 359–389 (2002).

[27] N. Masmoudi and K. Nakanishi, *Nonrelativistic limit from Maxwell-Klein-Gordon and Maxwell-Dirac to Poisson-Schrödinger*, Int. Math. Res. Not. 13, pp. 697–734 (2003).

[28] N. Masmoudi and K. Nakanishi, *From the Klein-Gordon-Zakharov system to the nonlinear Schrödinger equation*, J. Hyperbolic Differ. Equ. 2 (4), pp. 975–1008 (2005).

[29] N. Masmoudi and K. Nakanishi, *From the Klein-Gordon-Zakharov system to a singular nonlinear Schrödinger system*, Annales de Annales de l'I. H. P. (C) Non Linear Analysis Vol. 27, No. 4, pp. 1073–1096 (2010).

[30] N. Masmoudi and K. Nakanishi, *Energy convergence for singular limits of Zakharov type systems* Invent. math., Vol. 172, pp. 535–583 (2008). doi: 10.1007/s00222-008-0110-5

[31] N. Masmoudi and K. Nakanishi, *Uniqueness of Solutions for Zakharov Systems*, Funkcialaj Ekvacioj Vol. 52, No. 2, pp. 233–253 (2009).

[32] C. Muñoz, G. Ponce and J.-C. Saut, *On the long time behavoir of solutions to the Intermediate Long Wave equation*, arXiv:1910.03897.

[33] B. V. Sz. Nagy, *Über Integralgleichungen zwischen einer Funktion und ihrer Ableitung*. Acta Sci. Math. (Szeged) Vol. 10, pp. 64–74 (1941).

[34] M. Ohta, *Stability of solitary waves for Zakharov equations*, Dynamical Systems and Applications (WSSIAA), pp. 563–571 (1995). //doi.org/10.1142/9789812796417_0037.

[35] M. Ohta and G. Todorova, *Strong instability of standing waves for nonlinear Klein-Gordon equation and Klein-Gordon-Zakharov system*, SIAM Journal on Mathematical Analysis Vol. 38, No. 6, pp. 1912–1931 (2006).

[36] T. Ozawa, K.Tsutaya and Y. Tsutsumi, *Normal form and global solutions for the Klein-Gordon-Zakharov equations* Annales de l'I. H. P. (C), Vol. 12, No. 4, pp. 459–503 (1995).

[37] T. Ozawa, K.Tsutaya and Y. Tsutsumi, *Well-posedness in energy space for the Cauchy problem of the Klein-Gordon-Zakharov equations with different propagation speeds in three space dimensions*, Math. Ann. 313, pp. 127–140 (1999).

[38] T. Ozawa and Y. Tsutsumi, *The nonlinear Schrödinger limit and the initial layer of the Zakharov equations*, Proc. Japan Acad. Ser. A Math. Sci. 67, no. 4, pp. 113–116 (1991).

[39] T. Ozawa and Y. Tsutsumi, *Existence and Smoothing Effect of Solutions for the Zakharov Equations*, Publ RIMS, Kyoto Univ. 28, pp. 329–361 (1992).

[40] H. Pecher, *Global Well-Posedness below Energy Space for the 1-Dimensional Zakharov System*, Int. Math. Res. Not. IMRN, No. 19, pp. 1027–1056 (2001).

[41] J. R. Quintero and J. C. Corodero, *Inestability of the standing waves for the Benney-Roskes/Zakharov-Rubenchik system and blow-up for the Zakharov equations*, Discrete and Continuous dynamical, Series B, pp. 1213–1240 (2020). doi:10.3934/dcdsb.2019217

[42] P. A. Robinson, *Nonlinear wave collapse and strong turbulence* Rev. Mod. Phys. 69, pp 507–573 (1997).

[43] A. Shimomura, *Scattering theory for the Zakharov equation in three espace dimensions*, Commun. Contemp. Math. 6, No. 6, pp. 881–899 (2004). doi:10.1142/S0219199704001574

[44] C. Sulem and P. L. Sulem, *Quelques résulatats de régularité pour les équation de la turbulence de Langmuir*. C. R. Acad. Sci. Paris 289, pp. 173–176 (1979).

[45] C. Sulem, P.-L. Sulem, *The Nonlinear Schrödinger Equation. Self-Focusing and Wave Collapse*, Applied Mathematical Sciences Series Volume 139, Springer-Verlag New York (1999).

[46] S. H. Schochet and M. I. Weinstein, *The nonlinear Schrödinger limit of the Zakharov equations governing Langmuir turbulence*, Comm. Math. Phys. 106, no. 4, pp. 569–580 (1986).

[47] K. Tsutaya, *Global existance of small amplitude solutions for the Klein-Gordon-Zakharov equations*, Nonlinear Analysis, Theory, Methods & Applications, Vol. 27, No. 12, pp. 1373–1380 (1996).

[48] Y. Wu, *Orbital stability of solitary waves of Zakharov system*, J. Math. Phys. Vol. 35, No. 5, pp. 2413–2422 (1994). doi:10.1063/1.530512

[49] V. E. Zakharov, *Collapse of Langmuir Waves*, Zh. Eksp. Teor. Fiz. 62, pp. 1745–1759 (1972).

[50] X Zheng, Y Shang, P Xiaoming, *Orbital stability of periodic traveling wave solutions to the generalized Zakharov equations*, Acta Math. Sin. 37B(4), pp. 998–1018 (2017).

Part III

The Zakharov-Rubenchik / Benney-Roskes model

Chapter 4

On long-time behavior of solutions of the Zakharov-Rubenchik/Benney-Roskes system

Abstract. We study decay properties for solutions to the initial value problem associated with the one-dimensional Zakharov-Rubenchik/Benney-Roskes system. We prove time-integrability in growing compact intervals of size t^r, $r < 2/3$, centered on some characteristic curves coming from the underlying transport equations associated with the ZR/BR system. Additionally, we prove decay to zero of the local energy-norm in so-called far-field regions. Our results are independent of the size of the initial data and do not require any parity condition.

This work is contained in M. E. Martínez and J. M. Palacios, *On long-time behavior of solutions of the Zakharov-Rubenchik/Benney-Roskes system*, accepted in Nonlinearity, available in arXiv, 2021.

Contents

4.1 Introduction and main results

4.1.1 The model

In this work we seek to show decay properties for solutions of the initial value problem (IVP) associated with the Zakharov-Rubenchik/Benney-Roskes (ZR/BR) system in one space dimension

$$\begin{cases} \mathrm{i}\partial_t\psi + \omega\partial_x^2\psi = \gamma\big(\eta - \tfrac{1}{2}\alpha\rho + q|\psi|^2\big)\psi, \\ \theta\partial_t\rho + \partial_x\big(\eta - \alpha\rho\big) = -\gamma\partial_x(|\psi|^2), \\ \theta\partial_t\eta + \partial_x\big(\beta\rho - \alpha\eta\big) = \tfrac{1}{2}\alpha\gamma\partial_x(|\psi|^2), \\ \psi(0,x) = \psi_0(x),\ \rho(0,x) = \rho_0(x),\ \eta(0,x) = \eta_0(x). \end{cases} \tag{4.1}$$

Here $\psi(t,x)$ denotes a complex-valued function, while $\rho(t,x)$ and $\eta(t,x)$ are both real-valued functions, and $t, x \in \mathbb{R}$. All Greek letters $(\omega, \alpha, \beta, \gamma, \theta)$ denote real parameters, and in the sequel we shall always assume that

$$\omega > 0, \quad \beta > 0, \quad \gamma > 0, \quad \beta - \alpha^2 > 0, \quad 0 < \theta < 1, \quad \text{and} \quad q := \gamma + \frac{\alpha(\alpha\gamma - 1)}{2(\beta - \alpha^2)}.$$

Model (4.1) corresponds to the one-dimensional case of the most general system derived by Zakharov and Rubenchik [19] to describe the interaction of spectrally narrow high-frequency wave packets of small amplitude with low-frequency acoustic type oscillations. This system was also independently found by Benney and Roskes [1] in the context of gravity waves, and in the 3-dimensional case has the following form

$$\begin{cases} \mathrm{i}\partial_t\psi + \mathrm{i}v_g\partial_z\psi = -\frac{\omega''}{2}\partial_z^2\psi - \frac{v_g}{2k}\Delta_\perp\psi + (q|\psi|^2 + \beta\rho + \alpha\partial_z\eta)\psi, \\ \partial_t\rho + \rho_0\Delta\eta + \alpha\partial_z|\psi|^2 = 0, \\ \partial_t\eta + \frac{c^2}{\rho_0}\rho + \beta|\psi|^2 = 0, \end{cases} \tag{4.2}$$

where $\Delta_\perp = \partial_x^2 + \partial_y^2$. In this context, $\psi(t,x)$ stands for the amplitude of the carrying (high frequency) waves with wave number k, frequency $\omega = \omega(k)$ and $v_g = \omega'(k)$ stands for its group velocity. On the other hand, $\rho(t,x)$ and $\eta(t,x)$ correspond to the density fluctuation and the hydrodynamic potential respectively.

System (4.2) has also been derived in several other physical situations, such as for example, in the study of Alfvén waves (transverse oscillations of the magnetic fields) in the Magneto-Hydrodynamics equations (see for instance [2, 17]). Moreover, system (4.2) contains various important models as limiting cases, such as the classical (scalar) Zakharov system and the Davey-Stewartson systems. We refer to [3] for a rigorous justification of the Zakharov limit (supersonic limit) of the ZR/BR system. However, the rigorous proof of the Davey-Stewartson limit from system (4.2) remains still open.

In the one dimensional case the situation is a little better understood. In fact, in this case we can also consider the adiabiatic limit, that is, to take $\theta \to 0$ in (4.1), from where we can formally see that $\rho(t,x)$ and $\eta(t,x)$ satisfy now the following relations

$$\rho = -\frac{\gamma\alpha}{2(\beta-\alpha^2)}|\psi|^2, \quad \eta = -\gamma\frac{\beta-\alpha^2/2}{\beta-\alpha^2}|\psi|^2.$$

Then, we infer that the complex amplitude ψ solves the cubic nonlinear Schrödinger equation

$$\mathrm{i}\partial_t\psi + \omega\partial_x^2\psi = -\frac{\gamma\alpha}{3(\beta-\alpha^2)}|\psi|^2\psi.$$

A rigorous justification of such limit (for well-prepared initial data) was proved by Oliveira in [16]. Therefore, we can certainly see that the ZR/BR system is thus richer than those models.

On the other hand, the ZR/BR system (4.1) and (4.2) posses a Hamiltonian structure [19], and hence, it follows (at least formally) that the energy of system (4.1) is conserved along the trajectory, which in the one-dimensional case can be written as

$$\begin{aligned} E\big(\psi(t),\rho(t),\eta(t)\big) &:= \int_{\mathbb{R}} \Big(\omega|\psi_x|^2 + \tfrac{\gamma q}{2}|\psi|^4 + \tfrac{\beta}{2}\rho^2 + \tfrac{1}{2}\eta^2 + \tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2 - \alpha\rho\eta\Big)\mathrm{d}x \\ &= E(\psi_0,\rho_0,\eta_0). \end{aligned}$$

Moreover, the ZR/BR system (4.1) also conserves (formally) the mass and the momentum of the solution, which are given by the following relations (respectively)

$$\begin{aligned} M\big(\psi(t),\rho(t),\eta(t)\big) &:= \int |\psi(t,x)|^2\mathrm{d}x = M(\psi_0,\rho_0,\eta_0), \qquad \text{and,} \\ P\big(\psi(t),\rho(t),\eta(t)\big) &:= \mathrm{Im}\int_{\mathbb{R}} \psi\overline{\psi}_x - \theta\int_{\mathbb{R}}\rho(t,x)\eta(t,x)\mathrm{d}x = P(\psi_0,\rho_0,\eta_0). \end{aligned}$$

Additionally, related to these conservation laws, the ZR/BR system (4.1) is invariant under space-time translations, as well as invariant under phase rotations.

Regarding the existence of solitary waves, in the case $\beta-\alpha^2>0$, $\gamma>0$ and $\theta<1$, Oliveira has proved in [15] the existence and the orbital stability of solitary waves of the form

$$\big(\psi,\rho,\eta\big)(t,x) := \big(e^{i\lambda t}e^{icx/2\omega}R(x-ct), a(c)|R(x-ct)|^2, b(c)|R(x-ct)|^2\big), \tag{4.3}$$

where $\lambda\in\mathbb{R}$, $c\geqslant 0$ and $R(\cdot)$ is an positive, even and exponentially decaying complex-valued function, while $a(c)$ and $b(c)$ are given by the following formulas

$$a(c) := -\frac{\gamma(\beta-\frac{\alpha}{2}(c\theta+\alpha))}{\beta-(c\theta+\alpha)^2}, \quad b(c) := -\frac{\gamma(c\theta+\frac{1}{2}\alpha)}{\beta-(c\theta+\alpha)^2}.$$

In particular, the analysis carried out by Oliveira shows that a necessary condition for these solitary waves to exists is that the following two inequalities must be satisfied

$$a(c) - \frac{\alpha}{2}b(c) + q < 0 \quad \text{and} \quad \frac{c^2}{4\omega} - \lambda < 0. \tag{4.4}$$

On the other hand, recently in [7] Luong *et al.* studied the existence of the so-called bright and dark solitons for system (4.1). They proved their existence under some conditions on the coefficients of the equations (similar to the one in (4.4)). Then, they used these solitons to construct line-solitons for the higher dimensional case. However, none of these solitons belongs to the energy space since they do not decay at $\pm\infty$ (see [7] for further details).

Finally, concerning the well-posedness for system (4.1), Oliveira [15] proved local and global well-posedness for the one-dimensional case in $H^2(\mathbb{R})\times H^1(\mathbb{R})\times H^1(\mathbb{R})$. Later, Linares and Matheus [5] extended the result given by Oliveira showing local (and then global) well-posedness for inital data in the energy space $H^1(\mathbb{R})\times L^2(\mathbb{R})\times L^2(\mathbb{R})$. Additionally, a polinomial bound for the growth of the H^s-norm of ψ was stated in [5]. More specifically, they proved that, for smooth initial data, solutions to system (4.1) satisfies the following property:

$$\|\psi\|_{H^s(\mathbb{R})}\lesssim 1+|t|^{(s-1)^+}.$$

In fact, Linares and Matheus used this property to show that system (4.1) is globally well-posed in $H^k(\mathbb{R})\times H^{k-\frac{1}{2}}(\mathbb{R})\times H^{k-\frac{1}{2}}(\mathbb{R})$ for all $k\geqslant 0$. Moreover, regarding the higher dimensional cases, Ponce and Saut [18] have proved that (4.2) is locally well posed in $H^s(\mathbb{R}^{\mathrm{d}})\times H^{s-\frac{1}{2}}(\mathbb{R}^{\mathrm{d}})\times H^{s+\frac{1}{2}}(\mathbb{R}^{\mathrm{d}})$, for $s>\mathrm{d}/2$, where the space-dimension $\mathrm{d}=2,3$. Lastly, we mention that Luong *et al.* have recently proved the well-posedness (under some extra conditions) of system (4.2) in the background of a line-soliton [7].

4.1.2 Main results

In the remainder of this work we focus in decay properties for general solutions of (4.1) in the energy space. Our first main result states that there exists two specific characteristic curves such that, along them, there is an additional time-integrability property on growing compact sets.

Theorem 4.1. *Let $v_\pm:=\pm\theta^{-1}\big(\sqrt{\beta}\pm\alpha\big)$ fixed. Consider $(\psi,\rho,\eta)\in C(\mathbb{R},H^1\times L^2\times L^2)$ to be any solution to system (4.1) emanating from an initial data $(\psi_0,\rho_0,\eta_0)\in H^1\times L^2\times L^2$. Then, for any $c\in\mathbb{R}_+$, the following inequality holds*

$$\int_0^{+\infty}\frac{1}{\mu_*(t)}\int_{\Omega_\pm(t)}|\psi(t,x)|^2\mathrm{d}x\mathrm{d}t<+\infty,$$

where $\Omega_\pm(t):=\{x\in\mathbb{R}:\ -c\lambda(t)\leqslant x-v_\pm t\leqslant c\lambda(t)\}$, $\kappa:=10^{100}$ and

$$\lambda(t):=t^{2/3}\log\log^{-2/3}(\kappa+t)\quad\textit{and}\quad\mu_*(t):=t\log(\kappa+t)\log\log(\kappa+t).$$

Furthermore, we have the following scenarios:

1. *If $\pm\alpha<0$, then, the following inequality holds*

$$\int_0^{+\infty}\frac{1}{\mu_*(t)}\int_{\Omega_\pm(t)}\big(|\psi_x(t,x)|^2+|\psi(t,x)|^2+\rho^2(t,x)+\eta^2(t,x)\big)\mathrm{d}x\mathrm{d}t<+\infty.$$

In particular, we have that

$$\liminf_{t\to+\infty}\int_{\Omega_{\pm}(t)}\big(|\psi_x(t,x)|^2+|\psi(t,x)|^2+\rho^2(t,x)+\eta^2(t,x)\big)\mathrm{d}x=0.$$

2. *If $\alpha=0$, then, the following inequality holds*

$$\int_0^{+\infty}\frac{1}{\mu_*(t)}\int_{\Omega_0(t)}\Big(|\psi_x(t,x)|^2+|\psi(t,x)|^4+\eta^2(t,x)+\rho^2(t,x)\Big)\mathrm{d}x\mathrm{d}t<+\infty,$$

where λ and μ_* defined as above and $\Omega_0(t):=\{x\in\mathbb{R}:\ c\lambda(t)\leqslant|x|\leqslant C\lambda(t)\}$. In particular, the following is satisfied

$$\liminf_{t\to+\infty}\int_{\Omega_0(t)}\Big(|\psi_x(t,x)|^2+|\psi(t,x)|^4+\eta^2(t,x)+\rho^2(t,x)\Big)\mathrm{d}x=0.$$

Remark 4.1. In the previous statement, the condition $\pm\alpha<0$ must be understood according to the sets $\Omega_\pm$. In other words, if $+\alpha<0$, then both results for Ω_+ hold, while if $-\alpha<0$, then both results for Ω_- hold. Notice that if $\alpha<0$, the result for Ω_- is not necessarily true.

Remark 4.2. It is important to notice that, as soon as $\alpha\neq0$ we cannot deduce any time-integrability nor decay property on compacts sets centered at the origin. Of course, this is a consequence of (and consistent with) the existence of the standing-wave solution presented in (4.3). On the other hand, when $\alpha=0$, condition (4.4) does not allow standing-wave solutions to exists, more specifically, the first inequality in (4.4) is not satisfied when $c=\alpha=0$, and hence item 2 is not contradictory with the existence of such family of solutions.

Our second main result states that, in the so-called far-field region, solutions (in the energy space) must decay to zero.

Theorem 4.2. *Let $(\psi,\rho,\eta)\in C(\mathbb{R},H^1\times L^2\times L^2)$ be any solution to system (4.1) emanating from an initial data $(\psi_0,\rho_0,\eta_0)\in H^1\times L^2\times L^2$. Then, for any pair of constants $c_1,c_2>0$ the following properties holds:*

1. *Consider any non-negative function $\zeta\in C^1(\mathbb{R})$ satisfying that, there exists $\delta>0$ such that, for all $t>0$ it holds*

$$\zeta(t)\gtrsim t\log(\kappa+t)^{1+\delta}\quad\text{ and }\quad\zeta'(t)\gtrsim\log(\kappa+t)^{\delta+1}.$$

Then, setting $\Omega_\zeta(t):=\{x\in\mathbb{R}:\ c_1\zeta(t)\leqslant|x|\leqslant c_2\zeta(t)\}$, the following limit holds

$$\lim_{t\to\infty}\|\psi(t)\|_{L^2(\Omega_\zeta(t))}=0. \tag{4.5}$$

2. *Assume additionally that $(\psi,\rho,\eta)\in C(\mathbb{R},H^2\times H^1\times H^1)$ is a solution emanating from an initial data $(\psi_0,\rho_0,\eta_0)\in H^2\times H^1\times H^1$. Then, for any non-negative $\zeta\in C^1(\mathbb{R})$ satisfying that, there exists $\delta>0$ such that, for all $t>0$,*

$$\zeta(t)\gtrsim t^{2+\delta}\quad\text{ and }\quad\zeta'(t)\gtrsim t^{1+\delta},$$

the following decay for the local energy norm holds

$$\lim_{t\to\infty}\big(\|\psi(t)\|_{H^1(\Omega_\zeta(t))}+\|\rho(t)\|_{L^2(\Omega_\zeta(t))}+\|\eta(t)\|_{L^2(\Omega_\zeta(t))}\big)=0.$$

Remark 4.3. Note that none of the above theorems require any smallness assumption in terms of the initial data $\|\psi_0\|_{H^1} \ll 1$. Moreover, they do not require any parity assumption either (as their counterparts founded in [11, 12]), nor any extra decay hypotheses in terms of weighted Sobolev norms, such as $\|x\psi\|_{L^2} \ll 1$ for example.

Remark 4.4. One important difference between the results above and those in [11, 12] is that, in both of those works, the equations under study preserve the oddness of the initial data (for the Schrödinger component ψ), while system (4.1) does not. Hence, the analysis presented there assuming parity conditions on the initial data cannot be applied to system (4.1).

Remark 4.5. To the best of our knowledge, these are the first results dealing with global properties of system (4.1) in the one dimensional case.

Finally, it is worth mentioning that the techniques involved in the proof of Theorems 4.1 and 4.2 have already been used before in some other contexts. We refer to [11] for the use of some of these ideas in context of the one-dimensional Schrödinger equation, and to [12] for scalar Zakharov system (as well as the Klein-Gordon Zakharov system). On the other hand, for other type of systems that have served us for motivations we refer to [4, 6, 14]. However, as previously described, system (4.1) has some important differences with respect to the above cases (see Remark 4.4), what does not allow us to apply the same ideas. In particular, the presence of some transport equations in (4.1) breaks the symmetry properties used in previous works to study Schrödinger-type equations/systems with these specific techniques. Finally, it is important to mention that most of these ideas come from classical works, such as [8, 9, 10, 13].

4.2 Preliminary lemmas

4.2.1 Virial identities

In this section we seek to establish the key virial identities required in our analysis. In order to do that we consider the following weight function

$$\Phi(x) := \tanh(x), \quad \text{and hence} \quad \Phi' = \operatorname{sech}^2(x).$$

Additionally, we consider time-dependent scaling functions $\lambda_1(t)$, $\lambda_2(t)$ and $\mu(t)$ given by

$$\begin{aligned} \lambda_1(t) &:= (\kappa+t)^{2/3}\log\log^{-2/3}(\kappa+t), \\ \lambda_2(t) &:= (\kappa+t)^{2/3}\log\log^{1/3}(\kappa+t), \\ \mu(t) &:= (\kappa+t)^{1/3}\log(\kappa+t)\log\log^{5/3}(\kappa+t), \end{aligned} \tag{4.6}$$

where $\kappa := 10^{100}$. The role of $\mu(t)$ and $\lambda_{\mathrm{i}}(t)$ is to provide some extra time-decay so that we can somehow neglect bad terms (with no sign) that prevent us to conclude the properties claimed in the above theorems. Additionally, one can think of $\lambda_1(t)$ as the rate of growth of the set $\Omega_\pm(t)$ and $\Omega_0(t)$ defined in Theorem 4.1. The key idea for considering exactly these definitions for $\mu(t)$ and $\lambda_{\mathrm{i}}(t)$ is that

$$\frac{1}{\mu(t)\lambda_2(t)},\ \frac{\lambda_{\mathrm{i}}'(t)}{\mu(t)\lambda_{\mathrm{i}}(t)},\ \frac{\mu'(t)}{\mu^2(t)}\in L^1(\mathbb{R}_+),\qquad \text{while}\qquad \frac{1}{\mu(t)\lambda_1(t)}\notin L^1(\mathbb{R}_+).$$

In the sequel we shall exploit these two properties. Moreover, for the sake of simplicity we introduce the following useful notation

$$\upsilon_- := -\theta^{-1}\big(\sqrt{\beta}-\alpha\big)\qquad\text{and}\qquad \upsilon_+ := \theta^{-1}\big(\sqrt{\beta}+\alpha\big). \tag{4.7}$$

Then, with all of the above notations, we define the *modified mean* functionals $\mathcal{J}_1(t)$ and $\mathcal{J}_2(t)$, adapted to the curves $x-\upsilon_\pm t$, which are given by

$$\begin{aligned}
\mathcal{J}_1(t) &:= \frac{\theta}{\mu(t)}\int_{\mathbb{R}}\Big(\sqrt{\beta}\rho\big(t,x-\upsilon_-t\big)+\eta\big(t,x-\upsilon_-t\big)\Big)\Phi\left(\frac{x}{\lambda_1(t)}\right)\Phi'\left(\frac{x}{\lambda_2(t)}\right)\mathrm{d}x,\\
\mathcal{J}_2(t) &:= \frac{\theta}{\mu(t)}\int_{\mathbb{R}}\Big(\sqrt{\beta}\rho\big(t,x-\upsilon_+t\big)-\eta\big(t,x-\upsilon_+t\big)\Big)\Phi\left(\frac{x}{\lambda_1(t)}\right)\Phi'\left(\frac{x}{\lambda_2(t)}\right)\mathrm{d}x.
\end{aligned}$$

The reason why we evaluate solutions on these translated points $x-\upsilon_\pm t$ is to be able to take advantage of the characteristics of the underlying transport equations associated to (4.1). Moreover, another key quantity that shall play a fundamental role in our proof is the *modified momentum* functional $\mathcal{I}(t)$, which is given by

$$\mathcal{I}(t) := \frac{1}{\mu(t)}\mathrm{Im}\int_{\mathbb{R}}\psi(t,x)\overline{\psi_x}(t,x)\Phi\left(\frac{x}{\lambda_1(t)}\right)\mathrm{d}x-\frac{\theta}{\mu(t)}\int_{\mathbb{R}}\rho(t,x)\eta(t,x)\Phi\left(\frac{x}{\lambda_1(t)}\right)\mathrm{d}x.$$

For the sake of simplicity and the clarity of computations, we split the previous functional into two parts, namely,

$$\begin{aligned}
\mathcal{I}_1(t) &:= \frac{1}{\mu(t)}\mathrm{Im}\int_{\mathbb{R}}\psi(t,x)\overline{\psi_x}(t,x)\Phi\left(\frac{x}{\lambda_1(t)}\right)\mathrm{d}x,\\
\mathcal{I}_2(t) &:= \frac{\theta}{\mu(t)}\int_{\mathbb{R}}\rho(t,x)\eta(t,x)\Phi\left(\frac{x}{\lambda_1(t)}\right)\mathrm{d}x.
\end{aligned}$$

We anticipate that, thanks to the explicit form of Φ and the conservation of the momentum and the energy, if (ψ,ρ,η) is a solution to system (4.1) belonging to the class $C(\mathbb{R},H^1\times L^2\times L^2)$, then, all the modified functionals above $\mathcal{J}_1(t)$, $\mathcal{J}_2(t)$ and $\mathcal{I}(t)$ are well defined for all times $t\in\mathbb{R}$ (see Lemma 4.6 for further details).

The following three lemmas give us the first basic virial identities satisfied by the modified functionals $\mathcal{J}_1(t)$, $\mathcal{J}_2(t)$ and $\mathcal{I}(t)$ above.

Lemma 4.3. *Let $(\psi,\rho,\eta)\in C(\mathbb{R},H^1\times L^2\times L^2)$ be any solution to system (4.1). Then, for all $t\in\mathbb{R}$, the following identity holds*

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) &= \frac{2\omega}{\mu\lambda_1}\int|\psi_x|^2\Phi'-\frac{\omega}{2\mu\lambda_1^3}\int|\psi|^2\Phi'''+\frac{1}{2\mu\lambda_1}\int\eta^2\Phi'+\frac{\beta}{2\mu\lambda_1}\int\rho^2\Phi'\\
&\quad+\frac{\gamma q}{2\mu\lambda_1}\int|\psi|^4\Phi'-\frac{\alpha}{\mu\lambda_1}\int\rho\eta\Phi'+\frac{\gamma}{\mu\lambda_1}\int\big(\eta-\tfrac{\alpha}{2}\rho\big)|\psi|^2\Phi'-\frac{\theta\mu'}{\mu^2}\int\rho\eta\Phi\\
&\quad+\frac{\mu'}{\mu^2}\mathrm{Im}\int\psi\overline{\psi_x}\Phi+\frac{\lambda_1'}{\mu\lambda_1}\mathrm{Im}\int\left(\frac{x}{\lambda_1}\right)\psi\overline{\psi}_x\Phi'-\frac{\theta\lambda_1'}{\mu\lambda_1}\int\left(\frac{x}{\lambda_1}\right)\rho\eta\Phi'.
\end{aligned}\tag{4.8}$$

Proof. The proof is somehow straightforward and follows from direct computations; we shall only proceed formally. Notice that the following reasoning can be made rigorously by standards approximation and density arguments.

Directly differentiating the definition of the functional $\mathcal{I}_1$, using system (4.1) and performing several integration by parts we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_1(t) &= \frac{1}{\mu}\mathrm{Im}\int \psi_t\overline{\psi_x}\Phi + \frac{1}{\mu}\mathrm{Im}\int \psi\overline{\psi_{tx}}\Phi - \frac{\lambda_1'}{\mu\lambda_1}\mathrm{Im}\int\left(\frac{x}{\lambda_1}\right)\psi\overline{\psi_x}\Phi' - \frac{\mu'}{\mu^2}\mathrm{Im}\int\psi\overline{\psi_x}\Phi \\
&= \frac{2}{\mu}\mathrm{Im}\int \psi_t\overline{\psi}_x\Phi - \frac{1}{\mu\lambda_1}\mathrm{Im}\int\psi\overline{\psi_t}\Phi' - \frac{\lambda_1'}{\mu\lambda_1}\mathrm{Im}\int\left(\frac{x}{\lambda_1}\right)\psi\overline{\psi_x}\Phi' - \frac{\mu'}{\mu^2}\mathrm{Im}\int\psi\overline{\psi_x}\Phi \\
&= \frac{2}{\mu}\mathrm{Re}\int\Big(\omega\psi_{xx} - \gamma\big(\eta - \tfrac{\alpha}{2}\rho + q|\psi|^2\big)\psi\Big)\overline{\psi_x}\Phi - \frac{\mu'}{\mu^2}\mathrm{Im}\int\psi\overline{\psi_x}\Phi \\
&\quad + \frac{1}{\mu\lambda_1}\mathrm{Re}\int\Big(\omega\overline{\psi_{xx}} - \gamma\big(\eta - \tfrac{\alpha}{2}\rho + q|\psi|^2\big)\overline{\psi}\Big)\psi\Phi' - \frac{\lambda_1'}{\mu\lambda_1}\mathrm{Im}\int\left(\frac{x}{\lambda_1}\right)\psi\overline{\psi_x}\Phi' \\
&= -\frac{2\omega}{\mu\lambda_1}\int|\psi_x|^2\Phi' + \frac{\gamma}{\mu\lambda_1}\int\Big(\eta - \tfrac{\alpha}{2}\rho + \tfrac{q}{4}|\psi|^2\Big)|\psi|^2\Phi' - \frac{\mu'}{\mu^2}\mathrm{Im}\int\psi\overline{\psi_x}\Phi \\
&\quad + \frac{\gamma}{\mu}\int\big(\eta_x - \tfrac{\alpha}{2}\rho_x\big)|\psi|^2\Phi + \frac{\omega}{2\mu\lambda_1^3}\int|\psi|^2\Phi''' - \frac{\lambda_1'}{\mu\lambda_1}\mathrm{Im}\int\left(\frac{x}{\lambda_1}\right)\psi\overline{\psi}_x\Phi' \\
&\quad - \frac{\gamma}{\mu\lambda_1}\int\big(\eta - \tfrac{\alpha}{2}\rho + q|\psi|^2\big)|\psi|^2\Phi' \\
&= -\frac{2\omega}{\mu\lambda_1}\int|\psi_x|^2\Phi' - \frac{\mu'}{\mu^2}\mathrm{Im}\int\psi\overline{\psi_x}\Phi + \frac{\gamma}{\mu}\int\big(\eta_x - \tfrac{\alpha}{2}\rho_x\big)|\psi|^2\Phi \\
&\quad + \frac{\omega}{2\mu\lambda_1^3}\int|\psi|^2\Phi''' - \frac{\gamma q}{2\mu\lambda_1}\int|\psi|^4\Phi' - \frac{\lambda_1'}{\mu\lambda_1}\mathrm{Im}\int\left(\frac{x}{\lambda_1}\right)\psi\overline{\psi}_x\Phi'
\end{aligned}$$

Now we compute the time-derivative of the second functional $\mathcal{I}_2$. In fact, by direct differentiation again, using system (4.1) and performing several integration by parts we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_2 &= \frac{\theta}{\mu}\int\rho_t\eta\Phi + \frac{\theta}{\mu}\int\rho\eta_t\Phi - \frac{\theta\lambda_1'}{\mu\lambda_1}\int\left(\frac{x}{\lambda_1}\right)\rho\eta\Phi' - \frac{\theta\mu'}{\mu^2}\int\rho\eta\Phi \\
&= \frac{1}{2\mu\lambda_1}\int\eta^2\Phi' + \frac{\alpha}{\mu}\int\rho_x\eta\Phi + \frac{\gamma}{\mu}\int\eta_x|\psi|^2\Phi + \frac{\gamma}{\mu\lambda_1}\int\eta|\psi|^2\Phi' - \frac{\theta\lambda_1'}{\mu\lambda_1}\int\left(\frac{x}{\lambda_1}\right)\rho\eta\Phi' \\
&\quad + \frac{\beta}{2\mu\lambda_1}\int\rho^2\Phi' + \frac{\alpha}{\mu}\int\rho\eta_x\Phi - \frac{\alpha\gamma}{2\mu}\int\rho_x|\psi|^2\Phi - \frac{\alpha\gamma}{2\mu\lambda_1}\int\rho|\psi|^2\Phi' - \frac{\theta\mu'}{\mu^2}\int\rho\eta\Phi \\
&= \frac{1}{2\mu\lambda_1}\int\eta^2\Phi' - \frac{\alpha}{\mu\lambda_1}\int\rho\eta\Phi' + \frac{\gamma}{\mu}\int\eta_x|\psi|^2\Phi + \frac{\gamma}{\mu\lambda_1}\int\eta|\psi|^2\Phi' - \frac{\theta\mu'}{\mu^2}\int\rho\eta\Phi \\
&\quad + \frac{\beta}{2\mu\lambda_1}\int\rho^2\Phi' - \frac{\alpha\gamma}{2\mu}\int\rho_x|\psi|^2\Phi - \frac{\alpha\gamma}{2\mu\lambda_1}\int\rho|\psi|^2\Phi' - \frac{\theta\lambda_1'}{\mu\lambda_1}\int\left(\frac{x}{\lambda_1}\right)\rho\eta\Phi'.
\end{aligned}$$

Hence, gathering both previous identities we conclude the desired result. ∎

Lemma 4.4. *Let $(\psi,\rho,\eta)\in C(\mathbb{R},H^1\times L^2\times L^2)$ be any solution to system* (4.1)*. Then, for* *all $t \in \mathbb{R}$, the following identities hold:*

$$
\begin{aligned}
(1)\ \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_1(t) = {} & \frac{\gamma(2\sqrt{\beta}-\alpha)}{2\mu(t)\lambda_1(t)}\int \left|\psi\big(t,x-\upsilon_- t\big)\right|^2 \Phi'\left(\frac{x}{\lambda_1(t)}\right)\Phi'\left(\frac{x}{\lambda_2(t)}\right) \\
& + \frac{\gamma(2\sqrt{\beta}-\alpha)}{2\mu(t)\lambda_2(t)}\int \left|\psi\big(t,x-\upsilon_- t\big)\right|^2 \Phi\left(\frac{x}{\lambda_1(t)}\right)\Phi''\left(\frac{x}{\lambda_2(t)}\right) \\
& - \frac{\theta\mu'(t)}{\mu^2(t)}\int \big(\sqrt{\beta}\rho+\eta\big)(t,x-\upsilon_- t)\Phi\left(\frac{x}{\lambda_1(t)}\right)\Phi'\left(\frac{x}{\lambda_2(t)}\right) \\
& - \frac{\theta\lambda_1'(t)}{\mu(t)\lambda_1(t)}\int \left(\frac{x}{\lambda_1(t)}\right)\big(\sqrt{\beta}\rho+\eta\big)(t,x-\upsilon_- t)\Phi'\left(\frac{x}{\lambda_1(t)}\right)\Phi'\left(\frac{x}{\lambda_2(t)}\right) \\
& - \frac{\theta\lambda_2'(t)}{\mu(t)\lambda_2(t)}\int \left(\frac{x}{\lambda_2(t)}\right)\big(\sqrt{\beta}\rho+\eta\big)(t,x-\upsilon_- t)\Phi\left(\frac{x}{\lambda_1(t)}\right)\Phi''\left(\frac{x}{\lambda_2(t)}\right).
\end{aligned}
$$

$$
\begin{aligned}
(2)\ \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_2(t) = {} & \frac{\gamma(2\sqrt{\beta}+\alpha)}{2\mu(t)\lambda_1(t)}\int_{\mathbb{R}} \left|\psi\big(t,x-\upsilon_+ t\big)\right|^2 \Phi'\left(\frac{x}{\lambda_1(t)}\right)\Phi'\left(\frac{x}{\lambda_2(t)}\right) \\
& + \frac{\gamma(2\sqrt{\beta}+\alpha)}{2\mu(t)\lambda_2(t)}\int_{\mathbb{R}} \left|\psi\big(t,x-\upsilon_+ t\big)\right|^2 \Phi\left(\frac{x}{\lambda_1(t)}\right)\Phi''\left(\frac{x}{\lambda_2(t)}\right) \\
& - \frac{\theta\mu'(t)}{\mu^2(t)}\int \big(\sqrt{\beta}\rho-\eta\big)(t,x-\upsilon_+ t)\Phi\left(\frac{x}{\lambda_1(t)}\right)\Phi'\left(\frac{x}{\lambda_2(t)}\right) \\
& - \frac{\theta\lambda_1'(t)}{\mu(t)\lambda_1(t)}\int \left(\frac{x}{\lambda_1(t)}\right)\big(\sqrt{\beta}\rho-\eta\big)(t,x-\upsilon_+ t)\Phi'\left(\frac{x}{\lambda_1(t)}\right)\Phi'\left(\frac{x}{\lambda_2(t)}\right) \\
& - \frac{\theta\lambda_2'(t)}{\mu(t)\lambda_2(t)}\int \left(\frac{x}{\lambda_2(t)}\right)\big(\sqrt{\beta}\rho-\eta\big)(t,x-\upsilon_+ t)\Phi\left(\frac{x}{\lambda_1(t)}\right)\Phi''\left(\frac{x}{\lambda_2(t)}\right).
\end{aligned}
$$

Proof. Similarly to the previous lemma, we proceed by a direct computation. For the sake of simplicity, throughout this proof we will denote by Φ_{i}, Φ_{i}' and Φ_{i}'' the functions given by $\Phi_{\mathrm{i}}(x) := \Phi\big(\tfrac{x}{\lambda_{\mathrm{i}}(t)}\big)$, $\Phi_{\mathrm{i}}'(x) := \Phi'\big(\tfrac{x}{\lambda_{\mathrm{i}}(t)}\big)$ and $\Phi_{\mathrm{i}}''(x) := \Phi''\big(\tfrac{x}{\lambda_{\mathrm{i}}(t)}\big)$, with $\mathrm{i} = 1, 2$. Also, we shall only write ρ, η, Φ_1 and Φ_2, ommiting their arguments.

Indeed, taking the time derivative of the functional, using system (4.1) and performing some integration by parts we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_1(t) &= \frac{\sqrt{\beta}\theta}{\mu}\int \rho_t\Phi_1\Phi_2' + \frac{\beta-\alpha\sqrt{\beta}}{\mu}\int \rho_x\Phi_1\Phi_2' - \frac{\theta\sqrt{\beta}\mu'}{\mu^2}\int \rho\Phi_1\Phi_2' + \frac{\theta}{\mu}\int \eta_t\Phi_1\Phi_2' \\
&\quad + \frac{\sqrt{\beta}-\alpha}{\mu}\int \eta_x\Phi_1\Phi_2' - \frac{\theta\mu'}{\mu^2}\int \eta\Phi_1\Phi_2' - \frac{\sqrt{\beta}\theta\lambda_1'}{\mu\lambda_1}\int\left(\frac{x}{\lambda_1}\right)\rho\Phi_1'\Phi_2' \\
&\quad - \frac{\theta\lambda_1'}{\mu\lambda_1}\int\left(\frac{x}{\lambda_1}\right)\eta\Phi_1'\Phi_2' - \frac{\sqrt{\beta}\theta\lambda_2'}{\mu\lambda_2}\int\left(\frac{x}{\lambda_2}\right)\rho\Phi_1\Phi_2'' - \frac{\theta\lambda_2'}{\mu\lambda_2}\int\left(\frac{x}{\lambda_2}\right)\eta\Phi_1\Phi_2'' \\
&= \frac{\alpha\sqrt{\beta}}{\mu}\int \rho_x\Phi_1\Phi_2' - \frac{\sqrt{\beta}}{\mu}\int \eta_x\Phi_1\Phi_2' - \frac{\gamma\sqrt{\beta}}{\mu}\int (|\psi|^2)_x\Phi_1\Phi_2' + \frac{\alpha}{\mu}\int \eta_x\Phi_1\Phi_2' \\
&\quad + \frac{\beta-\sqrt{\beta}\alpha}{\mu}\int \rho_x\Phi_1\Phi_2' - \frac{\sqrt{\beta}\theta\mu'}{\mu^2}\int \rho\Phi_1\Phi_2' - \frac{\beta}{\mu}\int \rho_x\Phi_1\Phi_2' - \frac{\theta\mu'}{\mu^2}\int \eta\Phi_1\Phi_2' \\
&\quad + \frac{\alpha\gamma}{2\mu}\int (|\psi|^2)_x\Phi_1\Phi_2' + \frac{\sqrt{\beta}-\alpha}{\mu}\int \eta_x\Phi_1\Phi_2' - \frac{\sqrt{\beta}\theta\lambda_1'}{\mu\lambda_1}\int\left(\frac{x}{\lambda_1}\right)\rho\Phi_1'\Phi_2' \\
&\quad - \frac{\theta\lambda_1'}{\mu\lambda_1}\int\left(\frac{x}{\lambda_1}\right)\eta\Phi_1'\Phi_2' - \frac{\sqrt{\beta}\theta\lambda_2'}{\mu\lambda_2}\int\left(\frac{x}{\lambda_2}\right)\rho\Phi_1\Phi_2'' - \frac{\theta\lambda_2'}{\mu\lambda_2}\int\left(\frac{x}{\lambda_2}\right)\eta\Phi_1\Phi_2'' \\
&= \frac{\gamma\sqrt{\beta}}{\mu\lambda_1}\int |\psi|^2\Phi_1'\Phi_2' + \frac{\gamma\sqrt{\beta}}{\mu\lambda_2}\int |\psi|^2\Phi_1\Phi_2'' - \frac{\sqrt{\beta}\theta\mu'}{\mu^2}\int \rho\Phi_1\Phi_2' - \frac{\theta\mu'}{\mu^2}\int \eta\Phi_1\Phi_2' \\
&\quad - \frac{\alpha\gamma}{2\mu\lambda_1}\int |\psi|^2\Phi_1'\Phi_2' - \frac{\alpha\gamma}{2\mu\lambda_2}\int |\psi|^2\Phi_1\Phi_2'' - \frac{\sqrt{\beta}\theta\lambda_1'}{\mu\lambda_1}\int\left(\frac{x}{\lambda_1}\right)\rho\Phi_1'\Phi_2' \\
&\quad - \frac{\theta\lambda_1'}{\mu\lambda_1}\int\left(\frac{x}{\lambda_1}\right)\eta\Phi_1'\Phi_2' - \frac{\sqrt{\beta}\theta\lambda_2'}{\mu\lambda_2}\int\left(\frac{x}{\lambda_2}\right)\rho\Phi_1\Phi_2'' - \frac{\theta\lambda_2'}{\mu\lambda_2}\int\left(\frac{x}{\lambda_2}\right)\eta\Phi_1\Phi_2''.
\end{aligned}
$$

Thus, by gathering terms we conclude the proof of the lemma. The proof of the second formula follows the same arguments. Hence, we omit it. ■

Remark 4.6. We emphasize that none of these Virial Lemmas require the explicit definition of Φ nor the one for the scaling functions λ_{i} and μ that we gave at the beginning of this section. In fact, in Section 4.4 we shall exploit (4.8) for completely different definitions of Φ, λ and μ (as soon as all quantities are well-defined). However, unless stated otherwise, throughout the proof of Theorem 4.1 we shall always assume that we are referring to the functions defined at the beginning of this section.

4.2.2 Uniform boundedness of the energy norm

The following lemma is a direct consequence of the conservation laws and give us the time-uniform boundedness of the $H^1\times L^2\times L^2$-norm.

Lemma 4.5. *Let $(\psi,\rho,\eta)\in C(\mathbb{R},H^1\times L^2\times L^2)$ be a global solution emanating from an initial data $(\psi_0,\rho_0,\eta_0)\in H^1\times L^2\times L^2$. Then, there exists a constant $C\in\mathbb{R}_+$, depending only on the norm of the initial data, such that the following global bound holds*

$$\|\psi(t)\|_{H^1}^2+\|\rho(t)\|_{L^2}^2+\|\eta(t)\|_{L^2}^2\leqslant C,\qquad \forall t\in\mathbb{R}.$$

Proof. The idea of the proof is to use the conservation of the energy, re-constructing such conserved quantity from the energy norm. In fact, first of all let us recall that from the conservation of the energy we have

$$\int_{\mathbb{R}} \Big(\omega|\psi_x|^2 + \tfrac{\beta}{2}\rho^2 + \tfrac{1}{2}\eta^2 + \tfrac{\gamma q}{2}|\psi|^4 + \tfrac{\gamma}{2}(2\eta - \alpha\rho)|\psi|^2 - \alpha\rho\eta\Big)\mathrm{d}x = E(0). \tag{4.9}$$

Hence, essentially we have to show that we can control the last three addends with the first three of them. Specifically, taking advantage of the conservation of both mass and energy, we would like to find appropriate constants $c > 0$, $a, b \in \mathbb{R}$ such that

$$\|\psi(t)\|_{H^1}^2 + \|\rho(t)\|_{L^2}^2 + \|\eta(t)\|_{L^2}^2 \leqslant c\left(E(0)^a + M(0)^b\right).$$

First, we notice that to control the crossed term $\rho\eta$, it is enough to use Young inequality for products, from where we get

$$\alpha\int_{\mathbb{R}} \rho(t,x)\eta(t,x)\mathrm{d}x \leqslant \frac{\beta+\alpha^2}{4}\int_{\mathbb{R}}\rho^2(t,x)\mathrm{d}x + \frac{\alpha^2}{2(\beta+\alpha^2)}\int_R \eta^2(t,x)\mathrm{d}x. \tag{4.10}$$

Then, gathering the corresponding quadratic terms with respect to (ρ,η) appearing in the energy, we have

$$\int_{\mathbb{R}}\Big(\tfrac{\beta}{2}\rho^2(t,x) + \tfrac{1}{2}\eta^2(t,x) - \alpha\rho(t,x)\eta(t,x)\Big)\mathrm{d}x \geqslant \frac{\beta-\alpha^2}{4}\|\rho(t)\|_{L^2}^2 + \frac{\beta}{2(\beta+\alpha^2)}\|\eta(t)\|_{L^2}^2.$$

We continue by bounding the contribution of the L^4-norm of $\psi(t)$. Indeed, by using Gagliardo-Nirenberg interpolation inequality, as well as Young inequality in the resulting right-hand side, we obtain

$$\int_{\mathbb{R}} |\psi(t,x)|^4\mathrm{d}x \leqslant \|\psi(t)\|_{H^1}\|\psi(t)\|_{L^2}^3 \leqslant \varepsilon\|\psi_x(t)\|_{L^2}^2 + \varepsilon M(0) + \frac{1}{\varepsilon}M(0)^3.$$

Once again, due to the conservation of mass, it is enough to choose $\varepsilon \in (0,1)$ sufficiently small so that we can absorb $\varepsilon\|\psi_x(t)\|_{L^2}^2$ by using the first term in (4.9). Finally, it only remains to bound

$$\frac{\gamma}{2}\int_{\mathbb{R}}\big(2\eta(t,x) - \alpha\rho(t,x)\big)|\psi(t,x)|^2\mathrm{d}x.$$

However, notice that this term can be controlled by the previous ones. In fact, we have

$$\begin{aligned}\frac{\gamma}{2}\int_{\mathbb{R}}\big(2\eta(t,x) - \alpha\rho(t,x)\big)|\psi(t,x)|^2\mathrm{d}x \leqslant{}& \frac{\beta}{16(\beta+\alpha^2)}\|\eta(t)\|_{L^2}^2 + \frac{\beta-\alpha^2}{16}\|\rho(t)\|_{L^2}^2 \\ &+ \left(\frac{\gamma^2(\beta+\alpha^2)}{8\beta} + \frac{2\alpha^2\gamma^2}{\beta-\alpha^2}\right)\|\psi(t)\|_{L^4}^4.\end{aligned}$$

Therefore, gathering all the above estimates we conclude the proof of the lemma. ■

As a consequence of the previous lemma, we conclude the uniform boundedness of all the modified functionals.

Corollary 4.6. *Let $(\psi,\rho,\eta)\in C(\mathbb{R},H^1\times L^2\times L^2)$ be any solution to system* (4.1) *emanating from an initial data $(\psi_0,\rho_0,\eta_0)\in H^1\times L^2\times L^2$. Consider $\lambda_1(t)$, $\lambda_2(t)$ and $\mu(t)$ defined as in* (4.6)*. Then, the following bound holds*

$$\sup_{t\in(0,+\infty)}\Big(\big|\mathcal{J}_1(t)\big|+\big|\mathcal{J}_2(t)\big|+\big|\mathcal{I}_1(t)\big|+\big|\mathcal{I}_2(t)\big|\Big)<+\infty.$$

Proof. First of all, notice that the time-uniform boundedness of $\mathcal{I}_1$ and $\mathcal{I}_2$ follows directly from Hölder inequality as well as the previous Lemma. In the same fashion, to bound $\mathcal{J}_1$ we proceed by Hölder inequality. However, since $\mathcal{J}_1$ is of order 1 in (ρ,η), in this case we obtain

$$|\mathcal{J}_1(t)|\lesssim\frac{1}{\mu(t)}\|\rho(t)+\eta(t)\|_{L^2}\|\Phi\|_{L^\infty}\left\|\Phi'\big(\tfrac{\cdot}{\lambda_2(t)}\big)\right\|_{L^2}\lesssim\frac{\lambda_2^{1/2}(t)}{\mu(t)}<C, \tag{4.11}$$

where $C>0$ only depends on the initial data (ψ_0,ρ_0,η_0).

To conclude, we notice that the same procedure also provides a time-uniform bound for $\mathcal{J}_2(t)$. The proof is complete. ■

Remark 4.7. Inequality (4.11) is precisely the condition that does not allow us to choose $\lambda_1(t)$ growing any faster. In particular, this is the reason why we cannot choose $\lambda_1(t)=t^{1^-}$, for example.

4.3 Proof of Theorem 4.1

4.3.1 Time integrability of $|\psi|^2$

In this section we seek to use the previously found virial identities to prove the time integrability of the solution. In order to do that, we split the analysis in several steps. First, we shall show the time integrability (in the region given in Theorem 4.1) only for $|\psi(t,x)|^2$, which is proved in the following proposition.

Proposition 4.7. *Let $(\psi,\rho,\eta)\in C(\mathbb{R},H^1\times L^2\times L^2)$ be any solution to system* (4.1) *emanating from an initial data $(\psi_0,\rho_0,\eta_0)\in H^1\times L^2\times L^2$. Then, for $\lambda_1(t)$, $\lambda_2(t)$, $\mu(t)$ and $\upsilon_\pm$ defined as in* (4.6)-(4.7)*, the following inequality holds*

$$\int_0^{+\infty}\frac{1}{\mu(t)\lambda_1(t)}\int_{\mathbb{R}}\big|\psi(t,x-\upsilon_\pm t)\big|^2\operatorname{sech}^4\left(\frac{x}{\lambda_1(t)}\right)\mathrm{d}x\mathrm{d}t<+\infty. \tag{4.12}$$

Proof. Let us first consider the case of υ_-. The case for υ_+ follows from the same bounds up to trivial modifications. Indeed, we define

$$\begin{aligned}\mathcal{F}_-(t) := &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_1(t) - \frac{\gamma(2\sqrt{\beta}-\alpha)}{2\mu(t)\lambda_2(t)}\int \big|\psi(t,x-\upsilon_-t)\big|^2\Phi\left(\frac{x}{\lambda_1(t)}\right)\Phi''\left(\frac{x}{\lambda_2(t)}\right)\\ &+\frac{\theta\mu'(t)}{\mu^2(t)}\int\big(\sqrt{\beta}\rho+\eta\big)(t,x-\upsilon_-t)\Phi\left(\frac{x}{\lambda_1(t)}\right)\Phi'\left(\frac{x}{\lambda_2(t)}\right)\\ &+\frac{\theta\lambda_1'(t)}{\mu(t)\lambda_1(t)}\int\left(\frac{x}{\lambda_1(t)}\right)\big(\sqrt{\beta}\rho+\eta\big)(t,x-\upsilon_-t)\Phi'\left(\frac{x}{\lambda_1(t)}\right)\Phi'\left(\frac{x}{\lambda_2(t)}\right)\\ &+\frac{\theta\lambda_2'(t)}{\mu(t)\lambda_2(t)}\int\left(\frac{x}{\lambda_2(t)}\right)\big(\sqrt{\beta}\rho+\eta\big)(t,x-\upsilon_-t)\Phi\left(\frac{x}{\lambda_1(t)}\right)\Phi''\left(\frac{x}{\lambda_2(t)}\right)\\ =: {}&\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.\end{aligned}$$

Then, from Lemma 4.4 we infer that

$$\frac{\gamma(2\sqrt{\beta}-\alpha)}{2\mu(t)\lambda_1(t)}\int\big|\psi(t,x-\upsilon_-t)\big|^2\Phi'\left(\frac{x}{\lambda_1(t)}\right)\Phi'\left(\frac{x}{\lambda_2(t)}\right)\mathrm{d}x=\mathcal{F}_-(t).$$

Hence, the problem is reduced to prove that we can integrate $\mathcal{F}_-(t)$ on $(0,+\infty)$. In fact, first of all notice that, from Corollary 4.6 we infer that

$$\left|\int_0^{+\infty}\mathrm{I}(t)\mathrm{d}t\right|\lesssim\limsup_{t\to+\infty}\big|\mathcal{J}_1(t)-\mathcal{J}_1(0)\big|<+\infty.$$

Moreover, from Lemma 4.5 as well as the explicit definitions of $\mu(t)$ and $\lambda_2(t)$, it immediately follows that $\mathrm{II}\in L^1(\mathbb{R}_+)$. On the other hand, from Lemma 4.5 along with Hölder inequality, we can bound $\mathrm{III}(t)$ by

$$\big|\mathrm{III}(t)\big|\lesssim\frac{\mu'(t)\|\Phi'(\frac{\cdot}{\lambda_2(t)})\|_{L^2}}{\mu^2(t)}\lesssim\frac{\lambda_2^{1/2}\mu'(t)}{\mu^2(t)}\lesssim\frac{1}{(\kappa+t)\log(\kappa+t)\log\log^{3/2}(\kappa+t)}\in L^1(\mathbb{R}_+).$$

In the same fashion, applying Lemma 4.5 and Hölder inequality, we can bound $|\mathrm{IV}(t)|$ and $|\mathrm{V}(t)|$ pointwisely by integrable function as

$$\begin{aligned}\big|\mathrm{IV}(t)\big|&\lesssim\frac{\lambda_1^{1/2}(t)\lambda_1'(t)}{\mu(t)\lambda_1(t)}\lesssim\frac{1}{(\kappa+t)\log(\kappa+t)\log\log^2(\kappa+t)}\in L^1(\mathbb{R}_+),\\ \big|\mathrm{V}(t)\big|&\lesssim\frac{\lambda_2^{1/2}(t)\lambda_2'(t)}{\mu(t)\lambda_2(t)}\lesssim\frac{1}{(\kappa+t)\log(\kappa+t)\log\log^{3/2}(\kappa+t)}\in L^1(\mathbb{R}_+).\end{aligned}$$

Therefore, gathering all the above inequalities, we conclude the proof of (4.12) in the case of υ_-. Notice that the same proof (up to trivial modifications) also works for υ_+. The proof is complete. ■

Remark 4.8. The proof of the previous proposition does not depend on the value of $\alpha\in\mathbb{R}$, and hence, this concludes the first inequality in Theorem 4.1.

4.3.2 Time Integrability of the full solution

In this section we seek to extend the analysis to the full solution, that is, to include the corresponding integral terms associated to (ψ_x,ρ,η). From now on we split the analysis in two cases concerning the values of $\alpha\in\mathbb{R}$.

Case $\alpha \neq 0$

In order to take advantage of the previous analysis, we consider a different version of the modified momentum functional adapted to this region. More specifically, we define modified momentum functional adapted to the characteristics $x - v_{\pm}t$, that is,

$$\begin{aligned}\widetilde{\mathcal{I}}_{\pm}(t) := \frac{1}{\mu(t)}\mathrm{Im}\int_{\mathbb{R}} \psi(t, x - v_{\pm}t)\overline{\psi_x}(t, x - v_{\pm}t)\Phi\left(\frac{x}{\lambda_1(t)}\right)\mathrm{d}x \\ - \frac{\theta}{\mu(t)}\int_{\mathbb{R}} \rho(t, x - v_{\pm}t)\eta(t, x - v_{\pm}t)\Phi\left(\frac{x}{\lambda_1(t)}\right)\mathrm{d}x.\end{aligned}$$

Notice that, as a direct consequence of Lemma 4.3, we have the following identity

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{I}}_{\pm}(t) &= \frac{2\omega}{\mu\lambda_1}\int |\psi_x|^2\Phi' - \frac{\omega}{2\mu\lambda_1^3}\int |\psi|^2\Phi''' + \frac{1}{2\mu\lambda_1}\int \eta^2\Phi' + \frac{\beta}{2\mu\lambda_1}\int \rho^2\Phi' \\ &+ \frac{\gamma q}{2\mu\lambda_1}\int |\psi|^4\Phi' - \frac{\alpha}{\mu\lambda_1}\int \rho\eta\Phi' + \frac{\gamma}{\mu\lambda_1}\int \left(\eta - \tfrac{\alpha}{2}\rho\right)|\psi|^2\Phi' - \frac{\theta\mu'}{\mu^2}\int \rho\eta\Phi \\ &+ \frac{\mu'}{\mu^2}\mathrm{Im}\int \psi\overline{\psi_x}\Phi + \frac{\lambda_1'}{\mu\lambda_1}\mathrm{Im}\int \left(\frac{x}{\lambda_1}\right)\psi\overline{\psi}_x\Phi' - \frac{\theta\lambda_1'}{\mu\lambda_1}\int \left(\frac{x}{\lambda_1}\right)\rho\eta\Phi' \\ &\pm \frac{\sqrt{\beta} \pm \alpha}{\theta\mu\lambda_1}\mathrm{Im}\int \psi\overline{\psi}_x\Phi' \mp \frac{\sqrt{\beta} \pm \alpha}{\mu\lambda_1}\int \rho\eta\Phi',\end{aligned} \tag{4.13}$$

where we have used the fact that, since Φ is real-valued, we have

$$-\mathrm{Im}\int \psi\overline{\psi}_{xx}\Phi\left(\frac{x}{\lambda_1}\right) = \frac{1}{\lambda_1}\mathrm{Im}\int \psi\overline{\psi}_x\Phi'\left(\frac{x}{\lambda_1}\right).$$

With all of this at hand, we are ready to prove the time integrability of the full solution in weighted spaces along these characteristics. The following proposition concludes the proof the Theorem 4.1 in the case $\alpha \neq 0$.

Proposition 4.8. *Let* $(\psi, \rho, \eta) \in C(\mathbb{R}, H^1 \times L^2 \times L^2)$ *be any solution to system* (4.1) *emanating from an initial data* $(\psi_0, \rho_0, \eta_0) \in H^1 \times L^2 \times L^2$. *For* $\lambda_1(t)$, $\mu(t)$ *and* $v_{\pm}$ *defined as in* (4.6)-(4.7), *we have the following two cases:*

1. *If* $\alpha > 0$, *then the following holds*

$$\int_0^{+\infty} \frac{1}{\mu(t)\lambda_1(t)}\int_{\mathbb{R}} \left(|\psi_x|^2 + |\psi|^2 + \rho^2 + \eta^2\right)(t, x - v_- t)\,\mathrm{sech}^4\left(\frac{x}{\lambda_1(t)}\right)\mathrm{d}x\mathrm{d}t < +\infty.$$

2. *If* $\alpha < 0$, *then the following holds*

$$\int_0^{+\infty} \frac{1}{\mu(t)\lambda_1(t)}\int_{\mathbb{R}} \left(|\psi_x|^2 + |\psi|^2 + \rho^2 + \eta^2\right)(t, x - v_+ t)\,\mathrm{sech}^4\left(\frac{x}{\lambda_1(t)}\right)\mathrm{d}x\mathrm{d}t < +\infty.$$

Proof. We shall proceed in a similar fashion as in Proposition 4.7. However, notice that in this case we have several quadratic terms with no definite sign. The idea is to use Proposition 4.7 to deal with the terms involving $|\psi|^2$, and to absorb the crossed terms of the form $\rho\eta$

with the ones with ρ^2 and η^2. In fact, first of all notice that, thanks to Lemma 4.5 and the explicit definitions of μ and λ_1, it is not difficult to see that

$$\frac{1}{\mu\lambda_1^3}\int|\psi|^2\Phi''' \in L^1(\mathbb{R}_+), \qquad \frac{\mu'}{\mu^2}\int\rho\eta\Phi \in L^1(\mathbb{R}_+), \qquad \frac{\mu'}{\mu^2}\mathrm{Im}\int\psi\overline{\psi}_x\Phi \in L^1(\mathbb{R}_+).$$

Moreover, from Lemma 4.5 we also infer that we can integrate $\widetilde{\mathcal{I}}'(t)$ on $(0,+\infty)$. On the other hand, by Hölder inequality, the explicit definitions of μ and λ_1, as well as Lemma 4.5, we obtain

$$\begin{aligned}
&\frac{\lambda_1'}{\mu\lambda_1}\left|\mathrm{Im}\int\left(\frac{x}{\lambda_1}\right)\psi\overline{\psi}_x\Phi'\right| \lesssim \frac{\lambda_1'\|\psi\|^2_{L^\infty_t H^1_x}\|(\cdot)\Phi'\|_{L^\infty}}{\mu\lambda_1} \in L^1(\mathbb{R}_+), \qquad \text{and,}\\
&\frac{\lambda_1'}{\mu\lambda_1}\left|\int\left(\frac{x}{\lambda_1}\right)\rho\eta\Phi'\right| \lesssim \frac{\lambda_1'\|\rho\|_{L^\infty_t L^2_x}\|\eta\|_{L^\infty_t L^2_x}\|(\cdot)\Phi'\|_{L^\infty}}{\mu\lambda_1} \in L^1(\mathbb{R}_+).
\end{aligned}$$

Now, for the term involving $|\psi|^4$ in (4.13), we use Proposition 4.7, Lemma 4.5 as well as Sobolev embedding, from where we get[1]

$$\frac{1}{\mu\lambda_1}\int|\psi|^4\Phi' \lesssim \frac{1}{\mu\lambda_1}\|\psi\|^2_{L^\infty_t H^1_x}\int|\psi|^2\Phi' \in L^1(\mathbb{R}_+). \tag{4.14}$$

Besides, from Young inequality for products, it is not difficult to see that

$$\frac{2\omega}{\mu\lambda_1}\int|\psi_x|^2\Phi' \pm \frac{(\sqrt{\beta}\pm\alpha)}{\mu\lambda_1}\mathrm{Im}\int\psi\overline{\psi}_x\Phi' \geqslant \frac{\omega}{\mu\lambda_1}\int|\psi_x|^2\Phi' - \frac{(\sqrt{\beta}-\alpha)^2}{4\omega\mu\lambda_1}\int|\psi|^2\Phi'.$$

Thanks to Proposition 4.7, the last term in the right-hand side above, belongs to $L^1(\mathbb{R}_+)$. Also, from Young inequality for products again, we additionally infer that

$$\frac{\gamma}{\mu\lambda_1}\int\left(\eta-\tfrac{\alpha}{2}\rho\right)|\psi|^2\Phi' \geqslant -\frac{\varepsilon_1^*}{\mu\lambda_1}\int\eta^2\Phi' - \frac{\varepsilon_2^*}{\mu\lambda_1}\int\rho^2\Phi' - \frac{K}{\mu\lambda_1}\int|\psi|^4\Phi' \tag{4.15}$$

where $\varepsilon_1^*, \varepsilon_2^* > 0$ denote sufficiently small numbers that shall be fixed later. Here $K = K(\varepsilon_1^*, \varepsilon_2^*) > 0$ is a large number, however, proceeding in the same fashion as in (4.14) we infer that this term belongs to $L^1(\mathbb{R}_+)$ no matter what the value of K is.

Finally, it only remains to control the quadratic terms in ρ and η. Unfortunately, due to the factor $\mp(\sqrt{\beta}\pm 2\alpha)\rho\eta$ appearing in (4.13), we cannot obtain the required positivity in both directions $v_\pm$ for the terms involving ρ^2, η^2 and $\rho\eta$. Thus, we split the analysis in two cases regarding the sign of α.

Case $\alpha > 0$**:** We aim to prove the following claim that provide us the required positivity: There exists two constants $c_1, c_2 > 0$, such that

$$\frac{1}{2\mu\lambda}\int\eta^2\Phi' + \frac{\beta}{2\mu\lambda}\int\rho^2\Phi' + \frac{\sqrt{\beta}-2\alpha}{\mu\lambda}\int\rho\eta\Phi' \geqslant \frac{c_1}{\mu\lambda}\int\eta^2\Phi' + \frac{c_2}{\mu\lambda}\int\rho^2\Phi'. \tag{4.16}$$

[1]Notice that here we are actually using Proposition 4.7 with a different definition of $\Phi(x)$, so that $\Phi'(x) = \mathrm{sech}^4(x)$. By taking the integration constant equal to zero we get $\Phi \in L^\infty$.

In this case we shall take advantage of $\widetilde{\mathcal{I}}_-(t)$. In fact, first of all, notice that, if $\sqrt{\beta}-2\alpha=0$, then there is nothing to prove. Then, in the sequel we assume $\sqrt{\beta}-2\alpha\neq 0$. Indeed, consider the parameter $\varepsilon_1\in\mathbb{R}$ given by

$$\varepsilon_1 := \frac{1}{2}\left(1+\frac{\beta}{(\sqrt{\beta}-2\alpha)^2}\right)>0.$$

Then, by using Young inequality for products, with parameter given by ε_1, we infer

$$\begin{aligned}&\frac{1}{2\mu\lambda}\int\eta^2\Phi'+\frac{\beta}{2\mu\lambda}\int\rho^2\Phi'+\frac{\sqrt{\beta}-2\alpha}{\mu\lambda}\int\rho\eta\Phi'\\&\qquad\geqslant\frac{1}{2\mu\lambda}\big(1-\varepsilon_1^{-1}\big)\int\eta^2\Phi'+\frac{1}{2\mu\lambda}\big(\beta-(\sqrt{\beta}-2\alpha)^2\varepsilon_1\big)\int\rho^2\Phi'\end{aligned}$$

Moreover, notice that, since $\beta-\alpha^2>0$ and $\alpha>0$, we infer that $\varepsilon_1>1$, and hence $1-\varepsilon_1^{-1}>0$. On the other hand, by direct computations we see that

$$\beta-(\sqrt{\beta}-2\alpha)^2\varepsilon_1>\beta-\frac{(\sqrt{\beta}-2\alpha)^2\beta}{(\sqrt{\beta}-2\alpha)^2}=0.$$

Then, plugging the previous computations into (4.16), we conclude the proof of the claim.

Case $\alpha<0$: In this case we aim to prove the following claim that provide us the required positivity: There exists two constants $c_1,c_2>0$, such that

$$\frac{1}{2\mu\lambda}\int\eta^2\Phi'+\frac{\beta}{2\mu\lambda}\int\rho^2\Phi'-\frac{\sqrt{\beta}+2\alpha}{\mu\lambda}\int\rho\eta\Phi'\geqslant c_1\int\eta^2\Phi'+c_2\int\rho^2\Phi'. \tag{4.17}$$

In contrast with the previous case, in this case we shall take advantage of $\widetilde{\mathcal{I}}_+(t)$. In fact, first of all, notice that, if $\sqrt{\beta}+2\alpha=0$, then there is nothing to prove. Thus, in the sequel we assume $\sqrt{\beta}+2\alpha\neq 0$. Indeed, define $\varepsilon_2\in\mathbb{R}$ as

$$\varepsilon_2 := \frac{1}{2}\left(1+\frac{\beta}{(\sqrt{\beta}+2\alpha)^2}\right)>0.$$

Then, by using Young inequality for products, with parameter given by ε_2, we infer

$$\begin{aligned}&\frac{1}{2\mu\lambda}\int\eta^2\Phi'+\frac{\beta}{2\mu\lambda}\int\rho^2\Phi'-\frac{\sqrt{\beta}+2\alpha}{\mu\lambda}\int\rho\eta\Phi'\\&\qquad\geqslant\frac{1}{2\mu\lambda}\big(1-\varepsilon_2^{-1}\big)\int\eta^2\Phi'+\frac{1}{2\mu\lambda}\big(\beta-(\sqrt{\beta}+2\alpha)^2\varepsilon_2\big)\int\rho^2\Phi'.\end{aligned}$$

Moreover, by using both $\beta-\alpha^2>0$ and $\alpha<0$, proceeding in exactly the same fashion as in the previous claim, we conclude that both factors in front of each of the integral on the right-hand side of the latter inequality are strictly positive. Thus, we conclude the proof of the claim.

Finally, it only remains to set the definition of ε_1^* and ε_2^* in (4.15). Notice that we have to give a different definition depending on the case $\alpha\gtrless 0$. In fact, from the above analysis it follows that it is enough to consider

$$\begin{aligned}\varepsilon_{1,+}^* &:= 10^{-10}\big(\beta-(\sqrt{\beta}-2\alpha)^2\varepsilon_1\big) &\qquad \varepsilon_{2,+}^* &:= 10^{-10}(1-\varepsilon_1^{-1}),\\ \varepsilon_{1,-}^* &:= 10^{-10}\big(\beta-(\sqrt{\beta}+2\alpha)^2\varepsilon_2\big) &\qquad \varepsilon_{2,-}^* &:= 10^{-10}(1-\varepsilon_2^{-1}),\end{aligned}$$

where $\varepsilon_{\mathrm{i},\pm}^*$ stands for the case where α is positive or negative respectively. Hence, we conclude the proof of the proposition. ■

Case $\alpha = 0$

In the case when $\alpha = 0$, we can give a much simpler and shorter proof. In fact, in this case, from Lemma 4.3 we can easily deduce the following result.

Proposition 4.9. *Let $(\psi, \rho, \eta) \in C(\mathbb{R}, H^1 \times L^2 \times L^2)$ be any solution to system* (4.1) *emanating from an initial data $(\psi_0, \rho_0, \eta_0) \in H^1 \times L^2 \times L^2$. For $\lambda_1(t)$, $\mu(t)$ and $\upsilon_\pm$ defined as in* (4.6)-(4.7)*, the following inequality holds*

$$\int_0^{+\infty} \frac{1}{\mu(t)\lambda_1(t)} \int_{\mathbb{R}} \Big(|\psi(t,x)|^4 + |\psi_x(t,x)|^2 + \eta^2(t,x) + \rho^2(t,x) \Big) \operatorname{sech}^2 \left(\frac{x}{\lambda_1(t)} \right) \mathrm{d}x\mathrm{d}t < +\infty.$$

Proof. In fact, by using the standard modified momentum function $\mathcal{I}(t)$ (instead of $\widetilde{\mathcal{I}}(t)$ as before), we proceed in the same fashion as in the previous proposition, using Lemma 4.3 and noticing that, under our current assumptions, $q = \gamma > 0$, from where we infer that

$$\frac{3\gamma q}{4\mu\lambda_1} \int |\psi|^4 \Phi' + \frac{1}{2\mu\lambda_1} \int \eta^2 \Phi' + \frac{\gamma}{\mu\lambda_1} \int \eta |\psi|^2 \Phi' > \frac{\gamma^2}{100\mu\lambda_1} \int |\psi|^4 \Phi' + \frac{1}{100\mu\lambda_1} \int \eta^2 \Phi'.$$

Then, the proof follows by gathering the latter inequality with (4.8) and recalling that $\lambda_1^{-3} \in L^1(\mathbb{R}_+)$. In order to avoid over-repeated computations we omit the details. ■

4.4 Decay in far field regions

In this section we seek to prove pointwise decay in far field regions by taking advantage of some suitable virial identities, as before. The analysis is similar (in spirit) to that shown in the previous section. However, in this case, the idea will be somewhat the opposite, in the sense that now the important terms shall come from the derivative of the weight Φ, instead of the derivative of the solution, as in the previous section. To do so, we consider both the modified mass functional as well as the modified energy functional, which are given by (respectively)

$$\mathcal{M}_\pm(t) := \int_{\mathbb{R}} \Phi \left(\frac{\pm x + \zeta(t)}{\lambda(t)} \right) |\psi(t,x)|^2 \mathrm{d}x,$$

$$\mathcal{E}_\pm(t) := \int_{\mathbb{R}} \Phi \left(\frac{\pm x + \zeta(t)}{\lambda(t)} \right) \Big(\omega |\psi_x|^2 + \tfrac{\gamma q}{2} |\psi|^4 + \tfrac{\beta}{2} \rho^2 + \tfrac{1}{2} \eta^2 + \tfrac{\gamma}{2} (2\eta - \alpha\rho) |\psi|^2 - \alpha\rho\eta \Big) \mathrm{d}x.$$

Here, Φ stands for a smooth and bounded weight (not necessarily decaying at $\pm\infty$), which shall be completely different to the one chosen in the previous section (see (4.26)-(4.27) for the exact definition). Notice also that, in contrast with the modified mean functional, now we only require Φ belonging to $L^\infty(\mathbb{R})$ in order for $\mathcal{M}_\pm$ and $\mathcal{E}_\pm$ to be well-defined and uniformly bounded (for solutions in the energy space). Additionally, in this case we define the scaling $\lambda(t)$ and the shift $\zeta(t)$ as

$$\lambda(t) := (1+t)^{2+\delta}, \quad \zeta(t) \geqslant c_1\lambda(t), \quad \zeta'(t) \geqslant c_2\lambda'(t), \quad c_1, c_2 > 0, \ \delta > 0. \tag{4.18}$$

In a similar spirit as in the previous section, the main motivation to consider these specific definitions of λ and ζ is to obtain

$$\frac{1}{\lambda}, \frac{1}{\zeta} \in L^1(\mathbb{R}_+), \quad \text{however} \quad \frac{\lambda'}{\lambda}, \frac{\zeta'}{\lambda} \notin L^1(\mathbb{R}_+), \tag{4.19}$$

which shall allow us to neglect some bad terms (in some sense). We emphasize that, in this case, we are considering a scaling factor $\lambda(t)$ growing faster than linear (in contrast with the previous sections). This changes the behavior of some important terms (with respect to the above analysis) that we intend to take advantage of.

On the other hand, to simplify computations, we split the modified energy functional $\mathcal{E}$ into the following functionals

$$\begin{aligned} \mathcal{E}_{\pm,1}(t) &:= \int_{\mathbb{R}} \Phi\left(\frac{\pm x + \zeta(t)}{\lambda(t)}\right)\Big(\omega|\psi_x|^2 + \tfrac{\gamma q}{2}|\psi|^4\Big)\mathrm{d}x, \\ \mathcal{E}_{\pm,2}(t) &:= \int_{\mathbb{R}} \Phi\left(\frac{\pm x + \zeta(t)}{\lambda(t)}\right)\Big(\tfrac{\beta}{2}\rho^2 + \tfrac{1}{2}\eta^2 + \tfrac{\gamma}{2}(2\eta - \alpha\rho)|\psi|^2 - \alpha\rho\eta\Big)\mathrm{d}x. \end{aligned} \tag{4.20}$$

Before going further, let us compute the virial identities associated with our current functionals, that shall give us the fundamental information for the following analysis.

Lemma 4.10. *Let $(\psi, \rho, \eta) \in C(\mathbb{R}, H^1 \times L^2 \times L^2)$ be any solution to system* (4.1)*. Then, for all $t \in \mathbb{R}$, the following identity holds*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_\pm(t) = &-\frac{\lambda'}{\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x + \zeta}{\lambda}\right)\left(\frac{\pm x + \zeta}{\lambda}\right)|\psi|^2\mathrm{d}x + \frac{\zeta'}{\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x + \zeta}{\lambda}\right)|\psi|^2\mathrm{d}x \\ &\pm \frac{2\omega}{\lambda}\mathrm{Im}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x + \zeta}{\lambda}\right)\overline{\psi}\psi_x\mathrm{d}x. \end{aligned} \tag{4.21}$$

Proof. The proof follows from direct computations. We omit this proof. ■

Lemma 4.11. *Let $(\psi, \rho, \eta) \in C(\mathbb{R}, H^2 \times H^1 \times H^1)$ be any solution to system* (4.1)*. Then,* *for all $t \in \mathbb{R}$, the following identity holds,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\pm(t) = &\frac{\zeta'}{\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\omega|\psi_x|^2+\tfrac{\gamma q}{2}|\psi|^4\right)\mathrm{d}x\\ &-\frac{\lambda'}{\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\omega|\psi_x|^2+\tfrac{\gamma q}{2}|\psi|^4\right)\mathrm{d}x\\ &+\frac{\zeta'}{\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2+\tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2-\alpha\rho\eta\right)\mathrm{d}x\\ &-\frac{\lambda'}{\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2+\tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2-\alpha\rho\eta\right)\mathrm{d}x\\ &\pm\frac{2\omega^2}{\lambda}\mathrm{Im}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\overline{\psi}_x\psi_{xx}\mathrm{d}x\pm\frac{2\gamma q\omega}{\lambda}\mathrm{Im}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)|\psi|^2\overline{\psi}\psi_x\mathrm{d}x\\ &\mp\frac{\alpha}{\theta\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\beta|\rho|^2+|\eta|^2\right)\mathrm{d}x\pm\frac{\beta+\alpha^2}{\theta\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\rho\eta\mathrm{d}x\\ &\pm\frac{\gamma\left(\beta+\alpha^2/2\right)}{\theta\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\rho|\psi|^2\mathrm{d}x\mp\frac{3}{2}\frac{\gamma\alpha}{\theta\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\eta|\psi|^2\mathrm{d}x\\ &\mp\frac{\gamma^2\alpha}{\theta\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)|\psi|^4\mathrm{d}x\pm\frac{\gamma\omega}{\lambda}\mathrm{Im}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)(2\eta-\alpha\rho)\,\overline{\psi}\psi_x\mathrm{d}x.\end{aligned}\tag{4.22}$$

Proof. First of all, in order to simplify the computations, we split the derivative of $\mathcal{E}_\pm$ into the sum of the derivatives of $\mathcal{E}_{\pm,\mathrm{i}}$, $\mathrm{i}=1,2$, treating separately each of these functionals and then summing-up the corresponding results. In fact, directly differentiating $\mathcal{E}_{\pm,1}$, using that (ψ,ρ,η) solves system (4.1) and then performing several integration by parts, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\pm,1}(t) = &\frac{\zeta'}{\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\omega|\psi_x|^2+\tfrac{\gamma q}{2}|\psi|^4\right)\mathrm{d}x\\ &-\frac{\lambda'}{\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\omega|\psi_x|^2+\tfrac{\gamma q}{2}|\psi|^4\right)\mathrm{d}x\\ &\pm\frac{2\omega^2}{\lambda}\mathrm{Im}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\overline{\psi}_x\psi_{xx}\mathrm{d}x\pm\frac{2\gamma q\omega}{\lambda}\mathrm{Im}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)|\psi|^2\overline{\psi}\psi_x\mathrm{d}x\\ &+2\omega\gamma\mathrm{Im}\int_{\mathbb{R}}\Phi\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\overline{\psi}_x\psi\left(\eta-\tfrac{1}{2}\alpha\rho+q|\psi|^2\right)_x\mathrm{d}x\\ &+2\gamma q\omega\mathrm{Im}\int_{\mathbb{R}}\Phi\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(|\psi|^2\right)_x\overline{\psi}\psi_x\mathrm{d}x\\ =:\,&R_{\pm,1}+R_{\pm,2}+R_{\pm,3}+R_{\pm,4}+R_{\pm,5}+R_{\pm,6}.\end{aligned}\tag{4.23}$$

We now proceed with $\mathcal{E}_{\pm,2}$. In fact, in a similar fashion as before, directly differentiating $\mathcal{E}_{\pm,2}$, using that (ψ,ρ,η) solves (4.1), and then performing several integration by parts, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\pm,2}(t) &= \frac{\zeta'}{\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2+\tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2-\alpha\rho\eta\Big)\mathrm{d}x \\ &\quad-\frac{\lambda'}{\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2+\tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2-\alpha\rho\eta\Big)\mathrm{d}x \\ &\quad\mp\frac{\alpha}{\theta\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\beta|\rho|^2+|\eta|^2\Big)\mathrm{d}x\pm\frac{\beta+\alpha^2}{\theta\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\rho\eta\mathrm{d}x \\ &\quad\pm\frac{\gamma\left(\beta+\alpha^2/2\right)}{\theta\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\rho|\psi|^2\mathrm{d}x\mp\frac{3}{2}\frac{\gamma\alpha}{\theta\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\eta|\psi|^2\mathrm{d}x \\ &\quad\mp\frac{\gamma^2\alpha}{2\theta\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)|\psi|^4\mathrm{d}x\pm\frac{\gamma\omega}{\lambda}\mathrm{Im}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)(2\eta-\alpha\rho)\,\overline{\psi}\psi_x\mathrm{d}x \\ &\quad+\gamma\omega\mathrm{Im}\int_{\mathbb{R}}\Phi\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)(2\eta-\alpha\rho)_x\,\overline{\psi}\psi_x\mathrm{d}x.\end{aligned}\tag{4.24}$$

Finally, for the sake of simplicity let us define $R_{\pm,7}$ as the following quantity

$$R_{\pm,7}:=\gamma\omega\mathrm{Im}\int_{\mathbb{R}}\Phi\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)(2\eta-\alpha\rho)_x\,\overline{\psi}\psi_x\mathrm{d}x.$$

Then, it is not difficult to see that with these definitions we have the relation $R_{\pm,5}=R_{\pm,6}+R_{\pm,7}$. Therefore, summing up all the previous computations, and then using the above relation, we conclude the proof of (4.22). ■

4.4.1 Time integrability of the weighted L^2-norm

In this subsection we restrict ourselves to the simpler case of the time integrability of the weighted L^2-norm for the Schrödinger part of the solution $\psi(t,x)$. The integrability (and decay) of this weighted-norm is a fundamental part of the analysis since (as we shall see) it triggers the decay of the whole weighted energy norm. In fact, let us start by recalling the relation

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_{\pm}(t)\pm\frac{2\omega}{\lambda}\mathrm{Im}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta}{\lambda}\right)\overline{\psi}\psi_x\mathrm{d}x \\ &\quad=\frac{\lambda'}{\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta}{\lambda}\right)\left(\frac{\pm x+\zeta}{\lambda}\right)|\psi|^2\mathrm{d}x-\frac{\zeta'}{\lambda}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta}{\lambda}\right)|\psi|^2\mathrm{d}x.\end{aligned}\tag{4.25}$$

Then, notice that the left-hand side of the above relation is time-integrable on $\mathbb{R}_+$, provided that $\Phi'\in L^\infty(\mathbb{R})$. In fact, if Φ' is bounded, then from Hölder inequality, Lemma 4.5 and the fact that (4.19) holds, we have

$$\left|\frac{2}{\lambda}\mathrm{Im}\int_{\mathbb{R}}\Phi'\left(\frac{\pm x+\zeta}{\lambda}\right)\overline{\psi}\psi_x\mathrm{d}x\right|\lesssim\frac{1}{\lambda}\in L^1(\mathbb{R}_+).$$

Motivated by the time-integrability above, as well as identity (4.25), we shall give a suitable definition for $\Phi\in C^\infty(\mathbb{R})$ so that we are able to obtain a convenient sign-property in the right-hand side of (4.25).

To take advantage of the structure of the virial identities, from now on we consider Φ to be any non-increasing smooth function such that it satisfies the following conditions

$$\big\{\Phi(s) = 1,\ s \leqslant -1\big\}, \quad \big\{\Phi(s) = 0,\ s \geqslant 0\big\} \quad \text{and} \quad \{\Phi' \equiv -1 \text{ on } [-\tfrac{9}{10}, -\tfrac{1}{10}]\}. \tag{4.26}$$

Notice that, as a particular consequence of its definition, we have the following inequalities

$$\forall s \in \mathbb{R}, \quad \Phi'(s) \leqslant 0 \quad \text{and} \quad s\Phi'(s) \geqslant 0. \tag{4.27}$$

As already mentioned, we now focus in studying the right-hand side of (4.25). Notice that, thanks to (4.26)-(4.27) we infer that, for all $t \geqslant 0$, the following sign-properties are satisfied

$$\frac{\lambda'}{\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x + \zeta}{\lambda}\right)\left(\frac{\pm x + \zeta}{\lambda}\right)|\psi|^2\mathrm{d}x \geqslant 0 \quad \text{and} \quad -\frac{\zeta'}{\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x + \zeta}{\lambda}\right)|\psi|^2\mathrm{d}x \geqslant 0.$$

Consequently, due to the fact that the left-hand side of (4.25) is integrable in time, we can compute the time-integral over $\mathbb{R}_+$ and get

$$\int_0^\infty \left(\frac{\lambda'}{\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x + \zeta}{\lambda}\right)\left(\frac{\pm x + \zeta}{\lambda}\right)|\psi|^2\mathrm{d}x - \frac{\zeta'}{\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x + \zeta}{\lambda}\right)|\psi|^2\mathrm{d}x\right)\mathrm{d}t < \infty. \tag{4.28}$$

Then, gathering this latter inequality with the sign-property above, we deduce in particular

$$\int_0^\infty \frac{\zeta'(t)}{\lambda(t)}\int_{\mathbb{R}} \left|\Phi'\left(\frac{\pm x + \zeta(t)}{\lambda(t)}\right)\right| |\psi(t,x)|^2\mathrm{d}x\mathrm{d}t < \infty.$$

Remark 4.9. Notice that, as a particular consequence of the latter inequality, recalling also that $\lambda^{-1}\zeta' \notin L^1(\mathbb{R}_+)$, we infer the existence of a sequence of times $\{t_n\}_{n\in\mathbb{R}}$, satisfying $t_n \to \infty$, such that

$$\lim_{n\to+\infty}\int_{\Omega(t_n)} |\psi(t_n,x)|^2\mathrm{d}x = 0, \tag{4.29}$$

where the set $\Omega(t)$ can be defined, for example, as

$$\Omega(t) := \{x \in \mathbb{R} : \tfrac{1}{10}\lambda(t) + \zeta(t) \leqslant |x| \leqslant \tfrac{9}{10}\lambda(t) + \zeta(t)\}. \tag{4.30}$$

Moreover, notice that from the above

$$\int_0^\infty \frac{\lambda'(t)}{\lambda(t)}\int_{\Omega(t)} |\psi(t,x)|^2\mathrm{d}x < +\infty \quad \text{and} \quad \int_0^\infty \frac{\zeta'(t)}{\lambda(t)}\int_{\Omega(t)} |\psi(t,x)|^2\mathrm{d}x < +\infty. \tag{4.31}$$

Hence, from now on, we can use properties (4.29) and (4.31) without depending on weights Φ satisfying (4.26). In particular, in the sequel we shall use (4.29) for compactly supported weight functions encoding the same (or strictly contained) regions as in (4.30).

4.4.2 Decay of the L^2-norm

In this section we seek to prove the pointwise decay of L^2-norm of the Schrödinger component of the solution $\psi(t,x)$ restricted to the far-field regions. In fact, let us start by considering $\Psi \in C_0^\infty(\mathbb{R})$ to be any non-negative function such that

$$\mathrm{supp}(\Psi) \subset \left[-\tfrac{3}{4}, -\tfrac{1}{4}\right] \quad \text{with} \quad \Psi \equiv 1 \text{ on } [-\tfrac{3}{5}, -\tfrac{2}{5}]. \tag{4.32}$$

Notice that, in particular, this implies that $\mathrm{supp}(\Psi) \subset \mathrm{supp}(\Phi')$. Additionally, we assume that Ψ satisfies the following pointwise properties

$$\forall s \in \mathbb{R}, \quad \Psi(s) \lesssim |\Phi'(s)| \quad \text{and} \quad |\Psi'(s)| \lesssim |\Phi'(s)|. \tag{4.33}$$

Now, we re-write the previous virial identity (4.21) in terms of our new weight function Ψ (instead of using Φ). Then, using Lemma 4.5 it is not difficult to see that

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \Psi\left(\frac{x+\zeta}{\lambda}\right)|\psi|^2\mathrm{d}x\right| \\ \lesssim \frac{1}{\lambda} + \frac{\lambda'}{\lambda}\int_{\mathbb{R}}\left|\Psi'\left(\frac{\pm x+\zeta}{\lambda}\right)\left(\frac{\pm x+\zeta}{\lambda}\right)\right||\psi|^2\mathrm{d}x + \frac{\zeta'}{\lambda}\int_{\mathbb{R}}\left|\Psi'\left(\frac{\pm x+\zeta}{\lambda}\right)\right||\psi|^2\mathrm{d}x.\end{aligned} \tag{4.34}$$

Now, recall that, as stated in remark 4.9, there exists a sequence of time $\{t_n\}_{n\in\mathbb{N}}$, satisfying $t_n \to \infty$, such that (4.29) holds. As a consequence, we also have that

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}} \Psi\left(\frac{x+\zeta}{\lambda}\right)|\psi|^2\mathrm{d}x\right)(t_n) = 0.$$

Therefore, we can integrate both sides of (4.34) in time over the interval $[t, t_n]$, and then take the limit $t_n \to \infty$, what lead us to

$$\begin{aligned}\int_{\mathbb{R}} \Psi\left(\frac{\pm x+\zeta}{\lambda}\right)|\psi|^2\mathrm{d}x \lesssim \int_t^\infty \frac{\mathrm{d}s}{\lambda(s)} + \int_t^\infty \frac{\zeta'}{\lambda}\int_{\mathbb{R}}\left|\Psi'\left(\frac{\pm x+\zeta}{\lambda}\right)\right||\psi|^2\mathrm{d}x\mathrm{d}s \\ + \int_t^\infty \frac{\lambda'}{\lambda}\int_{\mathbb{R}}\left|\Psi'\left(\frac{\pm x+\zeta}{\lambda}\right)\left(\frac{\pm x+\zeta}{\lambda}\right)\right||\psi|^2\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Finally, by using both time-integrabilities in (4.31), we can take now the limit $t \to \infty$, from where we conclude that

$$\lim_{t\to\infty}\int_{\mathbb{R}} \Psi\left(\frac{\pm x+\zeta}{\lambda}\right)|\psi|^2\mathrm{d}x = 0.$$

Remark 4.10. Notice that the proof of the integrability (and subsequent decay) of the L^2-norm of ψ also works for other definitions of λ as well as other definitions of Φ and Ψ. For example, in the above analysis we have only used the fact that the scaling $\lambda(t)$ satisfies

$$\frac{1}{\lambda} \in L^1(\mathbb{R}_+) \quad \text{and} \quad \frac{\lambda'}{\lambda} \notin L^1(\mathbb{R}_+).$$

In consequence, the proof still holds for any λ with such property. As a result, we can take for example $\lambda(t) = ct^p$, for any $p > 2$ and any $c \in \mathbb{R}_+$, and then following the above computations we obtain the desired result.

4.4.3 Time-integrability of the full solution

In this section we seek to show the time integrability of the local energy norm for the remaining terms. Now, in order to make computations simpler, let us break down expression (4.22), so that we can write the cleaner formula

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\pm(t) + \mathcal{R}_\pm(t) = -\mathfrak{E}_\pm(t). \tag{4.35}$$

More specifically, we define the functionals $\mathfrak{E}_\pm$ and $\mathcal{R}_\pm$ given by

$$\begin{aligned}
\mathfrak{E}_\pm :=& \frac{\zeta'}{\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\omega|\psi_x|^2+\tfrac{\gamma q}{2}|\psi|^4\right)\mathrm{d}x\\
&-\frac{\lambda'}{\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\omega|\psi_x|^2+\tfrac{\gamma q}{2}|\psi|^4\right)\mathrm{d}x\\
&+\frac{\zeta'}{\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2+\tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2-\alpha\rho\eta\right)\mathrm{d}x\\
&-\frac{\lambda'}{\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2+\tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2-\alpha\rho\eta\right)\mathrm{d}x,\\
\mathcal{R}_\pm(t) :=& \pm\frac{2\omega^2}{\lambda}\mathrm{Im}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\overline{\psi}_x\psi_{xx}\mathrm{d}x \pm \frac{2\gamma q\omega}{\lambda}\mathrm{Im}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)|\psi|^2\overline{\psi}\psi_x\mathrm{d}x\\
&\mp\frac{\alpha}{\theta\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\beta|\rho|^2+|\eta|^2\right)\mathrm{d}x \pm \frac{\beta+\alpha^2}{\theta\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\rho\eta\mathrm{d}x\\
&\pm\frac{\gamma\left(\beta+\alpha^2/2\right)}{\theta\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\rho|\psi|^2\mathrm{d}x \mp\frac{3}{2}\frac{\gamma\alpha}{\theta\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\eta|\psi|^2\mathrm{d}x\\
&\mp\frac{\gamma^2\alpha}{\theta\lambda}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)|\psi|^4\mathrm{d}x \pm\frac{\gamma\omega}{\lambda}\mathrm{Im}\int_{\mathbb{R}} \Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)(2\eta-\alpha\rho)\,\overline{\psi}\psi_x\mathrm{d}x\\
=:& \mathcal{R}_{\pm,1}+\mathcal{R}_{\pm,2}+\cdots+\mathcal{R}_{\pm,8}.
\end{aligned}$$

On the other hand, notice that, since (ψ,ρ,η) is a solution to system (4.1) belonging to the class $C(\mathbb{R},H^2\times H^1\times H^1)$, then the energy associated to this solution $E\left(\psi(t),\rho(t),\eta(t)\right)$ is finite. This means that, because the weight Φ considered in the modified functional $\mathcal{E}$ is bounded, one has

$$\int_{\mathbb{R}_+}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\pm\mathrm{d}t<\infty.$$

Now, we treat the remaing term.

Let us consider $\Phi\in C^\infty(\mathbb{R})$ to be any non-increasing function such that (4.26) holds, and hence, satisfying also (4.27). Now, we intend to bound term by term the right-hand side of (4.22). In fact, first, since Φ' is bounded, using Young inequality and Sobolev embbeding along with Lemma 4.5, we see that

$$|\mathcal{R}_{\pm,2}|\leqslant\frac{2\gamma\omega|q|}{\lambda(t)}\|\psi\|_{L^\infty(\mathbb{R})}^2\left(\|\psi\|_{L^2(\mathbb{R})}^2+\|\psi_x\|_{L^2(\mathbb{R})}^2\right)\lesssim\frac{1}{\lambda(t)}\in L^1(\mathbb{R}_+)$$

Also, immediately from Proposition (4.5), we obtain

$$|\mathcal{R}_{\pm,3}|\leqslant\frac{|\alpha|}{\theta\lambda(t)}\int_{\mathbb{R}}\left|\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\right|\left(\beta|\rho|^2+|\eta|^2\right)\mathrm{d}x\lesssim\frac{1}{\lambda(t)}\in L^1(\mathbb{R}_+).$$

On the other hand, for $\mathcal{R}_{+,4}$, by using Young inequality for products and Lemma 4.5 we get

$$|\mathcal{R}_{\pm,4}|\leqslant\frac{\beta+\alpha^2}{2\theta}\frac{1}{\lambda(t)}\left(\|\rho(t)\|_{L^2(\mathbb{R})}^2+\|\eta(t)\|_{L^2(\mathbb{R})}^2\right)\lesssim\frac{1}{\lambda(t)}\in L^1(\mathbb{R}_+).$$

We point out that all the remaining terms $\mathcal{R}_{\pm,\mathrm{i}}$, $\mathrm{i}=5,...,8$, can be treated in the very same fashion as for the previous terms. In fact, from Hölder inequality, Sobolev embedding and then using Lemma 4.5 in the resulting right-hand side, we deduce that

$$\sum_{\mathrm{i}=5}^{8}|\mathcal{R}_{\pm,\mathrm{i}}|\lesssim\frac{1}{\lambda(t)}\Big(\|\eta(t)\|_{L^2(\mathbb{R})}^2+\|\rho(t)\|_{L^2(\mathbb{R})}^2+\|\psi(t)\|_{H^1(\mathbb{R})}^4\Big)\lesssim\frac{1}{\lambda(t)}\in L^1(\mathbb{R}_+).$$

Finally, to complete the analysis regarding the time-integrability of $\mathcal{R}_\pm$, it only remains to consider $\mathcal{R}_{\pm,1}$. In order to do that, we first need to recall one of the main results proven in [5] which give us the polynomial growth of the H^2-norm of $\psi(t)$. In fact, from [5, Proposition 1.1] we have that

$$\|\psi(t)\|_{H^2(\mathbb{R})}\lesssim 1+|t|^{1^+}. \tag{4.36}$$

As a consequence, recalling the explicit form of $\lambda(t)$ in (4.18), we conclude that

$$\frac{1}{\lambda(t)}\|\psi(t)\|_{H^2(\mathbb{R})}\in L^1(\mathbb{R}_+)$$

Therefore, by using Hölder inequality as well as Lemma 4.5 and (4.36), we infer that

$$|\mathcal{R}_{\pm,1}|\lesssim\frac{1}{\lambda(t)}\|\psi_x(t)\|_{L^2(\mathbb{R})}\|\psi_{xx}(t)\|_{L^2(\mathbb{R})}\in L^1(\mathbb{R}_+).$$

In conclusion, we can integrate over $\mathbb{R}_+$ in time both sides of (4.35), from where we obtain

$$\int_{\mathbb{R}_+}-\mathfrak{E}_\pm(t)\mathrm{d}t<\infty. \tag{4.37}$$

Now, let us break down this expression so that we can analyze the conflicting terms of $\mathfrak{E}_\pm(t)$ without sign. More specifically, we would like to absorbs or discard the part of the expression that does not constitute the weighted energy norm. In fact, in similar fashion as in the proof of Proposition 4.5, we have that

$$\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2-\alpha\rho\eta\geqslant\frac{\beta-\alpha^2}{4}|\rho|^2+\frac{\beta}{2(\beta+\alpha^2)}|\eta|^2>0,$$

where we used the fact that $\beta-\alpha^2>0$ and $\beta>0$. Also, we have that

$$\begin{aligned}\frac{\gamma}{2}\big(2\eta-\alpha\rho\big)|\psi|^2\leqslant{}&\frac{\beta-\alpha^2}{16}|\rho|^2+\frac{\beta}{16(\beta+\alpha^2)}|\eta|^2\\&+\left(\frac{\gamma^2(\beta+\alpha^2)}{8\beta}+\frac{2\alpha^2\gamma^2}{\beta-\alpha^2}\right)|\psi|^4.\end{aligned}$$

Gathering both equations above, we get

$$\begin{aligned}\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2-\alpha\rho\eta+\frac{\gamma}{2}\big(2\eta-\alpha\rho\big)|\psi|^2\geqslant{}&\frac{3(\beta-\alpha^2)}{16}|\rho|^2+\frac{7\beta}{16(\beta+\alpha^2)}|\eta|^2\\&-\left(\frac{\gamma^2(\beta+\alpha^2)}{8\beta}+\frac{2\alpha^2\gamma^2}{\beta-\alpha^2}\right)|\psi|^4.\end{aligned} \tag{4.38}$$

Finally, to deal with the remaining uncontrolled terms (involving the L^4-norm of ψ), we make use of (4.28). Indeed, notice that by Sobolev embedding,

$$\begin{aligned}
&\int_{\mathbb{R}}\left[\frac{\lambda'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)-\frac{\zeta'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\right]|\psi|^4\mathrm{d}x\\
&\leqslant \|\psi\|_{H^1(\mathbb{R})}^2\int_{\mathbb{R}}\left[\frac{\lambda'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)-\frac{\zeta'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\right]|\psi|^2\mathrm{d}x.
\end{aligned}$$

Then, thanks to (4.28), we can conlude that

$$\int_{\mathbb{R}}\left[\frac{\lambda'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)-\frac{\zeta'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\right]|\psi|^4\mathrm{d}x$$

is integrable in time over $\mathbb{R}_+$. Therefore, one can write the following

$$\begin{aligned}
&\int_{\mathbb{R}}\left[\frac{\lambda'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)-\frac{\zeta'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\right]\omega|\psi_x|^2\mathrm{d}x\\
&+\int_{\mathbb{R}}\left[\frac{\lambda'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)-\frac{\zeta'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\right]\tfrac{3(\beta-\alpha^2)}{16}|\rho|^2\mathrm{d}x\\
&+\int_{\mathbb{R}}\left[\frac{\lambda'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)-\frac{\zeta'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\right]\tfrac{7\beta}{16(\beta-\alpha^2)}|\eta|^2\mathrm{d}x\\
&\leqslant -\mathfrak{E}_{\pm}+K\int_{\mathbb{R}}\left[\frac{\lambda'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)-\frac{\zeta'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\right]|\psi|^4\mathrm{d}x,
\end{aligned}\tag{4.39}$$

where K is the absolute value of a constant depending on β,α,γ,q. This way, we conclude that the right-hand side of the inequality above can be integrated in time over $\mathbb{R}_+$. Moreover, since

$$\Phi'(s)s\geqslant 0\quad\text{and}\quad\Phi'(s)\leqslant 0\quad\forall s\in\mathbb{R},$$

we also infer that

$$\int_{\mathbb{R}_+}\frac{\zeta'}{\lambda}\int_{\mathbb{R}}\left|\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\right|\left(|\psi_x|^2+|\rho|^2+|\eta|^2\right)\mathrm{d}x\mathrm{d}t<\infty.$$

Remark 4.11. We notice that, thanks to the fact that $\frac{\zeta'}{\lambda}\notin L^1(\mathbb{R}_+)$, one infers that there exists a sequence $\{t_n\}$, with $\{t_n\}\to\infty$, such that

$$\left(\int_{\mathbb{R}}\left|\Phi'\left(\frac{\pm x+\zeta(t_n)}{\lambda(t_n)}\right)\right|\left(|\psi_x|^2+|\rho|^2+|\eta|^2\right)\mathrm{d}x\right)(t_n)\to 0,\ \text{as } t_n\to\infty.\tag{4.40}$$

4.4.4 Decay of the full solution

Finally, in this subsection we devote ourselves to prove decay of solutions in the energy space along the curves $\pm\zeta$. The idea is the same as for the decay of the L^2-norm in Subsection 4.4.2. We proceed by taking a convenient weight Ψ such that (4.32) holds. Then, we have that $\mathrm{supp}(\Psi)\subset\mathrm{supp}(\Phi')$ and (4.33) is satisfied. Next, we consider the virial identity 4.11

with the weight Ψ instead of Φ. Thus, taking into account the previous estimations stated in Subsection 4.4.3 along with the pointwise properties (4.32)-(4.33), we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\Psi\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\omega|\psi_x|^2+\tfrac{\gamma q}{2}|\psi|^4+\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2+\tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2-\alpha\rho\eta\Big)\mathrm{d}x\\
&\quad\leqslant\frac{\zeta'}{\lambda}\int_{\mathbb{R}}\Psi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\omega|\psi_x|^2+\tfrac{\gamma q}{2}|\psi|^4\Big)\mathrm{d}x\\
&\qquad-\frac{\lambda'}{\lambda}\int_{\mathbb{R}}\Psi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\omega|\psi_x|^2+\tfrac{\gamma q}{2}|\psi|^4\Big)\mathrm{d}x\mathrm{d}t\\
&\qquad+\frac{\zeta'}{\lambda}\int_{\mathbb{R}}\Psi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2+\tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2-\alpha\rho\eta\Big)\mathrm{d}x\\
&\qquad-\frac{\lambda'}{\lambda}\int_{\mathbb{R}}\Psi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2+\tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2-\alpha\rho\eta\Big)\mathrm{d}x\\
&\qquad+\frac{C}{\lambda(t)}\left(1+\|\psi(t)\|_{H^2(\mathbb{R})}\right),
\end{aligned}$$

where we recall that λ^{-1} and $\lambda^{-1}\|\psi\|_{H^2}$ are both time-integrable in $\mathbb{R}_+$. Moreover, the whole right-hand side of the last inequality is integrable. Indeed, because Ψ satisfies (4.32)-(4.33) then (4.39) along with Young inequality implies that

$$\begin{aligned}
&\frac{\zeta'}{\lambda}\int_{\mathbb{R}}\Psi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\omega|\psi_x|^2+\tfrac{\gamma q}{2}|\psi|^4\Big)\mathrm{d}x\\
&-\frac{\lambda'}{\lambda}\int_{\mathbb{R}}\Psi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\omega|\psi_x|^2+\tfrac{\gamma q}{2}|\psi|^4\Big)\mathrm{d}x\mathrm{d}t\\
&+\frac{\zeta'}{\lambda}\int_{\mathbb{R}}\Psi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2+\tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2-\alpha\rho\eta\Big)\mathrm{d}x\\
&-\frac{\lambda'}{\lambda}\int_{\mathbb{R}}\Psi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2+\tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2-\alpha\rho\eta\Big)\mathrm{d}x\\
&\leqslant C\left(-\mathfrak{E}_{\pm}+K\int_{\mathbb{R}}\left[\frac{\lambda'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)-\frac{\zeta'}{\lambda}\Phi'\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\right]|\psi|^4\mathrm{d}x\right).
\end{aligned}$$

Now, recall that we have already shown in the previous section that the right-hand side of the above inequality is time-integrable in $\mathbb{R}_+$. Therefore, we conclude that there exists a time-integrable function $g:\mathbb{R}\to\mathbb{R}$ such that we can write

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\Psi\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\omega|\psi_x|^2+\tfrac{\gamma q}{2}|\psi|^4+\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2+\tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2-\alpha\rho\eta\Big)\mathrm{d}x\right|\leqslant g(t).$$

Consequently, we are entitled to integrate over the time interval $[t,t_n]$ and, because of (4.40), taking $t_n\to\infty$, we get

$$\int_{\mathbb{R}}\Psi\left(\frac{\pm x+\zeta}{\lambda}\right)\Big(\omega|\psi_x|^2+\tfrac{\gamma q}{2}|\psi|^4+\tfrac{\beta}{2}\rho^2+\tfrac{1}{2}\eta^2+\tfrac{\gamma}{2}(2\eta-\alpha\rho)|\psi|^2-\alpha\rho\eta\Big)\mathrm{d}x\leqslant\int_t^{\infty}g(\tau)\mathrm{d}t.$$

Now, notice that, by using inequality (4.38) we can re-write the expression above in terms of the energy norm, as

$$\begin{aligned}\int_{\mathbb{R}} \Psi\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(\omega|\psi_x|^2+\tfrac{\gamma(\beta-\alpha^2)}{16}\rho^2+\tfrac{7\beta}{16(\beta+\alpha^2)}\eta^2\Big)(t,x)\mathrm{d}x \\ \lesssim \int_t^{\infty} g(\tau)\mathrm{d}t+\int_{\mathbb{R}} \Psi\left(\frac{\pm x+\zeta(t)}{\lambda}\right)|\psi(t)|^4\mathrm{d}x.\end{aligned}$$

Finally, notice that the latter integral involving $|\psi(t,x)|^4$ converges to zero as $t\to\infty$. In fact, this is a consequence of the decay of the L^2-norm (4.5) and Lemma (4.5), along with the Gagliardo-Nirenberg inequality, that allows us to bound the L^4-norm with the H^1-norm and L^2-norm. Then, to conclude, we take $t\to\infty$ in the latter inequality above, from where we obtain the decay

$$\lim_{t\to\infty}\int_{\mathbb{R}} \Psi\left(\frac{\pm x+\zeta(t)}{\lambda(t)}\right)\Big(|\psi_x(t,x)|^2+\rho^2(t,x)+\eta^2(t,x)\Big)(t,x)\mathrm{d}x=0.$$

The proof is complete.

Bibliography

[1] D. J. Benney, G. J. Roskes, *Wave instability*, Studies in Applied Math., 48 (1969), pp 455-472.

[2] S. Champeaux, D. Laveder, T. Passot, E L. Sulem, *Remarks on the parallel propagation of small-amplitude dispersive Alfvén waves*, Nonlinear Processes in Geophysics 6 (1999), 169

[3] J. C. Cordero, *Supersonic limit for the Zakharov-Rubenchik system*, J. Differential Equations 261 (2016), no. 9, 5260-5288.

[4] C. Kwak, C. Muñoz, *Asymptotic dynamics for the small data weakly dispersive one-dimensional Hamiltonian ABCD system*, Trans. Amer. Math. Soc. 373 (2020), no. 2, 1043-1107.

[5] F. Linares, C. Matheus, *Well posedness for the 1D Zakharov-Rubenchik system*, Adv. Differential Equations 14 (2009), no. 3-4, 261-288.

[6] F. Linares, A. Mendez, *On long time behavior of solutions of the Schrödinger-Korteweg-de Vries system*, preprint arXiv:2010.14614

[7] H. Luong, N. Mauser, J-C Saut, *On the Cauchy problem for the Zakharov-Rubenchik/Benney-Roskes system*, Commun. Pure Appl. Anal. 17 (2018), no. 4, 1573-1594.

[8] Y. Martel and F. Merle, A Liouville theorem for the critical generalized Korteweg-de Vries equation, J. Math. Pures Appl. (9) 79 (2000), no. 4, 339-425.

[9] Y. Martel and F. Merle, Asymptotic stability of solitons for subcritical generalized KdV equations, Arch. Ration. Mech. Anal. 157 (2001), no. 3, 219-254.

[10] Y. Martel and F. Merle, Asymptotic stability of solitons for subcritical gKdV equations revisited. Nonlinearity, 18 (2005), no. 1, 55-80.

[11] M. E. Martínez, *Decay of small odd solutions for long range Schrödinger and Hartree equations in one dimension*, Nonlinearity 33 (2020), no. 3, 1156-1182.

[12] M. E. Martínez, *On the decay problem for the Zakharov and Klein-Gordon Zakharov systems in one dimension*, preprint arXiv:2004.01070

[13] F. Merle and P. Raphaël, The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation, Ann. of Math. (2) 161 (2005), no. 1, 157-222.

[14] C. Muñoz, G. Ponce, J.-C. Saut, *On the long time behavoir of solutions to the Intermediate Long Wave equation*, preprint arXiv:1910.03897.

[15] F. Oliveira, *Stability of the solitons for the one-dimensional Zakharov-Rubenchik equation*, Phys. D 175 (2003), no. 3-4, 220-240.

[16] F. Oliveira, *Adiabatic limit of the Zakharov-Rubenchik equation*, Rep. Math. Phys. 61 (2008), no. 1, 13-27.

[17] T. Passot, P.-L. Sulem, C. Sulem, *Generalization of acoustic fronts by focusing wave-packets*, Physica D 94 (1996), 168-187.

[18] G. Ponce, J-C Saut, *Well-posedness for the Benney-Roskes/Zakharov-Rubenchik system*, Discrete Contin. Dyn. Syst. 13 (2005), no. 3, 811-825.

[19] V. E. Zakharov and A. M. Rubenchik, *Nonlinear interaction of high-frequency and low-frequency waves*, Prikl. Mat. Techn. Phys., 5 (1972), pp 84-98.

Part IV

The Zakharov Water Waves problem under variable bottom

Chapter 5

Existence of solitary waves in the Water Waves Zakharov system with slowly varying bottom

Abstract. We deal with the solitary wave problem for the Zakharov water waves system with surface tension and a non-flat bottom in one dimension. Amick-Kirchgässner [3] proved the existence of small solitary waves in the case of a finite flat bottom. However, in practical situations, the bottom is always non-constant. In this work, we consider a domain with a slightly varying (in space) bottom and prove the existence of a generalized solitary wave like solution. The techniques used in the proof of the main result are based on the construction of a multi-soliton like solution, introduced in [14].

This chapter is part of the work M. E. Martínez, *Existence of solitary waves in the Water Waves Zakharov system with slowly varying bottom*, preprint 2021.

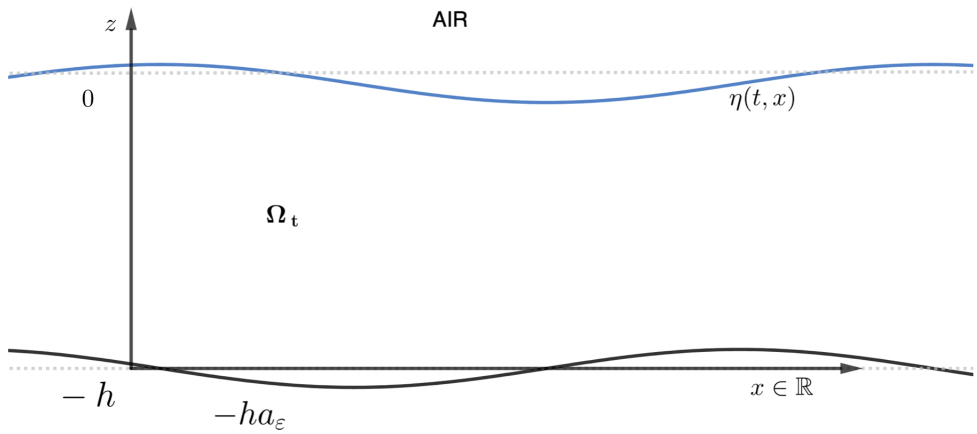
Contents

5.1 Introduction

Consider a fluid under the influence of gravity and with constant density, contained in a domain with rigid bottom and free surface:

$$\Omega_t = \left\{(x,z) \in \mathbb{R}^2 \text{ such that } -a(\varepsilon x)h \leqslant z \leqslant \eta(t,x)\right\},$$

where $h > 0$ is a fixed height, $\varepsilon > 0$ is a small parameter, $a : \mathbb{R} \to \mathbb{R}$ is a horizontal description of the bottom, and $\eta : \mathbb{R}^2 \to \mathbb{R}$ is the (unknown) free surface elevation. Usually, we will denote $a_\varepsilon(x) = a(\varepsilon x)$.



On the fluid, we assume that it is homogeneous, inviscid, incompressible and irrotational, which implies that its motion follows the classical inviscid irrotational constant density Euler equations. In particular, denoting by $\mathbf{u} \in \mathbb{R}^2$ the velocity of the fluid, the fact that the flow is irrotational implies the existence of a velocity potential Φ, a scalar mapping, that inside the fluid domain Ω_t satisfies,

$$\mathbf{u}(t,x,z) = (\partial_x \Phi(t,x,z), \partial_z \Phi(t,x,z)) = \nabla_{x,z}\Phi(t,x,z).$$

Since the motion of the fluid follows the *free surface Euler equations*, one finds that the velocity potential Φ satisfies the *free surface Bernoulli* formulation.

We assume that no particle of the fluid can cross the bottom or the surface, which leads to boundary conditions on the bottom

$$\partial_z \Phi(t,x,-a(\varepsilon x)h) = 0, \tag{5.1}$$

and (the kinematic condition) on the free surface

$$\partial_t \eta(t,x) + \partial_x \Phi(t,x,\eta(t,x)) \cdot \partial_x \eta(t,x) - \partial_z \Phi(t,x,\eta(t,x)) = 0. \tag{5.2}$$

The Zakharov water waves system arises from considering the free surface elevation η and the trace of the potential velocity on the surface $\Phi|_{z=\eta}$ fully determine the flow. Consequently, we are interested in the action of the flow on the free surface. In this particular case, we are dealing with the problem when the surface tension is present. Then, since the velocity potential Φ follows Bernoulli laws, taking into account the surface tension to eliminate the pressure term, we get the following equation on the surface of the fluid

$$\partial_t \Phi(t,x,\eta(t,x)) + \frac{1}{2}|\nabla_{x,z}\Phi(t,x,\eta(t,x))|^2 + g\eta(t,x) = b\partial_x\left(\frac{\partial_x \eta(t,x)}{\sqrt{1+|\partial_x\eta(t,x)|^2}}\right), \tag{5.3}$$

where b is the surface tension coefficient and g is the gravitational constant.

Finally, we also need to impose the water depth to be always bounded from below by a nonnegative constant. That is, there exist $h_{\min} > 0$ such that

$$a(\varepsilon x)h + \eta(t,x) \geqslant h_{\min}, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{5.4}$$

As it is customary to do, we shall write (5.1)-(5.2)-(5.3) as a system involving unknowns defined on the free surface only, $\eta(t,x)$ and $\varphi(t,x) = \Phi(t,x,\eta(x))$. With this in mind, we shall consider the Dirichlet-Neumann operator, first introduced in the bibliography by Craig-Sulem-Sulem [5, 6], defined as

$$\mathcal{G}[\eta,a] : \varphi \mapsto \sqrt{1+|\nabla\eta|^2}\ \partial_{\mathbf{n}}\Phi\big|_{z=\eta},$$

where $\mathbf{n}$ is the unit upward normal vector on the boundary of the fluid domain at the point $z=\eta(x)$ and Φ is the solution of the elliptic equation with moving domain

$$\begin{cases} \Delta_{x,z}\Phi = 0, \quad (x,z)\in\Omega_t, \\ \Phi\big|_{z=\eta(t,x)} = \varphi, \\ \partial_{\mathbf{n}}\Phi\big|_{z=-a(\varepsilon x)h} = 0. \end{cases} \tag{5.5}$$

Note the influence of the bottom in this last equation. As mentioned before, it was stated by Zakharov in [21] that, if one defines the trace of the velocity potential $\varphi = \Phi|_{z=\eta}$, then η and φ fully determine the flow. This ultimately allows us to write the system only in terms of the unknowns $(\eta(t,x),\varphi(t,x)) := (\eta(t,x),\Phi(t,x,\eta(t,x)))$. In consequence, the one-dimensional water waves problem reads

$$\begin{cases} \partial_t\eta = \mathcal{G}[\eta,a]\varphi \\ \partial_t\varphi = -\frac{1}{2}\left|\partial_x\varphi\right|^2 + \frac{1}{2}\dfrac{\left(\mathcal{G}[\eta,a]\varphi + \partial_x\varphi\partial_x\eta\right)^2}{1+\left|\partial_x\eta\right|^2} - g\eta + b\partial_x\left(\dfrac{\partial_x\eta}{\sqrt{1+\left|\partial_x\eta\right|^2}}\right) \end{cases} \tag{5.6}$$

where g is the gravitational constant, b is the tension surface coefficient and the velocity potential Φ is recovered by solving the elliptic problem (5.5). Denoting $\mathbf{U} = (\eta,\varphi)^\intercal$, we can write (5.6) as

$$\partial_t\mathbf{U} = \mathcal{F}(\mathbf{U}) \tag{5.7}$$

where the functional $\mathcal{F}$ is defined as

$$\mathcal{F}(\mathbf{U}) = \begin{pmatrix} \mathcal{G}[\eta,a]\varphi \\ -\frac{1}{2}\left|\partial_x\varphi\right|^2 + \frac{1}{2}\dfrac{\left(\mathcal{G}[\eta,a]\varphi + \partial_x\varphi\partial_x\eta\right)^2}{1+\left|\partial_x\eta\right|^2} - g\eta + b\partial_x\left(\dfrac{\partial_x\eta}{\sqrt{1+\left|\partial_x\eta\right|^2}}\right) \end{pmatrix}. \tag{5.8}$$

Moreover, as stated by Zakharov in [21], this system has a Hamiltonian structure in the variable $\mathbf{U}$. Indeed, let us define the Hamiltonian $\mathcal{H}$ as the total energy given by

$$\mathcal{H}(\eta,\varphi)=\frac{1}{2}\int_{\mathbb{R}}\left(\varphi\mathcal{G}[\eta,a]\varphi+g\eta^2+2b\left(\sqrt{1+|\partial_x\eta|^2}-1\right)\right)\mathrm{d}x, \tag{5.9}$$

Then, if I denotes, as usual, the identity matrix, it is possible to write (5.6) as

$$\partial_t\begin{pmatrix}\eta\\ \varphi\end{pmatrix}=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}\partial_\eta\mathcal{H}\\ \partial_\varphi\mathcal{H}\end{pmatrix}.$$

The Zakharov water waves problem, also commonly refered to as the Zakharov-Craig-Sulem formulation, is an important model in the theory of water waves equations and has gained an increasing interest over recent years. The assumptions in which the mathematical formulation is based on makes it suitable for applications, which might motivate its popoularity. Condition (5.4), along with the fact that the fluid is at rest at infinity, imply that system (5.6) describe the nonlinear dynamics of deep water gravity waves, avoiding the coast. Sometimes, surface tension is assumed to be zero, a reasonable approach, essentially since the surface tension in coastal oceanography is so small that it can be neglected. Nevertheless, the interest in considering $b\neq 0$ is the possibility to study capillary gravity waves, see [12] for some applications.

In addition, from the Zakharov-Craig-Sulem formulation can be deduced a great deal of canonical, simpler models. Indeed, among some of its assymptotics formulations are, for instance, Korteweg- de Vries and the family of Boussinesq equations for shallow water, or Benney-Roskes and Davey-Stewartson systems as deep-water, full-dispersion models (see [11]).

Regarding the well-posedness for the Zakharov water waves problem, when the surface tension can be neglected, global well-posedness is shown by Wu [19, 20] and Alazard-Burq-Zuily [2]. In [17, 4], Schneider-Wayne and Craig presented an early approach to the local well-posedness by relying on the fact that the KdV equation can be formally derived from the Zakharov water-waves problem in the limit of long waves, obtaining classical solutions up to a finite time. In the presence of surface tension, the 3-dimensional Zakharov water-waves problem is globally well-posed for small initial data (Germain-Masmoudi-Shatah [8, 9]). In [1], Alazard-Burq-Zuily described the Cauchy problem for the 2-dimensional case with surface tension in the space $H^{s+1/2}\times H^s$, $s>5/2$.

The study of solitary waves for equation (5.6) was mainly devoted to the flat-bottom case ($a\equiv 1$). Indeed, existence of solitary waves of the form $\mathbf{Q}_c(x-ct)=(\eta_c(x-ct),\varphi_c(x-ct))$ of speed $c\sim\sqrt{gh}$ was shown for suitable values of the parameters g,b and H [3]. The statement reads as follows

Theorem 5.1 (Amick-Kirchgässner [3]). *Suppose that g,b,h satisfy*

$$\frac{gh}{c^2}=1+\lambda^2,\quad \frac{b}{hc^2}>\frac{1}{3}. \tag{5.10}$$

Then, there exists ϵ_0 such that for every $\epsilon \in (0, \epsilon_0)$, there exists a solution of (5.6) *with* $a = 1$ *under the form*

$$\begin{aligned} \boldsymbol{Q}_c(x-ct) &= (\eta_c(x-ct), \varphi_c(x-ct)) \\ &= \left(h\eta_\lambda(h^{-1}(x-ct)), ch\varphi_\lambda(h^{-1}(x-ct))\right), \end{aligned}$$

with

$$\eta_\lambda(x) = \lambda^2\Theta_1(\lambda x, \lambda) \quad \varphi_\lambda(x) = \lambda\Theta_2(\lambda x, \lambda)$$

where Θ_1 *and* Θ_2 *satisfy*

$$\exists \mathrm{d} > 0, \quad \forall \alpha \geqslant 0, \quad \exists C_\alpha > 0, \quad \forall (x, \lambda) \in \mathbb{R} \times (0, \lambda_0), \quad |(\partial_x^\alpha \Theta_1(x, \lambda))| \leqslant C_\alpha \mathrm{e}^{-\mathrm{d}|x|}$$

and

$$\exists \mathrm{d} > 0, \quad \forall \alpha \geqslant 1, \quad \exists C_\alpha > 0, \quad \forall (x, \lambda) \in \mathbb{R} \times (0, \lambda_0), \quad |(\partial_x^\alpha \Theta_2(x, \lambda))| \leqslant C_\alpha \mathrm{e}^{-\mathrm{d}|x|}.$$

Moreover, Θ_1 *is even and* Θ_2 *is odd.*

Among solitary waves for the one dimensional (surface) case, there also exist multi-solitons solutions, as proven by Ming, Rousset and Tzvetkov [14]. More precisely, they were able to construct solutions that are time asymptotic to a sum of decoupling solitary waves, with different speed, assuming they are never near each other. We point out that such existence results for solitary waves and multi-solitons solutions for the problem with surface tension are given under the assumption that the bottom of the domain is *flat*. It is our goal to address the existence of soliton-like solutions for a *non-flat* bottom problem.

A rather characteristic property of the one-dimensional Zakharov water waves model is the fact that the surface of the fluid is invariant by translation. As a result, usual Lyapounov stability cannot be expected. Nevertheless, orbital stability of the solitary waves holds, as was proven by Mielke [13]. This is not the case for the problem in two dimensions. Indeed, Rousset and Tzvetkov showed in [16] (and improved in [14]) that solitary waves are not stable under 2-dimensional (transverse) perturbations.

As noted in [3, 16], the profiles $\Theta_1(x, \varepsilon)$ and $\Theta_2(x, \varepsilon)$ have smooth expansions in ε. Then, we are entitled to study the particular case $\varepsilon = 0$, for which we get

$$\Theta_1(x, 0) = \cosh^{-2}\left(\frac{x}{2(\beta/(Hc^2) - 1/3)^{1/2}}\right).$$

Thus, the KdV solitary wave is recovered.

5.1.1 Setting and main result

In this work, we are devoted to the study of the solitary wave, given by Amick-Kirchgassner (AK) for the flat bottom problem, when interacting with a bottom that changes (slightly) from a certain point in space. Since the bottom of the domain has a non-local influence on the solution $\mathbf{U}$ of system (5.6), one cannot assume that the AK solitary wave exists for the non-flat bottom problem. It is necessary, then, to prove the existence of a solution to (5.6) that behaves asymptotically like $\mathbf{Q}_c$ before it encounters the changing point in the bottom (as $t \to -\infty$) (see Figure 5.1). A second part of this study would be to explore how the description of the bottom impacts such solution as it travels towards the point of changing for the bottom and enters the interaction regime.

This manuscript is the first of two works, and deals with the first part of the problem: the existence of a pure solitary wave like solution, before it reaches the interaction point.

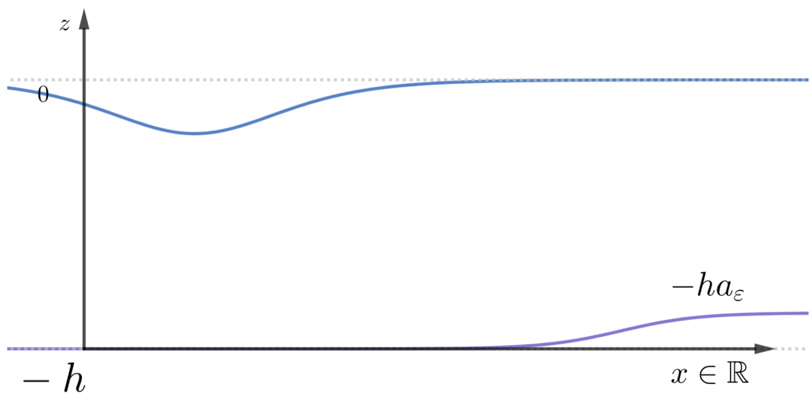


Figure 5.1: A solitary wave in nonflat bottom.

Throughout this paper, we will consider a *slightly changing bottom* described by the function $a_\varepsilon = a(\varepsilon\cdot) \in C_b^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$, where $\varepsilon > 0$ and a is assumed to satisfy the following conditions:

$$\begin{aligned} &\text{There exist } K > 0,\ 0 < \kappa < 1 \text{ and } \gamma_1, \gamma_2 > 0 \text{ such that:} \\ &\qquad 1 - \kappa < a(r) < 1, \quad \forall r \in \mathbb{R}, \\ &\qquad 1 - a(r) \leqslant K\mathrm{e}^{\gamma_2 r}, \quad \forall r \leqslant 0, \\ &\qquad \lim_{r \to -\infty} a(r) = 1,\ \lim_{r \to \infty} a(r) = 1 - \kappa, \\ &\qquad |a'(r)| < K\mathrm{e}^{-\gamma_1 |r|}, \quad \forall r \in \mathbb{R}, \\ &\qquad a' \text{ does not change sign.} \end{aligned} \tag{5.11}$$

From now on, we will denote by $\mathbf{Q}_c(x - ct) = (\eta_c(x - ct), \varphi_c(x - ct))^t$ the solitary wave given by Theorem 5.1. That is, $\mathbf{Q}_c$ satisfies

$$\left\{ \begin{aligned} &\partial_t \eta_c = \mathcal{G}[\eta_c, 1]\varphi_c \\ &\partial_t \varphi_c = -\tfrac{1}{2} |\partial_x \varphi_c|^2 + \tfrac{1}{2} \frac{(\mathcal{G}[\eta_c, 1]\varphi_c + \partial_x \varphi_c \partial_x \eta_c)^2}{1 + |\partial_x \eta_c|^2} - g\eta_c + b\partial_x \left(\frac{\partial_x \eta_c}{\sqrt{1 + |\partial_x \eta_c|^2}} \right) \end{aligned} \right. \tag{5.12}$$

and $\mathcal{G}[\eta_c, 1] = \varphi_c \mapsto \sqrt{1 + |\partial_x \eta_c|^2}\ \partial_{\mathbf{n}} \Phi_c\big|_{z = \eta_c}$ where Φ_c is the solution to the Laplace equation in a flat-bottom domain $\Omega_t^\flat := \{(x, z) \in \mathbb{R}^2 \text{ such that } -h \leqslant z \leqslant \eta_c(t, x)\}$,

$$\left\{ \begin{aligned} &\Delta_{x,z} \Phi_c = 0 \quad (x, z) \in \Omega_t^\flat \\ &\Phi_c\big|_{z = \eta_c} = \varphi_c \\ &\partial_{\mathbf{n}} \Phi\big|_{z = -h} = 0. \end{aligned} \right. \tag{5.13}$$

As mentioned before, our main goal is to prove the existence of a solution to the Zakharov water waves system (5.6) that behaves asymptotically like the solitary wave $\mathbf{Q}_c$ as time tends to $-\infty$. It will be convenient to define

$$\mathbf{R}(t,x) = \mathbf{Q}_c(x - ct + A),$$

where $A \gg 1$ is a parameter that allows us to stay a safe distance A from the point of change in the bottom. The precise result reads:

Theorem 5.2. *Let us fix $s \geqslant 0$. Suppose that the speed $c > 0$ satisfy (5.10) with a parameter λ. Then, there exists λ^* such that for every $\lambda \in (0, \lambda^*)$, and $A > 0$ sufficiently large (depending on ε), there exists a solution $\boldsymbol{U} = (\eta, \varphi)^t$ to (5.6) defined in the time interval $(-\infty, 0]$, that satisfies*

$$\boldsymbol{U} - \boldsymbol{R} \in \mathcal{C}_b((-\infty, 0], H^s(\mathbb{R}) \times H^s(\mathbb{R})),$$

and

$$\lim_{t\to\infty} \| \boldsymbol{U}(t) - \boldsymbol{R}(t) \|_{H^s(\mathbb{R})} = 0.$$

Remark 5.1. The hypothesis $\lambda \in (0, \lambda^*)$ is merely to ensure the existence of the solitary wave, given by Theorem 5.1. Nevertheless, one could replace such condition by simply assuming that there exist smooth, bounded and exponentially decaying solitary wave solutions.

Remark 5.2. The parameter $A \geqslant 0$, that presents a safe distance from the change in the bottom, is equivalent to consider a time interval existence like $(-\infty, T]$. For the second part of the problem, such T should define the moment where the soliton-like solution constructed here enters the interaction regime.

Remark 5.3. A fully detailed first step into Theorem 4.1, including explicit convergence estimates and decomposition of the soliton-like solution, can be found in Theorem 5.13.

The proof of Theorem 5.2 follows the techniques that Ming, Rousset and Tzvetkov introduced in [14] to prove the existence of multisoliton-like solutions for the one-dimensional, flat bottom problem. It consists on two main steps. Firstly, we construct an proximate solution of the form

$$\mathbf{U}_{ap}(t,x) = \mathbf{R}(t,x) + \sum_{j=1}^{N} \rho^j \mathbf{V}_j(t,x),$$

where $\mathbf{V}_j(t,x)$ are solutions to linear problems (linearization of (5.6) about the solitary wave) with exponentially decaying source terms. The idea is to understand how fast the fundamental solution of the linear problems grows so that it can be controlled by the decay of the sources (depending on A). To do so, the principal tool would be the use of spectral properties of the linear operator, proved in [14] and stated here in Section 5.2.3.

When plugging the approximate solution into the Zakharov water waves system (5.6), a remainder term $\mathbf{r}_c$ appears, with a faster decay for larger N. Finally, we construct an exact solution $\mathbf{U} = \mathbf{U}_{ap} + \mathbf{U}_r$, where $\mathbf{U}_r$ solves

$$\partial_t \mathbf{U}_r = \mathcal{F}(\mathbf{U}_{ap} + \mathbf{U}_r) - \mathcal{F}(\mathbf{U}_{ap}) - \mathbf{r}_{ap}.$$

Proving the existence of such equation means that $\mathbf{U}$ satisfies

$$\mathbf{U}(t) \to \mathbf{R}(t) \text{ as } t \to -\infty \text{ in } H^s.$$

The nonconstant bottom introduces several difficulties in our approach, as it presents a nonlinear interaction with the solution $\mathbf{U}$. To overcome this matter, we essentially split the solution into two parts: one of them heavily influenced by the bottom (that we control by staying away from the changing point) and another part, ruled by the solitary wave $\mathbf{Q}_c$. The first issue in this regard would be to undestand how the the description of the bottom affects the error produced by the solitary wave in the new regime $a \neq 1$. The need for a sufficiently fast decaying error imposes the smoothness condition and softness in the variation of the bottom. Such assumptions also prevent from breaking the properties on the linearized equation, as an abrupt change would, and allow as to find suitable estimations in the construction of the approximate solution.

5.2 Preliminaries

5.2.1 Study of the Dirichlet-Neumann operator

Firstly, let us recall the definition of the Dirichlet-Neumann operator

$$\mathcal{G}[\eta, a_\varepsilon] : \varphi \mapsto \sqrt{1+|\nabla\eta|^2}\, \partial_{\mathbf{n}}\Phi\big|_{z=\eta},$$

where Φ is the solution of the Laplace equation (5.5). Note that the velocity potential Φ, associated to a solution $\mathbf{U} = (\eta, \varphi)^t$ of (5.6), is defined in the domain Ω_t, whose surface is described by η. To be able to compare velocity potentials associated with different solutions (in particular, with $\mathbf{Q}_c$), it will be necessary to re-write the Laplace equation (5.5) and turn it into an elliptic problem in a flat domain $S := \mathbb{R} \times [-1, 0]$ (see [11, Subsection 2.2] for further details). Indeed, consider $\Sigma : S \to \Omega_t$ a difeomorphism such that

$$\Sigma(x,z) = (x, (ha(\varepsilon x) + \eta(t,x))\, z + \eta(t,x)), \quad (x,z) \in S.$$

Now, we can define the new unknown $\tilde{\Phi} = \Phi(\Sigma(x,z))$, solution to the following elliptic problem defined on the strip S,

$$\begin{cases} \nabla_{x,z} \cdot \left(P_a(\Sigma)\nabla_{x,z}\tilde{\Phi}\right) = 0 \quad \text{in } S, \\ \tilde{\Phi}|_{z=0} = \varphi, \\ \partial_{\mathbf{n}}^{P_a}\tilde{\Phi}|_{z=-1} = 0, \end{cases} \tag{5.14}$$

where the matrix $P_a(\Sigma)$ is defined as

$$P_a(\Sigma) := \begin{bmatrix} ha_\varepsilon(x) + \eta(t,x) & -(z+1)\partial_x\eta(t,x) - hza'_\varepsilon(x) \\ -(z+1)\partial_x\eta(t,x) - hza'_\varepsilon(x) & \dfrac{1 + |(z+1)\partial_x\eta(t,x) + hza'_\varepsilon(x)|^2}{ha_\varepsilon(x) + \eta(t,x)} \end{bmatrix}, \tag{5.15}$$

denoted from now on as $P_a = P_a(\Sigma)$ to simplify notation, and $\partial_{\mathbf{n}}^{P_a} = \mathbf{n} \cdot (P_a(\Sigma)\nabla_{x,z})$, for $\mathbf{n} = -\mathbf{e}_z$ the upward unit normal to the boundary $z = -1$.

When considering the notation $\partial_{\mathbf{n}}^{P_a} = (0,1)^t \cdot P_a \nabla_{x,z}$, we get that the Dirichlet-Neumann operator

$$\mathcal{G}[a_\varepsilon, \eta]\varphi = \partial_{\mathbf{n}}^{P_a} \tilde{\Phi}|_{z=0}$$

can be re-written in a more explicit fashion as

$$\mathcal{G}[a_\varepsilon, \eta]\varphi = -\left((z+1)\partial_x \eta_c + zha'_\varepsilon\right) \partial_x \tilde{\Phi}|_{z=0} + \frac{1 + |(z+1)\partial_x \eta + hza'_\varepsilon|^2}{ha_\varepsilon + \eta} \partial_z \tilde{\Phi}|_{z=0}. \tag{5.16}$$

Evaluations at $z = 0$ will be justified later on in terms of the regularity required on the involved functions.

The DN operator meets the following properties:

Proposition 5.3. *Let $\eta \in H^\infty(\mathbb{R})$ and $a \in C^\infty(\mathbb{R})$ defined by conditions* (5.11)*, with $ha_\varepsilon(x) + \eta(x) > h_{min} > 0$ for all $x \in \mathbb{R}$, $\varepsilon > 0$. Then:*

(a) $\mathcal{G}[\eta, a_\varepsilon]$ is symmetric on $L^2(\mathbb{R})$:

$$(\mathcal{G}[\eta, a_\varepsilon]\varphi, \psi) = (\varphi, \mathcal{G}[\eta, a_\varepsilon]\psi), \quad \forall \varphi, \psi \in H^{\frac{1}{2}}(\mathbb{R}).$$

(b) There exist $c > 0$, $C > 0$ such that,

$$|\left(\mathcal{G}[\eta, a_\varepsilon]\varphi, \psi\right)| \leqslant C|\mathfrak{B}\varphi|_{L^2}|\mathfrak{B}\psi|_{L^2}, \quad \forall \varphi, \psi \in H^{\frac{1}{2}}(\mathbb{R}), \tag{5.17}$$

$$(\mathcal{G}[\eta, a_\varepsilon]\varphi, \varphi) \geqslant c|\mathfrak{B}\varphi|_{L^2}^2, \quad \forall \varphi \in H^{\frac{1}{2}}(\mathbb{R}), \tag{5.18}$$

where $\mathfrak{B}$ is the Fourier multiplier

$$\mathfrak{B} = \left(1 - \partial_x^2\right)^{-\frac{1}{4}} \partial_x.$$

(c) The linear operator $\mathcal{G}[\eta, a_\varepsilon] : H^{s+1}(\mathbb{R}) \to H^s(\mathbb{R})$ is continuous for every $s \in \mathbb{R}$.

We refer to [11, Section 3] for the proof of Proposition (5.3), where the results are shown for a more general case (the bottom taking an arbitrary form).

Taking into account the shape of the solitary wave and (specifically, the shape of φ_c), it will be useful to understand the behavoir of the Dirichlet Neumann operator acting on smooth localized function. In particular, we shall see that it behaves as a space-derivative for exponentially decaying functions. Indeed, we give an estimation on its derivatives:

Proposition 5.4. *Assume that $\psi \in C_b^\infty(\mathbb{R})$ has the exponential decay*

$$\exists \mathrm{d} > 0, \forall \alpha \in \mathbb{N}, \exists\, C_\alpha, \forall x \in \mathbb{R}, |\partial_x^\alpha \psi(x)| \leqslant C_\alpha \mathrm{e}^{-\mathrm{d}(1+|x|^2)^{\frac{1}{2}}}. \tag{5.19}$$

Then, for any $\eta \in H^\infty(\mathbb{R})$ with $\min\limits_{x \in \mathbb{R}}\{ha_\varepsilon + \eta\} \geqslant h_{min} > 0$, $\mathcal{G}[\eta, a]\psi$ also decays exponentially fast, that is, for any $\alpha \in \mathbb{N} \cup \{0\}$, there exist a constant c_α depending on α and $0 < \delta < \mathrm{d}$ independent of α such that for every $x \in \mathbb{R}$,

$$|\partial_x^\alpha \left(\mathcal{G}[\eta, a_\varepsilon]\psi\right)(x)| \leqslant c_\alpha \mathrm{e}^{-\delta(1+|x|^2)^{\frac{1}{2}}}.$$

Proof. We divide the proof in several steps.

Step 1. Under the assumption $\psi \in C_b^\infty(\mathbb{R})$ and $\eta \in H^\infty(\mathbb{R})$, from (5.16), we can write the DN operator $G[\eta, a_\varepsilon]$ acting on ψ as

$$\mathcal{G}[\eta, a_\varepsilon]\psi = -\left((z+1)\partial_x\eta + zha'_\varepsilon\right)\partial_x\tilde{\Phi}|_{z=0} + \frac{1+\left|(z+1)\partial_x\eta + zha'_\varepsilon\right|^2}{ha_\varepsilon + \eta}\partial_z\tilde{\Phi}|_{z=0}, \tag{5.20}$$

where $\tilde{\Phi}$ solves the elliptic problem (5.14). Thus, to get the expected estimation, it shall be sufficient to prove exponential decay for derivatives of $\tilde{\Phi}$.

We begin the proof by making the decomposition

$$\tilde{\Phi}(x,z) = u(x,z) + v(x,z), \tag{5.21}$$

where u is the solution of the elliptic problem

$$\begin{cases} \Delta_{x,z}u = 0, & (x,z) \in S, \\ u(x,0) = \psi(x), \ \partial_{\mathbf{n}}^{P_a}u(x,-1) = 0. \end{cases} \tag{5.22}$$

This way, we get that v solves the elliptic problem with homogeneous boundary conditions

$$\begin{cases} \nabla_{x,z}\cdot(P_a\nabla_{x,z}v) = -\nabla_{x,z}\cdot(P_a\nabla_{x,z}u) & (x,z) \in S, \\ v(x,0) = 0, \ \partial_{\mathbf{n}}^{P_a}v(x,-1) = 0. \end{cases} \tag{5.23}$$

By solving an ODE, one can obtain an explicit expression of the Fourier transform in x of u,

$$\mathcal{F}_x(u)(\xi,z) = \frac{\cosh(\xi(z+1))}{\cosh(\xi)}\mathcal{F}_x(\psi)(\xi).$$

From this expression we also have that

$$\mathcal{F}_x(\partial_x u)(\xi,z) = \frac{\cosh(\xi(z+1))}{\cosh(\xi)}\mathcal{F}_x(\partial_x\psi)(\xi).$$

Step 2. As in [14], we can prove exponential decay of the solution u of (5.22) by following a Paley-Wiener type argument. First, we notice that the fact that $\partial_x\psi$ and all its derivatives have the exponential decay (5.19) $\mathcal{F}_x(\partial_x\psi)$ has an holomorphic extension for ξ satisfying $|\mathrm{Im}\ \xi| < \mathrm{d}$. Also, we find that, after integrating by parts, for $\delta \in (0, \mathrm{d})$,

$$|\mathcal{F}_x(\partial_x u)| \leqslant \frac{C_n}{1+|\xi|^n}, \quad (\xi,z) \in \mathbb{R}\times[-1,0], \ |\mathrm{Im}\ \xi| \leqslant \rho, \ \forall n \in \mathbb{N}$$

Since $\xi \to \frac{\cosh(\xi(z+1))}{\cos(\xi)}$ has an holomorphic bounded (uniformly on z) extension to $|\mathrm{Im}\xi| \leqslant \rho$ for any $\rho \in (0, \pi/2)$, using contour deformation, one has

$$\partial_x u(x,z) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\xi}\frac{\cosh(\xi(z+1))}{\cosh(\xi)}\mathcal{F}_x(\partial_x\psi)(\xi)\mathrm{d}\xi = \int_{\mathrm{Im}\xi=\rho} \mathrm{e}^{\mathrm{i}x\xi}\frac{\cosh(\xi(z+1))}{\cosh(\xi)}\mathcal{F}_x(\partial_x\psi)(\xi)\mathrm{d}\xi,$$

for any ρ such that $|\rho| < \min\{\mathrm{d}, \pi/2\}$. This implies that by taking $\rho = 2\delta\mathrm{sgn}(x)$, with δ sufficiently small, we find that

$$|\partial_x u| \lesssim C_\alpha \mathrm{e}^{-2\delta|x|} \quad \forall (x,z) \in S.$$

Now, the z-derivative can be estimated in a similar fashion. In fact, we have

$$\partial_z \mathcal{F}_x\left(u(x,z)\right) = \frac{\sinh\left(\xi(z+1)\right)}{\cosh(\xi)}\xi\mathcal{F}_x(\psi)(\xi) = \frac{\sinh\left(\xi(z+1)\right)}{\cosh(\xi)}\frac{1}{\mathrm{i}}\mathcal{F}_x\left(\partial_x\psi\right)(\xi).$$

Then, we argue as before and get that

$$|\partial_z u| \lesssim C_\alpha \mathrm{e}^{-2\delta|x|} \quad \forall (x,z) \in S.$$

Higher-order derivatives can be estimated using the same arguments, which implies the following

$$\forall \alpha, |\alpha| \geqslant 1, \quad \left|\partial^\alpha_{x,z} u\right| \lesssim C_\alpha \mathrm{e}^{-2\delta|x|} \quad \forall (x,z) \in S. \tag{5.24}$$

Step 3. Then, we are left to prove exponential decay for the solution v of (5.23). To do so, we rely on the decay properties of P_a (inherited from ψ (5.19)) and define the increasing weight

$$\omega(x) = \mathrm{e}^{\delta\langle x\rangle}, \quad 0 < \delta < \mathrm{d},$$

where we denoted by $\langle x\rangle = (1+x^2)^{\frac{1}{2}}$ and δ is to taken sufficiently small later. We get the following elliptic problem for $\tilde{v} = \omega v$:

$$\begin{cases} -\nabla_{x,z}\cdot\left(P_a\nabla_{x,z}\tilde{v}\right) - \left[\nabla_{x,z}\cdot P_a\nabla_{x,z},\omega\right]v = \omega\nabla_{x,z}\cdot\left(P_a\nabla_{x,z}u\right) \quad (x,z)\in S, \\ \tilde{v}(x,0) = 0, \quad \partial^{P_a}_{\mathbf{n}}\tilde{v}(x,-1) = \delta h\varepsilon a'(\varepsilon x)\tilde{v}(x,-1). \end{cases} \tag{5.25}$$

For the Neumann boundary condition we used the fact that on the boundary $z = -1$,

$$\partial^{P_a}_{\mathbf{n}} = ha'_\varepsilon\partial_x + \frac{1+|ha'_\varepsilon|^2}{ha_\varepsilon+\eta_c}\partial_z, \quad \text{on} \quad z=-1.$$

Now, we want to estimate the Sobolev norms of $\tilde{v}$. Using Divergence Theorem, denoting by $\Gamma = \{z=-1\}\cup\{z=0\}$, we can compute

$$\begin{aligned} &\int_S P_a\nabla_{x,z}\tilde{v}\cdot\nabla_{x,z}\tilde{v}\mathrm{d}x\mathrm{d}z - \int_\Gamma \mathbf{n}\cdot\left(P_a\nabla_{x,z}\tilde{v}\right)\tilde{v}\mathrm{d}\Gamma \\ &= \int_S P_a\nabla_{x,z}\tilde{v}\cdot\nabla_{x,z}\tilde{v}\mathrm{d}x\mathrm{d}z - \int_{\mathbb{R}} \delta H\varepsilon a'(\varepsilon x)\tilde{v}^2(x,-1)\mathrm{d}x \\ &= \int_S \left(\left[\nabla_{x,z}\cdot P_a\nabla_{x,z},\omega\right]v\right)\tilde{v}\mathrm{d}x\mathrm{d}z + \int_S \omega\nabla_{x,z}\cdot(P_a\nabla_{x,z}u)\tilde{v}\mathrm{d}x\mathrm{d}z. \end{aligned}$$

We write the following

$$\begin{aligned} \int_S P_a\nabla_{x,z}\tilde{v}\cdot\nabla_{x,z}\tilde{v}\mathrm{d}x\mathrm{d}z &= \int_S \left(\left[\nabla_{x,z}\cdot P_a\nabla_{x,z},\omega\right]\omega^{-1}\tilde{v}\right)\tilde{v}\mathrm{d}x\mathrm{d}z \\ &+ \int_S \omega\nabla_{x,z}\cdot(P_a\nabla_{x,z}u)\tilde{v}\mathrm{d}x\mathrm{d}z + \int_{\mathbb{R}} \delta H\varepsilon a'(\varepsilon x)\tilde{v}^2(x,-1)\mathrm{d}x \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

For the left-hand side, because P_a is coercive, we have that

$$\int_S P_a\nabla_{x,z}\tilde{v}\cdot\nabla_{x,z}\tilde{v}\ \mathrm{d}x\mathrm{d}z \geqslant c\|\nabla_{x,z}\tilde{v}\|^2_{L^2(S)}.$$

for a constant $c > 0$ that does not depend on δ. To estimate the right-hand side terms, starting with I_1, we can compute

$$\begin{aligned}[\nabla_{x,z} \cdot P_a \nabla_{x,z}, \omega] = &\,(ha(\varepsilon x) + \eta)\,[\partial_x^2, \omega] \\ &- ((z+1)\partial_x \eta + z\varepsilon h a'(\varepsilon x))\,[\partial_x, \omega]\partial_z.\end{aligned}$$

Since we know that $ha_\varepsilon + \eta \geqslant h_{min}$, the last equality implies that

$$|I_1| \leqslant C\left(\delta^2 \|\tilde{v}\|_{L^2(S)} + \delta \|\nabla_{x,z}\tilde{v}\|_{L^2(S)}\right)\|\tilde{v}\|_{L^2(S)}.$$

Secondly, to estimate the I_2, we use (5.24), and obtain

$$|I_2| \leqslant C_\delta\left(\delta\|\tilde{v}\|_{L^2(S)} + \|\nabla_{x,z}\tilde{v}\|_{L^2(S)}\right). \tag{5.26}$$

Indeed, from Divergence Theorem,

$$\begin{aligned} I_2 &= -\int_S P_a \nabla_{x,z} u \cdot \nabla_{x,z}(\omega\tilde{v})\,\mathrm{d}x\mathrm{d}z + \int_\Gamma \mathbf{n}\cdot(P_a\nabla_{x,z}u)\,\omega\tilde{v}\mathrm{d}\Gamma \\ &\leqslant \left\|P_a\nabla_{x,z}u\cdot\nabla_{x,z}\mathrm{e}^{\delta\langle x\rangle}\right\|_{L^2(S)}\|\tilde{v}\|_{L^2(S)} + \left\|(P_a\nabla_{x,z}u)\,\mathrm{e}^{\delta\langle x\rangle}\right\|_{L^2(S)}\|\nabla_{x,z}\tilde{v}\|_{L^2(S)},\end{aligned}$$

where we used that $\partial_{\mathbf{n}}^{P_a} u = 0$ on the boundary $z = -1$. Now, notice that, since (5.24) holds, we have that there exists $C_\delta > 0$ such that

$$\left\|P_a\nabla_{x,z}u\cdot\nabla_{x,z}\mathrm{e}^{\delta\langle x\rangle}\right\|_{L^2(S)} \leqslant \delta C_\delta \quad \text{and} \quad \left\|(P_a\nabla_{x,z}u)\,\mathrm{e}^{\delta\langle x\rangle}\right\|_{L^2(S)} \leqslant C_\delta,$$

so that (5.26) is implied. Then, from Young inequality, one has

$$|I_2| \leqslant \delta^2\|\tilde{v}\|^2_{L^2(S)} + \delta\|\nabla_{x,z}\tilde{v}\|^2_{L^2(S)} + C_\delta.$$

In a similar fashion, since $H\varepsilon a'(\varepsilon x)$ is (uniformly) bounded, after using Fundamental Theorem of Calculus on the variable z, one can obtain

$$I_3 \leqslant C\delta\int_{\mathbb{R}} \tilde{v}^2(x,-1)\mathrm{d}x \leqslant C\delta\,\|\tilde{v}\|_{L^2(S)}\,\|\nabla_{x,z}\tilde{v}\|_{L^2(S)}\,.$$

Arranging all this estimates together, we conclude that

$$\|\nabla_{x,z}\tilde{v}\|^2_{L^2(S)} \leqslant C\left(\delta^2\|\tilde{v}\|^2_{L^2(S)} + \delta\|\nabla_{x,z}\tilde{v}\|_{L^2(S)}\|\tilde{v}\|_{L^2(S)} + \delta\|\nabla_{x,z}\tilde{v}\|^2_{L^2(S)} + C_\delta.\right) \tag{5.27}$$

Now, we take advantage of the fact that the fluid has finite depth, meaning that the domain is bounded in the z direction, which allows us to apply Poincaré inequality. Indeed, S is bounded in the z direction, thus we are entitled to use Poincaré inequality (C independent of $\tilde{v}$):

$$\|\tilde{v}\|_{L^2(S)} \leqslant C\|\nabla_{x,z}\tilde{v}\|_{L^2(S)}.$$

Therefore, going back to (5.27), we can conclude that if δ is small enough, then $\|\nabla_{x,z}\tilde{v}\|^2_{L^2(\mathbb{R})} \leqslant C$. In consequence, summing up we have that

$$\|\tilde{v}\|_{L^2(S)} \leqslant C, \quad \|\nabla_{x,z}\tilde{v}\|_{L^2(S)} \leqslant C,$$

for $\delta > 0$ sufficiently small.

Step 4. Finally, for the higher order estimates of $\tilde{v}$, we use an induction argument. Indeed, we re-write equation (5.28) as

$$\begin{cases} -\nabla_{x,z} \cdot (P_a \nabla_{x,z} \tilde{v}) = F \quad (x,z) \in S, \\ \tilde{v}(x,0) = 0, \quad \partial_{\mathbf{n}}^{P_a} \tilde{v}(x,-1) = \delta h a_\varepsilon'(x) \tilde{v}(x,-1). \end{cases}$$

By standard elliptic regularity theory (for instance, regularity estimates in [11, Subsection 2.3] and [7, Subsection 6.3]), we know that,

$$\|\tilde{v}\|_{H^{s+2}(S)} \leqslant C \left(\|\tilde{v}\|_{H^1(S)} + \|F\|_{H^s(S)} \right).$$

Because of the estimate (5.24), we have that

$$\|F\|_{H^s(S)} \leqslant C \left(1 + \|\tilde{v}\|_{H^{s+1}(S)} \right).$$

Therefore, since we already proved the H^1 estimate, we can start from there and get by induction that $\|\tilde{v}\|_{H^s(S)} \leqslant C_s$, for every $s \geqslant 0$. By Sobolev embbeding, we conclude that

$$\forall \alpha, \ \left| \partial_{x,z}^\alpha \left(\mathrm{e}^{\delta \langle x \rangle} v(x,z) \right) \right| \leqslant C_\alpha.$$

Equivalently,

$$\forall \alpha, \ |\partial_{x,z}^\alpha v(x,z)| \leqslant C_\alpha \mathrm{e}^{-\delta \langle x \rangle}. \tag{5.28}$$

Plugging this and (5.24) into (5.20) finishes the proof. ■

Since in this work we study, essentially, the solitary wave that arises in the flat-bottom problem, it is natural to adapt the result to the case of the solitary wave. With this in mind, we give the following corollary:

Corollary 5.5. *Let $\psi \in C_b^\infty(\mathbb{R})$ with an exponential decay (5.19). Then, for $\eta \in H^\infty(\mathbb{R})$ such that $\min_{x \in \mathbb{R}} \{-h a_\varepsilon + \eta\} > h_{min}$ and $r \in \mathbb{R}$, $\mathcal{G}[\eta, a_\varepsilon](\psi(x-r))$ also satisfies an exponential decay; that is, for any $\alpha \in \mathbb{N} \cup \{0\}$ there exist C_α and $0 < \delta < \mathrm{d}$ (that does not depend on r) such that for all $x \in \mathbb{R}$,*

$$|\partial_x^\alpha \mathcal{G}[\eta, a_\varepsilon] \left(\psi(\cdot - r) \right)(x)| \leqslant C_\alpha \mathrm{e}^{-\delta (1+|x-r|^2)^{\frac{1}{2}}}.$$

Remark 5.4. The result still holds when considering the case $a = 1$. Indeed, in [14] it was proven that if $\psi \in C^\infty(\mathbb{R})$ satisfies (5.19), $\eta \in H^\infty(\mathbb{R})$ such that $\min_{x \in \mathbb{R}} \{-h + \eta\} > h_{min}$ and $r \in \mathbb{R}$, then $\forall \alpha \in \mathbb{N} \cup \{0\}$, $\exists C_\alpha$ and $0 < \delta < \mathrm{d}$ such that

$$|\partial_x^\alpha \mathcal{G}[\eta, 1] \left(\psi(\cdot - r) \right)(x)| \leqslant C_\alpha \mathrm{e}^{-\delta (1+|x-r|^2)^{\frac{1}{2}}}, \quad \forall x \in \mathbb{R}.$$

Remark 5.5. As noticed before, to prove Proposition 5.4 is actually to demostrate that for every $\alpha \in \mathbb{N}$,

$$|\partial_{x,z}^\alpha \tilde{\Phi}| \leqslant C_\alpha \mathrm{e}^{-\delta \langle x \rangle}.$$

Notice that the solitary wave $Q_c = (\eta_c, \varphi_c)^t$ (solution to system (5.12)) satisfies hypothesis of the result. Then, after considering an appropriate matrix to turn the Laplace equation into an elliptic problem in a flatten domain for the case $a = 1$, one can use similar computations to show that

$$|\partial_{x,z}^\alpha \Phi_{c,1}| \leqslant C_\alpha \mathrm{e}^{-\delta \langle x \rangle},$$

where $\Phi_{c,1}$ in the strip domain associated to the flat bottom DN operator $\mathcal{G}[\eta_c, 1]$.

5.2.2 Shape derivatives for the Dirichlet-Neumann operator

Neumann-Neumann operator and moving bottoms

In most of this work, we deal with the Zakharov system with a fix bottom (that is, depending only on space variable) described by a_ε. Nevertheless, it is possible to assume that the bottom is moving, as was done for instance in [1] or [10]. Such assumption implies a change in the domain and, consequently, a different condition on the bottom (which would now be a kinematic condition, like the one imposed on the surface). More precisely, if the bottom is described by a function $\overline{a}(t,x)$, $\overline{a} \in C^2(\mathbb{R}\times\mathbb{R})$, then the domain would be defined (for the one-dimensional surface case) as

$$\Omega_t = \left\{(x,z) \in \mathbb{R}^2, \quad -\overline{a}(t,x)h \leqslant z \leqslant \eta(t,x)\right\}$$

and the kinematic condition on the bottom (in terms of the velocity potential Φ) would read

$$\sqrt{1+h^2|\partial_x \overline{a}|^2}\partial_{\mathbf{n}}\Phi = -h\partial_t \overline{a} \quad \text{ on } \{z = -h\overline{a}(t,x)\},$$

where $\mathbf{n}$ is the unit normal vector to the fluid domain pointing upwards. Accordingly, Φ is now recovered as the solution to the Laplacian equation with *non-homogeneous* Neumann condition at the bottom,

$$\begin{cases} \Delta_{x,z}\Phi = 0 \quad \text{in } \mathcal{S}, \\ \Phi|_{z=\eta} = \varphi, \\ \sqrt{1+h^2|\partial_x \overline{a}|^2}\partial_{\mathbf{n}}\Phi|_{z=-h\overline{a}(t,x)} = -h\partial_t \overline{a}. \end{cases}$$

Taking into account the definition of the Dirichlet-Neumann operator, it is useful to decompose Φ into a *fix bottom* component, Φ_{fb}, and a *moving bottom* component, Φ_{mb}, as

$$\begin{cases} \Delta_{x,z}\Phi_{fb} = 0 \quad \text{in } \mathcal{S}, \\ \Phi_{fb}|_{z=\eta} = \varphi, \\ \sqrt{1+h^2|\partial_x \overline{a}|^2}\partial_{\mathbf{n}}\Phi_{fb}|_{z=-h\overline{a}(t,x)} = 0, \end{cases}$$

and

$$\begin{cases} \Delta_{x,z}\Phi_{mb} = 0 \quad \text{in } \mathcal{S}, \\ \Phi_{mb}|_{z=\eta} = 0, \\ \sqrt{1+H^2|\partial_x \overline{a}|^2}\partial_{\mathbf{n}}\Phi_{mb}|_{z=-h\overline{a}(t,x)} = -h\partial_t \overline{a}. \end{cases}$$

As a consequence, we now have that

$$\sqrt{1+|\partial_x \eta|^2}\partial_{\mathbf{n}}\Phi|_{z=\eta} = \mathcal{G}[\eta,a]\varphi + h\mathcal{G}^{NN}[\eta,\overline{a}]\partial_t\overline{a}$$

where the operator $\mathcal{G}^{NN}[\eta,\overline{a}]$ is defined as

$$\mathcal{G}^{NN}[\eta,\overline{a}] : \partial_t \overline{a} \mapsto \sqrt{1+|\partial_x\eta|^2}\partial_{\mathbf{n}}\Phi_{mb}|_{z=\eta}.$$

Equivalently, after using a diffeomorphism $\Sigma : S \to \Omega_t$ to flatten the domain of the elliptic problem, we can write

$$\mathcal{G}[\eta,\overline{a}]\varphi + \mathcal{G}^{NN}[\eta,\overline{a}]B = \mathbf{e}_z \cdot P\nabla_{x,z}\Phi|_{z=0}.$$

Shape derivatives

Since we are interested in the DN operator in terms of the bottom and the moving surface, let us consider Γ the set of all $(\eta, a) \in H^{3/2}(\mathbb{R})^2$ such that the following condition is satisfied

$$\exists h_0 > 0, \text{ such that } \forall x \in \mathbb{R}, \ -ha(x) + \eta(x) > h_0.$$

For $0 \leqslant s \leqslant 3/2$ and $\varphi \in \dot{H}^{s+1/2}$ (fixed), we want to study the operator

$$\begin{aligned} \mathcal{G}[\eta, a] : \Gamma \subset H^{3/2}(\mathbb{R})^2 &\to H^{s-1/2}(\mathbb{R}) \\ (\eta, a) &\mapsto \mathcal{G}[\eta, a]\varphi \end{aligned} \tag{5.29}$$

Given $(\zeta, b) \in H^{3/2}(\mathbb{R})$, we will denote by $D(\mathcal{G}[\eta, a]\varphi) \cdot (\zeta, b)$ the Fréchet derivative of (5.29) at (η, a) in the direction (ζ, b). We use an analogous definition for the derivative of $\mathcal{G}^{NN}[\eta, a]$.

We present the following Proposition, proved in [11, Theorem 3.1] and [10, Theorem 3.5 and Theorem 3.6], regarding shape derivatives of the Dirichlet-Neumann and Neumann-Neumann operator:

Proposition 5.6. *Assume that $0 \leqslant s \leqslant 3/2$ and $\varphi \in \dot{H}^{s+1/2}$.*

1. *For all $\zeta \in H^{3/2}(\mathbb{R})$, one has*

$$D_\eta \left(\mathcal{G}[\eta, a]\varphi\right) \cdot \zeta + D_\eta \left(\mathcal{G}^{NN}[\eta, a]B\right) \cdot \zeta = -\mathcal{G}[\eta, a](\zeta \tilde{Z}) - \partial_x(\tilde{v}\zeta),$$

with

$$\tilde{Z} = \tilde{Z}[\eta, \varphi, B] = \frac{\mathcal{G}[\eta, a]\varphi + \mathcal{G}^{NN}[\eta, a]B + \partial_x\eta\partial_x\varphi}{1 + |\partial_x\eta|^2}, \quad \tilde{v} = \tilde{v}[\eta, \varphi, B] = \partial_x\varphi - \tilde{Z}\partial_x\eta.$$

2. *For all $b \in H^{3/2}(\mathbb{R})$, one has*

$$D_a \left(\mathcal{G}[\eta, a]\varphi\right) \cdot b + D_a \left(\mathcal{G}^{NN}[\eta, a]B\right) \cdot b = -\mathcal{G}^{NN}[\eta, a]\partial_x(b\underline{v}),$$

with

$$\underline{Z} = \underline{Z}[\eta, a] = \frac{B + h\partial_x a\partial_x\varphi}{1 + h^2|\partial_x a|^2}, \quad \underline{v} = \underline{v}[\eta, \varphi] = \partial_x\varphi - \underline{Z}h\partial_x a.$$

Proposition 5.7. *For $\eta \in H^\infty(\mathbb{R})$ such that $ha_\varepsilon + \eta \geqslant h_{min}$, $\varphi, \zeta_1, \ldots \zeta_j \in H^{3/2}(\mathbb{R})$, and $s > 1/2$, we have that*

$$|D^j_\eta(\mathcal{G}[\eta, a_\varepsilon]\varphi) \cdot (\zeta_1, \ldots, \zeta_j)| \leqslant C_s|\mathfrak{B}\varphi|_{H^s}\prod_{i=1}^{j}|\zeta_i|_{H^{s+1}}.$$

5.2.3 Linearization around the solitary wave

In this subsection, we study the linearization of (5.6) about a solitary wave type solution

$$\mathbf{Q}_c(x - ct) = (\eta_c(x - ct), \varphi_c(x - ct))^T$$

to the flat-bottom problem given by [3]. The construction of the linearized problem follows the idea of [16], with a slightly different frame-work adapted to our case, particularly by the fact that the Dirichlet-Neumann operator depends on the description of the domain.

Since we study a solitary wave of speed c, it will be useful to change the frame from x to $x-ct$, so that the properties stated in this section can be property applied in the construction of the approximate solution (Section 5.4). In particular, we point out that, because we made the change of frame, the soliton is now a stationary wave. We proceed with the linearization.

From Proposition 5.6, given $\varphi \in \dot{H}^{s+1/2}$, $0 \leqslant s \leqslant 3/2$ and, the operator $\eta \mapsto \mathcal{G}[\eta, a_\varepsilon]\varphi$ acting on $H^{3/2}(\mathbb{R})$ satisfies

$$D_\eta \left(\mathcal{G}[\eta, a_\varepsilon]\varphi\right) \cdot \zeta = -\mathcal{G}[\eta, a_\varepsilon](\zeta \tilde{Z}) - \partial_x(\tilde{v}\zeta),$$

with

$$\tilde{Z}[\eta, \varphi] = \frac{\mathcal{G}[\eta, a_\varepsilon]\varphi + \partial_x\eta\partial_x\varphi}{1+|\partial_x\eta|^2}, \quad \tilde{v}[\eta, \varphi] = \partial_x\varphi - \tilde{Z}\partial_x\eta.$$

In addition, we have that

$$D_\eta \left(b\partial_x\left(\frac{\partial_x\eta}{\sqrt{1+|\partial_x\eta|^2}}\right)\right) \cdot \zeta = b\partial_x\left(\frac{\partial_x\zeta}{(1+|\partial_x\eta^2|)^{\frac{3}{2}}}\right).$$

With this in mind, let us denote by

$$Z_c = \tilde{Z}[\eta_c, \varphi_c] \text{ and } v_c = \tilde{v}[\eta_c, \varphi_c].$$

as well as define the following operator

$$\mathcal{P}_c = b\partial_x\left(\frac{\partial_x\cdot}{(1+|\partial_x\eta_c|^2)^{\frac{3}{2}}}\right).$$

Therefore, the linearization of equation (5.6) about the solitary wave $\mathbf{Q}_c$ after the change of frame reads

$$\begin{aligned}
\partial_t\eta &= c\partial_x\eta + \mathcal{G}[\eta_c, a_\varepsilon]\varphi - \mathcal{G}[\eta_c, a_\varepsilon]\left(Z_c\eta\right) - \partial_x(v_c\eta)\\
\partial_t\varphi &= c\partial_x\varphi - v_c\partial_x\varphi + Z_c\mathcal{G}[\eta_c, a_\varepsilon]\varphi - Z_c\mathcal{G}[\eta_c, a_\varepsilon]\left(Z_c\eta\right) - \left(g + Z_c\partial_x v_c\right)\eta + \mathcal{P}_c\eta.
\end{aligned}$$

Thus, we re-write the system above depending on the variable $\mathbf{U} = (\eta, \varphi)^t$ in a more compact fashion as $\partial_t\mathbf{U} = J\mathbb{A}[\mathbf{Q}_c]\mathbf{U}$, where

$$J = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$$

is skew-symmetric and

$$\mathbb{A}[\mathbf{Q}_c] = \begin{pmatrix} -\mathcal{P}_c + g + Z_c\mathcal{G}[\eta_c, a_\varepsilon](Z_c\cdot) + Z_c\partial_x v_c & (v_c - c)\partial_x - Z_c\mathcal{G}[\eta_c, a_\varepsilon]\\ -\partial_x\left((v_c - c)\cdot\right) - \mathcal{G}[\eta_c, a_\varepsilon](Z_c\cdot) & \mathcal{G}[\eta_c, a_\varepsilon]\end{pmatrix}$$

is a symmetric operator on $L^2 \times L^2$ (in this case, acting on $\mathbf{Q}_c$). To simplify this expression, let us introduce the change of unknowns $V_1 = \eta$, $V_2 = \varphi - Z_c\eta$, so that for $\mathbf{V} = (V_1, V_2)^t$ we obtain the following system

$$\begin{aligned}
\partial_t V_1 &= \mathcal{G}[\eta_c, a_\varepsilon]V_2 - \partial_x\left((v_c - c)V_1\right)\\
\partial_t V_2 &= \mathcal{P}_c V_1 - (v_c - c)\,\partial_x V_2 - \left(g + (v_c - c)\partial_x Z_c)\right)V_1.
\end{aligned}$$

Furthermore, we can set the symmetric operator, defined on $L^2 \times L^2$, in this case evaluated in $\mathbf{Q}_c$,

$$\mathbb{L}[\mathbf{Q}_c] = \begin{pmatrix} -\mathcal{P}_c + g + (v_c - c)\partial_x Z_c & (v_c - c)\partial_x \\ -\partial_x((v_c - c)\cdot) & \mathcal{G}[\eta_c, a_\varepsilon] \end{pmatrix}$$

and

$$R_c = \begin{pmatrix} 1 & 0 \\ -Z_c & 1 \end{pmatrix}$$

so that $\mathbf{V} = R_c\mathbf{U}$ and satisfies

$$\partial_t \mathbf{V} = J\mathbb{L}[Q_c]\mathbf{V}.$$

In order to make notation more comfortable, we shall simply write $\mathbb{A}_c$ and $\mathbb{L}_c$, instead of $\mathbb{A}_c[\mathbf{Q}_c]$ and $\mathbb{L}_c[\mathbf{Q}_c]$, respectively. The two operators $\mathbb{A}_c$ and $\mathbb{L}_c$ are related by the property

$$\mathbb{L}_c = \left(R_c^{-1}\right)^t \mathbb{A}_c R_c^{-1}.$$

Remark 5.6 (On the notation)**.** We use here the notation $\mathbb{A}_c$ and $\mathbb{L}_c$ when referring to the linearized operators *in the frame* $x - ct$, so that we can differentiate them from the linearization in the usual frame x, Λ_c and L_c, to appear in Section 5.4.

Finally, we define the space $X^0 = H^1(\mathbb{R}) \times \dot{H}_*^{\frac{1}{2}}(\mathbb{R})$, where $\dot{H}_*^{\frac{1}{2}}$ is a modified homogeneous Sobolev space defined by

$$\dot{H}_*^{\frac{1}{2}} = \left\{u \in \mathcal{S}'(\mathbb{R}) \text{ such that } \mathfrak{B}u \in L^2(\mathbb{R})\right\}$$

and $\mathfrak{B} = \left(1 - \partial_x^2\right)^{-\frac{1}{4}} \partial_x$ is the Fourier multiplier defined in Proposition 5.3. On $\dot{H}_*^{\frac{1}{2}}$ we consider the semi-norm

$$|u|_{\dot{H}_*^{\frac{1}{2}}} = |\mathfrak{B}u|_{L^2}.$$

Thus, for $\mathbf{U} = (U_1, U_2)^t \in X^0$ we define

$$|U|_{X^0} = |U_1|_{H^1} + |U_2|_{\dot{H}_*^{\frac{1}{2}}}.$$

Remark 5.7. Notice that the quadratic form associated to $\mathbb{L}_c$ is well-defined in X^0. Indeed, this is a consequence of Proposition 5.3.

In the rest of this subsection, we shall devote ourselves to the study of the operator $\mathbb{L}_c$ arising from the linearization of the Hamiltonian after the change of framework about the soliton $\mathbf{Q}_c$. In particular, we will be interested in proving a somewhat coercivity property for $\mathbb{L}_c$ to happen away from the change of bottom ($a \sim 1$). To do so, let us consider

$$\mathbb{L}_c^1 := \begin{pmatrix} -\mathcal{P}_c + g + (v_c^1 - c)\partial_x Z_c^1 & (v_c^1 - c)\partial_x \\ -\partial_x((v_c^1 - c)\cdot) & \mathcal{G}[\eta_c, 1] \end{pmatrix}$$

where

$$Z_c^1 := \frac{\mathcal{G}[\eta_c, 1]\varphi_c + \partial_x\eta_c\partial_x\varphi_c}{1 + |\partial_x\eta_c|^2}, \quad \text{and} \quad v_c^1 = \partial_x\varphi_c - Z_c^1\partial_x\eta_c.$$

This way, the effect of the change of bottom is mainly described by $\mathbb{L}_c^a := \mathbb{L}_c - \mathbb{L}_c^1$, whereas $\mathbb{L}_c^1$ only depends on the solitary wave $\mathbf{Q}_c$. In the same fashion, we will also consider

$$R_c^1 = \begin{pmatrix} 1 & 0 \\ Z_c^1 & 1 \end{pmatrix}$$

and define $\mathbb{A}_c^1$ such that the relation

$$\mathbb{L}_c^1 = \left((R_c^1)^{-1}\right)^t \mathbb{A}_c^1 (R_c^1)^{-1}$$

still holds.

The main appeal of doing this decomposition is the fact that $\mathbb{L}_c^1$ satisfies spectral properties in [14], as it is actually the operator associated to the linearization of the flat bottom problem around its solitary wave $\mathbf{Q}_c$. We present the following coercivity result for $\mathbb{L}_c^1$:

Proposition 5.8 (Ming-Rousset-Tzvetkov [14], Proposition 3.6)**.** *There exists $\varepsilon^* > 0$ such that for every $\varepsilon \in (0, \varepsilon^*]$, we have that for $\boldsymbol{Q}_c = (\eta_c, \varphi_c)$ that there exists $C > 0$ such that for every $\boldsymbol{U} = (\eta, \varphi)^t \in X^0$ satisfying*

$$(\boldsymbol{U}, JR_c^1 \partial_x \boldsymbol{Q}_c) = (\eta, \partial_x \eta_c) = 0, \tag{5.30}$$

$$(\mathbb{L}_c^1 \boldsymbol{U}, \boldsymbol{U}) \geqslant C |\boldsymbol{U}|_{X^0}^2. \tag{5.31}$$

We refer the reader to [14, Subsection 3.2] for the proof of Proposition 5.8. In particular, as first step to the proof, the authors in [14] state the weaker version of Proposition 5.8:

Lemma 5.9. *Let $\boldsymbol{U} = (\eta, \varphi)^t \in X^0$, $\boldsymbol{U} \notin \{(0, r),\ r \in \mathbb{R}\}$, satisfying*

$$(\boldsymbol{U}, J\partial_x \boldsymbol{Q}_c) = (\eta, J\partial_x \eta_c) = 0. \tag{5.32}$$

Then, $(\mathbb{A}_c^1 U, U) > 0$.

Remark 5.8. Notice that, because of the fact that $\left((R_c^1)^{-1}\right)^t J = JR_c^1$, then Lemma 5.9 is equivalent to prove that $(\mathbb{L}_c \mathbf{U}, \mathbf{U}) > 0$ for every $\mathbf{U} \in X^0$, $\mathbf{U} \notin \{(0, r),\ r \in \mathbb{R}\}$ such that (5.30) holds.

As a consequence of Proposition 5.8, one can get the following decomposition for $\mathbf{U} \in X^0$:

Proposition 5.10. *For every $\boldsymbol{U} \in X^0$ there exists a unique decomposition*

$$\boldsymbol{U} = \alpha JR_c^1 \partial_x \boldsymbol{Q}_c + \beta R_c^1 \partial_x \boldsymbol{Q}_c + \boldsymbol{V} \tag{5.33}$$

with $\boldsymbol{V} = (V_1, V_2)^t \in X^0$ such that

$$(\boldsymbol{V}, JR_c^1 \partial_x Q_c) = (V_1, \partial_x \eta_c) = 0. \tag{5.34}$$

Moreover, there exists $c_0 > 0$ and $C > 0$ such that for every $\boldsymbol{U} \in X^0$ written under the form (5.33)*, one has*

$$(\mathbb{L}_c^1 \boldsymbol{U}, \boldsymbol{U}) \geqslant c_0 |\boldsymbol{V}|_{X^0}^2 - C|\alpha|^2.$$

Remark 5.9. By choosing V orthogonal to $JR_c\partial_x Q_c$ and $(\partial_x\eta_c,0)^t$, it is possible to obtain a decomposition from Proposition 5.8. On the other hand, as noted in [16], in the proposition above, $\mathbf{V}$ is not orthogonal to the $R_c\partial_x\mathbf{Q}_c$ and decomposition (5.34) has better properties than the obtained from Proposition 5.8. This is due to the fact that $R_c\partial_x\mathbf{Q}_c$ is in the kernel of $\mathbb{L}_c^1$ while $(\partial_x\eta_c,0)^t$ is not.

We refer to [14, Proposition 3.11] for the proof of Proposition 5.10.

5.3 Error produced by the solitary wave in a non-flat bottom system

In this section, the goal is to analyse the error produced by plugging the solitary wave of the flat-bottom problem in the non-flat regime. More specifically, we shall prove that $\mathbf{Q}_c$ solves system (5.6) plus some residual (exponentially decaying) terms.

From now on, we will consider $\lambda \in (0,\lambda^*)$, so that we can can always assume the existence of the solitary wave

$$\mathbf{Q}_c(x-ct+A) = \left(\eta_c(x-ct+A),\varphi(x-ct+A)\right)^T,$$

in this case, translated a distance $A \gg 1$ form the change of bottom $a(\varepsilon x)\neq 1$ for every $t\leqslant 0$. As mentioned before, this solution satisfies the properties given in Theorem (5.1), meaning that the following estimations hold

$$\begin{aligned}
&\exists \mathrm{d}>0, \quad \forall\alpha\geqslant 0, \quad \exists C_\alpha>0, \quad \forall x\in\mathbb{R}, \quad |\partial_{t,x}^\alpha\eta_c|\leqslant C_\alpha \mathrm{e}^{-\mathrm{d}|x-ct+A|}.\\
&\exists \mathrm{d}>0, \quad \forall\alpha\geqslant 1, \quad \exists C_\alpha>0, \quad \forall x\in\mathbb{R}, \quad |\partial_{t,x}^\alpha\varphi_c|\leqslant C_\alpha \mathrm{e}^{-\mathrm{d}|x-ct+A|}.
\end{aligned} \tag{5.35}$$

From the definition of the Dirichlet-Neumann operator, we know that

$$\mathcal{G}[\eta_c,1]\varphi_c = \sqrt{1+|\partial_x\eta_c|^2}\partial_{\mathbf{n}}\Phi_{c,1}|_{z=\eta_c},$$

where $\Phi_{c,1}$ is the solution to the elliptic equation (5.13) on a flat-bottom regime. It will be convenient to adapt the Laplace equation (5.13) and turn it into an equation in the strip $S=\mathbb{R}\times[-1,0]$, so that both regime (with flat bottom and with slightly changing bottom) can be comparable. With this in mind, using a change of variables, we write

$$\mathcal{G}[\eta_c,1]\varphi_c = \begin{pmatrix}0\\1\end{pmatrix}\cdot P_1\nabla_{x,z}\Phi_{c,1}|_{z=0},$$

where $\Phi_{c,1}=\Phi_{c,1}(x,(H+\eta)z+\eta)$ solves

$$\begin{cases}\nabla_{x,z}\cdot P_1\nabla_{x,z}\Phi_{c,1}=0 \quad \text{in } S\\ \Phi_{c,1}|_{z=0}=\varphi_c\\ \partial_{\mathbf{n}}^{P_1}\Phi_{c,1}|_{z=-1}=0,\end{cases} \tag{5.36}$$

and P_1 is defined as

$$P_1 := \begin{bmatrix} h+\eta_c & -(z+1)\partial_x\eta_c \\ -(z+1)\partial_x\eta_c & \dfrac{1+|(z+1)\partial_x\eta_c|^2}{h+\eta_c} \end{bmatrix}.$$

Following the same idea and implementing the notation used in Subsection (5.2.1), we will write

$$\mathcal{G}[\eta_c, a_\varepsilon] = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot P_a \nabla_{x,z}\Phi_{c,a}|_{z=0},$$

where now $\Phi_{c,a}$ is the solution to the elliptic problem with boundary condition $\Phi_{c,a}(x, z = 0) = \varphi_c$ and a strip domain S (5.5) associated with the non-flat bottom problem, and P_a defined as in (5.15) after exchanging (η, φ) for $\mathbf{Q}_c$.

In this context, let us set

$$r(a) := \mathcal{G}[\eta_c, 1]\varphi_c - \mathcal{G}[\eta_c, a_\varepsilon]\varphi_c. \tag{5.37}$$

Consequently, since $\mathbf{Q}_c$ solves (5.12), we have that

$$\begin{cases} \partial_t\eta_c = \mathcal{G}[\eta_c, a_\varepsilon]\varphi_c + r_1(a) \\ \partial_t\varphi_c = -\frac{1}{2}\left|\partial_x\varphi_c\right|^2 + \frac{1}{2}\dfrac{\left(\mathcal{G}[\eta, 1]\varphi_c + \partial_x\varphi_c\partial_x\eta_c\right)^2}{1+\left|\partial_x\eta_c\right|^2} - g\eta_c + b\partial_x\left(\dfrac{\partial_x\eta_c}{\sqrt{1+\left|\partial_x\eta_c\right|^2}}\right) + r_2(a) \end{cases} \tag{5.38}$$

for

$$r_1(a) = r(a) \quad \text{and}$$
$$r_2(a) = \frac{1}{2}\frac{r(a)\left(\mathcal{G}[\eta_c, 1]\varphi_c + \mathcal{G}[\eta_c, a_\varepsilon]\varphi_c\right) + 2r(a)\partial_x\varphi_c\partial_x\eta_c}{1+|\partial_x\eta_c|^2}. \tag{5.39}$$

It is our goal in this section, then, to prove that the residual terms of the system above (5.38) are, indeed, decaying (exponentially) fast.

Proposition 5.11. *The remainder $\boldsymbol{r}(a) = (r_1(a), r_2(a))^t$ defined in (5.37)-(5.39) has an exponential decay in time. That is, there exist $0 < \delta_0 < \min\{\gamma\epsilon, \delta\}$ and $C_s > 0$ such that for every $s \geqslant 0$,*

$$|\boldsymbol{r}(a)|_{E^s} \leqslant C_s \mathrm{e}^{-\delta_0 A}\mathrm{e}^{\delta_0 ct}, \quad \textit{for all } t \leqslant 0.$$

Proof. First of all, we notice that, because the solution $\mathbf{Q}_c$ of (5.12) satisfies the decay property (5.35), then from the proof of Proposition 5.27 we know that for every $\alpha \in \mathbb{N}$, there exist C_α (depending on α) and $0 < \delta < \mathrm{d}$ (independent of α) such that the following estimates for $\Phi_{c,1}$ hold

$$|\partial^\alpha_{x,z}\Phi_{c,1}| \leqslant C_\alpha \mathrm{e}^{-\delta|x-ct+A|}, \quad \forall (x,z) \in S, \tag{5.40}$$

as stated in Remark 5.5. Since the same estimate holds for $\Phi_{c,a}$, we also have that

$$|\partial^\alpha_{x,z}\Phi_{\mathrm{d}}| \leqslant C_\alpha \mathrm{e}^{-\delta|x-ct+A|}, \quad \forall (x,z) \in S. \tag{5.41}$$

We make the following decomposition for $R(a)$,

$$\begin{aligned} r_1(a) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot P_1 \nabla_{x,z} \Phi_{c,1}|_{z=0} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot P_a \nabla_{c,a} \Phi_{c,a}|_{z=0} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (P_1 - P_a) \nabla_{x,z} \Phi_{c,1}|_{z=0} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot P_a \nabla_{x,z} (\Phi_{c,1} - \Phi_{c,a})|_{z=0}. \end{aligned}$$

From the definition of both P_1 and P_a, we have that the first term takes the form

$$\begin{aligned} &\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (P_1 - P_a) \nabla_{x,z} \Phi_{c,1}|_{z=0} \\ &\qquad = h a_\varepsilon' \partial_x \Phi_{c,1}|_{z=0} + \frac{2(z+1) z h \partial_x \eta_c a_\varepsilon' + |z h a_\varepsilon'|^2}{h a_\varepsilon + \eta_c} \partial_z \Phi_{c,1}|_{z=0}. \end{aligned}$$

Then, just as we did in the proof of Proposition 5.4, we rely on (5.40) to conclude the estimation needed for the first term of $R(a)$. Indeed, we present the following auxiliary lemma:

Lemma 5.12. *For $c > 0$, $\delta > 0$, $\gamma > 0$, $\varepsilon > 0$, and $\delta_0 \in (0, \min\{\gamma\varepsilon, \delta\})$, there exists $C > 0$ such that for every $A \geqslant 0$, $t \leqslant 0$,*

$$\int_{\mathbb{R}} \mathrm{e}^{-\gamma|\varepsilon x|} \mathrm{e}^{-\delta|x-ct+A|} \mathrm{d}x \leqslant C \mathrm{e}^{-\delta_0 A} \mathrm{e}^{\delta_0 ct}.$$

Proof Lemma 5.12. Denoting $\overline{\delta} = \min\{\gamma\varepsilon, \delta\} > 0$, we have that

$$\int_{\mathbb{R}} \mathrm{e}^{-\overline{\delta}|x|} \mathrm{e}^{-\overline{\delta}|x-ct+A|} \mathrm{d}x \leqslant C \mathrm{e}^{-\delta_0 A} \mathrm{e}^{\delta_0 ct}. \tag{5.42}$$

In fact, since $t \leqslant 0$, one can divide the space $\mathbb{R}$ into three subsets:

$$x \leqslant ct - A, \quad ct - A \leqslant x \leqslant 0, \quad x \geqslant 0.$$

We have that

$$\int_{x \leqslant ct-A} \mathrm{e}^{-\overline{\delta}|x|} \mathrm{e}^{-\overline{\delta}|x-ct+A|} \mathrm{d}x = \mathrm{e}^{\overline{\delta}(-ct+A)} \int_{x \leqslant ct-A} \mathrm{e}^{2\overline{\delta}x} \mathrm{d}x = \frac{\mathrm{e}^{-\overline{\delta}(-ct+A)}}{2\overline{\delta}}.$$

The case $x \geqslant 0$ follows from the same arguments. Finally, for the remaining case, we note that if $ct - A \leqslant x \leqslant 0$, then

$$\mathrm{e}^{-\overline{\delta}|x|} \mathrm{e}^{-\overline{\delta}|x-ct+A|} = \mathrm{e}^{\overline{\delta}x} \mathrm{e}^{-\overline{\delta}(x-ct+A)} = \mathrm{e}^{\overline{\delta}(ct-A)}$$

and, therefore, (5.42) follows. ∎

We return to the proof of Proposition 5.11. Taking into account (5.40), as a consequence of Lemma 5.12 we have that, for every $s \geqslant 0$, there exists C_s and δ_0 such that

$$\left| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (P_1 - P_a) \nabla_{x,z} \Phi_{c,1}|_{z=0} \right|_{E^s} \leqslant C_s \mathrm{e}^{-\delta_0 A} \mathrm{e}^{\delta_0 ct}, \quad t \leqslant 0.$$

Now, let us analyze the second term of $r_1(a)$. More precisely, we want to study $\Phi_{\mathrm{d}} = \Phi_{c,1} - \Phi_{c,a}$, which turns out to be the solution of the equation:

$$\begin{cases} \nabla_{x,z} \cdot P_a \nabla_{x,z} \Phi_{\mathrm{d}} = -\nabla_{x,z} \cdot P_a \nabla_{x,z} \Phi_{c,1} = \nabla_{x,z} \cdot (P_1 - P_a) \nabla_{x,z} \Phi_{c,1}, \quad \text{in } S, \\ \Phi_{\mathrm{d}}|_{z=0} = 0, \quad \partial_{\mathbf{n}}^{P_a} \Phi_{\mathrm{d}}|_{z=-1} = \partial_{\mathbf{n}}^{P_a} \Phi_{c,1}|_{z=-1}. \end{cases} \tag{5.43}$$

Taking into account the decay of the solitary wave and of the function describing the bottom, we consider the weight

$$\omega(t,x) = \mathrm{e}^{\overline{\delta}|x|} \mathrm{e}^{\overline{\delta}|x-ct+A|},$$

where $0 < \overline{\delta} < \frac{1}{2} \min\{\varepsilon\gamma, \delta\}$ is to be chosen (sufficiently small) later. Then, multiplying (5.43) by such weight ω, we obtain the equation

$$\begin{cases} \nabla_{x,z} \cdot P_a \nabla_{x,z} (\omega \Phi_{\mathrm{d}}) + [\omega, \nabla_{x,z} \cdot P_a \nabla_{x,z}] \Phi_{\mathrm{d}} = \omega \nabla_{x,z} \cdot (P_1 - P_a) \nabla_{x,z} \Phi_{c,1}, \quad \text{in } S, \\ \Phi_{\mathrm{d}}|_{z=0} = 0, \quad \partial_{\mathbf{n}}^{P_a} \Phi_{\mathrm{d}}|_{z=-1} = \partial_{\mathbf{n}}^{P_a} \Phi_{c,1}|_{z=-1}. \end{cases}$$

Now, we proceed by Divergence Theorem. Certainly, recall that

$$\partial_n^{P_a} = \mathbf{n} \cdot P_a \nabla_{x,z} = h a_\varepsilon' \partial_x + \frac{1 + |h a_\varepsilon'|^2}{h a_\varepsilon + \eta_c} \partial_z \quad \text{on the boundary } z = -1.$$

In fact, notice that when using the notation $\partial_{\mathbf{n}}^{P_1} = \mathbf{n} \cdot P_1 \nabla_{x,z}$, it is straightforward to see that $\partial_{\mathbf{n}}^{P_1} = \frac{1}{h+\eta_c} \partial_z$ on the boundary $z = -1$. Since $\partial_{\mathbf{n}}^{P_1} \Phi_{c,1}|_{z=-1} = 0$, this implies that $\partial_z \Phi_{c,1}|_{z=-1} = 0$. In particular, it means that

$$\begin{aligned} \partial_{\mathbf{n}}^{P_a} (\omega \Phi_{\mathrm{d}})|_{z=-1} &= \partial_{\mathbf{n}}^{P_a} (\omega \Phi_{c,1})|_{z=-1} - h a_\varepsilon' \overline{\delta} \omega \Phi_{c,a}|_{z=-1} - \omega \partial_{\mathbf{n}}^{P_a} (\Phi_{c,a})|_{z=-1} \\ &= h a_\varepsilon' \partial_x (\omega \Phi_{c,1})|_{z=-1} - h a_\varepsilon' \overline{\delta} \omega \Phi_{c,a}|_{z=-1} \\ &= h a_\varepsilon' \overline{\delta} \omega \Phi_{\mathrm{d}}|_{z=-1} + h a_\varepsilon' \omega \partial_x (\Phi_{c,1})|_{z=-1}. \end{aligned}$$

Then, Divergence Theorem gives us the following

$$\begin{aligned} -\int_S \nabla_{x,z} \cdot \Big(P_a \nabla_{x,z} (\omega \Phi_{\mathrm{d}}) \Big) \omega \Phi_{\mathrm{d}} \mathrm{d}x \mathrm{d}z &= \int_S P_a \nabla_{x,z} (\omega \Phi_{\mathrm{d}}) \cdot \nabla_{x,z} (\omega \Phi_{\mathrm{d}}) \, \mathrm{d}x \mathrm{d}z \\ &\quad - \int_{\mathbb{R}} h a_\varepsilon' \Big(\overline{\delta} \omega(x) \Phi_{\mathrm{d}}(t,x,-1) + \omega(x) \partial_x \Phi_{c,1}(t,x,-1) \Big) \omega(x) \Phi_{\mathrm{d}}(t,x,-1) \mathrm{d}x. \end{aligned}$$

Then, since P_a is coercive, we have that there exists $C_0 > 0$ such that

$$\begin{aligned} C_0 \int_S |\nabla_{x,z} (\omega \Phi_{\mathrm{d}})|^2 \mathrm{d}x \mathrm{d}z &\leqslant \int_S \Big([\omega, \nabla_{x,z} \cdot P_a \nabla_{x,z}] \Phi_{\mathrm{d}} \Big) \omega \Phi_{\mathrm{d}} \, \mathrm{d}x \mathrm{d}z \\ &\quad - \int_S \omega \nabla_{x,z} \cdot \Big((P_1 - P_a) \nabla_{x,z} \Phi_{c,1} \Big) \omega \Phi_{\mathrm{d}} \, \mathrm{d}x \mathrm{d}z \\ &\quad + \int_{\mathbb{R}} h a_\varepsilon' \Big(\overline{\delta} \omega(x) \Phi_{\mathrm{d}}(t,x,-1) + \omega(x) \partial_x \Phi_{c,1}(t,x,-1) \Big) \omega(x) \Phi_{\mathrm{d}}(t,x,-1) \mathrm{d}x \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{5.44}$$

Following the arguments of Proposition 5.4, to estimate I_1 we make use of the fact that the depth of the fluid is finite (bounded in the z direction), to apply Poincaré inequality. Indeed, we have that

$$[\omega, \nabla_{x,z} \cdot P_a \nabla_{x,z}] = (ha_\varepsilon + \eta_c) [\omega, \partial_x^2] \\ - ((z+1)\partial_x \eta_c + zha'_\varepsilon) [\omega, \partial_x] \partial_z.$$

This, along with uniform upper bounds for both $|a'|$ and $|\partial_x \eta_c|$ and the existence of $h_{max} > 0$ such that $|ha_\varepsilon + \eta_c| \leqslant h_{max}$ uniformly on (t, x), lead to

$$|I_1| \leqslant C \left(\overline{\delta}^2 \|\omega \Phi_\mathrm{d}\|^2_{L^2(S)^2} + \overline{\delta} \|\omega \partial_z \Phi_\mathrm{d}\|_{L^2(S)} \|\omega \Phi_\mathrm{d}\|_{L^2(S)} \right).$$

Thus, since S si bounded in the z direction, from Poincaré inequality, we have that

$$|I_1| \leqslant C \left(\overline{\delta}^2 + \overline{\delta} \right) \|\nabla_{x,z} (\omega \Phi_\mathrm{d})\|^2_{L^2(S)^2}.$$

Now, for I_2, from the conditions that define the bottom a (5.11) and the decay of $\Phi_{c,1}$ (5.40), we have that there exists $C_\delta > 0$, such that

$$|I_2| \leqslant C_{\overline{\delta}} \left(\overline{\delta} \|\omega \Phi_\mathrm{d}\|_{L^2(S)} + \|\nabla_{x,z} (\omega \Phi_\mathrm{d})\|_{L^2(S)} \right).$$

Indeed, using Divergence Theorem one more time, denoting by $\Gamma = \{z = 0\} \cup \{z = -1\}$ and $\mathbf{n}$ the outward unit normal,

$$I_2 = \int_S (P_1 - P_a) \nabla_{x,z} \Phi_{c,1} \cdot \nabla_{x,z} \left(\omega^2 \Phi_\mathrm{d} \right) \mathrm{d}x \mathrm{d}z - \int_\Gamma \mathbf{n} \cdot ((P_1 - P_a) \nabla_{x,z} \Phi_{c,1}) \omega^2 \Phi_\mathrm{d} \mathrm{d}\Gamma := I_{2,1} + I_{2,2}.$$

We analyse the line integral by writing

$$\int_\Gamma \mathbf{n} \cdot ((P_1 - P_a) \nabla_{x,z} \Phi_{c,1}) \omega^2 \Phi_\mathrm{d} = - \int_{\{z=-1\}} \mathbf{n} \cdot (P_a \nabla_{x,z} \Phi_{c,1}) \omega^2 \Phi_\mathrm{d},$$

where we used the fact that $\Phi_{c,1}$ solves (5.36), which in particular means that $\partial_\mathbf{n}^{P_1} \Phi_{c,1}|_{z=-1} = \partial_z \Phi_{c,1}|_{z=-1} = 0$. Thus, writing $\partial_\mathbf{n}^{P_a}$ in a more explicit fashion, we have that

$$\int_\Gamma \mathbf{n} \cdot \left((P_1 - P_a) \nabla_{x,z} \Phi_{c,1} \omega^2 \Phi_\mathrm{d} \right) = - \int_{\{z=-1\}} \left(ha'_\varepsilon \partial_x \Phi_{c,1} + \frac{1 + |ha'_\varepsilon|^2}{ha_\varepsilon + \eta_c} \partial_z \Phi_{c,1} \right) \omega^2 \Phi_\mathrm{d} \\ = - \int_{\{z=-1\}} ha'_\varepsilon \partial_x \Phi_{c,1} \omega^2 \Phi_\mathrm{d}.$$

Notice that because of the definition of the weight ω, its growth can be controlled by $\partial_x \Phi_{c,1}$ and $a'(\varepsilon x)$. Then, using Cauchy-Schwartz inequality, we have that there exists $C_{\overline{\delta}} > 0$ such that

$$\int_\Gamma \mathbf{n} \cdot \left((P_1 - P_a) \nabla_{x,z} \Phi_{c,1} \omega^2 \Phi_\mathrm{d} \right) \leqslant C_{\overline{\delta}} \|\omega \Phi_\mathrm{d}(z = -1)\|_{L^2(\mathbb{R})}.$$

By Fundamental Theorem of Calculus on the z variable and the fact that on $\{z = 0\}$, $\Phi_\mathrm{d}|_{z=0} = 0$, we conclude

$$I_{2,2} \leqslant C_{\overline{\delta}} \left(\|\omega \Phi_\mathrm{d}\|_{L^2(S)} + \|\nabla_{x,z} (\omega \Phi_\mathrm{d})\|_{L^2(S)} \right).$$

A similar reasoning is applied to the first term of I_2. Let us write

$$I_{2,1} = \int_S (P_1 - P_a)\,\nabla_{x,z}\Phi_{c,1}\omega \cdot \nabla_{x,z}(\omega\Phi_{\mathrm{d}})\,\mathrm{d}x\mathrm{d}z + \overline{\delta}\int_S (P_1 - P_a)\,\nabla_{x,z}\Phi_{c,1}\cdot(\omega,0)^t\,(\omega\Phi_{\mathrm{d}})\,\mathrm{d}x\mathrm{d}z.$$

Since

$$P_1 - P_a = \begin{bmatrix} h(1-a_\varepsilon) & xha'_\varepsilon \\ zha'_\varepsilon & \frac{1+|(z+1)\partial_x\eta_c|^2}{H+\eta_c} - \frac{1+|(z+1)\partial_x\eta_c + zha'_\varepsilon|^2}{ha_\varepsilon+\eta_c} \end{bmatrix},$$

estimating $I_{2,1}$ involves studying not only the interaction between the decaying functions $\partial^\alpha_{x,z}\Phi_{c,1}$ and $a'(\varepsilon x)$ with and the growing weight ω, but also the decay of $1 - a(\varepsilon x)$. Indeed, notice that

$$\begin{aligned} &\left|\frac{1+|(z+1)\partial_x\eta_c|^2}{h+\eta_c} - \frac{1+|(z+1)\partial_x\eta_c + zha'_\varepsilon|^2}{ha_\varepsilon+\eta_c}\right| \\ &\quad = \frac{(z+1)zh\partial_x\eta_c a'_\varepsilon + |zha'_\varepsilon|^2}{ha_\varepsilon+\eta_c} + \frac{\left(1+|(z+1)\partial_x\eta_c|^2\right)h(1-a_\varepsilon)}{(ha_\varepsilon+\eta_c)\,(h+\eta_c)}. \end{aligned}$$

Then, taking into account (5.11) and (5.40), we obtain

$$\begin{aligned} |I_2| &\leqslant C_{\overline{\delta}}\left(\left(\overline{\delta}+1\right)\|\omega\Phi_{\mathrm{d}}\|_{L^2(S)} + \|\nabla_{x,z}(\omega\Phi_{\mathrm{d}})\|_{L^2(S)}\right) \\ &\leqslant \left(\overline{\delta}^2 + \overline{\delta}\right)\|\omega\Phi_{\mathrm{d}}\|^2_{L^2(S)} + \overline{\delta}\|\nabla_{x,z}(\omega\Phi_{\mathrm{d}})\|^2_{L^2(S)} + C_{\overline{\delta}}. \end{aligned}$$

where we used Young inequality for the second inequality (notice that the constant $C_{\overline{\delta}}$ might change from line to line).

Consequently, Poincaré inequality gives us the following

$$|I_2| \leqslant C\left(\overline{\delta}^2 + \overline{\delta}\right)\|\nabla_{x,z}(\omega\Phi_{\mathrm{d}})\|^2_{L^2(S)} + C_{\overline{\delta}}.$$

Finally, for I_3, we have use a similar reasoning as before and obtain

$$\begin{aligned} |I_3| &\leqslant \overline{\delta}^2\|\omega\Phi_{\mathrm{d}}\|^2_{L^2(S)} + \overline{\delta}\|\nabla_{x,z}(\omega\Phi_{\mathrm{d}})\|^2_{L^2(S)} + C_{\overline{\delta}} \\ &\leqslant C\left(\overline{\delta}^2 + \overline{\delta}\right)\|\nabla_{x,z}(\omega\Phi_{\mathrm{d}})\|^2_{L^2(S)} + C_{\overline{\delta}}. \end{aligned}$$

Gathering all estimations computed above, we can conclude from (5.44) that

$$\|\nabla_{x,z}(\omega\Phi_{\mathrm{d}})\|^2_{L^2(S)} \leqslant C\left(\left(\overline{\delta}^2 + \overline{\delta}\right)\|\nabla_{x,z}(\omega\Phi_{\mathrm{d}})\|^2_{L^2(S)} + C_{\overline{\delta}}\right).$$

Therefore, if $\overline{\delta}$ is sufficiently small, we can write

$$\|\nabla_{x,z}(\omega\Phi_{\mathrm{d}})\|^2_{L^2(S)} \leqslant C.$$

As a consequence, after choosing $\overline{\delta} > 0$ sufficiently small, we conclude

$$\|\omega\Phi_{\mathrm{d}}\|_{L^2(S)} \leqslant C \quad\text{and}\quad \|\nabla_{x,z}(\omega\Phi_{\mathrm{d}})\|_{L^2(S)} \leqslant C.$$

As in the proof of Proposition 5.4, for the higher order estimates for $\omega\Phi_{\mathrm{d}}$, we use an induction and argument standard elliptic regularity theory. We can deduce that

$$\forall\alpha,\ \left|\partial^{\alpha}_{x,z}\left(\omega(x)\Phi_{\mathrm{d}}(x,z)\right)\right| \leqslant C_{\alpha}.$$

Equivalently,

$$\forall\alpha,\ |\partial^{\alpha}_{x,z}\Phi_{\mathrm{d}}(x,z)| \leqslant C_{\alpha}\mathrm{e}^{-\bar{\delta}|x|}\mathrm{e}^{-\bar{\delta}|x-ct+A|}.$$

■

5.4 Construction of an approximate solution

In this section, our main goal is to find an approximate solution $\mathbf{U}_{ap}$ of the system (5.6) that behaves asymptotically (as $t \to -\infty$) like the solitary wave of the flat-bottom problem $\mathbf{Q}_c$. In other words, we focus on the construction of a solution $\mathbf{U}_{ap} = \mathbf{Q}_c + \mathbf{V}$ such that is an approximate solution in the sense that

$$\partial_t\mathbf{U}_{ap} = \mathcal{F}(\mathbf{U}_{ap}) + \mathbf{r}_{ap},$$

where both $\mathbf{r}_{ap}$ and $\mathbf{V}$ decay to 0 as $t \to -\infty$ with exponential rate.

We will denote by $\rho = \mathrm{e}^{-\delta_0 A} > 0$, so that ρ becomes smaller for larger $A > 0$. In particular, for the remainder $\mathbf{r}(a)$, obtained when plugging $\mathbf{Q}_c$ into the nonflat-bottom problem, we define $\mathbf{r}_c = \frac{1}{\rho}\mathbf{r}(a)$ and write, for all $s \geqslant 0$,

$$|\mathbf{r}_c|_{E_s} \leqslant C_s\mathrm{e}^{\delta_0 ct} \quad \text{for } t \leqslant 0.$$

Let us define

$$\mathbf{V}(t,x) = \sum_{j=1}^{N}\rho^j\mathbf{V}_j(t,x),$$

for $\mathbf{V}_j$ still unknown (to be constructed) and $N \in \mathbb{N}$. If we make Taylor expansion of $\mathcal{F}$ around the solitary wave, we have that

$$\mathcal{F}(\mathbf{U}) = \mathcal{F}(\mathbf{Q}_c + \mathbf{V}) = \mathcal{F}(\mathbf{Q}_c) + \sum_{j=1}^{N}\frac{1}{j!}D^j\mathcal{F}[\mathbf{Q}_c](\mathbf{V},\ldots,\mathbf{V}) + \mathbf{r}_{N,\delta}(\mathbf{V}) \tag{5.45}$$

where the first derivative of $\mathcal{F}$ is $D\mathcal{F} = J\Lambda[\mathbf{Q}_c]$,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is skew-symmetric, and

$$\Lambda[\mathbf{Q}_c] = \begin{pmatrix} -\mathcal{P}_c + g + Z_c\mathcal{G}[\eta_c,a_\varepsilon](Z_c\cdot) + Z_c\partial_x v_c & v_c\partial_x - Z_c\mathcal{G}[\eta_c,a_\varepsilon] \\ -\partial_x(v_c\cdot) - \mathcal{G}[\eta_c,a_\varepsilon](Z_c\cdot) & G[\eta_c,a_\varepsilon] \end{pmatrix},$$

$$Z_c = \tilde{Z}[\eta_c,\varphi_c] = \frac{\mathcal{G}[\eta_c,a_\varepsilon]\varphi_c + \partial_x\eta_c\partial_x\varphi_c}{1+|\partial_x\eta_c|^2},$$

$$v_c = \tilde{v}[\eta_c,\varphi_c] = \partial_x\varphi_c - Z_c\partial_x\eta_c,$$

and $\mathcal{P}_c = \mathcal{P}[\eta_c, \varphi_c]$ for $\mathcal{P}$ defined as

$$\mathcal{P}[\eta_c, \varphi_c]\psi = b\partial_x \left(\left(1 + (\partial_x \eta_c)^2\right)^{-\frac{3}{2}} \partial_x \psi \right).$$

Then, going back to the equation, using the decomposition (5.45), one gets linear problems for every $\mathbf{V}_k$. Indeed, the system for $\mathbf{V}_1$ is

$$\partial_t \mathbf{V}_1 - J\Lambda[\mathbf{Q}_c]\mathbf{V}_1 = -\mathbf{r}_c.$$

For $\mathbf{V}_2$:

$$\partial_t \mathbf{V}_2 - J\Lambda[\mathbf{Q}_c]\mathbf{V}_2 = \frac{1}{2} D^2 \mathcal{F}[\mathbf{Q}_c](\mathbf{V}_1, \mathbf{V}_1).$$

And for any $\mathbf{V}_j$, $j \in \{2, \dots N\}$,

$$\partial_t \mathbf{V}_j - J\Lambda[\mathbf{Q}_c]\mathbf{V}_j = \sum_{p=1}^{j} \sum_{\substack{1 \leqslant j_1, \dots, j_p \leqslant j-1 \\ j_1 + \dots j_p = j}} \frac{1}{p!} D^p \mathcal{F}[\mathbf{Q}_c](\mathbf{V}_{j_1}, \dots, \mathbf{V}_{j_p}).$$

In this context, we present the main result of this section:

Theorem 5.13. *For every $N \in \mathbb{N}$, there exists*

$$\boldsymbol{U}_{ap} = \boldsymbol{Q}_c + \boldsymbol{V} = \boldsymbol{Q}_c + \sum_{j=1}^{N} \rho^j \, \boldsymbol{V}_j(t, x),$$

where $\boldsymbol{V}_j \in C^\infty(\mathbb{R}, H^\infty(\mathbb{R}))$ such that

$$|\, \boldsymbol{V}_j|_{E^s} \leqslant A^{(2j-1)/4} C_{s,j}(\delta_0) \mathrm{e}^{-j\delta_0 c|t|} \quad \forall t \leqslant 0. \tag{5.46}$$

In addition, $\boldsymbol{U}_{ap}$ is an approximate solution of (5.6) in the sense that the remainder $\boldsymbol{r}_{ap}$ defined as

$$\partial_t \boldsymbol{U}_{ap} - \mathcal{F}(\boldsymbol{U}_{ap}) = \boldsymbol{r}_{ap}$$

satisfies the exponential decay

$$|\boldsymbol{r}_{ap}|_{E^s} \leqslant A^{(2N+1)/4} C_{N,s}(\delta_0) \rho^{N+1} \mathrm{e}^{-(N+1)\delta_0 c|t|} \quad \forall t \leqslant 0.$$

Remark 5.10. We point out that $\rho = \mathrm{e}^{-\delta_0 A}$, which means that $A^{2N+1}\rho^{N+1}$ shall not grow to infinity for a larger A nor for a larger N. In other words, the constant of the decay of the remainder is controlled.

Going back to the linear equation satisfied by $\mathbf{V}_j$, since the source of such equations have exponential decay, we need to study the homogeneous linear equation. More precisely, we need to study (and control) the growth of its fundamental solution. Being able to control such growth would imply the decay of $\mathbf{V}_j$ and, eventually, of the remainder as well.

5.4.1 The homogeneous linear equation

In this subsection, we devote ourselves to the study of the linear homogeneous equation

$$\partial_t \mathbf{V} = J\Lambda[\mathbf{Q}_c]\mathbf{V}. \tag{5.47}$$

We perform the change of variables $\mathbf{U} = R\mathbf{V}$, with

$$R = \begin{pmatrix} 1 & 0 \\ -Z_c & 1 \end{pmatrix}$$

so that we obtain

$$\partial_t \mathbf{U} = JL[\mathbf{Q}_c]\mathbf{U}, \tag{5.48}$$

where

$$L[\mathbf{Q}_c] = \begin{pmatrix} -\mathcal{P}_c + g + w_c & v_c\partial_x \\ -\partial_x(v_c\cdot) & \mathcal{G}[\eta_c, a_\varepsilon] \end{pmatrix}$$

is self adjoint and $w_c = w[\mathbf{Q}_c] = v_c\partial_x Z_c + \partial_t Z_c$. From now on, for the sake of simplicity, we will denote $L_c = L[\mathbf{Q}_c]$. The operators L_c and Λ_c should not be confused with $\mathbb{L}_c$ and $\mathbb{\Lambda}_c$, defined in Subsection 5.2.3. The latters arise when linearizing about Q_c after a change of framework from x to $x - ct$, while the formers correspond to the linearization in the framework x. We also introduce the following notation for $\mathbf{U} = (U_1, U_2)^t$:

$$|\mathbf{U}(t)|^2_{X^k} = \sum_{0\leqslant\alpha,\beta\leqslant k} \left(\left|\partial_t^\alpha\partial_x^\beta U_1(t,\cdot)\right|^2_{H^1(\mathbb{R})} + \left|\partial_t^\alpha\partial_x^\beta U_2(t,\cdot)\right|^2_{\dot{H}^{1/2}_*(\mathbb{R})} \right) \tag{5.49}$$

$$|\mathbf{U}(t)|_{X^k_\infty} = \sup_{\alpha+\beta\leqslant k} \left|\partial_t^\alpha\partial_x^\beta \mathbf{U}(t,\cdot)\right|_{L^\infty}, \tag{5.50}$$

where $|\psi|^2_{\dot{H}^{1/2}_*(\mathbb{R})} = |\mathcal{B}\psi|^2_{L^2(\mathbb{R})}$ and $\mathcal{B} = (1-\partial_x^2)^{-1/4}\partial_x$.

As stated above, the main goal of this subsection is to understand the growth rate for the solution $\mathbf{V}$ of the homogeneous linear equation or, equivalently, the growth for $\mathbf{U}$, solution of (5.48). In this regard, the main result will be the following

Theorem 5.14. *For every $\lambda \in (0,\lambda^*)$, there exists $\overline{A}$ such that for all $A \geqslant \overline{A}$, the solution $\boldsymbol{U}$ of (5.48), satisfy for every $k \geqslant 0$,*

$$|\boldsymbol{U}(t)|_{X^k} + \sum_{\alpha\leqslant k} |\partial_t^\alpha U_2(t)|_{L^2} \leqslant A^{1/2}\left(|\boldsymbol{U}(\tau)|_{X^k} + \sum_{\alpha\leqslant k} |\partial_t^\alpha U_2(\tau)|_{L^2} \right)(1 - c\delta_0(t-\tau))^k \mathrm{e}^{-\delta_0 ct/2}, \tag{5.51}$$

for all $t \leqslant \tau \leqslant 0$.

Before we begin with the proof, we need to define the following partition of unity, so that it is possible to localize on one side the movement of the solitary waves and, on the other partition, the non-flatness of the bottom. Consider $\chi_0 \in C^\infty(\mathbb{R})$ such that

$$\chi(t,x) = \begin{cases} 1 & x \leqslant 0, \\ 0 & x \geqslant \frac{1}{2}. \end{cases}$$

Then, we define

$$
\begin{aligned}
\tilde{\chi}_1(t,x) &= \chi\left(\frac{x-\frac{c}{2}t+\frac{A}{4}}{\frac{A}{4}}\right), \quad \tilde{\chi}_a(t,x) = 1-\tilde{\chi}_1(t,x), \\
\chi_1(t,x) &= \frac{\tilde{\chi}_1}{(\tilde{\chi}_1^2+\tilde{\chi}_a^2)^{1/2}} \quad \chi_a(t,x) = \frac{\tilde{\chi}_a}{(\tilde{\chi}_1^2+\tilde{\chi}_a^2)^{1/2}}.
\end{aligned}
\tag{5.52}
$$

That way, we obtain $\chi_1^2+\chi_a^2=1$. With this partition in mind, from now on we shall make use of the notation

$$\mathbf{U}^1 = \chi_1\mathbf{U} \text{ and } \mathbf{U}^a = \chi_a\mathbf{U}.$$

Also, we have the following (very useful) properties regarding the interaction of a and $\mathbf{Q}_c$ with the partition:

Lemma 5.15. *Functions χ_{i}, $\mathrm{i}=1,a$ defined above satisfy:*

$$\forall\ |\alpha|\geqslant 1,\ \left|\partial_{t,x}^{\alpha}\chi_{\mathrm{i}}(t,x)\right| \lesssim \frac{1}{A^{|\beta|}},\ \mathrm{i}=1,a.$$

Moreover, for any $\delta>0$ we have that

$$\left|\mathrm{e}^{-\delta|x-ct+A|}\chi_a\right| \leqslant \frac{C_\delta}{A}, \quad \textit{and} \quad \left|\mathrm{e}^{-\delta|x|}\chi_1(t,x)\right| \lesssim \frac{1}{A}, \quad \forall t\leqslant 0, A>1,$$

Proof. We notice that $\mathrm{supp}(\chi_a) = \left\{(t,x) \text{ such that } x-\frac{c}{2}t+\frac{A}{4}\geqslant 0\right\}$. Then, since $t\leqslant 0$, $x-ct+A\geqslant x-\frac{c}{2}t+\frac{A}{4}\geqslant 0$. In particular

$$\left|\mathrm{e}^{-\delta|x-ct+A|}\chi_a(t,x)\right| \leqslant \left|\mathrm{e}^{-\delta(x-ct+A)}\right| \leqslant \mathrm{e}^{-\delta A} \lesssim \frac{1}{A}.$$

Similarly, for the second inequality, we notice that $\mathrm{supp}(\chi_1) = \left\{(t,x) \text{ such that } x\leqslant\frac{c}{2}t\right\}$, which means that $x\leqslant 0$ in the support of χ_1 and then implies that

$$\left|\mathrm{e}^{-\delta|x|}\chi_1(t,x)\right| \lesssim \left|\mathrm{e}^{-\delta A/8}\right| \lesssim \frac{1}{A}.$$

∎

Remark 5.11. In our computations, χ_1 will typically be paired with $a'(\varepsilon x)$ (or $1-a'(\varepsilon x)$) to prove decay estimates. Notice that every time we encounter both functions, we will automatically have that

$$|a'(\varepsilon x)\chi_2| \leqslant \frac{C_{\gamma\varepsilon}}{A}.$$

Then, making ε smaller, is similar to taking a larger (safer) distance A between the change of bottom and the solitary wave.

Also, we shall use the following norm equivalence:

Lemma 5.16. *There exists $C>0$ such that for every $A \geqslant 1$, we have that*

$$|\boldsymbol{U}|_{X^0}^2 \leqslant C\left(\sum_{\mathrm{i}=1,2} |\boldsymbol{U}^{\mathrm{i}}|_{X^0}^2 + \frac{1}{A}|\kappa(D)U_2|_{L^2}^2\right),$$

and

$$\sum_{\mathrm{i}=1,a} |\boldsymbol{U}^{\mathrm{i}}|_{X^0}^2 \leqslant C\left(|\boldsymbol{U}|_{X_0}^2 + \frac{1}{A}|\kappa(D)U_2|_{L^2}^2\right),$$

where $\hat{\kappa}(\xi)$ is a smooth cut-off function with $\hat{\kappa}(\xi)=1$ near $\xi=0$.

Proof. For the proof of this Lemma, we refer to [14, Lemma 5.6]. ■

5.4.2 Lower-order estimates

We consider the following energy functional $\mathcal{H}_0(\mathbf{U}) = (L_c\mathbf{U},\mathbf{U}) - c(\mathcal{A}\chi_1\mathbf{U},\chi_1\mathbf{U}) - c(\mathcal{A}\chi_a\mathbf{U},\chi_a\mathbf{U})$, where

$$\mathcal{A} = \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix}.$$

As we agreed in previous subsection, we will denote by $\mathbf{U}^1 = \chi_1\mathbf{U}$ and $\mathbf{U}^a = \chi_a\mathbf{U}$. In the same spirit, let us define

$$Z_c^1 := \frac{\mathcal{G}[\eta_c,1]\varphi_c + \partial_x\eta_c\partial_x\varphi_c}{1+|\partial_x\eta_c|^2}, \quad \text{and} \quad Z_c^a := Z_c - Z_c^1 = \frac{\mathcal{G}[\eta_c,a_\varepsilon]\varphi_c - \mathcal{G}[\eta_c,a_\varepsilon]\varphi_c}{1+|\partial_x\eta_c|^2},$$

so that Z_c^1 depends only on the solitary wave Q_c, whereas all the information of the bottom (and its change) relies on Z_c^a. An additional property that we obtain for such decomposition is that Z_c^1 inherits the decay of the solitary wave $\mathbf{Q}_c$. Indeed from Remark (5.5), we infer that for all $\alpha \in \mathbb{N}\cup\{0\}$, there exists $0<\delta<\mathrm{d}$ such that

$$|\partial_{t,x}^\alpha Z_c^1| \lesssim \mathrm{e}^{-\delta|x-ct+A|}. \tag{5.53}$$

In addition, following the reasoning applied to study the decay properties of $r(a)$ in Section 5.3, we also have that there exists $0<\overline{\delta}<\min\{\gamma\varepsilon,\delta\}$ such that

$$\left|\partial_{t,x}^\alpha Z_c^a\right| \lesssim \mathrm{e}^{-\overline{\delta}|x|}\mathrm{e}^{-\overline{\delta}|x-ct+A|}. \tag{5.54}$$

In a similar fashion, we decompose v_c and w_c as

$$v_c^1 = \partial_x\varphi_c - Z_c^1\partial_x\eta_c, \quad v_c^a = -Z_c^a\partial_x\eta_c,$$
$$w_c^1 = v_c^1\partial_x Z_c^1 + \partial_t Z_c^1, \quad w_c^a = \left(v_c^1+v_c^a\right)\partial_x Z_c^a + v_c^a\partial_x Z_c^1 + \partial_t Z_c^a.$$

Then, decay properties of v_c^1 and v_c^a take after Z_c^1 and Z_c^a, respectively, and the same goes for w_c^1 and w_c^a. Finally, we can also perform the following decomposition of the operator L_c:

$$L_c^1 = \begin{pmatrix} -\mathcal{P}_c+g+w_c^1 & v_c^1\partial_x \\ -\partial_x(v_c^1\cdot) & \mathcal{G}[\eta_c,1] \end{pmatrix}, \quad L_c^a = \begin{pmatrix} w_c^a & v_c^a\partial_x \\ -\partial_x(v_c^a\cdot) & \mathcal{G}[\eta_c,a_\varepsilon]-\mathcal{G}[\eta_c,1] \end{pmatrix}. \tag{5.55}$$

Then, the energy functional $\mathcal{H}_0(\mathbf{U})$ can now be decomposed into terms depending only on the solitary wave and terms that will follow the decay of a'. Indeed, we write

$$\mathcal{H}_1(\mathbf{U}) = (L_c^1\mathbf{U}, \mathbf{U}) + (L_c^a\mathbf{U}, \mathbf{U}) - c(\mathcal{A}\mathbf{U}^1, \mathbf{U}^1) - c(\mathcal{A}\mathbf{U}^a, \mathbf{U}^a).$$

The first step to prove Theorem 5.14 is to give an estimate for the lowest-order terms, that is, the case $k = 0$. It reads

Proposition 5.17. *Under the hypothesis of 5.14, we have the growth estimate:*

$$|\boldsymbol{U}(t)|_{X^0}^2 + |U_2(t)|_{L^2}^2 \leqslant C\left(A^{1/2}|\boldsymbol{U}(\tau)|_{X^0}^2 + |U_2(\tau)|_{L^2}^2\right) \mathrm{e}^{\delta_0 c|t-\tau|/4}, \quad \forall 0 \leqslant \tau \leqslant t.$$

We will prove Proposition 5.17 for the case $\tau = 0$, since it stands for the worst case scenario (if we let time go by sufficiently long, one could even obviate taking A large).

Decay rate (or upper bound) for the energy functional:

We want to show that there exists $C > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_0(\mathbf{U}) \leqslant \frac{C}{A}\left(|\mathbf{U}|_{X_0}^2 + |U_2|_{L^2}^2\right). \tag{5.56}$$

With such goal in mind, let us compute the derivative in time of the energy functional $\mathcal{H}_0(\mathbf{U})$:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_0(\mathbf{U}) = \frac{1}{2}&\left([\partial_t, L_c]\mathbf{U}, \mathbf{U}\right) + (L_c\mathbf{U}, \partial_t\mathbf{U}) - c\left(\mathcal{A}\chi_1(\partial_t\mathbf{U}), \chi_1\mathbf{U}\right)\\ &- c\left(\mathcal{A}\chi_a(\partial_t\mathbf{U}), \chi_a U\right) - c\left(\mathcal{A}(\partial_t\chi_1)\mathbf{U}, \chi_1\mathbf{U}\right) - c\left(\mathcal{A}(\partial_t\chi_a)\mathbf{U}, \chi_a\mathbf{U}\right)\\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.\end{aligned} \tag{5.57}$$

We start with the analysis of I_1 and I_2 and write

$$\begin{aligned}I_1 =& \frac{1}{2}\left([\partial_t, L_c](\chi_1^2 + \chi_a^2)\mathbf{U}, \mathbf{U}\right) = \frac{1}{2}\left([\partial_t, L_c]\chi_1^2\mathbf{U}, \mathbf{U}\right) + \frac{1}{2}\left([\partial_t, L_c]\chi_a^2\mathbf{U}, \mathbf{U}\right)\\ =& \frac{1}{2}\sum_{\mathrm{i}=1,a}\Big(\left([[\partial_t, L_c], \chi_\mathrm{i}]\,\chi_\mathrm{i}\mathbf{U}, \mathbf{U}\right) + \left([\partial_t, L_c^1]\,\chi_\mathrm{i}\mathbf{U}, \chi_\mathrm{i}\mathbf{U}\right) + \left([\partial_t, L_c^2]\,\chi_\mathrm{i}\mathbf{U}, \chi_\mathrm{i}\mathbf{U}\right)\Big).\end{aligned}$$

On the other hand, from (5.48), one gets that

$$I_2 = (L_c\mathbf{U}, \partial_t\mathbf{U}) = (L_c\mathbf{U}, JL_c\mathbf{U}) = 0.$$

In addition, also from (5.48), and since L_c is a self-adjoint operator, we have that

$$\begin{aligned}(\mathcal{A}\chi_1(\partial_t\mathbf{U}), \chi_1\mathbf{U}) &= (\mathcal{A}\chi_1 JL_c\mathbf{U}, \chi_1\mathbf{U}) = -\left(\partial_x(\chi_1 L_c\mathbf{U}), \chi_1\mathbf{U}\right)\\ &= (L_c\chi_1\mathbf{U}, \partial_x(\chi_1\mathbf{U})) + ([\chi_1, L_c]\mathbf{U}, \partial_x(\chi_1\mathbf{U}))\\ &= -\frac{1}{2}\left([\partial_x, L_c]\chi_1\mathbf{U}, \chi_1\mathbf{U}\right) + \left([\chi_1, L_c]\mathbf{U}, \partial_x(\chi_1\mathbf{U})\right).\end{aligned}$$

Now, notice that, since the operator L_c^1 only depends on $\mathbf{Q}_c$ ($\mathbf{Q}_c = Q_c(x-ct+A)$), to derivate in space is actually equivalent to computing a derivative in time:

$$\frac{c}{2}\left(\left[\partial_x, L_c^1\right]\chi_1\mathbf{U}, \chi_1\mathbf{U}\right) = -\frac{1}{2}\left(\left[\partial_t, L_c^1\right]\chi_1\mathbf{U}, \chi_1\mathbf{U}\right).$$

In consequence,

$$I_3 = -\frac{1}{2}\left([\partial_t, L_c^1]\chi_1\mathbf{U}, \chi_1\mathbf{U}\right) + \frac{c}{2}\left([\partial_x, L_c^a]\,\chi_1\mathbf{U}, \chi_1\mathbf{U}\right) - c\left([\chi_1, L_c]\mathbf{U}, \partial_x(\chi_1\mathbf{U})\right).$$

Also, in a similar fashion as for the I_3 term, we have that

$$I_4 = -\frac{1}{2}\left(\left[\partial_t, L_c^1\right]\chi_a\mathbf{U}, \chi_a\mathbf{U}\right) + \frac{c}{2}\left([\partial_x, L_c^a]\,\chi_a\mathbf{U}, \chi_a\mathbf{U}\right) - c\left([\chi_a, L_c]\mathbf{U}, \partial_x(\chi_a\mathbf{U})\right).$$

Therefore, we obtain

$$\begin{aligned} I_1 + I_2 + I_3 + I_4 =& \frac{1}{2}\sum_{\mathrm{i}=1,a}\Big(\left([[\partial_t, L_c], \chi_\mathrm{i}]\,\chi_\mathrm{i}\mathbf{U}, \mathbf{U}\right) + \left([\partial_t, L_c^a]\,\chi_\mathrm{i}\mathbf{U}, \chi_\mathrm{i}\mathbf{U}\right)\Big) \\ &+ \frac{c}{2}\left([\partial_x, L_c^a]\,\chi_1\mathbf{U}, \chi_1\mathbf{U}\right) - c\left([\chi_1, L_c]\mathbf{U}, \partial_x(\chi_1\mathbf{U})\right) \\ &+ \frac{c}{2}\left([\partial_x, L_c^a]\,\chi_a\mathbf{U}, \chi_a\mathbf{U}\right) - c\left([\chi_a, L_c]\mathbf{U}, \partial_x(\chi_a\mathbf{U})\right). \end{aligned} \tag{5.58}$$

Since we are invested in proving (5.56), we need to show that each term of (5.58) satisfy an inequality such as (5.56).

Throughout this part of the proof, to simplify notation, we will write

$$\mathcal{G}_{c,1} := \mathcal{G}[\eta_c, 1] \quad \text{and} \quad \mathcal{G}_{c,a} := \mathcal{G}[\eta_c, a_\varepsilon].$$

We start by noticing that one has the following commutators:

$$[\partial_t, L_c^a] = \begin{pmatrix} \partial_t w_c^a & (\partial_t v_c^a)\,\partial_x \\ -\partial_x\left(\partial_t v_c^a\cdot\right) & [\partial_t, \mathcal{G}_{c,a} - \mathcal{G}_{c,1}] \end{pmatrix}, \quad [\partial_x, L_c^a] = \begin{pmatrix} \partial_x w_c^a & (\partial_x v_c^a)\,\partial_x \\ -\partial_x\left(\partial_x v_c^a\cdot\right) & [\partial_x, \mathcal{G}_{c,a} - \mathcal{G}_{c,1}] \end{pmatrix}$$
$$[\partial_t, L_c] = \begin{pmatrix} -[\partial_t, \mathcal{P}_c] + \partial_t w_c & (\partial_t v_c)\,\partial_x \\ -\partial_x\left(\partial_t v_c\cdot\right) & [\partial_t, \mathcal{G}_{c,a}] \end{pmatrix}.$$

Then, to treat the terms involving the Dirichlet Neumann operators $\mathcal{G}_{c,a}$ and $\mathcal{G}_{c,1}$, it will be useful the following lemma:

Lemma 5.18. *There exists a constant C such that the following estimations hold*

$$\left([\partial_t, \mathcal{G}_{c,a} - \mathcal{G}_{c,1}]\chi_\mathrm{i} U_2, \chi_\mathrm{i} U_2\right) \leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2}^2 + |U_2|_{L^2}^2\right), \tag{5.59}$$

$$\left([\partial_x, \mathcal{G}_{c,a} - \mathcal{G}_{c,1}]\chi_\mathrm{i} U_2, \chi_\mathrm{i} U_2\right) \leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2}^2 + |U_2|_{L^2}^2\right), \tag{5.60}$$

$$\left([[\partial_t, \mathcal{G}_{c,a}], \chi_\mathrm{i}]\,\chi_\mathrm{i} U_2, U_2\right) \leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2}^2 + |U_2|_{L^2}^2\right), \tag{5.61}$$

for $\mathrm{i} = 1, a$, *where* χ_i *are defined in* (5.52) *and* $\mathfrak{B} = (1 + \partial_x^2)^{-\frac{1}{4}}\partial_x$.

For the proof of this result, see Appendix 5.5.

Let us analyze, for instance, the term $([[\partial_t, L_c], \chi_{\mathrm{i}}]\, \chi_{\mathrm{i}}\mathbf{U}, \mathbf{U})$. Writing in a more explicit fashion, we have that

$$([[\partial_t, L_c], \chi_{\mathrm{i}}]\, \chi_{\mathrm{i}}\mathbf{U}, \mathbf{U}) = ([[\partial_t, \mathcal{P}_c]\,, \chi_{\mathrm{i}}]\, \chi_{\mathrm{i}} U_1, U_1) + ([[\partial_t, \mathcal{G}_{c,a}]\,, \chi_{\mathrm{i}}]\, \chi_{\mathrm{i}} U_1, U_1)\,.$$

In particular, since

$$\mathcal{P}_c = b\partial_x \left(\frac{\partial_x \cdot}{(1+|\partial_x \eta_c|^2)^{-3/2}} \right),$$

and because of Lemma 5.15, we have that

$$|([[\partial_t, \mathcal{P}_c]\,, \chi_{\mathrm{i}}]\, \chi_{\mathrm{i}} U_1, U_1)| \leqslant \frac{C}{A} |U_1|_{H^1}^2.$$

Then, thanks to Lemma 5.18, we get that

$$([[\partial_t, L_c], \chi_{\mathrm{i}}]\, \chi_{\mathrm{i}}\mathbf{U}, \mathbf{U}) \leqslant \frac{C}{A} \left(|\mathbf{U}|_{X^0}^2 + |U_2|_{L^2}^2 \right).$$

Similarly, for the terms $([\partial_t, L_c^a]\chi_{\mathrm{i}}\mathbf{U}, \chi_{\mathrm{i}}\mathbf{U})$ we have that

$$\begin{aligned} ([\partial_t, L_c^a]\chi_{\mathrm{i}}\mathbf{U}, \chi_{\mathrm{i}}\mathbf{U}) &= (\partial_t w_c^a \chi_{\mathrm{i}} U_1, \chi_{\mathrm{i}} U_1) + (\partial_t v_c^a \partial_x(\chi_{\mathrm{i}} U_2), \chi_{\mathrm{i}} U_1) \\ &\quad - (\partial_x(\partial_t v_c^a \chi_{\mathrm{i}} U_1), \chi_{\mathrm{i}} U_2) + ([\partial_t, \mathcal{G}_{c,a} - \mathcal{G}_{c,1}]\chi_{\mathrm{i}} U_2, \chi_{\mathrm{i}} U_2)\,. \end{aligned}$$

Thus, taking into account the decay estimate for Z_c^a (5.54), inherited by w_c^a and v_c^a, along with Lemma 5.15, we obtain

$$|([\partial_t, L_c^a]\chi_{\mathrm{i}}\mathbf{U}, \chi_{\mathrm{i}}\mathbf{U})| \leqslant \frac{C}{A} \left(|\mathbf{U}|_{X^0}^2 + |U_2|_{L^2}^2 \right),$$

where we used (5.59) from Lemma 5.18 for the term involving the DN operator. The same way, using (5.60) from Lemma 5.18, we can control $([\partial_t, L_c^a]\chi_{\mathrm{i}}\mathbf{U}, \chi_{\mathrm{i}}\mathbf{U})$ and write

$$\left|\left([\partial_x, L_c^2]\chi_{\mathrm{i}}\mathbf{U}, \chi_{\mathrm{i}}\mathbf{U}\right)\right| \leqslant \frac{C}{A} \left(|\mathbf{U}|_{X^0}^2 + |U_2|_{L^2}^2 \right).$$

The rest of the term from (5.58) are treated in a similar fashion and we conclude

$$I_1 + I_2 + I_3 + I_4 \leqslant \frac{C}{A} \left(|\mathbf{U}|_{X^0}^2 + |U_2|_{L^2}^2 \right).$$

Finally, integrating by parts and making use of Lemma 5.15, we obtain

$$|I_5| + |I_6| \leqslant \frac{C}{A} \left(|U_1|_{H^1} + |U_2|_{L^2} \right).$$

Indeed, for instance, for I_5, we have that

$$\begin{aligned} (\mathcal{A}(\partial_t \chi_1)\mathbf{U}, \chi_1 \mathbf{U}) &= \big(\partial_x\big((\partial_t \chi_1) U_2\big), \chi_1 U_1\big) - \big(\partial_x\big((\partial_t \chi_1) U_1\big), \chi_1 U_2\big) \\ &= -\left((\partial_t \chi_1) U_2, \partial_x(\chi_1 U_1)\right) + \big(\partial_x\big((\partial_t \chi_1) U_1\big), \chi_1 U_2\big)\,. \end{aligned}$$

Then, Lemma 5.15 implies that

$$(\mathcal{A}(\partial_t\chi_1)\mathbf{U}, \chi_1\mathbf{U}) \leqslant \frac{C}{A}\left(|U_1|_{H^1}^2 + |U_2|_{L^2}^2\right).$$

Consequently, we proved that (5.56).

Lower bounds for the energy functional: By definition of the partition of unity (5.52), we have that

$$(L_c\mathbf{U}, \mathbf{U}) = \sum_{\mathrm{i}=1,a} \Big((L_c\chi_{\mathrm{i}}\mathbf{U}, \chi_{\mathrm{i}}\mathbf{U}) + ([L_c, \chi_{\mathrm{i}}]\chi_{\mathrm{i}}\mathbf{U}, \mathbf{U}) \Big).$$

Thus, we can write

$$\mathcal{H}_0(\mathbf{U}) = \sum_{\mathrm{i}=1,a} \Big((L_c\chi_{\mathrm{i}}\mathbf{U}, \chi_{\mathrm{i}}\mathbf{U}) + ([L_c, \chi_{\mathrm{i}}]\chi_{\mathrm{i}}\mathbf{U}, \mathbf{U}) - c(\mathcal{A}\chi_{\mathrm{i}}\mathbf{U}, \chi_{\mathrm{i}}\mathbf{U}) \Big) = I_1 + I_a + I_R,$$

where, using the notation $\chi_1\mathbf{U} = \mathbf{U}^1$ and $\chi_a\mathbf{U} = \mathbf{U}^a$, along with the decomposition (5.55), I_1, I_2, I_R are defined as

$$I_R := \sum_{\mathrm{i}=1,a} ([L_c, \chi_{\mathrm{i}}]\chi_{\mathrm{i}}\mathbf{U}, \mathbf{U}) + \left(L_c^a\mathbf{U}^1, \mathbf{U}^1\right) + \left(L_c^a\mathbf{U}^a, \mathbf{U}^a\right),$$
$$I_1 := \left(L_c^1\mathbf{U}^1, \mathbf{U}^1\right) - c\left(\mathcal{A}\mathbf{U}^1, \mathbf{U}^1\right) \text{ and } I_a := \left(L_c^1\mathbf{U}^1, \mathbf{U}^a\right) - c\left(\mathcal{A}\mathbf{U}^a, \mathbf{U}^a\right).$$

We begin by estimating I_R. Let us take first, for instance, $I_{R,1} := \left(L_c^a\mathbf{U}^1, \mathbf{U}^1\right)$. Notice that the function χ_1 is constructed so it follows the movement of the solitary wave $\mathbf{Q}_c$, while the operator L_c^a inherits its decay from a'. In this case, we set $t \to -\infty$, that is, we are under the assumption that the solitary wave does not encounter the change of bottom yet. Hence, χ_1 (its support) is moving away from the regime in which the change of bottom happens. In particular, this means that the term $I_{1,R}$ can actually be seen as a residual terms. More specifically, from the definition of L_c^a one has

$$I_{R,1} = \left(w_c^a U_1^1, U_1^1\right) + \left(v_c^a\partial_x U_2^1, U_1^1\right) - \left(\partial_x(v_c^a U_1^1), U_2^1\right) + \left((\mathcal{G}_{c,a} - \mathcal{G}_{c,1})U_2^1, U_2^1\right).$$

Thus, to deal with the term involving the DN operator, we present the following lemma proved in the Appendix 5.5:

Lemma 5.19. *There exists a constant C such that the following estimation holds*

$$\left|\left((\mathcal{G}_{c,a} - \mathcal{G}_{c,1})U_2^{\mathrm{i}}, U_2^{\mathrm{i}}\right)\right| \leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2}^2 + |U_2|_{L^2}^2\right), \quad \mathrm{i} = 1, a.$$

Then, Lemma 5.15 and Lemma 5.19 imply that

$$|I_{R,a,1}| \leqslant \frac{C}{A}\left(|\mathbf{U}|_{X^0}^2 + |U_2|_{L^2}^2\right).$$

On the other hand, to study $I_{R,a} := (L_c^1\mathbf{U}^a, \mathbf{U}^a)$, notice that the decay of Z_c^a (5.54) follows the behaviour of both the changing bottom a' and the solitary wave $\mathbf{Q}_c$. Moreover, such decay is inherited by v_c^a and w_c^a. In particular we have that, for $\alpha \in \mathbb{N} \cup \{0\}$,

$$\left|\partial_{t,x}^\alpha v_c^a\right| \lesssim \mathrm{e}^{-\overline{\delta}|x-ct+A|} \quad \text{and} \quad \left|\partial_{t,x}^\alpha w_c^a\right| \lesssim \mathrm{e}^{-\overline{\delta}|x-ct+A|}.$$

Hence, Lemma 5.19 also entails the estimation for $I_{R,a}$,

$$|I_{R,a}| \leqslant \frac{C}{A}\left(|\mathbf{U}|^2_{X^0} + |U_2|^2_{L^2}\right).$$

Finally, from Lemma 5.15, we have that

$$|([L_c, \chi_{\mathrm{i}}]\chi_{\mathrm{i}}\mathbf{U}, \mathbf{U})| \leqslant \frac{C}{A}\left(|\mathbf{U}|^2_{X^0} + |U_2|^2_{L^2}\right), \quad \mathrm{i} = 1, a,$$

which ultimately leads to

$$|I_R| \leqslant \frac{1}{A}C\left(|\mathbf{U}|^2_{X^0} + |U_2|^2_{L^2}\right). \tag{5.62}$$

Next, we deal with $\left(L^1_c\mathbf{U}^1, \mathbf{U}^1\right)$. Since L^1_c depends only on the solitary wave $\mathbf{Q}_c = \mathbf{Q}_c(x - ct + A)$, we notice again that to derivate in time Z^1_c is actually computing a derivative in space. In fact, we have that $\partial_t Z^1_c = -c\partial_x Z^1_c$. Taking this into account and writing w^1_c in a more explicit manner, we find

$$\begin{aligned}
L^1_c &= \begin{pmatrix} -\mathcal{P}_c + g + w^1_c & v^1_c\partial_x \\ -\partial_x(v^1_c\cdot) & \mathcal{G}_{c,1} \end{pmatrix} \\
&= \begin{pmatrix} -\mathcal{P}_c + g + v^1_c\partial_x Z^1_c + \partial_t Z^1_c & v^1_c\partial_x \\ -\partial_x(v^1_c\cdot) & \mathcal{G}_{c,1} \end{pmatrix} \\
&= \begin{pmatrix} -\mathcal{P}_c + g + (v^1_c - c)\partial_x Z^1_c & v^1_c\partial_x \\ -\partial_x(v^1_c\cdot) & \mathcal{G}_{c,1} \end{pmatrix}.
\end{aligned}$$

Thus, going back to our computations, one can write

$$I_1 = \left(\tilde{L}^1_c\mathbf{U}^1, \mathbf{U}^1\right)$$

where

$$\tilde{L}^1_c = \begin{pmatrix} -\mathcal{P}_c + g + (v^1_c - c)\partial_x Z^1_c & (v^1_c - c)\partial_x \\ -\partial_x((v^1_c - c)\cdot) & \mathcal{G}_{c,1} \end{pmatrix}.$$

Notice that the operator $\tilde{L}^1_c$ is actually written as $\mathbb{L}^1_c$ from Subsection 5.2.3, with the difference that the coefficients in $\tilde{L}^1_c$ depend on $Q_c = Q_c(x - ct + A)$. That is, if $\mathcal{T}_{x_0}$ is the translation operator,

$$\mathcal{T}_{x_0}\mathbf{U}(x) = \mathbf{U}(x + x_0)$$

then we have that

$$\mathcal{T}_{ct-A}\tilde{L}^1_c = \mathbb{L}^1_c\mathcal{T}_{ct-A}$$

Now, since $\mathcal{T}_{x_0}$ is an isometry in L^2 and X_0, Proposition 5.8 and Proposition 5.10 still hold for $\tilde{L}^1_c$. Then, for $\overline{\mathbf{U}}^1$ defined as

$$\overline{\mathbf{U}}^1(t, y) = \left(\mathcal{T}_{ct-A}\mathbf{U}^1\right)(t, y) = \chi_1(t, y + ct - A)\mathbf{U}(t, y + ct - A),$$

we have the unique decomposition given by Proposition 5.10

$$\overline{\mathbf{U}}^1(t, y) = \alpha^1(t)JR^1_c\mathbf{Q}'_c(y) + \beta^1(t)R^1_c\mathbf{Q}'(y) + \mathbf{W}^1(t, y)$$

where we denoted by $\mathbf{Q}_c'(x) = \partial_x Q_c(x)$ (standing wave) and $\mathbf{W}^1 \in X^0$ such that

$$(\mathbf{W}^1, JR_c^1\mathbf{Q}_c') = (W_1^1, \eta_c') = 0.$$

Moreover, there exists $k > 0$ and $C > 0$ such that for every $\mathbf{U} \in X^0$ written under the form (5.33), one has

$$\left(\mathbb{L}_c^1\overline{\mathbf{U}}^1, \overline{\mathbf{U}}^1\right) \geqslant k|\mathbf{W}^1|_{X^0}^2 - C|\alpha^1|^2.$$

Consequently, since we have that

$$\left(\tilde{L}_c^1\mathbf{U}^1, \mathbf{U}^1\right) = \left(\mathcal{T}_{-ct+A}\mathbb{L}_c^1\mathcal{T}_{ct-A}\mathbf{U}^1, \mathbf{U}^1\right) = \left(\mathbb{L}_c^1\mathcal{T}_{ct-A}\mathbf{U}^1, \mathcal{T}_{ct-A}\mathbf{U}^1\right) = \left(\mathbb{L}_c^1\overline{\mathbf{U}}^1, \overline{\mathbf{U}}^1\right),$$

we conclude the following

$$I_1 \geqslant k|\mathbf{W}^1|_{X^0}^2 - C|\alpha^1|^2. \tag{5.63}$$

In a similar way, for

$$\overline{\mathbf{U}}^a(t,y) = \mathcal{T}_{ct-A}\mathbf{U}^a(t,y) = \chi_a(y+ct-A)\mathbf{U}(t, y+ct-A)$$

we also have that

$$\overline{\mathbf{U}}^a(t,y) = \alpha^a(t)JR_c\mathbf{Q}_c'(y) + \beta^a(t)R_cQ_c'(y) + \mathbf{W}^a(t,y)$$

where $\mathbf{W}^a \in X^0$ satisfies

$$(\mathbf{W}^a, JR_cQ_c') = (W_1^a, \eta_c') = 0.$$

We obtain

$$I_2 \geqslant k|\mathbf{W}^a|_{X^0}^2 - C|\alpha^a|^2. \tag{5.64}$$

Then, gathering estimations (5.63), (5.64) and (5.62), one obtains

$$\mathcal{H}_0(\mathbf{U}) \geqslant k\left(|\mathbf{W}^1|_{X_0}^2 + |\mathbf{W}^a|_{X_0}^2\right) - C\left(|\alpha^1|^2 + |\alpha^a|^2\right) - \frac{\tilde{C}}{A}\left(|\mathbf{U}|_{X_0}^2 + |U_2|_{L^2}^2\right) \tag{5.65}$$

To finish the proof, we are left to compute estimations for $|U_2|_{L^2}$, α^{i} and β^{i}, $\mathrm{i} = 1, a$.

We start with $|U_2|_{L^2}$. Since $\mathbf{U}$ solves (5.48), we can write

$$\partial_t U_2 = \left(\mathcal{P}_c - w_c - g\right)U_1 - v_c\partial_x U_2.$$

In view of Lemma 5.16, let us choose $\kappa(D)$, $\kappa \in C_0^\infty(\mathbb{R})$ such that $\kappa(\xi) = 1$ around $\xi = 0$. Then, we have that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\kappa(D)U_2|_{L^2}^2 &= (\kappa(D)\partial_t U_2, \kappa(D)U_2)\\ &= \left(\kappa(D)\left(\mathcal{P}_c - w_c - g\right)U_1, \kappa(D)U_2\right) - \left(\kappa(D)v_c\partial_x U_2, \kappa(D)U_2\right).\end{aligned}$$

Given the fact that κ compactly supported and taking into account the defintion of $\mathcal{P}_c$ and a_c, this implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\kappa(D)U_2|_{L^2}^2 \leqslant C\left(|U_1|_{H^1} + |\mathfrak{B}U_2|_{L^2}\right)\left(|\mathfrak{B}U_2|_{L^2} + |\kappa(D)U_2|_{L^2}\right)$$

Thus, from Young inequality, for some $\vartheta > 0$ (to be chosen latter),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\kappa(D)U_2|^2_{L^2} \leqslant C\left(\frac{1}{\vartheta}|\mathbf{U}|_{X^0} + \vartheta|\kappa(D)U_2|_{L^2}\right) \tag{5.66}$$

We focus now on $|\alpha^{\mathrm{i}}|$, $\mathrm{i} = 1, 2$. Note that if if $\mathbf{U}$ is any function X^0 such that it can be decomposed as (5.33)-(5.34), then α and β are determined by

$$\alpha = \frac{(\mathbf{U}, JR^1_c\mathbf{Q}'_c)}{|JR^1_c\mathbf{Q}'_c|^2_{L^2}}, \quad \beta = \frac{(\mathbf{U}, (\eta'_c, 0)^t)}{(R^1_c\mathbf{Q}'_c, (\eta'_c, 0)^t)} - \alpha\left(JR^1_c\mathbf{Q}'_c, (\eta'_c, 0)^t\right) \tag{5.67}$$

In particular, for α_1, using the decomposition of $\mathbf{U}^1$ and the fact that $\mathbf{U}$ solves, (5.48), we find that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\alpha^1 &= \frac{1}{|JR^1_c\mathbf{Q}'_c|^2_{L^2}}\partial_t\left(\overline{\chi}_1(t)\overline{\mathbf{U}}(t), JR^1_c\mathbf{Q}'_c\right) \\
&= \frac{1}{|JR^1_c\mathbf{Q}'_c|}\Big(\left(\partial_t\overline{\chi}_1(t)\overline{\mathbf{U}}(t), JR^1_c\mathbf{Q}'_c\right) + \left(\overline{\chi}_1(t)(J\overline{L_c\mathbf{U}} + c\overline{\partial_x\mathbf{U}})(t), JR^1_c\mathbf{Q}'_c\right)\Big) \\
&= \frac{1}{|JR^1_c\mathbf{Q}'_c|}\Bigg(\left(\partial_t\overline{\chi}_1(t)\overline{U}(t), JR^1_c\mathbf{Q}'_c\right) + \left(\overline{\chi}_1(t)J\overline{L^a_c\mathbf{U}}(t), JR^1_c\mathbf{Q}'_c\right)\Bigg) \\
&\quad + \left(\overline{\chi}_1(t)(J\overline{L^1_c\mathbf{U}} + c\overline{\partial_x\mathbf{U}})(t), JR^1_c\mathbf{Q}'_c\right)\Bigg),
\end{aligned}$$

where we used the notation $\overline{f}(t,x) = f(t, x + ct - A)$. In addition, from the definition of $\tilde{L}^1_c$,

$$\begin{aligned}
&\left(\overline{\chi}_1(t)(J\overline{L^1_c\mathbf{U}} + c\overline{\partial_x\mathbf{U}})(t), JR^1_c\mathbf{Q}'_c\right) \\
&\quad = \left(\overline{\chi}_1(t)(J\overline{L^1_c\mathbf{U}} - cJ\overline{\partial_x\mathbf{U}})(t), JR^1_c\mathbf{Q}'_c\right) \\
&\quad = \left(\overline{\chi}_1(t)J\tilde{L}^1_c[\mathbf{Q}_c(y)]\overline{\mathbf{U}}(t), JR^1_c\mathbf{Q}'_c\right) \\
&\quad = \left(\Big[\overline{\chi}_1(t), J\tilde{L}^1_c[\mathbf{Q}_c(y)]\Big]\overline{\mathbf{U}}(t), JR^1_c\mathbf{Q}'_c\right) - \left(\overline{\chi}_1(t)\overline{\mathbf{U}}(t), \tilde{L}^1_c[\mathbf{Q}_c(y)]J^2R^1_c\mathbf{Q}'_c\right).
\end{aligned}$$

Note that

$$\left(\overline{L^1_c} - cJ\partial_x\right)J^2R^1_c\mathbf{Q}'_c = -\tilde{L}^1_cR^1_c\mathbf{Q}'_c = -L^{c,1}R^1_c\mathbf{Q}'_c = 0.$$

Thus,

$$\left(\overline{\chi}_1(t)(J\overline{L^1_c\mathbf{U}} + c\overline{\partial_x\mathbf{U}})(t), JR^1_c\mathbf{Q}'_c\right) = \left(\Big[\overline{\chi}_1(t), J\tilde{L}^1_c[\mathbf{Q}_c(y)]\Big]\overline{\mathbf{U}}(t), JR^1_c\mathbf{Q}'_c\right).$$

Then, thanks to Lemma 5.15, we conclude

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\alpha^1\right| \leqslant \frac{C}{A}\left(|\mathbf{U}|_{X^0} + |U_2|_{L^2}\right). \tag{5.68}$$

Following similar arguments, we also have for α^a,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\alpha^a\right| \leqslant \frac{C}{A}\left(|\mathbf{U}|_{X^0} + |U_2|_{L^2}\right). \tag{5.69}$$

At last, we propose to find estimates for β^{i}. From (5.67), we write

$$\beta^{\mathrm{i}} = \tilde{\beta}^{\mathrm{i}} - \alpha^{\mathrm{i}}\left(JR_c^1\mathbf{Q}_c', (\eta_c', 0)^t\right), \quad \mathrm{i} = 1, a,$$

where

$$\tilde{\beta}^{\mathrm{i}} = \frac{\left(\overline{\mathbf{U}}^{\mathrm{i}}, (\eta_c', 0)^t\right)}{(R_c^1\mathbf{Q}_c', (\eta_c', 0)^t)} = \frac{\left(\overline{\mathbf{U}}^{\mathrm{i}}, (\eta_c', 0)^t\right)}{|\eta_c'|_{L^2}^2}, \quad \mathrm{i} = 1, a.$$

Then, since (5.68) and (5.69) hold,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\beta^{\mathrm{i}}\right| \leqslant \left|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\beta}^{\mathrm{i}}\right| + \frac{C}{A}\left(|\mathbf{U}|_{X^0} + |U_2|_{L^2}\right), \quad \mathrm{i} = 1, a.$$

To estimate β^1, we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\beta}^1 &= \frac{\partial_t\left(\overline{\chi}_1(t)\overline{\mathbf{U}}(t), (\eta_c', 0)^t\right)}{|\eta_c'|_{L^2}^2} \\
&= \frac{(\partial_t\overline{\chi}_1(t)\mathbf{U}(t), (\eta_c', 0)^t)}{|\eta_c'|_{L^2}^2} + \frac{\left(\overline{\chi}_1(t)\left(\overline{\partial_t\mathbf{U}}(t) + c\overline{\partial_x\mathbf{U}}(t)\right), (\eta_c', 0)^t\right)}{|\eta_c'|_{L^2}^2} \\
&= \frac{(\partial_t\overline{\chi}_1(t)\mathbf{U}(t), (\eta_c', 0)^t)}{|\eta_c'|_{L^2}^2} + \frac{\left(\overline{\chi}_1(t)J\overline{L_c\mathbf{U}}(t), (\eta_c', 0)^t\right)}{|\eta_c'|_{L^2}^2} + c\frac{\left(\overline{\chi}_1(t)\overline{\partial_x\mathbf{U}}(t), (\eta_c', 0)^t\right)}{|\eta_c'|_{L^2}^2}.
\end{aligned}$$

In particular, we have that

$$\begin{aligned}
\left(\chi_1(t)J\overline{L_c\mathbf{U}}(t), (\eta_c', 0)^t\right) &= \left(\chi_1(t)J\overline{L_c^1\mathbf{U}}(t), (\eta_c', 0)^t\right) + \left(\chi_1(t)J\overline{L_c^a\mathbf{U}}(t), (\eta_c', 0)^t\right) \\
&= \left(\left[\chi_1(t), J\overline{L_c^1}\right]\overline{\mathbf{U}}(t), (\eta_c', 0)^t\right) + \left(J\overline{L_c^1}\overline{\mathbf{U}}^1(t), (\eta_c', 0)^t\right) \\
&\quad + \left(\chi_1(t)J\overline{L_c^a\mathbf{U}}(t), (\eta_c', 0)^t\right).
\end{aligned}$$

Also,

$$c\left(\overline{\chi}_1(t)\overline{\partial_x\mathbf{U}}(t), (\eta_c', 0)^t\right) = c\left(\partial_x\overline{\mathbf{U}}^1(t), (\eta_c', 0)^t\right) - c\left(\partial_x\overline{\chi}_1(t)\overline{\mathbf{U}}(t), (\eta_c', 0)^t\right)$$

Therefore, we have that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\beta}_1(t)\right| \leqslant C\left|\left(J\tilde{L}_c^1\overline{\mathbf{U}}^1, (\eta_1', 0)^t\right)\right| + \frac{C}{A}\left(|\mathbf{U}|_{X_0} + |U_2|_{L^2}\right).$$

To conclude, we use the decomposition of $\overline{\mathbf{U}}^1$ and the fact that $\tilde{L}_c^1R_c^1\mathbf{Q}_c' = 0$ and find that

$$\left|\left(J\tilde{L}_c^1\overline{\mathbf{U}}^1, (\eta_1', 0)^t\right)\right| \leqslant C\left(|\alpha^1| + |\mathbf{W}^1|_{X^0}\right).$$

Gathering all the estimations above, one obtains

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\beta}^1(t)\right| \leqslant C\left(|\alpha^1| + |\mathbf{W}^1|_{X^0} + \frac{1}{A}\left(|\mathbf{U}|_{X_0} + |U_2|_{L^2}\right)\right),$$

which, ultimately implies that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\beta^1(t)\right| \leqslant C\left(|\alpha^1| + |\mathbf{W}^1|_{X^0} + \frac{1}{A}\left(|\mathbf{U}|_{X_0} + |U_2|_{L^2}\right)\right). \tag{5.70}$$

Similarly, for β^a,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\beta^a(t)\right| \leqslant C\left(|\alpha^1| + |\mathbf{W}^1|_{X^0} + \frac{1}{A}\left(|\mathbf{U}|_{X_0} + |U_2|_{L^2}\right)\right). \tag{5.71}$$

Then, putting together (5.68), (5.69), (5.70) and (5.71), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\alpha^{\mathrm{i}}|^2 \leqslant C|\alpha^{\mathrm{i}}|\frac{1}{A}\left(|\mathbf{U}|_{X_0} + |U_2|_{L^2}\right), \quad \mathrm{i} = 1, a, \tag{5.72}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}|\beta^{\mathrm{i}}|^2 \leqslant C|\beta^{\mathrm{i}}|\left(\frac{1}{A}\left(|\mathbf{U}|_{X_0} + |U_2|_{L^2}\right) + |\alpha^{\mathrm{i}}| + |\mathbf{W}^{\mathrm{i}}|_{X_0}\right), \quad \mathrm{i} = 1, a. \tag{5.73}$$

End of the proof of Proposition 5.17:

To make the appropriate estimations, considering (5.56), (5.72) and (5.73), it will be convenient to define a somewhat different weighted energy

$$\tilde{\mathcal{H}}_0(\mathbf{U}(t)) = \frac{1}{2}A^{1/2}\mathcal{H}_0(\mathbf{U}(t)) + CA^{1/2}\left(|\alpha^1(t)|^2 + |\alpha^a(t)|^2\right) + \frac{1}{2}\left(|\beta^1(t)|^2 + |\beta^a(t)|^2\right).$$

We derivate in time $\tilde{\mathcal{H}}_0$ and obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{H}}_0(\mathbf{U}(t)) \leqslant \frac{C}{A^{1/4}}\Bigg[&\frac{1}{A^{1/4}}\left(|\mathbf{U}|^2_{X_0} + |U_2|^2_{L^2}\right) + \frac{1}{A^{1/4}}\left(|\alpha^1(t)| + |\alpha^a(t)|\right)\left(|\mathbf{U}|_{X_0} + |U_2|_{L^2}\right)\\ &+ \frac{1}{A^{3/4}}\left(|\beta^1(t)| + |\beta^a(t)|\right)\left(|\mathbf{U}|_{X_0} + |U_2|_{L^2}\right)\\ &+A^{1/4}\left(|\beta^1(t)| + |\beta^a(t)|\right)\left(|\alpha^1| + |\alpha^a| + |\mathbf{W}^1|_{X_0} + |\mathbf{W}^2|_{X_0}\right)\Bigg]\end{aligned}$$

Thanks to the decomposition of $\overline{U}^{\mathrm{i}}$ and Lemma 5.16, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{H}}_0(\mathbf{U}(t)) \leqslant &\frac{C}{A^{1/4}}\Big[A^{1/2}\left(|\mathbf{W}^1|^2_{X_0} + |\mathbf{W}^a|^2_{X_0}\right) + \frac{1}{A^{1/4}}|U_2|^2_{L^2}\\ &+ \left(|\beta^1|^2 + |\beta^a|^2\right) + A^{1/2}\left(|\alpha^1|^2 + |\alpha^a|^2\right)\Big].\end{aligned}$$

Next, we integrate in time over the time interval $[0,t]$, where $t \leqslant 0$, and we write

$$\tilde{\mathcal{H}}_0(\mathbf{U}(t)) \leqslant \tilde{\mathcal{H}}_0(\mathbf{U}(0)) - \frac{C}{A^{1/4}}\int_t^0 F_0(\tau) + \frac{1}{A^{1/4}}|\kappa(D)U_2(\tau)|_{L^2}\mathrm{d}\tau, \tag{5.74}$$

where

$$F_0(t) := A^{1/2}\left(|\mathbf{W}^1|^2_{X_0} + |\mathbf{W}^2|^2_{X_0}\right) + \left(|\beta^1|^2 + |\beta^2|^2\right) + A^{1/2}\left(|\alpha^1|^2 + |\alpha^2|^2\right).$$

On the other hand, from (5.65) and Lemma 5.16,

$$\begin{aligned}\tilde{\mathcal{H}}_0(\mathbf{U}(t)) \geqslant &\frac{1}{2}A^{1/2}k\left(|\mathbf{W}^1|^2_{X_0} + |\mathbf{W}^2|^2_{X_0}\right) + \frac{1}{2}A^{1/2}C\left(|\alpha^1(t)|^2 + |\alpha^2(t)|^2\right)\\ &+ \frac{1}{2}\left(|\beta^1(t)|^2 + |\beta^2(t)|^2\right) - \frac{1}{A^{1/2}}\tilde{C}\left(\sum_{\mathrm{i}=1,2}|\mathbf{U}^{\mathrm{i}}|^2_{X_0} + |\kappa(D)U_2|^2_{L^2}\right)\\ \geqslant &\,\tilde{c}F_0(t) - \frac{1}{A^{1/2}}\tilde{C}\left(|\kappa(D)U_2|^2_{L^2}\right).\end{aligned}$$

Gathering this inequality with (5.74), we deduce

$$F_0(t) - \frac{1}{A^{1/2}}\tilde{C}|\kappa(D)U_2|_{L^2}^2 \leqslant A^{1/2}|\mathbf{U}(0)|_{X^0}^2 - \frac{C}{A^{1/4}}\int_t^0 F_0(\tau) + \frac{1}{A^{1/4}}\left(|\kappa(D)U_2(\tau)|_{L^2}\right)\mathrm{d}\tau$$

In addition, as a consequence of (5.66), after integrating over the time interval $[0,t]$, $t \leqslant 0$, we find that

$$|\kappa(D)U_2(t)|_{L^2}^2 \leqslant |U_2(0)|_{L^2}^2 - C\int_t^0 \frac{1}{\vartheta}F_0(\tau) + \vartheta|\kappa(D)U_2(\tau)|_{L^2}\mathrm{d}\tau,$$

where $\vartheta > 0$ is to be chosen appropriately later. Then, for any $\upsilon > 0$, this leads to

$$\begin{aligned} F_0(t) + \left(\upsilon - \frac{\tilde{C}}{A^{1/2}}\right)|\kappa(D)U_2|_{L^2}^2 \leqslant & A^{1/2}|U(0)|_{X^0}^2 + \upsilon|U_2(0)|_{L^2}^2 \\ & - C\upsilon\int_t^0 \frac{1}{\vartheta}F_0(\tau) + \vartheta|\kappa(D)U_2(\tau)|_{L^2}\mathrm{d}\tau \\ & - \frac{C}{A^{1/4}}\int_t^0 F_0(\tau) + \frac{1}{A^{1/4}}\left(|\kappa(D)U_2(\tau)|_{L^2}\right)\mathrm{d}\tau. \end{aligned}$$

Let us choose υ and ϑ in order to satisfy the following:

$$\upsilon - A^{-1/2}C \geqslant \upsilon/2, \quad CA^{-1/4} \leqslant \delta_0 c/4, \quad A^{-1/4} \leqslant k/2, \quad \upsilon C\vartheta^{-1} \leqslant \delta_0 c/4 \quad \text{and} \quad \vartheta^2 \leqslant \upsilon/2,$$

where δ_0 is the constant that arises in Proposition 5.11. For instance, one can take

$$\upsilon = \frac{\delta_0 c}{64C^2}, \quad \vartheta = \frac{\sqrt{\upsilon}}{2} \quad \text{and} \quad A \gg 1,$$

to finally obtain

$$\begin{aligned} F_0(t) + \frac{\upsilon}{2}|\kappa(D)U_2|_{L^2}^2 &\leqslant CA^{1/2}|\mathbf{U}(0)|_{X^0}^2 + \upsilon|U_2(0)|_{L^2}^2 - \frac{\delta_0 c}{4}\int_t^0 F_0(\tau) + \frac{\upsilon}{2}|\kappa(D)U_2(\tau)|_{L^2}^2\mathrm{d}\tau \\ &\leqslant CA^{1/2}|\mathbf{U}(0)|_{X^0}^2 + \upsilon|U_2(0)|_{L^2}^2 + \frac{\delta_0 c}{4}\int_t^0 F_0(\tau) + \frac{\upsilon}{2}|\kappa(D)U_2(\tau)|_{L^2}^2\mathrm{d}\tau. \end{aligned}$$

Then, using Gronwall inequality, we get

$$F_0(t) + \frac{\upsilon}{2}|\kappa(D)U_2(t)|_{L^2}^2 \leqslant \left(CA^{1/2}|\mathbf{U}(0)|_{X^0}^2 + \upsilon|U_2(0)|_{L^2}^2\right)\mathrm{e}^{-\delta_0 ct/4}, \quad t \leqslant 0.$$

Then, from the definition of F, the decomposition of $\overline{\mathbf{U}}^{\mathrm{i}}$ and Lemma 5.16, we know that there exists $\overline{\upsilon}$ and $\tilde{\upsilon}$ such that

$$F_0(t) \geqslant \overline{\upsilon}\sum_{\mathrm{i}=1,a}|\mathbf{U}^{\mathrm{i}}(t)|_{X^0}^2 \geqslant \left(1 - \frac{C}{A}\right)\tilde{\upsilon}|\mathbf{U}(t)|_{X^0}^2 - \frac{\overline{\upsilon}}{A}|\kappa(D)U_2(t)|_{L^2}^2.$$

by choosing A large enough so that $\overline{\upsilon}/A \leqslant \upsilon/4$, we find

$$F_0(t) \geqslant \frac{1}{2}\tilde{\upsilon}|\mathbf{U}|_{X_0}^2 - \frac{\upsilon}{4}|\kappa(D)U_2|_{L^2}^2.$$

Then, we can conclude that

$$\frac{1}{2}\tilde{\upsilon}|\mathbf{U}|_{X_0}^2 - \frac{\upsilon}{4}|\kappa(D)U_2|_{L^2}^2 \leqslant \left(CA^{1/2}|\mathbf{U}(0)|_{X^0}^2 + \upsilon|U_2(0)|_{L^2}^2\right)\mathrm{e}^{-\delta_0 ct/4}, \quad \forall t \leqslant 0.$$

which finishes the proof of Proposition 5.17.

5.4.3 Proof of Theorem 5.13

We proceed by induction. We begin the proof with the case $k = 1$ and then rely on this analysis to explain the higher-order cases.

First-order energy estimates: To compute the derivates of the linear system (5.48), we shall make use of the following operator

$$D(\partial) = \partial_t + c\partial_x$$

so that we can take advantage of the properties of L_c^a. The reason why we take such operator is the fact that the solitary wave terms (precisely, L_c^1) get cancelled by $D(\partial)$ (such terms are evaluated in the argument $x - ct$). We apply this operator to both sides of the equation (5.48) and obtain

$$\partial_t D(\partial)\mathbf{U} - JL_c D(\partial)\mathbf{U} = [\partial_t, D(\partial)]\,\mathbf{U} - J\,[L_c, D(\partial)]\,\mathbf{U}. \tag{5.75}$$

In a more explicit fashion, taking into account the partition of unity (5.52), we also have that

$$[\partial_t, D(\partial)]\,\mathbf{U} = \sum_{\mathrm{i}=1,a} c(\partial_t \chi_{\mathrm{i}}^2)\partial_x \mathbf{U},$$

as well as

$$\begin{aligned} J\,[L_c, D(\partial)]\,\mathbf{U} &= J\,[L_c, \partial_t]\,\mathbf{U} + Jc\,[L_c, \partial_x]\,\mathbf{U} \\ &= J \sum_{\mathrm{i}=1,a} \Big(\chi_{\mathrm{i}}^2\left[L_c^1, \partial_t\right]\mathbf{U} + \chi_{\mathrm{i}}^2\,[L_c^a, \partial_t]\,\mathbf{U} + c\left[L_c, \chi_{\mathrm{i}}^2\right]\partial_x \mathbf{U} \\ &\qquad + \chi_{\mathrm{i}}^2 c\left[L_c^1, \partial_x\right]\mathbf{U} + \chi_{\mathrm{i}}^2 c\,[L_c^a, \partial_x]\,\mathbf{U}\Big) \end{aligned}$$

As we pointed out before, from the decomposition (5.55), we know that L_c^1 depends only on the solitary wave, and then we get

$$J\sum_{\mathrm{i}=1,a}\Big(\chi_{\mathrm{i}}^2\left[L_c^1, \partial_t\right]\mathbf{U} + \chi_{\mathrm{i}}^2 c\left[L_c^1, \partial_x\right]\mathbf{U}\Big) = J\sum_{\mathrm{i}=1,2}\chi_{\mathrm{i}}^2\left[L_c^1, \partial_t + c\partial_x\right]\mathbf{U} = 0.$$

This implies that

$$J\,[L_c, D(\partial)]\,\mathbf{U} = J\sum_{\mathrm{i}=1,a}\Big(\chi_{\mathrm{i}}^2\,[L_c^a, \partial_t]\,\mathbf{U} + c\left[L_c, \chi_{\mathrm{i}}^2\right]\partial_x\mathbf{U} + \chi_{\mathrm{i}}^2 c\,[L_c^a, \partial_x]\,\mathbf{U}\Big) := J\mathcal{S}\mathbf{U}.$$

Then, going back to (5.75), we obtain

$$\partial_t D(\partial)\mathbf{U} = JL_c D(\partial)\mathbf{U} + [\partial_t, D(\partial)]\,\mathbf{U} - J\mathcal{S}\mathbf{U} := JL_c D(\partial)\mathbf{U} + F_1(\mathbf{U}), \tag{5.76}$$

where $F_1(\mathbf{U}) := [\partial_t, D(\partial)]\,\mathbf{U} - J\mathcal{S}\mathbf{U}$ is the source term of the linear equation. In particular, using Duhamel Formula, the fact that we consider F_1 as the source means that

$$D(\partial)\mathbf{U}(t) = S_c(t,0)(D(\partial)U)(0) - \int_t^0 S_c(t,\tau)F_1(\tau)\mathrm{d}\tau,$$

where $S_c(t,\tau)$ here is the fundamental solution of the linear equation (5.48), and, because of Proposition 5.17, it satisfies

$$\|S_c(t,\tau)\|_{X^0\cap L^2\to X^0\cap L^2} \leqslant A^{1/2} C \mathrm{e}^{-\delta_0 ct/4}, \quad \forall 0 \leqslant \tau \leqslant t.$$

Consequently, we have the following

$$\begin{aligned} |D(\partial)\mathbf{U}(t)|_{X^0} + |D(\partial)U_2(t)|_{L^2} \leqslant CA^{1/2}\Bigg(&\mathrm{e}^{-\delta_0 ct/4}\Big(|\mathbf{U}(0)|_{X^1} + \sum_{\alpha=0,1} |\partial_t^\alpha U_2(0)|_{L^2}\Big) \\ &- \int_t^0 \mathrm{e}^{\delta_0 c(\tau-t)/4}\left(|F_1(\tau)|_{X^0} + |F_1(\tau)|_{L^2}\right) \mathrm{d}\tau\Bigg). \end{aligned} \tag{5.77}$$

Then, we need to estimate the source F_1.

For any norm $\|\cdot\|$ depending on the variable x, we shall use the notation

$$\|\langle\partial_t\rangle^k \mathbf{U}\| = \sum_{0\leqslant l\leqslant k} \|\partial_t^l \mathbf{U}\|.$$

To have some control on the higher order space derivatives in (5.48) using time derivatives, it will be useful the following lemma:

Lemma 5.20. *Any smooth solution of* (5.48) *satisfies the following a priori estimates*

$$\forall l \geqslant 0,\ m \geqslant 0,\ \exists C_{k,l},\ \textit{such that}\ |\partial_t^l (U_1,U_2)|_{H^{m+5/2}\times H^{m+2}} \leqslant C_{l,k} |\langle\partial_t\rangle^{l+1}(U_1,U_2)|_{H^{m+1}\times H^{m+1/2}}.$$

The proof of this lemma can be found in the Appendix.

Remark 5.12 (Consequences of Lemma)**.** Lemma (5.20) implies the following two estimations:

1. For any $\delta > 0$ we have that

$$|\partial_x \mathbf{U}|_{X^0} \leqslant \delta\left(|\partial_t \mathbf{U}|_{X^0} + |\partial_t U_2|_{L^2}\right) + C_\delta\left(|\mathbf{U}|_{X^0} + |\mathbf{U}|_{L^2}\right). \tag{5.78}$$

2. For any $\alpha,\ \beta$ such that $\alpha+\beta = k$, $\beta \geqslant 1$, and $\delta > 0$,

$$|\partial_t^\alpha \partial_x^\beta \mathbf{U}|_{X^0} \leqslant \delta\left(|\partial_t^k \mathbf{U}|_{X^0} + |\partial_t^k U_2|_{L^2}\right) + C_\delta\left(|\mathbf{U}|_{X^{k-1}} + |\langle\partial_t\rangle^k U_2|_{L^2}\right). \tag{5.79}$$

The first inequality can be deduced by taking $l = 0$ and $m = 0$,

$$|\partial_x (U_1,U_2)|_{H^{3/2}\times H^1} \leqslant |(U_1,U_2)|_{H^{5/2}\times H^2} \leqslant C_{l,k}|\langle\partial_t\rangle^1 (U_1,U_2)|_{H^1\times H^{1/2}},$$

and then using the interpolation inequality

$$|\partial_x \mathbf{U}|_{H^1\times H^{1/2}} \leqslant \delta |\partial_x \mathbf{U}|_{H^{3/2}\times H^1} + C_\delta |\mathbf{U}|_{L^2}. \tag{5.80}$$

for any $\delta > 0$. The second estimation can be deduced from Lemma 5.20 by choosing $l = \alpha$, $m = \beta - 1$

$$|\langle\partial_t\rangle^{\alpha}(U_1, U_2)|_{H^{\beta+3/2}\times H^{\beta+1}} \leqslant C|\langle\partial_t\rangle^{\alpha+1}(U_1, U_2)|_{H^{\beta}\times H^{\beta-1/2}}.$$

We iterate the last step until we get

$$|\partial_t^{\alpha}\partial_x^{\beta}(U_1, U_2)|_{H^{3/2}\times H^1} \leqslant |\langle\partial_t\rangle^{\alpha}(U_1, U_2)|_{H^{\beta+3/2}\times H^{\beta+1}} \leqslant C_k|\langle\partial_t\rangle^{k}(U_1, U_2)|_{H^{1}\times H^{1/2}}.$$

We conclude (5.79) after we use again the interpolation inequality (5.80).

Now, we proceed with the estimation of F_1. From its definition, we have that

$$\begin{aligned}|F_1(\mathbf{U})|_{X^0} &\leqslant |J\mathcal{S}\mathbf{U}|_{X^0} + |\left[\partial_t, D(\partial)\right]\mathbf{U}|_{X^0} \\ &\leqslant |J\sum_{\mathrm{i}=1,a}\chi_{\mathrm{i}}^2\left[L_c^a, \partial_t\right]\mathbf{U}|_{X^0} + |J\sum_{\mathrm{i}=1,a}c\left[L_c, \chi_{\mathrm{i}}^2\right]\partial_x\mathbf{U}|_{X^0} \\ &\quad + |J\sum_{\mathrm{i}=1,a}\chi_{\mathrm{i}}^2 c\left[L_c^a, \partial_x\right]\mathbf{U}|_{X^0} + |\left[\partial_t, D(\partial)\right]\mathbf{U}|_{X^0}\end{aligned}$$

First, using Lemma 5.18 and Lemma 5.15 (the same way we did for the 0-order estimates),

$$|F_1(\mathbf{U})|_{X^0} \leqslant \frac{C}{A}\left(|(\partial_x U_2, \partial_x^2 U_1)^t|_{X^0} + |\partial_x\mathbf{U}|_{X^0} + |\mathbf{U}|_{X^0}.\right)$$

Next, from Lemma 5.20, we obtain

$$|F_1(\mathbf{U})|_{X^0} \leqslant \frac{C}{A}\left(|\partial_t\mathbf{U}|_{X^0} + |\mathbf{U}|_{X^0} + |\partial_t U_2|_{L^2} + |U_2|_{L^2}.\right)$$

We also deduce that

$$|F_1(\mathbf{U})|_{L^2} \leqslant \frac{C}{A}\left(|\partial_t\mathbf{U}|_{X^0} + |\mathbf{U}|_{X^0} + |\partial_t U_2|_{L^2} + |U_2|_{L^2}.\right)$$

Thus, replacing such estimation in (5.77) and using (5.78)

$$\begin{aligned}|D(\partial)U(t)|_{X^0} + |D(\partial)U(t)|_{L^2} &\leqslant CA^{1/4}\mathrm{e}^{-\delta_0 ct/4}\Big(|U(0)|_{X^0} + |U(0)|_{L^2} + |\partial_t U(0)|_{L^2}\Big) \\ &\quad - \frac{C}{A^{3/4}}\int_t^0 \mathrm{e}^{\delta_0 c(\tau-t)/4}\Big(|\partial_t\mathbf{U}(\tau)|_{X^0} + |\mathbf{U}(\tau)|_{X^0} + |\partial_t U_2(\tau)|_{L^2} + |U_2(\tau)|_{L^2}\Big)\mathrm{d}\tau.\end{aligned}$$

From Proposition 5.17,

$$|\mathbf{U}(\tau)|_{X_0}^2 + |U_2(\tau)|_{L^2}^2 \leqslant CA^{1/2}\left(|\mathbf{U}(0)|_{X^0}^2 + |U_2(0)|_{L^2}^2\right)\mathrm{e}^{\delta_0 c\tau/4} \quad \forall 0 \leqslant \tau \leqslant t,$$

we conclude that

$$\begin{aligned}|\partial_t\mathbf{U}(t)|_{X^0} + |\partial_t\mathbf{U}(t)|_{L^2} &\leqslant CA^{1/4}(1 - A^{-3/4}t)\mathrm{e}^{-\delta_0 ct/4}\Big(|\mathbf{U}(0)|_{X^0} + |\mathbf{U}(0)|_{L^2} + |\partial_t U_2(0)|_{L^2}\Big) \\ &\quad - \frac{C}{A^{3/4}}\int_t^0 \mathrm{e}^{\delta_0 c(\tau-t)/4}\Big(|\partial_t\mathbf{U}(\tau)|_{X^0} + |\partial_t U_2(\tau)|_{L^2}\Big)\mathrm{d}\tau.\end{aligned}$$

Then, using Gronwall's inequality, we have that there exists $\overline{C} > 0$ such that

$$|\partial_t \mathbf{U}(t)|_{X^0} + |\partial_t U_2(t)|_{L^2} \leqslant CA^{1/4}(1 - A^{-3/4}t)\mathrm{e}^{-\delta_0 ct/4 - \overline{C}A^{-3/4}t}\left(|\mathbf{U}(0)|_{X^0} + \sum_{\alpha=0,1} |\partial_t^\alpha U_2(0)|_{L^2}\right).$$

The desired estimation can be deduced by taking A sufficiently large. Finally, in view of (5.78)

$$|\partial_x U|_{X^0} \leqslant \delta\left(|\partial_t U|_{X^0} + |\partial_t U_2|_{L^2}\right) + C_\delta\left(|U|_{X^0} + |U_2|_{L^2}\right).$$

and taking into account the estimation for the zero-order estimates given in Proposition 5.17, we conclude the result for the first-order case.

Higher-order energy estimates. We proceed by induction argument. Indeed, given any $k \geqslant 0$, we will assume that the estimate (5.51) holds for $k-1$-order case. As we did for the first order case, we start estimating $|D^k(\partial)\mathbf{U}|_{X^0}$. We have that

$$\partial_t D^k(\partial)\mathbf{U} = JL_c D^k(\partial) + F_k(\mathbf{U})$$

where the source term $F_k(U)$ is given by

$$F_k(\mathbf{U}) = \sum_{\mathrm{i}=0}^{k-1} D^{\mathrm{i}}(\partial)\left(J\left[D(\partial), L_c\right] + \left[\partial_t, D(\partial)\right]\right) D^{k-1-\mathrm{i}}(\partial)\mathbf{U}.$$

In particular, using Duhamel Formula, the fact that we consider F_k as the source means that

$$D^k(\partial)\mathbf{U}(t) = S_c(t,0)(D^k(\partial)\mathbf{U})(0) - \int_t^0 S_c(t,\tau)F_k(\mathbf{U})(\tau)\mathrm{d}\tau,$$

where $S_c(t,\tau)$ is considered again the fundamental solution of the linear equation (5.48). Hence, because of Proposition 5.17, we write

$$\begin{aligned}|D^k(\partial)\mathbf{U}(t)|_{X^0} + |D^k(\partial)\mathbf{U}(t)|_{L^2} \leqslant CA^{1/4}\Bigg(&\mathrm{e}^{-\delta_0 ct/4}\Big(|\mathbf{U}(0)|_{X^k} + \sum_{\alpha\leqslant k} |\partial_t^\alpha U_2(0)|_{L^2}\Big)\\ &- \int_t^0 \mathrm{e}^{\delta_0 c(\tau - t)/4}\left(|F_k(\mathbf{U})(\tau)|_{X^0} + |F_k(\mathbf{U})(\tau)|_{L^2}\right)\mathrm{d}\tau\Bigg).\end{aligned} \tag{5.81}$$

Then, we need to estimate the source F_k.

From the definition of F_k, we have that

$$\begin{aligned}&|F_k(\mathbf{U})|_{X^0}\\ &\quad\leqslant \sum_{\mathrm{i}=0}^{k-1}\left(|D^{\mathrm{i}}(\partial)\left(J\left[D(\partial), L_c\right]D^{k-1-\mathrm{i}}(\partial)\mathbf{U}|_{X^0} + |\left[\partial_t, D(\partial)\right]\right)D^{k-1-\mathrm{i}}(\partial)\mathbf{U}|_{X^0}\right)\\ &\quad\leqslant \sum_{\mathrm{i}=0}^{k-1}\sum_{j=1,a}\left(|\chi_j^2 JD^{\mathrm{i}}(\partial)\left[L_c^a, \partial_t\right]D^{k-1-\mathrm{i}}(\partial)\mathbf{U}|_{X^0} + c|JD^{\mathrm{i}}(\partial)\left[L_c, \chi_j^2\right]D^{k-1-\mathrm{i}}(\partial)\mathbf{U}|_{X^0}\right)\\ &\qquad + \sum_{\mathrm{i}=0}^{k-1}\left(\sum_{j=1,a} c|\chi_j^2 JD^{\mathrm{i}}(\partial)\left[L_c^a, \partial_x\right]D^{k-1-\mathrm{i}}(\partial)\mathbf{U}|_{X^0} + |D^{\mathrm{i}}(\partial)\left[\partial_t, D^k(\partial)\right]D^{k-1-\mathrm{i}}(\partial)\mathbf{U}|_{X^0}\right).\end{aligned}$$

Making a similar analysis as the previous from the first order estimatives,

$$|F_k(\mathbf{U})|_{X^0} + |F_k(\mathbf{U})|_{L^2} \leqslant \frac{C_k}{A} \sum_{\substack{\alpha+\beta=k \\ \alpha\neq k-1}} \left(|(\partial_t^\alpha \partial_x^\beta U_2, \partial_t^\alpha \partial_x^{\beta+1} U_1)^t|_{X^0} + |\partial_t^\alpha \partial_x^\beta \mathbf{U}|_{X^0}\right) + \frac{C}{A}|\mathbf{U}|_{X^{k-1}}.$$

Next, using (5.79),

$$|\partial_t^\alpha \partial_x^\beta \mathbf{U}|_{X^0} \leqslant \delta\left(|\partial_t^k \mathbf{U}|_{X^0} + |\partial_t^k U_2|_{L^2}\right) + C_\delta\left(|\mathbf{U}|_{X^{k-1}} + |\langle\partial_t\rangle^k U_2|_{L^2}\right),$$

$$|F_k(\mathbf{U})|_{X^0} + |F_k(\mathbf{U})|_{L^2} \leqslant \frac{C_k}{A}\left(|\partial_t^k \mathbf{U}|_{X^0} + |\mathbf{U}|_{X^{k-1}} + \sum_{\alpha\leqslant k} |\partial_t^\alpha U_2|_{L^2}\right).$$

Plugging such estimation into (5.81)

$$\begin{aligned}|D^k(\partial)\mathbf{U}(t)|_{X^0} + |D^k(\partial)\mathbf{U}(t)|_{L^2} &\leqslant CA^{1/4}\mathrm{e}^{-\delta_0 ct/4}\Big(|\mathbf{U}(0)|_{X^k} + \sum_{\alpha\leqslant k}|\partial_t^\alpha U_2(0)|_{L^2}\Big) \\ &- \frac{C_k}{A^{3/4}}\int_t^0 \mathrm{e}^{\delta_0 c(\tau-t)/4}\left(|\partial_t^k \mathbf{U}(\tau)|_{X^0} + |\mathbf{U}(\tau)|_{X^{k-1}} + \sum_{\alpha\leqslant k}|\partial_t^\alpha U_2(\tau)|_{L^2}\right)\mathrm{d}\tau.\end{aligned} \tag{5.82}$$

We claim that

$$|D^k(\partial)\mathbf{U}(t)|_{X^0} + |D^k(\partial)\mathbf{U}(t)|_{L^2} \geqslant |\partial_t^k \mathbf{U}(t)|_{X^0} + |\partial_t^k U_2(t)|_{L^2} - C\sum_{\alpha\leqslant k-1}|\partial_t^\alpha U(t)|_{L^2} + C|\mathbf{U}(t)|_{X^{k-1}}. \tag{5.83}$$

Indeed, this comes from (5.79) and the fact that

$$|D^k(\partial)\mathbf{U}(t)|_{X^0} + |D^k(\partial)U_2(t)|_{L^2} \geqslant |\partial_t^k \mathbf{U}(t)|_{X^0} + |\partial_t^k U_2(t)|_{L^2} - C\sum_{\substack{\alpha+\beta\leqslant k \\ \alpha\leqslant k-1}}|\partial_t^\alpha \partial_x^\beta U(t)|_{X^0} + C|\mathbf{U}(t)|_{X^{k-1}}.$$

Now, because we are using induction argument, we have that the desired estimation for the $k-1$ holds, that is, we have that

$$\begin{aligned}&|\mathbf{U}(t)|_{X^{k-1}} + \sum_{\alpha\leqslant k-1}|\partial_t^\alpha U_2(t)|_{L^2} \\ &\quad\leqslant CA^{1/4}\mathrm{e}^{-\delta_0 ct/4}\Big(|\mathbf{U}(0)|_{X^k} + \sum_{\alpha\leqslant k}|\partial_t^\alpha U_2(0)|_{L^2}\Big) \\ &\quad\quad - \frac{C_k}{A^{3/4}}\int_t^0 \mathrm{e}^{\delta_0 c(\tau-t)/4}\left(|\partial_t^k \mathbf{U}(\tau)|_{X^0} + |\mathbf{U}(\tau)|_{X^{k-1}} + \sum_{\alpha\leqslant k}|\partial_t^\alpha U_2(\tau)|_{L^2}\right)\mathrm{d}\tau.\end{aligned} \tag{5.84}$$

Going back to (5.82), (5.84) and (5.83) imply that

$$\begin{aligned}&|\partial_t^k \mathbf{U}(t)|_{X^0} + |\partial_t^k U_2(t)|_{L^2} \\ &\quad\leqslant C_{k-1}A^{1/4}(1 - A^{-3/4}t)^{k-1}\mathrm{e}^{-\delta_0 ct/4 - A^{-3/4}\overline{C}t}\Big(|\mathbf{U}(0)|_{X^k} + \sum_{\alpha\leqslant k}|\partial_t^\alpha U_2(0)|_{L^2}\Big) \\ &\quad\quad - \frac{C_k}{A^{3/4}}\int_t^0 \mathrm{e}^{\delta_0 c(\tau-t)/4}\left(|\partial_t^k \mathbf{U}(\tau)|_{X^0} + |\partial_t^k U_2(\tau)|_{L^2}\right)\mathrm{d}\tau.\end{aligned}$$

Then, thanks to Gronwall's inequality and Lemma 5.20, by taking A large enough, we conclude the proof.

5.4.4 Proof of Theorem 5.13

Now, we can go back to the construction of the approximate solution. We consider

$$\mathbf{V}(t,x) = \sum_{j=1}^{N} \rho^j \mathbf{V}_j(t,x),$$

with $\rho = \mathrm{e}^{-\delta_0 A} > 0$. Here $\mathbf{V}_1$ solves the system

$$\partial_t \mathbf{V}_1 - J\Lambda[\mathbf{Q}_c]\mathbf{V}_1 = -\mathbf{r}_c.$$

where, recall from Section 5.3, that for all $k \geqslant 0$,

$$|\mathbf{r}(a))|_{E_k} \leqslant C_k \mathrm{e}^{\delta_0 ct} \quad \forall\ t \leqslant 0.$$

Notice that, since $\mathbf{U}(t) = R\mathbf{V}(t)$ and R is invertible, from Theorem 5.14 we get

$$|S_c^\Lambda(t,\tau)\mathbf{V}|_{E^k} \leqslant A^{1/4} C_k(\delta_0)|\mathbf{V}|_{H^{s(k)}} \left(1 + |t-\tau|^k\right) \mathrm{e}^{\delta_0 c|t-\tau|/2}, \quad \forall t,\tau \geqslant 0,$$

where here S_c^Λ is the fundamental solution of the system (5.47). With this in mind, we choose the solution

$$\mathbf{V}_1(t,x) = -\int_{-\infty}^{t} S_c^\Lambda(t,\tau)\mathbf{r}_c(\tau)\mathrm{d}\tau.$$

From the estimation of the fundamental solution S_c^Λ, we get that $\mathbf{V}_1$ is well defined and satisfies

$$\begin{aligned}
|\mathbf{V}_1(t)|_{E^k} &\leqslant A^{1/4} C_k \int_{-\infty}^{t} (1 + |t-\tau|)^k \mathrm{e}^{\delta_0 c|t-\tau|/2} \sum_{j=0}^{k} \|\partial_t^j \mathbf{r}_c\|_{H^{s(k-1)}} \mathrm{d}\tau \\
&\leqslant A^{1/4} C_k \int_{-\infty}^{t} (1 + |t-\tau|)^k \mathrm{e}^{\delta_0 c|t-\tau|/2} \mathrm{e}^{-\delta_0 c|\tau|/2} \mathrm{d}\tau \\
&\leqslant A^{1/4} C_k(\delta_0) \mathrm{e}^{-\delta_0 c|t|} \quad \forall t \leqslant 0.
\end{aligned}$$

For the general case

$$\partial_t \mathbf{V}_j - D\mathcal{F}[\mathbf{Q}_c]\mathbf{V}_j = \sum_{p=1}^{j} \sum_{\substack{1 \leqslant j_1,\ldots,j_p \leqslant j-1 \\ j_1+\ldots j_p = j}} \frac{1}{p!} D^p \mathcal{F}[\mathbf{Q}_c](V_{j_1},\ldots,\mathbf{V}_{j_p}) := \mathbf{r}_j(t,x)$$

we use induction argument. Suppose that for every j such that $1 \leqslant j \leqslant l-1$,

$$|\mathbf{V}_j|_{E^k} \leqslant A^{(2j-1)/4} C_{k,j}(\delta_0) \mathrm{e}^{-j\delta_0 c|t|} \quad \forall t \leqslant 0.$$

For the source terms, we get the estimation

$$|\mathbf{r}_j(t,x)|_{E^k} \leqslant A^{(2l-1)/4} C_{k,j}(\delta_0) \mathrm{e}^{-j\delta_0 c|t|} \quad \forall t \leqslant 0.$$

Then, taking the solution

$$\mathbf{V}_j(t,x) = -\int_{-\infty}^{t} S_c^\Lambda(t,\tau)\mathbf{r}_j(\tau)\mathrm{d}\tau,$$

we conclude the proof of Theorem 5.13.

5.5 Construction of the exact solution

Now that the approximate solution of the Zakharov water waves system (5.6) is constructed, we need to find the exact solution $\mathbf{U} = \mathbf{U}_{ap} + \mathbf{U}_r$ from the constructed approximate solution $\mathbf{U}_{ap} = (\eta_{ap}, \varphi_{ap})$, where $\mathbf{U}_r$ is a remainder to be determined. In other words, for $\mathbf{U} = (\eta, \varphi)^t$ to satisfy (5.6), $\mathbf{U}_r$ needs to be the solution to

$$\partial_t \mathbf{U}_r = \mathcal{F}(\mathbf{U}_{ap} + \mathbf{U}_r) - \mathcal{F}(\mathbf{U}_{ap}) - \mathbf{r}_{ap}.$$

The Cauchy problem for such equation has a global solution, as we prove in the following proposition. The precise result reads:

Proposition 5.21. *Let $p \geqslant 2$. For N large enough and ρ sufficiently small (A sufficiently large) in the definition of $\boldsymbol{V}$, there exists a solution $\boldsymbol{U}_r = (\eta_r, \varphi_r)^t \in L^\infty\left((-\infty, 0], H^{m+4} \times H^{7/2}\right)$ to*

$$\begin{cases} \partial_t \boldsymbol{U}_r = \mathcal{F}(\boldsymbol{U}_{ap} + \boldsymbol{U}_r) - \mathcal{F}(\boldsymbol{U}_{ap}) - \boldsymbol{r}_{ap}, \\ \boldsymbol{U}_r(0) \text{ fixed,} \end{cases} \tag{5.85}$$

such that $h\|a_\varepsilon\|_{L^\infty} - \|\eta_{ap}\|_{L^\infty} - \|\eta_r\|_{L^\infty} \geqslant h_{min} > 0$, and

$$|\boldsymbol{U}_r|_{X^{m+4} \times X^{m+7/2}} \leqslant A^{(2N-1)/4} \rho^{N+1} \mathrm{e}^{-N\delta_0 c|t|} \quad \forall t \leqslant 0. \tag{5.86}$$

This proposition is proved in the Appendix 5.5

Once having proved the existence of $\mathbf{U}_r$, we obtain that $\mathbf{U} = \mathbf{U}_{ap} + \mathbf{U}_r$ exists and it is defined in $(-\infty, 0]$. Recall the definition $\mathbf{R}(t,x) = \mathbf{Q}_c(x - ct + A)$. We are left to prove

$$\lim_{t \to -\infty} |\mathbf{U}(t) - \mathbf{R}(t)|_{H^s} = 0. \tag{5.87}$$

From the definition of $\mathbf{U}$,

$$\mathbf{U} = \mathbf{R} + \sum_{j=1}^{N} \rho^j \mathbf{V}_j + \mathbf{U}_r.$$

The terms $\mathbf{V}_j$ and $\mathbf{U}_r$ satisfy a decay estimation each (deduced from (5.46) and (5.86)) for every $t \leqslant 0$:

$$|\mathbf{V}_l|_{H^s} \leqslant A^{(2l-1)/4} C_{s,l}(\delta_0) \mathrm{e}^{-l\delta_0 c|t|} \quad \text{and} \quad |\mathbf{U}_r|_{H^s} \leqslant A^{(2N-1)/4} \rho^{N+1} \mathrm{e}^{-N\delta_0 c|t|}.$$

Consequently, we conclude (5.87).

Appendix

A.1. Proof of Lemma 5.18

Proof of (5.59) *in Lemma 5.18. Case* i $=1$. First, recall that for any $u \in C_b^\infty(\mathbb{R})$, the Dirichlet-Neumann operator can be defined as $\mathcal{G}_{c,a}u = \partial_{\mathbf{n}}^{P_a} u_a^\flat\big|_{z=0}$, where $u_a^\flat$ satisfies

$$\begin{cases} \nabla_{x,z} \cdot P_a \nabla_{x,z} u_a^\flat = 0 \quad \text{in } S, \\ u_a^\flat|_{z=0} = u, \\ \partial_{\mathbf{n}}^{P_a} u_a^\flat|_{z=-1} = 0, \end{cases} \tag{5.88}$$

and $\partial_{\mathbf{n}}^{P_a} = \mathbf{n} \cdot P_a \nabla_{x,z}$, where $\mathbf{n} = -\mathbf{e}_z$ is the upward unit normal to the boundary in $z=-1$. Then, one can use Divergence Theorem to get the following Green formula:

$$(\mathcal{G}_{c,a}u, v) = \int_S P_a \nabla_{x,z} u_a^\flat \cdot \nabla_{x,z} v_a^\flat \mathrm{d}x\mathrm{d}z.$$

In particular, we have that

$$([\partial_t, \mathcal{G}_{c,a}]u, u) = \int_S \partial_t P_a \nabla_{x,z} u_a^\flat \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z - 2\int_S P_a \nabla_{x,z} \left((\partial_t u)_a^\flat - \partial_t u_a^\flat\right) \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z.$$

Similarly, for $\mathcal{G}_{c,1}$ we have that

$$([\partial_t, \mathcal{G}_{c,1}]u, u) = \int_S \partial_t P_1 \nabla_{x,z} u_1^\flat \cdot \nabla_{x,z} u_1^\flat \mathrm{d}x\mathrm{d}z - 2\int_S P_1 \nabla_{x,z} \left((\partial_t u)_1^\flat - \partial_t u_1^\flat\right) \cdot \nabla_{x,z} u_1^\flat \mathrm{d}x\mathrm{d}z,$$

where u_1^{e} is the solution to the elliptic problem associated to a flat-domain regime

$$\begin{cases} \nabla_{x,z} \cdot P_1 \nabla_{x,z} u_1^\flat = 0 \quad \text{in } S, \\ u_1^\flat|_{z=0} = u, \\ \partial_{\mathbf{n}}^{P_1} u_1^\flat|_{z=-1} = \partial_z u_1^\flat|_{z=-1} = 0. \end{cases} \tag{5.89}$$

Then we have that

$$\begin{aligned} ([\partial_t, \mathcal{G}_{c,a} - \mathcal{G}_{c,1}]u, u) = &\int_S \partial_t P_a \nabla_{x,z} u_a^\flat \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z - 2\int_S P_a \nabla_{x,z} \left((\partial_t u)_a^\flat - \partial_t u_a^\flat\right) \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z \\ &- \int_S \partial_t P_1 \nabla_{x,z} u_1^\flat \cdot \nabla_{x,z} u_1^\flat \mathrm{d}x\mathrm{d}z + 2\int_S P_1 \nabla_{x,z} \left((\partial_t u)_1^\flat - \partial_t u_1^\flat\right) \cdot \nabla_{x,z} u_1^\flat \mathrm{d}x\mathrm{d}z. \end{aligned}$$

Recall, from the definition of both P_1 and P_a, we have that

$$\partial_t P_1 = \begin{pmatrix} \partial_t \eta_c & -(z+1)\partial_t\partial_x \eta_c \\ -(z+1)\partial_t\partial_x\eta_c & B_1 \end{pmatrix}, \quad \partial_t P_a = \begin{pmatrix} \partial_t \eta_c & -(z+1)\partial_t\partial_x \eta_c \\ -(z+1)\partial_t\partial_x\eta_c & B_a \end{pmatrix},$$

where

$$B_1 = \frac{2(z+1)^2 \partial_x \eta_c \partial_{t,x}^2 \eta_c}{h+\eta_c} - \frac{\left(1 + |(z+1)\partial_x \eta_c|^2\right)\partial_t \eta_c}{(h+\eta_c)^2}, \quad \text{and}$$

$$B_a = \frac{2\left((z+1)\partial_x\eta_c + zha_\varepsilon'\right)(z+1)\partial_{t,x}^2\eta_c}{Ha_\varepsilon + \eta_c} - \frac{\left(1 + |(z+1)\partial_x\eta_c + zha_\varepsilon'|^2\right)\partial_t\eta_c}{(ha_\varepsilon + \eta_c)^2}.$$

Then, taking into account the fact that the solitary wave η_c satisfies the exponential decay [3]

$$|\partial_x^\alpha \eta_c (x - ct + A)| \leqslant \mathrm{e}^{-\mathrm{d}(1+|ct-x+A|^2)^{1/2}},$$

this means that we can estimate each entry of $\partial_t P_1$ and $\partial_t P_a$ by using a weight ω_1 defined as

$$\omega_1(t,x) = \mathrm{e}^{-\delta(1+|ct-x+A|^2)^{1/2}}, \tag{5.90}$$

where $0 < \delta < \min\{\varepsilon\gamma, \mathrm{d}\}$ to be chosen sufficiently small.

On the other hand, from the definition of P_a and P_1, we know that

$$P_a - P_1 = \begin{bmatrix} H(a(\varepsilon x) - 1) & -hza'_\varepsilon \\ -hza'_\varepsilon & E \end{bmatrix},$$

where

$$E = \frac{(z+1)zh\partial_x\eta_c a'_\varepsilon + |zha'_\varepsilon|^2}{ha_\varepsilon + \eta_c} + \frac{(1 + |(z+1)\partial_x\eta_c|^2)\, h(1-a_\varepsilon)}{(ha_\varepsilon + \eta_c)\,(h+\eta_c)}.$$

This implies that we also need to take into consideration the change of bottom a', coming into play $P_a - P_1$. Then, we define the weight

$$\omega_a(t,x) = \mathrm{e}^{-\delta(1+|x|^2)^{1/2}}, \tag{5.91}$$

where $0 < \delta < \mathrm{d}$ is the same used in the definition of ω_1.

Finally, with respect to $\partial_t(P_a - P_1)$, after computing the derivative in time, we get that the only entry of $\partial_t(P_a - P_a)$ that survives is $\partial_t E = B_a - B_1$, which depends both on the solitary wave η_c and the description of the change of bottom a'.

From now on, we shall assume that $u = \chi_1 U_2$ and take advantage of the fact that, from Lemma 5.15, there exists $C \geqslant 0$ such that $\omega_a \chi_1 \leqslant \frac{C}{A}$. Similarly, for the case $u = \chi_a U_2$, one only needs to take into account the weight ω_1 and and use symmetric arguments. With this in mind, we write

$$\begin{aligned}
&([\partial_t, \mathcal{G}_{c,a} - \mathcal{G}_{c,1}]u, u) \\
&\quad = \int_S \partial_t (P_a - P_1) \nabla_{x,z} u_a^\flat \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z + \int_S \partial_t P_1 \nabla_{x,z} \left(u_a^\flat - u_1^\flat\right) \cdot \nabla_{x,z} u_1^\flat \mathrm{d}x\mathrm{d}z \\
&\qquad + \int_S \partial_t P_1 \nabla_{x,z} u_a^\flat \cdot \nabla_{x,z} \left(u_a^\flat - u_1^\flat\right) \mathrm{d}x\mathrm{d}z - 2\int_S (P_a - P_1) \nabla_{x,z} \left((\partial_t u)_a^\flat - \partial_t u_a^\flat\right) \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z \\
&\qquad + 2\int_S P_1 \nabla_{x,z}\left((\partial_t u)_a^\flat - \partial_t u_a^\flat - (\partial_t u)_1^\flat + \partial_t u_1^\flat\right) \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z \\
&\qquad - 2\int_S P_1 \nabla_{x,z} \left((\partial_t u)_1^\flat - \partial_t u_1^\flat\right) \cdot \nabla_{x,z}\left(u_a^\flat - u_1^\flat\right)\mathrm{d}x\mathrm{d}z.
\end{aligned} \tag{5.92}$$

In fact, from (5.92), we have that

$$\begin{aligned}
|([\partial_t, \mathcal{G}_{c,a} - \mathcal{G}_{c,1}]\, u, u)| \leqslant{}& C\|\omega_a \nabla_{x,z} u_a^\flat\|_{L^2(S)} \left(\|\nabla_{x,z} u_a^\flat\|_{L^2(\mathcal{S})} + \left\|\nabla_{x,z}\left((\partial_t u_a)^\flat - \partial_t u_a^\flat\right)\right\|_{L^2(S)}\right) \\
&+ C\left\|\nabla_{x,z}\left((\partial_t u_a)^\flat - \partial_t u_a^\flat - (\partial_t u_1)^\flat + \partial_t u_1^\flat\right)\right\|_{L^2(S)} \|\nabla_{x,z} u_a^\flat\|_{L^2(S)} \\
&+ C\|\nabla_{x,z}\left(u_a^\flat - u_1^\flat\right)\|_{L^2(S)} \left(\|\nabla_{x,z} u_a^\flat\|_{L^2(S)} + \|\nabla_{x,z} u_1^\flat\|_{L^2(S)}\right. \\
&\qquad \left.\left\|\nabla_{x,z}\left((\partial_t u_a)^\flat - \partial_t u_a^\flat\right)\right\|_{L^2(S)}\right).
\end{aligned}$$

The rest of the proof, then, consists on estimating $\|\nabla_{x,z} u_j^\flat\|_{L^2(S)}$ and $\left\|\nabla_{x,z}\left((\partial_t u)_j^\flat - \partial_t u_j^\flat\right)\right\|_{L^2}$, for $j = 1, a$, $\|\omega_a \nabla_{x,z} u_a^\flat\|_{L^2}$, $\|\nabla_{x,z}\left(u_a^\flat - u_1^\flat\right)\|_{L^2}$ and $\|\nabla_{x,z}\big((\partial_t u_a)^\flat - \partial_t u_a^\flat - (\partial_t u_1)^\flat + \partial_t u_1^\flat\big)\|_{L^2}$.

1. **Computing the estimation for $\|\nabla_{x,z} u_j^\flat\|_{L^2(S)}$:**
   Let us denote by $u_j^\dagger$, $j = 1, a$, a function defined as

$$\forall z \in [-1, 0], \quad u^\dagger(\cdot, z) = \chi\left(z|D|\right) u, \tag{5.93}$$

   where χ is a smooth compactly supported function such that $\chi(0) = 1$.

   Thus, to decompose $u_j^\flat$ into $u_j^\flat = v_j + u^\dagger$, $j = 1, a$, means that v_j must solve

$$\begin{cases} \nabla_{x,z} \cdot P_j \nabla_{x,z} v_j = -\nabla_{x,z} \cdot P_j \nabla_{x,z} u^\dagger & \in \mathcal{S}, \\ v|_{z=0} = 0, \\ \partial_{\mathbf{n}}^{P_a} v|_{z=-1} = -\partial_{\mathbf{n}}^{P_a} u^\dagger|_{z=-1}. \end{cases}$$

   In particular, thanks to Divergence Theorem and the coercivity of P_j,

$$\|\nabla_{x,z} v_j\|_{L^2(S)} \leqslant C \left\|P_j \nabla_{x,z} u^\dagger\right\|_{L^2(S)} \leqslant C \left\|\nabla_{x,z} u^\dagger\right\|_{L^2(S)}.$$

   We write

$$\left\|\nabla_{x,z} u^\dagger\right\|_{L^2(S)} \leqslant \|\chi(z|D|)\partial_x u\|_{L^2(\mathbb{R}^2)} + \|\chi'(z|D|)|D|u\|_{L^2(\mathbb{R}^2)}.$$

   Now, we use the following auxiliary lemma

   Lemma 5.22. *For all real valued, compactly supported function χ, one has*

$$\|\chi(z|D|)u\|_{L^2}^2 \leqslant C(\chi) \left|\frac{1}{\left(1+|D|^2\right)^{1/4}} u\right|_{L^2}^2,$$

   where $C(\chi) > 0$ is a constant that only depends on the function χ.

   Proof. We write

$$\|\chi(z|D|)u\|_{L^2}^2 = \int_{\mathbb{R}} \int_{-1}^0 \left|\chi\left(z|\xi|\right)\right|^2 |\hat{u}(\xi)|^2 \mathrm{d}z\mathrm{d}\xi = \int_{\mathbb{R}} \int_{-1}^0 \frac{X(0) - X(-|\xi|)}{|\xi|} |\hat{u}(\xi)|^2 \mathrm{d}z\mathrm{d}\xi$$

   where X is the primitive of χ^2. In particular, since χ is compactly supported, X is bounded. Thus, we conclude the proof by noticing that

$$\left|\frac{X(0) - X(-|\xi|)}{|\xi|}\right| \leqslant C(\chi) \frac{1}{\left(1+|\xi|^2\right)^{1/2}}.$$

   ∎

   Lemma 5.22 implies that

$$\left\|\nabla_{x,z} u^\dagger\right\|_{L^2(S)}^2 \leqslant C \left|\mathfrak{B} u\right|_{L^2(\mathbb{R})}^2.$$

And then, since $u = \chi_1 U_2$,

$$\left\|\nabla_{x,z} u^\dagger\right\|^2_{L^2(S)} = C \left|\mathfrak{B}(\chi_1 U_2)\right|^2_{L^2(\mathbb{R})} \leqslant \left|[\mathfrak{B}, \chi_1]\, U_2\right|^2_{L^2} + \left|\chi_1 \mathfrak{B} U_2\right|^2_{L^2}.$$

Now, we make use of the following commutator estimation,

$$|[\mathfrak{B}, f]\, g|_{L^2} \lesssim |\partial_x f|_{L^\infty} |g|_{L^2}. \tag{5.94}$$

Indeed, this can be proved by noticing

$$\begin{aligned} |[\mathfrak{B}, f]\, g|_{L^2} &\leqslant \left| \frac{\partial_x f g}{(1-\partial_x^2)^{1/4}} + \left[(1-\partial_x^2)^{-1/4}, f\right] \partial_x g \right|_{L^2} \\ &\lesssim |\partial_x f|_{L^\infty} |g|_{L^2} + \left|\left[(1-\partial_x^2)^{-1/4}, f\right] \partial_x g\right|_{L^2}, \end{aligned}$$

and using [18, Proposition 3.6.B] to bound the second term. Thus, going back to our computations, (5.94) implies

$$\|\nabla_{x,z} v_j\|^2_{L^2(S)} \leqslant \left\|\nabla_{x,z} u^\dagger\right\|^2_{L^2(S)} \leqslant C \left(|U_2|^2_{L^2(\mathbb{R})} + |\mathfrak{B} U_2|^2_{L^2(\mathbb{R})} \right), \quad \mathrm{i} = 1, a.$$

Consequently, one has

$$\left\|\nabla_{x,z} u_j^\flat\right\|^2_{L^2(S)} \leqslant C \left(|U_2|^2_{L^2(\mathbb{R})} + |\mathfrak{B} U_2|^2_{L^2(\mathbb{R})} \right), \quad \mathrm{i} = 1, a. \tag{5.95}$$

2. **Computing the estimation for $\|\omega_2 \nabla u_a^\flat\|_{L^2}$:**

We have that $\omega_a u_a^\flat$ solves

$$\begin{cases} \nabla_{x,z} \cdot P_a \nabla_{x,z} \left(\omega_a u_a^\flat\right) = -[\omega_a, \nabla_{x,z} \cdot P_a \nabla_{x,z}] u_a^\flat \quad \in \mathcal{S}, \\ \omega_a u_a^\flat|_{z=0} = \omega_2 u, \\ \partial_{\mathbf{n}}^{P_a} \left(\omega_a u_a^\flat\right)|_{z=-1} = -[\omega_a, \partial_{\mathbf{n}}^{P_a}] u_a^\flat. \end{cases} \tag{5.96}$$

We can decompose

$$\omega_a u_a^\flat = m\left(z, |D|\right)\left(\omega_a u\right) + v, \tag{5.97}$$

where

$$m\left(z, |D|\right) = \frac{\cosh\left(|D|(z+1)\right)}{\cosh|D|},$$

is the solution to the homogeneous Laplace equation associated to (5.96) and v solves

$$\begin{cases} \nabla_{x,z} \cdot P_a \nabla_{x,z} v + [\omega_2, \nabla_{x,z} \cdot P_a \nabla_{x,z}]\, u_a^\flat = -\nabla_{x,z} \cdot P_a \nabla_{x,z} \left(m(\omega_2 u)\right) \quad \text{in } S, \\ v|_{z=0} = 0, \quad \partial_n^{P_a} v|_{z=-1} = -h a'_\varepsilon \delta v|_{z=-1}. \end{cases} \tag{5.98}$$

We argue as in the proof of Proposition 5.4 and of Proposition 5.11, relying on the possibility of using Poincaré inequality and the fact that $|\nabla_{x,z}\omega_2| \lesssim \delta\omega_2$ to obtain

$$\begin{aligned} \|\nabla_{x,z} v\|^2_{L^2(S)} \leqslant \tilde{C}\Big(&(1+\delta) \|\nabla_{x,z}(m(\omega_a u))\|_{L^2(S)} \\ &+ \left(\delta^2 + \delta\right) \|\omega_a u_a^\flat\|_{L^2(S)} + \delta \|\nabla_{x,z}(\omega_a u_a^\flat)\|_{L^2(S)}\Big) \|\nabla_{x,z} v\|_{L^2(S)}. \end{aligned}$$

The decomposition (5.97) implies that

$$\left(1-\tilde{C}\delta-\tilde{C}\delta^2\right)\|\nabla_{x,z}v\|^2_{L^2(S)}$$
$$\leqslant \tilde{C}\left((\delta^2+\delta)\|m(\omega_a u)\|_{L^2(S)}+(1+\delta)\left\|\nabla_{x,z}\left(m(\omega_a u)\right)\right\|_{L^2(S)}\right).$$

Then, for δ sufficiently small, there exists $C>0$ such that

$$\|\nabla_{x,z}v\|_{L^2(\mathcal{S})}\leqslant C\left((\delta^2+\delta)\|m(\omega_a u)\|_{L^2(\mathcal{S})}+(1+\delta)\left\|\nabla_{x,z}\left(m(\omega_a u)\right)\right\|_{L^2(\mathcal{S})}\right). \qquad (5.99)$$

Our goal is prove that

$$\|\omega_2 u_a^\flat\|_{L^2(S)}+\|\nabla_{x,z}(\omega_2 u_a^\flat)\|_{L^2(S)}\leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2(\mathbb{R})}+|U_2|_{L^2(\mathbb{R})}\right). \qquad (5.100)$$

Thus, in view of decomposition (5.98) and (5.99), it would be sufficient to show

$$\|\nabla_{x,z}m(\omega_a u)\|_{L^2(S)}+\|m(\omega_a u)\|_{L^2(S)}\leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2(\mathbb{R})}+|U_2|_{L^2(\mathbb{R})}\right). \qquad (5.101)$$

To do so, we argue as in a)., using the commutator estimate (5.94) and Lemma 5.15 to obtain (5.101). Consequently, we proved (5.100) which, in particular, leads to

$$\|\omega_a\nabla_{x,z}u_a^\flat\|_{L^2(S)}\leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2}+|U_2|_{L^2}\right). \qquad (5.102)$$

3. **Computing the estimation for $\left\|\nabla_{x,z}\left((\partial_t u)_j^\flat-\partial_t u_j^\flat\right)\right\|_{L^2}$:**

   For each $j=1,a$ (the first associated to the flat-bottom problem and the latter with a changing bottom problem), we have that $\left((\partial_t u)_j^\flat-\partial_t u_j^\flat\right)$ satisfies

$$\begin{cases}\nabla_{x,z}\cdot P_j\nabla_{x,z}\left((\partial_t u)_j^\flat-\partial_t u_j^\flat\right)=\nabla_{x,z}\cdot\partial_t P_j\nabla_{x,z}u_j^\flat & (x,z)\in S,\\ \left((\partial_t u)_j^\flat-\partial_t u_j^\flat\right)|_{z=0}=0, \quad \partial_{\mathbf{n}}^{P_j}\left((\partial_t u)_j^\flat-\partial_t u_j^\flat\right)|_{z=-1}=0.\end{cases}$$

   Then, using energy estimates, taking into account the coercivity of P_j, we write

$$\left\|\nabla_{x,z}\left((\partial_t u)_j^\flat-\partial_t u_j^\flat\right)\right\|^2_{L^2(\mathcal{S})}\leqslant \left\|\partial_t P_j\nabla_{x,z}u_j^\flat\right\|^2_{L^2(\mathcal{S})}.$$

   This, along with (5.95), implies that

$$\left\|\nabla_{x,z}\left((\partial_t u)_j^{\mathrm{e}}-\partial_t u_j^{\mathrm{e}}\right)\right\|_{L^2(S)}\leqslant C\left(|\mathfrak{B}U_2|_{L^2(\mathbb{R})}+|U_2|_{L^2(\mathbb{R})}\right).$$

4. **Computing the estimation for $\|\nabla_{x,z}\left(u_a^\flat-u_1^\flat\right)\|_{L^2}$:**

   We have that $u_a^\flat-u_1^\flat$ solves the following equation,

$$\begin{cases}\nabla_{x,z}\cdot P_1\nabla_{x,z}\left(u_a^\flat-u_1^\flat\right)=\nabla_{x,z}\cdot P_1\nabla_{x,z}u_a^\flat=-\nabla_{x,z}\cdot\left(P_a-P_1\right)\nabla_{x,z}u_a^\flat & (x,z)\in S,\\ \left(u_a^\flat-u_1^\flat\right)|_{z=0}=0, \quad \partial_{\mathbf{n}}^{P_1}\left(u_a^\flat-u_1^\flat\right)|_{z=-1}=\frac{1}{h+\eta_c}\partial_z u_a^\flat|_{z=-1}.\end{cases}$$

Then, multiplying by $u_a^\flat - u_1^\flat$ standard energy estimates lead to

$$\left\|\nabla_{x,z}\left(u_a^\flat - u_1^\flat\right)\right\|_{L^2(S)}^2 \leqslant C\left\|(P_a - P_1)\nabla_{x,z}u_a^\flat\right\|_{L^2(S)}^2 \leqslant C\|\omega_a\nabla_{x,z}u_a^\flat\|_{L^2(S)}^2,$$

which ultimately (in view of (5.102)) implies that

$$\left\|\nabla_{x,z}\left(u_a^\flat - u_1^\flat\right)\right\|_{L^2(S)} \leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2} + |U_2|_{L^2}\right). \tag{5.103}$$

5. **Computing the estimation for** $\|\nabla_{x,z}\big((\partial_t u_a)^{\mathrm{e}} - \partial_t u_a^{\mathrm{e}} - (\partial_t u_1)^{\mathrm{e}} + \partial_t u_1^{\mathrm{e}}\big)\|_{L^2}$ **:**

In order to make notation more simple, we shall consider $v = (\partial_t u_a)^{\mathrm{e}} - \partial_t u_a^{\mathrm{e}} - (\partial_t u_1)^{\mathrm{e}} + \partial_t u_1^{\mathrm{e}}$ and $f_j = \partial_t P_j \nabla u_j^\flat$, $j = 1, a$. Then, taking into account the equations that $(\partial_t u_a)^{\mathrm{e}} - \partial_t u_a^{\mathrm{e}}$ and $(\partial_t u_1)^{\mathrm{e}} + \partial_t u_1^{\mathrm{e}}$ solve, we get that v is the solution to the following formulation:

$$\begin{cases} \nabla_{x,z}\cdot P_1\nabla_{x,z}v = \nabla_{x,z}\cdot\Big(f_a - f_1 - (P_a - P_1)\nabla_{x,z}\left((\partial_t u)_a^\flat - \partial_t u_a^\flat\right)\Big), & (x,z)\in S,\\ v|_{z=0} = 0, \quad \partial_{\mathbf{n}}^{P_a}v|_{z=-1} = -\mathrm{e}_z\cdot\Big(f_a - f_1 - (P_a - P_1)\nabla_{x,z}\left((\partial_t u)_a^\flat - \partial_t u_a^\flat\right)\Big)|_{z=-1}. \end{cases}$$

Then, energy estimates lead to

$$\begin{aligned} \|\nabla_{x,z}v\|_{L^2(S)} &\leqslant C\Big(\left\|(P_a - P_1)\nabla_{x,z}\left((\partial_t u)_a^\flat - \partial_t u_a^\flat\right)\right\|_{L^2(S)} + \left\|\partial_t(P_a - P_1)\nabla_{x,z}u_a^\flat\right\|_{L^2(S)}\\ &\qquad + \left\|\partial_t P_1\nabla_{x,z}\left(u_a^\flat - u_1^\flat\right)\right\|_{L^2(S)}\Big)\\ &\leqslant C\Big(\left\|\omega_2\nabla_{x,z}\left((\partial_t u)_a^\flat - \partial_t u_a^\flat\right)\right\|_{L^2(S)} + \left\|\omega_2\nabla_{x,z}u_a^\flat\right\|_{L^2(S)}\\ &\qquad + \left\|\nabla_{x,z}\left(u_a^\flat - u_1^\flat\right)\right\|_{L^2(S)}\Big). \end{aligned}$$

Subsequently, in view of (5.102) and (5.103), it would be sufficient to prove

$$\left\|\omega_2\nabla_{x,z}\left((\partial_t u)_a^\flat - \partial_t u_a^\flat\right)\right\|_{L^2(S)} \leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2} + |U_2|_{L^2}\right), \tag{5.104}$$

so that we can conclude the desired estimation, that is,

$$\|\nabla_{x,z}v\|_{L^2(S)} \leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2} + |U_2|_{L^2}\right). \tag{5.105}$$

Let us show that (5.104) holds. Indeed, we reason as in the proof of (5.102) and write the following equation satisfied by $v_a = \omega_a\left((\partial_t u)_a^\flat - \partial_t u_a^\flat\right)$,

$$\begin{cases} \nabla_{x,z}\cdot P_a\nabla_{x,z}v_a + [\omega_a, \nabla_{x,z}\cdot P_a\nabla_{x,z}]\left((\partial_t u)_a^\flat - \partial_t u_a^\flat\right) = -\omega_a\nabla_{x,z}\cdot\partial_t P_a\nabla_{x,z}u_a^\flat & (x,z)\in S,\\ v_a|_{z=0} = 0, \quad \partial_{\mathbf{n}}^{P_j}v_a|_{z=-1} = -ha_\varepsilon'\delta v_a|_{z=-1}. \end{cases}$$

Thus, following similar computations as in step b)., we have that,

$$\|\nabla_{x,z}v_a\|_{L^2(S)}^2 \leqslant C\left(\left(\delta^2 + \delta\right)\|\nabla_{x,z}v_a\|_{L^2(S)}^2 + \|\omega_a\partial_t\nabla_{x,z}u_a^\flat\|_{L^2(S)}^2\right),$$

which, along with (5.102) leads to

$$\|v_a\|_{L^2(S)} + \|\nabla_{x,z}v_a\|_{L^2(S)} \leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2} + |U_2|_{L^2}\right).$$

and ultimately implies (5.104).

$\blacksquare$

Proof of (5.59) *in Lemma 5.18. Case* i = 2. Recall that

$$([\partial_t, \mathcal{G}_{c,a} - \mathcal{G}_{c,1}]u, u) = \int_S \partial_t P_a \nabla_{x,z} u_a^\flat \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z - 2\int_S P_a \nabla_{x,z} \left((\partial_t u)_a^\flat - \partial_t u_a^\flat\right) \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z$$
$$- \int_S \partial_t P_1 \nabla_{x,z} u_1^\flat \cdot \nabla_{x,z} u_1^\flat \mathrm{d}x\mathrm{d}z + 2\int_S P_1 \nabla_{x,z} \left((\partial_t u)_1^\flat - \partial_t u_1^\flat\right) \cdot \nabla_{x,z} u_1^\flat \mathrm{d}x\mathrm{d}z.$$

where P_1, $\partial_t P_1$ and $\partial_t P_a$ can be estimated with the weight ω_1 (5.90), whereas $P_a - P_1$ is estimated by ω_a. On the other hand, the decay of $\partial_t(P_a - P_1)$ and P_a depend both on the solitary wave η_c and the description of the change of bottom a'. Then, taking this into account, and considering $u = \chi_a U_2$, we write

$$|([\partial_t, \mathcal{G}_{c,a} - \mathcal{G}_{c,1}]u, u)| \leqslant C\left(\left\|\omega_1 \nabla_{x,z} u_a^\flat\right\|_{L^2(S)} + \left\|\nabla_{x,z}\left((\partial_t u)_a^\flat - \partial_t u_a^\flat\right)\right\|_{L^2(S)}\right)\left\|\nabla_{x,z} u_a^\flat\right\|_{L^2(S)}$$
$$+ C\left(\left\|\omega_1 \nabla_{x,z} u_1^\flat\right\|_{L^2(S)} + \left\|\nabla_{x,z}\left((\partial_t u)_1^\flat - \partial_t u_1^\flat\right)\right\|_{L^2(S)}\right)\left\|\nabla_{x,z} u_1^\flat\right\|_{L^2(S)}.$$

From (5.95), we already have that

$$\left\|\nabla_{x,z} u_{\mathrm{i}}^\flat\right\|_{L^2(S)} \leqslant C\left(|\mathfrak{B}U_2|_{L^2} + |U_2|_{L^2}\right), \quad \mathrm{i} = 1, a.$$

The estimation

$$\left\|\omega_1 \nabla_{x,z} u_{\mathrm{i}}^\flat\right\|_{L^2(S)} \leqslant C\left(|\mathfrak{B}U_2|_{L^2} + |U_2|_{L^2}\right), \quad \mathrm{i} = 1, a, \tag{5.106}$$

follows from the computations in the proof of (5.102) (swapping ω_a for ω_1) and Lemma 5.15. Finally, we have that

$$\|\nabla_{x,z}\left((\partial_t u)_{\mathrm{i}}^{\mathrm{e}} - \partial_t u_{\mathrm{i}}^{\mathrm{e}}\right)\|_{L^2(S)} \leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2(\mathbb{R})} + |U_2|_{L^2(\mathbb{R})}\right), \quad \mathrm{i} = 1, a$$

Indeed, notice that $\left((\partial_t u)_j^\flat - \partial_t u_j^\flat\right)$ solves

$$\begin{cases} \nabla_{x,z} \cdot P_j \nabla_{x,z}\left((\partial_t u)_j^\flat - \partial_t u_j^\flat\right) = \nabla_{x,z} \cdot \partial_t P_j \nabla_{x,z} u_j^\flat & (x,z) \in S, \\ \left((\partial_t u)_j^\flat - \partial_t u_j^\flat\right)|_{z=0} = 0, \quad \partial_n^{P_j}\left((\partial_t u)_j^\flat - \partial_t u_j^\flat\right)|_{z=-1} = 0. & \end{cases}$$

Energy estimates and the coercivity of P_j, gives us

$$\left\|\nabla_{x,z}\left((\partial_t u)_j^\flat - \partial_t u_j^\flat\right)\right\|_{L^2(S)}^2 \leqslant \left\|\partial_t P_j \nabla_{x,z} u_j^\flat\right\|_{L^2(S)}^2 \leqslant \left\|\omega_1 \nabla_{x,z} u_j^\flat\right\|_{L^2(S)}^2.$$

Then, thanks to (5.106), we conclude.

$\blacksquare$

Proof of (5.60) *in Lemma 5.18.* As we did for the proof of (5.59), we write

$$([\partial_x, \mathcal{G}_{c,a} - \mathcal{G}_{c,1}]u, u) = \int_S \partial_x P_a \nabla_{x,z} u_a^\flat \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z - 2\int_S P_a \nabla_{x,z}\left((\partial_x u)_a^\flat - \partial_x u_a^\flat\right) \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z$$
$$- \int_S \partial_x P_1 \nabla_{x,z} u_1^\flat \cdot \nabla_{x,z} u_1^\flat \mathrm{d}x\mathrm{d}z + 2\int_S P_1 \nabla_{x,z}\left((\partial_x u)_1^\flat - \partial_x u_1^\flat\right) \cdot \nabla_{x,z} u_1^\flat \mathrm{d}x\mathrm{d}z.$$

Notice that, since P_1 only depends on the solitary wave Q_c, we have that

$$c\partial_x P_1 = -\partial_t P_1.$$

In a similar way, given that most entries of P_a also depend on Q_c, to derivate such terms in time is actually to compute a derivative in space. In consequence, one can use the same arguments as in the proof of (5.59) to obtain the desired result.

The objects that actually create a different situation are the ones that involve the changing bottom $a'(\varepsilon x)$. For these terms, we integrate by parts to avoid dealing with $a''(\varepsilon x)$. Indeed, suppose we are in the case i = 1. After integration by parts, and considering again the weight ω_a (5.91), to conclude, one needs to find an estimation for $\left\|\omega_a \nabla_{x,z}\partial_x u_a^\flat\right\|$.

Let us prove that

$$\left\|\omega_a \nabla_{x,z}\partial_x u_a^\flat\right\| \leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2} + |U_2|_{L^2}\right).$$

We have that $\omega_a \partial_x u_a^\flat$ solves the equation

$$\begin{cases} \nabla_{x,z}\cdot P_a \nabla_{x,z}\left(\omega_a \partial_x u_a^\flat\right) = -[\omega_a, \nabla_{x,z}\cdot P_a\nabla_{x,z}]\partial_x u_a^\flat - \omega_a \nabla_{x,z}\cdot \partial_x P_a \nabla_{x,z} u_a^\flat & (x,z)\in\mathcal{S}, \\ \partial_x u_a^\flat|_{z=0} = \partial_x u, \quad \partial_n^{P_a}\left(\omega_a\partial_x u_a^\flat\right)|_{z=-1} = -[\omega_a, \partial_n \mathbf{n}^{P_a}]u_a^\flat. \end{cases}$$

Then, if we write

$$\omega_a \partial_x u_a^\flat = v + m(z,|D|)(\omega_a\partial_x u),$$

we have that v solves

$$\begin{cases} \nabla_{x,z}\cdot P_a\nabla_{x,z} v = f \quad (x,z)\in\mathcal{S}, \\ v|_{z=0} = 0, \quad \partial_{\mathbf{n}}^{P_a} v|_{z=-1} = -[\omega_a, \partial_n^{P_a}]u_a^\flat. \end{cases}$$

where

$$f = -[\omega_a, \nabla_{x,z}\cdot P_a\nabla_{x,z}]\partial_x u_a^\flat - \omega_a\nabla_{x,z}\cdot\partial_x P_a\nabla_{x,z}u_a^\flat - \nabla_{x,z}\cdot P_a\nabla_{x,z}\left(m(\omega_a\partial_x u)\right).$$

We argue as in the proof of Proposition 5.4, using energy estimates, Poincaré inequality, the coercivity of P_a and the fact that

$$\begin{aligned}[\omega_a, \nabla_{x,z}\cdot P_a\nabla_{x,z}] = &\left(Ha_\varepsilon + \eta\right)[\omega_a, \partial_x^2] \\ &- \left((z+1)\partial_x\eta + zHa_\varepsilon'\right)[\omega_a, \partial_x]\partial_z\end{aligned}$$

to get that, for $\delta > 0$ sufficiently small, there exists $C > 0$ such that

$$\begin{aligned}\|v\|_{L^2(S)} &\leqslant C\left\|\omega_a\partial_x P_a\nabla_{x,z}u_a^\flat\right\|_{L^2(S)} + \|P_a\nabla_{x,z}m(\omega_a\partial_x u)\|_{L^2(S)} \\ &\leqslant C\left\|\omega_a\nabla_{x,z}u_a^\flat\right\|_{L^2(S)} + \|\nabla_{x,z}m(\omega_a\partial_x u)\|_{L^2(S)}.\end{aligned}$$

To treat $\|\nabla_{x,z}m(\omega_a\partial_x u)\|_{L^2(S)}$, we use the argument in step a). of the proof of (5.98) and obtain

$$\|\nabla_{x,z}m(\omega_a\partial_x u)\|_{L^2(S)} \leqslant \frac{C}{A}\left(|\mathfrak{B}U_2|_{L^2} + |U_2|_{L^2}\right).$$

The above comes from

$$\|\nabla_{x,z}m(\omega_a\partial_x u)\|_{L^2(\mathcal{S})} \leqslant (\delta+1)|\mathfrak{B}(\omega_a\partial_x u)|_{L^2}$$

and 5.94, where we use integration by parts on to pass the derivative in space from u to ω_a, and so

$$\|\nabla_{x,z}(\omega_a\partial_x u)^\dagger\| \leqslant \|\chi(z|D|)\partial_x(\omega_a\partial_x u)\| + \|\chi'(z|D|)|D|\omega_a\partial_x u\|$$

$\blacksquare$

Proof of (5.61) *in Lemma 5.18.* We begin the proof noticing that we can write

$$\begin{aligned}([[\partial_t,\mathcal{G}_a],\chi_{\mathrm{i}}]\,\chi_{\mathrm{i}}U_2,U_2) = {}& \partial_t\left(\left(\mathcal{G}_a\chi_{\mathrm{i}}^2U_2\right),U_2\right) - \left(\left(\mathcal{G}_a\chi_{\mathrm{i}}^2U_2\right),\partial_tU_2\right) - \left(\mathcal{G}_c\partial_t(\chi_{\mathrm{i}}^2U_2),U_2\right)\\ &- \partial_t\left(\chi_{\mathrm{i}}\left(\mathcal{G}_a\chi_{\mathrm{i}}U_2\right),U_2\right) + \left(\partial_t\chi_{\mathrm{i}}\left(\mathcal{G}_a\chi_{\mathrm{i}}U_2\right),U_2\right)\\ &+ \left(\chi_{\mathrm{i}}\left(\mathcal{G}_a\chi_{\mathrm{i}}U_2\right),\partial_tU_2\right) + \left(\chi_{\mathrm{i}}\mathcal{G}_a\partial_t(\chi_{\mathrm{i}}U_2),U_2\right).\end{aligned}$$

As we did before, in the proof of (5.59), if $u,v \in \dot{H}^{1/2}(\mathbb{R})$ and $u^\flat, v^\flat$ define the solution to the elliptic equation (5.88) associated to u and v respectively, then, from Divergence Theorem,

$$(\mathcal{G}_a u,v) = \int_S P_a\nabla_{x,z}u^\flat\cdot\nabla_{x,z}v^\flat\mathrm{d}x\mathrm{d}z.$$

Therefore, in a more explicit fashion, one has

$$\begin{aligned}([[\partial_t,\mathcal{G}_a],\chi_{\mathrm{i}}]\,u,v) = {}& \partial_t\int_S P_a\nabla_{x,z}(\chi_{\mathrm{i}}u)^\flat\cdot\nabla_{x,z}v^\flat\mathrm{d}x\mathrm{d}z - \int_S P_a\nabla_{x,z}(\chi_{\mathrm{i}}u)^\flat\cdot\nabla_{x,z}(\partial_t v)^\flat\mathrm{d}x\mathrm{d}z\\ &- \int_S P_a\nabla_{x,z}(\partial_t(\chi_{\mathrm{i}}u))^\flat\cdot\nabla_{x,z}v^\flat\mathrm{d}x\mathrm{d}z - \partial_t\int_S \chi_{\mathrm{i}}P_a\nabla_{x,z}u^\flat\cdot\nabla_{x,z}v^\flat\mathrm{d}x\mathrm{d}z\\ &- \partial_t\int_S \nabla_{x,z}\chi_{\mathrm{i}}\cdot P_a\nabla_{x,z}u^\flat v^\flat\mathrm{d}x\mathrm{d}z + \int_S(\partial_t\chi_{\mathrm{i}})P_a\nabla_{x,z}u^\flat\cdot\nabla_{x,z}v^\flat\mathrm{d}x\mathrm{d}z\\ &+ \int_S \nabla_{x,z}(\partial_t\chi_{\mathrm{i}})\cdot P_a\nabla_{x,z}u^\flat v^\flat\mathrm{d}x\mathrm{d}z + \int_S \chi_{\mathrm{i}}P_a\nabla_{x,z}u^\flat\cdot\nabla_{x,z}\partial_t v^\flat\mathrm{d}x\mathrm{d}z\\ &+ \int_S \nabla_{x,z}\chi_{\mathrm{i}}\cdot P_a\nabla_{x,z}u^\flat\partial_t v^\flat\mathrm{d}x\mathrm{d}z + \int_S \chi_{\mathrm{i}}P_a\nabla_{x,z}(\partial_t u)^\flat\cdot\nabla_{x,z}v^\flat\mathrm{d}x\mathrm{d}z\\ &+ \int_S \nabla_{x,z}\chi_{\mathrm{i}}\cdot P_a\nabla_{x,z}(\partial_t u)^\flat v^\flat\mathrm{d}x\mathrm{d}z.\end{aligned}$$

Notice that, if we consider $v^\dagger = \chi(z|D|)v$, with χ a smooth, compactly supported cut-off function, so that $v^\dagger = v$ in the boundary $z=0$. In particular, if we write $v^\flat = v^\dagger + v_r$, then v_r solves the equation

$$\begin{cases}\nabla_{x,z}\cdot P_a\nabla_{x,z}v_r = -\nabla_{x,z}\cdot P_a\nabla_{x,z}v^\dagger \quad \text{in } \mathcal{S},\\ v_r|_{z=0} = 0,\\ \partial_n^{P_a}u_a^\flat|_{z=-1} = 0,\end{cases}$$

This means that, for any $u \in C_b^\infty$,

$$\int_{\mathcal{S}} P_a\nabla_{x,z}u^\flat\cdot\nabla_{x,z}v^\flat\mathrm{d}x\mathrm{d}z = \int_{\mathcal{S}} P_a\nabla_{x,z}u^\flat\cdot\nabla_{x,z}v^\dagger\mathrm{d}x\mathrm{d}z.$$

The advantage of considering such function is that we already know from the proof of (5.95) that

$$\left\|\nabla_{x,z}v^\dagger\right\|_{L^2(\mathcal{S})} \leqslant C|\mathfrak{B}v|_{L^2}. \tag{5.107}$$

In addition, we also have that $\partial_t v^\dagger = (\partial_t v)^\dagger$. Then, computing the derivate in time of the integrals above , we have

$$\begin{aligned}
&([[\partial_t, \mathcal{G}_a], \chi_{\mathrm{i}}]\, u, v) \\
&\quad = \int_S \Big(\partial_t P_a \nabla_{x,z} (\chi_{\mathrm{i}} u)^\flat \cdot \nabla_{x,z} v^\dagger - \chi_{\mathrm{i}} \partial_t P_a \nabla_{x,z} u^\flat \cdot \nabla_{x,z} v^\dagger - \nabla_{x,z}\chi_{\mathrm{i}} \cdot \partial_t P_a \nabla_{x,z} u^\flat v^\dagger \Big) \mathrm{d}x\mathrm{d}z \\
&\qquad + \int_S \Big(P_a \nabla_{x,z} \left(\partial_t (\chi_{\mathrm{i}} u)^\flat - (\partial_t(\chi_{\mathrm{i}} u))^\flat \right) \cdot \nabla_{x,z} v^\dagger - \chi_{\mathrm{i}} P_a \nabla_{x,z} \left(\partial_t u^\flat - (\partial_t u)^\flat \right) \cdot \nabla_{x,z} v^\dagger \Big) \mathrm{d}x\mathrm{d}z \\
&\qquad - \int_S \nabla_{x,z}\chi_{\mathrm{i}} \cdot P_a \nabla_{x,z} \left(\partial_t u^\flat - (\partial_t u)^\flat \right) v^\dagger \mathrm{d}x\mathrm{d}z \\
&\quad = \int_S \Big(\partial_t P_a \nabla_{x,z} \left((\chi_{\mathrm{i}} u)^\flat - \chi_{\mathrm{i}} u^\flat \right) \cdot \nabla_{x,z} v^\dagger + \partial_t P_a \nabla_{x,z}\chi_{\mathrm{i}} u^\flat \cdot \nabla_{x,z} v^\dagger \Big) \mathrm{d}x\mathrm{d}z \qquad (5.108) \\
&\qquad - \int_S \nabla_{x,z}\chi_{\mathrm{i}} \cdot \partial_t P_a \nabla_{x,z} u^\flat v^\dagger \mathrm{d}x\mathrm{d}z \\
&\qquad + \int_S P_a \nabla_{x,z} \left(\partial_t (\chi_{\mathrm{i}} u)^\flat - (\partial_t(\chi_{\mathrm{i}} u))^\flat - \chi_{\mathrm{i}} \partial_t u^\flat + \chi_{\mathrm{i}} (\partial_t u)^\flat \right) \cdot \nabla_{x,z} v^\dagger \mathrm{d}x\mathrm{d}z \\
&\qquad + \int_S P_a (\nabla_{x,z}\chi_{\mathrm{i}}) \cdot \left((\partial_t u^\flat - (\partial_t u)^\flat) \nabla_{x,z} v^\dagger - \nabla_{x,z} \left(\partial_t u^\flat - (\partial_t u)^\flat \right) v^\dagger \right) \mathrm{d}x\mathrm{d}z.
\end{aligned}$$

From now on, we consider the case $\mathrm{i} = 1$, and assume $u = \chi_1 U_2$, $v = U_2$. The case $\mathrm{i} = a$ can be obtained by symmetric arguments. Then, from (5.108), we have that

$$\begin{aligned}
([[\partial_t, \mathcal{G}_a], \chi_{\mathrm{i}}]\, u, v) \leqslant{} & C \Big(\left\| \nabla_{x,z} \left((\chi_1 u)^\flat - \chi_1 u^\flat \right) \right\|_{L^2(S)} + \| \partial_x \chi_1 u^\flat \|_{L^2(S)} \Big) \left\| \nabla_{x,z} v^\dagger \right\|_{L^2(S)} \\
& + C \left\| \nabla_{x,z} \left(\partial_t (\chi_{\mathrm{i}} u)^\flat - (\partial_t(\chi_{\mathrm{i}} u))^\flat - \chi_{\mathrm{i}} \partial_t u^\flat + \chi_{\mathrm{i}} (\partial_t u)^\flat \right) \right\|_{L^2(S)} \left\| \nabla_{x,z} v^\dagger \right\|_{L^2(S)} \\
& + C \left(\| \partial_x \chi_1 \nabla_{x,z} u^\flat \|_{L^2(S)} + \| \partial_x \chi_1 \nabla_{x,z} \left(\partial_t u^\flat - (\partial_t u)^\flat \right) \|_{L^2(S)} \right) \left\| v^\dagger \right\|_{L^2(S)} \\
& + C \| \partial_x \chi_1 \left(\partial_t u^\flat - (\partial_t u)^\flat \right) \|_{L^2(S)} \left\| \nabla_{x,z} v^\dagger \right\|_{L^2(S)}.
\end{aligned}$$

Now, we begin by noticing that, from (5.107), and Poincaré inequality,

$$\left\| v^\dagger \right\|_{L^2(S)} + \left\| \nabla_{x,z} v^\dagger \right\|_{L^2(S)} \leqslant C \left(|\mathfrak{B} U_2|_{L^2} + |U_2|_{L^2} \right).$$

Also, from Lemma 5.15 and step c). in the proof of (5.59)

$$\begin{aligned}
& \| \partial_x \chi_1 \left(\partial_t u^\flat - (\partial_t u)^\flat \right) \|_{L^2(S)} + \| \partial_x \chi_1 \nabla_{x,z} \left(\partial_t u^\flat - (\partial_t u)^\flat \right) \|_{L^2(S)} \\
& \quad \leqslant \frac{C}{A} \left(\| \partial_t u^\flat - (\partial_t u)^\flat \|_{L^2(S)} + \| \nabla_{x,z} \left(\partial_t u^\flat - (\partial_t u)^\flat \right) \|_{L^2(S)} \right) \\
& \quad \leqslant \frac{C}{A} \left(|\mathfrak{B} U_2|_{L^2} + |U_2|_{L^2} \right).
\end{aligned}$$

Also, using again Lemma 5.15 and step a). in the proof of (5.59),

$$\begin{aligned}
\| \partial_x \chi_1 u^\flat \|_{L^2(S)} + \| \partial_x \chi_1 \nabla_{x,z} u^\flat \|_{L^2(S)} & \leqslant \frac{C}{A} \left\| u^\flat \|_{L^2(S)} + \| \nabla_{x,z} u^\flat \|_{L^2(S)} \right) \\
& \leqslant \frac{C}{A} \left(|\mathfrak{B} U_2|_{L^2} + |U_2|_{L^2} \right).
\end{aligned}$$

In consequence, we are left to prove the following estimations

$$\left\| \nabla_{x,z} \left((\chi_1 u)^\flat - \chi_1 u^\flat \right) \right\|_{L^2(S)} \leqslant \frac{C}{A} \left(|\mathfrak{B} U_2|_{L^2} + |U_2|_{L^2} \right), \tag{5.109}$$

and, finally,

$$\left\| \nabla_{x,z} \left(\partial_t (\chi_1 u)^\flat - (\partial_t (\chi_1 u))^\flat - \chi_1 \partial_t u^\flat + \chi_1 (\partial_t u)^\flat \right) \right\|_{L^2(S)} \leqslant \frac{C}{A} \left(|\mathfrak{B} U_2|_{L^2} + |U_2|_{L^2} \right), \tag{5.110}$$

so that we can conclude the desired result.

We begin by noticing that $w = (\chi_1 u)^\flat - \chi_1 u^\flat$ solves the equation

$$\begin{cases} \nabla_{x,z} \cdot P_a \nabla_{x,z} w = [\chi_1, \nabla_{x,z} \cdot P_a \nabla_{x,z}] u^\flat \quad \text{in } S, \\ w|_{z=0} = 0, \\ \partial_n^{P_a} w|_{z=-1} = [\chi_1, \partial_{\mathbf{n}}^{P_a}] u^\flat|_{z=-1}. \end{cases}$$

Then, the usual energy estimate for this problems leads to

$$\left\| \nabla_{x,z} (\chi_1 u)^\flat - \chi_1 u^\flat \right\| \leqslant \frac{C}{A} \left(\|u^\flat\|_{L^2(\mathcal{S})} + \|\nabla_{x,z} u^\flat\|_{L^2(S)} \right) \leqslant \frac{C}{A} \left(|\mathfrak{B} U_2|_{L^2} + |U_2|_{L^2} \right),$$

where we used Lemma 5.15 for the first estimate. Next, we show (5.110). We denote now by $w = \partial_t (\chi_1 u)^\flat - (\partial_t (\chi_1 u))^\flat - \chi_1 \partial_t u^\flat + \chi_1 (\partial_t u)^\flat$, which solves in S,

$$\begin{aligned} \nabla_{x,z} \cdot P_a \nabla_{x,z} w = & - \nabla_{x,z} \cdot \partial_t P_a \nabla_{x,z} \left((\chi_1 u)^\flat - \chi_1 u^\flat \right) + [\chi_1, \nabla_{x,z} \cdot P_a \nabla_{x,z}] \left(\partial_t u^\flat - (\partial_t u)^\flat \right) \\ & + [\chi_1, \nabla_{x,z} \cdot P_a \nabla] u^\flat, \end{aligned}$$

and has the boundary conditions

$$w|_{z=0} = 0, \quad \partial_n^{P_a} w|_{z=-1} = 0.$$

Then, we have the estimate

$$\|\nabla_{x,z} w\|_{L^2(\mathcal{S})} \leqslant C \left(\left\| \nabla_{x,z} \left((\chi_1 u)^\flat - \chi_1 u^\flat \right) \right\|_{L^2(S)} + \frac{1}{A} \left\| \partial_t u^\flat - (\partial_t u)^\flat \right\|_{H^1(S)} + \frac{1}{A} \|u^\flat\|_{H^1(S)} \right).$$

Ergo, in view of the estimations computed above, we obtained the desired result. ∎

A.2. Proof of Lemma 5.19

Proof of Lemma 5.19. Using the same notation as in the proof of Lemma 5.18, from Divergence Theorem we have that, for every $u, v \in C_b^\infty(\mathbb{R})$,

$$(\mathcal{G}_{c,a} u, v) = \int_S P_a \nabla_{x,z} u_a^\flat \cdot \nabla_{x,z} v_a^\flat \mathrm{d}x \mathrm{d}z,$$

where $u_a^\flat$ and $v_a^\flat$ are the solutions to the elliptic problem without flat bottom (5.88) associated to u and v respectively. Similarly, for the flat-bottom problem, one has

$$(\mathcal{G}_{c,1} u, v) = \int_S P_1 \nabla_{x,z} u_1^\flat \cdot \nabla_{x,z} v_1^\flat \mathrm{d}x \mathrm{d}z.$$

In particular, we shall consider $u = \chi_1 U_2$, and write

$$\begin{aligned}
((\mathcal{G}_{c,a} - \mathcal{G}_{c,1})u, u) &= \int_S P_a \nabla_{x,z} u_a^\flat \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z - \int_S P_1 \nabla_{x,z} u_1^\flat \cdot \nabla_{x,z} u_1^\flat \mathrm{d}x\mathrm{d}z \\
&= \int_S (P_a - P_1) \nabla_{x,z} u_a^\flat \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z + \int_S P_1 \nabla_{x,z} (u_a^\flat - u_1^\flat) \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z \\
&\quad + \int_S P_1 \nabla_{x,z} u_1^\flat \cdot \nabla_{x,z} (u_a^\flat - u_1^\flat) \mathrm{d}x\mathrm{d}z.
\end{aligned}$$

Then, using the weight function ω_a defined in (5.91), we have that

$$\begin{aligned}
|((\mathcal{G}_{c,a} - \mathcal{G}_{c,1})u, u)| &\leqslant C \|\omega_2 \nabla_{x,z} u_a^\flat\|_{L^2(S)} \|\nabla_{x,z} u_a^\flat\|_{L^2(S)} \\
&\quad + C \|\nabla_{x,z}(u_a^\flat - u_1^\flat)\|_{L^2(S)} \left(\|\nabla_{x,z} u_a^\flat\|_{L^2(S)} + \|\nabla_{x,z} u_1^\flat\|_{L^2(S)} \right).
\end{aligned}$$

Consequenty, from (5.95), (5.102) and (5.103), we obtain the desired estimation.

On the other hand, if $u = \chi_a U_2$, for this particular case, from the proof of (5.95) and Lemma 5.15, we have that

$$\|\nabla_{x,z} u_1^\flat\|_{L^2(S)} \leqslant \frac{C}{A} \left(|\mathfrak{B} U_2|_{L^2} + |U_2|_{L^2} \right). \tag{5.111}$$

Hence, we decompose

$$\begin{aligned}
((\mathcal{G}_{c,a} - \mathcal{G}_{c,1})u, u) &= \int_S P_a \nabla_{x,z} u_a^\flat \cdot \nabla_{x,z} u_a^\flat \mathrm{d}x\mathrm{d}z - \int_S P_1 \nabla_{x,z} u_1^\flat \cdot \nabla_{x,z} u_1^\flat \mathrm{d}x\mathrm{d}z \\
&= \int_S P_a \nabla_{x,z} u_a^\flat \cdot \nabla_{x,z} (u_a^\flat - u_1^\flat) \mathrm{d}x\mathrm{d}z + \int_S P_a \nabla_{x,z} u_a^\flat \cdot \nabla_{x,z} u_1^\flat \mathrm{d}x\mathrm{d}z \\
&\quad - \int_S P_1 \nabla_{x,z} u_1^\flat \cdot \nabla_{x,z} u_1^\flat \mathrm{d}x\mathrm{d}z.
\end{aligned}$$

This implies that

$$\begin{aligned}
|((\mathcal{G}_{c,a} - \mathcal{G}_{c,1})u, u)| &\leqslant C \|\nabla_{x,z} u_a^\flat\|_{L^2(S)} \left(\|\nabla_{x,z}(u_a^\flat - u_1^\flat)\|_{L^2(S)} + \|\nabla_{x,z} u_1^\flat\|_{L^2(S)} \right) \\
&\quad + \|\omega_1 \nabla_{x,z} u_1^\flat\|_{L^2(S)} \|\nabla_{x,z} u_1^\flat\|_{L^2(S)}.
\end{aligned}$$

We conclude by using (5.95), (5.102) (with ω_1 instead of ω_a, which is the appropriate case when $u = \chi_a U_2$), (5.103) and (5.111). ■

A.3. Proof of Lemma 5.20

This result was actually proved by in [Ming-Rousset-Tzvetkov [14] Lemma 5.8]. Even though in [14], lemma 5.20 was intended for flat bottom, the idea is based on properties of the DN operator that both domains ($a_\varepsilon = 0$ and $a_\varepsilon \neq 0$) share. Nevertheless, for the sake of completeness, we give the proof here.

Before we start with the proof of Lemma 5.20, we present the following auxiliar Lemma:

Lemma 5.23. *The operator $\mathcal{G}[\eta_c, 1]$ verifies*

$$\mathcal{G}[\eta_c, a_\varepsilon] = |D_x| + \mathcal{G}_c^0(x, D_x), \tag{5.112}$$

where $\mathcal{G}_c^0$ is a bounded (pseudo-differential) operator on H^s of order 0.

We refer to [12, Theorem 3.10] for the proof of the Lemma, or [16, Lemma 3.5] for the proof in the case d = 3 and flat bottom.

We can re-write the linear equation (5.47) $\partial_t \mathbf{U} = J\Lambda_c \mathbf{U}$ as

$$\begin{cases} \mathcal{G}[\eta_c, a_\varepsilon]U_2 = \partial_t U_1 \partial_x(v_c U_1) \\ \mathcal{P}_c U_1 = \partial_t U_2 + (w_c + g)U_1 + v_c \partial_x U_2. \end{cases} \tag{5.113}$$

Notice that at the LHS of both equations we have elliptic operator of order 1 and 2. More precisely,

$$\mathcal{P}_c = b\partial_x \left(\frac{\partial_x \cdot}{(1 + |\partial_x \eta_c|^2)^{\frac{3}{2}}} \right).$$

is an elliptic operator of order 2 and $\mathcal{G}[\eta_c, a_\varepsilon]$, of order 1. Indeed, this is a consequence of Lemma 5.23 and the fact that η_c is smooth. From the second equation in (5.113), using the elliptic operator of order 2, we have that

$$|U_1|_{H^{m+5/2}} \leqslant C_m \left(|\partial_t U_2|_{H^{m+1/2}} + |U_2|_{H^{m+3/2}} + |U_1|_{H^m} \right).$$

Then, from the interpolation inequality

$$|U_1|_{H^m} \leqslant \delta |U_1|_{H^{m+5/2}} + C_\delta |U_1|_{L^2}, \tag{5.114}$$

we obtain

$$|U_1|_{H^{m+5/2}} \leqslant C_m \left(|\partial_t U_2|_{H^{m+1/2}} + |U_2|_{H^{m+3/2}} + |U_1|_{L^2} \right).$$

In a similar fashion, using the first equation and the fact that th eDN operator is of order one,

$$|U_2|_{H^{m+2}} \leqslant C_m \left(|\partial_t U_1|_{H^{m+1}} + |U_1|_{H^{m+2}} + |U_2|_{L^2} \right).$$

Then, taking δ sufficiently small,

$$|U_1|_{H^{m+5/2}} + |U_2|_{H^{m+2}} \leqslant C_m \left(|\partial_t U_1|_{H^{m+1}} + |\partial_t U_2|_{H^{m+1/2}} + |\mathbf{U}|_{L^2} \right),$$

which is the thesis of the lemma with $l = 0$.

For the rest of the cases $l \geqslant 0$, we proceed with induction argument. We derivate in time (5.113) and obtain, for the first equation, that

$$\mathcal{G}[\eta_c, a_\varepsilon]\partial_t^l U_2 = \partial_t^{l+1} U_1 + \partial_t^l \partial_x(v_c U_1) - [\partial_t^l, \mathcal{G}[\eta_c a_\varepsilon]]U_2 := F_1.$$

Then, using Proposition 5.7, we have that

$$|F_1|_H^{m+1} \leqslant C_{m,l} \left(|\partial_t^{l+1} U_1|_{H^{m+1}} + |\langle \partial_t \rangle^l U_1|_{H^{m+2}} + |\langle \partial_t \rangle^{l-1} U_2|_{H^{m+2}} \right)$$

Finally, the fact that the DN operator is elliptic, we get

$$|\partial_t U_2|_H^{m+2} \leqslant C_{m,l}\left(|\partial_t^{l+1}U_1|_{H^{m+1}} + |\langle\partial_t\rangle^l U_1|_{H^{m+2}} + |\langle\partial_t\rangle^{l-1}U_2|_{H^{m+2}}\right) \tag{5.115}$$

On the other hand, from the second equation in (5.113), derivating l-times in time, we have that

$$\mathcal{P}_c\partial_t^l U_1 = \partial_t^{l+1}U_2 + \partial_t^l((w_c+g)U_1) + \partial_t^l(v_c\partial_x U_2) - [\partial_t^l, \mathcal{P}_c]U_1 = F_2.$$

We obtain that

$$|F_2|_{H^{m+1/2}} \leqslant C_{m,l}\left(|\partial_t^{l+1}U_2|_{H^{m+1/2}} + |\langle\partial_t\rangle^l U_2|_{H^{m+3/2}} + |\langle\partial_t\rangle^l U_1|_{H^{m+5/2}}\right),$$

and by elliptic regularity,

$$|\partial_t^l U_1|_{H^{m+5/2}} \leqslant C_{m,l}\left(|\partial_t^{l+1}U_2|_{H^{m+1/2}} + |\langle\partial_t\rangle^l U_2|_{H^{m+3/2}} + |\langle\partial_t\rangle^l U_1|_{H^{m+5/2}}\right). \tag{5.116}$$

We combine (5.115) and (5.116) and obtain

$$|\partial_t^l \mathbf{U}|_{H^{m+5/2}\times H^{m+2}} \leqslant C_{m,l}\left(|\partial_t^{l+1}\mathbf{U}|_{H^{m+1}\times H^{m+1/2}} + |\partial_t^l\mathbf{U}|_{H^{m+2}\times H^{m+3/2}} + |\langle\partial_t\rangle^l\mathbf{U}|_{H^{m+5/2}\times H^{m+2}}\right).$$

We get the desired result after using the interpolation inequality

$$|\partial_t\mathbf{U}|_{H^{m+2}\times H^{m+3/2}} \leqslant \delta|\partial_t^l\mathbf{U}|_{H^{m+5/2}\times H^{m+2}} + C_\delta|\partial_t^l\mathbf{U}|_{L^2}$$

and the induction hypothesis.

A.4. Proof of Proposition 5.21

We will use approximate sequence of solutions $\{\mathbf{U}^n\}$ to prove there exists a global in time solution $\mathbf{U}_r$ of (5.85). Let $\{T_n\}$ be a strictly increasing sequence such that $T_n > 0$ and $T_\rightarrow\infty$ as $n\to\infty$. Assume that $\mathbf{U}^n$ is the solution to (5.85) in the time interval $[-T_n, -T]$, for $T_n, T \geqslant$ (possibly close together), which is possible (at least in a smaller time interval $[-T_n, -T]$, $0 \leqslant T \leqslant T_n$) because the water-waves problem is locally well-posed.

To prove global existence, we shall make use of an a priori estimate for the solution $\mathbf{U}_r$, stated in Proposition 5.24. This proposition can be shown using the same arguments as in [16, Theorem 7.1]. Indeed, it follows after an exhaustive study of the DN operator and estimations regarding its derivatives, estimations that still hold in our case. We shall give a outline of the proof after proving Proposition 5.21. It reads as follows:

Proposition 5.24. *Let $\boldsymbol{U}^n$ be a smooth of (5.85) on $[-T_n, -T]$, for $T_m, T \geqslant 0$, satisfying $h\|a_\varepsilon\|_{L^\infty} - \|\eta_{ap}\|_{L^\infty} - \|\eta_r\|_{L^\infty} \geqslant h_{\min} > 0$. Then, for any $m \geqslant 2$, $s \geqslant 5$, and $t \in [-T_n, -T]$, we have the estimate*

$$|\boldsymbol{U}^n(t)|^2_{X^{m+3}} \leqslant \omega\left(|\boldsymbol{r}_{ap}|_{X_t^{m+3}} + |\boldsymbol{U}_{ap}|_{X_t^{m+s}} + |\boldsymbol{U}^n|_{X_t^{m+3}}\right)$$
$$\left(|\boldsymbol{r}_{ap}|_{X_t^{m+3}} + \int_{-T_n}^t \left(|\boldsymbol{U}^n(\tau)|^2_{X^{m+3}} + |\boldsymbol{r}_{ap}(\tau)|_{X^{m+3}}\right)\mathrm{d}\tau\right)$$

where ω is a continuous increasing function.

Here, the seminorms $|\cdot|_{X^k_t}$ denote the seminorm in X^k defined in a finite interval space, in this case, defined in $[-T_n,-T]$.

We use Proposition 5.24 to prove that $\mathbf{U}^n$ is well defined in the whole time interval $[-T_n,0]$. Because of the decay estimate for $\mathbf{r}_{ap}$ and $\mathbf{U}_{ap}$, from Proposition 5.24, we have that for $t\in[-T_n,-T]$,

$$\begin{aligned}|\mathbf{U}^n(t)|^2_{X^{m+3}} \leqslant \omega\left(\tilde{C}_{m,N}+C_{m,N}\rho+|\mathbf{U}^n|_{X^{m+3}_t}\right)& \\ \left(\int_{-T_n}^t |\mathbf{U}^n(\tau)|^2_{X^{m+3}}\mathrm{d}\tau + A^{(2N+1)/4}C_{N,s}(\delta_0)\rho^{2(N+1)}\mathrm{e}^{-(N+1)\delta_0 c|t|}\right).\end{aligned}$$

Define

$$T^*=\inf\{T\in[0,T_n]:\forall t\in[-T_n,-T],\ |\mathbf{U}^n(t)|_{X^{m+3}}\leqslant 1,\ ha_\varepsilon-\|\eta_{ap}\|_{L^\infty}-\|\eta_r\|_{L^\infty}\geqslant h_{min}>0\}$$

In particular, if $t\in[-T_n,-T^*]$, we obtain

$$\begin{aligned}&|\mathbf{U}^n(t)|^2_{X^{m+3}}\\ &\leqslant \omega\left(\tilde{C}_{m,N}+C_{m,N}\rho\right)\left(\int_{-T_n}^t |\mathbf{U}^n(\tau)|^2_{X^{m+3}}\mathrm{d}\tau + A^{(2N+1)/4}C_{N,s}(\delta_0)\rho^{2(N+1)}\mathrm{e}^{-(N+1)\delta_0 c|t|}\right).\end{aligned}$$

We note that, using the equation above,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\omega\left(\tilde{C}_{m,N}+C_{m,N}\rho\right)t}\int_{-T_n}^t |\mathbf{U}^n(\tau)|^2_{X^{m+3}}\mathrm{d}\tau\right)\\ &\quad\leqslant \omega\left(\tilde{C}_{m,N}+C_{m,N}\rho\right)\mathrm{e}^{-\omega\left(\tilde{C}_{m,N}+C_{m,N}\rho\right)t}A^{(2N+1)/4}C_{N,s}(\delta_0)\rho^{2(N+1)}\mathrm{e}^{-(N+1)\delta_0 c|t|}.\end{aligned}\tag{5.117}$$

Now, since ω is continuous, we can take N large enough and ρ small enough (that is, taking A very large, because $\rho=\mathrm{e}^{-\delta_0 A}$) such that

$$-(N+1)\delta_0 c>\omega\left(\tilde{C}_{m,N}+C_{m,N}\rho\right).$$

This implies that we can integrate in (5.117), and obtain

$$\int_{-T_n}^t |\mathbf{U}^n(\tau)|^2_{X^{m+3}}\mathrm{d}\tau\leqslant A^{(2N+1)/4}C_{N,s}(\delta_0)\rho^{2(N+1)}\mathrm{e}^{-(N+1)\delta_0 c|t|},$$

which ultimately leads to the following estimation for any $t\in[-T_n,-T^*]$:

$$|\mathbf{U}^n(t)|^2_{X^{m+3}}\leqslant A^{(2N+1)/4}C_{N,s}(\delta_0)\rho^{2(N+1)}\mathrm{e}^{-(N+1)\delta_0 c|t|}.\tag{5.118}$$

Taking ρ large, so that $h\|a_\varepsilon\|_{L^\infty}-\|\eta_{ap}\|_{L^\infty}-\|\eta_r\|_{L^\infty}\geqslant h_{min}>0$ and also $A^{(2N+1)/4}C_{N,s}(\delta_0)\rho^{2(N+1)}<1$, by the definition of $-T^*\leqslant 0$, which means that we can extend $\mathbf{U}^n$ to the whole interval $[-T_n,0]$.

We are left to prove global existence of equation (5.85). We use compactness argument to find $\mathbf{U}_r$ as a limit of $\{\mathbf{U}^n\}$.

Let $\chi \in C_0^\infty(-1/2, 1/2)$ such that $\chi(\tau) = 1$ for $\tau \in (-1/4, 1/4)$. We define

$$\tilde{\mathbf{U}}^n(t) = \chi\left(\frac{t}{-T_n}\right)\mathbf{U}^n(t),$$

where $\mathbf{U}^n$ is extended as zero for $t \leqslant -T_n$. Consequently, derivating in time

$$\partial_t \tilde{\mathbf{U}}^n(t) = -\frac{1}{T_n}\chi'\left(\frac{x}{-T_n}\right)\mathbf{U}^n(t) + \chi'\left(\frac{x}{-T_n}\right)\partial_t \mathbf{U}^n(t).$$

Then, for $t \leqslant 0$, from (5.118) we get the following estimation for $\tilde{\mathbf{U}}^n$ and $\partial_t \tilde{\mathbf{U}}^n$,

$$|\tilde{\mathbf{U}}^n(t)|^2_{H^{m+4}\times H^{m+7/2}} \leqslant A^{(2N+1)/4} C_{N,s}(\delta_0)\rho^{2(N+1)}\mathrm{e}^{-(N+1)\delta_0 c|t|} \quad \text{and}$$

$$|\partial_t \tilde{\mathbf{U}}^n(t)|_{H^{m+3}\times H^{m+5/2}} \leqslant A^{(N+1)/4} C_{N,s}(\delta_0)\rho^{2(N+1)}\mathrm{e}^{-(N+1)\delta_0 c|t|}.$$

Finally, we obtain that there exists a subsequence $\{\tilde{\mathbf{U}}^{n_k}\}$ and a limit

$$\mathbf{U}_r \in L^\infty\left((-\infty, 0], H^{m+4}\times H^{m+7/2}\right)$$

such that

$$\tilde{\mathbf{U}}^{n_k} \to \mathbf{U}_r \quad \text{in} \quad C_{loc}\left((-\infty, 0], H^{m+3}_{loc}\times H^{m+5/2}_{loc}\right) \quad \text{as} \quad n_k \to \infty,$$

and

$$|\mathbf{U}_r(t)|^2_{H^{m+4}\times H^{m+7/2}} \leqslant A^{(2N+1)/4} C_{N,s}(\delta_0)\rho^{2(N+1)}\mathrm{e}^{-(N+1)\delta_0 c|t|} \text{ for } t \in (-\infty, 0]$$

We have concluded the proof of Proposition 5.21.

A.4. Proof of proposition 5.24

The idea is to derivate equation (5.85) at least three times so that a linearised equation (linearisation around $\mathbf{U}$) is obtained, and then focus on finding estimations from the already known $\mathbf{U}_{ap}$ and $\mathbf{r}_{ap}$. To make explanation simpler, let us consider briefly the sistem $\partial_t \mathbf{U} = \mathcal{F}(\mathbf{U})$, for a generic solution $\mathbf{U}$. As in [10], we shall compute three derivatives of $\partial_t \mathbf{U} = \mathcal{F}(\mathbf{U})$. We begin by derivating one time the first equation of the system (that is $\partial_t \eta = \mathcal{G}[\eta, a_\varepsilon]\varphi$), we have

$$\partial_t^2 \eta = \mathcal{G}[\eta, a_\varepsilon]\partial_t\varphi + D_\eta \mathcal{G}[\eta, a_\varepsilon]\varphi \cdot \partial_t \eta. \tag{5.119}$$

Using the shape derivatives for the DN operator, stated in Proposition 5.6, we obtain

$$\partial_t^2 \eta = \mathcal{G}[\eta, a_\varepsilon]\partial_t\varphi - \mathcal{G}[\eta, a_\varepsilon](\tilde{Z}[\eta, \varphi]\partial_t \eta) - \partial_x(\tilde{v}[\eta, \varphi]\partial_t \eta), \tag{5.120}$$

where

$$\tilde{Z}[\eta, \varphi] = \frac{\mathcal{G}[\eta, a_\varepsilon]\varphi + \partial_x\varphi\partial_x\eta}{1 + |\partial_x\eta|^2} \quad \text{and} \quad \tilde{v}[\eta, \varphi] = \partial_x\varphi - \tilde{Z}[\eta, \varphi]\partial_x\eta.$$

We note that equation (5.120) is actually the first equation of the linearised system around $\mathbf{U}$ (see Subsection 5.2.3). Derivating (5.119) two more times, we obtain the following

$$\partial_t\partial_t^3 \eta = \mathcal{G}[\eta, a_\varepsilon]\partial_t^3\varphi + D_\eta G[\eta, a_\varepsilon]\varphi \cdot \partial_t^3 \eta + \mathcal{R}_1[\mathbf{U}] + \mathcal{Q}_1[\mathbf{U}], \tag{5.121}$$

where the nonlinear terms are defined by

$$\mathcal{Q}_1[\mathbf{U}] = 3D_\eta \mathcal{G}[\eta, a_\varepsilon]\partial_t^2\varphi \cdot \partial_t\eta \tag{5.122}$$

and

$$\mathcal{R}_1[\mathbf{U}] = \sum D_\eta^n \mathcal{G}[\eta, a_\varepsilon]\partial_t^\gamma \varphi \cdot (\partial_t^{\beta_1}\eta, \dots, \partial_t^{\beta_n}\eta). \tag{5.123}$$

where the sum is taken on indices satisfying

$$1 \leqslant n \leqslant 3,\ \beta_1 + \dots \beta_n + \gamma = 3,\ \gamma \leqslant 1,\ 1 \leqslant \beta_{\rm i} < 3, \forall {\rm i}.$$

Taking into account the notation $\tilde{Z}, \tilde{v}$ for the shape derivative of the DN operator, (5.121) turns into

$$\partial_t\partial_t^3\eta = \mathcal{G}[\eta, a_\varepsilon]\partial_t^3\varphi - \mathcal{G}[\eta, a_\varepsilon](\tilde{Z}[\eta,\varphi]\partial_t^3\eta) - \partial_x(\tilde{v}[\eta,\varphi]\partial_t^3\eta) + \mathcal{R}_1[\mathbf{U}] + \mathcal{Q}_1[\mathbf{U}].$$

Just like (5.120), the first three terms of the RHS compose the first equation of the linearised system around $\mathbf{U}$, evaluated in $\partial\mathbf{U}$. This will be more evident once we have computed the derivative of the second equation of $\partial_t\mathbf{U} = \mathcal{F}(\mathbf{U})$. Indeed, after derivating one time, we have that

$$\partial_t^2\varphi = -\tilde{v}\partial_x\partial_t\varphi + \tilde{Z}\mathcal{G}[\eta, a_\varepsilon]\partial_t\varphi - \tilde{Z}\mathcal{G}[\eta, a_\varepsilon](\tilde{Z}\partial_t\eta) - \left(g + \tilde{Z}\partial_x\tilde{v}\right)\partial_t\eta + \mathcal{P}[\eta]\partial_t\eta$$

where, for simplicity, $\tilde{Z} = \tilde{Z}[\eta,\varphi]$, $\tilde{v} = \tilde{v}[\eta,\varphi]$, and

$$\mathcal{P}[\eta] = b\partial_x\left(\frac{\partial_x\cdot}{(1+|\partial_x\eta|^2)^{3/2}}\right).$$

Replicating the idea implemented above, we derivate two more times and obtain the following

$$\begin{aligned}\partial_t\partial_t^3\varphi = &- \tilde{v}\partial_x\partial_t^3\varphi + \tilde{Z}\mathcal{G}[\eta, a_\varepsilon]\partial_t^3\varphi - \tilde{Z}\mathcal{G}[\eta, a_\varepsilon](\tilde{Z}\partial_t^3\eta) \\ &- \left(g + \tilde{Z}\partial_x\tilde{v}\right)\partial_t^3\eta + \mathcal{P}[\eta]\partial_t^3\eta + \mathcal{Q}_2[\mathbf{U}] + \mathcal{R}_2[\mathbf{U}],\end{aligned}$$

for

$$\mathcal{Q}_2[\mathbf{U}] = 3\beta\partial_x\left(D\mathcal{P}[\eta]\cdot(\partial_t\partial_x\eta, \partial_t^2\partial_x\eta)\right) \tag{5.124}$$

and

$$\begin{aligned}\mathcal{R}_2[\mathbf{U}] = &-[\partial_t^2, \tilde{v}]\partial_x\partial_t\varphi + [\partial_t^2, \tilde{Z}G[\eta, a_\varepsilon]\partial_t\varphi]\cdot\partial_t\eta \\ &+ [\partial_t^2, \tilde{Z}\tilde{v}]\partial_x\partial_t\eta + D^2\mathcal{P}[\eta]\cdot(\partial_x\partial_t\eta, \partial_x\partial_t\eta, \partial_x\partial_t\eta).\end{aligned} \tag{5.125}$$

In particular, gathering both equations, we have an almost linear system

$$\partial_t\partial_t^3\mathbf{U} = J\left(\Lambda[\mathbf{U}]\partial_t^3\mathbf{U} + \mathcal{Q}[\mathbf{U}]\right) + \mathcal{R}[\mathbf{U}],$$

for $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)^t$ and $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)^t$.

Now, let us go back to our subject of interest, system (5.85). From now on, $\mathbf{U}$ shall denote $\mathbf{U} = \mathbf{U}_{ap} + \mathbf{U}_r$. After derivating three times, equation (5.85) turns into

$$\partial_t\partial_t^3 U_r = J\left(\Lambda[\mathbf{U}]\partial_t^3\mathbf{U}_r + \mathcal{Q}[\mathbf{U}] - \mathcal{Q}[\mathbf{U}_{ap}]\right) + \mathbf{S}. \tag{5.126}$$

where

$$\mathbf{S} = \mathcal{R}[\mathbf{U}] - \mathcal{R}[\mathbf{U}_{ap}] + J\left(\Lambda[\mathbf{U}] - \Lambda[\mathbf{U}_{ap}]\right)\partial\mathbf{U}_{ap} - \partial\mathbf{r}_{ap}.$$

Proposition 5.24 is actually a consequence of combining two a priori estimates, one regarding the nonlinear terms in (5.85) and another one, regarding a more explicit description of the behaviour of the solution to $\mathbf{U}_r$. Indeed, we have the following:

Proposition 5.25. *Let $\boldsymbol{U}_r$ be a smooth of* (5.85) *on* $[-T, 0]$, *for* $T \geqslant 0$, *satisfying* $h\|a_\varepsilon\|_{L^\infty} - \|\eta_{ap}\|_{L^\infty} - \|\eta_r\|_{L^\infty} \geqslant h_{min} > 0$. *Then, for any* $m \geqslant 2$, $s \geqslant 5$, *and* $t \in [-T, 0]$, *we have the estimate*

$$|\boldsymbol{U}_r(t)|^2_{X^{m+3}} \leqslant \omega\left(|\boldsymbol{r}_{ap}|_{X_t^{m+3}} + |\boldsymbol{U}_{ap}|_{X^{m+s}_{\infty,t}} + |\boldsymbol{U}_r|_{X_t^{m+3}}\right)$$
$$\left(|\boldsymbol{r}_{ap}|_{X_t^{m+3}} + \int_{-T_n}^{t}\left(|\boldsymbol{U}_r(\tau)|^2_{X^{m+3}} + |\boldsymbol{S}(\tau)|_{X^{m+3}}\right)\mathrm{d}\tau\right)$$

where ω is a continuous increasing function.

In Proposition 5.25, the seminorm $|\cdot|_{X_t^k}$ is to be understand as the X^k-seminorm defined in the time interval $[-T, 0]$.

To estimate the source $\mathbf{S}$, since we $\mathbf{r}_{ap}$ is already controlled, we need to understand $\mathcal{R}[\mathbf{U}] - \mathcal{R}[\mathbf{U}_{ap}] + J\left(\Lambda[\mathbf{U}] - \Lambda[\mathbf{U}_{ap}]\right)\partial\mathbf{U}_{ap}$. To this end, we have the following result:

Proposition 5.26. *For $m \geqslant 2$ and $s \geqslant 5$ we have the estimate*

$$|\mathcal{R}[\boldsymbol{U}] - \mathcal{R}[\boldsymbol{U}_{ap}] + J\left(\Lambda[\boldsymbol{U}] - \Lambda[\boldsymbol{U}_{ap}]\right)\partial_t^3\boldsymbol{U}_{ap}|_{X^m} \leqslant \omega(|\boldsymbol{U}_{ap}|_{X^{m+s}} + |\boldsymbol{U}|_{X_\infty^{m+3}})|\boldsymbol{U}_r|_{X^{m+3}}.$$

To simplify computations, from now on we shall denote $\partial^\alpha = \partial_t^\beta\partial_x^\gamma, \beta + \gamma = |\alpha|$. Also, throughout we shall use $\langle\partial\rangle^m = (\partial^\alpha)_{|\alpha|\leqslant m}$, so that if $|\cdot|$ is a (semi) norm, $\langle\partial\rangle^m u|$ denotes the sum of (semi) norms of all the components of $\langle\partial\rangle^m u$. In other word,

$$\sum_{|\alpha|\leqslant m}\|\partial^\alpha\cdot\|_{H^1} \approx \|\langle\partial\rangle^m\cdot\|_{H^1}$$

In the following subsection, we have some apriori estimates regarding the DN operator and the H^s spaces we are dealing with. In Subsection A.4.2 we give the proof of Proposition 5.26. Finally, in Subsection A.4.3, we focus on the proof of Proposition 5.25.

A.4.1 Useful estimations to prove Propositions 5.25 and 5.26

First, we shall give a few a priori estimates on the spaces H^s used in the proofs of Propositions 5.25 and 5.26

We shall make use of the following proposition:

Proposition 5.27. *For $m \geqslant 2$, if $\sigma = 1/2$ or $\sigma = 1$, we have*

$$\begin{aligned} \|\langle\partial\rangle^m (uv)\|_{H^\sigma(\mathbb{R})} &\leqslant C\|\langle\partial\rangle^m u\|_{H^\sigma(\mathbb{R})}\|\langle\partial\rangle^m v\|_{H^\sigma(\mathbb{R})}, \\ \|\langle\partial\rangle^m (uv)\|_{H^\sigma(\mathbb{R})} &\leqslant C\|\langle\partial\rangle^m u\|_{W^{1,\infty}(\mathbb{R})}\|\langle\partial\rangle^m v\|_{H^\sigma(\mathbb{R})}. \end{aligned} \tag{5.127}$$

and if $\sigma = 0, 1/2, 1$

$$\begin{aligned} |\mathfrak{B}\langle\partial\rangle^m (uv)|_{\dot{H}^{1/2}_*(\mathbb{R})} &\leqslant C|\langle\partial\rangle^m u|_{W^{1,\infty}}|\langle\partial\rangle^m v|_{\dot{H}^{1/2}_*(\mathbb{R})}, \\ |\mathfrak{B}\langle\partial\rangle^m (uv)|_{\dot{H}^{1/2}_*(\mathbb{R})} &\leqslant C|\langle\partial\rangle^m u|_{\dot{H}^{1/2}_*(\mathbb{R})}|\langle\partial\rangle^m v|_{\dot{H}^{1/2}_*(\mathbb{R})}. \end{aligned} \tag{5.128}$$

Proof of Proposition 5.27. The first estimation in (5.127) for $\sigma = 1$ follows from the fact that H^1 is an algebra in dimension $\mathrm{d} = 1$.

Let us assume that $\sigma = 1/2$ and consider α, β such that $\alpha \leqslant \beta$, $\alpha + \beta = m$.

When $\beta \leqslant m - 1$, we have that

$$\begin{aligned} \|\partial^\alpha u \partial^\beta v\|_{H^{1/2}(\mathbb{R})} &\lesssim \|\partial^\alpha u \partial^\beta v\|_{H^1(\mathbb{R})} \lesssim \|\partial^\alpha u\|_{H^1(\mathbb{R})}\|\partial^\beta v\|_{H^1(\mathbb{R})} \\ &\lesssim \|\langle\partial\rangle^m u\|_{L^2(\mathbb{R})}\|\langle\partial\rangle^m v\|_{L^2(\mathbb{R})} \\ &\lesssim \|\langle\partial\rangle^m u\|_{H^{1/2}(\mathbb{R})}\|\langle\partial\rangle^m v\|_{H^{1/2}(\mathbb{R})}. \end{aligned} \tag{5.129}$$

On the other hand, if $\beta = m$ and $\alpha = 0$, using Sobolev embedding inequality $\|\cdot\|_{L^\infty} \leqslant \|\cdot\|_{H^1}$, valid in one dimension, we have that

$$\|u\langle\partial\rangle^m v\|_{H^{1/2}(\mathbb{R})} \lesssim \|u\|_{L^\infty(\mathbb{R})}\|\langle\partial\rangle^m v\|_{H^{1/2}(\mathbb{R})} \lesssim \|u\|_{H^1(\mathbb{R})}\|\langle\partial\rangle^m v\|_{H^{1/2}(\mathbb{R})}.$$

Finally, since $m \geqslant 2$,

$$\|u\langle\partial\rangle^m v\|_{H^{1/2}(\mathbb{R})} \lesssim \|\langle\partial\rangle^m u\|_{H^{1/2}(\mathbb{R})}\|\langle\partial\rangle^m v\|_{H^{1/2}(\mathbb{R})}.$$

The estimation is completed by summing all the terms involving the norm $\|\langle\partial\rangle^m \cdot\|_{H^\sigma}$.

We prove now the second estimation in (5.127). Using Sobolev-Slobodeckij norm definition we have that

$$\|f\|^2_{H^{1/2}(\mathbb{R})} = \|f\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x) - f(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y.$$

Then, we compute

$$\begin{aligned} \|uv\|^2_{H^{1/2}} \lesssim{}& \|u\|^2_{L^\infty(\mathbb{R})}\|v\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v(x) - v(y)|^2|u(x)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(x) - u(y)|^2|v(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y \\ \lesssim{}& \|u\|^2_{L^\infty(\mathbb{R})}\|v\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^\infty(\mathbb{R})}\|v\|^2_{H^{1/2}(\mathbb{R})} + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(x) - u(y)|^2|v(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y. \end{aligned}$$

For the last integral, we write

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(x) - u(y)|^2|v(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}}\int_{|x-y|\leqslant 1} \frac{|u(x) - u(y)|^2}{|x-y|^2}\mathrm{d}x|v(y)|^2\mathrm{d}y + \int_{\mathbb{R}}\int_{|x-y|\geqslant 1} \frac{|u(x) - u(y)|^2}{|x-y|^2}\mathrm{d}x|v(y)|^2\mathrm{d}y \\ &\lesssim \|u\|_{C^1(\mathbb{R})}\|v\|_{L^2(\mathbb{R})} + \|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}. \end{aligned}$$

Hence, we find that

$$\|uv\|^2_{H^{1/2}(\mathbb{R})} \lesssim \|u\|^2_{L^\infty(\mathbb{R})}\|v\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^\infty(\mathbb{R})}\|v\|^2_{H^{1/2}(\mathbb{R})} + \|u\|_{C^1(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}.$$

Using Morrey's inequality, we obtain the following

$$\|uv\|^2_{H^{1/2}(\mathbb{R})} \lesssim \|u\|^2_{W^{1,\infty}(\mathbb{R})}\|v\|^2_{H^{1/2}(\mathbb{R})}.$$

We conclude bu using the same idea as above (separating the derivative into α, β and making a similar argument as in equation (5.129)). We leave the case H^1, since it follows from using similar reasoning.

Finally, let us prove the first equation (5.128). We consider α, β such that $\alpha \leqslant \beta$, $\alpha + \beta = m$.

Let us assume $\beta \leqslant m-1$. We have that

$$|\partial^\alpha u \partial^\beta v|_{H^{1/2}_*(\mathbb{R})} = \|\mathfrak{B}\partial^\alpha u \partial^\beta v\|_{L^2(\mathbb{R})} \lesssim \|\partial^\alpha u\|_{L^\infty(\mathbb{R})}\|\mathfrak{B}\partial^\beta v\|_{L^2(\mathbb{R})}.$$

Since $\alpha \neq 0$,

$$|\partial^\alpha u \partial^\beta v|_{H^{1/2}_*(\mathbb{R})} \lesssim \|\langle\partial\rangle^m u\|_{W^{1,\infty}(\mathbb{R})}|\langle\partial\rangle^m v|_{\dot{H}^{1/2}_*(\mathbb{R})}.$$

On the other hand, if $\beta = m$ and $\alpha = 0$,

$$|u\partial^m v|_{\dot{H}^{1/2}_*(\mathbb{R})} \lesssim |u|_{L^\infty(\mathbb{R})}|\langle\partial\rangle^m v|_{\dot{H}^{1/2}_*(\mathbb{R})} \lesssim \|u\|_{W^{1,\infty}(\mathbb{R})}|\langle\partial\rangle^m v|_{\dot{H}^{1/2}_*(\mathbb{R})}.$$

We focus on the second equation of (5.128). Take α, β such that $\alpha \leqslant \beta \leqslant m$.

Assume $\alpha \leqslant \beta \leqslant m-1$. From the definition of $\mathfrak{B}$, $|\mathfrak{B}\cdot|_{H^s} \leqslant |\partial_x^{1/2}\cdot|_{H^{s-1/4}}$. This implies the following

$$|\mathfrak{B}\partial^\alpha u \partial^\beta v|_{L^2} \lesssim |\mathfrak{B}^{1/2}\partial^\alpha u|_{H^{3/4}}|\mathfrak{B}^{1/2}\partial^\alpha v|_{H^{3/4}} \lesssim |\partial_x^{1/2}\partial^\alpha u|_{H^{1/2}}|\partial_x^{1/2}\partial^\alpha v|_{H^{1/2}}.$$

Notice that $|\partial_x^{1/2}\cdot|^2_{H^{1/2}} = |\cdot|^2_{\dot{H}^{1/2}} + |\cdot|^2_{\dot{H}^1}$. For $\gamma = 1/2$ or $\gamma = 1$, we claim the following:

$$|\partial_x^\gamma f|_{L^2} \lesssim |\mathfrak{B}f|_{L^2} + |\mathfrak{B}\partial_x^\gamma f|_{L^2}.$$

This inequality follows from analysing separately low and high frequencies. We conclude (5.128) by noticing that $\alpha, \beta < m$.

The case $\alpha = 0$ and $\beta = m$ follows using the idea for (5.127). ■

The most challenging terms to estimate will be the ones involving the DN operator, since it is a nonlinear term that interacts non-locally with the surface η. We give the following two results to help deal with said terms:

Proposition 5.28. *Assume η_0, η_1 such that $h\|a_\varepsilon\|_{L^\infty} - \|\eta_0\| - \|\eta_1\| > 0$. For every $m \geqslant 2$, and $\sigma = -1/2, 1, 1/2$, we have the estimates:*

$$|\langle\partial\rangle^m G[\eta_0 + \eta_1, a_\varepsilon]\psi|_{H^\sigma(\mathbb{R})} \leqslant \omega\left(|\langle\partial\rangle^m \eta_0|_{H^{5/2}(\mathbb{R})} + |\langle\partial\rangle^m \eta_1|_{X^{m+3}_\infty}\right)|\langle\partial\rangle^m \mathfrak{B}\psi|_{H^{\sigma+1/2}(\mathbb{R})} \quad (5.130)$$

and

$$\begin{aligned}
&|\langle\partial\rangle^m \left(D^n_\eta G[\eta_0+\eta_1, a_\varepsilon]\psi\cdot(\zeta_1,\ldots,\zeta_n)\right)|_{H^\sigma(\mathbb{R})}\\
&\quad\leqslant \omega\left(|\langle\partial\rangle^m\eta_0|_{H^{5/2}(\mathbb{R})}+|\langle\partial\rangle^m\eta_1|_{X^{m+3}_\infty}\right)|\langle\partial\rangle^m\mathfrak{B}\psi|_{H^{\sigma+1/2}(\mathbb{R})}\\
&\quad\quad\left(\prod_{j=1}^{l}|\zeta_j|_{X^{m+3}_\infty}\right)\left(\prod_{j=l+1}^{n}|\langle\partial\rangle^{m+1}\zeta_j|_{H^1}\right).
\end{aligned}\tag{5.131}$$

Moreover,

$$\begin{aligned}
&\|[\partial^\alpha,\mathcal{G}[\eta_0+\eta_1,a_\varepsilon]]\psi\|_{H^{-1/2}(\mathbb{R})}\\
&\quad\leqslant \omega\left(|\langle\partial\rangle^m\eta_0|_{H^{5/2}(\mathbb{R})}+|\langle\partial\rangle^m\eta_1|_{X^{m+3}_\infty}\right)|\langle\partial\rangle^{m-1}\psi|_{\dot{H}^{1/2}_*(\mathbb{R})}.
\end{aligned}\tag{5.132}$$

In addition, we also present the following result:

Proposition 5.29. *Assume η_0,η_1 such that $h\|a_\varepsilon\|_{L^\infty}-\|\eta_0\|-\|\eta_1\|>0$. For every $m\geqslant 0$, and $\sigma\in(1,2)$, we have the estimates:*

$$\|\langle\partial\rangle^m G[\eta_0+\eta_1,a_\varepsilon]\psi\|_{C^\sigma(\mathbb{R})}\leqslant\omega\left(\|\langle\partial\rangle^{m+3}\eta_0\|_{H^1(\mathbb{R})}+\|\langle\partial\rangle^m\eta_1\|_{C^{\sigma+1}}\right)\|\langle\partial\rangle^m\psi\|_{C^{\sigma+1}(\mathbb{R})}\tag{5.133}$$

and

$$\begin{aligned}
&\|\langle\partial\rangle^m\left(D^n_\eta G[\eta_0+\eta_1,a_\varepsilon]\psi\cdot(\zeta_1,\ldots,\zeta_n)\right)\|_{C^\sigma(\mathbb{R})}\\
&\quad\leqslant\omega\left(\|\langle\partial\rangle^{m+3}\eta_0\|_{H^1(\mathbb{R})}+\|\langle\partial\rangle^m\eta_1\|_{C^{\sigma+1}}\right)\left(\|\langle\partial\rangle^m\zeta_1\|_{C^{\sigma+1}}\ldots\|\langle\partial\rangle^m\zeta_n\|_{C^{\sigma+1}}\right)\|\langle\partial\rangle^m\psi\|_{C^{\sigma+1}(\mathbb{R})}.
\end{aligned}\tag{5.134}$$

Moreover, for $n>l\geqslant 0$,

$$\begin{aligned}
&\|\langle\partial\rangle^m\left(D^n_\eta G[\eta_0+\eta_1,a_\varepsilon]\psi\cdot(h_1,\ldots,h_n)\right)\|_{H^1(\mathbb{R})}\\
&\quad\leqslant\omega\left(\|\langle\partial\rangle^{m+3}\eta_0\|_{H^1(\mathbb{R})}+\|\langle\partial\rangle^m\eta_1\|_{X^{m+3}_\infty}\right)\\
&\quad\quad\left(\prod_{j=1}^{l}\|\zeta_j\|_{X^{m+3}_\infty}\right)\left(\prod_{j=l+1}^{n}\|\langle\partial\rangle^{m+1}\zeta_j\|_{H^1}\right)\|\langle\partial\rangle^m\psi\|_{X^{m+3}_\infty(\mathbb{R})}.
\end{aligned}\tag{5.135}$$

The proofs of Propositions 5.28 and 5.29 are obtained by adapting the demonstrations of results proved in [16, Proposition 7.3] and [16, Proposition 7.13]. Indeed, it is sufficient to use the fact that

$$\|\nabla\phi\|_{H^s}\leqslant C_{s,\mathrm{d}}|\mathfrak{B}\psi|_{H^s},$$

(see [11, Corollary 2.40]) where ϕ is the solution of the elliptic equation associated with ψ as the border of the domain.

Notice that Propositions 5.28 and 5.29 enable estimations for the DN operator by η_0,η_1, even if one of them is not particularly smooth. In our case, this will be very helpful, as we will find that η_r might not be (a priori) as regular as η_{ap}.

A.4.1 Proof of Proposition 5.26

We begin by estimating the first term, $\mathcal{R}[\mathbf{U}]-\mathcal{R}[\mathbf{U}_{ap}]$:

Proposition 5.30. *For $m \geqslant 2$ and $s \geqslant 5$,*

$$|\langle\partial\rangle^m \mathcal{R}_1[\boldsymbol{U}] - \mathcal{R}_1[\boldsymbol{U}_{ap}]|_{H^1} \leqslant \omega\left(|\boldsymbol{U}_{ap}|_{X^{m+s}_\infty} + |\boldsymbol{U}_r|_{X^{m+3}}\right)|\boldsymbol{U}_r|_{X^{m+3}},$$

$$|\langle\partial\rangle^m \mathcal{R}_2[\boldsymbol{U}] - \mathcal{R}_2[\boldsymbol{U}_{ap}]|_{\dot{H}^{1/2}_*} \leqslant \omega\left(|\boldsymbol{U}_{ap}|_{X^{m+s}_\infty} + |\boldsymbol{U}_r|_{X^{m+3}}\right)|\boldsymbol{U}_r|_{X^{m+3}}.$$

Proof. We split the proof into various steps, one for each term involved.

Step 1: We prove that

$$|\langle\partial\rangle^m\left(\mathcal{R}_1[\mathbf{U}] - \mathcal{R}_1[\mathbf{U}_{ap}]\right)|_{H^1} \leqslant \omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty} + |\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}. \tag{5.136}$$

Recall the definition of $\mathcal{R}_1$,

$$\mathcal{R}_1[\mathbf{U}] = \sum D^n_\eta \mathcal{G}[\eta, a_\varepsilon]\partial_t^\gamma \varphi \cdot (\partial_t^{\beta_1}\eta, \ldots, \partial_t^{\beta_n}\eta).$$

with $1 \leqslant n \leqslant 3,\ \beta_1 + \ldots \beta_n + \gamma = 3,\ \gamma \leqslant 1,\ 1 \leqslant \beta_{\rm i} < 3, \forall {\rm i}$. Then, we need to estimate terms as

$$\begin{aligned}\langle\partial\rangle^m\Bigg(&D^n_\eta\mathcal{G}[\eta_{ap}+\eta_r, a_\varepsilon]\partial_t^\gamma\left(\varphi_{ap} - \varphi_r\right)\cdot(\partial_t^{\beta_1}(\eta_{ap}+\eta_r), \ldots, \partial_t^{\beta_n}(\eta_{ap}+\eta_r))\\ &- D^n_\eta\mathcal{G}[\eta_{ap}, a_\varepsilon]\partial_t^\gamma\varphi_{ap}\cdot(\partial_t^{\beta_1}\eta_{ap}, \ldots, \partial_t^{\beta_n}\eta_{ap})\Bigg).\end{aligned} \tag{5.137}$$

Since $D^n_\eta\mathcal{G}[\eta, a_\varepsilon]\phi\cdot(\zeta_1, \cdots, \zeta_n)$ is lineal with respect to the variables ϕ and $\zeta_{\rm i}$, $1 \leqslant {\rm i} \leqslant n$, then (5.137) can be simplifed into terms such as

$$^m\left(D^n_\eta\left(\mathcal{G}[\eta_{ap}+\eta_r, a_\varepsilon] - D^n_\eta\mathcal{G}[\eta_{ap}, a_\varepsilon]\right)\partial_t^\gamma\varphi_{ap}\cdot(\partial_t^{\beta_1}\eta_{ap}, \ldots, \partial_t^{\beta_n}\eta_{ap})\right), \tag{5.138}$$

$$^m\left(D^n_\eta\mathcal{G}[\eta_{ap}+\eta_r, a_\varepsilon]\partial_t^\gamma\varphi_r\cdot(\partial_t^{\overline{\beta}_1}\eta_{ap}, \ldots, \partial_t^{\overline{\beta}_l}\eta_{ap}, \partial_t^{\overline{\beta}_{l+1}}\eta_r, \cdots \partial_t^{\overline{\beta}_n}\eta_r)\right),\ 0 \leqslant l \leqslant n, \tag{5.139}$$

$$^m\left(D^n_\eta\mathcal{G}[\eta_{ap}+\eta_r, a_\varepsilon]\partial_t^\gamma\varphi_{ap}\cdot(\partial_t^{\overline{\beta}_1}\eta_{ap}, \ldots, \partial_t^{\overline{\beta}_l}\eta_{ap}, \partial_t^{\overline{\beta}_{l+1}}\eta_r, \cdots \partial_t^{\overline{\beta}_n}\eta_r)\right),\ 0 \leqslant l \leqslant n-1, \tag{5.140}$$

and $\overline{\beta}_{\rm i} = \beta_{\sigma({\rm i})}$ for some permutation σ of $\{1, \cdots n\}$. Then, using Proposition 5.28, we obtain the desired estimation for (5.139) and (5.140). For (5.138), we can write

$$\begin{aligned}&D^n_\eta\left(\mathcal{G}[\eta_{ap}+\eta_r, a_\varepsilon] - D^n_\eta\mathcal{G}[\eta_{ap}, a_\varepsilon]\right)\partial_t^\gamma\varphi_{ap}\cdot(\partial_t^{\beta_1}\eta_{ap}, \ldots, \partial_t^{\beta_n}\eta_{ap})\\ &\quad= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} D^n_\eta\mathcal{G}[\eta_{ap}+s\eta_r, a_\varepsilon]\partial_t^\gamma\varphi_{ap}\cdot(\partial_t^{\beta_1}\eta_{ap}, \ldots, \partial_t^{\beta_n}\eta_{ap})\mathrm{d}s\\ &\quad= \int_0^1 D^{n+1}_\eta\mathcal{G}[\eta_{ap}+s\eta_r, a_\varepsilon]\partial_t^\gamma\varphi_{ap}\cdot(\partial_t^{\beta_1}\eta_{ap}, \ldots, \partial_t^{\beta_n}\eta_{ap}, \eta_r)\mathrm{d}s.\end{aligned}$$

Using Proposition (5.29) (in particular, (5.135)), we conclude (5.136).

Step 2: We prove that

$$|\langle\partial\rangle^{m+2}\left(\tilde{Z}[\mathbf{U}] - \tilde{Z}[\mathbf{U}_{ap}]\right)|_{\dot{H}^{1/2}_*} \leqslant \omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty} + |\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}, \tag{5.141}$$

and, for $\gamma \geqslant 2$, and ψ sufficiently smooth,

$$|\langle\partial\rangle^m \left(\partial^\gamma \tilde{Z}[\mathbf{U}]\psi\right)|_{\dot{H}^{1/2}_*} \leqslant \omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty} + |\mathbf{U}|_{X^{m+3}}\right)|\langle\partial\rangle^m\psi|_{\dot{H}^{1/2}_*}. \tag{5.142}$$

Going back to the proof of (5.141), we will denote

$$\tilde{Z}_1[\eta,\varphi] = \frac{\mathcal{G}[\eta,a_\varepsilon]\varphi}{1+|\partial_x\eta|^2}, \quad Z_2[\eta,\varphi] = \frac{\partial_x\eta\partial_x\varphi}{1+|\partial_x\eta|^2}.$$

We focus on $\tilde{Z}_1$, since it involves the DN operator, which implies more analysis. We have that

$$\begin{aligned}\tilde{Z}_1[\mathbf{U}] - \tilde{Z}_2[\mathbf{U}_{ap}] = &\left(\frac{1}{1+|\partial_x\eta|^2} - \frac{1}{1+|\partial_x\eta_{ap}|^2}\right)\mathcal{G}[\eta_{ap},a_\varepsilon]\varphi_{ap} \\ &+ \frac{1}{1+|\partial_x\eta|^2}\left(\mathcal{G}[\eta,a_\varepsilon] - \mathcal{G}[\eta_{ap},a_\varepsilon]\right)\varphi_{ap} + \frac{1}{1+|\partial_x\eta|^2}\mathcal{G}[\eta,a_\varepsilon]\varphi.\end{aligned} \tag{5.143}$$

To treat the first term in (5.143), we use Proposition 5.27 and find

$$\begin{aligned}&\left|\langle\partial\rangle^{m+2}\left(\left(\frac{1}{1+|\partial_x\eta|^2} - \frac{1}{1+|\partial_x\eta_{ap}|^2}\right)\mathcal{G}[\eta_{ap},a_\varepsilon]\varphi_{ap}\right)\right|_{\dot{H}^{1/2}_*} \\ &\quad\lesssim \left|\langle\partial\rangle^{m+2}\left(\frac{1}{1+|\partial_x\eta|^2} - \frac{1}{1+|\partial_x\eta_{ap}|^2}\right)\right|_{\dot{H}^{1/2}_*}\left|\langle\partial\rangle^{m+2}\left(\mathcal{G}[\eta_{ap},a_\varepsilon]\varphi_{ap}\right)\right|_{W^{1,\infty}}.\end{aligned}$$

We apply Proposition 5.27 several times and obtain

$$\left\|\langle\partial\rangle^{m+2}\left(\frac{1}{1+|\partial_x\eta|^2} - \frac{1}{1+|\partial_x\eta_{ap}|^2}\right)\right\|_{H^{1/2}} \lesssim \omega\left(\|\mathbf{U}_{ap}\|_{X^{m+s}} + \|\mathbf{U}\|_{X^{m+2}}\right)\|\mathbf{U}_r\|_{X^{m+3}}. \tag{5.144}$$

Also, from (5.133), we have

$$\left\|\langle\partial\rangle^{m+2}\left(\mathcal{G}[\eta_{ap},a_\varepsilon]\varphi_{ap}\right)\right\|_{W^{1,\infty}} \lesssim \omega\left(|\mathbf{U}_{ap}|_{X^{m+s}}\right).$$

Now, we focus, on the second term of (5.143). We notice that if ψ is sufficiently smooth,

$$\left|\langle\partial\rangle^{m+2}\left(\frac{1}{1+|\partial_x\eta|^2}\psi\right)\right|_{\dot{H}^{1/2}_*} \leqslant \omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty} + |\mathbf{U}_r|_{X^{m+3}}\right)|\psi|_{\dot{H}^{1/2}_*}. \tag{5.145}$$

Indeed, we can write

$$\begin{aligned}\left|\langle\partial\rangle^{m+2}\left(\frac{1}{1+|\partial_x\eta|^2}\psi\right)\right|_{\dot{H}^{1/2}_*} \leqslant &\left|\langle\partial\rangle^{m+2}\left(\left(\frac{1}{1+|\partial_x\eta|^2} + \frac{1}{1+|\partial_x\eta_{ap}|^2}\right)\psi\right)\right|_{\dot{H}^{1/2}_*} \\ &+ \left|\langle\partial\rangle^{m+2}\left(\frac{1}{1+|\partial_x\eta|^2}\psi\right)\right|_{\dot{H}^{1/2}_*}\end{aligned}$$

Then, we use Proposition 5.27 and obtain

$$\left|\langle\partial\rangle^{m+2}\left(\frac{1}{1+|\partial_x\eta|^2}\psi\right)\right|_{\dot{H}_*^{1/2}} \leqslant \left|\langle\partial\rangle^{m+2}\left(\frac{1}{1+|\partial_x\eta|^2}+\frac{1}{1+|\partial_x\eta_{ap}|^2}\right)\right|_{\dot{H}_*^{1/2}} \left|\langle\partial\rangle^{m+2}\psi\right|_{\dot{H}_*^{1/2}} + \left|\langle\partial\rangle^{m+2}\left(\frac{1}{1+|\partial_x\eta|^2}\right)\right|_{\dot{H}_*^{1/2}} \left|\langle\partial\rangle^{m+2}\psi\right|_{\dot{H}_*^{1/2}}$$

We conclude (5.145) after applying Proposition 5.27 several times, like we did for (5.144). In particular, estimation (5.145) (and the fact that $|\cdot|_{\dot{H}^{1/2}} \leqslant |\cdot|_{H^1}$) implies

$$\begin{aligned}&\left|\langle\partial\rangle^{m+2}\left(\frac{1}{1+|\partial_x\eta|^2}\left(\mathcal{G}[\eta,a_\varepsilon]-\mathcal{G}[\eta_{ap},a_\varepsilon]\right)\varphi_{ap}\right)\right|_{\dot{H}_*^{1/2}}\\ &\quad\leqslant \omega\left(|\mathbf{U}_{ap}|_{X_\infty^{m+s}}+|\mathbf{U}_r|_{X^{m+3}}\right)|\langle\partial\rangle^{m+2}\left(\left(\mathcal{G}[\eta,a_\varepsilon]-\mathcal{G}[\eta_{ap},a_\varepsilon]\right)\varphi_{ap}\right)|_{H^1}.\end{aligned}$$

Then, we need to understand the term $\left(\mathcal{G}[\eta,a_\varepsilon]-\mathcal{G}[\eta_{ap},a_\varepsilon]\right)\varphi_{ap}$. Since $\eta=\eta_r+\eta_{apa}$, we write

$$\partial^2\left(\mathcal{G}[\eta,a_\varepsilon]-\mathcal{G}[\eta_{ap},a_\varepsilon]\right)\varphi_{ap}=\partial^2\left(\int_0^1 D_\eta\mathcal{G}[s\eta_r+\eta_{ap},a_\varepsilon]\varphi_{ap}\cdot\eta_r \mathrm{d}s\right).$$

In addition, we have that $\partial^2\left(D_\eta\mathcal{G}[s\eta_r+\eta_{ap},a_\varepsilon]\varphi_{ap}\cdot\eta_r\right)$ can be express as a sum of terms such as

$$\int_0^1 D_\eta^n\mathcal{G}[s\eta_r+\eta_{ap},a_\varepsilon]\partial^\beta\varphi_{ap}\cdot(\eta_r^{\gamma_1},\partial^{\gamma_2}h_1,\cdots,\partial^{\gamma_n}h_{n-1})\mathrm{d}s,$$

$n\geqslant 1$, $h_{\mathrm{i}}\in\{\eta_r,\eta_{ap}\}$, $\gamma\leqslant 2$, $1\leqslant \mathrm{i}\leqslant n-1$. Thus, we can use equation (5.135) from Proposition 5.29, like we did in Step 1, to get

$$|\langle\partial\rangle^{m+2}\left(\left(\mathcal{G}[\eta,a_\varepsilon]-\mathcal{G}[\eta_{ap},a_\varepsilon]\right)\varphi_{ap}\right)|_{H^1}\leqslant\omega\left(|\mathbf{U}_{ap}|_{X_\infty^{m+s}}+|\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}},$$

which ultimately leads to

$$\left|\langle\partial\rangle^{m+2}\left(\frac{1}{1+|\partial_x\eta|^2}\left(\mathcal{G}[\eta,a_\varepsilon]-\mathcal{G}[\eta_{ap},a_\varepsilon]\right)\varphi_{ap}\right)\right|_{\dot{H}_*^{1/2}}\leqslant\omega\left(|\mathbf{U}_{ap}|_{X_\infty^{m+s}}+|\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}.$$

Finally, we deal with the third term in (5.143). To do so, we can argue as for the second term in (5.143), but using Proposition 5.28.

We have concluded that

$$\left|\langle\partial\rangle^{m+2}\left(\tilde{Z}_1[\mathbf{U}]-\tilde{Z}_1[\mathbf{U}_{ap}]\right)\right|_{\dot{H}_*^{1/2}}\leqslant\omega\left(|\mathbf{U}_{ap}|_{X_\infty^{m+s}}+|\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}.$$

We are left to prove that

$$\left|\langle\partial\rangle^{m+2}\left(\tilde{Z}_2[\mathbf{U}]-\tilde{Z}_2[\mathbf{U}_{ap}]\right)\right|_{\dot{H}_*^{1/2}}\leqslant\omega\left(|\mathbf{U}_{ap}|_{X_\infty^{m+s}}+|\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}},$$

which follows after using Proposition 5.27 several times. Hence, we have proven (5.141).

To prove (5.142), since $\gamma \geqslant 2$, we write

$$\left|\langle\partial\rangle^m \left(\partial^\gamma \tilde{Z}[\mathbf{U}]\psi\right)\right|_{\dot{H}_*^{1/2}} \leqslant \left|\langle\partial\rangle^m \left(\partial^\gamma \left(\tilde{Z}[\mathbf{U}] - \tilde{Z}[\mathbf{U}_{ap}]\right)\psi\right)\right|_{\dot{H}_*^{1/2}} + \left|\langle\partial\rangle^m \left(\partial^\gamma \tilde{Z}[\mathbf{U}_{ap}]\psi\right)\right|_{\dot{H}_*^{1/2}}.$$

For the first term, we use (5.141) along with Proposition 5.27. For the second term, we use Proposition 5.29. We conclude the proof of estimation (5.142).

Step 3: Use (5.141) *to show that*

$$|\langle\partial\rangle^{m+2}\left(\tilde{v}[\mathbf{U}] - \tilde{v}[\mathbf{U}_{ap}]\right)|_{\dot{H}_*^{1/2}} \leqslant \omega\left(|\mathbf{U}_{ap}|_{X_\infty^{m+s}} + |\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}, \tag{5.146}$$

and, for $\gamma \geqslant 2$ and ψ smooth enough,

$$|\langle\partial\rangle^{m}\left(\partial^\gamma \tilde{v}[\mathbf{U}]\psi\right)|_{\dot{H}_*^{1/2}} \leqslant \omega\left(|\mathbf{U}_{ap}|_{X_*^{m+s}} + |\mathbf{U}_r|_{X^{m+3}}\right)|\langle\partial\rangle^m\psi|_{X^{m+3}}. \tag{5.147}$$

From the definition of $\tilde{v}$, we have that

$$\tilde{v}[\mathbf{U}] - \tilde{v}[\mathbf{U}_{ap}] = \partial_x\varphi_r - \left(\left(\tilde{Z}[\mathbf{U}] - \tilde{Z}[\mathbf{U}_{ap}]\right)\partial_x\eta_{ap} + \tilde{Z}[\mathbf{U}]\partial_x\eta_r\right).$$

Thus, from (5.141) and (5.142), using again Proposition 5.27, we obtain (5.146). Estimation (5.147) is a result of following a similar reasoning as for equation (5.142), with (5.146).

Step 4: We prove that

$$\begin{aligned}&|\langle\partial\rangle^m\left(\left[\partial_t^2, \tilde{v}[\mathbf{U}]\right]\partial_t\partial_x\varphi - \left[\partial_t^2, \tilde{v}[\mathbf{U}_{ap}]\right]\partial_t\partial_x\varphi_{ap}\right)|_{\dot{H}_*^{1/2}}\\ &\qquad\leqslant \omega\left(|\mathbf{U}_{ap}|_{X_\infty^{m+s}} + |\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}},\end{aligned} \tag{5.148}$$

and

$$\begin{aligned}&|\langle\partial\rangle^m\left(\left[\partial_t^2, \tilde{Z}[\mathbf{U}]\mathcal{G}[\eta, a_\varepsilon]\right]\partial_t\varphi - \left[\partial_t^2, \tilde{Z}[\mathbf{U}_{ap}]\mathcal{G}[\eta_{ap}, a_\varepsilon]\right]\partial_t\varphi_{ap}\right)|_{\dot{H}_*^{1/2}}\\ &\qquad\leqslant \omega\left(|\mathbf{U}_{ap}|_{X_\infty^{m+s}} + |\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}.\end{aligned} \tag{5.149}$$

Estimation (5.149) involves the DN operator, a nonlinear (nonlocal) term that deals with the surface and bottom, which means that it is much more challenging than estimation (5.148). Therefore, we shall focus in (5.149); (5.148) follows using similar arguments.

The commutator $[\partial_t^2, \tilde{Z}[\mathbf{U}]\mathcal{G}[\eta, a_\varepsilon]]\partial_t\varphi$ can be expressed as a sum of terms like the following

$$\mathcal{O}[\mathbf{U}] = \partial^{\gamma_0}\tilde{Z}[\mathbf{U}]D_\eta^n\mathcal{G}[\eta, a_\varepsilon]\partial_t\partial^\beta\varphi\cdot\left(\partial^{\gamma_1}\eta, \dots, \partial^{\gamma_n}\right),$$

the parameters under the conditions $\beta \leqslant 1$ and $\gamma_{\mathrm{i}} \leqslant 2$, for $0 \leqslant \mathrm{i} \leqslant n$. Such decomposition work for $[\partial_t^2, \tilde{Z}[\mathbf{U}_{ap}]\mathcal{G}[\eta_{ap}, a_\varepsilon]]\partial_t\varphi_{ap}$ as well, substituting $\mathcal{O}[\mathbf{U}]$ by $\mathcal{O}[\mathbf{U}_{ap}]$. We write

$$\begin{aligned}\mathcal{O}[\mathbf{U}] - \mathcal{O}[\mathbf{U}_{ap}] &= \left(\partial^{\gamma_0}\tilde{Z}[\mathbf{U}] - \partial^{\gamma_0}\tilde{Z}[\mathbf{U}_{ap}]\right)D_\eta^n\mathcal{G}[\eta_{ap}, a_\varepsilon]\partial_t\partial^\beta\varphi_{ap}\cdot\left(\partial^{\gamma_1}\eta_{ap}, \dots, \partial^{\gamma_n}\eta_{ap}\right)\\ &\quad+ \partial^{\gamma_0}\tilde{Z}[\mathbf{U}]\left(D_\eta^n\mathcal{G}[\eta, a_\varepsilon] - D_\eta^n\mathcal{G}[\eta_{ap}, a_\varepsilon]\right)\partial_t\partial^\beta\varphi_{ap}\cdot\left(\partial^{\gamma_1}\eta_{ap}, \dots, \partial^{\gamma_n}\eta_{ap}\right)\\ &\quad+ \partial^{\gamma_0}\tilde{Z}[\mathbf{U}]D_\eta^n\mathcal{G}[\eta, a_\varepsilon]\partial_t\partial^\beta\varphi_{ap}\cdot\left(\partial^{\gamma_1}\eta_{ap}, \dots, \partial^{\gamma_n}\eta_{ap}\right) + \mathcal{O}_\Sigma\\ &:= \mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3 + \mathcal{O}_\Sigma,\end{aligned} \tag{5.150}$$

where $\mathcal{O}_{\sum}$ is a sum of terms such as

$$\mathcal{O}_{\sum_l} = \partial^{\gamma_0}\tilde{Z}[\mathbf{U}]D^n_\eta\mathcal{G}[\eta,a_\varepsilon]\partial_t\partial^\beta\varphi_{ap}\cdot\left(\partial^{\overline{\gamma}_1}\eta_{ap},\dots,\partial^{\overline{\gamma}_l}\eta_{ap},\partial^{\overline{\gamma}_{l+1}}\eta_r,\dots,\partial^{\overline{\gamma}_{l+1}}\eta_r,\right)$$

for $\overline{\gamma}_{\mathrm{i}}\in\{\gamma_1,\dots,\gamma_n\}$, for all $1\leqslant \mathrm{i}\leqslant n$ and l is such that $l\leqslant n-1$.

To deal with the first term of (5.150), we use Proposition 5.27, along with the second equation of Proposition 5.29, (5.134), and the estimation proved in Step 2, 5.141. Then, since γ_0, we have that

$$\begin{aligned}
&|\langle\partial\rangle^m\mathcal{O}_1|_{\dot{H}^{1/2}_*}\\
&\quad\lesssim \left\|\langle\partial\rangle^m D^n_\eta\mathcal{G}[\eta_{ap},a_\varepsilon]\partial_t\partial^\beta\varphi_{ap}\cdot(\partial^{\gamma_1}\eta_{ap},\dots,\partial^{\gamma_n}\eta_{ap})\right\|_{W^{1,\infty}}\left|\langle\partial\rangle^{m+2}\left(\tilde{Z}[\mathbf{U}]-\tilde{Z}[\mathbf{U}_{ap}]\right)\right|_{\dot{H}^{1/2}_*}\\
&\quad\leqslant \omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty}+|\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}.
\end{aligned}$$

For the third term in (5.150), we use (5.142), from the Step 2, and find that

$$|\langle\partial\rangle^m\mathcal{O}_3|_{\dot{H}^{1/2}_*}\leqslant\omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty}+|\mathbf{U}_r|_{X^{m+3}}\right)|\langle\partial\rangle^m\left(D^n_\eta\mathcal{G}[\eta,a_\varepsilon]\partial_t\partial^\beta\varphi_{ap}\cdot(\partial^{\gamma_1}\eta_{ap},\dots,\partial^{\gamma_n}\eta_{ap})\right)|_{\dot{H}^{1/2}_*}.$$

Then, we use (5.131) from Proposition 5.28 with $l=n$, and obtain

$$\begin{aligned}
&|\langle\partial\rangle^m\left(D^n_\eta\mathcal{G}[\eta,a_\varepsilon]\partial_t\partial^\beta\varphi_{ap}\cdot(\partial^{\gamma_1}\eta_{ap},\dots,\partial^{\gamma_n}\eta_{ap})\right)|_{\dot{H}^{1/2}_*}\\
&\quad\leqslant\omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty}+|\mathbf{U}_r|_{X^{m+3}}\right)\left|\langle\mathfrak{B}\partial\rangle^{m+1}\partial_t\partial^\beta\varphi_r\right|_{H^{1+1/2}}\\
&\quad\leqslant\omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty}+|\mathbf{U}_r|_{X^{m+3}}\right)\left|\langle\partial\rangle^{m+3}\varphi_r\right|_{\dot{H}^{1/2}_*}\\
&\quad\leqslant\omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty}+|\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}.
\end{aligned}$$

For the term $\mathcal{O}_2$, we reason as we did for $\mathcal{O}_3$. Finally, we estimate $\mathcal{O}_{\sum_l}$ by using equation (5.142) and (5.131) (as we did for $\mathcal{O}_3$) for the terms involving φ_r (recall that $\varphi=\varphi_r+\varphi_{ap}$), and (5.135) for the terms with φ_{ap}.

Step 5: We show the estimations

$$\begin{aligned}
&|\langle\partial\rangle^m\left(\left[\partial_t^2,\tilde{Z}[\mathbf{U}]\tilde{v}[\mathbf{U}]\right]\partial_t\partial_x\eta-\left[\partial_t^2,\tilde{Z}[\mathbf{U}_{ap}]\tilde{v}[\mathbf{U}_{ap}]\right]\partial_t\partial_x\eta_{ap}\right)|_{\dot{H}^{1/2}_*}\\
&\quad\leqslant\omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty}+|\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}},
\end{aligned}\tag{5.151}$$

and

$$\begin{aligned}
&|\langle\partial\rangle^m\left(\left[\partial_t^2,\tilde{Z}[\mathbf{U}]D_\eta\mathcal{G}[\eta,a_\varepsilon]\varphi\right]\cdot\partial_t\eta-\left[\partial_t^2,\tilde{Z}[\mathbf{U}_{ap}]D_\eta\mathcal{G}[\eta_{ap},a_\varepsilon]\varphi_{ap}\right]\cdot\partial_t\eta_{ap}\right)|_{\dot{H}^{1/2}_*}\\
&\quad\leqslant\omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty}+|\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}.
\end{aligned}\tag{5.152}$$

Again, as in Step 4, we shall only give the proof of (5.152), since (5.151) follows from similar (less challenging) arguments. The commutator $\left[\partial_t^2,\tilde{Z}[\mathbf{U}]D_\eta\mathcal{G}[\eta,a_\varepsilon]\partial^\beta\varphi\right]\cdot\partial_t\eta$ con be expanded as a sum of terms such as

$$\partial^{\gamma_0}\tilde{Z}[\mathbf{U}]D^n_\eta D_\eta\mathcal{G}[\eta,a_\varepsilon]\partial^\beta\varphi\cdot(\partial^{\gamma_1},\dots\partial^{\gamma_{n-1}}\partial_t\partial^\gamma\eta)\,,$$

with $n \geqslant 1$, $\gamma_0 + \dots \gamma_{n-1} + \beta + \gamma = 2$, $\gamma \leqslant 1$. Then, the proof of (5.152) is concluded using the estimations in Step 4.

Step 6: We prove that

$$\begin{aligned}&\left|\langle\partial\rangle^m \left(D^2\mathcal{P}[\eta]\cdot(\partial_x\partial_t\eta,\partial_x\partial_t\eta,\partial_x\partial_t\eta) - D^2\mathcal{P}[\eta_{ap}]\cdot(\partial_x\partial_t\eta_{ap},\partial_x\partial_t\eta_{ap},\partial_x\partial_t\eta_{ap})\right)\right|_{\dot{H}^{1/2}_*}\\ &\quad\leqslant \omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty} + |\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}.\end{aligned} \tag{5.153}$$

Estimation (5.153) follows from the definition of $\mathcal{P}$ and Moser type estimates.

Step 7: Conclude that

$$|\langle\partial\rangle^m\left(\mathcal{R}_2[\mathbf{U}] - \mathcal{R}_12[\mathbf{U}_{ap}]\right)|_{\dot{H}^{1/2}_*} \leqslant \omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty} + |\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}. \tag{5.154}$$

Recall the definition of $\mathcal{R}_2$:

$$\begin{aligned}\mathcal{R}_2[\mathbf{U}] = &-[\partial_t^2,\tilde{v}]\partial_x\partial_t\varphi + [\partial_t^2,\tilde{Z}G[\eta,a_\varepsilon]\partial_t\varphi]\cdot\partial_t\eta\\ &+ [\partial_t^2,\tilde{Z}\tilde{v}]\partial_x\partial_t\eta + D^2\mathcal{P}[\eta]\cdot(\partial_x\partial_t\eta,\partial_x\partial_t\eta,\partial_x\partial_t\eta).\end{aligned}$$

Then, (5.154) is a consequence of Steps 4, 5 and 6.

We have completed the proof of Proposition 5.30. ■

Now, we can return to the proof of Proposition 5.26, that is, the estimation for $\mathcal{R}[\mathbf{U}] - \mathcal{R}[\mathbf{U}_{ap}] + J\left(\Lambda[\mathbf{U}] - \Lambda[\mathbf{U}_{ap}]\right)\partial_t^3\mathbf{U}_{ap}$. Since we already dealt with the $\mathcal{R}$ terms in Proposition 5.30, we are left to prove that

$$|\langle\partial\rangle^m\left(J\left(\Lambda[\mathbf{U}] - \Lambda[\mathbf{U}_{ap}]\right)\partial_t^3\mathbf{U}_{ap}\right)_1|_{H^1} \leqslant \omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty} + |\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}, \tag{5.155}$$

and

$$|\langle\partial\rangle^m\left(J\left(\Lambda[\mathbf{U}] - \Lambda[\mathbf{U}_{ap}]\partial_t^3\mathbf{U}_{ap}\right)\right)_2|_{\dot{H}^{1/2}_*} \leqslant \omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty} + |\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}. \tag{5.156}$$

Recall from the definition of Λ that

$$\left(J\Lambda[\mathbf{U}]\partial_t^3\mathbf{U}_{ap}\right)_1 = -\partial_x(\tilde{v}[\mathbf{U}]\partial_t^3\eta_{ap}) - \mathcal{G}[\eta,a_\varepsilon]\left(\tilde{Z}[\mathbf{U}]\partial_t^3\eta_{ap}\right) + \mathcal{G}[\eta,a\varepsilon]\partial_t^3\varphi_{ap},$$

and

$$\begin{aligned}\left(J\Lambda[\mathbf{U}]\partial_t^3\mathbf{U}_{ap}\right)_2 = &-\tilde{v}[\mathbf{U}]\partial_x\partial_t^3\varphi + \tilde{Z}[\mathbf{U}]G[\eta,a_\varepsilon]\partial_t^3\varphi - \tilde{Z}[\mathbf{U}]\mathcal{G}[\eta,a_\varepsilon](\tilde{Z}[\mathbf{U}]\partial_t^3\eta)\\ &-\left(g + \tilde{Z}[\mathbf{U}]\partial_x\tilde{v}[\mathbf{U}]\right)\partial_t^3\eta + \mathcal{P}[\eta]\partial_t^3\eta.\end{aligned}$$

We observe that the terms are of the same nature as the ones we estimated above. Thus, just like we did previous estimations, to prove (5.156), it suffices to make use of Propositions 5.28, 5.29 and Steps 2 and 3 of Proposition 5.30.

A.4.3 Proof of Proposition 5.25

Before we begin the proof, we shall give some useful estimations regarding the subprincipal term $\mathcal{Q}[\mathbf{U}] - \mathcal{Q}[\mathbf{U}_{ap}]$.

Proposition 5.31. *For $m \geqslant 2$ and $s \geqslant 5$, we have that*

$$\|\langle\partial\rangle^m \left(\mathcal{Q}_1[\boldsymbol{U}] - \mathcal{Q}_1[\boldsymbol{U}_{ap}]\right)\|_{H^{-1/2}} \leqslant \omega\left(|\boldsymbol{U}_{ap}|_{X^{m+s}_\infty} + |\boldsymbol{U}_r|_{X^{m+3}}\right)|\boldsymbol{U}_r|_{X^{m+2}}$$

and

$$\|\langle\partial\rangle^m \left(\mathcal{Q}_2[\boldsymbol{U}] - \mathcal{Q}_2[\boldsymbol{U}_{ap}]\right)\|_{H^{-1}} \leqslant \omega\left(|\boldsymbol{U}_{ap}|_{X^{m+s}_\infty} + |\boldsymbol{U}_r|_{X^{m+3}}\right)|\boldsymbol{U}_r|_{X^{m+2}}.$$

The proof is essentially a consequence of Propositions 5.28, 5.29 and Steps 2 and 3 of Proposition 5.30. We shall omit it. However, see [88, Proposition 7.27] for more details.

As in [88], we shall give the energy estimates on the equation satisfied by $\mathbf{W} = R[\mathbf{U}]\partial_t^3\mathbf{U}_r$, where $R[\mathbf{U}]$ is defined as

$$R[\mathbf{U}] = \begin{pmatrix} 1 & 0 \\ -\tilde{Z}[\mathbf{U}] & 1 \end{pmatrix}.$$

We find that $\mathbf{W}$ solves the problem

$$\partial_t\mathbf{W} = J\left(L[\mathbf{U}]\mathbf{W} - JRJ\left(\mathcal{Q}[\mathbf{U}] - \mathcal{Q}[\mathbf{U}_{ap}]\right)\right) + R\mathbf{S}. \tag{5.157}$$

for $\mathbf{L}$ defined such as $L[\mathbf{U}] = \left(R[\mathbf{U}]^{-1}\right)^t \Lambda[\mathbf{U}]R[\mathbf{U}]^{-1}$.

We present the following result, regarding the quadratic form associated to $L[\mathbf{U}]$:

Proposition 5.32. *We have the estimates*

$$(L[\boldsymbol{U}]\,\boldsymbol{W}, \boldsymbol{W}) + \|\boldsymbol{W}\|_{L^2}^2 \geqslant \frac{|\boldsymbol{W}|_{X^0}}{\omega\left(|\boldsymbol{U}_{ap}|_{X^7_\infty} + |\boldsymbol{U}_r|_{X^5}\right)},$$

$$(L[\boldsymbol{U}]\,\boldsymbol{W}, \boldsymbol{v}) \leqslant \omega\left(|\boldsymbol{U}_{ap}|_{X^7_\infty} + |\boldsymbol{U}_r|_{X^5}\right)|\boldsymbol{W}|_{X^0}|\boldsymbol{V}|_{X^0}.$$

Moreover, for $m \geqslant 2$ and $s \geqslant 5$, if $\alpha \leqslant m$,

$$\left|\left(\left[\partial^\alpha_{t,x}, L[\boldsymbol{U}]\right]\boldsymbol{W}, \boldsymbol{V}\right)\right| \leqslant \omega\left(|\boldsymbol{U}_{ap}|_{X^{m+s}_\infty} + |\boldsymbol{U}_r|_{X^{m+3}}\right)|\boldsymbol{W}|_{X^m}|\boldsymbol{V}|_{X^0}.$$

This is, again, a consequence of of Propositions 5.28, 5.29 and Steps 2 and 3 of Proposition 5.30, along with Sobolev embedding inequalities. We shall omit it. See [88, Proposition 7.28] for more details.

Thanks to Step 2 of Proposition 5.30, for $s \geqslant 5$, we have that

$$|\mathbf{W}|_{X^m} \leqslant \omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty} + |\mathbf{U}_r|_{X^{m+3}}\right)|\mathbf{U}_r|_{X^{m+3}}$$

and

$$|\mathbf{U}_r|_{X^{m+3}} \leqslant \omega\left(|\mathbf{U}_{ap}|_{X^{m+s}_\infty} + |\mathbf{U}_r|_{X^{m+3}}\right)\left(|\mathbf{W}|_{X^m}|\mathbf{U}_r|_{X^3}\right).$$

Ultimately, this means taht it is equivalent to estimate $\mathbf{W}$ or $\partial_t^3\mathbf{U}_r$. We thus consider the equation solved by

$$\partial_t\partial^\alpha\mathbf{W} = J\left(L[\mathbf{U}]\partial^\alpha\mathbf{W} + [\partial^\alpha, L[\mathbf{U}]]\,\mathbf{W} - \partial^\alpha JRJ\left(\mathcal{Q}[\mathbf{U}] - \mathcal{Q}[\mathbf{U}_{ap}]\right)\right) + \partial^\alpha(R\mathbf{S}). \tag{5.158}$$

We take the L^2 scalar product of (5.158) with

$$\mathcal{M} = L[\mathbf{U}]\partial^\alpha\mathbf{W} + [\partial^\alpha, L[\mathbf{U}]]\,\mathbf{W} - \partial^\alpha JRJ\left(\mathcal{Q}[\mathbf{U}] - \mathcal{Q}[\mathbf{U}_{ap}]\right).$$

From the skew symmetry of J and the symmetry of $L[\mathbf{U}]$, we get the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(\partial^\alpha\mathbf{W}, L[\mathbf{U}]\partial^\alpha W\right) + \mathcal{I}^\alpha\right) = \mathcal{J}^\alpha, \tag{5.159}$$

where

$$\mathcal{I}^\alpha = \left(\partial^\alpha\mathbf{W}, [\partial^\alpha, L[\mathbf{U}]]\mathbf{W}\right) - \left(\partial^\alpha\mathbf{W}, \partial^\alpha\left(JR[\mathbf{U}]J\left(\mathcal{Q}[\mathbf{U}] - \mathcal{Q}[\mathbf{U}_{ap}]\right)\right),\right)$$

$$\begin{aligned}\mathcal{J}^\alpha = \frac{1}{2}&\left(\partial^\alpha\mathbf{W}, [\partial_t, L[\mathbf{U}]]\partial^\alpha\mathbf{W}\right) + \left(\partial^\alpha\mathcal{W}, \partial_t[\partial^\alpha, L[\mathbf{U}]]\mathbf{W}\right)\\ &- \left(\partial^\alpha\mathbf{W}, \partial_t\partial^\alpha\left(JR[\mathbf{U}]J\left(\mathcal{Q}[\mathbf{U}] - \mathcal{Q}[\mathbf{U}_{ap}]\right)\right)\right) + \left(\mathcal{M}, \partial^\alpha(R[\mathbf{U}]\mathbf{S})\right).\end{aligned}$$

We give the following proposition, that estimates each term arising in the energy identity (5.159).

Lemma 5.33. *We have the estimates*

$$|\mathcal{J}^\alpha| \leqslant \omega\left(|\boldsymbol{U}_{ap}|_{X_\infty^{m+s}} + |\boldsymbol{U}_r|_{X^{m+3}}\right)\left(|\boldsymbol{U}_r|_{X^{m+3}}|\boldsymbol{S}|_{X^m} + |\boldsymbol{U}_r|_{X^3}^2\right) \quad |\alpha| \leqslant m$$

and

$$|\mathcal{I}^\alpha| \leqslant \omega\left(|\boldsymbol{U}_{ap}|_{X_\infty^{m+s}} + |\boldsymbol{U}_r|_{X^{m+3}}\right)|\boldsymbol{U}_r|_{X^{m+3}}|\boldsymbol{U}_r|_{X^{m+2}} \quad |\alpha| \leqslant m.$$

Lemma follows from Propositions 5.31 and 5.32. For a proof in the case three dimensional can be found in [16, Lemma 7.31].

We also consider the following estimate

Lemma 5.34. *Let $\boldsymbol{U}_r$ be a solution of (5.85). Then, for $m \geqslant 2$ and $s \geqslant 5$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{U}(t)|_{X^3}^2 \leqslant \omega\left(|\boldsymbol{r}_{ap}|_{X^3} + |\boldsymbol{U}_{ap}|_{X_t^{m+s}} + |\boldsymbol{U}_r|_{X_t^{m+3}}\right)|\boldsymbol{U}_r|_{X^{m+3}}^2.$$

Moreover, we have the estimate

$$|\partial_t^l\boldsymbol{U}|_{L^2}^2 \leqslant \omega\left(|\boldsymbol{U}_{ap}|_{X_\infty^{m+s}} + |\boldsymbol{U}_r|_{X^{m+3}}\right)\left(|\partial_x\partial_t^{l-1}\boldsymbol{U}_r|_{X^0} + |\boldsymbol{U}_r|_{X^2}^2 + |\boldsymbol{r}_{ap}|_{X^m}^2\right) \quad 4 \leqslant l \leqslant m+3.$$

Proof. To prove the first estimation, it suffices ti multiply by $(\eta, \mathfrak{B}\varphi)^t$, integrate in $\mathbb{R}$ and use the same kind of estimates we have been using before, such as (5.127) Proposition 5.27 and Proposition 5.29.

For the second estimation, one should apply ∂_t^{l-1} to (5.126), taking the L^2-norm and using the typical estimations used in Propositions 5.28 and 5.29. ■

Now, let as define the energy

$$\tilde{\mathcal{H}}_\alpha(t) = \frac{1}{2}\left(\partial^\alpha \mathbf{W}, L[\mathbf{U}]\partial^\alpha \mathbf{W}\right) + \mathcal{I}^\alpha.$$

We also define for $m \geqslant 2$, $1 \leqslant l \leqslant m, \tau \in [t,0], t \leqslant 0$,

$$\tilde{\mathcal{H}}_m(\tau) = \sum_{1\leqslant l\leqslant m} \Gamma^{m-1}\tilde{\mathcal{H}}_{l,m}(\tau) + \Gamma|\mathbf{U}_r(\tau)|^2_{X^3},$$

for Γ large and

$$\tilde{\mathcal{H}}_{l,m} = \sum_{\substack{1\leqslant|\alpha|\leqslant l\\ \alpha'\neq 0}} \tilde{\mathcal{H}}_\alpha + \Gamma \sum_{\substack{1\leqslant|\alpha|\leqslant l\\ \alpha'=0}} \tilde{\mathcal{H}}_\alpha$$

for α' denoting the time derivative represented by α, that is, $\alpha = (\alpha', \alpha^x)$.

Lemma 5.35 ([16, Lemma 7.35])**.** *For every $t \leqslant 0$, there exists Γ such that for every $\tau \in (t,0)$, we have that*

$$\tilde{\mathcal{H}}_m(\tau) \geqslant \frac{|\boldsymbol{U}(\tau)|_{X^{m+3}}}{\omega\left(|\boldsymbol{U}_{ap}|_{X^{m+s}_{\infty,t}} + |\boldsymbol{U}_r|_{X^{m+3}_t}\right)}.$$

We now can complete the proof of Proposition 5.25. From (5.159) and Lemma 5.33, we get for $t \leqslant \tau \leqslant 0$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{H}}_m(\tau)\right| \leqslant \omega\left(|\mathbf{r}_{ap}|_{X^{m+s}_t}|\mathbf{U}_{ap}|_{X^{m+s}_{\infty,t}} + |\mathbf{U}_r|_{X^{m+3}_t}\right)\left(|\mathbf{U}_r(\tau)|^2_{X^{m+3}} + |\mathbf{S}(\tau)|^2_{X^m}\right).$$

Consequently, we can integrate in time for $[t,0]$ and use Lemma 5.35, along with the fact that

$$|\tilde{\mathcal{H}}_m(0)| \leqslant \omega\left(|\mathbf{r}_{ap}|_{X^{m+s}_0}|\mathbf{U}_{ap}|_{X^{m+s}_{\infty,0}} + |\mathbf{U}_r|_{X^{m+3}_0}\right)|\mathbf{r}_{ap}(0)|^2_{X^{m+3}}.$$

Bibliography

[1] T. Alazard, N. Burq, C. Zuily, *On the water-wave equations with surface tension* Duke Math. J. Volume 158, Number 3, pp. 413–499 (2011).

[2] T. Alazard, N. Burq, C. Zuily, *On the Cauchy problem for gravity water waves*, Invent. math. 198, pp. 71–163 (2014).

[3] C.J. Amick and K. Kirchgassner, *A theory of solitary water-waves in the presence of surface tension*, Arch. Ration. Mech. Analysis 105, pp. 1–49 (1989).

[4] W. Craig, *An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits*, Commun. Partial Differ. Equ. 10(8), pp. 787–1003 (1985).

[5] W. Craig and C. Sulem, *Numerical Simulation of Gravity Waves*, Journal of Computational Physics 108, pp. 73–83 (1993).

[6] W. Craig, C. Sulem and P.L. Sulem, *Nonlinear modulation of gravity waves: a rigorous approach*, Nonlinearity 5, pp. 497–522 (1992).

[7] L. C. Evans, *Partial Differential Equations, second edition*, Graduate Studes in Mathematics 19, Americal Mathematical Society, Providence, Rhodes Island (2010).

[8] P. Germain, N. Masmoudi and J. Shatah, *Global solutions for the gravity water waves equation in dimension 3*, C. R. Acad. Sci. Paris, Ser. I 347 pp. 897–902 (2012).

[9] P. Germain, N. Masmoudi and J. Shatah *Global existence for capillary water waves* Communications on Pure and Applied Mathematics 68 (4), pp. 625–687 (2015).

[10] T. Iguchi, *A mahematical analysis of tsunami generation in shallow water due to seabed deformation*, Proceedings of the Royal Society of Edinburgh 141A, pp. 551–608 (2011).

[11] D Lannes, *The Waves Problem*, Mathematical Surveys and Monographs 188, Ameriacan Mathematical Society, Providence, Rhodes Island (2013).

[12] D. Lannes, *A stability criterion for two fluid interfaces amd applications*, Arch Rational Mech Anal 208, pp. 481–567 (2013). doi:10.1007/s00205-012-0604-6

[13] A. Mielke, *On the energetic stability of solitary water waves*, Phil. Trans. R. Soc. Lond. A 360, pp. 2337–2358 (2002).

[14] M. Ming, F. Rousset and N. Tzvetkov *Multi-solitons and Related Solutions for the Water-waves System* SIAM J. Math. Anal., 47 (1), pp. 897–954 (2013).

[15] C. Muñoz, *Soliton dynamics for the generalized KdV equations in a slowly mediun*, Analysis and PDE, Vol. 4, No. 4, pp. 573–638 (2011).

[16] F. Rousset and N. Tzvetkov, *Transverse inestability of the line solitary water-waves*, Invent. Math. 184, pp. 275–388 (2011).

[17] G. Schneider, E. Wayne, *The long wave limit for the water wave problem I. The case of zero surface tension*, Commun. Pure. Appl. Math. 53(12), pp. 1475–1535 (2000).

[18] M. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Bitkhäuser (1991).

[19] S. Wu, *Almost global wellposedness of the 2–D full water wave problem*, Invent. Math. 177, pp. 45–135 (2009).

[20] S. Wu, *Global wellposedness of the 3–D full water wave problem*, Invent math 184, pp. 125–220 (2011).

[21] V. E. Zakharov, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. Applied Mech. Tech. Phys. 9, pp. 190–194 (1968).

Part V

Conclusion

Chapter 6

Conclusions and Perspectives

6.1 Conclusions

This work is concerned with the study of long time asymptotics for the following dispersive models, mainly related to fluid dynamics: Schrödinger, Hartree, Zakharov, Klein-Gordon Zakharov, Zakharov-Rubenchik/Benney-Roskes and Zakharov Water waves model.

The results obtained in the Part II of this work essentially consist of a deep analysis of virial technics to obtain the following:

- Decay of (small) odd solutions of the **Schrödinger equation**,
$$\mathrm{i}u_t + u_{xx} = \mu V(x)u + |u|^{p-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \quad \textbf{for } 1 < p < 5. \tag{6.1}$$
for which we were able to approach the super critical case $p < 3$.
- Decay of odd solutions of the defocusing **Hartree equation**,
$$\mathrm{i}u_t + u_{xx} = \sigma\left(|x|^{-\alpha} * |u|^2\right)u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \quad \sigma = 1, \tag{6.2}$$
using a virial method to deal with the one-dimensional case.
- Decay in compact intervals and in far field regions for **Zakharov system**:
$$\begin{aligned} \mathrm{i}u_t + \Delta u &= nu, && (t,x) \in \mathbb{R}\times\mathbb{R},\\ n_{tt} - \Delta n &= \Delta|u|^2, && (t,x) \in \mathbb{R}\times\mathbb{R}, \end{aligned}$$
and **Klein-Gordon Zakharov**
$$\begin{aligned} u_{tt} - \Delta u + c^2u &= -nu, && (t,x) \in \mathbb{R}\times\mathbb{R},\\ n_{tt} - \Delta n &= \Delta|u|^2 && (t,x) \in \mathbb{R}\times\mathbb{R}. \end{aligned}$$

In Part III of this thesis we studied Zakharov-Rubenchik/Benney Roskes:
$$\begin{aligned} \mathrm{i}\partial_t\psi + \omega\partial_x^2\psi &= \gamma\big(\eta - \tfrac{1}{2}\alpha\rho + q|\psi|^2\big)\psi, && (t,x) \in \mathbb{R}\times\mathbb{R}\\ \theta\partial_t\rho + \partial_x\big(\eta - \alpha\rho\big) &= -\gamma\partial_x(|\psi|^2), && (t,x) \in \mathbb{R}\times\mathbb{R}\\ \theta\partial_t\eta + \partial_x\big(\beta\rho - \alpha\eta\big) &= \tfrac{1}{2}\alpha\gamma\partial_x(|\psi|^2), && (t,x) \in \mathbb{R}\times\mathbb{R} \end{aligned}$$

where

$$\omega > 0, \quad \beta > 0, \quad \gamma > 0, \quad \beta - \alpha^2 > 0, \quad 0 < \theta < 1, \quad \text{and} \quad q := \gamma + \frac{\alpha(\alpha\gamma - 1)}{2(\beta - \alpha^2)}.$$

We proved decay for the energy norm in far field regions. Also, being able to use the underlying characteristics curves of the model, we gave decay properties in growing compact intervals, outside the light cone and around zero.

Part IV deals with the Zakharov Water Waves model:

$$\begin{cases} \partial_t \eta = G[\eta, a]\varphi \\ \partial_t \varphi = -\frac{1}{2}\left|\partial_X \varphi\right|^2 + \frac{1}{2}\dfrac{\left(G[\eta,a]\varphi + \nabla_X \varphi \cdot \nabla_X \eta\right)^2}{1 + \left|\nabla_X \eta\right|^2} - g\eta + \beta \nabla_X \cdot \left(\dfrac{\nabla_X \eta}{\sqrt{1 + \left|\nabla_X \eta\right|^2}}\right) \end{cases} \tag{6.3}$$

in a domain defined by

$$\Omega_t = \left\{(x,z) \in \mathbb{R}^2 \text{ such that } -h + A_\varepsilon \leqslant z \leqslant \eta(t,x)\right\}.$$

We proved the existence of soliton-like solutions of flat-bottom nature approaching a change in the domain.

6.2 Future Work

6.2.1 Decay for non-small odd solutions to semilinear Schrödinger equation

We already know that for the one dimensional Schrödinger equation (6.1), there is decay in fixed (non-growing) intervals for small, odd solutions. The oddness condition rules out of the result soliton solutions and breathers. On the other hand, the fact that we are considering fixed intervals, allows us to forget about not only travelling waves, but also solutions that can be written as a sum of solitary waves with different speeds (and sufficiently away from each other) plus radiation. This is due to the fact that, since such solitons are moving at a speed different to zero, then for any given fixed interval, one can wait sufficiently long and, eventually, all solitons move away from our space interval.

With this in mind, the smallness condition is not strictly necessary for ruling out non-decaying solutions. In fact, in our previous work, such condition is only used to control terms arising from the non-linear part of the equation. Consequently, **it is expected that decay still holds for non-small odd solutions to focusing NLS.**

6.2.2 Decay result for focusing Hartree equation

The focusing case of the Hartree equation (6.2) turns out to be more challenging, as is the case where solitary waves exist. However, solitons for the Hartree equation are even, which means that the oddness condition is sufficient to rule out such solutions. In consequence, **it is expected that decay for odd (and, possibly, small) solutions for the focusing Hartree equation** (6.2) **($\sigma = -1$) still holds.**

Fractional derivatives for the better understanding of Hartree's non-local linearity

In order to prove decay for (6.2) with $\sigma = -1$, we need to be able to control the non-local part of the equation. Then, it would be useful to take into account the decay that the non-local term presents in the definition of the weighted norm. Indeed, it should be sufficient to define the modified momentum as

$$\mathcal{P}(t) := \operatorname{Im} \int_{\mathbb{R}} \varphi\left(\frac{x}{\lambda}\right) u(t,x)\overline{u}_x(t,x)\mathrm{d}x.$$

where

$$\varphi(x) = \lambda \int_0^{x/\lambda} \left(1+s^2\right)^{-\frac{1}{2}(1+\nu)} \mathrm{d}s$$

for some $\nu > 0$ and $\lambda > 1$. Then, a weighted norm appropriate for such virial approach would be

$$\|u(t)\|^2_{H^1_\omega(\mathbb{R})} = \int_{\mathbb{R}} \left(1+x^2\right)^{-\frac{1}{2}(1+\nu)} \left(|u(t,x)|^2 + |u_x(t,x)|^2\right) \mathrm{d}x$$

Consequently, Hardy-Poincaré type inequalities, along with an appropriate definition of the fractional derivative and an intensive study of fractional analysis should help solve this problem.

6.2.3 Behavior of solitons for Zakharov Water Waves models under changes in the bottom of the fluid

After proving the existence of solitary waves for the non-flat bottom problem (6.3) before it encounters the changing point, a natural second step constitutes the study of the interaction between the constructed solution $(\eta, \varphi)^t$ and the change of the bottom, represented as a (sufficiently small) exponentially decaying function a_ε.

A similar problem was introduced by Muñoz [5] for the gKdV equation, where the the problem of existence and global behavior of solitons with a slowly varying (in space) perturbation was considered. In the mentioned work, virial identities were used to prove that such slowly varying media induce on the soliton dynamics large dispersive effects at large times. Because of the similarities between the dynamics of the Zakharov water-waves system and the KdV equation (and between the solitons themselves), it seems fitting to rely on the analysis given by Muñoz in [5].

An interesting new problem to consider would be the **study for the solitons dynamics under a slowly varying bottom for the 3-dimensional system** (1.17)**.** In Chapter 5, the construction of the approximate solution in is mainly based on properties of the operator that arises when linearising the equation (1.17) about the solitary wave. Roughly speaking, the linearised equation about the solitary wave reads

$$\partial_t \begin{pmatrix} \eta \\ \varphi \end{pmatrix} = J\Lambda[Q_c] \begin{pmatrix} \eta \\ \varphi \end{pmatrix},$$

where J is a skew-symmetric matrix and $\Lambda[Q_c]$ is a symmetric operator on $L^2 \times L^2$. One of the main ingredients would be a positiveness result for $\Lambda[Q_c]$ under the orthogonality condition $(J\partial_x Q_c, U) = 0$, for $U = (\eta, \varphi)^t$. Nevertheless, such result can already be derived from [6]. Then, it seems a natural second step to analyze the long-time behavior of solitary waves and their interaction with a change of bottom for the 3-dimensional case.

One of my main objectives in this area is to consider the collision of two small solitary waves moving in different directions. Using numerics, the interaction of two solitary waves for the case without surface tension was extensively studied by Craig, Guyenne, Hammack, Henderson and Sulem in [1]. This is a very influential paper, and establishing rigorous results regarding the collision itself would be of great importance.

Bibliography

[1] W. Craig, P. Guyenne, J. Hammack, D. Henderson and C. Sulem *Solitary water wave interactions*, Physycs of Fluids Vol. 18, No. 5, id. 057106-057106-25 (2006)

[2] W. Craig, C. Sulem, P.-L. Sulem, *Nonlinear modulation of gravity waves: A rigorous approach*, Nonlinearity 5, pp. 497–522 (1992).

[3] S. Cuccagna, N. Visciglia and V. Georgiev, *Decay and scattering of small solutions of pure power NLS in* $\mathbb{R}$ *with* $p > 3$ *and with a potential*, Comm. Pure Appl. Math, 67 (2014), pp. 957–981.

[4] J. Kato and F. Pusateri, *A new proof of long range scattering for critical nonlinear Schrödinger equations*, Differential Integral Equations 24 (2011), pp. 923–940.

[5] C. Munõz, *Soliton dynamics for the generalized KdV equations in a slowly mediun*, Analysis and PDE, Vol. 4, No. 4, pp. 573–638 (2011).

[6] F. Rousset and N. Tzvetkov, *Transverse instability of the line solitary water-waves*, Invent. Math. 184, pp. 257–388 (2011).

Chapter 7

Bibliography

Bibliography

[1] T. Alazard, N. Burq, C. Zuily, *On the water-wave equations with surface tension* Duke Math. J. Volume 158, Number 3, pp. 413–499 (2011).

[2] T. Alazard, N. Burq, C. Zuily, *On the Cauchy problem for gravity water waves*, Invent. math. 198, pp. 71–163 (2014).

[3] C.J. Amick and K. Kirchgassner, *A theory of solitary water-waves in the presence of surface tension*, Arch. Ration. Mech. Analysis 105, pp. 1–49 (1989).

[4] J. E. Barab *Nonexistence of asymptotically free solutions for a nonlinear Schroödinger equation*, Journal of Mathematical Physics 25, 3270 (1984); doi: 10.1063/1.526074

[5] D. J. Benney, G. J. Roskes, *Wave instability*, Studies in Applied Math., 48 (1969), pp 455-472.

[6] L. Bergé, *Wave collapse in physics: principles and applications to light and plasma waves*, Phys. Rep., Vol. 303, pp. 259–370 (1998).

[7] M. Bonforte, J. Dolbeault, G. Grillo, and J. L. Vázquez, *Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities*, Proc. Natl. Acad. Sci. USA 107 (2010), no. 38, 16459–16464.

[8] V. S. Buslaev, and G. Perelman, *Nonlinear scattering: the states that are closed to a soliton*, Journal of Mathematical Sciences, Vol. 77, no. 3 (1995), pp. 3161–3169.

[9] V. S. Buslaev, and C. Sulem, *On asymptotic stability of solitary waves for the nonlinear Schrödinger equations*, Annales de l'Institut Hneri Poincare (C) Non Linear Analysis, Vol. 20, Issue 3, pp. 419–475.

[10] T. Cazenave. *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, New York University, New York, 2003.

[11] T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. 85 (1982), no. 4, 549–561.

[12] T. Cazenave and F. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Anal. 14 (1990), no. 10, 807–836.

[13] C. Miao, G. Xu, and L. Zhao, *The Cauchy problem of the Hartree equation*, Journal of Partial Differential Equations 2121 (2018), pp. 22–44.

[14] S. Champeaux, D. Laveder, T. Passot, E L. Sulem, *Remarks on the parallel propagation of small-amplitude dispersive Alfvén waves*, Nonlinear Processes in Geophysics 6 (1999), 169

[15] L. Chen, *Orbital stability of solitary waves for the Klein-Gordon-Zakharov equations*, Acta Math. Appl. Sin., Vol. 15, No. 1, pp. 54–64 (1999).

[16] S. Cingolani, S. Secchi and M. Squassina, *Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities.*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 140(5) (2017), pp. 973-1009.

[17] J. C. Cordero, *Supersonic limit for the Zakharov-Rubenchik system*, J. Differential Equations 261 (2016), no. 9, 5260-5288.

[18] J. Colliander, *The initial value problem for the zakharov system*, Thesis, University of Illinois at Urbana-Champaign (1997).

[19] W. Craig, *An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits*, Commun. Partial Differ. Equ. 10(8), pp. 787–1003 (1985).

[20] W. Craig, P. Guyenne, J. Hammack, D. Henderson and C. Sulem *Solitary water wave interactions*, Physycs of Fluids Vol. 18, No. 5, id. 057106-057106-25 (2006)

[21] W. Craig, C. Sulem, P.-L. Sulem, *Nonlinear modulation of gravity waves: A rigorous approach*, Nonlinearity 5, pp. 497–522 (1992).

[22] S. Cuccagna (2001), *Stabilization of solutions to nonlinear Schrödinger equations*, Comm. Pure Appl. Math., 54 (2001), pp. 1110–1145.

[23] S. Cuccagna, *The Hamiltonian Structure of the Nonlinear Schrödinger Equation and the Asymptotic Stability of its Ground States*, Commun. Math. Phys. 305 (2011), pp. 279–331.

[24] S. Cuccagna, and M. Maeda *On stability of small solitons of the 1–D NLS with a trapping potential*, arXiv:1904.11869.

[25] S. Cuccagna and T. Mizumachi, *On Asymptotic Stability in Energy Space of Ground States for Nonlinear Schroödinger Equations*, Commun. Math. Phys. 284 (2008), pp. 51–77.

[26] S. Cuccagna, N. Visciglia and V. Georgiev, *Decay and scattering of small solutions of pure power NLS in $\mathbb{R}$ with $p > 3$ and with a potential*, Comm. Pure Appl. Math, 67 (2014), pp. 957–981.

[27] P. Deift, and X. Zhou, *Perturbation theory for the infinite-dimensional integrable system on the line. A case study*, Acta Math, Vol. 188, Number 2 (2002), pp. 163–262.

[28] N. Hayashi and P. I. Naumkin, *Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations*, Amer. J. Math. 120 (1998), pp. 369–389.

[29] Jean-Marc Delort, *Modified scattering for odd solutions of cubic nonlinear Schrödinger equations with potential in dimension one*, preprint 2016 $<$ hal $-$ 01396705 $>$.

[30] P. Germain, N. Masmoudi, J. Shatah *Global solutions for the gravity water waves equation in dimension 3*, C. R. Acad. Sci. Paris, Ser. I 347 pp. 897–902 (2012).

[31] P Germain, N Masmoudi, J Shatah *Global existence for capillary water waves* Communications on Pure and Applied Mathematics 68 (4), pp. 625–687 (2015)

[32] P. Germain, F. Pusateri and F. Rousset, *The nonlinear Schrödinger equation with a potential*, Annales de l'Institut Henri Poincaré C, Analyse non linéaire 35 (2018), pp. 1477–1530.

[33] J. Ginibre and T. Ozawa, *Long Range Scattering for Non-Linear Schrodinger and Hartree Equations in Space Dimension $n > 2$*, Commun. Math. Phys. 151 (1993), pp. 619–645.

[34] J. Ginibre and G. Velo, *Scattering theory for the Zakharov system*, Hokkaido Mathematical Journal 35, no. 4, pp. 865–892 (2006).

[35] J. Ginibre, and G. Velo, *On a class of nonlinear Schrödinger equations. I: The Cauchy problem*, J. Funct. Anal. 32 (1979), 1–32.

[36] R. Glassey, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. 18 (1977), no. 9, pp. 1794–1797.

[37] Z. Guo, S. Lee, K. Nakanishi and C. Wang, *Generalized Strichartz Estimates and Scattering for 3D Zakharov System*, Commun. Math. Phys. 331, pp. 239–259 (2014). doi:10.1007/s00220-014-2006-0

[38] Z. Guo and K. Nakanishi, *Small Energy Scattering for Zakharov*, Int. Math. Res. Not. IMRN, Vol. 2014, No. 9, pp. 2327–2342 (2013). doi:10.1093/imrn/rns296

[39] Z. Guo, K. Nakanishi and S. Wang, *Small energy scattering for the Klein-Gordon-Zakharov system with radial symmetry*, Math. Res. Lett., Vol. 21, No. 04, pp. 733–755 (2014).

[40] Z. Guo, K. Nakanishi and S. Wang, *Global Dynamics below the Ground State Energy for the Klein-Gordon-Zakharov System in the 3D Radial Case*, Communications in Partial Differential Equations, Vol. 39, pp. 1158–1184 (2014).

[41] Z. Guo, K. Nakanishi and S. Wang, *Global Dynamics below the Ground State Energy for the Klein-Gordon-Zakharov System in the 3D Radial Case*, Communications in Partial Differential Equations, Vol. 39, pp. 1158–1184, (2014). doi: 10.1080/03605302.2013.836715

[42] I. I. Gol'dman, V. D. Krivchenkov, B. T. Geilikman, E. Marquit, E. Lepa, *Problems in*

quantum mechanics, Authorised revised ed. Edited by B. T. Geilikman; translated from the Russian by E. Marquit and E. Lepa, Pergamon Press, 1961.

[43] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I.* J. Funct. Anal. 74 (1987), no. 1, 160–197.

[44] T. Kapitula, *On the stability of N-solitons in integrable systems*, Nonlinearity, 20 (2007) 879–907.

[45] J. Kato and F. Pusateri, *A new proof of long range scattering for critical nonlinear Schrödinger equations*, Differential Integral Equations 24 (2011), pp. 923–940.

[46] M. Kowalczyk, Y. Martel and C. Muñoz, *Kink dynamics in the ϕ^4 model: asymptotic stability for odd perturbations in the energy space*, J. Amer. Math. Soc. 30 (2017), 769–798.

[47] M. Kowalczyk, Y. Martel and C. Muñoz, *Nonexistence of small, odd breathers for a class of nonlinear wave equations*, Lett. Math. Phys. 107 (2017), pp. 921–931.

[48] M. Klaus, *On the Bound State of Schrödinger Operators in One Dimension*, Annals of Physics 108 (1977), pp. 288–300.

[49] M. Kowalczyk, Y. Martel and C. Muñoz, *Kink dynamics in the ϕ^4 model: asymptotic stability for odd perturbations in the energy space*, J. Amer. Math. Soc. 30 (2017), 769–798.

[50] M. Kowalczyk, Y. Martel and C. Muñoz, *Nonexistence of small, odd breathers for a class of nonlinear wave equations*, Lett. Math. Phys. 107 (2017), pp. 921–931.

[51] M. Kowalczyk, Y. Martel and C. Muñoz, *Soliton dynamic for the 1D NLKG equation with symmetry and in the absence of internal modes*, preprint: arXiv:1903.12460.

[52] E. Lieb and M. Loss, *Analysis*, 2nd. Ed., Graduate Studies in Mathematics, Vol. 14, Amer. Math. Soc., 2001.

[53] P. L. Lions, *The concentration-compactness principle in the Calculus of Variations. The locally compact case, part 1*, Annales de l'Institut Henri Poincaré (C) 1 (1984), pp. 109–145.

[54] Y. Martel, and F. Merle, *Multi-solitary waves for nonlinear Schrödinger equations*, Annales de l'IHP (C) Non Linear Analysis, 23 (2006), 849–864.

[55] F. Linares, C. Matheus, *Well posedness for the 1D Zakharov-Rubenchik system*, Adv. Differential Equations 14 (2009), no. 3-4, 261-288.

[56] H. Luong, N. Mauser, J-C Saut, *On the Cauchy problem for the Zakharov-Rubenchik/Benney-Roskes system*, Commun. Pure Appl. Anal. 17 (2018), no. 4, 1573-1594.

[57] Y. Martel, F. Merle and T.-P. Tsai, *Stability in H^1 of the sum of K solitary waves for some nonlinear Schrödinger equations*, Duke Math. J. 133 (2006), 405–466.

[58] N. Masmoudi and K. Nakanishi, *Nonrelativistic limit from Maxwell-Klein-Gordon and Maxwell-Dirac to Poisson-Schrödinger*, Int. Math. Res. Not. 13, pp. 697–734 (2003).

[59] N. Masmoudi and K. Nakanishi, *From the Klein-Gordon-Zakharov system to the nonlinear Schrödinger equation*, J. Hyperbolic Differ. Equ. 2 (4), pp. 975–1008 (2005).

[60] N. Masmoudi and K. Nakanishi, *From the Klein-Gordon-Zakharov system to a singular nonlinear Schrödinger system*, Annales de Annales de l'I. H. P. (C) Non Linear Analysis Vol. 27, No. 4, pp. 1073–1096 (2010).

[61] N. Masmoudi and K. Nakanishi, *Energy convergence for singular limits of Zakharov type systems* Invent. math., Vol. 172, pp. 535–583 (2008). doi: 10.1007/s00222-008-0110-5

[62] N. Masmoudi and K. Nakanishi, *Uniqueness of Solutions for Zakharov Systems*, Funkcialaj Ekvacioj Vol. 52, No. 2, pp. 233–253 (2009).

[63] F. Merle, *Blow-up Results of Viriel Type for Zakharov Equations*, Commun. Math. Phys. 175, pp. 433–455 (1996).

[64] F. Merle, *Lower Bounds for the Blowup Rate of Solutions of the Zakharov Equation in Dimension Two*, Commun. Pure Appl. Anal., Vol. XLIX, pp. 765–794 (1996).

[65] F. Merle and P. Raphaël, *The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation*, Ann. of Math. (2) **161** (2005), no. 1, 157–222.

[66] A. Mielke, *On the energetic stability of solitary water waves*, Phil. Trans. R. Soc. Lond. A 360, pp. 2337–2358 (2002).

[67] M. Ming, F. Rousset and N. Tzvetkov *Multi-solitons and Related Solutions for the Water-waves System* SIAM J. Math. Anal., 47 (1), pp. 897–954 (2013).

[68] V. Moroz and J. Van Schaftingen, *A guide to the Choquard equation*, J. J. Fixed Point Theory Appl. 19 (2017), pp. 773–813.

[69] C. Muñoz and G. Ponce, *Breathers and the dynamics of solutions to the KdV type equations*, preprint 2018, to appear in Comm. Math. Phys.

[70] C. Munõz, *Soliton dynamics for the generalized KdV equations in a slowly mediun*, Analysis and PDE, Vol. 4, No. 4, pp. 573–638 (2011).

[71] C. Muñoz, G. Ponce and J.-C. Saut, *On the long time behavoir of solutions to the Intermediate Long Wave equation*, arXiv:1910.03897.

[72] J. Murphy, and K. Nakanishi, *Failure of scattering to solitary waves for the long-range nonlinear Schrödinger equations*, preprint arXiv: 1906.01802.

[73] K. Nakanishi, and T. Ozawa, *Remarks on scattering for nonlinear Schrödinger equations*,

NoDEA 2002 Vol. 9, Issue 1, pp. 45–68.

[74] I. P. Naumkin, *Sharp asymptotic behavior of solutions for cubic nonlinear Schrödinger equations with a potential*, J. Math. Phys. 57 (2016), pp. 051501.

[75] T. V. Nguyen, *Existence of multi-solitary waves with logarithmic relative distances for the NLS equation*, Comptes Rendus Mathematique, 357 (2019), pp. 13–58.

[76] F. Oliveira, *Stability of the solitons for the one-dimensional Zakharov-Rubenchik equation*, Phys. D 175 (2003), no. 3-4, 220-240.

[77] F. Oliveira, *Adiabatic limit of the Zakharov-Rubenchik equation*, Rep. Math. Phys. 61 (2008), no. 1, 13-27.

[78] T. Ozawa, *Long range scattering for nonlinear Schrödinger equations in one space dimension*, Communications in Mathematical Physics (1991) DOI: 10.1007/BF02101876.

[79] M. Ohta and G. Todorova, *Strong instability of standing waves for nonlinear Klein-Gordon equation and Klein-Gordon-Zakharov system*, SIAM Journal on Mathematical Analysis Vol. 38, No. 6, pp. 1912–1931 (2006).

[80] T. Ozawa, K.Tsutaya and Y. Tsutsumi, *Normal form and global solutions for the Klein-Gordon-Zakharov equations* Annales de l'I. H. P. (C), Vol. 12, No. 4, pp. 459–503 (1995).

[81] T. Ozawa, K.Tsutaya and Y. Tsutsumi, *Well-posedness in energy space for the Cauchy problem of the Klein-Gordon-Zakharov equations with different propagation speeds in three space dimensions*, Math. Ann. 313, pp. 127–140 (1999).

[82] T. Ozawa and Y. Tsutsumi, *The nonlinear Schrödinger limit and the initial layer of the Zakharov equations*, Proc. Japan Acad. Ser. A Math. Sci. 67, no. 4, pp. 113–116 (1991).

[83] T. Ozawa and Y. Tsutsumi, *Existence and Smoothing Effect of Solutions for the Zakharov Equations*, Publ RIMS, Kyoto Univ. 28, pp. 329–361 (1992).

[84] T. Passot, P.-L. Sulem, C. Sulem, *Generalization of acoustic fronts by focusing wave-packets*, Physica D 94 (1996), 168-187.

[85] H. Pecher, *Global Well-Posedness below Energy Space for the 1-Dimensional Zakharov System*, Int. Math. Res. Not. IMRN, No. 19, pp. 1027–1056 (2001).

[86] D. H. Peregrine, *Water waves, nonlinear Schrödinger equations and their solutions*, J. Austral. Math. Soc. Ser. B 25, pp. 16–43 (1983).

[87] D. Ruiz, *On the Schrödinger-Poisson-Slater System: Behavior of Minimizers, Radial and Nonradial Cases*, Arch Rational Mech Anal 198 (2010), pp. 349–368.

[88] F. Rousset and N. Tzvetkov, *Transverse instability of the line solitary water-waves*, Invent. Math. 184, pp. 257–388 (2011).

[89] J. Satsuma, and N. Yajima, *Initial Value Problems of One-Dimensional Self-Modulation*

of Nonlinear Waves in Dispersive Media, Supplement of the Progress of Theoretical Physics, No. 55, 1974, 284–306.

[90] B. Simon. *The Bound State of Weakly Coupled Schrödinger Operators in One and Two Dimensions*, Annals of Physics 97 (1976), pp. 279–288.

[91] A. Soffer, and M. I. Weinstein, *Resonances, radiation damping and instabilitym in Hamiltonian nonlinear wave equations*, Invent. Math. March 1999, Volume 136, Issue 1, pp. 9–74.

[92] G. Schneider, E. Wayne, *The long wave limit for the water wave problem I. The case of zero surface tension*, Commun. Pure. Appl. Math. 53(12), pp. 1475–1535 (2000).

[93] A. Shimomura, *Scattering theory for the Zakharov equation in three espace dimensions*, Commun. Contemp. Math. 6, No. 6, pp. 881–899 (2004). doi:10.1142/S0219199704001574

[94] W. Strauss, *Nonlinear Scattering Theory*, Scattering Theory in Mathematical Physics, edited by J. A. Lavita and J-P. Marchand (Reidel, Dordrecht, Holland, 1974), pp. 53-78.

[95] E. C. Titchmarsh, *Eigenfunction Expansions Associated With Second-order Differential Equations*, Oxford University Press, 1946.

[96] C. Sulem and P. L. Sulem, *Quelques résulatats de régularité pour les équation de la turbulence de Langmuir*. C. R. Acad. Sci. Paris 289, pp. 173–176 (1979).

[97] C. Sulem, P.-L. Sulem, *The Nonlinear Schrödinger Equation. Self-Focusing and Wave Collapse*, Applied Mathematical Sciences Series Volume 139, Springer-Verlag New York (1999).

[98] B. Texier, *Derivation of the Zakharov Equations* Archive for Rational Mechanics and Analysis 184, pp. 121–183 (2008).

[99] Y. Tsutsumi, *Scattering problem for nonlinear Schroödinger equations*, Annales de l'IHP, section A, tome 43, No. 3 (1985), p. 321-347.

[100] Y. Tsutsumi, *L^2-solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcial. Ekvac. 30 (1987), no. 1, 115–125.

[101] M. I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39**, (1986) 51–68.

[102] S. Wu, *Almost global wellposedness of the 2–D full water wave problem*, Invent math 177, pp. 45–135 (2009).

[103] S. Wu, *Global wellposedness of the 3–D full water wave problem*, Invent math 184, pp. 125—220 (2011).

[104] Y. Wu, *Orbital stability of solitary waves of Zakharov system*, J. Math. Phys. Vol. 35,

No. 5, pp. 2413–2422 (1994). doi:10.1063/1.530512

[105] V. E. Zakharov, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. Applied Mech. Tech. Phys. 9, pp. 190–194 (1968).

[106] V. E. Zakharov, *Collapse of Langmuir Waves*, Zh. Eksp. Teor. Fiz. 62, pp. 1745–1759 (1972).

[107] V. E. Zakharov, and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, JETP, 34 (1): 62–69.

[108] V. E. Zakharov and A. M. Rubenchik, *Nonlinear interaction of high-frequency and low-frequency waves*, Prikl. Mat. Techn. Phys., 5 (1972), pp 84-98.

[109] V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, JETP, 34 (1), pp. 62–69.