UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

CONTRIBUTIONS TO LINEAR DYNAMICS, SWEEPING PROCESS AND REGULARITY OF LIPSCHITZ FUNCTIONS.

TESIS PARA OPTAR AL GRADO DE:
DOCTOR EN CIENCIAS DE LA INGENIERÍA
MENCIÓN MODELACIÓN MATEMÁTICA
EN COTUTELA CON LA UNIVERSITÉ DE BORDEAUX

SEBASTIÁN GABRIEL TAPIA GARCÍA

PROFESOR GUÍA:
ARIS DANIILIDIS
PROFESOR CO-GUÍA:
ROBERT DEVILLE

MIEMBROS DE LA COMISIÓN:
JÉRÔME BOLTE
STÉPHANE GAUBERT
SOPHIE GRIVAUX
JESÚS JARAMILLO AGUADO
ALEJANDRO MAASS SEPÚLVEDA
ÉTIENNE MATHERON

Este trabajo ha sido parcialmente financiado por ANID PFCHA/Doctorado Nacional 2018/21181905, FONDECYT 1171854 y CMM ANID PIA AFB170001, CMM ANID BASAL ACE210010 y CMM ANID BASAL FB210005

SANTIAGO DE CHILE
2021

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FECHA: 2021
PROF. GUÍAS: ARIS DANIILIDIS, ROBERT DEVILLE

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English version

Abstract:

This thesis deals with three topics related to linear operators defined on infinite dimensional spaces and two topics of real analysis and variational analysis in finite dimensional spaces.

The first chapter contains preliminaries on Banach space theory which will be relevant for the three topics related to linear operators. The second chapter is a characterization of some types of bounded linear operators in terms of the differentiability of Lipschitz functions. Our results include a characterization for the classes of finite rank, compact, limited and weakly compact operators. The third and fourth chapters are inscribed in linear dynamics on infinite dimensional spaces, studying epsilon-hypercyclicity and wild operators respectively. We establish an epsilon-hypercyclicity criterion based on which we can construct epsilon-hypercyclic operators in a large class of separable Banach spaces. With respect to wild operators, we establish results about their spectra and about the norm closure of the set of wild operators in the space of linear bounded operators. In addition, we introduce and explore the concept of asymptotically separated sets to construct linear operators with interesting dynamical properties. The fifth chapter is a generalization of the Kurdyka-Łojasiewicz inequality for multivalued maps which are not necessarily definable in an o-minimal structure. We characterize smooth multivalued functions which satisfy a certain desingularization of the coderivative in terms of the length of the solutions of the related sweeping process as well as the integrability of the oriented talweg. The last chapter is devoted to absolutely minimizing Lipschitz (AML) functions. The main contribution in this subject is a characterization of the regularity of planar AML functions in terms of the regularity of the underlying norm.

Key words: Linear operators, linear dynamic, epsilon-hypercyclicity, KŁ-inequality, sweeping process, regularity of functions.

Versión en español

Resumen:

En la presente tesis se estudian tres temas relacionados a la teoría de operadores lineales definidos en espacios de Banach de dimensión infinita y dos tópicos del análisis real y análisis variacional en espacios de dimensión finita.

El primer capítulo contiene los fundamentos de la teoría de espacios de Banach que serán utilizados durante los primeros tres temas abarcados por esta tesis. En segundo capítulo se caracterizan algunas clases de operadores lineales acotados con respecto a la diferenciabilidad de un conjunto de funciones Lipschitz. Nuestros resultados pueden ser utilizados para caracterizar los operadores de rango finito, compactos, limitados y débilmente compactos. El tercer y cuarto capítulo están enmarcados en la teoría de dinámicas lineales: estudiaremos la épsilon-hiperciclidad y los operadores salvajes. Se establece un criterio de épsilon-hiperciclicidad, el que permite la construcción de operadores épsilon-hipercíclicos (que no son hipercíclicos) en una gran clase de espacios de Banach separables. En lo que respecta a operadores salvajes, se obtienen resultados en las siguientes tres lineas: sus espectros, la (no) estabilidad para el producto en la clase de operadores salvajes y la adherencia en norma del conjunto de operadores salvajes en el espacio de los operadores lineales acotados. Más aún, se introduce y se explora el concepto de conjuntos asintóticamente separados, además de establecer su relación con la construcción de operadores lineales con ciertas propiedades dinámicas interesantes. El quinto capítulo es una generalización de la desigualdad de Kurdyka-Łojasiewicz para funciones multivaluadas que no son (necesariamente) definibles en una estructura o-mínima. Además de establecer una desigualdad tipo Kurdyka-Łojasiewicz, esta desigualdad es caracterizada para la clase de funciones multivaluadas lisas en términos del largo de curva de las órbitas del proceso de barrido, así como de la integrabilidad del la función talweg orientado. El último capítulo está dedicado a la funciones Lipschitz absolutamente mínimas (AML). La contribución principal es una caracterización de la regularidad de las funciones AML planares en función de la regularidad de la norma subyacente.

Palabras claves: Operadores lineales, dinámica lineal, épsilon-hiperciclicidad, desigualdad KŁ, procesos de barrido, regularidad de funciones.

Version française

Résumé:

Cette thèse traite de trois thèmes liés aux opérateurs linéaires définis sur des espaces de dimension infinie et de deux sujets de l'analyse réelle et de l'analyse variationelle dans des espaces de dimension finie.

Le premier chapitre contient les préliminaires de la théorie des espaces de Banach qui seront utilisés dans les trois premiers thèmes. Le deuxième chapitre est une caractérisation de certains types d'operateurs linéaires bornées en termes de la différentiabilité des fonctions lipschitziennes. Nos résultats incluent une caractérisation pour les opérateurs de rang fini, compacts, limités et faiblement compacts. Les troisième et quatrième chapitres concernent la dynamique linéaire : nous étudions respectivement l'epsilon-hypercyclicité et les opérateurs dits "sauvages". Nous établissons un critère d'epsilon-hypercyclicité avec lequel nous pouvons construire des opérateurs epsilon-hypercycliques dans une large classe d'espaces de Banach séparables. En ce qui concerne les opérateurs sauvages, nous obtenons quelques résultats sur leurs spectres et sur la fermeture en norme de l'ensemble des opérateurs sauvages dans l'espace des opérateurs linéaires bornés. De plus, nous introduisons et explorons le concept d'ensembles asymptotiquement séparés pour construire des opérateurs linéaires avec des propriétés dynamiques intéressantes. Le cinquième chapitre est une généralisation de l'inégalité de Kurdyka-Łojasiewicz pour les fonctions multivoques qui ne sont pas nécessairement définissables dans une structure o-minimale. Nous caractérisons les fonctions multivoques lisses qui satisfont une désingularisation de la codérivée en termes de longueur des orbites du processus de rafle associé ainsi que de l'intégrabilité du talweg orienté. Le dernier chapitre est consacré aux fonctions Lipschitz absolument minimales (AML). La contribution principale est une caractérisation de la régularité de les fonctions AML planaires en termes de la régularité de la norme sous-jacente.

Mots-clés: Opérateur linéaire, dynamique linéaire, epsilon-hyperciclicité, inégalité KŁ, processus de rafle, régularité des fonctions.

Acknowledgments

I'm deeply grateful to my advisors Aris Daniilidis and Robert Deville, without their teachings, wisdom and advice, this work would not be possible. I especially appreciate their dedication and availability to guide me during my years of thesis.

I'm honored that professors Jesús Jaramillo and Étienne Matheron have accepted to be rapporteurs of my thesis and I appreciate their useful comments to improve this work. Also, I'm honored that professors Jérôme Bolte, Stéphane Gaubert, Sophie Grivaux and Alejandro Maass have accepted to be part of the jury of this thesis.

I'm very thankful to professor Mohammed Bachir with whom we have collaborated several times the last 3 years.

I'm thankful to the administrative staff from both the Departamento de Ingeniería Matemática (U. de Chile) and the Institute de Mathématiques de Bordeaux (U. de Bordeaux) for their kindness and patience to explain all the respective administrative procedures to me. Particularly, I want to express my gratitude to Natacha Astromujoff, Silvia Mariano, Sylvaine Granier and Muriel Michel.

Thanks to my friends and colleagues that I met in France, especially to Carlos Mudarra, Miguel Bravo, Baptiste Huguet, Nicoletta Prencipe, Paul Freulon, Yeeye Jiang, Marcu-Antone Orsoni and Gastón Vergara.

Agradezco a mis amigos y colegas con quienes compartimos alegrías, objetivos y matemáticas durante estos años Francisco Venegas, Emilio Molina, Gonzalo Flores, Felipe Muñoz, Ma. Eugenia Martínez, Martín Ríos, Hugo Maturana, Juan José Maulén y Álvaro Bustos.

Agradezco a mis amigos Matias Ramírez, Benjamín Martínez, Sebastián Berrios, Diego Gramusset, Camila Romero, David Salas, Alejandra Cevo, Ricardo Arancibia y Juan Pablo Donoso por su apoyo y preocupación a lo largo de mis estudios y viajes fuera de Chile.

Finalmente, estoy agradecido de todo corazón con mi familia y su apoyo incondicional en las decisiones que he tomado a lo largo de mis estudios. Especialmente dedico este trabajo a mis padres, Gabriel y Guillermina, mis hermanos, Juan Francisco y Belén, mis tías y tío, Mary, Ruby y Mario, y mis abuelas y abuelo, Ruby, Alicia y Gabriel.

Contents

Notation

$\mathbb{R}, \mathbb{C}$	Real and complex fields
$\mathbb{K}$	Real or complex field
$\mathbb{K}^n$	n-dimensional euclidean space
X, Y, Z	Banach spaces
H	Hilbert space
X^*	Dual space of X
$X \oplus Y$	Direct sum of X and Y
$\mathcal{L}(X, Y)$	Space of bounded operators from X to Y
$\mathcal{L}(X)$	Space of bounded operators from X to X
$\lvert \cdot \rvert$	The module on $\mathbb{K}$
$\lVert \cdot \rVert_X$ or $\lVert \cdot \rVert$	The norm on X
$\mathrm{Lip}(f)$	Lipschitz constant of a function f
$\mathbb{D}$	Complex open unit disk
$\mathbb{T}$	Complex unit circle
$B_X, \overline{B}_X$	The open and closed unit ball of X centered at 0
S_X	The unit sphere of X
$B(x, r)$ or $B_r(x)$	The open ball of center x and radius r
B_r	The open ball of center 0 and radius r
$S : \mathbb{R} \rightrightarrows \mathbb{R}^n$	A multivalued map
$\mathrm{dom}(S)$	The effective domain of S
$N_C(x)$	Limiting normal cone of $C \subset \mathbb{R}^n$ at $x \in C$
$D^*S(t, x)$	Limiting coderivative of S at (t, x)
$\overline{A}$	Closure of A
$\mathrm{int}(A)$	Interior of A
∂A	Boundary of A
$\mathrm{core}(A)$	Algebraic interior of A
$\mathcal{V} \subset\subset \mathcal{U}$	$\mathcal{V}$ is contained in a compact subset of the open set $\mathcal{U}$

Introduction Générale

Cette thèse est divisée en 6 chapitres qui explorent différents aspects de la théorie des opérateurs, de l'analyse variationnelle et de l'analyse réelle. Dans le premier chapitre introductif, nous fournissons les fondements de la théorie des espaces de Banach qui seront utilisés dans les chapitres 2, 3 et 4 et qui ne sont pas nécessairement standard. Ensuite, nous continuons avec cinq chapitres qui traitent de cinq problèmes différents et qui peuvent être lus indépendamment. Du deuxième au quatrième chapitre, nous traitons de problèmes liés à la classification des opérateurs bornés et à la dynamique linéaire qui ne se produisent que dans des espaces de Banach de dimension infinie. D'autre part, dans le cinquième et le sixième chapitres, nous présentons deux sujets différents, où l'espace sous-jacent est de dimension finie, concernant respectivement le processus de rafle et la régularité des fonctions lipschitziennes.

Chapitre 2 : β-opérateurs et différentiabilité

L'étude de différentes notions de différentiabilité de fonctions est l'un des principaux sujets de la théorie de la régularité des fonctions définies dans les espaces de Banach réels de dimension infinie. Les notions les plus connues de différentiabilité sont celles au sens de Fréchet, Gâteaux et Hadamard. Ces notions de différentiabilité sont comprises dans le cadre plus abstrait de la différentiabilité donnée par une bornologie. Nous rappelons qu'une bornologie sur un espace de Banach X est une famille de sous ensembles bornés de X qui recouvre X, qui est stable par unions finies et qui est héréditaire pour l'inclusion. Par exemple, la famille des ensembles bornés (resp. ensembles finis, ensembles relativement compacts) est une bornologie qui peut être définie sur tout espace de Banach. La différentiabilité uniforme sur les éléments de cette famille est la différentiabilité au sens de Fréchet (resp. Gâteaux, Hadamard).

Soit β une bornologie sur X. Un opérateur borné $T \in \mathcal{L}(Y, X)$ est un β-opérateur si $TB(y, r) \in \beta$ pour tout $y \in Y$ et pour tout $r > 0$. Pour motiver ce chapitre, nous introduisons le concept suivant : un ensemble $A \subset X$ est dit *limité* si, pour toute suite $(x_n^*)_n \subset X^*$ qui converge pour la topologie faible* vers 0, nous avons que

$$\lim_{n\to\infty} \sup\{x_n^*(x) : \ x \in A\} = 0.$$

Comme la famille des ensembles limités est une bornologie, un opérateur $T : Y \to X$ est limité si $T(B_Y)$ est un sous ensemble limité de Y. M. Bachir dans [11] a caractérisé les opérateurs limités en terme de la différentiabilité des fonctions convexes:

Théorème (Bachir) *Soient X et Y deux espaces de Banach. Soit $T : Y \to X$ un opérateur linéaire borné. Alors, T est limité si et seulement si, pour toute fonction convexe et continue $f : X \to \mathbb{R}$ qui est Gâteaux différentiable en $x = Ty$, la fonction $f \circ T$ est Fréchet différentiable en y.*

Motivés par le résultat précédant, avec M. Bachir et G. Flores, nous avons établi dans [13] des caractérisations similaires pour les opérateurs compacts et pour les opérateurs de rang fini. Nous utiliserons la définition suivante :

Définition *Soit β une bornologie sur X. Nous dirons que β satisfait la propriété (S) si pour tout ensemble borné $A \subset X$ tel que $A \notin \beta$, il existe une suite $(x_n)_n \subset A$ et $\delta > 0$ tels que pour toute suite croissante $(n_k)_k \subset \mathbb{N}$ et pour toute suite $(y_k)_k$ qui satisfait $\|y_k - x_{n_k}\| \leq \delta$ pour tout $k \in \mathbb{N}$, l'ensemble $\{y_k : k \in \mathbb{N}\}$ n'appartient pas à β.*

Par exemple, la bornologie de Hadamard (des ensembles relativement compacts) satisfait la propriété (S). De la même manière, la bornologie des ensembles limités ainsi que celle des ensembles relativement faiblement-compacts satisfont la propriété (S) (Section 2.2).

Nous dirons qu'une bornologie β sur X est convexe si, pour tout $A \in \beta$, l'enveloppe convexe $\text{co}(A)$ et $x + \lambda A$ appartiennent à β, pour tous $x \in X$, $\lambda \in \mathbb{R}$. Notons que, si β est une bornologie convexe sur X, alors $T \in \mathcal{L}(Y, X)$ est un β-opérateur si et seulement si $T(B_Y) \in \beta$.

Le résultat suivant caractérise certaines classes d'opérateurs linéaires tels que les opérateurs limités, compacts et faiblement-compacts.

Théorème A *Soient X et Y deux espaces de Banach réels et soit β une bornologie convexe sur X qui satisfait la propriété (S). Soit $T \in \mathcal{L}(Y, X)$. Alors, T est un β-opérateur si et seulement si, pour toute fonction lipschitzienne $f : X \to \mathbb{R}$, β-différentiable en $x = Ty$, alors la fonction $f \circ T$ est Fréchet différentiable en y.*

Avant d'énoncer la caractérisation des opérateurs de rang fini, nous avons besoin de la définition suivante.

Définition *Soient X et Y deux espaces de Banach. Nous dirons qu'une fonction $f : X \to Y$ est finiment lipschitzienne si pour tout sous espace affine Z de X, la restriction $f|_Z$ est une fonction lipschitzienne.*

Par exemple, tout opérateur linéaire $f : X \to Y$ est finiment lipschitzien. Nous sommes maintenant en mesure d'énoncer la deuxième contribution principale de ce chapitre.

Théorème B *Soient X et Y deux espaces de Banach réels. Soit $T \in \mathcal{L}(Y, X)$. Alors T est de rang fini si et seulement si, pour toute fonction finiment lipschitzienne $f : X \to \mathbb{R}$, Gâteaux différentiable en $x = Ty$, la fonction $f \circ T$ est Fréchet différentiable en y.*

Chapitre 3 : Critère d'epsilon-hypercyclicité

Dans ce chapitre, le corps sous-jacent peut être fixé comme $\mathbb{R}$ ou $\mathbb{C}$. Une manière naturelle de classer les opérateur linéaires passe par la dynamique engendrée par l'action de l'opérateur sur l'espace sous-jacent. Autrement dit, si $T \in \mathcal{L}(X)$ et $x \in X$, nous étudierons les propriétés de l'orbite de x sous l'action de T, l'ensemble $\mathrm{Orb}_T(x) := \{T^n x : n \in \mathbb{N}\}$. Soit $T \in \mathcal{L}(X)$ fixé. Nous dirons que l'orbite de $x \in X$ sous l'action T est régulière si la suite $(\|T^n x\|)_n$ tend vers 0, ou tend vers l'infini ou bien reste uniformément loin de 0 et de l'infini. Cette notion de régularité provient de la remarque suivante : si X est un espace de dimension finie et $T \in \mathcal{L}(X)$, alors toute orbite de T est régulière. D'un autre côté, si X est un espace de dimension infinie, nous savons qu'il y a des opérateurs avec des orbites non-régulières. En effet, dans [85], S. Rolewicz a construit des opérateurs $T \in \mathcal{L}(X)$, où X est $c_0(\mathbb{N})$ ou $\ell^p(\mathbb{N})$, avec $p \in [1, \infty)$, pour lesquels il y a un vecteur $x \in X$ avec une orbite dense. Plus précisément, il a considéré l'opérateur $2B$, où $B : X \to X$ est le *décalage à gauche* sur X, i.e., si $(\mathrm{e}_n)_n$ est la base canonique de X, $B\mathrm{e}_0 = 0$ et $B\mathrm{e}_n = \mathrm{e}_{n-1}$ si $n \geq 1$. Un opérateur qui possède un vecteur avec une orbite dense s'appelle *hypercyclique* et le vecteur associé s'appelle vecteur hypercyclique. À la lumière de cet exemple, la question suivante est naturelle : à quel point les orbites irrégulières peuvent-elles être différentes ?

Dans cette ligne, ces dernières décennies, la communauté a intensifié les efforts pour mieux comprendre les phénomènes purement infini dimensionnel de la dynamique linéaire. Un outil notable pour déterminer si un opérateur est hypercyclique est le critère d'hypercyclicité suivant.

Théorème *[67, Critère d'hypercyclicité]*
Soit X un espace de Banach, réel ou complexe, séparable et soit $T \in \mathcal{L}(X)$. Supposons qu'il existe une suite croissante $(n(k))_k \subset \mathbb{N}$, deux ensembles denses dans X, $\mathcal{D}_1$ et $\mathcal{D}_2$, et une suite des fonctions $S_{n(k)} : \mathcal{D}_2 \to X$ tels que :

(1) $\lim_{k\to\infty} T^{n(k)}x = 0$ *pour tout* $x \in \mathcal{D}_1$.
(2) $\lim_{k\to\infty} S_{n(k)}y = 0$ *pour tout* $y \in \mathcal{D}_2$.
(3) $\lim_{k\to\infty} T^{n(k)}S_{n(k)}y = y$ *pour tout* $y \in \mathcal{D}_2$.

Alors T est hypercyclique.

Dans [15], C. Badea, S. Grivaux et V. Müller ont introduit le concept suivant :

Définition *Soit $\varepsilon \in (0,1)$, soit X un espace de Banach réel ou complexe et soit $T \in \mathcal{L}(X)$. L'opérateur T est dit ε-hypercyclique s'il existe $x \in X$ vérifiant*

$$\forall\, y \in X \setminus \{0\},\ \exists\, n \in \mathbb{N} \text{ tel que } \|T^n x - y\| \leq \varepsilon \|y\|.$$

Comme conséquence de la définition, tout opérateur hypercyclique est ε-hypercyclique pour tout $\varepsilon > 0$. Notons que pour tout opérateur linéaire $T \in \mathcal{L}(X)$, le vecteur 0 est un vecteur 1-hypercyclique. Les auteurs ont construit dans [15], pour tout $\varepsilon \in (0,1)$ fixé, un opérateur ε-hypercyclique qui n'est pas hypercyclique dans $\ell^1(\mathbb{N})$. Deux ans plus tard, F. Bayart a construit dans [16] un opérateur ε-hypercyclique qui n'est pas hypercyclique dans $\ell^2(\mathbb{N})$.

La contribution principale de ce chapitre est le critère d'epsilon-hypercyclicité suivant. Notons que les opérateurs construits dans [15, 16] satisfont ce critère.

Théorème C (Critère d'epsilon-hypercyclicité) *Soit X un espace de Banach réel ou complexe, soit $T \in \mathcal{L}(X)$ et soit $\varepsilon \in (0,1)$. On se donne $\mathcal{D}_1$ un sous ensemble dense dans X et $\mathcal{D}_2 := \{y_k : k \in \mathbb{N}\} \subset X$ tel que, pour tout $x \in X \setminus \{0\}$, il existe une infinité d'entiers $k \in \mathbb{N}$ tels que $y_k \in \overline{B}(x, \varepsilon\|x\|)$. Enfin, on se donne une suite croissante $(n(k))_k \subset \mathbb{N}$ et une suite des fonctions $S_{n(k)} : \mathcal{D}_2 \to X$ vérifiant:*

(1) $\lim_{k\to\infty} \|T^{n(k)}x\| = 0$ *pour tout* $x \in \mathcal{D}_1$,

(2) $\lim_{k\to\infty} \|S_{n(k)}y_k\| = 0$,

(3) $\lim_{k\to\infty} \|T^{n(k)}S_{n(k)}y_k - y_k\| = 0$.

Alors T est δ-hypercyclique pour tout $\delta > \varepsilon$.

Comme dans le cas du critère d'hypercyclicité, nous fournissons une preuve constructive et une preuve topologique (basée sur le théorème de Baire). De plus, à l'aide du Théorème C, nous pouvons construire des opérateurs dans des espaces de Banach plus généraux. En fait, nous obtenons le résultat suivant.

Théorème D *Soit X un espace de Banach séparable qui possède un sous espace complémenté isomorphe à $c_0(\mathbb{N})$ ou à $\ell^p(\mathbb{N})$, avec $p \in [1, \infty)$. Alors, pour tout $\varepsilon > 0$, il existe un opérateur dans X qui est ε-hypercyclique mais pas hypercyclique.*

Chapitre 4 : Opérateurs sauvages et ensembles asymptotiquement séparés

Soit X un espace de Banach réel ou complexe et soit T un opérateur borné dans X. Le théorème de Banach-Steinhaus implique que l'ensemble des points dont l'orbite est non-bornée (sous l'action de T) est vide ou dense dans X. Si X est un espace de dimension finie, pour tout $x \in X$, l'orbite $\{T^n x : n \in \mathbb{N}\}$ est non-bornée si et seulement si la suite $(\|T^n x\|)_n$ tend vers l'infini. Sur la base de ces observations, G. Prăjitură a proposé la conjecture suivante : l'ensemble

$$A_T := \{x \in X : \lim_{n\to\infty} \|T^n x\| = \infty\}$$

est-il toujours vide ou dense dans X? Deux années après, P. Hajek et R. Smith ont réfuté cette conjecture. En fait, pour tout espace X de dimension infinie avec une base symétrique, ils ont construit un opérateur borné T dans X tel que l'ensemble A_T est non-vide et non-dense dans X, voir [61]. J.M. Augé a construit dans [8] un opérateur borné qui réfute la conjecture de Prăjitură dans tout espace de Banach séparable de dimension infinie.

Nous avons besoin de la définition suivante :

Définition *Un ensemble $F \subset X$ est dit asymptotiquement séparé s'il existe une suite $(x_n^*)_n \subset X^*$ telle que*

i) $\liminf_{n\to\infty} |x_n^*(x)| = 0$, *pour tout* $x \in F$.

ii) $\lim_{n\to\infty} |x_n^*(x)| = \infty$, *pour tout* $x \in X \setminus F$.

Dans [8], l'existence d'un ensemble asymptotiquement séparé non-trivial $F \subset \mathbb{K}^2$, à savoir

$$\{(x,y) \in \mathbb{K}^2 : \ |x| \leq |y|\},$$

permet la construction d'un opérateur qui réfute la conjecture de Prăjitură.

Dans la première partie de ce chapitre, nous explorons les ensembles asymptotiquement séparés dans des espaces de Banach et nous en donnons des application à la dynamique linéaire. Le résultat suivant donne quelques exemples d'ensembles asymptotiquement séparés que nous avons trouvés dans ce travail.

Théorème E *Soit X un espace de Banach réel ou complexe et soit $F \subset X$. Supposons une de deux conditions suivantes :*

i) $\dim(X) < \infty$ *et F est union d'hyperplans linéaires tel que $F \setminus \{0\}$ est ouvert.*

ii) *X est séparable et F est égale à $\{0\}$ ou est un sous espace fermé de X.*

Alors F est asymptotiquement séparé. De plus, tout espace de Banach de dimension deux ou supérieure contient un sous ensemble asymptotiquement séparé qui est dense et tel que son complémentaire est dense aussi.

Pour un opérateur $T \in \mathcal{L}(X)$, nous définissons l'ensemble de points récurrents :

$$R_T := \{x \in X : \ \liminf_{n\to\infty} \|T^n x - x\| = 0\}.$$

Définition *Soit X un espace de Banach réel ou complexe. Un opérateur $T \in \mathcal{L}(X)$ est dit sauvage si $\{A_T, R_T\}$ est une partition de X en deux ensembles qui ont un intérieur non-vide.*

L'existence d'opérateurs sauvages dans tout espace de Banach séparable et de dimension infinie est démontrée dans [8]. Le prochain résultat montre le lien entre la dynamique linéaire et les ensembles asymptotiquement séparés. Ce théorème est une généralisation du résultat principal de [8].

Théorème F *Soit X un espace de Banach réel ou complexe, séparable et de dimension infinie. Soit V un sous espace fermé de X, complémenté et de codimension infinie. Soit $F \subseteq V$ un ensemble asymptotiquement séparé dans V. Alors, il existe un opérateur $T \in \mathcal{L}(X)$ tel que $R_T = P^{-1}(F)$ et $A_T = P^{-1}(V \setminus F)$, où $P \in \mathcal{L}(X)$ est une projection bornée sur V.*

Comme conséquence du Théorème F, un exemple intéressant d'ensemble asymptotiquement séparé implique l'existence d'un opérateur linéaire borné avec une dynamique intéressante.

En fait, si nous appliquons le Théorème E, le Théorème F et le fait que tout espace de Banach de dimension infinie peut être décomposé comme $V \oplus W$, avec V un sous espace de dimension finie qui est plus grande ou égale à deux, nous obtenons le corollaire suivant.

Corollaire *Soit X un espace de Banach séparable et de dimension infinie. Alors*

- *Il existe $T \in \mathcal{L}(X)$ tel que $\{A_T, R_T\}$ est une partition de X et les deux ensembles sont denses dans X.*
- *Il existe $T \in \mathcal{L}(X)$ tel que $\{A_T, R_T\}$ est une partition de X et $A_T \cup \{0\}$ est un sous espace de codimension finie.*
- *Il existe $T \in \mathcal{L}(X)$ sauvage tel que $A_R \cup \{0\}$ est fermé.*

Dans la seconde partie de ce chapitre nous étudions quelques propriétés des opérateurs sauvages. Plus précisément, le théorème suivant donne les résultats que nous avons obtenus dans les trois directions suivantes : la non-stabilité par produit de la classe des opérateurs sauvages, la construction d'opérateurs sauvages non-inversibles et la taille de l'adhérence en norme de l'ensemble des opérateurs sauvages.

Théorème G *Soit X un espace de Banach complexe, séparable et de dimension infinie. Alors :*

- *Il existe $T \in \mathcal{L}(X)$ sauvage tel que $T \oplus T$ n'est pas sauvage dans $X \oplus X$.*
- *Si X a une base symétrique, il existe $T \in \mathcal{L}(X)$ qui est sauvage mais qui n'est pas inversible.*
- *Si X a une base inconditionnelle $(\mathrm{e}_n)_n$, tout opérateur linéaire borné qui est diagonal par rapport à (e_n), avec valeurs propres de module 1, appartient à la adhérence en norme de l'ensemble des opérateurs sauvages dans $\mathcal{L}(X)$.*

Chapitre 5: désingularisation des processus de rafle lisses.

Ce chapitre est le début de la deuxième partie de cette thèse, dans laquelle nous étudions quelque aspects de l'analyse réelle et de l'analyse variationelle dans l'espaces de dimension finie. Dans ce chapitre $\mathbb{R}^n$ désigne l'espace euclidien de dimension n.

Il est bien connu que toute fonction lisse de classe $\mathcal{C}^1$, $f : \mathbb{R}^n \to \mathbb{R}$, qui est définissable dans une structure o-minimale, a un nombre fini de valeurs critiques. K. Kurdyka a montré dans [68] que si $\bar{r} \in f(\mathbb{R}^n)$ est une valeur critique et si $\mathcal{U}$ est un sous ensemble ouvert, borné et non-vide de $\mathbb{R}^n$, alors il existent $\rho > 0$ et une fonction lisse de classe $\mathcal{C}^1$, $\psi : [\bar{r}, \bar{r} + \rho] \to [0, +\infty)$, qui satisfont

$$\|\nabla(\psi \circ f)(x)\| \geq 1, \qquad \text{pour tout } \ x \in \mathcal{U} \text{ tel que } \ f(x) \in (\bar{r}, \bar{r} + \rho). \tag{1}$$

L'inégalité précédente généralise, au cas o-minimal, une inégalité du gradient démontré pour Łojasiewicz dans [70] pour la classe de fonctions $\mathcal{C}^1$ et sous analytique. L'expression (1) est maintenant connu sous le nom *l'inégalité de Kurdyka-Łojasiewicz* (en abrégé inégalité KŁ).

Les deux inégalités mentionnées ci-haut ont été étendues à des fonctions non-lisses (resp. sous analytiques et o-minimales), voir [24, 25]. Ces inégalités nous permettent un contrôle uniforme sur la longueur des orbites bornées du (sous)gradient, voir [71, 68, 24]. Ce contrôle reste vrai pour la longueur de courbes de gradient par morceaux, i.e., courbes obtenues par la concaténation d'un nombre au plus dénombrable de courbes de gradient $\{\gamma_i\}_{i \geq 1}$, où

$\gamma_{\mathrm{i}} \subset f^{-1}([r_{\mathrm{i}+1}, r_{\mathrm{i}}))$ et $\{r_{\mathrm{i}}\}_{\mathrm{i}}$ est une suite décroissante sur $(\bar{r}, \bar{r}+\rho)$, qui converge vers $\bar{r}$. Ce type de courbes possède un nombre au plus dénombrable de discontinuités.

En dehors du cadre o-minimal, l'inégalité KŁ(1) peut être fausse même pour des fonctions lisses de classe $\mathcal{C}^2$ [26, Section 4.3] ou même pour des fonctions lisses de classe $\mathcal{C}^\infty$ avec un valeur critique unique [76, p. 12]. J. Bolte, A. Daniilidis, O. Ley et L. Mazet, dans [26], ont considéré le problème de caractériser l'existence d'une fonction désingularisante ψ et de la validité de l'inégalité (1) pour une valeur critique isolée supérieurement $\bar{r}$ d'une fonction semiconvexe *coercive* f définie dans un espace de Hilbert (la fonction f n'étant nécessairement pas définissable dans un structure o-minimale).

Nous définissons maintenant la dynamique générée par une fonction multivoque

Définition *Soit $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ une fonction multivoque et soit $I \subset \mathrm{dom}(S)$ un intervalle non-trivial de $\mathbb{R}$. Une courbe absolument continue $\gamma : I \to \mathbb{R}^n$ est dite solution (orbite) du processus de rafle défini par S si*

$$\begin{cases} -\dot{\gamma}(t) \in N_{S(t)}(\gamma(t)), \ \forall_{p.p}\ t \in I, \\ \quad \gamma(t) \in S(t) \ \textit{pour tout}\ t \in I, \end{cases}$$

où $N_{S(t)}(\gamma(t))$ est le cône normal limite de $S(t)$ au point $\gamma(t)$. Nous désignons par $\mathcal{AC}(S, I)$ ($\mathcal{PAC}(S, I)$) l'ensemble des orbites absolument continues (absolument continues par morceaux respectivement) générées par le processus de rafle défini par S dans l'intervalle $I \subset \mathrm{dom}(S)$.

Soit $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ une fonction multivoque. Le graphe de S sera noté par

$$\mathcal{S} := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : \ x \in S(t)\}$$

Définition *Soit $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ une fonction multivoque. La codérivée de S à $(t, x) \in \mathcal{S}$ en $u \in \mathbb{R}^n$ est définie par :*

$$D^*S(t, x)(u) := \{a \in \mathbb{R} : \ (a, -u) \in N_{\mathcal{S}}(t, x)\}.$$

En 2017, A. Daniilidis et D. Drusvyatskiy, dans [41], ont montré que pour toute fonction multivoque $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$, définissable dans une structure o-minimale et pour toute valeur critique $\bar{t} \in \mathbb{R}$, il existe une fonction qui désingularise la codérivée $D^*S(t, \cdot)$ autour de $\bar{t}$. Ce résultat implique un contrôle uniforme sur la longueur des orbites bornées du processus de rafle défini par S. Ce résultat récupère les résultats de K. Kurdyka dans [68]. En fait, il suffit de considérer la fonction multivoque S définie par les sous-niveaux d'une fonction lisse et définissable f.

La contribution principale de ce chapitre est liée à la désingularisation de la codérivée appliquée aux fonctions multivoques qui ne sont pas nécessairement définissable dans une structure o-minimale. Nous avons besoin les définitions suivantes.

Définition *Soit $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ une fonction multivoque.*

i) *Pour tout* $(t,x)\in\mathcal{S}$, *le module asymétrique de la codérivée* $D^*S(t,x)$ *est défini par:*

$$\|D^*S(t,x)|^+ = \sup\{\max(a,0):\ a\in D^*S(t,x)(u),\ \|u\|\le 1\},$$

où nous faisons la convention $\sup(\emptyset)=0$.

ii) *Le talweg orienté de* S, *noté* $\varphi^\uparrow:\mathrm{dom}(S)\to\mathbb{R}\cup\{\infty\}$, *est défini par:*

$$\varphi^\uparrow(t) = \sup_{x\in S(t)}\{\|D^*S(t,x)|^+\},\quad \textit{pour tout } t\in\mathrm{dom}(S).$$

Notre travail s'inscrit dans le cadre de la définition et les hypothèses $(A1)$, $(A2)$ et $(A3)$ suivantes.

Définition *Nous dirons que* $S:\mathbb{R}\rightrightarrows\mathbb{R}^n$ *est un* processus de rafle lisse *si*
– $\mathcal{S}$ *est une sous variété lisse, connexe, fermé et de classe* $\mathcal{C}^1$ *de* $\mathbb{R}^{n+1}$, *de dimension au plus* n *; ou*
– $\mathcal{S}$ *est une sous variété à bord de dimension* $n+1$ *dont la frontière* $\partial\mathcal{S}$ *est une variété lisse de classe* $\mathcal{C}^1$ *et de dimension* n.

Hypothèses *Soit* $S:\mathbb{R}\rightrightarrows\mathbb{R}^n$ *une fonction multivoque et soit* $\mathrm{T}:=\sup(\mathrm{dom}(S))$. *Nous dirons que* S *satisfait*

$(A1)$ l'hypothèse d'existence *: pour tout* $(t,x)\in\mathcal{S}$, *avec* $\|D^*S(t,x)|^+<+\infty$, *il existe* $\delta_x>0$ *et au moins une orbite* $\gamma_x\in AC(S;[t,t+\delta_x))$ *tels que* $\gamma_x(t)=x$.

$(A2)$ l'hypothèse de régularité supérieure *en* $\bar{t}\in\mathrm{dom}(S)$, *avec* $\bar{t}<\mathrm{T}$ *: il existe* $\delta>0$ *tel que* $\varphi^\uparrow(t)<+\infty$ *pour tout* $t\in(\bar{t},\bar{t}+\delta)$.

$(A3)$ l'hypothèse de continuité *en* $\bar{t}\in\mathrm{dom}(S)$, *avec* $\bar{t}<\mathrm{T}$ *: il existe* $\delta>0$ *tel que la fonction multivoque* $t\rightrightarrows\partial\mathcal{S}\cap(\{t\}\cap\mathbb{R}^n)$ *est continue pour la métrique de Pompeiu-Hausdorff sur* $(\bar{t},\bar{t}+\delta)$ *(Peut-être il y a une discontinuité en* $\bar{t}$*).*

Maintenant, nous sommes prêts à énoncer notre résultat principal.

Théorème H *Soit* $S:\mathbb{R}\rightrightarrows\mathbb{R}^n$ *un processus de rafle lisse avec des valeurs bornées qui satisfait (A1). Soit* $\mathcal{T}:=\{t\in\mathrm{dom}(S):$ *(A2)–(A3) sont vérifiés* $t\}$. *Soit* $a\in\mathcal{T}$ *(typiquement une valeur critique de* D^*S*).*
Les assertions suivantes sont équivalentes :

a) ***(Désingularisation de la codérivée)*** *Il existe* $b>a$, $\rho>0$ *et un homeomorphisme* $\Psi:[0,\rho]\to[a,b]$, *qui est un* $\mathcal{C}^1$*-diffeomorphisme entre* $(0,\rho)$ *et* (a,b), *avec* $\Psi'(r)>0$ *pour tout* $r\in(0,\rho)$, *tels que:*

$$\|D^*(S\circ\Psi)(r,x)|^+\le 1,\qquad \textit{pour tout } r\in(0,\rho),\ \textit{pour tout } x\in S(\Psi(r)).$$

b) ***(Contrôle uniforme de la longueur des orbites)*** *Il existe* $b>a$ *et une fonction croissante* $\sigma:[a,b]\mapsto\mathbb{R}^+$, *avec* $\sigma(a)=0$, *tels que pour tous* $a\le t_1<t_2\le b$ *et* $\gamma\in\mathcal{AC}(S,[t_1,t_2])$, *il tient que :*

$$\ell(\gamma):=\int_{t_1}^{t_2}\|\dot\gamma\|\le\sigma(t_2)-\sigma(t_1).$$

c) ***(Contrôle uniforme de la longueur des orbites par morceaux)*** *Il existe $b > a$ et $M < \infty$ tels que pour toute $\gamma \in \mathcal{PAC}(S,[a,b])$ il tient que:*

$$\ell(\gamma) := \int_{t_1}^{t_2} \|\dot{\gamma}\| \leq M.$$

d) ***(Intégrabilité du talweg)*** *Il existe $b > a$ tel que*

$$\int_a^b \varphi^{\uparrow}(t) < \infty.$$

De plus, dans ce chapitre nous fournissons également une caractérisation de la désingularisation de la codérivée d'un processus de rafle lisse en terme de la dynamique discrète générée par la fonction multivoque donnée, à savoir, les suites générées par l'algorithme "Catching-up".

Chapitre 6 : Fonctions AML dans les espaces de dimension deux

À la différence du chapitre 5, dans ce chapitre, $(\mathbb{R}^n, \|\cdot\|)$ désigne un espace de dimension n équipé avec une norme $\|\cdot\|$ (non nécessairement euclidienne).

Nous nous proposons d'étudier la régularité des fonctions lipschitziennes. Soit $\mathcal{U} \subset \mathbb{R}^n$ un ensemble ouvert non-vide. Le théorème de Rademacher dit que si $f : \mathcal{U} \to \mathbb{R}$ est une fonction localement lipschitzienne, alors f est differentiable presque partout. Dans ce travail, nous nous concentrons sur la classe de fonctions définie comme suit :

Définition *Soit $(\mathbb{R}^n, \|\cdot\|)$ un espace normé de dimension n et soit $\mathcal{U} \subset \mathbb{R}^n$ un ouvert non-vide. Nous dirons qu'une fonction localement lipschitzienne $f : \mathcal{U} \to \mathbb{R}$ est $\|\cdot\|$-Lipschitz absolument minimale (en abrégé $\|\cdot\|$-AML) si pour tout ouvert non-vide $\mathcal{V} \subset\subset \mathcal{U}$ et pour toute fonction lipschitzienne $g : \overline{\mathcal{V}} \to \mathbb{R}$ telle que $g = f$ sur $\partial\mathcal{V}$, alors*

$$\mathrm{Lip}(g) \geq \mathrm{Lip}(f|_{\overline{\mathcal{V}}}).$$

S'il n'y a pas de confusion avec la norme $\|\cdot\|$, nous écririons simplement fonction AML.

L'existence de fonctions AML non-triviales et la régularité de ces fonctions font partie des principaux problèmes de cette théorie. Dans le cas euclidien, G. Aronsson a montré dans [6] que pour les fonctions $f : \mathcal{U} \to \mathbb{R}$ de classe $\mathcal{C}^2$, f est AML si et seulement si elle est une solution classique de l'équation du *Laplacien-infini*, i.e., la fonction satisfait

$$\triangle_\infty f := \sum_{\mathrm{i},j=1}^{n} \partial_{\mathrm{i}} f \partial_j f \partial^2_{\mathrm{i}j} f = 0, \text{ dans } \mathcal{U}. \tag{∞L}$$

En 1993, R. Jensen a démontré que la famille des fonctions AML coïncide avec les solutions de l'équation (∞L) au sens de viscosité. De plus, R. Jensen a démontré l'existence et l'unicité de la solution au sens de viscosité du problème de Cauchy gouverné pour l'équation (∞L) avec condition au bord continue, voir [64].

Maintenant, nous rappelons quelques résultats qui concernent la régularité de cette classe de fonctions. O. Savin a démontré dans [89] que les fonctions AML définies sur un ouvert de $(\mathbb{R}^2, \|\cdot\|_2)$ (équipé avec la norme euclidienne) sont continûment différentiables. Dans [48], L. Evans et C. Smart ont montré que les fonctions AML définies sur un ouvert de $(\mathbb{R}^n, \|\cdot\|_2)$ sont partout différentiables. Toutefois, la continuité de la dérivée reste une question ouverte pour $n \geq 3$.

Le théorème principal de ce chapitre est le suivant.

Théorème I *Soit X un espace normé de dimension 2. Les assertions suivantes sont équivalentes :*

a) La norme sous-jacente est différentiable sur $X \setminus \{0\}$.

b) Toute fonction AML définie sur un ouvert de X est continûment différentiable.

c) Toute fonction AML définie sur un ouvert de X est partout différentiable.

Au début de 2021, F. Peng, C. Wang et Y. Zhou ont généralisé le résultat de régularité de O. Savin aux fonctions absolument minimisantes par rapport à un Hamiltonien convexe définies sur un ouvert de $(\mathbb{R}^2, \|\cdot\|_2)$, voir [77]. Ce résultat généralise également notre Théorème I aussi. Toutefois, la preuve fournie dans [77] se base sur la structure euclidienne sous-jacente, alors que notre preuve est complètement non-euclidienne.

General Introduction

This thesis is divided into 6 chapters which explore different aspects of Operator Theory, Variational Analysis and Real Analysis. In the first introductory chapter, we present the fundamentals of the Banach space theory which will be used in Chapters 2, 3 and 4. Then, we continue with five chapters which deal with five different problems and can be read independently. From the second to the fourth chapter we deal with problems arising from the classification of bounded operators and linear dynamics which occur only in infinite dimensional Banach spaces. On the other hand, in the fifth and sixth chapter we present two different problems in finite dimensional spaces, concerning sweeping processes and regularity of Lipschitz functions respectively.

Chapter 2: β-operators and differentiability:

The study of distinct notions of differentiability has been one of the main topics in the theory of regularity of functions defined in general real Banach spaces. The most common notions of differentiability are given by the ones in the sense of Fréchet, Gâteaux and Hadamard. These notions of differentiability are enclosed in the more abstract setting of the differentiability given by a bornology. Let us recall that a bornology is a family of bounded subsets of X which is a covering of X, is stable under finite unions and hereditary under inclusion. For instance, the family of bounded sets (resp. finite sets, relatively compact sets) is a bornology that can be defined on each Banach space and the uniform differentiability with respect to these sets consists exactly on the Fréchet (resp. Gâteaux, Hadamard) differentiability. Let β be a bornology in X, we say that a bounded operator $T \in \mathcal{L}(Y, X)$ is a β-operator if $TB(y, r) \in \beta$ for all $y \in Y$ and $r > 0$. To motivate this chapter, let us introduce the following definition. A set $A \subset X$ is said *limited* if, for any sequence $(x_n^*)_n \subset X^*$, weakly*-convergent to 0, we have that

$$\lim_{n\to\infty} \sup\{x_n^*(x) : \; x \in A\} = 0.$$

Observe that the family of limited sets of a Banach space form a bornology. Moreover, a bounded linear operator $T : X \to Y$ is limited if and only if $T(B_X)$ is a limited subset of Y. In [11], M. Bachir characterized limited operators in terms of the differentiability of convex functions as follows:

Theorem (Bachir) *Let X and Y be two real Banach spaces. Let $T : Y \to X$ be a bounded operator. Then, T is limited if and only if, for every $f : X \to \mathbb{R}$ continuous convex function, $f \circ T$ is Fréchet differentiable at y whenever f is Gâteaux differentiable at $x = Ty$.*

Motivated by this result, with M. Bachir and G. Flores in [13], we obtained two characterizations, the first one for compact operators and the second one for finite rank operators. In order to state our results, let us introduce the following property on bornologies.

Definition *Let β be a bornology on X. We say that β satisfies the property (S) if for every bounded set $A \subset X$ such that $A \notin \beta$, there are a sequence $(x_n)_n \subset A$ and $\delta > 0$ such that for any increasing sequences $(n_k) \subset \mathbb{N}$ and for any sequence $(y_k)_k$ satisfying $\|y_k - x_{n_k}\| \leq \delta$, the set $\{y_k : k \in \mathbb{N}\}$ does not belong to β.*

It is not hard to see that the bornology of relatively compact sets satisfies property (S). Also, the bornology of limited sets and the one of relatively weakly-compact sets satisfy property (S), see Section 2.2.

We use the following notation: A bornology β on X is said to be convex if for any $A \in \beta$, the convex envelop $\operatorname{co}(A)$ and $x + \lambda A$ belong to β, for any $x \in X$ and $\lambda \in \mathbb{R}$. Observe that, if β is a convex bornology on X, $T \in \mathcal{L}(Y, X)$ is a β-operator if $T(B_Y) \in \beta$. The following result characterizes several kind of operators including limited, compact and weakly-compact operators.

Theorem A *Let X and Y be two real Banach spaces and let β be a convex bornology on X satisfying property (S). Let $T : Y \to X$ be a bounded linear operator. Then T is a β-operator if and only if for every Lipschitz function $f : X \to \mathbb{R}$, β differentiable at $x = Ty$, the function $f \circ T$ is Fréchet differentiable at y.*

In order to state a characterization for finite rank operators we introduce the following class of functions.

Definition *Let X and Y be two Banach spaces. A function $f : X \to Y$ is called finitely Lipschitz if for any finite dimensional affine subspace Z of X, the restriction $f|_Z$ is Lipschitz.*

For instance, any linear map from X to Y is finitely Lipschitz. Thus, the second main contribution of this chapter reads as follows:

Theorem B *Let X and Y be two real Banach spaces. Let $T : Y \to X$ be a bounded linear operator. Then T has finite rank if and only if for every finitely Lipschitz function $f : X \to \mathbb{R}$, Gâteaux differentiable at $x = Ty$, the function $f \circ T$ is Fréchet differentiable at y.*

Chapter 3: Epsilon-Hypercyclicity Criterion

In this chapter, the underlying scalar field can be fixed as $\mathbb{R}$ or $\mathbb{C}$. A natural way to classify linear operators is through the dynamic generated by the action of the operator on the underlying space. That is, for a given operator $T \in \mathcal{L}(X)$ and $x \in X$, the study of the properties of the orbit of x under the action of T: $\operatorname{Orb}_T(x) := \{T^n x : \ n \in \mathbb{N}\}$. Let us fix $T \in \mathcal{L}(X)$. It is considered that the orbit of $x \in X$ under T is regular if the sequence $(\|T^n x\|)_n$ either tends to 0 or tends to ∞ or remains bounded away from 0 and ∞ as n tends to ∞. This notion of regularity comes from the fact that if X is a finite dimensional Banach space, then any orbit generated by the action of any linear operator is regular. Moreover, it is known that there are operators defined on infinite dimensional spaces with non-regular orbits. Indeed, in [85], S. Rolewicz constructed operators $T \in \mathcal{L}(X)$, where X is $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$ with $p \in [1, \infty)$, such that there is a vector $x \in X$ with dense orbit. More precisely, he considered the operator $2B$, where $B : X \to X$ is the backward shift on X, i.e., if $(\mathrm{e}_n)_n$ is the canonical basis of X, $B\mathrm{e}_0 = 0$ and $B\mathrm{e}_n = \mathrm{e}_{n-1}$ for all $n \geq 1$. An operator which has a vector with dense orbit is called *hypercyclic* and the associated vector is called hypercyclic vector. Regarding this example as an operator with irregular orbits, a natural question arises: How different can be the irregular orbits?

In this line, the effort of the community to understand purely infinite dimensional phenomena in linear dynamics has increased during the last decades. A remarkable tool to determine if a given operator is hypercyclic is the so-called Hypercyclicity Criterion.

Theorem *[67, Hypercyclicity Criterion] Let X be a separable real or complex Banach space and let $T \in \mathcal{L}(X)$. If there exists an increasing sequence of integers $(n(k))_k \subset \mathbb{N}$, two dense sets $\mathcal{D}_1$, $\mathcal{D}_2 \subset X$ and sequence of maps $S_{n(k)} : \mathcal{D}_2 \to X$ such that:*

(1) $\lim_{k\to\infty} \|T^{n(k)}x\| = 0$ *for any* $x \in \mathcal{D}_1$.

(2) $\lim_{k\to\infty} \|S_{n(k)}y\| = 0$ *for any* $y \in \mathcal{D}_2$.

(3) $\lim_{k\to\infty} \|T^{n(k)}S_{n(k)}y - y\| = 0$ *for any* $y \in \mathcal{D}_2$.

Then T is hypercyclic.

In [15], C. Badea, S. Grivaux and V. Müller, introduced the following notion:

Definition *Let $\varepsilon \in (0,1)$, let X be a real or complex Banach space and let $T \in \mathcal{L}(X)$. The operator T is called ε-hypercyclic if there is $x \in X$ such that*

$$\forall\, y \in X \setminus \{0\},\ \exists\, n \in \mathbb{N},\ \|T^n x - y\| \leq \varepsilon \|y\|.$$

It is clear that each hypercyclic operator is ε-hypercyclic for all $\varepsilon > 0$. Also, the vector 0 would be a 1-hypercyclic vector for each bounded operator. In the mentioned paper, [15], for $\varepsilon \in (0,1)$ fixed, it is constructed an ε-hypercyclic operator which is not hypercyclic on $\ell^1(\mathbb{N})$. Two years later, F. Bayart in [16] constructed an ε-hypercyclic operator which is not hypercyclic on $\ell^2(\mathbb{N})$.

The main contribution of this chapter is a criterion for epsilon-hypercyclicity. As a remark, the operators constructed in [15, 16] satisfy the following criterion.

Theorem C (Epsilon-Hypercyclicity Criterion) *Let X be a separable real or complex Banach space, let $T \in \mathcal{L}(X)$ and let $\varepsilon \in (0,1)$. Let $\mathcal{D}_1$ be a dense set on X. Let $\mathcal{D}_2 := \{y_k : k \in \mathbb{N}\}$ be a countable subset of X. Assume further that for each $x \in X \setminus \{0\}$, there are infinitely many integers $k \in \mathbb{N}$ such that $y_k \in \overline{B}(x, \varepsilon\|x\|)$. Let $(n(k))_k \subset \mathbb{N}$ be an increasing sequence and let $S_{n(k)} : \mathcal{D}_2 \to X$ be a sequence of maps such that:*

(1) $\lim_{k\to\infty} \|T^{n(k)}x\| = 0$ *for all* $x \in \mathcal{D}_1$,
(2) $\lim_{k\to\infty} \|S_{n(k)}y_k\| = 0$,
(3) $\lim_{k\to\infty} \|T^{n(k)}S_{n(k)}y_k - y_k\| = 0$.

Then, T is δ-hypercyclic for all $\delta > \varepsilon$.

In the fashion of the Hypercyclic Criterion, we present a constructive proof and a topological proof of our criterion. Moreover, with the help of Theorem C, we are able to further enhance the construction and obtain ε-hypercyclic operators which are not hypercyclic. In fact, we obtain the following result.

Theorem D *Let X be a separable Banach space which admits a complemented isomorphic copy of $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$, with $p \in [1,\infty)$. Then, for any $\varepsilon > 0$, X admits an ε-hypercyclic operator which is not hypercyclic.*

Chapter 4: Wild operators and asymptotically separated sets

Let X be a real or complex Banach space and let T be a bounded operator on X. The Banach-Steinhaus Theorem implies that the set of points with unbounded orbits under the action of T must be either dense or empty. If X is a finite dimensional space and since the orbits of T are regular, an orbit $\{T^n x : n \in \mathbb{N}\}$ is unbounded if and only if the sequence $(\|T^n x\|)_n$ tends to infinity. Based on these observations, G. Prăjitură proposed the following conjecture: is the set

$$A_T := \{x \in X : \lim_{n\to\infty} \|T^n x\| = \infty\}$$

either empty or dense? Two year later, P. Hajek and R. Smith refuted this conjecture by constructing a bounded operator T on each infinite dimensional Banach space with a symmetric basis, such that the set A_T was a nonempty non-dense set, see [61]. In [8], J.M. Augé construct a counterexample for mentioned conjecture on each infinite dimensional separable Banach space. Before proceeding, we need the following definition.

Definition *A set $F \subset X$ is called asymptotically separated if there exist $(x_n^*)_n \subset X^*$ such that*

i) $\liminf_{n\to\infty} |x_n^*(x)| = 0$, *for all* $x \in F$.
ii) $\lim_{n\to\infty} |x_n^*(x)| = \infty$, *for all* $x \in X \setminus F$.

In [8], the existence of a non-trivial asymptotically separated set $F \subset \mathbb{K}^2$, namely

$$\{(x,y) \in \mathbb{K}^2 : \ |x| \leq |y|\},$$

allows the construction of an operator which refutes Prăjitură's conjecture. In the first part of this chapter we explore the asymptotically separated sets defined in both finite and infinite dimensional spaces and its consequences in linear dynamics. The following theorem summarizes some of the examples of asymptotically separated sets that can be found in this work.

Theorem E *Let X be a complex or real Banach space and let $F \subset X$.*

i) *If* $\dim(X) < \infty$ *and* F *is a union of linear hyperplanes such that* $F \setminus \{0\}$ *is open, or*

ii) *if* X *is separable and* F *is equal to* $\{0\}$ *or a closed subspace of* X*,*

then F *is asymptotically separated. Moreover, any Banach space of dimension at least* 2 *admits a dense asymptotically separated subset with dense complement.*

In order to continue, let us define the recurrent set of a linear operator $T \in \mathcal{L}(X)$ by

$$R_T := \{x \in X : \liminf_{n\to\infty} \|T^n x - x\| = 0\}.$$

Definition *Let* X *be a real or complex Banach space. An operator* $T \in \mathcal{L}(X)$ *is called wild if* A_T *and* R_T *form a partition of* X *and both sets have nonempty interior.*

In [8] it is proved that each separable infinite dimensional Banach space admits a wild operator. The link between linear dynamics and asymptotically separated sets comes from the following theorem, which is a generalization of the main result of [8].

Theorem F *Let* X *be a separable infinite dimensional real or complex Banach space. Let* V *be a complemented, infinite codimensional subspace of* X*. Let* $F \subseteq V$ *be an asymptotically separated subset in* V*. Then there exists an operator* $T \in \mathcal{L}(X)$ *such that* $R_T = P^{-1}(F)$ *and* $A_T = P^{-1}(V \setminus F)$*, where* $P \in \mathcal{L}(X)$ *is a projection onto* V*.*

In virtue of Theorem F, any (non-trivial) example of asymptotically separated set leads to the existence of a linear operator with interesting dynamics. In fact, combining Theorem E and Theorem F and the fact that any infinite dimensional Banach space X can be decomposed as $V \oplus W$, with V a finite dimensional subspace (of dimension at least 2), we obtain the following corollary.

Corollary *Let* X *be a separable infinite dimensional Banach space. Then:*

- *There is* $T \in \mathcal{L}(X)$ *such that* A_T *and* R_T *form a partition of* X *and both sets are dense.*
- *There is* $T \in \mathcal{L}(X)$ *such that* A_T *and* R_T *form a partition of* X *and* $A_T \cup \{0\}$ *is a finite codimensional subspace.*
- *There is* $T \in \mathcal{L}(X)$ *wild such that* $A_T \cup \{0\}$ *is closed.*

In the second part of this chapter we study some properties of wild operators. More precisely, the following theorem gives some results obtained in three different directions: the non-stability under products of the class of wild operators, the construction of non-invertible wild operators and the size of the norm-closure of the set of wild operators.

Theorem G *Let* X *be a separable infinite dimensional complex Banach space. Then*

- X *admits a wild operator* $T \in \mathcal{L}(X)$ *such that* $T \oplus T$ *is not wild on* $X \oplus X$*.*
- *if* X *has a symmetric basis, then* X *admits a non-invertible wild operator.*
- *if* X *has an unconditional basis* $(\mathrm{e}_n)_n$*, then each diagonal operator with respect to* (e_n)*, with only unitary eigenvalues, belongs to the norm-closure of the set of wild operators in* $\mathcal{L}(X)$*.*

Chapter 5: Desingularization of smooth sweeping processes

This chapter is the beginning of the second part of this thesis, in which we turn our attention to some aspects of Variational Analysis and Real Analysis in finite dimensional spaces. In this chapter $\mathbb{R}^n$ denotes the n-dimensional vector space endowed with the canonical euclidean norm.

It is well-known that every $\mathcal{C}^1$ smooth function $f : \mathbb{R}^n \to \mathbb{R}$ which is definable in some o-minimal structure has finitely many critical values. K. Kurdyka [68] showed that if $\bar{r} \in f(\mathbb{R}^n)$ is a critical value and $\mathcal{U}$ is a nonempty open bounded subset of $\mathbb{R}^n$, then there exist $\rho > 0$ and a $\mathcal{C}^1$-smooth function $\psi : [\bar{r}, \bar{r} + \rho] \to [0, +\infty)$ satisfying

$$\|\nabla(\psi \circ f)(x)\| \geq 1, \qquad \text{for all } \ x \in \mathcal{U} \text{ such that } \ f(x) \in (\bar{r}, \bar{r} + \rho). \tag{2}$$

The above inequality generalizes to o-minimal functions the Łojasiewicz gradient inequality (established in [70] for the class of $\mathcal{C}^1$ subanalytic functions) and is nowadays known as the Kurdyka-Łojasiewicz inequality (in short, KŁ-inequality).

Both the Łojasiewicz and the KŁ-inequality have been further extended to nonsmooth (subanalytic and respectively o-minimal) functions, see [24, 25]. These inequalities allow to control uniformly the lengths of the bounded (sub)gradient orbits, see [71, 68, 24]. The same is true for the lengths of the piecewise gradient curves, that is, curves obtained by concatenating countably many gradient curves $\{\gamma_\mathrm{i}\}_{\mathrm{i}\geq 1}$, where $\gamma_\mathrm{i} \subset f^{-1}([r_{\mathrm{i}+1}, r_\mathrm{i}))$ and $\{r_\mathrm{i}\}_\mathrm{i}$ is a strictly decreasing sequence in $(\bar{r}, \bar{r} + \rho)$ converging to $\bar{r}$. (These curves have at most countably many discontinuities.)

Outside the framework of o-minimality the KŁ-inequality (2) may fail even for $\mathcal{C}^2$-smooth functions [26, Section 4.3] or for $\mathcal{C}^\infty$-smooth function with a unique critical value [76, p. 12]. J.Bolte, A. Daniilidis, O. Ley and L. Mazet in [26] considered the problem of characterizing the existence of a desingularization function ψ and the validity of (2)-inequality for an upper isolated critical value $\bar{r}$ of a semiconvex *coercive* function f defined in a Hilbert space (where f is not necessarily a definable function).

In order to continue, let us introduce the dynamic generated by a multivalued function and its coderivative.

Definition *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a multivalued map and $I \subset \mathrm{dom}(S)$ be a nonempty interval of $\mathbb{R}$. We say that the absolutely continuous curve $\gamma : I \to \mathbb{R}^n$ is a solution (orbit) of the sweeping process defined by S if*

$$\begin{cases} -\dot{\gamma}(t) \in N_{S(t)}(\gamma(t)), \ \forall_{a.e.}\ t \in I, \\ \ \ \gamma(t) \in S(t) \textit{ for all } t \in I, \end{cases}$$

where $N_{S(t)}(\gamma(t))$ stands for the normal cone of $S(t)$ at $\gamma(t)$. We denote by $\mathcal{AC}(S, I)$ ($\mathcal{PAC}(S, I)$) the set of absolutely continuous (resp. piecewise absolutely continuous) orbits of the sweeping process defined by S on the interval $I \subset \mathrm{dom}(S)$.

For a multivalued map $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$, we denote its graph by $\mathcal{S} := \{(t, x) \in \mathbb{R}\times\mathbb{R}^n : \ x \in S(t)\}$

Definition *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a multivalued function. The (limiting) coderivative of S at $(t, x) \in \mathcal{S}$ in $u \in \mathbb{R}^n$ is defined as follows:*

$$D^*S(t, x)(u) := \{a \in \mathbb{R} : \ (a, -u) \in N_{\mathcal{S}}(t, x)\}.$$

Recently, A. Daniilidis and D. Drusvyatskiy [41] showed that every multivalued map $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ with definable graph admits a desingularization of its graphical coderivative $D^*S(t, \cdot)$ around any critical value $t \in \mathbb{R}$. This result yields a uniform bound for the lengths of all bounded orbits of the sweeping process defined by S. The aforementioned results of [41] are also covering the results of Kurdyka in [68] by considering a sweeping process mapping S related to the sublevel sets of the smooth definable function f.

The main contribution of this chapter is the desingularization of the coderivative for multivalued functions which are not necessarily definable in some the o-minimal structure. In order to state our main result, let us introduce some definitions.

Definition *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a multivalued function.*

i) *For every $(t, x) \in \mathcal{S}$, the asymmetric modulus of the coderivative $D^*S(t, x)$ is defined as follows:*

$$\|D^*S(t, x)|^+ = \sup\{\max(a, 0) : \ a \in D^*S(t, x)(u), \ \|u\| \leq 1\},$$

where we adopt the convention $\sup(\emptyset) = 0$.

ii) *The oriented talweg function of S denoted by $\varphi^\uparrow$ is defined as follows:*

$$\varphi^\uparrow(t) = \sup_{x\in S(t)} \{\|D^*S(t, x)|^+\}, \quad \textit{for all } t \in \mathrm{dom}(S).$$

The setting of our work is described in the following definition and the assumptions $(A1)$, $(A2)$ and $(A3)$ given below.

Definition *We say that $S : R \rightrightarrows \mathbb{R}^n$ is a* smooth sweeping process *if either*

– *$\mathcal{S}$ is a closed connected $\mathcal{C}^1$-smooth submanifold of $\mathbb{R}^{n+1}$ of dimension at most n ; or*

– *$\mathcal{S}$ is a connected smooth manifold with boundary of dimension $n + 1$ such that $\partial\mathcal{S}$ is a $\mathcal{C}^1$-smooth manifold of dimension n.*

Assumption *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a multivalued map and let $\mathrm{T} := \sup(\mathrm{dom}(S))$. We say that S satisfies the*

$(A1)$ existence assumption*: for every $(t, x) \in \mathcal{S}$ with $\|D^*S(t, x)|^+ < +\infty$, there exist $\delta_x > 0$ and at least one orbit $\gamma_x \in AC(S; [t, t + \delta_x))$ such that $\gamma_x(t) = x$.*

$(A2)$ upper regular assumption *at $\bar{t} \in \mathrm{dom}(S)$ with $\bar{t} < \mathrm{T}$: if there exists $\delta > 0$ such that $\varphi^\uparrow < +\infty$ on $(\bar{t}, \bar{t} + \delta)$.*

(A3) continuity assumption *at* $\bar{t} \in \operatorname{dom}(S)$ *with* $\bar{t} < \mathrm{T}$*: if there exists* $\delta > 0$ *such that the multivalued map* $t \rightrightarrows \partial\mathcal{S} \cap (\{t\} \cap \mathbb{R}^n)$ *is continuous for the Pompeiu-Hausdorff metric on* $(\bar{t}, \bar{t} + \delta)$ *(it may be discontinuous at* $\bar{t}$*).*

Now, we are ready to state our main result.

Theorem H *Let* $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ *be a smooth sweeping process with bounded values that satisfies (A1). Let* $\mathcal{T} := \{t \in \operatorname{dom}(S) :$ *(A2)–(A3) are fulfilled at* $t\}$*. Let* $a \in \mathcal{T}$ *(typically a critical value for* D^*S*).*

The following assertions are equivalent:

a) ***(Desingularization of the coderivative)*** *There exist* $b > a$, $\rho > 0$ *and a homeomorphism* $\Psi : [0, \rho] \to [a, b]$*, which is a* $\mathcal{C}^1$*-diffeomorphism between* $(0, \rho)$ *and* (a, b) *with* $\Psi'(r) > 0$ *for every* $r \in (0, \rho)$*, such that:*

$$\|D^*(S \circ \Psi)(r, x)|^+ \leq 1, \qquad \text{for all } r \in (0, \rho), \text{ for all } x \in S(\Psi(r)).$$

b) ***(Uniform length control for the absolutely continuous orbits)*** *There exist* $b > a$ *and an increasing continuous function* $\sigma : [a, b] \mapsto \mathbb{R}^+$ *with* $\sigma(a) = 0$ *such that for every* $a \leq t_1 < t_2 \leq b$ *and* $\gamma \in \mathcal{AC}(S, [t_1, t_2])$ *we have:*

$$\ell(\gamma) := \int_{t_1}^{t_2} \|\dot{\gamma}\| \leq \sigma(t_2) - \sigma(t_1).$$

c) ***(Length bound for the piecewise absolutely continuous orbits)*** *There exist* $b > a$ *and* $M < \infty$ *such that for every* $\gamma \in \mathcal{PAC}(S, [a, b])$ *we have:*

$$\ell(\gamma) := \int_{t_1}^{t_2} \|\dot{\gamma}\| \leq M.$$

d) ***(Integrability of the talweg)*** *There exists* $b > a$ *such that*

$$\int_a^b \varphi^{\uparrow}(t) < \infty.$$

Also, in this chapter we provide a characterization of the desingularization of the coderivative for smooth sweeping processes in terms of the discrete dynamic generated by the given multivalued map, namely, the sequences generated by the Catching-Up Algorithm.

Chapter 6: AML functions in two dimensional spaces:

In contrast to Chapter 5, in this chapter $(\mathbb{R}^n, \|\cdot\|)$ denotes an n-dimensional vector space $\mathbb{R}^n$ equipped with a (not necessarily Euclidean) norm $\|\cdot\|$.

This chapter is devoted to the regularity of Lipschitz functions. Let $\mathcal{U} \subset \mathbb{R}^n$ be a nonempty open set. By Rademacher Theorem, any locally Lipschitz function $f : \mathcal{U} \to \mathbb{R}$ is differentiable almost everywhere. In this work we study the regularity of the following subfamily of locally Lipschitz functions.

Definition *Let $(\mathbb{R}^n, \|\cdot\|)$ be a finite dimensional normed space and let $\mathcal{U} \subset \mathbb{R}^n$ a nonempty open set. We say that a locally Lipschitz function $f : \mathcal{U} \to \mathbb{R}$ is a $\|\cdot\|$-absolutely minimizing Lipschitz function ($\|\cdot\|$-AML for short) if for any nonempty open set $\mathcal{V} \subset\subset \mathcal{U}$ and any Lipschitz function $g : \overline{\mathcal{V}} \to \mathbb{R}$ such that $g|_{\partial\mathcal{V}} = f|_{\partial\mathcal{V}}$, then*

$$\mathrm{Lip}(g) \geq \mathrm{Lip}(f|_{\overline{\mathcal{V}}}).$$

If no confusion arises from the underlying norm on $\mathbb{R}^n$, we simple say AML functions.

The existence of non-trivial AML functions and their regularity are some of the main issues of this theory. In the Euclidean setting, Aronsson proved that a $\mathcal{C}^2$-smooth function, $f : \mathcal{U} \to \mathbb{R}$ is AML if and only if it is a classical solution of the equation governed by the *infinity-Laplacian*, i.e. the function f satisfies

$$\triangle_\infty f := \sum_{\mathrm{i},j=1}^{n} \partial_\mathrm{i} f \partial_j f \partial^2_{\mathrm{i}j} f = 0, \text{ on } \mathcal{U} \qquad (\infty\mathrm{L})$$

in the classical sense, see [6]. In 1993, Jensen showed that the family of AML functions coincides with the solutions of Equation (∞L) in the viscosity sense. Moreover, Jensen proved existence and uniqueness in the sense of viscosity of the Cauchy problem given by the Equation (∞L) with a continuous boundary condition, see [64].

Let us now summarize some results concerning the regularity of this class of functions. In the seminal paper [89], O. Savin proved that AML functions defined on open subsets of $(\mathbb{R}^2, \|\cdot\|_2)$ (equipped with an Euclidean norm) are continuously differentiable. In [48], L. Evans and C. Smart proved that AML functions defined on open subsets of $(\mathbb{R}^n, \|\cdot\|_2)$ are differentiability everywhere. However, the continuity of the differential remains open for $n \geq 3$.

The main theorem of this chapter reads as follows.

Theorem I *Let X be a 2 dimensional Banach space. The following statements are equivalent.*

a) The underlying norm is differentiable in $X \setminus \{0\}$.

b) Every AML function defined on an open subset of X is continuously differentiable.

c) Every AML function defined on an open subset of X is everywhere differentiable.

In the early 2021, F. Peng, C. Wang and Y. Zhou generalized Savin's result to absolutely minimizing functions under convex Hamiltonians defined on open sets of $(\mathbb{R}^2, \|\cdot\|_2)$. This result also generalizes our Theorem I. However, the proof presented in [77] relies in the underlying Euclidean structure of $(\mathbb{R}^2, \|\cdot\|_2)$, in contrast with our purely non-Euclidean technique to prove Theorem I.

Chapter 1

General Facts of Banach space Theory

The aim of this chapter is to give the fundamentals to develop the first part of this thesis. Functional Analysis and Banach space theory have been widely developed along the whole 20th century. We present some well-known facts of three distinct areas, namely: bornologies, bases and linear dynamics. In fact, Section 1.1 contains the basics for Chapter 2. On the other hand, Section 1.2 and Section 1.3 are the main bricks to Chapters 3 and 4. We point out that in Section 1.1 the usual framework is a real Banach space while in Section 1.2 and Section 1.3, the underlying scalar field can be either $\mathbb{R}$ or $\mathbb{C}$.

1.1 Bornologies, differentiability and linear operators

Let us start with the definition of a bornology on Banach spaces.

Definition 1.1 *[63, Page 18] Let X be a real Banach space. A bornology on X, denoted by β, is any nonempty family of bounded subsets of X that satisfies the following properties:*

1. *β is a covering of X, i.e. $X = \bigcup\{A : A \in \beta\}$,*
2. *β is hereditary under inclusion, i.e. if $A \in \beta$ and $B \subset A$, then $B \in \beta$,*
3. *β is stable under finite union, i.e. if $A, B \in \beta$, then $A \cup B \in \beta$.*

For a given Banach space, the most common bornologies are the following: Fréchet, the family of all bounded sets; Gâteaux, the family of all finite sets and (weakly-)Hadamard, the family of all relatively (weakly-)compact sets. One of the uses of bornologies is to define different notions of differentiability as follows.

Definition 1.2 *Let Ω be an nonempty open subset of X and let β be a bornology on X. A function $f : \Omega \to Y$ is said β differentiable at $x \in \Omega$, with differential $T \in \mathcal{L}(X, Y)$, if*

$$\lim_{t\to 0} \sup_{z\in A} \left\| \frac{f(x+tz) - f(x)}{t} - T(z) \right\| = 0, \ \forall A \in \beta.$$

We denote the β differential of f at x by $\mathrm{d}_\beta f(x) := T$.

Observe that, in the sense of Definition 1.2, Gâteaux and Fréchet differentiability correspond to the weakest and strongest notions of differentiability. A well known result asserts that if f is Fréchet differentiable at some point x, then f is continuous at x. However, this does not hold true whenever we interchange Fréchet for Gâteaux. In what follows, we present a folklore result on differentiability of Lipschitz functions. Recall that a function $f : \Omega \subset X \to Y$ is said K-Lipschitz if

$$\|f(x) - f(z)\| \leq K\|x - z\|, \text{ for all } x, y \in X.$$

The Lipschitz constant of f is the lowest constant K such that f is a K-Lipschitz function and it is denoted by $\mathrm{Lip}(f)$.

Proposition 1.3 *Let $f : \Omega \subset X \to Y$ be a Lipschitz function. Assume that f is Gâteaux differentiable at $x \in \mathrm{int}(\Omega)$. Then, f is Hadamard differentiable at x.*

Proof. Let $\mathrm{d}_G f(x)$ be the Gâteaux differential of f at x. Let us show that f is in fact Hadamard differentiable and the Hadamard differential $\mathrm{d}_H f(x)$ coincides with $\mathrm{d}_G f(x)$. Let A be a compact subset of X and $\varepsilon > 0$. Let A_ε be a finite ε-net of A. Then, since $\|\mathrm{d}_G f(x)\| \leq \mathrm{Lip}(f)$, for any $t \neq 0$, we compute

$$\begin{aligned}\sup_{z\in A}\left\|\frac{f(x+tz)-f(x)}{t} - \mathrm{d}_G f(x)(z)\right\| &\leq \sup_{z\in A}\left\|\frac{f(x+tz)-f(x+ta_z)}{t}\right\| \\ &\quad + \left\|\frac{f(x+ta_z)-f(x)}{t} - \mathrm{d}_G f(x)(a_z)\right\| + \|\mathrm{d}_G f(x)(a_z - z)\| \\ &\leq 2\varepsilon\mathrm{Lip}(f) + \sup_{z\in A}\left\|\frac{f(x+ta_z)-f(x)}{t} - \mathrm{d}_G f(x)(a_z)\right\|,\end{aligned}$$

where $a_z \in A_\varepsilon$ is chosen such that $\|z - a_z\| \leq \varepsilon$. Therefore, since A_ε is a finite set, sending t to 0 we obtain that

$$\lim_{t\to 0}\sup_{z\in A}\left\|\frac{f(x+tz)-f(x)}{t} - \mathrm{d}_G f(x)(z)\right\| \leq 2\varepsilon\mathrm{Lip}(f).$$

Finally, since $\varepsilon > 0$ is arbitrary, we conclude that $\mathrm{d}_G f(x)$ is the Hadamard differential of f at x. □

As a direct consequence we have:

Corollary 1.4 *Let $f : \Omega \subset X \to Y$ be a Lipschitz function. Assume that X is finite-dimensional. Then, Gâteaux and Fréchet differentiability coincide for f.*

According to [43], we can consider the Banach spaces of β differentiable functions.

Proposition 1.5 *[43, Section 2] Let X be a real Banach space and let β be a bornology on X. Then, the vector space $C^\beta_u(X)$ of bounded, Lipschitz continuous and everywhere β differentiable functions from X to $\mathbb{R}$ is a Banach space when it is equipped with the norm*

$$\begin{aligned}\|f\|_1 :&= \|f\|_\infty + \|\mathrm{d}_\beta f\|_\infty \\ &= \sup\{|f(x)| : \ x \in X\} + \sup\{\|\mathrm{d}_\beta f(x)\| : \ x \in X\},\end{aligned}$$

where $f \in C^\beta_u(X)$.

To end this section, and in the spirit of [63, Chapter 1], we introduce the following definition.

Definition 1.6 *Let X and Y be two Banach spaces and let β be a bornology on Y. We say that a linear operator $T : X \to Y$ is a β-operator if $TB(x,r) \in \beta$ for all $x \in X$ and all $r > 0$.*

Observe that, for any bornology β in Y, a β-operator $T : X \to Y$ is continuous. Indeed, this is due to the fact that $TB_X \in \beta$ and that any bornology is made up of bounded sets. That is, in this notation, an operator is bounded if and only if it is a Fréchet-operator. Further, the classifications of compact and weakly compact operator are related to the Hadamard and weakly-Hadamard bornology respectively.

1.2 Bases in Banach spaces

In this section, we introduce several notions of bases for real or complex Banach spaces. Bases allow us to treat certain Banach spaces as sequence spaces. According to [60], a sequence $(\mathrm{e}_n, \mathrm{e}_n^*)_n \subseteq X \times X^*$ is a biorthogonal system if $\mathrm{e}_n^*(\mathrm{e}_m) = 1$ if $n = m$, and $\mathrm{e}_n^*(\mathrm{e}_m) = 0$ if $n \neq m$. A complete survey about the importance of biorthogonal systems in Banach space theory can be found in [60]. In what follows, we introduce some important kinds of biorthogonal systems.

Definition 1.7 *A sequence $(\mathrm{e}_n)_n \subset X$ is called a Schauder basis of X if for any $x \in X$, there is a unique sequence of scalars $(a_n)_n \subset \mathbb{K}$ such that*

$$x = \sum_{n=1}^{\infty} a_n \mathrm{e}_n.$$

Observe that, if $(\mathrm{e}_n)_n$ is a Schauder basis of X, then it induces a canonical biorthogonal system. Namely, $(\mathrm{e}_n, \mathrm{e}_n^*)_n \subset X \times X^*$ where e_n^* is defined by the n-th scalar obtained in the series defining x in terms of $(\mathrm{e}_n)_n$. Also, notice that every Hamel basis of a finite dimensional Banach space is a Schauder basis in the sense of Definition 1.7. However, in [47], P. Enflo constructed an infinite dimensional separable Banach space that lacks of Schauder basis.

Definition 1.8 *[1, Definition 3.1.4] A Schauder basis $(\mathrm{e}_n)_n \subset X$ is said C-unconditional if, for all $N \in \mathbb{N}$,*

$$\left\| \sum_{n=1}^{N} a_n \mathrm{e}_n \right\| \leq K \left\| \sum_{n=1}^{N} b_n \mathrm{e}_n \right\|$$

whenever $a_1, ..., a_N, b_1, ..., b_N$ are scalars satisfying $|a_n| \leq |b_n|$ for all $n = 1, ..., N$. The Schauder basis $(\mathrm{e}_n)_n$ is said unconditional if it is C-unconditional for some $C > 0$.

In [56], W. Gowers and B. Maurey constructed a separable infinite dimensional Banach space X such that no sequence $(\mathrm{e}_n)_n \subset X$ is an unconditional basis in the infinite dimensional subspace $\overline{\mathrm{span}}(\mathrm{e}_n : n \in \mathbb{N})$. On the other hand, any orthonormal basis of a separable Hilbert space is 1-unconditional. More generally, the canonical basis of $\ell^p(\mathbb{N})$, with $p \in [1, \infty)$, or of

$c_0(\mathbb{N})$ is 1-unconditional. The following proposition is a well-known result concerning unconditional basis and renorming a Banach space.

Proposition 1.9 *Let X be a Banach space with an unconditional basis $(\mathrm{e}_n)_n \subset X$. Then, there exists a renorming on X such that $(\mathrm{e}_n)_n$ is a 1-unconditional basis.*

Let us introduce the last notion of biorthogonal system of this section.

Definition 1.10 *Let $(\mathrm{e}_n, \mathrm{e}_n^*)_n \subset X \times X^*$ be a biorthogonal system. We say that $(\mathrm{e}_n, \mathrm{e}_n^*)_n$ (or just $(\mathrm{e}_n)_n$) is a bounded M-basis if:*

1. $X = \overline{\operatorname{span}}(\mathrm{e}_n : n \in \mathbb{N})$,
2. $X^* = \overline{\operatorname{span}}^{\omega^*}(\mathrm{e}_n^* : n \in \mathbb{N})$, *and*
3. $\sup\{\|\mathrm{e}_n\|\|\mathrm{e}_n^*\| : n \in \mathbb{N}\} \leq K < \infty$.

The concept of bounded M-basis is weaker than the notion of Schauder basis. In fact, In [75] R. Ovsepian and A. Pełczyński showed that every separable Banach space admits a bounded M-basis. We present here the more precise formulation found in [60].

Theorem 1.11 *[60, Theorem 1.27] Let X be a separable real or complex Banach space. Then, X admits a bounded M-basis. Moreover, the constant K can be chosen as $1 + \varepsilon$ with $\varepsilon > 0$ arbitrarily small.*

1.3 Dynamics of linear operators

Linear dynamic is a rapidly increasing area of Functional Analysis which deals with the study of dynamical systems generated by the action of a bounded linear operator T on some topological vector space X. In this work we are interested in the case whenever X is an infinite dimensional Banach space. To fix notation, for $x \in X$, we say that the orbit of T at x is the set

$$\operatorname{Orb}_T(x) := \{T^n x : n \in \mathbb{N}\}.$$

In [85], S. Rolewicz stated that linear dynamics in finite dimensional spaces are regular in the following sense.

Theorem 1.12 *[85, Page 1] Let X be a finite dimensional real or complex normed space and let $T \in \mathcal{L}(X)$. Let $x \in X$. Then, one of the following assertions holds true:*

1. $\lim_{n\to\infty} \|T^n x\| = \infty$.
2. $\lim_{n\to\infty} \|T^n x\| = 0$.
3. $0 < \alpha < \|T^n x\| < \beta < \infty$ *for all* $n \in \mathbb{N}$.

Sketch of the proof. Let us give some comments of the proof for the case whenever X is a complex vector space. If X is a real vector space, we can do the same analysis with its complexification. Since T is a linear operator, it admits a Jordan canonical form M_T. That is, a block diagonal matrix such that each block is a Jordan block. Let $x = x^L + x^1 + x^G$ be the unique decomposition of x such that x^L belongs to the span of the union of the generalized eigenspaces associated to eigenvalues of T of modulus less than 1, x^1 belongs to the span of the union of the generalized eigenspaces of T associated to eigenvalues of modulus equal to 1 and x^G belongs to the span of the union of the generalized eigenspaces associated to eigenvalues of T of modulus greater than 1.

Now we proceed with the analysis of $\|T^n x\|$. It is straightforward that $\|T^n x\|$ tends to infinity as n tends to infinity if $x^G \neq 0$. Also, it is clear that $\|T^n x^L\|$ tends to 0 as n tends to infinity. So, to end the analysis, we can assume that $x = x^1$. That is, x belongs to the span of the union of the generalized eigenspaces associated to eigenvalues of T of modulus equal to 1. If x can be written as a sum of eigenvectors of T, then x satisfies (2). The last case is whenever x belongs to the span of the union of the generalized eigenspaces associated to eigenvalues of modulus 1 but it cannot be decomposed in eigenvectors. It is not difficult to see that x satisfies (1). □

Remark 1.13 *In the proof of Theorem 1.12, if $x^G \neq 0$, then $\|T^n x\|$ grows exponentially to infinity. On the other hand, if $x^G = 0$ and $x^1 \neq 0$ cannot be decomposed in eigenvectors of T, then $\|T^n x\|$ grows polynomially to infinity.*

To continue, we need some definitions to motivate our work.

Definition 1.14 *Let X be a real or complex Banach space and let $T \in \mathcal{L}(X)$. We say that T is cyclic if there is $x \in X$ such that*

$$\overline{\text{span}}(\text{Orb}_T(x)) = X.$$

The vector x is called a cyclic vector for T.

Observe that if X is a d-dimensional space and $\{\mathrm{e}_n : n = 1, ..., \mathrm{d}\}$ is a basis of X, the operator $T \in \mathcal{L}(X)$ defined by the following permutation of the basis:

$$T\mathrm{e}_\mathrm{d} = \mathrm{e}_1, \text{ and } T\mathrm{e}_n = \mathrm{e}_{n+1}, \text{ for all } n = 1, ..., \mathrm{d} - 1,$$

is cyclic and the vectors $\{\mathrm{e}_k,\ k = 1, ..., \mathrm{d}\}$ are cyclic vectors for T.

Definition 1.15 *Let X be a real or complex Banach space and let $T \in \mathcal{L}(X)$. We say that T is supercyclic if there is $x \in X$ such that*

$$\overline{\mathbb{K}\text{Orb}_T(x)} = X.$$

The vector x is called a supercyclic vector for T.

Observe that if X is a 2-dimensional space and $\{\mathrm{e}_1, \mathrm{e}_2\}$ is a basis of X, θ is an irrational multiple of π and $T \in \mathcal{L}(X)$ is the operator defined by:

$$T(x_1\mathrm{e}_1 + x_2\mathrm{e}_2) := x_1(\cos(\theta)\mathrm{e}_1 + \sin(\theta)\mathrm{e}_2) + x_2(-\sin(\theta)\mathrm{e}_1 + \cos(\theta)\mathrm{e}_2), \text{ for all } x_1, x_2 \in \mathbb{K}$$

is supercyclic and the vectors $\{\mathrm{e}_k,\ k = 1, ..., \mathrm{d}\}$ are supercyclic vectors for T. Let us continue with the last definition of this chapter.

Definition 1.16 *Let X be a Banach space and let T be a linear bounded operator on X. We say that T hypercyclic if there is $x \in X$ such that*

$$\overline{\mathrm{Orb}}_T(x) = X.$$

The vector x is called an hypercyclic vector for T.

Observe that, thanks to Theorem 1.12, there is no hypercyclic operators defined on finite dimensional spaces. However, in [85], S. Rolewicz showed examples of hypercyclic operators defined on infinite dimensional Banach spaces.

Example 1.17 *Let $p \in [1, \infty)$ and let $(\mathrm{e}_n)_n$ be the canonical coordinates of $\ell^p(\mathbb{N})$. Let $B \in \mathcal{L}(\ell^p(\mathbb{N}))$ defined by $B\mathrm{e}_1 = 0$ and $B\mathrm{e}_n = \mathrm{e}_{n-1}$ if $n \geq 2$. The operator B is commonly called the backward shift on $\ell^p(\mathbb{N})$. Then, λB is hypercyclic in $\ell^p(\mathbb{N})$ if and only if $|\lambda| > 1$.*

The examples given by S. Rolewicz seem to be the first examples constructed in Banach spaces. However, 40 years before in [16], Birkhoff showed an example of a hypercyclic operator constructed on a Fréchet space. Since [85], linear dynamics has been a prolific area in functional analysis. Indeed, these definitions are related to the following subspace/subset invariant problems: Does there exist a linear operator $T : X \to X$ without non-trivial invariant closed subspace/subset? In fact, an operator $T \in \mathcal{L}(X)$ has no non-trivial invariant closed subspace if and only if each non-zero vector of X is a cyclic vector for T. On the other hand, an operator T has no non-trivial invariant closed subset if and only if each non-zero vector of X is a hypercyclic vector for T. Let us give some comments about these problems. P. Enflo, in [47], showed the first operator with no non-trivial closed subspace in an infinite dimensional Banach space. In the same work, P. Enflo constructed the Banach space and the operator. C. Read, in [82], constructed an operator on $\ell^1(\mathbb{N})$ with no non-trivial invariant closed subsets. On the other hand, in [4], S. Argyros and R. Haydon constructed a separable infinite dimensional Banach space X such that every operator $T \in \mathcal{L}(X)$ has a non-trivial invariant subspace.

Nowadays, thanks to the joint effort of researchers to understand these kind of phenomena, such as the hypercyclicity, there is a vast literature in which we can find several distinct classifications that strengthen the concept of hypercyclicity, such as frequent hypercyclicity, weak mixing, among others. Further information can be found in [18].

We end this section by presenting an important tool to check if a given operator is hypercyclic, the so-called Hypercyclicity Criterion. It appeared for first time (in a particular case) in Kitai's PhD thesis [67]. In what follows, we present the version found in [18] applied to Banach spaces.

Theorem 1.18 *[18, Theorem 1.6] Let X be a separable real or complex Banach space and let $T \in \mathcal{L}(X)$. If there exist an increasing sequence of integers $(n(k))_k \subset \mathbb{N}$, two dense sets $\mathcal{D}_1$, $\mathcal{D}_2 \subset X$ and sequence of maps $S_{n(k)} : \mathcal{D}_2 \to X$ such that:*

1. $\lim_{k\to\infty} T^{n(k)}x = 0$ *for any* $x \in \mathcal{D}_1$.
2. $\lim_{k\to\infty} S_{n(k)}y = 0$ *for any* $y \in \mathcal{D}_2$.

3. $\lim_{k\to\infty} T^{n(k)} S_{n(k)} y = y$ *for any* $y \in \mathcal{D}_2$.

Then T *is hypercyclic.*

Let us show that the operator given in Example 1.17 satisfies the Hypercyclicity Criterion. Let $p \in [1, \infty)$, let $|\lambda| > 1$ and let $(\mathrm{e}_n)_n$ be the canonical basis of $\ell^p(\mathbb{N})$. Let B be the *backward shift* on $\ell^p(\mathbb{N})$ and let F be the *forward shift* on $\ell^p(\mathbb{N})$, that is, $F\mathrm{e}_n = \mathrm{e}_{n+1}$ for all $n \in \mathbb{N}$. Let us set $\mathcal{D}_1 = \mathcal{D}_2 := \mathrm{span}(\{\mathrm{e}_n : n \in \mathbb{N}\})$, the sequence $n(k) := k$ and the map $S_k := \lambda^{-k} F^k$ for all $k \in \mathbb{N}$. Then, we can easily apply the Hypercyclicity Criterion on λB. Thus, λB is a hypercyclic operator defined on $\ell^p(\mathbb{N})$.

Chapter 2

β-operators and differentiability

This chapter is devoted to understand the interplay between bornologies, linear operators and differentiability on real Banach spaces. To do this, we follow the structure of a result of M. Bachir given in [11], precisely, Theorem 2.2 below. The basics and the corresponding notation related to this chapter can be found in Chapter 1, Section 1.1. This chapter is partially based in [13], which is a joint work with M. Bachir and G. Flores. However, one of the main results of this chapter, Theorem 2.5, is a generalization of the main result of [13]. In this chapter all Banach spaces are real.

2.1 Introduction

It is a well-known fact that differentiability in the sense of bornologies (see Definition 1.2) implies distinct properties of the functions depending on the chosen bornology. In this framework, the most common bornologies are those of finite, relatively compact and bounded sets. Each one of them is related to some type of differentiability, namely, Gâteaux, Hadamard and Fréchet respectively, see [79]. To motivate this work we need the following definition which can be found in [30].

Definition 2.1 *Let X be a Banach space. A bounded subset A of X is said limited if for any weak* null sequence $(x_n)_n^*$, the following limit holds:*

$$\lim_{n\to\infty} \sup_{x\in A} |\langle x_n^*, x\rangle| = 0.$$

That is, sequentially weakly* convergence is uniform on A.

We know that every relatively compact subset in a Banach space is limited, but the converse is false in general. The family of limited subsets of a Banach space form a bornology, which will be called the limited bornology. Recalling Definition 1.6, for a bornology β on X and an operator $T \in \mathcal{L}(Y, X)$, we say that T is a β-operator if $T(B_Y(y, r)) \in \beta$ for all $y \in Y$ and all $r > 0$. The study of limited operator is an interesting line of research. Further information on limited sets and limited operators can be found in [30, 65, 74]. The next theorem, which is the motivation of this chapter, characterizes limited operators in terms of the differentiability of convex functions via the composition with the operator.

Theorem 2.2 *[11, Theorem 1] Let X and Y be two real Banach spaces, let $\mathcal{U}$ be a nonempty convex open subset of X and let $T \in \mathcal{L}(Y, X)$. Then, T is limited if and only if for every convex continuous function $f : \mathcal{U} \to \mathbb{R}$, $f \circ T$ is Fréchet-differentiable at $y \in Y$ whenever f is Gâteaux-differentiable at $Ty \in \mathcal{U}$.*

In this sense, a limited operator transforms (for convex functions) Gâteaux-differentiability (the weakest type) into Fréchet-differentiability (the strongest type) via composition. Bearing this in mind, we can go further. Mimicking the structure of Theorem 2.2, in [13], we proved that compact operators are characterized by the differentiability of Lipschitz functions. Another way to express these results is the fact that in infinite-dimensional spaces, what prevents a continuous convex function $f : X \to \mathbb{R}$ which is Gâteaux-differentiable at some point from being Fréchet-differentiable at the same point is the fact that the identity operator on X is not limited, whereas what prevents a general Lipschitz function which is Gâteaux-differentiable to be Fréchet-differentiable is the fact that the identity operator on X is not compact. The non-compactness of the identity operator in infinite dimensional spaces is the well known Riesz theorem. On the other hand, the fact that the identity operator is not limited in infinite dimensional spaces has been discovered independently by Josefson in [65] and Nissenzweig in [74].

To state the first main result of this chapter, we need the following two definitions.

Definition 2.3 *Let β be a bornology on X. We say that β is a convex bornology if for any $A \in \beta$:*

1. *the convex envelope of A, $\mathrm{co}(A)$, belongs to β, and*
2. *$x + \lambda A$ belongs to β for any $x \in X$ and any $\lambda \in \mathbb{R}$.*

For instance the limited, Hadamard, weak-Hadamard and Fréchet bornologies are convex bornologies.

Definition 2.4 *Let β be a bornology on X. We say that β satisfies the property (S) if for every bounded set $A \subset X$ such that $A \notin \beta$, there is a sequence $(x_n)_n \subset A$ and $\delta > 0$ such that for any increasing sequences $(n_k) \subset \mathbb{N}$ and for any sequence $(y_k)_k$ satisfying $\|y_k - x_{n_k}\| \leq \delta$ for all $k \in \mathbb{N}$, the set $\{y_k : k \in \mathbb{N}\}$ does not belong β.*

Although property (S) could seem artificial, in Section 2.2 it is established that the Hadamard, weakly-Hadamard and Limited bornologies satisfy it. Moreover, the Fréchet bornology trivially satisfies property (S). Our first main result of this chapter reads as follows.

Theorem 2.5 *Let X and Y be two Banach spaces and let β be a convex bornology on X satisfying property (S). Let $T \in \mathcal{L}(Y, X)$. Then T is a β-operator if and only if for every Lipschitz function $f : X \to \mathbb{R}$, β-differentiable at $x = Ty$, $f \circ T$ is Fréchet-differentiable at y.*

Remark 2.6 *Thanks to Proposition 2.11, Proposition 2.12 and Proposition 2.13, we can apply Theorem 2.5 to compact (recovering the main result of [13]), weakly-compact and limited operators. We point out that, to the best of our knowledge, this a new characterization for* *weakly-compact operators.*

To present our second main result of this chapter, let us introduce a new class of functions with still some Lipschitz flavour.

Definition 2.7 *We say that a function $f : \mathcal{U} \subset X \to Z$ is finitely Lipschitz if for every finite dimensional affine subspace Y of X such that $\mathcal{U} \cap Y \neq \emptyset$, $f|_{(Y\cap\mathcal{U})}$ is Lipschitz. We denote by* $\mathrm{FLip}(\mathcal{U}, Z)$ *the linear space of finitely Lipschitz functions from $\mathcal{U} \subset X$ to Z. In the case of $Z = \mathbb{R}$, we simply write* $\mathrm{FLip}(\mathcal{U})$.

Observe that finitely Lipschitz functions do not need to be continuous. Indeed, any linear functional from an infinite dimensional Banach space to another Banach space is finitely Lipschitz.

Theorem 2.8 *Let X and Y be two Banach spaces and let $T \in \mathcal{L}(Y, X)$. Then, T has finite rank if and only if for every Banach space Z and every continuous finitely Lipschitz function $f : X \to Z$, Gâteaux-differentiable at $x = Ty$, $f \circ T$ is Fréchet-differentiable at $y \in Y$.*

Remark 2.9 *Theorem 2.5 and Theorem 2.8 remain true if we change Z by $\mathbb{R}$ or if we restrict the domain of the Lipschitz functions (finitely-Lipschitz functions resp.) to some open subset of X.*

This chapter is organized as follows: In Section 2.2 we give some comments on property (S) and we construct a useful Lipschitz function. Section 2.3 is devoted to the proof of Theorem 2.5 and 2.8, together with some results on spaceability/lineability. In Section 2.4 we present an alternative (and simplified) proof of Theorem 2.2. Finally, we end this chapter with Section 2.5 which contains some applications of Theorem 2.5.

2.2 Property (S) and the construction of a Lipschitz function

We start this section by showing that if a given bornology satisfies property (S), then it must contain all relatively compact sets, see Definition 2.4. Then, we show that the Hadamard, limited and weak-Hadamard bornology satisfy property (S). We end this section with the construction of a Lipschitz function that will be used in the forthcoming sections.

Proposition 2.10 *Let X be a Banach space and let β be a bornology on X. If β satisfies property (S), then it contains the relatively compact sets of X.*

Proof. Let us proceed by contradiction. Let A be a relatively compact subset of X such that $A \notin \beta$. Let $(x_n)_n \subset A$ and $\delta > 0$ be the sequence and the positive number given by property (S). Since A is a relatively compact set, there is a subsequence $(x_{n_k})_k$ of $(x_n)_n$ convergent to $\overline{x} \in X$. Up to a subsequence, we can assume that $\|\overline{x} - x_{n_k}\| \leq \delta$ for all $k \in \mathbb{N}$. Thus, the sequence $(y_k)_k$ defined by $y_k = \overline{x}$ satisfies that $\|y_k - x_{n_k}\| \leq \delta$ for all $k \in \mathbb{N}$. Property (S) implies that $\{y_k : k \in \mathbb{N}\} = \{\overline{x}\} \notin \beta$, which is a contradiction. □

Proposition 2.10 implies that a bornology on a finite dimensional Banach space satisfies property (S) if and only if it is the Fréchet bornology. Thus, in this section X will always denote an infinite dimensional Banach space.

Proposition 2.11 *The Hadamard bornology on X satisfies property (S).*

Proof. Let $A \subset X$ be a bounded non-relatively compact set. Then there exists a sequence $(x_n)_n \subset A$ with no accumulation points. Up to a subsequence, which we denote by $(x_n)_n$, we find $\sigma > 0$ such that $\|x_n - x_m\| \geq \sigma$ for all $n \neq m$. Therefore, if we choose $\delta = \sigma/4$, then any sequence $(y_k)_k$ satisfying $\|y_k - x_{n_k}\| \leq \delta$, for some increasing sequence (n_k), has no accumulation points. Thus, the set $\{y_k : k \in \mathbb{N}\}$ is not relatively compact. $\square$

Proposition 2.12 *The Limited bornology on X satisfies property (S).*

Proof. Let $A \subset X$ be a bounded non-limited set. Then there is a weak*-null sequence $(x_n^*)_n \subset B_{X^*}$ which does not converge to 0 uniformly on A. Hence, up to a subsequence of $(x_n^*)_n$, there exist a sequence $(x_n)_n \subset A$ and $\sigma > 0$ such that $|x_n^*(x_n)| \geq \sigma$ for all $n \in \mathbb{N}$. Considering $\delta = \sigma/2$, we obtain that any vector $y_k \in B(x_{n_k}, \delta)$, with $(n_k)_k$ an increasing sequence, satisfies $|x_{n_k}^*(y_k)| \geq \sigma/2$. Therefore, the set $\{y_k : k \in \mathbb{N}\}$ is not limited. $\square$

The following result concerns the weak-Hadamard bornology. It can be found inside the proof of [28, Theorem 1].

Proposition 2.13 *The weak-Hadamard bornology on X satisfies property (S).*

Proof. Let $A \subset X$ be a bounded non-relatively weakly-compact set. By the Eberlein-Šmulian Theorem, there is a sequence $(x_n)_n \subset X$ with no weakly-convergent subsequence. By contradiction, suppose that no subsequence of (x_n) satisfies the statement of property (S). Then, there exist an increasing sequence $(n(1,j))_j \subset \mathbb{N}$ and a sequence $(z^1_{n(1,j)})_j$ weakly-convergent to z^1 such that $z^1_{n(1,j)} \in B(x_{n(1,j)}, 1)$ for all $j \in \mathbb{N}$. Inductively, for $k \geq 2$, there exist a subsequence $(n(k,j))_j$ of $(n(k-1,j))_j$ and $(z^k_{n(k,j)})_j$ weakly-convergent sequence to z^k such that $z^k_{n(k,j)} \in B(x_{n(k,j)}, 1/k)$ for all $j \in \mathbb{N}$. Let us show that the sequence $(z^k)_k$ converges in norm to some $z^\infty \in X$. Indeed, let $k < l$. Recalling that the norm is a weakly-lower semi continuous and that $(n(l,j))_j$ is a subsequence of $(n(k,j))_j$, we obtain

$$\|z^k - z^l\| \leq \liminf_j \|z^k_{n(l,j)} - z^l_{n(l,j)}\| \leq \liminf_j \|z^k_{n(l,j)} - x_{n(l,j)}\| + \|x_{n(l,j)} - z^l_{n(l,j)}\| \leq \frac{1}{k} + \frac{1}{l},$$

proving that $(z^k)_k$ is a norm-Cauchy sequence. We claim that $(x_{n(k,k)})_k$ weakly-converges to z^∞. Let $x^* \in S_{X^*}$ and $\varepsilon > 0$. Let $n_0 \in \mathbb{N}$ such that $n_0^{-1} \leq \varepsilon/3$. Thus, $\|z^k - z^\infty\| \leq \varepsilon/3$ for all $k \geq n_0$. Since the sequence $(z^{n_0}_{n(n_0,j)})_j$ is weakly-convergent to z^{n_0}, there is $m_0 \in \mathbb{N}$ such that $|\langle x^*, z^{n_0} - z^{n_0}_{n(n_0,j)}\rangle| \leq \varepsilon/3$ for all $j \geq m_0$. Observe that, for any $k > n_0$, there is $j_k \in \mathbb{N}$ such that $n(k,k) = n(n_0, j_k)$. Hence, for k large, we have that $j_k > m_0$ and then

$$
\begin{aligned}
|\langle x^*, x_{n(k,k)} - z^\infty\rangle| &\le |\langle x^*, x_{n(n_0,j_k)} - z^{n_0}_{n(n_0,j_k)}\rangle| + |\langle x^*, z^{n_0}_{n(n_0,j_k)} - z^{n_0}\rangle| + |\langle x^*, z^{n_0} - z^\infty\rangle| \\
&\le n_0^{-1} + \frac{\varepsilon}{3} + \|z^{n_0} - z^\infty\| \le \varepsilon,
\end{aligned}
$$

concluding that the sequence $(x_{n(k,k)})_k$ weakly-converges to z^∞. Therefore, this contradicts the fact that (x_n) has no weakly convergent subsequence. □

The last ingredient of the proof of Theorem 2.5 is the Lipschitz function constructed in Proposition 2.17 below. This function is used to prove Theorem 2.5 and we conveniently modify it to prove also Theorem 2.8 and Corollary 2.21. In [28] a similar construction is used to prove certain properties of non-reflexive spaces. Let us continue with the following definitions. For a set $A \subset X$, the cone generated by A is the set

$$
\mathrm{cone}(A) := \{\lambda x : x \in A,\ \lambda \ge 0\}.
$$

Definition 2.14 *Let X be a Banach space, let $(x_n)_n \subset X$ be a sequence such that $\|x_n\| = \|x_1\|$ for all $n \in \mathbb{N}$ and let $\sigma \in (0, \|x_1\|)$.*

1. *We say that $(x_n)_n$ is σ-separated if $\|x_n - x_m\| \ge \sigma$ for all $n, m \in \mathbb{N}$, with $n \ne m$.*
2. *We say that $(x_n)_n$ is σ-cone separated if the sets $\{\mathrm{cone}(B(x_n, \sigma)) \setminus \{0\} : n \in \mathbb{N}\}$ are pairwise disjoint.*

By definition, a σ-cone separated sequence is 2σ-separated. Conversely, we have the following result.

Proposition 2.15 *Let $(x_n)_n \subset X$ be a σ-separated sequence. Then $(x_n)_n$ is $\sigma/4$-cone separated.*

Proof. Let $x \in X \setminus \{0\}$ and $0 < \alpha < \|x\|$. Let us define the set

$$
P_\alpha(x) := \left\{ \frac{\|x\|}{\|y\|} y \ : \ y \in B(x, \alpha) \right\} \subset \partial B(0, \|x\|).
$$

Observe that $\mathrm{cone}(B(x,\alpha)) = \mathrm{cone}(P_\alpha(x))$. In what follows, we prove that $P_\alpha(x) \subset B(x, 2\alpha)$. Indeed, if $y \in B(x, \alpha)$, then:

$$
\begin{aligned}
\left\| x - \frac{\|x\|}{\|y\|} y \right\| &\le \|x - y\| + \left\| y - \frac{\|x\|}{\|y\|} y \right\| \\
&= \|x - y\| + |\|y\| - \|x\|| \le 2\|x - y\| < 2\alpha.
\end{aligned}
$$

So, $\mathrm{cone}(B(x,\alpha)) \cap \partial B(0, \|x\|) = \mathrm{cone}(P_\alpha(x)) \cap \partial B(0, \|x\|) = P_\alpha(x) \subset B(x, 2\alpha)$.

Let $(x_n)_n$ be a σ-separated sequence. Then, $\|x_n\| = \|x_1\|$ for all $n \in \mathbb{N}$ and $\sigma < \|x_1\|$. Since $P_{\sigma/4}(x_n) \subset B(x_n, \frac{\sigma}{2})$ for each $n \in \mathbb{N}$, the sets $\{P_{\sigma/4}(x_n) : n \in \mathbb{N}\}$ are pairwise disjoint. Indeed, if there are $n, m \in \mathbb{N}$, with $n \ne m$, such that $P_{\sigma/4}(x_n) \cap P_{\sigma/4}(x_m) \ne \emptyset$, then there is

$$
y \in P_{\sigma/4}(x_n) \cap P_{\sigma/4}(x_m).
$$

However, this implies that $\|x_n - x_m\| \leq \|x_n - y\| + \|y - x_m\| < \sigma$ which is a contradiction. Finally, since

$$\mathrm{cone}\{B(x_n, \sigma/4)\} = \mathrm{cone}\{P_{\sigma/4}(x_n)\},$$

we get that $(x_n)_n$ is $\sigma/4$-cone-separated, because the cones do not intersect in the sphere $\partial B(0, \|x\|)$.

$\square$

The core of a set $A \subset X$, denoted by $\mathrm{core}(A) \subset X$, is the set defined by

$$\mathrm{core}(A) := \{x \in A : \forall y \in S_X, \exists t > 0,\ (x - ty, x + ty) \subset A\}.$$

Proposition 2.16 *Let $f : X \to \mathbb{R}$ be a function. If $x \in \mathrm{core}(\{f = f(x)\})$, then f is Gâteaux-differentiable at x with Gâteaux-differential equals to $\mathrm{d}_G f(x) = 0$.*

Proof. Let $x \in X$ and let $A = \{y \in X :\ f(y) = f(x)\}$. Let us assume that $x \in \mathrm{core}(A)$. Then, for each $y \in X$, there exists $t_y > 0$ such that $f(x+ty) = f(x)$ for all $|t| < t_y$. Therefore,

$$\lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} = 0, \text{ for all } y \in X.$$

Thus, the directional derivative of f at x in any direction is equal to 0. Hence, f is Gâteaux-differentiable at x and $\mathrm{d}_G f(x) = 0$. $\square$

Finally, we present the mentioned Lipschitz function.

Proposition 2.17 *Let β be a convex bornology on X, distinct from the Fréchet bornology, satisfying property (S). Let $A \notin \beta$ be a nonempty symmetric bounded convex set. Then, there exist $\sigma > 0$ and a σ-separated sequence $(x_n)_n \subset A$ such that the Lipschitz function $f : X \to \mathbb{R}$ defined by*

$$f(x) := \mathrm{dist}\left(x, X \setminus \bigcup_{n=1}^{\infty} B\left(\frac{x_n}{n}, \frac{\sigma}{4n}\right)\right), \text{ for all } x \in X,$$

is β-differentiable at 0 but not Fréchet-differentiable at 0.

Proof. Let $(x_n)_n \subset A$ and let $\delta > 0$ given by property (S). Since A is bounded and β contains the relatively compact sets, we can assume, up to a subsequence, that $(\|x_n\|)_n$ converges to $\alpha > 0$. Further, maybe shrinking δ, taking again a subsequence and perturbing the sequence $(x_n)_n$, (recall that A is a symmetric convex set), we can assume that $\|x_n\| = \alpha$ for every $n \in \mathbb{N}$. Since the sequence $(x_n)_n$ does not have accumulation points, up to a subsequence, we can assume that $(x_n)_n$ is a σ-separated sequence, for some $\sigma > 0$. Let us redefine σ by $\sigma := \min\{\delta, \sigma\}$. Let $f : X \to \mathbb{R}$ be the 1-Lipschitz function on X defined by

$$f(x) := \mathrm{dist}\left(x, X \setminus \bigcup_{n=1}^{\infty} B\left(\frac{x_n}{n}, \frac{\sigma}{4n}\right)\right), \text{ for all } x \in X.$$

By Proposition 2.15 and Proposition 2.16, f is Gâteaux-differentiable at 0, with Gâteaux-differential equal to $\mathrm{d}_G f(0) = 0$. However, since $nf(x_n/n) = \sigma/4$, f is not Fréchet-differentiable at 0. Finally, it only remains to prove that f is β-differentiable at 0. We proceed by contradiction. Suppose, for some symmetric convex set $W \in \beta$, the ratio of differentiability does not converge uniformly on W. That is, there exist a null sequence $(t_k) \subset \mathbb{R}$, $(w_k)_k \subset W$ and $\varepsilon > 0$ such that:

$$\left|\frac{f(t_k w_k)}{t_k}\right| \geq \varepsilon, \ \forall n \in \mathbb{N}.$$

Since W is symmetric, we can assume that $(t_n)_n$ is a sequence of positive numbers. Also, since $f(t_k w_k) > 0$, there is a sequence $(n_k) \subset \mathbb{N}$ such that $t_k w_k \in B(x_{n_k}/n_k, \sigma/4n_k)$. Due to the fact that W is bounded and that $(t_k w_k)_k$ converges to 0, we can assume, up to a subsequence, that $(n_k)_k$ is increasing. We have two different cases now. If the sequence $(w_k)_k$ tends to 0, the set $\{w_k : \ k \in \mathbb{N}\}$ is relatively compact. However, the quotient of differentiability at a point of Gâteaux differentiability converges uniformly on relatively compact sets for Lipschitz functions (see Theorem 1.3). This contradicts the fact that $\varepsilon > 0$. Therefore, we can assume that $(w_k)_k$ is not a norm-null sequence and then, up to a subsequence, the sequence $(\|w_k\|)_k$ converges to some $\nu > 0$. Since $t_k w_k \in B(x_{n_k}/n_k, \sigma/4n_k)$, then $n_k t_k w_k \in B(x_{n_k}, \sigma/4)$, Therefore,

$$n_k t_k \in \left[\frac{\alpha}{\|w_k\|} - \frac{\sigma}{4\|w_k\|}, \frac{\alpha}{\|w_k\|} + \frac{\sigma}{4\|w_k\|}\right].$$

Thus, the sequence $(t_k n_k)_k$ accumulates in $[\frac{\alpha}{\nu} - \frac{\sigma}{4\nu}, \frac{\alpha}{\nu} + \frac{\sigma}{4\nu}]$. Passing through a subsequence, we assume that $(t_k n_k)_k$ converges to some $\lambda > 0$. Hence, there is $K \in \mathbb{N}$ such that $\lambda w_k \in B(x_{n_k}, \sigma)$, for all $k \geq K$. This is a contradiction with the property (S) for the bornology β, since $(\lambda w_k)_k \subset \lambda W \in \beta$ and $\|\lambda w_k - x_{n_k}\| \leq \sigma \leq \delta$ for all $k \geq K$. $\square$

2.3 Characterization of β-operators

This section is devoted to prove both theorems stated in the introduction of this chapter. Also, we present some results of spaceability and lineability related to the proposed characterizations. Let us start with the proof Theorem 2.5.

Proof of Theorem 2.5. The necessity is straightforward and it does not require property (S). Indeed, let $T : Y \to X$ be a β-operator and let $f : X \to Z$ be a Lipschitz function β-differentiable at $x = Ty$. We claim that the Fréchet-differential of $f \circ T$ at y is equal to $\mathrm{d}_\beta f(Ty) \circ T$. Indeed, observe that

$$\lim_{t\to 0} \sup_{u \in B_Y} \left\| \frac{f \circ T(y + tu) - f \circ T(y)}{t} - \mathrm{d}_\beta f(Ty)Tu \right\| =$$
$$\lim_{t\to 0} \sup_{v \in TB_Y} \left\| \frac{f(x - tv) - f(x)}{t} - \mathrm{d}_\beta f(x)v \right\| = 0,$$

where the last equality relies in the fact that $TB_Y \in \beta$. Conversely, we proceed by contradiction. Assume that T is not a β-operator. Then $TB_Y \notin \beta$. Since β is a convex bornology which satisfies property (S), Proposition 2.17 gives us a σ-separated sequence $(x_n) \subset TB_Y$ and a 1-Lipschitz function $f : X \to \mathbb{R}$ defined by

$$f(x) := \operatorname{dist}\left(x, X \setminus \bigcup_{n=1}^{\infty} B\left(\frac{x_n}{n}, \frac{\sigma}{4n}\right)\right),$$

which is β-differentiable 0. Let us see that $f \circ T$ is not Fréchet-differentiable at 0. Since $f \circ T$ is positive and $f \circ T(0) = 0$, the only candidate for Fréchet-differential of $f \circ T$ at 0 is the functional 0. Let $(y_n) \subset B_Y$ such that $Ty_n = x_n$. We notice that

$$\frac{f \circ T(\frac{y_n}{n}) - f \circ T(0)}{\frac{1}{n}} = nf(\frac{x_n}{n}) = \frac{\sigma}{4}, \ \forall n \in \mathbb{N},$$

showing that $f \circ T$ is not Fréchet-differentiable at 0. □

Let $T : Y \to X$ be a bounded non-β-operator where β is a convex bornology satisfying Property (S). Then, the set of Lipschitz functions, Gâteaux differentiable at 0 such that $f \circ T$ is not Fréchet differentiable at 0, denoted by $\mathcal{F}$, is dense in the space of Lipschitz, Gâteaux differentiable functions at 0 (for the topology generated by the Lipschitz seminorm). In what follows, we want to measure the size of the set $\mathcal{F}$ in an algebraic sense. To do this, let us introduce the following concepts. Let α be a cardinal number. A set $A \subset X$ is said α-lineable if $A \cup \{0\}$ contains a subspace of dimension α. A set $A \subset X$ is said α-spaceable, if $A \cup \{0\}$ contains a closed subspace of dimension α. The following corollary states that the set $\mathcal{F}$ is $\mathfrak{c}$-spaceable, meaning that it contains an isometric copy of a Banach space of dimension of the continuum. More on lineability and spaceability can be found in [5], [3], [59] and references therein.

Corollary 2.18 *Let X and Y be two Banach spaces. Let β be a convex bornology on X satisfying property (S). Let $T \in \mathcal{L}(Y, X)$ be a bounded non-β operator. The set of real-valued Lipschitz functions in $\mathrm{Lip}_0(X)$ which are β-differentiable at 0 but $f \circ T$ is not Fréchet-differentiable at 0, contains a subset isometric to $\ell^\infty(\mathbb{N})$, up to the function 0.*

Proof. Let $\sigma > 0$ and let $(x_n)_n \subset TB_Y$ be a σ-separated sequence given by Proposition 2.17. Let $(y_n)_n \subset B_Y$ such that $Ty_n = x_n$. For each $p \in \mathbb{N}$ prime number define the sets $B_{p,n} = B(\frac{x_{p^n}}{p^n}, \frac{\sigma}{4p^n})$ and $B_p := \cup_n B_{p,n}$. As in Proposition 2.17, for each $p \in \mathbb{N}$, we define $f_p : X \to \mathbb{R}$ by

$$f_p(x) = \operatorname{dist}(x, X \setminus B_p), \text{ for all } x \in X,$$

which is 1-Lipschitz, β-differentiable at 0 and the compositions $f_p \circ T$ is not Fréchet-differentiable at 0. In what follows, $(p_\mathrm{i})_\mathrm{i}$ stands for an enumeration of the prime numbers. By Proposition 2.15, the interior of the supports of the functions $\{f_{p_\mathrm{i}} : \mathrm{i} \in \mathbb{N}\}$ are pairwise disjoint. Therefore, $(f_{p_\mathrm{i}})_\mathrm{i} \subset \mathrm{Lip}_0(X)$ is a sequence of linearly independent functions. Moreover, if $\mu \in \ell^\infty(\mathbb{N})$, the function

$$f_\mu(x) := \sum_{\mathrm{i}=1}^{\infty} \mu_\mathrm{i} f_{p_\mathrm{i}}(x),$$

is well defined, because for each $x \in X$ there is at most one non-zero term in the series, and $\|\mu\|_\infty$-Lipschitz. Also, since $f_\mu|_{\mathrm{supp}(f_{p_{\mathrm{i}}})} = \mu_{\mathrm{i}} f_{p_{\mathrm{i}}}|_{\mathrm{supp}(f_{p_{\mathrm{i}}})}$ and that $\mathrm{Lip}(f_{p_{\mathrm{i}}}) = 1$ for all $\mathrm{i} \in \mathbb{N}$, we have that $\mathrm{Lip}(f_\mu) \geq |\mu_{\mathrm{i}}|$ for all $\mathrm{i} \in \mathbb{N}$. Thus, the operator $L : \ell^\infty(\mathbb{N}) \to \mathrm{Lip}_0(X)$ given by $L\mu = f_\mu$ is an isometry. Since $(x_n)_n$ is a σ-separated sequence, Proposition 2.15 implies that $0 \in \mathrm{core}(X \setminus \cup_k B_{p^k_{\mathrm{i}}})$ and by Proposition 2.16 $L\mu$ is Gâteaux-differentiable at 0. Moreover, an analogous argument of the proof of Theorem 2.5 shows that $L\mu$ is, in fact, β-differentiable at 0. However, if $\mu \in \ell^\infty(\mathbb{N})$ and $\mu \neq 0$, f_μ is not Fréchet-differentiable at 0. Indeed, if $\mu_k \neq 0$, then

$$\liminf_{n\to\infty} \frac{|(f_\mu \circ T)(y_{p^n_k}/p^n_k) - (f_\mu \circ T)(0)|}{1/p^n_k} = \liminf_{n\to\infty} p^n_k(|\mu_k f_{p_k}(x_{p^n_k}/p^n_k) - \mu_k f_{p_k}(0)|)$$

$$\geq \liminf_{n\to\infty} p^n_k \frac{|\mu_k|\sigma}{4p^n_k} = \frac{|\mu_k|\sigma}{4} > 0.$$

□

Now, we continue with the proof of Theorem 2.8.

Proof of Theorem 2.8. The necessity part goes along the lines of the necessity of Theorem 2.5. Let $T : Y \to X$ be a bounded finite rank operator and let $f : X \to Z$ be a finitely Lipschitz function, Gâteaux-differentiable at $x = Ty$. Since TY is a finite dimensional subspace of X, the function $g := f|_{TY}$ is Lipschitz and Fréchet-differentiable at Ty. Then, if $\mathrm{d}_F g(Ty)$ denotes the Fréchet-differential of g at Ty and $t \in \mathbb{R}$, with $t \neq 0$, we have that

$$\begin{aligned}
&\sup_{h\in B_Y} \frac{\|(f\circ T)(y+th) - (f\circ T)(y) - (\mathrm{d}_F g(Ty)\circ T)(th)\|}{|t|} \\
&\quad = \sup_{u\in TB_Y} \frac{\|f(Ty+tu) - f(Ty) - t\mathrm{d}_F g(Ty)(u)\|}{|t|} \\
&\quad = \sup_{u\in TB_Y} \frac{\|g(Ty+tu) - g(Ty) - t\mathrm{d}_F g(Ty)(u)\|}{|t|}
\end{aligned}$$

From the last line, since g is Fréchet-differentiable at Ty, we deduce that the first supremum tends to 0 as t tends to 0. Then $f \circ T$ is Fréchet-differentiable at y, with Fréchet-differential equal to $\mathrm{d}_G g(Ty) \circ T$.

In order to prove the sufficiency we proceed by contradiction. Suppose that $T : Y \to X$ is a bounded operator such that TY is infinite dimensional. By Riesz Theorem, there exists a bounded σ-separated sequence $(x_n)_n$ in TY, with $\sigma > 0$. Recall that $\|x_n\| = \|x_1\|$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, let $y_n \in Y$ such that $Ty_n = x_n$. Now, for $n \in \mathbb{N}$, we define the sets

$$B_n = B\left(\frac{x_n}{n\|y_n\|}, \frac{\sigma}{4n\|y_n\|}\right).$$

Since $(x_n)_n$ is a σ-separated sequence, by Proposition 2.15, we deduce that the family $(B_n)_n$ is pairwise disjoint. Thus, for each $n \in \mathbb{N}$, the function $f_n : X \to \mathbb{R}$ defined by

$$f_n(x) = \|y_n\|\mathrm{d}(x, X \setminus B_n), \quad \text{for all } x \in X$$

is $\|y_n\|$-Lipschitz and the interior of the supports of the functions f_n are pairwise disjoint. Consider now $f : X \to \mathbb{R}$ defined by

$$f(x) := \sup_n f_n(x) = \sum_{n=1}^{\infty} f_n(x), \text{ for all } x \in X.$$

It is easy to see that this function is well defined.

We claim that $f \in \mathrm{FLip}(X)$. Let V be a finite dimensional affine subspace of X and suppose that V intersects infinitely many different balls $(B_n)_n$, namely $(B_{n_k})_k$. Since the sequence of balls converges to the singleton $\{0\}$ and V is a closed affine subspace, we deduce that V is a linear subspace (i.e. $0 \in V$). Take any sequence $(v_k)_k$ such that $v_k \in V \cap B_{n_k}$. If we consider $v_k' := n_k\|y_k\|v_k$, we see that for $k, j \in \mathbb{N}$,

$$\sigma \leq \|x_{n_k} - x_{n_j}\| \leq \|x_{n_k} - v_k'\| + \|v_k' - v_j'\| + \|v_j' - x_{n_j}\| \leq \frac{\sigma}{4} + \|v_k' - v_j'\| + \frac{\sigma}{4}.$$

which implies that $(v_k')_k \subset V$ does not have accumulation points. This leads to a contradiction because V is finite dimensional and $(v_k')_k$ is bounded. Indeed,

$$\|v_k'\| \leq \|v_k' - x_{n_k}\| + \|x_{n_k}\| \leq \frac{\sigma}{4} + M,$$

where $M = \|x_n\|$ (for all n). Therefore, V intersects only finitely many different balls $(B_n)_n$, namely $(B_{n_k})_{k=1}^N$. Since the interior of the supports of the functions f_n are pairwise disjoint, it follows that

$$\mathrm{Lip}(f|_V) \leq \max\{\mathrm{Lip}(f_{n_k}) : k = 1, ..., N\} = \max\{\|y_{n_k}\| \ : \ k = 1, ..., N\},$$

which proves the claim, i.e., $f \in \mathrm{FLip}(X)$. Notice that f is continuous. In fact, we need to check the continuity of f only at 0 which easily follows from the fact that $f(\frac{x_n}{n\|y_n\|}) = \sigma/4n$ for all $n \in \mathbb{N}$ and that $f(0) = 0$. Indeed, let $m \in \mathbb{N}$ and $x \in X$ such that $f(x) > \sigma/4m$. Then, there is $m \in \{1, 2, ..., n-1\}$ such that $x \in B_m$. Therefore,

$$\|x\| \geq \frac{\|x_m\|}{m\|y_m\|} - \frac{\sigma}{4m\|y_m\|} \geq \inf_{k \leq n-1} \frac{4\|x_k\| - \sigma}{4k\|y_k\|} = \frac{4M - \sigma}{4(n-1)\sup_{k \leq n-1} \|y_k\|}.$$

Observe that 0 belongs to the core of the set $\{f = 0\}$. Thus, by Proposition 2.16, we deduce that f is Gâteaux-differentiable at 0, and $\mathrm{d}_G f(0) = 0$. However, we notice that

$$\liminf_{n \to \infty} \frac{f \circ T\left(\frac{y_n}{n\|y_n\|}\right) - f \circ T(0)}{\frac{1}{n}} = \liminf_{n \to \infty} n\|y_n\| \frac{\sigma}{4n\|y_n\|} - 0 = \frac{\sigma}{4} > 0,$$

which shows that $f \circ T$ is not Fréchet-differentiable at 0. $\square$

Finally, we state the following result of lineability. Its proof is analogous to the non-topological part of the proof presented for Corollary 2.18.

Corollary 2.19 *Let X and Y be two Banach spaces. Let $T \in \mathcal{L}(Y, X)$ be an operator with infinite rank. Then the set $\mathcal{F} \subset \mathrm{FLip}(X)$ of finitely Lipschitz functions f which are Gâteaux-differentiable at 0 but $f \circ T$ is not Fréchet-differentiable at 0, contains a subset algebraically isomorphic to $\ell^\infty(\mathbb{N})$, up to the function 0.*

We end this section with a smooth version of Theorem 2.5. Let $C_b^G(X)$ and $C_b^F(X)$ be the Banach spaces of bounded, Lipschitz continuous and everywhere Gâteaux-differentiable (everywhere Fréchet-differentiable resp.) functions. Recall that these spaces are Banach spaces endowed with the norm $\|f\|_1 := \|f\|_\infty + \|\mathrm{d}_\beta f\|_\infty$, where β is the Gâteaux bornology (Fréchet resp.), see [43]. We say that a Banach space X admits a smooth bump function if there exists $b \in C_b^G(X)$ with $b \neq 0$ and with bounded support. The existence of a smooth bump function is intimately related with the geometry of the underlying Banach space. Observe that if X admits a bump function $b \in C_b^G(X)$, then by choosing $x_0 \in \mathrm{int}(\mathrm{supp}(b))$ and $\lambda > 0$ large, the function $b(\lambda(\cdot - x_0))$ is a smooth bump function with $b(0) \neq 0$ and $\mathrm{supp}(b) \subset \overline{B}_X$. Further information on this subject can be found in [44, 79].

Proposition 2.20 *Let X be a Banach space and let β be a convex bornology, different from the Fréchet bornology, satisfying property (S). Assume further that X admits a smooth bump function $b \in C_b^G(X)$ with $b(0) = 1$ and $\mathrm{supp}(b) \subset \overline{B}_X$. Let $A \notin \beta$ be a bounded symmetric convex set. Then, there exist $\sigma > 0$ and a σ-separated sequence $(x_n)_n \subset A$ such that the Lipschitz function defined by*

$$f(\cdot) := \sum_{n=1}^{\infty} \frac{1}{n} b\left(\frac{4n}{\sigma}\left(\cdot - \frac{x_n}{n}\right)\right)$$

belongs to $C_b^G(X)$ and f is β-differentiable at 0 but not Fréchet-differentiable at 0.

Corollary 2.21 *Let X and Y be two Banach spaces and let $T \in \mathcal{L}(Y, X)$. Assume further that X admits a smooth bump function. Then, T is a compact operator if and only if for every $f \in C_b^G(X)$, $f \circ T \in C_b^F(X)$.*

The proof of Proposition 2.20 and of Corollary 2.21 are analogous to the one presented for Proposition 2.17 and Theorem 2.5 respectively.

2.4 Alternative proof of Theorem 2.2

In [11], the proof of Theorem 2.2 is quite technical. However, in this section we provide a simplified proof of this result. To start, we establish that Gâteaux-differentiability and limited-differentiability coincide for continuous convex functions. In fact, this partial result clarifies the picture about Theorem 2.2. Let us recall the following two results that can be found in [29].

Proposition 2.22 *[29, Proposition 8.1.1] Let β be a bornology on X. Let $(x_n^*)_n \subset X^*$ be a bounded sequence. Let $f : X \to \mathbb{R}$ be the convex function defined by:*

$$f(x) = \sup_n \left\{0, x_n^*(x) - \frac{1}{n}\right\}.$$

Then, f is β-differentiable at 0 if and only if $(x_n^*)_n \to_{\tau_\beta} 0$, where τ_β denotes the topology of the uniform convergence on β-sets on X^*.

Theorem 2.23 *[29, Theorem 8.1.3] Let X be a Banach space with bornologies $\beta_1 \subseteq \beta_2$. Then, the following assertions are equivalent:*

(a) τ_{β_1} and τ_{β_2} agree sequentially in X^*.

(b) β_1-differentiability and β_2-differentiability coincide for continuous convex functions.

The following proposition can be seen as the analogous for convex functions of the well known result of differentiability of Lipschitz functions, Theorem 1.3.

Proposition 2.24 *Let X be a Banach space. Gâteaux-differentiability and limited-differentiability coincide for continuous convex functions.*

Proof. We use Theorem 2.23 with the bornologies $\beta_1 =$ Gâteaux and $\beta_2 =$ limited. Let $(x_n^*)_n \subset X^*$ be a sequence τ_{β_1}-convergent to 0, i.e., $(x_n^*)_n$ is a weak*-null sequence. Let $A \subseteq X$ be limited set on X. By definition of limited set, we have that

$$\lim_{n\to\infty} \sup_{x\in A} |x_n^*(x)| = 0.$$

Since A was an arbitrary limited set on X, we have that $(x_n^*)_n$ converges to 0 for τ_{β_2}. Applying Theorem 2.23 we obtain the desired result. □

Now, we can present a simplified proof of Theorem 2.2. As it can be noticed, our approach is completely different from the one presented in [16].

Alternative proof of Theorem 2.2. Thanks to Proposition 2.24, the necessity of Theorem 2.2 is straightforward. In fact, it is analogous to the necessity of Theorem 2.5. Conversely, we proceed by contradiction. Let $T : Y \to X$ be a bounded non-limited operator. Then, there exists a weak*-null sequence $(x_n^*)_n \subset X^*$ and a sequence $(y_n)_n \subset B_Y$ such that $x_n^*(Ty_n) \geq 2$. Let us consider the function $f : X \to \mathbb{R}$ defined by:

$$f(x) = \max\left\{0, \sup\left\{x_n^*(x) - \frac{1}{n}\right\}\right\}, \text{ for all } x \in X,$$

which is Gâteaux-differentiable at 0, thanks to Proposition 2.22. Since f is a positive function and $f(0) = 0$, we know that $\mathrm{d}_G f(0) = 0$. In fact, the only candidate to Fréchet-differential to $f \circ T$ at 0 is also the functional 0. However, the computation

$$nf \circ T\left(\frac{y_n}{n}\right) \geq n\left(\frac{2}{n} - \frac{1}{n}\right) = 1,$$

shows that $f \circ T$ is not Fréchet-differentiable at 0. □

2.5 Some consequences of Theorem 2.5

2.5.1 Gelfand-Phillips spaces

Let us start with the definition of a Gelfand-Phillips space.

Definition 2.25 *A Banach space X is called a Gelfand-Phillips space, if all limited sets in X are relatively norm-compact.*

It is easy to see that X is a Gelfand-Phillips space, if and only if every limited operator with range in X is compact (see [45, Introduction, (C)]). In what follows, we present a characterization of Gelfand-Phillips spaces in terms of differentiability of Lipschitz functions. This result was discussed with M. Bachir.

Theorem 2.26 *Let X be a Banach space. Then, X is a Gelfand-Phillips space if and only if Gâteaux-differentiability and limited-differentiability coincide for real-valued Lipschitz functions on X.*

Proof. Suppose that X is a Gelfand-Phillips space, then limited sets and relatively compact sets coincide in X. Thus, Limited-differentiability and Hadamard-differentiability coincide for real-valued Lipschitz functions on X. Therefore, thanks to Theorem 1.3, Gâteaux-differentiability and limited-differentiability coincide for Lipschitz function defined on X. Conversely, suppose that Gâteaux-differentiability and Limited-differentiability coincide for real-valued Lipschitz functions defined on X. Let Y be any Banach space and let $T : Y \to X$ be a limited operator. We prove that T is a compact operator. To this end, let $f : \mathcal{U} \subset X \to \mathbb{R}$ be any Lipschitz function. Then, by the necessity of Theorem 2.5 with β equal to the limited bornology, $f \circ T$ is Fréchet-differentiable at $y \in Y$ whenever f is Gâteaux-differentiable at $Ty \in X$ ($\Longleftrightarrow$ Limited-differentiable at Ty). Hence, thanks to the sufficiency of Theorem 2.5 with β equal to the Hadamard bornology, T is a compact operator. Thus, each limited operator with range in X is compact. So X is a Gelfand-Phillips space. $\square$

2.5.2 A Banach-Stone like theorem

In order to present our second application of Theorem 2.5, we need the following definition and axioms which were introduced in [12] and [10] respectively.

Definition 2.27 (The property P^F) *Let (X, d) be a complete metric space and let $(A, \|\cdot\|)$ be a closed subspace of $C_b(X)$ (the space of all real-valued bounded continuous functions on X). We say that A satisfies property P^F if, for each sequence $(x_n)_n \subset X$, the two following assertions are equivalent:*

1. *The sequence $(x_n)_n$ converges in (X, d).*
2. *The associated sequence of Dirac masses $(\delta_{x_n})_n$ converges in $(A^*, \|\cdot\|_*)$, where the Dirac mass associated to a point $x \in X$, is the continuous linear functional $\delta_x : \varphi \in A \mapsto \varphi(x)$.*

Axioms. Let (X, d) be a complete metric space and let A be a space of functions included in $C_b(X)$. We say that the space A satisfies the axioms (A_1)-(A_4^F) if the space A satisfies:

(A_1) The space $(A, \|\cdot\|)$ is a Banach space such that $\|\cdot\| \geq \|\cdot\|_\infty$.

(A_2) The space A contains the constants.

(A_3) For each $n \in \mathbb{N}$ there exists a positive constant M_n such that for each $x \in X$ there exists a function $h_n : X \to [0, 1]$ such that $h_n \in A$, $\|h_n\| \leq M_n$, $h_n(x) = 1$ and $\mathrm{diam}(\mathrm{supp}(h_n)) < \frac{1}{n+1}$. This axiom implies in particular that the space A separates the points of X.

(A_4^F) The space A has the property P^F.

A simple adaptation of the proof given in [12, Proposition 2.5] shows that the spaces $C_b^\beta(X)$ have property P^F. In addition, if we assume that these spaces contain a bump function respectively, then they will satisfy the axiom (A_3). Thus, the spaces $C_b^G(X)$ and $C_b^F(X)$ satisfy the axioms (A_1)-(A_4^F) whenever they contain a bump function respectively, and so we can apply the extension of the Banach-Stone theorem established in [10, Corollary 1.3.]. In the sequel, we prove Theorem 2.28 which is a consequence of [10, Corollary 1.3.] and Corollary 2.21.

Theorem 2.28 *Let X and Y be two Banach spaces having a bump function in $C_b^G(X)$ and $C_b^F(Y)$ respectively. Then, the following assertions are equivalent.*

1. *There exists an isomorphism $\Phi : C_b^G(X) \to C_b^F(Y)$ such that $\|\Phi(f)\|_\infty = \|f\|_\infty$ and $\|\mathrm{d}_F(\Phi(f))\|_\infty = \|\mathrm{d}_G f\|_\infty$ for all $f \in C_b^G(X)$.*
2. *X and Y are isometrically isomorphic and of finite dimension.*

The proof will be given after the following lemma.

Lemma 2.29 *For every $a, b \in X$, we have*

$$\begin{aligned}\|a-b\| &= \sup_{f\in C_b^F(X)\setminus\{0\}, \|\mathrm{d}_F f\|_\infty>0} \frac{|f(a)-f(b)|}{\|\mathrm{d}_F f\|_\infty} \\ &= \sup_{f\in C_b^G(X)\setminus\{0\}, \|\mathrm{d}_G f\|_\infty>0} \frac{|f(a)-f(b)|}{\|\mathrm{d}_G f\|_\infty}.\end{aligned}$$

Proof. By Hahn-Banach theorem, there exists a unit vector $x_{a,b}^* \in X^*$ such that $\|a-b\| = x_{a,b}^*(a-b)$. For each $\omega > 0$, let $\alpha_\omega : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$-smooth, 1-Lipschitz function such that

$$\alpha_\omega(t) = \begin{cases} t \text{ if } |t| \le \omega, \\ \omega+1 \text{ if } t \ge \omega+2, \\ -\omega-1 \text{ if } t \le -\omega-2. \end{cases}$$

Let us consider the function $f_\omega(x) = \alpha_\omega \circ x_{a,b}^*(x)$, for all $x \in X$. We have that $f_\omega \in C_b^F(X)$ and is 1-Lipschitz for every $\omega > 0$. By choosing $\omega_0 \ge 2\max\{\|a\|, \|b\|\}$,

$$\begin{aligned}|f_{\omega_0}(a) - f_{\omega_0}(b)| &= |\alpha_{\omega_0} \circ x_{a,b}^*(a) - \alpha_{\omega_0} \circ x_{a,b}^*(b)| \\ &= |x_{a,b}^*(a) - x_{a,b}^*(b)| \\ &= |x_{a,b}^*(a-b)| \\ &= \|a-b\|.\end{aligned}$$

Moreover, $\|\mathrm{d}_F f_{\omega_0}\|_\infty = 1$. Since $\mathrm{Lip}(f) = \|\mathrm{d}_G f\|_\infty$ for functions in $C_b^G(X)$, it follows that

$$\begin{aligned}
\|a-b\| &\geq \sup_{f\in C_b^G(X)\setminus\{0\}, \|\mathrm{d}_G f\|_\infty>0} \frac{|f(a)-f(b)|}{\|\mathrm{d}_G f\|_\infty} \\
&\geq \sup_{f\in C_b^F(X)\setminus\{0\}, \|\mathrm{d}_F f\|_\infty>0} \frac{|f(a)-f(b)|}{\|\mathrm{d}_F f\|_\infty} \\
&\geq |f_{\omega_0}(a)-f_{\omega_0}(b)| \\
&= \|a-b\|.
\end{aligned}$$

□

PROOF OF THEOREM 2.28. Since Φ is an isometric isomorphism for the norm $\|\cdot\|_\infty$, thanks to [10, Corollary 1.3.], there exist an homeomorphism $T : Y \to X$ and a continuous function $\varepsilon : Y \to \{\pm 1\}$ such that $\Phi(f)(y) = \varepsilon(y) f\circ T(y)$ for all $f \in C_b^G(X)$ and all $y \in Y$. Since Y is a connected space, we have that ε is constant equal to 1 or -1. Replacing Φ by $-\Phi$ if necessary, we can assume without loss of generality that $\Phi(f) = f\circ T$ for all $f \in C_b^G(X)$. We are going to prove that T is an isometry. Let $y_1, y_2 \in Y$. Using Lemma 2.29 and the fact that $\|\mathrm{d}_F(\Phi(f))\|_\infty = \|\mathrm{d}_G f\|_\infty$ for all $f \in C_b^G(X)$, we have

$$\begin{aligned}
\|T(y_1)-T(y_2)\| &= \sup_{f\in C_b^G(X)\setminus\{0\}, \|\mathrm{d}_G f\|_\infty>0} \frac{|f(T(y_1))-f(T(y_1))|}{\|\mathrm{d}_G f\|_\infty} \\
&= \sup_{f\in C_b^G(X)\setminus\{0\}, \|\mathrm{d}_G f\|_\infty>0} \frac{|f\circ T(y_1)-f\circ T(y_2)|}{\|\mathrm{d}_G f\|_\infty} \\
&= \sup_{f\in C_b^G(X)\setminus\{0\}, \|\mathrm{d}_G f\|_\infty>0} \frac{|\Phi(f)(y_1)-\Phi(f)(y_2)|}{\|\mathrm{d}_F(\Phi(f))\|_\infty} \\
&= \sup_{g\in C_b^F(Y)\setminus\{0\}, \|\mathrm{d}_F g\|_\infty>0} \frac{|g(y_1)-g(y_2)|}{\|\mathrm{d}_F g\|_\infty} \\
&= \|y_1-y_2\|.
\end{aligned}$$

Thus, $T : Y \to X$ is a surjective isometry. From Mazur-Ulam theorem [93], T is an affine map, equivalently $T-T(0)$ is linear. Finally, $T-T(0)$ is a linear surjective isometry from Y onto X. So X and Y are isometrically isomorphic. On the other hand, since $f\circ T \in C_b^F(Y)$, whenever $f \in C_b^G(X)$, then $T-T(0)$ is a compact operator by Corollary 2.21. Therefore, thanks to the Riesz theorem, X and Y are finite dimensional. Thus, X and Y are finite dimensional and isometrically isomorphic spaces. The converse is clear. Indeed, since Gâteaux and Fréchet-differentiability coincides for Lipschitz functions in finite dimensional Banach space, we have that $C_b^G(X) = C_b^F(X)$. On the other hand, if $T : Y \to X$ is an isometric isomorphism, then the operator given by $\Phi(f) = f\circ T$ is an isomorphism between $C_b^F(X)$ and $C_b^F(Y)$ satisfying the two desired conditions. □

Proposition 2.30 *Let X and Y be two Banach spaces having a bump function in $C_b^G(X)$ and $C_b^F(Y)$ respectively. Let $T \in \mathcal{L}(Y, X)$. Then, the following assertions are equivalent.*

1. *T is a compact operator with dense range.*

2. *The operator* $\Phi : C_b^G(X) \to C_b^F(Y)$ *defined by* $\Phi(f) = f \circ T$ *is a well-defined injective bounded linear operator.*

Proof. Suppose that T is compact. Then $f \circ T \in C_b^F(Y)$ whenever $f \in C_b^G(X)$ by the necessity of Theorem 2.5, so Φ maps $C_b^G(X)$ into $C_b^F(Y)$. By the density of the range of T, Φ is injective. Then, it is clear that Φ is a bounded linear operator satisfying $\|\Phi(f)\|_\infty \leq \|f\|_\infty$ and $\|\mathrm{d}_F(\Phi(f))\|_\infty \leq \|\mathrm{d}_G f\|_\infty \|T\|$ for all $f \in C_b^G(X)$. Conversely, since Φ maps $C_b^G(X)$ into $C_b^F(Y)$, by Corollary 2.21 the operator T is compact. Suppose by contradiction that $\overline{T(Y)} \neq X$. There exists $x_0 \in X$ such that $x_0 \notin \overline{T(Y)}$. By the Hahn-Banach theorem, there exists a continuous linear map $x^* \in X^*$ such that $x^*(x_0) = 1$ and $x^*|_{\overline{T(Y)}} = 0$. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$-smooth function such that

$$\alpha(t) = \begin{cases} 2t - 1 \text{ if } 1 \leq t \leq 2 \\ 4 \text{ if } t \geq 4 \\ 0 \text{ if } t \leq 0. \end{cases}$$

Let us define $f_0(x) = \alpha \circ x^*$. Thus, $f_0 \in C_b^G(X)$ and we have $f_0 \circ T = 0$. Thus, $\Phi(f_0) = \Phi(0) = 0$ but $f_0 \neq 0$ since $f_0(x_0) = 1$. This contradicts the injectivity of Φ. $\square$

Chapter 3

Epsilon-Hypercyclicity Criterion

In this chapter we explore the epsilon-hypercyclicity phenomenon, a concept introduced in [15] and also developed in [16]. Concretely, we deal with the construction of epsilon-hypercyclic operators which are not hypercyclic. The main contribution of this chapter is a sufficient condition for a given operator to be epsilon-hypercyclic (Theorem 3.2). The basics and notation to carry out this chapter can be found in Chapter 1, Section 1.2 and Section 1.3. In what follows, the results are mainly based on the preprint [90]. Our results hold true in either real or complex Banach spaces.

3.1 Introduction

Let X be a separable infinite dimensional real or complex Banach space and let T be a linear bounded operator on X. During the last decades, the effort of the researchers to find dynamics of bounded operators which are purely infinite-dimensional phenomena but different from hypercyclicity has increased. For instance, N. Feldman in [51] solve negatively the following question: Let $\varepsilon > 0$. Does there exist a bounded non-hypercyclic operator T on X having a vector $x \in X$ such that $\mathrm{Orb}_T(x)$ meets every ball of radius ε? In this line, C. Badea, S. Grivaux and V. Müller introduced the following concept in [15].

Definition 3.1 *Let X be a Banach space. Let $\varepsilon \in (0,1)$. A bounded operator T on X is called ε-hypercyclic if there exists a vector x such that for all $y \in X \setminus \{0\}$, there exists $n \in \mathbb{N}$ for which*

$$\|T^n x - y\| \leq \varepsilon \|y\|.$$

The vector x is said to be an ε-hypercyclic vector for T.

Clearly, each hypercyclic operator is ε-hypercyclic for every $\varepsilon > 0$. So, we already know that ε-hypercyclic operators exist in every separable infinite dimensional Banach space, see [2] and [21]. Also, every linear operator is 1-hypercyclic. Indeed, the origin satisfies the inequality of ε-hypercyclicity for $\varepsilon = 1$. However, it remains open if each separable infinite dimensional Banach space admits an ε-hypercyclic which is not hypercyclic. Up to the best of our knowledge, in the literature we can find the construction of such an operator in $\ell^1(\mathbb{N})$ and $\ell^2(\mathbb{N})$, see [15] and [16] respectively. We point out that the construction of F. Bayart in

[16] is a nice modification of the one made by C. Badea, S. Grivaux and V. Müller in [15]. Also, we can find other mentions of ε-hypercyclicity in [17], [20] and [80].

In this work, we introduce the following ε-Hypercyclicity Criterion (Theorem 3.2) and we use it to prove that several classical Banach spaces that admits ε-hypercyclic operators which are not hypercyclic, see examples below.

Theorem 3.2 (ε-Hypercyclicity Criterion) *Let X be a separable real or complex Banach space, let $T \in \mathcal{L}(X)$ and let $\varepsilon \in (0,1)$. Let $\mathcal{D}_1$ be a dense subset of X. Let $\mathcal{D}_2 := \{y_k : k \in \mathbb{N}\}$ be a countable subset of X. Assume further that for each $x \in X \setminus \{0\}$, there are infinitely many integers $k \in \mathbb{N}$ such that $y_k \in \overline{B}(x, \varepsilon\|x\|)$. Let $(n(k))_k \subset \mathbb{N}$ be an increasing sequence and let $S_{n(k)} : \mathcal{D}_2 \to X$ be a sequence of maps such that:*

(1) $\lim_{k\to\infty} \|T^{n(k)}x\| = 0$ *for all* $x \in \mathcal{D}_1$,

(2) $\lim_{k\to\infty} \|S_{n(k)}y_k\| = 0$,

(3) $\lim_{k\to\infty} \|T^{n(k)}S_{n(k)}y_k - y_k\| = 0$.

Then, T is δ-hypercyclic for all $\delta > \varepsilon$.

We use the preceding criterion to extended the construction of ε-hypercyclic operators which are not hypercyclic made in [16] to more general spaces, including $c_0(\mathbb{N})$ and $\ell^p(\mathbb{N})$, with $p \in [1,\infty)$, see Theorem 3.13. Moreover, using a stability result on products, Proposition 3.19, we are able to extend further previous construction obtaining, Theorem 3.20, which implies:

Theorem 3.3 *Let X be a separable real or complex Banach space. Assume that X contains a complemented subspace isomorphic to $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$, for $p \in [1,\infty)$. Then, for any $\varepsilon \in (0,1)$, X admits an ε-hypercyclic operator which is not hypercyclic.*

Before proceeding with some corollaries, let us recall the following classical result of Sobzyck [89]: Let X be a separable Banach space and let E be a closed subspace of X. Let $T \in \mathcal{L}(E, c_0(\mathbb{N}))$. Then there exists a bounded operator $\tilde{T} \in \mathcal{L}(X, c_0(\mathbb{N}))$ such that $\tilde{T}|_E = T$. We denote by $C(K)$ the Banach space of continuous functions on the compact space K. This space is endowed with the norm of the maximum.

Corollary 3.4 *Let $\varepsilon \in (0,1)$. The following Banach spaces admit ε-hypercyclic operators which are not hypercyclic:*

1. *$\ell^p(X)$ for $p \in [1,+\infty)$ and $c_0(X)$ whenever X is a (finite or infinite dimensional) separable Banach space.*
2. *Any separable infinite dimensional L^p space.*
3. *Any separable infinite dimensional space X containing an isomorphic copy of $c_0(\mathbb{N})$. Particularly, all separable infinite dimensional $C(K)$ spaces enjoy this property.*

Proof. (1) and (2) are directs from Theorem 3.3. On the other hand, (3) is a consequence of Sobzyck's Theorem. Indeed, $c_0(\mathbb{N})$ must be complementable on X. Therefore, by Theorem 3.3, X admits an ε-hypercyclic operator which is not hypercyclic. Now, let us assume that

X is a separable infinite dimensional $C(K)$ space. Then, K must be an infinite metrizable compact set. Hence, X admits a complemented subspace isometric to $c_0(\mathbb{N})$. For details see [1, Proposition 4.3.11]. $\square$

This chapter is organized as follows. In Section 3.2 we provide a constructive proof of our ε-Hypercyclic Criterion. In Section 3.3, in the spirit of the Hypercyclic Criterion, we present a proof based on the Baire-category theorem of Theorem 3.2. In Section 3.4 we introduce some notation to extended the construction of ε-hypercyclic operators found in [16]. Section 3.5 is devoted to prove the main theorem about constructions of epsilon-hypercyclic operators which are not hypercyclic, Theorem 3.20. In Section 3.6, we discuss why a natural choice for an ε-Hypercyclicity Criterion is, in fact, equivalent to the Hypercyclicity Criterion. Finally, we end this chapter with some proofs of simple but useful facts about epsilon-hypercyclicity used through this chapter.

Notation: In only this chapter $0 \in \mathbb{N}$.

3.2 A constructive proof of the epsilon-Hypercyclicity Criterion

In the fashion of the Hypercyclicity Criterion, Theorem 1.18, for which we can find a constructive proof and a topological proof based on the Baire-category theorem, see [18, Chapter 1], we provide two distinct proofs of our criterion of epsilon-hypercyclicity. Let us start with the constructive proof.

Constructive proof of the ε-Hypercyclicity Criterion. Let us construct a δ-hypercyclic vector for T, for any $\delta > \varepsilon$. Let $(\eta_k)_k \subset \mathbb{R}^+$ be any sequence of positive numbers such that $(k^2\eta_k)_k$ converges to 0. Observe that the series $\sum_k \eta_k$ is convergent. Let $\{z_k : k \in \mathbb{N}\}$ be a countable dense subset of $X \setminus \{0\}$. Let $m_0 \in \mathbb{N}$ such that

$$\|z_0 - y_{m_0}\| \leq \varepsilon\|z_0\|, \ \|S_{n(m_0)}y_{m_0}\| < \eta_0 \text{ and } \|T^{n(m_0)}S_{n(m_0)}y_{m_0} - y_{m_0}\| < \eta_0.$$

By density of $\mathcal{D}_1$ and continuity of T, there is $x_0 \in \mathcal{D}_1$ such that $\|x_0\| < \eta_0$ and $\|T^{m_0}x_0 - y_{m_0}\| < \eta_0$. Let $k \geq 1$ and let us assume that $(x_\mathrm{i})_\mathrm{i} \subset \mathcal{D}_1$ and $(m_\mathrm{i})_\mathrm{i} \subset \mathbb{N}$ are already defined for all $\mathrm{i} \leq k-1$. Let $\rho_k > 0$ be a positive number such that $\|T^{n(m_\mathrm{i})}u\| \leq 2^{-k}$ for all $\|u\| \leq \rho_k$ and for all $\mathrm{i} < k$. Redefine $\eta_k := \min\{\eta_k, \rho_k\}$. Let m_k be an integer such that $m_k > m_{k-1}$, $\|T^{n(m_k)}x_\mathrm{i}\| < \eta_k$ for all $\mathrm{i} < k$,

$$\|z_k - y_{m_k}\| \leq \varepsilon\|z_k\|, \ \|S_{n(m_k)}y_{m_k}\| < \eta_k \text{ and } \|T^{n(m_k)}S_{n(m_k)}y_{m_k} - y_{m_k}\| < \eta_k.$$

By density of $\mathcal{D}_1$, there is $x_k \in \mathcal{D}_1$ such that $\|x_k\| < \eta_k$ and $\|T^{n(m_k)}x_k - y_{m_k}\| < \eta_k$.

Now, since $\|x_k\| < \eta_k$ for all $k \in \mathbb{N}$, the vector $\overline{x} = \sum_{k=0}^{\infty} x_k \in X$ is well defined. We claim that $\overline{x}$ is a δ-hypercyclic vector for T, for all $\delta > \varepsilon$. Indeed, let $j \in \mathbb{N}$. Then:

$$
\begin{aligned}
\|T^{n(m_j)}\overline{x} - z_j\| &\leq \sum_{k=0}^{j-1} \|T^{n(m_j)}x_k\| + \|T^{n(m_j)}x_j - y_{m_j}\| + \|y_{m_j} - z_j\| + \sum_{k=j+1}^{\infty} \|T^{n(m_j)}x_k\| \\
&\leq (j+1)\eta_j + \varepsilon\|z_j\| + \sum_{k=j+1}^{\infty} 2^{-k}.
\end{aligned}
$$

Now, let $z \in X$ and let $(j(l))_l$ be an increasing sequence such that $(z_{j(l)})_l$ converges to z. Then

$$
\begin{aligned}
\|T^{n(m_{j(l)})}\overline{x} - z\| &\leq \|T^{n(m_{j(l)})}\overline{x} - z_{j(l)}\| + \|z_{j(l)} - z\| \\
&\leq (j(l)+1)\eta_{j(l)} + \varepsilon\|z_{j(l)}\| + \sum_{k=j(l)+1}^{\infty} 2^{-k} + \|z_{j(l)} - z\|,
\end{aligned}
$$

expression which tends to $\varepsilon\|z\|$ as l tends to infinity. Therefore, if $\delta > \varepsilon$ and $z \neq 0$, for l large enough, we have that $\|T^{n(m_{j(l)})}\overline{x} - z\| \leq \delta\|z\|$. □

Remark 3.5 *From the proof, notice that if the sequence $(z_{j(l)})_l$ converges to z, then*

$$
\limsup_{l\to\infty} \|T^{n(m_{j(l)})}\overline{x} - z\| \leq \varepsilon\|z\|.
$$

Remark 3.6 *The previous criterion can be applied to the operators constructed in [15] and [16]. In fact, this can be done similarly as we do in the proof Theorem 3.13.*

In [86], M. de la Rosa and C. Read proved that there are hypercyclic operators which do not satisfy the Hypercyclic Criterion. Therefore, our epsilon-Hypercyclicity Criterion leads to the following natural question: Does there exist an epsilon-hypercyclic operator which does not satisfies the epsilon-Hypercyclicity Criterion?

By definition, every hypercyclic operator is ε-hypercyclic for all $\varepsilon > 0$. In [15] it is shown that the converse is also true, i.e. if an operator is ε-hypercyclic for each $\varepsilon > 0$, then it is hypercyclic. In this line, we have the following result.

Proposition 3.7 *Let X be a separable infinite dimensional Banach space and let $T \in \mathcal{L}(X)$. If T satisfies the Hypercyclicity Criterion, then it satisfies the ε-Hypercyclicity Criterion for each $\varepsilon > 0$.*

Proof. Let T be a bounded operator satisfying the Hypercyclicity Criterion. Let $\mathcal{D}_1, \mathcal{D}_2 \subset X$, $(n(k))_k \subset \mathbb{N}$ and $(S_{n(k)})_k$ given by the mentioned criterion. Since X is separable, without loss of generality, we can assume that $\mathcal{D}_2$ is a countable dense set. Let us enumerate $\mathcal{D}_2 := \{y_k : k \in \mathbb{N}\}$. To achieve the ε-Hypercyclic-Criterion we only need to construct a subsequence of $(n(k))_k$, namely $(m(k))_k$, which satisfies hypothesis (2) and (3) of Theorem 3.2. To this end, let us define $m(0) \in \{n(k) : k \in \mathbb{N}\}$ such that $\|S_{m(0)}y_0\| \leq 1$ and

$\|T^{m(0)}S_{m(0)}y_0 - y_0\| \le 1$. Let $k \ge 1$ and suppose that we have constructed $(m(j))_j$ for all $j \le k-1$. Let us fix $m(k) \in \{n(j) : j \in \mathbb{N}\}$ such that

$$m(k) > m(k-1), \qquad \|S_{m(k)}y_k\| \le k^{-1}, \qquad \text{and} \qquad \|T^{m(k)}S_{m(k)}y_k - y_k\| \le k^{-1}.$$

Now, it is straightforward that hypothesis (1), (2) and (3) of the ε-Hypercyclicity Criterion are satisfied for the sequence $(m(k))_k$ and the maps $(S_{m(k)})_k$. Finally, since $\mathcal{D}_2$ is dense, the intersection of $\mathcal{D}_2$ with any open set must be an infinite set. Therefore, T satisfies the ε-Hypercyclicity Criterion for each $\varepsilon > 0$. □

3.3 A topological proof of the Epsilon-Hypercyclic Criterion

Let us start with the following definitions.

Definition 3.8 *Let X be a separable Banach space, let $\varepsilon \in (0,1)$ and let $T \in \mathcal{L}(X)$. We define the sets:*

$$\varepsilon HC(T) := \{x \in X : \ x \text{ is an } \varepsilon\text{-hypercyclic vector for } T\}$$
$$\varepsilon^+ HC(T) := \{x \in X : \ x \text{ is an } \delta\text{-hypercyclic vector for } T, \ \forall\, \delta > \varepsilon\}.$$

Observe that $x \in \varepsilon^+ HC(T)$ if and only if

$$\inf_{n\in\mathbb{N}} \|T^n x - y\| \le \varepsilon \|y\|, \ \forall\, y \in X.$$

Also, notice that if T satisfies the ε-hypercyclicity criterion, then $\varepsilon^+ HC(T)$ is a nonempty set. Albeit simple, let us continue with the following proposition.

Proposition 3.9 *Let X be a separable Banach space, let $T \in \mathcal{L}(X)$ and let $\varepsilon \in (0,1)$. Then the sets $\varepsilon HC(T)$ and $\varepsilon^+ HC(T)$ are stable under non-zero scalar multiplication.*

Proof. Let us check first $\varepsilon HC(T)$. Let $\lambda \in \mathbb{K} \setminus \{0\}$. Let $x \in \varepsilon HC(T)$ and let $y \in X$ be a non-zero vector. Therefore, there exists $n \in \mathbb{N}$ such that

$$\|T^n x - \lambda^{-1} y\| \le \varepsilon \|\lambda^{-1} y\|.$$

Multiplying the last expression by λ, we conclude that $\lambda x \in \varepsilon HC(T)$. The second part follows by noticing that

$$\varepsilon^+ HC(T) = \bigcap_{\delta \in (\varepsilon, 1)} \delta HC(T).$$

□

In order to provide the Baire-category based on proof of our ε-hypercyclicity criterion, we need the following proposition.

Proposition 3.10 *Let X be a separable Banach space, let $T \in \mathcal{L}(X)$ and let $\varepsilon \in (0,1)$. Then*

$$\varepsilon^+ HC(T) := \bigcap_{k=1}^{\infty} \bigcap_{y \in \mathcal{D}} \bigcup_{n=0}^{\infty} T^{-n}\left(B\left(y, \|y\|\left(\varepsilon + \frac{1}{k}\right)\right)\right), \tag{3.1}$$

where $\mathcal{D}$ is any countable subset of $X \setminus \{0\}$, dense in X. Therefore, $\varepsilon^+ HC(T)$ is a G_δ set.

Proof. Let $\mathcal{D} \subset X \setminus \{0\}$ be any countable dense set.

Let $x \in \varepsilon^+ HC(T)$. Then, for any $y \in \mathcal{D}$ and for any $k \geq 1$, there exists $n_{y,k} \in \mathbb{N}$ such that

$$\|T^{n_{y,k}} x - y\| \leq \left(\varepsilon + \frac{1}{k}\right)\|y\|,$$

which is exactly the translation of the right-hand part of Equation (3.1).

Let $x \in X$ be a vector belonging to the set defined in the right-hand part of equality (3.1). Let $z \in X \setminus \{0\}$. The following computation shows that $x \in \varepsilon^+ HC(T)$:

$$\inf_{n \in \mathbb{N}} \|T^n x - z\| \leq \inf_{y \in \mathcal{D}} \inf_{n \in \mathbb{N}} \|T^n x - y\| + \|y - z\| \leq \inf_{y \in \mathcal{D}} \varepsilon\|y\| + \|y - z\| \leq \varepsilon\|z\|.$$

$\square$

Now we can provide an alternative proof of Theorem 3.2.

Topological proof of Theorem 3.2. Let $T \in \mathcal{L}(X)$ satisfying the ε-Hypercyclicity Criterion. Let us show that the set $\varepsilon^+ HC(T)$ is nonempty, and thus, T is δ-hypercyclic for every $\delta > \varepsilon$. Let $\mathcal{D} \subset X \setminus \{0\}$ be any countable dense set. By Proposition 3.10 we have that

$$\varepsilon^+ HC(T) := \bigcap_{k=1}^{\infty} \bigcap_{y \in \mathcal{D}} \bigcup_{n=0}^{\infty} T^{-n}\left(B\left(y, \|y\|\left(\varepsilon + \frac{1}{k}\right)\right)\right).$$

Therefore, thanks to Baire-category Theorem, if the set

$$A_{y,k} := \bigcup_{n \in \mathbb{N}} T^{-n}\left(B\left(y, \|y\|\left(\varepsilon + \frac{1}{k}\right)\right)\right)$$

is open and dense for each $y \in \mathcal{D}$ and each integer $k \geq 1$, then the set $\varepsilon^+ HC(T)$ is nonempty as well.

Let $y \in \mathcal{D}$ and $k \geq 1$. Since T is continuous, we only have to prove the density of the set $A_{y,k}$. Let $z \in X$ and let $r > 0$. We prove that $A_{y,k} \cap B(z,r) \neq \emptyset$. Let $x_1 \in \mathcal{D}_1$ be a vector such that $\|z - x_1\| < r/2$. By definition of $\mathcal{D}_2$ and hypothesis (1), (2) and (3) of the ε-Hypercyclicity Criterion, let $j \in \mathbb{N}$ large enough such that

- $\|T^{n(j)} x_1\| < \frac{1}{2k}\|y\|$
- $\|y - y_j\| \leq \varepsilon\|y\|$.

- $\|S_{n(j)}y_j\| < r/2$.
- $\|T^{n(j)}S_{n(j)}y_j - y_j\| < \frac{1}{2k}\|y\|$.

Let $x_2 = S_{n(j)}y_j$ and set $x := x_1 + x_2$. We claim that $x \in A_{y,k} \cap B(z,r)$. First, $\|x - z\| \leq \|x_1 - z\| + \|x_2\| < r$. Second, observe that $x \in A_{y,k}$ if and only if there exists $n \in \mathbb{N}$ such that $T^n(x) \in B(y, \|y\|(\varepsilon + k^{-1}))$. So, followed by the computation

$$\begin{aligned}\|T^{n(j)}x - y\| &\leq \|T^{n(j)}x_1\| + \|T^{n(j)}x_2 - y_j\| + \|y_j - y\| \\ &< \frac{1}{2k}\|y\| + \frac{1}{2k}\|y\| + \varepsilon\|y\| = \|y\|\left(\varepsilon + \frac{1}{k}\right),\end{aligned}$$

we prove that $x \in A_{y,k}$. Therefore $A_{y,k}$ is dense in X. Finally, the Baire-category Theorem finishes the proof of Theorem 3.2. □

As a direct consequence of the topological proof of Theorem 3.2 presented above we get:

Corollary 3.11 *Let X be a separable Banach space, let $T \in \mathcal{L}(X)$ and let $\varepsilon \in (0,1)$. Assume that T satisfies the ε-Hypercyclicity Criterion. Then, $\varepsilon^+HC(T)$ is a G_δ-dense set.*

3.4 Infinite direct sum of a Banach space

Let X be a Banach space. It is common to consider the Banach space of p-summable sequences on X, with $p \in [1,\infty)$ or space of sequences on X which converge to 0, i.e. the space $\ell^p(X)$, with $p \in [1,\infty)$, or $c_0(X)$ respectively. That is, for $p \in [1,\infty)$

$$\ell^p(X) := \{(x_n)_n \subset X^{\mathbb{N}} : \sum_{n=0}^{\infty} \|x_n\|^p < \infty\},$$

endowed with the norm $\|(x_n)_n\| = (\sum \|x_n\|^p)^{1/p}$. The definition of $c_0(X)$ is analogous. In what follows, we introduce a generalized version of these spaces.

Definition 3.12 *Let X and Y be two Banach spaces. Let $(f_n)_n \subset Y$ be a normalized 1-unconditional basis of Y. We denote by $\bigoplus_Y X$ the vector space defined by*

$$\bigoplus_Y X := \{(x_n)_n \in X^{\mathbb{N}} : \sum_{n=0}^{\infty} \|x_n\|_X f_n \in Y\}.$$

We endow this space with the norm $\|\cdot\|$ defined by $\|(x_n)_n\| = \|\sum_{n=0}^{\infty} \|x_n\|_X f_n\|_Y$. A standard procedure shows that $(\bigoplus_Y X, \|\cdot\|)$ is a Banach space.

The 1-unconditionality of $(f_n)_n \subset Y$ in Definition 3.12 implies the triangle inequality of the norm on $\bigoplus_Y X$. Clearly, the space constructed in Definition 3.12 depends on the chosen 1-unconditional basis $(f_n)_n$ of Y, but we omit it for sake of brevity. If X is either $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$, with $p \in [1,\infty)$, then X is isometric to $\bigoplus_X X$, whenever we use the canonical basis of X. Also, notice that for all $(x_n)_n \in \bigoplus_Y X$, the sequence $(\|x_n\|_X)_n$ converges to 0.

3.5 Construction of epsilon-hypercyclic operators

In this section we prove Theorem 3.20, which is an abstract version of Theorem 3.3. To do this, we show first the existence of ε-hypercyclic operators which are not hypercyclic in a particular class of Banach spaces, using mainly Bayart's ideas, [16]. Then, we extend further this construction using a result of stability under products, Proposition 3.19.

Theorem 3.13 *Let X and Y be two infinite dimensional separable Banach spaces. Assume that Y admits a 1-unconditional basis $(f_n)_n$ such that the associated backward shift operator is continuous. Then, for every $\varepsilon \in (0,1)$, the space $\bigoplus_Y X$, related to (f_n), admits an ε-hypercyclic operator which is not hypercyclic.*

The proof is divided in two main parts. In the first one, we formally construct the candidate of ε-hypercyclic operator, whereas in the second part we prove that the operator is well defined, ε-hypercyclic but not hypercyclic. Let us point out that, in the second part, step 3, we use our ε-Hypercyclicity Criterion.

Let us start the proof of Theorem 3.13.

First part: Let $(\mathrm{e}_n)_n \subset X$ be a normalized bounded M-basis given by the classical result of Ovsepian and Pełczyński, Theorem 1.11, and let $(\mathrm{e}_n^*)_n \subset X^*$ be the associated coordinates sequence. Let $b = \sup_n \|\mathrm{e}_n^*\| < \infty$. Let $\varepsilon \in (0,1)$, let $\alpha > 1$ and let $\mathrm{d} \in \mathbb{N}$, with $\mathrm{d} > 1$, such that $2\alpha^{-\mathrm{d}} b < \varepsilon$. Let $(\Delta_k)_k \subset \mathbb{N}$ be a rapidly increasing sequence which will be specified later on, in Proposition 3.17. Let $(n_k)_k$ and $(n'_k)_k$ be two increasing sequences defined by $n_0 = n'_0 = 0$, $n_k = n'_{k-1} + \mathrm{d} + 1 + \Delta_k$ and $n'_k = n_k + \mathrm{d} + 1 + \Delta_k$, for all $k \geq 1$. It is clear that $k \leq n'_{k-1}$ for all $k \geq 2$. For $k \in \mathbb{N}$ and $\sigma, \beta \in \mathbb{K}$, we define the diagonal operator, with respect to $(\mathrm{e}_n)_n$, $D_{k,\sigma,\beta}$ on X by:

$$D_{k,\sigma,\beta} = \sigma I\mathrm{d} + (1-\sigma)\mathrm{e}_0^* \otimes \mathrm{e}_0 + (\beta - \sigma)\mathrm{e}_{k^2}^* \otimes \mathrm{e}_{k^2}.$$

Since $D_{k,\sigma,\beta}$ is a rank 2 perturbation of $\sigma I\mathrm{d}$, it is a bounded operator with norm

$$\|D_{k,\sigma,\beta}\| \leq |\sigma|(1+2b) + |\beta| b + b.$$

Moreover, whenever σ and β are different from 0 and $k \geq 1$, $D_{k,\sigma,\beta}^{-1} = D_{k,\sigma^{-1},\beta^{-1}}$ easily follows. For each $k \in \mathbb{N}$, we define the operator $N_k := \mathrm{e}_{k^2}^* \otimes \mathrm{e}_0$, i.e. $N_k(x) = \mathrm{e}_{k^2}^*(x)\mathrm{e}_0$ for all $x \in X$. Notice that $(\|N_k\|)_k$ is uniformly bounded. Indeed, $\|N_k\| \leq b$, for all $k \in \mathbb{N}$. Also, for each $j \geq 1$, we define the operator S_j on X as follows. Let $k \in \mathbb{N}$ be the unique integer such that $n'_{k-1} < j \leq n'_k$, then we set:

$$S_j := \begin{cases} D_{k,\frac{1}{\alpha},\alpha} & n'_{k-1} + 1 \leq j \leq n'_{k-1} + \mathrm{d}, \\ D_{k,\frac{1}{\alpha},1} - N_k & j = n'_{k-1} + \mathrm{d} + 1, \\ D_{k,\frac{1}{\alpha},\frac{1}{\alpha}} & n'_{k-1} + \mathrm{d} + 2 \leq j \leq n'_{k-1} + \mathrm{d} + 1 + \Delta_k = n_k, \\ D_{k,\frac{1}{\alpha},\alpha} & n_k + 1 \leq j \leq n_k + \Delta_k, \\ D_{k,\frac{1}{\alpha},1} + N_k & j = n_k + \Delta_k + 1, \\ D_{k,\frac{1}{\alpha},\frac{1}{\alpha}} & n_k + \Delta_k + 2 \leq j \leq n_k + \mathrm{d} + \Delta_k, \\ D_{k,\alpha^{n'_k - n'_{k-1} - 1},\frac{1}{\alpha}} & j = n_k + \mathrm{d} + 1 + \Delta_k = n'_k. \end{cases}$$

Notice that each S_j is an upper-triangular operator on X with respect to the sequence $(\mathrm{e}_n)_n$. Further, observe that $(D_{k,\frac{1}{\alpha},1} \pm N_k)^{-1} = D_{k,\alpha,1} \mp N_k$. The following three properties are direct from the definition of the operators S_j.

$(\mathcal{Q}_0)$ $S_j\mathrm{e}_0 = \mathrm{e}_0$ for all $j \geq 1$.

$(\mathcal{Q}_1)$ $S_{n'_k} \cdots S_1 = Id$ for all $k \geq 1$.

$(\mathcal{Q}_2)$ For $k \geq 1$, $p \notin \{0, k^2\}$ and $\mathrm{i} \in \{n'_k, \cdots, n'_{k+1} - 1\}$, $S_1^{-1} \cdots S_{\mathrm{i}}^{-1}\mathrm{e}_p = \alpha^{\mathrm{i}-n'_k}\mathrm{e}_p$ holds.

Let us now formally define the operator T on $\bigoplus_Y X$. Let $z = (x_n)_n \in \bigoplus_Y X$, then:

$$Tz = (S_1^{-1}x_1, S_2^{-1}x_2, \cdots),$$

i.e., T is a backward shift on $\bigoplus_Y X$ with weights $(S_n^{-1})_n$.

Second part: *Step 1: T is a well-defined, bounded operator on $\bigoplus_Y X$.*

Proposition 3.14 *Each operator S_j is bounded and invertible. Moreover, the sequence $(\|S_j^{-1}\|)_j$ is uniformly bounded.*

Proof. Let $j \geq 1$. Assume that there are $k \in \mathbb{N}$ and $\sigma, \beta \in \mathbb{R}$ such that $S_j = D_{k,\sigma,\beta}$. Recalling that $S_j^{-1} = D_{k,\sigma^{-1},\beta^{-1}}$, we get that $\|S_j^{-1}\| \leq |\sigma^{-1}|(1 + 2b) + |\beta^{-1}|b + b$. Since $\sigma \in \{\alpha^{-1}, \alpha^{n'_k - n'_{k-1} - 1}\}$ and $\beta \in \{\alpha^{-1}, 1, \alpha\}$, we conclude that $\|S_j^{-1}\| \leq \alpha(1 + 3b) + b$, which is a constant independent of j. Otherwise, if $S_j = D_{k,\alpha^{-1},1} \pm N_k$, then $S_j^{-1} = D_{k,\alpha,1} \mp N_k$. Therefore, $\|S_j^{-1}\| \leq \|D_{k,\alpha,1}\| + b \leq \alpha(1 + 2b) + 2b$, which is a constant independent of j as well. $\square$

Let $(f_n)_n$ be the 1-unconditional basis on Y used to construct the space $\bigoplus_Y X$. Thanks to Proposition 3.14, we know that there exists a constant $C > 0$ such that $\|S_n^{-1}x\| \leq C\|x\|$ for all $x \in X$ and for all $n \in \mathbb{N}$. Let $K > 0$ be the norm of the backward shift operator associated to the basis $(f_n)_n$. Then, for $z = (x_n)_n \in \bigoplus_Y X$ we get

$$\|Tz\| = \left\| \sum_{n=1}^{\infty} \|S_n^{-1}x_n\|_X f_{n-1} \right\|_Y \leq K \left\| \sum_{n=0}^{\infty} C\|x_n\|_X f_n \right\|_Y = KC\|z\|,$$

which implies the well definition and continuity of T.

Step 2: T is not a hypercyclic operator.

Proposition 3.15 *The sequence $(\|S_jS_{j-1} \cdots S_1\|)_j$ is bounded by a constant $M(\mathrm{d})$ which depends only on* d.

Proof. Let $j \geq 1$ and let $k \in \mathbb{N}$ such that $n'_{k-1} \leq j < n'_k$. Then, by property $(\mathcal{Q}_1)$, $S_jS_{j-1} \cdots S_1 = S_j \cdots S_{n'_{k-1}+1}$. Let $X_1 = \mathrm{span}(\mathrm{e}_0, \mathrm{e}_{k^2})$ and let $X_2 = \overline{\mathrm{span}}(\mathrm{e}_n : n \neq 0, k^2)$. Observe that X is isomorphic to $X_1 \oplus X_2$. Indeed, let $P = \mathrm{e}_0^* \otimes \mathrm{e}_0 + \mathrm{e}_{k^2}^* \otimes \mathrm{e}_{k^2}$ and let $Q = I - P$. Then, P and Q are bounded parallel projections onto X_1 and X_2 respectively. In fact, $\|P\| \leq 2b$. Since $Id = P + Q$, we get that $\|S_j \cdots S_{n'_{k-1}+1}\| \leq \|S_j \cdots S_{n'_{k-1}+1}P\| + \|S_j \cdots S_{n'_{k-1}+1}Q\|$. Thanks to $(\mathcal{Q}_1)$ and $(\mathcal{Q}_2)$, it follows that $S_j \cdots S_{n'_{k-1}+1}Q = \alpha^{-(j-n'_{k-1})}Q$.

Thus, $\|S_j \cdots S_{n'_{k-1}+1}Q\| \leq \|Q\| \leq 1 + 2b$. On the other hand, regarding the operator $S_j \cdots S_{n'_{k-1}+1}P$, we can notice that

$$S_j \cdots S_{n'_{k-1}+1}P = (\mathrm{e}_1^* + \sigma_j \mathrm{e}_{k^2}^*) \otimes \mathrm{e}_1 + \beta_j \mathrm{e}_{k^2}^* \otimes \mathrm{e}_{k^2}, \text{ where } |\sigma_j|, \beta_j \in [0, \alpha^{\mathrm{d}}],$$

with which we conclude that $\|S_j \cdots S_{n'_{k-1}+1}P\| \leq b(1 + 2\alpha^{\mathrm{d}})$, a constant independent of j. Finally, the proof is finished choosing $M(\mathrm{d}) = 1 + 3b + 2b\alpha^{\mathrm{d}}$. $\square$

Let us check that T is a non-hypercyclic operator. Suppose that $z = (\mathrm{e}_0, 0, \cdots)$ is a cluster point of the orbit of some $w \in \bigoplus_Y X$, under the action of T. Therefore, there exists $(m_k)_k \subset \mathbb{N}$ an increasing sequence such that $(T^{m_k}w)_k$ converges to z. Hence, the first coordinate of $T^{m_k}w$, which is $S_1^{-1} \cdots S_{m_k}^{-1} w_{m_k}$, tends to e_0 as k tends to infinity. However, we have that

$$\begin{aligned} \|w_{m_k} - \mathrm{e}_0\| &= \|S_{m_k} \cdots S_1(S_1^{-1} \cdots S_{m_k}^{-1} w_{m_k} - \mathrm{e}_0)\| \\ &\leq M(\mathrm{d}) \|S_1^{-1} \cdots S_{m_k}^{-1} w_{m_k} - \mathrm{e}_0\|, \end{aligned} \tag{3.2}$$

which implies that $(w_{m_k})_k$ converges to e_0. This contradicts the fact that $w \in \bigoplus_Y X$ because $(\|w_k\|)_k$ does not converge to 0.

Remark 3.16 *The operator T is not δ-hypercyclic for any $\delta < 1/M(\mathrm{d})$. Indeed, let $w \in \bigoplus_Y X$. Thanks to triangle inequality and replacing the vector z by the vector λz in (3.2) we get that*

$$\begin{aligned} \frac{\|\lambda z\|}{M(\mathrm{d})} - \frac{\sup\{\|w_k\| : k \in \mathbb{N}\}}{M(\mathrm{d})} &\leq \frac{\|\lambda \mathrm{e}_0\|}{M(\mathrm{d})} - \frac{\|w_k\|}{M(\mathrm{d})} \\ &\leq \|S_1^{-1} \cdots S_k^{-1} w_k - \lambda \mathrm{e}_0\| \\ &\leq \|T^k w - \lambda z\|, \quad \text{for all } k \in \mathbb{N}. \end{aligned}$$

Let us fix $\delta < 1/M(\mathrm{d})$. Since $\sup\{\|w_k\| : k \in \mathbb{N}\}$ is finite, we can choose $\lambda \in \mathbb{K}$ with large modulus to show that w is not a δ-hypercyclic vector for T. Finally, since w is an arbitrary vector, T is not a δ-hypercyclic operator.

Step 3: T is an ε-hypercyclic operator.

Proposition 3.17 *There exist two sequences $(x^k)_k$, $(z^k)_k \subset \bigoplus_Y X$ such that*

(1) *$(x^k)_k$ is dense in $\bigoplus_Y X$,*
(2) *$\|z^k - x^k\| \leq 2\alpha^{-\mathrm{d}} b \|x^k\|$, for all $k \geq 2$, and*
(3) *$\|S_{n_k+j} \cdots S_{j+1} z_j^k\| \leq 2^{-k}$ for every $k \geq 2$ and for every $j = 0, \cdots, k-1$.*
(4) *For each k, there is $N_k \in N$ such that $z_j^k = 0$ for all $j \geq N_k$.*

Proof. Let $(x^k)_k \subset \bigoplus_Y X \setminus \{0\}$ be a sequence which satisfies the following two properties:

1. $\bigoplus_Y X = \overline{\{x^k : k \in \mathbb{N}\}}$.
2. For each $k \in \mathbb{N}$, $x^k = (x_0^k, \cdots, x_{k-1}^k, 0, \cdots)$, where each $x_j^k \in \operatorname{span}(\mathrm{e}_n : n \leq k-1)$.

Let $k \geq 2$. In order to define z^k, let us fix $j < k$ and $l \in \mathbb{N}$ such that $n'_{l-1} \leq j < n'_l$. We know that $l < k$. Let us define $v_j^k \in X$ by:

$$\alpha^{\mathrm{d}+j-n'_{l-1}} v_j^k = \begin{cases} \mathrm{e}_0^*(x_j^k)\mathrm{e}_{k^2} & \text{if } n'_{l-1} \leq j \leq n'_{l-1} + \mathrm{d}, \\ (\mathrm{e}_0^* + \alpha^{j-(n'_{l-1}+\mathrm{d}+1)}\mathrm{e}_{l^2}^*)(x_j^k)\mathrm{e}_{k^2} & \text{if } n'_{l-1} + \mathrm{d} + 1 \leq j \leq n'_{l-1} + \mathrm{d} + 1 + \Delta_l = n_l \\ (\mathrm{e}_0^* + \alpha^{\Delta_l-(j-n_l)}\mathrm{e}_{l^2}^*)(x_j^k)\mathrm{e}_{k^2} & \text{if } n_l + 1 \leq j \leq n_l + \Delta_l, \\ \mathrm{e}_0^*(x_j^k)\mathrm{e}_{k^2} & \text{if } j \geq n_l + \Delta_l + 1. \end{cases}$$

Set $v^k = (v_0^k, v_1^k, \cdots, v_{k-1}^k, 0, \cdots)$ and $z^k = x^k + v^k$. Observe that $\|v_j^k\| \leq 2\alpha^{-\mathrm{d}}b\|x_j^k\|$, for all $j \in \{0, 1, \cdots, k-1\}$. Since the space $\bigoplus_Y X$ is constructed with a 1-unconditional basis of Y, we conclude that $\|z^k - x^k\| \leq 2\alpha^{-\mathrm{d}}b\|x^k\|$. So, it only remains to prove property (3). Let $k \geq 2$ and $j \in \{0, \cdots, k-1\}$. For the sake of brevity, let us set $c_j^k \in \mathbb{K}$ by $v_j^k = c_j^k \mathrm{e}_{k^2}$. Let $l \in \mathbb{N}$ such that $n'_{l-1} \leq j < n'_l$. Then, we get

$$\begin{aligned} S_{n_k+j} \cdots S_{j+1} z_j^k &= S_{n_k+j} \cdots S_1 (S_1^{-1} \cdots S_j^{-1} z_j^k) \\ &= S_{n_k+j} \cdots S_{n'_{k-1}+1} (S_{n'_{l-1}+1}^{-1} \cdots S_j^{-1}(x_j^k + v_j^k)) \\ &= S_{n_k+j} \cdots S_{n'_{k-1}+1} (S_{n'_{l-1}+1}^{-1} \cdots S_j^{-1}(x_j^k) + \alpha^{j-n'_{l-1}} v_j^k), \\ &= S_{n_k+j} \cdots S_{n'_{k-1}+1} (S_{n'_{l-1}+1}^{-1} \cdots S_j^{-1}(x_j^k)) + \alpha^{j-n'_{l-1}}(-\alpha^{\mathrm{d}} c_j^k \mathrm{e}_0 + \alpha^{\mathrm{d}-\Delta_k+j} c_j^k \mathrm{e}_{k^2}). \end{aligned}$$

where the second equality comes from $(\mathcal{Q}_1)$, the third one is due to $(\mathcal{Q}_2)$ and the fact that $l < k$ and in the last line we have assumed that Δ_k is bigger than k. To continue, let us set the vector $h := S_{n'_{l-1}+1}^{-1} \cdots S_j^{-1}(x_j^k)$ and the operators $P = \mathrm{e}_0^* \otimes \mathrm{e}_0$ and $Q = I - P$. Then, since the operators $\{S_j : j \geq 1\}$ are upper-triangular with respect to the M-basis $(\mathrm{e}_n)_n$, we conclude that $Qh \in \operatorname{span}\{\mathrm{e}_n : 0 < n < k\}$. Thus, we get

$$\begin{aligned} S_{n_k+j} \cdots S_{j+1} z_j^k &= S_{n_k+j} \cdots S_{n'_{k-1}+1}(Ph + Qh) + \alpha^{j-n'_{l-1}}(-\alpha^{\mathrm{d}} c_j^k \mathrm{e}_0 + \alpha^{\mathrm{d}-\Delta_k+j} c_j^k \mathrm{e}_{k^2}), \\ &= [Ph - \alpha^{\mathrm{d}+j-n'_{l-1}} c_j^k \mathrm{e}_0] + \alpha^{-(n_k+j-n'_{k-1})} Qh + \alpha^{j-n'_{l-1}+\mathrm{d}-\Delta_k+j} c_j^k \mathrm{e}_{k^2}, \end{aligned}$$

where in the second line we have used property $(\mathcal{Q}_0)$ and that the operator S_j restricted to $\operatorname{span}(\mathrm{e}_n : 0 < n < k)$ is equal to $\alpha^{-1} I\mathrm{d}$ for all $j \in [n'_{k-1}+1, n'_k - 1]$. Since $n_k + j - n'_{k-1} = j + \Delta_k + \mathrm{d} + 1$ and $\|Qh\|$ does not depend on Δ_k, because $l < k$, the third term in the last expression tends to 0 as Δ_k tends to infinity. Also, since $l < k$ and $|c_j^k| = \|v_j^k\| \leq 2\alpha^{-\mathrm{d}}b\|x_j^k\|$ does not depend on Δ_k, the fourth term in the last expression tends to 0 as Δ_k tends to infinity. On the other hand, the coefficients c_j^k were chosen to cancel the expression enclosed in square brackets. Finally, if we choose Δ_k large enough (with $\Delta_k > k$), we can ensure that $\|S_{n_k+j} \cdots S_{j+1} z_j^k\| \leq 2^{-k}$. □

Proof of Theorem 3.13. We already know that T is a bounded non-hypercyclic operator on $\bigoplus_Y X$. Let us show that T is ε-hypercyclic, using the ε-Hypercyclicity Criterion, Theorem 3.2. Let $(x^k)_k$ and $(z^k)_k$ be sequences given by Proposition 3.17. Let us set

$$\mathcal{D}_1 := \{(y_{\mathrm{i}})_{\mathrm{i}} \in \bigoplus_Y X : \ \exists N \in \mathbb{N}, y_{\mathrm{i}} = 0, \ \forall \mathrm{i} \geq N\},$$

which is dense in $\bigoplus_Y X$. Let $\mathcal{D}_2 := \{z^k \in \bigoplus_Y X : \ k \geq 2\}$. Let $w \in \bigoplus_Y X$ be a vector different from 0 and let $(x^{m_k})_k$ be a subsequence of $(x^k)_k$ which converges to w. Let $\rho > 2\alpha^{-\mathrm{d}}b$.

We claim that, for k large enough, $z^{m_k} \in B(w, \rho\|w\|)$. In fact, applying Proposition 3.17 (2) we obtain that

$$\|w - z^{m_k}\| \leq \|w - x^{m_k}\| + \|x^{m_k} - z^{m_k}\| \leq \|w - x^{m_k}\| + 2\alpha^{-\mathrm{d}} b \|x^{m_k}\|,$$

Since $\rho > 2\alpha^{-\mathrm{d}} b$ and $(x^{m_k})_k$ converges to w, the claim is proved. Thus, there are infinitely many $k \in \mathbb{N}$ such that $z^k \in B(w, \rho\|w\|)$. Let $(n(k)) = n_k$ be the sequence constructed in the first part. Now, we check the three hypotheses of the ε-Hypercyclicity Criterion. Let us define the map $U : \mathcal{D}_1 \to \mathcal{D}_1$ as the formal right inverse of T. i.e. U is defined by

$$U(y_{\mathrm{i}})_{\mathrm{i}} = T^{-1}(y_{\mathrm{i}})_{\mathrm{i}} = (0, S_1 y_0, S_2 y_1, \cdots), \ \forall (y_{\mathrm{i}})_{\mathrm{i}} \in \mathcal{D}_1.$$

Let $U_{n(k)} := T^{-n(k)}$. With this, hypotheses (1) and (3) are straightforward. Indeed, let $y \in \mathcal{D}_1$. Since $(n(k))_k$ tends to infinity and T is a backward shift, we have that $T^{n(k)} y = 0$ for k large enough. Hypothesis (3) follows from the formula $T^{n(k)} U_{n(k)} = I\mathrm{d}$, which is valid in $\mathcal{D}_2$ thanks to Proposition (3.17) (d). Finally, hypothesis (2) is implied by Proposition 3.17 (c). Indeed, let $k \geq 2$. By triangle inequality we have that

$$\|U_{n(k)} z^k\| \leq \sum_{j=0}^{k-1} \|U_{n(k)}(0, \cdots, 0, z_j^k, 0, \cdots)\| \leq k 2^{-k},$$

expression which tends to 0 as k tends to ∞. Hence, T is a ρ'-hypercyclic operator for any $\rho' > \rho$. Finally, since ρ can be chosen arbitrary close to $2\alpha^{-\mathrm{d}} b$, and then $\rho < \varepsilon$, we finally get that T is an ε-hypercyclic operator. □

Remark 3.18 *Notice that the sequence $(\Delta_k)_k$ used in the construction of the ε-hypercyclic operator T can be replaced by any sequence of integers $(\Delta'_k)_k$ such that $\Delta_k \leq \Delta'_k$, for all $k \in \mathbb{N}$. Moreover, observe that the operator constructed in Theorem 3.13 satisfies the ε-Hypercyclicity Criterion associated to the sequence $(n_k)_k$, where*

$$n_k = (2k - 1)(\mathrm{d} + 1) + \Delta_k + 2 \sum_{j=1}^{k-1} \Delta_j, \ \forall k \geq 1.$$

In order to extend further our result we recall that there exist hypercyclic operators in each infinite dimensional separable Banach space, see [2] and [21]. Further, in [69], León-Saavedra and Montes-Rodríguez showed that the operator constructed in [21] satisfies the Hypercyclicity Criterion. The following proposition states the stability of the epsilon-hypercyclicity property on products of two operators which satisfy the Hypercyclicity Criterion and the Epsilon-Hypercyclicity Criterion respectively. In fact, Proposition 3.19 can be seen as a generalization of the necessity part of the following theorem of Bès and Peris, [22, Theorem 2.3]: $T \in \mathcal{L}(X)$ satisfies the Hypercyclicity Criterion if and only if $T \oplus T \in \mathcal{L}(X \oplus X)$ is hypercyclic.

Proposition 3.19 *Let X and Y be two separable infinite dimensional Banach spaces and let $\varepsilon \in (0, 1)$. Let $T \in \mathcal{L}(X)$ satisfying the Hypercyclicity Criterion. Let $S \in \mathcal{L}(Y)$ satisfying the ε-Hypercyclicity Criterion. Further, assume that the sequences of integers provided by both criteria are the same. Then, the operator $T \oplus S$ is δ-hypercyclic on $X \oplus Y$, for all $\delta > \varepsilon$, where $X \oplus Y$ is equipped with the norm of the maximum.*

Before proceeding with the proof of Proposition 3.19, we recall the following simple fact: If T satisfies the Hypercyclicity Criterion for some sequence $(n(k))_k$, then the operator satisfies this criterion for any subsequence $(n(k(j)))_j$.

Proof. Let $(n(k))_k$ be an increasing sequence of integers, let $\mathcal{D}_1^X, \mathcal{D}_2^X \subset X$ be two dense sets and let $(U_{n(k)})_k$ a sequence of maps, provided by the Hypercyclicity Criterion for T. Let $\mathcal{D}_1^Y \subset Y$ be a dense set, let $\mathcal{D}_2^Y := \{z_k : k \in \mathbb{N}\} \subset Y$ and let $(V_{n(k)})_k$ be a sequence of maps provided by the ε-Hypercyclicity Criterion for S, all of them related to the sequence $(n(k))_k$.

Let $(v_k)_k \subset Y \setminus \{0\}$ be a dense sequence such that $v_\mathrm{i} \neq v_j$ if $\mathrm{i} \neq j$. Summarizing the constructive proof of Theorem 3.2, we can obtain a subsequence of $(z_k)_k$, which we still denote by $(z_k)_k$, a sequence $(y_k)_k \subset \mathcal{D}_1^Y$ and a fast decreasing null sequence $(\eta_k)_k \subset \mathbb{R}^+$ such that:

- $\|v_k - z_k\| \leq \varepsilon \|v_k\|$, for all $k \in \mathbb{N}$,
- $\|y_k\| \leq \eta_k$ for all $k \in \mathbb{N}$,
- $\|S^{n(k)} y_\mathrm{i}\| \leq \eta_k$ for all $\mathrm{i} < k$,
- $\|S^{n(k)} y_k - z_k\| \leq \eta_k$ for all $k \in \mathbb{N}$.

Further, the vector $y = \sum_k y_k$ is δ-Hypercyclic, for all $\delta > \varepsilon$. For each $\mathrm{i} \in \mathbb{N}$, let us consider an increasing sequence $(k(\mathrm{i}, j))_j$ such that $v_{k(\mathrm{i},j)}$ converges to v_i, as j tends to infinity. Inductively, we define the sets $\mathbb{N}_0 := \{k(0, j) : j \in \mathbb{N}\}$ and, for $\mathrm{i} \geq 1$, $\mathbb{N}_\mathrm{i} := \{k(\mathrm{i}, j) : j \in \mathbb{N}\} \setminus \bigcup_{k<\mathrm{i}} \mathbb{N}_k$. Since the sequence $(v_k)_k$ is injective, the sets $\mathbb{N}_\mathrm{i}$ are infinite for each $\mathrm{i} \in \mathbb{N}$. Observe that, by Remark 3.5, the expression

$$\limsup_{t \in \mathbb{N}_k,\ t \to \infty} \|S^{n(t)} y - v_k\| \leq \varepsilon \|v_k\|, \tag{3.3}$$

holds true for each $k \in \mathbb{N}$.

Now, let us construct a hypercyclic vector x of T adapted to y in the following sense: the set $\{T^n x : n \in \mathbb{N}_j\}$ is dense in X for each $j \in \mathbb{N}$. Assume that $\mathcal{D}_2^X$ is a countable set and fix an enumeration of it, i.e. $\mathcal{D}_2^X = \{w_l : l \in \mathbb{N}\}$. Let us define the following total order on $\mathbb{N}^2$: for $(\mathrm{i}, j), (k, l) \in \mathbb{N}^2$, we write

$$(\mathrm{i}, j) \preceq (k, l) \quad \text{if} \quad \mathrm{i} + j < k + l \quad \text{or} \quad \mathrm{i} + j = k + l \wedge \mathrm{i} \geq k.$$

Let $m(0, 0) \in \mathbb{N}_0$ and $x_{0,0} \in \mathcal{D}_1^X$ such that $\|U_{m(0,0)} w_0\| \leq 1$, $\|x_{0,0}\| \leq 1$ and $\|T^{m(0,0)} x_0 - w_0\| \leq 1$. Now, let us proceed by induction. Let $k, l \in \mathbb{N}$. Suppose that we have constructed $m(\mathrm{i}, j)$ and $x_{\mathrm{i},j}$ for all $(\mathrm{i}, j) \prec (k, l)$. Let $m(k, l) \in \mathbb{N}_k$ and $x_{k,l} \in \mathcal{D}_1^X$ such that

- $m(k, l) > m(\mathrm{i}, j)$ for all $(\mathrm{i}, j) \prec (k, l)$.
- $\|T^{m(k,l)} x_{\mathrm{i},j}\| \leq \rho(k, l)$, for all $(\mathrm{i}, j) \prec (k, l)$,
- $\|T^{m(\mathrm{i},j)} x_{k,l}\| \leq 2^{-k-l}$, for all $(\mathrm{i}, j) \prec (k, l)$,
- $\|U_{m(k,l)} w_l\| < \rho(k, l)$,
- $\|x_{(k,l)}\| \leq \rho(k, l)$,
- $\|T^{m(k,l)} x_{(k,l)} - w_l\| \leq \rho(k, l)$,

where $\rho : \mathbb{N}^2 \to \mathbb{R}^+$ is a decreasing function (for $\preceq$) such that $(k+l)^3\rho(k,l)$ tends to 0 whenever (k,l) tends to infinity through the order $\preceq$. Thus, we claim that the vector $x = \sum_{\mathrm{i},j\geq 0} x_{\mathrm{i},j}$ is well defined and a hypercyclic vector for T. Moreover, for each $\mathrm{i} \in \mathbb{N}$, the set

$$\{T^{m(\mathrm{i},j)}x : j \in \mathbb{N}\} \text{ is dense in } X. \tag{3.4}$$

Indeed, the claim follows from the next computation and the fact that $(w_l)_l$ is dense in X. Let $(k,l) \in \mathbb{N}^2$. Then we get

$$\begin{aligned} \|T^{m(k,l)}x - w_l\| &\leq \sum_{(\mathrm{i},j)\prec(k,l)} \|T^{m(k,l)}x_{\mathrm{i},j}\| + \|T^{m(k,l)}x_{k,l} - w_l\| + \sum_{(k,l)\prec(\mathrm{i},j)} \|T^{m(k,l)}x_{\mathrm{i},j}\| \\ &\leq \rho(k,l)\left(\frac{(k+l+1)(k+l+2)}{2} + 1\right) + \sum_{(k,l)\prec(\mathrm{i},j)} 2^{-\mathrm{i}-j}, \end{aligned}$$

where the last expression tends to 0 as k tends to infinity. Observe that, by construction, the sequence $(m(\mathrm{i},j))_j \subset \mathbb{N}_\mathrm{i}$ for all $\mathrm{i} \in \mathbb{N}$.

Let us equip the product space $X \oplus Y$ with the norm of the maximum, i.e. $\|(a,b)\| = \max\{\|a\|_X, \|b\|_Y\}$, for all $(a,b) \in X \oplus Y$. Let us prove that the vector (x,y) is δ-hypercyclic for $T \oplus S$, for all $\delta > \varepsilon$. Combining (3.3) and (3.4), we obtain that

$$\inf_{j\in\mathbb{N}} \|(T \oplus S)^{m(\mathrm{i},j)}(x,y) - (a,v_k)\| \leq \varepsilon\|(0,v_k)\| \leq \varepsilon\|(a,v_k)\|,\ \forall\ a \in X, \forall\ k \in \mathbb{N}.$$

Let $(a,b) \in X \oplus Y \neq (0,0)$, using triangle inequality and the previous inequality we get

$$\inf_{n\in\mathbb{N}} \|(T \oplus S)^n(x,y) - (a,b)\| \leq \inf_{k\in\mathbb{N}} \varepsilon\|(a,v_k)\| + \|(a,b) - (a,v_k)\| \leq \varepsilon\|(a,b)\|.$$

Since $(v_k)_k$ is dense in Y and $(a,b) \neq (0,0)$, by definition of infimum we finally conclude that (x,y) is δ-hypercyclic for each $\delta > \varepsilon$. $\square$

Finally, the next result is an abstract version of the one presented in the introduction.

Theorem 3.20 *Let X be a separable Banach space. Assume that X admits an infinite dimensional complemented subspace V of the form $V = \bigoplus_Y Z$, where Y and Z satisfy the assumptions of Theorem 3.13. Then X admits an ε-hypercyclic operator which is not hypercyclic.*

Proof of Theorem 3.3. Theorem 3.3 is exactly Theorem 3.20 whenever the space V is either $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$, for $p \in [1,\infty)$. $\square$

Proof of Theorem 3.20. Let $\varepsilon > 0$. Let $V = \bigoplus_Y Z$ be the complemented subspace given by the statement. Let W be a topological complement of V on X. Without loss of generality, we assume that W is infinite dimensional. Otherwise, considering $(f_n)_n$ as the basis of Y used in the construction of V, we replace Y by $\overline{\mathrm{span}}(f_n : n \geq 1)$ and W by $W \oplus Z$, which is infinite dimensional. Let us consider T be any bounded hypercyclic operator on W that satisfies the Hypercyclicity Criterion. Let $(n_k)_k$ be a sequence of integers provided by the Hypercyclicity Criterion for T. By Theorem 3.13, there is an ε-hypercyclic operator S on

V which is not hypercyclic. Moreover, by Remark 3.18, we can choose S that satisfies the ε-Hypercyclicity Criterion for a sequence $(m_k)_k$ of the form

$$m_k = 2k(\mathrm{d}+1) + 2\sum_{j=1}^{k-1} \Delta_j + \Delta_k, \qquad \text{for all } k \geq 1.$$

Since, for each $j \in \mathbb{N}$, we can choose Δ_j as large as we want, we can (and shall) assume that the sequence $(m_k)_k$ is a subsequence of $(n_k)_k$. Therefore, since T also satisfies the Hypercyclicity Criterion for the sequence $(m_k)_k$, we can apply Proposition 3.19 to deduce that $S \oplus T$ is δ-hypercyclic on $V \oplus W$, for all $\delta > \varepsilon$. However, $S \oplus T$ is not hypercyclic. Indeed, notice that V and W are complemented spaces and both are invariant for $S \oplus T$. If $S \oplus T$ were hypercyclic, then both restriction, $S \oplus T|_V$ and $S \oplus T|_W$ would be hypercyclic as well. However, $S \oplus T|_V = S$, which is not hypercyclic.

□

3.6 A remark on the epsilon-Hypercyclicity Criterion

One of the main differences between the proposed ε-Hypercyclicity Criterion and the Hypercyclicity Criterion is the necessity of an enumeration of the set $\mathcal{D}_2$. In fact, there are several criteria having a structure similar to the Hypercyclicity Criterion, in which the corresponding set $\mathcal{D}_2$ is not necessarily enumerated. For instance, regarding the criteria for supercyclicity, cyclicity or frequent hypercyciclity stated in [18, Theorem 1.14, Exercise 1.4 and Theorem 6.18] respectively, the conditions at each point of $\mathcal{D}_2$ is identical. However, the next result says that we cannot naively avoid this technicality.

Proposition 3.21 *Let X be an infinite dimensional separable Banach space, let $T \in \mathcal{L}(X)$ and let $\varepsilon \in (0,1)$. Let $\mathcal{D}_1$ be a dense set in X. Let $\mathcal{D}_2$ be a subset of X such that $\mathcal{D}_2 \cap B(x, \varepsilon\|x\|)$ is nonempty for all $x \in X \setminus \{0\}$. Let $(n(k))_k \subset \mathbb{N}$ be an increasing sequence and let $S_{n(k)} : \mathcal{D}_2 \to X$ be a sequence of maps such that:*

1. *$\lim_{k\to\infty} \|T^{n(k)}x\| = 0$ for all $x \in \mathcal{D}_1$,*
2. *$\lim_{k\to\infty} \|S_{n(k)}y\| = 0$, for all $y \in \mathcal{D}_2$,*
3. *$\lim_{k\to\infty} \|T^{n(k)}S_{n(k)}y - y\| = 0$ for all $y \in \mathcal{D}_2$.*

Then, T satisfies the Hypercyclicity criterion.

Let us recall that an operator T on X is called cyclic if there exists a vector $x \in X$ such that $\mathrm{span}(\mathrm{Orb}_T(x))$ is dense in X, see Definition 1.14.

Proof. It is enough to show that $T \oplus T$ is a cyclic operator on $X \oplus X$. Indeed, if $T \oplus T$ is cyclic, then $T \oplus T$ is hypercyclic by [53, Proposition 4.1] and, finally, T satisfies the Hypercyclicity Criterion by [22, Theorem 2.3]. Since the argument is analogous to the one presented in the proof of Proposition 3.19, we present only a sketch of the proof. First, we fix a sequence $(v_k)_k \subset X \setminus \{0\}$ which is dense in X. Let us consider a countable partition $(\mathbb{N}_j)_j$ of $\mathbb{N}$ such that $\mathbb{N}_j$ is an infinite set for each $j \in \mathbb{N}$. By Remark 3.5, we can construct a vector $z_1 \in X$

and an increasing sequence $(k(\mathrm{i}))_{\mathrm{i}} \subset \mathbb{N}$ such that:

$$\limsup_{\mathrm{i}\in\mathbb{N}_j,\ \mathrm{i}\to\infty} \|T^{n(k(\mathrm{i}))} z_1 - v_j\| \le \varepsilon \|v_j\|,\ \forall\, j \in \mathbb{N}.$$

Now, we construct a vector z_2 adapted to z_1 in the following sense:

$$\liminf_{\mathrm{i}\in\mathbb{N}_j,\ \mathrm{i}\to\infty} \|T^{n(k(\mathrm{i}))} z_2 - x\| \le \varepsilon \|x\|,\ \forall x \in X \setminus \{0\}, \forall\, j \in \mathbb{N}.$$

Finally, (z_1, z_2) is an ε-hypercyclic vector for $T \oplus T$ on $X \oplus X$ whenever this space is endowed with the norm of the maximum. Hence, by Proposition 3.23 below, (z_1, z_2) is a cyclic vector for $T \oplus T$, and so T satisfies the Hypercyclicity Criterion. □

3.7 Elementary results

Proposition 3.22 *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be two isomorphic Banach spaces. Assume that, for each $\varepsilon > 0$, X_1 admits an ε-hypercyclic operator which is not hypercyclic. Then X_2 enjoys the same property.*

Proof. Let $T \in \mathcal{L}(X_1, X_2)$ be an isomorphism between X_1 and X_2. Let $\varepsilon \in (0,1)$ and let S be an ε-hypercyclic operator on X_1 which is not hypercyclic. We claim that TST^{-1} is a $\|T\|\|T^{-1}\|\varepsilon$-hypercyclic but not hypercyclic operator on X_2. Indeed, let $x \in X_1$ be an ε-hypercyclic vector for S. Let $y \in X_2$ and $n \in \mathbb{N}$ be an integer such that $\|S^n x - T^{-1}y\|_1 \le \varepsilon \|T^{-1}y\|_1$. Now, we can observer that

$$\|TS^nT^{-1}(Tx) - y\|_2 \le \|T\|\|S^n x - T^{-1}y\|_1 \le \|T\|\|T^{-1}\|\varepsilon\|y\|_2,$$

concluding that Tx is an $\|T\|\|T^{-1}\|\varepsilon$-hypercyclic vector for TST^{-1}. Finally, TST^{-1} cannot be hypercyclic since this property is preserved under conjugacy. □

Proposition 3.23 *Let T be an ε-hypercyclic operator on X, with $\varepsilon \in (0,1)$. Then T is a cyclic operator.*

Proof. It is a direct consequence of the following well-known result: for any closed subspace Y of X, different from X, and any $\delta > 0$ there exists a unit vector $z \in X \setminus Y$ such that $\operatorname{dist}(z, Y) \ge 1 - \delta$. Assume now, towards a contradiction, that T is a non-cyclic operator. Let $Y = \overline{\operatorname{span}}(\operatorname{Orb}_T(x))$, where x is an ε-hypercyclic vector for T. Let $\delta \in (\varepsilon, 1)$ and let $z \in X \setminus Y$ be a unit vector such that $\operatorname{dist}(z, Y) > \delta$. Therefore $B(z, \delta) \cap Y = \emptyset$. Hence, x is not a δ-hypercyclic vector, and thus, x cannot be an ε-hypercyclic vector, which is a contradiction. □

Chapter 4

Asymptotically separated sets and wild operators

In this chapter we continue the study of linear bounded operators whose dynamics is a purely infinite dimensional phenomenon. Precisely, we focus our attention to wild operators, a concept introduced in [8] by J.M. Augé, and we define and explore the asymptotically separated sets. The fundamentals and notation to go throughout this chapter can be found in Chapter 1, Section 1.2 and Section 1.3. In what follows, the whole chapter is mainly based on the work [91].

4.1 Introduction

Let X be a real or complex Banach space and let T be a bounded operator on X. If X is a finite dimensional vector space, the possible dynamics of T in terms of the sequence $(\|T^n x\|)_n$ have been completely determined in Theorem 1.12. The conclusion of the mentioned theorem is the following: if $\dim(X) < \infty$, for any $x \in X$, we have that either the sequence $(\|T^n x\|)_n$ goes to infinity, or goes to 0, or else is bounded away from 0 and infinity.

Let us denote by $A'_T \subset X$ the set of points with unbounded orbits under the action of T. It follows that A'_T is a G_δ-set. Moreover, thanks to the Banach-Steinhaus Theorem theorem, A'_T is in fact either empty or G_δ-dense. Indeed, if there are $x \in X$ and $\varepsilon > 0$ such that $B(x, \varepsilon)$ does not contain points with unbounded orbit, then we realize that the orbit of any vector $z \in B(0, \varepsilon)$ is bounded (it is enough to consider $T^n z = T^n(x + z) - T^n x$, for any $n \in \mathbb{N}$). Thus, the sequence $(\|T^n\|)_n$ is bounded and therefore, A'_T is empty.

In finite dimensional dynamics, a sequence $(\|T^n x\|)_n$ is unbounded if and only if it tends to infinity. Therefore, for any linear operator $T : \mathbb{K}^n \to \mathbb{K}^n$, the set:

$$A_T := \{x \in \mathbb{K}^n : \lim_{n\to\infty} \|T^n x\| = \infty\},$$

is either empty or dense, where $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$. This fact motivated the following conjecture proposed by G. Prăjitură, [81]: Let X be a Banach space and let $T \in \mathcal{L}(X)$.

Then the set A_T is either empty or dense. P. Hájek and R. Smith, in [61], constructed a bounded operator which refutes Prăjitură's conjecture on each infinite dimensional separable Banach space with symmetric basis. Two year later, this result was extended by J.M. Augé in [8] who constructed such an operator on each infinite dimensional separable Banach space. In order to continue, let us fix the set A_T and the recurrent set R_T of a linear operator $T \in \mathcal{L}(X)$ by:

$$A_T := \{x \in X : \lim_{n\to\infty} \|T^n x\| = \infty\} \quad \text{and} \quad R_T := \{x \in X : \liminf_{n\to\infty} \|T^n x - x\| = 0\}.$$

Definition 4.1 *Let T be a linear bounded operator defined on a Banach space X. We say that T is a wild operator (or that T has a wild dynamic) if both sets A_T and R_T have nonempty interior and form a partition of X. The set of wild operators on X is denoted by $\mathcal{W}(X)$. The set of linear bounded operators on X which refutes Prăjitură's conjecture is denoted by $RP(X)$.*

Let us state the main result of [8] which corresponds to the existence of wild operators on infinite dimensional separable Banach spaces.

Theorem 4.2 *[8, Theorem 1.1] Let X be an infinite dimensional separable (real or complex) Banach space. Then there exists a wild operator T on X. Moreover, T can be taken of the form $I + N$, where N is a nuclear operator.*

In section 4.3, we summarize the construction made in [8] but in a slightly generalized way.

Remark 4.3 *For an infinite dimensional separable Banach space X, it follows that $\mathcal{W}(X) \subsetneq RP(X)$. Indeed, let Y be a closed hyperplane of X, and let Z be a 1-dimensional subspace of X such that $X = Y \oplus Z$. Let $T \in \mathcal{W}(Y)$ given by Theorem 4.2. Let $\widetilde{T}$ be the operator on X defined by $\widetilde{T}(y + z) := T(y)$ where $y \in Y$ and $z \in Z$. We have that $A_{\widetilde{T}} = A_T + Z$ and $R_{\widetilde{T}} = R_T + \{0\}$, where the former set is not dense and the last one has empty interior. Therefore, $\widetilde{T} \in RP(X) \setminus \mathcal{W}(X)$.*

The following definition plays an important role in Section 4.3, in which these sets are used to construct operators with interesting dynamical properties.

Definition 4.4 *Let F be a subset of X. We say that F is asymptotically separated in X if there exists a sequence $(g_n)_n \subseteq X^*$ such that:*

(i) For all $x \notin F$, $\lim_{n\to\infty} |g_n(x)| = +\infty$.

(ii) For all $x \in F$, $\liminf_{n\to\infty} |g_n(x)| = 0$.

We say that the sequence $(g_n)_n$ asymptotically separates F.

It is clear that any asymptotically separated set must be a balanced G_δ cone, see forthcoming Proposition 4.5. Also, $\{0\}$ and $\mathbb{K}$ are the only asymptotically separated subsets in $\mathbb{K}$. Indeed, it is enough to consider the sequence $(n)_n$ and $(0)_n$ which asymptotically separate $\{0\}$ and $\mathbb{K}$ respectively.

Let us summarize the contributions of this chapter: we explore in depth the notion of asymptotically separated sets, considering both finite and infinite dimensional spaces. For instance, the following sets are asymptotically separated: in finite dimensional spaces, subsets which are union of linear hyperplanes and such that they are open after removing the origin; in separable Banach spaces, the set $\{0\}$ and each closed subspace. For general Banach spaces, we construct a dense asymptotically separated set with dense complement. In Section 4.3 we write down the proof of Theorem 4.2, stressing the importance of asymptotically separated sets. Thanks to this, we obtain operators T such that the sets A_T and R_T form a partition of the underlying Banach space. For instance, we obtain an operator T such that $\{A_T, R_T\}$ form a partition of X in two dense sets. In this case, both A_T and the set of vectors with unbounded recurrent orbits are dense in X. Moreover, we present several results concerning wild operators. Among them, we study the stability under products, invertibility and norm-approximation, see Theorem 4.42, Theorem 4.45 and Theorem 4.56 respectively.

The outline of this chapter is as follows: In Section 4.2 we study asymptotically separated sets. In Section 4.3, we show how the existence of a non-trivial asymptotically set helps in the construction of a linear bounded operator T on a separable infinite dimensional Banach spaces such that $\{A_T, R_T\}$ forms a partition of X. In Section 4.4, we investigate geometrical aspects of the set of recurrent points of a given operator, R_T. In Section 4.5, we focus our study to spectral properties of wild operators, and we construct a non-invertible wild operator whenever the ambient space admits a symmetric basis. Further, we present an example of a wild operator such that its spectrum is equal to the closed unit disk. Finally, Section 4.6 is devoted to the norm closure of the set $\mathcal{W}(X)$. In particular, whenever the ambient space is an infinite dimensional separable Hilbert space, every unitary operator can be approximated by wild operators.

Notation: For $A, B \subseteq X$, we write $\operatorname{dist}(A, B) = \inf\{\|x - y\| : \ x \in A, \ y \in B\}$.

4.2 Asymptotically separated sets

In this section we explore in depth which kind of sets can be asymptotically separated, see Definition 4.4. First of all, the notion of asymptotically separated set is stable under isomorphism: If $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on a given vector space X, then the asymptotically separated sets of $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ coincide. Therefore, whenever X is a finite dimensional space, we assume that it is endowed with an euclidean norm. Albeit simple, the next proposition delimits the sets which can be asymptotically separated.

Proposition 4.5 *Let $F \subset X$ be an asymptotically separated set. Then F is a balanced G_δ cone.*

Proof. Let $(g_n)_n \subset X^*$ be a sequence that asymptotically separates F. Then

$$F = \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \bigcup_{p \geq m} g_p^{-1} \left(-\frac{1}{n}, \frac{1}{n} \right),$$

which is clearly a G_δ set. Moreover, since both properties defining asymptotically separated sets are stable under non-zero scalar multiplication, F is a balanced cone. $\square$

Observe that both properties defining asymptotically separated sets are stable under non-zero scalar multiplication. Thus, to show that a given set $F \subset X$ is asymptotically separated in X we only need to check the property in $F \cap S_X$ and $S_X \setminus F$. In [8] it is observed that the closed set $\{z = (z_1, z_2) \in \mathbb{K}^2 : |z_1| \leq |z_2|\}$ is asymptotically separated. The following proposition shows that if $\mathcal{U}$ is the open set $\{z = (z_1, z_2) \in \mathbb{K}^2 : |z_1| < |z_2|\}$, then $\mathcal{U} \cup \{(0,0)\}$ is asymptotically separated as well.

Proposition 4.6 *Let $\mathcal{U} \subset \mathbb{K}^{\mathrm{d}}$ be a nonempty set which is union of linear hyperplanes and let $\vec{0} \in \mathbb{K}^{\mathrm{d}}$ be the origin. If $\mathcal{U} \setminus \{\vec{0}\}$ is open, then $\mathcal{U}$ is asymptotically separated.*

Proof. If $\mathcal{U} = \mathbb{K}^{\mathrm{d}}$, it is enough to consider the sequence $(g_n)_n \subset (\mathbb{K}^{\mathrm{d}})^*$ defined by $g_n = 0$ for all $n \in \mathbb{N}$. Thus, we assume that $\mathcal{U} \subsetneq \mathbb{K}^{\mathrm{d}}$. Let us set $S = S_{\mathbb{K}^{\mathrm{d}}}$. For $g \in (\mathbb{K}^{\mathrm{d}})^*$, we define the number

$$\alpha(g) = \mathrm{dist}(\ker(g) \cap S, \partial\mathcal{U} \cap S).$$

Notice that if $\ker(g) \subset \mathcal{U}$, then $\ker(g) \cap S$ is a compact set contained in $\mathcal{U} \cap S$. Since $\mathcal{U} \cap S$ is open in S and $\partial\mathcal{U} \cap S \neq \emptyset$, we get that $\alpha(g) > 0$. Let us define

$$\mathcal{U}_n := \{x \in \mathcal{U} : \ \exists g \in (\mathbb{K}^{\mathrm{d}})^*, x \in \ker(g) \subset \mathcal{U}, \ \alpha(g) > 2^{-n}\}.$$

It is clear that for k large enough, $\mathcal{U}_k \neq \emptyset$ and $\bigcup_n \mathcal{U}_n = \mathcal{U}$. Since $\mathcal{U}_n \cap S$ is totally bounded, we can find $E_n \subseteq \mathcal{U}_n \cap S$ a finite 4^{-n}-net of $\mathcal{U}_n \cap S$. Let $G_n \subseteq (\mathbb{K}^{\mathrm{d}})^*$ be a finite set such that each $g \in G_n$ satisfies that $\|g\| = 3^n$ and $\ker(g) \subset \mathcal{U}_n$, and that for each $x \in E_n$, there is $g_x \in G_n$ satisfying $g_x(x) = 0$. Notice that, if $g \in G_n$, then $\alpha(g) > 2^{-n}$. Indeed, this is due to the fact $\ker(g) \subset \mathcal{U}_n$.

Let $(g_n)_n$ be a sequence generated by an enumeration of $\bigcup_n G_n$, where we first find the elements of G_1, then the elements of G_2, and so on. We prove that this sequence asymptotically separates $\mathcal{U}$.

Let $x \in S$. If $x \in \mathcal{U}$, there exist $f \in (\mathbb{K}^{\mathrm{d}})^*$ and $N \in \mathbb{N}$ such that $x \in \ker(f)$ and $\alpha(f) > 2^{-N}$. Therefore, for each $n \geq N$, there exists $g \in G_n$ such that $\mathrm{d}(x, \ker(g)) \leq 4^{-n}$. Thus, recalling that $\|g\| = 3^n$ we get

$$|g(x)| = \|g\|\mathrm{dist}(x, \ker(g)) \leq \frac{3^n}{4^n},$$

which tends to 0 as n tends to infinity. On the other hand, if $x \notin \mathcal{U}$ and $g \in G_n$, we know that $x \notin \mathcal{U} \supseteq \mathcal{U}_n \supseteq \ker(g) \cap S$. Therefore, $\mathrm{dist}(x, \ker(g) \cap S) \geq \alpha(g) > 2^{-n}$. Finally, we get that

$$|g(x)| = \|g\|\mathrm{dist}(x, \ker(g)) \geq 3^n \frac{\mathrm{dist}(x, \ker(g) \cap S)}{\sqrt{2}} \geq \frac{3^n}{2^n\sqrt{2}},$$

which tends to ∞ as n tends to infinity. $\square$

In the previous proof, the euclidean structure of $\mathbb{K}^n$ is only used to obtain the last bound. We continue with a useful example.

Corollary 4.7 *Let $X = \mathbb{K}^{\mathrm{d}}$, where $\mathrm{d} \geq 2$. There exists an asymptotically separated set $\mathcal{U} \subset X$ with nowhere dense, nonempty complement such that $\mathcal{U} \setminus \{0\}$ is open.*

Proof. Let $(\mathrm{e}_k)_{k=1}^{\mathrm{d}} \subset \mathbb{K}^{\mathrm{d}}$ be a basis and let $(\mathrm{e}_k^*)_{k=1}^{\mathrm{d}} \subset (\mathbb{K}^{\mathrm{d}})^*$ be its associated coordinate functionals. Let $\mathcal{U} \subseteq \mathbb{K}^{\mathrm{d}}$ defined by

$$\mathcal{U} := (\mathbb{K}^{\mathrm{d}} \setminus \{x \in \mathbb{K}^{\mathrm{d}} : \mathrm{e}_1^*(x) \neq 0 \text{ and } \mathrm{e}_k^*(x) = 0, \forall k \neq 1\}).$$

Clearly, the complement of $\mathcal{U}$ is nowhere dense in X. We claim that $\mathcal{U}$ is asymptotically separated. Let us show that $\mathcal{U}$ is union of linear hyperplanes. Let $u \in \mathcal{U} \setminus \{0\}$. By definition of $\mathcal{U}$, there is $j \neq 1$ such that $\mathrm{e}_j^*(u) \neq 0$. Let us consider $x^* \in \mathbb{K}^*$ defined by

$$x^* := \mathrm{e}_1^*(u)\mathrm{e}_j^* - \mathrm{e}_j^*(u)\mathrm{e}_1^*.$$

An easy computation gives to us that $u \in \ker(x^*) \subset \mathcal{U}$. Thus, $\mathcal{U}$ is union of linear hyperplanes. Since $\mathcal{U} \setminus \{0\}$ is an open set, Proposition 4.6 finishes the proof. □

For closed sets, we have the following result.

Proposition 4.8 *Let F be a nonempty closed subset of a Banach space X which is union of linear hyperplanes. Suppose that there exists $Y \subset X^*$ such that $F \cap B_X$ is $\sigma(X, Y)$-compact in X and, for all $x \in F$, there exists $x^* \in Y$ such that $x \in \ker(x^*) \subseteq F$. Then F is asymptotically separated.*

Proof. Let us consider the set $H := \{x^* \in Y^* : \ker(x^*) \subseteq F\}$. Notice that, for every $\alpha,\ \varepsilon > 0$, the family of sets $\{x^{*-1}((-\varepsilon, \varepsilon)) : x^* \in H,\ \|x^*\| = \alpha\}$ is a $\sigma(X, Y)$-open covering of $F \cap B_X$. Let $(\varepsilon_n)_n \subseteq \mathbb{R}_+$ be a decreasing sequence converging to 0. Let $n \in \mathbb{N}$. By compactness, there is a finite sequence $(g_{n,k})_k \subseteq H$ such that $\{g_{n,k}^{-1}(-\varepsilon_n, \varepsilon_n) : k\}$ is a finite open covering of $F \cap B_X$ and $\|g_{n,k}\| = \varepsilon_n^{-1/2}$ for all k. Let us denote by $(g_m)_m$ the sequence obtained by the concatenation of the finite sequences $(g_{1,k})_k$, $(g_{2,k})_k$ and so on. We claim that F is asymptotically separated by (g_m). Let $x \in S_X$. If $x \in F$, then for each $n \in \mathbb{N}$, there is $k(n) \in \mathbb{N}$, depending on x, such that $|g_{n,k(n)}(x)| \leq \varepsilon_n$. Hence, $\liminf |g_m(x)| = 0$. On the other hand, if $x \in S_X \setminus F$, we have that for all $n \in \mathbb{N}$:

$$|g_{n,k}(x)| = \|g_{n,k}\| \mathrm{dist}(x, \ker(g_{n,k})) \geq \varepsilon_n^{-\frac{1}{2}} \mathrm{dist}(x, F),$$

which tends to infinity as n tends to infinity. □

Remark 4.9 *As a consequence of the proof of Proposition 4.8 and the Banach-Alaoglu Theorem, we deduce that for each dual space X, there exists a sequence $(x_n^*)_n \subset X^*$ such that $\lim_n \|x_n^*\| = \infty$ and which asymptotically separates X. Indeed, it is enough to consider Y as the canonical injection of the predual of X into X^*.*

Since weak* and norm topology coincide for finite dimensional spaces, we obtain a key Proposition of [8].

Corollary 4.10 *[8, Proposition 2.1] Let F be a nonempty closed subset of $\mathbb{K}^{\mathrm{d}}$ which is union of linear hyperplanes. Then F is asymptotically separated.*

Corollary 4.11 *Let X be a Banach space of dimension at least 2. Then there exists a sequence $(x_n^*) \subset X^*$ which asymptotically separates X and satisfies $\lim_n \|x_n^*\| = \infty$.*

Proof. Let Y be a two dimensional subspace of X and let $P : X \to X$ be a bounded projection onto Y. By Remark 4.9, there exists a sequence $(y_n^*) \subset Y^*$ such that asymptotically separates Y and the sequence $(\|y_n^*\|)_n$ goes to infinity. For each $n \in \mathbb{N}$, let $x_n^* \in X^*$ be any extension of y_n^* to the space X. Thus, the sequence $(x_n^* \circ P)_n$ asymptotically separates X and the sequence of norms diverges to infinity. □

Another result about closed asymptotically separated sets in general Banach spaces can be found in [9, Proposition 4.6.4].

Now, we present two examples of asymptotically separated set which are not enclosed by the previous results: the set containing only the origin is asymptotically separated in any separable Banach space and, secondly, the existence of a dense asymptotically separated set which has dense complement in any Banach space of dimension at least 2. Let us start with the finite dimensional case.

Proposition 4.12 *Let $X = \mathbb{K}^{\mathrm{d}}$, with $\mathrm{d} \geq 1$, and let $\vec{0}$ be the origin of X. Then the set $F = \{\vec{0}\}$ is asymptotically separated.*

Proof. If $X = \mathbb{K}$, then it is enough to consider the sequence $(n)_n$ which asymptotically separates $\{\vec{0}\}$. Let us assume that $\mathrm{d} \geq 2$. Let us consider the sequence $(g_n^*)_n \subseteq X^*$ defined by $g_n^* = \left(n^{\mathrm{d}}, n^{\mathrm{d}-1}, ..., n\right)$ for all $n \in \mathbb{N}$. We claim that the sequence $(g_n^*)_n$ asymptotically separates F. To this end, let $x = (x_1, ..., x_{\mathrm{d}}) \in S_X$. Let $j \in \{1, ..., \mathrm{d}\}$ such that $x_j \neq 0$ and $x_{\mathrm{i}} = 0$ for all $\mathrm{i} < j$. If $j = \mathrm{d}$, then the sequence $(|g_n(x)|)_n = (n|x_{\mathrm{d}}|)_n$, which tends to infinity. Thus, we assume that $1 \leq j < \mathrm{d}$. Since all the coordinates of x are bounded by 1, there exists $N \in \mathbb{N}$ such that

$$\frac{|x_j|}{\mathrm{d}n^{j-1}} \geq \frac{|x_k| + 1}{n^{k-1}}, \ \forall n \geq N, \ \forall k > j.$$

Now, for $n \geq N$, we can compute

$$\begin{aligned} |g_n(x)| &= n^{\mathrm{d}} \left| \sum_{k=j}^{\mathrm{d}} \frac{x_k}{n^{k-1}} \right| \geq n^{\mathrm{d}} \left(\frac{|x_j|}{n^{j-1}} - \sum_{k=j+1}^{\mathrm{d}} \frac{|x_k|}{n^{k-1}} \right) \\ &\geq n^{\mathrm{d}} \left(\sum_{k=j+1}^{\mathrm{d}} \frac{|x_j|}{\mathrm{d}n^{j-1}} - \frac{|x_k|}{n^{k-1}} \right) \geq \sum_{k=j+1}^{\mathrm{d}} n^{\mathrm{d}-k+1} \end{aligned}$$

expression which tends to infinity as n tends to infinity, finishing the proof. □

Before proceeding with the next proposition, we recall a result of Ovsepian and Pełczyński, [75]: Each separable Banach space admits a normalized bounded M-basis, see Theorem 1.11.

Proposition 4.13 *Let X be a separable real or complex Banach space and let $\vec{0}$ be the origin of X. Then the set $\{\vec{0}\}$ is asymptotically separated.*

Proof. If X is a finite-dimensional space, the result follows from Proposition 4.12. Thus, we assume that X is infinite dimensional. Let $(\mathrm{e}_n)_n \subseteq X$ be a normalized bounded M-basis of X with $(\mathrm{e}_n)^* \subseteq X^*$ its associated biorthogonal system. Let $C \geq 1$ such that $\sup \|\mathrm{e}_n^*\| \leq C$. For each $n \in \mathbb{N}$, let $(\alpha_{n,\mathrm{i}})_{\mathrm{i}} \subset \mathbb{R}$ be a sequence of positive real numbers such that:

1. $\lim\limits_{n\to\infty} \alpha_{n,1} = 0$.
2. $\dfrac{\alpha_{n,\mathrm{i}}}{n} > \sum\limits_{j=\mathrm{i}+1}^{\infty} \alpha_{n,j}$, for all $1 \leq \mathrm{i} \leq n$.

Now, for $n \in \mathbb{N}$, let us consider

$$x_n^* = \mathrm{e}_1^* + \sum_{\mathrm{i}=1}^{\infty} \alpha_{n,\mathrm{i}} \mathrm{e}_{\mathrm{i}+1}^* \in X^*,$$

which is well defined since the series is absolutely convergent. Let $\beta_n = n/\alpha_{n,n}$ and let $g_n = \beta_n x_n^*$ for each $n \in \mathbb{N}$. We claim that the sequence $(g_n)_n$ asymptotically separates $\{0\}$. Indeed, let $x \in S_X$. Since $\overline{\mathrm{span}}^{w^*}(\mathrm{e}_n^* : n \in \mathbb{N}) = X^*$ we know that $\mathrm{e}_n^*(x) \neq 0$ for some $n \in \mathbb{N}$. Let $j = \min\{n \in \mathbb{N} : \ \mathrm{e}_n^*(x) \neq 0\}$. If $j = 1$, then we get

$$|g_n(x)| \geq \beta_n \left(|\mathrm{e}_1^*(x)| - C\|x\| \sum_{\mathrm{i}=1}^{\infty} \alpha_{n,\mathrm{i}} \right) \geq \beta_n \left(|\mathrm{e}_1^*(x)| - \frac{(n+1)C}{n} \alpha_{n,1} \|x\| \right),$$

and, applying property (1) and using the fact that $(\beta_n)_n$ tends to ∞, we see that the sequence $(|g_n(x)|)_n$ tends to infinity as n tends to infinity. If $2 \leq j \leq n$, then:

$$\begin{aligned} |g_n(x)| &\geq \beta_n \left(\alpha_{n,j-1} |\mathrm{e}_j^*(x)| - \sum_{k \geq j+1} \alpha_{n,k-1} |\mathrm{e}_k^*(x)| \right) \\ &\geq \beta_n \left(\alpha_{n,j-1} |\mathrm{e}_j^*(x)| - C\|x\| \frac{\alpha_{n,j-1}}{n} \right) \\ &\geq n \frac{\alpha_{n,j-1}}{\alpha_{n,n}} \left(|\mathrm{e}_j^*(x)| - \frac{C\|x\|}{n} \right). \end{aligned}$$

where we have used triangle inequality in the first line and the boundedness of the biorthogonal system and property (2) in the second one. Finally, since $\alpha_{n,j-1}/\alpha_{n,n} \geq 1$ for all $n \geq j$, we deduce that the sequence $(|g_n(x)|)_n$ tends to infinity. □

Corollary 4.14 *Let X be a separable real or complex Banach space and let Y be a closed subspace of X. Then Y is asymptotically separated. In particular, every finite dimensional subspace of X is asymptotically separated.*

Proof. Let $Q : X \to X/Y$ be the canonical quotient linear map. By Proposition 4.13, let $(g_n)_n \in (X/Y)^*$ be a sequence that asymptotically separates the origin in X/Y. It follows that the sequence $(g_n \circ Q)_n \subset X^*$ asymptotically separates Y. □

We continue with the proof of the existence of a dense asymptotically separated set with dense complement in each Banach space of dimension at least 2. In fact, this result is a consequence of the following theorem.

Theorem 4.15 *Let $X = \mathbb{K}^2$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $(x_n^*)_n$ be a sequence in X^* and let $\mathcal{U} = X \setminus \bigcup_n \ker(x_n^*)$. Then the set $\mathcal{U} \cup \{0\}$ is asymptotically separated.*

Corollary 4.16 *Let X be a real or complex Banach space of dimension at least 2. Then there exists an asymptotically separated set $\mathcal{U} \subseteq X$ such that $\mathcal{U}$ and $\mathcal{U}^c$ are dense for the norm topology.*

Proof. Let $P : X \to X$ be a bounded projection onto any two dimensional subspace Y of X. Let $(y_n^*)_n \subseteq Y^*$ be a sequence of norm one linear functions, dense in S_{Y^*}. Then, by Theorem 4.15, the set

$$\mathcal{O} = \left(Y \setminus \bigcup_n \ker(y_n^*) \right) \cup \{0\}$$

is asymptotically separated by some sequence $(g_n)_n \subseteq Y^*$. It is clear that the set $\mathcal{U} = P^{-1}(\mathcal{O})$ is asymptotically separated in X by the sequence $(g_n \circ P)_n$. Moreover, by the Open mapping theorem, $\mathcal{U}$ and $X \setminus \mathcal{U}$ are dense in X since $\mathcal{O}$ and $Y \setminus \mathcal{O}$ are dense in Y. $\square$

To prove Theorem 4.15, let us start with the following elementary lemma.

Lemma 4.17 *Let $X = \mathbb{K}^2$. Let $x^* \in X^*$ be a norm one linear functional. Let $\alpha \in (0,1)$ and $G = X \setminus \bigcup\{\ker(y^*) : \ \|y^*\| = 1, \ \|x^* - y^*\| \le \alpha\}$. Then, $\mathrm{dist}(\ker(x^*) \cap S_X, G \cap S_X) = \alpha$. Moreover, if $\mathrm{dist}(y, \ker(x^*) \cap S_X) > \alpha$ and $\|y\| = 1$, then $y \in G$.*

Proof. Let us denote $S = S_X$. Let $z \in \ker(x^*) \cap S$ and let $y \in G \cap S$. Let us set $z = (z_1, z_2)$ and $y = (y_1, y_2)$. Let $z^*, y^* \in X^*$ be the linear functionals defined by $z^* = (-\overline{z}_2, \overline{z}_1)$ and $y^* = (-\overline{y}_2, \overline{y}_1)$. In the case whenever $\mathbb{K} = \mathbb{R}$, $\overline{v} = v$ for all $v \in \mathbb{K}$. It is clear that $\|z^*\| = \|y^*\| = 1$, $\|z - y\| = \|z^* - y^*\|$, $z \in \ker(z^*)$ and $y \in \ker(y^*)$. Since X is a two dimensional space, there exists $\lambda \in \mathbb{K}$, with $|\lambda| = 1$, such that $\lambda z^* = x^*$. Recalling that $y \in G$ and that $y \in \ker(\lambda y^*)$, we obtain that

$$\|z - y\| = \|z^* - y^*\| = \|x^* - \lambda y^*\| \ge \alpha,$$

showing that $\mathrm{dist}(z, G) \ge \alpha$. Since z is an arbitrary point in $\ker(x^*) \cap S$, $\mathrm{dist}(\ker(x^*) \cap S, G) \ge \alpha$. Let $\varepsilon > 0$ and $y \in S$ such that $\mathrm{dist}(y, \ker(x^*) \cap S) = \alpha + \varepsilon$. Let $z \in \ker(x^*) \cap S_X$. Then $\|\lambda z - y\| \ge \alpha + \varepsilon$, for all $|\lambda| = 1$. It follows that $\|\lambda z - \rho y\| \ge \alpha + \varepsilon$ for all $|\lambda| = |\rho| = 1$. Proceeding as before, since X is a two dimensional euclidean space, the last computation gives us that $\|x^* - y^*\| \ge \alpha + \varepsilon$, for all norm one linear functional $y^* \in Y^*$ such that $y \in \ker(y^*)$. Finally, by definition of G, we conclude that $y \in G$.

$\square$

Proof of Theorem 4.15. Without loss of generality, we assume that X is endowed with an Euclidean norm. Let us fix $S = S_X$. If $\mathcal{U} \cup \{0\} = \{0\}$, the result follows from Proposition 4.13. Let us assume that $\mathcal{U} \cup \{0\} \neq \{0\}$. Observe that, since $\dim(X) = 2$, for any two non-zero linear functionals, $x^*, y^* \in X^*$, we have that $\ker(x^*) = \ker(y^*)$ or $\ker(x^*) \cap \ker(y^*) = \{0\}$. Thus, we deduce that $\mathcal{U} \cup \{0\}$ can be written as union of linear hyperplanes. Therefore, if the set $\{\ker(x_n^*) : n \in \mathbb{N}\}$ is finite and $\mathcal{U} \neq \emptyset$, the result follows from Proposition 4.6. So, we assume that $(x_n^*)_n$ is a sequence of linear functionals of norm one such that $x_n^* \neq \lambda x_m^*$, for all $n \neq m$ and all $|\lambda| = 1$, i.e., $\ker(x_n^*) \neq \ker(x_m^*)$ for all $n \neq m$. Observe that, in this case, $\mathcal{U} \neq \emptyset$

Let us define $\alpha_1 = 2$ and, for $n \geq 2$

$$\alpha_n = \min\{\mathrm{dist}(\ker(x_j^*) \cap S, \ker(x_k^*) \cap S) : \ j, k \leq n, \ j \neq k\},$$

which is a strictly positive real number. By compactness of the unit sphere, we have that $(\alpha_n)_n$ is a decreasing sequence which converges to 0. Let us consider three sequences of positive numbers, $(\beta_n)_n$, $(\sigma_n)_n$, $(\gamma_n)_n \subseteq \mathbb{R}$, which will be specified later on. For each n, we consider

$$G_n = X \setminus \bigcup \left\{ \ker(x^*) : \ \|x^*\| = 1, \ \mathrm{dist}(x^*, \{x_k^* : k \leq n\}) \leq \frac{\alpha_n}{\beta_n} \right\}.$$

Let $(x_{n,k})_k$ be a finite σ_n-net of $G_n \cap S$. For each $x_{n,k}$, consider $g_{n,k} \in X^*$ such that $g_{n,k}(x_{n,k}) = 0$ and that $\|g_{n,k}\| = \gamma_n > 0$. Observe that $\ker(g_{n,k}) = \mathbb{K}x_{n,k} \subseteq G_n \cup \{\vec{0}\}$. Indeed, this is because $G_n \cup \{\vec{0}\}$ is union of linear hyperplanes and that, for each vector $x \neq \vec{0}$, there is only one linear hyperplane containing it (this argument holds true only in two dimensional spaces).

We claim that the sequence $(g_m)_m$ obtained by the concatenation of $(g_{1,k})_k$, $(g_{2,k})_k$, and so on, asymptotically separates $\mathcal{U}$ if the sequences $(\beta_n)_n$, $(\sigma_n)_n$ and $(\gamma_n)_n$ are correctly chosen. To this end, let $x \in S \setminus \mathcal{U}$. Then, there exists $N \in \mathbb{N}$ such that $x \in \ker(x_N^*)$. For every $n \geq N$, applying Lemma 4.17, we get that:

$$|g_{n,k}(x)| = \|g_{n,k}\| \mathrm{dist}(x, \ker(g_{n,k})) \geq \|g_{n,k}\| \frac{\mathrm{dist}(x, \ker(g_{n,k}) \cap S)}{\sqrt{2}} \geq \gamma_n \frac{\alpha_n}{\beta_n \sqrt{2}}, \tag{4.1}$$

which must tend to infinity when n tends to infinity. On the other hand, let $x \in \mathcal{U} \cap S$. Let us define

$$\rho_n(x) = \frac{\beta_n}{\alpha_n} \mathrm{dist}(x, \{y \in S : \ \exists k \leq n, \ x_k^*(y) = 0\}).$$

If $\limsup_n \rho_n(x) > 1$, then $x \in G_n$ for infinitely many $n \in \mathbb{N}$. Indeed, let $\varepsilon > 0$ and $(n_l)_l \subseteq \mathbb{N}$ be an increasing sequence such that $\rho_{n_l}(x) \geq 1 + \varepsilon$ for all $l \in \mathbb{N}$. Then, we get that

$$\mathrm{dist}(x, \{y \in S : \ \exists k \leq n_l, \ x_k^*(y) = 0\}) \geq (1+\varepsilon) \frac{\alpha_{n_l}}{\beta_{n_l}}. \text{ for all } l \in \mathbb{N}$$

Applying Lemma 4.17, we deduce that $x \in G_{n_l}$ for all $l \in \mathbb{N}$. Now, let us consider n such that $x \in G_n$. Picking $g_{n,k}$ such that $\mathrm{dist}(x, \ker(g_{n,k})) \leq \sigma_n$, we have that

$$|g_{n,k}(x)| = \|g_{n,k}\| \mathrm{dist}(x, \ker(g_{n,k})) \leq \gamma_n \sigma_n, \tag{4.2}$$

which must tend to 0 as n tends to infinity. Now, let us assume that $\limsup \rho_n(x) \leq 1$. If we suppose that

$$\frac{\alpha_n}{\beta_n} \leq \frac{\alpha_{n+1}}{6}, \tag{4.3}$$

then, by the triangle inequality and Lemma 4.17 we can show that eventually there is exactly one $\hat{x}_n^* \in \{x_k^* : k \leq n\}$ such that $\frac{\beta_n}{\alpha_n}\mathrm{dist}(x, \ker \hat{x}_n^* \cap S) = \rho_n(x)$. Moreover, the sequence $(\hat{x}_n^*)_n$ is stationary. Indeed, let and $N \in \mathbb{N}$ such that $\rho_n(x) < 2$ for all $n \geq N$. Let $n \geq N$. Then, we have that $\mathrm{dist}(x, \ker \hat{x}_n^* \cap S) \leq 2\alpha_n/\beta_n \leq \alpha_{n+1}/3$. Since $\alpha_{n+1} \leq \alpha_n$, if there are two linear functionals $\hat{x}_{n,1}^*$ and $\hat{x}_{n,2}^*$ in $\{x_k^* : k \leq n\}$ such that $\mathrm{dist}(x, \ker \hat{x}_{n,\mathrm{i}}^* \cap S) \leq 2\alpha_n/\beta_n$ for $\mathrm{i} = 1, 2$, then the triangle inequality implies that

$$\begin{aligned}\mathrm{dist}(\ker \hat{x}_{n,1}^* \cap S, \ker \hat{x}_{n,2}^* \cap S) &\leq \mathrm{dist}(x, \ker \hat{x}_{n,1}^* \cap S) + \mathrm{dist}(x, \ker \hat{x}_{n,2}^* \cap S)\\ &\leq 2\frac{\alpha_{n+1}}{3} < \alpha_n.\end{aligned}$$

Thus, by definition of α_n, $\hat{x}_{n,1}^* = \hat{x}_{n,1}^*$ and the choice of $\hat{x}_n^*$ is unique. Also, observe that since $(\alpha_n)_n$ converges to 0 as n tends to infinity, the sequence $(\mathrm{dist}(x, \ker \hat{x}_n^*))_n$ converges to 0 as well. Moreover, we deduce that $\hat{x}_n^* = \hat{x}_{n+1}^*$ for all $n \geq N$. Indeed, recalling that $\mathrm{dist}(x, \ker(\hat{x}_n^*) \cap S) \leq \alpha_{n+1}/3$ for all $n \geq N$, we have that

$$\mathrm{dist}(\ker(\hat{x}_n^*) \cap S, \ker(\hat{x}_{n+1}^*) \cap S) \leq \frac{\alpha_{n+1}}{3} + \frac{\alpha_{n+2}}{3} < \alpha_{n+1}, \text{ for all } n \geq N.$$

Therefore, by definition of α_{n+1}, $\hat{x}_n^* = \hat{x}_{n+1}^*$ for all $n \geq N$.

Now, since the sequence $(\mathrm{dist}(x, \ker(\hat{x}_n^*)))_n$ tends to 0, we conclude that $x \in \ker(\hat{x}_N^*) \subset S \setminus \mathcal{U}$. This contradicts the fact that $x \in \mathcal{U}$.

Summarizing, we have that the sequence $(g_n)_n$ asymptotically separates $\mathcal{U} \cup \{0\}$ whenever the conditions (4.1), (4.2) and (4.3) are satisfied, namely:

$$\lim_{n\to\infty} \gamma_n \frac{\alpha_n}{\beta_n} = \infty, \qquad \lim_{n\to\infty} \gamma_n \sigma_n = 0 \qquad \text{and} \qquad 6\alpha_n \leq \beta_n \alpha_{n+1}, \forall n \in \mathbb{N}.$$

So, if we consider $\beta_n = 6\alpha_n/\alpha_{n+1}$, $\gamma_n = n/\alpha_{n+1}$ and $\sigma_n = \alpha_{n+1}/n^2$, the three conditions are satisfied and the result is achieved. □

The final aim of this section is to try to advance in the following question: Let $\mathcal{U} \subset X$ be a G_δ set which is union of linear hyperplanes. Is it true that $\mathcal{U}$ is asymptotically separated? In the following, we present two result which corresponds to a partial positive result in $\mathbb{R}^2$ and a kind of negative result in $\ell^1(\mathbb{N})$.

Let us start with the case whenever the ambient space is $\mathbb{R}^2$. Before starting, let us introduce some important notation.

Notation 4.18 *For each $n \in \mathbb{N}$, let $\mathcal{U}_n \subset \mathbb{R}^2$ be a set union of linear hyperplanes such that $\mathcal{U}_n \setminus \{0\}$ is open. Let S be the unit sphere of $\mathbb{R}^2$ equipped with an euclidean norm. Assume that $(\mathcal{U}_n)_n$ is decreasing in the sense of inclusion. Then, for $n \in \mathbb{N}$, we denote by*

1. *$\{\mathcal{U}_{n,\mathrm{i}} : 1 \leq \mathrm{i} \leq m_n\}$, an enumeration of the connected components of $\mathcal{U}_n \cap S$, and by*
2. *$\{\mathcal{U}_{n,\mathrm{i},k} : 1 \leq k \leq m_{n,\mathrm{i}}\}$, an enumeration of the connected components of $\mathcal{U}_{n+1} \cap S$ contained in $\mathcal{U}_{n,\mathrm{i}}$, for each $1 \leq \mathrm{i} \leq m_n$.*

Now, we can state our result.

Theorem 4.19 *Let $(\mathcal{U}_n)_n$ be a decreasing sequence of subsets of $\mathbb{R}^2$ such that, for each $n \in \mathbb{N}$, $\mathcal{U}_n$ is a nonempty union of linear hyperplanes such that $\mathcal{U}_n \setminus \{0\}$ is open. According to Notation 4.18, assume that $m_1 \in \mathbb{N} \cup \{\infty\}$ and that $m_{n,\mathrm{i}}$ is finite for all $n \in \mathbb{N}$ and $1 \leq \mathrm{i} \leq m_n$. Then $\mathcal{U} = \bigcap_n \mathcal{U}_n$ is asymptotically separated.*

The hypothesis of Theorem 4.19 establishes that, for each $n \in \mathbb{N}$ and $1 \leq \mathrm{i} \leq m_n$, $\mathcal{U}_{n,\mathrm{i}}$ contains only finitely many connected components of $\mathcal{U}_{n+1} \cap S$, i.e., $\mathcal{U}_{n,\mathrm{i}} \cap (\mathcal{U}_{n+1} \cap S)$ is a finite union of arcs. On the other hand, for $n \in \mathbb{N}$, the set $\mathcal{U}_n \cap S$ may have infinitely many connected components. Theorem 4.19 allows us to recover all previous results whenever the ambient space is $\mathbb{R}^2$. Indeed, sets $\mathcal{U} \subset \mathbb{R}^2$ union of linear hyperplanes which are closed, or $\mathcal{U} \setminus \{0\}$ open, or $\mathcal{U} = \{0\}$, or the set stated in Theorem 4.15 satisfy the hypotheses of Theorem 4.19.

Before proving Theorem 4.19, we need some extra definitions. Notice that, in Notation 4.18, the sets $\mathcal{U}_{n,\mathrm{i}}$ and $\mathcal{U}_{n,\mathrm{i},k}$ are open arcs in S, so they can be identified (and we do) as open intervals in $(\mathbb{R}, \pmod{2\pi})$. In order to simplify notation, an open arc $I = \{(\cos(t), \sin(t)) : a < t < b\} \subsetneq S$, with $a, b \in \mathbb{R}$ and $a < b < a + 2\pi$, will be denoted by $I = (a, b)$. We write the length of I by $\lambda(I) = |b - a| \in [0, 2\pi]$. For $n \in \mathbb{N}$, we say that the n-(center, left, right) contraction of $I = (a, b)$ is the arc defined by:

$$\begin{aligned} F_n(I) &= \big((1-2^{-n})a + 2^{-n}b, 2^{-n}a + (1-2^{-n})b\big)\,, \\ F_{l,n}(I) &= \big((1-2^{-n})a + 2^{-n}b, b\big)\,, \\ F_{r,n}(I) &= \big(a, 2^{-n}a + (1-2^{-n})b\big)\,, \end{aligned}$$

respectively. It is clear that this definition does not depend on the representation of I whenever $I \neq S$. Finally, for an open arc $I \subsetneq S$, we define $l(I), r(I) \in \mathbb{R}$ as a selection such that $I = (l(I), r(I))$. i.e. $l(I)$ and $r(I)$ are the left and right extremity of I respectively.

Proof of Theorem 4.19. In the first step of this proof, we construct an auxiliary sequence $(A_m)_m$ of subsets of S such that:

i) For each $m \in \mathbb{N}$, A_m is compactly contained in $\mathcal{U}_m$,

ii) for each $m \in \mathbb{N}$, $\mathrm{dist}(A_m, \partial\mathcal{U}_m \cap S) > 0$ and

iii) the sequence $(A_m)_m$ satisfies

$$\mathcal{U} \cap S = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m.$$

The second step of the proof starts recalling ii). Then, for each m, we consider a finite sequence of linear functionals such that acts nicely on A_m. Finally, we show that the concatenation of the mentioned sequences asymptotically separates $\mathcal{U}$. A similar technique was used before in this chapter, for instance, in the proof of Proposition 4.6.

Let us mention a simple case that we treat it separately: If $\mathcal{U} = \mathbb{R}^2$, we simply consider the constant sequence equal to $0 \in (\mathbb{R}^2)^*$ which asymptotically separates $\mathbb{R}^2$. Then, we assume that $\mathcal{U}_1 \subsetneq \mathbb{R}^2$.

Let $\mathcal{U}_1 \cap S = \bigcup_{\mathrm{i}=1}^{m_1} \mathcal{U}_{1,\mathrm{i}}$, as in Notation 4.18, where $m_1 \in \mathbb{N} \cup \{\infty\}$. For each $n \geq 2$, let us assume that the sequence of arcs $(\mathcal{U}_{n,\mathrm{i}})_\mathrm{i}$ is ordered in such a way that first we find the sets contained in $\mathcal{U}_{1,1}$, followed by the ones contained in $\mathcal{U}_{1,2}$, and so on. This is possible because there are only finitely many connected components of $\mathcal{U}_n \cap S$ contained in $\mathcal{U}_{1,\mathrm{i}}$, for each $1 \leq \mathrm{i} \leq m_1$. For $l \in \mathbb{N}$, let $(p_{l,k})_k$ be the sequence of integers defined by

$$p_{l,k} := \max\left\{\mathrm{i} \in \mathbb{N} : \mathcal{U}_{l,\mathrm{i}} \subseteq \bigcup_{j \leq k} \mathcal{U}_{1,j}\right\} = \max\{\mathrm{i} \in \mathbb{N} : \mathcal{U}_{l,\mathrm{i}} \subseteq \mathcal{U}_{1,k}\}.$$

First step: Construction of the set A_n, for $n \in \mathbb{N}$. For A_1, we just consider $A_1 = F_1(\mathcal{U}_{1,1})$. Let us define A_2. For each $k \in \{1, ..., m_{1,1}\}$, we set $A_{1,1,k}$ depending on the relation between $\mathcal{U}_{1,1}$ and $\mathcal{U}_{1,1,k}$. There are four different cases: If $\mathcal{U}_{1,1} = \mathcal{U}_{1,1,k}$, then we set $A_{1,1,k} = F_2(\mathcal{U}_{1,1})$. If $\mathcal{U}_{1,1,k}$ is compactly contained in $\mathcal{U}_{1,1}$, then we set $A_{1,1,k} = \mathcal{U}_{1,1,k}$. If none of the previous cases hold, but $l(\mathcal{U}_{1,1}) = l(\mathcal{U}_{1,1,k})$, then we set $A_{1,1,k} = F_{l,2}(\mathcal{U}_{1,1,k})$. In the last case, whenever $r(\mathcal{U}_{1,1}) = r(\mathcal{U}_{1,1,k})$ but $l(\mathcal{U}_{1,1}) \neq l(\mathcal{U}_{1,1,k})$, we set $A_{1,1,k} = F_{r,2}(\mathcal{U}_{1,1,k})$. Taking this into account, we define A_2 by:

$$A_2 = \bigcup_{k=1}^{m_{1,1}} A_{1,1,k} \cup \bigcup_{k=1}^{m_{1,2}} F_2(\mathcal{U}_{1,2,k}).$$

Observe that the first union is indexed over all the connected components of $\mathcal{U}_2 \cap S$ which are contained in $\mathcal{U}_{1,1}$ whereas the second one is indexed over the connected components of $\mathcal{U}_2 \cap S$ which are contained in $\mathcal{U}_{1,2}$. Let us now write down the general case. Let $t \in \mathbb{N}$ with $t \geq 2$. For $\mathrm{i} \in \{1, ..., p_{t-1,t-1}\}$ and $k \in \{1, ..., m_{t-1,\mathrm{i}}\}$. We set $A_{t-1,\mathrm{i},k}$ depending on the relation between $\mathcal{U}_{t-1,\mathrm{i},k}$ and $\mathcal{U}_{t-1,\mathrm{i}}$, as it was done when we have defined A_2. Then, we set

$$A_{t-1,\mathrm{i},k} = \begin{cases} \mathcal{U}_{t-1,\mathrm{i},k} & \text{if } \mathcal{U}_{t-1,\mathrm{i},k} \text{ is compactly contained in } \mathcal{U}_{t-1,\mathrm{i}} \\ F_t(\mathcal{U}_{t-1,\mathrm{i},k}) & \text{if } \mathcal{U}_{t-1,\mathrm{i},k} = \mathcal{U}_{t-1,\mathrm{i}}, \\ F_{l,t}(\mathcal{U}_{t-1,\mathrm{i},k}) & \text{if } l(\mathcal{U}_{t-1,\mathrm{i},k}) = l(U_{t-1,\mathrm{i}}),\ r(\mathcal{U}_{t-1,\mathrm{i},k}) \neq r(\mathcal{U}_{t-1,\mathrm{i}}) \\ F_{r,t}(\mathcal{U}_{t-1,\mathrm{i},k}) & \text{if } l(\mathcal{U}_{t-1,\mathrm{i},k}) \neq l(\mathcal{U}_{t-1,\mathrm{i}}),\ r(\mathcal{U}_{t-1,\mathrm{i},k}) = r(\mathcal{U}_{t-1,\mathrm{i}}). \end{cases}$$

Finally, we define A_t by:

$$A_t := \bigcup_{\mathrm{i}=1}^{p_{t-1,t-1}} \bigcup_{k=1}^{m_{t-1,\mathrm{i}}} A_{t-1,\mathrm{i},k} \cup \bigcup_{\mathrm{i}=p_{t,t-1}+1}^{p_{t,t}} F_t(\mathcal{U}_{t,\mathrm{i}}).$$

By definition, A_t is compactly contained on $\mathcal{U}_m$. Moreover, since A_t is a finite union of arcs, we have that $\mathrm{dist}(A_m, \partial\mathcal{U}_m \cap S) > 0$. So, i) and ii) are already satisfied.

Let us prove iii), that is, $\mathcal{U} \cap S = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m$. First, for each $m \in \mathbb{N}$, observe that each set participating in the union defining A_m is a subset of $\mathcal{U}_m$, therefore $A_m \subset \mathcal{U}_m$. Since the sequence $(\mathcal{U}_n)_n$ is decreasing for the inclusion, then $A_m \subset \mathcal{U}_n$ for all $m \geq n$. t Therefore,

$\mathcal{U} \supseteq \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m$.

Conversely, let us fix $x \in \mathcal{U} \cap S$. We need to prove that x is contained in infinitely many sets A_n. Observe that for each $n \in \mathbb{N}$, there exists only one $\mathrm{i}(n) \in \mathbb{N}$, with $\mathrm{i}(n) \leq m_n$, such that $x \in \mathcal{U}_{n,\mathrm{i}(n)}$ ($\mathrm{i}(n)$ obviously depends on x). Let $\mathcal{U}_x = \bigcap_n \mathcal{U}_{n,\mathrm{i}(n)}$. Since each $\mathcal{U}_{n,\mathrm{i}(n)}$ is an interval, $\mathcal{U}_x$ is an interval as well. In fact, $\mathcal{U}_x$ is the connected component of $\mathcal{U} \cap S$ which contains x. Now, we divide our analysis in four cases. We recall that, for an interval $U \subset \mathbb{R}$, $\lambda(U)$ denotes its length.

First case: $x \in \mathrm{int}(\mathcal{U}_x)$. Let $\delta > 0$ such that $(x-\delta, x+\delta) \subset \mathcal{U}_x$. Let $N \in \mathbb{N}$ such that $\delta > 2^{-n}\lambda(\mathcal{U}_{n,\mathrm{i}(n)})$ for all $n \geq N$. Then, for every $n \geq \max\{N, \mathrm{i}(1)\}$, $x \in A_{n,\mathrm{i},k}$ whenever $x \in \mathcal{U}_{n,\mathrm{i},k}$. Thus, $x \in A_n$ for each $n \geq 1 + \max\{N, \mathrm{i}(1)\}$.

Second case: $x \in \partial\mathcal{U}_x$ and there exists $\delta > 0$ such that $[x, x+\delta) \subset \mathcal{U}_x$. Since the sequence $(l(\mathcal{U}_{n,\mathrm{i}(n)}))_n$ converges to $l(\mathcal{U}_x) = x$ and $l(\mathcal{U}_{n,\mathrm{i}(n)}) \neq x$ for all n, we deduce that there exists an increasing sequence $(n_j)_j$ such that $l(\mathcal{U}_{n_j,\mathrm{i}(n_j)}) \neq l(\mathcal{U}_{n_j-1,\mathrm{i}(n_j-1)})$ for all $j \in \mathbb{N}$. Let $N \in \mathbb{N}$ such that $\delta > 2^{-n}\lambda(\mathcal{U}_{n,\mathrm{i}(n)})$ for all $n \geq N$. Then, for every $n_j \geq \max\{N, \mathrm{i}(1)+1\}$, $x \in A_{n_j-1,\mathrm{i},k}$ whenever $x \in \mathcal{U}_{n_j-1,\mathrm{i},k}$. Thus, $x \in A_{n_j}$, for each $n_j \geq \max\{N, \mathrm{i}(1)+1\}$.

Third case: $x \in \partial\mathcal{U}_x$ and there exists $\delta > 0$ such that $(x-\delta, x] \subset \mathcal{U}_x$. It is analogous to the previous case.

Last case: $\mathcal{U}_x = \{x\}$. Let us proceed by contradiction. For each $n \in \mathbb{N}$, let $k(n) \in \{1, ..., m_{n,\mathrm{i}(n)}\}$ such that $x \in \mathcal{U}_{n,\mathrm{i}(n),k(n)}$ ($k(n)$ obviously depends on x). Observe that

$$\mathcal{U}_{n,\mathrm{i}(n),k(n)} = \mathcal{U}_{n+1,\mathrm{i}(n+1)}, \text{ for all } n \in \mathbb{N}.$$

Suppose that there exists $N \in \mathbb{N}$, such that $N \geq \mathrm{i}(1) + 1$ and $x \notin A_n$ for all $n \geq N$. Then, $\mathcal{U}_{n+1,\mathrm{i}(n+1)}$ can not be compactly contained in $\mathcal{U}_{n,\mathrm{i}(n)}$ whenever $n \geq N$. Otherwise, $A_{n,\mathrm{i}(n),k(n)} = \mathcal{U}_{n+1,\mathrm{i}(n+1)}$, and therefore $x \in A_{n+1}$. Let us fix $n \geq N$. Without loss of generality, we suppose that

$$\begin{aligned} &l(\mathcal{U}_{n,\mathrm{i}(n)}) = l(\mathcal{U}_{n+1,\mathrm{i}(n+1)}), \text{ and} \\ x \in &\left(l(\mathcal{U}_{n+1,\mathrm{i}(n+1)}), \left(1 - \frac{1}{2^{n+1}}\right) l(\mathcal{U}_{n+1,\mathrm{i}(n+1)}) + \frac{1}{2^{n+1}} r(\mathcal{U}_{n+1,\mathrm{i}(n+1)})\right], \end{aligned}$$

i.e. $x \notin A_{n+1}$. Recalling that both sequences $(l(\mathcal{U}_{m,\mathrm{i}(m)}))_m$ and $(r(\mathcal{U}_{m,\mathrm{i}(m)}))_m$ converge to x, we can define

$$M = \min\{m \in \mathbb{N} : \ m \geq n, \ l(\mathcal{U}_{m,\mathrm{i}(m)}) \neq l(\mathcal{U}_{m+1,\mathrm{i}(m+1)})\}.$$

It follows that $M \geq n+1$. We claim that $x \in A_{M+1}$, which would be a contradiction. By definition of M, we have that $A_{M-1,\mathrm{i}(M-1),k(M-1)} = F_{l,M}(\mathcal{U}_{M,\mathrm{i}(M)})$ and that $A_{M,\mathrm{i}(M),k(M)} = F_{r,M+1}(\mathcal{U}_{M+1,\mathrm{i}(M+1)})$. Since $x \notin A_M$, then $x \notin F_{l,M}(\mathcal{U}_{M,\mathrm{i}(M)})$ and we can deduce that:

$$x \notin \left[\frac{1}{2^M} l(\mathcal{U}_{M,\mathrm{i}(M)}) + \left(1 - \frac{1}{2^M}\right) r(\mathcal{U}_{M,\mathrm{i}(M)}), r(\mathcal{U}_{M,\mathrm{i}(M)})\right), \text{ and then}$$

$$x \notin \left[\frac{1}{2^{M+1}} l(\mathcal{U}_{M+1,\mathrm{i}(M+1)}) + \left(1 - \frac{1}{2^{M+1}}\right) r(\mathcal{U}_{M,\mathrm{i}(M)}), r(\mathcal{U}_{M,\mathrm{i}(M)})\right).$$

Finally, since $r(\mathcal{U}_{M,\mathrm{i}(M)}) = r(\mathcal{U}_{M+1,\mathrm{i}(M+1)})$, we conclude that

$$x \in F_{r,M+1}(\mathcal{U}_{M+1,\mathrm{i}(M+1)}) \subseteq A_{M+1}.$$

Second step: Construction of the sequence which asymptotically separates $\mathcal{U}$. First,since $\mathcal{U}_t$ is a symmetric set, $\mathcal{U}_t = -\mathcal{U}_t$, and $A_t \subset \mathcal{U}_t$, we have that

$$A_t \cup -A_t \subset \mathcal{U}_t \text{ and } \operatorname{dist}(A_t, \partial\mathcal{U}_t \cap S) = \operatorname{dist}(A_t \cup -A_t, \partial\mathcal{U}_t \cap S), \text{ for all } t \in \mathbb{N}.$$

So, after redefining $A_t := A_t \cup -A_t$, properties i), ii) and iii) still remain true (they were given at the beginning of the proof).

Thanks to ii), we know that the sequence $(\operatorname{dist}(A_t, \partial\mathcal{U}_t \cap S))_t$ is strictly positive. Let us consider a decreasing sequence $(\alpha_t)_t \subset \mathbb{R}$, convergent to 0, such that

$$0 < \alpha_t \leq \operatorname{dist}(A_t, \partial\mathcal{U}_t \cap S), \text{ for all } t \in \mathbb{N}.$$

For each $t \in \mathbb{N}$, let us consider X_t be a finite $\alpha_t/3^t$-net of A_t. Consider now the finite set $G_t \subset (\mathbb{R}^2)^*$ of all linear functionals g satisfying that $\|g\| = 2^t/\alpha_t$ and $g(x) = 0$ for some $x \in X_t$. In particular, for each $x \in X_t$ there is a linear functional $g \in G_t$ such that $g(x) = 0$. Let us define the sequence $(g_n)_n \subset (\mathbb{R}^2)^*$ by the concatenation of enumerations of the sets G_1, G_2 and so on. We claim that $(g_n)_n$ asymptotically separates $\mathcal{U}$.

Let $x \in S$. If $x \in \mathcal{U}$, then $\liminf_n |g_n(x)| = 0$. Indeed, let $t \in \mathbb{N}$ such that $x \in A_t$. Let $x' \in X_t$ and $g \in G_t$ such that $g(x') = 0$ and $\|x - x'\| \leq \alpha_t/3^t$. Then,

$$|g(x)| \leq \|g\|\|x - x'\| \leq \frac{2^t}{\alpha_t}\frac{\alpha_t}{3^t},$$

which tends to 0 as t tends to ∞. Recalling that $\mathcal{U} \cap S = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m$, x belongs to infinitely many sets A_t, so $\liminf_n |g_n(x)| = 0$. On the other hand, if $x \notin \mathcal{U}$, then $\lim_n |g_n(x)| = \infty$. Indeed, since $x \notin \mathcal{U}$, there exists $N \in \mathbb{N}$ such that $x \notin \mathcal{U}_n$, for all $n \geq N$. Thus, $x \notin A_n$ for all $n \geq N$ (recall that $A_n \subset \mathcal{U}_n$). By definition of α_t and since the sequence $(\mathcal{U}_n \cap S)_n$ is decreasing for the inclusion, we get that

$$\operatorname{dist}(x, A_t) \geq \operatorname{dist}(A_t, \partial\mathcal{U}_{t-1} \cap S) \geq \alpha_t, \ \forall t \geq N + 1.$$

Hence, for any $g \in G_t$, since $\ker(g) \cap S \subseteq A_t$ we get:

$$|g(x)| = \|g\|\operatorname{dist}(x, \ker(g)) \geq \frac{2^t}{\alpha_t}\frac{\operatorname{dist}(x, \ker(g) \cap S)}{\sqrt{2}} \geq \frac{2^t}{\sqrt{2}},$$

which tends to infinity as t tends to infinity.

$\square$

To present the last result of this section, we need the following definition. We say that a set $F \subseteq X$, union of linear hyperplanes, is inner-asymptotically separated if it is asymptotically separated by a sequence (g_n) such that $\ker(g_n) \subseteq F$, for all $n \in \mathbb{N}$. As a remark, all previous examples of asymptotically separated sets which are union of linear hyperplanes are, in fact, inner-asymptotically separated.

Theorem 4.20 *There exists a closed set $F \subseteq \ell^1(\mathbb{N})$, union of linear hyperplanes, which is not inner-asymptotically separated.*

Before proving Theorem 4.20, we need the following three lemmas.

Lemma 4.21 *There exists a norm-discrete, linearly independent subset of the sphere of $\ell^\infty(\mathbb{N})$ which is w^*-homeomorphic to the Cantor set $K \subseteq [0,1]$.*

Proof. This is a straightforward corollary of [88, Theorem 1], but we are going to use an explicit construction. Let us consider the following directed infinite binary tree $T = (V, E)$ where V denotes the nodes and E the edges. Let us write

$$V = \{(n, \mathrm{i}) : 0 \leq n < \infty,\ 0 \leq \mathrm{i} \leq 2^n - 1\},$$

where $(0,0)$ is the root of T. There is an edge from (n, i) to (m, j) if and only if $m = n+1$ and $j - 2\mathrm{i} \in \{0, 1\}$. Let us consider the bijection $\sigma : V \to \mathbb{N}$ defined by $\sigma(n, \mathrm{i}) = 2^n + \mathrm{i}$. Let $\alpha \subseteq V$ be a maximal branch of T, starting from $(0,0)$, and let us consider the set $A_\alpha := \{n \in \mathbb{N} : \exists x \in \alpha,\ n = \sigma(x)\}$. It is clear that there exist uncountable many different maximal branches, and that if $\alpha \neq \beta$ are maximal branches, then $A_\alpha \cap A_\beta$ is a nonempty finite set. By construction, there is a straightforward identification between the set $\{A_\alpha :$ α maximal branch$\}$ and the classical Cantor set $K \subseteq [0,1]$. From now on, we write the former set as $\{A_\alpha :\ \alpha \in K\}$. For each α, let us consider $x^*_\alpha \in \ell^\infty(\mathbb{N})$ as the indicator function of A_α. That is, if $(\mathrm{e}^*_n)_n$ is the canonical coordinate vectors of $\ell^\infty(\mathbb{N})$, then x^*_α corresponds to the w^*-limit defined by the series $\sum_{n \in A_\alpha} \mathrm{e}^*_n$. It is clear that the set $\{x^*_\alpha :\ \alpha \in K\}$ is a norm-discrete subset of the sphere of $\ell^\infty(\mathbb{N})$. Finally, the map $\mathrm{i} : \alpha \in K \mapsto x^*_\alpha \in \ell^\infty(\mathbb{N})$ is a one-to-one continuous function, where $\ell^\infty(\mathbb{N})$ is endowed with its w^*-topology. Since K is a Hausdorff compact space, we deduce that i is an homeomorphism onto its image. □

In the following, we use explicitly the set $\{x^*_\alpha\}_{\alpha \in K}$ constructed in Lemma 4.21.

Lemma 4.22 *The set $F = \bigcup_{\alpha \in K} \ker(x^*_\alpha) \subset \ell^1(\mathbb{N})$ is norm-closed and w-dense.*

Proof. Recalling that $\ell^\infty(\mathbb{N})$ is the dual space of $\ell^1(\mathbb{N})$, we notice that

$$F = \bigcup_\alpha \ker(x^*_\alpha) \subset \ell^1(\mathbb{N}).$$

Let us start proving that F is norm-closed in $\ell^1(\mathbb{N})$. Consider $x \in \ell^1(\mathbb{N})$ such that $\operatorname{dist}(x, F) = 0$. Consider a sequence $(x^*_n)_n \subseteq \{x^*_\alpha\}_{\alpha \in K}$ such that $\operatorname{dist}(x, \ker(x^*_n)) \leq 1/n$. Since the w^*-topology of $\ell^\infty(\mathbb{N})$ is metrizable on bounded sets, there is a subsequence of (x^*_n), which we still denote by (x^*_n), w^*-convergent to $x^* \in \{x^*_\alpha\}_{\alpha \in K}$. Since

$$|x^*_n(x)| = \|x^*_n\| \operatorname{dist}(x, x^*_n) \leq 1/n,$$

we conclude that $x^*(x) = 0$, and thus, $x \in F$. This shows that F is norm-closed. On the other hand, let $x \notin F$ and let $\mathcal{U}$ be a neighbourhood of 0 in the weak topology. We prove that $x+\mathcal{U}$ intersects F. Since $\mathcal{U}$ is an open set for the weak topology, there exist $\varepsilon > 0$ and a finite sequence $(f_{\mathrm{i}})_{\mathrm{i}} \subseteq S_{\ell^\infty(\mathbb{N})}$ such that $\bigcap_{\mathrm{i}} f_{\mathrm{i}}^{-1}(-\varepsilon, \varepsilon) \subseteq \mathcal{U}$. Then, $\mathcal{U}$ contains a finite codimensional subspace of $\ell^1(\mathbb{N})$ that we call V. Now, consider $\alpha \in K$. If $(x+V) \cap ker(x^*_\alpha) = \emptyset$, then $V \subseteq \ker(x^*_\alpha)$. So, assuming by contradiction that $x + V \cap F = \emptyset$, then $V \subseteq \bigcap_{\alpha \in K} \ker(x^*_\alpha)$, vector space which does not have finite codimension since the set $\{x^*_\alpha\}_\alpha$ is linearly independent. $\square$

Lemma 4.23 *Let $x^* \in \ell^\infty(\mathbb{N})$ such that $\ker(x^*) \subseteq F$. Then there exists $\beta \in \mathbb{C} \setminus \{0\}$ and $\alpha \in K$ such that $x^* = \beta x^*_\alpha$*

Proof. Let $(\mathrm{e}_n)_n$ be the canonical basis of $\ell^1(\mathbb{N})$. Observe that e_1 does not belong to F, then $x^*(\mathrm{e}_1) \neq 0$. Without loss of generality, we assume that $x^*(\mathrm{e}_1) = 1$. Let $x^*(\mathrm{e}_n) = x_n \in \mathbb{C}$. If $x_n \neq 0$, then $\mathrm{e}_1 - \mathrm{e}_n/x_n \in F$, which is only possible if $x_n = 1$. Then, the image of the canonical basis $(\mathrm{e}_n)_n$ of $\ell^1(\mathbb{N})$ by x^* is contained in $\{0, 1\}$. Let us prove that, for each $n \in \mathbb{N}$, $x^*(\mathrm{e}_{\sigma(n,\mathrm{i})})$ is equal to 1 for only one $\mathrm{i} < 2^n$. Let $n \in \mathbb{N}$. Suppose that for all $\mathrm{i} < 2^n$, $x^*(\mathrm{e}_{\sigma(n,\mathrm{i})}) = 0$. Then $\sum_{\mathrm{i}} \mathrm{e}_{\sigma(n,\mathrm{i})} \in \ker(x^*)$, which is a contradiction. Indeed, $\sum_{\mathrm{i}} \mathrm{e}_{\sigma(n,\mathrm{i})}$ does not belong to the kernel of any x^*_α. Suppose now that there exist two indexes i and j such that $x^*(\mathrm{e}_{\sigma(n,\mathrm{i})}) = x^*(\mathrm{e}_{\sigma(n,j)}) = 1$. Then, the vector $\mathrm{e}_1 - (\mathrm{e}_{\sigma(n,\mathrm{i})} + \mathrm{e}_{\sigma(n,j)})/2$ belongs to the kernel of x^*. Since $\ker(x^*) \subseteq F$, we conclude that necessarily $\mathrm{i} = j$.

Finally, let us consider a maximal branch of T, $\beta = \{(n, \mathrm{i}_n) : n < N\}$, starting from $(0,0)$ such that $x^*(\mathrm{e}_{\sigma((n,\mathrm{i}))}) = 1$. If $N = \infty$, then we are done, because clearly $\beta \in K$, and then $x^* \in \{x^*_\alpha\}_{\alpha \in K}$. If $N < \infty$, let (N, j) be the node in T such that $x^*(\mathrm{e}_{\sigma((N,j))}) = 1$. Then, the vector $\sum_{n<N} \mathrm{e}_{\sigma((n,\mathrm{i}_n))} - N\mathrm{e}_{\sigma(N,j)} \in \ker x^*$ but it does not belong to F, which yields a contradiction. $\square$

In the next proof, we use the following notation: for $x^* \in \ell^\infty(\mathbb{N})$, we say that its support is the set $\mathrm{supp}(x^*) := \{n \in \mathbb{N} : x^*(\mathrm{e}_n) \neq 0\}$, where $(\mathrm{e}_n)_n$ is the canonical basis of $\ell^1(\mathbb{N})$.

Proof of Theorem 4.20. Let us suppose that F is inner-asymptotically separated by $(g_n)_n \subseteq \ell^\infty(\mathbb{N})$. By Lemma 4.23, there are two sequences, $x^*_n \subseteq \{x^*_\alpha\}_{\alpha \in K}$ and $(\beta_n)_n \subseteq \mathbb{C}$, such that $g_n = \beta_n x^*_n$. Since $F \subsetneq \ell^1(\mathbb{N})$, then $(\beta_n)_n$ must tend to infinity. Since K is uncountable, there exists $\alpha \in K$ such that $x^*_\alpha \notin \{x^*_n : n \in \mathbb{N}\}$. Let $B = \{k \in \mathrm{supp}(x^*_\alpha) : \exists n \in \mathbb{N},\ k \in \mathrm{supp}(x^*_n)\}$. We have two cases, B is finite or $B = A_\alpha$. If B is finite, then let us consider $n \in \mathrm{supp}(x^*_\alpha) \setminus B$. The vector $\mathrm{e}_1 - \mathrm{e}_n$ belongs to F but $|g_n(\mathrm{e}_1 - \mathrm{e}_n)| = |\beta_n|$ which inferior limit is infinity instead of 0. Thus, we assume that $B = A_\alpha$. Let us consider an increasing sequence $(N_k)_k \subseteq \mathbb{N}$, such that, for each k, $|\beta_n| \geq 3^k$ if $n \geq N_k$. Let $(n_k)_k \subseteq \mathbb{N}$ be the sequence defined by induction as follows: set $n_0 = 1$ and, for $k \geq 1$

$$n_k = \min\{j \in \mathrm{supp}(x^*_\alpha) :\ j \notin \mathrm{supp}(x^*_n),\ \text{for all}\ \ n \leq N_k,\ j > n_{k-1}\}.$$

Consider $x = \mathrm{e}_1 - \sum_{k \geq 1} 2^{-k} \mathrm{e}_{n_k}$. Clearly, $x \in \ker(x^*_\alpha)$. However, if $t \in [N_k, N_{k+1})$, we have that

$$|g_t(x)| = |\beta_t||x^*_t(x)| \leq |\beta_t| \left(1 - \sum_{j=1}^{k} 2^{-j}\right) \geq 3^k 2^{-k},$$

expression which tends to infinity as t tends to infinity. Since $x \in F$, we get a contradiction. $\square$

Remark 4.24 *A natural question is whether or not F is asymptotically separated. Further, whether we can find such a set in any infinite dimensional Banach space or, in separable Banach spaces with non-separable dual. In [88] and [55] it can be found an abstract approach of Proposition 4.21. In fact, the construction in [88, Theorem 1] is a generalization of the set $\{x^*_\alpha\}_{\alpha\in K}$ used in Theorem 4.20.*

4.3 Construction of wild operators

In this section we deal with the construction of a wild operator in an arbitrary separable infinite dimensional Banach space X. We follow the ideas of [8, Section 3], but we emphasize the role of asymptotically separated sets. The main result of this section reads as follows:

Theorem 4.25 *Let X be a separable infinite dimensional real or complex Banach space. Let V be an complemented, infinite codimensional subspace of X. Let $F \subseteq V$ be an asymptotically separated subset in V. Then there exists an operator $T \in \mathcal{L}(X)$ such that $R_T = P^{-1}(F)$ and $A_T = P^{-1}(F^c)$, where $P \in \mathcal{L}(X)$ is a bounded projection onto V.*

Let us start with the following three corollaries, which are consequences of the examples of asymptotically separated sets obtained in Section 4.2.

Corollary 4.26 (Theorem 4.2) *Every infinite dimensional separable Banach space X admits an operator T such that A_T and R_T form a partition of X and both have nonempty interior. Moreover, the following two cases are possible:*

1. *R_T is closed (and therefore A_T is open), and*
2. *$R_T \setminus \{0\}$ is open (and therefore $A_T \cup \{0\}$ is closed).*

Proof. Let $V = \operatorname{span}\{v_1, v_2\} \subseteq X$ be a two dimensional subspace of X. The corollary follows thanks to Theorem 4.25 and the sets:

$$\{v \in V : \ v = av_1 + bv_2, \ |a| \le |b|\} \text{ and } \{v \in V : \ v = av_1 + bv_2, \ |a| < |b| \text{ or } v = 0\},$$

which are asymptotically separated thanks to Corollary 4.10 and Proposition 4.6 respectively. $\square$

Corollary 4.27 *Every infinite dimensional separable Banach space X admits an operator T such that A_T is a nonempty and nowhere dense set.*

Proof. Let V be a finite dimensional subspace of X such that $\dim(V) \ge 2$. It is enough to apply Theorem 4.25 with any asymptotically separated set of V given by Corollary 4.7. $\square$

Corollary 4.28 *Every infinite dimensional separable Banach space X admits an operator T such that A_T and R_T form a partition of X and both are dense.*

Proof. Let V be a finite dimensional subspace of X such that $\dim(V) \geq 2$. It is enough to apply Theorem 4.25 with any asymptotically separated set of V given by Corollary 4.16. $\square$

Remark 4.29 *Notice that, in Corollary 4.28, the set* $U_T = \{x \in X : \text{Orb}_T(x) \textit{ is unbounded}\}$ *contains* A_T*, and it is a dense* G_δ *of* X*. The set* R_T *is also a dense* G_δ*, so* $U_T \cap R_T$*, the set of points* $x \in X$ *such that* x *is recurrent under* T *and* $\text{Orb}_T(x)$ *is unbounded is a dense* G_δ *of* X*.*

4.3.1 The complex case

From now on, let X be a separable complex Banach space and let V be a subspace of X satisfying the hypothesis of Theorem 4.25. Let P be a bounded projection onto V. Let $W = (I\text{d} - P)(X)$, a topological complement of V in X. Let $Q = I\text{d} - P$ be the bounded projection onto W, parallel to V. The following easy proposition will help us in the forthcoming computations.

Proposition 4.30 *Let* $T,\ S,\ R \in \mathcal{L}(X)$ *such that* $T = S + R$*,* $RS = R$ *and* $R^2 = 0$*. Then* $x \in R_T$ *if* $\liminf_n \|S^n x - x\| \vee \|(I + S + ...S^{n-1})Rx\| = 0$*.*

Proof. It is clear since $T^n = S^n + (I + S + ...S^{n-1})R$. $\square$

Let us start with the proof of Theorem 4.25. Consider a normalized bounded M-basis $(\mathrm{e}_n)_n$ on W and its associated biorthogonal system $(\mathrm{e}_n^*)_n \subseteq W^*$ given by Theorem 1.11. Thus, $\overline{\text{span}}(\mathrm{e}_n : n \in \mathbb{N}) = W$, $\|\mathrm{e}_n\| = 1$ for all $n \in \mathbb{N}$, $\sup_n \|\mathrm{e}_n^*\| = K < \infty$ and $\mathrm{e}_n^*(\mathrm{e}_m) = \delta_{n,m}$ for all $n,\ m \in \mathbb{N}$, where $\delta_{m,n}$ stands for the Kronecker's symbol. Let us extended every e_n^* to X by 0 on V.

Let $F \subseteq V$ be a set asymptotically separated by $(f_n)_n \subseteq V^*$. If $F = V$, then $P^{-1}(F) = X$ and our theorem has a trivial solution, namely, $T = I\text{d}$, the identity operator. Thus, we assume that $F \subsetneq V$ and then the sequence $(\|f_n\|)_n$ must diverge to infinity. Let $(m_n)_{n \geq -1} \subseteq \mathbb{N}$ be a rapidly increasing sequence so that the following two conditions are satisfied:

$$m_n | m_{n+1} \text{ for all } n \geq \mathbb{N}, \quad \text{and} \quad \sum_{n \geq 1} \frac{m_{n-2}}{m_{n-1}} \|f_n\| < \infty, \tag{4.4}$$

where $m_0 = m_{-1} = 1$. Let $(\lambda_n)_n \subseteq \mathbb{C}$ be the sequence defined by $\lambda_n = \mathrm{e}^{\frac{\mathrm{i}\pi}{m_n}}$, for all $n \in \mathbb{N}$. Let us formally define the operators S and R on X by:

$$S = I\text{d} + \sum_{n=1}^{\infty} (\lambda_n - 1)(\mathrm{e}_n^* \circ Q) \otimes \mathrm{e}_n,\ R = \sum_{n=1}^{\infty} \frac{1}{m_{n-1}} (f_n \circ P) \otimes \mathrm{e}_n,$$

where $x^* \otimes x$ stands for the tensor product between $x^* \in X^*$ and $x \in X$, i.e., $x^* \otimes x$ is the 1-rank operator on X defined by $x^* \otimes x(y) = x^*(y)x$.

Proposition 4.31 *For a sequence* $(m_n)_n \subseteq \mathbb{N}$ *satisfying Condition* (4.4)*,* S *and* R *are well defined and bounded operators.*

Proof. For both operators, it is enough to show that their own series converges in the norm operator topology. Considering that the estimation $|\mathrm{e}^{\mathrm{i}t} - 1| \leq t$ holds true for $t \geq 0$, we can get

$$\begin{aligned}\sum_{n=1}^{\infty} \|(\lambda_n - 1)(\mathrm{e}_n^* \circ Q) \otimes \mathrm{e}_n\| &\leq \sum_{n=1}^{\infty} |\lambda_n - 1| \|\mathrm{e}_n^*\| \|Q\| \|\mathrm{e}_n\| \\ &\leq K\|Q\| \sum_{n=1}^{\infty} \frac{\pi}{m_n} < \infty,\end{aligned}$$

where the last inequality is a consequence of the second part of condition (4.4). This proves that the series defining S absolutely converges. Analogously, using the second part of condition (4.4) and that $\|\mathrm{e}_n\| = 1$ for all $n \in \mathbb{N}$, it can be shown that the series defining R is absolutely convergent as well. □

Straightforward computations give us that $RS = R$ and $R^2 = 0$. We claim that the bounded operator $T = S + R$ satisfies the statement of Theorem 4.25. For $l \in \mathbb{N}$, let us set $\lambda_{n,l} = \sum_{k=0}^{l-1} \lambda_n^k$. By induction, we get that:

$$T^l x = S^l x + (I + S + ... + S^{l-1})Rx = x + \sum_{n=1}^{\infty} (\lambda_n^l - 1)(\mathrm{e}_n^* \circ Q)(x)\mathrm{e}_n + \sum_{n=1}^{\infty} \frac{\lambda_{n,l}}{m_{n-1}} f_n(Px)\mathrm{e}_n.$$

Now we can prove the complex case of Theorem 4.25. In order to do this, we show that $R_T = P^{-1}(F)$ and $A_T = P^{-1}(V \setminus F)$. Let $x \in P^{-1}(F)$. Then

$$\begin{aligned}\|S^{2m_k}x - x\| &= \left\| \sum_{n=k+1}^{\infty} (\lambda_n^{2m_k} - 1)(\mathrm{e}_n^* \circ Q) \otimes \mathrm{e}_n(x) \right\| \\ &\leq \|x\| \sup_{j \in \mathbb{N}} \|\mathrm{e}_j\| \|\mathrm{e}^* j\| \|Q\| \sum_{n=k+1}^{\infty} |\lambda_n^{2m_k} - 1| \\ &\leq K\|x\|\|Q\| \sum_{n=k+1}^{\infty} \frac{2m_k\pi}{m_n},\end{aligned}$$

where in the first line we have used the first part of condition (4.4). i.e. $\lambda_n^{2m_k} = 1$ for all $n \leq k$. Thanks to the second part of condition (4.4), the last series converges. Therefore, the last expression tends to 0 as k tends to infinity. On the other hand, since $Px \in F$, let $(k_l)_l \subseteq \mathbb{N}$ be an increasing sequence such that $|f_{k_l}(Px)| \to 0$. Then, we compute

$$\begin{aligned}\|(I + S + ... + S^{2m_{k_l-1}-1})Rx\| &= \left\| \sum_{n=1}^{\infty} \frac{\lambda_{n,2m_{k_l-1}}}{m_{n-1}} f_n(Px)\mathrm{e}_n \right\| \\ &\leq \left\| \frac{\lambda_{k_l,2m_{k_l-1}}}{m_{k_l-1}} f_{k_l}(Px)\mathrm{e}_n \right\| + \sum_{n=k_l+1}^{\infty} \frac{|\lambda_{n,2m_{k_l-1}}|}{m_{n-1}} \|f_n(Px)\mathrm{e}_n\| \\ &\leq 2|f_{k_n}(Px)| + \|P\| \sum_{n=k_l+1}^{\infty} \frac{2m_{k_l-1}}{m_{n-1}} \|f_n\|,\end{aligned}$$

where in the last inequality we have used that $\|\mathrm{e}_n\| = 1$ and that $|\lambda_{k,t}| \leq t$, for any $t \in \mathbb{N}$. The second part of condition (4.4) implies that the last series converges. Therefore, the previous expression tends to 0 as l tends to infinity. By Proposition 4.30, we have proven that $P^{-1}(F) \subseteq R_T$. Finally, it only remains to prove that $P^{-1}(F^c) \subseteq A_T$. We start with the following fact.

Proposition 4.32 *[8, Fact 3.6] For $m_{k-1} \leq l \leq m_k$, $|\lambda_{k,l}| \geq \frac{2}{\pi} m_{k-1}$*

Proof. Notice that, since $\lambda_{k,l}$ is a geometric sum, we have that:

$$|\lambda_{k,l}| = \left| \frac{1 - \mathrm{e}^{\mathrm{i}\frac{l\pi}{m_k}}}{1 - \mathrm{e}^{\mathrm{i}\frac{\pi}{m_k}}} \right| = \left| \frac{\sin(\frac{l\pi}{2m_k})}{\sin(\frac{\pi}{2m_k})} \right|.$$

Now, since $\frac{2}{\pi}|y| \leq |sin(y)| \leq |y|$ for all $y \in [0, \frac{\pi}{2}]$, we obtain the desired inequality. □

Let $x \in P^{-1}(F^c)$ and let $l \in [m_k, m_{k+1})$. Then:

$$\begin{aligned} K\|Q\|\|T^l x\| &\geq |(\mathrm{e}_k^* \circ Q)(S^l x + (I + S + ... + S^{l-1})Rx)| \\ &\geq \frac{|\lambda_{k,l}|}{m_{k-1}} |f_k(Px)| - |\mathrm{e}_k^* \circ Q(S^l x)| \\ &\geq \frac{2}{\pi} |f_k(Px)| - K\|Q\|\|x\|. \end{aligned}$$

Since $|f_k(Px)|$ tends to infinity as k tends to infinity, we finally deduce that $x \in A_T$. This finishes the proof of Theorem 4.25. □

Remark 4.33 *Let X be a separable infinite dimensional complex Banach space, V be a complemented subspace X with infinite codimension, F be an asymptotically separated set in V and $(f_n)_n$ its related sequence of linear functionals. Then, we can observe that the previous proof shows that for each sequence (m_n) satisfying condition (4.4), there exists a bounded operator T on X, solution of Theorem 4.25, such that*

$$\liminf_{n \to \infty} \|T^{2m_n} x - x\| = 0, \text{ for all } x \in R_T = P^{-1}(F),$$

where P is a bounded projection onto V.

Remark 4.34 *As a by-product of the above construction, it can be shown that the identity operator on X belongs to the norm closure of $\mathcal{W}(X)$. Indeed, the norm of $T - I\mathrm{d}$ depends on the values of the sequence $(m_n)_n$ as it can be computed:*

$$\begin{aligned} \|T - I\mathrm{d}\| &\leq \left\| \sum_{n=1}^{\infty} (\lambda_n - 1)(\mathrm{e}_n^* \circ Q) \otimes \mathrm{e}_n + \sum_{n=1}^{\infty} \frac{1}{m_{n-1}} (f_n \circ P) \otimes \mathrm{e}_n \right\| \\ &\leq K\|Q\| \sum_{n=1}^{\infty} |\lambda_n - 1| + \|P\| \sum_{n=1}^{\infty} \frac{\|f_n\|}{m_{n-1}}. \end{aligned}$$

This fact is going to be useful in Section 4.6, which is dedicated to study the norm closure of $\mathcal{W}(X)$.

4.3.2 The real case

In this subsection we sketch the construction of an operator on a real Banach space which solves Theorem 4.25 as it was done in [8] to construct a wild operator.

In the context of Theorem 4.25 for real spaces, let us consider the following objects: Let $P \in \mathcal{L}(X)$ be a projection onto V and let $Q = Id - P$. Let $W := Q(X)$ which is a topological complement of V in X. Notice that W is an infinite dimensional closed subspace of X. Let $(\mathrm{e}_n)_n \subset X$ be a normalized bounded M-basis of W and let $(\mathrm{e}_n^*)_n \subset W^*$ be the associated biorthogonal system given by Theorem 1.11. Let $K := \sup_n \|\mathrm{e}_n\|$. Let $(f_n)_n \subset V^*$ be a sequence which asymptotically separates $F \neq V$. Then, $\lim_{n\to\infty} \|f_n\| = \infty$. Let $(m_n)_{n\geq -1} \subset \mathbb{N}$ be an increasing sequence such that $m_{-1} = m_0 = 1$,

$$m_n | m_{n+1}, \text{ for all } n \in \mathbb{N}, \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{m_{n-2}}{m_{n-1}} \|f_n\| < \infty. \tag{4.5}$$

Let $(\theta_k)_k \subset \mathbb{R}$ be the sequence defined by $\theta_k = \frac{\pi}{m_k}$, for all $k \in \mathbb{N}$.

Proposition 4.35 *The linear maps $S, R : X \to X$ defined by*

$$\begin{aligned} S :=& Id + \sum_{k=1}^{\infty} (\cos(\theta_k) - 1)(\mathrm{e}_{2k}^* \circ Q) \otimes \mathrm{e}_{2k} + \sin(\theta_k)(\mathrm{e}_{2k-1}^* \circ Q) \otimes \mathrm{e}_{2k} \\ &+ \sum_{k=1}^{\infty} (\cos(\theta_k) - 1)(\mathrm{e}_{2k-1}^* \circ Q) \otimes \mathrm{e}_{2k-1} - \sin(\theta_k)(\mathrm{e}_{2k}^* \circ Q) \otimes \mathrm{e}_{2k-1}, \\ R :=& \sum_{k=1}^{\infty} \frac{1}{m_{k-1}} (f_k \circ P) \otimes \mathrm{e}_{2k}, \end{aligned}$$

are well defined and continuous.

Proof. The proposition directly follows from the fact that the series involved in the definition of S and R converge in the norm topology of bounded operators. Indeed, for S it is enough to notice that

$$|sin(x)| \leq |x| \qquad \text{and} \qquad |\cos(x) - 1| \leq x^2, \text{ for all } x \in \mathbb{R},$$

and that $(\theta_k)_k$ decreases fast to 0 due to condition (4.5). On the other hand, the continuity of R follows directly from condition (4.5). $\square$

Observe that, for any $k \in \mathbb{N}$, the operator S restricted to $\mathrm{span}(\mathrm{e}_{2k-1}, \mathrm{e}_{2k})$ is a rotation of angle θ_k.

In order to sketch the proof of Theorem 4.25, we continue with some properties of the operator $T := S + R$. To see this, observe first that $R^2 = 0$ and $RS = R$. Thus,

$$T^n = S^n + (Id + S + ... + S^{n-1})R. \tag{4.6}$$

Proposition 4.36 *Let $(m_k)_k \subset \mathbb{N}$ be a sequence satisfying condition* (4.5). *Let $S \in \mathcal{L}(X)$ constructed by Proposition 4.35 and let $n \in \mathbb{N}$. Then*

$$\begin{aligned} S^n =& Id + \sum_{k=1}^{\infty} (\cos(n\theta_k) - 1)(\mathrm{e}_{2k}^* \circ Q) \otimes \mathrm{e}_{2k} + \sin(n\theta_k)(\mathrm{e}_{2k-1}^* \circ Q) \otimes \mathrm{e}_{2k} \\ &+ \sum_{k=1}^{\infty} (\cos(n\theta_k) - 1)(\mathrm{e}_{2k-1}^* \circ Q) \otimes \mathrm{e}_{2k-1} - \sin(n\theta_k)(\mathrm{e}_{2k}^* \circ Q) \otimes \mathrm{e}_{2k-1}. \end{aligned}$$

Moreover, $\lim_{j\to\infty} S^{2m_j} x = x$ *for all* $x \in X$.

PROOF. The formula for S^n follows by direct computation. Let us check the second part of the proposition. Let $x \in X$ and $j \in \mathbb{N}$. Then

$$\begin{aligned} S^{2m_j} x - x =& \sum_{k=1}^{\infty} (\cos(2m_j\theta_k) - 1)\mathrm{e}_{2k}^*(Q(x))\mathrm{e}_{2k} + \sin(2m_j\theta_k)\mathrm{e}_{2k-1}^*(Q(x))\mathrm{e}_{2k} \\ &+ \sum_{k=1}^{\infty} (\cos(2m_j\theta_k) - 1)\mathrm{e}_{2k-1}^*(Q(x))\mathrm{e}_{2k-1} - \sin(2m_j\theta_k)\mathrm{e}_{2k}^*(Q(x))\mathrm{e}_{2k-1}. \end{aligned}$$

Thus, since $\theta_k = \frac{\pi}{m_k}$ for all k, we have that

$$\begin{aligned} \|S^{2m_j} x - x\| \leq& \sum_{k=j+1}^{\infty} |\cos(2m_j\theta_k) - 1| \|\mathrm{e}_{2k}^*\| \|Q\| \|x\| + |\sin(2m_j\theta_k)| \|\mathrm{e}_{2k-1}^*\| \|Q\| \|x\| \\ &+ \sum_{k=j+1}^{\infty} |\cos(2m_j\theta_k) - 1| \|\mathrm{e}_{2k-1}^*\| \|Q\| \|x\| + |\sin(2m_j\theta_k)| \|\mathrm{e}_{2k}^*\| \|Q\| \|x\|. \\ \leq& \sum_{k=j+1}^{\infty} 2K\|Q\|\|x\| \left(\left(\frac{2m_j\pi}{m_k} \right)^2 + \frac{2m_j\pi}{m_k} \right) \end{aligned}$$

where the last expression tends to 0 as k tends to ∞ due to condition (4.5).

□

Thanks to Proposition 4.36, for $l \geq 0$, we have that

$$S^l R := \sum_{k=1}^{\infty} \frac{1}{m_{k-1}} (f_k \circ P) \otimes (\cos(l\theta_k)\mathrm{e}_{2k} - \sin(l\theta_k)\mathrm{e}_{2k-1}),$$

we obtain

$$\left(\sum_{l=0}^{n-1} S^l \right) R = \sum_{k=1}^{\infty} \frac{1}{m_{k-1}} (f_k \circ P) \otimes (\mu_{k,n}\mathrm{e}_{2k} - \eta_{k,n}\mathrm{e}_{2k-1}),$$

where $\mu_{k,t} = \sum_{l=0}^{t-1} \cos(l\theta_k)$ and $\eta_{k,t} = \sum_{l=0}^{t-1} \sin(l\theta_k)$ for all $k, t \in \mathbb{N}$.

In the following proposition we summarize some simple but useful estimations.

Proposition 4.37 *Let $k, t \in \mathbb{N}$ and let $x_{k,t} := \mu_{k,t} \mathrm{e}_{2k} - \eta_{k,t} \mathrm{e}_{2k-1}$. Then:*

i) $x_{k,t} = 0$ *for all t multiple of $2m_k$.*

ii) $\frac{1}{K} \max(|\mu_{k,t}|, |\eta_{k,t}|) \leq \|x_{k,t}\| \leq 2t$.

iii) $\|x_{k,t}\| \geq \frac{2}{\sqrt{2\pi}C} m_{k-1}$, *for all $t \in [m_{k-1}, m_k]$.*

Proof. Statement i) follows directly from the periodicity of the functions cos and sin. Statement ii) follows from the evaluation of $x_{k,t}$ in e^*_{2k-1} and e^*_{2k}. Statement iii) is a consequence of Proposition 4.32 and the estimation

$$\max(|\mu_{k,t}|, |\eta_{k,t}|) \geq \frac{1}{\sqrt{2}} \left| \sum_{l=0}^{t} \exp\left(\frac{\mathrm{i}\pi}{m_k} l\right) \right|$$

□

Finally, thanks to condition (4.5), equality (4.6), Proposition 4.36 and Proposition 4.37, we can proceed as in the complex case to prove that

$$R_T = P^{-1}(F) \quad \text{and} \quad A_T = P^{-1}(V \setminus F),$$

which finishes the sketch of the proof of the real part of Theorem 4.25.

4.4 Properties of wild operators

In this section, we investigate geometric aspects of the recurrent set R_T of a bounded operator T defined on a Banach space. This will be applied to the study of the recurrent points of wild operators.

Proposition 4.38 *Let X be a Banach space, let T be a bounded operator on X, let $x \in X$ and $\varepsilon > 0$. Suppose that for every $x_1,\ x_2 \in B(x, \varepsilon)$ there exists a strictly increasing sequence $(k_n)_n \subseteq \mathbb{N}$ such that $T^{k_n} x_\mathrm{i} \to x_\mathrm{i}$ for $\mathrm{i} = 1,\ 2$. Then $R_T = X$.*

Proof. Since $-R_T \subseteq R_T$, we deduce that $B(-x, \varepsilon) \subseteq R_T$. Let $x_1 \in B(x, \varepsilon)$, $x_2 \in B(-x, \varepsilon)$ and an increasing sequence $(k_n)_n \subset \mathbb{N}$ such that $T^{k_n} x_\mathrm{i} \to x_\mathrm{i}$ for $\mathrm{i} = 1,\ 2$. Then:

$$\lim_n T^{k_n}(x_1 + x_2) = x_1 + x_2.$$

Since $B(0, \varepsilon) \subseteq B(x, \varepsilon) + B(-x, \varepsilon)$, we deduce that $B(0, \varepsilon) \subseteq R_T$. Therefore, $R_T = X$. □

Remark 4.39 *Analogous to the preceding proof, if for $x_1,\ x_2 \in R_T$ there exists such a sequence $(k_n)_n \subseteq \mathbb{N}$, then we can prove that* $\mathrm{span}(x_1, x_2) \subseteq R_T$. *Due this simple fact, we are able to prove a certain non-stability on the class of wild operators. See Theorem 4.42.*

Corollary 4.40 *Let T be a bounded operator on X such that $R_T \neq X$ and* $\mathrm{int}(R_T) \neq \emptyset$. *Then there exists an uncountable set $C \subseteq R_T$ such that:*

$$\liminf_{n \to \infty} \max\{\|T^n x - x\|, \|T^n y - y\|\} > 0, \ \textit{for all } x, y \in C,\ x \neq y.$$

Proof. Since $R_T \neq X$ and $\mathrm{int}(R_T) \neq \emptyset$, by Proposition 4.38 we know that there exist two distinct vectors $x,\ y \in \mathrm{int}(R_T)$ such that any increasing sequence $(k_n) \subseteq \mathbb{N}$ does not satisfy both limits $T^{k_n}x \to x$ and $T^{k_n}y \to y$ simultaneously. In other words:

$$\liminf_{n\to\infty} \max\{\|T^n x - x\|, \|T^n y - y\|\} > 0. \tag{4.7}$$

Since $x \in \mathrm{int}(R_T)$, there exists $\varepsilon > 0$ such that, for every $\lambda \in \mathbb{C}$ with $|\lambda| < \varepsilon$, we have $x + \lambda y \in R_T$. We claim that the set $C := \{x + \lambda y :\ |\lambda| < \varepsilon\}$ proves Corollary 4.40.

Let $\lambda_1,\ \lambda_2 \in \mathbb{C}$ such that $\lambda_1 \neq \lambda_2$ and $|\lambda_{\mathrm{i}}| < \varepsilon$ for $\mathrm{i} = 1,\ 2$. Suppose that the inferior limit (4.7) is equal to 0 for $x + \lambda_{\mathrm{i}} y$ for $\mathrm{i} = 1,\ 2$. Then, there exists a strictly increasing sequence $(k_n)_n \subseteq \mathbb{N}$ such that:

$$\lim T^{k_n}(x + \lambda_{\mathrm{i}} y) = x + \lambda_{\mathrm{i}} y, \text{ for } \mathrm{i} = 1,\ 2.$$

This implies that $\lim T^{k_n} y = y$ by subtracting both expressions ($\mathrm{i} = 1$ and $\mathrm{i} = 2$), but it also implies that $\lim T^{k_n} x = x$, which is a contradiction. □

The following definition will be used only in the next result, Proposition 4.41. We say that a wild operator has standard form if it can be written as $T = S + R$, where $R^2 = 0$, $RS = R$ and the sequence $(\|S^n\|)_n$ is bounded.

In order to present an example in the complex case, let us recall from Subsection 4.3.1 the complementary subspaces V and W of X, the respective bounded projections P and Q, the sequence $(\lambda_n)_n \subset \mathbb{C}$ and the bounded M-basis $(\mathrm{e}_n)_n \subset W$. Let us check that, the operator $T := S + R \in \mathcal{L}(X)$ (a wild operator whenever F and $V \setminus F$ have nonempty interior relative to V) constructed in Subsection 4.3.1, the complex case, has standard form whenever $(\mathrm{e}_n)_n$ is a c-unconditional basis of W. Indeed, we only need to show that $(\|S^k\|)_k$ is bounded. Notice that, for any $x \in X$ and $k \in \mathbb{N}$, the following estimation holds:

$$\begin{aligned}\|S^k x\| &= \left\| x + \sum_{n=1}^{\infty} (\lambda_n^k - 1)(\mathrm{e}_n^* \circ Q(x))\mathrm{e}_n \right\| \leq \|x - Q(x)\| + \left\| \sum_{n=1}^{\infty} \lambda_n^k (\mathrm{e}_n^* \circ Q(x))\mathrm{e}_n \right\| \\ &\leq \|Px\| + c\|Qx\| \leq \|x\|(\|P\| + c\|Q\|).\end{aligned}$$

Thus, the sequence $(\|S^k\|)_k$ is bounded.

Proposition 4.41 *Let $T = S + R$ be a wild operator having standard form on X. Let us assume that* $\ker(R)$ *is a complemented subspace of X. Then, for any V closed subspace X, topological complement of* $\ker(R)$*, we have that $A_T = P^{-1}(V \cap A_T)$ and $R_T = P^{-1}(V \cap R_T)$, where $P \in \mathcal{L}(X)$ is the projection onto V, parallel to* $\ker(R)$.

Proof. Notice first that $T^n = S^n + (S^{n-1} + S^{n-2} + ... + I)R$. Let V be a topological complement of $\ker(R)$ on X and let $P : X \to X$ be the projection onto V parallel to $\ker(R)$. Since the operators $\{S^n :\ n \in \mathbb{N}\}$ are uniformly bounded, then $\|T^n x\|$ goes to infinity if and only if $\|(S^{n-1} + S^{n-2} + ... + I)Rx\|$ goes to infinity. Also, since $R = RP$, we can deduce that:

$$x \in A_T \text{ if and only if } Px \in A_T,$$

obtaining the characterization for A_T. Since A_T and R_T form a partition of X, this concludes the proof. $\square$

For a Banach space X and $k \in \mathbb{N}$, we write the product space containing k-tuples of elements of X by $\mathbf{X} = \bigoplus_{\mathrm{i}=1}^{k} X$. We may endow this space with any product norm and we do it with the norm of the maximum, i.e.

$$\|\mathbf{x}\| = \max\{\|x_{\mathrm{i}}\|,\ \mathrm{i} = 1, ..., k\},\ \text{for any } \mathbf{x} \in \mathbf{X},$$

where $\mathbf{x} = (x_1, x_2, ..., x_k)$. Further, for an operator $T : X \to X$ and $k \in \mathbb{N}$, we define the operator $\mathbf{T} = \bigoplus_{\mathrm{i}=1}^{k} T : \mathbf{X} \to \mathbf{X}$ by $\mathbf{T}(\mathbf{x}) = (Tx_1, Tx_2, ..., Tx_k)$ where $\mathbf{x} = (x_1, x_2, ..., x_k)$.

Theorem 4.42 *Let X be a separable infinite dimensional Banach space. There exists $T \in \mathcal{W}(X)$ such that $\boldsymbol{T} := T \oplus T$ is not a wild operator on $\boldsymbol{X} := X \oplus X$.*

Proof. Let V be a two dimensional subspace of X and let F be an asymptotically separated set in V such that F and $V \setminus F$ have nonempty interior relative to V. Let $T : X \to X$ be the operator constructed in Section 4.3, complex case, using V and F as the complemented subspace and the asymptotically separated set respectively. Reasoning by contradiction, let us assume that $\mathbf{T}$ is a wild operator on $\mathbf{X}$. Since $R_{\mathbf{T}}$ has nonempty interior in $\mathbf{X}$, there exist $\varepsilon > 0$ and $x_1, x_2 \in X$ such that $B(x_1, \varepsilon) \times B(x_2, \varepsilon) \subseteq R_{\mathbf{T}}$. Consider the bounded projection $P \in \mathcal{L}(X)$ onto V with which T is constructed. Then, $R_T = P^{-1}(F)$ and $A_T = P^{-1}(V \setminus F)$. We prove that $R_T \cap V = V$, which is a contradiction since $A_T \neq \emptyset$. By the Open Mapping theorem, we can choose $y_{\mathrm{i}} \in B(x_{\mathrm{i}}, \varepsilon)$ for each $\mathrm{i} = 1, 2$, such that the set $\{Py_{\mathrm{i}} :\ \mathrm{i} = 1, 2\}$ is linearly independent. Notice now that, since $\mathbf{y} = (y_1, y_2) \in R_{\mathbf{T}}$, we have that:

$$\liminf_{n\to\infty} \|\mathbf{T}^n\mathbf{y} - \mathbf{y}\| = 0 \implies \liminf_{n\to\infty} \max\{\|T^n y_{\mathrm{i}} - y_{\mathrm{i}}\| :\ \mathrm{i} = 1, 2\} = 0.$$

Therefore, there exists an increasing sequence $(k_n) \subseteq \mathbb{N}$ such that $(T^{n_j} y_{\mathrm{i}})_j$ tends to y_{i} as j tends to infinity, for $\mathrm{i} = 1, 2$. Thus, for each $(\lambda_1, \lambda_2) \in \mathbb{C}^k$ we have that

$$\lim_{j\to\infty} T^{n_j} \sum_{\mathrm{i}=1}^{2} \lambda_{\mathrm{i}} y_{\mathrm{i}} = \sum_{\mathrm{i}=1}^{2} \lambda_{\mathrm{i}} y_{\mathrm{i}}.$$

However, $P(\sum_{\mathrm{i}=1}^{2} \lambda_{\mathrm{i}} y_{\mathrm{i}}) = \sum_{\mathrm{i}=1}^{2} \lambda_{\mathrm{i}} P y_{\mathrm{i}} \in R_T \cap V$ for any $(\lambda_1, \lambda_2) \in \mathbb{C}^2$. Followed by the linear independence of $\{Py_1, Py_2\}$, we get that $R_T \cap V = V$. Thus, $F = V$ and therefore $R_T = X$, which is a contradiction. $\square$

Remark 4.43 *Let $T \in \mathcal{L}(X)$ be the operator used in the proof of Proposition 4.42. Notice that a similar argument (given in the proof of Proposition 4.42) we can show that, for any $\mathrm{d} \geq 2$, the operator $\bigoplus_{\mathrm{i}=1}^{\mathrm{d}} T$ is not a wild operator on $\bigoplus_{\mathrm{i}=1}^{\mathrm{d}} X$.*

The us continue with the last result of this section.

Proposition 4.44 *Let X be an infinite dimensional Banach space. Let T be a bounded operator on X and let $k \in \mathbb{N}$. If $\boldsymbol{T} := \bigoplus_{\mathrm{i}=1}^{k} T$ belongs to $\mathcal{W}(\bigoplus_{\mathrm{i}=1}^{k} X)$, then $\mathrm{int}(R_T) \cup \{0\}$ contains a subspace of dimension d.*

PROOF. Let us write $\mathbf{X} := \bigoplus_{\mathrm{i}=1}^{k} X$. If $\mathbf{T} \in \mathcal{W}(\mathbf{X})$, then $R_{\mathbf{T}}$ has nonempty interior in $\mathbf{X}$. Therefore, there exist $\varepsilon > 0$ and $(x_\mathrm{i})_{\mathrm{i}=1}^{k} \subset X$ such that

$$\Pi_{\mathrm{i}=1}^{\mathrm{d}} B(x_\mathrm{i}, \varepsilon) \subset R_{\mathbf{T}}.$$

Shrinking ε if necessary, we assume that $\{x_\mathrm{i} : \mathrm{i} = 1, ..., k\}$ is a linearly independent set. Let $Y = \mathrm{span}(x_\mathrm{i} : \mathrm{i} = 1, ..., k)$. We claim that $Y \subset \mathrm{int}(R) \cup \{0\}$. Indeed, let $y \in Y \setminus \{0\}$ and let $(\lambda_\mathrm{i})_\mathrm{i} \subset \mathbb{K}$ such that $y = \sum_{\mathrm{i}=1}^{k} \lambda_\mathrm{i} x_\mathrm{i}$. Let us assume that $\lambda_1 \neq 0$. As we did in the proof of Proposition 4.42, for all $x \in B(x_1, \varepsilon)$ there exists an increasing sequence $(n_j)_j \subset \mathbb{N}$ such that

$$\lim_{j\to\infty} \max\{\|T^{n_j} x - x\|, \ \|T^{n_j} x_\mathrm{i} - x_\mathrm{i}\| : \ \mathrm{i} = 2, ..., k\} = 0$$

Thus, thanks to the triangle inequality we obtain that

$$\lim_{j\to\infty} \|T^{n_j}(\lambda_1 x + \sum_{\mathrm{i}=2}^{k} \lambda_k x_k) - \lambda_1 x + \sum_{\mathrm{i}=2}^{k} \lambda_k x_k\| = 0.$$

Finally, since x is an arbitrary vector of $B(x_1, \varepsilon)$, we get that

$$B(0, \lambda_1 \varepsilon) + y = \lambda_1 B(x_1, \varepsilon) + \sum_{\mathrm{i}=2}^{k} \lambda_k x_k \subset R_T,$$

which shows that $y \in \mathrm{int}(R_T)$. □

4.5 Spectral properties of wild operators

This section is devoted to give some remarks about spectral properties of wild operators on complex Banach spaces. To the best of our knowledge, in the literature there is only the construction of invertible wild operators. Also, it is known that the spectrum of a wild operator must remain included in the unit disk. For the sake of completeness we present this result in Proposition 4.50. Along to this and to the next section, we use the concepts of unconditional and symmetric basis. For a precise definition of these bases, we refer to [1] and Chapter 1, Section 1.2.

We proceed with the construction of a non-invertible wild operator and then, we show the existence of a wild operator whose spectrum coincide exactly with the closed unit disk.

Theorem 4.45 *Any infinite dimensional Banach spaces having a symmetric basis admits a non-invertible wild operator.*

Before proceeding with the proof of the theorem, we need the following definition that can be found in [46, Chapter 4] and [38].

Definition 4.46 *Let X be a Banach space and let T be a bounded operator on X. We say that T is a rigid operator if there exists an increasing sequence $(n_k) \subseteq \mathbb{N}$ such that*

$$\lim_k T^{n_k} x = x, \ \forall x \in X.$$

Proposition 4.47 *Let X be an infinite dimensional (real or complex) Banach space having a symmetric basis and let $(m_n)_n \subset \mathbb{N}$ be an increasing sequence such that $nm_n | m_{n+1}$. Then there is a non-invertible rigid operator S on X such that for each $x \in X$, $\lim_n S^{2m_n} x = x$.*

PROOF. Let $(\mathrm{e}_n)_n$ be a normalized b-symmetric basis of X. Let us rewrite the basis as $(\mathrm{e}_{n,k})_{n=1}^{\infty}{}_{k=1}^{2m_n}$. For each $n \in \mathbb{N}$, consider the operator A_n on $\mathrm{span}(\{\mathrm{e}_{n,k} : k = 1, ..., 2m_n\})$, defined by:

$$A_n \mathrm{e}_{n,1} := \frac{1}{n}\mathrm{e}_{n,2},\ A_n \mathrm{e}_{n,\mathrm{i}} := \alpha_n \mathrm{e}_{n,\mathrm{i}+1}\ \forall \mathrm{i} = 2, ..., 2m_n - 1,\ A_n \mathrm{e}_{n,2m_n} := \alpha_n \mathrm{e}_1, \tag{4.8}$$

where $\alpha_n^{2m_n - 1} = n$. By the growth condition on $(m_n)_n$, it can be deduced that $2 > \alpha_n \geq \alpha_{n+1}$ for all $n \geq 2$, and that the sequence $(\alpha_{n+1}^{2m_n})_n$ is bounded by some strictly positive constant $C > 1$ independent of n. We define the operator S on $\mathrm{span}(\{\mathrm{e}_{n,k} : n, k\})$ by $S\mathrm{e}_{n,k} = A_n \mathrm{e}_{n,k}$. Since the sequence $(\alpha_n)_n$ is bounded and $(\mathrm{e}_n)_n$ is a symmetric basis, the operator S is bounded and can be continuously extended to X, extension which we still denote by S. By the Open mapping Theorem, the operator S can not be invertible. Indeed, we can notice that $\|S\mathrm{e}_{n,1}\| = 1/n$, but $\|\mathrm{e}_{n,1}\| = 1$. Let us check now that S is a rigid operator. Let $x \in X$ be a vector different from 0. Since $(\mathrm{e}_n)_n$ is a Schauder basis, then $x = \sum_n \sum_k x_{n,k} \mathrm{e}_{n,k}$. Recalling that $A_n^{2m_n}$ is the identity operator on $\mathrm{span}(\{\mathrm{e}_{n,k} : k\})$ and that $m_n | m_{n+1}$, we compute

$$\begin{aligned}
\|S^{2m_j} x - x\| &= \|\sum_{n=1}^{\infty}\sum_{k=1}^{2m_n} x_{n,k} A^{2m_j} \mathrm{e}_{n,k} - \sum_{n=1}^{\infty}\sum_{k=1}^{2m_n} x_{n,k}\mathrm{e}_{n,k}\| \\
&= \|\sum_{n=j+1}^{\infty}\sum_{k=1}^{2m_n} x_{n,k} A^{2m_j} \mathrm{e}_{n,k} - \sum_{n=j+1}^{\infty}\sum_{k=1}^{2m_n} x_{n,k}\mathrm{e}_{n,k}\| \\
&\leq (bC + 1)\|\sum_{n=j+1}^{\infty}\sum_{k=1}^{2m_n} x_{n,k}\mathrm{e}_{n,k}\|,
\end{aligned}$$

which tends to 0 as j tends to infinity. $\square$

Observe that Proposition 4.47 can be extend to Banach spaces containing an infinite dimensional complemented subspace with symmetric basis. Indeed, if Y is the subspace containing a symmetric basis, it is enough to extended the operator defined in Proposition 4.47 by the identity in any subspace which is a topological complement of Y. Now, we can proceed with the proof of Theorem 4.45.

PROOF OF THEOREM 4.45. Let X be an infinite dimensional Banach space with symmetric basis $(\mathrm{e}_n)_n$. Let us define $Y_1 = \overline{\mathrm{span}}(\{\mathrm{e}_{2n-1} : n \in \mathbb{N}\})$ and $Y_2 = \overline{\mathrm{span}}(\{\mathrm{e}_{2n} : n \in \mathbb{N}\})$. Since $(\mathrm{e}_n)_n$ is a symmetric basis, the natural associated projections onto Y_1 and Y_2 are continuous. Let $T \in \mathcal{W}(Y_1)$ be an operator constructed as in Section 4.3 associated to the a sequence $(m_n)_n$ that satisfies both growth conditions, condition (4.4) and $nm_n | m_{n+1}$ for all $n \in \mathbb{N}$. Let $S \in \mathcal{L}(Y_2)$ be an operator given by Proposition 4.47 related to the same sequence $(m_n)_n$. Thus, we define $U \in \mathcal{L}(X)$ by $U(x) = T(y_1) + S(y_2)$, where y_i is the projection of x on Y_i, for $\mathrm{i} = 1,\ 2$. We prove that U is a non-invertible wild operator on X. Indeed, let $y_1 \in R_T \subseteq Y_1$. By Remark 4.33, we know that $\liminf \|T^{2m_n} y_1 - y_1\| = 0$. Then, since $(S^{2m_n} y_2)_n$ converges to y_2 for each $y_2 \in Y_2$, we deduce that $R_T + Y_2 \subseteq R_U$. On the other hand, a straightforward computation shows that $A_T + Y_2 \subseteq A_U$, showing that U is a wild operator. Finally, since Y_1 and Y_2 are complementary subspaces of X, both spaces are U-invariant and the restriction $U|_{Y_2} = S \in \mathcal{L}(Y_2)$ is not invertible, we get that the operator U is not invertible either. $\square$

Remark 4.48 *In fact, Theorem 4.45 holds true in any infinite dimensional Banach space with an infinite dimensional complemented subspace with symmetric basis.*

Theorem 4.45 asserts that 0 can be part of the spectrum of a wild operator T. In fact, 0 is in the continuous spectrum of T. Moreover, it is easy to check that its point spectrum, $\sigma_p(T)$, must be a subset of $\mathbb{T}$. In [8] and [38], it is proved that each wild and rigid operators have spectral radius equals to 1 respectively. For the sake of completeness, we shall show how this is a consequence of the spectral radius formula and the following result of Müller and Vršovskŷ:

Proposition 4.49 *[73, Theorem 3] Let X be a real or complex Banach space. Let T be an operator on X. If*

$$\sum_{k=1}^{\infty} \frac{1}{\|T^k\|} < \infty,$$

then A_T is dense in X.

Lemma 4.50 *[8, Before of Proposition 4.1][38, Proposition 2.20] Let X be a complex Banach space and let T be a bounded operator such that R_T has nonempty interior. Then $r(T) = 1$. Particularly, wild operators and rigid operators have spectral radius equal to 1.*

Proof. Let $r = r(T)$. If $r < 1 - \varepsilon < 1$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\|T^n\| \leq (1-\varepsilon)^n$, which tends to 0. Then, for every point $x \in X$, its orbit under T would tend to 0, which is a contradiction since $R_T \neq \{0\}$.

If $r > 1 + \varepsilon > 1$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\|T^n\| \geq (1+\varepsilon)^n$. Then, using Proposition 4.49 we get a contradiction since the set of recurrent point R_T would have empty interior. $\square$

Then, for all $T \in \mathcal{W}(X)$ we have that $\sigma(T) \subseteq \overline{\mathbb{D}}$, and $\sigma(T) \cap \mathbb{T} \neq \emptyset$. But invoking Riesz decomposition theorem we can get the following result, which has been stated in [38] for rigid operators.

Proposition 4.51 *Let X be a complex Banach space X and let T be a bounded operator such that R_T has nonempty interior. Then each connected component of $\sigma(T)$ intersects the unit circle $\mathbb{T}$.*

Proof. Let C be a connected component of $\sigma(T)$. Suppose that $C \neq \sigma(T)$, then there exist two disjoint open sets U_1 and U_2 such that $U_\mathrm{i} \cap \sigma(T) \neq \emptyset$ for i $= 1,\ 2$, $\sigma(T) \subset U_1 \cup U_2$ and $C \subseteq U_1$. By Riesz decomposition theorem, there exist two T-invariant subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$, and the restrictions satisfy $\sigma(T|_{X_\mathrm{i}}) = U_\mathrm{i} \cap \sigma(T)$, for i $= 1,\ 2$. Since $r(T) = 1$, we have that $r(T|_{X_1}) \leq 1$. If $r(T|_{X_1})$ is strictly less than 1, as in the first case of Lemma 4.50, we can deduce that the set of recurrent point of T must be contained in X_2. Therefore, R_T would have empty interior. Finally, a standard argument finishes the proof. $\square$

Remark 4.52 *Proposition 4.51 is a spectral property that the operators in $FP(X)$ do not enjoy necessarily, unlike of Lemma 4.50 which remains true for them. In fact, the spectrum of the operator T constructed in Remark 4.3 is $\sigma(T) := \{exp(\mathrm{i}\pi/m_n) :\ n \in \mathbb{N}\} \cup \{0, 1\}$, whenever the wild part is constructed as in Section 4.3, associated to the sequence $(m_n)_n$. For details on the computation of $\sigma(T)$, see [8].*

Remark 4.53 *Proposition 4.51 shows that if T is a non-invertible wild operator or non-invertible rigid operator, then for all $r \in [0, 1]$, there exists $\lambda \in \mathbb{C}$, $|\lambda| = r$, such that $\lambda \in \sigma(T)$.*

Corollary 4.54 *Any infinite dimensional complex Banach space having a symmetric basis admits a non-invertible wild operator whose spectrum is exactly $\overline{\mathbb{D}}$. This result remains true for infinite dimensional Banach spaces which contain a complemented subspace having symmetric basis.*

Proof. Let X be an infinite dimensional Banach space having a symmetric basis. Let X_1 and X_2 be two infinite dimensional complementary subspaces of X, both having symmetric basis. Let V be a finite dimensional subspace of X_1 and F be an asymptotically separated set in V such that F and $V \setminus F$ have nonempty interior. Let $(f_n)_n \in V^*$ be a sequence of linear functional that asymptotically separates F. Let us consider an increasing sequence of integers $(m_n)_n$ that satisfies both condition (4.4) and $n!m_n|m_{n+1}$ for all $n \in \mathbb{N}$. Let $(\mathrm{e}_n)_n$ be a symmetric basis in X_2. Let us consider a countable partition of $\mathbb{N}$, $\{\mathbb{N}_\mathrm{i} :\ \mathrm{i}\}$, such that each $\mathbb{N}_\mathrm{i}$ is an infinite set. For $\mathrm{i} \in \mathbb{N}$, let us define the subspaces

$$X_{2,\mathrm{i}} = \overline{\mathrm{span}}(\mathrm{e}_n :\ n \in \mathbb{N}_\mathrm{i}) \quad \text{ and } \quad Y_{2,\mathrm{i}} = \overline{\mathrm{span}}(\mathrm{e}_n :\ j \in \mathbb{N} \setminus \{\mathrm{i}\}, n \in \mathbb{N}_j).$$

Observe that, for each $\mathrm{i} \in \mathbb{N}$, $X_{2,\mathrm{i}}$ and $Y_{2,\mathrm{i}}$ are complementary subspaces of X_2. Let $(n(\mathrm{i}, k))_k$ be the increasing enumeration of $\mathbb{N}_\mathrm{i}$. Let S_i be a bounded operator on $X_{2,\mathrm{i}}$ constructed as in Proposition 4.47 using the sequence $(m_{n(\mathrm{i},k)})_k$. Observe that, since $n!m_n|m_{n+1}$,it follows that $km_{n(\mathrm{i},k)}|m_{n(\mathrm{i},k+1)}$ for all i, $k \in \mathbb{N}$. Moreover, all the coefficients used to construct S_i are uniformly bounded by a constant independent of i. Let $(q_j)_j$ be an enumeration of the set $\{a + b\mathrm{i} \in \mathbb{D} :\ a, b \in \mathbb{Q}\}$. Let us fix $j \in \mathbb{N}$. By Remark 4.53, we know that there exists $\lambda \in \sigma(S_j)$ such that $|\lambda| = |q_j|$. Let ρ_j be a j-root of unity such that $\mathrm{dist}(q_j, \rho_j\sigma(S_j)) \leq 2\pi/j$. Now, we can define the bounded operator S on X_2 by the formula

$$S\mathrm{e}_n = \rho_j S_j \mathrm{e}_n, \text{ for all } n \in \mathbb{N}_j.$$

Since $(\mathrm{e}_n)_n$ is a symmetric basis in X_2, we can deduce that S is a bounded, rigid operator. Moreover, its associated sequence can be chosen as $(2m_n)_n$. Also, for each i, observe that $X_{2,\mathrm{i}}$ and $Y_{2,\mathrm{i}}$ are invariant subspaces for S. Let us now consider $T \in \mathcal{W}(X_1)$ be an operator constructed as in Section 4.3 associated to the sequence $(m_n)_n$. Analogous to the proof of Theorem 4.45, we define the bounded operator U on X by the formula $U(x) = Tx_1 + Sx_2$, where x_{i} is the projection of x onto X_{i}, for $\mathrm{i} = 1,\ 2$. As in Theorem 4.45, we conclude that U is a wild operator. Indeed, $A_U = A_T + X_2$ and $R_U = R_T + X_2$. Observe now that, for each i, $X_{2,\mathrm{i}}$ and $Y_{2,\mathrm{i}} + X_1$ are complementary subspaces on X and both spaces are U-invariant. Hence, noticing that $\rho_{\mathrm{i}} S_{\mathrm{i}} \in \mathcal{L}(X_{2,\mathrm{i}})$ is the restriction of U to the subspaces $X_{2,\mathrm{i}}$, we get that $\sigma(U) \supseteq \sigma(\rho_{\mathrm{i}} S_{\mathrm{i}})$, for all i. Therefore, $\sigma(U) \supseteq \overline{\bigcup_{\mathrm{i}} \sigma(\rho_{\mathrm{i}} S_{\mathrm{i}})} \supseteq \overline{\mathbb{D}}$. Finally, Lemma 4.50 finishes the proof.

□

Remark 4.55 *Let X be an infinite dimensional separable complex Banach space with symmetric basis. In the proof of Corollary 4.54 we show that X admits a bounded rigid operator S such that $\sigma(S) = \overline{\mathbb{D}}$.*

4.6 Approximation result

In [8], it is shown that the set $\mathcal{W}(X)$ is dense in $\mathcal{L}(X)$ for the strong operator topology and that $I\mathrm{d}$ is a cluster point of $\mathcal{W}(X)$ for the norm topology, see Remark 4.34. In [9, Proposition 4.4.1], we can find that any diagonal operator on a Banach space X having a symmetric basis can be approximated in norm by operators in $RP(X)$. We improve this result by showing that they can be approximated by wild operators. Moreover, thanks to the Weyl-von Neumann-Berg Theorem ([19] or [36, Theorem 39.4]), we can say more about the closure of $\mathcal{W}(X)$ whenever X is a separable infinite dimensional Hilbert space.

Theorem 4.56 *Let X be a separable infinite dimensional complex Banach space having a normalized unconditional basis $(\mathrm{e}_n)_n$. Then, the set of unitary diagonal operators with respect to $(\mathrm{e}_n)_n$ is contained in the norm closure of $\mathcal{W}(X)$, i.e. bounded linear operators D such that $D\mathrm{e}_n \in \mathbb{T}\mathrm{e}_n$ for all $n \in \mathbb{N}$. Moreover, if X is a separable infinite dimensional complex Hilbert space, each unitary operator belongs to the norm closure of $\mathcal{W}(X)$.*

Before proving Theorem 4.56 we need the following proposition.

Proposition 4.57 *Let X be an infinite dimensional complex Banach space with a normalized unconditional basis $(\mathrm{e}_n)_n$. Let D be a unitary diagonal operator, with respect to $(\mathrm{e}_n)_n$, on X having only finitely many eigenvalues. Then D belongs to the norm closure of $\mathcal{W}(X)$.*

Proof. Let $\{\alpha_{\mathrm{i}}\}_{\mathrm{i}=1}^N \subset \mathbb{T}$ be the set of eigenvalues of D. Let us assume that each α_{i} is a root of unity. Let us call $X^{\mathrm{i}} := \ker(D - \alpha_{\mathrm{i}} I\mathrm{d})$. Since X is infinite dimensional and $X = \oplus_{\mathrm{i}=1}^N X^{\mathrm{i}}$, we can assume that X^1 is infinite dimensional. To fix notation, for $x \in X$, we denote by x^{i} the canonical projection of x onto X^{i}. Recall that these projections are bounded since the basis $(\mathrm{e}_n)_n$ is unconditional. Let T_1 be a wild operator on X^1, constructed as in section 4.3. Let $(m_n)_n$ be the sequence of positive integers with which T_1 is constructed. We impose that $\alpha_{\mathrm{i}}^{m_1} = 1$, for all $\mathrm{i} \leq N$. Consider the bounded operator T defined by $Tx^1 = \alpha_1 T_1 x^1$ and $Tx^{\mathrm{i}} = Dx^{\mathrm{i}} = \alpha_{\mathrm{i}} x^{\mathrm{i}}$ for each i greater than 1. We can notice that:

$$
\begin{aligned}
A_T &= \{x \in X : \lim_{n\to\infty} \|T^n x\| = \infty\} = \{x \in X : \lim_{n\to\infty} \|T^n x^1\| = \infty\} \\
&= \{x \in X : \lim_{n\to\infty} |\alpha_1^n| \|T_1^n x^1\| = \infty\} = A_{T_1} + \bigoplus_{\mathrm{i}=2}^{N} X^{\mathrm{i}}.
\end{aligned}
$$

and also that:

$$
\begin{aligned}
R_T &= \{x \in X : \liminf_{n\to\infty} \|T^n x - x\| = 0\} \\
&\supseteq \{x \in X : \liminf_{n\to\infty} \max\{\|T^{2m_n} x^{\mathrm{i}} - x^{\mathrm{i}}\| : \mathrm{i} = 1, ..., N\} = 0\} \\
&\supseteq \{x \in X : \liminf_{n\to\infty} \|T_1^{2m_n} x^1 - x^1\| = 0\} = R_{T_1} + \bigoplus_{\mathrm{i}=2}^{N} X^{\mathrm{i}},
\end{aligned}
$$

and then T belongs to $\mathcal{W}(X)$. Bearing in mind Remark 4.34, for any $\varepsilon > 0$, there exists $T_1 \in \mathcal{W}(X^1)$ such that $\|U|_{X^1} - T_1\| \leq \varepsilon$. Finally, since $(\mathrm{e}_n)_n$ is an unconditional basis of X, we can deduce that $\|U - T\|_{\mathcal{L}(X)} \leq C\|U|_{X^1} - T^1\|_{\mathcal{L}(X^1)}$, for some positive constant $C > 0$ depending only on the unconditional constant of $(\mathrm{e}_n)_n$, achieving the result whenever the set of eigenvalues are roots of unity. A standard argument finishes the proof for the general case. $\square$

Proof of Theorem 4.56. Let D be an unitary diagonal operator on X with respect to the unconditional basis $(\mathrm{e}_n)_n$. For each $n \in \mathbb{N}$, let $\alpha_n \in \mathbb{T}$ be the complex number such that $D\mathrm{e}_n = \alpha_n \mathrm{e}_n$. For $k \in \mathbb{N}$, let D_k be the bounded diagonal operator on X defined by $D_k \mathrm{e}_n = \alpha_{k,n}\mathrm{e}_n$, where:

$$\alpha_{n,k}^k = 1, \ \arg(\alpha_n) - \arg(\alpha_{n,k}) \in [0, 2\pi/k), \ \forall n \in \mathbb{N}.$$

Since (e_n) is an unconditional basis, we have that D_k is a bounded operator. Moreover, since k-roots of unity are finite, by Proposition 4.57, D_k belongs to $\overline{\mathcal{W}(X)}$. Finally, if $(\mathrm{e}_n)_n$ is a b-unconditional basis, we can easily get that $\|D - D_k\| \leq 2b\pi/k$, achieving the first part of the theorem.

For the second part, let $U \in B(X)$ be an unitary operator, where X is a separable Hilbert space. Let $\varepsilon > 0$. Since U is a normal operator, invoking Weyl-von Neumann-Berg Theorem, there exist a diagonalizable operator D, a compact operator K such that $N = D + K$ and $\|K\| \leq \varepsilon$. Let $(\mathrm{e}_n)_n$ be the orthonormal sequence associated to D. Notice that $D = \sum_{\mathrm{i}=1}^{\infty} \alpha_n \mathrm{e}_n \otimes \mathrm{e}_n$ for some sequence $(\alpha_n) \subseteq \mathbb{C}$. Since $\|U\mathrm{e}_n\| = 1$ for all $n \in \mathbb{N}$, we get:

$$
\begin{aligned}
|\alpha_n|^2 &= \langle D\mathrm{e}_n, D\mathrm{e}_n \rangle \\
&= \|N\mathrm{e}_n\|^2 - 2\mathrm{Re}(\langle N\mathrm{e}_n, K\mathrm{e}_n \rangle) + \|K\mathrm{e}_n\|^2 \\
&\geq 1 - 2\|K\| \geq 1 - 2\varepsilon.
\end{aligned}
$$

On the other hand, $|\alpha_n| \leq \|D\| \leq \|N\| + \|K\| \leq 1 + \varepsilon$. Let us define the diagonal operator $\hat{D}$ by $\hat{D}\mathrm{e}_n = \hat{\alpha}_n \mathrm{e}_n$, where $\hat{\alpha}_n = \frac{\alpha_n}{|\alpha_n|}$. Notice that $\|D - \hat{D}\| \leq 1 - \sqrt{1 - 2\varepsilon}$, when $\varepsilon < 1/2$. Finally, since every unitary diagonal operator belongs to the norm closure of $\mathcal{W}(X)$, we conclude that U can be approximated by wild operators on X as well. $\square$

Remark 4.58 *We can notice that the first part of Theorem 4.56 is analogous to the following result: the set of complex bounded sequences* $x := (x_n)_n$ *such that* $\operatorname{card}(\{x_{\mathrm{i}} : \mathrm{i} \in \mathbb{N}\})$ *is finite, is dense in the space of bounded sequences* ℓ^∞.

Chapter 5

Desingularization of smooth sweeping processes

Chapter 5 and 6 form the second part of this thesis. From now on, we explore different issues on finite dimensional spaces. In the present chapter, we generalize the KŁ-inequality for real-analytic or semi-algebraic functions to multivalued map. This new inequality is a desingularization for the coderivative, which is an abstract notion of differentiability for multivalued maps, see Definition 5.3. Moreover, we provide several characterizations for multivalued maps which satisfy the mentioned inequality involving, for instance, length of the curves solutions of the related sweeping process, see Definition 5.2.

5.1 Kurdyka-Łojasiewicz inequality

It is well-known that every $\mathcal{C}^1$ smooth function $f : \mathbb{R}^n \to \mathbb{R}$ which is definable in some o-minimal structure has finitely many critical values. Kurdyka [68] showed that if $\bar{r} \in f(\mathbb{R}^n)$ is a critical value and $\mathcal{U}$ is a nonempty open bounded subset of $\mathbb{R}^n$, then there exist $\rho > 0$ and a $\mathcal{C}^1$-smooth function $\psi : [\bar{r}, \bar{r} + \rho] \to [0, +\infty)$ satisfying

$$\|\nabla(\psi \circ f)(x)\| \geq 1, \qquad \text{for all } \ x \in \mathcal{U} \text{ such that } \ f(x) \in (\bar{r}, \bar{r} + \rho). \tag{5.1}$$

The above inequality generalizes to o-minimal functions the Łojasiewicz gradient inequality (established in [70] for the class of $\mathcal{C}^1$ subanalytic functions) and is nowadays known as the Kurdyka-Łojasiewicz inequality (in short, KŁ-inequality). For definitions and properties of o-minimal functions the reader is referred to [94]. Both the Łojasiewicz and the KŁ-inequality have been further extended to nonsmooth (subanalytic and respectively o-minimal) functions, see [24, 25]. These inequalities allow to control uniformly the lengths of the bounded (sub)gradient orbits, see [71, 68, 24] .

One of the main features of Kurdyka's work [68] was to consider the so-called *talweg* function

$$m(r) = \inf_{x \in \mathcal{U}} \{||\nabla f(x)|| : \, f(x) = r\}, \qquad r \in (\bar{r}, \bar{r} + \rho), \tag{5.2}$$

which captures the worst behaviour (closer to criticality) of the gradient at the level set $[f = r]$. Kurdyka used the above function to defined the talweg set $\mathcal{V}(r)$ consisting of points

$x \in f^{-1}(r)$ with $||\nabla f(x)|| \leq 2m(r)$. He then made use of a definable version of the curve selection lemma to obtain a smooth curve $r \mapsto \theta(r) \in \mathcal{V}(r)$ which is directly linked to the *desingularizing* function ψ. A straightforward consequence of (5.1) is that the length of every bounded gradient curve $\dot{\gamma} = -\nabla f(\gamma)$ contained in $f^{-1}((\bar{r}, \bar{r} + \rho))$ is majorized by $\psi(\bar{r}+\rho)-\psi(0)$ (and therefore it is bounded). The same is true for the lengths of the piecewise gradient curves, that is, curves obtained by concatenating countably many gradient curves $\{\gamma_{\mathrm{i}}\}_{\mathrm{i}\geq 1}$, where $\gamma_{\mathrm{i}} \subset f^{-1}([r_{\mathrm{i}+1}, r_{\mathrm{i}}))$ and $(r_{\mathrm{i}})_{\mathrm{i}}$ is a strictly decreasing sequence in $(\bar{r}, \bar{r} + \rho)$ converging to $\bar{r}$. These curves have countably many discontinuities.

Outside the framework of o-minimality the KŁ-inequality (5.1) may fail even for $\mathcal{C}^2$-smooth functions [26, Section 4.3] or for $\mathcal{C}^\infty$-smooth function with a unique critical value [76, p. 12]. Bolte, Daniilidis, Ley and Mazet in [26] considered the problem of characterizing the existence of a desingularization function ψ and the validity of (5.1) for an upper isolated critical value $\bar{r}$ of a semiconvex *coercive* function f defined in a Hilbert space. (A function f is called coercive, if it has bounded sublevel sets. This assumption replaces the use of an open bounded set $\mathcal{U}$ in Kurdyka's result.) We reproduce below one of the main results of the aforementioned work, see [26, Theorem 20], for the special case where the function is smooth and defined in finite dimensions.

Theorem 5.1 (characterization of the KŁ-inequality) *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a $\mathcal{C}^2$-smooth (or more generally $\mathcal{C}^1$-smooth semi-convex) coercive function and $\bar{r} \in f(\mathbb{R}^n)$ an upper isolated critical value. The following statements are equivalent:*

a) ***(KŁ-inequality)*** *There exist $\rho > 0$ and a smooth function $\psi : [\bar{r}, \bar{r} + \rho) \to [0, \infty)$ such that*

$$\|\nabla(\psi \circ f)(x)\| \geq 1, \text{ for all } x \in f^{-1}((\bar{r}, \bar{r} + \rho)).$$

b) ***(Length control for gradient curves)*** *There exist $\rho > 0$ and a strictly increasing continuous function $\sigma : [\bar{r}, \bar{r} + \rho) \to [0, \infty)$ with $\sigma(\bar{r}) = 0$ such that*

$$\int_0^T \|\dot{\gamma}(t)\| \mathrm{d}t \leq \sigma(f(\gamma(0))) - \lim_{t\to T} \sigma(f(\gamma(t))), \qquad \textit{(we may have } T = +\infty\textit{)}$$

for all gradient curves $\gamma : [0, T) \to \mathbb{R}^n$ with $\gamma([0, T)) \subset f^{-1}((\bar{r}, \bar{r} + \rho))$.

c) ***(Length bound for piecewise gradient curves)*** *There exist $\rho, M > 0$ such that*

$$\int_0^T \|\dot{\gamma}(t)\| \mathrm{d}t \leq M,$$

for all piecewise gradient curves $\gamma : [0, T) \to \mathbb{R}^n$ with $\gamma([0, T)) \subset f^{-1}((\bar{r}, \bar{r} + \rho))$.

d) ***(Integrability condition)*** *There exists $\rho > 0$ such that the function*

$$r \mapsto \sup_{x\in f^{-1}(r)} \frac{1}{\|\nabla f(x)\|}, \qquad r \in (\bar{r}, \bar{r} + \rho),$$

is finite-valued and belongs to $\mathcal{L}^1(\bar{r}, \bar{r} + \rho)$.

Recently, Daniilidis and Drusvyatskiy [41] showed that every multivalued map $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ with definable graph admits a desingularization of its graphical coderivative $D^*S : \mathbb{R}^n \rightrightarrows \mathbb{R}$

around any critical value $t \in \mathbb{R}$. (Relevant definitions and a more precise statement are given in Section 5.2.3.) This result yields a uniform bound for the lengths of all bounded orbits of the sweeping process defined by S (see forthcoming Definition 5.2). The aforementioned results of [41] are also covering the results of Kurdyka in [68] by considering a sweeping process mapping S related to the sublevel sets of the smooth definable function f (*c.f.* Remark 5.9).

The main contributions of this work are the following:

- Without assuming o-minimality, we characterize the desingularization of the coderivative of a smooth sweeping process (see Definition 5.10) by establishing an analogous result to Theorem 5.1. This is the main result of this work, which is resumed in Section 5.3.2.
- Since the evolution of the sweeping process is not reversible in time, we introduce in Definition 5.3 an asymmetric version of the modulus for the coderivative of a multivalued map S, $\|D^*S(t,x)|^+$, that captures the orientation of the dynamics. (In [41], the prevailing assumption of o-minimality made it possible to work directly with the usual modulus.)
- We establish an asymmetric version of [84, Theorem 9.40] (sometimes known as the Mordukhovich Criterion) relating the asymmetric modulus of the coderivative to the *oriented calmenss* of the multivalued map (Proposition 5.26). We then obtain Theorem 5.17 (Section 5.3.3) which relates the desingularization of the coderivative with the length of discrete sequences given by the catching–up algorithm. (This algorithm can be perceived as the proximal algorithm over a function f whenever the multivalued map S is defined by the sublevel sets of f.)

The outline of this chapter is as follows: In Section 5.2, we fix our notation, we quote preliminary results of variational analysis required in the sequel. In Section 5.3, we fix our setting, explain our assumptions and state the two main results (Theorem 5.15 and Theorem 5.17). The proofs of these results together with other auxiliary results will be given in Section 5.4.

5.2 Notation and Preliminaries

The notation used along this chapter is standard and follows the lines of [84]. For any two nonempty sets $A,\ B \subset \mathbb{R}^n$, the excess of A over B is given by $\mathrm{ex}\,(A,B) := \sup\{\mathrm{d}(x,B) :\ x \in A\}$ and their Hausdorff-Pompeiu distance is defined by $\mathrm{dist}\,(A,B) := \max\{\mathrm{ex}\,(A,B),\, \mathrm{ex}\,(B,A)\}$.

Let $C \subseteq \mathbb{R}^n$ be a closed set and let $x \in \mathbb{R}^n$. The set of projections of x at C is defined by $\mathrm{Proj}_C(x) := \{y \in C : \|x-y\| = \mathrm{d}(x,C)\}$. The Fréchet normal cone to C at $x \in C$, denoted by $\hat{N}_C(x)$, is the set of vectors $v \in \mathbb{R}^n$ satisfying

$$\limsup_{\substack{y \in C \\ y \to x}} \frac{\langle v, y-x\rangle}{\|y-x\|} \leq 0.$$

The limiting normal cone to C at x, denoted by $N_C(x)$, consists of all vectors $v \in \mathbb{R}^n$ such that there exists a sequence $(x_i)_i \subset C$ and $v_i \in \hat{N}_C(x_i)$ satisfying $x_i \to x$ and $v_i \to v$.

5.2.1 Sweeping process dynamics

Let $S:\mathbb{R}\rightrightarrows\mathbb{R}^n$ be a multivalued map. The effective domain of S, denoted by $\mathrm{dom}(S)$, is the set $\{t\in\mathbb{R}:\ S(t)\neq\emptyset\}$. We denote by $\mathcal{S}=\mathrm{gph}(S)$ the graph of the multivalued map S, that is,

$$\mathcal{S}=\mathrm{gph}\,(S):=\{(t,x)\in\mathbb{R}^{n+1}:\ x\in S(t)\}.$$

Let us introduce the following dynamical system, known as sweeping process, determined by the multivalued function S. The definition implicitely implies that $\mathrm{dom}(S)$ has nonempty interior, and is often an interval (possibly unbounded). In particular, in our seeting (*c.f* Assumptions in Section 5.3.1) $\mathrm{dom}(S)$ will always be an interval (possibly unbounded).

Definition 5.2 (sweeping process dynamics) *Let $S:\mathbb{R}\rightrightarrows\mathbb{R}^n$ be a multivalued map and $I\subset\mathrm{dom}(S)$ be a nonempty interval of $\mathbb{R}$. We say that the absolutely continuous curve $\gamma:I\to\mathbb{R}^n$ is a solution (orbit) of the sweeping process defined by S if*

$$\begin{cases}-\dot\gamma(t)\in N_{S(t)}(\gamma(t)),\ \forall_{\text{a.e.}}\ t\in I,\\ \quad\gamma(t)\in S(t)\ \textit{for all}\ t\in I,\end{cases}\tag{5.3}$$

where $N_{S(t)}(\gamma(t))$ stands for the normal cone of $S(t)$ at $\gamma(t)$.

Notice that (5.3) can be formally satisfied by curves with possible discontinuities (the set of discontinuities has then to be of measure zero). For our purposes it is useful to consider the class of *piecewise absolutely continuous* curves, that is, curves $\gamma:I\to\mathbb{R}^n$ whose possible discontinuities are contained in a closed countable set D and being absolutely continuous on any interval subset of $I\diagdown D$. This latter set is open, therefore it is a countable union of disjoint intervals J_{i}, and γ is required to be absolutely continuous on each J_{i}.

Notation ($\mathcal{AC}(S,I)$, $\mathcal{PAC}(S,I)$). We denote by $\mathcal{AC}(S,I)$ (respectively $\mathcal{PAC}(S,I)$) the set of absolutely continuous (respectively, piecewise absolutely continuous) orbits of the sweeping process defined by S on the interval $I\subset\mathrm{dom}(S)$. The length of a (piecewise) absolutely continuous curve $\gamma:I\to\mathbb{R}^n$ is given by the formula

$$\ell(\gamma):=\int_I\|\dot\gamma(t)\|\mathrm{d}t.$$

5.2.2 Coderivative, (oriented) modulus and (oriented) talweg.

Let $S:\mathbb{R}\rightrightarrows\mathbb{R}^n$ be a multivalued map with closed values.

Definition 5.3 (Coderivative) *The (limiting) coderivative of S at $(t,x)\in\mathcal{S}$ in $u\in\mathbb{R}^n$ is defined as follows:*

$$D^*S(t,x)(u):=\{a\in\mathbb{R}:\ (a,-u)\in N_{\mathcal{S}}(t,x)\}.$$

Therefore $D^*S(t,x):\mathbb{R}^n\rightrightarrows\mathbb{R}$ is a multivalued map and

$$(u,a)\in\mathrm{gph}\,D^*S(t,x)\quad\text{ if and only if }\quad(a,-u)\in N_{\mathcal{S}}(t,x).$$

Since gph $D^*S(t,x)$ is a cone, the map $D^*S(t,x)$ is positively homogeneous and we can define its modulus via the formula:

$$\|D^*S(t,x)\|^+ := \sup_{\|u\|\leq 1} \{\, |a| : \, a \in D^*S(t,x)(u)\} \, .$$

Although the above definition of a modulus is classical and relates nicely to the Lipschitz continuity of S (*c.f.* [84, Theorem 9.40]), the symmetry of the absolute value of $\mathbb{R}$ (representing the time in our dynamics) does not fit to the non-reversible dynamics of the sweeping process. To remedy this, one needs to replace $|a|$ in the above formula by $a^+ := \max\{0, a\}$ which eventually gives rise to the following definition.

Definition 5.4 (Asymmetric modulus of coderivative) *For every $(t,x) \in \mathcal{S}$ we define the asymmetric modulus of the coderivative $D^*S(t,x)$ as follows:*

$$\|D^*S(t,x)|^+ = \sup\{a^+ : \; a \in D^*S(t,x)(u), \; \|u\| \leq 1\},$$

where we adopt the convention $\sup(\emptyset) = 0$.

The following example give some insight about the difference between the two moduli.

Example 5.5 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$-smooth function and set*

$$S(r) = [f \leq r] := \{x \in \mathbb{R}^n : \; f(x) \leq r\}, \qquad \textit{for all } r \in \mathbb{R}.$$

This defines a multivalued map $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ associated to f (the graph $\mathcal{S}$ of S is the epigraph of f). Let $x \in S(r)$.
If $f(x) < r$, then $x \in \operatorname{int}(S(r))$ and $N_{\mathcal{S}}(r,x) = \{0\}$, yielding $\|D^*S(r,x)\|^+ = \|D^*S(r,x)|^+ = 0$. On the other hand, since the normal space of gph(f) *at $(x, f(x))$ is exactly* $\mathbb{R}(\nabla f(x), -1)$, *if $f(x) = r$, then $N_{\mathcal{S}}(r,x) = \mathbb{R}_+(-1, \nabla f(x))$. Thus,*

$$\|D^*S(t,x)\|^+ = \frac{1}{\|\nabla f(x)\|}, \qquad \textit{but} \qquad \|D^*S(t,x)|^+ = 0.$$

We now define the *oriented talweg* function associated to the multivalued map $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$. This captures the worst case (larger value of the oriented modulus of the coderivative) on each set $S(t)$, $t \in \mathbb{R}$. This function will play an important role in our main result.

Definition 5.6 (oriented talweg) *The oriented talweg function of S denoted by $\varphi^{\uparrow}$ is defined as follows:*

$$\varphi^{\uparrow}(t) = \sup_{x \in S(t)} \{\|D^*S(t,x)|^+\}, \quad \textit{for all } t \in \operatorname{dom}(S).$$

Remark 5.7 (Asymmetric structures) *In [41] the usual talweg function φ has been considered, based on the (symmetric) modulus of the coderivative.*

$$\varphi(t) = \sup_{x \in S(t)} \{\|D^*S(t,x)\|^+\}, \quad \textit{for all } t \in \operatorname{dom}(S).$$

The difference between φ and $\varphi^{\uparrow}$ is that the modula $\|D^*S(t,x)\|^+$, $(t,x)\in\mathcal{S}$, are now replaced by their asymmetric versions $\|D^*S(t,x)|^+$. The reader might notice that $a^+ := \max\{0,a\}$ is a typical asymmetric norm of $\mathbb{R}$ and $\|D^*S(t,x)|^+$ can be seen as a natural asymmetrization of the modulus $\|D^*S(t,x)\|^+$. The use of asymmetric objects seems to be a natural tool in nonsmooth dynamics as well as in operations research (orientable graphs). More details on asymmetric structures can be found in [32] and [42].

5.2.3 Desingularization of the coderivative (definable case).

We now recall the main result of [41]. If $S:\mathbb{R}\rightrightarrows\mathbb{R}^n$ is a multivalued map with a closed *bounded* graph $\mathcal{S}$, then assuming that $\mathcal{S}$ is definable in some o-minimal structure, for every $a\in\mathbb{R}$ there exists $\rho>0$ and a strictly increasing, continuous function $\Psi\colon[0,\rho]\to\mathbb{R}$ that is $\mathcal{C}^1$-smooth on $(0,\rho)$, it satisfies $\Psi(0)=a$ and $\Psi'(r)>0$ for all $r\in(0,\rho)$ and

$$\|D^*(S\circ\Psi)(r,x)\|^+\le 1 \qquad \text{for all } r\in(0,\rho) \text{ and all } x\in S(\Psi(r)). \tag{5.4}$$

It is easily seen that Ψ is an homeomorphism between $[0,\rho]$ and $[a,b]$ where $b=\Psi(\rho)$ and a diffeomorphism between $(0,\rho)$ and (a,b). Inequality (5.4) has a particular interest when $a\in\mathbb{R}$ is a *critical value* of the coderivative D^*S of the sweeping process, that is,

$$\varphi(t)=\sup_{x\in S(t)}\|D^*S(t,x)\|^+=+\infty.$$

In this case we say that Ψ *desingularizes* the (modulus of the coderivative around the) critical value a. The assumption of o-minimality on S guarantees that the set of critical values is finite. In [41] it has further been established, as consequence of (5.4), that all bounded orbits of the sweeping process S have finite length and that the talweg function φ is integrable on $[a,b]$.

Let us notice that $\|D^*S(t,x)|^+\le\|D^*S(t,x)\|^+$ (and consequently $\varphi^{\uparrow}(t)\le\varphi(t)$) for all $t\in[a,b)$ and $x\in S(t)$. Therefore, we obtain the following.

Corollary 5.8 (desingularization of oriented coderivative – definable case) *If $S:\mathbb{R}\rightrightarrows\mathbb{R}^n$ is a multivalued map with a closed definable bounded graph, then for every $a\in\mathbb{R}$ (possibly critical for the oriented modulus) there exists $\rho>0$ and $b>a$ such that:*

(i). there exists an increasing homeomorphism $\Psi\colon[0,\rho]\to[a,b]$ which is $\mathcal{C}^1$-diffeomorphism on $(0,\rho)$ such that:

$$||D^*(S\circ\Psi)(r,x)|^+\le 1 \qquad \textit{for all } r\in(0,\rho) \textit{ and all } x\in S(\Psi(r)). \tag{5.5}$$

(ii). $\int_a^b\varphi^{\uparrow}(t)<\infty$ (the oriented talweg function is integrable).

Remark 5.9 *[Relation with the KŁ-inequality] (i). The described desingularization of the coderivative can be seen as a generalization of the KŁ-inequality for $\mathcal{C}^1$-smooth definable functions (established by Kurdyka in [68]) in the following sense: let $f:\mathbb{R}^n\to\mathbb{R}$ be a $\mathcal{C}^1$-smooth coercive function which is definable in some o-minimal structure. Then, the multivalued* *function*

$$\begin{cases} S_f : \mathbb{R} \rightrightarrows \mathbb{R}^n \\ S_f(t) = [f \le -t], \quad t \in \mathbb{R} \end{cases} \tag{5.6}$$

is o-minimal (it is definable in the same o-minimal structure as f) and the desingularization of its gradient described in (5.1) can be deduced from the desingularization coderivative of S and vice versa. We refer the reader to [41, Section 5.1] for more details.

(ii). In [41], the assumption that $\mathcal{S}$ is unbounded has not been considered, and similarly to (5.2)*, the supremum of the definition of $\varphi(t)$ had to be taken over $S(t) \cap \mathcal{U}$, where $\mathcal{U} \subset \mathbb{R}^n$ is a fixed open bounded set, which gives rise to a talweg function $\varphi_{\mathcal{S}}$ depending on $\mathcal{U}$. Even if Section 5.3 we deal with potentially unbounded sweeping processes, we do not need to make use of $\mathcal{U}$, thanks to the assumptions given in Section 5.3.1.*

5.3 Characterization of desingularization of the coderivative

We are interested in sweeping process mappings S that are not o-minimal (we shall assume smoothness of their graph instead). Under some mild assumptions, we shall characterize the existence of a desingularizing function Ψ that desingularizes the asymmetric modulus of the coderivative (*c.f.* Corollary 5.8). We give below our setting.

5.3.1 Assumptions, setting

Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a multivalued map with closed graph $\mathcal{S}$.

Definition 5.10 (smooth sweeping process) *We say that S is a* smooth *sweeping process if either*

– $\mathcal{S}$ is a closed connected $\mathcal{C}^1$-smooth submanifold of $\mathbb{R}^{n+1}$ of dimension at most n ; or

– $\mathcal{S}$ is a connected smooth manifold of full dimension with boundary and $\partial\mathcal{S}$ is a $\mathcal{C}^1$-smooth manifold of dimension n.

It is clear that the above assumption is satisfied if S is a sweeping process associated to a $\mathcal{C}^1$-smooth function f (*c.f.* Example 5.5 or Remark 5.9). As a consequence of this assumption we have the following result, which compares the modules versus the asymmetric modulus of D^*S.

Lemma 5.11 *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a smooth sweeping process and $(t,x) \in \mathcal{S}$. If either*

$$(a). \quad \mathcal{S} \text{ is a smooth manifold} \qquad \text{or} \qquad (b). \quad \|D^*S(t,x)|^+ > 0$$

we have

$$\|D^*S(t,x)|^+ = \|D^*S(t,x)\|^+.$$

Proof. If $\mathcal{S}$ is a smooth submanifold of $\mathbb{R}^{n+1}$, the requested equality holds true for every $(t,x) \in \mathcal{S}$ as a consequence of the fact that the limiting normal cone at any point coincides with the normal space of the manifold at the same point. On the other hand, if $\mathcal{S}$ is a manifold of full dimension with boundary such that $\partial\mathcal{S}$ is also a smooth manifold, then the normal cone $N_{\mathcal{S}}(t,x)$ is either $\{0\}$ or a ray generated by an outer pointing normal vector (s,y) of $\mathcal{S}$ at (t,x). The conclusion follows. □

Connectedness of $\mathcal{S}$ yields that $\mathrm{dom}(S)$ is an interval (possibly unbounded). We shall use the following notation:

$$\mathrm{T} = \sup(\mathrm{dom}(S)) \in \mathbb{R} \cup \{+\infty\}.$$

We also define the multivalued map $H_S : \mathbb{R} \rightrightarrows \mathbb{R}^{n+1}$ by

$$H_S(t) := \partial\mathcal{S} \cap (\{t\} \times \mathbb{R}^n), \quad \text{for all } t \in \mathbb{R}.$$

Assumption 5.12 *We say that S satisfies the:*

$(A1)$ existence assumption *if for every $(t,x) \in \mathcal{S}$ with $\|D^*S(t,x)|^+ < +\infty$, there exist $\delta_x > 0$ and at least one orbit $\gamma_x \in AC(S;[t,t+\delta_x))$ such that $\gamma_x(t) = x$.*

$(A2)$ upper regular assumption *at $\bar{t} \in \mathrm{dom}(S)$ with $\bar{t} < \mathrm{T}$, if there exists $\delta > 0$ such that $\varphi^{\uparrow}(t) < +\infty$ for all $t \in (\bar{t}, \bar{t}+\delta)$.*

$(A3)$ continuity assumption *at $\bar{t} \in \mathrm{dom}(S)$ with $\bar{t} < \mathrm{T}$, if there exists $\delta > 0$ such that the multivalued map H_S is continuous for the Pompeiu-Hausdorff metric on $(\bar{t}, \bar{t}+\delta)$ (it may be discontinuous at $\bar{t}$).*

Let us make some comments about the above assumptions:

Assumption (A1) ensures the existence of orbits issued from any non-critical point. This assumption is satisfied if the sweeping process is defined via (5.6) where f is a $\mathcal{C}^{1,1}$-smooth function, since in this case the existence of gradient orbits $\dot{\gamma} = -\nabla f(\gamma)$ is guaranteed, and these orbits are also orbits for the sweeping process S_f, up to a suitable change of variable, see Remark 5.9. Assumption (A1) is also fulfilled if S is a definable sweeping process, see [41, Section 6] or [57]. In the general case, classical existence results go back to the seminal work of J.J. Moreau [72] for convex-valued multifunctions which are Lipschitz continuous under the Hausdorff-Pompieu metric. Since then, several extensions have been obtained, see [33, 34, 66] and references therein.

Assumption (A2) is automatically satisfied in the definable case, since in this case the set of critical values is finite. In the general case, this assumption is analogous to the hypothesis made in [26, Section 3.3] that the critical values of f are upper isolated (see also statement of Theorem 5.1).

Assumption (A3) is the more restrictive, although it comes naturally from our setting. It is satisfied for the sweeping process S_f defined in (5.6) whenever f is convex or quasiconvex. In general, a smooth multivalued map $t \rightrightarrows S(t)$ is not necessarily monotone in the sense of set-inclusion and the sets $S(t)$ are not assumed convex (or of the same homology), therefore (A3) is required to guarantee a control on the behavior of the boundaries. In particular, the following result holds. (For the definitions of outer and inner semicontinuity of a multifunction the reader is referred to [84, Chapter 5].)

Proposition 5.13 *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a smooth sweeping process with bounded values and $a, b \in \mathbb{R}$ such that $(a, b) \subset \operatorname{dom}(S)$. If H_S is continuous on (a, b), then S is also continuous on (a, b).*

Proof. Let I be a nontrivial interval contained in a compact subset of (a, b). It is sufficient to prove that S is continuous on I. Since $\mathcal{S} \subset \mathbb{R}^{n+1}$ is closed and $S(t) = \mathcal{S} \cap (\{t\} \times \mathbb{R}^n)$, for every $t \in \mathbb{R}$, the map S has closed (therefore, compact) values and S is outer semicontinuous. Let us assume, towards a contradiction, that S is not continuous on I, that is, there exists $\bar{t} \in I$ such that S is not inner semicontinuous at $\bar{t}$. We deduce that there exist $\overline{x} \in S(\overline{t})$, $\varepsilon > 0$ and a sequence $(t_k)_k \subset \operatorname{dom}(S)$, converging to $\bar{t}$, such that

$$\mathrm{d}(\overline{x}, S(t_k)) \geq \varepsilon, \quad \text{for all } k \in \mathbb{N}.$$

The above easily yields that $(\bar{t}, \overline{x}) \in \mathcal{S} \setminus \operatorname{int}(\mathcal{S})$, that is, $(\bar{t}, \overline{x}) \in \partial\mathcal{S}$. However, since

$$\{t_k\} \times B(\overline{x}, \varepsilon) \cap \mathcal{S} = \emptyset,$$

this contradicts the continuity of H_S at $\bar{t}$. □

Remark 5.14 *In general, the converse of Proposition 5.13 is not true. To see this, set*

$$\mathcal{S} := (\mathbb{R} \times [-2, 2]) \setminus \left\{(t, x) \in \mathbb{R}^2 : \ (t-1)^2 + x^2 \leq 1\right\}$$

and consider the sweeping process $S : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$S(t) = \mathcal{S} \cap \left(\{t\} \times \mathbb{R}^2\right).$$

It follows easily that S is a smooth sweeping process. Moreover, S is continuous at every $t \in \mathbb{R}$, but H_S is discontinuous at 0.

5.3.2 Characterizations via continuous dynamics

Before we proceed, let us set

$$\mathcal{T} := \{t \in \operatorname{dom}(S) : \text{(A2)–(A3) are fulfilled at } t\}.$$

Observe that, if $t \in \mathcal{T}$, then there is $\delta > 0$ such that $[t, t + \delta) \subset \mathcal{T}$.
We are now ready to state the main result of this work. The proof will be given in Section 5.4.2.

Theorem 5.15 *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a smooth sweeping process with bounded values that satisfies (A1). Let $a \in \mathcal{T}$ (typically a critical value for D^*S).*
The following assertions are equivalent:

a) ***(Desingularization of the coderivative)*** *There exist $b > a$, $\rho > 0$ and a homeomorphism $\Psi : [0, \rho] \to [a, b]$, which is a $\mathcal{C}^1$-diffeomorphism between $(0, \rho)$ and (a, b) with $\Psi'(r) > 0$ for every $r \in (0, \rho)$, such that:*

$$\|D^*(S \circ \Psi)(r, x)|^+ \leq 1, \qquad \text{for all } r \in (0, \rho), \text{ for all } x \in S(\Psi(r)). \tag{5.7}$$

b) ***(Uniform length control for the absolutely continuous orbits)*** *There exist $b > a$ and an increasing continuous function $\sigma : [a,b] \mapsto \mathbb{R}^+$ with $\sigma(a) = 0$ such that for every $a \le t_1 < t_2 \le b$ and $\gamma \in \mathcal{AC}(S,[t_1,t_2])$ we have:*

$$\ell(\gamma) \le \sigma(t_2) - \sigma(t_1).$$

c) ***(Length bound for the piecewise absolutely continuous orbits)*** *There exist $b > a$ and $M < \infty$ such that for every $\gamma \in \mathcal{PAC}(S,[a,b])$ we have:*

$$\ell(\gamma) \le M.$$

d) ***(Integrability of the talweg)*** *There exists $b > a$ such that*

$$\int_a^b \varphi^{\uparrow}(t) < \infty.$$

5.3.3 Characterizations via discrete dynamics

We first need the following definition.

Definition 5.16 (piecewise catching-up sequence) *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a multivalued map with closed values.*
(i). A (finite or infinite) sequence $((t_{\mathrm{i}}, x_{\mathrm{i}}))_{\mathrm{i}\ge 0} \subset \mathcal{S}$ is called a catching-up sequence *for S if $(t_{\mathrm{i}})_{\mathrm{i}\ge 0}$ is strictly increasing and*

$$x_{\mathrm{i}+1} \in \mathrm{Proj}_{S(t_{\mathrm{i}+1})}(x_{\mathrm{i}}), \quad \textit{for } \mathrm{i} \ge 0.$$

(ii). A (finite or infinite) sequence of the form

$$(t_0^0, Y_0^0), (t_1^0, Y_1^0), \ldots, (t_{k_0}^0, Y_{k_0}^0), (t_0^1, Y_0^1), (t_1^1, Y_1^1), \ldots, (t_{k_1}^1, Y_{k_1}^1), \ldots$$

is called a piecewise catching-up sequence *for S if for every $j \ge 0$*

$$((t_{\mathrm{i}}^j, Y_{\mathrm{i}}^j))_{\mathrm{i}=0}^{k_j} \subset \mathcal{S} \ \text{ is a catching-up sequence for } S \text{ and } t_{k_j}^j = t_0^{j+1}.$$

Now we are ready to state our second result which complements Theorem 5.15.

Theorem 5.17 *The statements* (a)-(d) *of Theorem 5.15 are also equivalent to the following:*

e) ***(Uniform control of catching-up sequences)*** *There exist $b > a$ and a continuous increasing function $\sigma : [a,b] \to [0,\infty)$, with $\sigma(a) = 0$, such that for every catching-up sequence $((t_{\mathrm{i}}, x_{\mathrm{i}}))_{\mathrm{i}\ge 0} \subset \mathcal{S}$ with $\{t_{\mathrm{i}}\}_{\mathrm{i}\ge 0} \in (a,b)$, and every $k \ge 1$ we have*

$$\sum_{\mathrm{i}=0}^{k} \|x_{\mathrm{i}+1} - x_{\mathrm{i}}\| \le \sigma(t_k) - \sigma(t_0). \tag{5.8}$$

*f) **(Length bound for piecewise catching-up sequences)** There exist $b > a$ and $C < \infty$ such that for any piecewise catching-up sequence*

$$\{(t_{\mathrm{i}}^j, Y_{\mathrm{i}}^j) : j \geq 0, \mathrm{i} \in \{0, \dots, k_j\}\}$$

with

$$a < t_0^0 < t_1^0 < \dots < t_{k_0}^0 = t_0^1 < t_1^1 < \dots < b$$

we have:

$$\sum_{j \geq 0} \sum_{\mathrm{i}=0}^{k_j} \|Y_{\mathrm{i}+1}^j - Y_{\mathrm{i}}^j\| \leq C.$$

5.4 Proofs

In this section we give proofs to our two main results, Theorem 5.15 (Subsection 5.4.2) and Theorem 5.17 (Subsection 5.4.4). To do so, we shall need some auxiliary results (Subsection 5.4.1) and a new notion of oriented calmness (Subsection 5.4.3).

5.4.1 Auxiliary results

The first result concerns continuity of the moduli maps.

Lemma 5.18 (continuity of the (oriented) modulus on $\partial\mathcal{S}$) *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a smooth sweeping process. Then, the functions*

$$(t, x) \mapsto \|D^*S(t, x)|^+ \qquad \textit{and} \qquad (t, x) \mapsto \|D^*S(t, x)\|^+$$

are continuous on $\partial\mathcal{S}$ for the usual topology on $\mathbb{R} \cup \{+\infty\}$.

Proof. Let us start with the case $\mathcal{S}$ has no interior. Thanks to Lemma 5.11, we have that $\|D^*S(t, x)|^+$ coincides with $\|D^*S(t, x)\|^+$ for any $(t, x) \in \mathcal{S}$. Then, the continuity of both functions is a direct consequence of the continuity of the normal spaces of a smooth manifold in the Grasmannian. Let us continue with the case $\mathcal{S}$ has interior. Let $(t, x) \in \partial\partial\mathcal{S}$. Then $N_{\mathcal{S}}(t, x)$ coincide with the ray of exterior normal vectors of $\mathcal{S}$ at (t, x). Therefore, the continuity of $(t, x) \to \|D^*S(t, x)\|^+$ and $(t, x) \to \|D^*S(t, x)|^+$ follows from the continuity of the unit outer normal vector of a smooth manifold of full dimension with boundary. $\square$

The second result asserts continuity of the (oriented) talweg function. Let us recall from Subsection 5.3.1 that the multivalued function $H_S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ is defined by $H_S(t) := \partial\mathcal{S} \cap (\{t\} \times \mathbb{R}^n)$, for all $t \in \mathbb{R}$.

Lemma 5.19 (continuity of the (oriented) talweg function) *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a smooth sweeping process such that $S(t)$ is bounded for all $t \in \mathbb{R}$. Let $[a, b] \subset \mathrm{dom}(S)$ such that H_S is continuous for the Pompeiu-Hausdorff metric on $[a, b]$. Then the talweg functions $\varphi^\uparrow$ and φ are continuous on $[a, b]$, where the image space $\mathbb{R} \cup \{+\infty\}$ is considered with its usual topology.*

Proof. Set $K := H_S([a,b])$, which is a compact set. Since

$$\varphi^{\uparrow}(t) = \max_{x\in H_S(t)} \|D^*S(t,x)|^+ \qquad \text{(respectively, } \varphi(t) = \max_{x\in H_S(t)} \|D^*S(t,x)\|^+).$$

the result follows from Lemma 5.18 and the continuity of H_S. $\square$

Proposition 5.20 (diffeomorphic rescaling of time) *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a multivalued map and $\gamma \in \mathcal{AC}(S,(a,b))$. If $\Psi : (0,\rho) \to (a,b)$ is a $\mathcal{C}^1$-smooth diffeomorphism such that $\Psi'(r) > 0$ for all $r \in (0,\rho)$, then $\tilde{\gamma} = \gamma \circ \Psi$ is an orbit of the sweeping process defined by $\tilde{S} := S \circ \Psi$, that is, $\tilde{\gamma} \in \mathcal{AC}(\tilde{S},(0,\rho))$.*

Proof. It is straighforward that $\tilde{\gamma} = \gamma \circ \Psi$ is an absolutely continuous curve. Since Ψ is a bi-Lipschitz homeomorphism on each compact interval contained in $(0,\rho)$ we deduce that for any null subset A of (a,b) the set $\Psi^{-1}(A)$ is also null (with respect to the Lebesgue measure). If $\mathcal{I}$ is the set of points of differentiability of γ for which (5.3) holds, it follows that $\mathcal{J} := \Psi^{-1}((a,b)\setminus\mathcal{I})$ is a null set and for every $r \in (0,\rho)\setminus\mathcal{J}$ it holds:

$$\tilde{\gamma}'(r) = (\gamma\circ\Psi)'(r) = \gamma'(\Psi(r))\Psi'(r) \in N_{S(\Psi(r))}(\gamma(\Psi(r))),$$

yielding that $\tilde{\gamma}$ is an orbit solution of the sweeping process defined by $S \circ \Psi$. $\square$

In the sequel, given a curve $\gamma : I \to \mathbb{R}^n$ we define its lifting $\zeta : I \to \mathbb{R}^{n+1}$ by

$$\zeta(t) = (t,\gamma(t)), \qquad t \in I.$$

Proposition 5.21 (geometric facts) *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a smooth sweeping process. Fix $\bar{t} \in \operatorname{dom}(S)\setminus\{\mathrm{T}\}$ and $\bar{x} \in S(\bar{t})$. Then:*

a) If there is $\delta > 0$ such that $\bar{x} \in S(t)$, for all $t \in (\bar{t},\bar{t}+\delta)$, then $\alpha \leq 0$ for all $(\alpha,u) \in N_{\mathcal{S}}(\bar{t},\bar{x})$.

b) If $\|D^*S(\bar{t},\bar{x})|^+ > 0$, then for any $\tau > \bar{t}$ and $\gamma \in \mathcal{AC}(S,[\bar{t},\tau))$ with $\gamma(\bar{t}) = \bar{x}$, there exists $\delta > 0$ such that

$$\zeta(t) := (t,\gamma(t)) \in \partial\mathcal{S}, \qquad \textit{for all } t \in [\bar{t},\bar{t}+\delta).$$

c) If $\operatorname{int}(\mathcal{S})$ is nonempty and $N_{\mathcal{S}}(\bar{t},\bar{x}) = \mathbb{R}_+(\alpha,u)$ with $\alpha < 0$, then there is $\delta > 0$ such that $\bar{x} \in S(t)$ for all $t \in [\bar{t},\bar{t}+\delta)$.

Proof. (a). If $(\bar{t},\bar{x}) \in \operatorname{int}(\mathcal{S})$ then $N_{\mathcal{S}}(\bar{t},\bar{x}) = \{(0,0)\}$ and the conclusion follows trivially. In the case when $(\bar{t},\bar{x}) \in \partial\mathcal{S}$, since $\partial\mathcal{S}$ is a smooth manifold, the limiting normal cone $N_{\mathcal{S}}(\bar{t},\bar{x})$ is equal to the Fréchet normal cone and is contained in the normal space of $\partial\mathcal{S}$ at $(\bar{t},\bar{x})$. Therefore, for any $(\alpha,u) \in N_{\mathcal{S}}(\bar{t},\bar{x})$ and $t \in (\bar{t},\bar{t}+\delta)$, we have $(t,\bar{x}) \in \mathcal{S}$ and

$$\limsup_{t\searrow\bar{t}} \frac{\langle(\alpha,u),(t-\bar{t},\bar{x}-\bar{x})\rangle}{\|(t-\bar{t},\bar{x}-\bar{x})\|} = \alpha \leq 0.$$

(b). Let $\tau > \bar{t}$ and $\gamma \in \mathcal{AC}(S,[\bar{t},\tau))$ with $\gamma(\bar{t}) = \bar{x}$ and assume $\|D^*S(\bar{t},\bar{x})|^+ > 0$. Since $(t,y) \mapsto \|D^*S(t,y)|^+$ is continuous on $\partial\mathcal{S}$ (Lemma 5.18), there exists a neighborhood $\mathcal{V}$ of

$(\bar{t}, \bar{x})$ such that for all $(t, y) \in \mathcal{V} \cap \partial\mathcal{S}$ we have $\|D^*S(t, y)|^+ > 0$. Therefore, there is $\delta > 0$ such that $\|D^*S(\zeta(t))|^+ > 0$ and consequently, $\zeta(t) \in \partial\mathcal{S}$ for all $t \in [\bar{t}, \bar{t} + \delta)$.

(c). It follows from our assumption that $\dim(\partial\mathcal{S}) = n$ and (α, u) is a nonzero outer normal vector of $\mathcal{S}$ at $(\bar{t}, \bar{x})$. Without loss of generality, let us assume that (α, u) is a unit vector. Since $\mathrm{int}(\mathcal{S}) \neq \emptyset$, we deduce that $(\bar{t}, \bar{x}) - \lambda(\alpha, u) \in \mathcal{S}$ for all $\lambda > 0$ sufficiently small. Let us assume, reasoning to a contradiction, that there exists a decreasing sequence $(t_k)_k \subset \mathbb{R}$ converging to $\bar{t}$ such that $\bar{x} \notin S(t_k)$, for all $k \in \mathbb{N}$. Let us now take a decreasing sequence $(\lambda_k)_k \subseteq \mathbb{R}^+$ that converges to 0 and satisfies $(\bar{t}, \bar{x}) - \lambda_k(\alpha, u) \in \mathcal{S}$ for all k. Let $\boldsymbol{z}_k \in \mathbb{R}^{n+1}$ be any vector such that

$$\boldsymbol{z}_k \in \partial\mathcal{S} \bigcap \left[(t_k, \bar{x}),\, (\bar{t} - \lambda_k\alpha, \bar{x} - \lambda_k u) \right],$$

where $\left[(t_k, \bar{x}),\, (\bar{t} - \lambda_k\alpha, \bar{x} - \lambda_k u) \right]$ stands for the line segment joining the points $(t_k, \bar{x})$ and $(\bar{t} - \lambda_k\alpha, \bar{x} - \lambda_k u)$. It follows easily that $(\mathbf{z}_k)_k$ converges to $(\bar{t}, \bar{x})$ and that

$$\left\langle \frac{\mathbf{z}_k - (\bar{t}, \overline{x})}{\|\mathbf{z}_k - (\bar{t}, \overline{x})\|}, (\alpha, u) \right\rangle \leq \langle\, (1, 0), (\alpha, u)\, \rangle = \alpha.$$

Let $\mathbf{d}$ be any accumulation point of the sequence $(\boldsymbol{z}_k - (\bar{t}, x))/\|\boldsymbol{z}_k - (\bar{t}, x)\|$. Then, $\mathbf{d}$ belongs to the Bouligand tangent cone of $\partial\mathcal{S}$, which coincides with the tangent space of $\mathcal{S}$ at the same point. Therefore $\mathbf{d}$ should be orthogonal to the normal vector (α, u). However, $\langle \mathbf{d}, (\alpha, u) \rangle \leq \alpha < 0$, which leads to a contradiction. $\square$

The following lemma is crucial in the proof of our main theorem since it relates the value of the coderivative with the velocity of the orbit of the sweeping process. The proof follows closely the proof of [41, Theorem 4.1] where a similar result has been established for the usual modulus $\|D^*S(t, \gamma(t))\|^+$.

Lemma 5.22 *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a smooth sweeping process and $\gamma \in \mathcal{AC}(S, [a, b))$. Then,*

$$\|\dot{\gamma}(t)\| = \|D^*S(t, \gamma(t))|^+,$$

for all $t \in [a, b)$ such that $-\dot{\gamma}(t) \in N_{S(t)}(\gamma(t))$ and $\|D^*S(t, \gamma(t))|^+$ is finite.

Proof. Let $t \in [a, b)$ be a point of differentiability of γ such that $-\dot{\gamma}(t) \in N_{S(t)}(\gamma(t))$ and that $\|D^*S(t, \gamma(t))|^+$ is finite.

First case: $\dot{\gamma}(t) = 0$.
If $\zeta(t) := (t, \gamma(t)) \in \mathrm{int}(\mathcal{S})$, the desired equality holds trivially, while if $\zeta(t) \in \partial(\mathcal{S})$, then $\dot{\zeta}(t) = (1, 0)$ belongs to the tangent space of $\partial\mathcal{S}$ at $\zeta(t)$. Since S is a smooth sweeping process, the normal cone $N_{\mathcal{S}}(\zeta(t))$ is contained in the normal space of $\partial\mathcal{S}$ at $\zeta(t)$. Therefore,

$$\langle (1, 0), N_{\mathcal{S}}(\zeta(t)) \rangle = \{0\}.$$

Hence, if $(\alpha, u) \in N_{\mathcal{S}}(\zeta(t))$, then $\alpha = 0$. Thus, $\|D^*S(\zeta(t))|^+ = 0$.

Second case: $\dot{\gamma}(t) \neq 0$.

Then $\zeta(t) \in \partial\mathcal{S}$ and $\dot{\zeta}(t)$ belongs to the tangent space of $\partial\mathcal{S}$ at $\zeta(t)$. As in the first case, we obtain that

$$\langle (1, \dot{\gamma}(t)), N_{\mathcal{S}}(\zeta(t)) \rangle = \{0\}.$$

Hence, for every $(\alpha, u) \in N_{\mathcal{S}}(\zeta(t))$ with $\|u\| = 1$ we have $\alpha + \langle \dot{\gamma}(t), u \rangle = 0$. Thanks to Cauchy-Schwartz inequality, we obtain

$$\|\dot{\gamma}(t)\| \geq \|D^*S(\zeta(t)|^+.$$

By Proposition 5.21 (c), we can assume that

$$\sup_{\|u\|\leq 1} \{a : \, a \in D^*S(t,x)(u)\} \geq 0.$$

Setting $H = \{t\} \times \mathbb{R}^n$ we have $\{t\} \times S(t) = H \cap \mathcal{S}$. Due to the fact that $-\dot{\gamma}(t) \in N_{S(t)}(\gamma(t))$, we have:

$$(1, -\dot{\gamma}(t)) \in N_{\{t\}\times S(t)}(\zeta(t)).$$

In addition, since $\|D^*S(t, \gamma(t))|^+ < \infty$, we have that $(t, 0) \in N_{\mathcal{S}}(t, \gamma(t))$ only if $t = 0$. Hence, applying the calculus rule [84, Theorem 6.42], we get

$$N_{H\cap\mathcal{S}}(\zeta(t)) \subset N_H(\zeta(t)) + N_{\mathcal{S}}(\zeta(t)) = \mathbb{R} \times \{0\} + N_{\mathcal{S}}(\zeta(t)).$$

Therefore, the inclusion $(\lambda, -\dot{\gamma}(t)) \in N_{\mathcal{S}}(\zeta(t))$ holds for some $\lambda \in \mathbb{R}$. By orthogonality between normal and tangent vectors, we get that:

$$\langle (\lambda, -\dot{\gamma}(t)), (1, \dot{\gamma}(t)) \rangle = 0.$$

and thus $\lambda = \|\dot{\gamma}(t)\|^2$. After normalization, we obtain:

$$\left(\|\dot{\gamma}(t)\|, -\frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} \right) \in N_{\mathcal{S}}(\zeta(t)),$$

which readily yields $\|D^*S(t, \gamma(t))|^+ \geq \|\dot{\gamma}(t)\|$, as claimed. □

Let us finally quote the following result, which is a restatement of [26, Proposition 47]. For the sake of completeness, we present its proof.

Proposition 5.23 *Let $b > a$ and Γ be a collection of absolutely continuous curves γ defined in some nontrivial interval $J \subset (a, b)$ with values in $\mathbb{R}^n$, where $J = [\inf(J), \sup(J))$. Assume that for each $t \in (a, b)$ there exist $\varepsilon_t > 0$ and $\gamma_t \in \Gamma$ with $\operatorname{dom}(\gamma) = [t, t + \varepsilon_t)$. Moreover, assume that if $\gamma \in \Gamma$, then for any $t_1, t_2 \in (t, t + \varepsilon_t)$, the restriction $\gamma|_{[t_1,t_2)} \in \Gamma$. Then there exist a countable partition $\{I_n\}_{n\in\mathbb{N}}$ of (a, b) into intervals I_n of nonempty interior and a piecewise absolutely continuous curve $\gamma : (a, b) \to \mathbb{R}$ such that $\gamma|_{I_n} \in \Gamma$.*

Proof. Let $c \in (a, b)$. We first construct a curve with domain $[c, b)$ and then we deal with the interval (a, b).

First part: Let us consider Γ_c the family of curves on $\mathbb{R}^n$ such that $\gamma \in \Gamma_c$ if and only if $\mathrm{dom}(\gamma) = [c, t)$, with $t \in (c, \mathrm{d}]$, and γ is a concatenation of at most countable many curves in Γ. It is clear that Γ_c is nonempty since $\gamma_c \in \Gamma_c$.

Let us consider the partial order $\preceq$ on Γ_c defined as follows: for any $\gamma^1, \gamma^2 \in \Gamma_c$, $\gamma^1 \preceq \gamma^2$ if and only if $\mathrm{dom}(\gamma^1) \subseteq \mathrm{dom}(\gamma^2)$ and $\gamma^2|_{\mathrm{dom}(\gamma^1)} = \gamma^1$. Let us apply Zorn's Lemma in the partially ordered set $(\Gamma_c, \preceq)$. Let $(\gamma^\lambda)_{\lambda\in\Lambda} \subset \Gamma_c$ be a chain for $\preceq$. Since $\gamma^{\lambda_2}|_{\mathrm{dom}(\gamma^{\lambda_1})} = \gamma^{\lambda_1}$ for all $\lambda_1 < \lambda_2$, we can define $\widehat{\gamma} : \bigcup\{\mathrm{dom}(\gamma^\lambda) : \ \lambda \in \Lambda\} \to \mathbb{R}^n$ as follows:

$$\widehat{\gamma}(t) := \gamma^\lambda(t), \ \text{where } t \in \mathrm{dom}(\gamma^\lambda).$$

Let us check that $\widehat{\gamma} \in \Gamma_c$. It easily follows that there is $\mathrm{d} \in (c, b]$ such that $\mathrm{dom}(\widehat{\gamma}) = [c, \mathrm{d})$. Let $(t_n)_{n\geq 0} \subset (c, \mathrm{d})$ be a strictly increasing sequence, convergent to d, such that $t_0 = c$. Let us consider $(\lambda_n)_n \subset \Lambda$ be an increasing sequence such that, for all $n \geq 1$, $[c, t_n) \subset \mathrm{dom}(\gamma^{\lambda_n})$. Therefore, thanks to the properties of Γ, for all $n \geq 1$, the curve $\gamma^{\lambda_n}|_{[t_{n-1},t_n)}$ is a concatenation of countable many curves in Γ. Noticing now that $\widehat{\gamma}$ is the concatenation of family $\{\gamma^{\lambda_n}|_{[t_{n-1},t_n)} : \ n \in \mathbb{N}\}$, we conclude that $\widehat{\gamma}$ can be constructed as a concatenation of countable many curves in Γ. Thus, $\widehat{\gamma} \in \Gamma_c$ and it is an upper bound for the chain $(\gamma^\lambda)_\lambda$. Therefore, thanks to Zorn's lemma, there exists $\gamma_{c,b} \in \Gamma_c$, a maximal element for $\preceq$. If we suppose that $\mathrm{dom}(\gamma_{c,b}) = [c, \mathrm{d})$, with $\mathrm{d} < b$, then we can concatenate γ_{d} to $\gamma_{c,b}$ to contradict the maximality of $\gamma_{c,b}$. Therefore, $\mathrm{dom}(\gamma_{c,b}) = [c, b)$.

Second part: Let us construct a curve with domain equal to (a, b). Let $(c_n)_n \subset (a, b)$ be a decreasing sequence such that c_n tends to a as n tends to infinity and let $c_0 = b$. Applying the first part of the proof to each interval $[c_n, c_{n+1})$, with $n \in \mathbb{N}$, we obtain a curve $\gamma_n := \gamma_{c_n,c_{n-1}} : [c_n, c_{n-1}) \to \mathbb{R}^n$ which is made by the concatenation of countable many curves of Γ. Therefore, the curve $\gamma : (a, b) \to \mathbb{R}^n$ constructed by the concatenation of the curves $(\gamma_n)_n$ proves the proposition. Indeed, for each n, let us consider $\{I_{n,k} : k \in \mathbb{N}\}$ be a partition of intervals with nonempty interior of $\mathrm{dom}(\gamma_n)$ such that $\gamma_n|_{I_{n,k}} \in \Gamma$ for all $k \in \mathbb{N}$. Thus, the partition of (a, b) can be chosen as $\{I_{n,k} : \ n \in \mathbb{N}, \ k \in \mathbb{N}\}$. □

We are now ready to prove our main result.

5.4.2 Proof of Theorem 5.15

We prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (\mathrm{d}) \Rightarrow (a)$.

a) ⇒ b) : Let $\Psi : [0, \rho] \to [a, b]$ be given by (a). Let $\gamma \in \mathcal{AC}([t_1, t_2], S)$. Since Ψ is a $\mathcal{C}^1$-smooth function, $\partial\mathrm{gph}((S \circ \Psi)|_{(0,\rho)})$ is a smooth manifold. By Proposition 5.20, $\gamma \circ \Psi \in \mathcal{AC}([0, \rho), S \circ \Psi)$. Applying Lemma 5.22, we deduce that

$$\left|\frac{\mathrm{d}\,(\gamma \circ \Psi)}{\mathrm{d}r}(r)\right| = \|D^*(S \circ \Psi)(r, \gamma(\Psi(r)))|^+ \leq 1, \quad \forall_{a.e}\ r \in (a, \mathrm{d}).$$

Since Ψ is increasing and smooth, by change of variables we obtain:

$$\begin{aligned}\int_{t_1}^{t_2} \|\dot{\gamma}(\tau)\|\mathrm{d}\tau = \int_{\Psi^{-1}(t_1)}^{\Psi^{-1}(t_2)} \|\dot{\gamma}(\Psi(r))\|\, \dot{\Psi}(r)\mathrm{d}r &= \int_{\Psi^{-1}(t_1)}^{\Psi^{-1}(t_2)} \left\|\frac{\mathrm{d}\,(\gamma \circ \Psi)}{\mathrm{d}r}(r)\right\| \mathrm{d}r \\ &\leq \int_{\Psi^{-1}(t_1)}^{\Psi^{-1}(t_2)} \mathrm{d}r = \Psi^{-1}(t_2) - \Psi^{-1}(t_1).\end{aligned}$$

Therefore (b) is satisfied by setting $\sigma := \Psi^{-1}$.

b)* $\Rightarrow$ *c) : Since σ is an increasing function and $\sigma(a) = 0$, statement (c) follows by setting $M := \sigma(b)$.

***c)* $\Rightarrow$ d)** : Let $b > a$ and let $M > 0$ given by statement $c)$. Let $\varphi^{\uparrow} : (a, b) \to \mathbb{R} \cup \{+\infty\}$ be the oriented talweg function of S and let us assume, towards a contradiction, that for any $c \in (a, b)$ the function $\varphi^{\uparrow}$ is not integrable on (a, c). By Lemma 5.18, the function $(t, x) \mapsto \|D^*S(t, x)|^+$ is continuous on $\partial\mathcal{S}$. By assumptions (A2)–(A3), shrinking b if necessary, we may assume that $\varphi^{\uparrow}(t) < \infty$ for all $t \in (a, b)$ and that the multivalued map $t \rightrightarrows H_S(t)$ is continuous on (a, b). By Lemma 5.19, $\varphi^{\uparrow}$ is continuous on (a, b).

By Lemma 5.22, if J is a nontrivial interval of (a, b) then for any $\gamma \in \mathcal{AC}(S, J)$ we have $\|\dot{\gamma}(t)\| = \|D^*S(t, \gamma(t))|^+$ for almost every $t \in J$. Let $k \in \mathbb{N}$ and $t \in (a, b)$ and define a curve γ_t^k as follows:

- If $\varphi^{\uparrow}(t) = 0$, take $\gamma_t^k \in \mathcal{AC}(S, [t, \tau))$ be any curve such that $\tau - t < 1/k$.
- If $\varphi^{\uparrow}(t) > 0$, since $H_S(t)$ is compact, there exists $x \in S(t)$ such that $\|D^*S(t, x)|^+ = \varphi^{\uparrow}(t)$. Thanks to assumption (A1) and Lemma 5.19, we can take $\gamma_t^k \in \mathcal{AC}(S, [t, \tau))$, for some $\tau > t$, such that $\gamma(t) = x$ and

$$\|\dot{\gamma}_t^k(s)\| > \frac{k-1}{k}\varphi^{\uparrow}(s), \quad \text{for almost every } s \in (t, \tau).$$

Gluing together, thanks to Proposition 5.23, we obtain $\gamma^k \in \mathcal{PAC}(S, (a, b))$ such that for almost every $t \in (a, b)$

$$\varphi^{\uparrow}(t) \geq \|\dot{\gamma}^k(t)\| \geq f_k(t) := \begin{cases} 0, & \text{if } t \in A_k \\ \dfrac{k-1}{k}\varphi^{\uparrow}(t), & \text{if } t \in (a, b) \setminus A_k. \end{cases}$$

where $A = \{t \in (a, b) :\ \varphi^{\uparrow}(t) = 0\}$ and $A_k = (a, b) \cap (A + [0, 1/k])$ for all $k \in \mathbb{N}$.

The continuity of $\varphi^{\uparrow}$ yields that A is a closed set relatively to (a, b). Therefore, $A = \cap_{k\in\mathbb{N}} A_k$. Then, for all $t \in (a, b)$, $f_k(t) \nearrow \varphi^{\uparrow}(t)$ as k tends to infinity. Hence, by the Monotone Convergence Theorem, $(\int_a^b f_k)_k$ converges to $\int_a^b \varphi^{\uparrow}$, which is infinity. Thus, there is $K \in \mathbb{N}$ such that

$$\int_a^b \|\dot{\gamma}^K(t)\|\mathrm{d}t \geq \int_a^b f_K(t)\mathrm{d}t > M,$$

which contradicts statement (c) since $\gamma^K \in \mathcal{PAC}(S, (a, b))$.

d) $\Rightarrow$ *a)* : Let us assume that the oriented talweg function $\varphi^{\uparrow}$ is integrable on $[a, b]$ for some

$b > a$. As a consequence of assumptions (A2) and (A3), shrinking b if necessary, we may assume that $\varphi^{\uparrow}$ is continuous on $[a,b]$ and $\varphi^{\uparrow}(t) < \infty$ for all $t \in (a,b]$. Let $\overline{\varphi} = \max\{\varphi^{\uparrow}, 1\}$ which is an integrable continuous majorant of $\varphi^{\uparrow}$ and set

$$\theta(t) := \int_a^t \overline{\varphi}(s)\mathrm{d}s, \quad \text{for } t \in [a,b].$$

Since $\overline{\varphi}$ is positive and integrable on $[a,b]$, we set $\rho := \theta(b)$ and define $\Psi : [0,\rho] \to [a,b]$ as the inverse function of θ, that is, $\Psi(r) = \theta^{-1}(r)$. Since $\theta'(t) = \overline{\varphi}(t) \in [1,+\infty)$, for every $t \in (a,b]$, it follows that Ψ is $\mathcal{C}^1$-smooth on $(0,\rho)$, with derivative

$$\Psi'(r) = \frac{1}{\overline{\varphi}(\Psi(r))} \leq 1, \quad \text{for all } r \in (0,\rho).$$

Thus, Ψ is a Lipschitz homeomorphism between $[0,\rho]$ and $[a,b]$. Finally, using the chain rule for coderivatives [84, Theorem 10.37], we deduce that

$$\|D^* (S \circ \Psi)(r,x)|^+ \leq \frac{\|D^* S(\Psi(r),x)|^+}{\overline{\varphi}(\Psi(r))} \leq 1, \quad \text{for all } r \in (0,\rho).$$

The proof is complete. □

5.4.3 Oriented calmness

Before proceeding with the proof of Theorem 5.17, we need to introduce the modulus of oriented calmness and establish a result analogous to the Mordukhovich criterium for the oriented modulus of the coderivative. Let us first recall that the *Lipschitzian graphical modulus* of $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ at t for x is defined by

$$\begin{aligned}\operatorname{Lip} S(t,x) := \inf\{\kappa > 0 |\ \exists \varepsilon > 0,\ \delta > 0,\ \text{such that}\\ S(t_2) \cap B(x,\delta) \subset S(t_1) + \kappa|t_2 - t_1|B, \quad \text{for all } t_1, t_2 \in (t-\varepsilon, t+\varepsilon)\},\end{aligned}$$

where B stands for the open unit ball.

We recall that the multivalued function S has the Aubin property at t for x if and only if $\operatorname{Lip} S(t,x) < \infty$. More precisely, we have the following (see [84, Theorem 9.40]).

Theorem 5.24 *For every $(t,x) \in \mathcal{S}$ such that $\|D^* S(t,x)\|^+ < \infty$ it holds:*

$$\operatorname{Lip} S(t,x) = \|D^* S(t,x)\|^+.$$

Motivated by the above, we introduce the following graphical modulus.

Definition 5.25 (oriented calm modulus) *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a multivalued map and $(t,x) \in \mathcal{S}$. The oriented calm graphical modulus, denoted by* $\operatorname{calm}^{\uparrow} S$*, at t for x is defined by*

$$\begin{aligned}\operatorname{calm}^{\uparrow} S(t,x) := \inf\{\kappa > 0 |\ \exists \varepsilon > 0,\ \delta > 0,\ \textit{such that}\\ S(t) \cap B(x,\delta) \subset S(t_1) + \kappa|t_1 - t|B \textit{ for all } t_1 \in (t, t+\varepsilon)\}.\end{aligned}$$

Observe that, if S is a single-valued function and $\operatorname{calm}^{\uparrow} S(t,x) < \infty$, then S is calm at t to the right. More information on the notion of calmness for multivalued maps can be found in [62] and references therein. We are now ready to give the oriented version of Theorem 5.24.

Proposition 5.26 (oriented calm vs oriented modulus) *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a smooth sweeping process, $t \in \operatorname{dom}(S) \setminus \{\mathrm{T}\}$ and $x \in S(t)$ such that $\|D^*S(t,x)|^+ < +\infty$. Then*

$$\operatorname{calm}^{\uparrow} S(t,x) = \|D^*S(t,x)|^+.$$

Proof. Let us first notice that $\operatorname{calm}^{\uparrow} S(t,x) \le \operatorname{Lip} S(t,x)$. We consider two cases:

Case 1: $\|D^*S(t,x)|^+ = 0$.

If $\|D^*S(t,x)\|^+ = 0$, then $\operatorname{calm}^{\uparrow} S(t,x) = 0$. If $\|D^*S(t,x)\|^+ > 0$, then, by Lemma 5.11 $\mathcal{S}$ is a manifold of full dimension with boundary $\partial\mathcal{S}$ which is a smooth manifold of dimension n. Let us assume by contradiction that $\operatorname{calm}^{\uparrow} S(t,x) > 0$. Then, for every $k \in \mathbb{N}$ such that $k^{-1} < \operatorname{calm}^{\uparrow} S(t,x)$, there exists $y_k \in S(t) \cap B(x, 1/k)$ such that

$$y_k \notin S(t_k) + \left(\frac{t'_k - t}{k}\right) B, \text{ for some } t'_k \in (t,\ t + \frac{1}{k}).$$

Set $t_k := \inf\{r \in (t, t+\frac{1}{k}) :\ y_k \notin S(r)\}$. It is clear that $(t_k, y_k) \in \partial\mathcal{S}$ and that y_k is not right-locally stationary for S at t_k. Thus, by Proposition 5.21 (c), for every $k \in \mathbb{N}$ and $(\beta_k, v_k) \in N_{\mathcal{S}}(t_k, y_k)$, we have $\beta_k \ge 0$. Since $N_{\mathcal{S}}(t,x)$ is a ray and $\{(t_k, y_k)\}_k \to (t,x)$, the continuity of unit outer normal vectors of $\mathcal{S}$ on $\partial\mathcal{S}$ ensures that $\beta \ge 0$ whenever $(\beta, v) \in N_{\mathcal{S}}(t,x)$. This leads to the equality $\|D^*S(t,x)|^+ = \|D^*S(t,x)\|^+$, which is a contradiction. Therefore, $\operatorname{calm}^{\uparrow} S(t,x) = 0$.

Case 2: $\|D^*S(t,x)|^+ = \alpha > 0$.

In this case, we deduce from Lemma 5.11(b) that

$$\|D^*S(t,x)|^+ = \|D^*S(t,x)\|^+ = \operatorname{Lip} S(t,x) \ge \operatorname{calm}^{\uparrow} S(t,x).$$

By Lemma 5.18 and compactness of the unit ball of $\mathbb{R}^n$, there exists $u \in \mathbb{R}^n$ with $\|u\| = 1$ such that $(\alpha, u) \in N_{\mathcal{S}}(t,x)$. Let $(t_k)_{k\ge1} \subset \mathbb{R}$ be a decreasing sequence that converges to t. Let $(y_k)_{k\ge1} \subset \mathbb{R}^n$ be a sequence that satisfies $y_k \in \operatorname{Proj}(x, S(t_k))$ for each $k \in \mathbb{N}$. By compactness of the unit sphere of $\mathbb{R}^{n+1}$, up to a subsequence we deduce that

$$\lim_{k\to\infty} \frac{(t_k - t, y_k - x)}{\|(t_k - t, y_k - x)\|} = (\beta, v),$$

where (β, v) belongs to the tangent space of $\mathcal{S}$ at (t,x) and $\beta \ge 0$. Since $\mathcal{S}$ is a smooth sweeping process, it follows that

$$(\alpha, u) \perp (\beta, v) \qquad \text{yielding} \qquad \langle u, v\rangle = -\alpha\beta.$$

Since $\operatorname{calm}^{\uparrow} S(t,x) \le \|D^*S(t,x)|^+ < +\infty$, β must be a strictly positive number. Therefore

$$\lim_{k\to\infty} \frac{\|y_k - x\|}{t_k - t} = \frac{\|v\|}{\beta} \ge \frac{|\langle u, v\rangle|}{\beta} = \alpha,$$

implying that

$$\operatorname{calm}^{\uparrow} S(t,x) \ge \alpha = \|D^*S(t,x)|^+.$$

The proof is complete. □

Lemma 5.27 (controlling excess of $S(t_0)$) *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a smooth sweeping process and $[t_0, t_1] \subset \mathrm{dom}(S)$. Then*

$$\mathrm{ex}\,(S(t_0), S(t_1)) := \sup_{x \in S(t_0)} \mathrm{d}(x, S(t_1)) \leq \left(\sup_{t \in [t_0,t_1]} \varphi^{\uparrow}(t) \right) (t_1 - t_0)$$

and

$$\mathrm{dist}(S(t_0), S(t_1)) \leq \left(\sup_{t \in [t_0,t_1]} \varphi(t) \right) (t_1 - t_0).$$

Proof. Let us first notice that

$$K := \sup_{t \in [t_0,t_1]} \varphi^{\uparrow}(t) \; \geq \|D^*S(t,x)|^+ = \mathrm{calm}^{\uparrow} S(t,x), \quad \text{for all } t \in [t_0, t_1] \text{ and } x \in S(t)\,.$$

If $K = \infty$, there is nothing to prove. Let $K < +\infty$ and assume, towards a contradiction, that for some $\delta > 0$ we have

$$\mathrm{ex}\,(S(t_0), S(t_1)) > (K + \delta)(t_1 - t_0).$$

Let $\tau \in \mathbb{R}$ be defined by

$$\tau := \inf \{ \, t \in [t_0, t_1] : \; \mathrm{ex}\,(S(t_0), S(t)) > (K + \delta)(t - t_0) \, \}\,.$$

By Proposition 5.26 and the definition of the graphical modulus $\mathrm{calm}^{\uparrow}$, for each $x \in S(t_0)$, there is $\varepsilon_x > 0$ and $\delta_x > 0$ such that

$$S(t_0) \cap B(x, \delta_x) \subset S(t) + (K + \frac{\delta}{2})|t - t_0|B, \quad \text{for all } t \in [t_0, t_0 + \varepsilon_x).$$

Let $\tilde{\varepsilon}_x > 0$ be the supremum of all $\varepsilon > 0$ such that:

$$x \in S(t) + (K + \frac{\delta}{2})|t - t_0|B, \text{ for all } t \in [t_0, t_0 + \varepsilon).$$

If $\tau = t_0$, then there exists a sequence $(x_k)_k \subset S(\tau)$ such that $\tilde{\varepsilon}_{x_k} < 1/k$, for all $k \geq 1$. Since $S(\tau)$ is compact, the sequence $(x_k)_k$ has some cluster point $\overline{x} \in S(\tau)$. By Proposition 5.26, there exist $\varepsilon_{\overline{x}} > 0$ and $\delta_{\overline{x}} > 0$ such that

$$S(t_0) \cap B(\overline{x}, \delta_{\overline{x}}) \subset S(t) + (K + \frac{\delta}{2})|t - t_0|B, \text{ for all } t \in [t_0, t_0 + \varepsilon_{\overline{x}}).$$

which contradicts the maximality of $\tilde{\varepsilon}_{x_k}$, for k large enough. This establishes that $t_0 < \tau$. Proceeding in the same way, we can actually show that $\tau \geq t_1$. Indeed, assuming $\tau < t_1$, and using the same argument as above (with t_0 in the place of τ) together with the triangle inequality we get a contradiction. Therefore, for any $\delta > 0$ we have:

$$\mathrm{ex}(S(t_0), S(t_1)) \leq (K + \delta)(t_1 - t_0),$$

which finishes the first assertion of the lemma.

For the second part, we follow the same procedure to estimate the reverse excess $\mathrm{ex}\,(S(t_1), S(t_0))$, and conclude thanks to the fact that $\mathrm{dist}(S(t_0), S(t_1)) = \max\{\mathrm{ex}(S(t_0), S(t_1)), \mathrm{ex}(S(t_1), S(t_0))\}$. The details are left to the reader. $\square$

Now, we proceed with the proof of our second main result.

5.4.4 Proof of Theorem 5.17.

We recall from Section 5.4.2 the definition of $\mathcal{T}$ and fix $a \in \mathcal{T}$. We prove $(a) \Rightarrow$ (e) $\Rightarrow (f) \Rightarrow$ (d).

***a*) ⇒ e)** : Let $b > a$ such that the statements (a) to (d) of Theorem 5.15 hold true, and that the oriented talweg $\varphi^{\uparrow}$ takes finite values on (a, b) (*c.f.* Assumption (A2)). We set

$$\sigma(t) = \int_a^t \varphi^{\uparrow}(s)\mathrm{d}s, \quad t \in (a, b].$$

By (d) the above integral is well-defined and σ is continuous with $\sigma(a) = 0$. Let $\{(t_{\mathrm{i}}, x_{\mathrm{i}})\}_{\mathrm{i}\geq 0} \subset \mathcal{S}$ be any catching-up sequence for S. Let $k \geq 1$. We shall prove that (5.8) holds for k. By Proposition 5.13, S is continuous on the interval $[t_0, t_k]$ and by Lemma 5.19, $\varphi^{\uparrow}$ is continuous (and finite), hence Riemann integrable there. Let $\{s_j^{\mathrm{i}}\}_{j=0}^{k_{\mathrm{i}}}$ be a partition of the interval $[t_{\mathrm{i}}, t_{\mathrm{i}+1}]$, $\mathrm{i} \in \{0, \ldots, k-1\}$, with width

$$\max_{j\in\{0,\ldots,k_{\mathrm{i}}\}} |s_{j+1}^{\mathrm{i}} - s_j^{\mathrm{i}}| < \frac{1}{N}, \quad \text{for all } \mathrm{i} \in \{0, \ldots, k-1\}.$$

Notice that for every $\mathrm{i} \in \{0, \ldots, k-1\}$, we have $s_0^{\mathrm{i}} = t_{\mathrm{i}}$ and $s_{k_{\mathrm{i}}}^{\mathrm{i}} = t_{\mathrm{i}+1}$. We set

$$z_0^{\mathrm{i}} := x_{\mathrm{i}} \in S(t_{\mathrm{i}}) \qquad \text{and for each } \; j \in \{0, \ldots, k_{\mathrm{i}} - 1\} \text{ we pick } z_{j+1}^{\mathrm{i}} \in \mathrm{Proj}_{S(s_{j+1})}(z_j^{\mathrm{i}}).$$

Then using triangle inequality and the fact that

$$\|x_{\mathrm{i}+1} - x_{\mathrm{i}}\| = \mathrm{d}(x_{\mathrm{i}}, \underbrace{S(t_{\mathrm{i}+1})}_{=S(s_{k_{\mathrm{i}}}^{\mathrm{i}})})) \leq \|z_{k_{\mathrm{i}}}^{\mathrm{i}} - z_0^{\mathrm{i}}\|.$$

we deduce from Lemma 5.27 that:

$$\|x_{\mathrm{i}+1} - x_{\mathrm{i}}\| \leq \sum_{j=0}^{k_{\mathrm{i}}-1} \|z_{j+1}^{\mathrm{i}} - z_j^{\mathrm{i}}\| \leq \sum_{j=0}^{k_{\mathrm{i}}-1} \left(\sup_{t\in[s_j^{\mathrm{i}}, s_{j+1}^{\mathrm{i}}]} \varphi^{\uparrow}(t) \right) (s_{j+1} - s_j) \, .$$

Taking the limit as N tends to infinity, we obtain that

$$\|x_{\mathrm{i}+1} - x_{\mathrm{i}}\| \leq \int_{t_{\mathrm{i}}}^{t_{\mathrm{i}+1}} \varphi^{\uparrow}(s)\mathrm{d}s$$

and consequently,

$$\sum_{\mathrm{i}=0}^{k-1} \|x_{\mathrm{i}+1} - x_{\mathrm{i}}\| \leq \int_{t_0}^{t_k} \varphi^{\uparrow}(t)\mathrm{d}t = \sigma(t_k) - \sigma(t_0).$$

e) ⇒ *f*) : It follows directly by taking $M = \sigma(b)$.

***f*) ⇒ d)** : Let $b > a$ and $M > 0$ be given by statement (f). By (A2)–(A3), shrinking b if necessary, we may assume that $\varphi^{\uparrow}(t) < \infty$, for all $t \in (a, b)$ and ∂S is continuous on (a, b). Notice that for any compact interval $[c, \mathrm{d}] \subset (a, b)$, the function $\varphi^{\uparrow}$ is continuous and finite on $[c, \mathrm{d}]$, therefore Riemann integrable. We shall prove that its integral is bounded by M

(independently of the values of c and d).

To this end, let $t_0 \in [c, \mathrm{d}]$ and $N \in \mathbb{N}$. By compactness, there exists $x \in S(t_0)$ such that $\|D^*S(t_0, x)|^+ = \varphi^\uparrow(t_0)$. If $\varphi^\uparrow(t_0) < \frac{1}{N}$, we set $t_1 := \min\{t_0 + \frac{1}{N}, \mathrm{d}\}$, $x_0 = x$ and $y_0 \in \mathrm{Proj}_{S(t_1)}(x_0)$. Observe that

$$\|x_0 - y_0\| \geq 0 \geq (t_1 - t_0)\left(\varphi^\uparrow(t_0) - \frac{1}{N}\right).$$

If $\varphi^\uparrow(t) \geq \frac{1}{N}$, by Proposition 5.26, since $\mathrm{calm}^\uparrow S(t, x) = \varphi^\uparrow(t)$, there are $x_0 \in S(t_0)$ and $t_1 \in (t_0, \min\{t_0 + \frac{1}{N}, b\})$ such that any $y_0 \in \mathrm{Proj}_{S(t_1)}(x_0)$ satisfies

$$\|x_0 - y_0\| \geq (t_1 - t_0)\left(\varphi^\uparrow(t_0) - \frac{1}{N}\right).$$

Using transfinite induction we obtain an increasing net $\{t_\lambda\}_{\lambda \leq \Lambda} \subset [c, \mathrm{d}]$ indexed over a ordinal Λ, such that $t_0 = c$, $t_\Lambda = \mathrm{d}$, $0 < t_{\lambda+1} - t_\lambda \leq 1/N$ for all $\lambda < \Lambda$, and for any limit ordinal $\alpha \leq \Lambda$, $t_\alpha := \sup\{t_\lambda : \ \lambda < \alpha\}$. Also, we get a net $\{(x_\lambda, y_\lambda)\}_{\lambda \leq \Lambda}$ such that $y_\lambda \in \mathrm{Proj}_{S(t_{\lambda+1})}(x_\lambda)$ and

$$\|x_\lambda - y_\lambda\| \geq (t_{\lambda+1} - t_\lambda)\left(\varphi^\uparrow(t_\lambda) - \frac{1}{N}\right), \text{ for all } \lambda < \Lambda.$$

Observe that, since the intervals $(t_\lambda, t_{\lambda+1})_{\lambda<\Lambda}$ are pairwise disjoint and they intersect $\mathbb{Q}$, Λ is a countable ordinal. For every finite subset $F \subset \Lambda$ we have

$$\sum_{\lambda \in F} \|x_\lambda - y_\lambda\| \geq \sum_{\lambda \in F} (t_{\lambda+1} - t_\lambda)\varphi^\uparrow(t_\lambda) - \frac{\mathrm{d} - c}{N}.$$

Since $\{(t_\lambda, x_\lambda),\ (t_\lambda, y_\lambda) :\ \lambda \in F\}$ is a subsequence of a piecewise catching-up sequence for S, taking the supremum over all finite families F of Λ we get

$$M \geq \sum_{\lambda<\Lambda} \|x_\lambda - y_\lambda\| \geq \sum_{\lambda<\Lambda} (t_{\lambda+1} - t_\lambda)\varphi^\uparrow(t_\lambda) - \frac{\mathrm{d} - c}{N}.$$

Taking the limit as N goes to infinity we obtain:

$$M \geq \int_c^{\mathrm{d}} \varphi^\uparrow(t)\mathrm{d}t.$$

Since the above inequality is independent of the interval $[c, \mathrm{d}]$, we deduce that $\varphi^\uparrow$ is integrable on (a, b). □

5.5 A non-desingularizable smooth sweeping process

In Section 5.3 we state a characterization of the (5.7)-inequality for smooth sweeping processes by assuming assumptions $(A1)$, $(A2)$ and $(A3)$. In this last section, we provide a smooth sweeping process $S : \mathbb{R} \rightrightarrows \mathbb{R}$ which cannot be desingularized at 0 in the sense of inequality (5.7), i. e. there are no $\delta_1,\ \delta_2 > 0$ and $\Psi : (0,\delta_1) \to (0,\delta_2)$ diffeomorphism, with $\Psi'(t) > 0$ for all $t \in (0,\delta_1)$, such that

$$\|D^*S \circ \Psi(t,x)|^+ \leq 1, \text{ for all } t \in (0,\delta_1),\ x \in S(\Psi(t)).$$

In order to start the construction, let $b : \mathbb{R} \to \mathbb{R}$ be any positive $\mathcal{C}^\infty$-smooth function such that $b(0) = 1$ and $\operatorname{supp}(b) \subset [-1,1]$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$f(x) = \sum_{n=1}^{\infty} 2^{-n(n+2)} b\left(2^{n+2}\left(x - \frac{1}{2^n}\right)\right), \text{ for all } x \in \mathbb{R}. \tag{5.9}$$

Observe that the previous series is well defined because, for each $x \in \mathbb{R}$, at most one term is different from zero. In what follows, let us summarize some properties of f.

Fact 5.28 *Let $f : \mathbb{R} \to \mathbb{R}$ be the function constructed in* (5.9). *Then, f is non-negative and $\mathcal{C}^\infty$-smooth. Also,* $\operatorname{supp}(f) \subset [0, 5/8]$. *Moreover, for each $n \in \mathbb{N}$, there is a non-trivial interval $J_n \subset [2^{-n-1}, 2^{-n}]$ such that $f|_{J_n} \equiv 0$.*

Proof. By construction, the function f is non-negative. On the other hand, for $k \in \mathbb{N}$, we have that the k-th derivative of f is

$$f^{(k)}(x) = \sum_{n=1}^{\infty} 2^{(k-n)(n+2)} b\left(2^{n+2}\left(x - \frac{1}{2^n}\right)\right), \text{ for all } x \in \mathbb{R}.$$

The last fact is satisfied by choosing $J_n := 2^{-n-3}[5,6]$. □

Now, let us consider the curve $\gamma_1 : [-1,1] \to \mathbb{R}^2$ defined by

$$\gamma_1(t) := \left(\int_{-1}^{t} f(s)\mathrm{d}s, -t\right), \text{for all } t \in [-1,1].$$

Let $M := \int_{-1}^{1} f(s)\mathrm{d}s$ and let $\gamma_2 : [0,M] \to \mathbb{R}^2$ be the curve defined by $\gamma_2(t) = (t,-3)$.

Let $\mathcal{D} \subset \mathbb{R}^2$ be the closed bounded region delimited by $\operatorname{gph}(\gamma_1)$, $\operatorname{gph}(\gamma_2)$, $\{0\} \times [-3,1]$ and $\{M\} \times [-3,-1]$. Let $S : \mathbb{R} \rightrightarrows \mathbb{R}$ be any smooth sweeping process such that

$$\operatorname{gph}(S|_{[0,M]}) := \mathcal{D}.$$

Proposition 5.29 *The multivalued map $S : \mathbb{R} \rightrightarrows \mathbb{R}$ constructed above is a smooth sweeping process such that it cannot be desingularized at* 0 *in the sense of inequality* (5.7). *In particular, it does not satisfy assumptions* (A2) *and* (A3) *at* 0.

Proof. Let us denote by O the origin of $\mathbb{R}^2$. The proposition easily follows from the fact that the multivalued map S is decreasing on $[0,M]$, $O \in \partial\operatorname{gph}(S)$ and the set $\partial\operatorname{gph}(S)$ contains infinitely many vertical line segments on $(0,1)$ which accumulates at O. □

Chapter 6

AML functions in two dimensional spaces

In this final chapter we focus our study on the regularity of real-valued Lipschitz functions. We study absolutely minimizing Lipschitz functions (AML for short) defined in two dimensional normed spaces. The main contribution of this chapter is the characterization of the C^1-smoothness of AML functions in terms of the smoothness of the underlying norm. A more general result was obtained by F. Peng, C. Wang and Y. Zhou in [77], published in the early 2021. This work is discussed in the introduction of the chapter.

6.1 Introduction

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a nonempty, open and connected set, where $\mathbb{R}^n$ is equipped with an euclidean norm. Aronsson in [6] studied the class of $\mathcal{C}^2$-smooth infinite-harmonic functions defined on Ω. That is, classical solutions $u : \Omega \to \mathbb{R}$ of the equation given by the infinity-Laplacian, i.e.

$$\Delta_\infty u := \sum_{\mathrm{i},j=1}^{n} u_{x_\mathrm{i}} u_{x_j} u_{x_\mathrm{i} x_j} = 0 \tag{∞L}$$

We shall see below that solving the infinity-Laplacian is related to the following optimal Lipschitz extension problem: let $g : \partial\Omega \to \mathbb{R}$ be a continuous function. Find a function $u : \overline{\Omega} \to \mathbb{R}$ such that $g = u|_{\partial\Omega}$ and that, for every open set V compactly contained in Ω and for every function $h : \overline{V} \to \mathbb{R}$ such that $u|_{\partial V} = h|_{\partial V}$, the following estimate holds:

$$\mathrm{Lip}(u|_{\overline{V}}) \leq \mathrm{Lip}(h|_{\overline{V}}).$$

The above problem leads to the following definition.

Definition 6.1 *Let $(X, \|\cdot\|)$ be a finite dimensional Banach space and let Ω be a nonempty open subset of X. We say that a locally Lipschitz function $u : \Omega \subset X \to \mathbb{R}$ is a $\|\cdot\|$-absolute minimizing Lipschitz function ($\|\cdot\|$-AML function for short), if for every open set V compactly contained in Ω and for every function $g : \overline{V} \to \mathbb{R}$ such that $u|_{\partial V} = g|_{\partial V}$, the following estimate holds:*

$$\mathrm{Lip}(u|_{\overline{V}}) \leq \mathrm{Lip}(g|_{\overline{V}}).$$

If no confusion arises from the underlying norm on X, we just say AML functions.

Let us now present some results in the euclidean setting. Aronsson showed that $\mathcal{C}^2$-smooth infinity-harmonic functions coincide with $\mathcal{C}^2$-smooth AML functions, see [6]. Jensen proved that functions which are solutions of equation (∞L) in the viscosity sense are exactly the AML functions. Further, Jensen proved the existence and uniqueness of a viscosity solution of the Cauchy problem given by Equation (∞L) and a continuous boundary condition, see [64]. A link between this theory and the stochastic Tug-of-war game theory is presented in [78].

The regularity of AML functions is one of the main issue in this field. In the seminal paper [87], O. Savin proved that planar $\|\cdot\|_2$-AML functions are continuously differentiable, that is, AML functions defined on open subsets of $(\mathbb{R}^2, \|\cdot\|_2)$ are $\mathcal{C}^1$-smooth. L. Evans and C. Smart proved that planar $\|\cdot\|_2$-AML functions are $\mathcal{C}^{1,\alpha}$-smooth for some $\alpha > 0$, see [48]. Also, provided with tools from capacity theory, in [95] we can find an alternative proof of the smoothness of planar $\|\cdot\|_2$-AML functions. Further results assert that AML functions in (finite dimensional) euclidean spaces are at least everywhere differentiable, see [49, 50]. However, the continuity of the differential remains an open question in higher dimensions.

The main question we address here is the following.

Question 6.2 *If $(X, \|\cdot\|)$ is a finite dimensional normed space, which property of the norm guarantees the smoothness of all $\|\cdot\|$-AML functions defined on open subsets of X?*

In order to continue, let us recall that $B_r(x_0)$ and B_r stand for the open ball of radius r centered at x_0 and at the origin respectively. We also need the following definitions. Let $u : \Omega \subset X \to \mathbb{R}$ be an AML function and let $x \in \Omega$. For $r \in (0, \operatorname{dist}(x, \partial\Omega))$, we set

$$S(x,r)^+ = S_u(x,r)^+ := \max_{\|y-x\|=r} \frac{u(y) - u(x)}{r}.$$

By Corollary 6.12,

$$S(x) = S_u(x) := \lim_{r\to 0} S(x,r)^+ \text{ exists} \quad \text{ and } \quad 0 \le S(x) \le S^+(x,r).$$

Our main results read as follows:

Theorem 6.3 *Let X be a finite dimensional Banach space with differentiable norm and let Ω be a nonempty open subset of X. Let $u : \Omega \subset X \to \mathbb{R}$ be an AML function. Then for each $x \in \Omega$ and $r < \operatorname{dist}(x, \partial\Omega)$, there exists a vector $\mathrm{e}^*_{x,r} \in X^*$, with $\|\mathrm{e}^*_{x,r}\| = S(x)$, such that*

$$\max_{y\in \overline{B}_r(x)} \frac{|u(y) - u(x) - \mathrm{e}^*_{x,r}(y-x)|}{r} \to 0 \textit{ as } r \to 0.$$

Observe that, thanks to Theorem 6.3, in order to prove that an $\|\cdot\|$-AML-function is differentiable at some $x \in \Omega$, it is enough to prove that the net $(\mathrm{e}^*_{x,r})_r$ converges as r tends to 0. A nice example given by D. Preiss (mentioned in [39, 77]) shows that there is a Lipschitz function from $\mathbb{R}$ to $\mathbb{R}$ which is non-differentiable at 0, but with a net of linear maps $(\mathrm{e}^*_{0,r})_r$ satisfying the conclusion of Theorem 6.3. However, the convergence of the mentioned net, and moreover the continuity of the differential, is guaranteed by the following theorems.

Theorem 6.4 *Let X be a 2-dimensional normed space with differentiable norm. There exists a function $\delta : (0,\infty) \to (0,\infty)$ satisfying the following property: Given an AML function $u : B_1 \subset X \to \mathbb{R}$ such that $S(0) \neq 0$ and $\varepsilon > 0$, if there exists $\mathrm{e}^*_1 \in X^*$ such that*

$$\sup_{x\in B_1} |u(x) - \mathrm{e}^*_1(x)| \leq \delta(\varepsilon)\|\mathrm{e}^*_1\|,$$

then $\limsup_{r\to 0} \|\mathrm{e}^*_{0,r} - \mathrm{e}^*_1\| \leq \varepsilon\|\mathrm{e}^*_1\|$.

Theorem 6.5 *Let X be a 2 dimensional Banach space. The following statements are equivalent.*

a) The underlying norm is differentiable in $X \setminus \{0\}$.

b) Each AML function defined on an open subset of X is continuously differentiable.

c) Each AML function defined on an open subset of X is everywhere differentiable.

The proof of Theorem 6.5 relies on Theorem 6.3 and Theorem 6.4.

Proof of Theorem 6.5. The implication from *b*) to *c*) is trivial and the implication from *c*) to *a*) is given by Corollary 6.17, which asserts that the underlying norm of X, restricted to $X \setminus \{0\}$, is an AML function. So, we only have to prove that *a*) implies *b*).

Let $u : \Omega \subset X \to \mathbb{R}$ be an AML function. Let $x_0 \in \Omega$. Let us first prove that u is differentiable at x_0. Since we are only interested by the differentiability of u, replacing if necessary u by $Ru\left(\frac{\cdot - x_0}{R}\right) - u(x_0)$ for some $R > 0$, we can assume that $x_0 = 0$, $u(0) = 0$ and $\overline{B}_1 \subset \Omega$. By Theorem 6.3, there exists $(\mathrm{e}^*_r)_r \subset X^*$ such that $\|\mathrm{e}^*_r\| = S(0)$ for every $r < 1$ and

$$|u(x) - \mathrm{e}^*_r(x)| \leq r\sigma(r), \text{ for all } x \in \overline{B}_r,$$

where $\sigma : \mathbb{R}_+ \to \mathbb{R}$ is a positive function such that $\sigma(r)$ tends to 0 as r tends to 0. If $S(0) = 0$, then $\mathrm{e}^*_r = 0$ for all $r > 0$, so u is differentiable at 0, $u'(0) = 0$ and $\|u'(0)\| = S(0)$. We now assume $S(0) > 0$. Let us prove that e^*_r converges as r tends to 0. Let $\varepsilon > 0$. We fix $s = s(\varepsilon)$ such that $\sigma(s) \leq \delta(\varepsilon)S(0)$. The function $v := \frac{1}{s}u(s\cdot)$ is well defined on $\overline{B}_1$ and, for all $x \in \overline{B}_1$, we have $|v(x) - \mathrm{e}^*_s(x)| \leq \delta(\varepsilon)\|\mathrm{e}^*_s\|$. According to Theorem 6.4 applied to the function v, we get

$$\limsup_{r\to 0} \|\mathrm{e}^*_r - \mathrm{e}^*_s\| \leq \varepsilon\|\mathrm{e}^*_s\|.$$

If ℓ is any accumulation point of $(\mathrm{e}^*_{s(\varepsilon)})_\varepsilon$, the above inequality implies that for every $\varepsilon > 0$,

$$\limsup_{r\to 0} \|\mathrm{e}^*_r - \ell\| \leq \varepsilon S(0).$$

We have proved that $(\mathrm{e}^*_r)_r$ converges to ℓ. Therefore, u is differentiable at 0 and $u'(0) = \ell$. Moreover, since $\|\mathrm{e}^*_r\| = S(0)$ for all r, we have that $\|u'(0)\| = S(0)$. Using again Theorem 6.4, we get :

Claim. If u is any AML function defined on $\overline{B}_1$ such that $S(0) > 0$, e_1^* is a non zero linear functional and $|u(x) - \mathrm{e}_1^*(x)| \le \delta(\varepsilon)\|\mathrm{e}_1^*\|$ on $\overline{B}_1$, then $\|u'(0) - \mathrm{e}_1^*\| = \lim_{r\to\infty} \|\mathrm{e}_r^* - \mathrm{e}_1^*\| \le \varepsilon$.

Let us now check the continuity of u'. If $S(0) > 0$, fix $\varepsilon > 0$ and denote $\delta = \delta(\varepsilon)$. Let $0 < r_0 < \mathrm{dist}(0, \partial\Omega)$ such that, for all $r \le r_0$, $\sigma(r) \le \delta(\varepsilon)S(0)/2$. Fix $r < r_0$. The function $v(\cdot) = u(r\cdot)/r$ restricted to $\overline{B}_1$ satisfies

$$|v(x) - \mathrm{e}_r^*(x)| \le \frac{\delta}{2}S(0) = \frac{\delta}{2}\|\mathrm{e}_r^*\| \quad \text{for all } x \in \overline{B}_1.$$

By the above claim, we obtain that $\|u'(0) - \mathrm{e}_r^*\| \le \varepsilon\|\mathrm{e}_r^*\|$. Let $y \in \overline{B}_{r/2}$. If $w : \overline{B}_1 \to \mathbb{R}$ is defined by $w(\cdot) := \frac{2}{r}(u(\frac{r}{2}\cdot + y) - \mathrm{e}_r^*(y))$, we have $|w(x) - \mathrm{e}_r^*(x)| \le \delta\|\mathrm{e}_r^*\|$ on $\overline{B}_1$. The above claim shows that $\|w'(0) - \mathrm{e}_r^*\| = \|u'(y) - \mathrm{e}_r^*\| \le \varepsilon\|\mathrm{e}_r^*\|$, and hence, $\|u'(0) - u'(y)\| \le 2\varepsilon S(0)$. This proves the continuity of u' at 0.

Let us prove the continuity of u' in the case $S(0) = 0$. Let us fix $\varepsilon > 0$. By definition of $S(0)$, there exists $0 < r < \mathrm{dist}(0, \partial\Omega)$ such that $S(0, r)^+ < \varepsilon$. By the continuity of $S(\cdot, r)^+$, $S(x, r)^+ < \varepsilon$ in a neighbourhood W of 0. Finally, $\|u'(x)\| = S(x) \le S(x, r)^+ < \varepsilon$ for all $x \in W$.

□

Furthermore, as a consequence of Theorem 6.4 we obtain the following two corollaries. The proofs of these results are given in Section 6.5.

Corollary 6.6 *Let X be a two dimensional normed space with differentiable norm. There exists a function $\rho : [0, 1] \to \mathbb{R}$, satisfying $\lim_{t\to 0} \rho(t) = 0$, such that for any AML function $u : B_1 \to \mathbb{R}$, with $\mathrm{Lip}(u) \le 1$, the following inequality holds:*

$$\|u'(x) - u'(y)\| \le \rho(\|x - y\|), \text{ for all } x, y \in \overline{B}_{1/2}.$$

As a consequence of Corollary 6.6 we obtain

Corollary 6.7 *Let X be a two dimensional normed space. The underlying norm on X is differentiable if and only if every AML function $u : X \to \mathbb{R}$ with a linear growth at infinity, i. e.*

$$|u(x)| \le C(1 + \|x\|), \ \forall x \in X,$$

for some $C > 0$, is an affine function.

F. Peng, C. Wang and Y. Zhou, in [77], follow a different approach to study AML functions which encompasses Question 6.2. In the mention paper, it is considered as underlying space an n-dimensional euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$ and a convex Hamiltonian formulation of the AML-property. More precisely, let $H : \mathbb{R}^n \to \mathbb{R}$ be a coercive, convex function. It is said that a locally Lipschitz function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is an AM_H function if for any nonempty open set V, compactly contained in Ω, and for any Lipschitz function $g : \overline{V} \to \mathbb{R}$ such that $u|_{\partial V} = g|_{\partial V}$, the following estimate holds:

$$\operatorname{ess\,sup}(H(u'|_V)) \le \operatorname{ess\,sup}(H(g')).$$

Observe that previous essential suprema are well defined thanks to the Rademacher Theorem. Indeed, this theorem assert that any locally Lipschitz function defined on a nonempty open subset of $\mathbb{R}^n$ is differentiable almost everywhere.

Let $H : \mathbb{R}^n \to \mathbb{R}$ be a norm on $\mathbb{R}^n$. Then a locally Lipschitz function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is AM_H if and only if u is $\|\cdot\|$-AML, where H can be seen as the canonical dual norm of $\|\cdot\|$. Therefore, Question 6.2 can be generalized in the following sense: Which properties on the convex Hamiltonian H guarantees the smoothness of all AM_H functions defined on open subsets of $\mathbb{R}^n$?

The main results of [77] read as follow:

Theorem 6.8 *[77, Theorem 1.1] Let $n \geq 2$ and let $H : \mathbb{R}^n \to \mathbb{R}$ be a convex and coercive function such that the level set $H^{-1}(\{c\})$ does not contain any line segment for any $c \in \mathbb{R}$. Then, for any $u \in AM_H(\Omega)$, with $\Omega \subset \mathbb{R}^n$ a nonempty open set, any $x \in \Omega$ and any null sequence $(r_{\mathrm{i}})_{\mathrm{i}} \subset \mathbb{R}_+$, there is a subsequence $(r_{\mathrm{i}_k})_k$ and a vector $\mathrm{e} \in \mathbb{R}^n$ such that*

$$\max_{y \in \overline{B}_{r_{\mathrm{i}_k}}(x)} \frac{|u(y) - u(x) - \mathrm{e} \cdot (y - x)|}{r_{\mathrm{i}_k}} \to 0 \text{ as } k \to \infty,$$

where $H(\mathrm{e}) = \lim_{r \to 0} \operatorname{ess\,sup}(\|u'(y)\|' : \ y \in \overline{B}_r(x))$.

Theorem 6.9 *[77, Theorem 1.2] Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a convex and coercive convex function such that the level set $H^{-1}(\{c\})$ does not contain any line segment for any $c \in \mathbb{R}$. Then, for any $\Omega \subset \mathbb{R}^2$ nonempty open set and $u \in AM_H(\Omega)$, u is $\mathcal{C}^1$-smooth. Moreover, if $\Omega = \mathbb{R}^2$ and $u \in AM_H(\mathbb{R}^2)$ has a linear growth at infinity, then u is a linear function in $\mathbb{R}^2$.*

Even though the results of [77] can be applied to general normed spaces by setting H as the desired norm, the technique used to obtain these theorems relies on the euclidean structure of the ambient space $\mathbb{R}^2$. A notable difference between our approach and the one presented in [77] is the fact that in the mentioned work they avoid dealing with positively homogeneous convex functions while we work directly with them, see [77, point 2) in Section 1.1]. Also, probably since we deal only with norms and not with general convex, coercive functions as in [77], our proofs of Theorem 6.3 and Theorem 6.5 are shorter.

This chapter is organized as follows: In the next section we present some basic results of AML functions, several examples to motivate Question 6.2 and we introduce two moduli for the norm which turn to be important tools to prove Theorem 6.4. Section 6.3 is devoted to Theorem 6.3. In Section 6.4 we prove Theorem 6.4. We end this chapter with the proofs of Corollary 6.6 and Corollary 6.7.

Notations: For two functions $u, v : \Omega \subset X \to \mathbb{R}$, we denote by $[u < v]$ the set $\{x \in \Omega : \ u(x) < v(x)\}$. For two sets $V, \Omega \subset X$, we write $V \subset\subset \Omega$ whenever V is compactly contained in Ω.

6.2 Properties of AML functions and two dimensional spaces

This section is divided in three parts: we summarize some results of AML functions that are common in the literature, we give some examples to motivate our results and we introduce two modulus of the norm which will be used to prove Theorem 6.4. In the sequel, X denotes a finite dimensional real normed space and Ω a nonempty open subset of X.

6.2.1 Comparison with cones

The following geometric property is the main tool to work with AML functions.

Definition 6.10 *Let $u : \Omega \subset X \to \mathbb{R}$ be a continuous function. We say that u satisfies comparison with cones from above if for every bounded open set $V \subset\subset \Omega$, every $x_0 \in X$ and every $a, b \in \mathbb{R}$ for which*

$$u(x) \leq C(x) := a + b\|x - x_0\|$$

holds in $\partial(V \setminus \{x_0\})$, then $u \leq C$ in V as well. Analogously, we define comparison with cones from below. A function satisfies comparison with cones if it satisfies comparison with cones it from above and below.

In fact, the property of comparison with cones characterizes AML functions.

Proposition 6.11 *[7, Proposition 2.1][31, Theorem 6.4] Let $u : \Omega \subset X \to \mathbb{R}$ be a continuous function. Then u enjoys comparisons with cones if and only if it is AML.*

The next result is a consequence of Proposition 6.11. Its proof follows without changes from its euclidean counterpart found in [40].

Corollary 6.12 *[40, Lemma 2.4 & Lemma 2.7 (i)] Let $u : \Omega \subset X \to \mathbb{R}$ be an AML function. Then, for $r < \operatorname{dist}(x, \partial\Omega)$, the quantities:*

$$S^+(x, r) := \max_{y \in \partial B_r(x)} \frac{u(y) - u(x)}{r} \qquad \text{and} \qquad S^-(x, r) := -\min_{y \in \partial B_r(x)} \frac{u(y) - u(x)}{r}$$

are non negative. Moreover, for all $x \in \Omega$, the functions $S^+(x, \cdot)$ and $S^-(x, \cdot)$ are non decreasing in r and

$$\lim_{r \to 0} S^+(x, r) = \lim_{r \to 0} S^-(x, r).$$

If we denote by $S(x) = S_u(x)$ the common limit, we have

$$S(x) = \lim_{r \to 0} \sup_{y \in B_r(x)} \frac{u(y) - u(x)}{r}.$$

Notice that, since AML functions defined on open sets are locally Lipschitz, for any $r > 0$, the functions $S^+(\cdot, r)$ and $S^-(\cdot, r)$ are continuous in $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\}$.

Corollary 6.13 *Let $u : \Omega \subset X \to \mathbb{R}$ be an AML function and let $R > 0$ such that $\overline{B}_R \subset \Omega$. Assume that $u = \mathrm{e}_0^*$ on $\overline{B}_R$, where $\mathrm{e}_0^* \in X^*$ and $\mathrm{e}_0^* \neq 0$. If $\|x_0\| = R$, then $S(x_0) > 0$.*

Proof. Since $\overline{B}_R \subset \Omega$, there exists $y \in \partial B_1$ and $t > 0$ such that the segment $[x_0, x_0 + ty]$ is included in $\overline{B}_R$ and $\mathrm{e}_0^*(y) \neq 0$. If $\mathrm{e}_0^*(y) > 0$,

$$S(x_0) = \lim_{r \to 0} \max_{x \in \partial B_r(x_0)} \frac{u(x) - u(x_0)}{r} \geq \lim_{r \to 0} \frac{u(x_0 + ry) - u(x_0)}{r} = \mathrm{e}_0^*(y) > 0.$$

On the other hand, if $\mathrm{e}_0^*(y) < 0$,

$$S(x_0) = -\lim_{r \to 0} \min_{x \in \partial B_r(x_0)} \frac{u(x) - u(x_0)}{r} \geq -\lim_{r \to 0} \frac{u(x_0 + ry) - u(x_0)}{r} = -\mathrm{e}_0^*(y) > 0.$$

□

Corollary 6.14 *Let $u : \Omega \subset X \to \mathbb{R}$ be an AML function. Assume that there exist $x \in \Omega$, $W \subset \Omega$ neighborhood of x and a function $f : W \to \mathbb{R}$ satisfying $u \leq f$ in W. Then, $S(x) \leq \mathrm{Lip}(f)$ in the following cases:*

1. *$f(\cdot) = u(x) + c\| \cdot - x\|$ for some $c > 0$, or*
2. *f is an affine function on W and $f(x) = u(x)$.*

Proof. Both cases follow directly by computing $S(x)$ in terms of $S(x, \cdot)^+$.

□

6.2.2 Examples of AML functions

Albeit simple, the following proposition will allow us to provide several examples of AML functions.

Proposition 6.15 *Let $u : \Omega \subset X \to \mathbb{R}$ be a Lipschitz function. Assume that for every open set $V \subset\subset \Omega$ and for each $x \in V$, there exist $x_1, x_2 \in \partial V$, with the open segment (x_1, x_2) included in V, such that $x \in (x_1, x_2)$ and $u|_{[x_1,x_2]}$ is an affine function with slope equal to $\mathrm{Lip}(u)$. Then u is AML.*

Proof. If $\mathrm{Lip}(u) \equiv 0$, the conclusion follows trivially. So, we assume that $\mathrm{Lip}(u) > 0$. Let $V \subset\subset \Omega$ be a bounded open set. Let $g : \overline{V} \to \mathbb{R}$ be a function such that g and u coincide in ∂V. If $g \neq u$, without loss of generality there exists $x \in V$ such that $g(x) > u(x)$. Let $x_1, x_2 \in \partial V$ be two vectors such that $x \in (x_1, x_2) \subset V$, $u|_{[x_1,x_2]}$ is an affine function of slope $\mathrm{Lip}(u)$ and $u(x_2) > u(x) > u(x_1)$. Then, we get

$$\mathrm{Lip}(g) \geq \frac{g(x) - g(x_1)}{\|x - x_1\|} > \frac{u(x) - u(x_1)}{\|x - x_1\|} = \mathrm{Lip}(u).$$

Therefore, u is an AML function.

□

Corollary 6.16 *Let $P : \mathbb{R}^n \to \mathbb{R}$ and $Q : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projections onto the first coordinate and onto the last $n-1$ coordinates respectively. Let $u : (\mathbb{R}^n, \|\cdot\|_1) \to \mathbb{R}$ be a function defined by $u(x) = P(x) + g \circ Q(x)$, where $g : (\mathbb{R}^{n-1}, \|\cdot\|_1) \to \mathbb{R}$ is a 1-Lipschitz function. Then, u is AML.*

Proof. It is enough to apply Proposition 6.15 at segments included in lines of the form $x + \mathbb{R}e_1$, with $x \in \mathbb{R}^n$.

$\square$

Corollary 6.17 *Let $C \subset X$ be a closed convex set. Then, the function $u : X \setminus C \to \mathbb{R}$ defined by $u(x) = \operatorname{dist}(x, C)$ is AML. In particular, the restriction of the norm $\|\cdot\|$ to $X \setminus \{0\}$ is $\|\cdot\|$-AML.*

Proof. Let $x \in X \setminus C$ and let $y_x \in C$ be one projection of x to C. That is, $\|x - y_x\| = \min\{\|x - z\| : \ z \in C\}$. It is enough to apply Proposition 6.15 at segments included in half-lines of the form $y_x + \mathbb{R}_+(x - y_x)$, with $x \in X \setminus C$.

$\square$

Corollary 6.18 *Let X be a finite dimensional normed space with non-differentiable norm. Then there exists a $\|\cdot\|$-AML function $u : X \to \mathbb{R}$ such that $u(x) \le \|x\|$ for all $x \in X$ and u is not everywhere differentiable.*

Proof. Since the norm is not differentiable, we can find a norm one vector $z \in X$ and two distinct functionals, $u_1^*, u_2^* \in X^*$ of norm 1 such that $u_1^*(z) = u_2^*(z) = 1$. The function $u := \max\{u_1^*, u_2^*\} : X \to \mathbb{R}$ is not differentiable in the whole line $\mathbb{R}z$ and satisfies $u(x) \le \|x\|$ for all $x \in X$. To see that u is AML, it is enough to apply Proposition 6.15 at segments included in lines of the form $x + \mathbb{R}z$, with $x \in X$.

$\square$

Our final example shows that the set of smooth AML functions depends on the underlying norm.

Proposition 6.19 *Let $p > 2$. The function $u : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ defined by $u(x, y) = \|(x, y)\|_p$ is $\|\cdot\|_p$-AML but not $\|\cdot\|_2$-AML.*

Proof. By Corollary 6.17, we already know that u is $\|\cdot\|_p$-AML. Let us prove the second part of the proposition. Since $p > 2$, u is a $\mathcal{C}^2$ function. Then, u is $\|\cdot\|_2$-AML only if $\Delta_\infty u \equiv 0$ in the classical sense. However, $\Delta_\infty u((1/3)^{1/p}, (2/3)^{1/p}) = 3^{-4\frac{p-1}{p}} 2(p-1)(1 + 2^{2-\frac{4}{p}} - 2^{2-\frac{2}{p}})$, which is 0 only if $p = 2$.

$\square$

6.2.3 The moduli α and ρ

For $x^* \in X^*$ with $\|x^*\| = 1$, the face of the unit ball defined by x^* is the set

$$F_{x^*} := [x^* = 1] \cap \overline{B}_1,$$

and for $\beta > 0$, the slice of the closed unit ball defined by x^* and of depth β is the set

$$S(x^*, \beta) := [x^* > 1 - \beta] \cap \overline{B}_1$$

For $x^* \in X^*$ with $\|x^*\| = 1$ and $\alpha > 0$, we consider the following union of faces

$$H(x^*, \alpha) := \bigcup \{F_{h^*} : \ \|h^* - x^*\| \leq \alpha, \ \|h^*\| = 1\} \subset \partial B_1.$$

For $x^* \in X^* \setminus \{0\}$, we define $H(x^*, \alpha) := H(x^*/\|x^*\|, \alpha)$. The set $H(x^*, \alpha)$ is a compact subset of X^*. Now, we define, for x^* unit vector of X^* and $\beta > 0$,

$$\alpha(x^*, \beta) := \sup\{\alpha \in \mathbb{R} : \ H(x^*, \alpha) \subset S(x^*, \beta)\},$$

and $\alpha(\beta) := \inf\{\alpha(x^*, \beta) : \ \|x^*\| = 1\}$. Also, for $x^* \in X^*$, with $\|x^*\| = 1$, and $\sigma > 0$ we define

$$\rho(x^*, \sigma) := \sup\{\rho : \ S(x^*, \rho) \cap \partial B_1 \subset H(x^*, \sigma)\},$$

and $\rho(\sigma) := \inf\{\rho(x^*, \sigma) : \ \|x^*\| = 1\}$.

Let us present two examples: If X is an euclidean space, then $\alpha(x^*, \beta) = (2\beta)^{1/2}$ for every unit vector x^* and $\beta \in (0, 2)$. If $X = (\mathbb{R}^2, \|\cdot\|_\infty)$ and x^* is the unit linear map defined by $x^*((x_1, x_2)) = x_1$, then $\alpha(x^*, \beta) = 2$ for every $\beta > 0$. The next proposition generalizes the first example.

Proposition 6.20 *Let X be a finite dimensional normed space with differentiable norm. Then, for any unit vector $x^* \in X^*$, $\lim\limits_{\beta \to 0} \alpha(x^*, \beta) = 0$. In particular, $\lim\limits_{\beta \to 0} \alpha(\beta) = 0$.*

Proof. Let $x^* \in X^*$ of norm one and let $\varepsilon > 0$. Let $y^* \in X^*$ such that $\|x^* - y^*\| = \varepsilon$ and $\|y^*\| = 1$. Since X is a finite dimensional normed space, F_{y^*} is compact. Moreover, since the norm is differentiable, $F_{x^*} \cap F_{y^*} = \emptyset$. By continuity of x^* and compactness of F_{y^*}, there exists $c > 0$ such that

$$\max\{x^*(y) : \ y \in F_{y^*}\} = 1 - c.$$

Thus, if $\beta < c$, $F_{y^*} \not\subset S(x^*, \beta)$, and since $F_{y^*} \subset H(x^*, \varepsilon)$, we get

$$H(x^*, \varepsilon) \not\subset S(x^*, \beta).$$

Thus, $\alpha(x^*, \beta) \leq \varepsilon$ whenever $\beta < c$.

□

The following propositions will be used in the proof of Theorem 6.4.

Proposition 6.21 *Let X be a finite dimensional normed. Then, $\alpha(\beta) \geq \beta > 0$ for every $\beta > 0$.*

Proof. Let $x^*, y^* \in X^*$ be unit linear functionals such that $\|x^* - y^*\| < \beta$. Then,

$$x^*(y) = y^*(y) + (x^* - y^*)(y) > 1 - \beta, \ \text{for all } y \in F_{y^*}.$$

Thus, $F_{y^*} \subset S(x^*, \beta)$ and therefore, $\alpha(x^*, \beta) \geq \beta$.

□

Proposition 6.22 *Let X be a finite dimensional normed space with differentiable norm. For any $\sigma > 0$, $\rho(\sigma) > 0$. Therefore, for any unit vector $x^* \in X^*$, and for any unit vector $x \in X \setminus H(x^*, \sigma)$, $x^*(x) \leq 1 - \rho(\sigma)$ holds.*

Proof. Let $\sigma > 0$. Notice that, if $\sigma \geq 2$, there is nothing to prove since $H(x^*, \sigma) = \partial B_1$ for any $\|x^*\| = 1$. So, we assume that $\sigma < 2$. Let $x^* \in X^*$ with $\|x^*\| = 1$.

Step 1: $\rho(x^*, \sigma) > 0$. Let us define

$$G := \bigcup \{F_{y^*} : \|y^*\| = 1 \text{ and } \|x^* - y^*\| \geq \sigma\}.$$

Clearly, G is a compact set which depends on x^* and σ. Since X has differentiable norm, we have that $F_{x^*} \cap G = \emptyset$. Therefore, there exists $\rho > 0$ such that

$$\max\{x^*(h) : \; h \in G\} := 1 - \rho.$$

Thus, $\rho(x^*, \sigma) \geq \rho > 0$.

Step 2: For any $x^* \in X^*$ unit linear functional, there exists $\delta > 0$ and $c > 0$ such that

$$\rho(y^*, \sigma) > c \quad \text{for all } y^* \text{ such that } \|y^* - x^*\| < \delta \text{ and } \|y^*\| = 1.$$

Indeed, let $x^* \in X^*$ with $\|x^*\| = 1$. Let $\varepsilon \in (0, \sigma)$. By step 1, we know that $\rho := \rho(x^*, \sigma - \varepsilon) > 0$ and that $S(x^*, \beta) \cap \partial B_1 \subset H(x^*, \sigma - \varepsilon)$ whenever $\beta < \rho$. Define $\delta := \min\{\rho/2, \varepsilon\}$. If $\|x^* - y^*\| < \delta$, with $\|y^*\| = 1$, and if $\beta = \rho/2 + \|x^* - y^*\|$, we get that

$$S(y^*, \rho/2) \cap \partial B_1 \subset S(x^*, \beta) \cap \partial B_1 \subset H(x^*, \sigma - \varepsilon) \subset H(y^*, \sigma).$$

Therefore,

$$\rho(y^*, \sigma) \geq \frac{\rho}{2} > 0, \qquad \text{whenever } \|x^* - y^*\| < \delta.$$

Step 3: $\rho(\sigma) > 0$. Since X is finite dimensional, the conclusion follows directly from the compactness of the unit sphere of X^* and step 2.

□

6.3 Proof of Theorem 6.3

To prove Theorem 6.3, we mainly follow the ideas of [39] where we can find the proof of the theorem whenever X is an euclidean space. Let us start with some geometric facts which allow us to avoid the euclidean arguments used in the mentioned work. We point out that Proposition 6.23 and Proposition 6.25 hold true in general Banach spaces.

Proposition 6.23 *Let X be a normed space. Let $x \in \partial B_1$ and let $V = \partial B_1 \cap \partial B_2(x)$. Then for all $y \in V$, the segment $[-x, y]$ is contained in V.*

Proof. Let $y \in V$. Since $\overline{B}_1 \subset \overline{B}_2(x)$, there exists a closed hyperplane $[f^* = 1]$ which is tangent at y to both $\overline{B}_1$ and $\overline{B}_2(x)$ simultaneously. Observe that this implies that $\|f^*\| = 1$, $f^*(y) = 1$, $\|y\| = 1$ and $f^*(y - x) = 2$. Hence, we conclude that $f^*(-x) = 1$. Now, let $z \in [-x, y]$. Therefore, there is $\lambda \in [0, 1]$ such that $z = \lambda(-x) + (1 - \lambda)y$. By triangular inequality, we obtain that $\|z\| \leq 1$ and $\|z - x\| \leq 2$. By linearity of f^*, we get that $f^*(z) = 1$ and $f^*(z - x) = 2$. Therefore, $z \in V$.

□

Remark 6.24 *In Proposition 6.23, if X has a differentiable norm, then f^* is unique. Indeed, it must be the support functional of $-x$. Therefore, V is contained in $[f^* = 1]$.*

Before stating the next proposition, we recall that in finite dimensional normed spaces the notions of Gâteaux differentiability and Fréchet differentiability coincide for convex functions. Therefore, Proposition 6.25 can be used, for instance, in finite dimensional normed spaces with differentiable norm.

Proposition 6.25 *Let X be a Banach space. Let $u^+, u^- \in S_X$ such that the norm is Gâteaux differentiable at u^+ and u^- with differential u^* and $-u^*$ respectively. Let $f : X \to \mathbb{R}$ be a 1-Lipschitz function such that $f(tu^+) = t$ and $f(tu^-) = -t$ for all $t \geq 0$. Then $f \equiv u^*$.*

Proof. **First case:** Let us start with $v \in \ker(u^*)$. By the differentiability of the norm, there exists a sequence $(\varepsilon_n)_n \subset \mathbb{R}^+$ which tends to 0 as n tends to infinity and such that the following expression

$$\max\{|\|nu^+ - v\| - \|nu^+\||, |\|nu^- - v\| - \|nu^-\||\} \leq \varepsilon_n$$

holds true for all $n \in \mathbb{N}$. Now, if n is large enough, $n > f(v)$ and then

$$1 \geq \frac{f(nu^+) - f(v)}{\|nu^+ - v\|} \geq \frac{n - f(v)}{n + \varepsilon_n},$$

this implies $f(v) \geq -\varepsilon_n$ for all n large. Thus, $f(v) \geq 0$. For the reverse inequality, observe that

$$1 \geq \frac{|f(nu^-) - f(v)|}{\|nu^- - v\|} \geq \frac{n + f(v)}{n + \varepsilon_n},$$

holds true for all $n > 0$. Finally, we arrive to $f(v) \leq \varepsilon_n$, for all $n > 0$. Therefore $f(v) \leq 0$, implying that $f(v) = 0$.

Second case: Let $v \in X \setminus \ker(u^*)$. Without loss of generality, assume that $u^*(v) = \alpha > 0$. Let us consider the function $g : X \to \mathbb{R}$ defined by $g(x) = f(x + \alpha u^+) - \alpha$. Clearly, g is a 1-Lipschitz function such that $g(tu^+) = t$ for all $t \geq 0$. We claim that $g(tu^-) = -t$ for all $t > 0$. Indeed, let us fix $t > 0$. For $s > 0$, we have that $u^*(-\alpha u^+ + su^-) = -\alpha - s$. Thus, $\| - \alpha u^+ + su^-\| = \alpha + s$. Also, since $g(0) = 0$, $g(-\alpha u^+ + su^-) = -\alpha - s$ and g is 1-Lipschitz, g must be linear along the segment $[0, -\alpha u^+ + su^-]$, i.e.

$$g(\lambda(-\alpha u^+ + su^-)) = -\lambda(\alpha + s), \forall\lambda \in [0, 1], \forall s > 0.$$

If $s > t$, we can set $\lambda = t/s$. Using the continuity of g, sending s to infinity, we get that $g(tu^-) = -t$. Finally, the function g satisfies the hypothesis to apply the first case at the vector $v - \alpha u^+ \in \ker(u^*)$. Hence, we get that $g(v - \alpha u^+) = 0$. Thus, by definition of g, $f(v) = \alpha$, finishing the proof.

□

The following corollary directly follows from Proposition 6.25. In fact, this result is obtained in [39] whenever X is an euclidean space.

Corollary 6.26 *Let X be a Banach space. Let $u \in \partial B_1$ such that the norm is Gateaux differentiable at u with differential u^*. Let $f : X \to \mathbb{R}$ be a 1-Lipschitz function such that $f(tu) = t$ for all $t \in \mathbb{R}$. Then $f \equiv u^*$.*

Let us continue with the following lemma.

Lemma 6.27 *Let X be a finite dimensional Banach space and let Ω be a nonempty open subset of X. Let $u : \Omega \subset X \to \mathbb{R}$ be an AML function and let $x \in \Omega$. Then, the following assertions are equivalent:*

i) *For each $r \in (0, \mathrm{dist}(x, \partial\Omega))$, there exists a vector $(\mathrm{e}^*_{x,r})_r \subset X^*$, with $\|\mathrm{e}^*_{x,r}\| = S(x)$, such that*

$$\max_{y \in \overline{B}_r(x)} \frac{|u(y) - u(x) - \mathrm{e}^*_{x,r}(y - x)|}{r} \to 0 \text{ as } r \to 0.$$

ii) *For any decreasing sequence $(r_j)_j$, convergent to 0, there are a subsequence $(r_{j(k)})_k$ and $\mathrm{e}^* \in X^*$, with $\|\mathrm{e}^*\| = S(x)$, such that*

$$\max_{y \in \overline{B}_{r_{j(k)}}(x)} \frac{|u(y) - u(x) - \mathrm{e}^*(y - x)|}{r_{j(k)}} \to 0 \text{ as } k \to \infty.$$

Proof. i) $\Rightarrow$ ii). This is due to the compactness of closed bounded subsets of X^*.

ii) $\Rightarrow$ i). Reasoning by contradiction, if i) does not hold true, then there are $\varepsilon > 0$ and a sequence $(r_j)_j$, convergent to 0, such that

$$\max_{y \in \overline{B}_{r_j}(x)} \frac{|u(y) - u(x) - \mathrm{e}^*(y - x)|}{r_j} \geq \varepsilon \text{ for all } j \in \mathbb{N}, \text{ for all } \mathrm{e}^* \in X^*, \ \|\mathrm{e}^*\| = S(x).$$

This clearly contradicts statement ii). □

Now, we can provide the proof of Theorem 6.3.

Proof of Theorem 6.3. Let $x \in \Omega$. We prove Lemma 6.27 ii). Let $(r_j)_j \subset \mathbb{R}^+$ be a sequence which converges to 0. For each $j \in \mathbb{N}$, let us define $v_j : r_j^{-1}(\Omega - x) \to \mathbb{R}^*$ by

$$v_j(y) = \frac{u(x + r_j y) - u(x)}{r_j}.$$

For each compact subset K of X, the functions v_j are well defined on K for j large enough. Since u is a locally Lipschitz function, $(v_j)_j|_K$ form an equi-Lipschitz family vanishing at 0. So by Arzela-Ascoli Theorem, up to a subsequence, we can assume that the sequence (v_j) converges uniformly on compact subsets of X towards a Lipschitz function v vanishing at 0. i.e. $v(y) = \lim_j v_j(y)$ for any $y \in X$. If v linear, then we can take $\mathrm{e}^* = v$.

So, to prove Theorem 6.3, it remains to show that v is necessary linear. Let $S(x)$ be computed with the function u (see Corollary 6.12). Since a locally uniform limit of functions satisfying comparison with cones satisfies comparisons with cones, we can apply Corollary 6.12 on v as well. From now on, we define the quantities

$$L^+(y,r) := \max_{z\in\partial B_r(y)} \frac{v(z)-v(y)}{r}, \text{ and } L^-(y,r) := -\min_{z\in\partial B_r(y)} \frac{v(z)-v(y)}{r},$$

for $y \in X$ and $r > 0$. Also, we define the corresponding values

$$L^+(y) = \lim_{r\to 0} L^+(y,r) \quad \text{and} \quad L^-(y) = \lim_{r\to 0} L^-(y,r),$$

where $L(y) = L^+(y) = L^-(y)$, by Corollary 6.12.

Proposition 6.28 *Assume that* $\max\{L^-(y,r), L^+(y,r)\} \leq S(x)$ *for all* $r > 0$ *and for all* $y \in X$. *Further, assume that* $L^+(0) = S(x) = L^-(0)$. *Then* v *is linear.*

Proof. The first assumption implies that $\mathrm{Lip}(v) \leq S(x)$. Thanks to the monotonicity of $L^\pm(0,\cdot)$ (see Corollary 6.12) and the second assumption of the statement, we get that $S(x) \leq \min\{L^-(0,r), L^+(0,r)\}$, and therefore, $S(x) = L^+(0,r) = L^-(0,r)$. Further, this implies that $\mathrm{Lip}(v) \geq S(x)$, and then $\mathrm{Lip}(v) = S(x)$. By continuity of v and compactness of closed bounded sets, let $z_r^+, z_r^- \in \partial B_r$ be such that

$$L^\pm(0,r) = \pm\frac{v(z_r^\pm)-v(0)}{r} = \pm\frac{v(z_r^\pm)}{r}.$$

Therefore:

$$L^+(0,r) = L^-(0,r) = S(x) = \frac{v(z_r^+)-v(z_r^-)}{2r}.$$

Observe that the function v is an $S(x)$-Lipschitz function such that

$$v(z_r^+) - v(z_r^-) = 2S(x)r \leq S(x)\|z_r^+ - z_r^-\|. \tag{6.1}$$

Since $z_r^+, z_r^- \in \partial B_r$, $\|z_r^+ - z_r^-\| \leq 2r$, and together with (6.1), we get that $\|z_r^+ - z_r^-\| = 2r$ and that v is an affine function on $[z_r^-, z_r^+]$. Moreover, since $v(0) = 0$, we have that $v(z_r^+) = S(x)r$ and $v(z_r^-) = -S(x)r$.

Let $u^* \in X^*$ be the norm one linear functional such that $u^*(z_1^+) = 1$. By Proposition 6.23 and Remark 6.24, we deduce that $u^*(z_1^-) = -1$. Indeed, if $u^*(z_1^-) < -1$, then $z_1^- \notin \partial B_1 \cap \partial B_2(z_1^+)$. Thus, $\|z_1^+ - z_1^-\| < 2$, which contradicts the fact that $\mathrm{Lip}(v) = S(x)$. Let $r > 1$. Let us show that $u^*(z_r^+) = r$. Indeed, since $v(0) = 0$, $v(z_r^+) = S(x)r$ and v is $S(x)$-Lipschitz, v must be linear along the segment $[0, z_r^+]$. Therefore, the vector $z_r^+/\|z_r^+\|$ may take the place of z_1^+ because $v(z_r^+/\|z_r^+\|) = S(x)$. If $u^*(z_r^+) < r$, then $u^*(z_r^+/\|z_r^+\|) < 1$. By Proposition 6.23 and Remark 6.24, we get that

$$\left\| z_1^- - \frac{z_r^+}{\|z_r^+\|} \right\| < 2,$$

which contradicts the fact that $\mathrm{Lip}(v) = S(x)$. As a direct consequence of $u^*(z_r^+) = r$ we get that $u^*(z_r^-) = -r$.

Since X is a finite dimensional space, there exist a sequence $(r(n))_n \subset \mathbb{R}^+$ which goes to infinity and two vectors $q^+, q^- \in \partial B_1$ such that

$$\lim_{n\to\infty} \frac{z_{r(n)}^{\pm}}{\|z_{r(n)}^{\pm}\|} = q^{\pm}.$$

Clearly $u^*(q^+) = 1$ and $u^*(q^-) = -1$. As a consequence of the continuity of v and its linear behavior along the lines $[0, z_{r(n)}^+]$, with slope $S(x)$, we get that $v(tq^+) = tS(x)$ for all $t \geq 0$. Analogously, we get that $v(tq^-) = -tS(x)$. Finally, applying Proposition 6.25, we conclude that $v = S(x)u^*$. □

To finish the proof of Theorem 6.3, it remains to prove the hypothesis of Proposition 6.28. We point out that the this part of the proof follows as in the proof given in [39], where X is an euclidean space.

To this end, let us start with the case of the superscript $+$. Let $y \in X$ and $z \in \partial B_r(y)$ such that

$$L^+(y, r) = \frac{v(z) - v(y)}{r} = \lim_{j\to\infty} \frac{u(r_j z + x) - u(r_j y + x)}{r_j r}. \tag{6.2}$$

Since $r_j z \in \partial B_{r_j r}(r_j y)$ we get that

$$\frac{u(r_j z + x) - u(r_j y + x)}{r_j r} \leq S^+(r_j y + x, r_j r) \leq S^+(r_j y + x, R), \tag{6.3}$$

for $r_j r < R < \mathrm{dist}(r_j y + x, \partial U)$. Notice that in (6.3), the first and second inequality are due to the definition of S^+ and to Corollary 6.12 respectively. Combining (6.2), (6.3) and using the continuity of the function $S^+(\cdot, R)$, we get that

$$L^+(y, r) \leq \lim_{j\to\infty} S^+(r_j y + x, R) \leq S^+(x, R).$$

Finally, sending R to 0 we obtain that $L^+(y, r) \leq S(x)$. To prove the second hypothesis, let us consider $y = 0$. Then, we compute

$$L^+(0, r) = \max_{z\in\partial B_r} \frac{v(z)}{r} = \max_{z\in\partial B_r} \lim_{j\to\infty} \frac{u(r_j z + x) - u(x)}{r_j r}.$$

By compactness of ∂B_r and continuity of u, for each j there exists $z_j \in \partial B_r$ satisfying

$$u(r_j z_j + x) = \max_{z\in\partial B_r} u(r_j z + x).$$

Let us consider any cluster point $\overline{z}$ of $(z_j)_j \subset \partial B_r$. Let $(j(n))$ be a subsequence such that $z_{j(n)} \to \overline{z}$. Using the fact that u is Lipschitz in a neighborhood of x, we prove that

$$\begin{aligned} L^+(0,r) &\geq \lim_{n\to\infty} \frac{u(r_{j(n)}\overline{z}+x)-u(x)}{r_{j(n)}r} = \lim_{n\to\infty} \max_{z\in\partial B_r} \frac{u(r_{j(n)}z+x)-u(x)}{r_{j(n)}r} \\ &= \lim_{n\to\infty} S^+(x, r_{j(n)}r) = S(x). \end{aligned}$$

Therefore, sending r to 0 we get that $L^+(0) \geq S(x)$. Thus, $L^+(0) = S(x)$. The case with superscript $-$ is analogous. This ends the proof of Theorem 6.3.

□

6.4 Proof of Theorem 6.4

O. Savin, in [87], proved that every planar AML function is continuously differentiable whenever the underlying space is endowed with an euclidean norm. In the sequel, we generalize the technique developed in the mentioned paper to prove Theorem 6.4. For the sake of completeness, we provide the proofs of Proposition 6.29 and Lemma 6.32 which follow without significant changes from the work [87].

From now on, X denotes a 2 dimensional Banach space equipped with a differentiable norm. The proof of Theorem 6.4 uses Theorem 6.3 and the following two propositions.

Proposition 6.29 *[87, Lemma 1] Let $u : \Omega \subset X \to \mathbb{R}$ be an AML function where Ω is a nonempty open and convex set containing 0 and that u does not coincide with an affine function on any neighborhood of 0. Then, for every open subset W of Ω containing 0, there exist $y \in W$ and an affine function $g := \mathrm{e}^* + u(y) - \mathrm{e}^*(y)$, where $\mathrm{e}^* \in X^*$ satisfies $\|\mathrm{e}^*\| = S(y)$, such that the one of the sets $[u > g]$ and $[u < g]$ has at least two distinct connected components intersecting W.*

Proof. Let W be an open subset of Ω containing 0. Then, there exists a segment $[z_1, z_2] \subset W$ such that u is not affine restricted to it. Thus, there is an affine function ℓ on $[z_1, z_2]$ and a point $y \in (z_1, z_2)$ such that $u(y) = \ell(y)$ and

$$\begin{aligned} &u \geq \ell \text{ in } [z_1, z_2] \text{ and } u(z_\mathrm{i}) > \ell(z_\mathrm{i}), \text{ for i} = 1,2 \qquad \text{or} \\ &u \leq \ell \text{ in } [z_1, z_2] \text{ and } u(z_\mathrm{i}) < \ell(z_\mathrm{i}), \text{ for i} = 1,2. \end{aligned}$$

We treat the first case, the second one is similar. From Theorem 6.3, there exists vectors $\mathrm{e}^*_{y,r}$ such that $\|\mathrm{e}^*_{y,r}\| = S(y)$ and

$$\lim_{r\to 0} \max_{x\in \overline{B}_r(y)} \frac{|u(x)-u(y)-\mathrm{e}^*_{y,r}(x-y)|}{r} = 0.$$

By compactness, there is a sequence $(r_{\rm i})_{\rm i}$, which converges to 0, such that $\mathrm{e}^*_{y,r_{\rm i}} \to \mathrm{e}^*$. Therefore, $\|\mathrm{e}^*\| = S(y)$ and

$$\lim_{{\rm i}\to\infty} \max_{x\in\overline{B}_{r_{\rm i}}(y)} \frac{|u(x)-g(x))|}{r_{\rm i}} = 0. \tag{6.4}$$

where g is the affine function defined by $g(x) = \mathrm{e}^*(x) - \mathrm{e}^*(y) + u(y)$. Since $u \geq \ell$ in $[z_1, z_2]$ and $u(y) = \ell(y)$, the limit (6.4) implies that g coincides with ℓ in $[z_1, z_2]$, and then, $z_1, z_2 \in [u > g]$.

Reasoning by contradiction, suppose that there exists a polygonal line $\gamma \subset [u > g]$ connecting the point z_1 and z_2. Let Γ be the union of γ with the segment $[z_1, z_2]$, and U be the union of all bounded connected component of $X \setminus \Gamma$. Let $h^* \in X^*$ be a non-zero linear functional such that $h^*(z_2 - z_1) = 0$. Using the fact that $y \notin \gamma$ and replacing h^* by $-h^*$ if necessary, there exists $\delta > 0$ such that $\overline{B}_\delta(y) \cap [h^* > 0] \subset U$. Since γ is compactly contained in $[u > g]$, there exists $\varepsilon > 0$ such that $u \geq g + \varepsilon h^*$ on γ, hence also on Γ. We have $u \geq g + \varepsilon h^*$ on $\partial U \subset \Gamma$. Since u is an AML function, $u \geq g + \varepsilon h^*$ on U, so $u - g \geq \varepsilon h^*$ on $\overline{B}_\delta(y) \cap [h^* > 0]$. This contradicts the limit (6.4), finishing the proof. □

The assumptions of Proposition 6.30 are explained by the conclusions of Theorem 6.3 and Proposition 6.29.

Proposition 6.30 *Let $\rho > 0$. Let $u : B_\rho \subset X \to \mathbb{R}$ be an AML function and let $\mathrm{e}^*_1 \in X^*$ such that*

$$\sup\{|u(x) - \mathrm{e}^*_1(x)|,\ x \in B_\rho\} \leq \lambda\rho\|\mathrm{e}^*_1\|.$$

Further, assume that there exists $\mathrm{e}^* \in X^*$ such that $[u > \mathrm{e}^*]$ has at least two distinct connected components that intersect $\overline{B}_{\rho/6}$. Then, for $\varepsilon > 0$, there exists $\lambda(\varepsilon) > 0$ such that if $\lambda \leq \lambda(\varepsilon)$,

$$\|\mathrm{e}^* - \mathrm{e}^*_1\| \leq \varepsilon\|\mathrm{e}^*_1\|.$$

Proof. If $\mathrm{e}^*_1 = 0$, then u is identically 0 in B_ρ. Therefore, the second hypothesis cannot occur. So, without loss of generality, we assume that $\mathrm{e}^*_1 \neq 0$. Let $R = C(\varepsilon, X) > 0$ given by Lemma 6.31. Let us define $\lambda(\varepsilon) := \dfrac{1}{6C(\varepsilon, X)}$. If $w : B_{6R} \to \mathbb{R}$ is the function defined by

$$w(x) := \frac{6R}{\rho\|\mathrm{e}^*_1\|} u\left(\frac{\rho x}{6R}\right),$$

and if $\lambda < \lambda(\varepsilon)$, the function w satisfies

(H1) $\sup\{|w(x) - \frac{\mathrm{e}^*_1(x)}{\|\mathrm{e}^*_1\|}| :\ x \in B_{6R}\} \leq 1$.

(H2) The set $[w > \frac{\mathrm{e}^*}{\|\mathrm{e}^*_1\|}]$ has at least two distinct connected components that intersect B_R.

Therefore, Proposition 6.30 follows from Lemma 6.31 below.

□

Lemma 6.31 *For every $\varepsilon > 0$, there exists a constant $C(\varepsilon, X) > 0$ with the following property : Let $\varepsilon > 0$, $R \geq C(\varepsilon, X)$ and $u : B_{6R} \to \mathbb{R}$ be an AML function satisfying*

(H1) $\sup\{|u(x)-\mathrm{e}_1^*(x)| :\ x\in B_{6R}\}\le 1$ *for some* $\|\mathrm{e}_1^*\|=1$,

(H2) *There exists a linear functional* $\mathrm{e}^*\in X^*$ *such that the set* $[u>\mathrm{e}^*]$ *has at least two distinct connected components that intersect* $\overline{B}_R$.

Then

$$\|\mathrm{e}^*-\mathrm{e}_1^*\|\le\varepsilon.$$

Proof of Lemma 6.31. Let $f^*=\mathrm{e}_1^*-\mathrm{e}^*$ and let $\varepsilon>0$. Without loss of generality, assume that $f^*\neq 0$. By (H1), we have that

$$\begin{aligned}[f^*<-1]\cap B_{6R}&\subset[u<\mathrm{e}^*]\\ [f^*>1]\cap B_{6R}&\subset[u>\mathrm{e}^*]\end{aligned}$$

Thus, by hypothesis (H2), we can find a connected component $\mathcal{U}$ of $[u>\mathrm{e}^*]$ that intersects $\overline{B}_R$ and that is included in the strip $\mathcal{S}:=[|f^*|\le 1]$ of width $2\|f^*\|^{-1}$. If $R>\|f^*\|^{-1}$, the set $\mathcal{S}\cap\partial B_R$ is the union of two disjoint arcs of ∂B_R. Observe that $\mathcal{U}$ cannot be compactly contained in B_{6R}. Otherwise, it would contradict the AML property of u (comparing against e^* on $\overline{\mathcal{U}}$). Consider a polygonal line $\Gamma\subset\mathcal{U}\subset\mathcal{S}$ that starts in $\overline{B}_R$ and exits $\overline{B}_{6R}$. Let $x_0\in X$ be a vector such that $\|x_0\|=3R$ and $f^*(x_0)=0$. Replacing x_0 by $-x_0$ if necessary, we can assume that Γ intersects the two distinct arcs of $\mathcal{S}\cap\partial B_R(x_0)$. Let $v:\overline{B}_{2R}\to\mathbb{R}$ be the function defined by $v(\cdot):=u(\cdot+x_0)-\mathrm{e}_1^*(x_0)$. Observe that by (H1),

$$|v(x)-\mathrm{e}_1^*(x)|\le|u(x+x_0)-\mathrm{e}_1^*(x_0+x)|\le 1,\ \text{for all } x\in\overline{B}_{2R},$$

and that, due to the fact that $f^*(x_0)=0$, $y\in[v>\mathrm{e}^*]$ if and only if $y+x_0\in[u>\mathrm{e}^*]$. Therefore, replacing u by v (and x_0 by $-x_0$ if necessary), hypothesis (H1) and (H2) imply:

(H1') $\max\{|u(x)-\mathrm{e}_1^*(x)| :\ x\in\overline{B}_{2R}\}\le 1$ for some $\|\mathrm{e}_1^*\|=1$.

(H2') If $R>\|f^*\|^{-1}=\|\mathrm{e}_1^*-\mathrm{e}^*\|^{-1}$, the set $[u>\mathrm{e}^*]\cap\overline{B}_{2R}$ has a connected component $\mathcal{U}$, included in $\mathcal{S}=[|f^*|\le 1]$, which contains a polygonal line Γ connecting the two distinct arcs of $\mathcal{S}\cap\partial B_R$.

Lemma 6.31 follows from the next two lemmas.

Lemma 6.32 *[87, Lemma 3] Let* $0<\gamma<1$. *If* $R\ge C_1(\gamma):=20\gamma^{-2}$, *then*

$$\|\mathrm{e}^*\|\ge 1-\gamma.$$

In order to state the second lemma, let us recall that if $\mathrm{e}^*\neq 0$ and $\beta>0$, the set

$$H\left(\mathrm{e}^*,\frac{\beta}{2}\right):=\bigcup\left\{F_{h^*} :\ \left\|h^*-\frac{\mathrm{e}^*}{\|\mathrm{e}^*\|}\right\|\le\frac{\beta}{2},\ \|h^*\|=1\right\},$$

where $F_{h^*}:=[h^*=1]\cap\overline{B}_1$.

Lemma 6.33 *Let* u *be AML satisfying* (H1') *and* (H2'), *let* $\|\mathrm{e}^*\|\ge\gamma>0$ *and* $\beta>0$. *There exists* $C_2=C_2(\gamma,\beta)$ *such that if* $R\ge C_2$, *then*

$$\inf\left\{|f^*(h)| :\ h\in H\left(\mathrm{e}^*,\frac{\beta}{2}\right)\right\}<\gamma.$$

Let us finish the proof of Lemma 6.31. Since X is a finite dimensional space with differential norm, X^* is uniformly convex. Hence, there exists $\sigma(\varepsilon) > 0$ such that for any two unit vectors x^*, y^* in X^* satisfying $\|\frac{x^*+y^*}{2}\| > \sigma(\varepsilon)$, then $\|x^* - y^*\| < \varepsilon$. If $\gamma, \beta > 0$ are small, we have

$$\beta + \gamma < 1 - \sigma(\varepsilon/2). \tag{6.5}$$

and

$$\frac{\beta}{2} + \frac{\gamma+\beta}{1-\beta} < \frac{\varepsilon}{2}. \tag{6.6}$$

Let us fix $\beta = \beta(\varepsilon)$ and $\gamma = \gamma(\varepsilon)$ satisfying (6.5) and (6.6), and define

$$C(\varepsilon, X) := \max\{C_1(\gamma), C_2(\gamma, \beta)\}.$$

Assume that $R \geq C(\varepsilon, X)$. Lemma 6.32 implies that

$$\|\mathrm{e}^*\| \geq 1 - \gamma, \tag{6.7}$$

and Lemma 6.33 implies the existence of a unit vector $h^* \in X^*$ satisfying $\|h^* - \frac{\mathrm{e}^*}{\|\mathrm{e}^*\|}\| \leq \beta/2$ and a vector $h \in F_{h^*}$ such that $|f^*(h)| \leq \gamma$. So $h \in H(\mathrm{e}^*, \beta/2)$, and since Proposition 6.21 implies $\beta/2 < \beta \leq \alpha(\frac{\mathrm{e}^*}{\|\mathrm{e}^*\|}, \beta)$, we have

$$(1-\beta)\|\mathrm{e}^*\| < \mathrm{e}^*(h). \tag{6.8}$$

The condition $|f^*(h)| \leq \gamma$ implies that

$$\mathrm{e}^*(h) \leq \mathrm{e}_1^*(h) + \gamma \leq 1 + \gamma. \tag{6.9}$$

Conditions (6.7), (6.8) and (6.9) imply

$$\|\mathrm{e}_1^* + h^*\| \geq (\mathrm{e}_1^* + h^*)(h) \geq \mathrm{e}^*(h) - \gamma + 1 \geq (1-\beta)\|\mathrm{e}^*\| + 1 - \gamma \geq (1-\beta)(1-\gamma) + 1 - \gamma.$$

Thus, $\|\frac{\mathrm{e}_1^*+h^*}{2}\| \geq 1 - \gamma - \beta \geq \sigma(\varepsilon/2)$, therefore $\|\mathrm{e}_1^* - h^*\| \leq \varepsilon/2$. Conditions (6.7), (6.8) and (6.9) also imply

$$1 - \gamma \leq \|\mathrm{e}^*\| \leq \frac{1+\gamma}{1-\beta}.$$

So,

$$\|h^* - \mathrm{e}^*\| \leq \left\| h^* - \frac{\mathrm{e}^*}{\|\mathrm{e}^*\|} \right\| + |\|\mathrm{e}^*\| - 1\| \leq \frac{\beta}{2} + \frac{\gamma+\beta}{1-\beta} \leq \varepsilon/2.$$

Finally, we get that $\|\mathrm{e}^* - \mathrm{e}_1^*\| \leq \varepsilon$, finishing the proof of Lemma 6.31.

□

In the sequel, we prove Lemma 6.32, Lemma 6.33 and Theorem 6.4.

Proof of Lemma 6.32. Reasoning by contradiction, let us assume that $\|\mathrm{e}^*\| < 1 - \gamma$. Since $f^* = \mathrm{e}_1^* - \mathrm{e}^*$ and $\|\mathrm{e}_1^*\| = \|\mathrm{e}_1\| = \mathrm{e}_1^*(\mathrm{e}_1) = 1$, we have that $2 \geq \|f^*\| \geq f^*(\mathrm{e}_1) > \gamma$. Let $y_0 := -4\gamma^{-1}\mathrm{e}_1$, and let y_1 be the point of intersection of $\{t\mathrm{e}_1 : t \geq 0\}$ with the line $[f^* = 1]$. We have $\|y_1\| = f^*(\mathrm{e}_1)^{-1} < \gamma^{-1}$, so

$$4\gamma^{-1} < \|y_1 - y_0\| < 5\gamma^{-1}.$$

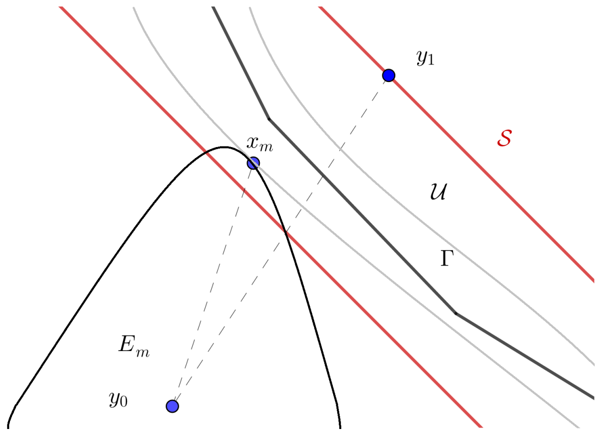

Figure 6.1: Lemma 6.32: The set E_m.

Since $R \geq C_1 := 20\gamma^{-2} > \|f^*\|^{-1}$, we can apply (H2'), and we also have $y_0 \in \overline{B}_R$ and $y_1 \in \overline{B}_R(y_0) \subset \overline{B}_{2R}$.

For $c \geq 0$, let V_c be the function defined on X by

$$V_c(x) := \mathrm{e}_1^*(y_0) + 1 + c\|x - y_0\|.$$

Notice that, for $c > \|\mathrm{e}^*\|$, the set

$$E_c := \{x \in X : V_c(x) \leq \mathrm{e}^*(x)\}$$

is convex and compact. We claim that $u(y_0) \leq V_c(y_0) < \mathrm{e}^*(y_0)$. Indeed, $y_0 \in \overline{B}_R$, so condition (H1') implies the first inequality. On the other hand, $V_c(y_0) = \mathrm{e}_1^*(y_0)+1 = \mathrm{e}^*(y_0)+f^*(y_0)+1 = \mathrm{e}^*(y_0) + 1 - 4f^*(\mathrm{e}_1)/\gamma$, this implies the second inequality.

Let $m := \max\{c > \|\mathrm{e}^*\| : E_c \cap \partial([u < \mathrm{e}^*] \cap \overline{B}_{2R}) \neq \emptyset\}$. The claim implies $y_0 \in E_c$ for every $c > 0$. Since the diameter of E_c tends to 0 as c tends to infinity, we conclude that E_c converges to $\{y_0\}$ in Hausdorff-Pompeiu distance. The claim also implies $u(y_0) < \mathrm{e}^*(y_0)$, so $m < \infty$. Now, we set

$$c_0 := 1 - \frac{2}{\|y_1 - y_0\|} > 1 - \frac{\gamma}{2} > \|\mathrm{e}^*\|.$$

The equality $\|y_1 - y_0\| = \mathrm{e}_1^*(y_1 - y_0)$ implies $V_{c_0}(y_1) = \mathrm{e}_1^*(y_0) + \|y_1 - y_0\| - 1 = \mathrm{e}_1^*(y_1) - f^*(y_1) = \mathrm{e}^*(y_1)$. Therefore, $y_1 \in E_{c_0}$, and we know that $y_0 \in E_{c_0}$, so the segment $[y_0, y_1]$ is included in the convex E_{c_0}. Since $[y_0, y_1]$ crosses all the strip $\mathcal{S}$, it must intersect the polygonal line Γ given by (H2') which is included in $[u > \mathrm{e}^*]$. Therefore, $E_{c_0} \cap \partial([u < \mathrm{e}^*] \cap \overline{B}_{2R}) \neq \emptyset$, this shows that $m \geq c_0$. The compact E_m is included in B_{2R}. Indeed, let $x \in E_m$. Since $\|\mathrm{e}^*\| < 1 - \gamma$, we get

$$0 \leq \mathrm{e}^*(x) - V_m(x) \leq \mathrm{e}^*(y_0) - V_m(y_0) + (1 - \gamma - m)\|x - y_0\|.$$

Using the inequalities $m > 1 - \gamma/2$ and $\mathrm{e}^*(y_0) - V_m(y_0) \leq 8/\gamma$, we obtain $\|x - y_0\| < 16\gamma^{-2}$, and so $\|x\| < 20\gamma^{-2} \leq R$. Therefore, if $x_m \in E_m \cap \partial([u < \mathrm{e}^*] \cap \overline{B}_{2R})$, then $\|x_m\| < R$, and so $x_m \in \partial[u < \mathrm{e}^*]$, and by continuity of u, we have that $u(x_m) = \mathrm{e}^*(x_m)$. Notice now, by definition of m, we have that $u \leq V_m$ in $\partial(E_m \setminus \{y_0\})$. Hence, by comparisons with cones, $u \leq V_m$ in E_m. Since V_m is an affine function restricted to $[x_m, y_0]$ and that $u(x_m) = V_m(x_m) \geq V_m(y_0)$, we get

$$S(x_m) = -\lim_{r\to 0} \min_{y \in \partial B_r(x_m)} \frac{u(y) - u(x_m)}{r} \geq -\lim_{r \to 0} \frac{V_m(y_r) - V_m(x_m)}{r} = m > c_0 \geq 1 - \frac{\gamma}{2}, \quad (6.10)$$

where y_r is the point of intersection of $\partial B_r(x_m)$ with $[x_m, y_0]$. However, we claim that $S(x_m) \leq \|\mathrm{e}^*\| + 2R^{-1}$. To this end, let $r > 0$ small and let U' be the open set relative to $\overline{B}_R(x_m)$ defined as the union of all connected components of $[u > \mathrm{e}^*] \cap \overline{B}_R(x_m)$ that intersect $\overline{B}_r(x_m)$. If $U' = \emptyset$, then $u \leq \mathrm{e}^*$ in $\overline{B}_r(x_m)$. Therefore, since $u(x_m) = \mathrm{e}^*(x_m)$, by Corollary 6.14 we get that $S(x_m) \leq \|\mathrm{e}^*\|$, this proves the claim in this case. If $U' \neq \emptyset$, from (H2') we have that $U' \subset \mathcal{S}$ provided that $r < \mathrm{dist}(x_m, \Gamma)$. For $x \in \partial U' \cap \overline{B}_R(x_m)$, we have that $u(x) = \mathrm{e}^*(x)$. For $x \in U' \cap \partial B_R(x_m)$,

$$u(x) \leq \mathrm{e}^*(x) + 2 \leq \mathrm{e}^*(x_m) + R\|\mathrm{e}^*\| + 2,$$

where the first inequality follows as in (6.13), in the proof of Lemma 6.33. Therefore, comparison with cones implies

$$u(x) \leq \mathrm{e}^*(x_m) + (\|\mathrm{e}^*\| + 2R^{-1})\|x - x_m\|, \text{ for all } x \in U' \cap \overline{B}_R(x_m). \quad (6.11)$$

Combining (6.11) and the fact that $u \leq \mathrm{e}^*$ in $\overline{B}_r(x_m) \setminus U'$, we get that the inequality (6.11) holds in $\overline{B}_r(x_m)$. By Corollary 6.14, we conclude that

$$S(x_m) \leq \|\mathrm{e}^*\| + 2R^{-1}.$$

Since $R \geq C_1 \geq 5\gamma^{-1}$, we arrive to $S(x_m) \leq 1 - \gamma/2$. The last inequality contradicts (6.10), finishing the proof of Lemma 6.32.

□

Proof of Lemma 6.33. If $\|f^*\| \leq \gamma$, the conclusion is direct. For this, let us assume that $\|f^*\| > \gamma$. If we further assume that $C_2 \geq 3/\gamma$, since $R \geq C_2$, the conclusion of hypothesis (H_2') is available for us. Reasoning by contradiction, we have

$$\inf\left\{|f^*(h)| : \ h \in H\left(\mathrm{e}^*, \frac{\beta}{2}\right)\right\} \geq \gamma. \quad (6.12)$$

Let e be a unit vector in X such that $\mathrm{e}^*(\mathrm{e}) = \|\mathrm{e}^*\|$, and let x_0 be the point of intersection of $\partial\mathcal{S}$ with the half line $\{-t\mathrm{e} : \ t > 0\}$. We have that $x_0 = -t_0\mathrm{e}$, where t_0 satisfies

$$1 = t_0|f^*(\mathrm{e})| \geq t_0\gamma.$$

So, $\|x_0\| = t_0 \leq 1/\gamma \leq C_2/3 \leq R/3$. Thus $-x_0 \in \overline{B}_R(x_0) \subset \overline{B}_{2R}$. Hypothesis (H1') and (H2') imply

$$\begin{aligned} |u(x) - \mathrm{e}^*(x)| \leq |u(x) - \mathrm{e}_1^*(x)| + |\mathrm{e}^*(x) - \mathrm{e}_1^*(x)| \leq 2, &\quad \text{for all } x \in \mathcal{U} \cap \overline{B}_R(x_0). \\ u(x) = \mathrm{e}^*(x) &\quad \text{for all } x \in \partial\mathcal{U} \cap \overline{B}_R(x_0). \end{aligned}$$

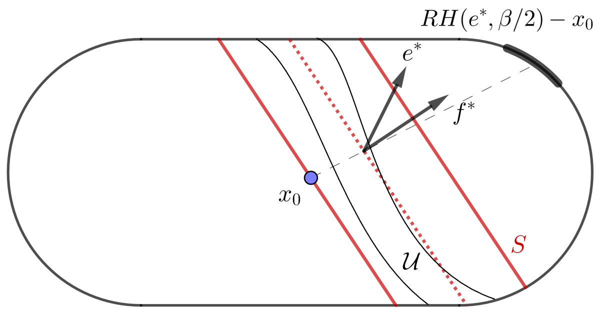

Figure 6.2: Lemma 6.33: Ball of radius R centered at x_0.

Hence, if $x \in \mathcal{U} \cap \partial B_R(x_0)$,

$$u(x) \leq \mathrm{e}^*(x) + 2 \leq \mathrm{e}^*(x_0) + \sup_{y \in \mathcal{S} \cap \partial B_R(x_0)} \mathrm{e}^*(y - x_0) + 2 \tag{6.13}$$

Since $R \geq C_2$, $|f^*(Rh)| \geq 3$ for every $h \in H(\mathrm{e}^*, \beta/2)$. Therefore, $|f^*(Rh - x_0)| \geq 2 > 1$ for every $h \in H(\mathrm{e}^*, \beta/2)$, i.e.

$$(\mathcal{S} \cap \partial B_R(x_0)) \cap (RH(\mathrm{e}^*, \beta/2) - x_0) = \emptyset.$$

By Proposition 6.22, with $\sigma = \beta/2$, we obtain $\rho \in (0, 1)$ depending on β, such that

$$\mathrm{e}^*(x - x_0) \leq (1 - \rho)\|\mathrm{e}^*\|\|x - x_0\|, \text{ for all } x \in \mathcal{S} \cap \partial B_R(x_0). \tag{6.14}$$

Let us assume now that $C_2 \geq 3/(\gamma\rho) > 3/\gamma$. Since $R \geq C_2$ and $\|\mathrm{e}^*\| \geq \gamma$, we get $R\|\mathrm{e}^*\|\rho \geq 2$. Combining (6.13) and (6.14) we get

$$u(x) \leq \mathrm{e}^*(x_0) + \|\mathrm{e}^*\|\|x - x_0\| \text{ for all } x \in \mathcal{U} \cap \partial B_R(x_0).$$

From comparisons with cones, we obtain

$$u(x) \leq \mathrm{e}^*(x_0) + \|\mathrm{e}^*\|\|x - x_0\|, \text{ for all } x \in \mathcal{U} \cap \overline{B}_R(x_0).$$

In particular,

$$u(x) \leq \mathrm{e}^*(x), \text{ for all } x \in \mathcal{U} \cap \overline{B}_R(x_0) \cap \{x_0 + t\mathrm{e} : \ t > 0\}.$$

This is a contradiction with (H_2') since $\mathcal{U} \cap \overline{B}_R(x_0) \cap \{x_0 + t\mathrm{e} : \ t > 0\}$ necessarily intersects Γ.

□

Now, we are able to present the proof of Theorem 6.4.

Proof of Theorem 6.4. If $\|\mathrm{e}_1^*\| = 0$, there is nothing to prove. If $\|\mathrm{e}_1^*\| \neq 0$, by homogeneity, we can assume $\|\mathrm{e}_1^*\| = 1$. By Theorem 6.3, there exists $(\mathrm{e}_{0,r}^*)_r \subset X^*$ such that $\|\mathrm{e}_{0,r}^*\| = S(0)$ for every r and

$$|u(x) - u(0) - \mathrm{e}_{0,r}^*(x)| \leq r\sigma(r), \text{ for all } x \in \overline{B}_r, \tag{6.15}$$

where $\sigma : \mathbb{R}_+ \to \mathbb{R}$ is a positive function such that $\sigma(r)$ tends to 0 as r tends to 0.

Let us fix $\varepsilon > 0$, we need to find $\delta := \delta(\varepsilon) > 0$ such that, if $|u(x) - \mathrm{e}_1^*(x)| \leq \delta$ in B_1, then

$$\limsup_{r\to 0} \|\mathrm{e}_{0,r}^* - \mathrm{e}_1^*\| \leq \varepsilon, \tag{6.16}$$

First case: Suppose that u is not identical to an affine function in any neighborhood of 0. We show that if $\delta \leq \delta_1(\varepsilon) = \min\{\lambda(\varepsilon/4)/4, 1/2\}$ where λ is the function given in Proposition 6.30, then (6.16) holds. Let $r \in (0, 1/2)$ such that $\sigma(r) \leq \lambda(\varepsilon/4)S(0)/4$. Thanks to Proposition 6.29, replacing u by $-u$ if necessary, there exist $y \in B_{r/24}$ and a linear functional $\mathrm{e}^* \in X^*$ satisfying $\|\mathrm{e}^*\| = S(y)$ and such that the set

$$\mathcal{O} = [u > \mathrm{e}^* + u(y) - \mathrm{e}^*(y)] \cap B_1$$

has at least two distinct connected components intersecting $B_{r/24}$. The function $v(\cdot) := u(\cdot + y) - u(y)$ is well defined on $B_{1/2}$. Let us check that v satisfies the hypothesis of Proposition 6.30. The set $[v > \mathrm{e}^*] = (\mathcal{O} - y) \cap B_{1/2}$ has at least two distinct connected components intersecting $\overline{B}_{r/12} \subset \overline{B}_{1/12}$. On the other hand, for $x \in \overline{B}_{1/2}$ we have

$$|v(x) - \mathrm{e}_1^*(x)| \leq |u(x + y) - \mathrm{e}_1^*(x + y)| + |u(y) - \mathrm{e}_1^*(y)| \leq 2\delta.$$

Since $2\delta \leq \lambda(\varepsilon/4)/2$, thanks to Proposition 6.30 applied with $\rho = 1/2$, we get

$$\|\mathrm{e}^* - \mathrm{e}_1^*\| \leq \frac{\varepsilon}{4}. \tag{6.17}$$

Since $|u(x) - \mathrm{e}_1^*(x)| \leq \delta$ in B_1 and $\|\mathrm{e}_1^*\| = 1$, we obtain

$$\|\mathrm{e}_{0,r}^*\| = S(0) \leq S^+\left(0, \frac{1}{2}\right) = 2 \max_{x\in\partial B_{1/2}} u(x) - u(0) \leq 1 + 4\delta. \tag{6.18}$$

We now apply Proposition 6.30 to the function v on $\overline{B}_{r/2}$. The set $[v > \mathrm{e}^*] \cap B_{r/2}$ has at least two distinct connected components which intersect $\overline{B}_{r/12}$. On the other hand, thanks to (6.15), for $x \in \overline{B}_{r/2}$ we have that

$$|v(x) - \mathrm{e}_{0,r}^*(x)| \leq |u(x + y) - u(0) - \mathrm{e}_{0,r}^*(x + y)| + |u(y) - u(0) - \mathrm{e}_{0,r}^*(y)| \leq 2r\sigma(r).$$

Since $2\sigma(r) \leq \lambda(\varepsilon/4)\|\mathrm{e}_{0,r}^*\|/2$, we get that

$$|v(x) - \mathrm{e}_{0,r}^*(x)| \leq \frac{r}{2}\lambda(\varepsilon/4)\|\mathrm{e}_{0,r}^*\|, \text{ for all } x \in \overline{B}_{r/2}.$$

Finally, we can apply Proposition 6.30 with $\rho = r/2$ to obtain

$$\|\mathrm{e}^* - \mathrm{e}_{0,r}^*\| \leq \frac{\varepsilon\|\mathrm{e}_{0,r}^*\|}{4} \leq \frac{(1 + 4\delta)\varepsilon}{4} \leq \frac{3\varepsilon}{4}, \tag{6.19}$$

Combining (6.17) with (6.19) we get that $\|\mathrm{e}_1^* - \mathrm{e}_{0,r}^*\| \leq \varepsilon$. Thus (6.16) is satisfied in this case whenever $\delta \leq \delta_1(\varepsilon)$.

Second case: Suppose that there exists $e_0^* \in X^*$ such that $u = e_0^*$ in a neighborhood of 0. Let

$$R = \max\{r \in (0,1];\ [u = e_0^*] \subset \overline{B}_r\}.$$

If $R \geq 1/2$, notice that $e_{0,r}^* = e_0^*$ satisfies (6.15) whenever $r \leq 1/2$. Assume $\delta \leq \varepsilon/2$ and $|u(x) - e_1^*(x)| < \delta$ in B_1. Since $u = e_0^*$ in $\overline{B}_{1/2}$, we get $\limsup_{r\to 0} \|e_{0,r}^* - e_1^*\| = \|e_0^* - e_1^*\| \leq \varepsilon$.

If $R < 1/2$, there exists $x_0 \in \partial B_R$ such that u is not identical to an affine function in any neighborhood of x_0. Let us define the AML function $v : \overline{B}_1 \to \mathbb{R}$ by

$$v(\cdot) := u(\frac{\cdot}{2} + x_0) - u(x_0).$$

Since v is not affine in any neighborhood of 0, we wish to apply the first case to the function v. According to Theorem 6.3, there exists $(e_{x_0,r}^*)_r \subset X^*$ such that $\|e_{0,r}^*\| = S(x_0)$ for every $r \in (0,1)$

$$|u(x) - u(x_0) - e_{x_0,r}^*(x - x_0)| \leq r\widetilde{\sigma}(r), \text{ for all } x \in \overline{B}_r(x_0),$$

where $\widetilde{\sigma} : \mathbb{R}_+ \to \mathbb{R}$ is a positive function such that $\widetilde{\sigma}(r)$ tends to 0 as r tends to 0. So, for $r \in (0, 1/2)$, we have

$$|v(x) - v(0) - \frac{e_{x_0,r}^*}{2}(x)| = |u(\frac{x}{2} + x_0) - u(x_0) - e_{x_0,r}^*(\frac{x}{2})| \leq r\widetilde{\sigma}(r), \quad \text{for all } x \in \overline{B}_r. \tag{6.20}$$

Let us suppose that $\delta \leq \delta_1(\varepsilon/2)/2$. Since $|u(x) - e_1^*(x)| \leq \delta$ in B_1, we have, for every $x \in \overline{B}_1$

$$|v(x) - \frac{e_1^*}{2}(x)| \leq |u(\frac{x}{2} + x_0) - e_1^*(\frac{x}{2} + x_0)| + |u(x_0) - e_1^*(x_0)| \leq \delta_1(\frac{\varepsilon}{2}). \tag{6.21}$$

Let us check that $S_v(0) > 0$. We know that $\|e_0^*\| = S_u(0) \neq 0$. Since $\|x_0\| = R$ and $u = e_0^*$ on $\overline{B}_R$ we can apply Corollary 6.13 to get

$$S_v(0) = \frac{S_u(x_0)}{2} > 0.$$

Conditions (6.20) and (6.21) allow to apply the first case to v. We get

$$\limsup_{r\to 0} \left\| \frac{e_1^*}{2} - \frac{e_{x_0,r}^*}{2} \right\| \leq \frac{\varepsilon}{2}. \tag{6.22}$$

Let us show now that $e_{x_0,r}^*$ tends to e_0^* as r tends to 0. Reasoning by contradiction, assume that there exists a null sequence $(r_i)_i$ such that e_{x_0,r_i}^* converges to some $h^* \neq e_0^*$. So, there exists $z \in \partial B_1$ and $t > 0$ such that the open segment $(x_0, x_0 + tz)$ is included in U and $(e_0^* - h^*)(z) \neq 0$. Finally, we compute

$$\lim_{i\to\infty} \frac{u(x_0 + r_i z) - u(x_0)}{r_i} - e_{x_0,r_i}^*(z) = e_0^*(z) - h^*(z) \neq 0,$$

which contradicts Theorem 6.3. Hence, $e_{x_0,r}^*$ converges to e_0^* and, from (6.22) we get

$$\limsup_{r\to 0} \|e_{0,r}^* - e_1^*\| = \|e_0^* - e_1^*\| = \limsup_{r\to 0} \|e_{x_0,r}^* - e_1^*\| \leq \varepsilon.$$

Thus, (6.16) is satisfied whenever $\delta(\varepsilon) = \min\{\delta_1(\varepsilon/2)/2, \varepsilon/2\} = \min\{\lambda(\varepsilon/8)/8, 1/4, \varepsilon/2\}$.

□

6.5 AML functions with linear growth

In this final section we prove that AML functions with linear growth at infinity are affine functions, Corollary 6.7. Let us start with Corollary 6.6.

Proof of Corollary 6.6. Let us proceed by contradiction. That is, for each $n \in \mathbb{N}$, there are a 1-Lipschitz AML function $u_n : B_1 \to \mathbb{R}$, $\varepsilon > 0$ and $x_n, y_n \in \overline{B}_{1/2}$ such that

$$\|u_n'(x_n) - u_n'(y_n)\| \geq \varepsilon \quad \text{and} \quad \|x_n - y_n\| < \frac{1}{n}.$$

By redefining $u_n(x) := 2(u_n\left(\frac{x}{2} + y_n\right) - u_n(y_n))$, we can assume that $y_n = 0$, $u_n(0) = 0$ and that

$$\|u_n'(x_n) - u_n'(0)\| \geq \varepsilon \quad \text{and} \quad \|x_n\| < \frac{2}{n}. \tag{6.23}$$

Since $(u_n)_n$ is a sequence of 1-Lipschitz functions such that $u_n(0) = 0$, by the Arzela-Ascoli Theorem, there is a subsequence, which we still denote by $(u_n)_n$, that converges uniformly on compact sets to a 1-Lipschitz function $u_\infty : B_1 \to \mathbb{R}$. Moreover, since $(u_n)_n$ is a sequence of AML functions, u_∞ is an AML function as well.

Thanks to Theorem 6.5, u_∞ is differentiable at 0. Since $u_\infty(0) = 0$, we have that

$$\max\{|u_\infty(x) - u_\infty'(0)(x)| : \ x \in \overline{B}_r\} \leq r\sigma(r), \ \text{for all } r \in (0,1),$$

where $\sigma(r)$ tends to 0 as r tends to 0. Let us fix $r \in (0,1)$. Then, due to the uniform convergence on compact sets, there is $N \in \mathbb{N}$ such that

$$\max\{|u_n(x) - u_\infty'(0)(x)| : \ x \in \overline{B}_{r/2}\} \leq 2r\sigma(r), \ \text{for all } n > N.$$

Therefore

$$\max\{|u_n(x+y) - u_n(x) - u_\infty'(0)(y)| : \ x \in \overline{B}_r\| \leq 2r\sigma(r), \ \forall y \in \overline{B}_{r/2}, \ \forall n > N. \tag{6.24}$$

As a consequence of Theorem 6.4, considering $r > 0$ such that $2\sigma(r) < \delta(\varepsilon/4)$, we have that inequality (6.24) implies

$$\|u_n'(x) - u_\infty'(0)\| \leq \frac{\varepsilon}{4}, \ \forall x \in B_{r/2}, \ \forall n > N.$$

The last expression contradicts (6.23) by choosing $n > N$ such that $4 < rn$. □

In order to prove Corollary 6.7, we need the following proposition that can be found in [77, Corollary 2.10].

Proposition 6.34 *Let X be a finite dimensional normed space. Let $u : X \to \mathbb{R}$ be an AML function with a linear growth at infinity. Then, u is Lipschitz.*

Proof. Let $C > 0$ such that

$$|u(x)| \leq C(1 + \|x\|), \text{ for all } x \in X.$$

Let us fix $x \in X$. Then, we have that

$$|u(y) - u(x)| \leq C(2 + \|y\| + \|x\|) \leq C(3 + 2\|x\|) \leq 3C\|x - y\|, \text{ for all } y \in \partial B(x, \|x\| + 1).$$

Therefore, by comparison with cones (Proposition 6.11), we have that

$$|u(y) - u(x)| \leq 3C\|x - y\|, \text{ for all } y \in B(x, \|x\| + 1).$$

Since $x \in X$ is arbitrary, we follows that u is $3C$-Lipschitz. □

Proof of Corollary 6.7. Let us start with the case whenever the underlying norm is not differentiable. Thanks to Corollary 6.18, there is an AML function defined on X with linear growth but not linear. On the other hand, if the underlying norm is differentiable, let $u : X \to \mathbb{R}$ be an AML function with linear growth at infinity. By Proposition 6.34, u is a Lipschitz function. Let $x_0 \in X$. For $R > 0$, let us consider the function

$$v_R(x) = \frac{1}{R}u(Rx), \text{ for all } x \in X.$$

Observe that $\mathrm{Lip}(v_R) = \mathrm{Lip}(u)$ for all $R > 0$. Thanks to Corollary 6.6 applied to v_R restricted to B_1, we have that

$$\|u'(0) - u'(x_0)\| = \|v'(0) - v'(R^{-1}x_0)\| \leq \mathrm{Lip}(u)\rho(R^{-1}\|x_0\|), \text{ for all } R > 2\|x_0\|.$$

Therefore, by sending R to infinity, we obtain that u' is constant. Thus, u is an affine function.

□

Conclusions

In this thesis, by mainly using tools from functional analysis and variational analysis we have obtained results in the following five different topics: classification of linear bounded operators, construction of ε-hypercyclic operators, wild operators, desingularization of the coderivative for multivalued functions (applied to sweeping process) and regularity of Lipschitz functions.

In Chapter 2 we have explored the interplay between linear bounded operators $T \in \mathcal{L}(Y, X)$ defined on infinite dimensional Banach spaces and the regularity properties of real-valued Lipschitz functions $f : X \to \mathbb{R}$ and the composition $f \circ T$. We have introduced an abstract property on bornologies β, that we called property (S), which is satisfied by the Fréchet, Hadamard, weakly-Hadamard and limited bornologies. Then, if β is a convex bornology on X, different from Fréchet, satisfying property (S), we have characterized β-operators $T \in \mathcal{L}(Y, X)$, in terms of the differentiability properties of f and $f \circ T$, where f runs over an appropriate set of functions. Our result characterizes compact, limited and weakly-compact operators. Also, we have introduced the notion of finitely Lipschitz functions to characterize finite-rank operators in a similar way.

In Chapter 3 we have investigated the ε-hypercyclicity. Following the constructions of ε-hypercyclic non-hypercyclic operators on $\ell^1(\mathbb{N})$ and $\ell^2(\mathbb{N})$, the former one given by C. Badea, S. Grivaux and V. Müller [15] and the last one given by F. Bayart [16], we have constructed ε-hypercyclic operators which are not hypercyclic in a larger class of infinite dimensional separable Banach spaces. In order to obtain our results, we have developed an ε-hypercyclic criterion, inspired in the well known hypercyclic criterion. Also, we have obtained a sufficient condition for which the product between an hypercyclic operator and a ε-hypercyclic operator remains ε-hypercyclic on the product space.

In Chapter 4, motivated by the construction of a wild operator made by J.M. Augé in [8], we have introduced and explored the notion of asymptotically separated set. Several examples of asymptotically separated sets, in both finite and infinite dimensional Banach spaces, are given. Moreover, we have established the connection between asymptotically separated sets and the construction of linear operators $T \in \mathcal{L}(X)$ such that the space X is partitioned in the sets $A_T := \{x \in X : \lim_n \|T^n\| = \infty\}$ and $R_T := \{x \in X : \liminf \|T^n x - x\| = 0\}$. We end this chapter with the following results on wild operators: there is a wild operator $(T \in \mathcal{L}(X))$ such that the product with itself $(T \oplus T \in \mathcal{L}(X \otimes X))$ is not wild on the product space, there are non-invertible wild operators defined on infinite dimensional spaces with symmetric basis and we study the norm-closure of the set of wild operators in the space of bounded linear operators.

In Chapter 5 we have proposed and characterized a generalization of the KŁ-inequality for multivalued maps which are not necessarily definable in an o-minimal structure. The KŁ-inequality, firstly obtained by S. Łojasiewicz [70] for real-analytic functions and then extended by K. Kurdyka [68] for $\mathcal{C}^1$-smooth definable functions, is a gradient inequality which can be seen as a desingularization of the gradient around a critical point. On the other hand, our inequality can be seen as the desingularization of the coderivative of a multivalued map. The results obtained in this chapter are mainly inspired by the following two works: A. Daniilidis and D. Drusvyatskiy in [41] established a generalization of the KŁ-inequality for multivalued maps which are definable in some o-minimal structure and J. Bolte, A. Daniilidis, O. Ley and L. Mazet in [26] explored the class of semi-convex functions such that satisfies a KŁ-like inequality, characterizing these functions in terms of the length of gradient orbits as well as in term of the integrability of the talweg. In the same line, under mild assumptions, we characterize a class of multivalued maps that satisfy our generalization of the KŁ-inequality for multivalued maps (or our desingularization for the coderivative) in terms of the length of the orbits given by the sweeping process governed by the same multivalued function, in terms of the integrability of the talweg function and also in terms of the length of the discrete sequences generated by the Catching-up algorithm.

In Chapter 6 we have studied regularity properties of the absolutely minimizing Lipschitz functions (for short AML) defined on open subsets of finite dimensional normed spaces. O. Savin in [87] proved that planar AML functions defined on open sets of a two dimensional euclidean space are continuously differentiable. We have provided a non-euclidean interpretation of the proof of the mentioned result of O. Savin to obtain that AML functions, defined on open subset of a two dimensional normed space X, are continuously differentiable if and only if the underlying norm is differentiable everywhere in $X \setminus \{0\}$.

Perspectives

The perspectives from this thesis are numerous. Here we list the most straightforward with respect to the results and techniques developed in this work:

In Chapter 2 we have characterized several classes of bounded linear operators in terms of the differentiability of a given class of functions. For compact, limited and weakly-compact operators we have used different subfamilies of Lipschitz functions. On the other hand, for finite-rank operators we have introduced the notion of finitely Lipschitz functions. A natural line of research is to find which other classes of linear operators can be characterized in terms of the differentiability properties of a given class of functions. For instance, it is interesting to know if it is possible such a characterization for the class of Hilbert-Schmidt operators defined on a Hilbert space, or more generally, for the Schatten p-class, with $p \in [1, \infty)$.

According to Chapter 3, the existence of ε-hypercyclic operators in general separable Banach spaces still remains open. Moreover, the development of the ε-hypercyclicity criterion leads to the following natural question: Does every ε-hypercyclic operator satisfy our ε-hypercyclicity criterion?

In Chapter 4 we have introduced and studied the asymptotically separated sets. We have provided several examples, nevertheless, a complete description of this kind of sets remains open. With respect to wild operators, there are at least two natural questions that follows from our work: the construction (if it exists) of a wild operator T such that $T \oplus T$ is wild in the product space and the construction (if it exists) of a non-invertible wild operator in a general separable infinite dimensional Banach space.

In Chapter 5 we have characterized smooth sweeping process maps that satisfy certain desingularization of their coderivative. An interesting line of research is the study of the same kind of desingularization but on more general multivalued maps (which are not necessarily definable on an o-minimal structure). For instance, for multivalued maps such that their graph is Whitney stratifiable.

In Chapter 6 we have proved that planar AML functions are continuously differentiable if the underlying norm is differentiable everywhere (except at 0). Since our technique strongly relays in some two dimensional arguments, it is not direct how to apply it to AML functions defined on open subsets of $\mathbb{R}^n$, with $n \geq 3$. However, and recalling that AML functions can be defined in any metric space, a natural framework to generalize our technique is on AML functions defined on open subsets of a two dimensional Finsler manifold.

Bibliography

[1] F. Albiac and N. Kalton. *Topics in Banach space theory*. Graduate Texts in Mathematics. Springer. 233, 2016.

[2] S. Ansari. *Existence of hypercyclic operators on topological vector spaces*. J. Funct. Anal., 148: 384–390, 1997.

[3] G. Araujo, L. Bernal-Gonzalez, G. Muñoz-Fernandez, J. A. Prado-Bassas and J. Seoane-Sepulveda. *Lineability in sequence and function spaces*. Studia Math., 237: 927-952, 2017.

[4] S. Argyros and R. Haydon. *A hereditarily indecomposable $\mathcal{L}_\infty$-space that solves the scalar-plus-compact problem*. Acta Math., 206: 1–54, 2011.

[5] R. Aron, L. Bernal-Gonzalez, D. Pellegrino, and J. Seoane Sepulveda. *Lineability: the search for linearity in mathematics*. Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2016.

[6] G. Aronsson. *Extension of functions satisfying Lipschitz conditions*. Ark. Mat., 6: 551–561, 1967.

[7] G. Aronsson, M. Crandall and P. Juutinen. *A tour of the theory of absolutely minimizing functions*. Bull. Amer. Math. Soc., 41(4): 439–505, 2004.

[8] J. M. Augé. *Linear operators with wild dynamics*. Proc. Amer. Math. Soc., 140(6): 2103-2116, 2012.

[9] J. M. Augé. *Quelque probleme de dynamique lineaire dans les espaces de Banach*. Ph.D. Thesis, Université de Bordeaux I, Bordeaux, 2012. (in French)

[10] M. Bachir. *An extension of the Banach-Stone theorem*. J. Aust. Math. Soc., 105: 1-23, 2018.

[11] M. Bachir. *Limited operators and differentiability*. North-Western European Journal of Mathematics, 3: 63-73 (2017).

[12] M. Bachir. *A non convexe analogue to fenchel duality*. J. Funct. Anal., 181: 300-312 (2001).

[13] M. Bachir, G. Flores and S. Tapia-García. *Compact and Limited operators*. Math. Nachr., 1-14, 2021.

[14] M. Bachir and G. Lancien. *On the composition of differentiable functions*. Canad. Math. Bull., 46: 481–494, 2003.

[15] C. Badea, S. Grivaux and V. Müller. *Epsilon-Hypercyclic operators*. Ergodic Theory & Dynamical Systems, 30: 1597-1606, 2009.

[16] F. Bayart. *Epsilon-Hypercyclic operators on a Hilbert Space*. Proc. Amer. Math. Soc., (11)138: 4037-4043, 2010.

[17] F. Bayart and G. Costakis. *Cyclic operators with finite support.* Israel J. Math., (1)193: 131–167, 2013.

[18] F. Bayart and É. Matheron. *Dynamics of linear operators.* Volume 179 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2009.

[19] I. Berg. *An extension of the Weyl-von Neumann theorem to normal operators*. Trans. Amer. Math. Soc., 160: 365-371, 1971.

[20] T. Bermúdez, A. Bonilla and N. Feldman. *On convex-cyclic operators.* J. Math. Anal. Appl., 434(2): 1166–1181, 2016.

[21] L. Bernal-González. *On hypercyclic operators on Banach spaces.* Proc. Amer. Math. Soc., 127 (4): 1003–1010, 1999.

[22] J. Bès and A,Peris. *Hereditarily hypercyclic operators.* J. Funct. Anal., 167(1): 94–112, 1999.

[23] G. D. Birkhoff. *Démonstration d'un théorème élémentaire sur les fonctions entières*. C. R. Acad. Sci. Paris, 189: 473–475, 1929.

[24] J. Bolte, A. Daniilidis and A. Lewis. *The Łojasiewicz inequality for nonsmooth subanalytic functions with applications to subgradient dynamical systems*. SIAM J. Optim., 17: 1205-1223, 2007.

[25] J. Bolte, A. Daniilidis, A. Lewis and M. Shiota. *Clarke subgradients of stratifiable functions.* SIAM J. Optim., 18: 556-572, 2007.

[26] J. Bolte, A. Daniilidis, O. Ley and L. Mazet. *Characterizations of Łojasiewicz inequalities: subgradient flows, talweg, convexity*, Trans. Amer. Math. Soc. 362(6): 3319-3363, 2010.

[27] J. M. Borwein and M. Fabian. *On convex functions having points of Gâteaux differentiability which are not points of Féchet-differentiability.* Can. J. Math.Vol. 45(6): 1121-1134, 1993.

[28] J. Borwein, M. Fabian and J. Vanderwerff. *Characterizations of Banach Spaces via Convex and other Locally Lipschitz Functions*. Acta Math. Vietnam., 22, 1996.

[29] J. Borwein and J. Vanderwerff. *Convex functions: constructions, characterizations and counterexamples.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2010.

[30] J. Bourgain and J. Diestel. *Limited operators and strict cosingularity.* Math. Nachr., 119: 55-58, 1984.

[31] T. Champion and L. De Pascale. *Principles of comparison with distance functions for absolute minimizers.* J. Convex Anal., 14: 515-541, 2007.

[32] S. Cobzas. *Functional analysis in asymmetric normed spaces.* Springer Science & Business Media, 2012.

[33] G. Colombo and V. Goncharov. *The sweeping processes without convexity.* Set-Valued Anal., 7(4): 357–374, 1999.

[34] G. Colombo and C. Kozaily. *Existence and uniqueness of solutions for an integral*

perturbation of Moreau's sweeping process. J. Convex Anal., 27(1): 229-238, 2020.

[35] G. Colombo and M. Monteiro. *Sweeping by a continuous prox-regular set*. J. Differential Equations, 187(1): 46-62, 2003.

[36] J. Conway. *A course in operator theory*. Volume 21 of American Mathematical Society, Providence, RI, 2000.

[37] R. Correa, D. Salas and L. Thibault. *Smoothness of the metric projection onto non-convex bodies in Hilbert spaces*. J. Math. Anal. Appl., 457(2): 1307–1332, 2018.

[38] G. Costakis, A. Manoussos and I. Parissis. *Recurrent linear operators*. Complex Anal. Oper. Theory, 8(8): 1601-1643, 2014.

[39] M. Crandall and L. Evans. *A remark on infinity harmonic functions*. Proceedings of the USA-Chile Workshop on Nonlinear Analysis (Viña del Mar-Valparaiso, 2000). Electron. J. Differ. Equ. Conf., 6: 123-129, 2001.

[40] M. Crandall, L. Evans and R. Gariepy. *Optimal Lipschitz extensions and the infinity Laplacian*. Calc. Var. Partial Differential Equations, 13(2): 123–139, 2001.

[41] A. Daniilidis and D. Drusvyatskiy. *Sweeping by a tame process*. Annales de l'Institut Fourier., 67(5): 2201-2223, 2017.

[42] A. Daniilidis, J.-M. Supelcre and F. Venegas. *Asymmetric free spaces and canonical asymmetrizations*. Studia Math., 261, 2021 (in press).

[43] R. Deville, G. Godefroy, and V. Zizler. *A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions*. J. Funct. Anal., 111: 197-212, 1993.

[44] R. Deville, G. Godefroy and V. Zizler. *Smoothness and renormings in Banach spaces*. Pitman Monographs and Surveys in Pure and Applied Mathematics, 1993.

[45] Drewnowski L., *On Banach spaces with the Gelfand-Phillips property*. Math. Z., 193: 405-411, 1986.

[46] T. Eisner. *Stability of operators and operator semigroups*. Oper. Theory Adv. Appl. Birkhäuser Verlag, Basel. 209, 2010.

[47] P. Enflo. *On the invariant subspace problem for Banach spaces*. Acta Math., 158(3-4): 213–313, 1987.

[48] L. Evans. and O. Savin. $\mathcal{C}^{1,\alpha}$ *regularity for infinity harmonic functions in two dimensions*. Calc. Var. Partial Differential Equations 32: 325–347, 2008.

[49] L. Evans and C. Smart. *Adjoint methods for the infinity Laplacian partial differential equation*. Arch. Ration. Mech. Anal., 201: 87–113, 2011.

[50] L. Evans and C. Smart. *Everywhere differentiability of infinity harmonic functions*. Calc. Var. Partial Differential Equations, 42: 289–299, 2011.

[51] N. Feldman. *Perturbations of hypercyclic vectors*. J. Math. Anal. Appl., 273: 67–74, 2002.

[52] R. Fry and S. McManus. *Smooth bump functions and the geometry of Banach spaces: a brief survey*. Expo. Math., 20: 143-183, 2002.

[53] S. Grivaux. *Hypercyclic operators, mixing operators, and the bounded steps problem*. J. Operator Theory, 54(1): 147–168, 2005.

[54] S. Grivaux and M. Roginskaya. *On Read's type operators on Hilbert spaces*. Int. Math. Res. Not. IMRN, 2008.

[55] G. Godefroy and M. Talagrand. *Espaces de Banach représentables*. Israel J. Math., 41(4): 321-330, 1982.

[56] W. T. Gowers and B. Maurey. *The unconditional basic sequence problem*. J. Amer. Math. Soc., 6(4): 851-874, 1993.

[57] B. Georgiev and N. Ribarska. *On sweeping process with the cone of limiting normals*. Set-Valued Var. Anal., 21: 673-689, 2013.

[58] K. Grosse-Erdmann and A. Manguillot. *Linear chaos*. Springer Science & Business Media, 2011.

[59] I. Gurariy and L. Quarta. *On lineability of sets of continuous functions*. J. Math. Anal. Appl., 294: 62-72, 2004.

[60] P. Hájek, V. Montecinos, J. Vanderwerff and V. Zizler. *Biorthogonal Systems in Banach Spaces*. Springer Science & Business Media, 2007.

[61] P. Hájek and R. Smith. *Operator machines on directed graphs*. Integral Equations Operator Theory, 67(19): 15-31, 2010.

[62] R. Henry and J. Outrata. *A subdifferential condition for calmness of multifunctions*. J. Math. Anal. Appl. 258: 110-130, 2001.

[63] H. Hogbe-Nlend. *Bornologies and Functional Analysis: introductory course on the theory of duality topology-bornology and its use in functional analysis*. Elsevier, 1977.

[64] R. Jensen. *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient*. Arch. Ration. Mech. Anal., 123: 51–74, 1993.

[65] B. Josefson. *Weak sequential convergence in the dual of a Banach space does not imply norm convergence*. Ark. Mat., 13: 79-89, 1975.

[66] A. Jourani and E. Vilchez. *Positively α-far sets and existence results for generalizaed pertubed sweeping processes*. J. Convex Anal., 23(3): 775–821, 2016.

[67] C. Kitai. *Invariant closed sets for linear operators*. Ph.D. Thesis, University of Toronto, Toronto, 1982.

[68] K. Kurdyka. *On gradients of functions definable in o-minimal structures*, Annales de l'Institut Fourier, 48(3): 769-783, 1998.

[69] F. León-Saavedra and A. Montes-Rodríguez. *Linear structure of hypercyclic vectors*. J. Funct. Anal., 148: 524-545, 1997.

[70] S. Łojasiewicz. *Une propriété topologique des sous-ensembles analytiques réels*. Colloques Internationaux du CNRS. Les equations aux dérivées partielles, vol 117, ed. B. Malgrange (Paris 1962), Publications du CNRS, Paris, 1963.

[71] S. Łojasiewicz. *Sur les trajectoires du gradient d'une fonction analytique*. Seminari di Geometria. (Bologna 1982/83), Università degli Studi di Bologna, Bologna (1984), 115-117.

[72] J. J. Moreau. *Rafle par un convexe variable. I.* Travaux du Séminaire d'Analyse Convexe, Vol. I, Exp. No. 15, 43 pp. Secrétariat des Math., Publ. No. 118, U.É.R. de Math., Univ. Sci. Tech. Languedoc, Montpellier, 1971.

[73] V. Müller and J. Vrsovský. *On orbits-reflexive operators*. J. Lond. Math. Soc., 79(2): 497-510, 2009.

[74] A. Nissenzweig. *ω^*-sequential convergence*. Israel J. Math., 28: 266-272, 1975.

[75] R. Ovsepian and A. Pełczyński. *On the existence of a fundamental total and bounded biorthogonal sequence in every separable Banach space, and related construction of uniformly bounded orthonormal system in L^2*. Studia Math., 54(2): 149-159, 1975.

[76] J. Palis and W. de Melo. *Geometric theory of dynamical systems*, Springer, 1982.

[77] F. Peng, C. Wang and Y. Zhou. *Regularity of absolute minimizers for continuous convex Hamiltonians*. J. Differential Equations, 274: 1115-1164, 2021.

[78] Peres, Yuval and Schramm, Oded and Sheffield, Scott and Wilson, David B. *Tug-of-war and the infinity Laplacian*, J. Amer. Math. Soc., 22(1): 167-210, 2009.

[79] R. R. Phelps. *Convex Functions, Monotone Operators and Differentiability*. Lecture Notes in Mathematics 1364. Springer-Verlag, Berlin, 1993.

[80] G. Prăjitură. *Irregular vectors of Hilbert space operators*. J. Math. Anal. Appl., 354(2): 689–697, 2009.

[81] G. Prăjitură. *The geometry of an orbit*. In Operator theory live, volume 12 of Theta Ser. Adv. Math., 145-154. Theta, Bucharest, 2010.

[82] C. Read. *The invariant subspace problem for a class of Banach spaces. ii. Hypercyclic operators*. Israel J. Math., 63(1):1-40, 1988.

[83] C. Read. *The invariant subspace problem on some Banach spaces with separable dual*. Proc. London Math. Soc., 58(3): 583–607, 1989.

[84] T. Rockafellar and R. Wets. *Variational Analysis*, Springer-Verlag Berlin Heidelberg, 2nd edition, 1998.

[85] S. Rolewicz. *On orbits of elements*. Studia Math., 32: 17-22, 1969.

[86] M. de la Rosa and C. Read. *A hypercyclic operator whose direct sum $T \oplus T$ is not hypercyclic*. J. Operator Theory, 61(2): 369–380, 2009.

[87] O. Savin. *C^1 regularity for infinity harmonic functions in two dimensions*. Arch. Ration. Mech. Anal., 176(3): 351-361, 2005.

[88] C. Stegall. *The Randon-Nikodym property in conjugate Banach spaces*. Trans. Amer. Math. Soc., 206: 213-223, 1975.

[89] A. Sobczyk, *Projection of the space (m) on its subspace (c_0)*. Bull. Am. Math. Soc., 47: 938–947, 1941.

[90] S. Tapia-García. *An Epsilon-Hypercyclicity criterion and its application on classical Banach spaces*. Submitted. Preprint: https://arxiv.org/abs/2103.08075.

[91] S. Tapia-García. *Asymptotically separated sets and wild dynamics*. Trans. Amer. Math. Soc. To appear 2021.

[92] S. Tapia-García. *Regularity of AML functions in two dimensional normed spaces*. Submitted.

[93] J. Väisälä. *A proof of the Mazur-Ulam theorem*. American Mathematical Monthly, 110(7): 633-635 (2003).

[94] L. van den Dries. *Tame topology and o-minimal structures*. London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 248, 1998.

[95] Y. Zhang and Y. Zhou. $\mathcal{C}^{1,\alpha}$ *regularity for infinity harmonic functions in two dimensions, revisited*. Proc. Amer. Math. Soc., 148: 1187-1193, 2020.