UNIVERSIDAD DE CHILE

# Quotients of the Bruhat-Tits tree by function field analogs of the Hecke congruence subgroups

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por

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Marzo, 2022

Director de Tesis: **Dr. Luis Arenas Carmona.**

Codirector de Tesis: **Dr. Giancarlo Lucchini Arteche.**

FACULTAD DE CIENCIAS

UNIVERSIDAD DE CHILE

INFORME DE APROBACIÓN

TESIS DOCTORADO

Se informa a la Escuela de Postgrado de la Facultad de Ciencias que la Tesis de Doctorado presentada por el candidato

**Claudio Abraham Bravo Castillo**

Ha sido aprobada por la Comisión de Evaluación de la Tesis como requisito para optar al grado de Doctor en Ciencias con mención en Matemáticas, en el examen de Defensa de Tesis rendido el día 4 de Marzo del 2022.

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Dr. Luis Lomelí ______________________

*Dedicado a mis padres*
*Claudio Bravo Opazo*
*y Sonia Castillo Vera.*

# BIOGRAFÍA

Nací el 13 de Mayo de 1994 en la cuidad de Cauquenes, Región del Maule, en el seno de una típica familia chilena. A la edad de 5 años ingresé a la Escuela Independencia. En tercero básico, mis padres decidieron cambiarme al Liceo Inmaculada Concepción. En dicha institución realicé la mayoría de mis estudios. Debo confesar que tuve una infancia bastante feliz, en la que no existían las sobrecargas estudiantiles, si no más bien una constante estimulación a observar el mundo natural y encontrar mis propias respuestas.

Fue en la enseñanza media en donde comenzó mi camino científico. Pienso yo, que la primera persona que notó algo de talento en mi fue la profesora Cecilia Moya. Ella fué mi profesora de Química, por quién guardo un gran respeto y aprecio. Fué gracias a sus enseñanzas y motivaciones que comencé a participar de las Olimpiadas Chilenas de Química, en donde obtuve medalla de Oro en 2010. Paralelamente, comencé a asistir a los cursos de verano de la Universidad de Chile. Mi primer curso fué Matemáticas I. Hasta este punto todo me hacía pensar que estudiaría algo vinculado a ambas disciplinas, la Química y las Matemáticas. Pero en Febrero de 2010, el terremoto del día 27 hizo que las cosas cambiaran. Producto de dicho acontecimiento, mi familia se trasladó a la cuidad de San Javier de Loncomilla. Por ende, continué mis estudios medios en el Instituto Regional del Maule. Es en este punto, y a raíz de mi alejamiento de la Química, que decidí sumergirme completamente en las Matemáticas. Es por esto que realizé los cursos de Matemáticas II y III en la Universidad de Chile y finalmente me decidí ingresar a la Licenciatura en Matemáticas en la misma casa de estudios.

En la Licenciatura en Matemáticas gocé con cada clase. En la segunda mitad de la Licenciatura comencé a sentir gran interés por la Teoría de Números. Fué por esto que decidí perdir una unidad de invertigación al Dr. Luis Arenas. Una vez egresado de la Licenciatura, dedicí continuar con mis estudios de Magister, siendo dirigido por el Dr. Arenas en la tesis de dicho postgrado.

Terminado el Magister, en 2018 comencé mis estudios doctorales. En un principio estos fueron desarrollados sin contratiempos y con una sana regularidad. A inicios de 2019, siendo aprobado mi proyecto de tesis, comenzaba dicho trabajo siendo dirigido por el Dr. Arenas y el Dr. Lucchini-Arteche, a quienes agradezco profundamente por su dedicación, compromiso y consejo sincero. En el cuarto semestre de doctorado, tuve oportunidad de realizar una pasantía de investigación en la École Normale Supérieure de Lyon y la École Polytechnique. Este fue un período de gran crecimiento personal y científico, pues tuve la fortuna de conocer a nuevos investigadores, algunos de los cuales se transformaron en colaboradores. Luego de esto el camino no ha sido fácil. La pandemia de Covid-19 ha dificultado la comunicación directa. No obstante, se ha continuado con las labores académicas con gran compromiso y este trabajo es una muestra de ello. El trabajo duro todo lo vence.

# AGRADECIMIENTOS

En este camino muchas personas han participado dando su trabajo, consejo y ánimos. Puede que a algunas de estas no logre hacer notar en este breve espacio, sin embargo estoy muy agradecido de ellos(as) y muy especialmente de las que se ilustran a continuación.

Quiero agradecer, en primera instancia, a mi familia, a mis padres Claudio y Sonia quienes han sido mi apoyo, mi sustento y mi ejemplo a seguir. Agradezco también a Dios, porque sin prueba alguna, tengo plena confianza de que guía mis pasos. Y por cierto, agradezco a mi nación, a Chile, por haber confiado en mis habilidades y haberme otrogado la ayuda económica para realizar mis estudios de pre y postgrado.

A mis maestros y profesores que tuve desde la enseñanza básica hasta el día de hoy, agradezco sinceramente. Si uno de ellos hubiese fallado, muy posiblemente yo no estaría acá. Es en este marco que quiero destacar a mi director de tesis Dr. Luis Arenas y codirector D. Giancarlo Lucchini, por su paciencia, su ejemplo y compromiso. En ambos he encontrado a verdaderos maestros, por quienes guardo un gran respeto y aprecio. A mis profesores de la facultad, especialmente a la profesora Anita Rojas, Gonzalo Robledo, Manuel Pinto, Alicia Labra, Eduardo Friedman y Nicolás Libedinsky, quienes me han brindado consejo y ayudado enormemente. Agradecer también a mi profesora de Quimica Celicia Moya, quien fue la primera en confiar en mi como futuro científico.

Agradezco a los doctores integrantes de la comisión evaluadora de esta tesis, Dr. Eduardo Friedman, Dr. Diego Izquierdo y Dr. Luis Lomelí, por su tiempo y trabajo.

Finalmente quiero agradecer a mis amigos, a los que conocía de antes y con los que tuve la fortuna de coincidir en este trayecto por la Universidad. A los buenos muchachos con los que compartí experiencias, conversaciones, éxitos y frustaciones en estos años, a todos ellos mi más cordial y profundo agradecimiento.

## RESUMEN

Sea $\mathcal{C}$ una curva projectiva, suave y geométricamente conexa definida sobre un cuerpo finito $\mathbb{F}$. Para cada punto cerrado $P_\infty$ de $\mathcal{C}$, sea $R$ el anillo de funciones que son regulares fuera de $P_\infty$, y sea $K$ la completacion en $P_\infty$ del cuerpo de funciones de $\mathcal{C}$. Con el objetivo de estudiar grupos de la forma $\mathrm{GL}_2(R)$, Serre describe en [Se80, Chapter II] el grafo cociente $\mathrm{GL}_2(R)\backslash\mathfrak{t}$, donde $\mathfrak{t}$ es el árbol de Bruhat-Tits definido a partir de $\mathrm{SL}_2(K)$. En particular, Serre demuestra que $\mathrm{GL}_2(R)\backslash\mathfrak{t}$ es la union de un grafo finito con un número finito de subgrafos con forma de rayo, llamados cúspides. No es dificil ver que esta propiedad es heredada por subgrupos de índice finito.

En este trabajo describimos el grafo cociente $\mathrm{H}\backslash\mathfrak{t}$ asociado a la acción sobre $\mathfrak{t}$ del grupo $\mathrm{H} = \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}_2(R) : c \equiv 0 \,(\mathrm{mod}\, I) \right\}$, donde $I$ es un ideal de $R$. Más específicamente, damos una fórmula explícita para el número de cúspides de $\mathrm{H}\backslash\mathfrak{t}$. Luego, usando la teoría de Bass-Serre, describimos la estructura combinatorial de H. Estos grupos juegan, en el contexto de cuerpos de funciones, el mismo rol que los subgrupos de congruencia de Hecke de $\mathrm{SL}_2(\mathbb{Z})$. Los grupos estudiados por Serre corresponden al caso donde el ideal $I$ coincide con el anillo $R$.



## ABSTRACT

Let $\mathcal{C}$ be a smooth, projective and geometrically connected curve defined over a finite field $\mathbb{F}$. For each closed point $P_\infty$ of $\mathcal{C}$, let $R$ be the ring of functions that are regular outside $P_\infty$, and let $K$ be the completion at $P_\infty$ of the function field of $\mathcal{C}$. In order to study groups of the form $\mathrm{GL}_2(R)$, Serre describes in [Se80, Chapter II] the quotient graph $\mathrm{GL}_2(R)\backslash\mathfrak{t}$, where $\mathfrak{t}$ is the Bruhat-Tits tree defined from $\mathrm{SL}_2(K)$. In particular, Serre shows that $\mathrm{GL}_2(R)\backslash\mathfrak{t}$ is the union of a finite graph and a finite number of ray shaped subgraphs, which are called cusps. It is not hard to see that finite index subgroups inherit this property.

In this work we describe the associated quotient graph $\mathrm{H}\backslash\mathfrak{t}$ for the action on $\mathfrak{t}$ of the group $\mathrm{H} = \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}_2(R) : c \equiv 0 \,(\mathrm{mod}\, I) \right\}$, where $I$ is an ideal of $R$. More specifically, we give a explicit formula for the cusp number of $\mathrm{H}\backslash\mathfrak{t}$. Then, by using Bass-Serre Theory, we describe the combinatorial structure of H. These groups play, in the function field context, the same role as the Hecke congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. The groups studied by Serre correspond to the case where the ideal $I$ coincides with the ring $R$.

Contents

## 1. Introduction

**Group actions on symmetric spaces and Bass-Serre theory.** Classically, we study discrete subgroups $\Gamma$ of a real Lie group $G$ by their action on symmetric spaces, that is homogeneous spaces of the form $\mathcal{X} = K\backslash G$, where $K$ is a maximal compact subgroup of $G$. When $G = \mathrm{SL}_2(\mathbb{R})$, there exists a well-known theory that describes the action of $\Gamma$ on the upper half-plane $\mathcal{X} = \mathrm{SO}_2(\mathbb{R})\backslash G$ by Moebius transformations.

A program initiated by Bruhat and Tits establishes an analogy with the preceding case in the context of $p$-adic Lie groups. In this sense, Bruhat and Tits introduce certain simplicial complexes called buildings which play the role of symmetric spaces for $p$-adic groups. For $G = \mathrm{SL}_n(\mathbb{Q}_p)$ the building is a contractible $(n-1)$-dimensional complex, whose vertex set is parameterized by the quotient $\mathbb{SL}_n(\mathbb{Z}_p)\backslash G$ (cf. [AbB08, §6.9]). When $n = 2$, this complex is in fact a tree, i.e. a connected graph without cycles. In the literature, this graph is called the Bruhat-Tits tree and it is extensively studied in Serre's book [Se80]. In fact, it was Serre's observation that a discrete subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ is torsion-free if and only if it acts freely on a tree. This observation gives another proof of a theorem due to Ihara, which states that any discrete torsion-free subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ is free (cf. [Ih66, Theorem 1]). This prompted Serre to develop a general theory of groups acting on trees (cf. [Se80, Chapter I, §3]).

The goal of Bass-Serre theory (cf. [Se80, Chapter I, §5]) is to study the combinatorial group theory with the tools and techniques of the theory of groups acting on trees. In this sense, there exist two classical constructions in combinatorial group theory that we can analyse from the point of view trees. One of these is the amalgamated product of two groups (cf. [Se80, Chapter I, §1.2]), and the other is the HNN extension (cf. [Se80, Chapter I, §1.4]). Such tools arise in topology, for example, in the description of the fundamental group of a 3-fold which splits along a connected incompressible surface. Furthermore, in each of these two group-theoretical constructions there is a naturally defined tree $\mathfrak{t}$ on which the group acts, so that the quotient by the group action is respectively an edge or a loop. In the topological setting, this simply imitates the action of the fundamental group on the universal covering space.

By considering actions of more general groups on trees, one is naturally led to the more complex notion of a graph of groups. By a graph of groups $(\tilde{\Gamma}, Y)$ we mean a graph $Y$ together with assignments of groups $v \to \tilde{\Gamma}_v$, $e \to \tilde{\Gamma}_e$ to the vertices and edges of a graph $Y$ subject to certain compatibility conditions stated in §9. In this context Serre introduces the fundamental group $\pi_1(\tilde{\Gamma}, Y)$ associated to a graph of groups $(\tilde{\Gamma}, Y)$. This group is an amalgamated product of the fundamental group $\pi_1(Y)$ with another group $S(\tilde{\Gamma})$ defined as the sum of the groups $\tilde{\Gamma}_v$, amalgamated along the groups $\tilde{\Gamma}_e$. In this sense, amalgams of two groups and HNN extensions correspond to the fundamental group of an edge and a loop respectively. As such, these are the fundamental blocks with which the fundamental group is described.

Let $\Gamma$ be a group acting on a tree $\mathfrak{t}$ and let $Y = \Gamma\backslash\mathfrak{t}$ be the associated quotient graph. Let $T$ be a maximal tree in $Y$ and let $j : T \to \mathfrak{t}$ be a lift. Then, we can define a graph of groups by setting $\tilde{\Gamma}_v$ as the stabilizer of the vertex $j(v)$, and defining

$\tilde{\Gamma}_e$ by means of the edges. The main result of Bass-Serre theory is that, in this context, $\Gamma$ is isomorphic to the fundamental group $\pi_1(\tilde{\Gamma}, Y)$. This general theorem gives a method to systematically study the structure of many groups acting on trees. However, this requires characterizing the quotient graph $Y$ associated to the action of $\Gamma$ on $\mathfrak{t}$.

In [Se80, Chapter 2], Serre constructs the Bruhat-Tits tree $\mathfrak{t} = \mathfrak{t}(K)$ associated to the group $\mathrm{SL}_2(K)$ for a complete field $K$. The action of $\mathrm{SL}_2(K)$ on $\mathfrak{t}$ can actually be extended to an action of $\mathrm{GL}_2(K)$. Later, Serre studies the group $\mathrm{GL}_2(k)$, where $k$ is the function field of a smooth projective curve. He also studies its subgroup $\mathrm{GL}_2(R)$, where $R$ is the coordinate ring of an affine open set of the curve with a unique point $P_\infty$ at infinity. This closed point gives rise to a discrete valuation $\nu$ on $k$ and hence we have an action of $\mathrm{GL}_2(k)$ on the Bruhat-Tits tree associated to the completion $K$ of $k$ at $P_\infty$. In this situation, the author gives a reinterpretation of the vertices of this tree as vector bundles of rank two over the curve which are trivial on the affine part. Then, in order to study the structure of $\mathrm{GL}_2(R)$, Serre abundantly studies the structure of the quotient graph $\mathrm{GL}_2(R)\backslash\mathfrak{t}$. As a consequence, Serre gets an amalgamated free product structure on $\mathrm{GL}_2(R)$. Moreover, by describing some spectral sequences on homology, he gets some structural results on the homology groups of $\mathrm{GL}_2(R)$ and of its finite index subgroups.

There exists a very useful generalization of the Bass-Serre theory to the context of buildings, due to Bridson and Haeflinger. This theory is written in terms of the concept of small categories without loops (cf. [BH91, Chapter III.C]). In analogy with the Bass-Serre theory, we can apply Bridson and Haeflinger's construction in order to study presentations of groups acting on buildings.

**Statement of Serre's results and further developments.** In this section we summarize some classical results about the quotient structure of buildings by the action of certain groups of "arithmetic nature". Then we discuss the structural consequences for the groups themselves.

Let $\mathcal{C}$ be a smooth, projective, geometrically connected curve over a field $\mathbb{F}$. For each closed point $P_\infty$ of $\mathcal{C}$, let $R$ be the coordinate ring of the affine curve obtained by removing $P_\infty$ from $\mathcal{C}$. Let $k$ be the function field of $\mathcal{C}$. As we already recalled, one of the first families of examples studied by describing their actions on trees has been the family of arithmetic subgroups $\mathbf{G}(R) \subset \mathbf{G}(k)$ for $\mathbf{G} = \mathrm{GL}_2$. Indeed, in order to study these arithmetic groups, Serre gave the following description of the quotient graphs.

**Theorem 1.1.** [Se80, Chapter II, Theorem 9] Let $\mathfrak{t}$ be the local Bruhat-Tits tree defined by the group $\mathrm{SL}_2$ at the completion $K$ associated to the valuation induced by $P_\infty$ (cf. §2). Then, the graph $\mathbf{G}(R)\backslash\mathfrak{t}$ is combinatorially finite, i.e. is obtained by attaching a finite number of infinite half lines, called cuspidal rays, to a certain finite graph $Y$. The set of such cuspidal rays is indexed by the elements of the Picard group $\mathrm{Pic}(R) = \mathrm{Pic}(\mathcal{C})/\langle \overline{P_\infty} \rangle$.

Then, using Bass-Serre Theory, Serre concludes the following structural result for the groups $\mathbf{G}(R)$ defined above.

**Theorem 1.2.** [Se80, Chapter II, Theorem 10] There exists a finitely generated group $H$, and a family $\{(I_\sigma, \mathcal{P}_\sigma, \mathcal{B}_\sigma)\}_{\sigma \in \mathrm{Pic}(R)}$ where:

1. $I_\sigma$ is an $R$-fractional ideal and $\mathcal{P}_\sigma = (\mathbb{F}^* \times \mathbb{F}^*) \ltimes I_\sigma$,
2. $\mathcal{B}_\sigma$ is a group with canonical injections $\mathcal{B}_\sigma \to H$ and $\mathcal{B}_\sigma \to \mathcal{P}_\sigma$,

such that $\mathbf{G}(R)$ is isomorphic to the sum of $\mathcal{P}_\sigma$, for $\sigma \in \mathrm{Pic}(R)$, and $H$, amalgamated along their common subgroups $\mathcal{B}_\sigma$ according to the above injections.

Moreover, Serre describes the structure of $\mathbf{G}(R)$ as an amalgamated sum in certain cases, by explicitly describing the corresponding quotient graphs. This work considers, for example, the cases $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$, for $\deg(P_\infty) \in \{1, 2, 3, 4\}$, or when $\mathcal{C}$ is a curve of genus 0 without rational points and $\deg(P_\infty) = 2$. The case $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$, and $\deg(P_\infty) = 1$ reduces to a classical result, now called Nagao's Theorem (cf. [Na59]). In this context the corresponding quotient graph is a ray. Also, Arenas-Carmona in [A16] extends the study of Serre's and Nagao's explicit examples, by determining the quotient graphs when the closed point $P_\infty$ has degree 5 or 6, and giving a method for further computations. In general, we can apply Theorem 1.2 to show that $\mathbf{G}(R)$ is never finitely generated.

In order to prove Theorem 1.1, Serre makes an extensive use of the theory of vector bundles of rank 2 over $\mathcal{C}$. On the other hand, Mason in [Ma01] gives a more elementary approach which involves substantially less algebraic geometry. This point of view only requires the Riemann-Roch Theorem and some basic notions about Dedekind rings. The price to pay for this simplicity is that one is not able to prove the finiteness of the diameter of graph $Y$ in Theorem 1.1. However, Mason applies this result on quotient graphs in order to study the lowest index non-congruence subgroups of $\mathbf{G}(R)$.

In a more general context, let $K$ be the completion of $k$ at $P_\infty$ and let $\mathbf{G}$ be an arbitrary reductive algebraic $k$-group. We can define a poly-simplicial complex $\mathcal{X}(\mathbf{G}, K)$ associated to the group $\mathbf{G}$ and the field $K$. This topological space is called the building of $\mathbf{G}(K)$ and, as we said in the previous section, this notion generalizes the definition of the Bruhat-Tits tree. When $R = \mathbb{F}[t]$ and $\mathbf{G}$ is split over $k$, there exists a result that generalizes Nagao's theorem, which describes the structure of the quotient space $\overline{\mathcal{X}} = \mathbf{G}(\mathbb{F}[t]) \backslash \mathcal{X}$ associated to the action of $\mathbf{G}(\mathbb{F}[t])$ on $\mathcal{X} = \mathcal{X}(\mathbf{G}, \mathbb{F}((t^{-1})))$. This result is due to Soulé and described in [So77, Theorem 1]. Soulé shows that $\overline{\mathcal{X}}$ is isomorphic to a sector $Q_0 \subset \mathcal{X}$, which is the analog of a ray in the general building context. Then, in the same article, the author describes $\mathbf{G}(\mathbb{F}[t])$ as an amalgam. This structural result can be extended to the context where $\mathbf{G}$ is an isotrivial $k$-group, i.e. a reductive $k$-group that splits in the composite field $\ell = \mathbb{L}k$, for a finite extension $\mathbb{L}$ of $\mathbb{F}$. This problem has been developed by Margaux in [Mar09]. Indeed, Margaux manages to prove the same result as Soulé, where obviously he replaces the condition "split" by "isotrivial".

In the particular case where $\mathbb{F}$ is a finite field, one of the strongest results about the structure of quotient buildings that exists in the literature is due to Bux, Köhl and Witzel in [BKW13]. This is written in terms of a certain thin subspace of $\mathcal{X}$ that covers the quotient space.

**Theorem 1.3.** [BKW13, Proposition 13.6] Assume that $\mathbb{F}$ is finite. Let $\mathbf{G}$ be an isotropic and non-commutative algebraic $k$-group and let $\mathcal{X} = \mathcal{X}(\mathbf{G}, K)$ be the building associated to $\mathbf{G}$ and $K$. Let $\mathcal{S}$ be a finite set of places of $k$ containing $P_\infty$ and denote by $\mathcal{O}_{\mathcal{S}}$ the ring of $\mathcal{S}$-integers of $k$. Choose a particular realization $\mathbf{G}_{\mathrm{real}}$ of $\mathbf{G}$ as an algebraic set of some affine space. Given this realization, we define $G$ as the group of $\mathcal{O}_{\mathcal{S}}$-points of $\mathbf{G}_{\mathrm{real}}$. Then, there exists a constant $L$ and finitely many sectors $Q_1, \cdots, Q_s$ such that

(1) The $G$-translates of the $L$-neighborhood of $\bigcup_{i=1}^{s} Q_i$ cover $\mathcal{X}$.
(2) For $i \neq j$, the $G$-orbits of $Q_i$ and $Q_j$ are disjoint.

The group $G$ defined in the previous theorem is called an $\mathcal{S}$-arithmetic subgroup of $\mathbf{G}$. Of course, the $\mathcal{S}$-arithmetic group $G$ depends on the chosen realization of $\mathbf{G}$. Indeed, for any two choices leading to $\mathcal{S}$-arithmetic subgroups of $\mathbf{G}$ there is a common subgroup of finite index in either. This phenomenon, in particular, implies that the details of the conclusion in the previous result depend strongly on the given realization. Fortunately, when $\mathbf{G}$ is split over $k$, there exists an intrinsic and canonical way to define $G$ which we can exploit in order to characterize the quotient structures of buildings. In this case, the mathematical translation of Theorem 1.3 to the language of quotients of buildings, and its applications to the structure of arithmetic groups, is work in progress with B. Loisel. On the other hand, with L. Arenas-Carmona, B. Loisel and G. Lucchini Arteche, we extend Theorem 1.1 in [ABLL] to the context of special unitary groups of split rank one, which are the smallest quasi-split non-split reductive groups. In the future, we hope to combine techniques developed in the preceding two projects in order to describe the quotients of buildings by general groups of $R$-points of quasi-split groups, where $R$ is as above. We also hope that this will allow us to understand the combinatorial structure of the aforementioned groups, as well as their homology and cohomology groups. Finally, at the moment of presenting this thesis, with A. Hébert, D. Izquierdo and B. Loisel, we are working on extensions of some of the these results to higher-dimensional local fields.

**On the main problem of this thesis.** In this section we present the main goal of this thesis, and state our main results in this direction.

Widely speaking, the goal of this work is to study a certain family of congruence subgroups of $\mathbf{G}(R)$, for $\mathbf{G} = \mathrm{GL}_2$ and $\mathbb{F}$ a finite field, through the analysis of the associated group actions on trees. This objective is natural since this family is a direct analog of the Hecke congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ in the function field context.

In order to introduce the family of groups that concern us, we use the same definitions and notations as in the previous section. In particular, we denote by $\mathcal{C}$ a smooth, projective, geometrically connected $\mathbb{F}$-curve and by $k$ its function field. Since in the sequel we use the spinor genera theory in some proofs, and this theory is set in the context where the ground field $\mathbb{F}$ is finite, we assume this throughout and we denote its cardinality by $q$. Recall that a $\mathcal{C}$-order $\mathfrak{D}$ on the matrix algebra $\mathbb{M}_2(k)$ is a locally free sheaf of $\mathcal{O}_{\mathcal{C}}$-algebras whose generic fiber is $\mathbb{M}_2(k)$. Analogously, an $R$-order is a locally free $R$-algebra. As we explain in §5, any $R$-order can be extended to a $\mathcal{C}$-order by choosing an arbitrary local order at the point $P_\infty \in \mathcal{C}$. We say that a $\mathcal{C}$-order is maximal if its completions are maximal at all places of $\mathcal{C}$. By definition, a split maximal order is an order $\mathrm{GL}_2(k)$-conjugate to the sheaf

$$\mathfrak{D}_D = \begin{pmatrix} \mathcal{O}_{\mathcal{C}} & \mathfrak{L}^D \\ \mathfrak{L}^{-D} & \mathcal{O}_{\mathcal{C}} \end{pmatrix},$$

where $D$ is an arbitrary divisor on $\mathcal{C}$, and where $\mathfrak{L}^D$ is the invertible sheaf defined in every open set $U \subseteq \mathcal{C}$ by

$$\mathfrak{L}^D(U) = \{f \in k : \mathrm{div}(f)|_U + D|_U \geqslant 0\}\,.$$

In general, an Eichler $\mathcal{C}$-order is a sheaf-theoretical intersection of two maximal $\mathcal{C}$-orders. We define a specific family of Eichler $\mathcal{C}$-orders $\mathfrak{E}_D$ by taking

$$\mathfrak{E}_D = \mathfrak{D}_D \cap \mathfrak{D}_0,$$

where $D$ is an effective divisor. Let $U_0$ be the open set in $\mathcal{C}$ defined as the complement of $\{P_\infty\}$. We define $\mathrm{H}_D$ as the group of invertible elements in $\mathfrak{E}_D(U_0)$. In other words

$$\mathrm{H}_D = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R) : c \equiv 0 \,(\mathrm{mod}\ I_D) \right\}, \tag{1.1}$$

where $I_D$ is the $R$-ideal defined as $I_D = \mathfrak{L}^{-D}(U_0)$. Then, the family of groups $\mathcal{H} = \{\mathrm{H}_D : D \text{ effective divisor}\}$ plays the same role as the Hecke congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ in the function field context. At this point we can be more precise. The objective of this work is to characterize the quotient graph associated to the action of $\mathrm{H}_D$ on the Bruhat-Tits tree $\mathfrak{t}$, to subsequently describe the combinatorial structure of $\mathrm{H}_D$.

Note that $\mathrm{H}_D$ naturally contains the kernel of $\mathbf{G}(R) \to \mathbf{G}(R/I_D)$. This implies, as we prove in Corollary 3.8 (which follows from a lemma by Serre in [Se70]), that the quotient graph $\mathrm{H}_D\backslash\mathfrak{t}$ is combinatorially finite, and the number of cuspidal rays of $\mathfrak{t}_D = \mathrm{H}_D\backslash\mathfrak{t}$ is equal to the number of $\mathrm{H}_D$-orbits in $\mathbb{P}^1(k)$. The previous observation is useful in the context where $D$ has small degree. In fact, an explicit example can be written in the context where $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$, $D$ is a closed point $P$ and $\deg(P_\infty) = \deg(P) = 1$. Indeed, by using these hypotheses on $\mathcal{C}$ and $D$ we can show that $\{0, \infty\}$ is a set of $\mathrm{H}_D$-orbits in $\mathbb{P}^1(k)$. Unfortunately, this set of $\mathrm{H}_D$-orbits is really hard to characterize in the general case. Another obstruction for a direct computation of $\mathfrak{t}_D$ is that $\mathrm{H}_D$ is not a normal subgroup of $\mathbf{G}(R)$. In particular, $\mathbf{G}(R)\backslash\mathfrak{t}$ is not always a quotient of $\mathrm{H}_D\backslash\mathfrak{t}$.

In order to present our main result we introduce some additional notations. For any divisor $D$ on $\mathcal{C}$, we denote by $\overline{D}$ its linear equivalence class. Also, we denote by $\lfloor a \rfloor$ the largest integer not exceeding $a \in \mathbb{R}$. Observe that, when $D = 0$, we have $\mathrm{H}_D = \mathrm{GL}_2(R)$. In particular, the next theorem can be considered as a partial generalization of Serre's result on the structure of quotient graphs.

**Theorem 1.4.** Let $D$ be an effective divisor, which we write as $D = \sum_{i=1}^r n_i P_i$, where the points $P_1, \dots, P_r, P_\infty$ are all different. Then, the graph $\mathfrak{t}_D = \mathrm{H}_D\backslash\mathfrak{t}$ is obtained by attaching a finite number of rays, called cuspidal rays, to a certain finite graph $Y \subset \mathfrak{t}_D$. The cardinality $\mathfrak{c}_D$ of the set of such cuspidal rays satisfies

$$\mathfrak{c}_D \leqslant c(\mathrm{H}_D) := 2^r |g(2)| \left| \frac{2\mathrm{Pic}(\mathcal{C}) + \langle \overline{P_\infty} \rangle}{\langle \overline{P_\infty} \rangle} \right| \left( 1 + \frac{1}{q-1} \prod_{i=1}^r \left( q^{\deg(P_i)\lfloor \frac{n_i}{2} \rfloor} - 1 \right) \right), \tag{1.2}$$

where $g(2)$ is the maximal exponent-2 subgroup of $\mathrm{Pic}(R)$. Moreover, equality holds when $g(2)$ is trivial and each $n_i$ is odd.

Note that $g(2)$ is trivial in various cases: for example, when $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$ and $P_\infty$ has odd degree, or when $\mathcal{C}$ is an elliptic curve with no non-trivial 2-torsion rational points.

Let us assume $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$, and assume that $P_\infty$ is the point at infinity, which corresponds to the valuation induced by $\nu = -\deg$ on $k = \mathbb{F}(t)$. Let $D_0 =$

$\sum_{i=1}^{r} P_i - rP_\infty$ where $P_1, \cdots P_r, P_\infty$ are different degree-one points on $\mathcal{C}$, so that $\deg(D_0) = 0$. In particular, we can assume that there exists a square free polynomial $N = \prod_{i=1}^{n}(t-\lambda_i) \in \mathbb{F}[t]$ that generates the principal divisor $D_0$. Note that $\mathrm{H}_{D_0} = \mathrm{H}_D$, where $D = \sum_{i=1}^{r} P_i$ as above. In this case, we can give a more explicit description than Theorem 1.4 for the $\mathrm{H}_D$-action on $\mathfrak{t}$. In fact, we can characterize a fundamental region $\mathfrak{r}_D \subseteq \mathfrak{t}$ containing precisely one vertex from each $\mathrm{H}_D$-orbit in $\mathfrak{t}$. The description of the edge set of $\mathfrak{r}_D$ is however more involved, since we can have loops in the quotient graph $\mathfrak{t}_D$. See §8 for more details. In order to present our result, we introduce some definitions. Two rays in $\mathfrak{t}$ are called equivalent if they contain a common subray. Then, by an end of $\mathfrak{t}$, also called a visual limit, we mean an equivalence class of rays. Since we are assuming that $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$ and $P_\infty$ is the point at infinity, the set of ends in $\mathfrak{t}$ corresponds with $\mathbb{P}^1\big(\mathbb{F}((t^{-1}))\big)$. For any collection $\mathfrak{c}$ of ends of $\mathfrak{t}$ there exists a minimal subtree $\mathfrak{t}_{\mathfrak{c}} \subseteq \mathfrak{t}$ containing a set of representatives of $\mathfrak{c}$. Indeed, $\mathfrak{t}_{\mathfrak{c}}$ is defined as the union of a suitable set of representatives of $\mathfrak{c}$.

**Theorem 1.5.** Let $N = (t-\lambda_1)\cdots(t-\lambda_n) \in \mathbb{F}[t]$ be a square-free polynomial with all of its roots in $\mathbb{F}$ and let $D = \mathrm{div}(N)$. Let $\mathfrak{s}$ be the smallest subtree containing the ends $0$, $\infty$ and $1/M$, for every proper monic nonconstant divisor $M$ of $N$. Then the congruence subgroup

$$\mathrm{H}_D = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}[t]) \middle| c \equiv 0 \pmod{N} \right\}$$

has a fundamental region $\mathfrak{r}_D$ of the form $\mathfrak{s} \cup \mathfrak{f}$ for a finite graph $\mathfrak{f}$.

The previous results give a more precise description than [Se70, §3.3, Lemma 8], which in fact only says that the set of cuspidal rays is finite. Indeed, in Theorem 1.4, we have a control on the number of cusps, and, in the case where $g(2)$ is trivial and each $n_i$ is odd, we have an explicit expression to compute it. Moreover, by using Theorem 1.5 we can characterize a set of representatives for all but finitely many vertex classes. In particular, we get a set of representatives for the action of $\mathrm{H}_D$ on the ends of $\mathfrak{t}$. This is equivalent to describing the $\mathrm{H}_D$-orbits in $\mathbb{P}^1(k)$, which can be difficult to compute directly when $D$ has a large degree.

Now, by using Bass-Serre theory and Theorem 1.4 we can deduce the following general result on the combinatorial structure of $\mathrm{H}_D$. This can be considered as a partial generalization of Theorem 1.2, and as a more detailed description than [Se70, §3.3, Lemma 8].

**Theorem 1.6.** In the notation of Theorem 1.4, assume that each $n_i$ is odd and $g(2) = \{e\}$. Then, there exist a finitely generated group $H$, two sets of indices, denoted by $\mathbf{D}$ and $\mathbf{I}$, and a family $\{(I_\sigma, \mathcal{P}_\sigma, \mathcal{B}_\sigma) : \sigma \in \mathbf{D} \sqcup \mathbf{I}\}$, where

1. $\mathrm{Card}(\mathbf{D}) = 2^r[2\mathrm{Pic}(\mathcal{C}) + \langle \overline{P_\infty} \rangle : \langle \overline{P_\infty} \rangle]$, and $\mathrm{Card}(\mathbf{I}) = c(\mathrm{H}_D) - \mathrm{Card}(\mathbf{D})$,
2. $I_\sigma$ is an $R$-ideal contained in $I_D$
3. $\mathcal{P}_\sigma = (\mathbb{F}^* \times \mathbb{F}^*) \ltimes I_\sigma$, for any $\sigma \in \mathbf{D}$, while $\mathcal{P}_\sigma = \mathbb{F}^* \times I_\sigma$, for any $\sigma \in \mathbf{I}$,
4. $\mathcal{B}_\sigma$ is a group with canonical injections $\mathcal{B}_\sigma \to H$ and $\mathcal{B}_\sigma \to \mathcal{P}_\sigma$, for any $\sigma \in \mathbf{D} \sqcup \mathbf{I}$,

such that $\mathrm{H}_D$ is isomorphic to the sum of $\mathcal{P}_\sigma$, for $\sigma \in \mathbf{D} \sqcup \mathbf{I}$, and $H$, amalgameted along their common subgroups $\mathcal{B}_\sigma$ according to the above injections.

In [Se70, Chapter II, §2.8] Serre gives a series of results that relate the homology of congruence subgroups of $\mathbf{G}(R)$ with the structure of its quotient graphs. We can apply them to our context. In particular, the following result, which gives a more precise description of the abelianization of $\mathrm{H}_D$, is a consequence of Theorem 1.4.

**Theorem 1.7.** With the same notation and hypotheses of Theorem 1.6, there exists an homomorphism $\phi : (\mathrm{H}_D)_{\mathrm{ab}} \to \bigoplus_{\sigma \in \mathbf{D} \cup \mathbf{I}} (\mathcal{P}_\sigma)_{\mathrm{ab}}$ whose kernel and cokernel are finitely generated.

## 2. Preliminaries on the Bruhat-Tits tree

2.1. **Conventions and notations for graphs.** We recall some basic definitions on graphs and trees. We define a graph $\mathfrak{g}$ as a pair of sets $\mathrm{V} = \mathrm{V}(\mathfrak{g})$ and $\mathrm{E} = \mathrm{E}(\mathfrak{g})$, and three functions $s, t : E \to V$ and $r : E \to E$ satisfying the identities $r(a) \neq a$, $r\big(r(a)\big) = a$ and $s\big(r(a)\big) = t(a)$, for every $a \in E$. In all that follows $V$ and $E$ are called vertex and edge set, respectively, and the functions $s, t$ and $r$ are called respectively source, target and reverse. Our definition is chosen in a way that allows the existence of multiple edges and loops. Two vertices $v, w \in \mathrm{V}$ are neighbors if there exists an edge $e \in \mathrm{E}$ satisfying $s(e) = v$ and $t(e) = w$. The valency of a vertex $v$ is the cardinality of its set of neighboring vertices. A simplicial map $\gamma : \mathfrak{g} \to \mathfrak{g}'$ between graphs is a pair of functions $\gamma_V : \mathrm{V}(\mathfrak{g}) \to \mathrm{V}(\mathfrak{g}')$ and $\gamma_E : \mathrm{E}(\mathfrak{g}) \to \mathrm{E}(\mathfrak{g}')$ preserving the source, target and reverse functions. We say that a simplicial map $\gamma : \mathfrak{g} \to \mathfrak{g}'$ is an isomorphism if there exists another simplicial map $\gamma' : \mathfrak{g}' \to \mathfrak{g}$ such that $\gamma_V \circ \gamma'_V = \mathrm{id}_{\mathrm{V}(\mathfrak{g})}$, $\gamma'_V \circ \gamma_V = \mathrm{id}_{\mathrm{V}(\mathfrak{g}')}$, $\gamma_E \circ \gamma'_E = \mathrm{id}_{\mathrm{E}(\mathfrak{g})}$ and $\gamma'_E \circ \gamma_E = \mathrm{id}_{\mathrm{E}(\mathfrak{g}')}$. The group of automorphisms $\mathrm{Aut}(\mathfrak{g})$ of a graph $\mathfrak{g}$ is the set of isomorphism from $\mathfrak{g}$ to $\mathfrak{g}$, with the composition as a group law.

We say that a group $\Gamma$ acts on a graph $\mathfrak{g}$ is there exists an homomorphism from $\Gamma$ to $\mathrm{Aut}(\mathfrak{g})$. A group action of $\Gamma$ on a graph $\mathfrak{g}$ has no inversions if $g \cdot a \neq r(a)$, for every edge $a$ and every element $g \in \Gamma$. An action without inversions defines a quotient graph in the usual sense. Indeed, if $\Gamma$ acts on $\mathfrak{g}$ without inversions, then the vertex set of $\Gamma\backslash\mathfrak{g}$ corresponds to $\Gamma\backslash V$, and the edge set corresponds to $\Gamma\backslash E$.

Let $\mathfrak{g}$ be a graph. A finite line in $\mathfrak{g}$, usually denoted by $\mathfrak{p}$, is a subgraph whose vertex and edge sets are $\mathrm{V}(\mathfrak{p}) = \{v_i\}_{i=0}^{n}$ and $\mathrm{E}(\mathfrak{p}) = \{e_i, r(e_i)\}_{i=0}^{n-1}$, where $s(e_i) = v_i$ and $t(e_i) = v_{i+1}$, for all index $0 \leqslant i \leqslant n-1$. The length of $\mathfrak{p}$ is, by definition, $n = \mathrm{Card}(\mathrm{V}(\mathfrak{p})) - 1 = \mathrm{Card}(\mathrm{E}(\mathfrak{p}))/2$. We often emphasize the vertices $v_0$, the initial vertex of $\mathfrak{p}$, and $v_r$, the final vertex of $\mathfrak{p}$, by saying "$\mathfrak{p}$ is a path connecting $v_0$ with $v_r$". A graph $\mathfrak{g}$ is connected if, given two vertices $v, w \in \mathrm{V}(\mathfrak{g})$, there exists finite path $\mathfrak{p}$ connecting them. We define a ray $\mathfrak{r}$ in $\mathfrak{g}$ by replacing $n$ and $n-1$ by $\infty$ in the definition of finite line. A cycle in $\mathfrak{g}$ is a finite line with an additional pair of edges connecting $v_n$ with $v_0$. We define a tree as a connected graph without cycles.

A maximal path in $\mathfrak{g}$ is a doubly infinite line, i.e. the union of two rays intersecting only in one vertex. Let $\mathfrak{r}_1$ and $\mathfrak{r}_2$ be two rays whose vertex sets are respectively denoted by $V_1 = \{v_i : i \in \mathbb{Z}_{\geqslant 0}\}$ and $V_2 = \{v'_i : i \in \mathbb{Z}_{\geqslant 0}\}$. We say that $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are equivalent if there exists $t, i_0 \in \mathbb{Z}_{\geqslant 0}$ such that $v_i = v'_{i+t}$, for all $i \geqslant i_0$. In this case we write $\mathfrak{r}_1 \sim \mathfrak{r}_2$. We define the visual limit, also called the end set, $\partial_\infty(\mathfrak{g})$ of $\mathfrak{g}$ as the set of equivalence classes of rays $\mathfrak{r}$ in $\mathfrak{g}$. We denote the class of $\mathfrak{r}$ by $\partial_\infty(\mathfrak{r})$. By a cuspidal ray in a graph $\mathfrak{g}$, we mean a ray such that every non-initial vertex has valency two in $\mathfrak{g}$. A cusp in $\mathfrak{g}$ is an equivalence class of cuspidal rays in $\mathfrak{g}$. We denote the cusp set of $\mathfrak{g}$ by $\partial^\infty(\mathfrak{g})$. We say that a graph is combinatorially finite if it is the union of a finite graph and a finite number of cuspidal rays. In particular, when a graph is combinatorially finite its visual limit is also finite.

2.2. **The Bruhat-Tits tree.** Let $k$ be the function field of a smooth, projective, geometrically connected curve $\mathcal{C}$ defined over a field $\mathbb{F}$. Let $K$ be the completion of $k$ with respect to a discrete valuation $\nu : k^* \to \mathbb{Z}$, and let $\mathcal{O}$ be its ring of integers. Recall that a tree is a connected graph without cycles.

An example of tree is the Bruhat-Tits building $\mathfrak{t} = \mathfrak{t}(K)$ associated to the reductive group $\mathrm{SL}_2$ and the field $K$. In order to introduce this tree, we have to fix some definitions concerning lattices. Let $\pi \in \mathcal{O}$ be a fixed uniformizing parameter of $K$. A lattice in a $K$-vector space $V$ is a finitely generated $\mathcal{O}$-submodule of $V$, which generates $V$ as a vector space. Assume that $V$ is a two-dimensional $K$-vector space. Then, every lattice on $V$ is free of rank 2. The group $K^*$ acts on the set of lattices by homothetic transformations. The vertex set of $\mathfrak{t}(K)$ can be defined as the set of homothetic classes of lattices in $V$. We adopt this convention. Let $\Lambda$ and $\Lambda'$ be two lattices in $V$. By the Invariant Factor Theorem of Algebraic Number Theory, there is an $\mathcal{O}$-basis $\{e_1, e_2\}$ of $\Lambda$ and integers $a, b$ such that $\{\pi^a e_1, \pi^b e_2\}$ is an $\mathcal{O}$-basis for $\Lambda'$. The set $\{a, b\}$ does not depend on the choice of basis for $\Lambda, \Lambda'$. Moreover, if we replace $\Lambda$ by $x\Lambda$, and $\Lambda'$ by $y\Lambda'$, where $x, y \in K^*$, then $\{a, b\}$ changes into $\{a+c, b+c\}$, where $c = \nu(y/x)$. So, the integer $|a-b|$ is called the distance between the classes $[\Lambda]$ and $[\Lambda']$. We define one pair of mutually reverse edges in $\mathfrak{t}(K)$ for each pair of lattice classes at distance one. This defines a graph, which can be proved to be a tree (cf. [Se80, Chapter II, §1, Theorem 1]). The group $\mathrm{GL}_2(K)$ acts on $\mathfrak{t}$ by $g \cdot [\Lambda] = [g(\Lambda)]$, for any $\mathcal{O}$-lattice $\Lambda \subset K^2$ and any $g \in \mathrm{GL}_2(K)$. This induces an action of $\mathrm{PGL}(V) = \mathrm{PGL}_2(K)$ on $\mathfrak{t}$.

An order in $\mathbb{M}_2(K)$ is a lattice with a ring structure induced by the multiplication of $\mathbb{M}_2(K)$. We say that an order is maximal when it fails to be contained in any other order. One can reinterpret the Bruhat-Tits tree for $\mathrm{SL}_2$ in several ways. One of these arises from the following remark. There exists a bijective map from the vertex set of $\mathfrak{t}(K)$ to the set of maximal orders in $\mathbb{M}_2(K)$. Indeed, this function is defined by $[\Lambda] \mapsto \mathrm{End}_{\mathcal{O}}(\Lambda)$, which is valid, since the endomorphism rings of $\Lambda$ and $x\Lambda$ coincide for any $x \in K^*$. Moreover, under this identification, two maximal orders $\mathfrak{D}$ and $\mathfrak{D}'$ are neighbors if the pair $\{\mathfrak{D}, \mathfrak{D}'\}$ is $\mathrm{GL}_2(K)$-conjugate to the pair $\left\{ \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}, \begin{pmatrix} \mathcal{O} & \pi^{-1}\mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} \end{pmatrix} \right\}$.

Another reinterpretation of the Bruhat-Tits tree for $\mathrm{SL}_2$ comes from the topological structure of $K$, which is very useful in order to have a concrete representation of its visual limit. We denote by $B_a^{|r|}$ the closed ball in $K$ whose center is $a$ and radius is $|\pi^r|$. Then, we can define the function $\Sigma$ between the set of closed balls and the set of maximal orders in $\mathbb{M}_2(K)$ by $B_a^{|r|} \mapsto \mathrm{End}_{\mathcal{O}}(\Lambda_B)$, where $\Lambda_B = \left\langle \binom{a}{1}, \binom{\pi^r}{0} \right\rangle$. It follows from [AAC18, §4] that $\Sigma$ is bijective. Thus, this induces a correspondence between the vertex set of $\mathfrak{t} = \mathfrak{t}(K)$ and the set of closed balls in $K$. Moreover, if we say that two balls are neighbors whenever one is a proper maximal sub-ball of the other, then $\Sigma$ induces an isomorphism of graphs. In other words, under the previous definition, we have that two balls $B$ and $B'$ are neighbors precisely when $\Sigma(B)$ and $\Sigma(B')$ are neighbors. So, by using this reinterpretation of the Bruhat-Tits tree in terms of balls, it is straightforward that any ray $\mathfrak{r}$ in $\mathfrak{t}$ satisfies either $V(\mathfrak{r}) = \left\{ B_a^{|r+n|} : n \in \mathbb{Z}_{\geqslant 0} \right\}$, for certain $a \in K$ and $r \in \mathbb{Z}$, or $V(\mathfrak{r}) = \left\{ B_0^{|r-n|} : n \in \mathbb{Z}_{\geqslant 0} \right\}$, for certain $r \in \mathbb{Z}$. In the first case, the visual limit of $\mathfrak{r}$ can be identified with $a \in K$, and, in the second, we identify it with the point at infinity $\infty$. This brief remark shows that the visual limit of the Bruhat-Tits tree $\mathfrak{t} = \mathfrak{t}(K)$ is in natural correspondence with the $K$-points of the projective line $\mathbb{P}^1$. In all that follows, the equivalence classes of rays in $\partial_\infty(\mathfrak{t})$ are called ends of $\mathfrak{t}$. This set of ends is acted on naturally by the group $\mathrm{GL}_2(K)$, via Moebius transformations with coefficients in $K$. In fact, this action is compatible with the previously defined action of $\mathrm{GL}_2(K)$ on lattices, or the subsequent action on balls induced by the former (cf. [AAC18, §4]). It follows from the density of $k$ in $K$, that for any finite line $\mathfrak{p}$ of $\mathfrak{t}$, there is a ray containing $\mathfrak{p}$ whose end corresponds to a rational element $s \in \mathbb{P}^1(k) \subset \mathbb{P}^1(K)$.

It is shown in [Se80, Chapter II, §1.3] that there exists a bipartition of the vertex set of the Bruhat-Tits tree that is respected by every subgroup $\Gamma \subset \mathrm{GL}_2(K)$ satisfying $\det(\Gamma) \subseteq \mathcal{O}^* K^{*2}$. This implies that such subgroups act on $\mathfrak{t}$ without inversions. In particular, we can define a quotient graph for these groups. This applies to every finite index subgroup $\Gamma$ of $\mathrm{GL}_2(R)$.

## 3. On combinatorially finite quotients of the Bruhat-Tits tree

We keep the notation from last section. Here we give a detailed description of the quotient graphs of the Bruhat-Tits tree $\mathfrak{t}$ by certain subgroups of $\mathrm{GL}_2(k)$. In order to do this, we introduce the following notion.

**Definition 3.1.** Let $H$ be a subgroup of $\mathrm{GL}_2(k)$. We say that $H$ *closes enough umbrellas* if there exists a finite family of rays $\mathfrak{R}_H = \{\mathfrak{r}_i\}_{i=1}^{\gamma} \subset \mathfrak{t}$, each with a vertex set $\{v_n(i)\}_{n>0}^{\infty}$, where $v_n(i)$ and $v_{n+1}(i)$ are neighbors, satisfying each of the following statements:

(a) The set of ends of all rays in $\mathfrak{R}_H$ is a representative system of $H\backslash\mathbb{P}^1(k)$.
(b) $H\backslash\mathfrak{t}$ is obtained by attaching all the images $\overline{\mathfrak{r}_i} \subseteq H\backslash\mathfrak{t}$ to a certain finite graph $Y_H$.
(c) No $\mathfrak{r}_i$ contains a pair of vertices in the same $H$-orbit, and $\overline{\mathfrak{r}_i} \cap \overline{\mathfrak{r}_j} = \emptyset$, for each $i \neq j$.
(d) For each index $i$ and each $n > 0$, we have $\mathrm{Stab}_H(v_n(i)) \subseteq \mathrm{Stab}_H(v_{n+1}(i))$.
(e) $\mathrm{Stab}_H(v_n(i))$ acts transitively on the set of neighboring vertices in $\mathfrak{t}$ of $v_n(i)$, other than $v_{n+1}(i)$.

In particular, if $H$ closes enough umbrellas, then $H\backslash\mathfrak{t}$ is combinatorially finite. Moreover, for any ray $\mathfrak{r} \subset \mathfrak{t}$ whose visual limits belongs to $\mathbb{P}^1(k)$, there exists a subray $\mathfrak{r}' \subseteq \mathfrak{t}$ satisfying conditions (d) and (e). Note that the notion of "closing umbrellas" corresponds to these two statements, while (a), (b) and (c) convey the idea of "closing *enough* umbrellas", so as to have a good quotient graph.

We say that a subgroup $H$ of $\mathrm{GL}_2(k)$ is net when each element in $H$ fails to admit a root of unit different than one as an eigenvalue. It follows from [Se70, Chapter II, §2.1- §2.3] that every net subgroup of $\mathrm{GL}_2(k)$ closes enough umbrellas. We say that two groups are commensurable if they have a common finite index subgroup. The notion of "closing enough umbrellas" behaves well when we pass to commensurable groups as is shown in the following results. This is probably known to experts but, as far as we are aware, a precise reference does not exist.

**Theorem 3.2.** Let $H$ be a discrete subgroup of $\mathrm{GL}_2(k)$. Let $H' \subseteq \mathrm{GL}_2(k)$ be a group that is commensurable with $H$. If $H$ closes enough umbrellas, then $H'$ also closes enough umbrellas.

In order to prove this theorem, we analyze separately in the following two proposition the cases of subgroups of $H$ and of groups containing $H$.

**Proposition 3.3.** Let $H$ be a subgroup of $\mathrm{GL}_2(k)$. Assume that $H$ closes enough umbrellas. Let $H' \subset \mathrm{GL}_2(k)$ be a group containing $H$ as a finite index normal subgroup. Then, $H'$ also closes enough umbrellas.

In order to prove the proposition, we need the following lemma.

**Lemma 3.4.** Let $\mathfrak{g}$ be a combinatorially finite graph, and let $G$ be a finite group acting without inversions on this graph. Then, each cuspidal ray $\mathfrak{r}$ in $\mathfrak{g}$ has a finite number of vertices in the same $G$-orbit. In particular, $\mathfrak{r}$ has a subray whose image in $G\backslash\mathfrak{g}$ is a cuspidal ray, and hence $G\backslash\mathfrak{g}$ is a combinatorially finite graph.

*Proof.* By definition we have that there exists a set of rays $\mathfrak{R} = \{\tilde{\mathfrak{r}}_i\}_{i=1}^{\gamma}$ all contained in $\mathfrak{g}$, such that $\mathfrak{g}$ is obtained by attaching all $\tilde{\mathfrak{r}}_i$ to a certain finite graph $Y$. Let $\tilde{\mathfrak{r}}$ be a ray in $\mathfrak{R}$. Since $G$ acts simplicially on $\mathfrak{g}$, for each $g \in G$, the graph $g \cdot \tilde{\mathfrak{r}}$ is also a ray in $\mathfrak{g}$. Then, since $\mathfrak{g}$ is combinatorially finite, $g \cdot \tilde{\mathfrak{r}}$ has the same visual limit as some ray in $\mathfrak{R}$. First assume that $\partial_\infty(\tilde{\mathfrak{r}}) = \partial_\infty(g \cdot \tilde{\mathfrak{r}})$. Then, $\mathfrak{r}^\circ := \tilde{\mathfrak{r}} \cap (g \cdot \tilde{\mathfrak{r}})$ is a ray. Since each non-initial vertex of $\tilde{\mathfrak{r}}$ and $g \cdot \tilde{\mathfrak{r}}$ has valency two, we get $\mathfrak{r}^\circ = \tilde{\mathfrak{r}}$ or $\mathfrak{r}^\circ = g \cdot \tilde{\mathfrak{r}}$. In other words, $\tilde{\mathfrak{r}} \subseteq g \cdot \tilde{\mathfrak{r}}$ or $\tilde{\mathfrak{r}} \supseteq g \cdot \tilde{\mathfrak{r}}$. Assume that $\tilde{\mathfrak{r}} \subseteq g \cdot \tilde{\mathfrak{r}}$, then

$$\tilde{\mathfrak{r}} \subseteq g \cdot \tilde{\mathfrak{r}} \subseteq \cdots \subseteq g^k \cdot \tilde{\mathfrak{r}} \subseteq g^{k+1} \cdot \tilde{\mathfrak{r}}, \text{ for all } k \in \mathbb{Z}_{\geqslant 0}.$$

Since $G$ is finite, we get that $\tilde{\mathfrak{r}} = g \cdot \tilde{\mathfrak{r}}$. By an analogous argument we also prove that $\tilde{\mathfrak{r}} = g \cdot \tilde{\mathfrak{r}}$, when $\tilde{\mathfrak{r}} \supseteq g \cdot \tilde{\mathfrak{r}}$. We conclude that $g$ fixes every vertex in this case.

Now, assume that the visual limit of $g \cdot \tilde{\mathfrak{r}}$ is not $\partial_\infty(\tilde{\mathfrak{r}})$. Then, $\tilde{\mathfrak{r}} \cap (g \cdot \tilde{\mathfrak{r}})$ is a finite graph. So, for each index $i$, we define the ray $\tilde{\mathfrak{r}}_i'$ as the unique unbounded connected component of

$$\tilde{\mathfrak{r}}_i \smallsetminus \left( \bigcup_{\substack{h \in G \\ \partial_\infty(\tilde{\mathfrak{r}}_i) \neq \partial_\infty(h \cdot \tilde{\mathfrak{r}}_i)}} \tilde{\mathfrak{r}}_i \cap (h \cdot \tilde{\mathfrak{r}}_i) \right).$$

By definition, and by the final statement in last paragraph, the ray $\tilde{\mathfrak{r}}_i'$ does not have two vertices in the same $G$-orbit. Since $\tilde{\mathfrak{r}}_i$ and $\tilde{\mathfrak{r}}_i'$ differ by a finite graph, the first assertion follows. In order to prove the last assertion, we say that $\tilde{\mathfrak{r}}_i'$ and $\tilde{\mathfrak{r}}_j'$ are $G$-equivalent if $\partial_\infty(\tilde{\mathfrak{r}}_i') = \partial_\infty(g \cdot \tilde{\mathfrak{r}}_j')$, for some $g = g(i,j) \in G$. So, we define $\mathfrak{r}_i'' \subseteq G\backslash\mathfrak{g}$ as the intersection of the images by $\pi : \mathfrak{g} \to G\backslash\mathfrak{g}$ of all rays $\tilde{\mathfrak{r}}_j'$ in the $G$-equivalence class of $\tilde{\mathfrak{r}}_i'$. We claim that $\mathfrak{r}_i''$ is a cuspidal ray in $G\backslash\mathfrak{g}$. Indeed, any element $g \in G$ sending $\partial_\infty(\tilde{\mathfrak{r}}_i')$ to $\partial_\infty(\tilde{\mathfrak{r}}_j')$ gives an injective simplicial correspondence between the vertices in either ray, whose image contains a pre-image in $\mathfrak{g}$ of $\mathfrak{r}_i''$. This correspondence is independent on the choice of $g$, since a different choice $g'$ defines an element $g'g^{-1}$ fixing every vertex in $\tilde{\mathfrak{r}}_i'$. This proves the claim. Finally, let us define $Y''$ as the union of $\pi(Y)$ with all $\pi(\tilde{\mathfrak{r}}_j) \smallsetminus \mathfrak{r}_i''$, for all pairs $(i,j)$ whose corresponding rays $\tilde{\mathfrak{r}}_i'$ and $\tilde{\mathfrak{r}}_j'$ are $G$-equivalent. Thus, $G\backslash\mathfrak{g}$ is obtained by attaching all $\mathfrak{r}_i''$ to the finite graph $Y''$. □

*Proof of Proposition 3.3.* By hypothesis there exists a family of rays $\mathfrak{R}_H = \{\mathfrak{r}_j\}_{j=1}^{\gamma}$ satisfying (a), (b), (c), (d) and (e) in Definition 3.1. For all index $j$, we denote by $\xi_j$ the visual limit of $\mathfrak{r}_j$, and by $\{v_n(\xi_j)\}_{n=1}^{\infty}$ the vertex set of $\mathfrak{r}_j$. So, we have $\mathbb{P}^1(k) = H \cdot \{\xi_j\}_{j=1}^{\gamma}$.

Let $\{\omega_i\}_{i=1}^{\delta}$ be a set of representatives of $H'\backslash\mathbb{P}^1(k)$. Then, each $\omega_i$ can be written as $\omega_i = h \cdot \xi_j$ for some suitable index $j = j(i)$ and some suitable element $h = h(i) \in H$. Thus, we define $\widehat{\mathfrak{r}}_i'$ as the intersection of $h \cdot \mathfrak{r}_j$ with the unique ray in $\mathfrak{t}$ joining $B_0^{|0|}$ with $\omega_i$. Let us write $\mathrm{V}(\widehat{\mathfrak{r}}_i') = \{v_n(\omega_i)\}_{n=1}^{\infty}$, where $v_n(\omega_i)$ and $v_{n+1}(\omega_i)$ are neighbors. By definition, for each vertex $v_n(\omega_i)$, there exists $m = m(n) \in \mathbb{Z}_{>0}$ such that $v_n(\omega_i) = h \cdot v_m(\xi_j)$. Thus, we have $\mathrm{Stab}_H(v_n(\omega_i)) = h\mathrm{Stab}_H(v_m(\xi_j))h^{-1}$, where $H \subseteq H'$. Hence, condition (e) follows.

Let $G$ be the finite group $H'/H$. Note that $H'\backslash\mathbb{P}^1(k) = G\backslash(H\backslash\mathbb{P}^1(k))$. Moreover, note that the quotient graph $H'\backslash\mathfrak{t}$ is the quotient of the combinatorially finite graph $H\backslash\mathfrak{t}$ by the finite group $G$. Then, it follows from Lemma 3.4 that $H'\backslash\mathfrak{t}$ is combinatorially finite, and that for each ray $\overline{\mathfrak{r}_j}$ in $H\backslash\mathfrak{t}$ there exists a subray $\tilde{\mathfrak{r}_j}^{\circ}$ not containing two vertices in the same $G$-orbit.

Let $\mathfrak{r}_j^\circ \subseteq \mathfrak{r}_j \subset \mathfrak{t}$ be a lift of $\tilde{\mathfrak{r}_j}^\circ$. So, for each index $i$, we define $\mathfrak{r}_i'$ as the intersection of $\widehat{\mathfrak{r}}_i'$ with $h(i) \cdot \mathfrak{r}_j^\circ$. We write $\mathrm{V}(\mathfrak{r}_i') = \{v_n(\omega_i)\}_{n=N_i}^\infty$, where $N_i > 0$. Then, for each $n \geqslant N_i + 1$, the vertices $v_{n-1}(\omega_i)$ and $v_{n+1}(\omega_i)$ are not in the same $H'$-orbit. So, since, by condition (e), all other neighbors are in the same $\mathrm{Stab}_{H'}(v_n(\omega_i))$-orbit as $v_{n-1}(\omega_i)$, we see that $\mathrm{Stab}_{H'}(v_n(\omega_i))$ stabilizes $v_{n+1}(\omega_i)$, i.e. condition (d) holds.

In order to check condition (c) on $\mathfrak{R}_{H'} := \{\mathfrak{r}_i'\}_{i=1}^\delta$, we just have to prove the projections $\overline{\mathfrak{r}_i'}$ and $\overline{\mathfrak{r}_l'}$ to $H' \backslash \mathfrak{t}$ do not intersect when $i \neq l$ in $\{1, \cdots, \delta\}$. Indeed, it follows from Lemma 3.4, and the construction of the rays $\mathfrak{r}_i'$, that $\overline{\mathfrak{r}_i'} \cap \overline{\mathfrak{r}_l'} \neq \varnothing$ if and only if $\overline{\mathfrak{r}_i'} = \overline{\mathfrak{r}_l'}$, and also if and only if their visual limits coincide. By definition, the last assertion does not hold if $i \neq l$. Finally, condition (b) is an immediate consequence of Lemma 3.4, and condition (a) is immediate by construction. $\square$

**Proposition 3.5.** Let $H$ be a discrete subgroup of $\mathrm{GL}_2(k)$. Assume that $H$ closes enough umbrellas. Then, any finite index subgroup $H_0$ of $H$ closes enough umbrellas.

*Proof.* Since any finite index subgroup of $H$ contains a normal subgroup, by Proposition 3.3 we may assume that $H_0$ is normal in $H$. By hypothesis there exists a family of rays $\mathfrak{R}_H = \{\mathfrak{r}_j\}_{j=1}^\gamma$ satisfying (a), (b), (c), (d) and (e) in Definition 3.1. For all index $j$, we denote by $\xi_j$ the visual limit of $\mathfrak{r}_j$, and by $\{v_n(\xi_j)\}_{n=1}^\infty$ the vertex set of $\mathfrak{r}_j$. So, we have $\mathbb{P}^1(k) = H \cdot \{\xi_j\}_{j=1}^\gamma$.

Let $\{\mu_i\}_{i=1}^\beta$ be a set of representatives of $H_0 \backslash \mathbb{P}^1(k)$. Then, each $\mu_i$ can be written as $\mu_i = h \cdot \xi_j$ for some suitable index $j = j(i)$ and some suitable element $h = h(i) \in H$. Thus, we define $\widehat{\mathfrak{r}}_i = h \cdot \mathfrak{r}_j$, i.e. $\mathrm{V}(\widehat{\mathfrak{r}}_i) = \{v_n(\mu_i)\}_{n=1}^\infty$, where $v_n(\mu_i) = h \cdot v_n(\xi_j)$. So, we have

$$\mathrm{Stab}_{H_0}(v_n(\mu_i)) = H_0 \cap h\mathrm{Stab}_H(v_n(\xi_j))h^{-1}.$$

In particular, condition (d) for $H_0$ follows immediately.

Now, we check condition (c) for $H_0$. Indeed, assume that there exist $v_0 \in \mathrm{V}(\widehat{\mathfrak{r}}_k)$, $w_0 \in \mathrm{V}(\widehat{\mathfrak{r}}_l)$ and $h_0 \in H_0$ such that $h_0 \cdot v_0 = w_0$. Write $v_0 = h(k) \cdot v$ and $w_0 = h(l) \cdot w$, with $v \in \mathrm{V}(\mathfrak{r}_{j(k)})$ and $w \in \mathrm{V}(\mathfrak{r}_{j(l)})$. Then $h \cdot v = w$ with $h = h(l)^{-1} h_0 h(k) \in H$, which contradicts condition (c) for $H$. So, condition (c) for $H_0$ follows.

Let $G$ be the finite group $H/H_0$. Let $\pi : \mathrm{Stab}_H(v_n(\mu_i)) \to G$ be the map defined by composing the natural inclusion $\mathrm{Stab}_H(v_n(\mu_i)) \to H$ with the projection $H \to G$. Since, for each $n \in \mathbb{Z}_{\geqslant 1}$ we have $\ker(\pi) = \mathrm{Stab}_{H_0}(v_n(\mu_i))$, we obtain from condition (d) for $H$ the chain of contentions

$$\mathrm{Stab}_H(v_1(\mu_i))/\mathrm{Stab}_{H_0}(v_1(\mu_i)) \subseteq \cdots \subseteq \mathrm{Stab}_H(v_n(\mu_i))/\mathrm{Stab}_{H_0}(v_n(\mu_i)) \subseteq \cdots$$

Then, since $G$ is a finite set, there exists $t_0 = t_0(i) \in \mathbb{Z}_{\geqslant 1}$ such that, for each $n \geqslant t_0$

$$\mathrm{Stab}_H(v_n(\mu_i))/\mathrm{Stab}_{H_0}(v_n(\mu_i)) = \mathrm{Stab}_H(v_{n+1}(\mu_i))/\mathrm{Stab}_{H_0}(v_{n+1}(\mu_i)). \tag{3.1}$$

Recall that, since $K$ is locally compact, we have that $\mathrm{Stab}_{\mathrm{GL}_2(K)}(v_n(\mu_i))$ is compact. Then, for each discrete subgroup $D$, for instance $H$ or $H_0$, we get that $\mathrm{Stab}_D(v_n(\mu_i))$ is finite. Then, Equation (3.1) implies that, for each $n > t_0$

$$|\mathrm{Stab}_{H_0}(v_n(\mu_i))/\mathrm{Stab}_{H_0}(v_{n-1}(\mu_i))| = |\mathrm{Stab}_H(v_n(\mu_i))/\mathrm{Stab}_H(v_{n-1}(\mu_i))|.$$

In particular, the injective map

$$\psi : \mathrm{Stab}_{H_0}(v_n(\mu_i))/\mathrm{Stab}_{H_0}(v_{n-1}(\mu_i)) \to \mathrm{Stab}_H(v_n(\mu_i))/\mathrm{Stab}_H(v_{n-1}(\mu_i)),$$

induced by the inclusion $\iota : \mathrm{Stab}_{H_0}(v_n(\mu_i)) \to \mathrm{Stab}_H(v_n(\mu_i))$, is a bijection. It follows from condition (e) for $H$ and the orbit-stabilizer relation that the set

$\mathrm{Stab}_H(v_n(\mu_i))/\mathrm{Stab}_H(v_{n-1}(\mu_i))$ parametrizes all the neighboring vertices in $\mathfrak{t}$ of $v_n(\mu_i)$ other than $v_{n+1}(\mu_i)$. So, since $\psi$ is a bijection, we deduce that the set $\mathrm{Stab}_{H_0}(v_n(\mu_i))/\mathrm{Stab}_{H_0}(v_{n-1}(\mu_i))$ also parametrizes the aforementioned set of vertices. In other words, up to replacing $\widehat{\mathfrak{r}}_i$ by the ray $\mathfrak{r}'_i$ defined by the vertex set $\{v_n(\mu_i)\}_{i=t_0+1}^{\infty}$, condition (e) follows for $H_0$.

Now, note that the graph $H\backslash\mathfrak{t}$ is the quotient of the graph $H_0\backslash\mathfrak{t}$ by the finite group $G$. In particular, the pre-image of the finite graph $Y_H$ by the projection $H_0\backslash\mathfrak{t} \to H\backslash\mathfrak{t}$ is a finite graph. So, since $\widehat{\mathfrak{r}}_i \smallsetminus \mathfrak{r}'_i$ is also a finite graph, we conclude that condition (b) holds for $H_0$. Condition (a) for $H_0$ follows from definition. Thus, we conclude the proof. □

*Proof of Theorem 3.2.* Let $H_0$ be a common finite index subgroup containing in $H$ and $H'$. By replacing $H_0$ by a smaller subgroup if needed, we can assume that $H_0$ is normal in $H'$. Then, it follows from Proposition 3.5 that $H_0$ closes enough umbrellas. By applying Proposition 3.3 to $H_0$ and $H'$, we conclude that $H'$ also closes enough umbrellas. □

As in §2.2, let $k$ be the function field of a smooth, projective, geometrically connected curve $\mathcal{C}$ defined over a field $\mathbb{F}$. Let $Q$ be a closed point in $\mathcal{C}$, and set $U' = \mathcal{C} \smallsetminus \{Q\}$. Denote by $R'$ the ring of regular functions on $U'$. Let $\nu_Q$ be the discrete valuation map defined from the closed point $Q$. Let us denote by $k_Q$ the completion of $k$ with respect to $\nu_Q$. In the remaining of this section, we give a detailed description of certain quotient of the Bruhat-Tits tree $\mathfrak{t} = \mathfrak{t}(k_Q)$ defined from $\mathrm{SL}_2$ and $k_Q$. In order to do this, let us introduce the following definition:

**Definition 3.6.** A $\mathcal{C}$-order of maximal rank $\mathfrak{R}$ is a locally free sheaf of $\mathcal{O}_{\mathcal{C}}$-algebras whose generic fiber is $\mathbb{M}_2(k)$. We say that a $\mathcal{C}$-order $\mathfrak{D}$ is maximal when it is maximal with respect to inclusion. An Eichler $\mathcal{C}$-order $\mathfrak{E}$ is the sheaf-theoretical intersection of two maximal $\mathcal{C}$-orders.

**Example 3.7.** Let us denote by $\mathfrak{D}_0$ the sheaf $\mathfrak{D}_0 = \mathbb{M}_2(\mathcal{O}_{\mathcal{C}})$. Then $\mathfrak{D}_0$ is a maximal $\mathcal{C}$-order. Moreover $\mathfrak{D}_0(U')^* = \mathrm{GL}_2(R')$.

**Corollary 3.8.** Let $H \subset \mathrm{GL}_2(k)$ be a group commensurable with $\mathrm{GL}_2(R')$. Then $H$ closes enough umbrellas. In particular, for any Eichler $\mathcal{C}$-order $\mathfrak{E}$, we have that $\tilde{H} = \mathfrak{E}(U)^*$ and $\tilde{\Gamma} = \mathrm{Stab}_{\mathrm{GL}_2(k)}(\mathfrak{E}(U))$ close enough umbrellas.

*Proof.* First, it follows from [Se80, Chapter II, §2.1- §2.3] that $\mathrm{GL}_2(R')$ closes enough umbrellas. Then, it follows from Theorem 3.2 that any group $H \subset \mathrm{GL}_2(k)$ commensurable with $\mathrm{GL}_2(R')$ closes enough umbrellas.

Now, we claim that $\tilde{H}$ and $\tilde{\Gamma}$ are commensurable with $\mathrm{GL}_2(R')$. Indeed, let $\mathfrak{D}$ be a maximal $\mathcal{C}$-order containing $\mathfrak{E}$. Let us fix $\tilde{\Gamma}_0 = \mathrm{Stab}_{\mathrm{GL}_2(k)}(\mathfrak{D}(U'))$. Note that $\tilde{\Gamma}_0$ and $\tilde{\Gamma}$ are commensurable, since they contain the respective finite index subgroups $\tilde{H}_0 = \mathfrak{D}(U')^*$ and $\tilde{H}$, where $\tilde{H}$ is a finite index subgroup of $\tilde{H}_0$ (cf. [A16, Theorem 1.2]). Moreover, note that $\tilde{H}_0$ belongs to the same commensurability class as $\mathrm{GL}_2(R')$, since $\mathfrak{D} \cap \mathfrak{D}_0$ is a finite index Eichler $\mathcal{C}$-order simultaneously contained in $\mathfrak{D}$ and $\mathfrak{D}_0$. □

**Remark 3.9.** Theorem 3.2 and Corollary 3.8 can be easily extended to subgroups of $\mathrm{PGL}_2(k)$.

## 4. Preliminaries on divisors and vector bundles

Let $\mathcal{C}$ be a smooth, projective, geometrically connected curve over a finite field $\mathbb{F}$. By definition, a divisor on $\mathcal{C}$ is a formal finite linear combination $n_1P_1 + \cdots + n_rP_r$ of distinct closed points $P_1, \cdots, P_r \in \mathcal{C}$ with integer coefficients $n_1, \cdots, n_r \in \mathbb{Z}$, for some $r \in \mathbb{N}$. Obviously, the divisors on $\mathcal{C}$ form an abelian group under coefficient-wise addition. We denote it by $\mathrm{Div}(\mathcal{C})$. A divisor $D = n_1P_1 + \cdots + n_rP_r$ as above is called effective, and written $D \geqslant 0$, if each $n_i \geqslant 0$. If $D_1$ and $D_2$ are two divisors such that $D_2 - D_1$ is effective, then we write $D_2 \geqslant D_1$ or $D_1 \leqslant D_2$.

Each closed point $P$ in $\mathcal{C}$ defines a discrete valuation $\nu_P$ on the global function field $k = \mathbb{F}(\mathcal{C})$. Let $k_P$ be the completion of $k$ at $P$, i.e. the completion of $k$ with respect to $\nu_P$. Let $\mathcal{O}_P$ be the ring of integers of $k_P$, and fix a uniformizing parameter $\pi_P \in \mathcal{O}_P$. Then, we define the degree of the point $P$ as the degree of the finite extension $\mathbb{F}(P) = \mathcal{O}_P/\pi_P\mathcal{O}_P$ of $\mathbb{F}$. More generally, the degree of a divisor $D = n_1P_1 + \cdots + n_rP_r$ as above is the integer $\deg(D) := n_1 \deg(P_1) + \cdots n_r \deg(P_r) \in \mathbb{Z}$. Thus defined, the degree is a group homomorphism $\deg : \mathrm{Div}(\mathcal{C}) \to d\mathbb{Z}$, where $d$ is the gcd of $\deg(Q)$ with $Q \in \mathcal{C}$. Its kernel is denoted by $\mathrm{Div}^0(\mathcal{C})$. Moreover, every element $f \in k^*$ defines a divisor $\mathrm{div}(f) \in \mathrm{Div}^0(\mathcal{C})$. We define the Picard group $\mathrm{Pic}(\mathcal{C})$ as the quotient of $\mathrm{Div}(\mathcal{C})$ by the subgroup $\mathrm{div}(k^*) = \{\mathrm{div}(f) : f \in k^*\}$. Hence, one has the exact sequence:

$$0 \to J(\mathbb{F}) \to \mathrm{Pic}(\mathcal{C}) \to d\mathbb{Z} \to 0, \tag{4.1}$$

where $J(\mathbb{F}) = \mathrm{Div}^0(\mathcal{C})/\mathrm{div}(k^*)$, also denoted $\mathrm{Pic}^0(\mathcal{C})$, corresponds to the set of $\mathbb{F}$-points of the Jacobian variety of $\mathcal{C}$ (cf. [Se80, Chapter II, §2.2]). Since $\mathbb{F}$ is finite, the group $J(\mathbb{F})$ is also finite (cf. [Se80, Chapter II, §2.2]).

Let $\mathcal{A}$ be the affine line considered as an algebraic variety. A vector bundle on $\mathcal{C}$ is a variety which "locally looks like a direct product of $\mathcal{C}$ with a vector space". Formally, a vector bundle of rank $s$ over $\mathcal{C}$ is an algebraic variety $\mathcal{B}$ over $\mathbb{F}$ equipped with a morphism $\pi : \mathcal{B} \to \mathcal{C}$ such that there exists a covering $\mathcal{C} = \bigcup_{i \in I} U_i$ by Zariski open sets satisfying

(a) For each $i \in I$ there exists an isomorphism $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathcal{A}^s$ satisfying that the composition $\pi \circ \phi_i^{-1} : U_i \times \mathcal{A}^s \to U_i$ is the projection onto the first coordinate, and
(b) For each $i, j \in I$ there exists an $(s \times s)$-matrix $A_{ij}$, whose entries are regular functions in $U_i \cap U_j$, satisfying that the composition

$$\phi_{ij} : \phi_j \circ \phi_i^{-1}|_{(U_i \cap U_j) \times \mathcal{A}^s} : (U_i \cap U_j) \times \mathcal{A}^s \to (U_i \cap U_j) \times \mathcal{A}^s,$$

takes the form $\phi_{ij}(x, v) = (x, A_{ij}(x)v)$.

We call the tuple $(U_i, \phi_i, \phi_{ij})$ a trivialization of the respective vector bundle. If $s = 1$, we say that $(\mathcal{B}, \pi)$ is a line bundle. Let $(\mathcal{B}, \pi)$ be a vector bundle of rank $s$ with trivialization $(U_i, \phi_i, \phi_{ij})$. Define $(\mathcal{B}', \pi')$, $s'$ and $(U_i', \phi_i', \phi_{ij}')$ analogously. A morphism of vector bundles $f : \mathcal{B} \to \mathcal{B}'$ is a $\mathcal{C}$-morphism, i.e. such that the following diagram commutes

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathcal{B}' \arrow[dl, "\pi'"] \\ & \mathcal{C} & \end{tikzcd},$$

and satisfying that, for any $i \in I$ and $i' \in I'$, the algebraic morphism $\pi^{-1}(U_i \cap U_{i'}) \to \pi^{-1}(f(U_i \cap U_i'))$ has the form $\mathrm{id} \times f_{ij}$, for some linear map $f_{ij} : \mathcal{A}^s \to \mathcal{A}^s$.

For one-dimensional vector bundles, elements of the Picard group $\mathrm{Pic}(\mathcal{C})$ correspond to isomorphism classes of line bundles over $\mathcal{C}$.

For one-dimensional vector bundles, elements of the Picard group $\mathrm{Pic}(\mathcal{C})$ correspond to isomorphism classes of line bundles over $\mathcal{C}$. This bijection is induced by the following map. Let $D \in \mathrm{Div}(\mathcal{C})$ and let $\mathfrak{L}^D$ be the sheaf defined in every open set $U \subseteq \mathcal{C}$ by

$$\mathfrak{L}^D(U) = \{f \in k : \mathrm{div}(f)|_U + D|_U \geqslant 0\}. \tag{4.2}$$

Then, we can show that $\mathfrak{L}^D$ is a locally free sheaf of rank one. Thus, it defines a line bundle on $\mathcal{C}$. If we define a group structure on the set of classes via tensor products, then the previously defined map is actually a group isomorphism.

Naturally associated to a line bundle $\mathfrak{L}^D$ we can define the maximal $\mathcal{C}$-order $\mathfrak{D}_D$ (cf. Definition 3.6) as follows

$$\mathfrak{D}_D = \mathrm{End}_{\mathcal{O}_\mathcal{C}}\begin{pmatrix} \mathcal{O}_\mathcal{C} \\ \mathfrak{L}^{-D} \end{pmatrix} = \begin{pmatrix} \mathcal{O}_\mathcal{C} & \mathfrak{L}^D \\ \mathfrak{L}^{-D} & \mathcal{O}_\mathcal{C} \end{pmatrix}. \tag{4.3}$$

In this thesis we will study a special family of intersections of two maximal orders as above. This family consists in objects of the form

$$\mathfrak{E}_D = \mathfrak{D}_0 \cap \mathfrak{D}_D. \tag{4.4}$$

More specifically, one of our main goals is to understand the quotient graph $\mathfrak{t}_D = \mathrm{H}_D \backslash \mathfrak{t}(k_{P_\infty})$, where $\mathrm{H}_D = \mathfrak{E}_D(\mathcal{C} \smallsetminus \{P_\infty\})$. See Theorems 1.4, 1.5, 1.6 and 1.7 for more details.

4.1. **An interpretation of the Riemann-Roch Theorem.** Let $D$ be a divisor on $\mathcal{C}$, and let $U$ be a open affine set. The Rieman-Roch Theorem states that $\dim_\mathbb{F}(\mathfrak{L}^D(U))$ is bounded by a constant depending on the degree of $D$ and the genus $g$ of $\mathcal{C}$ (cf. [St93, §1, Theorem 1.5.17]). Indeed, we have the following statements:

- $\dim_\mathbb{F}(\mathfrak{L}^D(U)) \geqslant \deg(D) + 1 - g$, and
- if $\deg(D) \geqslant 2g - 1$, then $\dim_\mathbb{F}(\mathfrak{L}^D(U)) = \deg(D) + 1 - g$.

Let $P_\infty$ be a fixed closed point in $\mathcal{C}$, and let $R$ be the ring of functions that are regular outside $P_\infty$. Then $R$ is a Dedekind domain whose quotient field is $k$. Let $\nu : k \to \mathbb{Z} \cup \{\infty\}$ be the discrete valuation induced by $P_\infty$. We recall some elementary properties, which follow immediately from the product formula and the hypothesis that $\mathcal{C}$ is geometrically connected (which implies that $\mathbb{F}$ is algebraically closed in $k$):

- $\nu_\infty(x) \leqslant 0$, for all $x \in R \smallsetminus \{0\}$,
- for any $x \in R$, we have $\nu_\infty(x) = 0$ if and only if $x \in \mathbb{F}^*$, and
- $R^* = \mathbb{F}^*$.

Each closed point $P \in \mathcal{C}$ other than $P_\infty$ corresponds to a prime ideal $I(P) = \{x \in R : \nu_P(x) \geqslant 1\}$ of $R$, and conversely. Furthermore, every non-zero fractional $R$-ideal $J$ has a decomposition $J = \prod_{P \neq P_\infty} I(P)^{n_P}$, and an associated divisor $D_J = \sum_{P \neq P_\infty} n_P P$. We define $\deg(J) := \deg(D_J)$. For any $m \in \mathbb{N}$, we define

$$J[m] := \mathfrak{L}^{-D_J + mP_\infty}(U_0) = \{x \in J : \nu(x) \geqslant -m\}\,,$$

where $U_0 = \operatorname{Spec}(R) = \mathcal{C} \smallsetminus \{P_\infty\}$. We denote by $g$ the genus of $\mathcal{C}$. Then, by the Riemann-Roch Theorem, the set $J[m]$ is a finite-dimensional vector space over $\mathbb{F}$, and when $\deg(-D_J + mP_\infty) \geqslant 2g - 1$ we have

$$\dim_{\mathbb{F}}(J[m]) = \deg(-D_J + mP_\infty) + 1 - g.$$

It follows from a simple computation that

$$\deg(-D_J + mP_\infty) = -\deg(J) + m\deg(P_\infty),$$

whence we finally get,

$$\dim_{\mathbb{F}}(J[m]) = -\deg(J) + m\mathrm{deg}(P_\infty) + 1 - g, \tag{4.5}$$

when $m\mathrm{deg}(P_\infty) \geqslant \deg(J) + 2g - 1$. In all that follows, by abuse of language, we refer to Equation (4.5) as the Riemann-Roch Theorem.

# 5. Spinor class fields

In this section we introduce the basic definitions and results about completions, spinor genera and spinor class fields of orders. See [A12] for details.

We denote by $|\mathcal{C}|$ the set of closed points in the smooth projective geometrically connected curve $\mathcal{C}$, and we fix $P_\infty \in |\mathcal{C}|$. Let $U_0$ be the affine open set $\mathcal{C} \smallsetminus \{P_\infty\}$. For every point $P \in |\mathcal{C}|$, we denote by $k_P$ the completion at $P$ of the function field $k = k(\mathcal{C})$, and by $\mathcal{O}_P$ the ring of integers of the former. For any open set $U \subseteq \mathcal{C}$, we define the adèle ring $\mathbb{A}_U$ of $U$ as the subring of $\prod_{P\in|U|} k_P$ containing all elements $a = (a_P)_P$ for which all but a finite number of coordinates $a_P$ belong to $\mathcal{O}_P$. We also define the idèle group $\mathbb{I}_U$ as the group of invertible adèles $\mathbb{A}_U^*$. We write $\mathbb{A} = \mathbb{A}_{\mathcal{C}}$ and $\mathbb{I} = \mathbb{I}_{\mathcal{C}}$.

A $\mathcal{C}$-lattice or $\mathcal{C}$-bundle in a finite dimensional $k$-vector space $V$ is a locally free subsheaf of the constant sheaf $V$. For any sheaf of groups $\Lambda$ on $\mathcal{C}$ we denote by $\Lambda(U)$ its group of $U$-sections. In particular, this convention applies to $\mathcal{C}$-lattices. By definition, the completion at $P$ of $\Lambda$, denoted $\Lambda_P$, is the topological closure of $\Lambda(U)$ in $V_P := V \otimes_k k_P$ for an arbitrary affine open subset $U$ containing $P$. Thus defined, $\Lambda_P$ is independent of the choice of $U$. Note that, for every affine open subset $U \subseteq \mathcal{C}$, the $\mathcal{O}_{\mathcal{C}}(U)$-module $\Lambda(U)$ is an $\mathcal{O}_{\mathcal{C}}(U)$-lattice. The same property holds for orders. As in the affine context, every $\mathcal{C}$-lattice is determined by its local completions $\Lambda_P$, where $P$ runs over the set of closed points $|\mathcal{C}|$, in the following sense:

(a) For any two lattices $\Lambda$ and $\Lambda'$ in $V$, we have $\Lambda_P = \Lambda'_P$ for almost all $P$,
(b) if $\Lambda_P = \Lambda'_P$ for all $P$, then $\Lambda = \Lambda'$, and
(c) every family $\{\Lambda''(P)\}_P$ of local lattices satisfying $\Lambda''(P) = \Lambda_P$ for almost all $P$ is the family of completions of a global lattice $\Lambda''$ in $V$.

In particular, the same results hold for orders. We define the adelization $W_{\mathbb{A}}$ of a finite dimensional vector space $W$ over $k$ as $W_{\mathbb{A}} = W \otimes_k \mathbb{A}$. This is a $k$-vector space isomorphic to $\mathbb{A}^{\dim_k W}$. In particular, this definition applies to $W = \mathrm{End}_k(V)$. We also define the adelization of a lattice $\Lambda$ by $\Lambda_{\mathbb{A}} = \prod_{P\in|\mathcal{C}|} \Lambda_P$, which is an open and compact subgroup of $V_{\mathbb{A}}$. For every $\mathcal{C}$-lattice $\Lambda$ and every element

$$a \in \mathrm{End}_{\mathbb{A}}(V_{\mathbb{A}}) = \big(\mathrm{End}_k(V)\big)_{\mathbb{A}},$$

the adelic image $L = a\Lambda$ is the unique $\mathcal{C}$-lattice satisfying $L_{\mathbb{A}} = a\Lambda_{\mathbb{A}}$. To each $\mathcal{C}$-lattice $\Lambda$ in $k^2$, we associate the $\mathcal{C}$-order $\mathfrak{D}_\Lambda = \mathrm{End}_{\mathcal{O}_X}(\Lambda)$ in the matrix algebra $\mathbb{M}_2(k)$, which is defined on every open set $U \subseteq \mathcal{C}$ by

$$\mathfrak{D}_\Lambda(U) = \left\{ a \in \mathbb{M}_2(k) \middle| a\Lambda(U) \subseteq \Lambda(U) \right\}.$$

This is a maximal $\mathcal{C}$-order in $\mathbb{M}_2(k)$ (cf. Definition 3.6). Moreover, every maximal $\mathcal{C}$-order in the two-by-two matrix algebra equals $\mathfrak{D}_\Lambda$, for some $\mathcal{C}$-lattice $\Lambda$ in $k^2$. In particular, if we fix a maximal $\mathcal{C}$-order $\mathfrak{D}$, then any other maximal $\mathcal{C}$-order in $\mathbb{M}_2(k)$ is equal to $\mathfrak{D}' = a\mathfrak{D}a^{-1}$, for some $a \in \mathrm{GL}_2(\mathbb{A})$. In general, if we fix an $\mathcal{C}$-order $\mathfrak{D}$ of maximal rank, then we can define the genus $\mathrm{gen}(\mathfrak{D})$ of $\mathfrak{D}$ as the set of all $\mathcal{C}$-orders $a\mathfrak{D}a^{-1}$, for $a \in \mathbb{M}_2(\mathbb{A})^*$. So, the previous statement is equivalent to the fact that the set of maximal $\mathcal{C}$-orders is a genus, which we denote by $\mathbb{O}_0$.

Let $\mathfrak{D}$ be a $\mathcal{C}$-order of maximal rank, i.e. of rank 4. Let $U$ be either an affine open set of $\mathcal{C}$ or the full set $\mathcal{C}$. We define the $U$-spinor class field of $\mathfrak{D}$ as the field corresponding, via class field theory, to the subgroup $k^*H(\mathfrak{D},U) \subseteq \mathbb{I} = \mathbb{A}^*$, where

$$H(\mathfrak{D},U) = \left\{\det(a)|a \in \mathbb{M}_2(\mathbb{A})^*,\ a\mathfrak{D}(V)a^{-1} = \mathfrak{D}(V),\ \forall\ V \overset{\circ}{\subseteq} U\right\}. \tag{5.1}$$

The symbol $\overset{\circ}{\subseteq}$ above denotes an open subset. This field depends only on the genus $\mathbb{O} = \mathrm{gen}(\mathfrak{D})$ of $\mathfrak{D}$, and we denote it by $\Sigma(\mathbb{O},U)$. When $U = \mathcal{C}$ we simplify the notation by using $\Sigma = \Sigma(\mathbb{O})$. Let $\mathbb{I} \to \mathrm{Gal}(\Sigma/k)$, $t \mapsto [t,\Sigma/k]$ be the Artin map on the idèle group (cf. [Ne99, Chapter VI, §5, p. 387]). There exists a well-defined distance map $\rho : \mathbb{O} \times \mathbb{O} \to \mathrm{Gal}\big(\Sigma/k\big)$, given by $\rho(\mathfrak{D},\mathfrak{D}') = [\det(a),\Sigma/k]$, where $a \in \mathrm{GL}_2(\mathbb{A})$ is any adelic element satisfying $\mathfrak{D}' = a\mathfrak{D}a^{-1}$. The distance map has a multiplicative property, in the sense that, for any tuple $(\mathfrak{D},\mathfrak{D}',\mathfrak{D}'') \in \mathbb{O}^3$, it satisfies $\rho(\mathfrak{D},\mathfrak{D}'') = \rho(\mathfrak{D},\mathfrak{D}')\rho(\mathfrak{D}',\mathfrak{D}'')$. The kernel of $\rho$ consists of the pairs $(\mathfrak{D},\mathfrak{D}')$ such that $\mathfrak{D}(U)$ and $\mathfrak{D}'(U)$ are $\mathrm{GL}_2(\mathcal{O}_{\mathcal{C}}(U))$-conjugate for every affine open subset $U \subseteq \mathcal{C}$. In the case of maximal orders, the map defined above is $\rho_0 : \mathbb{O}_0^2 \to \mathrm{Gal}\big(\Sigma(\mathbb{O}_0)/k\big)$, and it can be characterized as follows: The image of $\rho_0$ for a pair $(\mathfrak{D},\mathfrak{D}')$ of maximal orders is given by the formula $\rho_0(\mathfrak{D},\mathfrak{D}') = [[D(\mathfrak{D},\mathfrak{D}'),\Sigma(\mathbb{O}_0)/k]]$, where $D \mapsto [[D,\Sigma(\mathbb{O}_0)/k]]$ is the Artin map on divisors and the divisor $D(\mathfrak{D},\mathfrak{D}')$ is defined in Equation (6.1).

## 6. Eichler orders and grids

In this section notation is as in §5. In the local algebra $\mathbb{M}_2(k_P)$, any two $\mathcal{O}_P$-maximal orders are simultaneously $\mathrm{GL}_2(k_P)$-conjugate to the orders $\mathfrak{D}_P = \begin{pmatrix} \mathcal{O}_P & \mathcal{O}_P \\ \mathcal{O}_P & \mathcal{O}_P \end{pmatrix}$ and $\mathfrak{D}'_P = \begin{pmatrix} \mathcal{O}_P & \pi_P^d \mathcal{O}_P \\ \pi_P^{-d} \mathcal{O}_P & \mathcal{O}_P \end{pmatrix}$ for some $d \in \mathbb{Z}_{\geqslant 0}$, where $\pi_P$ is a local uniformizing parameter in $k_P$. So, we define the local distance $d_P$ between maximal orders in $\mathbb{M}_2(k_P)$ by taking $d_{\mathfrak{p}}(\mathfrak{D}_P, \mathfrak{D}'_P) = d$, where $d$ is as above. As we introduce in Definition 3.6, an Eichler $\mathcal{C}$-order, or simply an Eichler order, is the intersection of two maximal $\mathcal{C}$-orders. This is a local definition in the sense that $\mathfrak{D}_P \cap \mathfrak{D}'_P = (\mathfrak{D} \cap \mathfrak{D}')_P$ for every pair of orders. Moreover, locally, for any Eichler order $\mathfrak{E}_P$ there exists a unique pair of maximal orders whose intersection is $\mathfrak{E}_P$. The level of a local Eichler order is, by definition, the distance between the maximal orders defining it. Globally, there exists a well-defined distance map on the set of maximal $\mathcal{C}$-orders, whose image on a pair $(\mathfrak{D}, \mathfrak{D}')$ is the effective divisor

$$D = D(\mathfrak{D}, \mathfrak{D}') = \sum_{P \in |\mathcal{C}|} d_P(\mathfrak{D}_P, \mathfrak{D}'_P)P. \tag{6.1}$$

In particular, there exists a global level $\lambda(\mathfrak{E}_{\Lambda,\Lambda'})$ defined, on an Eichler $\mathcal{C}$-order $\mathfrak{E}_{\Lambda,\Lambda'} = \mathfrak{D}_\Lambda \cap \mathfrak{D}_{\Lambda'}$, as the distance $D(\mathfrak{D}_\Lambda, \mathfrak{D}_{\Lambda'})$. A useful property of the level function is that two local Eichler orders are $\mathrm{GL}_2(k_P)$-conjugate if and only if their local levels coincide. This property can be interpreted in terms of genera by saying that two Eichler $\mathcal{C}$-orders belong to the same genus precisely when they have the same global level. So, for any effective divisor $D$, there exists a genus of Eichler $\mathcal{C}$-orders of level $D$, which is denoted by $\mathbb{O}_D$.

It follows, from the characterization of the Bruhat-Tits tree in terms of maximal orders (cf. §2), that there exists a bijective map between the set of local Eichler orders $\mathfrak{E}$ of level $\kappa$ and the set of finite lines $\mathfrak{p}$ of length $\kappa$ in the Bruhat-Tits tree. Formally, a local Eichler order $\mathfrak{E}$ corresponds to the finite line $\mathfrak{p} = \mathfrak{s}(\mathfrak{E})$ whose vertices are the maximal orders containing $\mathfrak{E}$. Let $\mathfrak{E}$ be an Eichler $\mathcal{C}$-order of level $D = \sum_P \alpha_P P$. Let us denote by $S(\mathfrak{E})$ the product of finite lines $S(\mathfrak{E}) = \prod_P \mathfrak{s}(\mathfrak{E}_P)$, where $P$ runs over the set of places at which $\alpha_P > 0$. This is called the grid of $\mathfrak{E}$. It follows from Property (c) in §5 that the set of maximal $\mathcal{C}$-orders containing $\mathfrak{E}$ corresponds to the vertex set in $S(\mathfrak{E})$. Moreover, it is easy to see that this correspondence is compatible with the action of $\mathrm{PGL}_2(k)$ on Eichler $\mathcal{C}$-orders by conjugation. To compare different orders, we fix an effective divisor $D = \sum_P \alpha_P P$, and a finite set of places $T \supseteq \mathrm{Supp}(D)$. Denote by $\mathrm{Eich}(D,T)$ the set of Eichler $\mathcal{C}$-orders of level $D$ satisfying $\mathfrak{E}_Q = \mathbb{M}_2(\mathcal{O}_Q)$ for $Q \notin T$. Then, the grid corresponding to an Eichler $\mathcal{C}$-order in $\mathrm{Eich}(D,T)$ can be seen naturally as a subcomplex of the product of Bruhat-Tits trees $\prod_{P \in T} \mathfrak{t}(k_P)$. Any grid of the form $S(\mathfrak{E})$, for $\mathfrak{E} \in \mathrm{Eich}(D,T)$ is called a concrete $D$-grid. Note that the group $G_T = \mathrm{GL}_2(\mathcal{O}_{\mathcal{C}}(\mathcal{C} \smallsetminus T))$ acts on the set of concrete $D$-grids by conjugation. Indeed, we can define this action as the extension of the conjugacy action of $G_T$ on the set of maximal $\mathcal{C}$-orders to $D$-grids, which is valid since $G_T$ acts simplicially on each local tree. The orbits of concrete $D$-grids by this action are called abstract $D$-grids. Any representative of an abstract grid is called a concrete representative. Note that all these definitions depend on the set $T$. This is why it is important to consider the following result.

**Proposition 6.1.** [ABp, Proposition 3.1] Let $D$ be an effective divisor. Then, there exists a finite set of places $T$ containing $\mathrm{Supp}(D)$ such that every $\mathrm{PGL}_2(k)$-conjugacy class of Eichler $\mathcal{C}$-orders contains a representative in $\mathrm{Eich}(D,T)$.

Let $Q$ be a closed point of $\mathcal{C}$, and write $D = D' + \alpha_Q Q$, where $D'$ is supported away from $Q$. Then, a concrete $D$-grid $S(\mathfrak{E})$ is a paralellotope having two concrete $D'$-grids as opposite faces. These opposite faces are called the $Q$-faces of the $D$-grid $S(\mathfrak{E})$. We say that two concrete $D'$-grids $S$ and $S'$ are $Q$-neighbors if there exists a concrete $D$-grid $\tilde{S}$, with $D = D' + Q$ and $Q \notin \mathrm{Supp}(D')$, such that $S$ and $S'$ are the $Q$-faces of $\tilde{S}$. Let $\mathfrak{D}$ be a maximal $\mathcal{C}$-order corresponding to a vertex $v$ in $S$. Then, there exists one and only one $Q$-neighbor $v'$ among the vertices of $S'$. We call it the $Q$-neighbor of $v$ in $\tilde{S}$.

As an intermediate step to prove Theorem 1.4, we characterize a quotient graph of $\mathfrak{t}$ other than $\mathfrak{t}_D = \mathrm{H}_D\backslash\mathfrak{t}$. In order to introduce this quotient structure, fix $D$ an effective divisor, and let $\mathbb{O}_D$ be the genus containing all Eichler $\mathcal{C}$-orders of level $D$. Let $Q \in |\mathcal{C}|$ be a closed point not contained in $\mathrm{Supp}(D)$. Let $V_0$ be the affine open set $\mathcal{C} \smallsetminus \{Q\}$. Then, any order in $\mathbb{O}_D$ is maximal at $Q$, i.e. its completion at $Q$ is maximal. For any $\mathfrak{E} \in \mathbb{O}_D$, we define the C-graph $C_Q(\mathfrak{E}) = \Gamma\backslash\mathfrak{t}$, where $\mathfrak{t} = \mathfrak{t}(k_Q)$, and $\Gamma$ is the stabilizer of $\mathfrak{E}(V_0)$ in $\mathrm{PGL}_2(k)$. Note that, it follows from Corollary 3.8 that $C_Q(\mathfrak{D})$ is combinatorially finite. Two Eichler $\mathcal{C}$-orders $\mathfrak{E}$ and $\mathfrak{E}'$ such that $\rho(\mathfrak{E},\mathfrak{E}')$ belongs to the group generated by $[[Q,\Sigma(\mathbb{O}_D)/k]]$ define isomorphic quotient graphs. Indeed, it follows from [A12, §2] that, if $\rho(\mathfrak{E},\mathfrak{E}') \in \langle[[Q,\Sigma(\mathbb{O}_D)/k]]\rangle$, then $\mathfrak{E}(U_0)$ and $\mathfrak{E}'(U_0)$ are $\mathrm{GL}_2(k)$-conjugate. In this case we write $\mathfrak{E} \sim \mathfrak{E}'$. We denote by $\mathfrak{Sp}(\mathbb{O}_D, Q)$ the quotient set of $\mathbb{O}_D$ by the previous equivalence relation. The classifying graph $C_Q(\mathbb{O}_D)$ is the disjoint union of the finitely many C-graphs corresponding to all elements in $\mathfrak{Sp}(\mathbb{O}_D, Q)$. In particular, it is combinatorially finite.

All definitions and conventions introduced in §2 apply to $C_Q(\mathbb{O}_D)$ by adapting them to the context of disjoint union of graphs. In particular, by the cusp set of $C_Q(\mathbb{O}_D)$ we mean the disjoint union of the cusp sets of all connected components of $C_Q(\mathbb{O}_D)$. In the following section we study the combinatorial structure of the classifying graphs of Eichler orders. With this in mind, we make frequent use of the next result:

**Proposition 6.2.** [ABp, Proposition 3.2] Let $D$ be an effective divisor supported away from the place $Q$. The vertices of the classifying graph $C_Q(\mathbb{O}_D)$ are in bijection with the abstract $D$-grids, while its pairs of mutually reverse edges are in bijection with the abstract $(D+Q)$-grids. The endpoints of an edge are the vertices of $C_Q(\mathbb{O}_D)$ corresponding to the $Q$-faces of the grid corresponding to that edge.

We finish this section by recalling some results about the spinor class field associated to the genus of Eichler $\mathcal{C}$-orders of level $D$. As in §1 and §5, let $U_0 = \mathcal{C}\backslash\{P_\infty\}$ be an affine open set. Let $D$ be a divisor, which we write as $D = \sum_{i=1}^r n_i P_i$, where $P_i \neq P_\infty$. The spinor class field $\Sigma_D = \Sigma(\mathbb{O}_D)$ (resp. $\Sigma(\mathbb{O}_D, U_0)$), for Eichler $\mathcal{C}$-orders of level $D$, is the maximal subfield of $\Sigma_0 = \Sigma(\mathbb{O}_0)$ (resp. $\Sigma(\mathbb{O}_0, U_0)$) splitting at every place $P_i$ for which $n_i$ is odd. See [A13, Theorem 1.2] for more details.

**Proposition 6.3.** Let $J = \{i : n_i \text{ is odd}\}$. The Galois group $\mathrm{Gal}(\Sigma_D/k)$ is isomorphic to the abelian group $\mathrm{Pic}(\mathcal{C})/(2\mathrm{Pic}(\mathcal{C}) + \langle \overline{P_j} : j \in J\rangle)$. Using the same notation, $\mathrm{Gal}(\Sigma(\mathbb{O}_D, U_0)/k)$ is isomorphic to $\mathrm{Pic}(\mathcal{C})/(2\mathrm{Pic}(\mathcal{C}) + \langle\overline{P_\infty}\rangle + \langle \overline{P_j} : j \in J\rangle)$.

*Proof.* Let $L/F$ be a finite abelian extension (i.e. Galois with abelian Galois group) of global fields. It follows from [Ne99, Chapter VI, §6, Theorem 6.1 and Corollary 6.6] that there exists an isomorphism from $\mathrm{Gal}(L/F)$ to $\mathbb{I}_F/F^*H(L)$, where $\mathbb{I}_F$ is the idèle group of $F$, and $H(L) := \{N_{L/F}(a) : a \in \mathbb{I}_L\}$ is the kernel of the Artin map, which satisfies the following properties:

(i) $Q$ is unramified in $L/F$ if and only if $\mathcal{O}_Q^* \subseteq F^*H(L)$.

(ii) $Q$ splits completely in $L/F$ if and only if $F_Q^* \subseteq F^*H(L)$.

Apply this when $F$ is the global function field $k = \mathbb{F}(\mathcal{C})$ and $L$ is $\Sigma_D$. Recall that $H(\Sigma_0)$ equals $H(\mathbb{M}_2(\mathcal{O}_\mathcal{C}), \mathcal{C})$ as in Equation (5.1). Then, since the localization of $\mathbb{M}_2(\mathcal{O}_\mathcal{C})$ at $Q$ is $\mathbb{M}_2(\mathcal{O}_Q)$, it is easy to see that $k_Q^{*2}\mathcal{O}_Q^* \subseteq H(\Sigma_0)$, for all closed points $Q \in \mathcal{C}$. In particular, since $\Sigma_D \subseteq \Sigma_0$, we obtain $k_Q^{*2}\mathcal{O}_Q^* \subseteq H(\Sigma_0) \subseteq H(\Sigma_D)$. So, if we write $\mathbb{I}_{k,\infty} := \prod_{Q \in \mathcal{C}} \mathcal{O}_Q^*$, then $\mathbb{I}_k^2\mathbb{I}_{k,\infty}$ is contained in $k^*H(\Sigma_D)$. And, since $\mathbb{I}_k/k^*\mathbb{I}_{k,\infty} \cong \mathrm{Pic}(\mathcal{C})$, the Galois group of $\Sigma_D/k$ is a quotient of $\mathrm{Pic}(\mathcal{C})/2\mathrm{Pic}(\mathcal{C})$.

Let us write $e(Q)$ for the idèle whose coordinate at $Q$ is $\pi_Q$ and any other coordinate equals one. Since $\Sigma_D/k$ splits at $P_j$, with $j \in J$, we deduce from (ii) that all $e(P_j)$, with $j \in J$, belong to $k^*H(\Sigma_D)$. In particular, we obtain the inclusion

$$k^*\mathbb{I}_k^2\mathbb{I}_{k,\infty}\langle e(P_j) : j \in J\rangle \subseteq k^*H(\Sigma_D).$$

Furthermore, the maximality condition on $\Sigma_D$ implies equality. We conclude that

$$\mathrm{Gal}(\Sigma_D/k) \cong \frac{\mathbb{I}_k}{k^*\mathbb{I}_k^2\mathbb{I}_{k,\infty}\langle e(P_j) : j \in J\rangle} \cong \frac{\mathrm{Pic}(\mathcal{C})}{2\mathrm{Pic}(\mathcal{C}) + \langle \overline{P_j} : j \in J\rangle}.$$

Moreover, we can analogously prove that $\mathfrak{G} = \mathrm{Gal}(\Sigma(\mathbb{O}_D, U_0)/k)$ is isomorphic to $\mathrm{Pic}(\mathcal{C})/(2\mathrm{Pic}(\mathcal{C}) + \langle \overline{P_\infty}, \overline{P_j} : j \in J\rangle)$, by noting that to compute $\mathfrak{G}$ we no longer need a condition at the place $P_\infty$, so the corresponding local stabilizer must be replaced by the full local group $\mathrm{GL}_2(k_{P_\infty})$. □

Moreover, it follows from [A16, Proposition 6.1] that the corresponding distance function $\rho_D$ on the genus of Eichler $\mathcal{C}$-orders of level $D$ is related to $\rho_0$ through restriction, i.e.

$$\rho_D(\mathfrak{E}_{\Lambda,\Lambda'}, \mathfrak{E}_{L,L'}) = \rho_0(\mathfrak{D}_\Lambda, \mathfrak{D}_L)\Big|_{\Sigma(D)}, \tag{6.2}$$

for any four $\mathcal{C}$-lattices $\Lambda, \Lambda', L$ and $L'$ (cf. §5).

## 7. On quotient graphs of Eichler groups

The objective of this section is to prove Theorem 1.4. To do so, we extensively use the following remark. As we said in §2, every subgroup of $\mathrm{GL}_2(k_{P_\infty})$ acts on $\mathfrak{t}$ via its image in $\mathrm{PGL}_2(k_{P_\infty})$. In particular, the topological space $\mathfrak{t}_D$ equals the quotient of $\mathfrak{t} = \mathfrak{t}(k_{P_\infty})$ by the projective image $\mathrm{PH}_D$ of

$$\mathrm{H}_D = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R) : c \equiv 0 \,(\mathrm{mod}\ I_D) \right\},$$

where $I_D$ is the $R$-ideal defined as $I_D = \mathfrak{L}^{-D}(U_0) = \mathfrak{L}^{-D}(\mathcal{C} \smallsetminus \{P_\infty\})$.

We start this section by presenting a proof of Theorem 1.4 assuming the following result, which is implied by Proposition 7.24 below.

**Proposition 7.1.** The number of cusps of any connected component of $C_{P_\infty}(\mathbb{O}_D)$ is the same, and it equals

$$c(D) = \alpha(D)[2\mathrm{Pic}(\mathcal{C}) + \langle \overline{P_{a_1}}, \cdots, \overline{P_{a_u}}, \overline{P_\infty} \rangle : \langle \overline{P_\infty} \rangle], \tag{7.1}$$

where

$$\alpha(D) = 1 + \frac{1}{q-1} \prod_{i=1}^{r} \left( q^{\deg(P_i) \lfloor \frac{n_i}{2} \rfloor} - 1 \right),$$

and $P_{a_1}, \cdots, P_{a_u}$ are the closed points in $\mathcal{C}$ whose coefficients in $D = \sum_{i=1}^{r} n_i P_i$ are odd.

7.1. **A proof of Theorem 1.4.** As we just said, we can replace $\mathrm{H}_D$ by its image $\mathrm{PH}_D$ in $\mathrm{PGL}_2(k)$ to compute the cusp number of $\mathfrak{t}_D$. First, we prove inequality (1.2). Set $\Gamma = \mathrm{Stab}_{\mathrm{PGL}_2(k)}(\mathfrak{E}_D(U_0))$. On one hand, it follows from Proposition 7.1 that the cusp number of $\Gamma \backslash \mathfrak{t}$ is equal to $c(D)$. On the other hand, it follows from [A16, Theorem 1.2] that

$$[\Gamma : \mathrm{PH}_D] = \frac{2^r |g(2)|}{[\Sigma(\mathbb{O}_0, U_0) : \Sigma(\mathbb{O}_D, U_0)]},$$

where $g(2)$ is the maximal exponent-2 subgroup of $\mathrm{Pic}(R)$. So, we obtain from Proposition 6.3,

$$[\Gamma : \mathrm{PH}_D] = \frac{2^r |g(2)|}{[2\mathrm{Pic}(\mathcal{C}) + \langle \overline{P_{a_1}}, \cdots, \overline{P_{a_u}}, \overline{P_\infty} \rangle : 2\mathrm{Pic}(\mathcal{C}) + \langle \overline{P_\infty} \rangle]}. \tag{7.2}$$

Recall now that, by Corollary 3.8, the set of cups of $\mathfrak{t}_D$ (resp. $\Gamma \backslash \mathfrak{t}$) is parametrized by $\mathbb{P}^1(k)/\mathrm{PH}_D$ (resp. $\mathbb{P}^1(k)/\Gamma$). Then, the cusp number of $\mathfrak{t}_D$ cannot exceed $c(\mathrm{H}_D) = c(D)[\Gamma : \mathrm{PH}_D]$ and inequality (1.2) follows. Now, we assume that each $n_i$ is odd and $g(2)$ is trivial. Then, we have to prove that the cusp number of $\mathfrak{t}_D$ is exactly $c(\mathrm{H}_D)$. This is a consequence of the following lemma.

**Lemma 7.2.** Assume that each $n_i$ is odd and that $g(2)$ is trivial. Then, there are exactly $[\Gamma : \mathrm{PH}_D]$ cusps in $\mathfrak{t}_D$ with the same image in $\Gamma \backslash \mathfrak{t}$.

*Proof.* Let $\Theta = \Theta(\eta)$ be the set of cusps of $\mathfrak{t}_D$ whose image in $\Gamma \backslash \mathfrak{t}$ is the cusp $\eta$. Then, $\mathrm{Card}(\Theta)$ is strictly less than $[\Gamma : \mathrm{PH}_D]$ precisely when there exists an element $\overline{g} \in \Gamma/\mathrm{PH}_D$ stabilizing an element of $\Theta$. Since the set of cups of $\mathfrak{t}_D$ (resp. $\Gamma \backslash \mathfrak{t}$) is

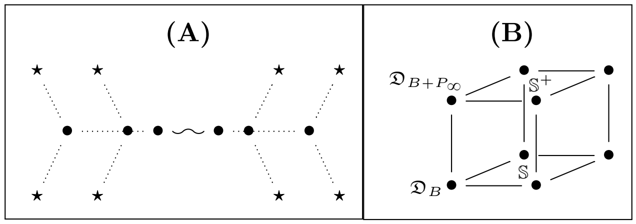


FIGURE 1. Figure **(A)** shows the Bruhat-Tits tree $\mathfrak{t}(k_{P_i})$, where $\mathfrak{p}_i$ corresponds to the finite central line and where the middle edge of $\mathfrak{p}_i$ is represented by a cursive edge. Figure **(B)** shows a concrete $(D + P_\infty)$-grid, or equivalently two $P_\infty$-neighboring $D$-grids.

parametrized by $\mathbb{P}^1(k)/\mathrm{PH}_D$ (resp. $\mathbb{P}^1(k)/\Gamma$), if we prove that, for any $s \in \mathbb{P}^1(k)$, we have

$$\mathrm{Stab}_\Gamma(s) \subset \mathrm{PH}_D, \tag{7.3}$$

then, the result follows. Let $g \in \Gamma$ and assume that $g$ stabilizes some class of rays corresponding to $s \in \mathbb{P}^1(k)$. By definition, $g\mathfrak{E}_D(U_0)g^{-1} = \mathfrak{E}_D(U_0)$. So, as we saw in §6, $g$ acts on the concrete $D$-grid $S_D$ associated to $\mathfrak{E}_D$ as an automorphism. In particular, we have

(1) $g \in \mathrm{Stab}((\mathfrak{D}_0)_Q)$ for every $Q \neq P_1, \cdots, P_r, P_\infty$, and
(2) $g(\mathfrak{D}_0)_{P_i} g^{-1} = (\mathfrak{D}_{\epsilon_i D})_{P_i}$ with $\epsilon_i \in \{0,1\}$ for any $P_i$ in the support of $D$, i.e., $g$ can either pointwise fix the line in $\mathfrak{t}(k_{P_i})$ joining $(\mathfrak{D}_0)_{P_i}$ with $(\mathfrak{D}_D)_{P_i}$, or flip it.

If some $\epsilon_i = 1$, then $g$ acts on $\mathfrak{t}_i = \mathfrak{t}(k_{P_i})$ without fixing any point of the finite path $\mathfrak{p}_i = \mathfrak{s}(\mathfrak{E}(U_0)_{P_i})$ whose length is odd. Let $\mathfrak{t}_i^d$ be the topological space obtained from $\mathfrak{t}_i$ by removing the central edge of $\mathfrak{p}_i$. Then, the action of $g$ on $\mathfrak{t}_i$ exchanges the two connected components of $\mathfrak{t}_i^d$. See Figure 1**(A)**. We conclude that $g$ fixes no visual limit in $\mathfrak{t}_i$, whence it fixes no element in $\mathbb{P}^1(k)$, which contradicts the hypothesis on $g$. On the other hand, if every $\epsilon_i = 0$, we have

$$g \in \Gamma_0 = \mathrm{Stab}_{\mathrm{PGL}_2(k)}(\mathfrak{D}_0(U_0)) \cap \mathrm{Stab}_{\mathrm{PGL}_2(k)}(\mathfrak{D}_D(U_0)) = \mathrm{Stab}_{\mathrm{PGL}_2(k)}(\mathfrak{E}_D(U_0)).$$

We claim that $\Gamma_0/\mathrm{PH}_D \hookrightarrow g(2)$, and since $g(2)$ is trivial, we get $g \in \mathrm{PH}_D$, which concludes the proof.

To prove the claim, we follow [A16, Theorem 1.2]. For $E \in \{0, D\}$ we denote by $\Lambda_E$ the $\mathcal{O}_\mathcal{C}(U_0)$-lattice satisfying $\mathfrak{D}_E(U_0) = \mathrm{End}_{\mathcal{O}_\mathcal{C}(U_0)}(\Lambda_E)$. Let $h \in \Gamma_0$ be an arbitrary element, and fix $h_0 \in \mathrm{GL}_2(k)$ a lift of $h$. Then, by definition, we get $h_0 \in \mathrm{Stab}_{\mathrm{GL}_2(k)}(\mathfrak{D}_E(U_0))$, for $E \in \{0, D\}$. Then, there exists $b_E \in \mathbb{I}_{U_0}$ such that $h_0\Lambda_E = b_E\Lambda_E$ (recall §5). By taking determinant in the preceding equality, we deduce that $b_E^2\mathcal{O}_\mathcal{C}(U_0) = \det(h_0)\mathcal{O}_\mathcal{C}(U_0)$, whence we deduce $b := b_0 = b_D$, since $\mathcal{O}_\mathcal{C}(U_0)$ is a Dedekind domain. Recall that $\mathrm{Pic}(A)$ is isomorphic to the ideal class group $\mathbb{I}_{U_0}/(k^* \prod_{Q \in |U_0|} \mathcal{O}_Q^*)$. Moreover, note that the class $[b]$ of $b = b(h_0)$ in $\mathrm{Pic}(A)$ only depends on the class $h = [h_0] \in \mathrm{PGL}_2(k)$. Indeed, if we change the representative $h_0$ of $h$ by $\lambda h_0$, then we obtain $[b(\lambda h_0)] = [b(h_0) \cdot \mathrm{div}(\lambda)] = [b(h_0)]$. In all that follows we denote by $[b]$ the class of any $b = b(h_0)$, which only depends on $h$. Let us define $\Xi : \Gamma_0 \to \mathrm{Pic}(A)$ as the function satisfying $\Xi(h) = [b] \in \mathrm{Pic}(A)$. On one hand, note that $2\Xi(h) = [b^2] = [\mathrm{div}(\det(h_0))] = 0$. In particular, we have

$\mathrm{Im}(\Xi) \subseteq g(2)$. On the other hand, if $\Xi(h) = 0$, then $b = \mathrm{div}(\lambda)$, for some $\lambda \in k^*$. This implies that $h_0\lambda^{-1} \in \mathrm{Aut}(\Lambda_E) = \mathfrak{D}_E(U_0)^*$, for all $E \in \{0, D\}$. Thus, we get $h_0\lambda^{-1} \in \mathfrak{E}_D(U_0)^*$, whence $h \in \mathrm{PH}_D$. Hence, we conclude $\Gamma_0/\mathrm{PH}_D$ injects into $g(2)$. □

This concludes the proof of Theorem 1.4. In the remainder of this section, we prove Proposition 7.24, which is a stronger version of Proposition 7.1. In §7.2, we study vertices in the classification graph $C_{P_\infty}(\mathbb{O}_D)$. To do so, we present the concept of *semi-decomposition datum* of $D$-grids. Then, in §7.3, we analyze the topological structure of the classification graph. Specifically, we define and characterize a ramified covering $C_{P_\infty}(\mathbb{O}_D) \to C_{P_\infty}(\mathbb{O}_0)$ via the characterization of vertices in $C_{P_\infty}(\mathbb{O}_D)$ obtained in §7.2. Finally, we use what is known about the classifying graph $C_{P_\infty}(\mathbb{O}_0)$, which is summarized in the following result.

**Theorem 7.3.** [A14, Theorem 1.2] The classifying graph $C_{P_\infty}(\mathbb{O}_0)$ is combinatorially finite, and it has exactly $\mathrm{Pic}(R) \cong \mathrm{Pic}(\mathcal{C})/\langle P\rangle$ cuspidal rays. The vertices corresponding to conjugacy classes of the form $[\mathfrak{D}_D]$, for some divisor $D$ on $\mathcal{C}$, are located in the cuspidal rays of $C_{P_\infty}(\mathbb{O}_0)$. Conversely, almost every vertex in a cuspidal ray $\mathfrak{r}_\sigma$ of $C_{P_\infty}(\mathbb{O}_0)$, with $\sigma \in \mathrm{Pic}(A)$, corresponds to a $\mathrm{GL}_2(k)$-conjugacy class of the form $[\mathfrak{D}_{B+nP_\infty}]$, where $B = B(\sigma)$ depends only on $\sigma$.

In Theorem 7.3 and hereafter, by "almost every" we mean all but finitely many.

7.2. **On the decomposition of grids.** Here the main goal is to establish and prove a "decomposition criterion" for grids, and subsequently for Eichler orders.

We fix the following notation for the rest of this section. Let $D$ be an effective divisor, and write $D = \sum_{i=1}^r n_i P_i$ where the points $P_1, \cdots, P_r, P_\infty$ are all different. Denote by $d_D$ the degree of $D$. Using Proposition 6.1 we fix a finite set of places $T$ such that every every $\mathrm{GL}_2(k)$-conjugacy class of Eichler $\mathcal{C}$-orders contains a representative in $\mathrm{Eich}(D, T)$.

For any pair of divisors $(B, B')$ such that $B + B'$ is effective, consider the Eichler $\mathcal{C}$-order

$$\mathfrak{E}[B, B'] := \begin{pmatrix} \mathcal{O}_{\mathcal{C}} & \mathfrak{L}^{-B'} \\ \mathfrak{L}^{-B} & \mathcal{O}_{\mathcal{C}} \end{pmatrix},$$

whose level is $B + B'$. Any $\mathrm{GL}_2(k)$-conjugate of such an order is called split. We denote an abstract grid by $\mathbb{S}$, and we often choose a representative of this class by writing $S \in \mathbb{S}$, or any verbal analog.

**Definition 7.4.** For any basis $\beta \subset k^2$ we denote by $A(\beta)$ the matrix whose columns are the vectors in $\beta$. A maximal $\mathcal{C}$-order $\mathfrak{D}$ is called $\beta$-split if $A(\beta)\mathfrak{D}A(\beta)^{-1} = \mathfrak{D}_E$, for some divisor $E$. We say that a $D$-grid $S$ is $\beta$-split if every vertex $S$ is $\beta$-split as an order. This is equivalent to

$$A(\beta)SA(\beta)^{-1} = S\left(\mathfrak{E}\left[B, B + D\right]\right), \tag{7.4}$$

for some divisor $B$. A corner of a $D$-grid $S$ is a vertex of $S$ having a unique $P$-neighbor, for each $P$ in the support of $D$. Let $D' \leqslant D$ be an effective divisor. A $D'$-corner of $S$ is a $D'$-grid $S' \subset S$ containing a corner of $S$.

**Definition 7.5.** A semi-decomposition datum of $S$ is a 3-tuple $(\beta, B, D')$, where:

(a) $\beta$ is a basis of $k^2$,
(b) $B$ and $D'$ are two divisors on $\mathcal{C}$ satisfying $2D \geqslant 2D' \geqslant D$,
(c) there exists a corner of $S$ of the form $v_0 = A(\beta)^{-1}\mathfrak{D}_B A(\beta)$,

(d) there is a $\beta$-split $D'$-corner $\Pi_{\mathrm{SD}} \subseteq S$ whose set of vertices is

$$\mathrm{V}(\Pi_{\mathrm{SD}}) = \left\{ A(\beta)^{-1}\mathfrak{D}_E A(\beta) : B \leqslant E \leqslant B + D' \right\},$$

(e) no vertex outside $\Pi_{\mathrm{SD}}$ is $\beta$-split.

If $D' = D$, then $(\beta, B, D')$ is called a total decomposition datum. The basis $\beta$ is called the semi-decomposition basis of $S$. The subgrid $\Pi_{\mathrm{SD}}$ is called the decomposed subgrid of $S$ associated to the datum. The degree of a semi-decomposition datum $(\beta, B, D')$ is the degree of $B$.

**Example 7.6.** Assume that $D$ is multiplicity free. Then, condition (b) in Definition 7.5 implies that any semi-decomposition datum of a concrete $D$-grid is a total decomposition datum.

Note that the pair of divisors $(B, D')$ in the previous definition depends only on the $\mathrm{GL}_2(k)$-conjugacy class of $D$-grids. Indeed, if $(\beta, B, D')$ is a semi-decomposition datum of $S$, and $S = GS'G^{-1}$ with $G \in \mathrm{GL}_2(k)$, then $(\beta', B, D')$ is a semi-decomposition datum of $S'$, where $\beta' = G(\beta)$. Furthermore, we have $A(\beta') = GA(\beta)$. This allows us to extend the definition of semi-decomposition data to abstract $D$-grids. However, in order to (partially) extend the notion of degree, we need the following result.

**Lemma 7.7.** Let $S$ be a concrete $D$-grid. Let $(\beta, B, D')$ and $(\beta^\circ, B^\circ, D^{\circ'})$ be two semi-decomposition data of $S$ with positive degree. Then $B$ and $B^\circ$ are linearly equivalent.

*Proof.* Set $A = A(\beta)$, $A^\circ = A(\beta^\circ)$, and let $G$ be the base change matrix from $\beta$ to $\beta^\circ$. We start by showing that we can restrict our proof to the context where $D$ is multiplicity free. Indeed, $A^\circ S (A^\circ)^{-1}$ is the image of the concrete $D$-grid $ASA^{-1}$ by the conjugation map induced by $G$. Let us write $D = \sum_{i=1}^r n_i P_i$, and let $\{P_{b_1}, \cdots, P_{b_s}\}$ be the set of such points with an odd coefficient $n_i$. Set $D_c = P_{b_1} + \cdots + P_{b_s}$. Let $\Delta$ be the subcomplex of $S$ whose vertex set is

$$\mathrm{V}(\Delta) = \left\{ A^{-1}\mathfrak{D}_E A : B_0 \leqslant E \leqslant B_0 + D_c \right\},$$

where $B_0 = B + \sum_{i=1}^r \lfloor \frac{n_i}{2} \rfloor P_i$. Note that the intersection of the maximal $\mathcal{C}$-orders corresponding to the vertex set of $\Delta$ is an Eichler $\mathcal{C}$-order of level $D_c$. Equivalently, we get that $\Delta$ is a concrete $D_c$-grid. Since $\mathrm{PGL}_2(k)$ acts simplicially on each grid, we get $GA\Delta A^{-1}G^{-1} = A^\circ \Delta (A^\circ)^{-1}$. Moreover, we have that

$$\mathrm{V}(\Delta) = \left\{ (A^\circ)^{-1}\mathfrak{D}_E A^\circ : B_0^\circ \leqslant E \leqslant B_0^\circ + D_c \right\},$$

where $B_0^\circ = B^\circ + \sum_{i=1}^r \lfloor \frac{n_i}{2} \rfloor P_i$. Thus, we conclude that the $D_c$-grid $\Delta$ has two induced positive degree total-decomposition data $(\beta, B_0, D_c)$ and $(\beta^\circ, B_0^\circ, D_c)$. Note that if $B_0$ and $B_0^\circ$ are linearly equivalent, then $B$ and $B^\circ$ are also. Therefore, by replacing $S$ by $\Delta$, we can assume that $D$ is multiplicity free.

Now, assume that $D$ is multiplicity free. In this case, every vertex in $ASA^{-1}$ and $A^\circ S (A^\circ)^{-1}$ correspond to a split maximal $\mathcal{C}$-order (cf. Example 7.6). Set $\mathfrak{D}_{B''} = G\mathfrak{D}_B G^{-1}$. Then one of the following holds:

(1) $B''$ is principal,
(2) $\mathfrak{L}^{-B''}(\mathcal{C}) = \{0\}$, or
(3) $\mathfrak{L}^{B''}(\mathcal{C}) = \{0\}$.

If $B''$ is principal, then $\mathfrak{D}_{B''}(\mathcal{C}) = \mathbb{M}_2(\mathbb{F})$. On the other hand, since $\deg(B) > 0$, we obtain $\mathfrak{L}^{-B}(\mathcal{C}) = \{0\}$. Thus, we conclude $\mathfrak{D}_B(\mathcal{C}) = \left(\begin{smallmatrix} \mathbb{F}^* & \mathfrak{L}^B(\mathcal{C}) \\ 0 & \mathbb{F}^* \end{smallmatrix}\right)$, which is not a simple algebra. So, the first case is impossible.

Now, assume the second case, i.e. $\mathfrak{L}^{-B}(\mathcal{C}) = \mathfrak{L}^{-B''}(\mathcal{C}) = \{0\}$. Then, it follows from [A14, §4, Proposition 4.1] that one of the following conditions holds:

(a) $G = \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$ and $B - B'' = \mathrm{div}(x^{-1}z)$, or

(b) $G = \left(\begin{smallmatrix} 0 & x \\ z & 0 \end{smallmatrix}\right)$ and $B + B'' = \mathrm{div}(x^{-1}z)$.

In case (a), for any divisor $0 \leqslant E \leqslant D$ we have that

$$G\mathfrak{D}_{B+E}G^{-1} \subseteq \begin{pmatrix} \mathcal{O}_{\mathcal{C}} - yz^{-1}\mathfrak{L}^{-B-E} & \mathfrak{J} \\ xz^{-1}\mathfrak{L}^{-B-E} & \mathcal{O}_{\mathcal{C}} + yz^{-1}\mathfrak{L}^{-B-E} \end{pmatrix}, \tag{7.5}$$

for some invertible sheaf $\mathfrak{J}$, where the other coefficients are optimal. Since

$$xz^{-1}\mathfrak{L}^{-B-E} = \mathfrak{L}^{-B-\mathrm{div}(xz^{-1})-E} = \mathfrak{L}^{-B''-E},$$

and $G\mathfrak{D}_{B+E}G^{-1}$ is a split maximal $\mathcal{C}$-order, we deduce that $G\mathfrak{D}_{B+E}G^{-1} = \mathfrak{D}_{B''+E}$, for any divisor $0 \leqslant E \leqslant D$. This implies that $B'' = B^\circ$, and then $B$ and $B^\circ$ are linearly equivalent.

In case (b), for any divisor $0 \leqslant E \leqslant D$ we have that

$$G\mathfrak{D}_{B+E}G^{-1} = \begin{pmatrix} \mathcal{O}_{\mathcal{C}} & xz^{-1}\mathfrak{L}^{-B-E} \\ zx^{-1}\mathfrak{L}^{B+E} & \mathcal{O}_{\mathcal{C}} \end{pmatrix}. \tag{7.6}$$

Then, it follows directly from the previous equation that $G\mathfrak{D}_{B+E}G^{-1} = \mathfrak{D}_{B''-E}$, for any divisor $0 \leqslant E \leqslant D$. Therefore $B'' = B^\circ + D$, whence $B$ is linearly equivalent to $-B^\circ - D$. But, the last condition contradicts the hypotheses of positive degree on $B$ and $B^\circ$. We conclude that only case (a) can hold.

Finally, if $\mathfrak{L}^{B''} = \{0\}$ we can replace $B''$ by $-B''$ in the preceding argument. $\square$

**Definition 7.8.** Let $\mathbb{S}$ be an abstract $D$-grid. A semi-decomposition datum of $\mathbb{S}$ is a pair $(B, D')$, where $(\beta, B, D')$ is a semi-decomposition datum of some concrete representative $S \in \mathbb{S}$. When $D' = D$, we say that $(B, D')$ is a total decomposition datum of $\mathbb{S}$. When $\deg(B) > 0$, the degree of this datum is by definition $\deg(B)$, which is well-defined by Lemma 7.7.

Let $\beta = \{e_1, e_2\}$ be a basis of $k^2$. We say that a two-dimensional vector bundle $\mathfrak{L}$ on $\mathcal{C}$ is $\beta$-split if $\mathfrak{L} = \mathfrak{L}^B e_1 \oplus \mathfrak{L}^C e_2$, where $B$ and $C$ are divisors on $\mathcal{C}$. Then, a maximal $\mathcal{C}$-order $\mathfrak{D}_\Lambda$ splits in the base $\beta$ if and only if at least one (and therefore every) vector bundle in the class $[\Lambda]$ is $\beta$-split. Moreover, either condition is equivalent to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{D}_\Lambda(\mathcal{C})$. More generally, an Eichler $\mathcal{C}$-order is split precisely when it contains a non-trivial idempotent as a global section, or equivalently, when the corresponding grid has a total-decomposition datum. In fact, we have a more precise result that follows immediately from the current paragraph and [ABp, Theorem 1.2]:

**Proposition 7.9.** Let $\mathfrak{E}$ be an Eichler $\mathcal{C}$-order of level $D$. Then the following statements are equivalent:

(1) $\mathfrak{E}$ is split,
(2) $S = S(\mathfrak{E})$ has a total-decomposition datum,
(3) The ring of global sections $\mathfrak{E}(\mathcal{C}) \subseteq \mathbb{M}_2(k)$ contains a non trivial idempotent matrix,

(4) There exists a nontrivial idempotent matrix of $\mathbb{M}_2(k)$ contained in the ring of global sections $\mathfrak{D}(\mathcal{C})$ for every maximal $\mathcal{C}$-order $\mathfrak{D}$ corresponding to a vertex of $S$.

Assume moreover that $D$ is multiplicity free. Then, for almost all conjugacy classes in $\mathbb{O}_D$, the orders in this class are split.

**Example 7.10.** Assume that $D = 0$. Let $Q$ be a closed point on $\mathcal{C}$, and set $R' = \mathcal{O}_{\mathcal{C}}(\mathcal{C} \smallsetminus \{Q\})$. Then, each concrete $D$-grid consists in precisely one vertex, which represents a maximal $\mathcal{C}$-order. According to Theorem 7.3, almost all abstract 0-grids admit a total decomposition datum, and these 0-grids correspond to vertices located in a finite union of rays in $C_Q(\mathbb{O}_0)$, which are parametrized by $\mathrm{Pic}(R') \cong \mathrm{Pic}(\mathcal{C})/\langle\overline{Q}\rangle$. Moreover, almost every vertex in a cuspidal ray $\mathfrak{r}_\sigma$ of $C_{P_\infty}(\mathbb{O}_0)$, with $\sigma \in \mathrm{Pic}(A)$, corresponds to a $\mathrm{GL}_2(k)$-conjugacy class of the form $[\mathfrak{D}_{B+nP_\infty}]$, where $B = B(\sigma)$ depends only on $\sigma$. In particular, given a divisor $D_0$, almost all classes of maximal $\mathcal{C}$-orders have a representative of the form $\mathfrak{D}_E$ with $\deg(E) > \deg(D_0)$.

The rest of this sub-section is exclusively devoted to proving the following proposition, which generalizes the previous example.

**Proposition 7.11.** Let $D$ be an effective divisor and $d_D = \deg(D)$. Then, for almost every abstract $D$-grid $\mathbb{S}$ there exists a semi-decomposition datum $(B, D')$ of degree $d$ for $\mathbb{S}$ with $d > d_D$. Moreover, $D'$ is unique and the class of $B$ in $\mathrm{Pic}(\mathcal{C})$ is unique. In particular, $d$ is unique.

In order to prove Proposition 7.11 we extensively work with some subgrids defined by a certain stratification, which we formalize in Definition 7.12 and Lemma 7.13.

**Definition 7.12.** Let $S$ be a concrete $D$-grid.

We define the $P_i$-axis $S(P_i) \subseteq S$ as the finite line in $S$ whose vertex set is $\{v_j\}_{j=0}^{n_i}$, where $v_0$ is as in Definition 7.5, and where $v_j$ is $P_i$-neighbor of $v_{j+1}$, whenever $0 \leqslant j \leqslant n_{i-1}$.

Write $D = D_0 + nQ$, with $D_0$ supported away from $Q$ and $n > 0$. Then, the vertex set of $S$ can be naturally written as the disjoint union of the vertex sets of $n+1$ different $D_0$-grids, denoted by $S_0, \cdots, S_n$. We do this in a way such that $S_0$ is a $Q$-face of $S$, and $S_i$ is a $Q$-neighbor of $S_{i+1}$, for each $i \in \{0, \cdots, n-1\}$. These $D_0$-grids are called the $Q$-strata of $S$. See Figure 2**(B)**. The numbering of the strata can be inverted if necessary.

The sequence of strata $\{S_0, \cdots, S_n\}$ defines a finite line in $\mathfrak{t}(k_Q)$ of length $n$. The following lemma characterizes, for almost every grid, the image of this line in the classifying graph.

**Lemma 7.13.** Let us write $D = D_0 + nQ$ as above. Then, for almost every abstract $D$-grid $\mathbb{S}$, there exists $S \in \mathbb{S}$ such that the corresponding line $\mathfrak{c}(S)$ in $\mathfrak{t}(k_Q)$ is defined by vertices $z_0, \cdots, z_s, \zeta_{s+1}, \cdots \zeta_n \in \mathfrak{t}(k_Q)$ satisfying:

(i) vertices in each pair $(z_i,\, z_{i+1})$, $(\zeta_i,\, \zeta_{i+1})$, or $(z_s,\, \zeta_{s+1})$ are neighbors,
(ii) $s > \lfloor \frac{n+1}{2} \rfloor$,
(iii) $z_0, \cdots, z_s$ are pairwise non-$\Gamma$-equivalent vertices,
(iv) $z_0, \cdots, z_s$ are on the maximal path joining $\infty$ with some $\epsilon \in k$, and
(v) $\zeta_{s+i}$ and $z_{s-i}$ are $\Gamma$-equivalent, for any $i \in \{1, \cdots, s\}$.

In particular, the image $\mathfrak{c}(\mathbb{S})$ of $\mathfrak{c}(S)$ in $C_Q(\mathbb{O}_{D_0})$ is a line of length $s$ contained in a cuspidal ray.

*Proof.* Let $\mathbb{S}$ be an abstract $D$-grid, and let $S \in \mathbb{S}$ be a concrete representative. Let $\mathfrak{c} = \mathfrak{c}(S)$ be the finite line in $\mathfrak{t}(k_Q)$ corresponding to $S$. The image $\mathfrak{c}(\mathbb{S})$ of $\mathfrak{c}$ in $C_Q(\mathbb{O}_{D_0})$ is a line of length $s \leqslant n$, which only depends on $\mathbb{S}$ by Proposition 6.2.

Now, as we noted in §6 (see the paragraphs before Proposition 6.2) the graph $C_Q(\mathbb{O}_{D_0})$ is combinatorially finite. Thus, there exist finitely many lines of length at most $n$ that are not contained in a cuspidal ray. Hence, we can assume that $\mathfrak{c}(\mathbb{S})$ is contained in a cuspidal ray $\mathfrak{r}$ of $C_Q(\mathbb{O}_{D_0})$. Let $\mathfrak{X} = \Gamma\backslash\mathfrak{t}(k_Q)$ be the connected component in $C_Q(\mathbb{O}_{D_0})$ containing $\mathfrak{r}$. Let $T$ be a maximal subtree of $\mathfrak{X}$, let $j : T \to \mathfrak{t}(k_Q)$ be a lift of $T$, and let $\pi : \mathfrak{t}(k_Q) \to \mathfrak{X}$ be the canonical projection. By Corollary 3.8 we may assume that the visual limit $\epsilon = \epsilon(\mathfrak{r})$ of $j(\mathfrak{r})$ belongs to $\mathbb{P}^1(k)$, and, up to changing the lift, we can assume $\epsilon \neq \infty$. Moreover, there exists a subray $\mathfrak{r}_0 \subset j(\mathfrak{r})$ such that:

- $\mathrm{V}(\mathfrak{r}_0) = \{w_i\}_{i=0}^{\infty}$, where $w_i$ and $w_{i+1}$ are adjacent,
- $\mathrm{Stab}_\Gamma(w_i) \subset \mathrm{Stab}_\Gamma(w_{i+1})$, and
- $\mathrm{Stab}_\Gamma(w_i)$ acts transitively on the set of neighboring vertices of $w_i$ other than $w_{i+1}$.

Note that the last statement implies that all neighbors of $w_i$, besides $w_{i+1}$, are in the same $\Gamma$-orbit. By induction on $L \geqslant 1$, we can show that, for any $i \geqslant 0$ and for any vertex $v$ at distance $\leqslant L$ of $w_{i+L}$ such that the line connecting $v$ and $w_{i+L}$ does not contain $w_{i+L+1}$, $v$ is in the same $\mathrm{Stab}_\Gamma(w_{i+L})$-orbit than $w_i \in V(\mathfrak{r}_0)$.

Define $\mathfrak{r}_0'$ as the subray of $\mathfrak{r}_0$ with vertices are $\{w_i\}_{i=n}^{\infty}$. In particular, this last statement applies to $L < n$ and every vertex in $\mathfrak{r}_0'$. Since $\mathfrak{r} \smallsetminus \pi(\mathfrak{r}_0')$ is a finite line, arguing as above, we may assume that $\pi(\mathfrak{c})$ is contained in $\pi(\mathfrak{r}_0')$. This implies that $\mathfrak{c}$ is $\Gamma$-equivalent to a finite line that intersects $\mathfrak{r}_0'$ in a finite line of length $s \leqslant n$ (recall that $\mathfrak{t}(k_Q)$ is a tree), so we may assume that this is the case for $\mathfrak{c}$. Let us write $\mathrm{V}(\mathfrak{c}) = \{\varpi_i\}_{i=0}^{n}$, where $\varpi_i$ and $\varpi_{i+1}$ are adjacent, and $\mathrm{V}(\mathfrak{c} \cap \mathfrak{r}_0') = \{\varpi_i\}_{i=t}^{s+t+1}$, where $\varpi_{s+t+1}$ is the vertex that is the closest to $\epsilon$. In particular, there exists $k > n$ such that, for each $i \in \{t, \cdots, s+t+1\}$, the vertex $\varpi_i$ equals $w_{i+k}$. Since $t < n$ and $w_k$ and $\varpi_0$ are both at distance $t$ from $\varpi_t = w_{t+k}$, we see that $w_k$ is $\mathrm{Stab}_\Gamma(\varpi_t)$-equivalent to $\varpi_0$. Then, up to replacing $\mathfrak{c}$ by a $\Gamma$-equivalent line, we may assume that $t = 0$.

We claim that we can assume that $s + 1 \geqslant \lfloor \frac{n+1}{2} \rfloor$. Indeed, if $s + 1 < \lfloor \frac{n+1}{2} \rfloor$ we argue as follows. Since $n - s - 1 < n$ and $w_{2s+2-n+k}$ and $\varpi_n$ are both at distance $n - s - 1$ from $\varpi_{s+1} = w_{s+1+k}$, we note that $w_{2s+2-n+k}$ is $\mathrm{Stab}_\Gamma(\varpi_{s+1})$-equivalent to $\varpi_n$. Equivalently, there exists $\gamma \in \Gamma$ such that $\gamma \cdot \{\varpi_i\}_{i=s+1}^{n} \subset \mathfrak{r}_0'$, and we may replace $\mathfrak{c}$ by $\gamma \cdot \mathfrak{c}$.

Write $V(\mathfrak{c}) = \{z_0, \cdots, z_s, \zeta_{s+1}, \cdots \zeta_n\}$, which satisfies conditions (i), (ii) in Lemma 7.13. Condition (iii) follows from the fact that the image of $\pi(\mathfrak{c}) = \mathfrak{c}(\mathbb{S})$ belongs to a cuspidal ray. Condition (iv) is immediate since we can always extend a ray to a maximal path reaching infinity. Finally, condition (v) follows by the same argument used above. $\square$

We prove the existence of semi-decomposition data by induction on $r$. In order to be able to use the inductive hypothesis we need the following result.

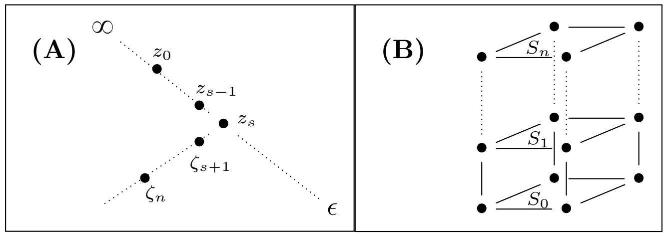


FIGURE 2. Figure **(A)** shows the finite line $\mathfrak{c}(S)$ as in Lemma 7.13, while Figure **(B)** shows the strata of a concrete $D$-grid. In the latter, vertical neighbors are $P_1$-neighbors.

**Lemma 7.14.** Let $D$ be an effective divisor, and write $D = D_0 + nQ$, where $D_0$ is supported away from $Q$ and $n > 0$. Let $\mathbb{S}_0$ be an abstract $D_0$-grid. Then the set of abstract $D$-grids $\mathbb{S}$ such that there exist $S_0 \in \mathbb{S}_0$ and $S \in \mathbb{S}$ with $S_0 \subset S$, is finite.

*Proof.* Let $q'$ be the cardinality of the residue field of $k_Q$. First we claim that any concrete $D_0$-grid is contained in finitely many concrete $D$-grids. In order to prove the claim, let us fix an Eichler $\mathcal{C}$-order $\mathfrak{E}_0$ of level $D_0$. Let $\mathfrak{E}$ be an Eichler $\mathcal{C}$-order of level $D$ contained in $\mathfrak{E}_0$. Since, for each $P \neq Q$, the $P$-coefficient of $D$ and $D_0$ is the same, we have $(\mathfrak{E}_0)_P = (\mathfrak{E})_P$, for all $P \neq Q$. Moreover, we have that $(\mathfrak{E}_0)_Q$ is a maximal order, while $(\mathfrak{E})_Q$ is a local Eichler order of level $n$. Denote by $v_0$ the vertex in $\mathfrak{t}(k_Q)$ corresponding to $(\mathfrak{E}_0)_Q$. Then $(\mathfrak{E})_Q$ corresponds to the intersection of all maximal orders in a finite line of length $n$ in $\mathfrak{t}(k_Q)$ containing $v_0$. Moreover, it follows from the local-global principles (b) and (c) in §5 that there exists a bijective map between the set of Eichler $\mathcal{C}$-orders $\mathfrak{E}$ of level $D$ contained in $\mathfrak{E}_0$, and the set of finite lines of length $n$ in $\mathfrak{t}(k_Q)$ containing $v_0$. In particular, there are finitely many Eichler $\mathcal{C}$-orders $\mathfrak{E}$ of level $D$ contained in $\mathfrak{E}_0$. Since there exists a bijective map between the set of concrete $D$-grids containing $S_0 = S(\mathfrak{E}_0)$, and the set of Eichler $\mathcal{C}$-orders $\mathfrak{E}$ of level $D$ contained in $\mathfrak{E}_0$ (cf. §6), the claim follows.

Let us define $\mathbb{W}$ as the set of abstract $D$-grids $\mathbb{S}$ such that there exist $S_0 \in \mathbb{S}_0$ and $S \in \mathbb{S}$ with $S_0 \subseteq S$. Fix a concrete representative $S_0^\circ \in \mathbb{S}_0$. We define $W$ as the set of concrete $D$-grids containing $S_0^\circ$. Then, it follows from the previous claim that $W$ is finite. Now, we claim that the map $\phi : W \to \mathbb{W}$, defined by $\phi(S) = [S]$, is surjective. Indeed, let $\mathbb{S} \in \mathbb{W}$. Then, by definition, there exist $S_0' \in \mathbb{S}_0$ and $S' \in \mathbb{S}$ with $S_0' \subseteq S'$. Since $S_0^\circ$ and $S_0'$ belong to $\mathbb{S}_0$, there exists $\gamma \in \mathrm{GL}_2(k)$ such that $S_0^\circ = \gamma \cdot S_0'$. Thus, the $D$-grid $S = \gamma \cdot S'$ contains $S_0^\circ$. This implies that $S \in W$, and, by definition, we have $\phi(S) = [\gamma \cdot S'] = [S'] = \mathbb{S}$. So, the claim follows. In particular, since $\phi$ is surjective, we conclude that $\mathbb{W}$ is finite, which completes the proof. $\square$

We are now ready to prove Proposition 7.11.

*Proof of Proposition 7.11.* Recall that $D = \sum_{i=1}^{r} n_i P_i$. We prove the existence of semi-decomposition data by induction on $r \in \mathbb{N}$. Note that, if $r = 0$, then $D$ is trivial, and, in such case, the result follows from Example 7.10. Now we prove the result for $r > 0$. We may assume that $P_1$ has minimal degree in the support of $D$. Set $U' = \mathcal{C} \smallsetminus \{P_1\}$. Then, we can write $D = n_1 P_1 + D_0$, where $D_0$ is supported in $U'$. Let $\mathbb{S}$ be an abstract $D$-grid, and fix a concrete representative $S \in \mathbb{S}$. Let $S_0, \cdots, S_{n_1}$

be the $D_0$-grids in the $P_1$-strata of $S$. See Definition 7.12 and Figure 2**(B)**. These define a line $\mathfrak{c}(S)$ in $\mathfrak{t}(k_{P_1})$, and we may assume that it satisfies statements (i), (ii), (iii), (iv) and (v) in Lemma 7.13, and that its image $\mathfrak{c}(\mathbb{S})$ is contained in a cuspidal ray of $C_{P_1}(\mathbb{O}_{D_0})$. Moreover, we can enumerate the strata of $S$ in a way such that $S_i$ corresponds to $z_i \in \mathrm{V}\big(\mathfrak{t}(k_{P_1})\big)$, for each $i \in \{0, \cdots, s\}$. Now, by the inductive hypothesis and Lemma 7.14, we may assume that there exists a semi-decomposition datum $(\beta, B, D_0')$ of degree $\deg(B) > d_D > d_{D_0}$ for $S_0$. We claim that $(\beta, B, sP_1 + D_0')$ is a semi-decomposition datum for $S$. Let $A = A(\beta)$ be the base change matrix, as defined in Definition 7.5. Then, by definition of a semi-decomposition datum, $\mathrm{V}(S_0)$ contains the vertex set $\mathrm{V}(\Pi(S_0)) = \big\{A\mathfrak{D}_E A^{-1} : B \leqslant E \leqslant B + D_0'\big\}$. We already know that the vertex $\pi_V(z_i)$ corresponding to the class $\mathbb{S}_i$ of $S_i$ has valency two in $C_{P_1}(\mathbb{O}_{D_0})$. Denote by $\mathbb{S}_{-1} \neq \mathbb{S}_1$ the other $P_1$-neighbor of $\mathbb{S}_0$. Then, we can describe a decomposed subgrid of some concrete representatives of $\mathbb{S}_1$ and $\mathbb{S}_{-1}$. Indeed, we know that the $D_0'$-grid $\Pi(S_0)$ is $P_1$-neighbor to the $D_0'$-grids $\nabla_{-1}, \nabla_1$, whose vertex sets are respectively

$$\mathrm{V}(\nabla_{-1}) = \{A^{-1}\mathfrak{D}_E A : B - P_1 \leqslant E \leqslant B - P_1 + D_0'\},$$

and

$$\mathrm{V}(\nabla_1) = \{A^{-1}\mathfrak{D}_E A : B + P_1 \leqslant E \leqslant B + P_1 + D_0'\}.$$

Then, we can complete $\nabla_{-1}, \nabla_1$ in order to obtain two $D_0$-grids, denoted respectively by $S_{-1}$ and $S_1$, which are $P_1$-neighbors to $S_0$. We claim that $S_1$ and $S_{-1}$ belong to different abstract $D_0$-grids. Indeed, if $S_1$ and $S_{-1}$ define the same abstract $D_0$-grid, then each concrete $D_0$-grid in the class has two positive degree semi-decomposition data of the form $(\beta_1, B - P_1, D_0')$ and $(\beta_2, B + P_1, D_0')$. Note that $\deg(B) > \deg(P_1)$ by hypothesis, whence $\deg(B+P_1) > 0$ and $\deg(B-P_1) > 0$. Thus, by Lemma 7.7 we deduce that $B + P_1$ is linearly equivalent to $B - P_1$, which is impossible. So, the claim follows.

Now, we claim that $S_1 \in \mathbb{S}_1$ and $S_{-1} \in \mathbb{S}_{-1}$. In order to prove this, let $\mathfrak{E}_0'$ and $\mathfrak{E}_{-1}'$ be the Eichler $\mathcal{C}$-orders defined as the intersection of all the maximal orders corresponding to vertices of the respective grids $S_0$ and $S_{-1}$. Since $S_0$ and $S_{-1}$ are $D_0$-grids, the level of $\mathfrak{E}_0'$ and $\mathfrak{E}_{-1}'$ is $D_0$. In particular, these Eichler $\mathcal{C}$-orders are maximal at $P_1$. By definition $\Pi(S_0) \subseteq S_0$ and $\nabla_{-1} \subseteq S_{-1}$. This implies that

$$\mathfrak{E}_0' \subseteq \bigcap_{B \leqslant E \leqslant B + D_0'} A^{-1}\mathfrak{D}_E A, \quad \text{and} \quad \mathfrak{E}_{-1}' \subseteq \bigcap_{B - P_1 \leqslant E \leqslant B - P_1 + D_0'} A^{-1}\mathfrak{D}_E A.$$

Thus, if $m$ is the coefficient of $P_1$ in $B$, then $(\mathfrak{E}_0')_{P_1} \subseteq A^{-1}(\mathfrak{D}_{mP_1})_{P_1}A$ and $(\mathfrak{E}_1')_{P_1} \subseteq A^{-1}(\mathfrak{D}_{(m-1)P_1})_{P_1}A$. Moreover, since $(\mathfrak{E}_0')_{P_1}$ and $(\mathfrak{E}_{-1}')_{P_1}$ are maximal, we get that $(\mathfrak{E}_0')_{P_1} = A^{-1}(\mathfrak{D}_{mP_1})_{P_1}A$ and $(\mathfrak{E}_{-1}')_{P_1} = A^{-1}(\mathfrak{D}_{(m-1)P_1})_{P_1}A$. Since the vertex $v_{-1}$ in $\mathfrak{t}(k_{P_1})$ corresponding to $S_{-1}$ is the projection of some (any) maximal $\mathcal{C}$-order in $\mathrm{V}(S_{-1})$ by localizing at $P_1$, it coincides with $A^{-1}(\mathfrak{D}_{(m-1)P_1})_{P_1}A$. This implies that $v_{-1}$ is the unique vertex in the maximal path joining the ends $\epsilon$ and $\infty$ that is further from $\epsilon$ than $z_0$. Thus, we deduce that $S_{-1} \in \mathbb{S}_{-1}$, whence we also obtain $S_1 \in \mathbb{S}_1$. We conclude from this that $(\beta, B+P_1, D_0')$ is a semi-decomposition datum of degree $d_{D_0}$ of $S_1$. So, by an inductive argument, we can show that, for each $i \in \{0, \cdots, n_1\}$, we have that $(\beta, B + iP_1, D_0')$ is a semi-decomposition datum of degree $d_{D_0}$ for $S_i$. Therefore, $(\beta, B, sP_1 + D_0')$ is a semi-decomposition datum for $S$.

We are only left to prove the condition $\deg(B) > d_D$. Let $\mathbb{S}$ be an abstract $D$-grid as above, and set $S \in \mathbb{S}$. Let us denote by $v_{\mathbb{S}} \in \mathrm{V}(C_{P_\infty}(\mathbb{O}_D))$ the vertex corresponding to $\mathbb{S}$. Then we know that almost every vertex $v_{\mathbb{S}}$ belongs to a cuspidal

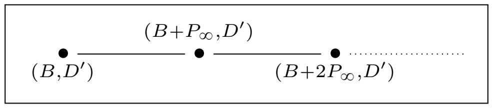


FIGURE 3. A cuspidal ray $\mathfrak{r} \subseteq C_{P_\infty}(\mathbb{O}_D)$. Each vertex in $\mathrm{V}(\mathfrak{r})$ represents an abstract $D$-grid with a semi-decomposition datum of the form $(B + nP_\infty, D')$.

ray $\mathfrak{r} \subset C_{P_\infty}(\mathbb{O}_D)$. In particular, almost every vertex $v_{\mathbb{S}}$ has exactly two $P_\infty$-neighbors. Let $v_{\mathbb{S}}^+$ be the neighbor of $v_{\mathbb{S}}$ that is closest to the end of $\mathfrak{r}$. Then, the argument here above shows that, for almost every abstract $D$-grid $\mathbb{S}$, with a semi-decomposition datum $(B, D')$, the abstract grid corresponding to $v_{\mathbb{S}}^+$ has the semi-decomposition datum $(B + P_\infty, D')$. See Figure 3. Recalling that there are finitely many cuspidal rays in $C_{P_\infty}(\mathbb{O}_D)$, we see that, up to increasing $n \in \mathbb{Z}$ at the divisor $B + nP_\infty$, for almost every abstract grid, and any concrete representative, there is a semi-decomposition datum of degree $> d_D$.

We prove now the uniqueness of semi-decomposition data. Let $\mathbb{S}$ be an abstract $D$-grid as above, and set $S \in \mathbb{S}$ be a concrete representative. In particular, $S$ has a semi-decomposition datum $(\beta, B, D')$ of degree $> d_D$. Write $D' = \sum_{i=1}^r s_i P_i$, where $s_i \leqslant \lfloor \frac{n_i+1}{2} \rfloor$. We can characterize the isomorphism class of non-$\beta$-split vertices in $\mathrm{V}(S)$, i.e. the vertices in $\mathrm{V}(S) \smallsetminus \mathrm{V}(\Pi_{\mathrm{SD}})$. Indeed, condition (v) in Lemma 7.13 implies that any non-$\beta$-split maximal $\mathcal{C}$-order in $\mathrm{V}(S)$ is isomorphic to a split maximal $\mathcal{C}$-order in $\Pi_{\mathrm{SD}}$. Then, the vertex set $\{v_j\}_{j=0}^{n_i}$ of the $P_i$-axis of $S$ satisfies that $v_j \cong \mathfrak{D}_{B+jP_i}$, if $j \leqslant s_i$, and $v_j \cong \mathfrak{D}_{B+(2s_i-j)P_i}$, if $j \geqslant s_i$. On the other hand, by hypothesis we have $\deg(B) > d_D \geqslant 0$. Then, Lemma 7.7 implies that $\mathfrak{D}_{B+(2s_i-j)P_i}$ is not isomorphic to $\mathfrak{D}_{B+jP_i}$, for any $j \geqslant s_i$. This implies that the integers $\{s_i\}_{i=1}^r$ are unique, and then $D'$ is unique. The uniqueness of the class of $B$ follows from Lemma 7.7. $\square$

An immediate corollary of the end of the preceding proof is the following statement, which we state here for further reference.

**Corollary 7.15.** Let $\mathbb{S}$ be an abstract $D$-grid corresponding to a vertex in a cuspidal ray in $C_{P_\infty}(\mathbb{O}_D)$. Let $(B, D')$ be a semi-decomposition datum of $\mathbb{S}$ of degree $> d_D$. Let $\mathbb{S}^+$ be the abstract $D$-grid corresponding to the unique neighbor in $C_{P_\infty}(\mathbb{O}_D)$ that is closer to the end of the cuspidal ray. Then $(B + P_\infty, D')$ is a semi-decomposition datum of $\mathbb{S}^+$.

**Remark 7.16.** Any semi-decomposed grid with a sufficiently negative degree datum is totally decomposed. Indeed, let $S$ be a $D$-grid, and let $(\beta, B, D')$ be a semi-decomposition datum for $S$. Replacing $S$ by another representative in the same class if needed, we can assume that $\beta$ is the canonical basis, or equivalently that $A(\beta) = \mathrm{Id}$. Let us write $D = \sum_{i=1}^r n_i P_i$ and $D' = \sum_{i=1}^r s_i P_i$. For each $i \in \{1, \cdots, r\}$, we denote by $S(P_i) \subset \mathfrak{t}(k_{P_i})$ the $P_i$-axis of $S$. Note that $S(P_i)$ is a length-$n_i$ line. Moreover, if we write $\mathrm{V}(S(P_i)) = \{v_j\}_{j=0}^{n_i}$, then, for any $0 \leqslant j \leqslant s_i$, the vertex $v_j$ corresponds to the local maximal order $(\mathfrak{D}_{B+jP_i})_{P_i}$. Thus, all vertices in $\{v_j\}_{j=0}^{s_i}$ are located on the maximal path $\mathfrak{f}(0, \infty)$ joining the the visual limits 0 and $\infty$ in $\partial_\infty(\mathfrak{t}(k_{P_i})) = \mathbb{P}^1(k_{P_i})$. Let $\mathfrak{E}$ be the Eichler $\mathcal{C}$-order corresponding to $\mathbb{S}$, i.e., assume that $\mathbb{S} = [S(\mathfrak{E})]$. Set $U_i = \mathcal{C} \smallsetminus \{P_i\}$ and $\Gamma = \mathrm{Stab}_{\mathrm{PGL}_2(k)}(\mathfrak{E}(U_i))$.

Arguing as in the proof of Lemma 7.13 we can prove that, for each $s_i+1 \leqslant j \leqslant n_i$, $v_j$ is $\Gamma$-equivalent to a vertex in $\mathfrak{f}(0,\infty)$ not in $\{v_l\}_{l=0}^{j-1}$. Thus, we get that $s_i = n_i$, and hence any semi-decomposition datum $(\beta, B, D')$ of $S$ is a total decomposition datum.

Finally, note that, if $\mathfrak{I} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then $S$ is in the same class as $\mathfrak{I}A(\beta)SA(\beta)^{-1}\mathfrak{I}$, whose total-decomposition datum $(\beta_0, D-B, D)$ has a positive degree.

7.3. **On the combinatorial structure of the classifying graph.** Here the main goal is to compute the cusp number of the classifying graph $C_{P_\infty}(\mathbb{O}_D)$, which corresponds to Proposition 7.1. Actually, we prove a more precise result stated in Proposition 7.24 below. To do this, we use the existence and uniqueness of semi-decomposition data of almost every abstract $D$-grid proved in the previous section. More specifically, using Proposition 7.11 we define a natural simplicial map $\tilde{\mathfrak{d}} : C_{P_\infty}(\mathbb{O}_D) \smallsetminus Y \to C_{P_\infty}(\mathbb{O}_0)$, where $Y$ is a finite subgraph, which is a regular cover on the cuspidal rays of $C_{P_\infty}(\mathbb{O}_0)$. Then, we compute the number of pre-images of a given cuspidal ray and we apply Theorem 7.3 in order to obtain the desired result.

**Definition 7.17.** Let $S$ be a concrete $D$-grid with a semi-decompostion datum $(\beta, B, D')$ of positive degree. We denote by $\delta(S)$ the $\mathrm{PGL}_2(k)$-conjugacy class of the maximal $\mathcal{C}$-order $\mathfrak{D}_B$. Let $\mathbb{S}$ be an abstract $D$-grid. Assume that $\mathbb{S}$ has a semi-decomposition datum $(B, D')$ with positive degree. We define the principal corner of $\mathbb{S}$ as $\mathfrak{d}(\mathbb{S}) := \delta(S)$, where $S \in \mathbb{S}$.

It follows from Lemma 7.7 that, if $S, S^\circ \in \mathbb{S}$ have respective semi-decomposition data $(\beta, B, D')$ and $(\beta^\circ, B^\circ, D'^\circ)$ with positive degree, then $\mathfrak{D}_B$ is $\mathrm{GL}_2(k)$-conjugate to $\mathfrak{D}_{B^\circ}$. Hence the previous definition is valid.

**Definition 7.18.** Let $\mathbb{S}$ be an abstract $D$-grid with a semi-decomposition datum $(B, D')$. Let us write $D = \sum_{i=1}^r n_i P_i$ and $D' = \sum_{i=1}^r s_i P_i$. We define the semi-decomposition vector associated to the previous datum as $l = l(\mathbb{S}) := (s_1, \cdots, s_r)$. Note that $(B, D')$ is a total decomposition datum exactly when $l = (n_1, \cdots, n_r)$.

Now, it follows from Proposition 7.11 that there exists a finite graph $Y \subset C_{P_\infty}(\mathbb{O}_D)$ such that, for each vertex $v \in \mathrm{V}(C_{P_\infty}(\mathbb{O}_D) \smallsetminus Y)$, the corresponding abstract $D$-grid $\mathbb{S} = \mathbb{S}(v)$ has a representative with a semi-decomposition datum of degree $> \deg(D)$. Then, $\mathfrak{d}$ induces a well-defined function $\tilde{\mathfrak{d}} : \mathrm{V}(C_{P_\infty}(\mathbb{O}_D) \smallsetminus Y) \to \mathrm{V}(C_{P_\infty}(\mathbb{O}_0))$. Let $\mathbb{S}$ be an abstract $D$-grid with a semi-decomposition datum $(B, D')$ of degree $> \deg(D)$. Let $\mathbb{S}^+$ be the abstract $D$-grid corresponding to the unique neighbor $v_{\mathbb{S}}^+$ of $v_{\mathbb{S}}$ in $C_{P_\infty}(\mathbb{O}_D)$ that is closer to the end of the cuspidal ray containing $v_{\mathbb{S}}$. By Corollary 7.15, $(B + P_\infty, D')$ is a semi-decomposition datum of $\mathbb{S}^+$. In this case, we get $\mathfrak{d}(\mathbb{S}^+) = [\mathfrak{D}_{B+P_\infty}]$. See Figure 1**(B)**. In particular, this implies that the function $\tilde{\mathfrak{d}}$ sends neighboring vertices into neighboring vertices. So, we can extend $\tilde{\mathfrak{d}}$ to a simplicial map from $C_{P_\infty}(\mathbb{O}_D) \smallsetminus Y$ to $C_{P_\infty}(\mathbb{O}_0)$, which we also denote by $\tilde{\mathfrak{d}}$. The following proposition describes the fibers of $\tilde{\mathfrak{d}}$.

**Proposition 7.19.** Let $\mathfrak{D} = \mathfrak{D}_B$ be a split maximal $\mathcal{C}$-order satisfying $\deg(B) > \deg(D)$. Assume that $[\mathfrak{D}] \neq \mathfrak{d}(\mathbb{S})$ for $\mathbb{S}$ corresponding to a vertex in $Y$. Then, $\tilde{\mathfrak{d}}^{-1}([\mathfrak{D}])$ contains

1. Exactly one totally decomposed $D$-grid, and
2. Exactly $\frac{1}{q-1}\prod_{s_i \neq n_i}(q^{\deg(P_i)}-1)q^{\deg(P_i)(n_i-s_i-1)}$ semi-decomposed $D$-grids whose semi-decomposed vector is $(s_1, \cdots, s_r) \neq (n_1, \cdots, n_r)$.

In order to prove this proposition we have to use the following lemma.

**Lemma 7.20.** Let $\mathbb{S}$ and $\mathbb{S}^\circ$ two $D$-grids with semi-decomposition data of degree $> \deg(D)$ such that $\mathfrak{d}(\mathbb{S}) = \mathfrak{d}(\mathbb{S}^\circ)$ and $l(\mathbb{S}) = l(\mathbb{S}^\circ)$. Then there exists concrete $D$-grids $S \in \mathbb{S}$ and $S^\circ \in \mathbb{S}^\circ$ such that their respective decomposed subgrids $\Pi_{\mathrm{SD}}$ and $\Pi_{\mathrm{SD}}^\circ$ are equal.

*Proof.* Let $\mathbb{S}$ and $\mathbb{S}^\circ$ be two abstract $D$-grids such that $\mathfrak{d}(\mathbb{S}) = \mathfrak{d}(\mathbb{S}^\circ)$. Set $S_0 \in \mathbb{S}$ and $S_0^\circ \in \mathbb{S}^\circ$, and let $(\beta, B, D')$ and $(\beta^\circ, B^\circ, D')$ be their respective semi-decomposition data. Let $\Pi_{\mathrm{SD}} \subset S_0$ and $\Pi_{\mathrm{SD}}^\circ \subset S_0^\circ$ be the respective decomposed subgrids, and write $A = A(\beta)$ and $A^\circ = A(\beta^\circ)$. By hypothesis $\mathfrak{d}(\mathbb{S}) = \mathfrak{d}(\mathbb{S}^\circ)$, whence we have that $\delta(S_0) = \mathfrak{D}_B$ is $\mathrm{GL}_2(k)$-conjugate to $\delta(S_0^\circ) = \mathfrak{D}_{B^\circ}$. Moreover, by hypothesis, $\deg(B), \deg(B^\circ) > \deg(D) \geqslant 0$. So, by [A14, §4, Proposition 4.1], there exists $f \in k^*$ such that $B - B^\circ = \mathrm{div}(f)$. Set $G = \left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(k)$ so that $G\mathfrak{D}_{B^\circ}G^{-1} = \mathfrak{D}_B$. We claim that $S := (GA)S_0(GA)^{-1} \in \mathbb{S}$ and $S^\circ := (A^\circ)S_0^\circ(A^\circ)^{-1} \in \mathbb{S}^\circ$ have the same decomposed subgrids. Indeed, for any divisor $E$, satisfying $0 \leqslant E \leqslant D'$, we have

$$G\mathfrak{D}_{B^\circ+E}G^{-1} = \begin{pmatrix} \mathcal{O}_{\mathcal{C}} & f^{-1}\mathfrak{L}^{-B^\circ-E} \\ f\mathfrak{L}^{B^\circ+E} & \mathcal{O}_{\mathcal{C}} \end{pmatrix} = \mathfrak{D}_{B+E}.$$

In particular, we obtain that $(GA)\Pi_{\mathrm{SD}}(GA)^{-1} = (A^\circ)\Pi_{\mathrm{SD}}^\circ(A^\circ)^{-1}$, i.e. the decomposed subgrid of $S$ and $S^\circ$ are equal. $\square$

*Proof of Proposition 7.19.* Let $\mathbb{S}$ and $\mathbb{S}^\circ$ be two abstract $D$-grids such that $\mathfrak{d}(\mathbb{S}) = \mathfrak{d}(\mathbb{S}^\circ) = [\mathfrak{D}_B]$, where $\deg(B) > \deg(B)$. It follows from Lemma 7.20 that there exist $S \in \mathbb{S}$ and $S^\circ \in \mathbb{S}^\circ$ with the same decomposed subgrid $\Pi_{\mathrm{SD}}$. So, if all $s_i = n_i$, then $S = S'$, whence $\mathbb{S} = \mathbb{S}^\circ$. On the other hand, we can always consider the totally decomposed $D$-grid whose vertices are $\mathfrak{D}_{B+E}$, where $0 \leqslant E \leqslant D$. Thus (1) follows. More generally, $\mathbb{S} = \mathbb{S}^\circ$ if and only if there exists $g \in \mathrm{GL}_2(k)$ such that

(1D) $g\Pi_{\mathrm{SD}}g^{-1} = \Pi_{\mathrm{SD}}$, and
(2D) $g(S \smallsetminus \Pi_{\mathrm{SD}})g^{-1} = S^\circ \smallsetminus \Pi_{\mathrm{SD}}$,

where we recall that no vertex in $S \smallsetminus \Pi_{\mathrm{SD}}$ and $S^\circ \smallsetminus \Pi_{\mathrm{SD}}$ is split in the canonical basis. As $\Pi_{\mathrm{SD}}$ is totally decomposed, by [A14, §4, Proposition 4.1], we have $g = \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$, where $\mathrm{div}(x^{-1}z) = 0$, i.e. $x^{-1}z \in \mathbb{F}^*$. Let $m_i$ be the multiplicity of $P_i$ in $B$. Then, Condition (1D) is equivalent to the following statement: *For every $i \in \{1, \cdots, r\}$, the action of $g \in \mathrm{GL}_2(k)$ on the tree $\mathfrak{t}_i = \mathfrak{t}(k_{P_i})$ point-wisely stabilizes the finite path $\mathfrak{c}_i$ whose vertex set is*

$$\left\{ \begin{pmatrix} \mathcal{O}_{P_i} & \pi_i^{-m_i-t}\mathcal{O}_{P_i} \\ \pi_i^{m_i+t}\mathcal{O}_{P_i} & \mathcal{O}_{P_i} \end{pmatrix} : t \in \{0, \cdots, s_i\} \right\}, \tag{7.7}$$

*where $\pi_i \in \mathcal{O}_{P_i}$ is a local uniformizing parameter.* Moreover, this condition is equivalent to $\nu_{P_i}(y) \geqslant -m_i$. On the other hand, when $s_i \neq n_i$, the localizations at $P_i$ of orders in $S$ that are different from the localizations of vertices in $\Pi_{\mathrm{SD}}$, correspond to the vertices of a line $\widehat{\mathfrak{p}}_i$ of length $(n_i - s_i - 1)$ in $\mathfrak{t}(k_{P_i})$, which does not intersect the maximal path joining 0 and $\infty$. See Figure 2**(A)**. Hence, vertices in $\widehat{\mathfrak{p}}_i$ are in correspondence with the local rings of endomorphisms of the lattices

$$\begin{pmatrix} a \\ 1 \end{pmatrix}\mathcal{O}_{P_i} + \begin{pmatrix} \pi_i^{-m_i-s_i-j} \\ 0 \end{pmatrix}\mathcal{O}_{P_i}, \tag{7.8}$$

where $a = a_1\pi_i^{-m_i-s_i+1} + \cdots + a_j\pi_i^{-m_i-s_i+j}$, and where, for any $k > 1$, we have $a_k \in \mathbb{F}(P_i) = \mathcal{O}_{P_i}/\pi_i\mathcal{O}_{P_i}$, while $a_1 \in \mathbb{F}(P_i)^*$. The same characterization holds for the localizations at $P_i$ of orders in $S^\circ$, by replacing $a_i$ by $a_i^\circ \in \mathbb{F}(P_i)$. Since $\nu_{P_i}(y) \geqslant -m_i$, Condition (2D) is equivalent to $a_j = a_j^\circ(x^{-1}z)$, for each $j \in \{1, \cdots, n_i - s_i\}$. Since two $D$-grids are equal if and only if all their $P_i$-projections coincide, it follows that $\mathrm{Card}(\tilde{\mathfrak{d}}^{-1}[\mathfrak{D}_B])$ equals the number of $\mathbb{F}^*$-homothety classes in

$$\prod_{s_i \neq n_i} \left(\mathbb{F}(P_i)^* \times \mathbb{F}(P_i)^{n_i - s_i - 1}\right),$$

which is $\frac{1}{q-1}\prod_{s_i \neq n_i}(q^{\deg(P_i)} - 1)q^{\deg(P_i)(n_i - s_i - 1)}$. □

Let us denote by $\alpha(D)$ the positive integer

$$\alpha(D) = 1 + \frac{1}{q-1}\prod_{i=1}^{r}\left(q^{\deg(P_i)\lfloor \frac{n_i}{2} \rfloor} - 1\right).$$

**Lemma 7.21.** Let $B$ be a divisor, whose degree is greater than $\deg(D)$, and assume that $[\mathfrak{D}_B] \neq \mathfrak{d}(\mathbb{S})$ for $\mathbb{S}$ corresponding to a vertex in $Y$. Then, $\tilde{\mathfrak{d}}^{-1}([\mathfrak{D}_B])$ has $\alpha(D)$ elements. In particular, there are $\alpha(D)$ different cuspidal rays in $C_{P_\infty}(\mathbb{O}_D)$ whose initial vertex corresponds to an abstract $D$-grid $\mathbb{S}$ satisfying that $\mathfrak{d}(\mathbb{S}) = [\mathfrak{D}_B]$.

*Proof.* It follows directly from Proposition 7.19 that $\tilde{\mathfrak{d}}^{-1}([\mathfrak{D}_B])$ contains one and only one split abstract $D$-grid, and, for each possible semi-decomposition vector $l = (s_1, \cdots, s_r) \neq (n_1, \cdots, n_r)$, there are precisely $\frac{1}{q-1}\prod_{i=1}^{r}(q^{\deg(P_i)} - 1)q^{\deg(P_i)(n_i - s_i - 1)}$ non-split abstract $D$-grids whose semi-decomposition vector is $l$. By definition of semi-decomposition data we know that $l \in \Upsilon = \{(s_1, \cdots, s_r) : n_i \geqslant s_i \geqslant \lfloor \frac{n_i+1}{2} \rfloor\}$. Then, $\tilde{\mathfrak{d}}^{-1}([\mathfrak{D}_B])$ contains

$$1 + \sum_{\Upsilon'} \frac{1}{q-1} \prod_{s_i \neq n_i} (q^{\deg(P_i)} - 1)q^{\deg(P_i)(n_i - s_i - 1)} = 1 + \frac{1}{q-1}\prod\left(q^{\deg(P_i)\lfloor \frac{n_i}{2} \rfloor} - 1\right)$$

different abstract $D$-grids, where $\Upsilon' := \Upsilon \smallsetminus \{(n_1, \cdots, n_r)\}$. Hence, the first claim is proved. Now, recall that, by Corollary 7.15, if $\mathbb{S}$ and $\mathbb{S}^\circ$ are two $P_\infty$-neighboring $D$-grids satisfying $\mathfrak{d}(\mathbb{S}) = [\mathfrak{D}_B]$ and $\mathfrak{d}(\mathbb{S}^\circ) = [\mathfrak{D}_{B^\circ}]$, then $(B, D')$ and $(B + P_\infty, D')$ are respective semi-decomposition data of $\mathbb{S}$ and $\mathbb{S}^\circ$. So, it follows that $\mathbb{S}$ and $\mathbb{S}^\circ$ have the same semi-decomposition vectors. Thus, the second claim follows. □

Recall that, by definition, the number of cusps of a disjoint finite union of graphs is the sum of the number of cusps in each of its connected components.

**Proposition 7.22.** The number of cusps of $C_{P_\infty}(\mathbb{O}_D)$ equals $\mathrm{Card}(\mathrm{Pic}(R))\alpha(D)$.

*Proof.* It follows from Theorem 7.3 that the vertices of $C_{P_\infty}(\mathbb{O}_0)$ corresponding to the $\mathrm{GL}_2(k)$-classes of split maximal $\mathcal{C}$-orders $\mathfrak{D}_B$ are located in a finite disjoint union of infinite lines or half lines in $C_{P_\infty}(\mathbb{O}_0)$. If we assume that a infinite line is the union of two half lines, then the number of such half lines is equal to the number of cusps of $C_{P_\infty}(\mathbb{O}_0)$, which coincide with $\mathrm{Pic}(R)$.

Recall that we can remove a finite set of vertices in the classifying graphs in order to simplify some arguments. Thus, we can consider only vertices associated to abstract $D$-grids with a semi-decomposition datum of degree $> \deg(D)$ (cf. Proposition 7.11). Recall also that $\tilde{\mathfrak{d}}$ is a simplicial map from the disjoint union of a set of cuspidal rays in $C_{P_\infty}(\mathbb{O}_D)$, representing all classes of cuspidal rays, to an analog set in $C_{P_\infty}(\mathbb{O}_0)$. In particular, the image of a cuspidal ray of the former set under $\tilde{\mathfrak{d}}$ is also a cuspidal ray. So, $\tilde{\mathfrak{d}}$ can be seen as a function between such cusps. In particular, to compute the cusp number in $C_{P_\infty}(\mathbb{O}_D)$ it suffices to compute the number of pre-images of each cusp in $C_{P_\infty}(\mathbb{O}_0)$. Moreover, this can be reduced to computing the number of pre-images of any vertex in a cuspidal ray of $C_{P_\infty}(\mathbb{O}_0)$. Hence, the proposition follows from Lemma 7.21. $\square$

We say that a cusp $\eta \in \partial^\infty(C_{P_\infty}(\mathbb{O}_D))$ is split if it is represented by a cuspidal ray formed only by vertices corresponding to totally decomposed $D$-grids. In any other case we say that the cusp is non-split. Note that the arguments given in the proof above imply that $\widetilde{\mathfrak{d}}$ induces a natural function $\widetilde{\mathfrak{d}}^\infty : \partial^\infty(C_{P_\infty}(\mathbb{O}_D)) \to \partial^\infty(C_{P_\infty}(\mathbb{O}_0))$ between the respective cusp sets of $C_{P_\infty}(\mathbb{O}_D)$ and $C_{P_\infty}(\mathbb{O}_0)$. Then, Lemma 7.19.(1) and Lemma 7.21 imply that for each $\eta \in \partial^\infty(C_{P_\infty}(\mathbb{O}_0))$ there exists a unique split cusp in $(\widetilde{\mathfrak{d}}^\infty)^{-1}(\eta)$, and $\alpha(D)-1$ non split cusps.

**Remark 7.23.** We can characterize the unique split cusp in $(\widetilde{\mathfrak{d}}^\infty)^{-1}(\eta)$. Indeed, let $\eta'_B \in \partial^\infty(C_{P_\infty}(\mathbb{O}_D))$ be the class of the cuspidal ray $\mathfrak{r}_{B,D} \subseteq C_{P_\infty}(\mathbb{O}_D)$, whose vertices correspond to the $\mathrm{PGL}_2(k)$-classes of decomposable Eichler $\mathcal{C}$-orders $\mathfrak{E}_{B,n}$ (equiv. the totally-decomposable abstract $D$-grids $\mathbb{S} = [S(\mathfrak{E}_{B,n})]$) defined by

$$\mathfrak{E}_{B,n} = \begin{pmatrix} \mathcal{O}_{\mathcal{C}} & \mathfrak{L}^{B+nP_\infty} \\ \mathfrak{L}^{-B-nP_\infty+D} & \mathcal{O}_{\mathcal{C}} \end{pmatrix}, \quad n \geqslant 0. \tag{7.9}$$

Then, the image by $\widetilde{\mathfrak{d}}^\infty$ of $\eta'_B$ is the class $\eta_B \in \partial^\infty(C_{P_\infty}(\mathbb{O}_0))$ of the cuspidal ray $\mathfrak{r}_B \subseteq C_{P_\infty}(\mathbb{O}_0)$, whose vertices correspond to $\{[\mathfrak{D}_{B+nP_\infty}] : n \geqslant 0\}$. This is the unique split cusp in $(\widetilde{\mathfrak{d}}^\infty)^{-1}(\eta)$. Moreover, it follows from Theorem 7.3 that, if we fix a representative set $\Delta_R \subset \mathrm{Div}(\mathcal{C})$ of $\mathrm{Pic}(R)$, then $\{\eta_B : B \in \Delta_R\}$ is a representative set of $\partial^\infty(C_{P_\infty}(\mathbb{O}_0))$.

We are finally ready to prove Proposition 7.1 as an immediate consequence of the following result, which concludes the proof of Theorem 1.4.

**Proposition 7.24.** The number of cusps of any connected component of $C_{P_\infty}(\mathbb{O}_D)$ is the same, and it equals

$$c(D) = \alpha(D)[2\mathrm{Pic}(\mathcal{C}) + \langle \overline{P_{a_1}}, \cdots, \overline{P_{a_u}}, \overline{P_\infty} \rangle : \langle \overline{P_\infty} \rangle], \tag{7.10}$$

where $P_{a_1}, \cdots, P_{a_u}$ are the closed points in $\mathrm{Supp}(D) \subseteq \mathcal{C}$ whose coefficients are odd. Moreover, there are $c(D)/\alpha(D)$ split cusps in any connected component of $C_{P_\infty}(\mathbb{O}_D)$.

*Proof.* We will use the bijection between the set of abstract $D$-grids and the set of Eichler $\mathcal{C}$-orders of level $D$ (cf. Proposition 6.1). In particular, we will assign to any Eichler $\mathcal{C}$-order of level $D$ a vertex in $C_{P_\infty}(\mathbb{O}_D)$. Let $\mathfrak{E}$ and $\mathfrak{E}'$ be two split Eichler $\mathcal{C}$-orders associated to two vertices $v, v'$ in the same connected component of $C_{P_\infty}(\mathbb{O}_D)$. We denote by $\mathbb{S}$ and $\mathbb{S}'$ be the respective abstract $D$-grids corresponding to $\mathfrak{E}$ and $\mathfrak{E}'$. Assume that $\mathfrak{d}(\mathbb{S}) = [\mathfrak{D}_B]$, $\mathfrak{d}(\mathbb{S}') = [\mathfrak{D}_{B'}]$, and $\deg(B), \deg(B') > \deg(D)$. Let $\Sigma_D = \Sigma(\mathbb{O}_D)$ be the spinor class field of Eichler $\mathcal{C}$-orders of level $D$. By definition of $C_{P_\infty}(\mathbb{O}_D)$, the vertices $v$ and $v'$ are in the same connected component if and only if $\rho(\mathfrak{E}, \mathfrak{E}') \in \{\mathrm{id}_{\Sigma_D}, [[P_\infty, \Sigma_D/k]]\}$. It follows from the Equation (6.2) that $\rho(\mathfrak{E}, \mathfrak{E}') = \rho(\mathfrak{D}_B, \mathfrak{D}_{B'}) = [[B - B', \Sigma_D/k]]$. Hence, by Proposition 6.3, $v$ and $v'$ are in the same connected component precisely when $\overline{B - B'} \in 2\mathrm{Pic}(\mathcal{C}) + \langle \overline{P_{a_1}}, \cdots, \overline{P_{a_u}}, \overline{P_\infty} \rangle$.

Moreover, since the cuspidal ray $\mathfrak{r} \subseteq C_{P_\infty}(\mathbb{O}_D)$ containing $v$ consists of vertices corresponding to $[\mathfrak{D}_{B+nP_\infty}]$ (cf. Corollary 7.15), we see that $\overline{B + \langle P_\infty \rangle}$ determines the image in $C_{P_\infty}(\mathbb{O}_0)$ of $\mathfrak{r}$. This tells us that there are $[2\mathrm{Pic}(\mathcal{C}) +$

$\left\langle \overline{P_{a_1}}, \cdots, \overline{P_{a_u}}, \overline{P_\infty} \right\rangle : \langle \overline{P_\infty} \rangle]$ possible images via $\widetilde{\mathfrak{d}}^\infty$ for a cusp in a given connected component of $C_{P_\infty}(\mathbb{O}_D)$. Since by Lemma 7.21 there are $\alpha(D)$ non equivalent cuspidal rays in $C_{P_\infty}(\mathbb{O}_D)$ covering each cuspidal ray in $C_{P_\infty}(\mathbb{O}_0)$, we get that there are $c(D)$ different cusps in any connected component of $C_{P_\infty}(\mathbb{O}_D)$. This proves the first statement.

For the last statement, it follows from Lemma 7.19.(1) that there exists only one split cuspidal ray in each fiber of $\widetilde{\mathfrak{d}}^\infty$. Thus, it suffices to count the number of cusps in the image of $\widetilde{\mathfrak{d}}^\infty$ in any given connected component. As we proved above, this number is precisely $c(D)/\alpha(D)$. □

## 8. On some explicit examples

In the current section, the main goal is to prove Theorem 1.5, and to subsequently present some explicit computations of quotient graphs $\mathfrak{t}_D$ associated to the action of Eichler groups $\mathrm{H}_D$, for small values of $\deg(D)$. We deduce from Theorem 1.4 that the computation of the number of cusps gets more involved as $\deg(D)$, $\deg(P_\infty)$, or the genus of $\mathcal{C}$ increases. Naturally, we expect the same behavior from the finite graphs $Y \subset \mathfrak{t}_D$ (cf. Theorem 1.4). For these reasons, in this section we work in the most elementary non-trivial context possible. In other words, we set $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$ and $P_\infty$ to be the point at infinity. In all that follows we denote by $P[i]$ the degree-one closed point corresponding to $i \in \mathbb{F} \cup \{\infty\}$. In particular, we have that $\mathrm{div}(t-i) = P[i] - P[\infty]$ and $P_\infty = P[\infty]$, whence $R = \mathbb{F}[t]$, and $K = k_{P_\infty} = \mathbb{F}((t^{-1}))$. In this context $\mathcal{O}_{P_\infty} = \mathbb{F}[[t^{-1}]]$ is the ring of integers of $K$, $\nu = \nu_{P_\infty} = -\deg$ and $\pi = 1/t$ is a uniformizing parameter. We also consistently fix the absolute value $x \mapsto |x| = |x|_{P_\infty}$. In all that follows, we identify the Bruhat-Tits tree for $\mathrm{SL}_2(K)$ (cf. §2) with the tree $\mathfrak{b} = \mathfrak{b}(K)$, whose vertices are the closed balls in $K$, and two of them are neighbors if one is a proper sub-ball of the other.

8.1. **Conventions on fundamental regions.** It follows from [Se80, §3 and §4] that in order to define quotient graphs in full generality, i.e., in the context where a group acts with some edge inversions, it is convenient to work with the barycentric subdivision. Here, in order to define the fundamental domains associated to non-simply connected quotient graphs, we adopt this convention. Then, in order to define a fundamental domain in the Bruhat-Tits tree, we begin by performing a finite number of "surgeries" on the quotient graph $\mathfrak{q}$ to turn it into a tree. See Figures 4**(A)** and 4 **(B)**. By a surgery we mean the process of replacing an edge by a pair of half edges, provided that the resulting graph is still connected. After surgery, we get a tree $\mathfrak{q}'$, we fix a vertex $v \in \mathrm{V}(\mathfrak{q}')$ that corresponds to a "real" vertex in $\mathfrak{q}$, and then we choose a pre-image $\tilde{v}$ in the Bruhat-Tits tree. Successively, we consistently lift the path from $v$ to any vertex or non-vertex, where a non-vertex is the lift in a half edge in $\mathfrak{q}'$. The union of the images of such liftings is the *fundamental domain* under consideration. See Figure 4**(C)**. Note that any structural result on a quotient graph can be translated into a result on its corresponding fundamental region. Moreover, this correspondence is perfect, in the sense that the quotient graph can be recovered from the fundamental domain and the pairs of corresponding non-vertices. Indeed, this is done by gluing the latter in an obvious manner. In particular, any combinatorial or topological result on the fundamental region can be also interpreted in terms of the corresponding quotient graph. We can define the ends of a fundamental region as the visual limit of its rays. This point of view allows us to explicitly describe the ends of the fundamental regions or quotient graphs in terms of representatives of the $\mathrm{H}_D$-orbits in $\mathbb{P}^1(k)$.

8.2. **A Proof of Theorem 1.5.** Let $N = (t-\lambda_1)\cdots(t-\lambda_n)$. We can write $\mathrm{div}(t-\lambda_i) = P[\lambda_i] - P[\infty]$, where $P[j]$ is a degree-one closed point on $\mathbb{P}^1_{\mathbb{F}}$. Let $D = \sum_{i=1}^r P[\lambda_i]$ be the corresponding multiplicity free divisor on $\mathcal{C}$. Then, since $n_1 = \cdots = n_r = 1$ and $\deg(P[\infty]) = \deg(P[\lambda_1]) = \cdots = \deg(P[\lambda_r]) = 1$, it follows from Theorem 1.4 that the quotient graph $\mathfrak{t}_D$ as exactly $2^n$ cusps. Thus, in

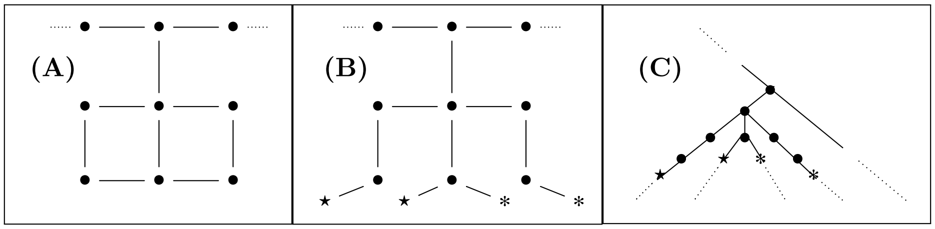


FIGURE 4. In the Figure, **(A)** represents a quotient graph, **(B)** shows the tree obtained from the previous graph by the process of surgery, and finally **(C)** represents the corresponding choice of a fundamental domain.

order to prove Theorem 1.5, it suffices to prove that the restriction of the canonical projection $\pi : \mathfrak{t}(K) \cong \mathfrak{b}(K) \to \mathfrak{t}_D$ to the tree $\mathfrak{s}$ is an injection. Consequently, the result follows from next lemma, which we prove following the techniques in [Ma01]:

**Lemma 8.1.** The vertices in $\mathfrak{s}$ are in different $\mathrm{H}_D$-orbits.

*Proof.* Note that, when $N = 1$, the lemma reduces to Nagao's Theorem (cf. [Na59]). So, we assume throughout that $n \geqslant 1$. As in §2 we write $B_x^{|t|}$ for the ball of radius $|\pi|^t$ centered at $x \in K$, where $\pi = \pi_{P_\infty}$ is a uniformizing parameter. This ball corresponds to the local maximal order $\mathrm{End}_{\mathcal{O}_{P_\infty}}(\Lambda_B)$, where $\Lambda_B = \left\langle \binom{a}{1}, \binom{\pi^r}{0} \right\rangle$. In particular, $B_0 := B_0^{|0|}$ corresponds to the local maximal order $\mathbb{M}_2(\mathcal{O}_{P_\infty})$.

Let $B_1 = B_{x_1}^{|r_1|}$ and $B_2 = B_{x_2}^{|r_2|}$ be two vertices in $\mathfrak{t}_D$, where each center is either 0 or the multiplicative inverse of a proper monic nonconstant divisor of $N = \prod_{i=1}^n (t - \lambda_i)$. Assume that there exists a matrix $g = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \mathrm{H}_D$ satisfying $g \cdot B_1 = B_2$. We must prove that $B_1 = B_2$. Set $h_1 = \begin{pmatrix} x_1 & \pi^{r_1} \\ 1 & 0 \end{pmatrix}$ and $h_2 = \begin{pmatrix} x_2 & \pi^{r_2} \\ 1 & 0 \end{pmatrix}$, so that we have both $B_1 = h_1 \cdot B_0$ and $B_2 = h_2 \cdot B_0$. Since $K^*\mathrm{GL}_2(\mathcal{O}_{P_\infty})$ is the stabilizer of $B_0$ in $\mathrm{GL}_2(K)$, we must have $h_2^{-1} g h_1 \in \lambda \mathrm{GL}_2(\mathcal{O}_{P_\infty})$, for some $\lambda \in K^*$. By taking determinants, we get $2\nu(\lambda) = r_1 - r_2$, where $\nu$ is the valuation corresponding to $P_\infty$. Hence, $r_1 - r_2$ is an even integer and $\pi^{\frac{r_2 - r_1}{2}} h_2^{-1} g h_1 \in \mathrm{GL}_2(\mathcal{O}_{P_\infty})$. After a simple computation we have $\pi^{\frac{r_2-r_1}{2}} h_2^{-1} g h_1$ equals

$$\begin{pmatrix} \pi^{\frac{r_2-r_1}{2}}(d + Ncx_1) & \pi^{\frac{r_2+r_1}{2}} Nc \\ \pi^{\frac{-r_1-r_2}{2}}(ax_1 - dx_2 + b - Ncx_1x_2) & \pi^{\frac{r_1-r_2}{2}}(a - Ncx_2) \end{pmatrix}. \tag{8.1}$$

We conclude that $\pi^{\frac{r_1-r_2}{2}}(a - Ncx_2), \pi^{\frac{r_2-r_1}{2}}(d + Ncx_1) \in \mathcal{O}_{P_\infty}$. On the other hand, the polynomials $a - Ncx_2$ and $d + Ncx_1$ either vanish or have non-positive valuations. This leaves us three alternatives:

(i) $r := r_1 = r_2$, and $\nu(a - Ncx_2) = \nu(d + Ncx_1) = 0$,
(ii) $a = Ncx_2$ or
(iii) $d = -Ncx_1$.

The last two alternatives imply $\det(g) \notin \mathbb{F}^*$, so (i) must hold. The result follows if $x_1 = x_2$, as this implies $B_1 = B_2$. The same holds if $r \leqslant 0$ since $\nu(x_1), \nu(x_2) > 0$

and hence $B_1 = B_0^{|r|} = B_2$. We assume in the sequel that $x_1 \neq x_2$ and $r > 0$. In particular, one of the $x_1$, $x_2$ is not zero. From Equation (8.1) and (i) we deduce the following facts:

(a) $a - Ncx_2 =: a_0 \in \mathbb{F}^*$,
(b) $d + Ncx_1 =: d_0 \in \mathbb{F}^*$,
(c) $Nc \in \pi^{-r}\mathcal{O}_{P_\infty}$, or equivalently $\deg(Nc) \leqslant r$, and
(d) $a_0x_1 - d_0x_2 + b + Ncx_1x_2 = ax_1 - dx_2 + b - Ncx_1x_2 \in \pi^r\mathcal{O}_{P_\infty}$.

Firstly assume that either $\nu(Ncx_1x_2) > 0$ or $x_1x_2 = 0$, then the dominant term on the left hand side of identity (d) is $b \in \mathbb{F}[t]$, unless it vanishes. As $r > 0$ we must conclude the latter. It follows that $g = \left(\begin{smallmatrix} a & 0 \\ Nc & d \end{smallmatrix}\right)$, in particular $a, d \in \mathbb{F}^*$. Then, it follows from (a) and (b) that $Ncx_2, Ncx_1 \in \mathbb{F}$, and then $c = 0$, as at least one element in $\{x_1, x_2\}$ is the inverse of a nonconstant proper monic divisor of $N$. From the preceding considerations, we get the identity $B_{x_2}^{|r|} = B_2 = g \cdot B_1 = B_{ax_1/d}^{|r|}$. This implies that $dx_2 - ax_1 \in \pi^r\mathcal{O}_{P_\infty}$. Note that if $\nu(x_1) \geqslant r$, then $0 \in B_1$, whence we can assume that $x_1 = 0$. This implies that $dx_2 \in \pi^r\mathcal{O}_{P_\infty}$, i.e. $\nu(x_2) \geqslant r$. Thus, we conclude $B_1 = B_0^{|r|} = B_2$. In any other case $\nu(x_1), \nu(x_2) < \nu(N) < r$, whence we deduce $\nu(x_1) = \nu(x_2)$ and $a = d$. We conclude $x_1 - x_2 \in \pi^r\mathcal{O}_{P_\infty}$, hence $B_1 = B_2$.

Finally, assume that both $x_1, x_2 \neq 0$ and $\nu(Ncx_1x_2) \leqslant 0$. We can assume $r > \max\{\nu(x_1), \nu(x_2)\}$ since other case we could redefine $x_1$ or $x_2$ as 0 and return to the preceding case. Let

$$\epsilon = b + Ncx_1x_2 \in -a_0x_1 + d_0x_2 + \pi^r\mathcal{O}_{P_\infty} \subseteq \pi\mathcal{O}_{P_\infty}. \tag{8.2}$$

By a simple computation, we get $\det(g) = a_0d_0 - \xi \in \mathbb{F}^*$, where $\xi = Nc(a_0x_1 - d_0x_2 + \epsilon) \in \mathbb{F}$. If $\xi = 0$, we have that $c = 0$ or

$$Nc + bx_1^{-1}x_2^{-1} = \epsilon(x_1x_2)^{-1} = d_0x_1^{-1} - a_0x_2^{-1}. \tag{8.3}$$

If $c = 0$, then $b \in \pi\mathcal{O}_{P_\infty}$ by (8.2), so that $b = 0$ and we argue as in the previous paragraph. Otherwise, Equation (8.3) and conditions (a) and (b) imply that $x_1^{-1}$ divides $x_2^{-1}$ and conversely, as each divides $N$, whence $B_1 = B_2$.

Assume now that $\xi \neq 0$, so by applying, in the given order, (c), the definition of $\xi$, the definition of $\epsilon$, and (d), we prove the following chain of inequalities:

$$r \geqslant -\nu(Nc) = \nu(a_0x_1 - d_0x_2 + \epsilon) = \nu(a_0x_1 - d_0x_2 + b + Ncx_1x_2) \geqslant r,$$

whence $\nu(a_0x_1 - d_0x_2 + \epsilon) = -\nu(Nc) = r$. In this case we have

$$r = \nu(a_0x_1 - d_0x_2 + \epsilon) = \nu(x_1x_2) + \nu(a_0x_2^{-1} - d_0x_1^{-1} + \epsilon(x_1x_2)^{-1}) \leqslant \nu(x_1x_2),$$

as the second term is a polynomial. On the other hand, the hypothesis $\nu(Ncx_1x_2) \leqslant 0$ implies $\nu(x_1x_2) = \nu(Ncx_1x_2) - \nu(Nc) \leqslant r$. Thus, $r = \nu(x_1x_2)$ and

$$\sigma := a_0x_2^{-1} - d_0x_1^{-1} + \epsilon(x_1x_2)^{-1} = a_0x_2^{-1} - d_0x_1^{-1} + b(x_1x_2)^{-1} + Nc,$$

is a nonzero constant polynomial. But $\sigma$ is divisible by $\gcd(x_1^{-1}, x_2^{-1})$, and therefore $\gcd(x_1^{-1}, x_2^{-1}) = 1$. If $\epsilon \neq 0$ we conclude that $b(x_1x_2)^{-1} + Nc$ is a multiple of $(x_1x_2)^{-1}$. By the strong triangular inequality, $\nu(\sigma) = 0$ implies

$$\nu(a_0x_1^{-1} - d_0x_2^{-1}) = \nu(b(x_1x_2)^{-1} + Nc) \leqslant \nu((x_1x_2)^{-1}).$$

By conditions (a) and (b), the preceding inequality is impossible by a degree argument. To finish the proof we consider $\epsilon = 0$, in which case $\nu(a_0x_1^{-1} - d_0x_2^{-1}) = 0$. As the polynomials $1/x_1$ and $1/x_2$ are monic, this is only possible when $a_0 = d_0$. Then condition (d) implies $\nu(x_1 - x_2) \geqslant r$. We conclude that $B_1 = B_2$. $\square$

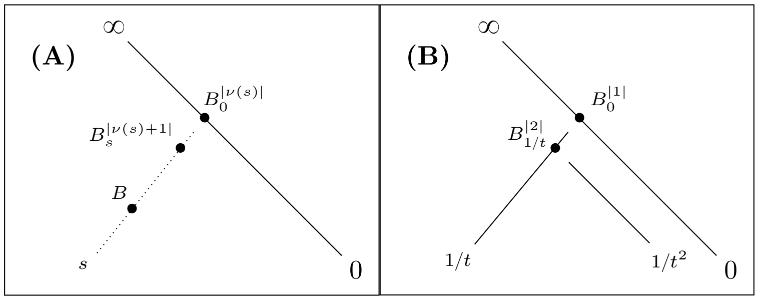


FIGURE 5. In **(A)** continuous line represents the double ray $\mathfrak{p}_{0,\infty}$, which is a fundamental region for the action of $\mathrm{H}_D$ on $\mathfrak{b}(K)$, when $D = P[0]|_{U_0}$. On the other hand, Figure **(B)** shows a fundamental region for the action of $\mathrm{H}_D$ on $\mathfrak{b}(K)$, when $D = (P[0] + P[1])|_{U_0}$.

8.3. **Small Examples.** Here we compute some examples of fundamental regions (or equivalently, quotient graphs) in the context where $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$, $P_\infty = P[\infty]$ and $\deg(D)$ is small.

**Example 8.2.** Assume that $N = t$, or equivalently assume that $D = \mathrm{div}(t)|_{U_0} = P[0]|_{U_0}$. We denote by $\mathfrak{p}_{a,b} \subset \mathfrak{b}(K)$ the double ray joining two different elements $a, b \in \mathbb{P}^1(k)$. Then, Theorem 1.5 implies that the union of $\mathfrak{p}_{0,\infty}$ with a finite graph is a fundamental region for the action of $\mathrm{H}_D$ on $\mathfrak{b}(K)$. More precisely, we claim that $\mathfrak{p}_{0,\infty}$ alone is a fundamental region in this case. See Figure 5**(A)**. In order to prove this claim, we introduce the following algorithm. For each $f \in R$, let us write:

$$\tau_f = \begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \quad \sigma_f = \begin{pmatrix} 1 & 0 \\ -tf & 1 \end{pmatrix} = I \cdot \tau_{tf} \cdot I.$$

Note that $\tau_f, \sigma_f \in \mathrm{H}_D$, for any $f \in R$. Let $s \in k$ be a finite rational element, and let $B = B_s^{|r|}$ be any vertex in the double infinity ray $\mathfrak{p}_{s,\infty}$. We claim that $B$ is in the $\mathrm{H}_D$-orbit of some vertex of $\mathfrak{p}_{0,\infty}$. So, first note that, if $\nu(s) \geqslant r$ or $s = 0$, then $B = B_0^{|r|}$, whence the claim holds immediately. Thus, assume that $s \neq 0$ and that $\nu(s) < r$. In all that follows we extensively use the fact that for any $p \in \mathbb{Z}$ and any pair $x, s \in k$ we have that

$$\tau_x \cdot B_s^{|p|} = B_{s-x}^{|p|}, \quad I \cdot B_0^{|p|} = B_0^{|-p|}, \quad \text{and} \quad I \cdot B_s^{|p|} = B_{1/s}^{|p-2\nu(s)|}, \text{ if } 0 \notin B_s^{|p|}.$$

Fix the uniformizing parameter $1/t \in \mathcal{O}_{P_\infty}$. Assume that $\nu(s) \leqslant 0$. Then, we can write $s = f_0 + \epsilon_0$, where $f_0 \in R = \mathbb{F}[t]$ and $\nu(\epsilon_0) \geqslant 1$. Let us define $\mathfrak{c}_0$ as the unique finite path in $\mathfrak{b}(K)$ joining $B_s^{|\nu(s)+1|}$ with $B_s^{|\nu(\epsilon_0)|}$, when $\epsilon_0 \neq 0$, and define it as the ray joining $B_s^{|\nu(s)+1|}$ with the end $s$, in the remaining case. Note that, since $\nu(s) < \nu(\epsilon_0)$, the path $\mathfrak{c}_0$ is non-trivial in any case. Moreover, note that $\tau_{f_0} \cdot B_s^{|p|} = B_{\epsilon_0}^{|p|}$ belongs to $\mathrm{vert}(\mathfrak{p}_{0,\infty})$ if and only if $\nu(\epsilon_0) \geqslant p$. In particular, we obtain $\tau_{f_0} \cdot \mathfrak{c}_0 \subset \mathfrak{p}_{0,\infty}$, i.e., each vertex in $\mathfrak{c}_0$ is in the $\mathrm{H}_D$-orbit of a vertex in $\mathfrak{p}_{0,\infty}$. If $B \notin \mathrm{vert}(\mathfrak{c}_0)$, we have not proven that $B \in \mathrm{H}_D \cdot \mathfrak{p}_{0,\infty}$, but we have proven that there exist vertices satisfying this condition in the path joining $B_s^{|\nu(s)+1|}$ to $B$. Since $\tau_{f_0} \cdot \mathfrak{p}_{s,\infty} = \mathfrak{p}_{\epsilon_0,\infty}$, where $\nu(\epsilon_0) \geqslant 1$, in the latter case we replace $B$ by $\tau_{f_0} \cdot B = B_{\epsilon_0}^{|r|}$, which leads us to the last case.

Now, assume that $\nu(s) \geqslant 1$. Then, $1/s = tf_0 + \epsilon_0$, where $f_0 \in R$ and $\nu(\epsilon_0) \geqslant 1$. So, in analogy with the previous case, let $\mathfrak{c}_0'$ be the unique finite path in $\mathfrak{b}(K)$ joining $B_s^{|\nu(s)+1|}$ with $B_s^{|2\nu(s)+\nu(\epsilon_0)|}$, when $\epsilon_0 \neq 0$, and define it as the ray joining $B_s^{|\nu(s)+1|}$ with $s$, in the remaining case. Assuming $p > \nu(s)$, we get

$$\sigma_{f_0} \cdot B_s^{|p|} = (I \cdot \tau_{tf_0} \cdot I) \cdot B_s^{|p|} = (I \cdot \tau_{tf_0}) \cdot B_{1/s}^{|p-2\nu(s)|} = I \cdot B_{\epsilon_0}^{|p-2\nu(s)|}.$$

So, since $I \cdot \mathfrak{p}_{0,\infty} = \mathfrak{p}_{0,\infty}$, we conclude that $\sigma_{f_0} \cdot B_s^{|p|} \in \mathrm{vert}(\mathfrak{p}_{0,\infty})$ precisely when $B_{\epsilon_0}^{|p-2\nu(s)|} \in \mathrm{vert}(\mathfrak{p}_{0,\infty})$, or equivalently, when $\nu(\epsilon_0) + 2\nu(s) \geqslant p$. Thus, we get that $\sigma_{f_0} \cdot \mathfrak{c}_0' \subset \mathfrak{p}_{0,\infty}$. Again, if $B \notin \mathrm{vert}(\mathfrak{c}_0')$, we have not yet proven that $B \in \mathrm{H}_D \cdot \mathfrak{p}_{0,\infty}$. But, as in the first case, we have proven that there exist vertices in the path joining $B_s^{|\nu(s)+1|}$ to $B$, which satisfy this condition. This shows that $\sigma \cdot B$ is closer to $\mathfrak{p}_{0,\infty}$ than $B$. In particular, we can keep applying either case until $B$ belongs to $\mathfrak{p}_{0,\infty}$, and we are done, i.e., we have shown that the each $\mathrm{H}_D$-orbit of vertices has a representative in $\mathrm{vert}(\mathfrak{p}_{0,\infty})$. Moreover, Theorem 1.5 shows that $\mathfrak{p}_{0,\infty}$ does not have vertices in the same $\mathrm{H}_D$-orbit. Since $\mathrm{H}_D$ acts without inversions, we conclude that $\mathfrak{t}_D$ is isomorphic to $\mathfrak{p}_{0,\infty}$, whence it is a fundamental region for the corresponding group action.

**Remark 8.3.** A similar method allows us to give another proof of Nagao's Theorem (cf. [Na59]).

**Remark 8.4.** We define a continued fraction with coefficient sets $S \subseteq K$ and $T \subseteq K^*$ as a sequence $(s_n)_{n=1}^{\infty}$, satisfying

$$s_n = f_0 + \cfrac{b_1}{f_1 + \cfrac{b_2}{f_{n-1} + \ddots \frac{b_n}{f_n}}},$$

where each $f_n \in S$, $b_n \in T$. When this sequence converges in $K$ we say that its limit has an expression as an infinite continued fraction. In [Pau02] Paulin interprets the existence of continued fractions with coefficient in $S = \mathbb{F}[t]$ and $T = \{1\}$, in terms of the action of $\mathrm{H}_0 = \mathrm{Gl}_2(\mathbb{F}[t])$ on the tree $\mathfrak{t}(K)$. These continued fractions express any element in $K$. Moreover, it is well-known that the elements $s_n$ in the sequence are the best rational Diophantine approximations of elements in $K$. In our setting, we can use a generalization of the previously introduced algorithm in order to extend Paulin's results. More specifically, with Arenas-Carmona we have shown (unpublished) the existence of continued fractions approximating all elements in $K$, associated to the action of certain arithmetical subgroups of $\mathrm{GL}_2(K)$ whose action in $\mathfrak{t}(K)$ has a small fundamental region. In particular, this applies to the case where the arithmetical subgroup is $\mathrm{H}_D$ with $D = P[0]|_{U_0}$.

**Example 8.5.** Here we exhibit a fundamental region for the action of $\mathrm{H}_D$, when $D = \mathrm{div}(t(t-1))|_{U_0} = (P[0] + P[1])|_{U_0}$ and $\mathbb{F} = \mathbb{F}_2$. In order to achieve this, we introduce a different method from the one introduced in the previous example. Indeed, the key step in this example is compute the valency of the image in $\mathfrak{t}_D$ of some vertices in $\mathfrak{b}(K)$.

Fix a vertex $x \in \mathrm{vert}(\mathfrak{s})$, where $\mathfrak{s} \subset \mathfrak{t}(K)$ is the smallest subtree containing all ends in $\{\infty, 0, 1/t, 1/(t-1)\}$, as in Theorem 1.5. We denote by $\mathfrak{v}^1(x)$ the star of $x$, i.e, the full subgraph of $\mathfrak{t}(K)$ whose vertices are precisely $x$ and its neighbors. In order to prove that $\mathfrak{s}$ is a fundamental region, we just need to show that every edge in $\mathfrak{v}^1(x)$ is in the same $\mathrm{H}_D$-orbit as some edge in $\mathfrak{s}$. Fix $s \in \{0, 1/t, 1/(t-1)\}$, and assume $x = x_n$ for some $n \in \mathbb{Z}$, where $x_i = B_s^{|i|} \in \mathfrak{p}_{s,\infty}$. Furthermore, assume that $n > \nu(s)$ when $s \neq 0$, and make no assumption on $n$ when $s = 0$. Note that every vertex in $\mathfrak{s}$ is accounted for in this way. Let $r_n$ be the edge connecting $x_n$ to $x_{n+1}$.

We begin our analysis by assuming that $s = 0$ and $-n = m \geqslant 0$. In this case we claim that the image of $x$ has valency two in $\mathfrak{t}_D$. First recall that $\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{H}_D$ precisely when $a, b, d \in \mathbb{F}[t]$, $c \in t(t-1)\mathbb{F}[t]$ and $ad - bc \in \mathbb{F}^*$. On the other hand, $\tau$ stabilizes $x$ if and only if $\nu(a), \nu(d) \geqslant 0$, $\nu(c) \geqslant m$ and $\nu(b) \geqslant -m$. Since the valuation of a non-constant polynomial is strictly negative, the two previous conditions imply that $\tau \in \mathrm{Stab}_{\mathrm{H}_D}(x_n)$ precisely when $a, d \in \mathbb{F}$, $c = 0$ and $b \in \mathbb{F}[t]_m := \{f \in \mathbb{F}[t] : \deg(f) \leqslant m\}$. In other words

$$\mathrm{Stab}_{\mathrm{H}_D} = \begin{pmatrix} \mathbb{F}^* & \mathbb{F}[t]_m \\ 0 & \mathbb{F}^* \end{pmatrix} := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \mathbb{F}^*, b \in \mathbb{F}[t]_m \right\}.$$

So, the set of unipotent elements in $\mathrm{Stab}_{\mathrm{H}_D}(x_n)$ equals

$$U_m := \begin{pmatrix} 1 & \mathbb{F}[t]_m \\ 0 & 1 \end{pmatrix} := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{F}[t]_m \right\}.$$

For each $i \in \mathbb{Z}_{\geqslant 0}$, let us introduce the group

$$\Delta_i = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \pi^{-i}\mathcal{O}_{P_\infty} \right\}.$$

It follows from [ABp, §5.1] that $\Delta_m$ acts transitively on the set of edges in $\mathfrak{v}^1(x)$ other than $r_n$. Moreover, $\Delta_{m-1}$ acts trivially on this set. Note that $U_m$ covers $\Delta_m/\Delta_{m-1}$, since $\pi^{-m}\mathcal{O}_{P_\infty}/\pi^{1-m}\mathcal{O}_{P_\infty} \cong \mathbb{F}(P_\infty) = \mathbb{F}$. Therefore, $U_m$ acts transitively on the set of edges in $\mathfrak{v}^1(x)$ other than $r_n$, whence the claim follows.

Now, assume that either $s = 0$ and $n > 0$ or $s \neq 0$ and $n > \nu(s) = 1$. In this case, we extend the previous arguments in order to show that the image of $x$ in $\mathfrak{t}_D$ has valency two, except when $s = 0$ and $n = 1$, or $s \in \{1/t, 1/(t-1)\}$ and $n = 2$. Indeed, let us fix

$$g_s = \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, we obtain $x = g_s \cdot y_n$, where $y_i := B_0^{|-i|}$. Thus, to prove the preceding statement, we just need to show that the set of unipotent elements in $g_s^{-1}\mathrm{Stab}_{\mathrm{H}_D}(x)\, g_s$ covers $\Delta_n/\Delta_{n-1}$. We compute these stabilizer subgroups next.

Indeed, note that $\tau \in g_s^{-1}\mathrm{Stab}_{\mathrm{H}_D}(x)\, g_s$ precisely when $\tau \in g_s^{-1}\mathrm{H}_D g_s$ and $\tau \in \mathrm{Stab}_{\mathrm{GL}_2(K)}(y_n)$. Let us write $\tau = g_s^{-1} g g_s$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{H}_D$. Then $\tau \in g_s^{-1}\mathrm{H}_D g_s$ if and only if

$$\tau = \begin{pmatrix} d + cs & c \\ (a-d)s - cs^2 + b & a - cs \end{pmatrix} \in \mathrm{Stab}_{\mathrm{GL}_2(K)}(y_n).$$

Equivalently, we have that $\nu(d + cs), \nu(a - cs) \geqslant 0$, $\nu(c) \geqslant -n$ and $\nu((a-d)s - cs^2 + b) \geqslant n$. If $s = 0$, then these previous conditions hold precisely when $a, d \in \mathbb{F}^*$, $b = 0$ and $c \in t(t-1)\mathbb{F}[t]_{n-2}$. So, we get

$$g_s^{-1}\mathrm{Stab}_{\mathrm{H}_D}(x)\, g_s = \left\{ \begin{pmatrix} a & c \\ 0 & d \end{pmatrix} : a, d \in \mathbb{F}^*,\ c \in t(t-1)\mathbb{F}[t]_{n-2} \right\}.$$

In particular, the set of unipotent elements in $g_s^{-1}\mathrm{Stab}_{\mathrm{H}_D}(x)\, g_s$ is exactly

$$U_n(0) := \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} : c \in t(t-1)\mathbb{F}[t]_{n-2} \right\}.$$

Thus, the image of $x = B_0^{|n|}$, $n \geqslant 2$ by the canonical projection has valency two in $\mathfrak{t}_D$, since $t(t-1)\mathbb{F}[t]_{n-2}$ covers $\pi^{-n}\mathcal{O}/\pi^{1-n}\mathcal{O} \cong \mathbb{F}(P_\infty) = \mathbb{F}$

Now, assume $s \in \{1/t, 1/(t-1)\}$ and $n \geqslant 3$. Note that $cs \in \mathbb{F}[t]$ in either case. So, we have that $a_0 = a - cs$ and $d_0 = d + cs$ are two polynomials with non-negative valuations, whence $a_0, d_0 \in \mathbb{F}$. Moreover, we have that $(a_0 - d_0)s + cs^2 + b = (a-d)s - cs^2 + b \in \pi^n\mathcal{O}$. Since $\nu(s) = 1$, we conclude that the polynomial $(a_0 - d_0)s^{-1} + c + bs^{-2}$ belongs $\pi^{n-2}\mathcal{O}$, whence it is zero. So, we can write $(a_0 - d_0) + cs + bs^{-1} = 0$. Thus, since $\mathbb{F} = \mathbb{F}_2$, we have $a_0 = d_0 = 1$, whence we conclude $c = bs^{-2} \in t(t-1)\mathbb{F}[t]$. We deduce that:

$$g_s^{-1}\mathrm{Stab}_{\mathrm{H}_D}(x)\, g_s = \left\{ \left( \begin{smallmatrix} 1 & t(t-1)s^{-1}f \\ 0 & 1 \end{smallmatrix} \right) : f \in \mathbb{F}[t]_{n-3} \right\}.$$

In particular, any element $g_s^{-1}\mathrm{Stab}_{\mathrm{H}_D}(x_n)\, g_s$ is unipotent, whence it covers $\Delta_n/\Delta_{n-1}$, since $t(t-1)s^{-1}\mathbb{F}[t]_{n-3}$ covers $\pi^{-n}\mathcal{O}/\pi^{1-n}\mathcal{O} \cong \mathbb{F}(P_\infty) = \mathbb{F}$.

It now follows from Theorem 1.5 that $\mathfrak{s}$ is contained in a fundamental region for the action of $\mathrm{H}_D$ on $\mathfrak{t}(K)$. Moreover, the previous analysis shows that $\mathfrak{s}$ contains a fundamental region, since the valency of $B_0^{|1|}$ and $B_{1/t}^{|2|}$ are both exactly three, since $\mathbb{F} = \mathbb{F}_2$. We conclude that $\mathfrak{s}$ is a fundamental region.

**Remark 8.6.** In Example 8.5, we can check that when $\mathbb{F} \neq \mathbb{F}_2$, the group

$$\left\{ \left( \begin{smallmatrix} 1 & t(t-1)s^{-1}f \\ 0 & 1 \end{smallmatrix} \right) : f \in \mathbb{F}[t]_{n-3} \right\},$$

is also contained in $g_s^{-1}\mathrm{Stab}_{\mathrm{H}_D}(x)g_s$. In particular $x$ has valency two again. Thus, the only part where the property $\mathbb{F} = \mathbb{F}_2$ is actually used is in the final paragraph, where the equality allows us to prove that there are no edges coming out from $B_0^{|1|}$ or $B_{1/t}^{|2|}$ that are not contained in $\mathfrak{s}$.

We can refine the above method to explicitly compute more complex quotient graphs up to certain degree. To do so, the key step is to properly use the Riemann-Roch equality (4.5). There exist several published computations employing this technique for $D = 0$. See [A14] and [Ma01] for more details.

**Remark 8.7.** Note that Examples 8.2 and 8.5 show simply connected quotient graphs. In [MS13, §6], Mason and Schweizer proposed the question:

*When is the quotient graph* $\mathrm{GL}_2(R)\backslash\mathfrak{t}(K)$ *a tree ?*

They indicated that the theory of Drinfeld modular curves provides a complete answer when $\mathbb{F}$ is finite (cf. [MS03]). An interesting question is if the same theory can be properly used in order to extend these results to the Hecke congruence subgroups $\mathrm{H}_D$. This can be eventually studied in order to give some complementary results to Theorem 1.4, 1.6 and 1.5.

# 9. Stabilizers and amalgams

In this section we analyze the structure of $\mathrm{H}_D$ as an amalgam. More specifically, the main goal of this section is to prove Theorems 1.6 and 1.7. For this reason, in all that follows we assume that $g(2)$ is trivial and that each $n_i$ is an odd positive integer. We also assume that $\mathrm{Supp}(D) \neq \varnothing$, since any other case can be reduced to Serre's result. In order to prove the aforementioned results we extensively use Bass-Serre theory (cf. [Se80, Chapter I, §5]).

Let $\mathbf{C}$ be a set indexing all the cusps in $\mathfrak{t}_D$. It follows from Theorem 1.4 that $\mathfrak{t}_D$ is the union of a finite graph $Y$ with a finite number of cuspidal rays, namely $\mathfrak{r}(\sigma)$, for $\sigma \in \mathbf{C}$. Moreover, the same results implies the following identity:

$$\mathrm{Card}(\mathbf{C}) = c(\mathrm{H}_D) = 2^r |g(2)| \left| \frac{2\mathrm{Pic}(\mathcal{C}) + \langle \overline{P_\infty} \rangle}{\langle \overline{P_\infty} \rangle} \right| \left( 1 + \frac{1}{q-1} \prod_{i=1}^{r} \left( q^{\deg(P_i)\lfloor \frac{n_i}{2} \rfloor} - 1 \right) \right).$$

Now, we choose a maximal tree $T$ of $\mathfrak{t}_D$ and a lift $j : T \to \mathfrak{t} = \mathfrak{t}(K)$. Note that each cuspidal ray $\mathfrak{r}(\sigma)$ of $\mathfrak{t}_D$ is contained in $T$, whence $j(\mathfrak{r}(\sigma))$ is a ray of $\mathfrak{t}$. More explicitly, we can fix a tree $T$ by taking the union of the cuspidal rays $\mathfrak{r}(\sigma)$ with a maximal tree in the finite graph $Y \subseteq \mathfrak{t}_D$.

9.1. **Review of Bass-Serre Theory.** Let us recall some definitions from §2. Let $(s, t, r)$ and $(\tilde{s}, \tilde{t}, \tilde{r})$ be the triplets indicating source, target and reverse maps for the graphs $\mathfrak{t}$ or $\mathfrak{t}_D$ respectively. An orientation on $\mathfrak{t}_D$ is a subset $O$ of $\mathrm{E}(\mathfrak{t}_D)$ such that $\mathrm{E}(\mathfrak{t}_D)$ is the disjoint union of $O$ and $\tilde{r}(O)$. In order to simplify some of the subsequent definitions, let us fix an orientation $O$ for $\mathfrak{t}_D$, and set $o(y) = 0$, if $y \in O$, while $o(y) = 1$, if $y \notin O$, i.e., if $\tilde{r}(y) \in O$.

We extend $j$ to a function $j : \mathrm{E}(\mathfrak{t}_D) \to \mathrm{E}(\mathfrak{t})$ satisfying the relation

$$j(\tilde{r}(y)) = r(j(y)), \tag{9.1}$$

as follows: For each $y \in O \smallsetminus \mathrm{E}(T)$, we choose $j(y)$ so that $s(j(y)) \in \mathrm{V}(j(T))$. For the remaining edges we define $j(y)$ by the relation (9.1). Note that we have $s(j(y)) = j(\tilde{s}(y))$, for all $y \in O$. In general, however, the corresponding relation for the target does not hold. Next, for each $y \in O \smallsetminus \mathrm{E}(T)$ we choose $g_y \in \mathrm{H}_D$ satisfying $t(j(y)) = g_y \cdot j(\tilde{t}(y))$. This is always possible since $t(j(y))$ and $j(\tilde{t}(y))$ have the same image $\tilde{t}(y)$ in the quotient set $\mathrm{V}(\mathfrak{t}_D)$. Now, we extend the map $y \mapsto g_y$ to all edges in $\mathfrak{t}_D$ by setting $g_y = \mathrm{id}$, for all $y \in \mathrm{E}(T)$, and for all remaining edges $g_{r(y)} = g_y^{-1}$. Note that the latter relation holds for each pair of reverse edges. Therefore, for each edge $y$ in the quotient graph, we get $s(j(y)) = g_y^{-o(y)} j(\tilde{s}(y))$ and $t(j(y)) = g_y^{1-o(y)} j(\tilde{t}(y))$.

For each vertex $\overline{v} \in \mathrm{V}(\mathfrak{t}_D)$, we define $\mathrm{Stab}_{\mathrm{H}_D}(\overline{v})$ as the stabilizer in $\mathrm{H}_D$ of the lift $j(\overline{v})$. An analogous convention applies to an edge $y$. Thus, for each pair $(\overline{v}, y)$ where $\overline{v} = \tilde{t}(y)$, we have a morphism $f_y : \mathrm{Stab}_{\mathrm{H}_D}(y) \to \mathrm{Stab}_{\mathrm{H}_D}(\overline{v})$ defined by $g \mapsto g_y^{o(y)-1} g g_y^{1-o(y)}$. This function is well defined since

$$g_y^{o(y)-1} \mathrm{Stab}_{\mathrm{H}_D}\big(j(y)\big) g_y^{1-o(y)} \subseteq \mathrm{Stab}_{\mathrm{H}_D}\Big(j\big(\tilde{t}(y)\big)\Big).$$

Thus, the data presented above allow us to define the graph of groups $(\mathfrak{h}_D, \mathfrak{t}_D) = (\mathfrak{h}_D, T, \mathfrak{t}_D)$ associated to the action of $\mathrm{H}_D$ on $\mathfrak{t}$ (cf. [Se80, Chapter I, §4.4]).

Now, we can define the fundamental group associated to this graph of groups. Indeed, let $F(\mathfrak{h}_D, \mathfrak{t}_D)$ be the group generated by all $\mathrm{Stab}_{\mathrm{H}_D}(\overline{v})$, where $\overline{v} \in \mathrm{V}(\mathfrak{t}_D)$, and elements $a_y$, for each $y \in \mathrm{E}(\mathfrak{t}_D)$, subject to the relations

$$a_{\tilde{r}(y)} = a_y^{-1}, \text{ and } a_y f_y(b) a_y^{-1} = f_{\tilde{r}(y)}(b), \ \forall y \in \mathrm{E}(\mathfrak{t}_D), \ \forall b \in \mathrm{Stab}_{\mathrm{H}_D}(y).$$

The fundamental group $\pi_1(\mathfrak{h}_D) = \pi_1(\mathfrak{h}_D, \mathfrak{t}_D)$ is, by definition, the quotient of $F(\mathfrak{h}_D, \mathfrak{t}_D)$ by the normal subgroup generated by the elements $a_y$ for $y \in \mathrm{E}(T)$. In other words, if we denote by $h_y$ the image of $a_y$ in $\pi_1(\mathfrak{h}_D, \mathfrak{t}_D)$, then the group $\pi_1(\mathfrak{h}_D, \mathfrak{t}_D)$ is generated by all $\mathrm{Stab}_{\mathrm{H}_D}(\overline{v})$, where $\overline{v} \in \mathrm{V}(\mathfrak{t}_D)$, and the elements $h_y$, for $y \in \mathrm{E}(\mathfrak{t}_D)$, subject to the relations

$$h_{\tilde{r}(y)} = h_y^{-1}, \quad h_y f_y(b) h_y^{-1} = f_{\tilde{r}(y)}(b), \quad \text{and} \quad h_z = \mathrm{id},$$

for all $(z, y) \in \mathrm{E}(T) \times \mathrm{E}(\mathfrak{t}_D)$ and for all $b \in \mathrm{Stab}_{\mathrm{H}_D}(y)$. It can be proven that the group $\pi_1$ is independent, up to isomorphism, of the choice of the graph of groups $\mathfrak{h}_D$, and in particular of the tree $T \subset \mathfrak{t}_D$.

As mentioned in §1, Bass-Serre Theory implies that all subgroups of $\mathrm{GL}_2(K)$ can be described from their actions on $\mathfrak{t}(K)$ (cf. [Se80, Chapter I, §5.4]). More specifically, they are isomorphic to their corresponding fundamental groups, as defined above (cf. [Se80, Chapter I, §5, Theorem 13]). In our case, $\mathrm{H}_D$ isomorphic to the fundamental group $\pi_1(\mathfrak{h}_D) = \pi_1(\mathfrak{h}_D, \mathfrak{t}_D)$.

Let $\mathfrak{r}(\sigma)$ be a cuspidal ray in $\mathfrak{t}_D$. We denote by $\mathcal{P}_\sigma$ the fundamental group $\pi_1(\mathfrak{h}_D|_{\mathfrak{r}(\sigma)})$ of the restriction of $\mathfrak{h}_D$ to $\mathfrak{r}(\sigma)$. Analogously, we define $H = \pi_1(\mathfrak{h}_D|_Y)$, for $Y$ as in Theorem 1.4. For each $\sigma$ as above, let $\mathcal{B}_\sigma$ be the vertex stabilizer in $\mathrm{H}_D$ of the unique vertex in $Y \cap \mathfrak{r}(\sigma)$. We have canonical injections $\mathcal{B}_\sigma \to \mathcal{P}_\sigma$ and $\mathcal{B}_\sigma \to H$. Now, as Serre points out in [Se80, Chapter II, §2.5, Theorem 10], if we have a graph of groups $\mathfrak{h}$, which is obtained by "gluing" two graphs of groups $\mathfrak{h}_1$ and $\mathfrak{h}_2$ by a tree of groups $\mathfrak{h}_{12}$, then there exist two injections $\iota_1 : \mathfrak{h}_{12} \to \mathfrak{h}_1$ and $\iota_2 : \mathfrak{h}_{12} \to \mathfrak{h}_2$, such that $\pi_1(\mathfrak{h})$ is isomorphic to the sum of $\pi_1(\mathfrak{h}_1)$ and $\pi_1(\mathfrak{h}_2)$, amalgamated along $\pi_1(\mathfrak{h}_{12})$ according to $\iota_1$ and $\iota_2$. In our context, we conclude that $\mathrm{H}_D$ is isomorphic to the sum of $\mathcal{P}_\sigma$, for all $\sigma$, and $H$, amalgamated along their common subgroups $\mathcal{B}_\sigma$ according to the above injections. Since $\mathfrak{r}(\sigma) \subseteq T$, each $\mathcal{P}_\sigma$ coincides with the direct limit of the vertex stabilizers defined by all $v \in \mathrm{V}(\mathfrak{r}(\sigma))$. In all that follows, we exploit this property of the groups $\mathcal{P}_\sigma$, in order to describe them in more detail.

We start with some comments on the previous choice that simplify our work. Note that, in our context, each vertex stabilizer is finite, since it is the intersection of a compact set with a discrete subgroup of $\mathrm{GL}_2(k)$. So, replacing $\mathfrak{r}(\sigma)$ by another equivalent cuspidal ray has no effect in the statements in Theorem 1.6. Hence, to prove the aforementioned theorem, we only need to describe the groups $\mathcal{P}_\sigma$ for a suitable set of cusp rays. We describe a convenient choice in what follows.

9.2. **On vertex stabilizers.** For any closed point $Q$ in $\mathcal{C}$, any $s \in k$ and any $n \in \mathbb{Z}$, let $\mathfrak{D}(s, n, Q)$ be the $\mathcal{O}_Q$-maximal order defined by

$$\mathfrak{D}(s, n, Q) = \begin{pmatrix} 1 & 0 \\ s & \pi_Q^n \end{pmatrix} \mathbb{M}_2(\mathcal{O}_Q) \begin{pmatrix} 1 & 0 \\ s & \pi_Q^n \end{pmatrix}^{-1}. \tag{9.2}$$

We denote by $\mathcal{O}$ the ring of local integers at $P_\infty$, and we fix a uniformizing parameter $\pi \in \mathcal{O}$. We define $v_n(s) \in \mathrm{V}(\mathfrak{t})$ as the vertex corresponding to the $\mathcal{O}$-maximal order $\mathfrak{D}(s, n, P_\infty)$. For any $s \in \mathbb{P}^1(k)$, we define the $R$-ideal $\mathcal{Q}_s$ by $\mathcal{Q}_s = R \cap s^{-1}R \cap s^{-2}R$, when $s \in k^*$, and by $\mathcal{Q}_s = R$, when $s \in \{0, \infty\}$. Let us write $R(n) = \{a \in R :$

$\nu(a) \geqslant -n\}$. Recall that $I_D$ denotes the $R$-ideal $\mathfrak{L}^{-D}(U_0)$, where $U_0 = \mathcal{C} \smallsetminus \{P_\infty\}$ (cf. Equation (1.1)). So, the next lemma follows immediately from [Ma01, Lemma 3.2] and [Ma01, Lemma 3.4].

**Lemma 9.1.** Assume first that $s = 0$ and $n < 0$. Then

$$\mathrm{Stab}_{\mathrm{H}_D}(v_n(0)) = \left\{ \begin{pmatrix} \alpha & c \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in \mathbb{F}^* \text{ and } c \in R(-n) \right\}. \tag{9.3}$$

On the other hand, if $s = 0$ and $n \geqslant 1$, then

$$\mathrm{Stab}_{\mathrm{H}_D}(v_n(0)) = \left\{ \begin{pmatrix} \alpha & 0 \\ c & \beta \end{pmatrix} : \alpha, \beta \in \mathbb{F}^* \text{ and } c \in R(n) \cap I_D \right\}. \tag{9.4}$$

Finally, if $s \in k \smallsetminus \mathbb{F}$ and $n \deg(P_\infty) > \deg(\mathcal{Q}_s)$, then the element $g \in \mathrm{GL}_2(k)$ belongs to $\mathrm{Stab}_{\mathrm{H}_D}(v_n(s))$ if and only if it has the form

$$g = A(\alpha, \beta, c) = \begin{pmatrix} \beta - sc & (\alpha - \beta)s + s^2 c \\ -c & \alpha + sc \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -s \end{pmatrix}^{-1} \begin{pmatrix} \alpha & c \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -s \end{pmatrix}, \tag{9.5}$$

where

(a) $\alpha, \beta \in \mathbb{F}^*$, and
(b) $c \in R(n) \cap I_D \cap Rs^{-1} \cap ((\beta - \alpha)s^{-1} + Rs^{-2})$.

In all cases, the stabilizer group $\mathrm{Stab}_{\mathrm{H}_D}(v_n(s))$ contains triangularizable matrices only. Moreover, in (9.4) and (9.5), the matrix group

$$\mathrm{Sb}_s(n) := \{A(\alpha, \alpha, c) : \alpha \in \mathbb{F}^*, c \in I_D \cap R(n) \cap \mathcal{Q}_s\},$$

which is always isomorphic to $\mathbb{F}^* \times (R(n) \cap \mathcal{Q}_s \cap I_D)$, is contained in $\mathrm{Stab}_{\mathrm{H}_D}(v_n(s))$.

Let $\mathbf{C}$ be a set indexing all the cusps in $\mathfrak{t}_D$. For each $\sigma \in \mathbf{C}$, let $\mathfrak{r}(\sigma)$ be a cuspidal ray in $\mathfrak{t}_D$ representing the corresponding cusp, and let $j(\mathfrak{r}(\sigma))$ be its lift to $\mathfrak{t}$. We can assume that its visual limit $\xi = \xi_\sigma$ is in $\mathbb{P}^1(k)$ by Corollary 3.8, and we assume moreover that one of them is $\infty$. It follows from the previous lemma that there exists a ray $\mathfrak{r}'(\sigma)$, equivalent to $\mathfrak{r}(\sigma)$, with a vertex set $\{\overline{v_i} : i = 1, \cdots, \infty\}$, where any pair of neighboring vertices $(\overline{v_i}, \overline{v_{i+1}})$ has the property $\mathrm{Stab}_{\mathrm{H}_D}(j(\overline{v_i})) \subset \mathrm{Stab}_{\mathrm{H}_D}(j(\overline{v_{i+1}}))$. Then, up to changing the representing cuspidal ray for each class, we can assume that the previous inclusion holds for each $\mathfrak{r}(\sigma)$ in $\mathfrak{t}_D$. In particular, in this case, the direct limit $\mathcal{P}_\sigma$ coincides with the increasing union $\bigcup_{i=1}^{\infty} \mathrm{Stab}_{\mathrm{H}_D}(v_i)$. Thus, in order to describe $\mathcal{P}_\sigma$, we only need to study the stabilizers of some unbounded subset $\{v_{\alpha(i)}\}_{i=1}^{\infty} \subseteq \mathrm{V}(j(\mathfrak{r}(\sigma)))$. So, let $\mathfrak{r}(\infty) \subseteq \mathfrak{t}_D$ be the cuspidal ray at infinity, i.e., the projection on $\mathfrak{t}_D$ of a ray whose vertex set is $\{v_{-n}(0) : n \geqslant N_0\}$, for certain suitable integer $N_0$. Then, Lemma 9.1 directly shows that $\mathcal{P}_\infty$ is isomorphic to $(\mathbb{F}^* \times \mathbb{F}^*) \ltimes R$.

When $\mathfrak{r}(\sigma)$ is different from $\mathfrak{r}(\infty)$ we have to work with other tools. We do this in what follows. Let us fix a cuspidal ray $\mathfrak{r}(\sigma)$ different from $\mathfrak{r}(\infty)$. Then, the vertex set of $j(\mathfrak{r}(\sigma))$ equals $\{v_n(\xi) : n \geqslant N_0\}$, where $\xi$ and $N_0$ depend on $\sigma$. It follows from Lemma 9.1 that the maximal unipotent subgroup of each $\mathrm{Stab}_{\mathrm{H}_D}(v_n(\xi))$ is isomorphic to the intersection of $R(n)$ with the $R$-ideal $\mathcal{Q}_\xi \cap I_D$. Therefore, since $\bigcup_{n > N_0} R(n) = R$, it follows that the maximal unipotent subgroup of $\mathcal{P}_\sigma$ is isomorphic to the $R$-ideal $\mathcal{Q}_\xi \cap I_D$. Thus, by Equation (9.5), in order to describe $\mathcal{P}_\sigma$, we only need to characterize the eigenvalues of some elements in $\mathrm{Stab}_{\mathrm{H}_D}(v_n(\xi))$.

We start by recalling some relevant definitions from §6. As before, $\Gamma \subset \mathrm{PGL}_2(k)$ denotes the stabilizer of $\mathfrak{E}_D(U_0)$, where $\mathfrak{E}_D$ is the Eichler $\mathcal{C}$-order defined in (4.4). So, for each $v \in \mathrm{V}(\mathfrak{t})$, its image $\overline{v} \in \mathrm{V}(\Gamma \backslash \mathfrak{t})$ represents one and only one $\mathcal{C}$-Eichler order of level $D$, or equivalently one abstract $D$-grid. Next, we make explicit this correspondence for any vertex $v = v_n(\xi)$. It follows from condition (c) in §5 that, given the family of local orders $\{\mathfrak{E}(P) : P \in |\mathcal{C}|\}$ defined by

($E_1$) $\mathfrak{E}(P_i) = \mathfrak{D}(0, 0, P_i) \cap \mathfrak{D}(0, n_i, P_i)$, for any $i \in \{1, \cdots, n\}$,
($E_2$) $\mathfrak{E}(P_\infty) = \mathfrak{D}(\xi, n, P_\infty)$, and
($E_3$) $\mathfrak{E}(Q) = \mathfrak{D}(0, 0, Q)$, for any $Q \neq P_1, \cdots, P_n, P_\infty$,

there exists an Eichler $\mathcal{C}$-order $\mathfrak{E} = \mathfrak{E}[v]$ such that $\mathfrak{E}_P = \mathfrak{E}(P)$, for all $P \in |\mathcal{C}|$. Then, the abstract $D$-grid corresponding to $v$ is $\mathbb{S}(v) = \left[S(\mathfrak{E}[v])\right]$. Observe that the level of $\mathfrak{E}$ is equal to $D = \sum_{i=1}^{n} n_i P_i$.

Now, by definition, $g \in \mathrm{Stab}_{\mathrm{H}_D}(v_n(\xi))$ if and only if $g \in \mathrm{H}_D = \mathfrak{E}_D(U_0)^*$ and $g \in \mathrm{Stab}_{\mathrm{GL}_2(k)}(v_n(\xi))$. Observe that $g \in \mathfrak{E}_D(U_0)^*$ is equivalent to $g \in \mathfrak{E}_D(U_0)$ and $\det(g) \in R^* = \mathbb{F}^*$. The following result describes the normalizer of local maximal orders. See [A16, §3] for details.

**Lemma 9.2.** For any closed point $Q$, the normalizer in $\mathrm{GL}_2(k_Q)$ of a local maximal order $\mathfrak{D}(s, n, Q)$ is $k_Q^* \mathfrak{D}(s, n, Q)^*$.

*Proof.* Since any $\mathcal{O}_Q$-maximal order is the ring of endomorphisms of an $\mathcal{O}_Q$-lattice, we can write $\mathfrak{D}(s, n, Q) = \mathrm{End}_{\mathcal{O}_Q}(\Lambda)$, for some lattice $\Lambda = \Lambda(s, n, Q)$ of $k_Q \times k_Q$. Then $g \in \mathrm{GL}_2(k_Q)$ normalizes $\mathfrak{D}(s, n, Q)$ if and only if $\mathrm{End}_{\mathcal{O}_Q}(\Lambda) = \mathrm{End}_{\mathcal{O}_Q}(g(\Lambda))$. So, since two lattices have the same endomorphism rings precisely when they belong to the same homothety class, we have that $\mathrm{End}_{\mathcal{O}_Q}(\Lambda) = \mathrm{End}_{\mathcal{O}_Q}(g(\Lambda))$ precisely when $g(\Lambda) = \lambda\Lambda$, for some $\lambda \in k_Q^*$, i.e. $\lambda^{-1} g \in \mathrm{End}_{\mathcal{O}_Q}(\Lambda)^* = \mathfrak{D}(s, n, Q)^*$. □

We deduce from the preceding lemma that $g \in \mathfrak{E}_D(U_0)^*$ if and only if the following conditions hold:

- $\det(g) \in \mathbb{F}^*$,
- $g$ normalizes $\mathfrak{D}(0, m, P_i)$, for each $P_i \in \mathrm{Supp}(D)$ and $m \in \{0, \cdots, n_i\}$, and
- $g$ normalizes $\mathfrak{D}(0, 0, Q)$, for every $Q \neq P_1, \cdots, P_r, P_\infty$.

Thus, we conclude that $g$ belongs to $\mathrm{Stab}_{\mathrm{H}_D}(v_n(\xi))$ precisely when it satisfies conditions ($A_1$) – ($A_4$) below:

($A_1$) $\det(g) \in \mathbb{F}^*$,
($A_2$) $g$ normalizes $\mathfrak{D}(0, m, P_i)$, for each $P_i \in \mathrm{Supp}(D)$ and $m \in \{0, \cdots, n_i\}$,
($A_3$) $g$ normalizes $\mathfrak{D}(0, 0, Q)$, for every $Q \neq P_1, \cdots, P_r, P_\infty$, and
($A_4$) $g$ normalizes $\mathfrak{D}(\xi, n, P_\infty)$.

Recall that we only have to describe the vertex stabilizers for an arbitrary unbounded set of vertex of $j(\mathfrak{r}(\sigma))$. Then, by changing $v_n(\xi)$ to $v_{n+1}(\xi)$ if needed, we can assume without loss of generality that the type of $v_n(\xi)$ coincides with the type of $v_0(0)$. Thus, there exists $h_{P_\infty} \in \mathrm{SL}_2(k_{P_\infty})$ such that $h_{P_\infty} \cdot v_n(\xi) = v_0(0)$, i.e., $h_{P_\infty} \mathfrak{D}(\sigma, n, P_\infty) h_{P_\infty}^{-1} = \mathfrak{D}(0, 0, P_\infty)$. Now, it follow from Lemma 9.2 that the $\mathrm{GL}_2(k_Q)$-normalizer of local maximal orders $\mathfrak{D}_Q$ are open. So, by the Strong Approximation Theorem applied on the open set $\mathcal{C} \smallsetminus \mathrm{Supp}(D)$, there exists $h = h(v) \in \mathrm{SL}_2(k)$ satisfying $h\mathfrak{D}(\sigma, n, P_\infty)h^{-1} = \mathfrak{D}(0, 0, P_\infty)$ and normalizing each $\mathfrak{D}(0, 0, Q)$, for $Q \neq P_1, \cdots, P_r, P_\infty$. For each $P_i \in \mathrm{Supp}(D)$, let $\mathfrak{s}_i$ be the finite line in $\mathfrak{t}(k_{P_i})$ whose vertex set is $\{h^{-1}\mathfrak{D}(0, m, P_i)h : m \in \{0, \cdots, n_i\}\}$. We define $S = S(v)$ as the concrete $D$-grid $S = \prod_{i=1}^{n} \mathfrak{s}_i$. Note that $S = hS(\mathfrak{E}[v])h^{-1}$ is another representative of $\mathbb{S} = \mathbb{S}(v)$. Thus, we deduce from conditions ($A_1$) – ($A_4$) that, $g$ belongs to $\mathrm{Stab}_{\mathrm{H}_D}(v_n(\xi))$ if and only if $\widetilde{g} = hgh^{-1} \in \mathrm{GL}_2(k)$ satisfies the following:

($B_1$) $\det(\widetilde{g}) \in \mathbb{F}^*$,

($B_2$) $\widetilde{g}$ normalizes each maximal $\mathcal{C}$-order in the vertex set of $S$, and
($B_3$) $\widetilde{g}$ normalizes each local maximal order $\mathfrak{D}(0,0,Q)$, where $Q \neq P_1, \cdots, P_r$.

Let $v_{n+2}(\xi)$ be the vertex at distance two from $v_n(\xi)$ towards $\xi$. Since $C_{P_\infty}(\mathbb{O}_D)$ is combinatorially finite, for $n \gg 1$, we have $\overline{v_n(\xi)} \neq \overline{v_{n+2}(\xi)}$. Moreover, it follows from Proposition 7.11 and Corollary 7.15 that there exists a suitable integer $N_\sigma$ such that for all $n > N_\sigma$ the actual $D$-grid $S = S(v_n(\sigma))$ has a semi-decomposition datum with positive degree.

First, assume that $S$ has a total-decomposition datum $(\beta, B, (n_i)_{i=1}^r)$. Let $A = A(\beta)$ be be the base change matrix from the canonical basis to $\beta$, and let $\mathfrak{E} = A^{-1}\mathfrak{E}[B, B+D]A$ be the split Eichler $\mathcal{C}$-order, defined as the intersection of all maximal orders in the vertex set of $S$. Then, it follows from Proposition 7.9, that there exists a global idempotent matrix $\epsilon_1 \in \mathfrak{E}(\mathcal{C})$. Since $\mathcal{T} := \mathbb{F}^*\epsilon_1 + \mathbb{F}^*(\mathrm{id} - \epsilon_1)$ is contined in $\mathfrak{E}(\mathcal{C})^*$, any matrix in $\mathcal{T}$ normalizes the Eichler $\mathcal{C}$-order $\mathfrak{E}$. In other words, any matrix $g \in \mathcal{T}$ satisfies the properties ($B_1$), ($B_2$) and ($B_3$). Thus, for any pair of elements $a, b \in \mathbb{F}^*$ there is a matrix $g_{a,b} \in \mathrm{Stab}_{\mathrm{H}_D}(v_n(\xi))$ whose eigenvalues are $a$ and $b$. Then, since the group generated by $\{g_{a,b} : a, b \in \mathbb{F}^*\}$ and the maximal unipotent subgroup of $\mathrm{Stab}_{\mathrm{H}_D}(v_n(\xi))$ equals

$$\{A(\alpha, \beta, c) : \alpha, \beta \in \mathbb{F}^*, c \in R(n) \cap \mathcal{Q}_\xi \cap I_D\},$$

we conclude from Lemma 9.1 that

$$\mathrm{Stab}_{\mathrm{H}_D}(v_n(\xi)) \cong (\mathbb{F}^* \times \mathbb{F}^*) \ltimes (R(n) \cap Q_\xi \cap I_D). \tag{9.6}$$

Now, assume that $S$ has a non total semi-decomposition datum $(\beta, B, (s_i)_{i=1}^r)$, and let $A = A(\beta)$ as before. Then $A^{-1}\widetilde{g}A$ normalizes each maximal $\mathcal{C}$-order in the vertex set of $A^{-1}SA$. In particular, the matrix $A^{-1}\widetilde{g}A$ fixes $\mathfrak{D}_B$, with $\deg(B) > 0$, whence $A^{-1}\widetilde{g}A = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ with $x^{-1}z \in \mathbb{F}^*$ (cf. [A14, §4, Proposition 4.1]). So, by taking $S^\circ = S$ in the proof of Proposition 7.19, we deduce that $x = z$. Thus, the eigenvalues of $\widetilde{g}$ are equal. The same holds for $g$. Therefore, we conclude from Lemma 9.1 that $\mathrm{Stab}_{\mathrm{H}_D}(v_n(\xi)) = \mathrm{Sb}_\xi(n)$ in this case. In particular, we have that

$$\mathrm{Stab}_{\mathrm{H}_D}(v_n(\xi)) \cong \mathbb{F}^* \times (R(n) \cap Q_\xi \cap I_D). \tag{9.7}$$

9.3. **End of proof of Theorem 1.6 and 1.7.** Note that, it follows from Corollary 7.15 that we can assume that the semi-decomposition vectors of $\mathbb{S}(v_n(\xi))$ and $\mathbb{S}(v_{n+2}(\xi))$ are equal. Then, by the arguments presented above, we deduce that:

- $\mathrm{Stab}_{\mathrm{H}_D}(v_{n+2}(\xi)) \cong (\mathbb{F}^* \times \mathbb{F}^*) \ltimes (R(n+2) \cap Q_\xi)$, if $\mathbb{S}(v_{n+2}(\xi))$ is split, and
- $\mathrm{Stab}_{\mathrm{H}_D}(v_{n+2}(\xi)) \cong \mathbb{F}^* \times (R(n+2) \cap Q_\xi)$, if not.

An inductive argument shows that, for each $t \in \mathbb{Z}_{\geqslant 0}$, $\mathrm{Stab}_{\mathrm{H}_D}(v_{n+2t}(\xi))$ is isomorphic to

- $(\mathbb{F}^* \times \mathbb{F}^*) \ltimes (R(n+2t) \cap Q_\xi \cap I_D)$, if $\mathbb{S}(v_{n+2t}(\xi))$ is split, and
- $\mathbb{F}^* \times (R(n+2t) \cap Q_\xi \cap I_D)$, if not.

Now, we say that $\mathfrak{r}(\sigma)$ is split when it only contains vertices corresponding to split abstract grids. Since we can assume that $\mathbb{S}(v_n(\xi))$ and $\mathbb{S}(v_{n+t}(\xi))$ correspond to vertices at distance $t > 0$ in the same cuspidal ray of $C_{P_\infty}(\mathbb{O}_D)$, we get from Corollary 7.15 that they have the same semi-decomposition vectors. In particular, if $\mathfrak{r}(\sigma)$ is not split, then every vertex in $\mathfrak{r}(\sigma)$ corresponds to a nonsplit abstract grid. Thus, we conclude

$$\mathcal{P}_\sigma \cong \begin{cases} (\mathbb{F}^* \times \mathbb{F}^*) \ltimes (Q_\xi \cap I_D) & \text{if } \mathfrak{r}(\sigma) \text{ is split,} \\ \mathbb{F}^* \times (Q_\xi \cap I_D) & \text{if not.} \end{cases} \tag{9.8}$$

Since each $n_i$ is odd by hypothesis, it follows from Proposition 7.24 that there are

$$c(D)/\alpha(D) = [2\mathrm{Pic}(\mathcal{C}) + \langle \overline{P_1}, \cdots, \overline{P_r}, \overline{P_\infty} \rangle : \langle \overline{P_\infty} \rangle],$$

split cusps in $\Gamma \backslash \mathfrak{t}$. Moreover, it follows from Lemma 7.2 that there are exactly $[\Gamma : \mathrm{PH}_D]$ cusps of $\mathfrak{t}_D$ with the same image in $\Gamma \backslash \mathfrak{t}$. Thus, we conclude from Equation (7.2) that there are $2^r[2\mathrm{Pic}(\mathcal{C}) + \langle \overline{P_\infty} \rangle : \langle \overline{P_\infty} \rangle]$ elements $\sigma \in \mathbf{C}$ such that $\mathfrak{r}(\sigma)$ is split. Then, Theorem 1.6 follows. Furthermore, Theorem 1.7 is an immediate application of [Se80, Chapter II, Proposition 14, Corollary 1] in this context.

**Example 9.3.** Let $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$, $R = \mathbb{F}[t]$, and let $D$ be the principal divisor of $t$, as in Example 8.2. In this case $\mathfrak{t}_D$ is isomorphic to the double ray in $\mathfrak{t}$ whose vertex set is $\{v_n(0) : n \in \mathbb{Z}\}$. See Example 8.2 for more details. Let $\mathbb{F}[t]_n$ be the set of polynomials whose degree is less or equal than $n$. Then, it follows from Lemma 9.1 that

- $\mathrm{Stab}_{\mathrm{H}_D}(v_n(0)) = \begin{pmatrix} \mathbb{F}^* & \mathbb{F}[t]_{-n} \\ 0 & \mathbb{F}^* \end{pmatrix}$, if $n < 0$,
- $\mathrm{Stab}_{\mathrm{H}_D}(v_n(0)) = \begin{pmatrix} \mathbb{F}^* & 0 \\ 0 & \mathbb{F}^* \end{pmatrix}$, if $n = 0$,
- $\mathrm{Stab}_{\mathrm{H}_D}(v_n(0)) = \begin{pmatrix} \mathbb{F}^* & 0 \\ t\mathbb{F}[t]_{n-1} & \mathbb{F}^* \end{pmatrix}$, if $n > 0$.

We conclude that $\mathrm{H}_D$ is isomorphic to the sum of $\left(\begin{smallmatrix} \mathbb{F}^* & \mathbb{F}[t] \\ 0 & \mathbb{F}^* \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mathbb{F}^* & 0 \\ t\mathbb{F}[t] & \mathbb{F}^* \end{smallmatrix}\right)$ amalgamated along the diagonal group $\mathbb{F}^* \times \mathbb{F}^*$

**Example 9.4.** Assume that $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$, $\mathbb{F} = \mathbb{F}_2$, $R = \mathbb{F}[t]$, and let $D$ be the principal divisor of $t(t+1)$, as in Example 8.5. So, it is not hard to see from the computations in Example 8.5:

- $\mathrm{Stab}_{\mathrm{H}_D}(v_n(0)) = \begin{pmatrix} \mathbb{F}^* & \mathbb{F}[t]_{-n} \\ 0 & \mathbb{F}^* \end{pmatrix}$, if $n < 0$,
- $\mathrm{Stab}_{\mathrm{H}_D}(v_n(0)) = \begin{pmatrix} \mathbb{F}^* & 0 \\ 0 & \mathbb{F}^* \end{pmatrix}$, if $n = 0, 1$,
- $\mathrm{Stab}_{\mathrm{H}_D}(v_n(0)) = \begin{pmatrix} \mathbb{F}^* & 0 \\ t(t+1)\mathbb{F}[t]_n & \mathbb{F}^* \end{pmatrix}$, if $n > 1$,
- $\mathrm{Stab}_{\mathrm{H}_D}(v_2(1/t)) = \{\mathrm{id}\}$, and
- $\mathrm{Stab}_{\mathrm{H}_D}(v_n(s)) = \mathrm{id} + \mathbb{F}[t]_{n-3} \begin{pmatrix} s_c^{-1}s^{-1} & -s_c^{-1} \\ s_c^{-1}s^{-2} & -s_c^{-1}s^{-1} \end{pmatrix}$, for each rational $s \in \{1/t, 1/(t+1)\}$, $s_c \in \{1/t, 1/(t+1)\} \smallsetminus \{s\}$ and $n > 2$.

So, let us define:

- $\mathcal{P}_\infty = \begin{pmatrix} \mathbb{F}^* & \mathbb{F}[t] \\ 0 & \mathbb{F}^* \end{pmatrix}$,
- $\mathcal{P}_0 = \begin{pmatrix} \mathbb{F}^* & 0 \\ t(t+1)\mathbb{F}[t] & \mathbb{F}^* \end{pmatrix}$, and
- $\mathcal{P}_{1/t} = \mathrm{id} + \mathbb{F}[t] \begin{pmatrix} t(t+1) & -(t+1) \\ t^2(t+1) & -t(t+1) \end{pmatrix}$, and
- $\mathcal{P}_{1/(t+1)} = \mathrm{id} + \mathbb{F}[t] \begin{pmatrix} t(t+1) & -t \\ t(t+1)^2 & -t(t+1) \end{pmatrix}$.

Then, since $\mathbb{F}^* = \{1\}$, we conclude that $\mathrm{H}_D$ is isomorphic to the free product of groups $\mathcal{P}_\sigma$, where $\sigma \in \{0, \infty, 1/t, 1/(t+1)\}$.

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