UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

ESTUDIO DE PROBLEMAS DE DISEÑO ÓPTIMO POR EL MÉTODO DE REGULARIDAD EN ECUACIONES NO LINEALES.

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA MENCIÓN MODELACIÓN MATEMÁTICA EN COTUTELA CON LA UNIVERSIDAD DE SEVILLA.

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Este trabajo ha sido parcialmente financiado por CONICYT PFCHA/DOCTORADO BECAS CHILE/2018 - 21182101, CMM ANID PIA AFB170001, CMM ANID BASAL ACE210010 y CMM ANID BASAL FB210005.
Esta tesis fue parcialmente apoyada por la infraestructura de supercómputo del NLHPC (ECM-02).

SANTIAGO DE CHILE
2021

SUMMARY FOR THE THESIS
TO OPT TO THE DEGREE OF: PHD IN ENGINEERING SCIENCES
MENTION IN MATHEMATICAL MODELING
IN COTUTELLE WITH THE UNIVERSITY OF SEVILLE.
BY: DONATO MAXIMILIANO VÁSQUEZ VARAS
DATE: 2021
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ESTUDIO DE PROBLEMAS DE DISEÑO ÓPTIMO POR EL MÉTODO DE REGULARIDAD EN ECUACIONES NO LINEALES.

This thesis is devoted to the study of an optimal design problem, which is the maximization of the internal energy for the solution of a p-Laplacian equation for a two-phase material. The control variable is the region to be filled with restricted amount of the best material. In general this type of problems has no solution and therefore it is necessary to work with a relaxed formulation. We obtain a relaxed formulation for this problem using the homogenization theory.

By means of the relaxation by homogenization we get a relaxed formuation, which in turns allow us to obtain some smoothness results. Namely, we show that the flux is in the Sobolev space $H^1(\Omega)^N$ and that the optimal proportion of the materials is differentiable in the orthogonal direction to the flux for the solutions of the relaxed problem. This allows us to prove that the non relaxed problem does not have any solution when $f = 1$ and the domain is smooth, bounded and simply connected.

For the relaxed formulation we develope two algorithms, a feasible directions method and an alternating minimization method. We show the convergence for both of them and we provide an estimate for the error. When $p > 2$ both methods are only well defined for a finite-dimensional approximation, because of this we also study the difference between solving the finite-dimensional and the infinite-dimensional problems. Although the error bounds for both methods are similar, numerical experiments show that the alternating minimization method works better than the feasible directions one.

We also study the problem of minimizing the first eigenvalue of the p-Laplacian operator for a two-phase material. We prove that there exists a relation between this problem and the maximization of the energy. Through this relation we provide a relaxed formulation of the problem and prove some smoothness results for these solutions. As a consequence we show that if Ω is of class $C^{1,1}$, simply connected with connected boundary, then the unrelaxed problem has a solution if and only if Ω is a ball. We provide an algorithm to approximate the solutions of the relaxed problem and perform some numerical simulations.

RESUMEN DE LA TESIS PARA OPTAR
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ESTUDIO DE PROBLEMAS DE DISEÑO ÓPTIMO POR EL MÉTODO DE REGULARIDAD EN ECUACIONES NO LINEALES.

Esta tesis está dedicada al estudio de un problema de diseño óptimo, el cual corresponde a la maximización de la energía interna para la solución de una ecuación del tipo p-Laplaciano para un material con dos fases. La variable de control es la región a ser rellenada por una cantidad restringida de material. En general este tipo de problemas no tiene un única solución y por lo tanto es necesario trabajar con una formulación relajada. En este caso la solución relajada es obtenida utilizando teoría de homogeneización.

Mediante el método de relajación por homogeneización se obtiene un problema relajado, el cual a su vez permite obtener algunos resultados de suavidad. Este es, se demuestra que el flujo asociado al problema, está en el espacio $H^1(\Omega)^N$ y que la proporción óptima de materiales es derivable en las direcciones ortogonales al flujo para las soluciones del problema relajado. Esto permite probar el problema no relajado no tiene solución cuando $f = 1$ y el dominio es suave, acotado y simplemente conexo.

Para la formulación relajada se desarrolan dos algoritmos, uno de direcciones factibles y otro de optimización alternada. Se demuestra la convergencia y se obtienen estimaciones para del error en ambos casos. Cuando $p > 2$ ambos métodos solo están bien definidos para una aproximación finito dimensional del problema. Aunque las estimaciones del error para ambos métodos son similares, a través de experimentos numéricos se aprecia que el método de optimización alternada funciona mejor que el de direcciones factibles.

También se estudia el problema de minimizar el primer valor propio del p-Laplaciano para un material con dos fases. Se demuestra que existe una relación entre este problema y el de la maximización de la energía. A través de esta relación se obtiene una relajación del problema y se prueban algunos resultados de suavidad para las soluciones de este problema. Como consecuencia se demuestra que si Ω es de clase $C^{1,1}$, simplemente conexo y con borde conexo, entonces el problema no relajado tiene un solución si y solo si Ω es una bola. Se desarrolla además un algoritmo para aproximar las soluciones del problema relajado y se realizan algunas simulaciones numéricas con este algoritmo.

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Introduction

0.1. Statement of the problem.

The p-Laplacian operator appears in many applied problems like non linear diffusion, machine learning, etc. In two dimensions the p-Laplacian is used to model the *torsional creep* (see [8] and [30] for a deeper explanation), namely, considering a cylindrical bar of cross section $\Omega \subset \mathbb{R}^2$ subject to a constant torsion, the problem is to find the function $u : \overline{\Omega} \to \mathbb{R}$ (which is called stress potential) such that

$$-\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = 1, \quad u \in W_0^{1,p}(\Omega)$$

where $p \in (1,\infty)$. This phenomenon occurs when a material is subjected to an extreme conditions such as very high pressures or temperatures. Under these conditions the material behaves in a plastic way. This situation is modeled using the p-Laplacian operator for the stress potential. When $p \to \infty$ the material behaves in a perfect plastic way. The case when $p \to 1$ is related to a geometric minimization problem. Here, we study the case $p \in (1,\infty)$ in two or more dimensions, specifically we study the maximization of energy for a mixture of two materials.

The general objective of this work is to study the maximization of the potential energy of a mixing of two non linear materials that fills a domain $\Omega \subset \mathbb{R}^N$, namely:

$$\begin{gathered} \sup \int_\Omega |\nabla u_\omega|^p(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\mathrm{d}x \\ \omega \subset \Omega \text{ measurable }, |\omega| \leqslant \kappa, \end{gathered} \tag{1}$$

where u_ω is the unique solution of the problem:

$$-\,\mathrm{div}\left(|\nabla u|^{p-2}\nabla u(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\right) = f \text{ in } \Omega,\ u \in W_0^{1,p}(\Omega), \tag{2}$$

$0 < \alpha < \beta$ are constants, $p \in (1,\infty)$, $0 < \kappa < |\Omega|$, $f \in W^{-1,p'}(\Omega)$ and $p' = \frac{p}{p-1}$, which is the Holder conjugate of p. The constants α and β represent the diffusion coefficients of the two materials that we want to mix, and ω is the region of Ω where we place the material which corresponds to the diffusion coefficient α. We pointed out that if we do not consider the restriction $|\omega| < \kappa$, then trivially the solution would be $\omega = \Omega$, so this corresponds to an economic constraint that enforce us to use some portion of the material characterized by the diffusion coefficient β.

In general, this kind of problem does not have a solution or at least is not possible to use the direct method of calculus of variations to prove the existence of a solution. This is because in order to use the direct method, we need to provide a topology such that the minimizing sequences are compacts and the objective function is lower semi-continuous in this topology. To overcome this difficulty we use the method of homogenization (see [3], [44], [52]) to get a relaxed formulation of the problem. Thanks to this relaxation we are able to study several properties of the solutions. Moreover, it allow us to develop two algorithms to find a solution in a finite dimensional approximation of the relaxed problem.

In the following sections we introduce some concepts such as relaxation by homogenization and we summarize some of the noteworthy aspects of the present work.

0.2. Relaxation by homogenization.

Let us recall the direct method in calculus of variation, for a general problem of the form

$$\inf_{x\in X} G(x), \tag{3}$$

where $G : X \to \mathbb{R}$ is bounded from below and X is not empty. The direct method consists in providing a topology for X such that G is lower semicontinuous and the minimizing sequences are compacts. For problem (1), a possibility to get such topology is to identify every measurable subset ω of Ω with its characteristic function χ_ω and then to use the weak$-*$ topology of $L^\infty(\Omega)$. This ensure the compactness of the minimizing sequences. Moreover, due to the non linearity of the problem, the function is not lower semicontinuous in general. Moreover, for a sequence $\omega_n \subset \Omega$ we only have the existence of $\theta \in L^\infty(\Omega;[0,1])$. not necessarily a characteristic function, such that $\chi_{\omega_n} \overset{*}{\rightharpoonup} \theta$. On the other hand, if we consider u_{ω_n} the solution of (2) and $u \in W_0^{1,p}(\Omega)$ such that u_{ω_n} converges to u in the weak topology of $W_0^{1,p}(\Omega)$, we do not have in general that satisfies (2) with χ_ω replaced by θ. To overcome these difficulties we first obtain a relaxed formulation through homogenization for (1). Briefly, we say that the problem

$$\inf_{x\in X} \tilde{G}(x) \tag{4}$$

is a relaxation of (3) if:

- X is a dense subset of $\tilde{X}$ and the restriction of $\tilde{G}$ to X is equal to G.
- G is lower semicontinuous, i.e. if $x_n \in X$ converges to $x \in X$, then

$$\tilde{G}(x) \leqslant \liminf_{n\to\infty} G(x_n).$$

- For every $x \in \tilde{X}$ there exists a sequence $x_n \in X$ such that

$$\tilde{G}(x) = \liminf_{n\to\infty} G(x_x).$$

If (4) is a relaxation of (3), then (4) has a solution and the infimum of (3) agrees with the minimum of (4). Moreover, $\tilde{x} \in \tilde{X}$ is a solution of (4) if and only if there exists a minimizing sequence of (3) converging to $\tilde{x}$. We will use the theory of Homogenization to obtain a relaxed formulation for problem (1). This theory deals with the behavior of composite materials and more precisely with the asymptotic behaviour of the sequences of mixtures of several materials. In this regard, a key concept in homogenization theory is the H-Convergence of monotone operators (see [45, 50, 52] for the case p=2 and [46] for the general case). To introduce this concept let us consider a sequence of continuous operators $A_n : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ which are strictly monotone, i.e.

$$\begin{aligned}&(A_n(\xi_1, x) - A_n(\xi_2, x)) \cdot (\xi_1 - \xi_2) > C\left(|\xi_1| + |\xi_2\right)^{\min(p-2,0)} |\xi_1 - \xi_2|^{\max(p,2)}\\ &\forall\ \xi_1, \xi_2 \in \mathbb{R}^N,\ \xi_1 \neq \xi_2, \quad \text{a.e. } x \in \Omega.\end{aligned} \tag{5}$$

and satisfy

$$|A_n(\xi, x)| \leqslant C|\xi|^{p-1}, \quad \forall \xi \in \mathbb{R}^N, \qquad \text{a.e. } \Omega \text{ with } C > 0. \tag{6}$$

We say that A_n H-converges to a continuous and strictly monotone operator $A_0 : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ (A_0 is the H-limit of A_n) if for every $f \in W^{-1,p'}(\Omega)$, the solutions $u_n \in W_0^{1,p}(\Omega)$ of

$$-\text{div}(A_n(\nabla u_n, x)) = f \text{ in } \Omega,\ u_n \in W_0^{1,p}(\Omega),$$

are such that

$$u_n \rightharpoonup u_0 \in W_0^{1,p}(\Omega), \quad A_n(\nabla u_n, x) \rightharpoonup A_0(\nabla u_0, x) \in L^{p'}(\Omega)^N,$$

with u_0 the solution of

$$-\text{div}(A_0(\nabla u, x)) = f \text{ in } \Omega,\ u \in W_0^{1,p}(\Omega).$$

In our case, we are interested in

$$A_n(\xi, x) = (\alpha\chi_{\omega_n}(x) + \beta\chi_{\Omega\setminus\omega_n}(x))|\xi|^{p-2}\xi.$$

The H-converges of this kind of operators have been extensively studied when $p = 2$. In fact, the set of H-limits for such sequence A_n when $\xi_{\omega_n} \overset{*}{\rightharpoonup} \theta \in L^\infty(\Omega)$ is completely characterized by the eigenvalues of the $H-$limit (see [44] and section 2.2.3 in [3]). When $p \neq 2$ such result is not known, but it is proved in in Chapter 1, which is given by

$$\begin{cases} \max \displaystyle\int_\Omega \frac{|\nabla u|^p}{(\theta\alpha^{\frac{1}{1-p}} + (1-\theta)\beta^{\frac{1}{1-p}})^{p-1}} \mathrm{d}x \\ -\text{div}\left(\dfrac{|\nabla u|^{p-2}}{(\theta\alpha^{\frac{1}{1-p}} + (1-\theta)\beta^{\frac{1}{1-p}})^{p-1}} \nabla u\right) = f \text{ in } \Omega, \\ u \in W_0^{1,p}(\Omega),\ \theta \in L^\infty(\Omega; [0,1]),\ \displaystyle\int_\Omega \theta \mathrm{d}x \leqslant \kappa, \end{cases} \tag{7}$$

where θ represents the proportion of the material α. We also get the optimality conditions for this problem, which allow us to get some smoothness results. As consequence we proved that if Ω is simple connected with smooth and connected boundary and f is constant, then problem (1) has a solution only if Ω is a ball. Let us also prove that the relaxed problem (7) can be reformulated as

$$\begin{aligned}&\min_{u,\theta} \left\{ \frac{1}{p} \int_\Omega \frac{|\nabla u|^p}{(1 + c\theta)^{p-1}} \mathrm{d}x - \left\langle \tilde{f}, u \right\rangle \right\}\\ &u \in W_0^{1,p}(\Omega),\ \theta \in L^\infty(\Omega; [0,1]),\ \int_\Omega \theta \mathrm{d}x \leqslant \kappa,\end{aligned} \tag{8}$$

with $c = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{1-p}} - 1$ and $\tilde{f} = f/\beta$. This provides a convex problem, which is used in Chapter 2 to develop a gradient descent algorithm to solve it.

0.3. Numerical simulations.

In Chapter 2 we develop two methods to solve problem (8) in a finite dimension setting by replacing $L^\infty(\Omega)$ and $W_0^{1,p}(\Omega)$ by finite dimensional spaces. The first one is based on the feasible direction method and the other is based on the optimality conditions of the problem (8). We detail the description of this algorithms in Chapter 2. Moreover, we prove the convergence and we estimate the rates of convergence.

To implement the algorithms we consider a polyhedral domain Ω in $\mathbb{R}^N$. Then, for a regular mesh $\mathcal{T}_h$ of $\overline{\Omega}$ composed by N-simplexes (see e.g. [48]), with maximum diameter $h > 0$, we consider the Lagrange finite element spaces

$$V_h = \left\{v \in C_0(\Omega) : \ v\big|_\tau \in \mathbb{P}_1(\tau), \quad \forall \tau \in \mathcal{T}_h\right\} \tag{9}$$
$$\Theta_h = \left\{\vartheta \in L^\infty(\Omega) : \ \vartheta\big|_\tau \in \mathbb{P}_0(\tau), \quad \forall \tau \in \mathcal{T}_h\right\}, \tag{10}$$

where $\mathbb{P}_0(\tau)$ denotes the space of constant functions in τ, and $\mathbb{P}_1(\tau)$ the space of affine functions in τ. Replacing $L^\infty(\Omega)$ and $W_0^{1,p}(\Omega)$ by Θ_h and V_h in (8) we obtain the finite dimensional problem:

$$\begin{gathered} \min_{u,\theta}\left\{\frac{1}{p}\int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}}\mathrm{d}x - \left\langle \tilde{f}, u\right\rangle\right\} \\ u \in V_h, \ \theta \in \Theta_h \cap L^\infty(\Omega;[0,1]), \ \int_\Omega \theta \mathrm{d}x \leqslant \kappa \end{gathered} \tag{11}$$

We prove that the value of the finite dimensional approximation (11) converges to the value of the problem (8). Additionally, assuming that there exists a solution $(\hat{u}, \hat{\theta}) \in W_0^{1,p}(\omega) \times L^\infty(\Omega)$ such that $\hat{\theta}$ is a function of bounded variation, we provide a convergence rate. Finally, using the finite dimensional approximation we perform some numerical experiments.

0.4. Minimization of the first eigenvalue

An interesting and applied problem related to (1), is the minimization of the first eigenvalue of the p-Laplacian for a two phase material, namely:

$$\begin{aligned} &\min_{\omega,u}\int_\Omega |\nabla u|^p(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}) \\ &\omega \subset \Omega \text{ measurable}, \ |\omega| \leqslant \kappa, \\ &\int_\Omega u^p \mathrm{d}x = 1, \ u \in W_0^{1,p}.(\Omega) \end{aligned} \tag{12}$$

We prove in Chapter 3 the following result which provides a strong relationship between problems (1) and (12). For a matrix $A \in L^\infty(\Omega)^{N\times N}$ we have the following equality:

$$\lambda_1(p,A)^{\frac{1}{1-p}} = \begin{cases} \displaystyle\max_{f,u}\int_\Omega |A\nabla u|^{p-2}A\nabla u\cdot\nabla u \mathrm{d}x \\ -\mathrm{div}(|A\nabla u|^{p-2}A\nabla u) = f, \ u \in W_0^{1,p}(\Omega), \ \|f\|_{L^{p'}(\Omega)} \leqslant 1. \end{cases} \tag{13}$$

Here $\lambda_1(p,A)$ is the first eigenvalue of the operator $u \in W_0^{1,p}(\Omega) \to -\mathrm{div}(|A\nabla u|^{p-2}A\nabla u) \in W^{-1,p}(\Omega)$:

$$\lambda_1(p,A) := \min_{\substack{u\in W_0^{1,p}(\Omega)\\ \|u\|_{L^p(\Omega)}=1}} \int_\Omega |A\nabla u|^{p-2}A\nabla u\cdot\nabla u\,\mathrm{d}x.$$

The equality (13) and the relaxation result for (1) that we get in Chapter 1 allow us to get a relaxed formulation for problem (12):

$$\begin{gathered}\min_{\omega,u}\int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}}\mathrm{d}x\\ \theta\in L^\infty(\Omega;[0,1]),\ \int_\Omega \theta\mathrm{d}x\leqslant\kappa,\\ \int_\Omega u^p\,\mathrm{d}x=1,\ u\in W_0^{1,p}.(\Omega)\end{gathered}\tag{14}$$

Through this relaxed formulation and the results proved in Chapter 1 we get a smoothness result for problem (14) and analogously to problem (1), we prove that when Ω is simply connected with connected and $C^{1,1}$ boundary, then problem (12) has a solution if and only if Ω is a ball.

It is noteworthy to mention the interpretation of (13). It shows that the minimization of the first eigenvalue is equivalent to solve the problem (1) for every f with $\|f\|_{L^{p'}(\Omega)}\leqslant 1$ and then to minimize in f. This can be seen as a robust optimization problem.

Finally, in Chapter 3 we also provide a numerical algorithm to solve the relaxed problem (14), but due to the non-convexity of problem we only prove the convergence to a critical point of the problem. In order to implement the algorithm, we discretize the problem by replacing $L^\infty(\Omega)$ and $W_0^{1,p}(\Omega)$ by $\Theta_h(10)$ and $V_h(9)$, respectively. Using the relation between the problems (8) and (14) and the convergence results for (8), we prove the convergence of the value of the discrete problem to the value of the continuous problem and furthermore we give a bound for the difference between both values.

Chapter 1

The Maximization of the p-Laplacian Energy

Abstract: We consider the optimal arrangement of two diffusion materials in a bounded open set $\Omega \subset \mathbb{R}^N$ in order to maximize the energy. The diffusion problem is modeled by the p-Laplacian operator. It is well known that this type of problems has no solution in general and then that it is necessary to work with a relaxed formulation. In the present paper we obtain such relaxed formulation using the homogenization theory, i.e. we replace both materials by microscopic mixtures of them. Then we get some uniqueness results and a system of optimality conditions. As a consequence we prove some regularity properties for the optimal solutions of the relaxed problem. Namely, we show that the flux is in the Sobolev space $H^1(\Omega)^N$ and that the optimal proportion of the materials is derivable in the orthogonal direction to the flux. This will imply that the unrelaxed problem has no solution in general. Our results extend those obtained by the first author for the Laplace operator.

1.1. Introduction

The present paper is devoted to study an optimal design problem for a diffusion process in a two-phase material modeled by the p-Laplacian operator. Namely, we are interested in the control problem

$$\left\{\begin{array}{c} \max\limits_{\omega} \displaystyle\int_{\Omega} \big(\alpha\mathcal{X}_\omega + \beta\left(1-\mathcal{X}_\omega\right)\big)|\nabla u|^p \mathrm{d}x \\ -\operatorname{div}\big(\left(\alpha\mathcal{X}_\omega + \beta\left(1-\mathcal{X}_\omega\right)\right)|\nabla u|^{p-2}\nabla u\big) = f \quad \text{in } \Omega \\ u \in W_0^{1,p}(\Omega), \quad \omega \subset \Omega \text{ mesurable }, \quad |\omega| \leqslant \kappa, \end{array}\right. \tag{1.1}$$

with Ω a bounded open set in $\mathbb{R}^N$, $N \geqslant 2$, $p \in (1,\infty)$, $\alpha, \beta, \kappa > 0$, $\alpha < \beta$, $\mathcal{X}_\omega$ the characteristic function of the set ω, and $f \in W^{-1,p'}(\Omega)$, with p' is the Holder conjugate of p $\left(p' = \frac{p}{p-1}\right)$.

In (1.1) the equation is understood to hold in the sense of distributions, combined with $u \in W_0^{1,p}(\Omega)$, denoting by u^α and u^β the values of u in ω and $\Omega \setminus \omega$ respectively and assuming

ω smooth enough, this means that the interphase conditions on $\partial\omega$ are given by

$$u^\alpha = u^\beta,\ \alpha|\nabla u^\alpha|^{p-2}\nabla u^\alpha\cdot\nu = \beta|\nabla u^\beta|^{p-2}\nabla u^\beta\cdot\nu \ \text{ on } \partial\omega\cap\Omega$$

in the sense of the traces in $W^{1/p',p}(\partial\omega)$ and $W^{-1/p',p'}(\partial\omega)$ respectively. Here ν denotes a unitary normal vector on $\partial\omega$.

Physically the constants α and β represent two diffusion materials that we are mixing in order to maximize the corresponding functional, which in (1.1) represent the potential energy. The control variable is the set ω where we place the material α. If we do not impose any restriction on the amount of this material, it is simple to check that the solution of (1.1) is the trivial one given by $\omega = \Omega$. Thus, the interesting problem corresponds to $\kappa < |\Omega|$, i.e. the material α is better than β but it is also more expensive and therefore, we do not want to use a large amount of it in the mixture. The case corresponding to $p = 2$ has been studied in several papers (see e.g. [13], [28], [44]) where some classical applications are the optimal mixture of two materials in the cross-section of a beam in order to minimize the torsion, and the optimal arrangement of two viscous fluids in a pipe. For $p \in (1,2)\cup(2,\infty)$ the p-Laplacian operator models the torsional creep in the cross-section of a beam [30] and therefore problem (1.1) corresponds to find the material which minimizes the torsion for the mixture of two homogeneous materials in non-linear elasticity.

It is well known that a control problem in the coefficients like (1.1) has no solution in general ([42], [43]). In fact, some counterexamples to the existence of solution for (1.1) with $p = 2$ can be found in [13] and [44]. Thus, it is necessary to work with a relaxed formulation. One way to obtain this formulation is to use the homogenization theory ([3], [44], [52]). The idea is to replace the material $\alpha\mathcal{X}_\omega + \beta(1-\mathcal{X}_\omega)$ in (1.1) by microscopic mixtures of α,β with a certain proportion $\theta = \theta(x) \in [0,1]$, $x \in \Omega$. The new materials do not only depend on the proportion of each original material but also on their microscopical distribution. In the case $p = 2$, this relaxed formulation has been obtained in [44]. Here we show that a relaxed formulation for (1.1) is given by

$$\left\{\begin{array}{l} \displaystyle\max_\theta\left\{\frac{1}{p}\int_\Omega\left(\theta\alpha^{\frac{1}{1-p}} + (1-\theta)\beta^{\frac{1}{1-p}}\right)^{1-p}|\nabla u|^p\mathrm{d}x\right\} \\ -\mathrm{div}\Big(\left(\theta\alpha^{\frac{1}{1-p}} + (1-\theta)\beta^{\frac{1}{1-p}}\right)^{1-p}|\nabla u|^{p-2}\nabla u\Big) = f \ \text{ in } \Omega \\ u \in W_0^{1,p}(\Omega), \quad \theta \in L^\infty(\Omega;[0,1]), \quad \displaystyle\int_\Omega \theta(x)\,\mathrm{d}x \leqslant \kappa, \end{array}\right. \tag{1.2}$$

which is equivalent to the Calculus of Variations problem

$$\left\{\begin{array}{l} \displaystyle\min_\theta\left\{\frac{1}{p}\int_\Omega\left(\theta\alpha^{\frac{1}{1-p}} + (1-\theta)\beta^{\frac{1}{1-p}}\right)^{1-p}|\nabla u|^p\mathrm{d}x - \langle f,u\rangle\right\} \\ u \in W_0^{1,p}(\Omega), \quad \theta \in L^\infty(\Omega;[0,1]), \quad \displaystyle\int_\Omega \theta(x)\,\mathrm{d}x \leqslant \kappa, \end{array}\right. \tag{1.3}$$

where here and in what follows, $\langle f,u\rangle$ denotes the duality product of f and u as elements of $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$ respectively.

Our main results extend those obtained in [13] (see also [44]) for $p = 2$ relative to the uniqueness and regularity of a solution for (1.2). Namely, we prove that although it is not clear that (1.3) has a unique solution (u,θ), the flux

$$\sigma = \Big(\frac{\theta}{\alpha^{\frac{1}{p-1}}} + \frac{1-\theta}{\beta^{\frac{1}{p-1}}}\Big)^{1-p} |\nabla u|^{p-2}\nabla u$$

is unique. Moreover, assuming $\Omega \in C^{1,1}$ and $f \in L^q(\Omega) \cap W^{1,1}(\Omega)$, with $q > N$, we have that σ belongs to $H^1(\Omega)^N \cap L^\infty(\Omega)$. This is related to some regularity results for the p- Laplacian operator obtained in [35]. We also prove that every solution (u,θ) of (1.3) satisfies

$$u \in W^{1,\infty}(\Omega), \quad \partial_{\mathrm{i}}\theta\, \sigma_j - \partial_j\theta\, \sigma_{\mathrm{i}} \in L^2(\Omega),\ 1 \leqslant \mathrm{i}, j \leqslant N, \tag{1.4}$$

where σ_{i} denotes the i-th component of the vector function σ, i.e. θ is derivable in the orthogonal subspace to σ. The existence of first derivatives for σ and θ will imply that we cannot hope in general an existence result for the unrelaxed problem (1.1). Namely, the existence of a solution for (1.1) is equivalent to the existence of a solution for (1.3) where θ only takes the values zero and one, but then the derivatives of θ in (1.4) vanish. Assuming Ω simply connected with connected boundary, we show that this implies $\sigma = |\nabla w|^{p-2}\nabla w$, with w the unique solution of

$$\begin{cases} -\mathrm{div}\,(|\nabla w|^{p-2}\nabla w) = f \ \text{ in } \Omega \\ w \in W_0^{1,p}(\Omega). \end{cases}$$

Similarly to the result obtained in ([13], [44]), we prove that this is only possible if Ω is a ball.

We finish this introduction remembering that the results obtained in the present paper are also related to those given in [12] where, for $p = 2$, it is considered the minimization in (1.1) instead of the maximization. Problem (1.1) is also related to the minimization of the first eigenvalue for the p-Laplacian operator (see [13], [14], [19], [20], [39] for $p = 2$), problem which we hope to study in a later work.

1.2. Position of the problem. Relaxation and equivalent formulations

For a bounded open set $\Omega \subset \mathbb{R}^N$, three positive constants α, β, κ with $0 < \alpha < \beta$, $\kappa < |\Omega|$, and a distribution $f \in W^{-1,p'}(\Omega)$, $p > 1$, we are interested in the control problem

$$\begin{cases} \displaystyle\max_{\omega} \int_\Omega \big(\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega}\big)\, |\nabla u_\omega|^p \mathrm{d}x \\ \omega \subset \Omega \text{ measurable},\ |\omega| \leqslant \kappa \\ -\mathrm{div}\,\big((\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega})|\nabla u_\omega|^{p-2}\nabla u_\omega\big) = f \text{ in } \Omega,\ u_\omega \in W_0^{1,p}(\Omega). \end{cases} \tag{1.5}$$

Here α and β represent the diffusion coefficients of two materials, where the diffusion process is modeled by the p-Laplacian operator. The problem consists in maximizing the potential energy.

Using u_ω as test function in the state equation we have

$$\int_\Omega \big(\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega}\big)\, |\nabla u_\omega|^p \mathrm{d}x = \langle f, u_\omega\rangle,$$

By the above equality and since $p' = \frac{p}{p-1}$ we have

$$\begin{aligned}
&\int_\Omega \left(\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega}\right) |\nabla u_\omega|^p \mathrm{d}x \\
&= -p'\left(\frac{1}{p}\int_\Omega \left(\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega}\right) |\nabla u_\omega|^p \mathrm{d}x - \int_\Omega \left(\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega}\right) |\nabla u_\omega|^p \mathrm{d}x\right) \\
&= -p'\left(\frac{1}{p}\int_\Omega \left(\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega}\right) |\nabla u_\omega|^p \mathrm{d}x - \langle f, u_\omega\rangle\right)
\end{aligned}$$

which combined with u_ω, unique solution of the minimization problem

$$\min_{u\in W_0^{1,p}(\Omega)} \left\{\frac{1}{p}\int_\Omega (\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega})|\nabla u|^p \mathrm{d}x - \langle f, u\rangle\right\},$$

gives the equivalent formulation for problem (1.5):

$$\left\{\begin{array}{l}
\displaystyle\min_{\omega,u}\left\{\frac{1}{p}\int_\Omega (\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega})|\nabla u|^p \mathrm{d}x - \langle f, u\rangle\right\} \\
u \in W_0^{1,p}(\Omega), \quad \omega\subset\Omega \text{ measurable}, \quad |\omega| \leqslant \kappa.
\end{array}\right. \tag{1.6}$$

It is known that the maximum in (1.5) or the minimun in (1.6) are not achieved, i.e., that (1.5) (or (1.6)) has no solution in general. Namely, for $p = 2$ and $f = 1$, it has been proved in [13] and [44] that if Ω is smooth, with connected smooth boundary, and (1.5) has a solution, then Ω is a ball. Some other classical counterexamples to the existence of solution for problems related to (1.5) can be found in [42] and [43]. Due to this difficulty it is then necessary to find a relaxed formulation for (1.5). This is done by the following theorem

Theorem 1.1 *A relaxed formulation of problem* (1.6) *is given by*

$$\left\{\begin{array}{l}
\displaystyle\min_{\theta,u}\left\{\frac{1}{p}\int_\Omega \left(\theta\alpha^{\frac{1}{1-p}} + (1-\theta)\beta^{\frac{1}{1-p}}\right)^{1-p} |\nabla u|^p \mathrm{d}x - \langle f, u\rangle\right\} \\
\displaystyle u \in W_0^{1,p}(\Omega), \quad \theta\in L^\infty(\Omega;[0,1]), \quad \int_\Omega \theta \mathrm{d}x \leqslant \kappa,
\end{array}\right. \tag{1.7}$$

in the following sense:

1. *Problem* (1.7) *has a solution.*
2. *The infimum for problem* (1.6) *agrees with the minimum for* (1.7).
3. *Every minimizing sequence* (u_n, ω_n) *for* (1.6) *has a subsequence still denoted by* (u_n, ω_n) *such that*
$$u_n \rightharpoonup u \ \text{ in } W_0^{1,p}(\Omega), \quad \mathcal{X}_{\omega_n} \overset{*}{\rightharpoonup} \theta \ \text{ in } L^\infty(\Omega), \tag{1.8}$$
with (u, θ) *solution of* (1.7).
4. *For every pair* $(u, \theta) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega;[0,1])$ *there exist* $u_n \in W_0^{1,p}(\Omega)$, $\omega_n \subset \Omega$ *measurable, with* $|\omega_n| \leqslant \kappa$ *such that* (1.8) *holds and such that*

$$\lim_{n\to\infty}\int_\Omega (\alpha\mathcal{X}_{\omega_n} + \beta\mathcal{X}_{\Omega\setminus\omega_n})|\nabla u_n|^p \mathrm{d}x = \int_\Omega \left(\theta\alpha^{\frac{1}{1-p}} + (1-\theta)\beta^{\frac{1}{1-p}}\right)^{1-p} |\nabla u|^p \mathrm{d}x. \tag{1.9}$$

Remark 1.1 *Such as we will see in the proof of Theorem 1.1, the relaxed materials in* (1.7) *are obtained as a simple lamination in a parallel direction to* ∇u*. In this context, a laminated material corresponds to a particular distribution of two materials, which depends exclusively on one direction, say* $\xi \in \mathbb{R}^N$*, which is represented by a function* $\varphi \in L^\infty(\Omega; [0,1])$ *with a generic form as follows:*

$$\varphi(x) = g(\xi \cdot x) \quad \forall x \in \Omega,$$

where g *is a real-valued function. (see sections 2.3.5 and 2.2.1 in [3] for more details on laminated materials).*

Proof of Theorem 1.1. Using that the function $J : \mathbb{R}^N \times (0, \infty) \to \mathbb{R}$ defined by

$$J(\xi, t) = \frac{|\xi|^p}{t^{p-1}}, \quad \forall\, (\xi, t) \in \mathbb{R}^N \times (0, \infty), \tag{1.10}$$

is convex, and the sequential compactness of the bounded sets in $W_0^{1,p}(\Omega) \times L^\infty(\Omega)$ with respect to the weak-$*$ topology, it is immediate to show that (1.7) has at least a solution and that every minimizing sequence (u_n, θ_n) for (1.7) has a subsequence which converges in $W_0^{1,p}(\Omega) \times L^\infty(\Omega)$ weak-$*$ to a minimum.

Since problem (1.6) consists in minimizing the same functional than the one in (1.7), but on the smaller set

$$\Big\{(u, \mathcal{X}_\omega) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega; [0,1]) : \ \omega \subset \Omega, \ \int_\Omega \mathcal{X}_\omega \,\mathrm{d}x \leqslant \kappa\Big\},$$

it is clear that the infimum in (1.6) is bigger or equal than the minimum in (1.7). Thus, taking into account that the convergence of the minimizing sequences stated above will imply statement (3), we deduce that it is enough to prove statement (4) to complete the proof of Theorem 1.1. For this purpose, we introduce the functions (the index $\sharp$ means periodicity) $H \in L^\infty((0,1) \times \mathbb{R}) \cap C^0([0,1]; L^1_\sharp(0,1))$, $G \in W^{1,\infty}((0,1) \times \mathbb{R}) \cap C^0([0,1]; W^{1,1}_\sharp(0,1))$, by

$$H(q, r) = \sum_{k=-\infty}^{\infty} \mathcal{X}_{[k,k+q)}(r), \quad G(q, r) = qr - \int_0^r H(q, s)\,\mathrm{d}s, \quad \forall\, q, r \in [0,1] \times \mathbb{R}. \tag{1.11}$$

Now, for a pair $(u, \theta) \in C^1_c(\Omega) \times C^0(\overline{\Omega})$ with

$$\int_\Omega \theta \,\mathrm{d}x < \kappa,$$

and $\delta > 0$, we consider a family of cubes Q_i, $1 \leqslant \mathrm{i} \leqslant n_\delta$, of side δ such that

$$\overline{\Omega} \subset \bigcup_{\mathrm{i}=1}^{n_\delta} Q_\mathrm{i}, \quad |Q_\mathrm{i} \cap Q_j| = 0, \ \text{ if } \mathrm{i} \neq j,$$

and a partition of the unity in $\overline{\Omega}$ by functions $\psi_\mathrm{i} \in C^\infty_c(\mathbb{R}^N)$, with

$$\sup(\psi_\mathrm{i}) \subset Q_\mathrm{i} + B(0, \delta), \ \psi_\mathrm{i}(x) \geqslant 0, \ 1 \leqslant \mathrm{i} \leqslant n_\delta \text{ and } \sum_{\mathrm{i}=1}^{n_\delta} \psi_\mathrm{i}(x) = 1, \ \forall x \in \Omega.$$

Then, we take

$$q_{\rm i}=\frac{1}{\delta^N}\int_{Q_{\rm i}}\theta\,{\rm d}x,\qquad \xi_{\rm i}=\frac{1}{\delta^N}\int_{Q_{\rm i}}\nabla u\,{\rm d}x,\quad \zeta_{\rm i}=\begin{cases}\xi_{\rm i} & \text{if } \xi_{\rm i}\neq 0\\ {\rm e} & \text{if } \xi_{\rm i}=0,\end{cases}$$

with ${\rm e}\in\mathbb{R}^N\setminus\{0\}$ fixed, and we introduce, for every $\varepsilon>0$, the sets $\omega_{\delta,\varepsilon}\subset\Omega$ and the functions $u_{\delta,\varepsilon}\in W^{1,\infty}(\Omega)$, with compact support by

$$\mathcal{X}_{\omega_{\delta,\varepsilon}}=\sum_{\rm i=1}^{n_\delta}H\Big(q_{\rm i},\frac{\zeta_{\rm i}\cdot x}{\varepsilon}\Big)\mathcal{X}_{Q_{\rm i}},\quad u_{\delta,\varepsilon}=u+\varepsilon\sum_{\rm i=1}^{n_\delta}\psi_{\rm i}\frac{G\big(q_{\rm i},\frac{\xi_{\rm i}\cdot x}{\varepsilon}\big)\big(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}}q_{\rm i}+\beta^{\frac{1}{1-p}}(1-q_{\rm i})}.$$

Using the result (see e.g. [2])

$$\Phi\big(x,\frac{x\cdot\xi}{\varepsilon}\big)\overset{*}{\rightharpoonup}\int_0^1\Phi(x,s)\,{\rm d}s\ \text{ in } L^\infty(\Omega),\tag{1.12}$$

for every $\Phi\in C^0(\overline{\Omega};L^1_\sharp(0,1))\cap L^\infty(\Omega\times\mathbb{R})$ and every $\xi\in\mathbb{R}^N\setminus\{0\}$, we have that $\omega_{\delta,\varepsilon}$ satisfies

$$\mathcal{X}_{\omega_{\delta,\varepsilon}}\overset{*}{\rightharpoonup}\theta_\delta:=\sum_{\rm i=1}^{n_\delta}q_{\rm i}\mathcal{X}_{Q_{\rm i}}\ \text{ in } L^\infty(\Omega),\quad \text{when } \varepsilon\to 0,\tag{1.13}$$

where thanks to θ uniformly continuous, we also have

$$\theta_\delta\to\theta\ \text{ in } L^\infty(\Omega;[0,1]),\quad \text{when } \delta\to 0.\tag{1.14}$$

In particular, since the integral of θ is strictly smaller than κ, we deduce that for every $\delta>0$ small enough, there exists $\varepsilon_\delta>0$ such that

$$|\omega_{\delta,\varepsilon}|<\kappa,\quad \forall\, 0<\varepsilon<\varepsilon_\delta.\tag{1.15}$$

Since $q(q-1)\leqslant G(q,r)\leqslant 0$, for every $q\in[0,1]$ and every $r\in\mathbb{R}$, we also have the existence of $C>0$ such that

$$\|u_{\delta,\varepsilon}-u\|_{C^0(\overline{\Omega})}\leqslant C\varepsilon,\qquad \forall\,\varepsilon,\delta>0\tag{1.16}$$

and taking into account that u has compact support and that $G(q,0)=0$, we deduce that, for δ small enough, $u_{\delta,\varepsilon}$ has compact support and thus belongs to $W^{1,p}_0(\Omega)$. Moreover, thanks to (1.12) (observe that there is not problem if $\xi_{\rm i}=0$ because then $G(q_{\rm i},\frac{\xi_{\rm i}\cdot x}{\varepsilon})=0$ for every $x\in\mathbb{R}^N$)

$$\begin{aligned}\nabla u_{\delta,\varepsilon}&=\nabla u+\sum_{\rm i=1}^{n_\delta}\frac{\big(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}}q_{\rm i}+\beta^{\frac{1}{1-p}}(1-q_{\rm i})}\Big(\varepsilon\nabla\psi_{\rm i}G\big(q_{\rm i},\frac{\xi_{\rm i}\cdot x}{\varepsilon}\big)+\psi_{\rm i}\big(q_{\rm i}-H\big(q_{\rm i},\frac{\xi_{\rm i}\cdot x}{\varepsilon}\big)\big)\xi_{\rm i}\Big)\\&\overset{*}{\rightharpoonup}\nabla u\ \text{ in } L^\infty(\Omega)\ \text{ when } \varepsilon\to 0,\quad \forall\,\delta>0.\end{aligned}$$

Therefore

$$u_{\delta,\varepsilon}\overset{*}{\rightharpoonup}u\ \text{ in } W^{1,\infty}(\Omega)\cap W^{1,p}_0(\Omega)\ \text{ when } \varepsilon\to 0,\quad \forall\,\delta>0\ \text{ small engouh.}\tag{1.17}$$

On the other hand, using the above expression of $\nabla u_{\delta,\varepsilon}$, and denoting $H_{\rm i}(s)=H(q_{\rm i},s)$, we can use (1.12) combined with $H(q,s)=1$ if $s\in(0,q)$, $H(q,s)=0$ if $s\in(q,1)$, and $\xi_{\rm i}=0$ is $\zeta_{\rm i}\neq\xi_{\rm i}$ to deduce

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_\Omega\big(\alpha\mathcal{X}_{\omega_{\delta,\varepsilon}}+\beta(1-\mathcal{X}_{\omega_{\delta,\varepsilon}})\big)|\nabla u_{\delta,\varepsilon}|^p{\rm d}x\\
&=\sum_{\rm i=1}^{n_\delta}\int_{Q_{\rm i}}\int_0^1\big(\alpha H_{\rm i}(s)+\beta(1-H_{\rm i}(s))\big)\left|\nabla u+\sum_{\rm i=1}^{n_\delta}\psi_{\rm i}\frac{\big(q_{\rm i}-H_{\rm i}(s)\big)\big(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}}q_{\rm i}+\beta^{\frac{1}{1-p}}(1-q_{\rm i})}\xi_{\rm i}\right|^p{\rm d}s{\rm d}x\\
&=\sum_{\rm i=1}^{n_\delta}\int_{Q_{\rm i}}\alpha q_{\rm i}\left|\nabla u+\frac{(q_{\rm i}-1)\big(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}}q_{\rm i}+\beta^{\frac{1}{1-p}}(1-q_{\rm i})}\xi_{\rm i}\right|^p{\rm d}x\\
&+\sum_{\rm i=1}^{n_\delta}\int_{Q_{\rm i}}\beta(1-q_{\rm i})\left|\nabla u+\frac{q_{\rm i}\big(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}}q_{\rm i}+\beta^{\frac{1}{1-p}}(1-q_{\rm i})}\xi_{\rm i}\right|^p{\rm d}x.
\end{aligned}$$

Thanks to the uniform continuity of θ and ∇u, we can also take the limit when δ tends to zero in the right-hand side of the above equality to get

$$\begin{aligned}
&\lim_{\delta\to0}\Bigg(\sum_{\rm i=1}^{n_\delta}\int_{Q_{\rm i}}\alpha q_{\rm i}\left|\nabla u+\frac{(q_{\rm i}-1)\big(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}}q_{\rm i}+\beta^{\frac{1}{1-p}}(1-q_{\rm i})}\xi_{\rm i}\right|^p{\rm d}x\\
&\qquad+\sum_{\rm i=1}^{n_\delta}\int_{Q_{\rm i}}\beta(1-q_{\rm i})\left|\nabla u+\frac{q_{\rm i}\big(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}}q_{\rm i}+\beta^{\frac{1}{1-p}}(1-q_{\rm i})}\xi_{\rm i}\right|^p{\rm d}x\Bigg)\\
&=\int_\Omega\Bigg(\alpha\theta\left|1+\frac{(\theta-1)(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}})}{\alpha^{\frac{1}{1-p}}\theta+\beta^{\frac{1}{1-p}}(1-\theta)}\right|^p\\
&\qquad+\beta(1-\theta)\left|1+\frac{\theta(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}})}{\alpha^{\frac{1}{1-p}}\theta+\beta^{\frac{1}{1-p}}(1-\theta)}\right|^p\Bigg)|\nabla u|^p{\rm d}x\\
&=\int_\Omega\Big(\theta\alpha^{\frac{1}{1-p}}+(1-\theta)\beta^{\frac{1}{1-p}}\Big)^{1-p}|\nabla u|^p{\rm d}x.
\end{aligned}\tag{1.18}$$

Let us now use that for $\varepsilon<1$, $\nabla u_{\delta,\varepsilon}$ is bounded in $L^\infty(\Omega)^N$, independently of δ and ε, and $\chi_{\omega_{\delta,\varepsilon}}\in\{0,1\}$. Thus, there exists $C\geqslant1$ such that

$$\|\mathcal{X}_{\omega_{\delta,\varepsilon}}\|_{L^\infty(\Omega)}\leqslant1,\quad\|\partial_ju_{\delta,\varepsilon}\|_{L^\infty(\Omega)}\leqslant C,\ 1\leqslant j\leqslant N,\qquad\forall\,\varepsilon,\delta>0,\ 0<\varepsilon<1.$$

Here, we recall that the closed ball $\overline{B}_C$ of center 0 and radius C in $L^\infty(\Omega)$, endowed with the weak-$*$ topology is metrizable. Taking d a suitable distance, and using (1.13), (1.15) and (1.17), we can choose for every $\delta>0$, $\varepsilon(\delta)>0$ such that

$${\rm d}(\mathcal{X}_{\omega_{\delta,\varepsilon(\delta)}},\theta_\delta)<\delta,\quad|\omega_{\delta,\varepsilon(\delta)}|<\kappa,\quad{\rm d}(\partial_ju_{\delta,\varepsilon(\delta)},\partial_ju)<\delta,\ 1\leqslant j\leqslant N,$$

$$\left|\begin{aligned}&\int_\Omega\big(\alpha\mathcal{X}_{\omega_{\delta,\varepsilon(\delta)}}+\beta(1-\mathcal{X}_{\omega_{\delta,\varepsilon(\delta)}})\big)|\nabla u_{\delta,\varepsilon(\delta)}|^p{\rm d}x\\
&-\sum_{\rm i=1}^{n_\delta}\int_\Omega\alpha q_{\rm i}\left|\nabla u+\frac{(q_{\rm i}-1)\big(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}}q_{\rm i}+\beta^{\frac{1}{1-p}}(1-q_{\rm i})}\xi_{\rm i}\right|^p{\rm d}x\\
&-\sum_{\rm i=1}^{n_\delta}\int_\Omega\beta(1-q_{\rm i})\left|\nabla u+\frac{q_{\rm i}\big(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}}q_{\rm i}+\beta^{\frac{1}{1-p}}(1-q_{\rm i})}\xi_{\rm i}\right|^p{\rm d}x\end{aligned}\right|<\delta.\tag{1.19}$$

Then, taking into account (1.14) and (1.18), we get

$$\mathcal{X}_{\omega_{\delta,\varepsilon(\delta)}} \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(\Omega), \quad |\omega_{\delta,\varepsilon(\delta)}| < \kappa, \quad u_{\delta,\varepsilon(\delta)} \overset{*}{\rightharpoonup} u \text{ in } W^{1,\infty}(\Omega) \cap W_0^{1,p}(\Omega),$$

$$\lim_{\delta\to 0} \int_\Omega \left(\alpha \mathcal{X}_{\omega_{\delta,\varepsilon(\delta)}} + \beta(1 - \mathcal{X}_{\omega_{\delta,\varepsilon(\delta)}})\right)|\nabla u_{\delta,\varepsilon(\delta)}|^p \mathrm{d}x = \int_\Omega \left(\theta\alpha^{\frac{1}{1-p}} + (1-\theta)\beta^{\frac{1}{1-p}}\right)^{1-p} |\nabla u|^p \mathrm{d}x.$$

This proves assertion (4) for u, θ smooth and $\int_\Omega \theta\,\mathrm{d}x < \kappa$. The general result follows by density. □

Remark 1.2 *We can express problem (1.7) in a simpler way defining*

$$c := \left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}} - 1 > 0, \qquad \tilde{f} := \frac{f}{\beta}, \tag{1.20}$$

which provides

$$\begin{cases} \displaystyle\min_{\theta,u} \left\{ \frac{1}{p} \int_\Omega \frac{|\nabla u|^p}{(1+c\,\theta)^{p-1}} \mathrm{d}x - < \tilde{f}, u > \right\} \\ u \in W_0^{1,p}(\Omega), \quad \theta \in L^\infty(\Omega;[0,1]), \quad \displaystyle\int_\Omega \theta \mathrm{d}x \leqslant \kappa. \end{cases} \tag{1.21}$$

For simplicity, in the following we will redefine f as $\tilde{f}$.

1.3. Uniqueness results and optimality conditions for the relaxed problem

Since in problem (1.21) the cost functional is not strictly convex, the uniqueness of solution is not clear. However, let us prove in Proposition 1.1 that the flux

$$\hat{\sigma} := \frac{|\nabla \hat{u}|^{p-2}}{(1+c\,\hat{\theta})^{p-1}} \nabla \hat{u}, \tag{1.22}$$

with $(\hat{u}, \hat{\theta})$ a solution of (1.21) is uniquely defined. The result follows from a dual formulation of (1.21) as a min-max problem. In the case $p = 2$, a similar result has been obtained in [44].

Proposition 1.1 *For every solution $(\hat{u}, \hat{\theta}) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega;[0,1])$ of (1.21), the flux $\hat{\sigma}$ defined by (1.22) is the unique solution of*

$$\min_{\substack{-\operatorname{div}\sigma = f \\ \sigma \in L^{p'}(\Omega)^N}} \max_{\substack{\theta \in L^\infty(\Omega;[0,1]) \\ \int_\Omega \theta\,\mathrm{d}x \leqslant \kappa}} \int_\Omega (1 + c\,\theta)|\sigma|^{p'} \mathrm{d}x. \tag{1.23}$$

The function $\hat{\theta}$ solves the problem

$$\max_{\substack{\theta \in L^\infty(\Omega;[0,1]) \\ \int_\Omega \theta\,\mathrm{d}x \leqslant \kappa}} \min_{\substack{-\operatorname{div}\sigma = f \\ \sigma \in L^{p'}(\Omega)^N}} \int_\Omega (1 + c\,\theta)|\sigma|^{p'} \mathrm{d}x, \tag{1.24}$$

and the minimum value in (1.23) agrees with the maximum in (1.24).

PROOF. For $\theta \in L^\infty(\Omega;[0,1])$, we define $\sigma_\theta \in L^{p'}(\Omega)^N$ as the unique solution of

$$\min_{\substack{-\mathrm{div}\sigma=f\\ \sigma\in L^{p'}(\Omega)^N}} \int_\Omega (1+c\,\theta)|\sigma|^{p'}\mathrm{d}x.$$

The uniqueness of σ_θ is ensured by the strictly convexity of the problem. Then, taking into account that σ_θ satisfies

$$p'\int_\Omega (1+c\theta)|\sigma_\theta|^{p'-2}\sigma_\theta\cdot\eta\,\mathrm{d}x = 0, \quad \forall\,\eta\in L^{p'}(\Omega), \quad \text{with } \mathrm{div}\,\eta = 0,$$

we deduce the existence of $u_\theta \in W_0^{1,p}(\Omega)$ such that $(1+c\theta)|\sigma_\theta|^{p'-2}\sigma_\theta = \nabla u_\theta$ in Ω. Using also that $-\mathrm{div}\,\sigma_\theta = f$ in Ω, we get that u_θ is the unique solution of

$$-\mathrm{div}\left(\frac{|\nabla u_\theta|^{p-2}}{(1+c\theta)^{p-1}}\nabla u_\theta\right) = f \ \text{ in } \Omega, \quad u_\theta \in W_0^{1,p}(\Omega),$$

or equivalently, of the minimization problem

$$\min_{u\in W_0^{1,p}(\Omega)}\left\{\frac{1}{p}\int_\Omega \frac{|\nabla u|^p}{(1+c\,\theta)^{p-1}}\mathrm{d}x - \langle f,u\rangle\right\},$$

which combined with

$$\frac{1}{p}\int_\Omega \frac{|\nabla u_\theta|^p}{(1+c\,\theta)^{p-1}}\mathrm{d}x - \langle f,u_\theta\rangle = -\frac{1}{p'}\int_\Omega (1+c\,\theta)|\sigma_\theta|^{p'}\mathrm{d}x,$$

proves that $(\hat u,\hat\theta)$ is a solution of (1.21) if and only if $\hat\theta$ is a solution of the max-min problem (1.24), and $(\hat\theta,\hat\sigma)$, with $\hat\sigma$ defined by (1.22), is a saddle point. From the von Neumann Min-Max Theorem [54, Theorem 2.G and Proposition 1 in Chapter 2], we get that the minimum in (1.23) agrees with the maximum in (1.24), and that $\hat\sigma$ is a solution of (1.23). Taking into account that the functional

$$\sigma\in L^{p'}(\Omega)^N \mapsto \max_{\substack{\theta\in L^\infty(\Omega;[0,1])\\ \int_\Omega \theta\mathrm{d}x\leqslant\kappa}} \int_\Omega (1+c\,\theta)|\sigma|^{p'}\mathrm{d}x$$

is strictly convex, as a maximum of a family of strictly convex functions, we deduce the uniqueness of $\hat\sigma$. □

The following theorem provides a system of optimality conditions for the convex problem (1.7). It proves in particular that $\hat u$ is the solution of a nonlinear Calculus of Variations problem which does not contain the proportion $\hat\theta$. We refer to Section 4 in [28] for a related result in the case $p=2$.

Theorem 1.2 *A pair $(\hat u,\hat\theta)\in W_0^{1,p}(\Omega)\times L^\infty(\Omega;[0,1])$ is a solution of (1.21) if and only if there exists $\hat\mu \geqslant 0$ such that $\hat u$ is a solution of*

$$\min_{u\in W_0^{1,p}(\Omega)}\left(\int_\Omega F(|\nabla u|)\mathrm{d}x - \langle f,u\rangle\right), \tag{1.25}$$

with $F \in C^1([0,\infty)) \cap W^{2,\infty}_{loc}(0,\infty)$, the convex function defined by

$$F(0)=0, \qquad F'(s)=\begin{cases} s^{p-1} & \text{if } 0 \leqslant s < \hat\mu \\ \hat\mu^{p-1} & \text{if } \hat\mu \leqslant s \leqslant (1+c)\hat\mu \\ \dfrac{s^{p-1}}{(1+c)^{p-1}} & \text{if } (1+c)\hat\mu < s, \end{cases} \tag{1.26}$$

and $\hat\mu$, $\hat\theta$ are related by

- *If $\hat\mu = 0$ then*

$$\hat\theta = 1 \quad \text{a.e. in } \big\{|\nabla \hat u| > 0\big\}, \qquad \int_\Omega \hat\theta \, \mathrm{d}x \leqslant \kappa. \tag{1.27}$$

- *If $\hat\mu > 0$, then*

$$\hat\theta = \begin{cases} 0 & \text{if } 0 \leqslant |\nabla \hat u| < \hat\mu \\ \dfrac{1}{c}\left(\dfrac{|\nabla \hat u|}{\hat\mu} - 1\right) & \text{if } \hat\mu \leqslant |\nabla \hat u| < (1+c)\hat\mu \\ 1 & \text{if } (1+c)\hat\mu < |\nabla \hat u|, \end{cases} \qquad \int_\Omega \hat\theta \, \mathrm{d}x = \kappa. \tag{1.28}$$

Proof. Applying Kuhn-Tucker's theorem to the convex problem (1.7), we get that $(\hat u, \hat\theta)$ is a solution if and only if there exists $\hat\mu \geqslant 0$ such that $(\hat u, \hat\theta)$ solves

$$\min_{\substack{u \in W^{1,p}_0(\Omega) \\ \theta \in L^\infty(\Omega;[0,1])}} \left\{ \int_\Omega \Big(\frac{1}{p} \frac{|\nabla u|^p}{(1+c\,\theta)^{p-1}} + \frac{c\hat\mu^p}{p'}\theta \Big) \mathrm{d}x - < f, u > \right\}, \tag{1.29}$$

and

$$\int_\Omega \hat\theta \, \mathrm{d}x \leqslant \kappa, \qquad \hat\mu \left(\int_\Omega \hat\theta \, \mathrm{d}x - \kappa \right) = 0. \tag{1.30}$$

Differentiating in (1.29) we have that $(\hat u, \hat\theta)$ is a solution of (1.29) if and only if

$$\int_\Omega \frac{|\nabla \hat u|^{p-2} \nabla \hat u \cdot \nabla \hat v}{(1+c\hat\theta)^{p-1}} \mathrm{d}x = \langle f, v \rangle, \quad \forall\, v \in W^{1,p}_0(\Omega), \tag{1.31}$$

$$\int_\Omega \Big(\hat\mu^p - \frac{|\nabla \hat u|^p}{(1+c\hat\theta)^p} \Big) (\theta - \hat\theta) \mathrm{d}x \geqslant 0, \quad \forall\, \theta \in L^\infty(\Omega;[0,1]). \tag{1.32}$$

Condition (1.31) is equivalent to $\hat u$ solution of the minimum problem

$$\min_{u \in W^{1,p}_0(\Omega)} \left\{ \frac{1}{p} \int_\Omega \frac{|\nabla u|^p}{(1+c\hat\theta)^{p-1}} \mathrm{d}x - \langle f, u \rangle \right\}, \tag{1.33}$$

while (1.32) is equivalent to $\hat\theta$ satisfying (1.27) or (1.28) depending on whether $\hat\mu = 0$ or $\hat\mu > 0$. Replacing this value of $\hat\theta$ in (1.29) we have the equivalence between (1.33) and (1.25). □

Remark 1.3 *Using* (1.27) *or* (1.28) *and expression* (1.22) *of* $\hat{\sigma}$, *we have that* $\hat{\theta}$ *satisfies*

$$\hat{\theta}(x) = \begin{cases} 1 & \text{if } |\hat{\sigma}| > \hat{\mu} \\ 0 & \text{if } |\hat{\sigma}| < \hat{\mu}. \end{cases} \tag{1.34}$$

Moreover, Theorem 1.2 implies $\hat{\mu} = 0$ *if and only if the unique solution* $\tilde{u}$ *of*

$$\min_{u \in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p} \int_\Omega \frac{|\nabla u|^p}{(1+c)^{p-1}} \mathrm{d}x - \langle f, u \rangle \right\},$$

satisfies

$$\left| \{ x \in \Omega : \ |\nabla \tilde{u}| > 0 \} \right| \leqslant \kappa,$$

where in this case $\hat{u} = \tilde{u}$.

1.4. Regularity for the relaxed problem

In the present section we study the regularity of the solutions of problem (1.21). As a consequence we show that the unrelaxed problem (1.6) has no solution in general. We begin by stating the main results. The corresponding proofs are given later.

Theorem 1.3 *Let* $\Omega \subset \mathbb{R}^N$ *be a* $C^{1,1}$ *bounded open set and* $(\hat{u}, \hat{\theta}) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega; [0,1])$ *be a solution of* (1.21), *then, for* $\hat{\sigma}$ *defined by* (1.22) *and* $\hat{\mu}$ *given by Theorem 1.2 we have:*

1. *If* $f \in W^{-1,q}(\Omega)$, $p' \leqslant q < \infty$, *then* $\nabla \hat{u} \in L^{q(p-1)}(\Omega)^N$ *and there exists* $C > 0$, *which only depends on* p, q, N *and* Ω *such that*

$$\|\nabla \hat{u}\|_{L^{q(p-1)}(\Omega)^N} \leqslant C\big(\|f\|^{\frac{1}{p-1}}_{W^{-1,q}(\Omega)} + \hat{\mu} \big). \tag{1.35}$$

2. *If* $f \in L^q(\Omega)$ *with* $q > N$, *then there exists* $C > 0$ *which only depends on* p, q, N *and* Ω *such that*

$$\|\nabla \hat{u}\|_{L^\infty(\Omega)^N} \leqslant C\big(\|f\|^{\frac{1}{p-1}}_{L^q(\Omega)} + \hat{\mu} \big). \tag{1.36}$$

3. *If* $f \in W^{1,1}(\Omega) \cap L^{2(1+r)}(\Omega)$, *with* $r \geqslant 0$ *or* $f \in W^{1,2(1+r)}(\Omega)$ *with* $r \in (-1/2, 0)$, *then the function* $|\hat{\sigma}|^r \hat{\sigma}$ *is in* $H^1(\Omega)^N$ *and there exists* $C > 0$, *which only depends on* $p, q, N, \hat{\mu}$ *and* Ω *such that*

$$\big\| |\hat{\sigma}|^r \sigma \big\|_{H^1(\Omega)^N} \leqslant \begin{cases} C\left(1 + \|f\|_{W^{1,1}(\Omega)} + \|f\|^{2(1+r)}_{L^{2(1+r)}(\Omega)} \right) & \text{if } r \geqslant 0 \\ C\left(1 + \|f\|_{W^{1,2(1+r)}(\Omega)} \right) & \text{if } -\frac{1}{2} < r < 0. \end{cases} \tag{1.37}$$

Moreover

$$\hat{\sigma} \quad \text{is parallel to } \nu \text{ on } \partial\Omega, \tag{1.38}$$

with ν *the unitary outside normal to* $\partial\Omega$.

4. For $1 \leqslant \mathrm{i}, j \leqslant N$ and $f \in W^{1,1}(\Omega) \cap L^2(\Omega)$

$$\partial_{\mathrm{i}}\hat{\theta}\hat{\sigma}_j - \partial_j\hat{\theta}\hat{\sigma}_{\mathrm{i}} = (1 + c\hat{\theta})(\partial_j\hat{\sigma}_{\mathrm{i}} - \partial_{\mathrm{i}}\hat{\sigma}_j)\mathcal{X}_{\{|\hat{\sigma}|=\hat{\mu}\}} \in L^2(\Omega). \tag{1.39}$$

Moreover, if $\hat{\theta}$ only takes a finite number of values a.e. in Ω, then

$$\partial_{\mathrm{i}}\hat{\theta}\hat{\sigma}_j - \partial_j\hat{\theta}\hat{\sigma}_{\mathrm{i}} = 0, \quad 1 \leqslant \mathrm{i}, j \leqslant N, \quad \mathrm{curl}(|\hat{\sigma}|^{p'-2}\hat{\sigma}) = 0 \quad \text{in } \Omega. \tag{1.40}$$

where, for a distribution from Ω into $\mathbb{R}^N$, the curl *operator is defined as* $\mathrm{curl}(\Phi) := \frac{1}{2}\left(\nabla\Phi - \nabla\Phi^\top\right)$.

Remark 1.4 *As in [13] we can also obtain some local regularity results for $\hat{u}$, $\hat{\theta}$ and $\hat{\sigma}$ but, for the sake of simplicity, we have preferred to only state and prove the global regularity result.*

Remark 1.5 *If we assume that f belongs to $W^{1,1}(\Omega) \cap L^2(\Omega)$, that the unrelaxed problem* (1.6) *has a solution $(\hat{u}, \hat{\theta})$, and that Ω is simply connected, then (1.40) proves the existence of $w \in W^{1,p}(\Omega)$ such that $\hat{\sigma} = |\nabla w|^{p-2}\nabla w$ a.e in Ω. By* (1.38)*, we must also have $\hat{u}$ constant in each connected component of $\partial\Omega$. Assuming then that $\partial\Omega$ has only a connected component and taking into account that w is defined up to an additive constant, we get*

$$\hat{\sigma} = |\nabla w|^{p-2}\nabla w, \quad w \text{ solution of } \begin{cases} -\mathrm{div}\,(|\nabla w|^{p-2}\nabla w) = f & \text{in } \Omega \\ w = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.41}$$

We will show that this implies that the unrelaxed problem has no solution in general.

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^N$ be a connected open set of class $C^{1,1}$ with connected boundary and $f = 1$. If there exists a solution of* (1.1)*, then Ω is a ball.*

Remark 1.6 *In the case $p = 2$, Theorem 1.4 has been proved in [44] assuming that* (1.1) *has a smooth solution and in [13] in the general case.*

The proof of Theorem 1.3 will follow from the following Lemma.

Lemma 1.1 *Let $\Omega \subset \mathbb{R}^N$ be a C^2 bounded open set and $G : [0, \infty) \to [0, \infty)$ be a C^1 function such that there exist $\lambda, \mu > 0$ and $p > 1$ satisfying*

$$G(s) = s^{p-2}, \qquad \forall\, s \geqslant \mu, \tag{1.42}$$

$$0 \leqslant G(s) + G'(s)s, \quad G(s) \leqslant \lambda s^{p-2}, \qquad \forall\, s \geqslant 0. \tag{1.43}$$

Let $u \in C^2(\overline{\Omega})$ be such that there exists $f \in C^{1,1}(\overline{\Omega})$ satisfying

$$-\,\mathrm{div}\Big(G(|\nabla u|)\nabla u\Big) = f \text{ in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.44}$$

Then, the following estimates hold:

1. *For every $q \in (p', \infty)$, there exists $C > 0$ depending only on p, q and Ω, such that*

$$\|\nabla u\|_{L^{q(p-1)}(\Omega)^N} \leqslant C\Big(\|f\|_{W^{-1,q}(\Omega)}^{\frac{1}{p-1}} + \mu\Big). \tag{1.45}$$

2. *For every $q > N$ there exists $C > 0$ depending only on p, q and Ω such that*

$$\|\nabla u\|_{L^{\infty}(\Omega)^N} \leqslant C\Big(\|f\|_{L^{q}(\Omega)}^{\frac{1}{p-1}} + \mu\Big). \tag{1.46}$$

3. *For every $\gamma > -1$, there exists $C > 0$ depending only on p, N, λ, γ and Ω such that*

$$\begin{aligned}&\int_\Omega |\nabla u|^\gamma\Big(\frac{G'\big(|\nabla u|\big)}{|\nabla u|}\big|\nabla^2 u\nabla u\big|^2 + G\big(|\nabla u|\big)\big|\nabla^2 u\big|^2\Big)\,\mathrm{d}x \\ &\leqslant C\mu^{p+\gamma} + C\mu^{1+\gamma}\|f\|_{W^{1,1}(\Omega)} + C\|f\|_{L^{\frac{p+\gamma}{p-1}}(\Omega)}^{\frac{p+\gamma}{p-1}},\end{aligned} \qquad \text{if } \gamma \geqslant p-2, \tag{1.47}$$

$$\begin{aligned}&\int_\Omega |\nabla u|^\gamma\Big(\frac{G'\big(|\nabla u|\big)}{|\nabla u|}\big|\nabla^2 u\nabla u\big|^2 + G\big(|\nabla u|\big)\big|\nabla^2 u\big|^2\Big)\,\mathrm{d}x \\ &\leqslant C\mu^{p+\gamma} + C\|f\|_{W^{1,\frac{p+\gamma}{p-1}}(\Omega)},\end{aligned} \qquad \text{if } -1 < \gamma < p-2. \tag{1.48}$$

Proof. In order to prove (1.45), we write (1.44) as

$$-\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = f - \mathrm{div}\Big(|\nabla u|^{p-2}\nabla u - G(|\nabla u|)\nabla u\Big) \ \text{ in } \Omega,$$

where the last term in the right-hand side is bounded in $W^{-1,\infty}(\Omega)$ by $C\mu^{p-1}$. Then the result follows from Theorem 2.3 in [38].

For the rest of the proof let us differentiate equation (1.44) with respect to x_{i}. This gives

$$-\,\mathrm{div}\Big(L\nabla\partial_{\mathrm{i}} u\Big) = \partial_{\mathrm{i}} f \ \text{ in } \Omega, \tag{1.49}$$

with

$$L = \frac{G'\big(|\nabla u|\big)}{|\nabla u|}\nabla u \otimes \nabla u + G\big(|\nabla u|\big)I. \tag{1.50}$$

Observe that L is non-negative thanks to (1.43).

In order to estimate $\partial_{\mathrm{i}} u$ from (1.49), we also need to add some boundary conditions. For this purpose, fixed $\bar{x} \in \partial\Omega$, we use that there exist $\delta > 0$ and functions $\tau^1, \ldots, \tau^N \in C^1(B(\bar{x}, \delta))^N$ such that for every $x \in B(\bar{x}, \delta)$

$$\left\{\begin{array}{l}\big\{\tau^1(x), \ldots, \tau^N(x)\big\} \ \text{ is an orthonormal basis of } \mathbb{R}^N,\\ \tau^N(x) \text{ agrees with the unitary outside normal vector to } \Omega \text{ on } \partial\Omega \cap B(\bar{x}, \delta).\end{array}\right. \tag{1.51}$$

Using that

$$\nabla u = \sum_{\mathrm{i}=1}^N \big(\nabla u \cdot \tau^{\mathrm{i}}\big)\tau^{\mathrm{i}} \ \text{ a.e. in } B(\bar{x}, \delta),$$

and (1.44), we get

$$-\sum_{\mathrm{i}=1}^{N}\operatorname{div}\big(G(|\nabla u|)\tau^{\mathrm{i}}\big)\nabla u\cdot\tau^{\mathrm{i}}-\sum_{\mathrm{i}=1}^{N}\nabla\big(\nabla u\cdot\tau^{\mathrm{i}}\big)\cdot\tau^{\mathrm{i}}G(|\nabla u|)=f\ \text{ in }\Omega, \tag{1.52}$$

where thanks to u vanishing on $\partial\Omega$, we have

$$\nabla u=(\nabla u\cdot\tau^N)\tau^N,\quad \nabla u\cdot\tau^{\mathrm{i}}=0,\quad \nabla(\nabla u\cdot\tau^{\mathrm{i}})\cdot\tau^{\mathrm{i}}=0\ \text{ on }\partial\Omega,\ 1\leqslant \mathrm{i}\leqslant N-1.$$

Thus, developping (1.52), we get

$$-L\nabla^2u\tau^N\cdot\tau^N=f+G(|\nabla u|)\Big(\operatorname{div}\tau^N I+\big(\nabla\tau^N\big)^t\Big)\tau^N\cdot\nabla u\ \text{ on }\partial\Omega\cap B(\bar{x},\delta).$$

By the arbitrariness of $\bar{x}$, we then deduce the existence of a vector function $h\in L^\infty(\partial\Omega)^N$, which only depends on Ω, such that ∇u satisfies the boundary conditions

$$\begin{cases}\nabla u=|\nabla u|s\nu, & s\in\{0,1\}\text{ a.e. on }\partial\Omega,\\ -L\nabla^2u\nu\cdot\nu=f+G(|\nabla u|)h\cdot\nabla u & \text{on }\partial\Omega,\end{cases} \tag{1.53}$$

with ν the unitary outside normal on $\partial\Omega$.

Let us now prove (1.45). We reason similarly to [23]. For

$$w=|\nabla u|^2, \tag{1.54}$$

and $k>\mu^p$, we multiply (1.49) by $\big(w^{\frac{p}{2}}-k\big)^+\partial_{\mathrm{i}}u\in H^1(\Omega)$ and integrate by parts. Adding in i and taking into account (1.53), we get

$$\begin{aligned}&\frac{p}{4}\int_{\{w^{\frac{p}{2}}\geqslant k\}}w^{\frac{p-2}{2}}L\nabla w\cdot\nabla w\,\mathrm{d}x+\sum_{\mathrm{i}=1}^{N}\int_\Omega\big(w^{\frac{p}{2}}-k\big)^+L\nabla\partial_{\mathrm{i}}u\cdot\nabla\partial_{\mathrm{i}}u\,\mathrm{d}x\\
&=-\int_{\partial\Omega}s|\nabla u|\big(f+G(|\nabla u|)h\cdot\nabla u\big)\big(w^{\frac{p}{2}}-k\big)^+\mathrm{d}s(x)+\int_\Omega\nabla f\cdot\nabla u\big(w^{\frac{p}{2}}-k\big)^+\mathrm{d}x\\
&=-\int_{\partial\Omega}s|\nabla u|G(|\nabla u|)h\cdot\nabla u\big(w^{\frac{p}{2}}-k\big)^+\mathrm{d}s(x)-\int_\Omega f\Delta u\big(w^{\frac{p}{2}}-k\big)^+\mathrm{d}x\\
&-\frac{p}{2}\int_{\{w^{\frac{p}{2}}\geqslant k\}}w^{\frac{p-2}{2}}f\nabla u\cdot\nabla w\,\mathrm{d}x,\end{aligned}$$

which thanks to $k>\mu$, (1.42) and (1.50) proves

$$\begin{aligned}&\int_{\{w^{\frac{p}{2}}\geqslant k\}}w^{p-2}|\nabla w|^2\mathrm{d}x+\int_\Omega\big(w^{\frac{p}{2}}-k\big)^+w^{\frac{p-2}{2}}\big|\nabla^2u\big|^2\mathrm{d}x\\
&\leqslant C\int_{\partial\Omega}w^{\frac{p}{2}}\big(w^{\frac{p}{2}}-k\big)^+\mathrm{d}s(x)+C\int_\Omega|f|\big|\nabla^2u\big|\big(w^{\frac{p}{2}}-k\big)^+\mathrm{d}x+C\int_{\{w^{\frac{p}{2}}\geqslant k\}}w^{\frac{p-1}{2}}|f||\nabla w|\mathrm{d}x,\end{aligned}$$

and then, using Young's inequality

$$\begin{aligned}&\int_{\{w^{\frac{p}{2}}\geqslant k\}}w^{p-2}|\nabla w|^2\mathrm{d}x+\int_\Omega\big(w^{\frac{p}{2}}-k\big)^+w^{\frac{p-2}{2}}\big|\nabla^2u\big|^2\mathrm{d}x\\
&\leqslant C\int_{\partial\Omega}w^{\frac{p}{2}}\big(w^{\frac{p}{2}}-k\big)^+\mathrm{d}s(x)+C\int_{\{w^{\frac{p}{2}}\geqslant k\}}|f|^2w\,\mathrm{d}x.\end{aligned} \tag{1.55}$$

In the first term on the right-hand side we use that, thanks to the compact embedding of $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\int_{\partial\Omega} |v| \mathrm{d}s(x) \leqslant C_\varepsilon \int_\Omega |v| \mathrm{d}x + \varepsilon \int_\Omega |\nabla v| \mathrm{d}x, \qquad \forall\, v \in W^{1,1}(\Omega).$$

Therefore there exists a constant C depending on p and ε such that

$$\int_{\partial\Omega} w^{\frac{p}{2}} \big(w^{\frac{p}{2}} - k\big)^+ \mathrm{d}s(x) \leqslant C \int_\Omega w^{\frac{p}{2}} \big(w^{\frac{p}{2}} - k\big)^+ \mathrm{d}x + \varepsilon \int_{\{w^{\frac{p}{2}} \geqslant k\}} w^{p-1} |\nabla w| \mathrm{d}x.$$

Replacing this inequality in (1.55), taking ε small enough, and using Young's inequality, we get

$$\int_{\{w^{\frac{p}{2}} \geqslant k\}} w^{p-2} |\nabla w|^2 \mathrm{d}x \leqslant C \int_{\{w^{\frac{p}{2}} \geqslant k\}} w^p \mathrm{d}x + C \int_{\{w^{\frac{p}{2}} \geqslant k\}} |f|^2 w \, \mathrm{d}x,$$

which by Sobolev's inequality and f in $L^q(\Omega)$ provides

$$\left(\int_\Omega \left|\big(w^{\frac{p}{2}} - k\big)^+\right|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant C \int_{\{w^{\frac{p}{2}} \geqslant k\}} w^p \mathrm{d}x + C \|f\|^2_{L^q(\Omega)} \left(\int_{\{w^{\frac{p}{2}} \geqslant k\}} w^{\frac{q}{q-2}} \mathrm{d}x\right)^{\frac{q-2}{q}}, \tag{1.56}$$

with

$$2^* = \frac{2N}{N-2} \ \text{ if } N > 2, \quad 2^* \in (2, \infty) \ \text{ if } N = 2.$$

Now, we use that $q > N$ allows us to take $r > 1$ large enough to have

$$\frac{2^*}{2}\Big(\frac{q-2}{q} - \frac{1}{r}\Big) > 1, \qquad \frac{2^*}{2}\Big(1 - \frac{p}{r}\Big) > 1.$$

For such r, we use Hölder's inequality in (1.56) to get

$$\begin{aligned} \left(\int_\Omega \left|\big(w^{\frac{p}{2}} - k\big)^+\right|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}} &\leqslant C \left(\int_\Omega w^r \mathrm{d}x\right)^{\frac{p}{r}} \left|\{w^{\frac{p}{2}} \geqslant k\}\right|^{1-\frac{p}{r}} \\ &\quad + C \|f\|^2_{L^q(\Omega)} \left(\int_\Omega w^r \mathrm{d}x\right)^{\frac{1}{r}} \left|\{w^{\frac{p}{2}} \geqslant k\}\right|^{\frac{q-2}{q} - \frac{1}{r}} \end{aligned}$$

which by (1.45) with $q = 2r/(p-1)$ and

$$\|f\|_{W^{-1,\frac{2r}{p-1}}(\Omega)} \leqslant C \|f\|_{L^q(\Omega)},$$

implies

$$\left(\int_\Omega \left|\big(w^{\frac{p}{2}} - k\big)^+\right|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant C \Big(\|f\|^{\frac{1}{p-1}}_{L^q(\Omega)} + \mu\Big)^{2p} \left|\{w^{\frac{p}{2}} \geqslant k\}\right|^{\min\left(1-\frac{p}{r}, \frac{q-2}{q} - \frac{1}{r}\right)}.$$

Taking $h > k$ and defining φ by

$$\varphi(k) = \left|\{w^{\frac{p}{2}} \geqslant k\}\right|,$$

we have then proved

$$\varphi(h)^{\frac{2}{2^*}} \leqslant \frac{C\big(\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}+\mu\big)^{2p}}{(h-k)^2}\varphi(k)^{\min\left(1-\frac{p}{r},\frac{q-2}{q}-\frac{1}{r}\right)}, \quad \text{for } h>k\geqslant \mu^p,$$

where C only depends on p, N, and Ω. Lemma 4.1 in [51] then proves (1.46).

Let us now prove (1.47). Defining w by (1.54), we take $(w+\varepsilon)^{\frac{\gamma}{2}}\partial_{\mathrm{i}}u$, with $\varepsilon>0$, $\gamma>-1$, as test function in (1.44). Using (1.53), we get

$$\begin{aligned}&\frac{\gamma}{4}\int_\Omega (w+\varepsilon)^{\frac{\gamma-2}{2}}L\nabla w\cdot\nabla w\,\mathrm{d}x+\sum_{\mathrm{i}=1}^N\int_\Omega (w+\varepsilon)^{\frac{\gamma}{2}}L\nabla\partial_{\mathrm{i}}u\cdot\nabla\partial_{\mathrm{i}}u\,\mathrm{d}x\\ &=-\int_{\partial\Omega}s|\nabla u|\big(f+G(|\nabla u|)h\cdot\nabla u\big)(w+\varepsilon)^{\frac{\gamma}{2}}\mathrm{d}s(x)+\int_\Omega\nabla f\cdot\nabla u(w+\varepsilon)^{\frac{\gamma}{2}}\mathrm{d}x.\end{aligned} \tag{1.57}$$

In this inequality, we observe that the integrand in the left-hand side is nonnegative due to

$$\begin{aligned}&2w\sum_{\mathrm{i}=1}^N L\nabla\partial_{\mathrm{i}}u\cdot\nabla\partial_{\mathrm{i}}u-L\nabla w\cdot\nabla w\\ &=2|\nabla u|^2\sum_{\mathrm{i}=1}^N L\nabla\partial_{\mathrm{i}}u\cdot\nabla\partial_{\mathrm{i}}u-2L(\nabla^2u\nabla u)\cdot(\nabla^2u\nabla u)\geqslant 0 \ \text{ a.e. in } \Omega,\end{aligned} \tag{1.58}$$

and $\gamma>-1$. This allows us to use the Fatou Lemma on the left-hand side and the dominated convergence theorem on the right-hand side, when ε tends to zero, to deduce

$$\begin{aligned}&\frac{\gamma}{4}\int_\Omega w^{\frac{\gamma-2}{2}}L\nabla w\cdot\nabla w\,\mathrm{d}x+\sum_{\mathrm{i}=1}^N\int_\Omega w^{\frac{\gamma}{2}}L\nabla\partial_{\mathrm{i}}u\cdot\nabla\partial_{\mathrm{i}}u\,\mathrm{d}x\\ &\leqslant-\int_{\partial\Omega}s|\nabla u|\big(f+G(|\nabla u|)h\cdot\nabla u\big)w^{\frac{\gamma}{2}}\mathrm{d}s(x)+\int_\Omega\nabla f\cdot\nabla u w^{\frac{\gamma}{2}}\mathrm{d}x.\end{aligned} \tag{1.59}$$

Let us first cosider the case $\gamma\geqslant p-2$. Defining $T\in W^{1,\infty}(0,\infty)$ by

$$T(s)=\begin{cases}0 & \text{if } 0\leqslant s\leqslant\mu^2\\ \dfrac{s}{\mu^2}-1 & \text{if } \mu^2\leqslant s\leqslant 2\mu^2\\ 1 & \text{if } s\geqslant 2\mu^2,\end{cases}$$

we decompose the last term in (1.59) as

$$\int_\Omega\nabla f\cdot\nabla u\,w^{\frac{\gamma}{2}}\mathrm{d}x=\int_\Omega\nabla f\cdot(1-T(w))\nabla u w^{\frac{\gamma}{2}}\mathrm{d}x+\int_\Omega\nabla f\cdot T(w)\nabla u w^{\frac{\gamma}{2}}\mathrm{d}x.$$

Integrating by parts the last term, replacing in (1.59) and using Young's inequality, $h\in L^\infty(\partial\Omega)$, and (1.42), we deduce

$$\begin{aligned}&\int_\Omega w^{\frac{\gamma-2}{2}}L\nabla w\cdot\nabla w\,\mathrm{d}x+\sum_{\mathrm{i}=1}^N\int_\Omega w^{\frac{\gamma}{2}}L\nabla\partial_{\mathrm{i}}u\cdot\nabla\partial_{\mathrm{i}}u\mathrm{d}x\leqslant\mu^{1+\gamma}\int_{\partial\Omega}|f|\,\mathrm{d}s(x)\\ &+C\int_{\partial\Omega}w^{\frac{p+\gamma}{2}}\,\mathrm{d}s(x)+\mu^{1+\gamma}\int_\Omega|\nabla f|\mathrm{d}x+C\int_\Omega|f|^2w^{\frac{\gamma-p+2}{2}}\mathrm{d}x+C\mu^{1+\gamma}\int_\Omega|f|\,\mathrm{d}x.\end{aligned} \tag{1.60}$$

For the second term on the right-hand side we use the continuous embedding of $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$ and Young's inequality to get

$$\begin{aligned}
&\int_{\partial\Omega} w^{\frac{p+\gamma}{2}}\,\mathrm{d}s(x) \leqslant C\mu^{p+\gamma} + \int_{\partial\Omega} \left|(w-\mu^2)^+\right|^{\frac{p+\gamma}{2}}\,\mathrm{d}s(x)\\
&\leqslant C\mu^{p+\gamma} + C\int_{\Omega} w^{\frac{p+\gamma}{2}}\mathrm{d}x + C\int_{\{w\geqslant\mu^2\}} w^{\frac{p+\gamma-2}{2}}|\nabla w|\,\mathrm{d}x\\
&\leqslant C\mu^{p+\gamma} + C\Big(1+\frac{1}{\delta}\Big)\int_{\Omega} w^{\frac{p+\gamma}{2}}\,\mathrm{d}x + C\delta\int_{\{w\geqslant\mu^2\}} w^{\frac{p+\gamma-4}{2}}|\nabla w|^2\mathrm{d}x,
\end{aligned} \tag{1.61}$$

with $\delta>0$ arbitrary. Taking δ small enough, replacing in (1.60) and using Hölder's inequality we have

$$\begin{aligned}
&\int_{\Omega} w^{\frac{\gamma-2}{2}} L\nabla w\cdot\nabla w\,\mathrm{d}x + \sum_{\mathrm{i}=1}^{N}\int_{\Omega} w^{\frac{\gamma}{2}} L\nabla\partial_{\mathrm{i}}u\cdot\nabla\partial_{\mathrm{i}}u\mathrm{d}x \leqslant \mu^{1+\gamma}\int_{\partial\Omega}|f|\,\mathrm{d}s(x)\\
&+C\mu^{p+\gamma} + C\int_{\Omega} w^{\frac{p+\gamma}{2}}\mathrm{d}x + \mu^{1+\gamma}\int_{\Omega}|\nabla f|\mathrm{d}x + C\int_{\Omega}|f|^{\frac{p+\gamma}{p-1}}\mathrm{d}x + C\mu^{1+\gamma}\int_{\Omega}|f|\,\mathrm{d}x.
\end{aligned}$$

Using (1.45) with $q=\frac{p+\gamma}{p-1}$ and the continuous imbedding of $L^q(\Omega)$ into $W^{-1,q}(\Omega)$, combined with (1.58) and

$$\sum_{\mathrm{i}=1}^{N} L\nabla\partial_{\mathrm{i}}u\cdot\nabla\partial_{\mathrm{i}}u = \frac{G'\big(|\nabla u|\big)}{|\nabla u|}\left|\nabla^2u\nabla u\right|^2 + G\big(|\nabla u|\big)\left|D^2u\right|^2, \quad \text{a.e. in } \Omega, \tag{1.62}$$

we conclude (1.47).

We now assume $-1<\gamma<p-2$. In this case we estimate the right-hand side in (1.59) as follows:

For the first term, using (1.61), we have for $\delta<1$

$$\begin{aligned}
&\left|\int_{\partial\Omega} s|\nabla u|\big(f+G(|\nabla u|)h\cdot\nabla u\big)w^{\frac{\gamma}{2}}\mathrm{d}s(x)\right| \leqslant C\int_{\partial\Omega}\Big(|f|w^{\frac{\gamma+1}{2}} + w^{\frac{p+\gamma}{2}}\Big)\mathrm{d}s(x)\\
&\leqslant C\int_{\partial\Omega}|f|^{\frac{p+\gamma}{p-1}}\mathrm{d}s(x) + C\int_{\partial\Omega} w^{\frac{p+\gamma}{2}}\mathrm{d}s(x)\\
&\leqslant C\int_{\partial\Omega}|f|^{\frac{p+\gamma}{p-1}}\mathrm{d}s(x) + C\mu^{p+\gamma} + \frac{C}{\delta}\int_{\Omega} w^{\frac{p+\gamma}{2}}\mathrm{d}x + C\delta\int_{\{w\geqslant\mu^2\}} w^{\frac{p+\gamma-4}{2}}|\nabla w|^2\mathrm{d}x.
\end{aligned} \tag{1.63}$$

For the second term on the right-hand side of (1.59), we just use Hölder's inequality to get

$$\left|\int_{\Omega}\nabla f\cdot\nabla u\, w^{\frac{\gamma}{2}}\mathrm{d}x\right| \leqslant C\int_{\Omega}|\nabla f|^{\frac{p+\gamma}{p-1}}\mathrm{d}x + C\int_{\Omega} w^{\frac{p+\gamma}{2}}\mathrm{d}x. \tag{1.64}$$

Using (1.63) with δ small enough, and (1.64) in (1.59), and then using (1.45) with $q=\frac{p+\gamma}{p-1}$, we conclude (1.48). $\square$

Remark 1.7 *Since the constant in the previous theorem only depends on the norm in L^∞ of the first derivative of the functions $\{\tau^{\mathrm{i}}\}_{\mathrm{i}=1}^N$ defined in (1.51), we can relax the conditions $u\in C^2(\bar\Omega)$ and Ω of class C^2 to $u\in C^{1,1}(\bar\Omega)$ and Ω of class $C^{1,1}$ by a density argument.*

Remark 1.8 *As a simple case, Lemma 1.1 can be applied to the p-Laplacian operator,* $G(s) = |s|^{p-2}$. *Indeed, since here* $\mu = 0$ *it is simple to check that the proof above does not use the assumption* $f \in W^{1,1}(\Omega)$ *in (1.47). Thus, it shows that for* $f \in W^{-1,p'}(\Omega) \cap L^{\frac{p+\gamma}{p-1}}(\Omega)$, *if* $\gamma \geqslant p-2$ *or* $f \in W^{-1,p'}(\Omega) \cap W^{1,\frac{p+\gamma}{p-1}}(\Omega)$ *if* $-1 < \gamma < p-2$, *there exists a solution* u *of* (1.44) *such that*

$$|\nabla u|^{\frac{p+\gamma-2}{2}}|\nabla^2 u| \ \text{ belongs to } L^2(\Omega),$$

i.e. $|\nabla u|^{\frac{p+\gamma}{2}}$ *belongs to* $H^1(\Omega)$. *In particular, it proves that* u *belongs to* $H^2(\Omega)$ *if* $p < 3$ *and* f *belongs to* $W^{1,\frac{2}{p-1}}(\Omega)$. *This is a known result which can be found in [22]. It also proves that for* $f \in L^{2(1+r)}(\Omega)$ *if* $r \geqslant 0$, *or* $f \in W^{1,2(1+r)}(\Omega)$ *if* $-1/2 < r < 0$ *the flux* $\sigma = |\nabla u|^{p-2}\nabla u$ *satisfies that* $|\sigma|^r D\sigma$ *belongs to* $L^2(\Omega)^{N\times N}$, *or equivalently, that* $|\sigma|^r\sigma$ *belongs to* $H^1(\Omega)^N$. *The case* $r = 0$ *has been proved in [35].*

Proof of Theorem 1.3. Let us assume the right-hand side f in (1.21) smooth enough, which by $\hat u$ solution of (1.25) implies that $\hat u \in C^{0,\alpha}(\Omega)$ for some $\alpha > 0$ (see e.g. [23]) and satisfies

$$-\operatorname{div}\left(\frac{F'(|\nabla \hat u|)}{|\nabla \hat u|}\nabla \hat u\right) = f \ \text{ in } \Omega, \quad u \in W_0^{1,p}(\Omega). \tag{1.65}$$

For $\varepsilon > 0$ small and F defined by (1.26), we take $F_\varepsilon : [0,\infty) \to [0,\infty)$ of class $C^2([0,\infty))$ such that for some $k > 0$, it satisfies

$$\begin{cases} F_\varepsilon(0) = 0, \quad F'_\varepsilon(s) \geqslant \dfrac{s^{p-1}}{2(1+c)^{p-1}}, \quad \varepsilon \leqslant F''_\varepsilon(s) \leqslant \varepsilon + ks^{p-2}, \quad \forall\, s \geqslant 0, \\ \qquad F_\varepsilon(s) = F(s), \ \forall\, s \geqslant (1+c)\hat\mu, \quad \lim\limits_{\varepsilon\to 0}\|F_\varepsilon - F\|_{L^\infty(0,\infty)} = 0. \end{cases} \tag{1.66}$$

The existence of this approximation is ensured by Theorem 2.1 and Remark 3.1 in [25]. Then, we define u_ε as the unique solution of

$$\min_{u \in W_0^{1,p}(\Omega)\cap L^2(\Omega)} \left\{\int_\Omega F_\varepsilon(|\nabla u|)\mathrm{d}x + \frac{1}{2}\int_\Omega |u-\hat u|^2\mathrm{d}x - \int_\Omega f\, u\,\mathrm{d}x\right\}. \tag{1.67}$$

and therefore

$$-\operatorname{div}\left(\frac{F'_\varepsilon(|\nabla u\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon\right) + u_\varepsilon - \hat u = f \ \text{ in } \Omega. \tag{1.68}$$

Since

$$\int_\Omega F_\varepsilon(|\nabla u_\varepsilon|)\mathrm{d}x + \frac{1}{2}\int_\Omega |u_\varepsilon - \hat u|^2\mathrm{d}x - \int_\Omega f u_\varepsilon\,\mathrm{d}x \leqslant \int_\Omega F_\varepsilon(|\nabla \hat u|)\mathrm{d}x - \int_\Omega f\hat u\,\mathrm{d}x,$$

we have that u_ε is bounded in $W_0^{1,p}(\Omega) \cap L^2(\Omega)$ and thus, up to a subsequence, it converges weakly in $W_0^{1,p}(\Omega) \cap L^2(\Omega)$ to a certain function u_0. Taking into account the uniform convergence of F_ε to F, and F convex, we can pass to the limit in the above inequality to deduce

$$\begin{aligned} &\int_\Omega F(|\nabla u_0|)\mathrm{d}x + \frac{1}{2}\int_\Omega |u_0 - \hat u|^2\mathrm{d}x - \int_\Omega f u_0\,\mathrm{d}x \\ &\leqslant \liminf_{\varepsilon\to 0}\left(\int_\Omega F_\varepsilon(|\nabla u_\varepsilon|)\mathrm{d}x + \frac{1}{2}\int_\Omega |u_\varepsilon - \hat u|^2\mathrm{d}x - \int_\Omega f\, u_\varepsilon\,\mathrm{d}x\right) \\ &\leqslant \int_\Omega F(|\nabla \hat u|)\mathrm{d}x - \int_\Omega f\hat u\,\mathrm{d}x, \end{aligned}$$

which combined with $\hat{u}$ solution of (1.25) shows $u_0=\hat{u}$ and

$$\lim_{\varepsilon\to 0}\int_\Omega F\big(|\nabla u_\varepsilon|\big)\mathrm{d}x=\lim_{\varepsilon\to 0}\int_\Omega F_\varepsilon\big(|\nabla u_\varepsilon|\big)\mathrm{d}x=\int_\Omega F\big(|\nabla \hat{u}|\big)\mathrm{d}x. \tag{1.69}$$

On the other hand, the assumptions of F_ε imply that

$$\sigma_\varepsilon=:\frac{F'_\varepsilon\big(|\nabla u_\varepsilon|\big)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon$$

is bounded in $L^{p'}(\Omega)^N$, and then by (1.68), for a subsequence, there exists $\sigma_0\in L^{p'}(\Omega)^N$ such that

$$\sigma_\varepsilon\rightharpoonup\sigma_0\ \text{ in } L^{p'}(\Omega)^N,\quad -\mathrm{div}(\sigma_0)=f \text{ in } \Omega. \tag{1.70}$$

Taking $V\in L^p(\Omega)^N$ and using the convexity of F_ε, we have

$$\int_\Omega\frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon\cdot\big(V-\nabla u_\varepsilon\big)\mathrm{d}x\leqslant\int_\Omega\big(F_\varepsilon(|V|)-F_\varepsilon(|\nabla u_\varepsilon|)\big)\mathrm{d}x,$$

which can also be written as

$$\begin{aligned}&\int_\Omega\Big(\frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon-\frac{F'_\varepsilon(|\nabla \hat{u}|)}{|\nabla \hat{u}|}\nabla \hat{u}\Big)\cdot\nabla(\hat{u}-u_\varepsilon)\,\mathrm{d}x\\&+\int_\Omega\frac{F'_\varepsilon(|\nabla \hat{u}|)}{|\nabla \hat{u}|}\nabla\hat{u}\cdot\nabla(\hat{u}-u_\varepsilon)\,\mathrm{d}x+\int_\Omega\frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon\cdot\big(V-\nabla\hat{u}\big)\,\mathrm{d}x\\&\leqslant\int_\Omega\big(F_\varepsilon(|V|)-F_\varepsilon(|\nabla u_\varepsilon|)\big)\mathrm{d}x.\end{aligned}$$

From (1.65), (1.69) and (1.70) we can pass to the limit in this inequality to deduce

$$\int_\Omega\sigma_0\cdot\big(V-\nabla\hat{u}\big)\,\mathrm{d}x\leqslant\int_\Omega\big(F(|V|)-F(|\nabla\hat{u}|)\big)\mathrm{d}x,\quad\forall\, V\in L^p(\Omega)^N.$$

Taking $V=\nabla\hat{u}+tW$, with $W\in L^p(\Omega)^N$, $t>0$, dividing by t and passing to the limit when t tends to zero, we get

$$\int_\Omega\sigma_0\cdot W\,\mathrm{d}x\leqslant\int_\Omega\frac{F'(|\nabla\hat{u}|)}{|\nabla\hat{u}|}\nabla\hat{u}\cdot W\,\mathrm{d}x,\quad\forall\, W\in L^p(\Omega)^N,$$

which shows

$$\sigma_0=\frac{F'(|\nabla\hat{u}|)}{|\nabla\hat{u}|}\ \text{ a.e. in } \Omega.$$

We have thus proved

$$u_\varepsilon\rightharpoonup\hat{u}\ \text{ in } W^{1,p}_0(\Omega),\qquad\frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon\rightharpoonup\frac{F'(|\nabla\hat{u}|)}{|\nabla\hat{u}|}\nabla\hat{u}\ \text{ in } L^{p'}(\Omega)^N.$$

Assuming $\Omega\in C^{2,\alpha}$ we can apply for example Theorem 15.12 in [26] to deduce that u_ε belongs to $C^{2,\alpha}(\overline{\Omega})$. On the other hand, we have that $G_\varepsilon\in C^1([0,\infty))$ defined by

$$G_\varepsilon(s)=\frac{F'_\varepsilon(s)}{s}\ \text{ if } s>0,\quad G_\varepsilon(0)=0,$$

satisfies

$$\frac{G_\varepsilon'\big(|\nabla u_\varepsilon|\big)}{|\nabla u_\varepsilon|}\big|\nabla^2 u_\varepsilon \nabla u_\varepsilon\big|^2 + G_\varepsilon\big(|\nabla u_\varepsilon|\big)\big|\nabla^2 u_\varepsilon\big|^2$$

$$= \frac{F_\varepsilon'(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\Big(|\nabla u_\varepsilon|^2 - \frac{|\nabla^2 u_\varepsilon \nabla u_\varepsilon|^2}{|\nabla u_\varepsilon|^2}\Big) + F_\varepsilon''(|\nabla u_\varepsilon|)\frac{|\nabla^2 u_\varepsilon \nabla u_\varepsilon|^2}{|\nabla u_\varepsilon|^2},$$

while

$$|D\sigma_\varepsilon|^2 = \frac{F_\varepsilon'(|\nabla u_\varepsilon|)^2}{|\nabla u_\varepsilon|^2}\Big(|\nabla u_\varepsilon|^2 - \frac{|\nabla^2 u_\varepsilon \nabla u_\varepsilon|^2}{|\nabla u_\varepsilon|^2}\Big) + F_\varepsilon''(|\nabla u_\varepsilon|)^2\frac{|\nabla^2 u_\varepsilon \nabla u_\varepsilon|^2}{|\nabla u_\varepsilon|^2},$$

Then, the assumptions of F_ε imply the existence of a constant $C > 0$, which only depends on the constant k in (1.66) such that

$$|D\sigma_\varepsilon|^2 \leqslant C\big(\varepsilon + |\nabla u_\varepsilon|^{p-2}\big)\left(\frac{G_\varepsilon'\big(|\nabla u_\varepsilon|\big)}{|\nabla u_\varepsilon|}\big|\nabla^2 u_\varepsilon \nabla u\big|^2 + G_\varepsilon\big(|\nabla u_\varepsilon|\big)\big|\nabla^2 u_\varepsilon\big|^2\right).$$

Using Lemma 1.1 and

$$|\nabla u_\varepsilon| \leqslant 2^{\frac{1}{p-1}}(1+c)|\sigma_\varepsilon|^{\frac{1}{p-1}},$$

we conclude (1.35), (1.36) and (1.37) for f and Ω smooth. The general case follows by an approximation argument.

Let us now show (1.39). First, we recall that since we are assuming $f \in W^{1,1}(\Omega) \cap L^2(\Omega)$, we have σ in $H^1(\Omega)^N$. Using that (1.22) implies

$$\nabla \hat{u} = (1 + c\hat{\theta})|\hat{\sigma}|^{p'-2}\hat{\sigma} \quad \text{a.e. in } \Omega,$$

and taking $\mathrm{i}, j \in \{1, \ldots, N\}$, and $\Phi \in C_c^\infty(0,\infty)$, such that $\Phi = 1$ in a neighborhood of $\hat{\mu}$, we get in the distributional sense

$$\begin{aligned}
&\partial_j \hat{u} \partial_\mathrm{i}[\Phi(|\hat{\sigma}|)] - \partial_\mathrm{i} \hat{u} \partial_j[\Phi(|\hat{\sigma}|)] = \partial_\mathrm{i}\big(\partial_j \hat{u}\, \Phi(|\hat{\sigma}|)\big) - \partial_j\big(\partial_\mathrm{i} \hat{u}\, \Phi(|\hat{\sigma}|)\big) \\
&= \partial_\mathrm{i}\Big((1 + c\hat{\theta})|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)\hat{\sigma}_j\Big) - \partial_j\Big((1 + c\hat{\theta})|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)\hat{\sigma}_\mathrm{i}\Big) \\
&= c\partial_\mathrm{i}\hat{\theta}\, |\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)\hat{\sigma}_j - c\partial_j\hat{\theta}\, |\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)\hat{\sigma}_\mathrm{i} \\
&+(1 + c\hat{\theta})\Big(\partial_\mathrm{i}\big(\Phi(|\hat{\sigma}|)|\hat{\sigma}|^{p'-2}\hat{\sigma}_j\big) - \partial_j\big(\Phi(|\hat{\sigma}|)|\hat{\sigma}|^{p'-2}\hat{\sigma}_\mathrm{i}\big)\Big),
\end{aligned} \tag{1.71}$$

which using that the support of Φ is compact and that σ belongs to $H^1(\Omega)^N$ shows

$$|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)\big(\partial_\mathrm{i}\hat{\theta}\, \hat{\sigma}_j - \partial_j\hat{\theta}\, \hat{\sigma}_\mathrm{i}\big) \in L^2(\Omega). \tag{1.72}$$

Now we recall that

$$\hat{\theta} = 0 \ \text{ in } \{|\hat{\sigma}| < \hat{\mu}\}, \quad \hat{\theta} = 1 \ \text{ in } \{|\hat{\sigma}| > \hat{\mu}\}.$$

This implies that for every $\Psi \in C_c^\infty((0,\infty) \setminus \{\hat{\mu}\})$ we have

$$|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)\big(\partial_\mathrm{i}\hat{\theta}\, \hat{\sigma}_j - \partial_j\hat{\theta}\, \hat{\sigma}_\mathrm{i}\big) = |\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)\big(\partial_\mathrm{i}\hat{\theta}\, \hat{\sigma}_j - \partial_j\hat{\theta}\, \hat{\sigma}_\mathrm{i}\big)\big(1 - \Psi(|\hat{\sigma}|)\big).$$

By (1.72) we can take $\hat{\Psi} = \hat{\Psi}_\delta$ with

$$0 \leqslant \hat{\Psi}_\delta \leqslant 1, \quad \hat{\Psi}_\delta(\hat{\mu}) = 0, \quad \hat{\Psi}_\delta(s) \to 1, \ \forall\, s \neq \hat{\mu},$$

to deduce that

$$|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)\big(\partial_{\mathrm{i}}\hat{\theta}\,\hat{\sigma}_j-\partial_j\hat{\theta}\,\hat{\sigma}_{\mathrm{i}}\big)$$

vanishes a.e. in $\{|\hat{\sigma}|\neq\hat{\mu}\}$ and then that

$$|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)\big(\partial_{\mathrm{i}}\hat{\theta}\,\hat{\sigma}_j-\partial_j\hat{\theta}\,\hat{\sigma}_{\mathrm{i}}\big)=\hat{\mu}^{p'-2}\Phi(\hat{\mu})\big(\partial_{\mathrm{i}}\hat{\theta}\,\hat{\sigma}_j-\partial_j\hat{\theta}\,\hat{\sigma}_{\mathrm{i}}\big)\mathcal{X}_{\{|\hat{\sigma}|=\hat{\mu}\}}.$$

On the other hand, recalling that $\nabla|\hat{\sigma}|=0$ a.e. in $\{|\hat{\sigma}|=\hat{\mu}\}$, we can return to (1.71) to conclude (1.39).

Assertion (1.40) now follows from Proposition 2.1 in [9]. which shows that

$$\partial_{\mathrm{i}}\hat{\theta}\hat{\sigma}_j-\partial_j\hat{\theta}\hat{\sigma}_{\mathrm{i}}\in L^2(\Omega),$$

implies

$$\partial_{\mathrm{i}}\hat{\theta}\hat{\sigma}_j-\partial_j\hat{\theta}\hat{\sigma}_{\mathrm{i}}=0\ \text{ a.e. in } \{\hat{\theta}=c\},\quad \forall\, c\in[0,1].$$

□

Proof of Theorem 1.4. Let $\hat{\omega}$ a mesurable subset of Ω, and $\hat{u}\in W_0^{1,p}(\Omega)$ be such that $(\chi_{\hat{\omega}},\hat{u})$ is a solution of (1.21) with $\tilde{f}=f$. By Remark 1.5, we have

$$\big(\alpha\mathcal{X}_{\hat{\omega}}+\beta\mathcal{X}_{\Omega\setminus\hat{\omega}}\big)\,\nabla\hat{u}=\nabla w,$$

with w the unique solution of

$$\begin{cases} -\operatorname{div}\big(|\nabla w|^{p-2}\nabla w\big)=1 \text{ in } \Omega\\ w\in W_0^{1,p}(\Omega).\end{cases}\tag{1.73}$$

Thanks to Theorem 1.1 in [33] and the fisrt corollary in [23] we know that w is in $C^{1,\beta}(\Omega)$ for some $\beta\in(0,1)$, and (see [41]) that it is analytic in $\{|\nabla w|>0\}$. Using Theorem 1.1 in [35] (or Theorem 1.3) we also have that $\hat{\sigma}=|\nabla w|^{p-2}\nabla w$ is in $H^1(\Omega)^N$. Thus, $-\operatorname{div}\hat{\sigma}=0$ a.e. in $\{\hat{\sigma}=0\}$, which combined with w solution of (1.73) implies that $\nabla w\neq 0$ a.e. in Ω. Analogouly, let us prove that for every $\lambda>0$, the set $\{|\nabla w|=\lambda\}$ has zero measure. For this purpose we observe that a.e. in $\{|\nabla w|=\lambda\}$, we have

$$0=\Delta|\nabla w|^p=p\lambda^{p-2}\big(|\nabla^2 w|^2+(\Delta\nabla w)\cdot\nabla w\big),$$

but a.e. in $\{|\nabla w|=\lambda\}$, we also have

$$0=\nabla\operatorname{div}(|\nabla w|^{p-2}\nabla w)=\lambda^{p-2}\nabla\Delta w=\lambda^{p-2}\Delta\nabla w.$$

Therefore $\nabla^2 w=0$ a.e. in $\{|\nabla w|=\lambda\}$, which combined with

$$-\lambda^{p-2}\Delta w=-\operatorname{div}(|\nabla w|^{p-2}\nabla w)=1\ \text{ a.e. in } \{|\nabla w|=\lambda\},$$

implies that the set $\{|\nabla w|=\lambda\}$ has zero measure. Now, we recall that thanks to (1.34), the constant $\hat{\mu}$ in Theorem 1.2 satisfies

$$\{x\in\Omega:\ |\nabla w|>\hat{\mu}\}\subset\hat{\omega}\subset\{x\in\Omega:\ |\nabla w|\geqslant\hat{\mu}\},$$

while Theorem 1.2 implies $|\hat{\omega}| = \kappa$. So, using that $|\{|\nabla w| = \hat{\mu}\}| = 0$, we get (up to a set of null measure)

$$\omega = \{x \in \Omega : \ |\nabla w| < \hat{\mu}\}, \tag{1.74}$$

and $|\hat{\omega}| < |\Omega|$. Then, taking a connected component O of the open set $\{x \in \Omega : \ |\nabla w| > \hat{\mu}\}$, we can repeat the argument in [14] to deduce that $O \Subset \Omega$ is an analytic manifold with connected boundary such that

$$\begin{cases} -\operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right) = 1 \text{ in } O \\ w, \ \dfrac{\partial w}{\partial \nu} \text{ are constant on } \partial O. \end{cases} \tag{1.75}$$

From Serrin's Theorem ([49]), this proves that O is an open ball and that w is a radial function in O with respect to its center. Taking into account the analyticity of w in $\{|\nabla w| \neq 0\}$, the unique continuation principle shows that Ω is a ball. □

1.5. Conclusion Section

In the present paper we have studied the optimal design of a two-phase material modeled by the p-Laplacian operator posed in a bounded open set $\Omega \subset \mathbb{R}^N$. The goal is to maximize the potential energy (problem (1.1)) when we only dispose of a limited amount of the best material. Since the problem has not solution in general, we have obtained a relaxed formulation (problems (1.2) and (1.3)) where instead of taking in every point of Ω one of both materials, we use a microscopic mixture where the proportion θ of the best material takes values in the whole interval $[0,1]$. This new formulation is obtained using homogenization theory. Reasoning by duality, we have also obtained a new formulation of the minimization problem as a min-max problem (problems (1.23) and (1.24)). As a consequence we show that although the relaxed problem has not uniqueness in general, the flux $\hat{\sigma}$ is unique.

The optimal conditions for the relaxed problem show that the state function $\hat{u}$ is the solution of a nonlinear Calculus of Variation problem (1.25). Since the second derivative of the function F in this problem is not uniformly elliptic, the corresponding Euler-Lagrange equation does not provide in general the existence of second derivatives for $\hat{u}$. However it allows us to show that if the data es smooth enough then, for every $r > -1/2$, the function $|\hat{\sigma}|^r\hat{\sigma}$ is in the Sobolev space $H^1(\Omega)^N \cap L^\infty(\Omega)^N$. Moreover, the optimal proportion $\hat{\theta}$ is derivable in the orthogonal directions to $\nabla\hat{u}$. As an application of these results, we show that the original problem has a solution in a smooth open set Ω with a connected boundary if and only if Ω is a ball.

The results obtained in the present paper extend those obtained by other authors in the case of the Laplacian operator (see e.g. [13], [18], [28], [44]).

Chapter 2

Numerical Maximization of the p-Laplacian energy of a two-phase material

Abstract: For a diffusion problem modeled by the p-Laplacian operator, we are interested in obtaining numerically the two-phase material which maximizes the internal energy. We assume that the amount of the best material is limited. In the framework of a relaxed formulation we present two algorithms, a feasible directions method and an alternating minimization method. We show the convergence for both of them and we provide an estimate for the error. Since for $p > 2$ both methods are only well defined for a finite-dimensional approximation, we also study the difference between solving the finite-dimensional and the infinite-dimensional problems. Although the error bounds for both methods are similar, numerical experiments show that the alternating minimization method works well than the feasible directions one.

2.1. Introduction

The aim of the present work is the numerical resolution of an optimal design problem. It corresponds to the maximization of the energy for a non-linear diffusion process in a two-phase material modeled by the p-Laplacian operator. Namely, we are interested in the control problem

$$\left\{\begin{array}{c}\displaystyle\max_{\omega}\frac{1}{p}\int_{\Omega}\left(\alpha\mathcal{X}_{\omega}+\beta\left(1-\mathcal{X}_{\omega}\right)\right)|\nabla u|^{p}\mathrm{d}x\\ -\mathrm{div}\left(\left(\alpha\mathcal{X}_{\omega}+\beta\left(1-\mathcal{X}_{\omega}\right)\right)|\nabla u|^{p-2}\nabla u\right)=f\quad\text{in }\Omega\\ u\in W_{0}^{1,p}(\Omega),\quad\omega\subset\Omega\text{ mesurable },\quad|\omega|\leqslant\kappa,\end{array}\right. \tag{2.1}$$

with Ω a bounded open set in $\mathbb{R}^N$, $N \geqslant 2$, $p \in (1, \infty)$, $\alpha, \beta, \kappa > 0$, $\alpha < \beta$, and $f \in W^{-1,p'}(\Omega)$. Here α and β are the diffusion constants corresponding to the two materials that we want to mix in order to maximize the corresponding functional. If we do not impose any restrictions on the amount of material α (i.e. $\kappa \geqslant |\Omega|$) then, the solution is the trivial one given by $\omega = \Omega$. Thus, the interesting case corresponds to $\kappa < |\Omega|$. This problem has been extensively studied for $p = 2$ ([3], [13], [19], [28], [31], [32], [44]). In this case, it models for example the optimal rearrangement of two materials in the cross section of a beam in order to minimize its torsion (in this application $f = 1$). Analogously, for $p \in (1,2) \cup (2,\infty)$ the p-Laplacian operator models the torsional creep in the cross-section of a beam [30]. Therefore problem (2.1) corresponds to find the two-phase material which minimizes the torsion in non-linear elasticity, assuming that the amount of the best material is limited. As it is usual for this type of problems ([42], [43]), it has no solution in general ([13], [15], [44]). Thus, it is necessary to work with a relaxed formulation which can be obtained from the homogenization theory ([3], [45], [52]). In the present case, it has been proved in [15] ([44] for $p = 2$) that such relaxation is given by

$$\begin{cases} \displaystyle\max_{u,\theta} \frac{1}{p} \int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}} \mathrm{d}x \\ \displaystyle -\mathrm{div}\Big(\frac{|\nabla u|^{p-2}}{(1+c\theta)^{p-1}} \nabla u\Big) = \frac{1}{\beta} f \quad \text{in } \Omega \\ \displaystyle u \in W_0^{1,p}(\Omega), \quad \theta \in L^\infty(\Omega;[0,1]), \quad \int_\Omega \theta \, \mathrm{d}x \leqslant \kappa, \end{cases} \tag{2.2}$$

with $c = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}} - 1$. In this formulation, the materials α and β have been replaced by mixtures of them obtained by laminations. The new control variable θ represents the proportion of the best material α used in the mixture.

The problem can also be formulated in a simple way as the following Calculus of Variation problem

$$\begin{cases} \displaystyle\min_{u,\theta} \left\{ \frac{1}{p} \int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}} \mathrm{d}x - \frac{1}{\beta} \langle f, u \rangle \right\} \\ \displaystyle \theta \in L^\infty(\Omega;[0,1]), \quad u \in W_0^{1,p}(\Omega), \quad \int_\Omega \theta \mathrm{d}x \leqslant \kappa, \end{cases} \tag{2.3}$$

The numerical resolution of (2.3) for $p = 2$ has been the subject of several works. In this way, some numerical simulations have been carried out in [28] and [31] using a multi-grid method. In [3] and [53], it has been shown the convergence of the alternating minimization algorithm, using the optimality conditions. In [7], it has been studied the convergence of a projected gradient method.

For $p \neq 2$, the use of the optimality conditions implies the resolution of the p-Laplacian equation in each iteration. This is a problem which has been considered for example in [27] and [29] using a steepest descent method . We also refer to [36] where a reformulation of the p-Laplacian is given in order to use an augmented Lagrangian method. In these works, the order of convergence is linear in the best case.

In the present paper we introduce two algorithms to solve (2.5). The first one is based on the Frank-Wolfe algorithm, also known as the feasible direction method. The second one is an alternating minimization method. In both of them we choose a descent direction in $H_0^1(\Omega)$ instead of $W_0^{1,p}(\Omega)$ and we solve a linear problem instead of a p-Laplacian which, as we said above, is very expensive from a computational point of view. For $p > 2$, this forces us to work with a discretized version of the problem because $H_0^1(\Omega)$ is not contained in $W_0^{1,p}(\Omega)$.

We prove the convergence of both methods obtaining estimates for the rate of convergence.

In the best of the cases ($p \geqslant 2$) we only have a convergence of order 1/i, with i the number of iterations. This is due to the non strict convexity of the problem. In this sense we can observe that solving the minimum in θ in problem (2.3) and using Kuhn-Tucher theorem, we can rewrite (2.3) as (see [13], [15], [28], [31])

$$\max_{\mu \geqslant 0} \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega F(\mu, \nabla u)\,\mathrm{d}x - \frac{\mu^p(p-1)c}{p}\kappa - \frac{1}{\beta}\langle f, u\rangle \right\} \tag{2.4}$$

with

$$F(\mu,\xi) = \begin{cases} \dfrac{1}{p}\dfrac{|\xi|^p}{(1+c)^{p-1}} + \dfrac{\mu^p(p-1)c}{p} & \text{if } \mu \leqslant \dfrac{|\xi|}{(1+c)} \\ \mu^{p-1}|\xi| - \dfrac{\mu^p(p-1)}{p} & \text{if } \dfrac{|\xi|}{(1+c)} < \mu < |\xi| \\ \dfrac{1}{p}|\xi|^p & \text{if } \mu \geqslant |\xi|. \end{cases}$$

Observe that F is non strictly convex in ξ and it is not differentiable with respect to μ.

We also prove the convergence of the solutions of the discretized problem towards the solutions of the continuous one. Even more, taking a regular sequence of triangulations in Ω of diameter $h > 0$, and discretizing $W_0^{1,p}(\Omega)$ and $L^\infty(\Omega)$ by the the usual P_1 and P_0 finite elements respectively, we show that the difference between the minimum for the continuous and the discretized problem is of order h. In order to prove this result we assume the existence of a solution (u,θ) for (2.3) such that u is in $W^{1,\infty}(\Omega)$, ∇u belongs to $BV(\Omega)^N$ and θ belongs to $BV(\Omega)$. Some smoothness results for problem (2.3) can be found in [13] and [31] for $p = 2$ and [15] for $p \in (1,\infty)$, we also refer to [12] for the relaxed problem corresponding to take minimum in (2.1) instead of the maximum one. These smoothness results imply that u is in $W^{1,\infty}(\Omega)$, the flow $\sigma = |\nabla u|^{p-2}\nabla u/(1+c\theta)^{p-1}$ is in $H^1(\Omega)^N$ and the derivatives of θ in the direction of σ are in $L^2(\Omega)$. However this is not enough to get ∇u and θ BV-functions. Nevertheless, this assumption seems to be satisfied in the numerical experiments.

The paper is organized as follows:

In section 2.2 we recall some known results for problem (2.3) which have been proved in [15] (see [13], [44], for $p = 2$).

In section 2.3 we state the main results of the paper.

Section 2.4 is devoted to prove the results in Section 2.3.

Finally in Section 2.5 we illustrate the results of the paper with some numerical simulations. They show that the alternating minimization method converges faster than the feasible direction method.

2.2. Previous results

As we mentioned in the introduction, our aim in the present paper is to numerically solve the optimal design problem (2.1). Since it has no solution in general, we work with the relaxed formulation (2.3), which renaming f/β by f to simplify the notation, can be written as

$$\min\left\{\mathcal{F}(\theta,u):\ \theta\in L^\infty(\Omega;[0,1]),\quad u\in W_0^{1,p}(\Omega),\quad \int_\Omega \theta\,\mathrm{d}x\leqslant\kappa\right\},\tag{2.5}$$

with

$$\mathcal{F}(\theta,u)=\frac{1}{p}\int_\Omega\frac{|\nabla u|^p}{(1+c\,\theta)^{p-1}}\,\mathrm{d}x-\langle f,u\rangle.\tag{2.6}$$

Here Ω is a bounded open set of $\mathbb{R}^N$, $N\geqslant 2$, $p\in(1,\infty)$, $c>0$, $\kappa\in(0,|\Omega|)$ and f is a distribution in $W^{-1,p'}(\Omega)$. Since $\mathcal{F}$ is convex in (θ,u) and coercive in u, $W_0^{1,p}(\Omega)$ is reflexive and $L^\infty(\Omega;[0,1])$ is bounded, and then sequentially compact for the weak-$*$ topology in $L^\infty(\Omega)$, the existence of solution is straightforward. However $\mathcal{F}$ is not strictly convex and therefore the uniqueness is not clear.

The relaxed formulation (2.5) has been obtained in [15]. In this paper we have also obtained some optimality conditions and some equivalent formulations. As a consequence we got some uniqueness and smoothness results (see [13], [28], [31], [44] for related results in the case $p=2$).

Thanks to the convexity of $\mathcal{F}$, Kuhn-Tucker's theorem easily provides the following system of optimality conditions ([15])

Proposition 2.1 *A pair $(\hat\theta,\hat u)$ is a solution of (2.5) if and only if there exists $\hat\mu\geqslant 0$ such that:*

If $\hat\mu=0$, then

$$\hat\theta=1\ \text{a.e. in}\ \{x\in\Omega:\ \nabla\hat u(x)\neq 0\},\qquad \big|\{x\in\Omega:\ \nabla\hat u(x)\neq 0\}\big|\leqslant\kappa,\tag{2.7}$$

$$\left\{\begin{array}{l}-\mathrm{div}\left(\dfrac{|\nabla\hat u|^{p-2}}{(1+c)^{p-1}}\nabla\hat u\right)=f\quad\text{in}\ \Omega\\[2ex] \hat u=0\quad\text{on}\ \partial\Omega.\end{array}\right.\tag{2.8}$$

If $\hat\mu>0$, then

$$\hat\theta=\max\left\{0,\min\left\{1,\frac{1}{c}\left(\frac{|\nabla\hat u|}{\hat\mu}-1\right)\right\}\right\},\quad \int_\Omega\hat\theta\,\mathrm{d}x=\kappa,\tag{2.9}$$

$$\left\{\begin{array}{l}-\mathrm{div}\left(\dfrac{|\nabla\hat u|^{p-2}}{(1+c\hat\theta)^{p-1}}\nabla\hat u\right)=f\quad\text{in}\ \Omega\\[2ex] \hat u=0\quad\text{on}\ \partial\Omega.\end{array}\right.\tag{2.10}$$

Remark 2.1 *The expression of $\hat\theta$ in Proposition 2.1 is obtained by solving (see [15], [44])*

$$\min\left\{\int_\Omega\frac{|\nabla\hat u|^p}{(1+c\theta)^{p-1}}\mathrm{d}x:\ \theta\in L^\infty(\Omega;[0,1]),\ \int_\Omega\theta\,\mathrm{d}x\leqslant\kappa\right\}.\tag{2.11}$$

The constant $\hat{\mu} \geqslant 0$ is a Lagrange multiplier corresponding to the constraint $\int_\Omega \theta \,\mathrm{d}x \leqslant \kappa$.

We observe that for an arbitrary function $\hat{u} \in W_0^{1,p}(\Omega)$ (not necessarily a solution for (2.5)), the solutions of (2.11) can be explicitly obtained using Kuhn-Tucker's theorem which shows that $\hat{\theta}$ is a solution if and only if there exists $\hat{\mu} \geqslant 0$ such that

$$\hat{\mu}\left(\int_\Omega \hat{\theta}\mathrm{d}x - \kappa\right) = 0,$$

and $\hat{\theta}$ is a solution of

$$\min\left\{\int_\Omega \frac{|\nabla \hat{u}|^p}{(1+c\theta)^{p-1}}\mathrm{d}x + \hat{\mu}\int_\Omega \hat{\theta}\,\mathrm{d}x :\ \theta \in L^\infty(\Omega;[0,1])\right\}. \qquad (2.12)$$

This provides the following rule to solve (2.11):

If $\hat{u}$ is such that $|\{\nabla \hat{u} \neq 0\}| \leqslant \kappa$, then $\hat{\theta}$ is any function in $L^\infty(\Omega;[0,1])$ satisfying

$$\hat{\theta} = 1 \quad a.e.\ in\ \{x \in \Omega :\ \nabla \hat{u}(x) \neq 0\}, \qquad \int_\Omega \hat{\theta}\,\mathrm{d}x \leqslant \kappa.$$

In the other case, denoting for $\mu > 0$

$$\theta_\mu := \max\left\{0, \min\left\{1, \frac{1}{c}\left(\frac{|\nabla \hat{u}|}{\mu} - 1\right)\right\}\right\},$$

and defining $G : (0,\infty) \to \big[0, |\Omega|\big]$ by

$$G(\mu) = \int_\Omega \theta_\mu \mathrm{d}x, \quad \forall \mu \in (0,\infty),$$

we have that the set of solutions of (2.11) is given by

$$\big\{\theta_\mu \in L^\infty(\Omega;[0,1]) :\ \mu > 0, \quad G(\mu) = \kappa\big\}. \qquad (2.13)$$

Remark that the equation $G(\mu) = \kappa$ has a solution (not unique in general) due to G decreasing, continuous, and

$$\lim_{\mu\to 0} G(\mu) = \big|\{x \in \Omega :\ \nabla \hat{u}(x) \neq 0\}\big|, \qquad \lim_{\mu\to\infty} G(\mu) = 0.$$

Numerically, the equation $G(\mu) = \kappa$ can be easily solved using for example a dichotomy method.

In [15] (see [44] for $p = 2$) it has also been proved that introducing the flow

$$\sigma = \frac{|\nabla u|^{p-2}}{(1+c\theta)^{p-1}}\nabla u,$$

we have that (2.5) is equivalent to the min-max problem

$$\min_{\substack{-\operatorname{div}\sigma = f\\ \sigma \in L^{p'}(\Omega)^N}} \ \max_{\substack{\theta \in L^\infty(\Omega:[0,1])\\ \int_\Omega \theta\,\mathrm{d}x \leqslant \kappa}} \int_\Omega (1+c\,\theta)|\sigma|^{p'}\mathrm{d}x = \max_{\substack{\theta \in L^\infty(\Omega:[0,1])\\ \int_\Omega \theta\,\mathrm{d}x \leqslant \kappa}} \ \min_{\substack{-\operatorname{div}\sigma = f\\ \sigma \in L^{p'}(\Omega)^N}} \int_\Omega (1+c\,\theta)|\sigma|^{p'}\mathrm{d}x. \quad (2.14)$$

Taking into account that the functional

$$\sigma \in L^2(\Omega)^N \to \max_{\substack{\theta\in L^\infty(\Omega:[0,1])\\ \int_\Omega \theta\, dx\leqslant\kappa}} \int_\Omega (1+c\,\theta)|\sigma|^{p'} dx$$

is strictly convex, we get the uniqueness of the optimal flow. Moreover, using (2.4), we get the following smoothness results for the solutions of (2.5).

Theorem 2.1 *For every solution $(\hat{\theta}, \hat{u})$ of (2.5) the flow $\hat{\sigma}$ defined by*

$$\hat{\sigma} = \frac{|\nabla \hat{u}|^{p-2}}{(1+c\hat{\theta})^{p-1}} \nabla \hat{u} \tag{2.15}$$

is uniquely defined

If f belongs to $W^{1,1}(\Omega) \cap L^q(\Omega)$, $q > N$, and Ω is a $C^{1,1}$ domain, then $\hat{\sigma}$ belongs to $H^1(\Omega)^N \cap L^\infty(\Omega)^N$. Moreover, there exists $C > 0$ which only depends on N, p, c and Ω such that

$$\|\hat{\sigma}\|_{H^1(\Omega)^N \cap L^\infty(\Omega)^N} \leqslant C \left(\|f\|_{W^{1,1}(\Omega)\cap L^q(\Omega)} + \hat{\mu} \right), \tag{2.16}$$

with $\hat{\mu}$ given by Proposition 2.1.

The function $\hat{\theta}$ satisfies

$$\hat{\theta}(x) = \begin{cases} 1 & \text{if } |\hat{\sigma}| > \hat{\mu} \\ 0 & \text{if } |\hat{\sigma}| < \hat{\mu}, \end{cases} \tag{2.17}$$

and decomposing $\hat{\sigma} = (\hat{\sigma}_1, ..., \hat{\sigma}_N)$, we have

$$\partial_{\rm i} \hat{\theta} \hat{\sigma}_j - \partial_j \hat{\theta} \hat{\sigma}_{\rm i} = (1+c\hat{\theta})(\partial_j \hat{\sigma}_{\rm i} - \partial_{\rm i} \hat{\sigma}_j)\chi_{\{|\hat{\sigma}|=\hat{\mu}\}} \in L^2(\Omega), \quad 1 \leqslant {\rm i}, j \leqslant N. \tag{2.18}$$

Remark 2.2 *Theorem 2.1 has been proved in [15] where some other regularity results depending on the smoothness of f have been obtained. The case $p = 2$, has been first proved in [13]. Observe that $\hat{\sigma}$ in $L^\infty(\Omega)^N$ implies that $\hat{u}$ belongs to $W^{1,\infty}(\Omega)$. This was previously shown in [31] for $p = 2$.*

2.3. Algorithms and main results

In this section we present two variants of a descent algorithm to numerically solve problem (2.5). We also show the convergence of both algoritms.

A first attempt to construct an algorithm is to use an alternate method consisting in minimizing in u, then in θ and so on. That is, assuming an approximation $(u_{\rm i}, \theta_{\rm i})$ of a solution of (2.5), we compute $u_{{\rm i}+1}$ as a solution of

$$\min_{v\in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p} \int_\Omega \frac{|\nabla v|^p}{(1+c\theta_{\rm i})^{p-1}} - \langle f, v\rangle \right\} \tag{2.19}$$

and then $\theta_{\rm i+1}$ as a solution of

$$\min_{\substack{\theta\in L^\infty(\Omega;[0,1])\\ \int_\Omega \theta\,{\rm d}x\leqslant\kappa}}\left\{\frac{1}{p}\int_\Omega \frac{|\nabla u_{\rm i+1}|^p}{(1+c\theta)^{p-1}}{\rm d}x\right\}. \tag{2.20}$$

This method works well if $p=2$, but for $p\neq 2$, problem (2.19) is a p-Laplacian problem which is very expensive to solve from the computational point of view due to the nonlinearity of the corresponding Euler-Lagrange equation.

Instead of using the above alternate method, we can also try to use a gradient method, i.e. an iterative method where the iterations are defined through $u_{\rm i+1}=u_{\rm i}+t_{\rm i}v_{\rm i+1}$, $\theta_{\rm i+1}=\theta_{\rm i}+s_{\rm i}(\vartheta_{\rm i+1}-\theta_{\rm i})$ for some $t_{\rm i},s_{\rm i}\in(0,1)$, with $(v_{\rm i+1},\vartheta_{\rm i+1})$ a solution of

$$\left\{\begin{array}{l}\displaystyle\min_{\|v\|_{W_0^{1,p}(\Omega)}\leqslant 1}\left\{\frac{1}{p}\int_\Omega\frac{|\nabla u_{\rm i}|^{p-2}}{(1+c\theta_{\rm i})^{p-1}}\nabla u_{\rm i}\cdot\nabla v\,{\rm d}x-\langle f,v\rangle\right\},\\[2ex] \displaystyle\max_{\substack{\vartheta\in L^\infty(\Omega;[0,1])\\ \int_\Omega\vartheta\,{\rm d}x\leqslant\kappa}}\int_\Omega\frac{|\nabla u_{\rm i}|^p}{(1+c\theta_{\rm i})^p}\,\vartheta\,{\rm d}x,\end{array}\right. \tag{2.21}$$

but the minimization in v also implies the resolution of a p-Laplacian problem. To avoid this difficulty, we can replace the constraint $\|v\|_{W_0^{1,p}(\Omega)}\leqslant 1$ by $\|v\|_{H_0^1(\Omega)}\leqslant 1$. This is a feasible direction method. In each iteration we look for the direction of maximum descent of $\mathcal{F}$ in the convex set:

$$\left\{(v,\vartheta)\in H_0^1(\Omega)\times L^\infty(\Omega;[0,1])\ :\ \|v\|_{H_0^1(\Omega)}\leqslant 1,\ \int_\Omega\vartheta{\rm d}x\leqslant\kappa\right\}.$$

The maximum direction with respect to ϑ is simple to calculate. Namely, reasoning as in Remark 2.1 we have:

If $|\{|\nabla u_{\rm i}|>0\}|\leqslant\kappa$, then $\vartheta_{\rm i}$ is any function in $L^\infty(\Omega;[0,1])$ such that

$$\chi_{\{|\nabla u_{\rm i}|>0\}}\leqslant\vartheta_{\rm i},\qquad\int_\Omega\vartheta_{\rm i}\,{\rm d}x\leqslant\kappa. \tag{2.22}$$

In another case, we introduce $H:(0,\infty)\to[0,|\Omega|]$ by

$$H(\mu)=\big|\{x\in\Omega:\ |\nabla u_{\rm i}(x)|>(1+c\theta_{\rm i})\mu\},\quad\forall\,\mu\geqslant 0.$$

Then, H is a decreasing function, continuous on the right and satisfying

$$\lim_{\mu\to 0^+}H(\mu)=\big|\{x\in\Omega:\ |\nabla u_{\rm i}(x)|>0\}\big|,\qquad\lim_{\mu\to\infty}H(\mu)=0.$$

This assures the existence of $\mu_{\rm i}>0$ (not unique in general) such that

$$H(\mu_{\rm i})\leqslant\kappa\leqslant\lim_{\mu\to\mu_{\rm i}^-}H(\mu),$$

which can be easily numerically obtained by a dichotomy rule. For such $\mu_{\rm i}$, the maximum direction in θ in (2.21), $\vartheta_{\rm i}$, is given by any function in $L^\infty(\Omega;[0,1])$ such that

$$\chi_{\{|\nabla u_{\rm i}|>(1+c\theta_{\rm i})\mu_{\rm i}\}}\leqslant\vartheta_{\rm i},\qquad\int_\Omega\vartheta_{\rm i}\,{\rm d}x=\kappa. \tag{2.23}$$

A similar result holds if we use a finite-dimensional approximation consisting in choosing θ taking constant values in the elements of a given mesh. On the other hand, the maximum descent direction with respect to v is unique and it is the solution of a linear equation. However, we observe that for $p > 2$, the sequence of functions $\{u_i\}$ generated by the method is not in $W_0^{1,p}(\Omega)$. Thus, the algorithm has only a sense using a finite-dimensional space instead of $L^\infty(\Omega) \times W_0^{1,p}(\Omega)$. In such case, all the norms are equivalent. However it would be necessary to prove the convergence of the solutions of the discretized problem to the continuous one.

With these considerations, we are going to be interested in the following problem

$$\min\left\{\mathcal{F}(\theta,u):\quad \theta\in\Theta,\quad u\in V,\quad \int_\Omega \theta\,\mathrm{d}x\leqslant\kappa,\quad \theta\in[0,1]\text{ a.e. in }\Omega\right\}, \tag{2.24}$$

with Θ and V finite-dimensional subspaces of $L^\infty(\Omega)$ and $H_0^1(\Omega)\cap W_0^{1,p}(\Omega)$ respectively. As in the continuous problem, it is not clear that (2.24) has a unique solution but for every solution (θ^*, u^*), the flow

$$\sigma^* = \frac{|\nabla u^*|^{p-2}}{(1+c\theta^*)^{p-1}}\nabla u^*, \tag{2.25}$$

is unique because it is a solution of (see (2.14))

$$\min\left\{\max_{\substack{\theta\in L^\infty(\Omega:[0,1])\\ \int_\Omega\theta\,\mathrm{d}x\leqslant\kappa}}\int_\Omega(1+c\,\theta)|\sigma|^{p'}\mathrm{d}x:\ \sigma\in L^{p'}(\Omega)^N,\quad \int_\Omega\sigma\cdot\nabla v\,\mathrm{d}x=\langle f,v\rangle,\ \forall\, v\in V\right\},$$

where the function to minimize is strictly convex.

As an example of practical interest we can consider a regular triangular mesh $\mathcal{T}_h$ of $\overline{\Omega}$ with maximum diameter $h > 0$ and the Lagrange finite element spaces

$$\Theta_h = \left\{v=\sum_{\tau\in\mathcal{T}_h}\alpha_\tau\mathcal{X}_\tau\ :\alpha_\tau\in\mathbb{R},\quad\forall\tau\in\mathcal{T}_h\right\} \tag{2.26}$$

$$V_h = \left\{v\in C_0^0(\Omega)\ :\ v|_\tau\in\mathbb{P}_1(\tau),\quad\forall\tau\in\mathcal{T}_h\right\}, \tag{2.27}$$

with $\mathbb{P}_1(\tau)$ the space of affine functions in τ.

Since the minimization of $\mathcal{F}$ in θ for u fixed is simple to carry out in practice (see (2.1) for the infinite-dimensional case, the finite-dimensional one is analogous) we can also consider a variant of the previous algorithm consisting in directly computing the minimum in θ in each iteration.

With these considerations, we present the following two algorithms:

Algorithm 1.

Initialization: $\mathrm{i}=1$, $\theta_0\in\Theta$, $u_0\in V$, $a,b\in(0,1)$.

1 : Set $v_{\rm i}$ solution of

$$\int_\Omega \nabla v_{\rm i} \cdot \nabla \phi \, {\rm d}x = \langle f, \phi \rangle - \int_\Omega \frac{|\nabla u_{\rm i}|^{p-2}}{(1+c\theta_{\rm i})^{p-1}} \nabla u_{\rm i} \cdot \nabla \phi \, {\rm d}x, \quad \forall \phi \in V. \tag{2.28}$$

2 : Choose the step length by $t_{\rm i} = b^j$ (Armijo's rule), with j the smallest non-negative integer such that

$$\mathcal{F}(\theta_{\rm i}, u_{\rm i} + t_{\rm i} v_{\rm i}) \leqslant \mathcal{F}(\theta_{\rm i}, u_{\rm i}) - a t_{\rm i} \int_\Omega |\nabla v_{\rm i}|^2 {\rm d}x \tag{2.29}$$

and set $u_{\rm i+1} = u_{\rm i} + t_{\rm i} v_{\rm i}$.

3 : Set $\vartheta_{\rm i}$ a solution of

$$\max \left\{ \int_\Omega \frac{|\nabla u_{\rm i+1}|^p}{(1+c\theta_{\rm i})^p} \vartheta \, {\rm d}x : \quad \vartheta \in \Theta, \ 0 \leqslant \vartheta \leqslant 1 \text{ a.e. in } \Omega, \ \int_\Omega \vartheta {\rm d}x \leqslant \kappa \right\}. \tag{2.30}$$

4 : Choose $s_{\rm i} = b^k$, with k the smallest non-negative integer such that

$$\mathcal{F}(\theta_{\rm i} + s_{\rm i}(\vartheta_{\rm i} - \theta_{\rm i}), u_{\rm i+1}) \leqslant \mathcal{F}(\theta_{\rm i}, u_{\rm i+1}) - a s_{\rm i} \frac{c(p-1)}{p} \int_\Omega \frac{|\nabla u_{\rm i+1}|^p}{(1+c\theta_{\rm i})^p} (\vartheta_{\rm i} - \theta_{\rm i}) \, {\rm d}x \tag{2.31}$$

and set $\theta_{\rm i+1} = \theta_{\rm i} + s_{\rm i}(\vartheta_{\rm i} - \theta_{\rm i})$.

Algorithm 2.

Initialization: ${\rm i} = 0$, $u_0 \in V$, $a, b \in (0,1)$.

1 : Set $v_{\rm i} \in V$ the solution of (2.28).

2 : Choose the step length by $t_{\rm i} = b^j$ with j the smallest non-negative integer such that (2.29) is satisfied, and set $u_{\rm i+1} = u_{\rm i} + t_{\rm i} v_{\rm i}$.

3 : Set $\theta_{\rm i+1}$ a solution of

$$\min \left\{ \int_\Omega \frac{|\nabla u_{\rm i}|^p}{(1+c\vartheta)^{p-1}} {\rm d}x : \quad \vartheta \in \Theta, \ 0 \leqslant \vartheta \leqslant 1 \text{ a.e. in } \Omega, \ \int_\Omega \vartheta \, {\rm d}x \leqslant \kappa \right\}. \tag{2.32}$$

Remark 2.3 *Since by definition (2.28) of $v_{\rm i}$, we have*

$$\lim_{t \to 0} \frac{\mathcal{F}(\theta_{\rm i}, u_{\rm i} + t v_{\rm i}) - \mathcal{F}(\theta_{\rm i}, u_{\rm i})}{t} = - \int_\Omega |\nabla v_{\rm i}|^2 {\rm d}x,$$

and $a < 1$, we get that

$$\mathcal{F}(\theta_{\rm i}, u_{\rm i} + t v_{\rm i}) - \mathcal{F}(\theta_{\rm i}, u_{\rm i}) \leqslant -a t \int_\Omega |\nabla v_{\rm i}|^2 {\rm d}x,$$

for $0 < t$ small enough. This proves the existence of $t_{\rm i}$ satisfying (2.29). A similar argument shows the existence of $s_{\rm i}$ in (2.31).

Remark 2.4 *If $p = 2$, then $t_i = 1$ for both algorithms and every $i \geqslant 0$. The second method agrees in this case with the one given in [3] Theorem 5.1.5, and [53].*

Our main result is given by theorem 2.2 below which provides the convergence for both algorithms. Before stating it, we need the following definition.

Definition 2.5 *For $p > 1$, we define $\gamma_p > 0$ by*

$$\begin{cases} \|v\|_{H_0^1(\Omega)} \leqslant \gamma_p \|v\|_{W_0^{1,p}(\Omega)} & \text{if } 1 < p < 2 \\ \|v\|_{W_0^{1,p}(\Omega)} \leqslant \gamma_p \|v\|_{H_0^1(\Omega)} & \text{if } p \geqslant 2. \end{cases} \qquad \forall\, v \in V. \tag{2.33}$$

Remark 2.6 *Clearly $\gamma_2 = 1$, while for $p \neq 2$ and V replaced by a sequence of finite dimensional spaces V_h such that*

$$\lim_{h \to 0} \min_{v \in V_h} \|v - v_h\|_{W_0^{1,p}(\Omega)} = 0, \quad \forall\, v \in W_0^{1,p}(\Omega),$$

we have that $\gamma_p = \gamma_{p,h}$ tends to infinity when h goes to zero. For example, in the case where the spaces V_h are given by (2.27), with $\mathcal{T}_h$ a sequence of regular meshes of diameter h, we have

$$\gamma_{p,h} = O\left(\frac{1}{h^{N|\frac{1}{2} - \frac{1}{p}|}} \right). \tag{2.34}$$

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $p \in (1, \infty)$, $f \in W^{-1,p'}(\Omega)$, and Θ, V finite-dimensional subspaces of $L^\infty(\Omega)$ and $W_0^{1,p}(\Omega)$ respectively. Taking $(\theta_i, u_i) \in \Theta \times V$, the sequence defined by Algorithm 1 or Algorithm 2, denoting by $\mathcal{F}^*$ the minimum value of (2.24), and by e_i the sequence of errors*

$$e_i = \mathcal{F}(\theta_i, u_i) - \mathcal{F}^* \geqslant 0, \quad i \geqslant 0, \tag{2.35}$$

we have that e_i is a decreasing sequence and that there exists $C > 0$ depending on $a, b, u_0, \theta_0, f, c, N$ and p such that

$$e_i \leqslant \begin{cases} C\gamma_p^p\, i^{-\frac{1}{p-1}} & \text{if } 1 < p < 2 \\ C\gamma_p^4\, i^{-1} & \text{if } p \geqslant 2. \end{cases} \qquad \forall i \geqslant 1. \tag{2.36}$$

Moreover, the sequence

$$\sigma_i = \frac{|\nabla u_i|^{p-2}}{(1 + c\theta_i)^{p-1}} \nabla u_i, \tag{2.37}$$

converges strongly to σ^* defined by (2.25) in $L^{p'}(\Omega)^N$. Namely, there exists $C > 0$ as above such that

$$\int_\Omega (|\sigma^*| + |\sigma_i|)^{p-2} |\sigma^* - \sigma|^2 dx \leqslant \begin{cases} C\big(1 + \gamma_p \|u_0\|_{W_0^{1,p}(\Omega)}\big)(e_i - e_{i+1})^{\frac{1}{p'}} & \text{if } 1 < p < 2 \\ C\big(|1 + \gamma_p^2 \|u_0\|^{p-1}_{W_0^{1,p}(\Omega)}\big)(e_i - e_{i+1})^{\frac{1}{2}} & \text{if } p \geqslant 2. \end{cases} \tag{2.38}$$

Remark 2.7 *In the continuous case $V = W_0^{1,p}(\Omega)$, and $1 < p < 2$, the classical regularity results for the Poisson equation show that the solution v_{i} of (2.28) satisfies the estimate*

$$\|v_{\mathrm{i}}\|_{W_0^{1,p'}(\Omega)} \leqslant C\big(\|u_{\mathrm{i}}\|_{W_0^{1,p}(\Omega)} + \|f\|_{W^{-1,p'}(\Omega)}\big), \tag{2.39}$$

with $C > 0$ depending only on p and Ω. Thanks to this result we can deduce that in this case (2.36) holds true with γ_p replaced by one. Similar results to (2.39) also hold for special choices of spaces V, see e.g. [11], Theorem 8.5.3. With these choices we can eliminate the dependence in γ_p of estimate (2.36) for $1 < p < 2$.

Remark 2.8 *In the case of the p-Laplacian problem, i.e.*

$$\min_{u \in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p} \int_\Omega |\nabla u|^p \mathrm{d}x - \langle f, u \rangle \right\},$$

we can consider the following algorithm, similar to Algorithms 1 and 2:

Initialization: $\mathrm{i} = 0$, $u_0 \in V$, $a, b \in (0, 1)$.

1 : Set $v_{\mathrm{i}} \in V$ the solution of

$$\int_\Omega \nabla v_{\mathrm{i}} \cdot \nabla \phi \, \mathrm{d}x = \langle f, \phi \rangle - \int_\Omega |\nabla u_{\mathrm{i}}|^{p-2} \nabla u_{\mathrm{i}} \cdot \nabla \phi \, \mathrm{d}x, \quad \forall \phi \in V. \tag{2.40}$$

2 : Choose the step length by $t_{\mathrm{i}} = b^j$ with j the smallest non-negative integer such that

$$\begin{aligned} &\frac{1}{p} \int_\Omega |\nabla (u_{\mathrm{i}} + t_{\mathrm{i}} v_{\mathrm{i}})|^p \mathrm{d}x - \langle f, u_{\mathrm{i}} + t_{\mathrm{i}} v_{\mathrm{i}} \rangle \\ &\leqslant \frac{1}{p} \int_\Omega |\nabla u_{\mathrm{i}}|^p \mathrm{d}x - \langle f, u_{\mathrm{i}} \rangle - a t_{\mathrm{i}} \int_\Omega |\nabla v_{\mathrm{i}}|^2 \mathrm{d}x, \end{aligned}$$

and set $u_{\mathrm{i}+1} = u_{\mathrm{i}} + t_{\mathrm{i}} v_{\mathrm{i}}$.

Then, a similar reasoning to the one used below to prove Theorem 2.2 shows the estimates

$$\mathcal{F}(u_{\mathrm{i}}) - \mathcal{F}^* \leqslant \begin{cases} \dfrac{C\gamma_p^{\frac{2p}{2-p}}}{\mathrm{i}^{\frac{2(p-1)}{2-p}}} & \text{if } p < 2 \\ C^{\mathrm{i}} & \text{if } p = 2 \\ \dfrac{C\gamma_p^{\frac{2p}{p-2}}}{\mathrm{i}^{\frac{p}{p-2}}} & \text{if } p > 2, \end{cases} \tag{2.41}$$

with $C < 1$ for $p = 2$. Similarly to Remark 2.7 the dependence of the estimate on γ_p can be suppressed for $1 < p < 2$ in the continuous case, or V finite-dimensional but satisfying further assumptions. Observe that estimates (2.41) are better than the ones obtained in Theorem 2.2. This is due to the strict convexity of the p-Laplacian operator, which does not hold in our case.

We finish this section studying the convergence of the solutions of the discrete problem to the solutions of the continuous one. Next result is an immediate consequence of the convexity of $\mathcal{F}$ and therefore is given without proof.

Proposition 2.2 *Assume two sequences of spaces $\Theta_h \subset L^\infty(\Omega)$ and $V_h \subset W_0^{1,p}(\Omega)$ such that*

- *For every $\theta \in L^\infty(\Omega)$, with $\theta \geqslant 0$, there exists a sequence $\theta_h \in \Theta_h$ such that*

$$0 \leqslant \theta_h \leqslant \|\theta\|_{L^\infty(\Omega)}, \qquad \int_\Omega \theta_h \mathrm{d}x \leqslant \int_\Omega \theta \mathrm{d}x, \qquad \theta_h \to \theta \ \ \text{in } L^1(\Omega). \tag{2.42}$$

- *For every $u \in W_0^{1,p}(\Omega)$, there exists a sequence $u_h \in V_h$ such that*

$$u_h \to u \ \ \text{in } W_0^{1,p}(\Omega). \tag{2.43}$$

Then, defining $\mathcal{F}_h^*$ as the value of the minimum in (2.24) with Θ and V replaced by Θ_h and V_h respectively, and $\hat{\mathcal{F}}$ as the value of the minimum in (2.19), we have

$$\lim_{h\to 0} \mathcal{F}_h^* = \hat{\mathcal{F}}. \tag{2.44}$$

Moreover, defining σ_h^* by (2.25), with (θ^*, u^*) any solution of (2.24) for $\Theta = \Theta_h$, $V = V_h$, we have

$$\sigma_h^* \to \hat{\sigma} \ \ \text{in } L^{p'}(\Omega)^N, \tag{2.45}$$

with $\hat{\sigma}$ defined by (2.15).

An example of spaces satisfying properties (2.42) and (2.43) is given by (2.26) and (2.27). In this case we have the following improvement

Theorem 2.3 *Assume Ω a polygonal open set, $f \in W^{-1,\infty}(\Omega) \cap L^1(\Omega)$ and that there exists a solution $(\hat{\theta}, \hat{u})$ of (2.5), such that*

$$\hat{\theta} \in BV(\Omega), \quad \hat{u} \in W^{1,\infty}(\Omega), \quad \nabla \hat{u} \in BV(\Omega)^N. \tag{2.46}$$

We also consider a regular sequence $\mathcal{T}_h$ of triangulations in Ω by N-simplexes and define the spaces Θ_h and V_h by (2.26) and (2.27) respectively. Then, there exists $C > 0$, depending on Ω, p, and the functions $\hat{\theta}$, $\hat{u}$, such that denoting by $\hat{\mathcal{F}}$ and $\mathcal{F}_h^*$ the minimum values of (2.5) and (2.24) respectively with $\Theta = \Theta_h$ and $V = V_h$, we have

$$\hat{\mathcal{F}} \leqslant \mathcal{F}_h^* \leqslant \hat{\mathcal{F}} + Ch, \quad \forall\, h > 0. \tag{2.47}$$

Moreover, the functions σ_h^* and σ^* defined as in Proposition 2.2, satisfy

$$\int_\Omega (|\sigma^*| + |\sigma_h^*|)^{p-2} |\sigma^* - \sigma_h^*|^2 \mathrm{d}x \leqslant Ch. \tag{2.48}$$

Remark 2.9 *In Theorem 2.1 we recalled some smoothness results for problem (2.5). Contrary to Theorem 2.3, they assumed that Ω is $C^{1,1}$ instead of a polygonal set. Indeed, assuming*

Ω a smooth convex set, Theorem 2.3 could still be applied, taking a sequence of regular meshes for polygonal subsets of Ω which fulfill Ω as the limit. Even with this assumption we do not know that θ and ∇u are in $BV(\Omega)$ and $BV(\Omega)^N$, but numerical simulations usually provide solutions which seem to satisfy these assumptions.

From (2.34), (2.36) and (2.47), we get

Corollary 2.1 *In the assumptions of Theorem 2.3, we have the estimates*

$$0 \leqslant \mathcal{F}(\theta_{\mathrm{i},h}, u_{\mathrm{i},h}) - \hat{\mathcal{F}} \leqslant \begin{cases} C\Big(\dfrac{1}{h^{N(1-\frac{p}{2})}\mathrm{i}^{p-1}} + h\Big) & \text{if } 1 < p < 2 \\ C\Big(\dfrac{1}{h^{2N(1-\frac{2}{p})}\mathrm{i}} + h\Big) & \text{if } p \geqslant 2, \end{cases} \tag{2.49}$$

Here $\hat{\mathcal{F}}$ denotes the minimum value of (2.5), $(\theta_{\mathrm{i},h}, u_{\mathrm{i},h})$ is the i-th pair obtained by any of the algorithms, and Θ_h, V_h are defined by (2.26) and (2.27) respectively.

2.4. Convergence proof

We dedicate this section to prove the results stated in the previous one. In order to simplify the proof of Theorem 2.2, we start with the following lemma.

Lemma 2.1 *Assume $p \in (1, \infty)$, then we have*

1. *There exists $C > 0$, depending only of p such that for every $\xi, \eta \in \mathbb{R}^N$, we get*

$$\big||\eta|^p - |\xi|^p - p|\xi|^{p-2}\xi \cdot (\eta - \xi)\big| \leqslant \begin{cases} C|\xi - \eta|^p & \text{if } p < 2 \\ C\big(|\xi| + |\eta|\big)^{p-2}|\xi - \eta|^2 & \text{if } p \geqslant 2. \end{cases} \tag{2.50}$$

2. *There exists $C > 0$, depending only on p and c, such that for every $q, r \in [0, 1]$, we get*

$$\left|\frac{1}{(1 + cr)^{p-1}} - \frac{1}{(1 + cq)^{p-1}} + \frac{(p-1)c(r-q)}{(1 + cq)^p}\right| \leqslant C|r - q|^2. \tag{2.51}$$

Proof. In order to show (2.50), we first recall the following property of the function $\xi \in \mathbb{R}^N \mapsto |\xi|^{p-2}\xi \in \mathbb{R}^N$: There exists $c_p > 0$, such that for every $\xi, \eta \in \mathbb{R}^N$, we have

$$\big||\eta|^{p-2}\eta - |\xi|^{p-2}\xi\big| \leqslant \begin{cases} c_p|\xi - \eta|^{p-1} & \text{if } p < 2 \\ c_p\big(|\xi| + |\eta|\big)^{p-2}|\eta - \xi| & \text{if } p \geqslant 2. \end{cases} \tag{2.52}$$

By the mean value theorem, for every $\xi, \eta \in \mathbb{R}^N$, there exists $\lambda \in (0, 1)$, such that

$$|\eta|^p - |\xi|^p = p\big|\lambda\xi + (1 - \lambda)\eta\big|^{p-2}\big(\lambda\xi + (1 - \lambda)\eta\big) \cdot (\eta - \xi), \tag{2.53}$$

where thanks to (2.52), we have

$$\begin{aligned}
&\left| \left|\lambda\xi + (1-\lambda)\eta\right|^{p-2}\big(\lambda\xi + (1-\lambda)\eta\big) - |\xi|^{p-2}\xi \right| \\
&\leqslant \begin{cases} c_p|\xi-\eta|^{p-1} & \text{if } p<2 \\ 2^{p-2}c_p\big(|\xi|+|\eta|\big)^{p-2}|\eta-\xi| & \text{if } p \geqslant 2. \end{cases}
\end{aligned} \tag{2.54}$$

This proves (2.50). Let us now show (2.51). As above, for every $q, r \in [0,1]$, the mean value theorem provides the existence of $\lambda \in (0,1)$ such that

$$\frac{1}{(1+cr)^{p-1}} - \frac{1}{(1+cq)^{p-1}} = -\frac{(p-1)c(r-q)}{\big(1+c(\lambda q + (1-\lambda)r)\big)^p}, \tag{2.55}$$

where

$$\left| \frac{1}{\big(1+c(\lambda q+(1-\lambda)r)\big)^p} - \frac{1}{(1+cq)^p} \right| = \frac{\left|(1+cq)^p - \big(1+c(\lambda q+(1-\lambda)r)\big)^p\right|}{\big(1+c(\lambda q+(1-\lambda)r)\big)^p(1+cq)^p}.$$

Using here the mean value theorem in the numerator, that the denominator is bigger or equal than 1, and that $q, r \in [0,1]$, we get

$$\begin{aligned}
&\left| \frac{1}{\big(1+c(\lambda q+(1-\lambda)r)\big)^p} - \frac{1}{(1+cq)^p} \right| \leqslant pc^p(2+c(q+r))^{p-1}|q-r| \\
&\leqslant p2^{p-1}c^p(1+c)^{p-1}|q-r|.
\end{aligned} \tag{2.56}$$

Inequalities (2.55) and (2.56) show (2.51). □

The proof of Theorem 2.2 also uses the following lemma which has been obtained in [29], Lemma 1.

Lemma 2.2 *Assume $\nu > 0$, $\gamma > 1$ and a sequence of positive numbers λ_n such that*

$$\lambda_n - \lambda_{n+1} \geqslant \nu\lambda_n^\gamma, \quad \forall\, n \geqslant 0.$$

Then, for $r = 1/(\gamma-1)$, we have

$$\lambda_n \leqslant \frac{1}{n^r}\max\left\{\lambda_0, \Big(\frac{2^r-1}{\nu}\Big)^r\right\}. \tag{2.57}$$

Proof of Theorem 2.2. Let us first prove estimate (2.36) for Algorithm 1.

For every $\mathrm{i} \geqslant 0$, estimate (2.50), Hölder's inequality and definition (2.28) of v_i imply the existence of $C > 0$ depending only on p such that:

If $1 < p < 2$

$$\begin{aligned}
&\mathcal{F}(\theta_\mathrm{i}, u_\mathrm{i} + tv_\mathrm{i}) - \mathcal{F}(\theta_\mathrm{i}, u_\mathrm{i}) \\
&\leqslant t\left(\int_\Omega \frac{|\nabla u_\mathrm{i}|^{p-2}}{(1+c\theta_\mathrm{i})^{p-1}}\nabla u_\mathrm{i}\cdot\nabla v_\mathrm{i}\,\mathrm{d}x - \langle f, v_\mathrm{i}\rangle\right) + Ct^p\int_\Omega |\nabla v_\mathrm{i}|^p\mathrm{d}x \\
&= -t\|v_\mathrm{i}\|^2_{H^1_0(\Omega)} + Ct^p\|v_\mathrm{i}\|^p_{W_0^{1p}(\Omega)}.
\end{aligned} \tag{2.58}$$

If $p \geqslant 2$

$$\begin{aligned}\mathcal{F}(\theta_{\rm i}, u_{\rm i}+tv_{\rm i}) - \mathcal{F}(\theta_{\rm i}, u_{\rm i}) &\leqslant t\left(\int_\Omega \frac{|\nabla u_{\rm i}|^{p-2}}{(1+c\theta_{\rm i})^{p-1}}\nabla u_{\rm i}\cdot\nabla v_{\rm i}\,{\rm d}x - \langle f, v_{\rm i}\rangle\right)\\ &\quad +Ct^2\Big(\|u_{\rm i}\|_{W_0^{1,p}(\Omega)} + \|u_{\rm i}+tv_{\rm i}\|_{W_0^{1,p}(\Omega)}\Big)^{p-2}\|v_{\rm i}\|^2_{W_0^{1,p}(\Omega)}\\ &= -t\|v_{\rm i}\|^2_{H_0^1(\Omega)} + Ct^2\Big(\|u_{\rm i}\|_{W_0^{1,p}(\Omega)} + \|u_{\rm i}+tv_{\rm i}\|_{W_0^{1,p}(\Omega)}\Big)^{p-2}\|v_{\rm i}\|^2_{W_0^{1,p}(\Omega)}.\end{aligned} \tag{2.59}$$

Now, we observe that if $t_{\rm i} < 1$ then, by definition of $t_{\rm i}$, we have

$$\mathcal{F}(\theta_{\rm i}, u_{\rm i}+bt_{\rm i}v_{\rm i}) - \mathcal{F}(\theta_{\rm i}, u_{\rm i}) > -abt_{\rm i}\|v_{\rm i}\|^2_{H_0^1(\Omega)}.$$

Combined with (2.58) or (2.59) this proves the existence of $\tau > 0$ which only depends on a, b and p such that

$$t_{\rm i} \geqslant \begin{cases} \min\left\{1, \tau\dfrac{\|v_{\rm i}\|^{\frac{2}{p-1}}_{H_0^1(\Omega)}}{\|v_{\rm i}\|^{p'}_{W_0^{1,p}(\Omega)}}\right\} & \text{if } 1<p<2\\[2ex] \min\left\{1, \tau\dfrac{\|v_{\rm i}\|^2_{H_0^1(\Omega)}}{\Big(\|u_{\rm i}\|_{W_0^{1,p}(\Omega)} + \|u_{\rm i+1}\|_{W_0^{1,p}(\Omega)}\Big)^{p-2}\|v_{\rm i}\|^2_{W_0^{1,p}(\Omega)}}\right\} & \text{if } p\geqslant 2.\end{cases} \tag{2.60}$$

On the other hand, inequality (2.51) implies the existence of another constant $C > 0$ depending only on p and c such that

$$\begin{aligned}&\mathcal{F}(\theta_{\rm i} + s(\vartheta_{\rm i}-\theta_{\rm i}), u_{\rm i+1}) - \mathcal{F}(\theta_{\rm i}, u_{\rm i+1})\\ &\leqslant -s\frac{c(p-1)}{p}\int_\Omega \frac{|\nabla u_{\rm i+1}|^p}{(1+c\theta_{\rm i})^p}(\vartheta_{\rm i}-\theta_{\rm i})\,{\rm d}x + Cs^2\int_\Omega |\nabla u_{\rm i+1}|^p|\vartheta_{\rm i}-\theta_{\rm i}|^2{\rm d}x,\end{aligned}$$

which reasonling as above, implies the existence of $\lambda > 0$ depending only on a, b, c and p such that

$$s_{\rm i} \geqslant \min\left\{1, \lambda\frac{\displaystyle\int_\Omega \frac{|\nabla u_{\rm i+1}|^p}{(1+c\theta_{\rm i})^p}(\vartheta_{\rm i}-\theta_{\rm i}){\rm d}x}{\displaystyle\int_\Omega |\nabla u_{\rm i+1}|^p|\vartheta_{\rm i}-\theta_{\rm i}|^2{\rm d}x}\right\}. \tag{2.61}$$

Using that

$$\begin{aligned}{\rm e}_{\rm i} - {\rm e}_{\rm i+1} &= \mathcal{F}(\theta_{\rm i}, u_{\rm i}) - \mathcal{F}^* - \big(\mathcal{F}(\theta_{\rm i+1}, u_{\rm i+1}) - \mathcal{F}^*\big)\\ &= \mathcal{F}(\theta_{\rm i}, u_{\rm i+1}) - \mathcal{F}(\theta_{\rm i+1}, u_{\rm i+1}) + \mathcal{F}(\theta_{\rm i}, u_{\rm i}) - \mathcal{F}(\theta_{\rm i}, u_{\rm i+1}),\end{aligned}$$

inequalities (2.29), (2.31), (2.60) and (2.61) and

$$\|v_{\rm i}\|_{W_0^{1,p}(\Omega)} \leqslant \begin{cases} |\Omega|^{\frac{1}{p}-\frac{1}{2}}\|v_{\rm i}\|_{H_0^1(\Omega)} & \text{if } 1<p<2\\ \gamma_p\|v_{\rm i}\|_{H_0^1(\Omega)} & \text{if } p\geqslant 2,\end{cases}$$

we deduce the existence of $C > 0$ depending only on a, b, c, p and $|\Omega|$, such that:

If $1<p<2$

$$\mathrm{e_i}-\mathrm{e_{i+1}} \geqslant C \min\left\{1, \frac{\left(\int_\Omega \frac{|\nabla u_{\mathrm{i}+1}|^p}{(1+c\theta_\mathrm{i})^p}(\vartheta_\mathrm{i}-\theta_\mathrm{i})\mathrm{d}x\right)^2}{\int_\Omega |\nabla u_{\mathrm{i}+1}|^p|\vartheta_\mathrm{i}-\theta_\mathrm{i}|^2\mathrm{d}x} + \|v_\mathrm{i}\|^{p'}_{H^1_0(\Omega)}\right\}. \tag{2.62}$$

If $p \geqslant 2$,

$$\begin{aligned}&\mathrm{e_i}-\mathrm{e_{i+1}}\\ &\geqslant C\min\left\{1, \frac{\left(\int_\Omega \frac{|\nabla u_{\mathrm{i}+1}|^p}{(1+c\theta_\mathrm{i})^2}(\vartheta_\mathrm{i}-\theta_\mathrm{i})\mathrm{d}x\right)^2}{\int_\Omega |\nabla u_{\mathrm{i}+1}|^2|\vartheta_\mathrm{i}-\theta_\mathrm{i}|^2\mathrm{d}x} + \frac{\gamma_p^{-4}\|v_\mathrm{i}\|^2_{W^{1,p}_0(\Omega)}}{\left(\|u_\mathrm{i}\|_{W^{1,p}_0(\Omega)}+\|u_{\mathrm{i}+1}\|_{W^{1,p}_0(\Omega)}\right)^{p-2}}\right\}.\end{aligned} \tag{2.63}$$

In particular, $\mathrm{e_i}$ is a non-negative and non-increasing sequence and therefore a converging sequence. In particular $\mathrm{e_i}-\mathrm{e_{i+1}}$ tends to zero. Moreover, $\mathrm{e_i}$ non-increasing implies that $\mathcal{F}(\theta_\mathrm{i}, u_\mathrm{i})$ and then $\|u_\mathrm{i}\|_{W^{1,p}_0(\Omega)}$ are bounded.

On the other hand, thanks to the convexity of $\mathcal{F}$, $u_{\mathrm{i}+1}=u_\mathrm{i}+t_\mathrm{i}v_\mathrm{i}$, with $0\leqslant t_\mathrm{i}\leqslant 1$, and definitions (2.28) and (2.30) of v_i and ϑ_i respectively, we have

$$\begin{aligned}\mathrm{e_i} = \mathcal{F}(\theta_\mathrm{i}, u_\mathrm{i}) - \mathcal{F}(\theta^*, u^*) &\leqslant \int_\Omega \frac{|\nabla u_\mathrm{i}|^{p-2}}{(1+c\theta_\mathrm{i})^{p-1}}\nabla u_\mathrm{i}\cdot\nabla(u_\mathrm{i}-u^*)\,\mathrm{d}x + \langle f, u_\mathrm{i}-u^*\rangle\\ &-\frac{c(p-1)}{p}\int_\Omega \frac{|\nabla u_{\mathrm{i}+1}|^p}{(1+c\theta_\mathrm{i})^p}(\theta_\mathrm{i}-\theta^*)\mathrm{d}x + Ct_\mathrm{i}\int_\Omega \left(|\nabla u_\mathrm{i}|+|\nabla u_{\mathrm{i}+1}|\right)^{p-1}|\nabla v_\mathrm{i}|\mathrm{d}x\\ &\leqslant \int_\Omega \nabla v_\mathrm{i}\cdot\nabla(u_\mathrm{i}-u^*)\,\mathrm{d}x - \frac{c(p-1)}{p}\int_\Omega \frac{|\nabla u_{\mathrm{i}+1}|^p}{(1+c\theta_\mathrm{i})^p}(\theta_\mathrm{i}-\vartheta_\mathrm{i})\mathrm{d}x\\ &+C\int_\Omega \left(|\nabla u_\mathrm{i}|+|\nabla u_{\mathrm{i}+1}|\right)^{p-1}|\nabla v_\mathrm{i}|\mathrm{d}x,\end{aligned} \tag{2.64}$$

where C only depends on p and c. Combined with Hölder's inequality, (2.62), (2.63), and

$$\min\left\{\|u^*\|_{W^{1,p}_0(\Omega)}, \min_{\mathrm{i}\geqslant 0}\|u_\mathrm{i}\|_{W^{1,p}_0(\Omega)}\right\} \leqslant C\|u_0\|_{W^{1,p}_0(\Omega)},$$

which is a consequence of $\mathrm{e_i}$ non-increasing and the definition of u^*, we conclude

$$\mathrm{e_i} \leqslant \begin{cases} C\gamma_p\|u_0\|_{W^{1,p}_0(\Omega)}(\mathrm{e_i}-\mathrm{e_{i+1}})^{\frac{p-1}{p}} & \text{if } 1<p<2\\ C\gamma_p^2\|u_0\|^p+1_{W^{1,p}_0(\Omega)}(\mathrm{e_i}-\mathrm{e_{i+1}})^{\frac{1}{2}} & \text{if } p\geqslant 2.\end{cases} \tag{2.65}$$

This inequality allows us to use Lemma 2.2 to get (2.36) for the first algorithm.

For Algorithm 2, using again (2.58) or (2.59) we get that (2.60) still holds true. Combined with

$$\mathcal{F}(\theta_{\mathrm{i}+1}, u_{\mathrm{i}+1}) \leqslant \mathcal{F}(\theta_\mathrm{i}, u_{\mathrm{i}+1}),$$

we have analogously to (2.62) and (2.63)

$$
\mathrm{e_i} - \mathrm{e_{i+1}} \geqslant \begin{cases} C \min\left\{1, \|v_{\mathrm{i}}\|^{p'}_{H^1_0(\Omega)}\right\} & \text{if } 1 < p < 2 \\ C \min\left\{1, \dfrac{\gamma_p^{-4}\|v_{\mathrm{i}}\|^2_{W^{1,p}_0(\Omega)}}{\big(\|u_{\mathrm{i}}\|_{W^{1,p}_0(\Omega)} + \|u_{\mathrm{i+1}}\|_{W^{1,p}_0(\Omega)}\big)^{p-2}}\right\} & \text{if } p \geqslant 2. \end{cases} \tag{2.66}
$$

Using then that by convexity, $\theta_{\mathrm{i+1}}$ solution of (2.32) is equivalent to $\theta_{\mathrm{i+1}}$ solution of

$$
\max\left\{\int_\Omega \frac{|\nabla u_{\mathrm{i}}|^p}{(1+c\theta_{\mathrm{i+1}})^p}\,\vartheta\,\mathrm{d}x : \quad \vartheta \in \Theta,\ 0 \leqslant \vartheta \leqslant 1 \text{ a.e. in } \Omega,\ \int_\Omega \vartheta \mathrm{d}x \leqslant \kappa\right\},
$$

we have similarly to (2.64)

$$
\mathrm{e_i} \leqslant \int_\Omega \nabla v_{\mathrm{i}} \cdot \nabla(u_{\mathrm{i}} - u^*)\,\mathrm{d}x + C\int_\Omega \big(|\nabla u_{\mathrm{i-1}}| + |\nabla u_{\mathrm{i}}|\big)^{p-1}|\nabla v_{\mathrm{i-1}}|\mathrm{d}x,
$$

Using here

$$
\mathrm{e_{i-1}} = \mathrm{e_i} + \mathrm{e_i} - \mathrm{e_{i-1}}, \qquad \mathrm{e_{i-1}} - \mathrm{e_i}, \mathrm{e_i} - \mathrm{e_{i+1}} \leqslant \mathrm{e_{i-1}} - \mathrm{e_{i+1}},
$$

and taking into account (2.66), we conclude similarly to (2.65)

$$
\mathrm{e_{i-1}} \leqslant \begin{cases} C\gamma_p\big(\|u_0\|_{W^{1,p}_0(\Omega)} + 1\big)(\mathrm{e_{i-1}} - \mathrm{e_{i+1}})^{\frac{p-1}{p}} & \text{if } 1 < p < 2 \\ C\gamma_p^2\big(\|u_0\|^p_{W^{1,p}_0(\Omega)} + 1\big)(\mathrm{e_{i-1}} - \mathrm{e_{i+1}})^{\frac{1}{2}} & \text{if } p \geqslant 2. \end{cases} \tag{2.67}
$$

which, by Lemma Lemma 2.2, proves that (2.36) also holds true for the second algorithm.

Let us now estimate the difference between σ_{i} and σ^*. To simplify the exposition, we just prove the result for Algorithm 1, the proof for Algorithm 2 is completely similar.

We consider a solution (θ^*, u^*) of (2.24). Then, (θ^*, σ^*) is a solution of the the discrete version of (2.14). Combined with the strict convexity properties of the function $\xi \in \mathbb{R}^N \mapsto |\xi|^p \in \mathbb{R}$, we get

$$
\begin{aligned}
&\int_\Omega (1+c\theta^*)|\sigma^*|^{p'}\mathrm{d}x \geqslant \int_\Omega (1+c\theta_{\mathrm{i}})|\sigma^*|^{p'}\mathrm{d}x \\
&\geqslant \int_\Omega (1+c\theta_{\mathrm{i}})|\sigma_{\mathrm{i}}|^{p'}\mathrm{d}x + p'\int_\Omega (1+c\theta_{\mathrm{i}})|\sigma_{\mathrm{i}}|^{p'-2}\sigma_{\mathrm{i}} \cdot (\sigma^* - \sigma_{\mathrm{i}})\,\mathrm{d}x \\
&\quad +\rho\int_\Omega \big(|\sigma^*| + |\sigma_{\mathrm{i}}|\big)^{p-2}|\sigma^* - \sigma_{\mathrm{i}}|^2\mathrm{d}x \\
&= \int_\Omega (1+c\vartheta_{\mathrm{i}})|\sigma_{\mathrm{i}}|^{p'}\mathrm{d}x + p'\int_\Omega (1+c\theta_{\mathrm{i}})|\sigma_{\mathrm{i}}|^{p'-2}\sigma_{\mathrm{i}} \cdot (\sigma^* - \sigma_{\mathrm{i}})\,\mathrm{d}x \\
&\quad +\rho\int_\Omega \big(|\sigma^*| + |\sigma_{\mathrm{i}}|\big)^{p-2}|\sigma^* - \sigma_{\mathrm{i}}|^2\mathrm{d}x + c\int_\Omega (\theta_{\mathrm{i}} - \vartheta_{\mathrm{i}})|\sigma_{\mathrm{i}}|^{p'}\mathrm{d}x,
\end{aligned} \tag{2.68}
$$

with ρ a positive constants which only depend on p. Simimilarly, using that ϑ_{i} a solution of (2.30) we have

$$
\begin{aligned}
&\int_\Omega (1+c\vartheta_{\mathrm{i}})|\sigma_{\mathrm{i}}|^{p'}\mathrm{d}x \geqslant \int_\Omega (1+c\theta^*)|\sigma^*|^{p'}\mathrm{d}x \\
&\quad +p'\int_\Omega (1+c\theta^*)|\sigma^*|^{p'-2}\sigma^* \cdot (\sigma_{\mathrm{i}} - \sigma^*)\,\mathrm{d}x + \rho\int_\Omega \big(|\sigma^*| + |\sigma_{\mathrm{i}}|\big)^{p-2}|\sigma^* - \sigma_{\mathrm{i}}|^2\mathrm{d}x
\end{aligned} \tag{2.69}
$$

From (2.68) and (2.69), we deduce

$$
\begin{aligned}
0 \geqslant p' \int_\Omega \Big((1+c\theta_{\rm i})|\sigma_{\rm i}|^{p'-2}\sigma_{\rm i} - (1+c\theta^*)|\sigma^*|^{p'-2}\sigma^*\Big)\cdot(\sigma^*-\sigma_{\rm i})\,{\rm d}x \\
+2\rho\int_\Omega \big(|\sigma^*|+|\sigma_{\rm i}|\big)^{p-2}|\sigma^*-\sigma_{\rm i}|^2{\rm d}x + c\int_\Omega(\theta_{\rm i}-\vartheta_{\rm i})|\sigma_{\rm i}|^{p'}{\rm d}x,
\end{aligned}
\tag{2.70}
$$

Now, we use that

$$
\begin{aligned}
&\Big((1+c\theta_{\rm i})|\sigma_{\rm i}|^{p'-2}\sigma_{\rm i} - (1+c\theta^*)|\sigma^*|^{p'-2}\sigma^*\Big)\cdot(\sigma^*-\sigma_{\rm i}) \\
&= \Big(\frac{|\nabla u^*|^{p-2}}{(1+c\theta^*)^{p-1}}\nabla u^* - \frac{|\nabla u_{\rm i}|^{p-2}}{(1+c\theta_{\rm i})^{p-1}}\nabla u_{\rm i}\Big)\cdot\nabla(u_{\rm i}-u^*).
\end{aligned}
$$

which taking into account (2.28), and that (θ^*,σ^*) satisfies the discrete version of (2.11) prove

$$
\int_\Omega \Big((1+c\theta_{\rm i})|\sigma_{\rm i}|^{p'-2}\sigma_{\rm i} - (1+c\theta^*)|\sigma^*|^{p'-2}\sigma^*\Big)\cdot(\sigma^*-\sigma_{\rm i})\,{\rm d}x = \int_\Omega \nabla(u_{\rm i}-u^*)\cdot\nabla v_{\rm i}\,{\rm d}x. \tag{2.71}
$$

Replacing this equality in (2.70) and recalling $u_{{\rm i}+1} = u_{\rm i} + t_{\rm i}v_{\rm i}$, with $0 \leqslant t_{\rm i} \leqslant 1$, we get

$$
\begin{aligned}
2\rho\int_\Omega \big(|\sigma^*|+|\sigma_{\rm i}|\big)^{p'-2}|\sigma^*-\sigma_{\rm i}|^2{\rm d}x \leqslant c\int_\Omega \frac{|\nabla u_{{\rm i}+1}|^p}{(1+c\theta_{\rm i})^p}(\vartheta_{\rm i}-\theta_{\rm i}){\rm d}x \\
+C\int_\Omega \big(|\nabla u_{\rm i}|+|\nabla u^*|\big)^{p-1}|\nabla v_{\rm i}|{\rm d}x + p'\int_\Omega\big(|\nabla u_{\rm i}|+|\nabla u^*|\big)|\nabla v_{\rm i}|{\rm d}x
\end{aligned}
\tag{2.72}
$$

with C depending only on p and c. By (2.62) and (2.63) we then conclude (2.38). □

Proof of Theorem 2.3. For $(\hat\theta,\hat u)$ the solution of (2.5) which satisfies (2.46), $\hat\sigma$ defined by (2.15) and $h>0$, we introduce $\hat\theta_h \in \Theta_h$, $\hat\sigma_h \in \Theta_h^N$ and $\hat u_h \in V_h$ by

$$
\hat\theta_h = \frac{1}{|\tau|}\int_\tau \hat\theta\,{\rm d}x, \quad \hat\sigma_h = \frac{1}{|\tau|}\int_\tau \hat\sigma\,{\rm d}x, \quad \forall\,\tau\in\mathcal{T}_h, \tag{2.73}
$$

$$
\hat u_h(x_{\rm i}) = \hat u(x_{\rm i}), \quad \forall\, x_{\rm i} \text{ vertex of } \mathcal{T}_h. \tag{2.74}
$$

Thanks to (2.46) and the regularity of $\mathcal{T}_h$, there exists $C>0$ such that

$$
h\|\hat u_h\|_{W^{1,\infty}(\Omega)} + \|\hat u_h - \hat u\|_{L^\infty(\Omega)} + \|\hat u_h - \hat u\|_{W_0^{1,1}(\Omega)} + \|\hat\theta_h - \hat\theta\|_{L^1(\Omega)} \leqslant Ch. \tag{2.75}
$$

The definition of $\mathcal{F}$, the mean value theorem and these estimates imply

$$
\begin{aligned}
\Big|\mathcal{F}(\hat\theta,\hat u) - \mathcal{F}(\hat\theta_h,\hat u_h)\Big| \leqslant C\Big(\|\nabla\hat u\|_{L^\infty(\Omega)^N} + \|\nabla\hat u_h\|_{L^\infty(\Omega)^N}\Big)^{p-1}\|\nabla(u-u_h)\|_{L^1(\Omega)^N} \\
+\|\nabla\hat u\|^p_{L^\infty(\Omega)^N}\|\hat\theta-\hat\theta_h\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)}\|\hat u - \hat u_h\|_{L^\infty(\Omega)} \leqslant Ch.
\end{aligned}
$$

Then, since the definitions of $\hat{\mathcal{F}}$ and $\mathcal{F}_h^*$ imply

$$
\mathcal{F}(\hat\theta,\hat u) = \hat{\mathcal{F}} \leqslant \mathcal{F}_h^* \leqslant \mathcal{F}(\hat\theta_h,\hat u_h),
$$

we conclude (2.47). On the other hand, we consider (θ_h^*, u_h^*) a solution of (2.24) with Θ and V replaced by Θ_h and V_h. We define $\hat{\sigma}$ by (2.15), σ_h^* by

$$\sigma_h^* = \frac{|\nabla u_h^*|^{p-2}}{(1+c\theta_h^*)^{p-1}} \nabla u_h^*, \tag{2.76}$$

and we recall that thanks to (2.14), we have

$$\int_\Omega (1+c\hat{\theta})|\hat{\sigma}|^{p'} \mathrm{d}x = \max \left\{ \int_\Omega (1+c\theta)|\hat{\sigma}|^{p'} \mathrm{d}x : \ \theta \in L^\infty(\Omega; [0,1]), \ \int_\Omega \theta \, \mathrm{d}x \leqslant \kappa \right\}.$$

Thus, we deduce

$$\begin{aligned} &\int_\Omega (1+c\hat{\theta})|\hat{\sigma}|^{p'} \mathrm{d}x \geqslant \int_\Omega (1+c\theta_h^*)|\hat{\sigma}|^{p'} \mathrm{d}x \geqslant \int_\Omega (1+c\theta_h^*)|\sigma_h^*|^{p'} \mathrm{d}x \\ &+p' \int_\Omega (1+c\theta_h^*)|\hat{\sigma}_h^*|^{p'-2}\hat{\sigma}_h^* \cdot (\hat{\sigma} - \hat{\sigma}_h^*) \mathrm{d}x + \rho \int_\Omega (|\hat{\sigma}| + \hat{\sigma}_h^*|)^{p-2} |\hat{\sigma} - \hat{\sigma}_h^*|^2 \mathrm{d}x, \end{aligned} \tag{2.77}$$

for some $\rho > 0$, which only depends on p. Using the definitions of $\hat{\sigma}$ and $\hat{\sigma}^*$ and that $(\hat{\theta}, \hat{u})$, (θ^*, u_h^*) are solutions of (2.5) and (2.24) we have

$$\int_\Omega (1+c\hat{\theta})|\hat{\sigma}|^{p'} \mathrm{d}x = \int_\Omega \frac{|\nabla \hat{u}|^p}{(1+c\hat{\theta})^{p-1}} \mathrm{d}x = -p'\hat{\mathcal{F}},$$

$$\int_\Omega (1+c\theta_h^*)|\sigma_h^*|^{p'} \mathrm{d}x = \int_\Omega \frac{|\nabla u_h^*|^p}{(1+c\theta_h^*)^{p-1}} \mathrm{d}x = -p'\mathcal{F}_h^*,$$

$$p' \int_\Omega (1+c\theta_h^*)|\hat{\sigma}_h^*|^{p'-2}\hat{\sigma}_h^* \cdot (\hat{\sigma} - \hat{\sigma}_h^*) \mathrm{d}x = p' \int_\Omega (\hat{\sigma} - \hat{\sigma}_h^*) \cdot \nabla u_h^* \, \mathrm{d}x = 0.$$

Replacing these equalities in (2.77) and taking into account (2.47) we get (2.48). □

2.5. Numerical experiments

In this section we present some simulations for the numerical resolution of (2.5) using the two algorithms presented in Section 2.3. The implementation has been carried out in python using the finite element solver Fenics [6].

In our numerical experiments, we have taken $N = 2$, Ω the unit disc, $c = 1, f = 1$ and $\kappa = 1$. In this case, the solution of (2.5) is explicitly given by

$$\hat{\theta}(x) = \begin{cases} 1 & \text{if } |x| < \pi^{-\frac{1}{2}} \\ 0 & \text{if } |x| > \pi^{-\frac{1}{2}}, \end{cases} \qquad \hat{u}(x) = \begin{cases} \dfrac{1}{2^{\frac{1}{p-1}} p'} \left(1 + \pi^{-\frac{p'}{2}} - 2|x|^{p'}\right) & \text{if } |x| < \pi^{-\frac{1}{2}} \\ \dfrac{1}{2^{\frac{1}{p-1}} p'} \left(1 - |x|^{p'}\right) & \text{if } |x| > \pi^{-\frac{1}{2}}, \end{cases}$$

and thus

$$\hat{\mathcal{F}} = \mathcal{F}(\hat{\theta}, \hat{u}) = -\frac{\pi}{p'(2+p')2^{\frac{1}{p-1}}} \left(1 - \frac{1}{2\pi^{1+\frac{p'}{2}}}\right).$$

We solve the problem for meshes of different diameter h and $p = 1.2, 2, 100$.

The stop criterion for the first algorithm is

$$\int_\Omega |\nabla v_\mathrm{i}|^2 \mathrm{d}x + \frac{c(p-1)}{p} \int_\Omega \frac{|\nabla u_{\mathrm{i}+1}|^p}{(1+\theta_\mathrm{i})^p} (\vartheta_\mathrm{i} - \theta_\mathrm{i})\, \mathrm{d}x \leqslant 10^{-7} \quad \text{or } \mathrm{i} \geqslant 2000, \tag{2.78}$$

while for the second one it is given by

$$\int_\Omega |\nabla v_\mathrm{i}|^2 \mathrm{d}x + \frac{c(p-1)}{p} \int_\Omega \frac{|\nabla u_{\mathrm{i}+1}|^p}{(1+\theta_{\mathrm{i}+1})^p} (\theta_{\mathrm{i}+1} - \theta_\mathrm{i}) \mathrm{d}x \leqslant 10^{-7} \quad \text{or } \mathrm{i} \geqslant 2000. \tag{2.79}$$

Observe that in both cases replacing 10^{-7} by 0 would mean that $(\theta_\mathrm{i}, u_\mathrm{i})$ satisfies the optimality conditions for (2.5) and then, by the convexity of $\mathcal{F}$, that $(\theta_\mathrm{i}, u_\mathrm{i})$ is a solution for (2.24).

Depending on p, h, and the choice of the algorithm, we present in figure 2.1 the convergence history of the objective function, the Lagrange multiplier μ, and the stop criterion ($\|DF\|$ denotes the left-hand sides in (2.78) and (2.79) respectively). Observe that for $p = 1.2$ and $p = 2$, the rate of convergence for both algorithms is similar. However for $p = 100$, Algorithm 2 converges faster than Algorithm 1. Although our estimates depend on h, we do not observe this dependence in the numerical experiments for $p = 2$. This is because $\gamma_2 = 1$ and therefore according to remark 2.4 the step length is constant and all the bounds in Theorem 2.2 do not depend on the mesh size.

In figure 2.2 we represent the solutions $(\theta_\mathrm{i}, u_\mathrm{i})$ depending on p but only for the finest mesh. Observe that the solutions obtained are very similar for both algorithms.

In figure 2.3 we show the time spent in the resolution of the numerical experiments. We observe that the iterations are faster calculated for Algorithm 2 than for Algorithm 1. When the diameter of the mesh decreases, the time increases for both algorithms in the same way. Moreover, for p large Algorithm 2 needs fewer iterations than Algorithm 1 while for p small both algorithms use more or less the same number of iterations.

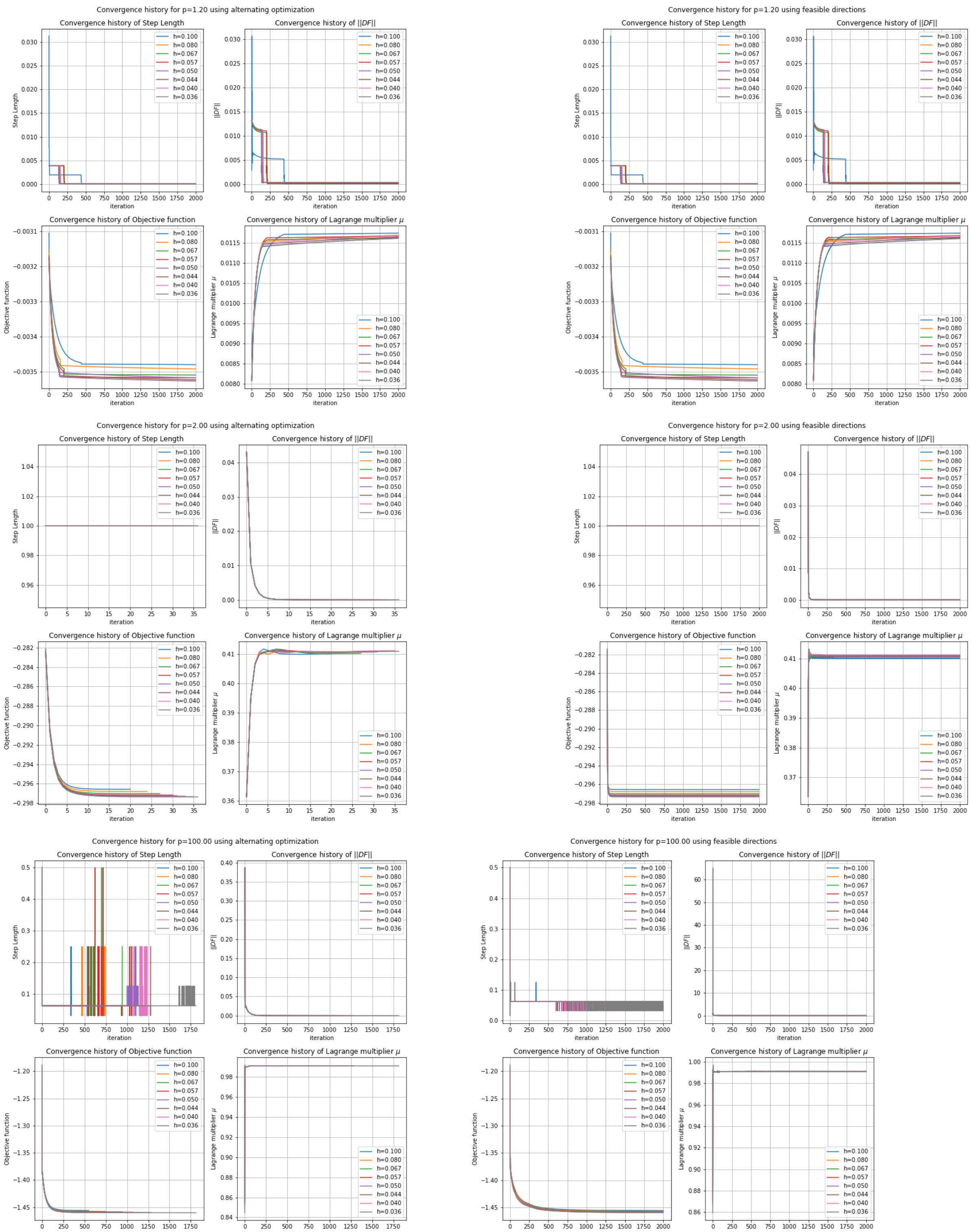


Figure 2.1: Convergence history for each p and mesh.

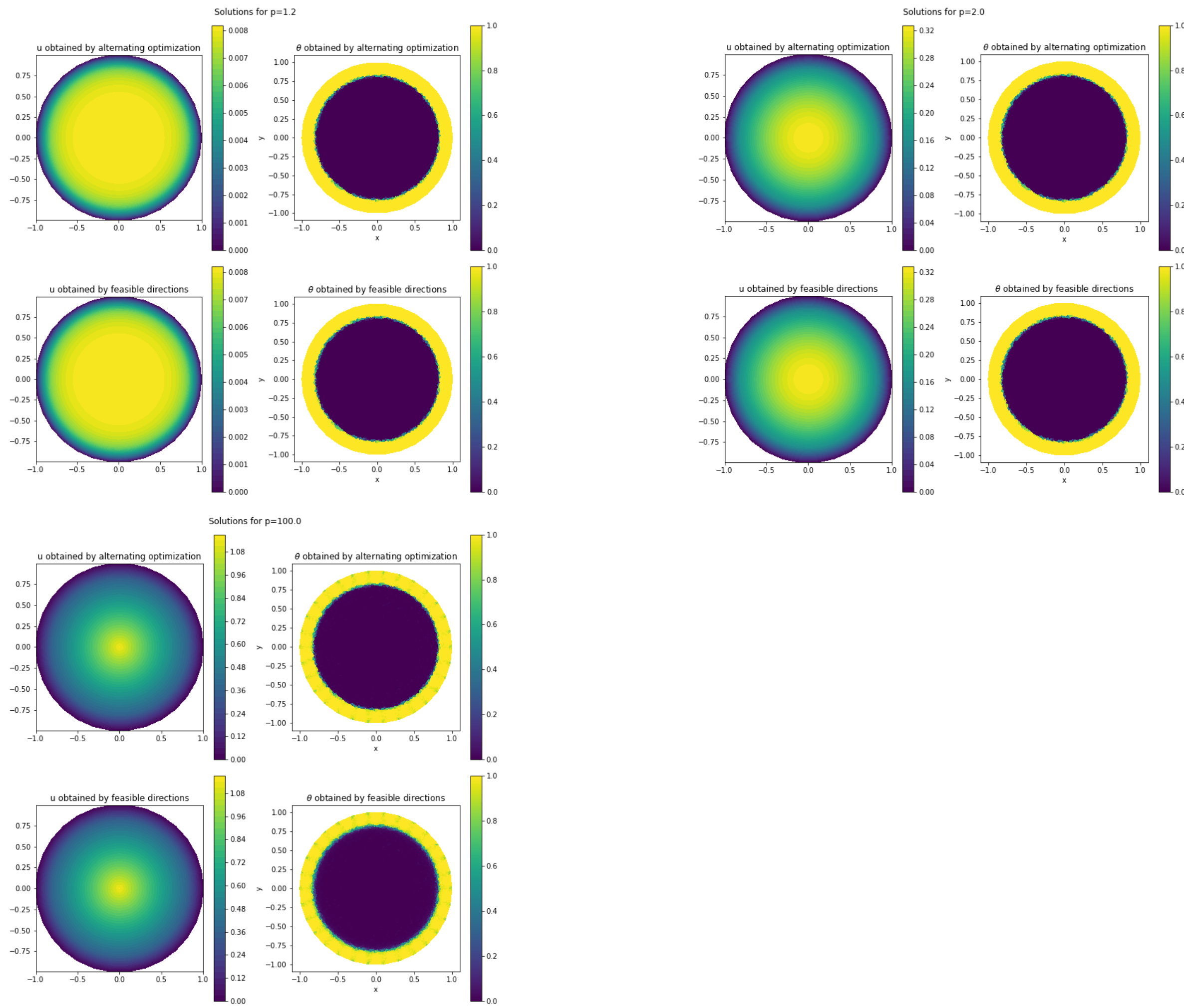


Figure 2.2: Solutions for the finest mesh.

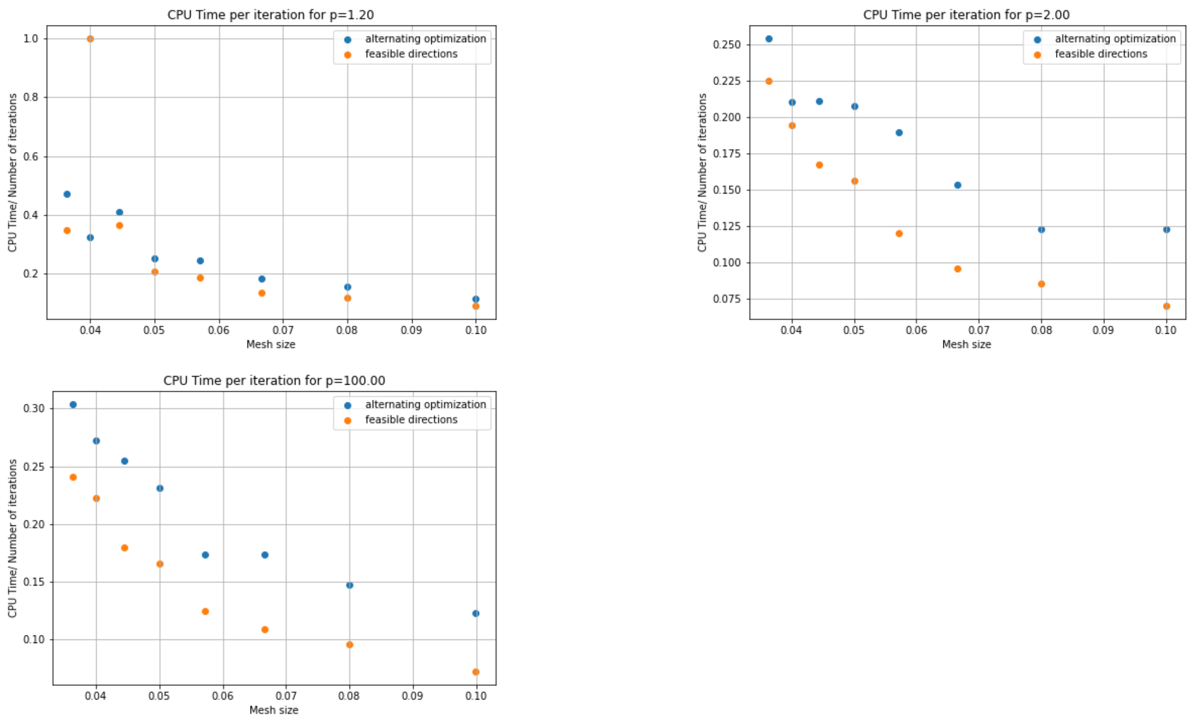


Figure 2.3: Mean CPU time by iteration in seconds for each p.

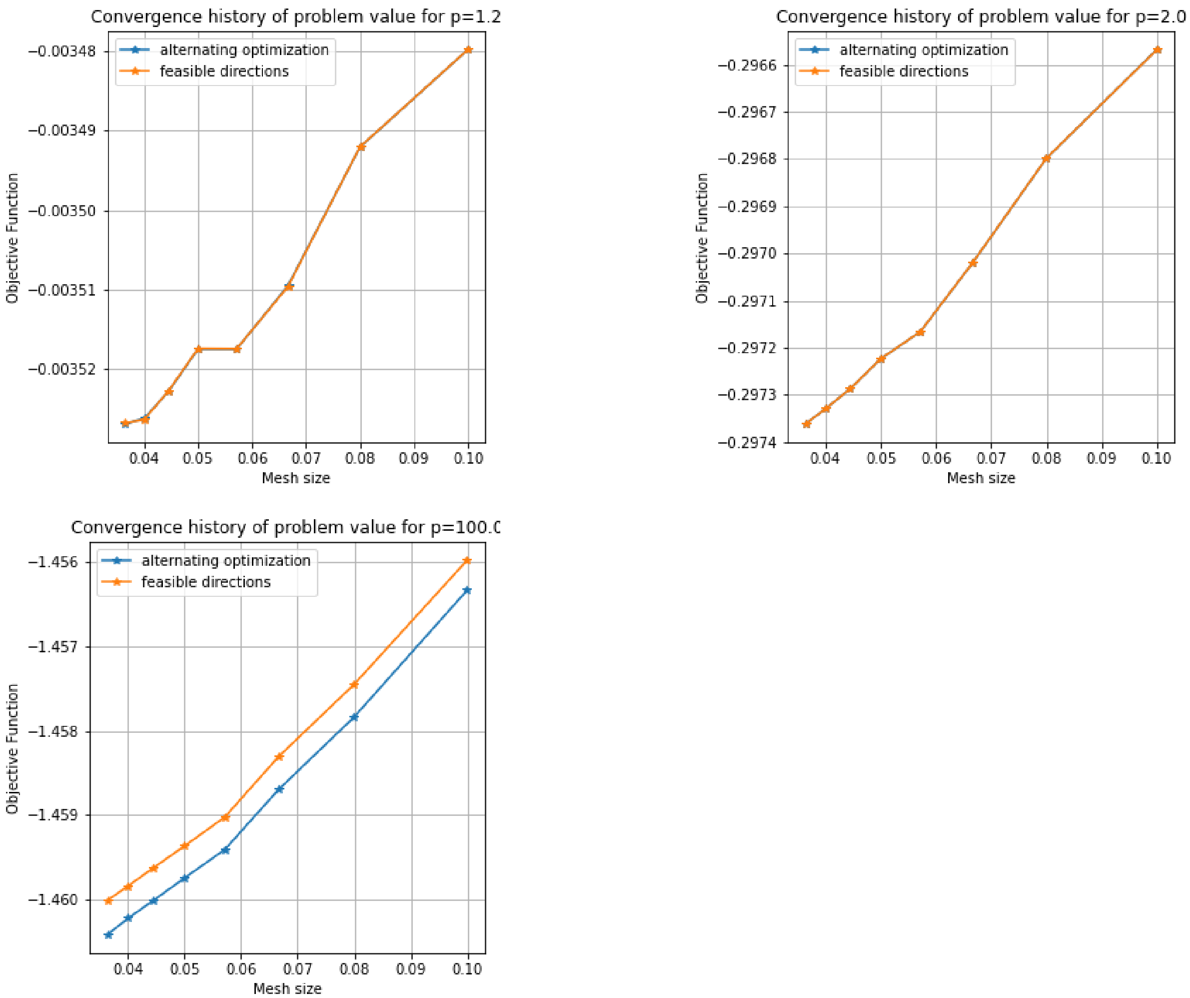


Figure 2.4: Convergence rate of minimum value as function of the mesh size for each p.

2

Chapter 3

Minimization of the p-Laplacian first eigenvalue for a two-phase material

Abstract: We study the problem of minimizing the first eigenvalue of the p-Laplacian operator for a two-phase material in a bounded open domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$ assuming that the amount of the best material is limited. We provide a relaxed formulation of the problem and prove some smoothness results for these solutions. As a consequence we show that if Ω is of class $C^{1,1}$, simply connected with connected boundary, then the unrelaxed problem has a solution if and only if Ω is a ball. We also provide an algorithm to approximate the solutions of the relaxed problem and perform some numerical simulations.

3.1. Introduction

The present paper is devoted to study the optimal design problem of obtaining the two-phase material which minimizes the first eigenvalue of the p-Laplacian operator with Dirichlet conditions assuming that the amount of the best material is limited. Namely, for a bounded open set $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, two positive constants $\alpha < \beta$ which represent the diffusion coefficients of the two materials, and a constant $\kappa \in (0, |\Omega|)$ which corresponds to the maximal amount of the best material, we are interested in the minimization problem

$$\left\{\begin{array}{l} \displaystyle\min_{\omega,u} \int_\Omega \left(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\right) |\nabla u|^p \mathrm{d}x \\ \omega \subset \Omega \text{ measurable}, \quad |\omega| \leqslant \kappa \\ \displaystyle u \in W_0^{1,p}(\Omega), \quad \int_\Omega |u|^p = 1, \end{array}\right. \tag{3.1}$$

with $p \in (1, \infty)$. Observe that without the restriction $|\omega| \leqslant \kappa$ the solution is the trivial one $\omega = \Omega$. The case $p = 2$ has been extensively studied from both the theorical and numerical point of view (see e.g. [13], [14], [19], [20], [21], [32], [37], [40]), but for $p \neq 2$ the problem is new in our knowledge.

It is well known ([43]) that this type of problems has not a solution in general. More concretely, in the case $p = 2$ and related with the results obtained in [13] and [44], it has been proved in [14] that if Ω is simply connected with connected boundary, then problem (3.1) has a solution if and only if Ω is a ball. This makes necessary to work with a relaxed formulation which is usually obtained by using homogenization theory ([3], [44], [52]). The idea is to replace the diffusion material with coefficients $(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})$ corresponding to take in every point of Ω the material α or the material β, by a general mixture of both materials where in each point x, we use the material α with proportion $\theta(x) \in [0, 1]$ and the material β with proportion $1 - \theta(x)$. The corresponding homogenized material obtained in this way does not only depend on the proportion but also on the disposition of both materials. Taking into account the results in [15], we deduce that for a given proportion $\theta = \theta(x)$, the optimal disposition of the materials is given by a simple laminate in the direction of the flux. This provides the relaxed formulation of (3.1):

$$\begin{cases} \displaystyle\min_{\theta,u} \int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}} \mathrm{d}x \\ \theta \in L^\infty(\Omega; [0, 1]), \quad u \in W_0^{1,p}(\Omega) \\ \displaystyle\int_\Omega \theta \mathrm{d}x \leqslant \kappa, \quad \int_\Omega |u|^p \mathrm{d}x = 1, \end{cases} \tag{3.2}$$

with $c = (\beta/\alpha)^{\frac{1}{p-1}} - 1 > 0$. We also show that the relaxed problem admits the equivalent formulation:

$$\begin{cases} \displaystyle\min_{\theta,u,f} \int_\Omega \Big(\frac{1}{p}\frac{|\nabla u|^p}{(1+c\theta)^{p-1}} - fu\Big) \mathrm{d}x \\ \theta \in L^\infty(\Omega; [0, 1]), \quad u \in W_0^{1,p}(\Omega), \quad f \in L^{p'}(\Omega) \\ \displaystyle\int_\Omega \theta \mathrm{d}x \leqslant \kappa, \quad \int_\Omega |f|^{p'} \mathrm{d}x \leqslant 1 \end{cases} \tag{3.3}$$

This allows us to use the results in [15] (see also [13] for $p = 2$) to deduce some smoothness properties for the solutions of (3.2). Namely, assuming $\Omega \in C^{1,1}$ we show that every solution (u, θ) of (3.2) is such that (ν denotes the outward unitary normal on $\partial\Omega$)

$$\begin{cases} u \in W^{1,\infty}(\Omega), \quad \sigma := \dfrac{|\nabla u|^{p-2}}{(1+c\theta)^{p-1}} \nabla u \in H^1(\Omega)^N, \\ \partial_\mathrm{i}\theta\sigma_j - \partial_j\theta\sigma_\mathrm{i} \in L^2(\Omega), \ \forall \mathrm{i}, j \in \{1, \cdots, N\}. \end{cases} \tag{3.4}$$

Observe that (3.1) has a solution if and only if (3.2) has a solution of the form (χ_ω, u) with $\omega \subset \Omega$ measurable. In this case, we show that the derivability condition on θ given in (3.4) implies that $\mathrm{curl}(|\sigma|^{p'-2}\sigma)$ vanishes. Thanks to this, we extend the result in [16] for $p = 2$, showing that if Ω is smooth and simply connected with connected boundary, then problem (3.1) has a solution if and only if Ω is a ball.

In the second part of the paper we carry out the numerical study of problem (3.3). The algorithm that we propose (see Section 3.3) solves in each iteration a problem of the form

$$\begin{cases} \displaystyle\min_{\theta,u} \int_\Omega \Big(\frac{1}{p}\frac{|\nabla u|^p}{(1+c\theta)^{p-1}} - fu\Big) \mathrm{d}x \\ \theta \in L^\infty(\Omega; [0, 1]), \quad u \in W_0^{1,p}(\Omega), \quad \displaystyle\int_\Omega \theta \mathrm{d}x \leqslant \kappa, \end{cases} \tag{3.5}$$

for a certain function f which changes with the iteration. The numerical resolution of (3.5) has been considered in [16], where we propose two algorithms and we show the convergence to a critical point. Problem (3.5) has been studied by several authors because it appears for example in the optimization of two isotropic materials posed in the cross section of a beam in order to minimize the torsion. In this way, a theoretical study of (3.5) for arbitrary $p>1$ has been carried out in [15]. In the case $p=2$ we refer to [3], [7], [13] [28], [31], [44] and [53] to the study of (3.5) from both, the theoretical and numerical point of view.

3.2. Relaxation and smoothness results.

Our purpose in the present section is to study problem (3.1). As we said in the introduction, the first difficulty is that it has not solution in general and therefore it is interesting to get a relaxed formulation which will consists in replacing the mixtures of materials $\alpha\chi_{\omega_n}+\beta\chi_{\Omega\setminus\omega_n}$ in (3.1) by more general mixtures where the material α is used with proportion $\theta=\theta(x)\in[0,1]$ and the material β with proportion $1-\theta$.

Theorem 3.1 *A relaxed formulation of problem (3.1) is given by*

$$\left\{\begin{array}{l}\displaystyle\min_{\theta,u}\int_\Omega\left(\theta\alpha^{\frac{1}{1-p}}+(1-\theta)\beta^{\frac{1}{1-p}}\right)^{1-p}|\nabla u|^p\mathrm{d}x\\ \qquad\theta\in L^\infty(\Omega;[0,1]),\quad u\in W_0^{1,p}(\Omega)\\ \qquad\displaystyle\int_\Omega\theta\mathrm{d}x\leqslant\kappa,\quad\int_\Omega|u|^p\mathrm{d}x=1,\end{array}\right.\tag{3.6}$$

in the sense of Murat-Tartar [44], p. 140, which implies the following four statements:

(1) Problem (3.6) has at least one solution.

(2) The infimum for problem (3.1) agrees with the minimum for problem (3.6).

(3) If (ω_n,u_n) is a minimizing sequence for (3.1), then u_n is bounded in $W_0^{1,p}(\Omega)$. Taking a subsequence, still denoted by (u_n,ω_n), such that there exists $(u,\theta)\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ with

$$u_n\rightharpoonup u\ \text{in}\ W_0^{1,p}(\Omega),\quad\chi_{\omega_n}\overset{*}{\rightharpoonup}\theta\ \text{in}\ L^\infty(\Omega)\tag{3.7}$$

we have that (θ,u) is a solution for (3.6) and

$$\lim_{n\to\infty}\int_\Omega(\alpha\chi_{\omega_n}+\beta\chi_{\Omega\setminus\omega_n})|\nabla u_n|^p\mathrm{d}x=\int_\Omega(\theta\alpha^{\frac{1}{p-1}}+(1-\theta)\beta^{\frac{1}{p-1}})^{1-p}|\nabla u|^p\mathrm{d}x.\tag{3.8}$$

(4) For every pair $(\theta,u)\in L^\infty(\Omega;[0,1])\times W_0^{1,p}(\Omega)$ with $\int_\Omega\theta\mathrm{d}x\leqslant\kappa$, and $\|u\|_{L^p(\Omega)}=1$, there exist $\omega_n\subset\Omega$ measurable, with $|\omega_n|\leqslant\kappa$ and $u_n\in W_0^{1,p}(\Omega)$, with $\|u_n\|_{L^p(\Omega)}=1$ such that (3.7) *and* (3.8) *hold.*

Proof. The first statement is a consequence of the convexity of the function $J:\mathbb{R}^N\times(0,\infty)\mapsto\mathbb{R}$ defined by

$$J(\xi,t)=\frac{|\xi|^p}{t}\quad\forall(\xi,t)\in\mathbb{R}^N\times(0,\infty),\tag{3.9}$$

combined with the Rellich-Kondrachov compactness theorem. Statement (4) is a consequence of Theorem 2.1 in [15]. Using again the convexity of J, it implies statement (3) and then statement (2). $\square$

Denoting

$$c := \left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}} - 1 > 0, \tag{3.10}$$

problem (3.6) can be written as (3.2). From now on, we will consider the problem in this form.

For a distribution f at least in the space $W^{-1,p'}(\Omega)$, we have studied in [15] the optimal design problem

$$\begin{cases} \displaystyle\min_{\theta,u}\left\{\frac{1}{p}\int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}}\mathrm{d}x - \langle f,u\rangle\right\} \\ \displaystyle\theta\in L^\infty(\Omega;[0,1]),\quad \int_\Omega \theta\mathrm{d}x \leqslant \kappa,\quad u\in W^{1,p}_0(\Omega). \end{cases} \tag{3.11}$$

Similarly to the result for $p = 2$ obtained in [13], let us show that both problems (3.2) and (3.11) are strongly related. Namely, problem (3.2) consists in solving (3.11) for every $f \in L^{p'}(\Omega)$ with $\|f\|_{L^{p'}(\Omega)} \leqslant 1$ and then to minimize in f. This is given by the following proposition

Proposition 3.1 *Problem (3.2) is equivalent to*

$$\begin{cases} \displaystyle\min_{\theta,u,f}\left\{\int_\Omega\left(\frac{1}{p}\frac{|\nabla u|^p}{(1+c\theta)^{p-1}} - fu\right)\mathrm{d}x\right\} \\ \displaystyle\theta\in L^\infty(\Omega;[0,1]),\quad \int_\Omega \theta\mathrm{d}x \leqslant \kappa,\quad u\in W^{1,p}_0(\Omega),\quad f\in L^{p'}(\Omega),\quad \int_\Omega |f|^{p'}\mathrm{d}x\leqslant 1, \end{cases} \tag{3.12}$$

in the following sense:

If $\hat\lambda$ denotes the minimum in (3.2) and $\hat I$ the minimum in (3.12), then

$$\hat\lambda = \left(-\frac{1}{p'\hat I}\right)^{p-1}. \tag{3.13}$$

Moreover, if $(\hat\theta,\hat u)$ is a solution of (3.2) then $(\hat\theta, \hat\lambda^{\frac{1}{1-p}}\hat u, |\hat u|^{p-2}\hat u)$ is a solution of (3.12). Reciprocally, if $(\hat\theta,\hat u,\hat f)$ is a solution of (3.12), then $\hat f = \hat\lambda|\hat u|^{p-2}\hat u$ and $(\hat\theta, \hat\lambda^{\frac{1}{p-1}}\hat u)$ is a solution of (3.2).

The above proposition is a consequence of the following lemma which in the case $p = 2$ was proved in [17] (see also [13]).

Lemma 3.1 *For $p\in(1,\infty)$ and $A\in L^\infty(\Omega)^{N\times N}$ symmetric and uniformly elliptic, the first eigenvalue*

$$\lambda_1(p,A) := \min_{\substack{u\in W^{1,p}_0(\Omega)\\ \|u\|_{L^p(\Omega)}=1}} \int_\Omega |A\nabla u|^{p-2}A\nabla u\cdot\nabla u\,\mathrm{d}x \tag{3.14}$$

of the operator $u \in W_0^{1,p}(\Omega) \mapsto -\mathrm{div}(|A\nabla u|^{p-2}A\nabla u) \in W^{-1,p'}(\Omega)$ *is characterized by*

$$\begin{aligned}\lambda_1(p,A)^{\frac{1}{1-p}} = \quad & \max_{f,u}\left\{\int_\Omega |A\nabla u|^{p-2}A\nabla u\cdot\nabla u\mathrm{d}x\right\} \\ & -\mathrm{div}(|A\nabla u|^{p-2}A\nabla u) = f,\ u\in W_0^{1,p}(\Omega),\ \|f\|_{L^{p'}(\Omega)}\leqslant 1.\end{aligned} \tag{3.15}$$

Moreover, if $f \in L^{p'}(\Omega)$ *is a solution of (3.15), then* $|f|^{p'-2}f$ *is an eigenfunction for* $\lambda_1(p,A)$*. Reciprocally, if* u *is an eigenfunction for* $\lambda_1(p,A)$*, then* $f = \|u\|_{L^p(\Omega)}^{1-p}|u|^{p-2}u$ *is a solution of (3.15).*

Proof. Let $f \in L^{p'}(\Omega)$ be with $\|f\|_{L^{p'}(\Omega)} \leqslant 1$ and let u be the unique solution of

$$-\mathrm{div}(|A\nabla u|^{p-2}A\nabla u) = f \ \text{ in } \Omega, \quad u\in W_0^{1,p}(\Omega). \tag{3.16}$$

Then, thanks to the definition of $\lambda_1(p,A)$, we get

$$\int_\Omega |A\nabla u|^{p-2}A\nabla u\cdot\nabla u\,\mathrm{d}x \geqslant \lambda_1(p,A)\int_\Omega |u|^p\mathrm{d}x.$$

On the other hand, using u as test function in (3.16) and taking into account that $\|f\|_{L^{p'}(\Omega)} \leqslant 1$, we have

$$\int_\Omega |A\nabla u|^{p-2}A\nabla u\cdot\nabla u\,\mathrm{d}x = \int_\Omega fu\,\mathrm{d}x \leqslant \|u\|_{L^p(\Omega)}. \tag{3.17}$$

These two inequalities prove

$$\int_\Omega |A\nabla u|^{p-2}A\nabla u\cdot\nabla u\,\mathrm{d}x \leqslant \|u\|_{L^p(\Omega)} \leqslant \left(\frac{1}{\lambda_1(p,A)}\int_\Omega |A\nabla u|^{p-2}A\nabla u\cdot\nabla u\,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{3.18}$$

and therefore

$$\int_\Omega |A\nabla u|^{p-2}A\nabla u\cdot\nabla u\mathrm{d}x \leqslant \lambda_1(p,A)^{\frac{1}{1-p}}. \tag{3.19}$$

By the arbitrariness of f, we have then proved that the right-hand side of (3.15) is smaller or equal than the left-hand side. To get the opposite inequality we consider an eigenfunction $v \in W_0^{1,p}(\Omega)$ with $\|v\|_{L^p(\Omega)} = 1$ and define $\tilde f = |v|^{p-2}v$, $\tilde u = \lambda_1(p,A)^{\frac{1}{1-p}}v$. Then, $\tilde f$ and $\tilde u$ satisfy

$$-\mathrm{div}(|A\nabla\tilde u|^{p-2}A\nabla\tilde u) = \tilde f \text{ in } \Omega, \quad \tilde u\in W_0^{1,p}(\Omega), \quad \|\tilde f\|_{L^{p'}(\Omega)} = 1,$$

$$\int_\Omega |A\nabla\tilde u|^{p-2}A\nabla\tilde u\cdot\nabla\tilde u\mathrm{d}x = \lambda_1(p,A)^{\frac{1}{1-p}}$$

This shows that there exists the maximum in the right-hand side of (3.15) and it agrees with $\lambda_1(p,A)^{\frac{1}{1-p}}$.

To finish the proof it remains to prove that every solution of the maximum problem in (3.15) is an eigenfunction of the operator $u \to -\mathrm{div}(|A\nabla u|^{p-2}A\nabla u)$. To do so, we consider $\hat f \in L^{p'}(\Omega)$ a solution to (3.15), and define $\hat u$ as the unique solution of

$$-\,\mathrm{div}(|A\nabla\hat u|^{p-2}A\nabla\hat u) = \hat f \text{ in } \Omega, \quad \hat u\in W_0^{1,p}(\Omega). \tag{3.20}$$

By (3.17) and (3.18) we get

$$\int_\Omega \hat{f}\hat{u}\,\mathrm{d}x = \|\hat{u}\|_{L^p(\Omega)} = \|\hat{f}\|_{L^{p'}(\Omega)} \cdot \|\hat{u}\|_{L^p(\Omega)} = \lambda_1(p,A)^{\frac{1}{1-p}}. \tag{3.21}$$

We have then shown that Hölder's inequality is an equality for the product $\hat{f}\hat{u}$ and thus that there exists $\lambda > 0$ such that $\hat{f} = \lambda|\hat{u}|^{p-2}\hat{u}$. The last equality in (3.21) combined with $\|\hat{f}\|_{L^{p'}(\Omega)} = 1$ implies now $\lambda = \lambda_1(p,A)$. Thus $\hat{u}$ and then $\hat{f}$ is an eigenfunction for $\lambda_1(p,A)$. □

Proof of Proposition 3.1. The result is a quite simple consequence of Lemma 3.1 and the fact that for every $f \in L^{p'}(\Omega)$, we have

$$\min_{u\in W_0^{1,p}(\Omega)} \int_\Omega \left(\frac{1}{p}\frac{|\nabla u|^p}{(1+c\theta)^{p-1}} - fu\right)\mathrm{d}x = -\frac{1}{p'}\int_\Omega \frac{|\nabla u_f|^p}{(1+c\theta)^{p-1}}\,\mathrm{d}x,$$

with u_f the solution of

$$-\mathrm{div}\left(\frac{|\nabla u|^{p-2}}{(1+c\theta)^{p-1}}\nabla u\right) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

□

Problem (3.11) has been studied in [15] (see also [13], [28], [44], [31] for $p = 2$), where they were obtained the corresponding optimality conditions and some smoothness properties. These smoothness results also hold for the solutions of (3.2) because by Proposition 3.1 every solution $(\hat{\theta},\hat{u})$ of (3.2) is a solution of (3.11) with $f = \hat{\lambda}|u|^{p-1}u$ and $\hat{\lambda}$ the value of the minimum in (3.2). Using a bootstrap argument and that every eigenfunction for the minimum eigenvalue of the operator $u \mapsto -\mathrm{div}((1+c\theta)^{1-p}|\nabla u|^{p-2}\nabla u)$ cannot change sign we immediately have the following result.

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with $C^{1,1}$ boundary, and (θ,u)a solution of (3.2). Then, we have*

- *The function u is in $W^{1,\infty}(\Omega)$ and except for a change of sign, it is strictly positive in Ω.*
- *The function*

$$\sigma = \frac{|\nabla u|^{p-2}}{(1+c\theta)^{p-1}}\nabla u \tag{3.22}$$

satisfies

$$|\sigma|^r\sigma \in H^1(\Omega)^N,\ \forall\, r > -\frac{1}{2}, \qquad \sigma = (\sigma\cdot\nu)\nu \quad \text{on } \partial\Omega, \tag{3.23}$$

with ν the unitary outward normal on $\partial\Omega$.

- *The function θ satisfies*

$$\int_\Omega \theta\,\mathrm{d}x = \kappa, \tag{3.24}$$

and there exists $\mu > 0$ *such that*

$$\theta = \max\left\{0, \min\left\{1, \frac{1}{c}\left(\frac{|\nabla u|}{\mu} - 1\right)\right\}\right\} \text{ a.e. in } \Omega. \tag{3.25}$$

Moreover, it satisfies

$$\partial_i \theta \sigma_j - \partial_j \theta \sigma_i = (1 + c\theta)\left(\partial_j \sigma_i - \partial_i \sigma_j\right) \chi_{\{|\sigma| = \mu\}} \in L^2(\Omega), \ \forall i, j \in \{1, ..., N\}. \tag{3.26}$$

- *If the solution is unrelaxed, i.e.* $\theta \in \{0, 1\}$ *a.e. in* Ω*, then*

$$\operatorname{curl}(|\sigma|^{p'-2}\sigma) = 0 \text{ a.e. in } \Omega. \tag{3.27}$$

A consequence of this regularity is the following non existence result for problem (3.1). The proof is similar to the one in [14], where it is proved the result for $p = 2$. Therefore we schematize some parts. We also refer to [13], [15], [44] for related results relative to problem (3.12).

Theorem 3.3 *Let* $\Omega \subset \mathbb{R}^N$ *be an open, bounded and simply connected set with* $C^{1,1}$ *connected boundary and assume that there exists a solution of* (θ, u) *of* (3.2) *with* $\theta = \chi_\omega$ *for some* $\omega \subset \Omega$ *measurable. Then,* Ω *is a ball and* u *and* ω *have a radial structure.*

Remark 3.1 *The existence of radial solutions for problem* (3.1) *has been first proved in [4]. The structure of these radial functions in the case* $p = 2$ *is a problem that has been considered for example in [19], [20], [32] and [40].*

Proof of Theorem 3.3. We define σ by (3.22). By (3.27) and Ω simply connected, we know that there exists $w \in W^{1,\infty}(\Omega)$, with $|\nabla w|^s \nabla w \in H^1(\Omega)^N$, for every $s > (p-3)/2$ such that

$$\nabla w = |\sigma|^{p'-2}\sigma \iff |\nabla w|^{p-2}\nabla w = \sigma. \tag{3.28}$$

Using also the second assertion in (3.23), we get that ∇w is ortogonal to the boundary on $\partial\Omega$. Since the boundary is connected, this means that w is constant on the boundary. Taking into account that w is defined up to a constant, we can then take w vanishing on $\partial\Omega$, which proves that w can be chosen as the unique solution of

$$\begin{cases} -\operatorname{div}(|\nabla w|^{p-2}\nabla w) = \lambda u^{p-1} \text{ in } \Omega \\ w = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{3.29}$$

Since u belongs to $W^{1,\infty}(\Omega)$, the first corollary in [23] and the Calderon-Zygmund theory show that for some $\gamma \in (0, 1)$,

$$w \in C^{1,\gamma}(\overline{\Omega}) \cap W^{3,q}_{loc}(\Omega \setminus \{\nabla w = 0\}), \quad \forall q \geqslant 1. \tag{3.30}$$

Now, taking $\mu > 0$ such that (3.25) holds, let us prove that for every $x_0 \in \Omega$, with

$$|B(x_0, r) \cap \omega| > 0, \quad |B(x_0, r) \setminus \omega| > 0, \quad \forall r > 0, \tag{3.31}$$

there exists a connected open set $\mathcal{O}$ strictly contained in Ω of class $W^{3,q}$ for every $q \geqslant 1$ such that

$$x_0 \in \partial\mathcal{O}, \quad |\nabla w| = \mu \ \text{ on } \partial\mathcal{O}, \quad w = w(x_0) \ \text{ on } \partial\mathcal{O}. \tag{3.32}$$

Moreover, all the points in $\partial\mathcal{O}$ satisfy (3.31). We define the set

$$\hat{\Upsilon} = \Big\{ x \in \Omega : \ w(x) = w(x_0), \quad |\nabla w(x_0)| > \frac{\mu}{2} \Big\}.$$

which using the implicit function theorem is a $(N-1)$-dimensional sub-manifold of class $W^{3,q}(\Omega)$ for every $q \geqslant 1$. We also define Υ as the connected component of $\hat{\Upsilon}$ containing x_0. Taking into account (3.28) and (3.22) we conclude (see Lemma 2.6 in [14]) that for any compact set $K \subset \Upsilon$, there exist a neighborhood U of K, $\tau > 0$ and $h \in W^{1,\infty}(w(x_0) - \tau, w(x_0) + \tau)$ such that

$$u(x) = h(w(x)), \ \text{ a.e. in } U \Rightarrow \nabla u = h'(w)\nabla w \ \text{ a.e. in } U.$$

Since we also know that $\nabla u = (1 + c\chi_\omega)\nabla w$ a.e. in Ω, we deduce as an application of the coarea formula that there exists $\mathcal{N} \subset (w(x_0) - \tau, w(x_0) + \tau)$ of null measure such that for every $s \in (w(x_0) - \tau, w(x_0) + \tau) \setminus \mathcal{N}$, we have ($H_{N-1}$ denotes the $(N-1)$-Hausdorff measure)

$$h'(s) \in \big\{1 + c, 1\big\}, \qquad \begin{cases} h'(s) = 1 + c \Rightarrow H_{N-1}\big(w^{-1}(s) \cap U \setminus \omega\big) = 0 \\ h'(s) = 1 \Rightarrow H_{N-1}\big(w^{-1}(s) \cap U \cap \omega\big) = 0, \end{cases}$$

Combined with (3.31) this shows that $|\nabla w| = \mu$ on K, which by the arbitrariness of K proves $|\nabla w| = \mu$ on Υ. Thanks to this equality we can now show that the open manifold Υ is also closed and then by the Jordan-Brouwer theorem, that Υ is the boundary of an open set $\mathcal{O}$ satisfying (3.32). Now, we can prove that the interior of the intersection of all the connected open sets $\mathcal{O}$ satisfying $|\nabla w| = \mu$ and w constant on $\partial\mathcal{O}$ is also in these conditions. To simplify the notation, we still denote such intersection as $\mathcal{O}$. Observe that this set cannot contain any point x_0 satisfying (3.31), which taking into account that it is connected, implies that $\mathcal{O}$ is contained in ω or in $\Omega \setminus \omega$. Since w is constant on $\partial\mathcal{O}$ it must contain at least a point where ∇w vanishes. By (3.25), this proves that it is the condition $\mathcal{O} \subset \Omega \setminus \omega$ which holds true. Therefore, $\nabla w = \nabla u$ a.e. in $\mathcal{O}$ and thus $w = u + a$ in $\mathcal{O}$ for some $a \in \mathbb{R}$. From (χ_ω, u) solution of (3.11), the definition (3.22) of σ and (3.28), we have that w satisfies

$$\begin{cases} -\text{div}\,\big(|\nabla w|^{p-2}\nabla w\big) = \hat{\lambda}|w - a|^{p-2}(w - a) & \text{in } \mathcal{O} \\ w, \dfrac{\partial w}{\partial \nu} \quad \text{constant on } \partial\mathcal{O}, \end{cases} \tag{3.33}$$

with $\hat{\lambda}$ the value of the minimum in (3.11). Since $\mathcal{O}$ is of class C^2, we can apply Serrin's Theorem ([49]) to deduce that $\mathcal{O} = B(\bar{x}, r)$ for some $\bar{x} \in \mathcal{O}$ and some $r > 0$. Moreover, w is a radial function with respect to $\bar{x}$ in $\mathcal{O}$. Let $R > 0$ be the ball defined by

$$B(\bar{x}, R) = \bigcup_{\substack{\mathcal{O} \subset B(\bar{x}, s) \\ w \text{ radial in } B(\bar{x}, s)}} B(\bar{x}, s).$$

If $w = 0$ on $\partial B(\bar{x}, R)$, then $\Omega = B(\bar{x}, R)$ since Ω is simply connected with connected boundary. In another case, by Hopf's Lemma for the p-Laplacian operator (Theorem 1 in [24], Theorem 5.5 in [47]) we have $w = c > 0$ and $\nabla w \neq 0$ on $\partial B(\bar{x}, R)$. Thus, by Lemma 2.6 in [14] there exists a neighborhood U of $\partial B(\bar{x}, R)$ and a Lipschitz function $h : (c-\delta, c+\delta) \to \mathbb{R}$, with $\delta > 0$, such that $u(x) = h(w(x))$ in U. Then, we take $\varepsilon > 0$ small enough to have $B(\bar{x}, R+\varepsilon) \setminus B(\bar{x}, R-\varepsilon) \subset U$ and such that there exists a solution $\phi \in W^{3,\infty}(R-\varepsilon, R+\varepsilon)$ of the Cauchy problem

$$\begin{cases} -\left(r^{N-1}|\phi'|^{p-2}\phi'\right) = r^{N-1}h(\phi) \quad \text{in } (R-\varepsilon, R+\varepsilon) \\ \phi(R) = c, \quad \phi'(R) = \dfrac{\partial w}{\partial n}\Big|_{\partial B(\bar{x},R)}. \end{cases} \tag{3.34}$$

The function $v(x) = \phi(|x-\bar{x}|)$ is a radial functions with respect to $\bar{x}$ and satisfies

$$\begin{cases} -\operatorname{div}(|\nabla v|^{p-2}\nabla v) = h(v) \text{ in } U. \\ \qquad w = v \text{ in } U \cap B(\bar{x}, R). \end{cases}$$

By lemma 3.2 below, we have $w = v$ in U. Thus w is a radial function in $B(\bar{x}, R+\varepsilon)$, in contradiction with the definition of $B(\bar{x}, R)$. Thus Ω is a ball centered in $\bar{x}$ and w is a radial function. By (3.22), (3.28) and (3.25) we also have that ω has a radial structure and that u is a radial function. □

In the proof of Theorem 3.3 we have used the following unique continuation lemma

Lemma 3.2 *Let $\Omega \subset \mathbb{R}^N$ be a connected open set and $h : (a,b) \to \mathbb{R}$ be a Lipschitz function. Assume $u_1 \in C^3(\Omega)$ and $u_2 \in W^{2,\infty}_{loc}(\Omega)$ such that*

$$-\operatorname{div}(|\nabla u_{\mathrm{i}}|^{p-2}\nabla u_{\mathrm{i}}) = h(u_{\mathrm{i}}) \text{ in } \Omega, \quad \mathrm{i} = 1,2 \tag{3.35}$$

and

$$\inf_{x\in\Omega} |\nabla u_{\mathrm{i}}(x)| > 0, \quad \mathrm{i} = 1,2. \tag{3.36}$$

If $u_1 = u_2$ in a open subset of Ω, then $u = v$ in Ω.

Proof. Thanks to the smoothness of u_1, u_2, we can rewrite (3.35) as:

$$A_{\mathrm{i}} : D^2 u_{\mathrm{i}} = \frac{h(u_{\mathrm{i}})}{|\nabla u_{\mathrm{i}}|^{p-2}} \text{ in } \Omega, \quad \mathrm{i} = 1,2, \tag{3.37}$$

with

$$A_{\mathrm{i}} = I + (p-2)\frac{\nabla u_{\mathrm{i}} \otimes \nabla u_{\mathrm{i}}}{|\nabla u_{\mathrm{i}}|^2}, \quad \mathrm{i} = 1,2.$$

Then, taking $v = u_1 - u_2$, subtracting the equations (3.37), and taking into account the smoothness properties of u_1 and u_2, we have that for every open set $\mathcal{O} \Subset \Omega$, there exists a constant M depending on $\mathcal{O}$ such that

$$\left|A_1 : D^2 v\right| \leqslant M\big(|\nabla v| + |v|\big) \text{ in } \mathcal{O}.$$

Since A_1 is uniformly elliptic and has coefficients of class C^2, this allows us to use the unique continuation principle in [5] to conclude $v = 0$ and then $u_1 = u_2$ in $\mathcal{O}$ for every open set $\mathcal{O} \Subset \Omega$. Thus, $u_1 = u_2$ in Ω. □

3.3. Numerical approximation.

The present section is devoted to define a numerical algorithm for the resolution of problem (3.2) and to prove the convergence to a critical point. We also show the convergence of the solution of a discrete version of (3.2) to the solutions of the continuous one and we provide some numerical experiments.

Algorithm.

Initialization: A strictly positive function $u_0 \in W_0^{1,p}(\Omega)$ with $\|u_0\|_{L^p(\Omega)} = 1$, $\theta_0 \in L^\infty(\Omega; [0,1])$ with $\|\theta_0\|_{L^1(\Omega)} \leqslant \kappa$ and $\mathrm{i} = 0$.

1 : Set

$$\lambda_{\mathrm{i}} = \int_\Omega \frac{|\nabla u_{\mathrm{i}}|^p}{(1+c\theta_{\mathrm{i}})^{p-1}} \mathrm{d}x. \tag{3.38}$$

2 : Set $(\theta_{\mathrm{i}+1}, \tilde{u}_{\mathrm{i}+1})$ a solution of

$$\begin{gathered} \min_{\vartheta, v} \int_\Omega \Big(\frac{1}{p} \frac{|\nabla v|^p}{(1+c\vartheta)^{p-1}} - \lambda_{\mathrm{i}} u_{\mathrm{i}}^{p-1} v \Big) \mathrm{d}x \\ \vartheta \in L^\infty(\Omega; [0,1]), \quad v \in W_0^{1,p}(\Omega), \quad \int_\Omega \vartheta \mathrm{d}x \leqslant \kappa. \end{gathered} \tag{3.39}$$

3 : Set

$$u_{\mathrm{i}+1} = \frac{\tilde{u}_{\mathrm{i}+1}}{\|\tilde{u}_{\mathrm{i}+1}\|_{L^p(\Omega)}}. \tag{3.40}$$

Theorem 3.4 *The sequence λ_{i} defined by the above algorithm is decreasing and converges to $\lambda \geqslant \hat{\lambda}$. Moreover, the sequence u_{i} is bounded in $W_0^{1,p}(\Omega)$. Taking a subsequence of* i, *still denoted by* i, *such that there exist $u \in W_0^{1,p}(\Omega)$ and $\theta \in L^\infty(\Omega; [0,1])$, with*

$$u_{\mathrm{i}} \rightharpoonup u \ \text{ in } W_0^{1,p}(\Omega), \qquad \theta_{\mathrm{i}} \overset{*}{\rightharpoonup} \theta \ \text{ in } L^\infty(\Omega), \tag{3.41}$$

we also have that

$$\tilde{u}_{\mathrm{i}},\ u_{\mathrm{i}-1} \rightharpoonup u \ \text{ in } W_0^{1,p}(\Omega). \tag{3.42}$$

and (θ, u) is a solution of

$$\begin{gathered} \min_{\vartheta, v} \Big\{ \int_\Omega \Big(\frac{1}{p} \frac{|\nabla v|^p}{(1+c\vartheta)^{p-1}} - \lambda u^{p-1} v \Big) \mathrm{d}x \Big\} \\ \vartheta \in L^\infty(\Omega; [0,1]), \quad v \in W_0^{1,p}(\Omega), \quad \int_\Omega \vartheta \mathrm{d}x \leqslant \kappa. \end{gathered} \tag{3.43}$$

Remark 3.2 *The fact that (θ, u) is a solution of* (3.43) *is equivalent (see Theorem 3.1 in [15]) to (θ, u) a critical point for* (3.11) *in the sense that it satisfies the optimality conditions*

$$-\operatorname{div}\Big(\frac{|\nabla u|^{p-2}}{(1+c\theta)^{p-1}} \nabla u \Big) = \lambda u^{p-1} \ \text{ in } \Omega, \tag{3.44}$$

$$\|u\|_{L^p(\Omega)}=1,\quad \lambda=\int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}}\mathrm{d}x,\quad u>0 \text{ in } \Omega,\quad \int_\Omega \theta=\kappa, \tag{3.45}$$

and there exists $\mu>0$ such that θ and u satisfy (3.25). *Moreover, u is positive implies that λ is the first eigenvalue of the operator $v\in W_0^{1,p}(\Omega)\mapsto -\mathrm{div}((1+c\theta)^{1-p}|\nabla v|^{p-2}\nabla v)\in W^{-1,p'}(\Omega)$.*

Proof of Theorem 3.4. First we observe that $(\theta_{\mathrm{i}+1},\tilde u_{\mathrm{i}+1})$ solution of (3.39) implies that $\tilde u_{\mathrm{i}+1}$ is a solution of

$$-\operatorname{div}\Big(\frac{|\nabla \tilde u_{\mathrm{i}+1}|^{p-2}}{(1+c\theta_{\mathrm{i}+1})^{p-1}}\nabla \tilde u_{\mathrm{i}+1}\Big)=\lambda_{\mathrm{i}}u_{\mathrm{i}}^{p-1}\quad \text{in } \Omega. \tag{3.46}$$

In particular this allows us to use the strong maximum principle to deduce by induction that u_{i} is striclty positive in Ω for every i.

Let us obtain some estimates: Multiplying (3.46) by $\tilde u_{\mathrm{i}+1}$ and taking into account (3.38) and (3.40) we get

$$\lambda_{\mathrm{i}+1}\|\tilde u_{\mathrm{i}+1}\|^p_{L^p(\Omega)}=\int_\Omega \frac{|\nabla \tilde u_{\mathrm{i}+1}|^p}{(1+c\theta_{\mathrm{i}+1})^{p-1}}\mathrm{d}x=\lambda_{\mathrm{i}}\int_\Omega u_{\mathrm{i}}^{p-1}\tilde u_{\mathrm{i}+1}\,\mathrm{d}x. \tag{3.47}$$

On the other hand, (3.39) proves

$$\int_\Omega\Big(\frac{1}{p}\frac{|\nabla \tilde u_{\mathrm{i}+1}|^p}{(1+c\theta_{\mathrm{i}+1})^{p-1}}-\lambda_{\mathrm{i}}u_{\mathrm{i}}^{p-1}\tilde u_{\mathrm{i}+1}\Big)\mathrm{d}x\leqslant \int_\Omega\Big(\frac{1}{p}\frac{|\nabla u_{\mathrm{i}}|^p}{(1+c\theta_{\mathrm{i}})^{p-1}}-\lambda_{\mathrm{i}}u_{\mathrm{i}}^{p}\Big)\mathrm{d}x, \tag{3.48}$$

which using the second equality in (3.47) combined with (3.38) and (3.40) can be written as

$$-\frac{\|\tilde u_{\mathrm{i}+1}\|^p_{L^p(\Omega)}}{p'}\lambda_{\mathrm{i}+1}\leqslant -\frac{1}{p'}\lambda_{\mathrm{i}}\iff \lambda_{\mathrm{i}}\leqslant \lambda_{\mathrm{i}+1}\|\tilde u_{\mathrm{i}+1}\|^p_{L^p(\Omega)}. \tag{3.49}$$

Using Hölder's inequality in (3.47) and $\|u_{\mathrm{i}}\|_{L^p(\Omega)}=1$, we also have

$$\lambda_{\mathrm{i}+1}\|\tilde u_{\mathrm{i}+1}\|^{p-1}_{L^p(\Omega)}\leqslant \lambda_{\mathrm{i}}, \tag{3.50}$$

$$\|\tilde u_{\mathrm{i}+1}\|^p_{W_0^{1,p}(\Omega)}\leqslant \lambda_{\mathrm{i}}(1+c)^{p-1}\|\tilde u_{\mathrm{i}+1}\|_{L^p(\Omega)}. \tag{3.51}$$

From (3.49), (3.50) and

$$\lambda_{\mathrm{i}}\geqslant \hat\lambda>0, \tag{3.52}$$

with $\hat\lambda$ the value of the minimum in (3.2), which is a consequence of (3.38), we have

$$1\leqslant \|\tilde u_{\mathrm{i}+1}\|_{L^p(\Omega)}. \tag{3.53}$$

Thus (3.50) shows

$$\lambda_{\mathrm{i}+1}\leqslant\lambda_{\mathrm{i}}. \tag{3.54}$$

Inequality (3.54) proves that the sequence λ_{i} is decreasing and therefore it converges to a limit λ which by (3.52) is bigger or equal than $\hat\lambda$. Inequalities (3.50) and (3.53) then show

$$\exists \lim_{\mathrm{i}\to\infty}\|\tilde u_{\mathrm{i}}\|_{L^p(\Omega)}=1. \tag{3.55}$$

By $\lambda_{\rm i}$ bounded, (3.51) and (3.55), we have that $\tilde u_{\rm i}$ and $u_{\rm i}$ are bounded in $W_0^{1,p}(\Omega)$, and that for a subsequence of i, still denoted by i, there exists $u \in W_0^{1,p}(\Omega)$ such that

$$u_{\rm i} \rightharpoonup u \ \text{ in } W_0^{1,p}(\Omega), \quad \tilde u_{\rm i} \rightharpoonup u \ \text{ in } W_0^{1,p}(\Omega), \quad \|u\|_{L^p(\Omega)} = 1. \tag{3.56}$$

Taking another subsequence if necessary, we can also assume that there exist $\theta \in L^\infty(\Omega;[0,1])$ and $z \in W_0^{1,p}(\Omega)$ such that

$$\theta_{\rm i} \overset{*}{\rightharpoonup} \theta \ \text{ in } L^\infty(\Omega), \quad u_{\rm i-1} \rightharpoonup z \ \text{ in } W_0^{1,p}(\Omega), \quad \|z\|_{L^p(\Omega)} = 1. \tag{3.57}$$

By (3.39) with i replaced by $\rm i-1$, we have

$$\int_\Omega \Big(\frac{1}{p}\frac{|\nabla u_{\rm i}|^p}{(1+c\theta_{\rm i})^{p-1}} - \lambda_{\rm i-1} u_{\rm i-1}^{p-1} u_{\rm i}\Big)\mathrm{d}x \leqslant \int_\Omega \Big(\frac{1}{p}\frac{|\nabla v|^p}{(1+c\vartheta)^{p-1}} - \lambda_{\rm i-1} u_{\rm i-1}^{p-1} v\Big)\mathrm{d}x, \tag{3.58}$$

for every $v \in W_0^{1,p}(\Omega)$ and every $\vartheta \in L^\infty(\Omega;[0,1])$, with $\|\vartheta\|_{L^1(\Omega)} \leqslant \kappa$. Using the convexity of J defined by (3.9), (3.56), (3.57) and the Rellich-Kondrachov compactness theorem, we can pass to the limit in (3.58) to deduce

$$\int_\Omega \Big(\frac{1}{p}\frac{|\nabla u|^p}{(1+c\theta)^{p-1}} - \lambda z^{p-1} u\Big)\mathrm{d}x \leqslant \int_\Omega \Big(\frac{1}{p}\frac{|\nabla v|^p}{(1+c\vartheta)^{p-1}} - \lambda z^{p-1} v\Big)\mathrm{d}x, \tag{3.59}$$

for every $v \in W_0^{1,p}(\Omega)$ and every $\vartheta \in L^\infty(\Omega;[0,1])$, with $\|\vartheta\|_{L^1(\Omega)} \leqslant \kappa$. Thanks to the Rellich-Kondrachov compactness theorem, we can also pass to the limit in (3.47) with i replaced by $\rm i-1$ to deduce

$$1 = \int_\Omega u^{p-1} z \,\mathrm{d}x,$$

where $\|u\|_{L^p(\Omega)} = \|z\|_{L^p(\Omega)} = 1$. Thus, Hölder's inequality is an equality and then, using u and z positive, we deduce $u = z$. Using this equality in (3.59), we finish the proof of the theorem. □

Remark 3.3 *Problem* (3.39) *is a particular case of* (3.11) *with* $f = \lambda_{\rm i} u_{\rm i}^{p-1}$. *The numerical resolution of this problem has been studied in [16] where we have given two algorithms and proved the convergence. We also refer to [3], [7], [28], [31] and [53] for some other results referred to the case* $p = 2$.

In order to implement the above algorithm is necessary to work with a discrete version of (3.2), where the spaces $L^\infty(\Omega)$ and $W_0^{1,p}(\Omega)$ are replaced by finite-dimensional spaces. Moreover, the algorithms proposed in [16] need to work with a discrete version of the problem at least for $p > 2$ because in this case $W_0^{1,p}(\Omega)$ is not included in $H_0^1(\Omega)$. As an example of discretization, let us assume that Ω is a polyhedral domain in $\mathbb{R}^N$. Then, for a regular mesh $\mathcal{T}_h$ of $\overline{\Omega}$ composed by N-simplexes (see e.g. [48]), with maximum diameter $h > 0$, let us define the Lagrange finite element spaces

$$V_h = \big\{ v \in C_0(\Omega) : \ v\big|_\tau \in \mathbb{P}_1(\tau), \quad \forall \tau \in \mathcal{T}_h \big\} \tag{3.60}$$

$$\Theta_h = \big\{ \vartheta \in L^\infty(\Omega) : \ \vartheta\big|_\tau \in \mathbb{P}_0(\tau), \quad \forall \tau \in \mathcal{T}_h \big\}, \tag{3.61}$$

where $\mathbb{P}_0(\tau)$ denotes the space of constant functions in τ, and $\mathbb{P}_1(\tau)$ the space of affine functions in τ. Using these spaces, we consider the discrete version of (3.2)

$$\begin{cases} \displaystyle\min_{u,\theta} \int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}} \mathrm{d}x \\ \theta \in \Theta_h, \quad 0 \leqslant \theta \leqslant 1 \text{ a.e. in } \Omega, \quad \displaystyle\int_\Omega \theta \mathrm{d}x \leqslant \kappa \\ u \in V_h, \quad \displaystyle\int_\Omega |u|^p \mathrm{d}x = 1, \end{cases} \tag{3.62}$$

Clearly, Theorem 3.4 still holds for this discrete problem where now the weak and strong convergences are the same because we are working in finite dimension. An important question is if the solutions of the discrete problem converge to the solutions of the continuous one when h tends to zero. The following theorem provides a positive answer to this question.

Theorem 3.5 *Assume a polyhedral open set $\Omega \subset \mathbb{R}^N$ and a sequence $\mathcal{T}_h$ of regular mesh whose diameter h tends to zero. Then, the value λ_h of the minimum in* (3.62) *converges to the value $\hat{\lambda}$ of the minimum in* (3.2). *Moreover, if (u_h, θ_h) is a solution of* (3.43), *then u_h is bounded in $W_0^{1,p}(\Omega)$ and extracting a subsequence of h, still denoted by h, such that there exist $\theta \in L^\infty(\Omega; [0,1])$ and $u \in W_0^{1,p}(\Omega)$ with*

$$u_h \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \qquad \theta_h \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(\Omega), \tag{3.63}$$

we have that (θ, u) is a solution of (3.2).

If we also asume that there exists a solution $(\hat{\theta}, \hat{u})$ of (3.2), such that

$$\hat{u} \in W^{1,\infty}(\Omega), \quad \nabla \hat{u} \in BV(\Omega)^N, \quad \hat{\theta} \in BV(\Omega), \tag{3.64}$$

then, there exists $C > 0$, depending on Ω, p, $\hat{\theta}$, and $\hat{u}$, such that

$$\hat{\lambda} \leqslant \lambda_h \leqslant \hat{\lambda} + Ch, \quad \forall\, h > 0. \tag{3.65}$$

Proof. By the definitions of λ_h and $\hat{\lambda}$, it is clear that $\lambda_h \geqslant \hat{\lambda}$. On the other hand, the classical finite element theory shows that for every solution $(\theta, u) \in L^\infty(\Omega; [0,1])$ of (3.2), there exist $\tilde{\theta}_h \in \Theta_h$ and $\tilde{u}_h \in V_h$ such that

$$\tilde{\theta}_h \to \theta \ \text{ in } L^1(\Omega), \quad \tilde{\theta}_h \overset{*}{\rightharpoonup} \theta \ \text{ in } L^\infty(\Omega), \quad \int_\Omega \tilde{\theta}_h \, \mathrm{d}x \leqslant \kappa. \tag{3.66}$$

$$\tilde{u}_h \to u \ \text{ in } W_0^{1,p}(\Omega), \quad \|\tilde{u}_h\|_{L^p(\Omega)} = 1. \tag{3.67}$$

Therefore,

$$\hat{\lambda} = \int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}} \mathrm{d}x = \lim_{h\to 0} \int_\Omega \frac{|\nabla \tilde{u}_h|^p}{(1+c\tilde{\theta}_h)^{p-1}} \mathrm{d}x \geqslant \limsup_{h\to 0} \lambda_h,$$

which combined with $\lambda_h \geqslant \hat{\lambda}$ proves that λ_h converges to $\hat{\lambda}$.

Let us now consider a solution (u_h, θ_h) of (3.43). Taking into account

$$\frac{\|u_h\|^p_{W_0^{1,p}(\Omega)}}{(1+c)^{p-1}} \leqslant \int_\Omega \frac{|\nabla u_h|^p}{(1+c\theta_h)^{p-1}} \mathrm{d}x = \lambda_h,$$

we get that u_h is bounded in $W_0^{1,p}(\Omega)$. Extracting a subsequence of h such that (3.63) holds, we have that

$$\|u\|_{L^p(\Omega)} = 1, \quad \theta \in L^\infty(\Omega;[0,1]), \quad \int_\Omega \theta \, \mathrm{d}x \leqslant \kappa,$$

while the convexity of J given by (3.9) shows

$$\hat{\lambda} \leqslant \int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}} \mathrm{d}x \leqslant \lim_{h\to 0} \int_\Omega \frac{|\nabla u_h|^p}{(1+c\theta_h)^{p-1}} \mathrm{d}x = \lim_{h\to 0} \lambda_h = \hat{\lambda}.$$

This proves that (θ, u) is a solution of (3.2).

If we now assume that there exists a solution $(\hat{\theta}, \hat{u})$ of (3.2) satisfying (3.64), then, defining $\tilde{\theta}_h \in \Theta_h$, $\tilde{u}_h \in V_h$ by

$$\tilde{\theta}_h\Big|_\tau = \frac{1}{|\tau|} \int_\tau \theta \, \mathrm{d}x, \quad \forall \tau \in \mathcal{T}_h, \qquad \tilde{u}_h = \frac{\pi_h u}{\|\pi_h u\|_{L^p(\Omega)}},$$

with $\pi_h u$ the usual projection operator in the finite element space V_h, i.e.

$$\pi_h u \in V_h, \quad (\pi_h u)(x_\mathrm{i}) = u(x_\mathrm{i}), \ \forall \, x_\mathrm{i} \text{ vertex of } \mathcal{T}_h,$$

we have that (3.66), (3.67) hold and, thanks to (3.64),

$$\|\tilde{u}_h - u\|_{W_0^{1,p}(\Omega)} + \|\tilde{\theta}_h - \theta\|_{L^1(\Omega)} \leqslant Ch.$$

Thus, we have

$$\int_\Omega \left| \frac{|\nabla u|^p}{(1+c\theta)^{p-1}} - \frac{|\nabla \tilde{u}_h|^p}{(1+c\tilde{\theta}_h)^{p-1}} \right| \mathrm{d}x \leqslant Ch,$$

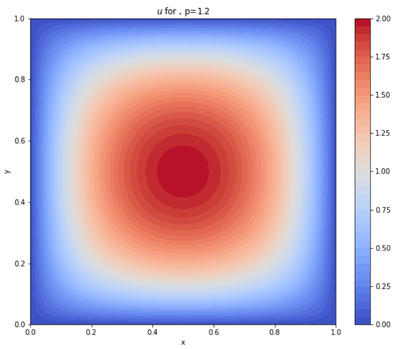
which implies

$$\hat{\lambda} = \int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}} \mathrm{d}x \leqslant \lambda_h \leqslant \int_\Omega \frac{|\nabla \tilde{u}_h|^p}{(1+c\tilde{\theta}_h)^{p-1}} \mathrm{d}x \leqslant \hat{\lambda} + Ch.$$

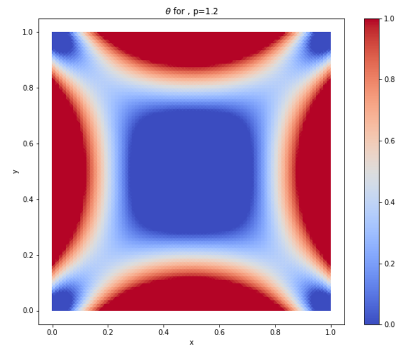
□

Remark 3.4 *Observe that the smoothness properties of the solutions of* (3.2) *do not imply* (3.64)*, however this seems to be satisfied in the numerical experiments.*

We implement the algorithm in Python, using the finite element solver Fenics [6]. We solve the problem in the square $(0,1)^2 \subset \mathbb{R}^2$ for five different values of p, $c = 1$ and $\kappa = 1/2$. The corresponding contour lines for the optimal functions u and θ are given by the following pictures.

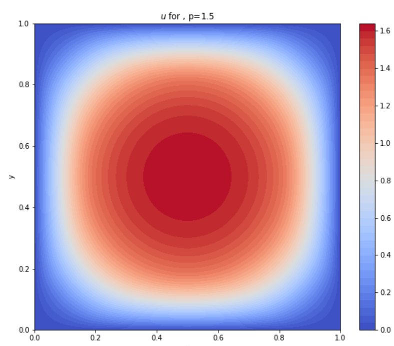


(a) u, p=1.2.

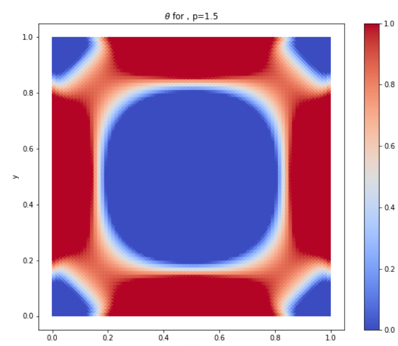


(b) θ, p=1.2.

Figure 3.1: Solutions for $p = 1.2$.

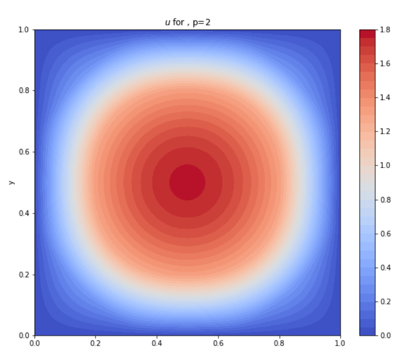


(a) u, p=1.5.

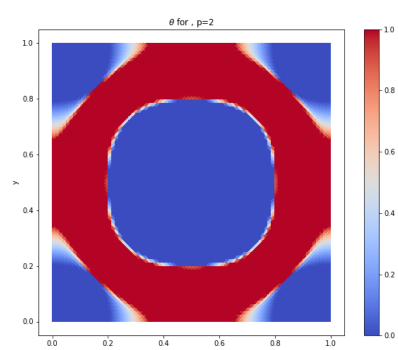


(b) θ, p=1.5.

Figure 3.2: Solutions for $p = 1.5$.

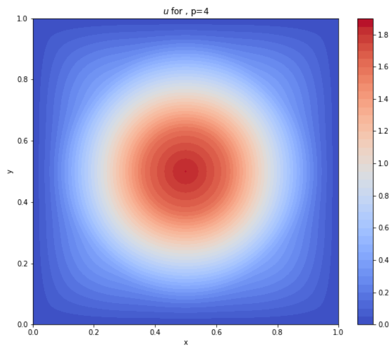


(a) u, p=2.

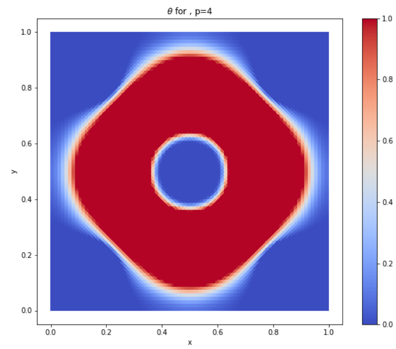


(b) θ, p=2.

Figure 3.3: Solutions for $p = 2$.

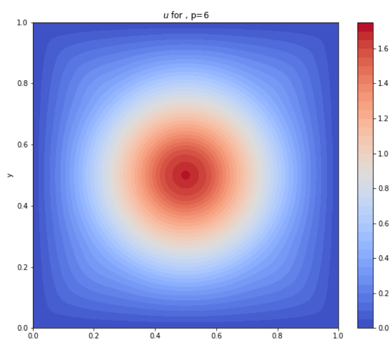


(a) u, p=4.

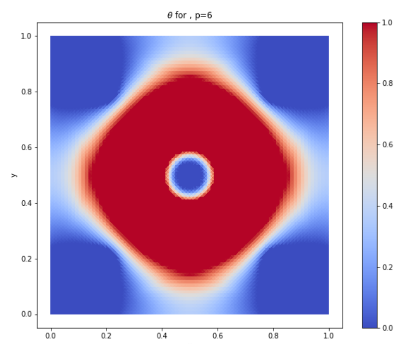


(b) θ, p=4.

Figure 3.4: Solutions for $p = 4$.

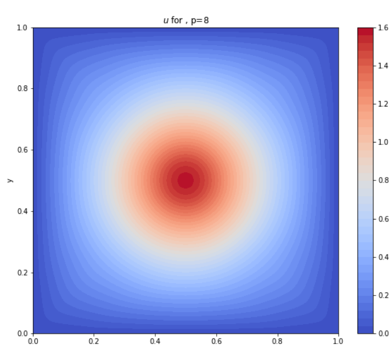


(a) u, p=6.

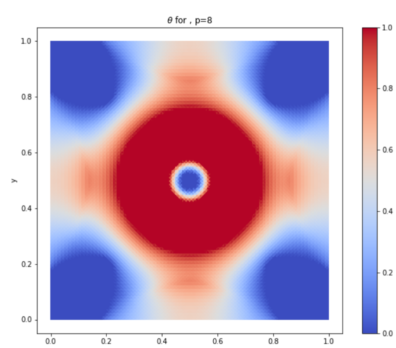


(b) θ, p=6.

Figure 3.5: Solutions for $p = 6$.

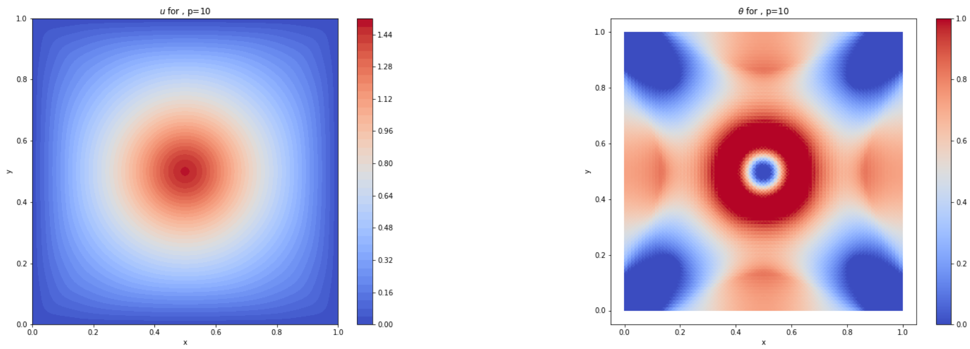


(a) u, p=8.



(b) θ, p=8.

Figure 3.6: Solutions for $p = 8$.



(a) u, p=10.

(b) θ, p=10.

Figure 3.7: Solutions for $p = 10$.

Conclusions

In this work we have studied the problem of maximizing the energy of the p-Laplacian operator for a two-phase material in a bounded open set by means of the optimization by the homogenization method.

In Chapter 1 we have obtained a relaxed formulation using the homogenization theory. We have proved that, although the relaxed problem does not have a unique solution in general, the flux $\hat{\sigma}$ is unique. The relaxed formulation (2.5) allowed us to get optimality conditions which we gave in Theorem 1.2. We have shown that if the data is smooth enough, then, for every $r > -1/2$ the function $|\hat{\sigma}|^r\hat{\sigma}$ is in the Sobolev space $H^1(\Omega)^N \cap L^\infty(\Omega)^N$. Moreover, the optimal proportion $\hat{\theta}$ is derivable in the orthogonal directions to $\nabla\hat{u}$. Using these results, we have proved that the original problem has a solution in a smooth open set Ω with connected boundary if and only if Ω is a ball.

In Chapter 2, we provided two algorithms to solve the finite dimensional approximation of (2.5), given by (2.24). In Theorem (2.2) have estimated the rates of convergence for both algorithms. We have also proved that the flux σ_{i}, obtained by any of both algorithms, converges strongly in $L^{p'}(\Omega)$ to the solution of (2.25). These results were known for $p = 2$, but we have extended them to any $p \in (1, \infty)$. Moreover, we have estimated the error between the value of the discretized and continuous problems (2.24) and (2.5), where $\Theta = \Theta_h$ and $V = V_h$ are finite elements spaces given by (2.26) and (2.27) respectively, assuming the existence of a solution $(\hat{\theta}, \hat{u})$ such that $\hat{\theta} \in BV(\Omega)$ and $\nabla\hat{u} \in BV(\Omega)^N$.

As an application of the results proved in Chapters 1 and 2 we have studied in Chapter 3 problem (3.1), which corresponds to the minimization of the first eigenvalue of the p-Laplacian for a two phase material. We obtained the homogenized formulation (3.2) of this problem using proposition 3.1 which in turns allowed us to apply a bootstrap argument to obtain the regularity and characterization results in Theorem 3.2. As in chapter 1, the regularity Theorem allows us to prove that the problem only has a solution in an open domain, bounded and simply connected if it is a ball. Additionally, based on the power method and the alternate optimization algorithm developed in Chapter 2, we developed and proved the convergence of an algorithm converges to a critical point of the problem. Analogously to the convergence Theorem (2.3), we have estimated the error between the finite dimensional problem (3.62) and the continuous one (3.2).

Finally, the results obtained through this thesis can be used as the basis for future research. For example, it is interesting to study the case when the material represented by the coefficient β is much worse than the corresponding to α. Mathematically, this happens when β goes to infinity. In this regard, by formal calculations we conjecture that the value of the relaxed formulation (1.7) of (1.1) converges to the value of

$$\left\{\begin{array}{c}\min_{\theta,u}\left\{\frac{\alpha}{p}\int_\Omega\frac{|\nabla u|^p}{\theta^{p-1}}\mathrm{d}x-\langle f,u\rangle\right\}\\ u\in W_0^{1,p}(\Omega),\quad \theta\in L^\infty(\Omega;[0,1]),\quad \int_\Omega\theta\mathrm{d}x\leqslant\kappa.\end{array}\right. \tag{3.68}$$

Moreover, if this problem has a solution $(\theta,u)\in L^\infty(\Omega;[0,1])\times W_0^{1,p}(\Omega)$ such that $\theta=\chi_\omega$ for a measurable set $\omega\subset\Omega$, then we need to have $\nabla u=0$ in $\Omega\setminus\omega$. Thus u must be constant in the connected components of $\Omega\setminus\omega$. This means that $\Omega\setminus\omega$ is filled with a extremely rigid material which is consistent with the fact that $\beta\to\infty$. From the results of Chapters 1 and 2 we can expect to be able to prove some smoothness results for the solutions of (3.70).

Another interesting extension is to study the case when $p\to\infty$, which corresponds to study the mixture of two plastic materials. It is important to observe that in this case we need to consider a sequence of coefficients α_p and β_p depending on p such that α_p^{1-p} and α_p^{1-p} are converging sequences. Arguing as in [10] and [1] and assuming that α_p^{1-p} and β_p^{1-p} converges to α_∞ and β_∞ when $p\to\infty$, we conjecture that the value of the problem

$$\left\{\begin{array}{l}\min_{\theta,u}\left\{\frac{1}{p}\int_\Omega\frac{|\nabla u|^p}{(\alpha_p^{1-p}\theta+\beta_p^{1-p}(1-\theta))^{p-1}}\mathrm{d}x-\langle f,u\rangle\right\}\\ u\in W_0^{1,p}(\Omega),\quad \theta\in L^\infty(\Omega;[0,1]),\quad \int_\Omega\theta\mathrm{d}x\leqslant\kappa,\end{array}\right.$$

converges to the value of

$$\left\{\begin{array}{c}\min_{\theta,u}-\langle f,u\rangle\\ u\in W_0^{1,\infty}(\Omega),\quad \theta\in L^\infty(\Omega;[0,1]),\quad \int_\Omega\theta\mathrm{d}x\leqslant\kappa,\\ |\nabla u(x)|(\alpha_\infty\theta(x)+\beta_\infty(1-\theta(x)))\leqslant 1\ a.\text{e. } x\in\Omega.\end{array}\right. \tag{3.69}$$

Therefore problem (3.71) can be seen as the homogenized version of the $\infty-$Laplacian equation. Of course, the rigorous demonstration of this convergence needs further analysis, however problem (3.71) make it possible to study the mixture of two inelastic materials characterized by the constants α_∞ and β_∞ respectively.

Conclusiones

En este trabajo se ha estudiado el problema de maximizar la energía del operador p-Laplaciano para un material de dos fases en un conjunto abierto y acotado por medio del método de optimización por homogeneización.

En el Capítulo 1 se ha obtenido una formulación relajada utilizando teoría de homogeneización. Se ha demostrado que, aunque el problema relajado no tiene una única solución en general, el flujo $\hat{\sigma}$ es único. La formulación relajada (2.5) permitió obtener condiciones de optimalidad las cuales son dadas en el Teorema 1.2. Se ha mostrado que si los datos son suficientemente suaves, entonces, para todo $r > -1/2$ la función $|\hat{\sigma}|^r\hat{\sigma}$ está en el espacio de Sobolev $H^1(\Omega)^N \cap L^\infty(\Omega)^N$. Más aún, la proporción óptima es derivable en las direcciones ortogonales a $\nabla\hat{u}$. Utilizando estos resultados, se ha probado que el problema original tiene un solución en un conjunto Ω abierto, suave, simplemente conexo y con borde conexo si y solo si Ω es una bola.

En el Capítulo 2, se presentaron dos algoritmos para resolver una aproximación finito dimensional de (2.5), dada por (2.24). En el Teorema (2.2) se han estimado los ratios de convergencia para ambos algoritmos. Se ha demostrado también que el flujo σ_{i}, obtenido por cualquiera de los dos algoritmos, converge fuertemente en $L^{p'}(\Omega)$ a la solución de (2.25). Estos resultados eran conocidos para $p = 2$, pero en este trabajo han sido extendidos a cualquier $p \in (1, \infty)$. Más aún, se ha estimado el error entre los valores de los problemas desratizado y continuo (2.24) y (2.5), donde $\Theta = \Theta_h$ y $V = V_h$ son espacios de elementos finitos dados por (2.26) y (2.27), respectivamente, asumiendo la existencia de una solución $(\hat{\theta}, \hat{u})$ tal que $\hat{\theta} \in BV(\Omega)$ y $\nabla\hat{u} \in BV(\Omega)^N$.

Como una aplicación de los resultados demostrados en los Capítulos 1 y 2, se ha estudiado en el Capítulo 3 el problema (3.1), el cual corresponde a la minimización del primer valor propio del p-Laplaciano para un material con dos fases. Se obtuvo la formulación homogeneizada (3.2) para este problema utilizando la proposición 3.1, que a su vez permitió aplicar un un argumento tipo 'bootstrap' para obtener los resultados de regularidad caracterización en el Teorema 3.2. Al igual que en el Capítulo 1, el resultado de regularidad permitió demostrar que el problema no relajado tiene solución en un conjunto abierto, acotado y simplemente conexo con borde simplemente conexo y suave si y solo si es una bola. Adicionalmente, basado en el método de la potencia y el método de optimización alternada desarrollado en el Capítulo 2, se desarrollo y demostró la convergencia de un algoritmo que converge a un punto crítico del problema. Análogamente al Teorema de convergencia (2.3), se estimó el error entre el problema finito dimensional (3.62) y el continuo (3.2).

Finalmente, los resultados obtenido a lo largo de esta tesis pueden ser usados como base para investigación futura. Por ejemplo, es interesante estudiar el caso cuando el material representado por el coeficiente β es mucho peor que el correspondiente a α. Matemáticamente, esto pasa cuando β tiende a infinito. Con respecto a esto, mediante cálculos formales se conjetura que el valor del problema relajado(1.7) converge al valor del problema dado por

$$\left\{\begin{array}{c}\displaystyle\min_{\theta,u}\left\{\frac{\alpha}{p}\int_\Omega\frac{|\nabla u|^p}{\theta^{p-1}}\mathrm{d}x-\langle f,u\rangle\right\}\\ u\in W_0^{1,p}(\Omega),\quad\theta\in L^\infty(\Omega;[0,1]),\quad\displaystyle\int_\Omega\theta\mathrm{d}x\leqslant\kappa.\end{array}\right.\tag{3.70}$$

Más aún, si este problema tiene un solución$(\theta,u)\in L^\infty(\Omega;[0,1])\times W_0^{1,p}(\Omega)$ tal que $\theta=\chi_\omega$ para un conjunto medible $\omega\subset\Omega$, entonces se debe tener que $\nabla u=0$ en $\Omega\setminus\omega$. Entonces, u debe ser constante en las componentes conexas de $\Omega\setminus\omega$. Esto significa que $\Omega\setminus\omega$ es rellenado con un material extremadamente rígido, lo cual es consistente con el hecho de que $\beta\to\infty$. Por los resultados obtenidos en los Capítulos 1 y 2 es esperable que se puedan obtener algunos resultados de regularidad para las soluciones de (3.70).

Otra extensión interesante es estudiar el caso cuando $p\to\infty$, lo cual corresponde a estudiar la mezcla de dos materiales perfectamente plásticos. Es importante observar que en este caso es necesario considerar una secuencia de coeficientes α_p y β_p tales que α_p^{1-p} y α_p^{1-p} son sucesiones convergentes. Argumentando como en [10] y [1] y asumiendo que α_p^{1-p} y β_p^{1-p} convergen a α_∞ y β_∞ cuando $p\to\infty$,se puede conjeturar que el valor del problema

$$\left\{\begin{array}{l}\displaystyle\min_{\theta,u}\left\{\frac{1}{p}\int_\Omega\frac{|\nabla u|^p}{(\alpha_p^{1-p}\theta+\beta_p^{1-p}(1-\theta))^{p-1}}\mathrm{d}x-\langle f,u\rangle\right\}\\ u\in W_0^{1,p}(\Omega),\quad\theta\in L^\infty(\Omega;[0,1]),\quad\displaystyle\int_\Omega\theta\mathrm{d}x\leqslant\kappa,\end{array}\right.$$

converge al valor de

$$\left\{\begin{array}{c}\displaystyle\min_{\theta,u}-\langle f,u\rangle\\ u\in W_0^{1,\infty}(\Omega),\quad\theta\in L^\infty(\Omega;[0,1]),\quad\displaystyle\int_\Omega\theta\mathrm{d}x\leqslant\kappa,\\ |\nabla u(x)|(\alpha_\infty\theta(x)+\beta_\infty(1-\theta(x)))\leqslant 1\ a.\mathrm{e.}\ x\in\Omega.\end{array}\right.\tag{3.71}$$

Por lo tanto el problema (3.71) puede ser visto como una versión homogeneizada de la ecuación del $\infty-$Laplacian. Por supuesto, la demostración rigurosa de esta convergencia necesita un análisis más profundo, sin embargo, el problema (3.71) hace posible estudiar la mezcla de dos materiales inelasticos caracterizados por las constantes α_∞ and β_∞, respectivamente.

Bibliography

[1] A. Addou, A. Lidouh, B. Seddoug. Elasto-plastic torsion problem as an infinity Laplace's equation. Electron. J. Differ. Equ. **2006**(156), 2006, 1—7.

[2] G. Allaire. *Homogenization and two-scale convergence.* SIAM J. Math. Anal. 23 (1992), 1482–1518.

[3] G. Allaire. *Shape optimization by the homogenization method.* Appl. Math. Sci. 146, Springer-Verlag, New York, 2002.

[4] A. Alvino, G. Trombetti, P.L. Lions. On optimization problems with prescribed rearrangements. *Nonlinear Anal.* **13**(2), 1989, 185–220.

[5] N. Aronszajn. A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. *J. Math. Pures Appl.* **36**(9), 1957, 235–249.

[6] M.S. Alnæs, J. Blechta, J. Hake, A. Johansson, B. Kehlet, A. Logg, C. Richardson, J. Ring, M.E. Rognes, G.N. Wells. *The fenics project version 1.5.* Archive of Numerical Software **3**(100), (2015).

[7] L. Blank, C. Rupprecht. *An extension of the projected gradient method to a Banach space setting with application in structural topology optimization.* SIAM J. Control Optim. **55**(3), 1481–1499 (2017).

[8] T. Bhattacharya, E. DiBenedetto, J. Manfredi *Limits as $p \to \infty$ of $\Delta_p u = f$ and related extremal problems* Rend. Sem. Mat. Univ. Politec. Torino (1989).

[9] J. M. Bernard. *Steady transport equation in the case where the normal component of the velocity does not vanish on the boundary.* SIAM J.Math. Anal. 44 (2012), 993-1018.

[10] M. Bocea, M. Mihăilescu *On a family of inhomogeneous torsional creep problems.* Proc. Amer. Math. Soc. **145** (2017), 4397-4409

[11] S.C. Brenner, L.R. Scott. *The mathematical theory. of finite element methods (third edition).* Text in Appl. Math. **15**, Springer, New York, 2008.

[12] J. Casado-Díaz. *Some smoothness results for the optimal design of a two-composite material which minimizes the energy.* Calc. Var. Partial Differ. Equ. 53 (2015), 649-673.

[13] J. Casado-Díaz. *Smoothness properties for the optimal mixture of two isotropic materials: The compliance and eigenvalue problems.* SIAM J. Control Optim. 53 (2015), 2319-2349.

[14] J. Casado-Díaz. *A characterization result for the existence of a two-phase material minimizing the first eigenvalue.* Ann. I.H. Poincaré-AN (2017), 1215-1226.

[15] J. Casado-Díaz, C. Conca, D. Vásquez-Varas. *The maximization of the p-Laplacian energy for a two-phase material. To appear.*

[16] J. Casado-Díaz, C. Conca, D. Vásquez-Varas. Numerical maximization of the p-Laplacian energy of a two phase material. *To appear.*

[17] A. Cherkaev, E. Cherkaeva. *Stable optimal design for uncertain loading conditions.* In *Homogenization: in memory of Serguei Kozlov*, ed. by V. Berdichevsky, V. Jikov, G. Papanicolau. Series on advances in math. for appl. sci. 50. World Scientific, Singapore, 1999, 193-213.

[18] M. Chipot, L. Evans. *Linearisation at infinity and Lipschitz estimates for certain problems in the calculus of variations.* Proc. Roy. Soc. Edinburgh A 102 (1986), 291-303.

[19] C. Conca, A.Laurain, R. Mahadevan. *Minimization of the groung state for two phase conductors in low constrast regime.* SIAM J. Appl. Math. 72 (2012), 1238-1259.

[20] C. Conca, R. Mahadevan, L. Sanz. *An extremal eigenvalue problem for a two-phase conductor in a ball.* Appl. Math. Optim. 60 (2009), 173-184.

[21] S. Cox, R. Lipton. Extremal eigenvalue problems for two-phase conductors. *Arch. Rational Mech. Anal.* **136**(2), 1996, 101–117.

[22] L. Damascelli, B. Sciunzi. *Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations.* J. Differential Equations, 206 (2004), 483–515.

[23] E. Di Benedetto. $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations.* Nonlinear Anal., 7 (1983), 827–850.

[24] J. García-Melián, J. Sabina de Lis. Maximum and comparison principles for operators involving the p-Laplacian. *J. Math. Anal. Appl.*, **218**(1), 1998, 49–65.

[25] M. Ghomi, *The problem of optimal smoothing for convex functions*, Proc. Amer. Math. Soc. 130 (2002), 2255–2259.

[26] D. Gilbarg, N.S. Trudinger. *Elliptic partial differential equations of second order.* Springer-Verlag, Berlin 2001.

[27] S. González-Andrade. *A preconditioned descent algorithm for variational inequalities of the second kind involving the p-Laplacian operator.* Comput. Optim. Appl. **66**(1), 123–162 (2017).

[28] J. Goodman, R.V. Kohn, L. Reyna. *Numerical study of a relaxed variational problem for optimal design.* Comput. Methods Appl. Mech. Engrg. 57 (1986), 107-127.

[29] Y.Q. Huang, R. Li, W. Liu. *Preconditioned descent algorithms for p-Laplacian.* J. Sci. Comput. **32**(2), 343–371 (2007).

[30] B. Kawohl. *On a family of torsional creep problems.* J. Reine Angew. Math. 410 (1990), 1-22.

[31] B. Kawohl, J. Stara, G. Wittum. *Analysis and numerical studies of a problem of shape design.* Arc. Rational Mech. Anal. 114 (1991), 343-363.

[32] A. Laurain. *Global minimizer of the ground state for two phase conductors in low contrast regime.* ESAIM Control Optim. Calc. Var. 20 (2014), 362-388.

[33] G.M. Lieberman.*Boundary regularity for solutions of degenerate elliptic equations.* Nonlinear Anal.Theor. Meth. Appl. 12 (1988), 1203–1219.

[34] A. Logg. *Efficient representation of computational meshes.* International J. Comput. Sci. Engrg. **4**(4), 283–295 (2009).

[35] H. Lou. *On singular sets of local solutions to p-Laplace equations.* Chin. Ann. Math. B, 29 (2008), 521–530.

[36] R. Glowinski, A. Marrocco. *Sur l'approximation, par éléments finis d'ordre un, et la résolution, par pénalisation-dualité d'une classe de problèmes de Dirichlet non linéaires.* Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér. **9** (R-2), 41–76 (1975).

[37] K. Matsue, H. Naito. Numerical studies of the optimization of the first eigenvalue of the heat diffusion in inhomogeneous media. *Japan. J. Ind. Appl. Math.* **32**(2), 2015, 489–512.

[38] G. Mingione. *Nonlinear Aspects of Calderón-Zygmund Theory.* Jahresber. Dtsch. Math.r-Ver. 112 (2010), 159–191.

[39] A. Mohammadi, M. Yousefnezhad. *Optimal groud state energy of two phase conductors.* Electron. J. Differ. Equ. 171 (2014), 1-8.

[40] S.A. Mohammadi, M. Yousefnezhad. Optimal ground state energy of two-phase conductors. *Electron. J. Diff. Eqns.* **171**, 2014, 8 pp.

[41] C.B. Morrey, Jr. *Multiple integral problems in the calculus of variations and related topics.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. 14 (1960), 1-61.

[42] F. Murat. *Un contre-exemple pour le problème du contrôle dans les coefficients.* C.R.A.S Sci. Paris A 273 (1971), 708-711.

[43] F. Murat. *Théorèmes de non existence pour des problèmes de contrôle dans les coeffi-*

cients. C.R.A.S Sci. Paris A 274 (1972), 395-398.

[44] F. Murat, L. Tartar. *Calcul des variations et homogénéisation*. In *Les méthodes de l'homogénéisation: Théorie et applications en physique*, Eyrolles, Paris, 1985, 319–369. English translation : F. Murat, L. Tartar. *Calculus of variations and homogenization.* In *Topics in the Mathematical Modelling of Composite Materials*, (L. Cherkaev, R.V. Kohn eds.), Birkhaüser, Boston (1998).

[45] F. Murat. *H-convergence*, Séminaire d'Analyse Fonctionnelle et Numérique, 1977-78, Université d'Alger, multicopied, 34 pp. English translation : F. Murat, L. Tartar, *H-convergence.* In *Topics in the Mathematical Modelling of Composite Materials*, (L. Cherkaev, R.V. Kohn eds.), Birkhaüser, Boston, 21–43 (1998).

[46] V. Chiadó Piat, G. Dal Maso, A. Defranceschi. *G-convergence of monotone operator*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **7** 3, 123–160(1990).

[47] P. Pucci, J. Serrin. The maximum principle. Progress in Nolinear Differential Equations and Their Applications. *Birkhäuser,* Basel, 2007.

[48] P.A. Raviart, J.M. Thomas. Introduction á l'analyse numérique des équations aus dérivées partielles. *Masson.* Paris, 1983.

[49] J. Serrin. *A symmetry problem in potential theory* Arch. Rational Mech. Anal. 43 (1971), 304-318.

[50] S. Spagnolo. *Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche.* Ann. Scuola Norm. Sup. Pisa Cl. Sci., 22 (1968), 571-597.

[51] G. Stampacchia. *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus.* Ann. Inst. Fourier, Grenoble 15 (1965), 189-258.

[52] L. Tartar. *The general theory of homogenization. A personalized introduction.* Springer, Berlin Heidelberger, 2009.

[53] A.M. Toader. *The convergence of an algorithm in numerical shape optimization. C. R. Acad. Sci. Paris Sér. I Math.*, **323**(11), 195–198 (1997).

[54] E. Zeidler. *Applied Functional Analysis: Main Principles and Their Applications.* Appl Math. Sci. 109, Springer-Verlag, New-York, 1991.