UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

ON DISCRETE CARLEMAN ESTIMATES: APPLICATIONS TO CONTROLLABILITY, STABILITY AND INVERSE PROBLEMS.

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA,
MENCIÓN MODELACIÓN MATEMÁTICA.

ARIEL ALONZO PÉREZ CONTRERAS

PROFESOR GUÍA:
JAIME ORTEGA PALMA

PROFESOR CO-GUÍA:
RODRIGO LECAROS LIRA

MIEMBROS DE LA COMISIÓN:
AXEL OSSES ALVARADO
CLAUDIO MUÑOZ CERÓN
MARIA DE LA LUZ DE TERESA DE OTEYZA
ENRIQUE ZUAZUA IRIONDO

Este trabajo ha sido parcialmente financiado por: Agencia Nacional de Investigación y Desarrollo (ANID), Beca de Doctorado Nacional Chile/2017 - 21170495.
Centro de Modelamiento Matemático (CMM) Proyectos ANID PIA AFB170001, BASAL ACE210010 , BASAL FB210005 y Proyecto Fondecyt 1201125, 11180874.

SANTIAGO DE CHILE
2022

ABSTRACT OF MEMORY TO OBTAIN THE DEGREE OF DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA
BY: ARIEL PÉREZ
DATE: 2022
ADVISOR: JAIME H. ORTEGA PALMA
CO-ADVISOR: RODRIGO LECAROS LIRA

ON DISCRETE CARLEMAN ESTIMATES: APPLICATIONS TO CONTROLLABILITY, STABILITY AND INVERSE PROBLEMS.

The aim of this thesis is the study of discrete and semi-discrete systems of Partial Differential Equations, via the finite-differences method, in order to understand some controllability, stability, and inverses problems in the discrete case. We focus on the conditions to ensure that the results from the continuous setting hold, in their respective discretization, since the usual approach based on the discretization of a controlled system or an inverse problem, via finite-differences, does not inherit the properties of the continuous.

We begin in Chapter 1 by introducing the issues that this thesis is about. In the first case we consider a unique continuation problem in the finite difference scheme setting. Then, we discuss the controllability formulation for a semi-discrete approximation of a parabolic controlled system. Finally, related with the discrete unique continuation, we state the discrete Calderon inverse problem for partial data.

Discrete calculus for uniform meshes are discussed in Chapter 2. We introduce the notation of discrete and semi-discrete meshes. Then, we set the discrete operator that allows us to approximate the partial differential operator considered in the next chapters. For the discrete difference and average operator we prove a discrete integration-by-parts formula. Finally, we establish fundamental estimate for several application of the discrete operator on the Carleman weight function. This is based on our works [14, 37].

In Chapter 3 we focus on the stability estimate for the semi-discrete linearized Benjamin-Bona-Mahony (BBM) equation, which is based on [37]. First, we study the continuous case. We obtain a stability estimate for the continuous linearized BBM equation, via Carleman estimate for the Laplacian operator, which implies a unique continuation property for this linearized equation. Then, following the continuous strategy, we prove a discrete Carleman estimate for a finite-difference approximation of the Laplacian operator with boundary observation. This yields a stability estimate for BBM equation when the space operator is discretized and the time is kept as a continuous variable (semi-discrete approximation case).

Based on [14], we apply in Chapter 4 the discrete calculus formulas for uniform meshes from Chapter 2 to establish a semi-discrete Carleman estimate for a semi-discrete fourth-order parabolic equation. As an application, following the Hilbert uniqueness method, we analyze the control/observation properties of space numerical approximation schemes of a linear fourth-order parabolic equation. These controllability results are uniform concerning the discretization parameter.

The discrete Calderon inverse problem with partial data is considered in Chapter 5. We extend the discrete calculus from Chapter 3 for arbitrary dimension. This, enables to prove a discrete Carleman estimate for Laplacian operator, defined on a family of non-uniform meshes obtained as the smooth image of a uniform grid, with boundary observations. This Chapter is based on [19].

The last Chapter is devoted to a brief discussion on some perspectives above the main results presented in this thesis.

RESUMEN DE LA TESIS PARA OPTARAL GRADO DE DOCTOR EN CIENCIASDE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA
POR:ARIEL ALONZO PÉREZ CONTRERAS
FECHA: 2022
PROF. GUÍA: JAIME ORTEGA PALMA Y RODRIGO LECAROS LIRA

SOBRE ESTIMACIONES DE CARLEMAN DISCRETAS: APLICACIONES A LA CONTROLABILIDAD, ESTABILIDAD Y PROBLEMAS INVERSOS.

El objetivo central de esta tesis es el estudio de sistemas discretos y semidiscretos de Ecuaciones en Derivadas Parciales, mediante el método de diferencias finitas, para entender de mejor forma algunos problemas de controlabilidad, estabilidad y problemas inversos en el caso discreto. Con este objetivo, nos centramos en analizar las condiciones necesarias para asegurar que los resultados del caso continuo sigan siendo válidas en su respectiva discretización, ya que el enfoque habitual basado en la discretización de un sistema controlado o un problema inverso, vía diferencias finitas, no necesariamente hereda las propiedades del caso continuo.

En el Capítulo 1 hacemos una breve introducción a los diferentes temas de esta tesis. En el primer caso consideramos un problema de continuación única en el entorno del esquema de diferencias finitas. Luego, discutimos la formulación de controlabilidad para una aproximación semidiscreta de un sistema controlado parabólico. Finalmente, planteamos el problema inverso discreto de Calderón para datos parciales.

El cálculo discreto para mallas uniformes se discute en el Capítulo 2. Introducimos la notación de mallas discretas y semidiscretas. Así, establecemos el operador discreto que nos permite aproximar el operador diferencial parcial considerado en los próximos capítulos. Para los operadores discreto de diferencia y promedio, probamos una fórmula discreta de integración por partes. Finalmente, establecemos una estimación fundamental para varias aplicaciones de los operadores discreto sobre la función de peso de Carleman. Estos resultados se basan en nuestros trabajos [14, 37].

En el Capítulo 3 nos centramos en la estimación de la estabilidad para la ecuación semidiscreta linealizada de Benjamin-Bona-Mahony (BBM), que se basa en [37]. En primer lugar, estudiamos el caso continuo. Obtenemos una estimación de estabilidad para la ecuación continua linealizada de BBM, a través de la estimación de Carleman para el operador laplaciano. A continuación, siguiendo la estrategia del caso continuo, demostramos una estimación discreta de Carleman para una aproximación por diferencia finita del operador Laplaciano con observación en la frontera. De esta forma se obtiene una estimación de estabilidad para la ecuación BBM semi-discreta.

Basándonos en [14], aplicamos en el Capítulo 4 las fórmulas de cálculo discreto para mallas uniformes del Capítulo 2 para establecer una estimación de Carleman semidiscreta para una ecuación parabólica de cuarto orden semidiscreta. Como aplicación, siguiendo el método de unicidad de Hilbert, analizamos las propiedades de control/observación de los esquemas de aproximación numérica espacial de una ecuación parabólica lineal de cuarto orden.

El problema inverso discreto de Calderón con datos parciales se considera en el Capítulo 5. Extendemos el cálculo discreto del Capítulo 3 para dimensión arbitraria. Esto, permite demostrar una estimación discreta de Carleman para el operador Laplaciano, definido en una familia de mallas no uniformes obtenidas como la imagen suave de una malla uniforme, con observaciones de borde. Los resultados de este capítulo se basan en el trabajo [19]. Finalmente, el Capítulo 6 está dedicado a una breve discusión sobre algunas perspectivas sobre los principales resultados presentados en esta tesis.

Acknowledgments

I would first like to thank my supervisors Jaime H. Ortega and Rodrigo Lecaros. I have been very lucky to work with them during the three years of my PhD and I have greatly benefited from their guidance and help. Their advice and encouraging support have been essential to obtaining the results presented in this thesis.

I would then like to thank the Department of Mathematics at Universidad de Chile, Centro de Modelamiento Matemático (CMM) Project ANID ACE210010 Basal FB210005, National Agency for Research and Development (ANID) Doctorado Nacional Chile/2017 - 21170495, and Fondecyt 1201125 for the finalcial support.

Finally, the most personal thoughts are heading towards my family and friends: thank you.

Contents

1. Introduction **1**
1.1. Unique continuation property . . . 1
1.2. Null-controllability of the heat equation . . . 4
1.2.1. The Hilbert Uniqueness Method and its penalised version . . . 6
1.2.2. $\phi(h)-$null controllability . . . 7
1.3. Discrete Calderón problem . . . 8

2. Discrete calculus for uniform meshes **10**
2.1. Some preliminaries on discrete formulations . . . 10
2.1.1. Definition of primal and dual meshes . . . 10
2.1.2. Definition of the discrete operator . . . 11
2.1.3. Discrete calculus formulas . . . 12
2.2. Some discrete calculus results . . . 14

3. Stability estimate for the semi-discrete linearized Benjamin-Bona-Mahony equation **22**
3.1. Discrete Carleman estimates with boundary observation . . . 23
3.2. Stability estimate for space semi-discrete BBM equation . . . 24
3.2.1. The continuous case . . . 25
3.3. Proof of the stability estimate . . . 25
3.3.1. Proof of Theorem 3.2 . . . 26
3.4. Discrete Carleman estimate . . . 28
3.4.1. The continuous case . . . 28
3.4.2. Proof of the discrete Carleman estimate . . . 30
3.4.2.1. Proof Theorem 3.1 . . . 30
3.5. Proof of intermediate results . . . 34
3.5.1. Proof of Lemma 3.2 . . . 34
3.5.2. Proof of Lemma 3.3 . . . 34
3.5.3. Proof of Lemma 3.4 . . . 36
3.5.4. Proof of Lemma 3.5 . . . 37
3.5.5. Proof of Lemma 3.6 . . . 39

4. Carleman estimates and controllability for a semi-discrete fourth-order parabolic equation **41**
4.1. Semi-discrete Carleman estimate for uniform meshes . . . 45
4.1.1. Conjugate operator . . . 45
4.1.2. An estimate for the left-hand side . . . 46
4.1.3. An estimate for the cross-term . . . 47

4.1.4. Conclusion 48
4.2. $\phi(h)-$null controllability 49
4.2.1. Observability inequalities 49
4.2.2. Controllability results 53
4.3. Appendix 55
4.3.1. Estimate of I_{11} and I_{41}. 55
4.3.2. Estimate of I_{12} 57
4.3.3. Estimate of I_{13} 59
4.3.4. Estimate of I_{14} 61
4.3.5. Estimate of I_{21} 62
4.3.6. Estimate of I_{22} 65
4.3.7. Estimate of I_{23} 67
4.3.8. Estimate of I_{24} 69
4.3.9. Estimate of I_{31} 71
4.3.10. Estimate of I_{32} 71
4.3.11. Estimate of I_{33} 73
4.3.12. Estimate of I_{34} 74
4.3.13. Estimate of I_{42} 75
4.3.14. Estimate of I_{43} 77
4.3.15. Estimate of I_{44} 79
4.3.16. Proof of Lemma 4.2 80
4.3.17. Proof of Lemma 4.3 81
4.3.18. Proof of Lemma 4.4 81

5. Discrete Carleman estimates for inverse problem with partial data 84
5.1. Some preliminaries 85
5.2. Carleman estimate with boundary observations 88
5.2.1. Carleman estimate 89
5.3. Application to an inverse problem with partial data 98

6. Some conclusions and commentaries 103
6.1. On Chapter 2 103
6.2. On Chapter 3 103
6.3. On Chapter 4 104

References 105

List of Figures

1.1. Geometry of system 1.2 2
1.2. Discrete propagation. 3
1.3. The subset ω does not contain any diagonal node. 6
2.1. Primal meshes. 11
2.2. Dual meshes. 11
5.1. Representation of the primal and dual meshes in dimension two. 86

Chapter 1

Introduction

This chapter briefly describes some ideas on the called Unique Continuation Property (UCP) and controllability for linear partial differential equations, and also we present the famous Calderon's Inverse Problem. Then, we focus on its discrete formulation using a finite difference scheme and the main difficulties facing these issues.

1.1. Unique continuation property

A Unique Continuation Property (UCP), for a linear partial differential operator P, is described by D. Tataru in [46] as an extension of the behavior of the solution of $Pu = 0$ from a smaller set to a bigger one. This property is useful to study control or inverse problems, where through measurement of a part of the domain, we try to control or recover some property of a partial differential equation solution. For instance, it is known (see [1] by G. Alessandrini et al.) that the uniqueness of the elliptic Cauchy problem is equivalent to the unique continuation property for

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ in } \omega, \end{aligned} \tag{1.1}$$

where $\omega \subset \Omega \subseteq \mathbb{R}^n$ is an open nonempty subset, that is, the only solution for the system (1.1) is the null function. As a consequence of this property for harmonic functions it follows that, for an open nonempty subset of the boundary $\gamma \subset \partial\Omega$, the unique solution of the system

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \gamma, \\ \partial_\nu u &= 0 \text{ on } \gamma, \end{aligned} \tag{1.2}$$

where $\partial_\nu u$ stands for the normal derivative of u, it is $u = 0$. We refer to Theorem 2.22 and Corollary 2.23 from [17] by M. Choulli for more details. In literature, this property is also known as data assimilation or propagation of smallness.

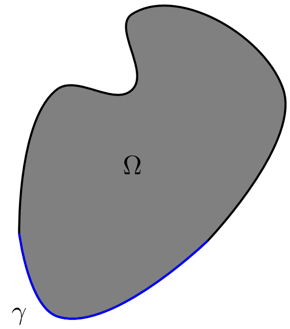


Figure 1.1: Geometry of system 1.2

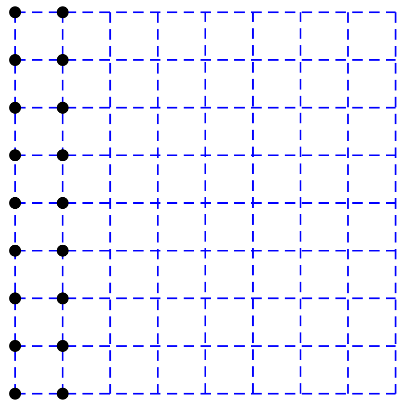
One could expect that the unique continuation property still holds for a discrete approximation of the problem (1.1) or (1.2). It is known that in general this is no true. Let us consider a naive example of this situation. Defining a uniform partition of the interval $(0,1) \subset \mathbb{R}$ by $\mathcal{N} := \{x_i \mid x_i := ih,\ i \in \{1,2,\ldots,N\}\}$, where $h := 1/(N+1)$ for $N \in \mathbb{N}$ given; we can consider a regular partition of the square $(0,1) \times (0,1)$ by $\mathcal{N} \times \mathcal{N}$. Thus, with the 5 point finite difference scheme, a discrete approximation of the Laplacian operator in $\mathcal{N} \times \mathcal{N}$ takes the form

$$\Delta_h u := \frac{1}{h^2} \left(u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i.j-1} - 4u_{i,j} \right).$$

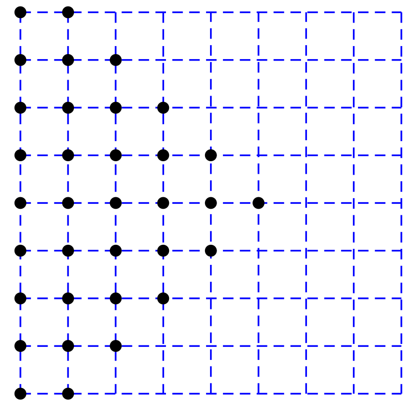
Therefore, denoting by $\overline{\mathcal{N}} := \mathcal{N} \cup \{0,1\}$ a discrete system similar to system (1.2) is

$$\begin{aligned} \Delta_h u &= 0 \text{ in } \mathcal{N} \times \mathcal{N}, \\ u &= 0 \text{ on } \{0\} \times \overline{\mathcal{N}} \\ u &= 0 \text{ on } \{h\} \times \overline{\mathcal{N}}. \end{aligned} \tag{1.3}$$

In the following figures, we can see the propagation of the boundary condition of the system (1.3). The Figure 1.2.a represents the nodes where the function u is equal to zero, and using that u is a discrete harmonic function ($\Delta_h u = 0$), Figure 1.2.b shows that this information does not propagate to the whole mesh.



(a) Boundary condition of the system (1.3)



(b) Propagation of boundary condition from system (1.3)

Figure 1.2: Discrete propagation.

In the case where a discrete harmonic function u is equal to zero on all the boundary nodes and also on a row or column one node next to the boundary, the function u will be zero in the whole mesh domain. This situation is proven by E. Zuazua and D. Chenais in [16].

Recently, there have been some results about the propagation of smallness for the discrete Laplacian operator. Using the representation of harmonic functions in the continuous setting, in [28] M. Guadie and E. Malinnikova prove a three-sphere inequality with correction term for discrete harmonic functions on the lattice. It is also shown that any discrete harmonic function on a cube could be extended through a discrete harmonic polynomial, representing another example where the unique continuation fails in the discrete setting. We might also mention the work [22] by A. Fernández-Bertolin et al. where, through a three sphere inequality for discrete magnetic Schrödinger operator, a three-sphere inequality with an additional term is established for finite difference approximation of Laplace operator. Both papers notice that the continuous three-sphere inequality is recovery, and therefore the unique continuation property holds when the discrete parameter goes to zero.

In [49], M. Yamamoto established a UCP for BBM-like equation

$$\partial_t u(x,t) - \partial_x^2 \partial_t u(x,t) = p(x,t)\partial_x u(x,t) + q(x,t)u(x,t), \quad (x,t) \in (0,1) \times (0,T), \tag{1.4}$$

where $p \in L^\infty((0,T) \times (0,1))$ and $q \in L^\infty(0,T; L^2(0,1))$. It was shown that the solution of (1.4) will vanish in $(0,1) \times (0,T)$, provided $u(1,t) = \partial_x u(1,t) = 0$, for all $t \in (0,T)$ and $u(x,0) = 0$ for $x \in (0,1)$. As in the harmonic case, it is not clear that this UCP still holds for a semi-discrete approximation in space of (1.4).

In Chapter 3, however, it is possible to obtain a quantitative UCP under the restriction over the mesh size for a semi-discrete linearized Benjamin-Bona-Mahony equation, via a stability estimate of the solution. The space semi-discrete approximation of equation (1.4) by using the centered finite difference method with respect to the space variable, for $i \in$

$\{1, 2, \dots, N\}$ and $t \in (0,T)$, is given by

$$\partial_t u_i(t) - \frac{\partial_t u_{i+1}(t) - 2\partial_t u_i(t) + \partial_t u_{i-1}(t)}{h^2} = p_i(t)\frac{u_{i+1}(t) - u_{i-1}(t)}{2h} + q_i(t)u_i(t),$$

where $N \in \mathbb{N}$; and the space discretization parameter is defined by $h := 1/(N+1)$. We consider the pairs (x_i, t) with $t \in (0,T)$, $T > 0$, and $x_i = ih$, for $i = 1, \dots, N$, thus, $u_i(t)$ stands for $u(x_i, t)$. The development of this Chapter combine the main ideas from [49] and [7], where for one side the result from [49] is refined and some estimates presented by F. Boyer et al. in [7] are extended, which are proved in Chapter 2, to apply the continuous strategy. It is worth to mention that, considering the previous examples, we exploit the 1-d case.

1.2. Null-controllability of the heat equation

In this Section, we will focus on the null-controllability for the heat equation. Let us consider $\Omega \subset \mathbb{R}^d$ a smooth bounded domain and $\omega \subset\subset \Omega$ a non-empty subset. Giving $T > 0$, the non-homogeneous heat equation in $Q := \Omega \times (0,T)$ with control supported in ω is giving by

$$\begin{cases} \partial_t y - \Delta y = v\chi_\omega, & \text{in } Q \\ y(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ y(x,0) = y^0(x), & \text{in } \Omega. \end{cases} \tag{1.5}$$

In (1.5), y stands for the state and χ_ω is the indicator function of the set ω. Denoting by

$$\mathcal{L}_T(v|y^0) = y_{v,y^0}(T),$$

the solution of the system (1.5) at time $T > 0$. We define, for $\delta \geq 0$, the (possibly empty) closed convex set

$$\mathrm{R}(y^0, \delta) := \left\{ v \in L^2(0,T;\omega) \mid \text{such that } \left\| \mathcal{L}_T(v|y^0) \right\|_{L^2(Q)} \leq \delta \right\}.$$

Definition 1.1

- *We say that Problem* (1.5) *is approximately null-controllable at time T, from the initial data $y^0 \in L^2(\Omega)$, if*
  $$R(y^0, \delta) \neq \emptyset, \ \forall \delta > 0.$$
  If this holds for any $y^0 \in L^2(\Omega)$, we simply say that problem (1.5) *is approximately null-controllable at time T.*

- *We say that problem* (1.5) *is null-controllable at time T from the initial data $y^0 \in L^2(\Omega)$, if*
  $$R(y^0, 0) \neq \emptyset.$$
  If this holds for any $y^0 \in L^2(\Omega)$, we simply say that the problem is null-controllable at time T.

The null-controllability of the system (1.5) was proved by G. Lebeau- L. Robbiano in [36], and independently by A. Fursikov- O. Yu. Imanuvilov in [26]. The precise statement is the following Theorem.

Theorem 1.1 ([26, 36]) *Let $\omega \neq \emptyset$ and $T > 0$. For all $y^0 \in L^2(\Omega)$, there exists a control $v \in L^2(Q)$ such that $y(T) = 0$ and $\|v\|_{L^2(Q)} \leq C \|y^0\|_{L^2(Q)}$, where $C > 0$ only depends on Ω, ω and T.*

Based on the strategy from [26], the null-controllability of the system (1.5) is equivalent to the observability inequality

$$\|q(x,T)\|_{L^2(\Omega)} \leq C \int_{\omega \times (0,T)} |q|^2, \tag{1.6}$$

for some constant $C > 0$, and q being solution of the following adjoint backward in time system of (1.5) defined by

$$\begin{cases} -\partial_t q - \Delta q = 0, & \text{in } Q, \\ q(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ q(x,T) = q^T(x), & \text{in } \Omega, \end{cases} \tag{1.7}$$

moreover, the control of the Theorem 1.1 is giving by $v = \chi_\omega(x) q(x,t)$.

A useful tool to prove observability inequalities are the called Carleman's estimates which are, for $q \in C_0^\infty(\Omega)$, a L^2-weighted estimates of the form

$$\|e^{s\varphi} \mathcal{P} q\|_{L^2(\Omega)} \geq C \|e^{s\varphi} q\|_{L^2(\Omega)},$$

where $\mathcal{P}$ is a differential operator, φ is called Carleman weight function and $s > 0$ is a large parameter. In 1939, T. Carleman introduced these type of energy weighted estimates to prove a UCP for second-order elliptic partial differential equation in [12], when coefficients fail to be analytic. Nowadays, it has become an efficient tool to prove UCP, to study controllability, observability, and stabilization for partial differential equations. We refer to the work of X. Fu et al. [25], and references therein (For instance the review of M. Yamamoto [50] or the work [17] by M. Choulli) where the authors present a unified approach to Carleman estimates for second-order partial differential equations and their applications to control theory and inverse problems.

Now, we consider a semi-discrete approximation in space of the system (1.5). We expect that the null-controllability still holds for this new semi-discrete system. Following the duality argument due to A.V. Fursikov and O.Y. Imanuvilov from [26], the observability inequality (1.6) should be hold for the respect semi-discrete approximation of the adjoint (1.7), but a counterexample due to O. Kavian, presented by E. Zuazua in [53], shows that the direct application of the continuous strategy does not work.

Being more precise, let us consider a regular partition of the square $(0,1) \times (0,1)$ defined by $\mathcal{N} \times \mathcal{N}$, where $\mathcal{N}$ is a regular partition of the interval $(0,1)$ given by $\mathcal{N} := \{x_i \mid x_i := ih,\ i \in \{1, 2, \ldots, N\}\}$ for $N \in \mathbb{N}$ and $h := 1/(N+1)$. Using the finite-difference scheme, the adjoint semi-discrete system of the heat equation can be written as

$$\begin{aligned} -\partial_t q - \Delta_h q &= 0 \quad \mathcal{N} \times \mathcal{N}, \\ q_{i,0} = q_{i,M+1} &= 0 \quad \forall x_i \in \mathcal{N}, \\ q_{0,j} = q_{N+1,j} &= 0 \quad \forall x_j \in \mathcal{M}. \end{aligned} \tag{1.8}$$

where we have used the five point approximation of the Laplacacian operator in uniform meshes given by

$$\Delta_h q := \frac{1}{h^2}\left(q_{i+1,j} + q_{i,j+1} + q_{i-1,j} + q_{i.j-1} - 4q_{i,j}\right).$$

We note that for

$$\tilde{q}_{i,j} := \begin{cases} 1, & i = j \text{ even} \\ 0, & i \neq j \\ -1, & i = j \text{ odd.} \end{cases}$$

It follows that

$$\Delta_h \tilde{q}_{i,j} = \begin{cases} -\frac{4}{h^2}, & i = j \text{ even} \\ 0, & i \neq j \\ \frac{4}{h^2}, & i = j \text{ odd.} \end{cases} = -\frac{4}{h^2}\tilde{q}_{i,j}.$$

Then $\tilde{q}_{i,j}$ is an eigenfunction of Δ_h. Thus, $\overline{q}_{i,j} := \exp\left(-4(T-t)/h^2\right)\tilde{q}_{i,j}$ solves (1.8) since $\partial_t \overline{q} = \frac{4}{h^2}\overline{q}_{i,j}$. Therefore, if any diagonal nodes do not belong to ω yields $\left\|\overline{q}_{i,j}\right\|_{L^2_h((\omega)\times(0,T))} = 0$, but the norm of $\overline{q}(x,0)$ has size $\exp(-C/h^2)$. This implies that a semi-discrete observability inequality like (1.6) is not possible for a semi-discrete in space approximation of the system (1.5). Figure (1.3) illustrates this situation.

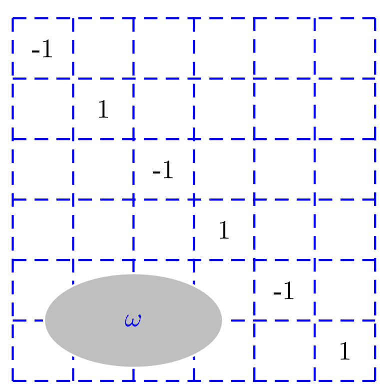


Figure 1.3: The subset ω does not contain any diagonal node.

1.2.1. The Hilbert Uniqueness Method and its penalised version

For the system (1.5), the Hilbert uniqueness method (HUM) consists in finding the control with the minimal $L^2(\omega \times (0,T))$-norm. In the case where $\mathrm{R}(y^0, 0) \neq \emptyset$, the control v^0 that minimize

$$F(v^0) = \inf_{v \in \mathrm{Adm}(y^0,0)} F(v), \tag{1.9}$$

where

$$F(v) := \frac{1}{2}\int_0^T \int_\omega |v|^2 \tag{1.10}$$

is called the HUM null-control associated with the initial data y^0. Usually, the minimization problem (1.9) is not solved directly since the space where the functional (1.10) is coercive, is challenging to describe. For this reason, it is convenient to deal with a penalised version of (1.10), that is, for any $\varepsilon > 0$ we define the following quadratic functional

$$F_\varepsilon(v) := \frac{1}{2}\int_0^T \int_\omega |v|^2 + \frac{1}{2\varepsilon}\left\| \mathcal{L}_T(v|y^0) \right\|^2_{L^2(Q)}, \forall v \in L^2(\omega \times (0,T)). \tag{1.11}$$

Now, our task is to minimize (1.11) onto the whole space $L^2(\omega \times (0,T))$. We note that for all $\varepsilon > 0$ the functional (1.11) is strictly convex, continuous and coercive then has a unique minimum onto $L^2(\omega \times (0,T))$. To find this minimum, applying the Fenchel-Rockafellar theory, we introduce for all $\varepsilon > 0$ the following functional

$$J_\varepsilon(q^T) := \frac{1}{2}\int_0^T \int_\omega |q|^2 + \frac{\varepsilon}{2}\int_\Omega |q^T|^2 + \int_\Omega q(x,0)y^0, \forall q^T \in L^2(\Omega), \tag{1.12}$$

where q is the solution of the system (1.7). Then, if v_ε and q_ε^T stand for the minimun of F_ε and J_ε respectively we have $F_\varepsilon(v_\varepsilon) = -J_\varepsilon(q_\varepsilon^T)$ (see Proposition 1.5 from [6]). Moreover, $v_\varepsilon(x,t) = \chi_\omega(x) q_\varepsilon^T(x,t)$ with $(x,t) \in \omega \times (0,T)$, and assuming that the observability inequality (1.6) holds it follows that $v_\varepsilon \longrightarrow v^0$ strongly in $L^2(\omega \times (0,T))$.

1.2.2. $\phi(h)-$null controllability

There is a uniform result due to A. Lopez and E. Zuazua presented in [39] for a semi-discretization in space of the 1D heat equation with constant coefficient using Fourier representation of the solutions, and in [54] for the 2D although some geometrical assumptions are needed in contrast to the continuous formulation. Moreover, E. Zuazua pointed out in [53] that this technique is not enough to achieve controllability results for semilinear heat equation or coefficients depending on time. Thus, mimicking the Fursikov-Imanuvilov strategy, discrete and semi-discrete Carleman estimates have been developed to obtain results in that direction.

It is well known, for instance the work developed by F. Boyer et al. [10], that we cannot expect the aforementioned classical notion of null-controllability since the semi-discrete system may not be even approximately controllable. Moreover, even if that property holds, it is very hard to prove some uniform behavior with respect to the discretization parameter h in order to say that our semi-discrete control problem approximates the continuous one. Thus, we are interested in the ϕ-controllability of the system (4.3), that is, to obtain uniformly bounded controls such that the norm of the semi-discrete solution at time T, $y(T)$, is approximately of the size $\sqrt{\phi(h)}$, where ϕ is a real-valued function that tend to zero when space discretization parameter tends to zero. This is done considering the parameter $\varepsilon := \phi(h)$ in the HUM penalised method. Thus, relaxed observability inequalities in the series of works [7, 9, 10, 35] by F. Boyer et al. it have been achieved to state relaxed ϕ-controllability results.

Several recent works have been concerned with discrete and semi-discrete Carleman estimates for second-order differential operators. The hyperbolic case has been developed for the one-dimensional case by L. Baudouin and S. Ervedoza [3], to study the stability of an inverse problem to recover a potential term in a semi-discrete wave equation in the one-dimensional setting. Then, in [4], this result is extended for arbitrary dimension. The elliptic case has been developed by F. Boyer et al. in [7] for the one-dimensional case to establish a relaxed observability estimate for the associated semi-discrete parabolic equation; and in [20] S. Ervedoza and F. de Gournay study the Laplacian operator in arbitrary dimension to prove the stability for the discrete Calderon problem, with limiting Carleman weight function. Semi-discrete Carleman estimates for parabolic operators have been established by F. Boyer and J. Le Rousseau in [10] for multidimensional Cartesian grids, moreover, in [42], T. N. T. Nguyen studied in the one-dimensional setting a semi-discrete parabolic operator with discontinuous diffusion coefficient, both of them obtain relaxed controllability results for their respective systems.

In the aforementioned works, the discretization was based on a finite difference scheme. Moreover, the Carleman parameter cannot be arbitrarily large, which is related to the discretization step size, in contrast to the continuous setting. Let us finally mentioned that recently in [30], a fully discrete Carleman estimates for parabolic operator have been obtained by V. Hernández-Santamaría and P. González Casanova, where the spatial and time discrete step-size parameters are connected to the Carleman parameter.

One of the main difficulties in the development of discrete or semi-discrete Carleman estimates is to compute multiple discrete operators such as D_h and A_h on the Carleman weight functions. For this reason, we establish Theorem 2.1 (see Section 2.2) to reduce some tedious computation, and it represents an extension of the results presented by F. Boyer et al. in [7] related to discrete estimate on the weight function. Semi-discrete estimates for the Carleman weight function are also needed in the development of the Chapter 4, these results are stated in the last two theorems from the Chapter 4, and also represent an extension of the estimated used to obtain the Carleman estimate in [10].

1.3. Discrete Calderón problem

The Carderón's problem, presented in [11] by A. Calderón, consists to determinate the electrical conductivity of a medium from measurements of the voltage and the current on the boundary. Its mathematical formulation is the following. Let us consider $\Omega \subset \mathbb{R}^d$ a smooth domain, with $d \geq 2$. Then, we consider the Dirichlet problem

$$\begin{cases} \operatorname{div}(\gamma \nabla u) + qu = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

where u stands for the voltage potential, and the coefficient γ is a positive definite symmetric matrix when the medium is anisotropic and scalar in the case of isotropic medium. Thus, the Calderón's problem is to determinate γ from boundary measurements of the voltage and the current on the boundary. To this end, we define the map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \longmapsto H^{-1/2}(\partial\Omega)$, known as Dirichlet-to-Neumann map

$$\Lambda_\gamma(f) := \gamma \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}, \tag{1.14}$$

where u is solution of (1.13) and $\dfrac{\partial u}{\partial \nu}$ is the normal derivative of u. Firstly, we expect that if two measurements coincide $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then $\gamma_1 = \gamma_2$ on the boundary and in the interior of the domain Ω, this is the uniqueness or identifiability result and it represents the injectivity of the Dirichlet-to-Neumann map (DN map). For a bounded C^1 domain $\Omega \subset \mathbb{R}^d$, with $d \geq 2$, it is known that for the positive functions $\gamma_1, \gamma_2 \in C(\overline{\Omega})$ such that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, it follows that $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$. To state a similar result although for Ω instead of $\partial\Omega$, we differentiate the case $d = 2$ and $d \geq 3$ due to the technique required. The uniqueness holds for $\gamma_1, \gamma_2 \in C^2(\Omega)$ if $d \geq 3$ and for $\gamma_1, \gamma_2 \in L^\infty(\Omega)$ when $d = 2$. We refer to [47] by G. Uhlmann, and references therein for details.

Furthermore, whether to measurements are similar is expected that its conductivities also be similar; this is its stability, which is the continuity of the inverse of the DN map where under some (optimal) assumptions is logarithmic. There are subcases concerned about whether the measurement is made on the whole boundary $\partial\Omega$ (full data) or any nonempty open subset of $\partial\Omega$ (local data). Focusing on $d \geq 3$, the full data case is studied by J. Sylvester and G. Uhlmann in [45], and the partial data case is considered by V. Isakov in [32] and by K. Knudsen and M. Salo in [34]. See also the survey [33], by C. Kening and M. Salo, for further references.

The results in dimension $d \geq 3$, in the isotropic case, are based on the density of products of solutions of the Schrödinger equation

$$\begin{cases} \Delta v + q' v = 0 & \text{in } \Omega \\ v = \gamma^{1/2} g & \text{on } \partial\Omega, \end{cases} \tag{1.15}$$

with $q' := \gamma^{-1} q + \gamma^{-1/2} \Delta(\gamma^{1/2})$. This is obtained considering the Liouville's transform, given by $v = \gamma^{1/2} u$, reducing the original system (1.13) to (1.15). This density is proved by complex geometrical optics (CGO) solutions where CGO solutions are constructed by suitable Carleman estimate.

Following this approach, for the full data case, in [20] S. Ervedoza and F. de Gournay considered a discrete approximation of the system (1.15) given by

$$\begin{cases} \Delta_h u_h + q_h u_h = 0 & \text{in } \mathring{\mathcal{W}}_h \\ u_h = g_h & \text{on } \partial\mathcal{W}_h. \end{cases} \tag{1.16}$$

Using a discrete Carleman estimate for discrete Laplace operator they construct discrete CGO solution, and then they prove a stability estimates for (1.16), uniformly with respect to the mesh size h. In Chapter 5, we consider the discrete Calderón problem with partial data. We prove a stability estimate under some illuminations conditions. Similar to [20], the proof is based on a discrete Carleman estimate with boundary observation.

Chapter 2

Discrete calculus for uniform meshes

This chapter presents the definition of the discrete sets that will be used throughout this thesis. Firstly, we define the discrete difference and average operators to approximate the continuous differential operator using the finite difference method. Then, for these discrete operators, we establish some calculus formulas mimicking the continuous one. Some of the most useful it is a discrete version of the classical integration-by-parts formulas, for both discrete operators. We note that the same results could be considered in the semi-discrete setting, due to the temporal variable does not play any significant role in the respective proofs. Since two of the following Chapters are in the one dimensional setting, we will focus the formulas for that case. It is worth to mention that all the results from this Chapter can be straightforward set in the multi-dimensional case, this fact will be consider in Chapter 5.

2.1. Some preliminaries on discrete formulations

In this section, we first introduce the notation of meshes and operators that will be used throughout this thesis. Then, we establish discrete calculus formulas, product rule and integration-by-parts for the discrete operators.

2.1.1. Definition of primal and dual meshes

We introduce the following regular partition of the interval $[0, 1]$ as

$$\mathcal{M} := \{x_i \mid x_i := ih,\ i = 0, 1, \ldots, N+1\},$$

for $N \in \mathbb{N}$ given and $h := 1/(N+1)$. We consider any sets of points $\mathcal{W} \subset \mathcal{M}$, then we define the following dual meshes $\mathcal{W}'$ and $\mathcal{W}^*$ as

$$\mathcal{W}' := \tau_+(\mathcal{W}) \cap \tau_-(\mathcal{W}), \quad \mathcal{W}^* := \tau_+(\mathcal{W}) \cup \tau_-(\mathcal{W}), \tag{2.1}$$

where

$$\tau_\pm(\mathcal{W}) := \left\{x \pm \frac{h}{2} \mid x \in \mathcal{W}\right\}.$$

For repeated computation of this sets we denote

$$\overline{\mathcal{W}} = \mathcal{W}^{**} := (\mathcal{W}^*)^*$$

and

$$\mathring{\mathcal{W}} = \mathcal{W}'' := (\mathcal{W}')'.$$

We note that if $\overline{\mathring{\mathcal{W}}} = \mathcal{W}$, then for two consecutive points $x_i, x_{i+1} \in \mathcal{W}$ we have $x_{i+1} - x_i = h$. Thus, any subset $\mathcal{W} \subset \mathcal{M}$ that verifies $\overline{\mathring{\mathcal{W}}} = \mathcal{W}$ will be called regular mesh. Finally, we define the boundary of a regular mesh $\mathcal{W}$ as $\partial\mathcal{W} := \overline{\mathcal{W}} \setminus \mathcal{W}$. These sets defined above can be seen more clearly in Figure 2.1 and Figure 2.2.

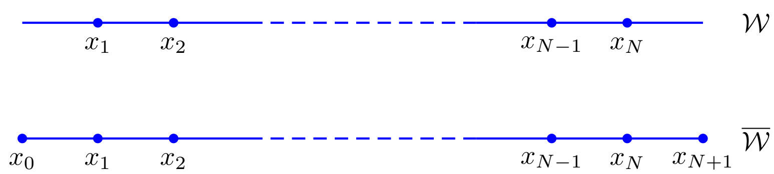


Figure 2.1: Primal meshes.

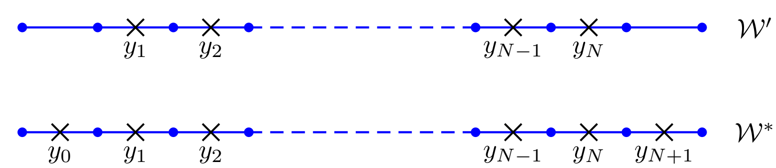


Figure 2.2: Dual meshes.

We introduce, using (2.1), the semi-discrete sets. Let us consider $T > 0$, we define $Q := \mathcal{W} \times (0, T)$. We also define the dual semi-discrete sets by $Q' := \mathcal{W}' \times (0, T)$ and $Q^* := \mathcal{W}^* \times (0, T)$. Similarly, the semi-discrete boundary is given by $\partial Q = \partial\mathcal{W} \times (0, T)$.

2.1.2. Definition of the discrete operator

We define the average operator A_h and the difference operator D_h by

$$A_h(u)(x, t) := \frac{\tau_+ u(x, t) + \tau_- u(x, t)}{2},$$

$$D_h(u)(x, t) := \frac{\tau_+ u(x, t) - \tau_- u(x, t)}{h},$$

where $\tau_\pm u(x, t) := u\left(x \pm \frac{h}{2}, t\right)$.

We denote by $C(Q)$ the set of real-valued functions defined in Q. For $u \in C(Q)$, we define its $L_h^\infty(Q)$-norm as

$$\|u\|_{L_h^\infty(Q)} := \max_{(x,t)\in Q} \{|u(x,t)|\} .$$

To introduce the boundary conditions, we define the outward normal for $(x,t) \in \partial Q$ as

$$n_h(x,t) := \begin{cases} 1 & (\tau_-(x),t) \in Q_h^* \text{ and } (\tau_+(x),t) \notin Q_h^*, \\ -1 & (\tau_-(x),t) \notin Q_h^* \text{ and } (\tau_+(x),t) \in Q_h^*, \\ 0 & \text{otherwise.} \end{cases}$$

We indicate by ∂Q^+ (resp. ∂Q^-) the set of points such that $n_h(x,t) = 1$ (resp. $n_h(x,t) = -1$), we also introduce the trace operator for $u \in C(Q_h^*)$ as

$$\forall (x,t) \in \partial Q,\ t_r(u) := \begin{cases} \tau_- u(x,t) & n_h(x,t) = 1, \\ \tau_+ u(x,t) & n_h(x,t) = -1, \\ 0 & n_h(x,t) = 0. \end{cases}$$

2.1.3. Discrete calculus formulas

Now, we will present some important results in our discrete operators. Let us consider a regular mesh $\mathcal{W}$. Let us also recall that $C(\mathcal{W})$ is the set of function from $\mathcal{W}$ to $\mathbb{R}$. For the difference and average operator we have the following properties.

Lemma 2.1 ([20], Lemma 2.1) *For any $u, v \in C(\mathcal{W})$, we have for the difference operator*

$$D_h(u\,v) = D_h u\, A_h v + A_h u\, D_h v, \ \text{on } \mathcal{W}^*, \tag{2.2}$$

Similarly, the average of the product gives

$$A_h(u\,v) = A_h u\, A_h v + \frac{h^2}{4} D_h u\, D_h v, \ \text{on } \mathcal{W}^*. \tag{2.3}$$

Finally, on $\mathring{\mathcal{W}}$ we have

$$u = A_h^2 u - \frac{h^2}{4} D_h^2 u. \tag{2.4}$$

As a direct consequence of Lemma 2.1 we have the following result.

Corollary 2.1 *Let $\mathcal{W} \subseteq \mathcal{M}$ be a regular mesh.*

- *For $u \in C(\mathcal{W})$,*

$$A_h(u^2) = (A_h u)^2 + \frac{h^2}{4} (D_h u)^2 , \text{on } \mathcal{W}^*. \tag{2.5}$$

In particular, for all $u \in C(\mathcal{W})$,

$$A_h(u^2) \geq (A_h u)^2 , \ \text{on } \mathcal{W}^*. \tag{2.6}$$

- *For* $u \in (\mathcal{W})$
$$D_h(u^2) = 2D_h u\, A_h u. \tag{2.7}$$

Now, for a regular set $\mathcal{W} \subseteq \Omega$, we define the discrete integral for $u \in C(\mathcal{W})$ as

$$\int_{\mathcal{W}} u := h^d \sum_{x \in \mathcal{W}} u(x),$$

and the following L^2 inner product in $C(\mathcal{W})$

$$\langle u, v\rangle_{\mathcal{W}} := \int_{\mathcal{W}} u\, v, \quad u\,, v \in C(\mathcal{W}),$$

with the associated norm
$$\|u\|_{L^2(\mathcal{W})} := \sqrt{\langle u, u\rangle_{\mathcal{W}}}.$$
Let us finally introduce the discrete integration on the boundary for $u \in C(\partial\mathcal{W})$ as

$$\int_{\partial\mathcal{W}} u := \sum_{x \in \partial\mathcal{W}} u(x).$$

The previous definition can be considered for semi-discrete domains $Q := \mathcal{W} \times (0, T)$ since the temporal variable does not play any mayor role. Thus, following the notation previously introduced we establish a discrete integral by parts formula for the discrete average and difference operator.

Proposition 2.1 *Let Q be a semi-discrete regular mesh. For $u \in C(\overline{Q})$ and $v \in C(Q^*)$ we have*
$$\int_Q u\, D_h v = -\int_{Q^*} D_h u\, v + \int_{\partial Q} u\, t_r(v) n \tag{2.8}$$
and
$$\int_Q u\, A_h v = \int_{Q^*} A_h u\, v - \frac{h}{2}\int_{\partial Q} u\, t_r(v). \tag{2.9}$$

Proof. Since the temporal variable does not play any significant role, the proof is based on the discrete space setting from [37, Proposition 2.4]. □

Let us note that the main difference with the results by F. Boyer et al. in the works [9, 10] it is that we do not make any distinction of the discrete operator. Indeed, in those papers Boyer at al. define a difference and average operator for the dual and primal mesh. In our case, we indicate the specific meshes where the integration is considered.

Finally, we establish some semi-discrete integration by parts formulas involving second-order discrete operators.

Corollary 2.2 *Let Q be a semi-discrete regular mesh. For $u, v \in C(\overline{Q})$ we have*

$$\int_Q u\, D_h^2 v = \int_{\overline{Q}} v\, D_h^2 u - \int_{\partial Q^*} D_h u\, t_r(v) n + \int_{\partial Q} u\, t_r(D_h v)\, n,$$

$$\int_Q u\, A_h^2 v = \int_{\overline{Q}} v\, A_h^2 u - \frac{h}{2}\int_{\partial Q^*} A_h u\, t_r(v) - \frac{h}{2}\int_{\partial Q} u\, t_r(A_h v),$$

and

$$\int_Q u\, D_h A_h v = -\int_{\overline{Q}} v\, D_h A_h u + \frac{h}{2}\int_{\partial Q^*} D_h u\, t_r(v) + \int_{\partial Q} u\;\, t_r(A_h v) n$$
$$= -\int_{\overline{Q}} v\, D_h A_h u - \frac{h}{2}\int_{\partial Q} u\, t_r(D_h v) + \int_{\partial Q^*} A_h u\, t_r(v) n.$$

Proof. Repeated application of Proposition 2.1 enables us to write the claimed semi-discrete integral formulas. ☐

2.2. Some discrete calculus results

In this section, we establish some previous estimates for the Carleman weight function that will be used in the thesis to obtain a discrete and semi-discrete Carleman estimate. Recall that our weight function is defined as $e^{s\varphi}$ for $s \geq 1$, with $\varphi = e^{\lambda\psi}$, where $\psi \in C^k$ for k sufficiently large and $\lambda \geq 1$. Our goal is to generalize the results presented previously in Section 3, obtained by F. Boyer et al. in [7], related to discrete operations performed on the Carleman weight functions, considering estimates and expansions for higher order discrete operators.

For easier comparison, we use the same notation by setting $r = e^{s\varphi}$ and $\rho = r^{-1}$, these positive parameters s and h will be large and small respectively and limited by the condition $sh \leq 1$. The proofs are similar in spirit to those given in [7].

We denote by $\mathcal{O}_\lambda(sh)$ the functions that verify

$$\|\mathcal{O}_\lambda(sh)\|_{L^\infty(Q_h)} \leq C_\lambda sh,$$

for some constant C_λ depending on λ. By $\mathcal{O}(1)$ we denote bounded functions and by $\mathcal{O}_\lambda(1)$ a bounded function once λ is fixed.

We say that α is a multi-index if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ and for $y \in \mathbb{R}^n$ we write:

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \quad \partial^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}, \quad y^\alpha = y_1^{\alpha_1} \ldots y_n^{\alpha_n}.$$

Proposition 2.2 *Let us consider $n \in \mathbb{N}$. Let f be a $(n+2)-$times differentiable and g a twice differentiable functions on $\mathbb{R}$. Then*

$$A_h^n g = g + R_{A_h^n}(g),$$
$$D_h^n f = f^{(n)} + R_{D_h^n}(f).$$

where $R_{D_h^n}$ and $R_{A_h^n}$ are given by

$$R_{D_h^n}(f) := h^2 \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{(n-2k)}{2}\right)^{n+2} \int_0^1 \frac{(1-\sigma)^{n+1}}{(n+1)!} f^{(n+2)}(\cdot + \frac{(n-2k)h}{2}\sigma) d\sigma$$

and

$$R_{A_h^n}(g) := \frac{h^2}{2^{n+2}} \sum_{k=0}^{n} \binom{n}{k} (n-2k)^2 \int_0^1 (1-\sigma) g^{(2)}(\cdot + \frac{(n-2k)h}{2}\sigma) d\sigma.$$

PROOF. The proof of this proposition follows from Taylor expansion

$$g(x+y)=\sum_{j=0}^{i-1}\frac{y^j}{j!}h^{(j)}(x)+y^i\int_0^1\frac{(1-\sigma)^{i-1}}{(i-1)!}g^{(i)}(x+\sigma y)d\sigma. \tag{2.10}$$

First, we use (2.10) with $i=2$ and $y=\frac{(n-2k)h}{2}$ to obtain

$$\tau_+^{n-2k}g=g+\frac{(n-2k)h}{2}g'+\left(\frac{(n-2k)h}{2}\right)^2\int_0^1(1-\sigma)g^{(2)}(\cdot+\frac{(n-2k)h}{2}\sigma)d\sigma.$$

Then, it follows that

$$\begin{aligned}A_h^n g=&\frac{1}{2^n}\left(\tau_++\tau_-\right)^n g\\ =&\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\tau_+^{n-k}\tau_-^k g\\ =&\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\tau_+^{n-2k}g\\ =&\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(g+\frac{(n-2k)h}{2}g'\right)+R_{A^n}(g)\end{aligned}$$

Now, using $\sum_{k=0}^{n}\binom{n}{k}=2^n$ and $\sum_{k=0}^{n}\binom{n}{k}k=n2^{n-1}$ we write

$$A_h^n g=g+R_{A_h^n}(g).$$

On the other hand, applying (2.10) for f with $i=n+2$ and $y=\frac{(n-2k)h}{2}$, we have

$$\tau_+^{n-2k}f=\sum_{j=0}^{n+1}\frac{1}{j!}\left(\frac{(n-2k)h}{2}\right)^j f^{(j)}+\left(\frac{(n-2k)h}{2}\right)^{n+2}\int_0^1\frac{(1-\sigma)^{n+1}}{(n+1)!}f^{(n+2)}(\cdot+\frac{(n-2k)h}{2}\sigma)d\sigma.$$

Thus, for the difference operator we get

$$
\begin{aligned}
D_h^n f =& \frac{1}{h^n}\left(\tau_+ - \tau_-\right)^n f \\
=& \frac{1}{h^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k \tau_+^{n-k}\tau_-^k f \\
=& \frac{1}{h^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k \tau_+^{n-2k} f \\
=& \frac{1}{h^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k \sum_{j=0}^{n+1}\frac{1}{j!}\left((n-2k)\frac{h}{2}\right)^j f^{(j)} \\
& + R_{D_h^n}(f) \\
=& \frac{1}{h^n}\sum_{j=0}^{n+1}\frac{1}{j!}h^j \sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{n}{2}-k\right)^j f^{(j)} + R_{D_h^n}(f)
\end{aligned}
$$

Now, using $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^n = n!$ and $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\frac{n}{2}-k)^{n+1} = 0$, we obtain

$$
\begin{aligned}
D_h^n f =& \frac{1}{h^n}\frac{1}{(n)!}h^n \sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{n}{2}-k\right)^n f^{(n)} \\
& + \frac{1}{h^n}\frac{1}{(n+1)!}h^{n+1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{n}{2}-k\right)^{n+1} f^{(n+1)} + R_{D^n}(f) \\
=& f^{(n)} + R_{D_h^n}(f),
\end{aligned}
$$

and the proof is complete. □

Corollary 2.3 *Let f be a $(n+4)-$times differentiable function defined on $\mathbb{R}$ and $m, n \in \mathbb{N}$, then*

$$
A_h^m D_h^n f = f^{(n)} + R_{A_h^m}(f^{(n)}) + R_{D_h^n}(f) + R_{A_h^m D_h^n}(f),
$$

where

$$
R_{A_h^m D_h^n}(f) := \sum_{k,k'=0}^{n,m} a_{k,k'}\int_0^1\int_0^1 \frac{(1-\sigma)^{n+1}}{(n+1)!}(1-\sigma')f^{(n+4)}(x+(n-2k)\frac{h}{2}\sigma + \frac{(m-2k')h}{2}\sigma')d\sigma' d\sigma
$$

with

$$
a_{k,k'} := \frac{h^4}{2^m}\binom{m}{k'}\binom{n}{k}(-1)^k\left(\frac{(n-2k)}{2}\right)^{n+2}\frac{(m-2k')^2}{4}.
$$

Proof. It is enough to see that,

$$
\begin{aligned}
A_h^m(D_h^n f) =& A_h^m f^{(n)} + A_h^m(R_{D_h^n}(f)) \\
=& f^{(n)} + R_{A_h^m}(f^{(n)}) + R_{D_h^n}(f) + R_{A_h^m D_h^n}(f).
\end{aligned}
$$

□

Note that $R_{A_h^m D_h^n} = R_{A_h^m} \circ R_{D_h^n} = R_{D_h^n} \circ R_{A_h^m}$. Now, we consider two fundamental estimates for our weight function. The proofs of these results can be found in [7]. We consider $\alpha = (\alpha_t, \alpha_x) \in \mathbb{N}^2$ multi-indices.

Lemma 2.2 *Let α and β be multi-indices. We have*

$$\begin{aligned}\partial^\beta(r\partial^\alpha\rho) =& |\alpha|^{|\beta|}(-s\varphi)^{|\alpha|}\lambda^{|\alpha+\beta|}(\partial_x\psi)^{\alpha+\beta} \\ &+ |\alpha||\beta|(s\varphi)^{|\alpha|}\lambda^{|\alpha+\beta|-1}\mathcal{O}(1) + s^{|\alpha|-1}|\alpha|(|\alpha|-1)\mathcal{O}_\lambda(1) \\ =& \mathcal{O}_\lambda(s^{|\alpha|}).\end{aligned} \tag{2.11}$$

Moreover, let $\sigma \in [0,1]$ and $sh \leq 1$, then $\partial^\beta(r(x)(\partial^\alpha\rho)(x+\sigma h)) = s^{|\alpha|}\mathcal{O}_\lambda(1)$.

Corollary 2.4 *Let α, β and δ be multi-indices. We have*

$$\begin{aligned}\partial^\delta\left(r^2(\partial^\alpha\rho)\partial^\beta\rho\right) =& |\alpha+\beta|^{|\delta|}(-s\varphi)^{|\alpha+\beta|}\lambda^{|\alpha+\beta+\delta|}(\partial_x\psi)^{\alpha+\beta+\delta} \\ &+ |\delta||\alpha+\beta|(s\varphi)^{|\alpha+\beta|}\lambda^{|\alpha+\beta+\delta|-1}\mathcal{O}(1) \\ &+ s^{|\alpha+\beta|-1}\left(|\alpha|(|\alpha|-1) + |\beta|(|\beta|-1)\right)\mathcal{O}_\lambda(1) \\ =& \mathcal{O}_\lambda(s^{|\alpha+\beta|}).\end{aligned}$$

Corollary 2.3 and Lemma 2.2 yield.

Proposition 2.3 *Let α be a multi-index and $n, m \in \mathbb{N}$. Provided $sh \leq 1$, we have*

$$rA_h^m D_h^n \partial^\alpha\rho = r\partial_x^n\partial^\alpha\rho + s^{|\alpha|+n}\mathcal{O}_\lambda((sh)^2) = s^{|\alpha|+n}\mathcal{O}_\lambda(1).$$

Proof. From Corollary 2.3 we write

$$rA_h^m D_h^n\partial^\alpha\rho = r\partial_x^n\partial^\alpha\rho + rR_{A_h^m}(\partial_x^n\partial^\alpha\rho) + rR_{D_h^n}(\partial^\alpha\rho) + rR_{A_h^m D_h^n}(\partial^\alpha\rho)$$

By Lemma 2.2 we have

$$r(x)(\partial_x^{n+2}\partial^\alpha\rho)(x+(n-2k)h\sigma/2) = \mathcal{O}_\lambda(s^{|\alpha|+n+2})$$

and

$$r(x)\partial_x^{n+4}\partial^\alpha\rho(x+(n-2k)h\sigma/2) = \mathcal{O}_\lambda(s^{|\alpha|+n+4}).$$

Then

$$\begin{aligned}rR_{A_h^m}(\partial_x^n\partial^\alpha\rho) &= s^{|\alpha|+n}\mathcal{O}_\lambda((sh)^2), \\ rR_{D_h^n}(\partial^\alpha\rho) &= s^{|\alpha|+n}\mathcal{O}_\lambda((sh)^2), \\ rR_{A_h^m D_h^n}(\partial^\alpha\rho) &= s^{|\alpha|+n}\mathcal{O}_\lambda((sh)^4),\end{aligned}$$

which yields the result. □

Lemma 2.3 *Let α and β multi-index and $n \in \mathbb{N}$. Provided $sh \leq 1$, we have*

$$A_h^m D_h^n\left(\partial^\beta\left(r\partial^\alpha\rho\right)\right) = \partial_x^n\partial^\beta\left(r\partial^\alpha\rho\right) + h^2\mathcal{O}_\lambda(s^{|\alpha|})$$

Let $\sigma \in [0,1]$, we have $A_h^m D_h^n \partial^\beta \left(r(x) \partial^\alpha \rho(x+\sigma h) \right) = \mathcal{O}_\lambda(s^{|\alpha|})$.

Proof. By Corollary 2.3 we write

$$\begin{aligned} A_h^m D_h^n(\partial^\beta(r\partial^\alpha\rho)) =& \partial_x^n \partial^\beta(r\partial^\alpha\rho) + R_{A_h^m}(\partial_x^n\partial^\beta(r\partial^\alpha\rho)) \\ &+ R_{D_h^n}(\partial^\beta(r\partial^\alpha\rho)) + R_{A_h^m D_h^n}(\partial^\beta(r\partial^\alpha\rho)). \end{aligned}$$

By using Lemma 2.2, we have

$$\begin{aligned} (\partial_x^n\partial^\beta(r\partial^\alpha\rho))(x+(n-2k)h\sigma/2) &= \mathcal{O}_\lambda(s^{|\alpha|}) \\ (\partial_x^{n+2}\partial^\beta(r\partial^\alpha\rho))(x+(n-2k)h\sigma/2) &= \mathcal{O}_\lambda(s^{|\alpha|}) \\ (\partial_x^{n+4}\partial^\beta(r\partial^\alpha\rho))(x+(n-2k)h\sigma/2) &= \mathcal{O}_\lambda(s^{|\alpha|}), \end{aligned}$$

which concludes the proof of the first result.

On the other hand, we set $\nu(x,\sigma h) := r(x)\rho(x+\sigma h)$ and $\mu_\alpha := r\partial^\alpha\rho$. Since $r\rho = 1$ it follows that $r(x)\partial^\alpha\rho(x+\sigma h) = \nu(x,\sigma h)\mu_\alpha(x+\sigma h)$. Note that, by continuous Leibniz rule, $\partial_x^n\partial^\beta(\nu\mu_\alpha)$ is a linear combination of terms of the form $\partial^{\beta'}\nu\partial^{\beta''}\mu_\alpha$, with $\beta'+\beta'' = n+\beta$ and by Lemma 2.2 we write $\partial^{\beta'}\nu = \mathcal{O}_\lambda(1)$ and $\partial^{\beta''}\mu_\alpha = \mathcal{O}_\lambda(s^{|\alpha|})$. Besides, it holds for the terms $\partial_x^{n+2}\partial^\beta(\nu\mu_\alpha)$ and $\partial_x^{n+4}\partial^\beta(\nu\mu_\alpha)$ as well. Therefore, applying Corollary 2.3 to $\nu\mu_\alpha$ we obtain

$$A_h^m D_h^n(\partial^\beta(r(x)\partial^\alpha\rho(x+\sigma h))) = \mathcal{O}_\lambda(s^{|\alpha|}) + h^2\mathcal{O}_\lambda(s^{|\alpha|}),$$

and the proof is complete. □

Lemma 2.4 *Let α, β, δ be multi-indices and $n,m \in \mathbb{N}$. Provided $sh \leq 1$, we have:*

$$\begin{aligned} A_h^m D_h^n \partial^\delta\left(r^2\left(\partial^\alpha\rho\right)\partial^\beta\rho\right) =& \partial_x^n\partial^\delta\left(r^2\left(\partial^\alpha\rho\right)\partial^\beta\rho\right) + h^2\mathcal{O}_\lambda(s^{|\alpha|+|\beta|}) \\ =& \mathcal{O}_\lambda(s^{|\alpha|+|\beta|}). \end{aligned}$$

Let $\sigma, \sigma' \in [0,1]$. We have

$$A_h^m D_h^n \partial^\delta\left(r(x)^2\left(\partial^\alpha\rho(x+\sigma h)\right)\partial^\beta\rho(x+\sigma' h)\right) = \mathcal{O}_\lambda(s^{|\alpha|+|\beta|}).$$

Proof. Applying Corollary 2.3 to $\partial^\delta\left(r^2\left(\partial^\alpha\rho\right)\partial^\beta\rho\right)$ we obtain

$$\begin{aligned} A_h^m D_h^n \partial^\delta\left(r^2\left(\partial^\alpha\rho\right)\partial^\beta\rho\right) =& \partial_x^n\partial^\delta\left(r^2\left(\partial^\alpha\rho\right)\partial^\beta\rho\right) \\ &+ R_{A_h^m}\left(\partial_x^n\partial^\delta\left(r^2\left(\partial^\alpha\rho\right)\partial^\beta\rho\right)\right) \\ &+ R_{D_h^n}\left(\partial^\delta\left(r^2\left(\partial^\alpha\rho\right)\partial^\beta\rho\right)\right) \\ &+ R_{A_h^m D_h^n}\left(\partial^\delta\left(r^2\left(\partial^\alpha\rho\right)\partial^\beta\rho\right)\right). \end{aligned}$$

Then, the first result follows from Corollary 2.4. For the second one, we proceed similarly as the proof of the second result of Lemma 2.3, that is, we apply Corollary 2.3 to $\nu^2\mu_\alpha\mu_\beta$, then we use continuous Leibniz rule and Lemma 2.3 to conclude. □

Lemma 2.5 *Let α be a multi-index. For $j, k, m, n \in \mathbb{N}$ and for $sh \le 1$, we have*

$$A_h^j D_h^k \partial^\alpha \left(r A_h^m D_h^n \rho \right) = \partial_x^k \partial^\alpha (r \partial_x^n \rho) + s^n \mathcal{O}_\lambda \left((sh)^2 \right) = s^n \mathcal{O}_\lambda(1).$$

Proof. By Corollary 2.3 we write

$$\partial^\alpha (r A_h^m D_h^n \rho) = \partial^\alpha \left(r \partial_x^n \rho) + \partial^\alpha (r R_{A_h^m}(\partial_x^n \rho) \right) + \partial^\alpha \left(r (R_{D_h^n}(\rho)) + \partial^\alpha (r R_{A_h^m D_h^n}(\rho) \right).$$

Then, applying again Corollary 2.3 to the first term of the above expression, we have

$$\begin{aligned} A_h^j D_h^k \partial^\alpha (r A_h^m D_h^n \rho) =& A_h^j D_h^k \partial^\alpha (r \partial_x^n \rho) + A_h^j D_h^k \partial^\alpha (r R_{A_h^m}(\partial_x^n \rho)) \\ &+ A_h^j D_h^k \partial^\alpha (r (R_{D_h^n}(\rho)) + A_h^j D_h^k \partial^\alpha (r R_{A_h^m D_h^n}(\rho)) \\ =& \partial_x^k \partial^\alpha (r \partial_x^n \rho) + R_{A_h^j}(\partial^\alpha (r \partial_x^n \rho)) + R_{D_h^k}(\partial^\alpha (r \partial_x^n \rho)) \\ &+ A_h^j D_h^k \partial^\alpha (r (R_{D_h^n}(\rho)) + A_h^j D_h^k \partial^\alpha (r R_{A_h^m D_h^n}(\rho)). \end{aligned}$$

Thus, by Lemma 2.2, we obtain

$$A_h^j D_h^k \partial^\alpha (r A_h^m D_h^n \rho) = \partial_x^k \partial^\alpha (r \partial_x^n \rho) + s^n \mathcal{O}_\lambda ((sh)^2),$$

which is the desired result. □

Theorem 2.1 *Let α, β be multi-indices and $j, k, l, m, n, p \in \mathbb{N}$. Provided $sh \le 1$, we have*

$$\begin{aligned} A_h^p D_h^l \partial^\beta (r^2 A_h^j D_h^k (\partial^\alpha \rho) A_h^m D_h^n (\rho)) =& \partial_x^l \partial^\beta \left(r^2 \partial_x^k \partial^\alpha \rho \partial_x^n \rho \right) + s^{n+k+|\alpha|} \mathcal{O}_\lambda ((sh)^2) \\ =& s^{n+k+|\alpha|} \mathcal{O}_\lambda(1). \end{aligned}$$

Proof. We have

$$\begin{aligned} A_h^m D_h^n (\rho) =& \partial_x^n \rho + R_{A_h^m}(\partial_x^n \rho) + R_{D_h^n}(\rho) + R_{A_h^m D_h^n}(\rho), \\ A_h^j D_h^k (\partial^\alpha \rho) =& \partial_x^k \partial^\alpha \rho + R_{A_h^j}(\partial_x^k \partial^\alpha \rho) + R_{D_h^k}(\partial^\alpha \rho) + R_{A_h^j D_h^k}(\partial^\alpha \rho). \end{aligned}$$

Thus, combining the the above estimate with Lemma 2.4, we conclude

$$A_h^p D_h^l \partial^\beta (r^2 A_h^j D_h^k (\partial^\alpha \rho) A_h^m D_h^n (\rho)) = \partial_x^l \partial^\beta \left(r^2 \partial_x^k \partial^\alpha \rho \partial_x^n \rho \right) + s^{n+k+|\alpha|} \mathcal{O}_\lambda ((sh)^2).$$

□

Let us finally mention that the results of this section can be extended for time-dependent case. For instance, if we consider a weight function of the form $r(x,t) = e^{s\theta(t)\varphi(x)}$ then the condition $sh \le 1$ must be replaced by $sh(\max_{[0,T]} \theta(t)) \le 1$ which implies that $s\theta(t)h \le 1$.

Indeed, we introduce now a weight functions that will be considered in the semi-discrete Carleman estimate for the semi-discrete fourth-order of parabolic equation.

$$r(x,t) := e^{s(t)\varphi(x)},\ \rho(x,t) = \frac{1}{r(x,t)},\ x \in \overline{\tilde{\Omega}},\ t \in (-\delta T, T + \delta T),$$

with

$$s(t) := \lambda \theta(t), \quad \lambda > 0, \quad \theta(t) := \frac{1}{(t + \delta T)(T + \delta T - t)},$$

where the parameter δ is chosen such that $0 < \delta < \frac{1}{2}$ to avoid the singularities at time $t = 0$ and $t = T$. Notice that

$$\max_{t\in[0,T]} \theta(t) = \theta(0) = \theta(T) = \frac{1}{T^2\delta(1+\delta)} \leq \frac{1}{T^2\delta},$$

and $\min_{t\in[0,T]} \theta(t) \geq \frac{1}{T^2}$. Other useful remark is that

$$\frac{d\theta}{dt} = (2t - T)\theta^2.$$

Lemma 2.6 ([10], Lemma 2.8 and 2.9) *For $\alpha, \beta \in \mathbb{N}$ we have*

$$\begin{aligned}\partial_x^\beta(\rho\partial_x^\alpha r) =& s^\alpha \partial_x^\beta((\partial_x\varphi)^\alpha) + s^{\alpha-1}\alpha(\alpha-1)\mathcal{O}(1) = \mathcal{O}(s^\alpha),\\ \partial_t(\rho\partial_x^\alpha r) =& s^\alpha T\theta\mathcal{O}(1).\end{aligned}$$

Let $\sigma \in [0,1]$, provided $\lambda h(\delta T)^1 \leq 1$ we have

$$\begin{aligned}\partial_x^\beta\Big(\rho(x)\partial_x^\alpha r(x+\sigma h)\Big) =& \mathcal{O}(s^\alpha),\\ \partial_t\left(\rho(x,t)(\partial_x^\alpha r)(x+\sigma h, t)\right) =& s^\alpha T\theta\mathcal{O}(1).\end{aligned}$$

Corollary 2.5 ([10], Corollary 2.9) *Let $\alpha, \beta, \gamma \in \mathbb{N}$. Provided $\lambda h(T\delta)^{-1} \leq 1$, we have*

$$\begin{aligned}\partial_x^\gamma\left(\rho^2\partial_x^\alpha r\partial_x^\beta r\right) =& s^{\alpha+\beta}\partial_x^\gamma\left((\partial_x\varphi)^{\alpha+\beta}\right) + s^{\alpha+\beta-1}\mathcal{O}(1) = \mathcal{O}(s^{\alpha+\beta}),\\ \partial_x^\gamma\left(\rho\partial_x^\alpha r\partial_x(\rho\partial_x^\beta r)\right) =& s^{\alpha+\beta}\partial_x^\gamma\left((\partial_x\varphi)^\alpha\ \partial_x(\partial_x\varphi)^\beta\right) + s^{\alpha+\beta-1}\mathcal{O}(1) = \mathcal{O}(s^{\alpha+\beta}).\end{aligned}$$

The following theorems give us an estimate of several computations for the discrete average and derivative operators applied on our new weight function.

Theorem 2.2 *We define, for $m, n \in \mathbb{N}$, the space discrete operator $\partial_h^{m,n} := A_h^m D_h^n$. Then, for $\alpha, \beta \in \mathbb{N}$ and $\lambda h(\delta T^2)^{-1} \leq 1$, the following estimate holds*

$$\begin{aligned}\partial_h^{p,l}\partial^\beta(\rho^2\partial_h^{j,k}(\partial^\alpha r)\partial_h^{m,n}(r)) =& \partial_x^l\partial^\beta\left(\rho^2\partial_x^k\partial^\alpha r\partial_x^n r\right) + s^{n+k+\alpha}\mathcal{O}((sh)^2)\\ =& s^{n+k+\alpha}\mathcal{O}(1),\end{aligned}$$

where $j, k, l, m, n, p \in \mathbb{N}$.

Proof. The proof the same methodology from [37, Theorem 4.1] in the time independence case, since the temporal variable does not play any major role. Indeed, we note that the condition $\lambda h(\delta T^2)^{-1} \leq 1$ implies $s(t)h \leq 1$ for all $t \in [0,T]$. This last condition is the main hypothesis of Theorem 4.1 from [37]. □

Theorem 2.3 *For $\alpha \in \mathbb{N}$ and $\lambda h(\delta T^2)^{-1} \leq 1$, we have*

$$\partial_t\left(\rho\partial_h^{m,n}\partial_x^\alpha r\right) = \partial_t\left(\rho\partial_x^n\partial^\alpha r\right) + Ts^{\alpha+n}\theta(t)\mathcal{O}((sh)^2) = T\theta s^{\alpha+n}\mathcal{O}(1) \tag{2.12}$$

and

$$\partial_t \partial_h^{j,k}\left(\rho \partial_h^{m,n} \partial_x^{\alpha} r\right) = T\theta s^{\alpha+n}\mathcal{O}(1). \tag{2.13}$$

PROOF. From Corollary 2.3 we have

$$\rho\partial_h^{m,n}\partial_x^{\alpha} r = \rho\partial_x^n\partial_x^{\alpha} r + \rho R_{A^m}(\partial_x^n\partial_x^{\alpha} r) + \rho R_{D^n}(\partial_x^{\alpha} r) + \rho R_{A^m D^n}(\partial_x^{\alpha} r).$$

We note that

$$\partial_t(\rho(x,t)(\partial_x^{n+2}\partial^{\alpha} r)(x + (n-2)h\sigma/2, t)) = Ts^{\alpha+n+2}\theta(t)\mathcal{O}(1), \tag{2.14}$$

due to Lemma 2.6. This gives $\partial_t(\rho R_{A_h^m}(\partial_x^n\partial^{\alpha} r)) = Ts^{\alpha+n}\theta(t)\mathcal{O}((sh)^2)$.

Similarly, we obtain

$$\partial_t(\rho R_{D_h^n}(\partial_x^n\partial^{\alpha} r)) = Ts^{\alpha+n}\theta(t)\mathcal{O}((sh)^2)$$

and

$$\partial_t(\rho R_{A_h^m D_h^n}(\partial^{\alpha} r)) = Ts^{\alpha+n}\theta(t)\mathcal{O}((sh)^4).$$

Combining these last three estimates with (2.14) establishes (2.12).The rest of the proof runs as before. We apply Corollary 2.3 to $\rho\partial_h^{m,n}\partial_x r$ to obtain (2.13), and the proof is complete. □

Chapter 3

Stability estimate for the semi-discrete linearized Benjamin-Bona-Mahony equation

In this chapter, we are interested in a linearized version of the Benjamin-Bona-Mahony equation (BBM)

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{3.1}$$

proposed by T. Benjamin, J. L. Bona and J. Mahony in [5] as a model for propagation of one-dimensional, unidirectional, small amplitude long waves in non-linear dispersive media.

In the last years, several authors have widely studied dispersive equations in the context of controllability and inverse problems. Nevertheless, the BBM equation presents several particularities due to the structure of the operator. In particular, the infinitesimal generator of the semigroup is given by $(I - \partial_x^2)^{-1}\partial_x$, which is a compact operator, in opposition with the most common situation in PDE's where the generator is an unbounded operator among others.

We note some interesting results about the unique continuation property (UCP) for BBM for the continuous case. We refer to the reader to those works and their references for a more detailed discussion: L. Rosier and B.-Y. Zhang in [44] developed a UCP for (3.1) on a periodic domain. Moreover, in [18] P. L. da Silva and I. L. Freire give an alternative proof using geometrical arguments for the periodic case, and for the case when (3.1) is solved in $\mathbb{R}$. In [51], X. Zhang and E. Zuazua considered a linearized BBM equation with space-dependent potential

$$u_t - u_{xxt} = [\alpha(x)u]_x + \beta(x)u, \quad (x,t) \in (0,1) \times (0,T). \tag{3.2}$$

In that work, the authors established that the only solution of (3.2), such that $u(0,t) = u(1,t) = 0$, is the trivial one $u \equiv 0$ provided that both α and β do not vanish on some open subset of $(0,1)$. Furthermore, if $\alpha(x) = -1$ and $\beta(x) = 0$ in (3.2), S. Micu proved in [40] a UCP with the additional boundary condition $u_x(1,t) = 0$, and study controllabily results. On the other hand, in [49], M. Yamamoto established a UCP for BBM-like equation with time and space dependent potential

$$\partial_t u(x,t) - \partial_x^2 \partial_t u(x,t) = p(x,t)\partial_x u(x,t) + q(x,t)u(x,t), \quad (x,t) \in (0,1) \times (0,T), \tag{3.3}$$

where $p \in L^\infty((0,T) \times (0,1))$ and $q \in L^\infty(0,T; L^2(0,1))$. It was shown that the solution of (3.3) will vanish in $(0,1) \times (0,T)$, provided $u(1,t) = \partial_x u(1,t) = 0$ for all $t \in (0,T)$ and

$u(x,0)=0$ for $x\in(0,1)$. The main tool to prove this result is a Carleman estimate for the Laplacian operator. Through a more refined version of this Carleman estimate, a stability estimate can be formulated for equation (3.3) (see Section 3.4).

In this chapter, we are interested whether a unique continuation property, as in the work of M. Yamamoto [49], still holds for a semi-discrete approximation in space of (3.3). In this sense, for $N\in\mathbb{N}$ given, we set the space discretization parameter $h := 1/(N+1)$. We consider the pairs (x_i,t) with $t\in(0,T)$, $T>0$, and $x_i = ih$, for $i=1,\dots,N$. Thus, the space semi-discrete approximation of equation (3.3) by using the centered finite difference method with respect to the space variable is given by

$$\partial_t u_i(t) - \frac{\partial_t u_{i+1}(t) - 2\partial_t u_i(t) + \partial_t u_{i-1}(t)}{h^2} = p_i(t)\frac{u_{i+1}(t)-u_{i-1}(t)}{2h} + q_i(t)u_i(t), \tag{3.4}$$

for $i\in\{1,2,\dots,N\}$ and $t\in(0,T)$, where $u_i(t)$ stands for $u(x_i,t)$.

3.1. Discrete Carleman estimates with boundary observation

For the discrete Carleman estimate, we consider the weight function of the form $r(x,t) = e^{s\varphi(x,t)}$ for $s\geq 1$, with $\varphi(x,t) = e^{\lambda\psi(x,t)}$ where ψ is a continuous function whose domain of definition $\overline{\Omega}$ is contained in an enlarged smooth open and connected neighborhood $\tilde{\Omega}$. We also assume $\psi\in C^k\left(\overline{\tilde{\Omega}}\right)$ with k large enough such that it satisfies the following property

$$\partial_x\psi(x,t) > 0, \quad (x,t)\in\overline{\tilde{\Omega}}\times(0,T). \tag{3.5}$$

The assumption of the higher-order derivatives is needed to obtain the estimates on the weight function presented in Section 2.2, in contrast to the continuous case. We will use the same notation for the sample of the continuous function on the discrete or semi-discrete sets. We now state a uniform Carleman estimate for the operator D_h^2 with boundary observation. Although for the Carleman estimate just the condition (3.5) is needed, to achieve the UCP property for the system (3.4) we also consider the following assumption

$$\partial_t\psi(x,t) < 0, \quad (x,t)\in\overline{\tilde{\Omega}}\times(0,T). \tag{3.6}$$

It is not difficult to find a function that verifies conditions (3.5) and (3.6), for instance the following function,

$$\psi(x,t) := (x-x_0)^2 - t^2,\ x_0<0. \tag{3.7}$$

Theorem 3.1 *(Discrete Carleman estimate)*
Let ψ be a function verifying (3.5) *and $T>0$. Then, for the parameter $\lambda_0\geq 1$ sufficiently large, there exist $s_0(\lambda_0)\geq 1$, $h_0>0$, $\varepsilon_0>0$ and $C=C(\varepsilon_0,s_0,\lambda_0)$ independent of $h>0$ such*

that

$$C\left(\left\|e^{s\varphi}D_h^2 v_h\right\|^2_{L_h^2(Q)} + s\int_{\partial Q^+}\varphi\partial_x\psi t_r(e^{2s\varphi})t_r(|D_h v_h|^2) + s^3\int_{\partial Q^+}(\varphi\partial_x\psi)^3 t_r(e^{2s\varphi})t_r(A_h(|v_h|^2))\right) \geq s^3\left\|e^{s\varphi}v_h\right\|^2_{L_h^2(Q)} + s\left\|e^{s\varphi}D_h v_h\right\|^2_{L_h^2(Q^*)}, \quad (3.8)$$

for all $h \in (0, h_0)$*,* $s \in (s_0, \varepsilon_0/h)$ *and* v_h *defined in* $Q := \mathring{\mathcal{M}} \times (0, T)$*.*

The proof of this result will be presented in Section 3.4.2.

3.2. Stability estimate for space semi-discrete BBM equation

With the notation introduced in Chapter 2, we can rewrite the semi-discretization (3.4) as

$$\partial_t u - D_h^2\partial_t u = p_h D_h A_h u + q_h u \text{ in } Q := \mathring{\mathcal{M}} \times (0, T), \quad (3.9)$$

where $p_h, q \in L_h^\infty(Q)$. In (3.9) $u(t)$ provides an approximation of $u(x_h, t)$, u being the solution of the continuous equation (3.3). $\partial_t u$ stands for the first order differentiation with respect to t and the operators D_h and D_h^2 are the classical central finite-difference approximation of the space derivatives. We assume that there exists a constant $M > 0$ independent of h, such that, $\max\{\|p_h\|_{L_h^\infty(Q)}, \|q_h\|_{L_h^\infty(Q)}\} \leq M$.

Theorem 3.2 *(Stability for space semi-discrete BBM equation)*
Let ψ *be a function verifying* (3.5) *and* (3.6)*, and* $T > 0$*. For* $\lambda_0 \geq 1$ *sufficiently large, there exist* $s_0(\lambda_0, M) \geq 1$*,* $h_0 > 0$ *depending on* M*,* $\varepsilon_0 > 0$ *and a constant* $C > 0$*, independent of* $h > 0$*, such that, the following estimate holds*

$$\begin{aligned} s^3\|e^{s\varphi}u\|^2_{L_h^2(Q)} + s^3\|e^{s\varphi}\partial_t u\|^2_{L_h^2(Q)} + s\|e^{s\varphi}D_h\partial_t u\|^2_{L_h^2(Q_h^*)} &\leq Cs\int_{\partial Q^+}\varphi\partial_x\psi t_r(e^{2s\varphi})t_r(|D_h\partial_t u|^2) \\ &+ Cs^3\int_{\partial Q^+}(\varphi\partial_x\psi)^3 t_r(e^{2s\varphi})t_r(A_h(|\partial_t u|^2)), \end{aligned} \quad (3.10)$$

for $0 < h \leq h_0$ *and* $s \in (s_0, \varepsilon_0/h)$ *and* $u(x_h, 0) = 0$ *in* $\mathring{\mathcal{M}}$*.*

As a consequence of Theorem 3.2, we have the following Unique Continuation Property for semi-discrete BBM equation (3.9).

Corollary 3.1 *(UCP for a semi-discrete BBM equation)*
There exists $h_0 > 0$ *depending on* M *such that if* $u = 0$ *on* $\{1\} \times (0, T)$*,* $D_h u = 0$ *on* $\{1 - h/2\} \times (0, T)$ *and* $u(\cdot, 0) = 0$ *in* $\mathring{\mathcal{M}}_h$*; then* $u(x_h, t) = 0$ *in* Q *for all* $h \in (0, h_0)$*.*

The methodology of the proof of Theorem 3.2 is similar to that set out by M. Yamamoto in [49] to obtain the UCP for equation (3.3) however, it cannot be followed straightly from the proof of the continuous cases since the parameter s cannot be taken arbitrary large. As we mentioned above, this parameter is related to the mesh size. Thus, a semi-discrete version of the Carleman estimate used in [49] is not enough. For this reason, we develop a more refined Carleman estimate, (3.17), and its semi-discrete counterpart, see Theorem 3.1. We follow as close as possible the ideas from its continuous formulation. For this reason we state a stability estimate for (3.3). The main tool for the proof is a Carleman estimate for Laplacian operator. For sake of exposition we postpone that proof, see Section 3.4. It is worth to mention that we refined the result presented in [49] (see Section 3.4.1).

3.2.1. The continuous case

We consider $Q := (0,1) \times (0,T)$, for $T > 0$, and we define the classical inner product

$$(u,v)_{L^2(Q)} := \int_Q u\, v\, dxdt$$

and its respective L^2-norm $\|u\|^2_{L^2(Q)} = (u,v)_{L^2(Q)}$.

Following the methodology from [49] we can obtain a stability estimate for (3.3). The proof is based on the Carleman estimate (3.17) and Lemma 6.4.2 from V. Isakov [31].

Theorem 3.3 *Let $\partial_x^j \partial_t^k u \in C([0,1]\times[0,T])$ with $j = 0,1,2$ and $k = 0,1$. For $\lambda_0 > 0$ sufficiently large, there exist constants $s_0 \geq 0$ and $C(s_0,\lambda_0,\psi) > 0$, such that*

$$s^3 \left\|e^{s\varphi}u\right\|^2_{L^2(Q)} + s^3 \left\|e^{s\varphi}\partial_t u\right\|^2_{L^2(Q)} + s \left\|e^{s\varphi}\partial_x\partial_t u\right\|^2_{L^2(Q)} \leq C s^3 \int_0^T \left((\partial_x\psi)^3\varphi^3 e^{2s\varphi}|\partial_t u|^2\right)(1,t)\, dt + Cs\int_0^T \left(\partial_x\psi\varphi e^{2s\varphi}|\partial_x\partial_t u|^2\right)(1,t)\, dt, \tag{3.11}$$

for all $s \geq s_0$, and u verify (3.3) *with $u(x,0) = 0$ for all $x \in (0,1)$.*

As a Corollary of Theorem 3.3, it follows the main result presented in [49].

Corollary 3.2 *Let $\partial_x^j\partial_t^k u \in C([0,1]\times[0,T])$ with $0,1,2$ and $k = 0,1$. If u is solution of* (3.3) *such that $u(1,t) = \partial_x u(1,t) = 0$ for all $t \in (0,T)$ and $u(x,0) = 0$ in $(0,1)$, then $u(x,t) = 0$ in $(0,1)\times(0,T)$.*

3.3. Proof of the stability estimate

As we mentioned above, the proof of Theorem 3.2 is based on the continuous setting strategy. Then, we write down a space semi-discrete version of Lemma 6.4.2 from [31], which is a Poincaré weighted inequality.

Lemma 3.1 *Let $\varphi \in C^1(\overline{Q})$ be such that $\dfrac{\partial\varphi}{\partial t} \leq 0$, then*

$$\int_Q \left|\int_0^t u(x,\sigma)d\sigma\right|^2 e^{2s\varphi(x,t)} \leq T^2 \int_Q (u(x,t))^2 e^{2s\varphi(x,t)},$$

for all $u \in L^2_h(Q)$.

PROOF. We have

$$\int_0^T \left|\int_0^t u(x,\sigma)d\sigma\right|^2 e^{2s\varphi(x,t)}dt \le \int_0^T t\int_0^t (u(x,\sigma))^2 e^{2s\varphi(x,t)}d\sigma dt$$
$$= \int_0^T \int_\sigma^T te^{2s\varphi(x,t)}(u(x,\sigma))^2 dt d\sigma$$
$$\le T^2 \int_0^T (u(x,\sigma))^2 e^{2s\varphi(x,\sigma)}d\sigma,$$

and integrating over $\mathring{\mathcal{M}}_h$ we complete the proof. □

3.3.1. Proof of Theorem 3.2

In this Section, we will give the proof of the Theorem 3.2.

PROOF. Note that by using (3.9), we have

$$D_h^2 \partial_t u = \partial_t u - p_h D_h A_h u - q_h u \ \text{ in } Q.$$

Thus, applying the Carleman estimate (3.8) to $v_h = \partial_t u$ we have

$$\begin{aligned} s^3 \|e^{s\varphi}\partial_t u\|^2_{L_h^2(Q)} + s\|e^{s\varphi}D_h\partial_t u\|^2_{L_h^2(Q_h^*)} \le & s\int_{\partial\mathring{\mathcal{M}}_h^+} \varphi\partial_x\psi t_r(e^{2s\varphi})t_r(|D_h\partial_t u|^2) \\ & + s^3\int_{\partial\mathring{\mathcal{M}}_h^+} (\varphi\partial_x\psi)^3 t_r(e^{2s\varphi})t_r(A_h(|\partial_t u|^2)) \\ & + \|e^{s\varphi}(\partial_t u - pD_hA_hu - Qu)\|^2_{L_h^2(Q)}, \end{aligned} \tag{3.12}$$

for $0 < h \le h_0$, $s \ge s_0$ and $sh < \varepsilon_0$. On the other hand, we note that

$$D_hA_hu(x,t) = \int_0^t D_hA_h\partial_t u(x,\sigma)d\sigma$$

and

$$u(x,t) = \int_0^t \partial_t u(x,\sigma)d\sigma,$$

since $u(x,0) = 0$ in $\mathring{\mathcal{M}}_h$. Then, by Lemma 3.1 we have

$$\begin{aligned} \|e^{s\varphi}(\partial_t u - pD_hA_hu - Qu)\|^2_{L_h^2(Q)} \le & CT^2\|p\|^2_{L_h^\infty(Q)}\|e^{s\varphi}D_hA_h\partial_t u\|^2_{L_h^2(Q)} \\ & + C(1 + T^2\|Q\|^2_{L_h^\infty(Q)})\|e^{s\varphi}\partial_t u\|^2_{L_h^2(Q)} \\ \le & CT^2M^2\|e^{s\varphi}D_hA_h\partial_t u\|^2_{L_h^2(Q)} \\ & + C(1 + T^2M^2)\|e^{s\varphi}\partial_t u\|^2_{L_h^2(Q)}. \end{aligned} \tag{3.13}$$

Now, we focus on the first term of the right-hand side above. Using (2.6) and a discrete integration by parts for the discrete average operator we obtain

$$\begin{aligned}
\|e^{s\varphi}D_hA_h\partial_t u\|^2_{L^2_h(Q)} &= \int_Q e^{2s\varphi}\left(D_hA_h\partial_t u\right)^2 \\
&\leq \int_Q e^{2s\varphi}A_h\left((D_h\partial_t u)^2\right) \\
&= \int_{Q_h^*} A_h(e^{2s\varphi})\left(D_h\partial_t u\right)^2 - \frac{h}{2}\int_{\partial Q} e^{2s\varphi}t_r\left((D_h\partial_t u)^2\right) \\
&\leq \int_{Q_h^*} A_h(e^{2s\varphi})\left(D_h\partial_t u\right)^2.
\end{aligned}$$

From Proposition 2.3 we have $A_h(e^{2s\varphi}) \leq C_\lambda e^{2s\varphi}$, we thus obtain

$$\|e^{s\varphi}D_hA_h\partial_t u\|^2_{L^2_h(Q)} \leq C_\lambda \|e^{s\varphi}D_h\partial_t u\|_{L^2_h(Q^*)}. \tag{3.14}$$

Combining (3.12), (3.13) and (3.14) we get

$$\begin{aligned}
s^3\|e^{s\varphi}\partial_t u\|^2_{L^2_h(Q)} + s\|e^{s\varphi}D_h\partial_t u\|^2_{L^2_h(Q^*)} \leq & s\int_{\partial Q^+}\varphi\partial_x\psi t_r(e^{2s\varphi})t_r(|D_h\partial_t u|^2) \\
&+ s^3\int_{\partial Q^+}(\varphi\partial_x\psi)^3 t_r(e^{2s\varphi})t_r(A_h(|\partial_t u|^2)) \\
&+ (1+T^2M^2)\|e^{s\varphi}\partial_t u\|^2_{L^2_h(Q)} \\
&+ C_\lambda T^2M^2\|e^{s\varphi}D_h\partial_t u\|^2_{L^2_h(Q^*)}.
\end{aligned} \tag{3.15}$$

We note that by choosing some $s \geq T^{2/3}(1+M^2)^{1/3} + T^2M$, the last term on the right-hand side from (3.15) can be absorbs it by its left-hand side. Thus, recalling the hypothesis on s from the Carleman estimates, by choosing $s_1 := \max\{s_0, k(M,T)\} \geq s_0$ large enough we obtain

$$\begin{aligned}
s^3\|e^{s\varphi}\partial_t u\|^2_{L^2_h(Q)} + s\|e^{s\varphi}D_h\partial_t u\|^2_{L^2_h(Q_h^*)} \leq & s\int_{\partial Q^+}\varphi\partial_x\psi t_r(e^{2s\varphi})t_r(|D_h\partial_t u|^2) \\
&+ s^3\int_{\partial Q^+}(\varphi\partial_x\psi)^3 t_r(e^{2s\varphi})t_r(A_h(|\partial_t u|^2)),
\end{aligned}$$

provided $s \geq s_1$, where $k(M,T) := T^{2/3}\left(1+M^2\right)^{1/3} + T^2M^2$. Now, we need to connect the condition over the Carleman parameter s with the mesh size h. Defining

$$h_1 := \frac{\varepsilon_0}{s_1},$$

it follows that for

$$0 < h \leq \min\{h_0, h_1\},$$

we have

$$sh \leq \varepsilon_0$$

provided

$$s \in (s_1, \varepsilon_0/h),$$

which concludes the proof. □

As a consequence, we obtain the UCP presented in Corollary 3.1 for semi-discrete BBM equation (3.9).

3.4. Discrete Carleman estimate

In this section, we establish a discrete Carleman estimate with boundary observation for a finite difference approximation of the Laplacian operator in the one-dimensional setting. In order to do so, it is natural to look closer at the continuous version of such estimates. For this purpose, we follow the methodology of A. V. Fursikov and O. Y. Imanuvilov [26] to obtain a Carleman estimate for Laplacian operator in the continuous setting, which is similar to the estimate obtained by Yamamoto in [49]. The main difference with the one in [49] to our estimate is that we do not consider a density argument, and we thus obtain a Carleman estimate with boundary observation. Then, following the methodology in [7] we establish a discrete Carleman estimate.

3.4.1. The continuous case

The proof of the following Carleman estimate has two steps. First, we consider the conjugate operator defined by $P_\varphi u := e^{s\varphi}\partial_x^2(e^{-s\varphi}u)$. In this case, our Carleman weight function is defined as $e^{s\varphi}$ for $s>0$ with $\varphi = e^{\lambda\psi}$, where $\lambda > 0$, and satisfy

$$\partial_x\psi(x,t) > 0, \quad (x,t)\in\overline{Q}. \tag{3.16}$$

Then, we split P_φ into the operators P_1 and P_2, and it is estimated the scalar product $(P_1u, P_2u)_{L^2(Q)}$.

Theorem 3.4 *(Carleman estimate) Let $\psi\in C(\mathbb{R}^2)$, and for any $t\in(0,T)$ let $\psi(\cdot,t)\in C^4(\mathbb{R})$ such that $\partial_x\psi(x,t)>0$ for $(x,t)\in\overline{Q}$. For the parameter $\lambda_0>0$ sufficiently large, there exists $s_0(\lambda_0)\geq 0$, and $C(s_0,\lambda_0,\psi)>0$, such that*

$$\begin{aligned}
&C\int_Q e^{2s\varphi}|\partial_x^2 v|^2 + s^3\lambda_0^3\int_0^T \left((\partial_x\psi)^3\varphi^3 e^{2s\varphi}|v|^2\right)(1,t) + s\lambda_0\int_0^T\left(\partial_x\psi\varphi e^{2s\varphi}|\partial_x v|^2\right)(1,t)\\
&\geq s^3\lambda_0^4\int_Q(\partial_x\psi)^4\varphi^3 e^{2s\varphi}|v|^2 + s\lambda_0^2\int_Q(\partial_x\psi)^2\varphi e^{2s\varphi}|\partial_x v|^2,
\end{aligned}\tag{3.17}$$

for all $s\geq s_0$.

Proof. We set $u = e^{s\varphi}v$. Then the conjugate operator can be expanded as follows

$$\begin{aligned}
P_\varphi u =& e^{s\varphi}\partial_x^2(e^{-s\varphi}u)\\
=& e^{s\varphi}\left(\partial_x\left(\partial_x(e^{-s\varphi})u + e^{-s\varphi}\partial_x u\right)\right)\\
=& e^{s\varphi}\left(\partial_x^2(e^{-s\varphi})u + \partial_x(e^{-s\varphi})\partial_x u + \partial_x(e^{-s\varphi})\partial_x u + e^{-s\varphi}\partial_x^2 u\right)\\
=& e^{s\varphi}\partial_x^2(e^{-s\varphi})u + 2e^{s\varphi}\partial_x(e^{-s\varphi})\partial_x u + \partial_x^2 u.
\end{aligned}\tag{3.18}$$

Adding $-s\partial_x^2(\varphi)u$, (3.18) can be written as

$$P_\varphi u - s\partial_x^2(\varphi)u = P_1u + P_2u, \tag{3.19}$$

where $P_1 u := \partial_x^2 u + e^{s\varphi}\partial_x^2(e^{-s\varphi})u$ and $P_2 u := 2e^{s\varphi}\partial_x(e^{-s\varphi})\partial_x u - s\partial_x^2(\varphi)u$. Besides, from (3.19), we have

$$\left\|P_\varphi u - s\partial_x^2(\varphi)u\right\|_{L^2(Q)}^2 = \|P_1 u\|_{L^2(Q)}^2 + \|P_2 u\|_{L^2(Q)}^2 + 2\left(P_1 u, P_2 u\right)_Q . \tag{3.20}$$

Note that

$$\left\|P_\varphi u - s\partial_x^2(\varphi)u\right\|_{L^2(Q)}^2 \leq C_\varphi \left(\|P_\varphi u\|_{L^2(Q)}^2 + s^2\|u\|_{L^2(Q)}^2\right), \tag{3.21}$$

since $\partial_x^2\varphi$ is bounded in Q. On the other hand, defining $C_1 u := \partial_x^2 u$, $C_2 u := e^{s\varphi}\partial_x^2(e^{-s\varphi})u$, $B_1 u := 2e^{s\varphi}\partial_x(e^{-s\varphi})\partial_x u$ and $B_2 u := -s\partial_x^2(\varphi)u$ we have

$$\left(P_1 u, P_2 u\right)_{L^2(Q)} = \sum_{i,j=1}^{2}\left(C_i, B_j\right)_{L^2(Q)} . \tag{3.22}$$

We note that, integrating by parts in space, (3.22) can be rewritten as

$$\begin{aligned}
\left(P_1 u, P_2 u\right)_{L^2(Q)} =& 2s^3\int_Q (\partial_x\varphi)^2\partial_x^2\varphi|u|^2 + \int_Q s^2\left((\partial_x^2\varphi)^2 - (\partial_x\varphi)^2 - \partial_x\varphi\partial_x^2\varphi\right)|u|^2 - \frac{s}{2}\int_Q \partial_x^4(\varphi)|u|^2 \\
&+ 2s\int_Q \partial_x^2(\varphi)|\partial_x u|^2 - s\int_0^T \partial_x(\varphi)|\partial_x u|^2\Big|_0^1 + \frac{s}{2}\int_0^T \partial_x^3(\varphi)|u|^2\Big|_0^1 \\
&+ \int_0^T (-s^3(\partial_x\varphi)^3 + s^2\partial_x\varphi\partial_x^2\varphi)|u|^2\Big|_0^1 - s\int_0^T u\partial_x u\partial_x^2(\varphi)\Big|_0^1 .
\end{aligned}$$

Now, using the Young's inequality on the last integral above, we have

$$\begin{aligned}
\left(P_1 u, P_2 u\right)_{L^2(Q)} \geq& 2s^3\int_Q (\partial_x\varphi)^2\partial_x^2\varphi|u|^2 + \int_Q s^2\left((\partial_x^2\varphi)^2 - (\partial_x\varphi)^2 - \partial_x\varphi\partial_x^2\varphi\right)|u|^2 - \frac{s}{2}\int_Q \partial_x^4(\varphi)|u|^2 \\
&+ 2s\int_Q \partial_x^2(\varphi)|\partial_x u|^2 - s\int_0^T \partial_x(\varphi)|\partial_x u|^2\Big|_0^1 + \frac{s}{2}\int_0^T \partial_x^3(\varphi)|u|^2\Big|_0^1 \\
&- \frac{s}{2}\int_0^T (\partial_x^2\varphi)^2|u|^2\Big|_0 - \frac{s}{2}\int_0^T (\partial_x^2\varphi)^2|u|^2\Big|_1 - \frac{s}{2}\int_0^T |\partial_x u|^2\Big|_0 - \frac{s}{2}\int_0^T |\partial_x u|^2\Big|_1 \\
&+ \int_0^T (-s^3(\partial_x\varphi)^3 + s^2\partial_x\varphi\partial_x^2\varphi)|u|^2\Big|_0^1 .
\end{aligned}$$

For λ large enough, there exist $C_{\lambda_0} > 0$ and $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$ we obtain

$$\begin{aligned}
C_{\lambda_0}\left(P_1 u, P_2 u\right)_{L^2(Q)} \geq& s^3\lambda^4\int_Q (\partial_x\psi)^4\varphi^3|u|^2 + s^2\lambda^4\int_Q (\partial_x\psi)^4\varphi^2|u|^2 - s\lambda^4\int_Q (\partial_x\psi)^4\varphi|u|^2 \\
&+ s\lambda^2\int_Q (\partial_x\psi)^2\varphi|\partial_x u|^2 - s\lambda\int_0^T \partial_x\psi\varphi|\partial_x u|^2\Big|_0^1 + s\lambda^3\int_0^T (\partial_x\psi)^3\varphi|u|^2\Big|_0^1 \\
&- s^3\lambda^3\int_0^T (\partial_x\psi)^3\varphi^3|u|^2\Big|_0^1 + s^2\lambda^3\int_0^T (\partial_x\psi)^3\varphi^2|u|^2\Big|_0^1 \\
&- s\int_0^T (\lambda\partial_x\psi)^2\varphi|u|^2\Big|_0 - s\int_0^T (\lambda\partial_x\psi)^2\varphi|u|^2\Big|_1 .
\end{aligned}$$

Now, if we fix $\lambda = \lambda_0$, there exist $C_{s_0,\lambda_0} > 0$ and $s_0(\lambda_0) > 0$ such that

$$\begin{aligned} C_{s_0,\lambda_0} \left(P_1 u, P_2 u\right)_{L^2(Q)} \geq & s^3\lambda^4 \int_Q (\partial_x\psi)^4\varphi^3|u|^2 + s\lambda^2 \int_Q (\partial_x\psi)^2\varphi|\partial_x u|^2 \\ & - s^3\lambda^3 \int_0^T (\partial_x\psi)^3\varphi^3|u|^2\Big|_0^1 - s\lambda \int_0^T \partial_x\psi\varphi|\partial_x u|^2\Big|_0^1, \end{aligned} \tag{3.23}$$

for $s \geq s_0$. Note that $|e^{s\varphi}\partial_x v|^2 = |s\, u\, \partial_x\varphi + \partial_x u|^2 \leq C_\varphi(s^2|u|^2 + |\partial_x u|^2)$. Thus, from (3.20), (3.21) and (3.23) we obtain for λ large enough

$$\begin{aligned} C_{\lambda_0,s_0,\varphi} \int_Q e^{2s\varphi}|\partial_x^2 v|^2 \geq & s^3\lambda^4 \int_Q (\partial_x\psi)^4\varphi^3 e^{2s\varphi}|v|^2 + s\lambda^2 \int_Q (\partial_x\psi)^2\varphi|\partial_x v|^2 \\ & - s^3\lambda^3 \int_0^T (\partial_x\psi)^3\varphi^3 e^{2s\varphi}|v|^2\Big|_0^1 - s\lambda \int_0^T \partial_x\psi\varphi e^{2s\varphi}|\partial_x v|^2\Big|_0^1, \end{aligned}$$

which proves the required result. □

Note that taking $x_0 > 1$, the observation data in (3.17) can be switched to the point $(0,t)$ for $t \in (0,T)$.

3.4.2. Proof of the discrete Carleman estimate

Now, we establish a discrete Carleman estimate for the discrete operator D_h^2. Note that this is the discrete Laplacian in one-dimensional setting. There are Carleman estimates for this kind of operator (see F. Boyer et al. [7],[10] and S. Ervedoza et al. [20]). The main difference respect to our estimate is the fact that we consider boundary observation, due to the choice of the weight function. Indeed, our Carleman weight function is defined as $e^{s\varphi}$ for $s \geq 1$, with $\varphi = e^{\lambda\psi}$ where $\psi \in C^k$ for k sufficiently large and $\lambda \geq 1$. We also assume that

$$\partial_x\psi(x,t) > 0, \quad (x,t) \in Q. \tag{3.24}$$

We follow a classical scheme based on conjugating the original operator with a well chosen exponential function.

3.4.2.1. Proof Theorem 3.1

We make the change of variable $u_h = e^{s\varphi}v_h$. Our first task is to obtain an expression for $P_{h,\varphi} := e^{s\varphi}D_h^2(e^{-s\varphi}u_h)$ with the change of variable proposed. Repeated application of (2.2) yields

$$P_{h,\varphi} = e^{s\varphi}D_h^2(e^{-s\varphi})A_h^2 u + 2e^{s\varphi}A_h D_h(e^{-s\varphi})D_h A_h u_h + e^{s\varphi}A_h^2(e^{-s\varphi})D_h^2 u_h. \tag{3.25}$$

We define the following coefficients $\alpha_1 := e^{s\varphi}A_h^2(e^{-s\varphi})$, $\alpha_2 := e^{s\varphi}D_h^2(e^{-s\varphi})$ and $\beta_1 :=$

$e^{s\varphi}A_hD_h(e^{-s\varphi})$. On the other hand, we set

$$
\begin{aligned}
C_1u_h &:= \alpha_1 D_h^2 u_h,\\
C_2u_h &:= \alpha_2 A_h^2 u_h,\\
B_1u_h &:= 2\beta_1 D_hA_hu_h,\\
B_2u_h &:= -s(\partial_x^2\varphi)u_h.
\end{aligned}
$$

Equation (3.25) thus reads $P_{h,\varphi}u_h - s(\partial_x^2\varphi)u_h = P_1u_h + P_2u_h$, where

$$
\begin{aligned}
P_1u_h &:= C_1u_h + C_2u_h\\
P_2u_h &:= B_1u_h + B_2u_h.
\end{aligned}
$$

We write

$$\left\|P_{h,\varphi} - s(\partial_x^2\varphi)u_h\right\|^2_{L^2_h(Q_h)} = \|P_1u_h\|^2_{L^2_h(Q_h)} + \|P_2u_h\|^2_{L^2_h(Q_h)} + 2\langle P_1u_h, P_2u_h\rangle_{Q_h}. \tag{3.26}$$

Since $\partial_x^2\varphi$ is bounded, we have

$$\left\|P_{h,\varphi} - s(\partial_x^2\varphi)u_h\right\|^2_{L^2_h(Q_h)} \leq C\left(\|P_\varphi u_h\|^2_{L^2_h(Q_h)} + s^2\|u_h\|^2_{L^2_h(Q_h)}\right). \tag{3.27}$$

Now, we will estimate the scalar product

$$\langle P_1u_h, P_2u_h\rangle_{Q_h} = \sum_{i,j=1}^{2}\langle C_iu_h, B_ju_h\rangle_{Q_h}. \tag{3.28}$$

For each term of (3.28), we obtain the following results.

Lemma 3.2 *For $sh \leq 1$, we have*

$$\langle C_1u_h, B_1u_h\rangle_{Q_h} = \int_{Q_h^*} s\lambda^2\varphi(\partial_x\psi)^2|D_hu_h|^2 + \int_{Q_h^*} s\lambda\varphi\partial_x^2\psi|D_hu_h|^2 - X_1 + Y_1,$$

where

$$X_1 := \int_{Q_h^*} s\mathcal{O}_\lambda((sh)^2)|D_hu_h|^2$$

and

$$Y_1 := \int_{\partial Q_h}(-s\lambda\varphi\partial_x\psi + s\mathcal{O}_\lambda((sh)^2))t_r(|Du_h|^2)n_h.$$

Lemma 3.3 *For $sh \leq 1$, we have*

$$\langle C_1u_h, B_2u_h\rangle_{Q_h} \geq \int_{Q_h^*} s\lambda^2(\partial_x\psi)^2\varphi|D_hu_h|^2 + \int_{Q_h^*} s\lambda\varphi\partial_x^2\psi|D_hu_h|^2 - X_2 + Y_2,$$

where

$$X_2 := \int_{Q_h} s\mathcal{O}_\lambda(1)|u_h|^2 + \int_{Q_h^*} s\mathcal{O}_\lambda(h^2 + (sh)^2)|D_hu_h|^2$$

and

$$Y_2 := \int_{\partial Q_h} s\mathcal{O}_\lambda(1)|u_h|^2 - \int_{\partial Q_h} s^2\mathcal{O}_\lambda(1)|u_h|^2 - \int_{\partial Q_h} \mathcal{O}_\lambda(1)t_r(|D_h u_h|^2).$$

Lemma 3.4 *For $sh \leq 1$, we have*

$$\langle C_2 u_h, B_1 u_h\rangle_{Q_h} = 3\int_{Q_h} s^3\lambda^4\varphi^3(\partial_x\psi)^4|u_h|^2 + \int_{Q_h}(s\lambda\varphi)^3\mathcal{O}(1)|u_h|^2 - X_3 + Y_3,$$

where

$$X_3 := \int_{Q_h} s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_\lambda((sh)^2)|u_h|^2 - \int_{Q_h^*} s\mathcal{O}_\lambda((sh)^2)|D_h u_h|^2$$

and

$$\begin{aligned} Y_3 := &\int_{\partial Q_h}\left(-(s\lambda\varphi\partial_x\psi)^3 + s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_\lambda((sh)^2)\right)t_r(A_h(|u_h|^2))n_h \\ &- \int_{\partial Q_h} s\mathcal{O}_\lambda((sh)^2)t_r(|D_h u_h|^2)n_h. \end{aligned}$$

Lemma 3.5 *For $sh \leq 1$, we have*

$$\langle C_2 u_h, B_2 u_h\rangle_{Q_h} \geq -\int_{Q_h} s^3\lambda^4\varphi^3(\partial_x\psi)^4|u_h|^2 + \int_{Q_h} s^3\lambda^3\varphi^2(\partial_x\psi)^2\partial_x^2\psi|u_h|^2 - X_4$$

where

$$X_4 := \int_{Q_h}\left(s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_\lambda((sh)^2)\right)|u_h|^2 + \int_{Q_h} s\mathcal{O}_\lambda(sh)|u_h|^2 + \int_{Q_h^*} s\mathcal{O}_\lambda((sh)^2)|D_h u_h|^2,$$

and

$$Y_4 := \int_{\partial Q_h} s\mathcal{O}_\lambda(1)|u_h|^2 + \int_{\partial Q_h} s\mathcal{O}_\lambda((sh)^2)|u_h|^2 + \int_{\partial Q_h} s\mathcal{O}_\lambda((sh)^2)t_r(|D_h u_h|^2)n_h.$$

The proof of Lemmas 3.2-3.5 can be found in Section 3.5.

Combining the aforementioned Lemmas, for $sh \leq 1$ there exist $\lambda_1 \geq 1$ and ε small enough such that for $\lambda \geq \lambda_1$ and $0 < sh \leq \min\{\varepsilon_1(\lambda), 1\} = \varepsilon_1(\lambda)$, there exists a constant $C_{\lambda_1,\varepsilon_1} > 0$ such that

$$C_{\lambda_1,\varepsilon_1}\langle P_1 u_h, P_2 u_h\rangle_{Q_h} \geq \int_{Q_h^*} s(\partial_x\psi)^2\varphi|D_h u_h|^2 + \int_{Q_h} s^3\varphi^3(\partial_x\psi)^4|u_h|^2 + \sum_{i=1}^4 Y_i - X_i. \quad (3.29)$$

Thus, from (3.26), (3.27) and (3.29) we get

$$\begin{aligned} C_{\lambda_1,\varepsilon_1}\left(\|P_{h,\varphi}\|^2_{L^2_h(Q_h)} + s^2\|u_h\|^2_{L^2_h(Q_h)}\right) + \sum_{i=1}^4 X_i \geq &\sum_{i=1}^4 Y_i + s^3\int_{Q_h}\varphi^3(\partial_x\psi)^4|u_h|^2 \\ &+ s\int_{Q_h^*}(\partial_x\psi)^2\varphi|D_h u_h|^2. \end{aligned}$$

On the other hand, we have to deal with the boundary terms. To do this, we can estimate separately the right and left boundary observation. Indeed, let us denote by Y_i^- and Y_i^+ the left and the right boundary observation of the term Y_i, respectively. Once λ is fixed, for s large enough there exist positive constants C_0 and C_1 such that

$$C_0 s \int_{\partial Q_h^-} \varphi \partial_x \psi t_r(|D_h u_h|^2) n_h + C_0 s^3 \int_{\partial Q_h^-} (\varphi \partial_x \psi)^3 t_r(A_h(|u_h|^2)) \leq \sum_{i=1}^4 Y_i^-,$$
$$\sum_{i=1}^4 Y_i^+ \leq C_1 s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(|D_h u_h|^2) + C_1 s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(A_h(|u_h|^2)).$$

Therefore, if we fix $\lambda = \lambda_1$, we can choose ε_0 and h_0 sufficiently small, with $0 < \varepsilon_0 \leq \varepsilon_1(\lambda_1)$, and $s_0 \geq 1$ sufficiently large, such that for $s \geq s_0$, $0 < h \leq h_0$, and $sh \leq \varepsilon_0$ we obtain

$$\begin{aligned} C_{\lambda_1,\varepsilon_0,s_0} \left(\|P_{h,\varphi}\|^2_{L^2_h(Q_h)} \right) \geq{}& s^3 \|u_h\|^2_{L^2_h(Q_h)} + s \|D_h u_h\|^2_{L^2_h(Q_h^*)} \\ & - C_1 s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(|D_h u_h|^2) n_h - C_1 s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(A_h(|u_h|^2)) \\ & + C_0 s \int_{\partial Q_h^-} \varphi \partial_x \psi t_r(|D_h u_h|^2) - C_0 s^3 \int_{\partial Q_h^-} (\varphi \partial_x \psi)^3 t_r(A_h(|u_h|^2)). \end{aligned} \tag{3.30}$$

Finally, we return to the variable v_h. To this end, we need the following Lemma.

Lemma 3.6 *For $sh \leq 1$, we have*

$$s \|e^{s\varphi} D_h v_h\|^2_{L^2_h(Q_h^*)} \leq C \left(s \|D_h u_h\|^2_{L^2_h(Q_h^*)} + s^3 \|u_h\|^2_{L^2_h(Q_h)} \right) + s^2 \mathcal{O}(sh) \int_{\partial Q_h} |u_h|^2, \tag{3.31}$$

$$\begin{aligned} s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(|D_h u_h|^2) \leq{}& C s^3 \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(e^{2s\varphi} A_h(|v_h|^2)) \\ & + C s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(e^{2s\varphi} |D_h v_h|^2), \end{aligned} \tag{3.32}$$

$$\begin{aligned} s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(A_h(|u_h|^2)) \leq{}& s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(e^{2s\varphi} A_h(|v_h|^2)) \\ & + s \mathcal{O}((sh)^2) \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(e^{2s\varphi} |D_h v_h|^2). \end{aligned} \tag{3.33}$$

For a proof see Section 3.5.

Combining (3.30) with Lemma 3.6, we can choose $\tilde{\varepsilon} > 0$ and $\tilde{h} > 0$ sufficiently small, with $0 < \tilde{h} \leq h_0$, $0 < \tilde{\varepsilon} \leq \varepsilon_0$, and $\tilde{s}$ sufficiently large, such that for $s \geq \tilde{s}$, $0 < h \leq \tilde{h}$, and $sh \leq \tilde{\varepsilon}$, we obtain

$$\begin{aligned} s^3 \|e^{s\varphi} v_h\|^2_{L^2_h(Q_h)} + s \|e^{s\varphi} D_h v_h\|^2_{L^2_h(Q_h^*)} \leq{}& C_{\tilde{\varepsilon},\tilde{s}} \Bigg(\left\|e^{s\varphi} D_h^2 v_h\right\|^2_{L^2_h(Q_h)} + s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(e^{2s\varphi}) t_r(|D_h v_h|^2) \\ & + s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(e^{2s\varphi}) t_r(A_h(|v_h|^2)) \Bigg), \end{aligned}$$

where we have dropped the left boundary observation, and the proof is complete.

3.5. Proof of intermediate results

In this section, we will prove some technical results used in the development of the discrete Carleman estimate. We consider $sh \leq 1$ in the following lemmas in order to ensure that every Lemma from Section 2.2 holds. Recall that our Carleman weight function defined as $r(x) := e^{s\varphi(x)}$ for $s \geq 1$, with $\varphi(x) = e^{\lambda\psi(x)}$ where $\psi \in C^k$ for k sufficiently large and $\lambda \geq 1$. We denote $\rho := r^{-1}$ and ψ verifies $\partial_x\psi > 0$ in Q. The proof we develop in each Lemma is standard in the following sense. We begin rewritten the semi-discrete integral, if necessary, using some identity related to the discrete operators from Corollary 2.1. Then we apply a semi-discrete integration by parts from Proposition 2.1 to identify the leader terms of the Carleman estimate. Finally, thanks to Theorem 2.1, we can obtain the estimate claimed in each Lemma.

3.5.1. Proof of Lemma 3.2

Recalling the definition of C_1 and B_1, setting $\gamma_{11} := \beta_1\alpha_1$ and $I_{11} := \langle C_1u, B_1u\rangle_Q$, we write

$$I_{11} := \int_Q 2\gamma_{11}D_h^2uD_hA_hu.$$

From Lemma 2.1 the semi-discrete integral I_{11} can be rewritten as

$$I_{11} = \int_Q \gamma_1 D_h\left(|D_hu|^2\right).$$

Using Proposition 2.1, for I_{11} we obtain

$$I_{11} = -\int_{Q^*} D_h(\gamma_1)|D_hu|^2 + \int_{\partial Q} \gamma_1 t_r(|D_hu|^2)n.$$

The proof is completed by showing that

$$\begin{aligned} D_h(\gamma_1) &= -s\varphi\lambda^2(\partial_x\psi)^2 - s\lambda\varphi\partial_x^2\psi + s\mathcal{O}_\lambda((sh)^2), \\ \gamma_1 &= -s\lambda\varphi\partial_x\psi + s\mathcal{O}_\lambda((sh)^2), \end{aligned}$$

which follows from Proposition 2.1 and Corollary 2.2.

3.5.2. Proof of Lemma 3.3

Set $I_{12} := \langle C_1u, B_2u\rangle_Q$. From the definition of the operators C_1 and B_2, we have

$$I_{12} := -s\int_Q \partial_x^2\varphi\alpha_1uD_h^2u$$

A semi-discrete integration by parts, Proposition 2.1, yields

$$I_{12} = s\int_{Q^*} D_h(\partial_x^2\varphi\alpha_1u)D_hu - s\int_{\partial Q} \partial_x^2\varphi\alpha_1ut_r(Du)n := I_{12}^{(a)} - I_{12}^{(b)}.$$

Let us focus on $I_{12}^{(a)}$. We note that thanks to Lemma 2.1, $I_{12}^{(a)}$ can be rewritten as

$$I_{12}^{(a)} = s\int_{Q^*} D_h(\partial_x^2\varphi\alpha_1)A_h u D_h u + s\int_{Q^*} A_h(\partial_x^2\varphi\alpha_1)|D_h u|^2 := I_{12}^{(a_1)} + I_{12}^{(a_2)}.$$

To estimate the term $I_{12}^{(a_2)}$, due to Lemma 2.1, we write

$$A_h(\alpha_1\partial_x^2\varphi) = A_h(\alpha_1)A_h(\partial_x^2\varphi) + \frac{h^2}{4}D_h(\alpha_1)D_h(\partial_x^2\varphi). \tag{3.34}$$

By using Proposition 2.2 we can obtain the following estimates

$$\begin{aligned} A_h(\partial_x^2\varphi) =& \partial_x^2\varphi + \mathcal{O}_\lambda(h^2),\\ D_h(\partial_x^2\varphi) =& \partial_x^3\varphi + \mathcal{O}_\lambda(h^2). \end{aligned}$$

Moreover, Lemma 2.5 leads to

$$\begin{aligned} A_h(\alpha_1) =& 1 + \mathcal{O}_\lambda((sh)^2),\\ D_h(\alpha_1) =& \mathcal{O}_\lambda((sh)^2). \end{aligned}$$

The previous estimates enables us to write (3.34) as

$$A_h(\alpha_1\partial_x^2\varphi) = \partial_x^2\varphi + \mathcal{O}_\lambda(h^2 + (sh)^2) = \lambda^2(\partial_x\psi)^2\varphi + \lambda\varphi\partial_x^2\psi + \mathcal{O}_\lambda(h^2 + (sh)^2).$$

Therefore, $I_{12}^{(a_2)}$ can be estimated as

$$I_{12}^{(a_2)} = s\lambda^2\int_{Q^*}(\partial_x\psi)^2\varphi|D_h u|^2 + \int_{Q^*} s\lambda\varphi\partial_x^2\psi|D_h u|^2 + \int_{Q^*} s\mathcal{O}_\lambda(h^2 + (sh)^2)|D_h u|^2. \tag{3.35}$$

On the other hand, by using (2.6), $I_{12}^{(a_1)}$ can be rewritten as

$$I_{12}^{(a_1)} = \frac{s}{2}\int_{Q^*} D_h\left(\alpha_1\partial_x^2\varphi\right)D_h\left(|u|^2\right).$$

A semi-discrete integration by parts with respect to the difference operator D_h leads to

$$I_{12}^{(a_1)} = -\frac{s}{2}\int_Q D_h^2\left(\partial_x^2\varphi\alpha_1\right)|u|^2 + \frac{s}{2}\int_{\partial Q} t_r(D_h(\partial_x^2\varphi\alpha_1))|u|^2 n.$$

By using (2.1), it follows that

$$D_h^2(\partial_x^2\varphi\alpha_1) = D_h^2(\partial_x^2\varphi)A_h^2(\alpha_1) + 2D_hA_h(\partial_x^2\varphi)A_hD_h(\alpha_1) + A_h^2(\partial_x^2\varphi)D_h^2(\alpha_1). \tag{3.36}$$

Now, applying Lemma 2.5 to $\alpha_1 := e^{s\varphi}A_h^2(e^{-s\varphi})$, we have

$$\begin{aligned} A_h^2(\alpha_1) =& \mathcal{O}_\lambda(1),\\ A_hD_h(\alpha_1) =& \mathcal{O}_\lambda(1),\\ D_h^2(\alpha_1) =& \mathcal{O}_\lambda(1). \end{aligned}$$

Moreover, applying Proposition 2.2 to $\partial_x^2\varphi$, we get

$$\begin{aligned} D_h^2(\partial_x^2\varphi) =&\partial_x^4\varphi + \mathcal{O}_\lambda(h^2) = \mathcal{O}_\lambda(1), \\ D_h A_h(\partial_x^2\varphi) =&\partial_x^3\varphi + \mathcal{O}_\lambda(h^2) = \mathcal{O}_\lambda(1), \\ A_h^2(\partial_x^2\varphi) =&\partial_x^2\varphi + \mathcal{O}_\lambda(h^2) = \mathcal{O}_\lambda(1). \end{aligned}$$

Thus, (3.36) can be estimated as

$$D_h^2(\alpha_1\partial_x^2\varphi) = \mathcal{O}_\lambda(1).$$

Similarly, we get

$$D_h(\alpha_1\partial_x^2\varphi) = \mathcal{O}_\lambda(1).$$

Hence, for $I_{12}^{(a_1)}$ we obtain

$$I_{12}^{(a_1)} = -s\int_Q \mathcal{O}_\lambda(1)|u|^2 + s\int_{\partial Q}\mathcal{O}_\lambda(1)|u|^2. \tag{3.37}$$

Finally, by using the Young's inequality, $I_{12}^{(b)}$ can be bounded as

$$|I_{12}^{(b)}| \le s^2\int_{\partial Q}|\mathcal{O}_\lambda(1)||u|^2 + \int_{\partial Q}|\mathcal{O}_\lambda(1)|t_r(|D_h u|^2). \tag{3.38}$$

Therefore, collecting the estimates (3.35), (3.37) and (3.38) ,I_{12} can be estimated as

$$I_{12} \ge s\lambda^2\int_{Q^*}(\partial_x\psi)^2\varphi|D_h u|^2 + \int_{Q^*}s\lambda\varphi\mathcal{O}(1)|D_h u|^2 - X_2 + Y_2,$$

where X_2 and Y_2 are given by

$$X_2 := s\int_Q \mathcal{O}_\lambda(1)|u|^2 + \int_{Q^*}s\mathcal{O}_\lambda(h^2+(sh)^2)|D_h u|^2$$

and

$$Y_2 := s\int_{\partial Q}\mathcal{O}_\lambda(1)|u|^2 - s^2\int_{\partial Q}\mathcal{O}_\lambda(1)|u|^2 - \int_{\partial Q}\mathcal{O}_\lambda(1)t_r(|D_h u|^2),$$

which is our claim.

3.5.3. Proof of Lemma 3.4

Setting $\gamma_{21} := \alpha_2\beta_1$ and $I_{21} := \langle C_2 u, B_1 u\rangle_Q$. Let us compute

$$I_{21} = \int_Q 2\gamma_{21}A_h^2 u D_h A_h u.$$

By using Lemma 2.1 the above semi-discrete integral can be rewritten as

$$I_{21} = \int_Q \gamma_{21}D_h((A_h u)^2).$$

A semi-discrete integration by parts with respect to the difference operator yields

$$
\begin{aligned}
I_{21} =& -\int_{Q^*} D_h(\gamma_{21})(A_h u)^2 + \int_{\partial Q} \gamma_{21} t_r \left((Au)^2\right) n \\
:=& I_{21}^{(a)} + I_{21}^{(b)}.
\end{aligned}
\tag{3.39}
$$

Let us first estimate $I_{21}^{(a)}$. Note that (2.5) leads to

$$
I_{21}^{(a)} = -\int_{Q^*} D_h(\gamma_{21}) A_h(u^2) + \frac{h^2}{4}\int_{Q^*} D_h(\gamma_{21})|D_h u|^2.
$$

Then, by Proposition 2.1 we obtain

$$
I_{21}^{(a)} = -\int_Q A_h D_h(\gamma_{21})|u|^2 - \frac{h}{2}\int_{\partial Q} t_r(D_h(\gamma_{21}))|u|^2 + \frac{h^2}{4}\int_{Q^*} D_h(\gamma_{21})|D_h u|^2.
$$

Recalling that $\alpha_2 := e^{s\varphi} D_h^2(e^{-s\varphi})$ and $\beta_1 := e^{s\varphi} A_h D_h(e^{-s\varphi})$ we have

$$
A_h D_h(\gamma_{21}) = -3s^3\lambda^4\varphi^3(\partial_x\psi)^4 + (s\lambda\varphi)^3\mathcal{O}(1) + s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_\lambda((sh)^2)
$$

and $D_h(\gamma_{21}) = s^3\mathcal{O}_\lambda(1)$, by virtue of Proposition 2.1 and Corollary 2.2. Hence, for $I_{21}^{(a)}$ we obtain the following estimate

$$
\begin{aligned}
I_{21}^{(a)} =& 3s^3\lambda^4\int_Q \varphi^3(\partial_x\psi)^4|u|^2 + \int_Q (s\lambda\varphi)^3\mathcal{O}(1)|u|^2 - \int_Q (s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_\lambda((sh)^2)|u|^2 \\
&- \int_{Q^*} s\mathcal{O}_\lambda((sh)^2)|D_h u|^2 - s^2\int_{\partial Q}\mathcal{O}_\lambda(sh)|u|^2.
\end{aligned}
\tag{3.40}
$$

On the other hand, $I_{21}^{(b)}$ can be estimated as

$$
I_{21}^{(b)} = \int_{\partial Q} -(s\lambda\varphi\partial_x\psi)^3 + s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_\lambda((sh)^2) t_r(A_h(|u|^2))n,
\tag{3.41}
$$

since $\gamma_{21} = -(s\lambda\varphi\partial_x\psi)^3 + s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_\lambda((sh)^2)$, due to Proposition 2.1. Thus, combining (3.39) with (3.40) and (3.41) the Lemma follows.

3.5.4. Proof of Lemma 3.5

Let $I_{22} := \langle C_2 u, B_2 u\rangle_Q$. By definition of C_2 and B_2, let us estimate the semi-discrete integral

$$
I_{22} = -s\int_Q \alpha_2 \partial_x^2\varphi A_h^2(u)u.
$$

To this end, by using (2.5), I_{22} can be rewritten as

$$
I_{22} = -s\int_Q \alpha_2\partial_x^2\varphi|u|^2 + \frac{-sh^2}{4}\int_Q \alpha_2\partial_x^2\varphi u D_h^2 u := I_{22}^{(a)} + I_{22}^{(b)}.
$$

Since $sh \le 1$, from Proposition 2.3 and Lemma 2.2 we have for $\alpha_2 := e^{s\varphi} D_h^2(e^{-s\varphi})$ the following estimate

$$
\alpha_2 = (s\lambda\varphi)^2(\partial_x\psi)^2 + s\mathcal{O}_\lambda(1) + s^2\mathcal{O}_\lambda((sh)^2).
$$

Furthermore, noting that $\partial_x^2\varphi = \lambda^2(\partial_x\psi)^2\varphi + \lambda\varphi\partial_x^2\psi$, with the previous estimate for α_2 we obtain

$$\alpha_2\partial_x^2\varphi = s^2\lambda^4\varphi^3(\partial_x\psi)^4 + s^2\lambda^3\varphi^2(\partial_x\psi)^2\partial_x^2\psi + s\mathcal{O}_\lambda(1) + s^2\mathcal{O}_\lambda((sh)^2) = s^2\mathcal{O}_\lambda(1). \tag{3.42}$$

Then, $I_{22}^{(a)}$ is estimated as

$$I_{22}^{(a)} = -s^3\lambda^4\int_Q \varphi^3(\partial_x\psi)^4|u|^2 - \int_Q \left(-s^3\lambda^3\varphi^2(\partial_x\psi)^2\partial_x^2\psi + s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_\lambda((sh)^2)\right)|u|^2. \tag{3.43}$$

Similarly, for $I_{22}^{(b)}$, a semi-discrete integration by parts yields

$$I_{22}^{(b)} = \frac{sh^2}{4}\int_{Q^*} D_h(\alpha_2\partial_x^2\varphi u)D_h u - \frac{sh^2}{4}\int_{\partial Q}\alpha_2\partial_x^2\varphi u t_r(D_h u)n := I_{22}^{(b_1)} - I_{22}^{(b_2)}.$$

Let us estimate $I_{22}^{(b_2)}$. Note that by using (3.42) and Young's inequality, $I_{22}^{(b_2)}$ can be bounded as

$$|I_{22}^{(b_2)}| \leq s\int_{\partial Q}|\mathcal{O}_\lambda((sh)^2)||u|^2 n + s\int_{\partial Q}|\mathcal{O}_\lambda((sh)^2)|t_r(|D_h u|^2).$$

Now, let us focus on $I_{22}^{(b_1)}$. Using Lemma 2.1 we write $D_h(|u|^2) = 2D_h u A_h u$. Thus, $I_{22}^{(b_1)}$ can be written as

$$I_{22}^{(b_1)} = \frac{sh^2}{8}\int_{Q^*} D_h(\alpha_2\partial_x^2\varphi)D_h(|u|^2) + \frac{sh^2}{4}\int_{Q^*} A_h(\alpha_2\partial_x^2\varphi)|D_h u|^2.$$

We now use a semi-discrete integration by parts on the first integral above to obtain

$$I_{22}^{(b_1)} = -\frac{sh^2}{8}\left(\int_Q D_h^2(\alpha_2\partial_x^2\varphi)|u|^2 + \int_{\partial Q}|u|^2 t_r(D_h(\alpha_2\partial_x^2\varphi))n\right) + \frac{sh^2}{4}\int_{Q^*} A_h(\alpha_2\partial_x^2\varphi)|D_h u|^2.$$

To obtain an estimate for $I_{22}^{(b_1)}$ we claim that

$$A_h(\alpha_2\partial_x^2\varphi) = s^2\mathcal{O}_\lambda(1), \tag{3.44}$$
$$D_h^2(\alpha_2\partial_x^2\varphi) = s^2\mathcal{O}_\lambda(1), \tag{3.45}$$
$$D_h(\alpha_2\partial_x^2\varphi) = s^2\mathcal{O}_\lambda(1). \tag{3.46}$$

Indeed, to prove the estimate (3.44) we use Lemma 2.1 to write

$$A_h(\alpha_2\partial_x^2\varphi) = A_h(\alpha_2)A_h(\partial_x^2\varphi) + \frac{h^4}{2}D_h(\alpha_2)D_h(\partial_x^2\varphi).$$

Then, thanks to Lemma 2.5, we obtain

$$A_h(\alpha_2) = s^2\mathcal{O}_\lambda(1),$$
$$D_h(\alpha_2) = s^2\mathcal{O}_\lambda(1).$$

Moreover, using Proposition 2.2 we have

$$\begin{aligned} A_h(\partial_x^2\varphi) =& \partial_x^2\varphi + h^2\mathcal{O}_\lambda(1), \\ D_h(\partial_x^2\varphi) =& \partial_x^3\varphi + h^2\mathcal{O}_\lambda(1), \end{aligned}$$

and since $\partial_x^2\varphi = \mathcal{O}_\lambda(1)$, (3.44) follows. For the estimate (3.45), applying (2.1) it follows that

$$D_h^2\left(\alpha_2\partial_x^2\varphi\right) = D_h^2(\alpha_2)A_h^2(\partial_x^2\varphi) + 2A_hD_h(\alpha_2)A_hD_h(\partial_x^2\varphi) + D_h^2(\partial_x^2\varphi)A_h^2(\alpha_2) \tag{3.47}$$

Similarly, by using Lemma 2.5 and Proposition 2.2 we have

$$\begin{aligned} D_h^2(\alpha_2) =& s^2\mathcal{O}_\lambda(1),\ A_hD_h(\alpha_2) = s^2\mathcal{O}_\lambda(1),\ A_h^2(\alpha_2) = s^2\mathcal{O}_\lambda(1), \\ A_h^2(\partial_x^2\varphi) =& \partial_x^2\varphi + h^2\mathcal{O}_\lambda(1),\ A_hD_h(\partial_x^2\varphi) = \partial_x^3\varphi + h^2\mathcal{O}_\lambda(1), \end{aligned}$$

These estimates and (3.47) establishes (3.45). The same methodology works for (3.46).
We thus have, from (3.44)-(3.46), the following estimate for $I_{22}^{(b_1)}$

$$I_{22}^{(b_1)} = -s\int_Q \mathcal{O}_\lambda(sh)|u|^2 + s\int_{Q^*} \mathcal{O}_{\lambda,\epsilon}((sh)^2)|D_hu|^2 + s\int_{\partial Q} \mathcal{O}_\lambda(1)|u|^2. \tag{3.48}$$

Therefore, combining (3.43) with (3.48) proves the Lemma.

3.5.5. Proof of Lemma 3.6

We begin proving the first inequality (3.31) of our Lemma. Recalling that $v_h = u_he^{-s\varphi}$, thanks to Lemma 2.1 and Young's inequality, we have

$$\begin{aligned} \|e^{s\varphi}D_hv_h\|^2_{L_h^2(Q_h^*)} \leq& \left\|e^{s\varphi}D_h(u_h)A_h(e^{-s\varphi})\right\|^2_{L_h^2(Q_h^*)} + \left\|e^{s\varphi}D_h(e^{-s\varphi})A_h(u_h)\right\|^2_{L_h^2(Q_h^*)} \\ :=& J_1 + J_2, \end{aligned} \tag{3.49}$$

Let us first estimate J_2. Using (2.6) and a discrete integration by part respect to the average operator we obtain

$$J_2 = \int_{Q_h} A_h((e^{s\varphi}D_h(e^{-s\varphi}))^2)|u_h|^2 + \frac{h}{2}\int_{\partial Q_h} t_r((e^{s\varphi}D_h(e^{-s\varphi}))^2)|u_h|^2. \tag{3.50}$$

Then, by virtue of Proposition 2.3, J_2 can be estimated as follows

$$J_2 \leq s\int_{Q_h}|u_h|^2 + s\mathcal{O}_\lambda(sh)\int_{Q_h}|u_h|^2.$$

It remains to proves that

$$J_1 \leq \mathcal{O}_\lambda(1)\int_{Q_h}|D_hu|^2,$$

which it follows from Proposition 2.3, and the proof for (3.31) is complete.

To prove the inequality (3.32), we note that

$$D_hu_h(h/2,t) = D_h(e^{s\varphi})A_hv_h(h/2,t) + D_h(v_h)A_h(e^{s\varphi})(h/2,t),$$

due to Lemma 2.1. Hence, Young's inequality and Proposition 2.3 yield

$$e^{-2s\varphi}|D_h u_h|^2(h/2,t) \le C_\lambda \left(s^2|A_h v_h|^2(h/2,t) + |D_h v_h|^2(h/2,t)\right),$$

which establishes inequality (3.32).

We proceed similarly for (3.33). From (2.5) we have

$$A_h(|u_h|^2) = |A_h u_h|^2 + \frac{h^2}{4}|D_h u_h|^2.$$

Repeated application of Lemma 2.1 and Young's inequality lead to

$$A_h(|u_h|^2) \le C\left(A_h(v_h^2)|A_h e^{s\varphi}|^2 + h^4|D_h v_h|^2|D_h e^{s\varphi}|^2 + h^2|D_h v_h|^2|A_h e^{s\varphi}|^2 + h^2|D_h e^{s\varphi}|^2|A_h v|^2\right).$$

Then, using Proposition 2.3 we obtain

$$e^{-2s\varphi}A_h(u_h^2) \le (\mathcal{O}_\lambda(1) + \mathcal{O}_\lambda((sh)^2)A_h(|v_h|^2) + (h^2 + h^2\mathcal{O}_\lambda((sh)^2))|D_h v_h|^2,$$

which completes the proof.

Chapter 4

Carleman estimates and controllability for a semi-discrete fourth-order parabolic equation

The boundary controllability of fourth-order parabolic equations has been addressed in recent literature. However, there are no results concerning their numerical approximation and the behavior of discrete controls when the discretization parameter goes to zero. This Chapter is intended to cover this gap by studying this issue when the space operator is discretized and the time is kept as a continuous variable (semi-discrete approximation case). The proof is based on a relaxed observability inequality for the corresponding semi-discrete adjoint system and a suitable semi-discrete Carleman estimate.

Since the works by Fattorini and Russell [21], by Lebeau and Robbiano [36], and by Fursikov and Imanuvilov [27], a huge literature has been devoted to the null controllability of parabolic equations. Internal or boundary controls have been considered in one or higher dimensions. Nowadays, the focus at the continuous level is put on nonlinear effects and the study of coupled systems where a variety of amazing new phenomena appears. Concerning the numerical approximation of these control systems, it is well known that the null controllability property is very hard to address. We found the papers [13, 23, 24, 41] where different numerical methods are proposed to deal with this ill-posed problem.

On the other hand, there are some works proposing relaxed conditions replacing null controllability. These relaxed definitions are very useful in order to obtain control well behaved when the discretization parameter goes to zero. In this direction, we found first the papers [35, 43] where a semigroup approach is applied imposing the analiticity of the discrete semigroups. Then, inspired by the Lebeau-Robbiano approach [36] we found the paper [7] where the authors obtain some Carleman estimate for discrete elliptic operator and obtain null-controllability results. Following the historical development of the topic, in [9] the authors established a Fursikov-Imanuvilov approach for semi-discrete parabolic systems. It is worth to mention that the moment method has also been applied at the semi-discrete level to obtain uniform control properties [2].

In this chapter we address this kind of questions for a fourth-order parabolic system inspired in the works by Boyer and his collaborators, see in particular [9]. More precisely, let

$T > 0$, $\Omega = (0,1)$, $a \in L^\infty(\Omega \times (0,T))$ and consider the system

$$\begin{cases} \partial_t y + \partial_x^4 y + ay = 0, & \forall (x,t) \in \Omega \times (0,T), \\ y(0,t) = u_1(t), \quad y(1,t) = 0, & \forall t \in (0,T), \\ \partial_x y(0,t) = u_2(t), \ \partial_x y(1,t) = 0, & \forall t \in (0,T), \end{cases} \tag{4.1}$$

where the state is given by $y = y(x,t)$ and the time-dependent functions u_1 and u_2 are boundary controls. This equation represents the main linear terms appearing in the Cahn-Hilliard equation and the Kuramoto-Sivashinsky equation, among many others.

Let us recall the main property we are interested in: the system (4.1) is said to be null controllable at time T if for any given state y_0, there exist some controls u_1, u_2 such that the solution $y(x,t)$ of (4.1) with the initial condition $y(x,0) = y_0(x)$ satisfies $y(x,T) = 0$. Following the duality between controllability and observability, it is well-known that the null controllability of (4.1) is equivalent to prove an observability inequality for the corresponding adjoint system, given by

$$\begin{cases} -\partial_t q + \partial_x^4 q + aq = 0, & \forall (x,t) \in \Omega \times (0,T), \\ q(0,t) = 0, \ q(1,t) = 0, & \forall t \in (0,T), \\ \partial_x q(0,t) = 0, \ \partial_x q(1,t) = 0, & \forall t \in (0,T), \\ q(x,T) = q_T(x), & \forall x \in \Omega. \end{cases} \tag{4.2}$$

This strategy has been successfully applied in [15], where the proof of the corresponding observability inequality uses Carleman estimates.

As already mentioned, the main purpose of this work is to study null controllability and observability properties for semi-discrete approximations of (4.1) and (4.2), respectively. It is worth to mention that up to our knowledge, this is the first work devoted to this question for fourth-order parabolic equations.

Let us start by writing the corresponding semi-discrete system. For given $N \in \mathbb{N}$, we set the space discretization parameter $h := 1/(N+1)$. We consider the pairs (x_i, t) with $t \in (0,T)$, $T > 0$, and $x_i = ih$, $i = 0, 1, \dots, N$. Applying the centered finite difference method to the space variable for the system (4.1), and considering two additional points $x_{-1} := x_0 - h$ and $x_{N+2} := x_{N+1} + h$ to discretize the boundary conditions, we obtain the following semi-discrete system

$$\begin{cases} \partial_t y_i(t) + \frac{1}{h^4}\left(y_{i+2}(t) - 4y_{i+1}(t) + 6y_i(t) - 4y_{i-1}(t) + y_{i-2}(t)\right) + a_i(t) y_i(t) = 0, \\ \text{for } i = 1, \dots, N, \ t \in (0,T). \\ y_{-1}(t) = u_1(t) - h u_2(t), \\ y_0(t) = u_1(t), \\ y_N(t) = 0, \\ y_{N+1}(t) = 0. \end{cases} \tag{4.3}$$

Similarly to the continuous case, we could wonder if for any initial condition, there exist controls u_1, u_2 such the solution $y(x,t)$ of system (4.3) satisfies $y(x,T) = 0$. If these controls exist we would say that the system (4.3) is null-controllable. It is well known, see [10], that we cannot expect the aforementioned classical notion of null-controllability since the semi-discrete system may not be even approximately controllable. Moreover, even if that property holds, it is very hard to prove some uniform behavior with respect to the discretization parameter h in order to say that our semi-discrete control problem approximates the continuous one. Thus, we are interested in the ϕ-controllability of the system (4.3), that is, to obtain uniformly bounded controls such that the norm of the semi-discrete solution at time T, $y(T)$, is approximately of the size $\sqrt{\phi(h)}$, where ϕ is a real-valued function that tend to zero when space discretization parameter tends to zero.

With the notation from Chapter 2, the operator

$$\mathcal{P} := \partial_t + \partial_x^4$$

has the usual consistent space finite-difference approximation given by

$$\mathcal{P}_h := \partial_t + D_h^4,$$

defined for $y \in C(Q)$ with $Q := \mathring{\mathcal{M}} \times (0,T)$. Thus, the controlled semi-discrete system (4.3) can be written as

$$\begin{cases} \partial_t y + D_h^4 y + ay = 0 \text{ in } Q, \\ y(0,t) = u_1(t), \quad y(1,t) = 0, \\ D_h y(-\frac{h}{2},t) = u_2(t), \ D_h y(1+\frac{h}{2},t) = 0, \\ y(0) = y_0. \end{cases} \tag{4.4}$$

Following the continuous methodology and the penalized Hilbert uniqueness method [6], we can establish a controllability result for (4.4) by proving an observability estimate for its adjoint system

$$\begin{cases} -\partial_t q + D_h^4 q + ay = 0 \text{ in } Q, \\ q(0,t) = 0, \quad q(1,t) = 0, \\ D_h q(-\frac{h}{2},t) = 0, \ D_h q(1+\frac{h}{2},t) = 0, \\ q(T) = q_T. \end{cases} \tag{4.5}$$

The methodology that we will apply requires a semi-discrete Carleman estimate. Let us assume that the domain $\overline{\Omega}$ is contained in an enlarged smooth open and connected neighborhood $\tilde{\Omega}$.

We choose a function $\varphi \in C^k(\overline{\tilde{\Omega}})$ with k large enough such that it satisfies the following properties

$$0 < \eta \leq \frac{d^k\varphi}{dx^k}(x), \ \forall x \in \overline{\Omega}, \ \text{for } k = 0,1 \tag{4.6}$$

and

$$\frac{d^2\varphi}{dx^2}(x) \leq -\eta < 0, \quad \forall x \in \overline{\Omega}. \tag{4.7}$$

The assumption of the higher-order derivatives for φ is needed to obtain the estimates on the weight function presented in Section 2.2, in contrast to the continuous case. We will use the same notation for the sample of the continuous function on the semi-discrete sets.
We introduce now some weight functions that will be considered in the remainder of this article

$$r(x,t) := e^{s(t)\varphi(x)}, \ \rho(x,t) = \frac{1}{r(x,t)}, \ x \in \overline{\tilde{\Omega}}, \ t \in (-\delta T, T+\delta T), \tag{4.8}$$

with

$$s(t) := \lambda\theta(t), \quad \lambda > 0, \quad \theta(t) := \frac{1}{(t+\delta T)(T+\delta T - t)},$$

where the parameter δ is chosen such that $0 < \delta < \frac{1}{2}$ to avoid the singularities at time $t = 0$ and $t = T$. Notice that

$$\max_{t\in[0,T]} \theta(t) = \theta(0) = \theta(T) = \frac{1}{T^2\delta(1+\delta)} \leq \frac{1}{T^2\delta}, \tag{4.9}$$

and $\min_{t\in[0,T]} \theta(t) \geq \frac{1}{T^2}$. Other useful remark is that

$$\frac{d\theta}{dt} = (2t - T)\theta^2. \tag{4.10}$$

In this context, we can now state our first result, a uniform Carleman estimate for the semi-discrete fourth-order parabolic operator $\mathcal{P}_h^\star = -\partial_t + D_h^4$ defined on $Q := \mathring{\mathcal{M}} \times (0,T)$.

Theorem 4.1 *We define the function φ according to (4.6)–(4.7). There exist C, $\lambda_0 \geq 1$, $h_0 > 0$, $\epsilon_0 > 0$ such that the following estimate holds*

$$\begin{aligned}
&\int_Q s^7 e^{-2s\varphi}|w|^2 + \int_{Q^*} s^5 e^{-2s\varphi}|D_h w|^2 + \int_{\overline{Q}} s^3 e^{-2s\varphi}|D_h^2 w|^2 + \int_{Q^*} s e^{-2s\varphi}|D_h^3 w|^2 \\
&+ \int_Q s^{-1} e^{-2s\varphi}|\partial_t w|^2 \leq C\Big(\|e^{-s\varphi}\mathcal{P}_h^\star w\|^2_{L^2_h(Q)} + \int_0^T s^3 e^{-2s\varphi}|D_h^2 w|^2\Big|_0 \\
&+ \int_0^T s e^{-2s\varphi}|D_h^3 w|^2\Big|_{h/2} + h^{-4}\int_{\mathcal{M}} e^{-2s\varphi}|w|^2\Big|_{t=0} + h^{-4}\int_{\mathcal{M}} e^{-2s\varphi}|w|^2\Big|_{t=T}\Big),
\end{aligned} \tag{4.11}$$

for all $\lambda \geq \lambda_0(T+T^2)$, $0 < h \leq h_0$ and $\lambda h(\delta T^2)^{-1} \leq \varepsilon_0$ and for all $w \in C^\infty(0,T;C(\mathcal{M}))$ satisfying $w = 0$ on $\partial\mathring{\mathcal{M}}$ and $D_h w = 0$ on $\partial\mathcal{M}^*$.

Let us notice that the last two terms in the right hand side are new with respect to the continuous case. These additional terms are expected in the semi-discrete case as they also appear for second-order parabolic equations. The main idea of the proof is a combination of the finite-difference setting for the derivation of discrete Carleman estimate as in [7, 42] for second order parabolic operator and the strategy in [15] to obtain the Carleman estimate in the continuous framework for the Kuramoto-Sivashinsky equation.

As in the continuous case, this Carleman estimate implies an observability inequality for system (4.5) which is the following. There exists $C_{\text{obs}} > 0$ such that for any $q(T) \in L^2_h(\mathcal{M})$ we have that the solution of (4.5) satisfies

$$\|q(0)\|^2_{L^2_h(\mathcal{M})} \leq C^2_{\text{obs}}\left(\int_0^T |D_h^2 q|^2\Big|_0 + \int_0^T |D_h^3 q|^2\Big|_{h/2} + e^{-\frac{C_1}{h}}\|q(T)\|^2_{L^2_h(\mathcal{M})}\right),$$

with $C_{\text{obs}} = e^{C_2\left(1+\frac{1}{T}+T\|a\|_{L^\infty_h(Q)}+\|a\|^{2/7}_{L^\infty_h(Q)}\right)}$. Here again we have an extra term with respect to the continuous case, which is the last one in previous inequality. Because of this phenomenon, this weak observability implies what we call a ϕ-controllability result, given by the following result.

Theorem 4.2 *Let h_0 be given by Theorem 4.1. There exists C_1, $C_2 > 0$ such that if $h \leq$*

$\min(h_0, h_1)$ *with*

$$h_1 = C_1\left(1+\frac{1}{T}+\|a\|_\infty^{2/7}\right)^{-1},$$

then for any initial data y_0*, there exist control functions* (u_1, u_2) *such that the solution to* (4.4) *satisfies*

$$\begin{aligned}\|y(T)\|_{L_h^2(\mathcal{M})} \leq & C_0 e^{-C_2/h}\|y_0\|_{L_h^2(\mathcal{M})},\\ \|u_1\|^2_{L^2(0,T)}+\|u_2\|^2_{L^2(0,T)} \leq & C_0\|y_0\|_{L_h^2(\mathcal{M})}.\end{aligned}$$

4.1. Semi-discrete Carleman estimate for uniform meshes

In this Section, we proof the semi-discrete Carleman estimate (4.11). For the sake of presentation, we split the proof into three steps. We follow a classical scheme based on conjugating the original operator with a well-chosen exponential function. We write the conjugate operator into symmetric part S_h, antisymmetric part P_h and an additional term R_h to reduce some computations. Then, we estimate the cross-inner product between these operators, which is divided into a sequence of Lemmas; and as a final stage we return to the original variable.

4.1.1. Conjugate operator

We recall the notation of our semi-discrete mesh $Q := \mathring{\mathcal{M}} \times (0,T)$. We also consider the notation $Q^* = \mathcal{M}^* \times (0,T)$, and $Q^{**} = \overline{\mathcal{M}} \times (0,T)$. At first, under the substitution $w = rv$, the semi-discrete differential operator becomes

$$\mathcal{P}_h^\star := -\partial_t w + D_h^4 w \text{ in } Q,$$

becomes

$$\rho\mathcal{P}_h^\star = -\rho\partial_t(rv) + \rho D_h^4(rv). \tag{4.12}$$

Direct calculation yields $\rho\partial_t(rv) = \partial_t v + \rho\partial_t(r)v = \partial_t v + \lambda\partial_t\theta\,\varphi\,v$, and repeated application of Lemma 2.1 gives

$$\begin{aligned}D_h^4(rv) =& D_h^4(r)\,A_h^4(v) + 4D_h^3A_h(r)\,D_hA_h^3(v) + 6A_h^2D_h^2(r)\,D_h^2A_h^2(v)\\ &+ 4A_h^3D_h(r)\,D_h^3A_h(v) + A_h^4(r)\,D_h^4(v).\end{aligned}$$

Equation (4.12) thus reads $S_hv + P_hv = \rho\mathcal{P}_h(rv) + R_hv$, where $S_hv = S_1v + S_2v + S_3v + S_4v$ and $P_hv = P_1v + P_2v + P_3v + P_4v$ with

$$\begin{aligned}S_hv =& 6\rho A_h^2D_h^2r\,D_h^2A_h^2v + \rho D_h^4r\,A_h^4v + \rho A_h^4r\,D_h^4v + 6A_hD_h(\rho A_h^2D_h^2r)\,A_hD_hv,\\ P_hv =& -\partial_tv + 4\rho A_hD_h^3r\,D_hA_h^3v + 4\rho A_h^3D_hr\,D_h^3A_hv + 2A_hD_h(\rho\,A_hD_h^3r)\,v,\\ R_hv =& \lambda\varphi\,\partial_t\theta\,v + 6A_hD_h(\rho\;A_h^2D_h^2r)\,A_hD_hv + 2A_hD_h(\rho\,A_hD_h^3r)\,v.\end{aligned} \tag{4.13}$$

As usual when looking for Carleman estimates, $\rho\mathcal{P}_h(rv)$ is decomposed into symmetric part S_hv and antisymmetric part P_hv. We refer to [7, 27, 42] for this decomposition. Here, we introduced the additional terms P_4v and S_4v which also is provided as the case of continuous

Kuramoto-Shivashinsky equation [15]. These additional terms are crucial. The aim of these additions is to eliminate difficult calculations in the inner product $\langle P_h v, S_h v\rangle_{L_h^2(Q)}$.

Then, by using the triangle inequality, we write

$$C\left(\|\rho\mathcal{P}_h^\star\|^2_{L_h^2(Q)} + \|R_h v\|^2_{L_h^2(Q)}\right) \geq \|\rho\mathcal{P}_h^\star + R_h v\|^2_{L_h^2(Q)} \\ = \|P_h v\|^2_{L_h^2(Q)} + \|S_h v\|^2_{L_h^2(Q)} + 2\int_Q P_h v\, S_h v. \tag{4.14}$$

The next step is to provide an estimate for the left-hand side of (4.14).

4.1.2. An estimate for the left-hand side

We need the following estimation for $\|R_h v\|^2_{L_h^2(Q)}$. In fact, using the triangular inequality and applying Theorem 2.2 we obtain

Lemma 4.1 *For $\lambda h(T^2\delta)^{-1} \leq 1$, we have*

$$\|R_h v\|^2_{L_h^2(Q)} \leq C\left(\left\|s^3 v\right\|^2_{L_h^2(Q)} + \left\|s^2 D_h v\right\|^2_{L_h^2(Q)} + \|Ts\theta\varphi v\|^2_{L_h^2(Q)}\right). \tag{4.15}$$

Proof. Using triangle inequality, we note that

$$\|R_h v\|^2_{L_h^2(Q)} \leq C\left(\|Ts\theta\varphi\, v\|^2_{L_h^2(Q)} + \left\|6A_h D_h(\rho\, A_h^2 D_h^2 r)\; A_h D_h v\right\|^2_{L_h^2(Q)} \right. \\ \left. + \left\|2A_h D_h(\rho\, A_h D_h^3 r)\, v\right\|^2_{L_h^2(Q)}\right).$$

Now, applying Theorem 2.2, we obtain

$$|6A_h D_h(\rho\, A_h^2 D_h^2 r)| \leq C|s|^2 \\ |2A_h D_h(\rho A_h D_h^3 r)| \leq C|s|^3.$$

Then, for R_h, we have

$$\|R_h v\|^2_{L_h^2(Q)} \leq C\left(\|Ts\theta\varphi\, v\|^2_{L_h^2(Q)} + \left\|s^2\, A_h D_h v\right\|^2_{L_h^2(Q)} + \left\|s^3\, v\right\|^2_{L_h^2(Q)}\right). \tag{4.16}$$

On the other hand, we note that

$$\left\|s^2\, A_h D_h v\right\|^2_{L_h^2(Q)} := \int_Q s^4 |A_h D_h v|^2 \leq \int_Q s^4 A_h(|D_h v|^2),$$

thanks to inequality (2.6). Integrating by parts with respect to the average operator A_h, we get

$$\int_Q s^4 A_h(|D_h v|^2) = \int_{Q^*} s^4 |D_h v|^2 - \frac{h}{2}\int_{\partial Q} s^2 t_r(|D_h v|^2) \leq \int_{Q^*} s^4 |D_h v|^2.$$

Thus,

$$\left\|s^2\, A_h D_h v\right\|^2_{L_h^2(Q)} \leq \left\|s^2 D_h v\right\|^2_{L_h^2(Q^*)}. \tag{4.17}$$

Hence, combining (4.16) and (4.17) proves the required result. □

4.1.3. An estimate for the cross-term

In this stage, we provide an estimate for the terms of the form

$$\int_Q P_h v\, S_h v = \sum_{i=1}^{4}\sum_{j=1}^{4} I_{ij}, \tag{4.18}$$

where I_{ij} stands for the inner product in $L_h^2(Q)$ between the i^{th}-term of $P_h v$ and the j^{th}-term of $S_h v$. For the sake of presentation, we present the estimate and proof of each term from (4.18) in Section 4.3. Combining the aforementioned results from Section 4.3, we obtain

$$\int_Q P_h v\, S_h v = \sum_{i=1}^{4}\sum_{j=1}^{4} I_{ij} \geq \sum_{k=0}^{3} I(D_h^k v) + \sum_{\substack{i,j=1\\ j\neq 4}}^{4} X_{ij} + Y + Y^t,$$

where

$$\begin{aligned} I(v) :=& 8\int_Q s^5 \partial_x^2\left((\partial_x\varphi)^4\partial_x^2\varphi\right)|v|^2 + \int_Q s^4\mathcal{O}(T)|v|^2 - 8\int_Q s^7(\partial_x\varphi)^6\partial_x^2\varphi|v|^2 \\ &+ 3\int_Q s^3\partial_x^4\left((\partial_x\varphi)^2\partial_x^2\varphi\right)|v|^2 - 36\int_Q s^5\partial_x(\partial_x(\varphi^3)\partial_x^2(\varphi^2))|v|^2, \end{aligned}$$

$$\begin{aligned} I(D_h v) :=& -18\int_{Q^*} s^5(\partial_x\varphi)^4\partial_x^2\varphi|D_h v|^2 + 6\int_{Q^*} s^3\partial_x^2((\partial_x\varphi)^2\partial_x^2\varphi)|D_h v|^2 \\ &- s^2\mathcal{O}(T)\int_{Q^*}|D_h v|^2, \end{aligned}$$

$$I(D_h^2 v) := -60\int_{\overline{Q}} s^3(\partial_x\varphi)^2\partial_x^2\varphi|D_h^2 v|^2,$$

$$I(D_h^3 v) := -2\int_{Q^*} s\partial_x^2\varphi|D_h^3 v|^2,$$

$$Y := Y_{12}+Y_{13}+Y_{22}+Y_{23}+Y_{32}+Y_{33}+Y_{42}+Y_{43}$$

and

$$Y^t := Y_{11}+Y_{21}+Y_{31}.$$

We can choose $\varepsilon_1 > 0$ and $h_1 > 0$, with $0 < \varepsilon_1 \leq 1$ and a sufficiently large $\lambda_1 \geq 1$, such that for $\lambda \geq \lambda_1$, $0 < h \leq h_1$ and $\lambda_1 h_1(\delta T^2)^{-1} \leq \varepsilon_1$, there exists a constant $C_{\lambda_1,\varepsilon_1} > 0$ such that

$$\begin{aligned} &\int_Q s^7|v|^2 + \int_{Q^*} s^5|D_h v|^2 + \int_{\overline{Q}} s^3|D_h^2 v|^2 + \int_{Q^*} s|D_h^3 v|^2 + \int_{\partial Q} s^5 t_r(|D_h v|^2)n \\ &+ \int_{\partial Q} s^3|D_h^2 v|^2 n + \int_{\partial Q} s t_r(|D_h^3 v|^2)n - \int_{\mathcal{M}^*} s^2|D_h v|^2\Big|_{t=0} - \int_{\mathcal{M}^*} s^2|D_h v|^2\Big|_{t=T} \\ &- \int_{\mathcal{M}} s^4|v|^2\Big|_{t=0} - \int_{\mathcal{M}} s^4|v|^2\Big|_{t=T} \leq C_{\lambda_1,\varepsilon_1}\left(\sum_{k=0}^{3} I(D_h^k v) + \sum_{\substack{i,j=1\\ j\neq 4}}^{4} X_{ij} + Y + Y^t\right). \end{aligned} \tag{4.19}$$

4.1.4. Conclusion

At this stage, we absorb the right-hand side established in Lemma 4.1, and then we return to the variable v. To this end, we will need Lemmas 4.2-4.4 which will be proved in Section 4.3.

Combining Lemma 4.1 and the inequalities (4.19) and (4.14), we can choose ε_2 and h_0 sufficiently small, with $0 < \varepsilon_3 < \varepsilon_2$, $0 < h_0 \leq h_1$ and $\lambda \geq 1$ sufficiently large such that for $\lambda \geq \lambda_1(T + T^2)$, $0 < h < h_0$ and $\lambda h(\delta T^2)^{-1} \leq \varepsilon_3$ there exists a constant $C_{\lambda,\varepsilon_3} > 0$ such that

$$C_{\lambda,\varepsilon}\Bigg(\|\rho\mathcal{P}_h^\star\|^2_{L^2_h(Q)} + \int_0^T s^5|D_h v|^2\Big|_{h/2} + \int_0^T s^3|D_h^2 v|^2\Big|_0 + \int_0^T s|D_h^3 v|^2\Big|_{h/2} + h^{-4}\int_{\mathcal{M}}|v|^2\Big|_{t=0}$$
$$+h^{-4}\int_{\mathcal{M}}|v|^2\Big|_{t=T}\Bigg) \geq \|P_h v\|^2_{L^2_h(Q)} + \|S_h v\|^2_{L^2_h(Q)} + \int_Q s^7|v|^2 + \int_{Q^*} s^5|D_h v|^2 \quad (4.20)$$
$$+\int_{\overline{Q}} s^3|D_h^2 v|^2 + \int_{Q^*} s|D_h^3 v|^2 + \int_Q s^{-1}\mathcal{O}((sh)^2)|\partial_t v|^2,$$

where we have used that

$$|D_h^2 v|^2 \leq Ch^{-4}\left(|v|^2 + |\tau_+^2 v|^2 + |\tau_-^2 v|^2\right)$$

and

$$|D_h v|^2 \leq Ch^{-2}\left(|\tau_+ v|^2 + |\tau_- v|^2\right),$$

and we have dropped the right-hand side boundary terms.

On the other hand, in order to absorb the term

$$\int_Q s^{-1}\mathcal{O}((sh)^2)|\partial_t v|^2,$$

we need the following result.

Lemma 4.2 *For $\lambda \geq \lambda_1(T + T^2)$ and $\lambda h(\delta T^2)^{-1} \leq 1$, there exists a constant $C > 0$ such that*

$$\left\|s^{-1/2}\partial_t v\right\|^2_{L^2_h(Q)} \leq C\left(\|P_h v\|^2_{L^2_h(Q)} + \int_{Q^*} s^5|D_h v|^2 + \int_{Q^*} s|D_h^3 v|^2 + \int_Q s^7|v|^2\right).$$

Proof. See 4.3.16 in Appendix. ☐

Thus, with $0 < \varepsilon_0 < \varepsilon_1$ sufficiently small and $\lambda h(\delta T^2)^{-1} \leq \varepsilon_0$, we obtain

$$C\Bigg(\|\rho\mathcal{P}_h^\star\|^2_{L^2_h(Q)} + \int_0^T s^5|D_h v|^2\Big|_{h/2} + \int_0^T s^3|D_h^2 v|^2\Big|_0 + \int_0^T s|D_h^3 v|^2\Big|_{h/2} + h^{-4}\int_{\mathcal{M}}|v|^2\Big|_{t=0}$$
$$+ h^{-4}\int_{\mathcal{M}}|v|^2\Big|_{t=T}\Bigg) \geq \int_Q s^7|v|^2 + \int_{Q^*} s^5|D_h v|^2 + \int_{\overline{Q}} s^3|D_h^2 v|^2 + \int_{Q^*} s|D_h^3 v|^2 + \int_Q s^{-1}|\partial_t v|^2.$$
$$(4.21)$$

We now proceed to return to the variable v. Recalling that $w := rv$, and using Lemma 2.1, we obtain

Lemma 4.3 *For $\lambda h(\delta T^2)^{-1} \leq 1$, we have*

$$\begin{aligned}
\left\|s^{-\frac{1}{2}}\rho\partial_t w\right\|^2_{L^2_h(Q)} \leq & C\left(\|sv\|^2_{L^2_h(Q)} + \left\|s^{-\frac{1}{2}}\partial_t v\right\|^2_{L^2_h(Q)}\right) \\
\|\rho D_h w\|^2_{L^2_h(Q^*)} \leq & C\left(\|sv\|^2_{L^2_h(Q)} + \|D_h v\|^2_{L^2_h(Q^*)}\right), \\
\left\|s^3\rho D_h^2 w\right\|^2_{L^2_h(Q)} \leq & C\left(\left\|s^{\frac{7}{2}}v\right\|^2_{L^2_h(Q)} + \left\|s^{\frac{5}{2}}D_h v\right\|^2_{L^2_h(Q^*)} + \left\|s^{\frac{3}{2}}D_h^2 v\right\|^2_{L^2_h(\overline{Q})}\right), \\
\left\|\rho s D_h^3 w\right\|^2_{L^2_h(Q^*)} \leq & C\left(\left\|s^{\frac{3}{2}}v\right\|^2_{L^2_h(Q)} + \left\|s^{\frac{5}{2}}D_h v\right\|^2_{L^2_h(Q^*)} + \left\|sD_h^2 v\right\|^2_{L^2_h(\overline{Q})} + \left\|s^{\frac{1}{2}}D_h^3 v\right\|^2_{L^2_h(Q^*)}\right).
\end{aligned}$$

Proof. See 4.3.17 in Appendix. □

Moreover, for the boundary terms we have the following result.

Lemma 4.4 *For $\lambda h(\delta T^2)^{-1} \leq 1$, we have the following*

$$\begin{aligned}
\int_0^T s^5|D_h v|^2\bigg|_{h/2} \leq & C\int_0^T s^3\mathcal{O}((sh)^2)|\rho\, D_h^2 w|^2\bigg|_0, \\
\int_0^T s^3|D_h^2 v|^2\bigg|_0 \leq & C\int_0^T s^3|\rho\, D_h^2 w|^2\bigg|_0, \\
\int_0^T s|D_h^3 v|^2\bigg|_{h/2} \leq & C\left(\int_0^T s^3|\rho\, D_h^2 w|^2\bigg|_0 + \int_0^T s|\rho D_h^3 w|^2\bigg|_{h/2}\right).
\end{aligned}$$

Proof. See 4.3.18 in Appendix. □

Therefore, the Carleman estimate (4.11) follows using Lemma 4.3 and 4.4 in (4.21).

4.2. $\phi(h)-$null controllability

In this Section, by using the Carleman estimate (4.11) we deduce boundary control properties for linear semi-discrete fourth-order parabolic system. As usual, the proof is based on a relaxed observability estimate.

4.2.1. Observability inequalities

For a potential $a \in L^\infty_h(Q)$, we consider the following linear semi-discrete fourth-order parabolic system

$$\begin{cases}
\partial_t y + D_h^4 y + ay = 0 \text{ in } Q, \\
y(t,0) = u_1(t),\ y(t,1) = 0, \\
D_h y(t, -\frac{h}{2}) = u_2(t),\ D_h y(t, 1+\frac{h}{2}) = 0, \\
y(0) = y_0.
\end{cases} \tag{4.22}$$

We prove a relaxed observability estimate for the adjoint system of (4.22) given by

$$\begin{cases} -\partial_t q + D_h^4 q + aq = 0 \text{ in } Q, \\ q(0,t) = 0, \ q(1,t) = 0, \\ D_h q(-h/2,t) = 0, \ D_h q(1+h/2,t) = 0, \\ q(T) = q_T. \end{cases} \tag{4.23}$$

The Carleman estimate (4.11) that we proved in the previous Section allows us to obtain an observability inequality for the system (4.23). We follow the strategy from [10].

Proposition 4.1 *There exist positive constants h_0, C_0, C_1 and C_2 such that for all $T > 0$, under the condition $h \le \min\{h_0, h_1\}$ with*

$$h_1 := C_0 \left(1 + \frac{1}{T} + \|a\|_{L_h^\infty(Q)}^{2/7}\right)^{-1},$$

any semi-discrete solution of (4.23) satisfies

$$\|q(0)\|_{L_h^2(\mathcal{M})}^2 \le C_{obs}^2 \left(\int_0^T |D_h^2 q|^2 \Big|_0 + \int_0^T |D_h^3 q|^2 \Big|_{h/2} + e^{-\frac{C_1}{h}} \|q(T)\|_{L_h^2(\mathcal{M})}^2 \right),$$

with

$$C_{obs} = e^{C_2\left(1+\frac{1}{T}+T\|a\|_{L_h^\infty(Q)}+\|a\|_{L_h^\infty(Q)}^{2/7}\right)}.$$

Proof. Let us first consider the change of variable $\tilde{q} = e^{\|a\|_\infty (t-T)} q$. By using the Carleman estimate (4.11) to the solution of the semi-discrete system (4.23) we have

$$\begin{aligned} \int_Q s^7 e^{-2s\varphi} q \le C \Bigg(\|e^{-s\varphi} a\, q\|_{L_h^2(Q)}^2 + \int_0^T s^3 e^{-2s\varphi} |D_h^2 q|^2 \Big|_0 + \int_0^T s e^{-2s\varphi} |D_h^3 q|^2 \Big|_{h/2} \\ + h^{-4} \int_{\mathcal{M}} e^{-2s\varphi} |q|^2 \Big|_{t=0} + h^{-4} \int_{\mathcal{M}} e^{-2s\varphi} |q|^2 \Big|_{t=T} \Bigg), \end{aligned} \tag{4.24}$$

for all $\lambda \ge \lambda_0(T + T^2)$ and $0 < h \le h_0$. We note that by taking

$$\lambda \ge CT^2 \|a\|_{L_h^\infty(Q)}^{2/7}, \tag{4.25}$$

we can absorb the first term in the right-hand side of (4.24) by its the left-hand side, obtaining

$$\begin{aligned} \int_Q s^7 e^{-2s\varphi} q \le C \Bigg(\int_0^T s^3 e^{-2s\varphi} |D_h^2 q|^2 \Big|_0 + \int_0^T s e^{-2s\varphi} |D_h^3 q|^2 \Big|_{h/2} \\ + h^{-4} \int_{\mathcal{M}} e^{-2s\varphi} |q|^2 \Big|_{t=0} + h^{-4} \int_{\mathcal{M}} e^{-2s\varphi} |q|^2 \Big|_{t=T} \Bigg), \end{aligned} \tag{4.26}$$

for $\lambda \ge \lambda_1(T + T^2 + T^2 \|a\|_{L_h^\infty(Q)}^{2/7})$, where we have combined the hypothesis on λ for Carleman estimate and the condition (4.25).

On the other hand, we claim that

$$\|q(0)\|^2_{L^2_h(\mathcal{M})} \leq \|q(t)\|^2_{L^2_h(\mathcal{M})}, \ t \in (0,T). \tag{4.27}$$

In fact, multiplying the main equation of the system (4.23) by q and integrate over $R := \mathcal{M} \times (0,t)$, with $t \in (0,T)$, it follows that

$$-\int_R q\,\partial_t q + \int_R q\,D_h^4 q + \int_R a\,q^2 = 0.$$

Then, applying Corollary 2.2 to the second integral above, we have

$$0 = -\frac{1}{2}\int_R \partial_t(|q|^2) + \int_R |D_h^2 q|^2 + \int_{\partial R} q\,t_r(D_h^3 q) + D_h^2 q\,t_r(D_h q)\,n + \int_R a\,q^2.$$

The boundary conditions on q yield

$$0 = -\frac{1}{2}\int_{\mathcal{M}} |q(t)|^2 + \frac{1}{2}\int_{\mathcal{M}} |q(0)|^2 + \int_R |D_h^2 q|^2 + \int_R a\,q^2,$$

and the claim follows.

In order to get the norm of $q(x,0)$ at the left-hand side of (4.26) we focus our analysis on $(\frac{T}{4}, \frac{3T}{4})$. We note that

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\mathcal{M}} s^7 e^{-2s\varphi}|q(t)|^2 \leq \int_Q s^7 e^{-2s\varphi}|q(t)|^2. \tag{4.28}$$

Now, recalling that θ is decreasing on $(0,T/2)$ and creasing on $(T/2,T)$, by using (4.27) and (4.26) it follows that

$$\begin{aligned}\int_Q s^7 e^{-2s\varphi}|q(t)|^2 &\geq \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\mathcal{M}} s^7 e^{-2s\varphi}|q(t)|^2 \\ &\geq \int_{\frac{T}{4}}^{\frac{3T}{4}} \lambda^7 \theta\left(\frac{T}{2}\right)^7 e^{-2\lambda\theta\left(\frac{T}{4}\right)\sup\psi} \|q(0)\|^2_{L^2_h(\mathcal{M})}.\end{aligned}$$

Thus, noting that $\theta\left(\frac{T}{2}\right) \geq C\frac{1}{T^2}$ and $2\theta\left(\frac{T}{4}\right)\sup\varphi \geq C''\frac{1}{T^2}$, we obtain

$$\int_Q s^7 e^{-2s\varphi}|q(t)|^2 \geq C T e^{-C''\frac{\tau}{T^2}} \|q(0)\|^2_{L^2_h(\mathcal{M})}, \tag{4.29}$$

as $\tau \geq \tau_0 T^2$.

Now, as $\theta(T) = \theta(0) \geq (T^2\alpha)^{-1}$, by using (4.27) we have

$$\begin{aligned}\int_{\mathcal{M}} e^{-s\varphi}|q|^2\Big|_{t=0} + \int_{\mathcal{M}} e^{-s\varphi}|q|^2\Big|_{t=T} &\leq C' e^{-C\frac{\lambda}{\alpha T^2}} \left(\|q(0)\|^2_{L^2_h(\mathcal{M})} + \|q(T)\|^2_{L^2_h(\mathcal{M})}\right) \\ &\leq C' e^{-C\frac{\lambda}{\alpha T^2}} \|q(T)\|^2_{L^2_h(\mathcal{M})}.\end{aligned} \tag{4.30}$$

Besides, we write

$$\int_0^T s^3 e^{-2s\varphi}|D_h^2 q|^2\Big|_0 + \int_0^T s e^{-2s\varphi}|D_h^3 q|^2\Big|_{h/2} \leq C e^{-\lambda\eta\theta\left(\frac{T}{2}\right)} \left(\int_0^T |D_h^2 q|^2\Big|_0 + \int_0^T |D_h^3 q|^2\Big|_{h/2}\right)$$
$$\leq C e^{-C'\frac{\lambda}{T^2}} \left(\int_0^T |D_h^2 q|^2\Big|_0 + \int_0^T |D_h^3 q|^2\Big|_{h/2}\right), \tag{4.31}$$

since $C'T^{-2} \leq \min_{t\in[0,T]} \theta(t)$.
Thus, combining (4.24), (4.29), (4.30) and (4.31) we obtain

$$T \|q(0)\|_{L_h^2(\mathcal{M})} \leq C e^{C\frac{\lambda}{T^2}} \left(\int_0^T |D_h^2 q|^2\Big|_0 + \int_0^T |D_h^3 q|^2\Big|_{h/2}\right) + h^{-4} e^{\frac{\lambda}{T^2}\left(C - \frac{C'}{\alpha}\right)} \|q(T)\|^2_{L_h^2(\mathcal{M})}.$$

For $0 < \alpha \leq \alpha_1 \leq \alpha_0$ with α_1 sufficiently small we obtain

$$T \|q(0)\|_{L_h^2(\mathcal{M})} \leq C e^{C\frac{\lambda}{T^2}} \left(\int_0^T |D_h^2 q|^2\Big|_0 + \int_0^T |D_h^3 q|^2\Big|_{h/2}\right) + h^{-4} e^{-C''\frac{\tau}{T^2\alpha}} \|q(T)\|^2_{L_h^2(\mathcal{M})}. \tag{4.32}$$

Recalling that the condition for the Carleman estimate $\frac{\lambda h}{\alpha T^2} \leq \varepsilon_0$ must be fulfilled for $\alpha \leq \alpha_1$. We fix $\lambda = \lambda_1(T + T^2 + T^2 \|a\|^{2/7})$ and we define

$$h_1 := \frac{\varepsilon_0}{\lambda_1} \alpha_1 \left(1 + \frac{1}{T} + \|a\|_\infty^{2/7}\right)^{-1},$$

which gives

$$\frac{\lambda h_1}{\alpha_1 T^2} = \varepsilon_0.$$

Choosing $h \leq \min\{h_0, h_1\}$ and $\alpha = \alpha_1 \frac{h}{h_1} \leq \alpha_1$ we then find $\frac{\lambda h}{\alpha T^2} = \varepsilon_0$. As $\frac{\lambda}{T^2\alpha} = \frac{\varepsilon_0}{h}$, we obtain from (4.32)

$$\|q(0)\|_{L_h^2(\mathcal{M})} \leq C e^{C''(1+\frac{1}{T}+\|a\|_\infty^{2/7})} \left(\int_0^T |D_h^2 q|^2\Big|_0 + \int_0^T |D_h^3 q|^2\Big|_{h/2}\right) + h^{-4} e^{-C''\frac{\varepsilon_0}{h}} \|q(T)\|^2_{L_h^2(\mathcal{M})}.$$

which gives

$$\|q(0)\|_{L_h^2(\mathcal{M})} \leq C e^{C''(1+\frac{1}{T}+\|a\|_\infty^{2/7})} \left(\int_0^T |D_h^2 q|^2\Big|_0 + \int_0^T |D_h^3 q|^2\Big|_{h/2}\right) + e^{-\frac{C'''}{h}} \|q(T)\|^2_{L_h^2(\mathcal{M})}.$$

The Proposition follows recalling the change of variable that we have been considered. $\square$

4.2.2. Controllability results

The previous observability inequality yields a controllability result for the linear semi-discrete system (4.22). The analysis is based on the Hilbert uniqueness method introduced by Lions in [38] for the continuous case. For our semi-discrete setting we follow the strategy developed in [10] by Boyer and Le Rousseau (see also [6]).

PROOF THEOREM 4.2. The observability inequality of Proposition 4.1 gives

$$\|q(0)\|^2_{L^2_h(\mathcal{M})} \leq C_{\text{obs}} \left(\int_0^T |D_h^2 q|^2 \Big|_0 + \int_0^T |D_h^3 q|^2 \Big|_{h/2} \right) + \epsilon \|q_T\|^2_{L^2_h(\mathcal{M})},$$

with $C_{\text{obs}} = e^{C_2\left(1+\frac{1}{T}+T\|a\|_{L_h^\infty(Q)}+\|a\|^{2/7}_{L_h^\infty(Q)}\right)}$ and $\epsilon = e^{-C_1/h}$. We introduce the penalized functional

$$\begin{aligned} J(q_T) :=& \frac{1}{2} \left(\int_0^T |D_h^2 q|^2 \Big|_0 + \int_0^T |D_h^3 q|^2 \Big|_{h/2} \right) + \frac{\epsilon}{2} \|q_T\|^2_{L^2_h(\mathcal{M})} + \langle y_0, q(0)\rangle_{L^2_h(\mathcal{M})} \\ =& \frac{1}{2} \|W\|^2_{L^2(0,T)} + \frac{\epsilon}{2} \|q_T\|^2_{L^2_h(\mathcal{M})} + \langle y_0, q(0)\rangle_{L^2_h(\mathcal{M})}, \end{aligned}$$

where

$$\|W\|^2_{L^2(0,T)} := \int_0^T |D_h^2 q|^2 \Big|_0 + \int_0^T |D_h^3 q|^2 \Big|_{h/2}.$$

We note that, thanks to Cauchy-Schwarz's inequality we have

$$J(q_T) \geq \frac{1}{2} \|W\|^2_{L^2(0,T)} + \frac{\epsilon}{2} \|q_T\|^2_{L^2_h(\mathcal{M})} - \|y_0\|_{L^2_h(\mathcal{M})} \|q(0)\|_{L^2_h(\mathcal{M})}$$

Then, by using the Young's inequality we obtain

$$J(q_T) \geq \frac{1}{2} \|W\|^2_{L^2(0,T)} + \frac{\epsilon}{2} \|q_T\|^2_{L^2_h(\mathcal{M})} - \frac{1}{4C_{\text{obs}}} \|q(0)\|^2_{L^2_h(\mathcal{M})} - C_{\text{obs}} \|y(0)\|^2_{L^2_h(\mathcal{M})}.$$

Thus, by using the observability inequality we conclude

$$\begin{aligned} J(q_T) \geq& \frac{1}{4} \|W\|^2_{L^2(0,T)} + \frac{\epsilon}{4} \|q_T\|^2_{L^2_h(\mathcal{M})} - C^2_{\text{obs}} \|y(0)\|_{L^2_h(\mathcal{M})} \\ \geq& \frac{\epsilon}{4} \|q_T\|^2_{L^2_h(\mathcal{M})} - C^2_{\text{obs}} \|y(0)\|^2_{L^2_h(\mathcal{M})}. \end{aligned}$$

Therefore, the functional J is coercive. Besides, J is smooth, strictly convex on a finite dimensional space, thus it admits a unique minimizer $q_T = q_T^{\text{opt}}$. We denote by q_T^{opt} the associated solution of the dual system (4.23). The Euler-Lagrange equation associated with this minimization problem reads

$$\langle W^{\text{opt}}, W\rangle_{L^2(0,T)} + \epsilon \langle q_T^{\text{opt}}, q_T\rangle_{L^2_h(\mathcal{M})} = -\langle y_0, q(0)\rangle, \tag{4.33}$$

for any $q_T \in \mathbb{R}^n$, with the associated solution $q(t)$ of the dual system (4.23). By setting the control $(u_1(t), u_2(t)) := (D_h^3 q^{\text{opt}}|_{\frac{h}{2}}, -D_h^2 q^{\text{opt}}|_0)$, we consider the solution y to the controlled problem as follows

$$\begin{cases} \partial_t y + D_h^4 y + ay = 0, \\ y(0,t) = u_1(t), \ y(1,t) = 0, \\ D_h y(t, -\frac{h}{2}) = u_2(t), \ D_h y(t, 1+\frac{h}{2}) = 0, \\ y(0) = y_0. \end{cases} \tag{4.34}$$

By multiplying by q, solution of the system (4.23), the main equation of (4.34) and applying Corollary 2.2, we have

$$\begin{aligned} 0 &= \int_{\mathcal{M}} q\, y \Big|_0^T + \int_{\partial Q^*} \Big(D_h y\, t_r(D_h^2 q) - D_h q\, t_r(D_h^2 y) \Big)\, n + \int_{\partial Q} \Big(q\, t_r(D_h^3 y) - y\, t_r(D_h^3 q) \Big)\, n \\ &= \int_{\mathcal{M}} q\, y \Big|_0^T - \int_0^T D_h y\, t_r(D_h^2 q) \Big|_{-\frac{h}{2}} + \int_0^T y\, t_r(D_h^3 q) \Big|_0 . \end{aligned}$$

Then, we deduce

$$\langle y(T), q_T \rangle_{L_h^2(\mathcal{M})} - \langle y_0, q(0) \rangle_{L_h^2(\mathcal{M})} + \int_0^T D_h^2 q^{\text{opt}} D_h^2 q \Big|_0 + \int_0^T D_h^3 q^{\text{opt}}\, D_h^3 q \Big|_{\frac{h}{2}} = 0,$$

for any $q_T \in \mathbb{R}^n$. From (4.33) we conclude that

$$y(T) = -\epsilon q_T^{\text{opt}}. \tag{4.35}$$

Now, we take $q_T = q_T^{\text{opt}}$ in (4.33) to obtain

$$\begin{aligned} \|W\|_{L^2(0,T)}^2 + \epsilon \left\| q_T^{\text{opt}} \right\|_{L_h^2(\mathcal{M})}^2 &= - \langle y_0, q^{\text{opt}}(0) \rangle_{L_h^2(\mathcal{M})} \\ &\leq \|y_0\|_{L_h^2(\mathcal{M})} \left\| q^{\text{opt}}(0) \right\|_{L_h^2(\mathcal{M})} . \end{aligned} \tag{4.36}$$

The observability inequality, for q^{opt}, enables us to write

$$\left\| q^{\text{opt}}(0) \right\|_{L_h^2(\mathcal{M})}^2 \leq C_{\text{obs}}^2 \left(\|W\|_{L^2(0,T)}^2 + \epsilon \left\| q_T^{\text{opt}} \right\|_{L_h^2(\mathcal{M})}^2 \right). \tag{4.37}$$

Thus, combining (4.36) with (4.37) it follows that

$$\epsilon^{1/2} \left\| q_T^{\text{opt}} \right\|_{L_h^2(\mathcal{M})} \leq C_{\text{obs}} \|y_0\|_{L_h^2(\mathcal{M})} \tag{4.38}$$

and

$$\|W\|_{L^2(0,T)} \leq C_{\text{obs}} \|y_0\|_{L_h^2(\mathcal{M})} .$$

Finally, from (4.35) and (4.38) we get

$$\|y(T)\|_{L_h^2(\mathcal{M})} \leq C_{\text{obs}} e^{-C/h} \|y_0\|_{L_h^2(\mathcal{M})}$$

and

$$\|W\|_{L^2(0,T)} \leq C_{\text{obs}} \|y_0\|_{L_h^2(\mathcal{M})} ,$$

and proof is complete. $\square$

Let us mention that the previous result can be stated for a function $h \mapsto \phi(h)$ such that

$$\liminf_{h\to 0} \frac{\phi(h)}{e^{-C/h}} > 0. \tag{4.39}$$

Indeed, from (4.39) we note that there exists $h^* > 0$ such that $e^{-C/h} \leq \phi(h)$ for all $h \leq h^*$. Then, for $h \leq \min\{h_0, h_1, h^*\}$, Proposition 4.1 holds for such ϕ and we can follow the same steps from the proof of Theorem 4.2 to achieve the $\phi-$controllability for the system (4.22).

4.3. Appendix

In this section, we present the estimates that were used in Section 4.1 to prove the Carleman estimate (4.11). We follow the steps from the continuous version obtained in [15]. The proof we develop in each Lemma is standard in the following sense. We begin rewritten the semi-discrete integral, if necessary, using some identity related to the discrete operators. Then we apply a discrete integration by parts from Proposition 2.1 to identify the leader terms of the Carleman estimate. Finally, thanks to Theorem 2.2 or 2.3, we can obtain the estimate claimed in each Lemma.

4.3.1. Estimate of I_{11} and I_{41}.

Lemma 4.5 *For $\lambda h(\delta T^2)^{-1} \leq 1$, we have*

$$I_{11} + I_{41} \geq -\int_{Q^*} s^2\mathcal{O}(T)|D_h v|^2 + X_{11} - Y_{11},$$

where

$$X_{11} := \int_{Q^*} s\mathcal{O}((sh)^2)|D_h v|^2 - \int_{\overline{Q}} s\mathcal{O}((sh)^2)|D_h^2 v|^2 - \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2 \\ - \int_Q s^{-1}\mathcal{O}((sh)^2)|\partial_t v|^2$$

and

$$Y_{11} := \int_{\mathcal{M}^*} \left(s^2(\partial_x\varphi)^2 + s\mathcal{O}(1) + s^2\mathcal{O}((sh)^2) + \mathcal{O}((sh)^2)\right) |D_h v|^2\Big|_{t=0} \\ + \int_{\mathcal{M}^*} \left(s^2(\partial_x\varphi)^2 + s\mathcal{O}(1) + s^2\mathcal{O}((sh)^2) + \mathcal{O}((sh)^2)\right) |D_h v|^2\Big|_{t=T}.$$

Proof. Consider $q_{11} := \rho A_h^2 D_h^2 r$, then I_{11} can be written as

$$I_{11} = -6\int_Q q_{11}\partial_t v\, A_h^2 D_h^2 v.$$

According to Corollary 2.2, it follows that

$$I_{11} = 6\int_{\overline{Q}} A_h D_h(q_{11}\partial_t v)\, A_h D_h v,$$

since $\partial_t v = 0$ on ∂Q and $D_h(q_{11}\partial_t v) = 0$ on ∂Q^*. Using Lemma 2.1 we have

$$I_{11} = 6\int_{\overline{Q}} \left(A_h^2(\partial_t v) + \frac{h^2}{4} D_h^2(\partial_t v) \right) D_h A_h(q_{11})\, A_h D_h v + 6\int_{\overline{Q}} \left(A_h^2(q_{11}) + \frac{h^2}{4} D_h^2(q_{11}) \right) D_h A_h(\partial_t v)\, A_h D_h v,$$

and noting that $v + \frac{h^2}{2} D_h^2 v = A_h^2(v) + \frac{h^2}{4} D_h^2 v$ and $\frac{1}{2}\partial_t(|D_h A_h v|^2) = D_h A_h(\partial_t v\,) A_h D_h v$, we obtain

$$I_{11} = 6\int_{\overline{Q}} D_h A_h(q_{11})\, A_h D_h v\, \partial_t v + 3h^2 \int_{\overline{Q}} D_h^2(\partial_t v)\, D_h A_h(q_{11})\, A_h D_h v + 3\int_{\overline{Q}} \left(q_{11} + \frac{h^2}{2} D_h^2(q_{11}) \right) \partial_t(|D_h A_h v|^2).$$

We apply Corollary 2.2 and an integration by parts with respect to the temporal variable to get

$$I_{11} = 6\int_Q D_h A_h q_{11}\, A_h D_h v\, \partial_t v + 3h^2 \int_Q D_h^2\left(D_h A_h q_{11}\, A_h D_h v\right) \partial_t v - 3\int_{\overline{Q}} \partial_t \left(q_{11} + \frac{h^2}{2} D_h^2(q_{11}) \right) |D_h A_h v|^2 + 3\int_{\overline{\mathcal{M}}} \left(q_{11} + \frac{h^2}{2} D_h^2(q_{11}) \right) |D_h A_h v|^2 \Bigg|_0^T,$$

since $\partial_t v|_{\partial Q} = \partial_t v|_{\partial \overline{Q}} = 0$. We note that from Lemma 2.1 it follows that

$$\begin{aligned} I_{11} =& 6\int_Q D_h A_h q_{11}\, A_h D_h v\, \partial_t v \\ &+ 3h^2 \int_Q \left(D_h^3 A_h q_{11}\, A_h^3 D_h v + 2 D_h^2 A_h^2 q_{11}\, D_h^2 A_h^2 v + D_h A_h^3 q_{11}\, A_h D_h^3 v \right) \partial_t v \\ &- 3\int_{\overline{Q}} \partial_t \left(q_{11} + \frac{h^2}{2} D_h^2(q_{11}) \right) |D_h A_h v|^2 + 3\int_{\overline{\mathcal{M}}} \left(q_{11} + \frac{h^2}{2} D_h^2(q_{11}) \right) |D_h A_h v|^2 \Bigg|_0^T . \end{aligned}$$

Additionally, we set $q_{41} := A_h D_h(\rho A_h^2 D_h^2 r)$ we thus write

$$I_{41} := -6\int_Q q_{41} A_h D_h v\, \partial_t v.$$

We note that $q_{41} = D_h A_h q_{11}$, then

$$\begin{aligned} I_{11} + I_{41} =& 3h^2 \int_Q \left(D_h^3 A_h q_{11}\, A_h^3 D_h v + 2 D_h^2 A_h^2 q_{11}\, D_h^2 A_h^2 v + D_h A_h^3 q_{11}\, A_h D_h^3 v \right) \partial_t v \\ &- 3\int_{\overline{Q}} \partial_t \left(q_{11} + \frac{h^2}{2} D_h^2(q_{11}) \right) |D_h A_h v|^2 + 3\int_{\overline{\mathcal{M}}} \left(q_{11} + \frac{h^2}{2} D_h^2(q_{11}) \right) |D_h A_h v|^2 \Bigg|_0^T . \end{aligned}$$

From Theorem 2.2 and Corollary 2.5 we have

$$\begin{aligned} D_h A_h(q_{11}) =& s^2\mathcal{O}(1),\ D_h^2(q_{11}) = s^2\mathcal{O}(1),\ \partial_t(q_{11}) = s^2\mathcal{O}(T), \\ D_h^2 A_h^2(q_{11}) =& s^2\mathcal{O}(1),\ D_h A_h^3(q_{11}) = s^2\mathcal{O}(1),\ D_h^3 A_h(q_{11}) = s^2\mathcal{O}(1) \\ q_{11} =& s^2(\partial_x\varphi)^2 + s\mathcal{O}(1) + s^2\mathcal{O}((sh)^2),\ \partial_t(D_h^2 q_{11}) = s^2\mathcal{O}(T). \end{aligned}$$

Therefore, for $I_{11}+I_{41}$ we get the following estimation

$$I_{11}+I_{41}=\int_Q \mathcal{O}((sh)^2)\left(A_h^3 D_h v + 2D_h^2 A_h^2 v + A_h D_h^3 v\right)\partial_t v - \int_{\overline{Q}} s^2 \mathcal{O}(T)|D_h A_h v|^2 + \int_{\mathcal{M}} \left(s^2(\partial_x\varphi)^2 + s\mathcal{O}(1) + s^2\mathcal{O}((sh)^2) + \mathcal{O}((sh)^2)\right)|D_h A_h v|^2\Big|_0^T .$$

Now, using Young's inequality on the first integral of the above expression we obtain

$$\begin{aligned} I_{11}+I_{41} \geq & -\int_Q s\mathcal{O}((sh)^2)\left(|A_h^3 D_h v|^2 + |D_h^2 A_h^2 v|^2 + |A_h D_h^3 v|^2\right) - \int_Q s^{-1}\mathcal{O}((sh)^2)|\partial_t v|^2 \\ & + \int_{\mathcal{M}} \left(s^2(\partial_x\varphi)^2 + s\mathcal{O}(1) + s^2\mathcal{O}((sh)^2) + \mathcal{O}((sh)^2)\right)|D_h A_h v|^2\Big|_0^T \\ & - \int_{\overline{Q}} s^2\mathcal{O}(T)|D_h A_h v|^2. \end{aligned}$$

In addition to this, we note that the integral $-\int_Q s\mathcal{O}((sh)^2)|A_h D_h^3 v|^2$ can be bound by using the inequality (2.6) as follows

$$-\int_Q s\mathcal{O}((sh)^2)|A_h D_h^3 v|^2 \leq -\int_Q s\mathcal{O}((sh)^2)A_h(|D_h^3 v|^2).$$

Then, a discrete integration by parts involving the average operator yields

$$-\int_Q s\mathcal{O}((sh)^2)|D_h^3 v|^2 + \frac{h}{2}\int_{\partial Q} s\mathcal{O}((sh)^2)t_r(|D_h^3 v|^2) \geq -\int_Q s\mathcal{O}((sh)^2)|D_h^3 v|^2,$$

since $s\mathcal{O}((sh)^2)|D_h^3 v|^2 \geq 0$. In the same manner, a similar bound can be drawn for the integrals with the terms $|A_h^3 D_h v|^2$, $|D_h A_h v|^2$ and $|D_h^2 A_h^2 v|^2$ to obtain

$$\begin{aligned} I_{11}+I_{41} \geq & \int_{Q^*} s\mathcal{O}((sh)^2)|D_h v|^2 - \int_{\overline{Q}} s\mathcal{O}((sh)^2)|D_h^2 v|^2 \\ & - \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2 - \int_Q s^{-1}\mathcal{O}((sh)^2)|\partial_t v|^2 - \int_{Q^*} s^2\mathcal{O}(T)|D_h v|^2 \\ & - \int_{\mathcal{M}^*} \left(s^2(\partial_x\varphi)^2 + s\mathcal{O}(1) + s^2\mathcal{O}((sh)^2) + \mathcal{O}((sh)^2)\right)|D_h v|^2\Big|_{t=0} \\ & - \int_{\mathcal{M}^*} \left(s^2(\partial_x\varphi)^2 + s\mathcal{O}(1) + s^2\mathcal{O}((sh)^2) + \mathcal{O}((sh)^2)\right)|D_h v|^2\Big|_{t=T}, \end{aligned}$$

where we have used that $D_h v = 0$ on ∂Q^*, which completes the proof. □

4.3.2. Estimate of I_{12}

Lemma 4.6 *For $\lambda h(\delta T^2)^{-1} \leq 1$, we have*

$$I_{12} \geq -60\int_{Q^*} s^5(\partial_x\varphi)^4\partial_x^2\varphi|D_h v|^2 + X_{12} + Y_{12},$$

where

$$X_{12} := \int_{Q^*}\left(s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2)\right)|D_h v|^2 - \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3(v)|^2 + \int_{Q^*} s\mathcal{O}((sh)^4)|D_h^3 v|^2$$

and

$$Y_{12} := \int_{\partial Q} \Big(s^5(\partial_x\varphi)^5 + s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2)\Big)\, t_r(|D_h v|^2)n - \int_{\partial Q} s^5\mathcal{O}((sh)^2)t_r(|D_h v|^2)$$
$$- \int_{\partial Q} s\mathcal{O}((sh)^2)|D_h^3 v|^2 + \int_{\partial Q} s\mathcal{O}((sh)^4)t_r(|D_h^3 v|^2)n.$$

Proof. Let us set $q_{12} := \rho^2 A_h^2 D_h^2 r\, A_h D_h^3 r$. Then, I_{12} is defined as

$$I_{12} := 24 \int_Q q_{12} D_h^2 A_h^2 v\, D_h A_h^3 v.$$

Using Lemma 2.1 we have the identity $D_h(|D_h A_h^2 v|^2) = 2D_h^2 Ah^2 v\, D_h A_h^3 v$. Then, I_{12} can be written as

$$I_{12} = 12 \int_Q q_{12} D_h(|D_h A_h^2 v|^2).$$

A discrete integration by part concerning to the difference operator D_h yields

$$I_{12} = -12 \int_{Q^*} D_h(q_{12})|A_h^2 D_h v|^2 + 12 \int_{\partial Q} q_{12} t_r(|A_h^2 D_h v|^2)n.$$

Now, using the identity $A_h^2(D_h v) = D_h v + \frac{h^2}{4} D_h^3 v$, due to 2.4, we obtain

$$I_{12} = -\, 12 \int_{Q^*} D_h(q_{12})|D_h v|^2 - 6h^2 \int_{Q^*} D_h(q_{12}) D_h v\, D_h^3 v - \frac{3h^4}{4} \int_{Q^*} D_h(q_{12})|D_h^3 v|^2$$
$$+ 12 \int_{\partial Q} q_{12} t_r(|A_h^2 D_h v|^2)n.$$

According to Theorem 2.2 and Corollary 2.5 we have

$$q_{12} = \rho^2 \partial_x^2(r)\partial_x^3(r) + s^5\mathcal{O}((sh)^2) = s^5(\partial_x\varphi)^5 + s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2),$$
$$D_h(q_{12}) = \partial_x\Big(\rho^2\partial_x^2(r)\partial_x^3(r)\Big) + s^5\mathcal{O}((sh)^2) = 5s^5(\partial_x\varphi)^4\partial_x^2\varphi + s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2).$$

Moreover,

$$\frac{3h^4}{4} q_{12} = s\mathcal{O}((sh)^4),\ -6h^2 D_h(q_{12}) = s^3\mathcal{O}((sh)^2) \text{ and } -\frac{3h^4}{4} D_h(q_{12}) = s\mathcal{O}((sh)^4).$$

We thus obtain

$$I_{12} = -60 \int_{Q^*} s^5(\partial_x\varphi)^4\partial_x^2\varphi|D_h v|^2 + Z_{12} + W_{12},$$

where

$$Z_{12} := \int_{Q^*} \Big(s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2)\Big)\, |D_h v|^2 + \int_{Q^*} s^3\mathcal{O}((sh)^2) D_h v D_h^3(v) + \int_{Q^*} s\mathcal{O}((sh)^4)|D_h^3(v)|^2$$

and

$$W_{12} := \int_{\partial Q} q_{12} t_r(|D_h A_h^2 v|^2)n..$$

We now proceed to find a lower bounds to Z_{12} and W_{12}. We begin by applying the Young's inequality to the second term of Z_{12} which yields

$$Z_{12} \geq \int_{Q^*} \left(s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2) \right) |D_h v|^2 - \int_{Q^*} s^5\mathcal{O}((sh)^2)|D_h v|^2$$
$$- \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2 + \int_{Q^*} s\mathcal{O}((sh)^4)|D_h^3 v|^2.$$

Additionally, to estimate W_{12} we note that by Lemma 2.1 we have the identity $D_h A_h^2 v = D_h v + \frac{h^2}{4} D_h^3 v$. Then, using this identity and the Young's inequality, for W_{12} we get

$$W_{12} \geq \int_{\partial Q} \left(s^5(\partial_x\varphi)^5 + s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2) \right) t_r(|D_h v|^2)n - \int_{\partial Q} s^5\mathcal{O}((sh)^2)t_r(|D_h v|^2)$$
$$- \int_{\partial Q} s\mathcal{O}((sh)^2)|D_h^3 v|^2 + \int_{\partial Q} s\mathcal{O}((sh)^4)t_r(|D_h^3 v|^2)n.$$

Consequently, I_{12} can be bounded as we claimed. □

4.3.3. Estimate of I_{13}

Lemma 4.7 *For $\lambda h(\delta T^2)^{-1} \leq 1$ we have*

$$I_{13} \geq -36 \int_{\overline{Q}} s^3(\partial_x\varphi)^2\partial_x^2\varphi|D_h^2 v|^2 + X_{13} + Y_{13},$$

where

$$X_{13} := \int_{\overline{Q}} \left(s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2) \right) |D_h^2 v|^2 + \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2$$

and

$$Y_{13} := \int_{\partial Q} \left(12s^3 \left(\partial_x\varphi\right)^3 + s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2) \right) |D_h^2 v|\, n + \int_{\partial Q} s\mathcal{O}((sh)^2)t_r \left(|D_h^3 v| \right) n$$
$$- \int_{\partial Q} s^3\mathcal{O}((sh))|D_h^2 v|^2 - \int_{\partial Q} s\mathcal{O}((sh))t_r \left(|D_h^3 v|^2 \right) + \int_{\partial Q^*} s^2\mathcal{O}((sh))t_r(|D_h^2 v|^2).$$

Proof. Define $q_{13} := \rho^2 A_h^2 D_h^2 r\, A_h^3 D_h r$. We thus have

$$I_{13} := 24 \int_Q q_{13} D_h^2 A_h^2 v\, D_h^3 A_h v I_{13} = 12 \int_Q q_{13} D_h(|D_h^2 A_h v|^2),$$

where we have used the identity $D_h(|D_h^2 A_h v|^2) = 2D_h(D_h^2 A_h v)\, A_h(D_h^2 A_h v)$ due to Lemma 2.1. A discrete integration by parts taking into account the difference operator D_h gives

$$I_{13} = -\,12 \int_{Q^*} D_h(q_{13})|A_h D_h^2 v|^2 + 12 \int_{\partial Q} q_{13}\ t_r(|A_h D_h^2 v|^2)n.$$

We note that by Lemma 2.1 the following identity $|A_h D_h^2 v|^2 = A_h(|D_h^2 v|^2) - \frac{h^2}{4}|D_h^3 v|^2$ holds. Then, I_{13} can written as

$$I_{13} = -\,12 \int_{Q^*} D_h(q_{13})A_h(|D_h^2 v|^2) + 3h^2 \int_{Q^*} D_h(q_{13})|D_h^3 v|^2 + 12 \int_{\partial Q} q_{13}\ t_r(|A_h D_h^2 v|^2)n.$$

A discrete integration by parts concerning the average operator A_h yields

$$\begin{aligned}I_{13} =& -12\int_{\overline{Q}} A_h D_h(q_{13})|D_h^2 v|^2 + 6h\int_{\partial Q^*} D_h(q_{13}) t_r(|D_h^2 v|^2)\\&+3h^2\int_{Q^*} D_h(q_{13})|D_h^3 v|^2 + 12\int_{\partial Q} q_{13} t_r(|A_h D_h^2 v|^2) n.\end{aligned}$$

Next, we note that

$$\int_{\partial Q} q_{13} t_r\left(|A_h D_h^2 v|^2\right) n = \int_{\partial Q} q_{13}|D_h^2 v|^2 n + \frac{h^2}{4}\int_{\partial Q} q_{13} t_r\left(|D_h^3 v|^2\right) n - h\int_{\partial Q} q_{13} D_h^2 v\, t_r\left(|D_h^3 v|\right),$$

since $t_r(|A_h D_h^2 v|^2)n = |D_h^2 v|^2 n + \frac{h^2}{4} t_r(|D_h^3 v|^2) - hD_h^2 v\, t_r(|D_h^3 v|)$. Thus, with the integral on the boundary above, for I_{13} we get

$$\begin{aligned}I_{13} =& -12\int_{\overline{Q}} A_h D_h(q_{13})|D_h^2 v|^2 + 3h^2\int_{Q^*} D_h(q_{13})|D_h^3 v|^2 + 12\int_{\partial Q} q_{13}|D_h^2 v|^2 n\\&+3h^2\int_{\partial Q} q_{13} t_r\left(|D_h^3 v|^2\right) n - 12h\int_{\partial Q} q_{13} D_h^2 v\, t_r\left(D_h^3 v\right) + 6h\int_{\partial Q^*} D_h(q_{13}) t_r(|D_h^2 v|^2).\end{aligned}$$

By virtue of Theorem 2.2 and Corollary 2.5, we have

$$\begin{aligned}q_{13} =& s^3\left(\partial_x\varphi\right)^3 + s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2),\\D_h(q_{13}) =& 3s^3(\partial_x\varphi)^2\partial_x^2\varphi + s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2),\\A_h D_h(q_{13}) =& 3s^3(\partial_x\varphi)^2\partial_x^2\varphi + s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2).\end{aligned}$$

Moreover, $3h^2 D_h(q_{13}) = s\mathcal{O}((sh)^2)$ and $-6hD_h(q_{13}) = s^2\mathcal{O}(sh)$. We thus have

$$I_{13} = -36\int_{\overline{Q}} s^3(\partial_x\varphi)^2\partial_x^2\varphi|D_h^2 v|^2 + X_{13} + W_{13},$$

where

$$X_{13} := \int_{\overline{Q}}\left(s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2)\right)|D_h^2 v|^2 + \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2$$

and

$$\begin{aligned}W_{13} :=& \int_{\partial Q}\left(12s^3\left(\partial_x\varphi\right)^3 + s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2)\right)|D_h^2 v|^2 n + \int_{\partial Q} s\mathcal{O}((sh)^2) t_r\left(|D_h^3 v|^2\right) n\\&-\int_{\partial Q} s^2\mathcal{O}(sh) D_h^2 v\, t_r\left(D_h^3 v\right) + \int_{\partial Q^*} s^2\mathcal{O}(sh) t_r(|D_h^2 v|^2).\end{aligned}$$

The only point remains to get an estimate for W_{13}. Using Young's inequality on the third term of W_{13} we obtain

$$\begin{aligned}W_{13} \leq& \int_{\partial Q}\left(12s^3\left(\partial_x\varphi\right)^3 + s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2)\right)|D_h^2 v|^2 n + \int_{\partial Q} s\mathcal{O}(sh) t_r\left(|D_h^3 v|\right)\\&-\int_{\partial Q} s^3\mathcal{O}((sh))|D_h^2 v|^2 + \int_{\partial Q^*} s^2\mathcal{O}((sh)) t_r(|D_h^2 v|^2).\end{aligned}$$

Hence, combining (4.3.3) with the above bound for W_{13}, I_{13} can be bounded as we required. □

4.3.4. Estimate of I_{14}

Lemma 4.8 *For $\lambda h(\delta T^2)^{-1} \leq 1$ we have*

$$I_{14} \geq -36 \int_{Q^*} s^5 (\partial_x \varphi)^4 \partial_x^2 \varphi |D_h v|^2 + 18 \int_Q s^5 \partial_x^2 \left((\partial_x \varphi)^4 \partial_x^2 \varphi \right) |v|^2 + X_{14},$$

where

$$\begin{aligned} X_{14} := & \int_{Q^*} \left(s^4 \mathcal{O}(1) + s^5 \mathcal{O}((sh)^2) \right) |D_h v|^2 + \int_Q \left(s^4 \mathcal{O}(1) + s^5 \mathcal{O}((sh)^2) \right) |v|^2 \\ & - \int_{Q^*} s\mathcal{O}((sh)^2) |D_h^3 v|^2 - \int_Q s^5 \mathcal{O}((sh)^2)|v|^2 - \int_{Q^*} s^5 \mathcal{O}((sh)^2)|D_h v|^2. \end{aligned}$$

Proof. Let us set $q_{14} := \rho A_h^2 D_h^2(r)\, A_h D_h(\rho A_h D_h^3(r))$. So, I_{14} is defined as

$$I_{14} := 12 \int_Q q_{14} D_h^2 A_h^2(v) v.$$

A discrete integration by parts involving the difference operator yields

$$I_{14} = -\, 12 \int_{Q^*} D_h(q_{14} v) D_h A_h^2(v),$$

since $v = 0$ on ∂Q. By Lemma 2.1 and the identity $D_h A_h^2(v) = D_h v + \frac{h^2}{4} D_h^3 v$, it follows that

$$\begin{aligned} I_{14} = & -\, 12 \int_{Q^*} A_h(q_{14}) |D_h v|^2 - 6 \int_{Q^*} D_h(q_{14}) D_h(v^2) \\ & - 3h^2 \int_{Q^*} A_h(q_{14}) D_h v D_h^3(v) - 3h^2 \int_{Q^*} D_h(q_{14}) A_h v D_h^3 v. \end{aligned}$$

Since $t_r(v) = 0$ on ∂Q^*, a discrete integration by parts concerning the difference operator on the second integral above gives

$$\begin{aligned} I_{14} = & -\, 12 \int_{Q^*} A_h(q_{14}) |D_h v|^2 + 6 \int_{\overline{Q}} D_h^2(q_{14}) |v|^2 \\ & - 3h^2 \int_{Q^*} A_h(q_{14}) D_h v D_h^3(v) - 3h^2 \int_{Q^*} D_h(q_{14}) A_h v D_h^3 v. \end{aligned}$$

Thanks to Theorem 2.2 and Corollary 2.5 we write

$$\begin{aligned} A_h(q_{14}) = & 3s^5 (\partial_x \varphi)^4 \partial_x^2 \varphi + s^4 \mathcal{O}(1) + s^5 \mathcal{O}((sh)^2), \\ D_h(q_{14}) = & 3s^5 \partial_x \left((\partial_x \varphi)^4 \partial_x^2 \varphi \right) + s^4 \mathcal{O}(1) + s^5 \mathcal{O}((sh)^2), \\ D_h^2(q_{14}) = & 3s^5 \partial_x^2 \left((\partial_x \varphi)^4 \partial_x^2 \varphi \right) + s^4 \mathcal{O}(1) + s^5 \mathcal{O}((sh)^2). \end{aligned}$$

Moreover, we have $3h^2 D_h(q_{14}) = s^3 \mathcal{O}((sh)^2)$ and $3h^2 A_h(q_{14}) = s^3 \mathcal{O}((sh)^2)$. Hence, for I_{14} we obtain the following estimation

$$I_{14} = -36 \int_{Q^*} s^5 (\partial_x \varphi)^4 \partial_x^2 \varphi |D_h v|^2 + 18 \int_{\overline{Q}} s^5 \partial_x^2 \left((\partial_x \varphi)^4 \partial_x^2 \varphi \right) |v|^2 + Z_{14}$$

where

$$Z_{14} := \int_{Q^*} \left(s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2)\right)|D_h v|^2 + \int_{\overline{Q}} \left(s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2)\right)|v|^2$$
$$+ \int_{Q^*} s^3\mathcal{O}((sh)^2)A_h v D_h^3 v + \int_{Q^*} s^3\mathcal{O}((sh)^2)D_h v D_h^3(v).$$

The task is now to estimate the last two terms of Z_{14}. We use the Young's inequality on these last two terms to write

$$Z_{14} \geq \int_{Q^*} \left(s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2)\right)|D_h v|^2 + \int_{\overline{Q}} \left(s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2)\right)|v|^2 - \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2$$
$$- \int_{Q^*} s^5\mathcal{O}((sh)^2)|A_h v|^2 - \int_{Q^*} s^5\mathcal{O}((sh)^2)|D_h v|^2 - \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2.$$

Furthermore, using (2.6) we have

$$Z_{14} \geq \int_{Q^*} \left(s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2)\right)|D_h v|^2 + \int_{\overline{Q}} \left(s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2)\right)|v|^2$$
$$- \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2 - \int_{Q^*} s^5\mathcal{O}((sh)^2)A_h(|v|^2) - \int_{Q^*} s^5\mathcal{O}((sh)^2)|D_h v|^2.$$

Finally, a discrete integration by parts with respect to the average operator A_h and using that $t_r(v) = 0$ on ∂Q^* lead to

$$Z_{14} \geq \int_{Q^*} \left(s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2)\right)|D_h v|^2 + \int_{\overline{Q}} \left(s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2)\right)|v|^2$$
$$- \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2 - \int_{\overline{Q}} s^5\mathcal{O}((sh)^2)|v|^2 - \int_{Q^*} s^5\mathcal{O}((sh)^2)|D_h v|^2$$
$$:= X_{14},$$

and the Lemma follows. □

4.3.5. Estimate of I_{21}

Lemma 4.9 *For $\lambda h(\delta T^2)^{-1} \leq 1$, we have*

$$I_{21} \geq \int_Q s^4\mathcal{O}(T)|v|^2 + X_{21} - Y_{21},$$

where

$$X_{21} := \int_Q \left(s^5\mathcal{O}((sh)^2) + s^2\mathcal{O}((sh)^2)\right)|v|^2 + \int_{Q^*} s^5\mathcal{O}((sh)^2)|D_h v|^2$$

and

$$Y_{21} := \int_{\mathcal{M}} \left(s^4(\partial_x\varphi)^4 + s^2\mathcal{O}((sh)^2) + s^4\mathcal{O}((sh)^2)\right)|v|^2\Big|_{t=0} + \int_Q \mathcal{O}((sh)^2)s^{-1}|\partial_t v|^2$$
$$+ \int_{\mathcal{M}} \left(s^4(\partial\varphi)^4 + s^2\mathcal{O}((sh)^2) + s^4\mathcal{O}((sh)^2)\right)|v|^2\Big|_{t=T}.$$

Proof. Set $q_{21} := \rho D_h^4(r)$. Let us compute $I_{21} := -\int_Q q_{21}A_h^4(v)\partial_t v$.

Using Corollary 2.2, we see that

$$I_{21} = -\int_{Q} A_h^2(q_{21}\partial_t v) A_h^2 v,$$

since $\partial_t v = 0$ on ∂Q, and $A_h \partial_t v = D_h \partial_t v = 0$ on ∂Q^*. By using Lemma 2.1, it follows that

$$\begin{aligned} I_{21} =& -\frac{1}{2}\int_{Q} A_h^2(q_{21})\partial_t(|A_h^2 v|^2) - \frac{h^2}{2}\int_{Q} D_h A_h(q_{21}) D_h A_h(\partial_t v) A_h^2 v \\ & - \frac{h^4}{16}\int_{Q} D_h^2(q_{21}) D_h^2(\partial_t v) A_h^2 v. \end{aligned}$$

We note that $\frac{h^2}{4} D_h^2(\partial_t v) = A_h^2(\partial_t v) - \partial_t v$, being a consequence of Lemma 2.1. This gives

$$\begin{aligned} I_{21} =& -\frac{1}{2}\int_{Q} A_h^2(q_{21})\partial_t(|A_h^2 v|^2) - \frac{h^2}{2}\int_{Q} D_h A_h(q_{21}) D_h A_h(\partial_t v) A_h^2 v \\ & - \frac{h^2}{8}\int_{Q} D_h^2(q_{21})\partial_t(|A_h^2 v|^2) + \frac{h^2}{4}\int_{Q} D_h^2(q_{21})\partial_t v A_h^2 v. \end{aligned}$$

According to Lemma 2.1, we also have $D_h A_h(\partial_t v) A_h^2(v) = \frac{1}{2}\partial_t(D_h(|A_h v|^2)) - A_h^2(\partial_t v) D_h A_h v$. Thus, I_{21} can be written as

$$\begin{aligned} I_{21} =& -\frac{1}{2}\int_{Q} A_h^2(q_{21})\partial_t(|A_h^2 v|^2) - \frac{h^2}{4}\int_{Q} D_h A_h(q_{21})\partial_t\left(D_h\left(|A_h v|^2\right)\right) \\ & + \frac{h^2}{2}\int_{Q} D_h A_h(q_{21}) A_h^2(\partial_t v) D_h A_h v - \frac{h^2}{8}\int_{Q} D_h^2(q_{21})\partial_t(|A_h^2 v|^2) + \frac{h^2}{4}\int_{Q} D_h^2(q_{21})\partial_t v A_h^2 v. \end{aligned}$$

An integration by parts with respect to the temporal variable yields

$$\begin{aligned} I_{21} =& \frac{1}{2}\int_{Q} \partial_t(A_h^2(q_{21}))|A_h^2 v|^2 - \frac{1}{2}\int_{\mathcal{M}} A_h^2(q_{21})|A_h^2 v|^2\Big|_0^T + \frac{h^2}{4}\int_{Q} \partial_t(D_h A_h(q_{21})) D_h\left(|A_h v|^2\right) \\ & - \frac{h^2}{4}\int_{\mathcal{M}} D_h A_h(q_{21}) D_h\left(|A_h v|^2\right)\Big|_0^T + \frac{h^2}{4}\int_{Q} D_h^2(q_{21})\partial_t v A_h^2 v \\ & + \frac{h^2}{2}\int_{Q} D_h A_h(q_{21}) A_h^2(\partial_t v) D_h A_h v + \frac{h^2}{8}\int_{Q} \partial_t(D_h^2(q_{21}))|A_h^2 v|^2 - \frac{h^2}{8}\int_{\mathcal{M}} D_h^2(q_{21})|A_h^2 v|^2\Big|_0^T. \end{aligned}$$

Now, using a discrete integration by parts, taking into account the difference operator D_h, on the third and fourth integral of I_{21} we have

$$\begin{aligned} I_{21} =& \frac{1}{2}\int_{Q} \partial_t(A_h^2 q_{21})|A_h^2 v|^2 - \frac{1}{2}\int_{\overline{\mathcal{M}}} A_h^2(q_{21})|A_h^2 v|^2\Big|_0^T - \frac{h^2}{4}\int_{Q^*} \partial_t(D_h^2 A_h q_{21})|A_h v|^2 \\ & + \frac{h^2}{4}\int_{\mathcal{M}^*} D_h^2 A_h(q_{21})|A_h v|^2\Big|_0^T + \frac{h^2}{2}\int_{Q} D_h A_h(q_{21}) A_h^2(\partial_t v) D_h A_h v \\ & + \frac{h^2}{8}\int_{Q} \partial_t(D_h^2 q_{21})|A_h^2 v|^2 - \frac{h^2}{8}\int_{\overline{M}} D_h^2(q_{21})|A_h^2 v|^2\Big|_0^T + \frac{h^2}{4}\int_{Q} D_h^2(q_{21})\partial_t v A_h^2 v, \end{aligned}$$

where we have used that $t_r(A_h v) = 0$ on $\partial\overline{Q}$ and $\partial\overline{\mathcal{M}}$. Moreover, using the identity $A_h^2 v =$

$v + \frac{h^2}{4} D_h^2 v$ we obtain

$$\begin{aligned} I_{21} =& \frac{1}{2}\int_Q \partial_t(A_h^2(q_{21}))|v|^2 + \frac{h^2}{4}\int_Q \partial_t(A_h^2(q_{21}))vD_h^2 v + \frac{h^4}{32}\int_Q \partial_t(A_h^2(q_{21}))|D_h v|^2 \\ &- \frac{1}{2}\int_{\mathcal{M}} A_h^2(q_{21})|A_h^2 v|^2\Big|_0^T - \frac{h^2}{4}\int_{Q^*} D_h(\partial_t(D_h A_h(q_{21})))|A_h v|^2 \\ &+ \frac{h^2}{4}\int_{\mathcal{M}^*} D_h^2 A_h(q_{21})|A_h v|^2\Big|_0^T + \frac{h^2}{2}\int_Q D_h A_h(q_{21})A_h^2(\partial_t v)D_h A_h v \\ &+ \frac{h^2}{8}\int_Q \partial_t(D_h^2(q_{21}))|A_h^2 v|^2 - \frac{h^2}{8}\int_{\mathcal{M}} D_h^2(q_{21})|A_h^2 v|^2\Big|_0^T + \frac{h^2}{4}\int_Q D_h^2(q_{21})\partial_t v A_h^2 v. \end{aligned}$$

Now, we shall use the Theorem 2.2 and Corollary 2.5 to write the following estimates

$$\begin{aligned} q_{21} =& s^4\mathcal{O}(1),\ A_h^2(q_{21}) = s^4\mathcal{O}(1),\ D_h A_h(q_{21}) = s^4\mathcal{O}((sh)^2), \\ D_h(q_{21}) =& s^4\mathcal{O}((sh)^2),\ D_h^2 A_h(q_{21}) = s^4\mathcal{O}((sh)^2),\ \partial_t(q_{21}) = Ts^4\theta\mathcal{O}(1), \\ \partial_t(A_h^2 q_{21}) =& s^4\theta\mathcal{O}(T),\ \partial_t(D_h^2 q_{21}) = s^4\theta\mathcal{O}(T),\ \partial_t(D_h^2 A_h q_{21}) = s^4\mathcal{O}(T). \end{aligned}$$

We thus can use the above estimates to rewrite I_{21} as

$$I_{21} = \int_Q s^4\mathcal{O}(T)|v|^2 + Z_{21} + W_{21}, \tag{4.40}$$

where

$$\begin{aligned} Z_{21} :=& \int_Q s^2\mathcal{O}((sh)^2)vD_h^2 v + \int_Q \mathcal{O}((sh)^4)|D_h v|^2 + \int_Q s^2\mathcal{O}((sh)^2)\partial_t v A_h^2 v \\ &+ \int_Q s^2\mathcal{O}((sh)^2)|A_h^2 v|^2 + \int_Q s^2\mathcal{O}((sh)^2)A_h^2(\partial_t v)D_h A_h v + \int_{Q^*} s^2\mathcal{O}((sh)^2)|A_h v|^2 \end{aligned}$$

and

$$\begin{aligned} W_{21} :=& \int_{\mathcal{M}} \left(s^4(\partial_x\varphi)^4 + s^3\mathcal{O}(1) + s^4\mathcal{O}((sh)^2)\right)|A_h^2 v|^2\Big|_0^T \\ &+ \int_{\mathcal{M}^*} s^2\mathcal{O}((sh)^2)|A_h v|^2\Big|_0^T + \int_{\mathcal{M}} s^4\mathcal{O}((sh)^2)|A_h^2 v|^2\Big|_0^T. \end{aligned}$$

What is left is to get estimates for Z_{21} and W_{21}. For Z_{12}, we claim that

$$\begin{aligned} |Z_{21}| \leq& \int_Q s^3\mathcal{O}((sh)^2)|v|^2 + \int_Q s\mathcal{O}((sh)^2)|D_h^2 v|^2 + \int_Q \mathcal{O}((sh)^4)|D_h v|^2 \\ &+ \int_Q \mathcal{O}((sh)^2)s^{-1}|\partial_t v|^2 + \int_Q s^5\mathcal{O}((sh)^2)|v|^2 + \int_{Q^*} s^2\mathcal{O}((sh)^2)|v|^2 \\ &+ \int_{Q^*} \mathcal{O}((sh)^2)s^{-1}|\partial_t v)|^2 + \int_{Q^*} s^5\mathcal{O}((sh)^2)|D_h v|^2 + \int_Q s^2\mathcal{O}((sh)^2)|v|^2. \end{aligned} \tag{4.41}$$

Indeed, we first examine the third integral from Z_{21}, we say $Z_{12}^{(3)} := \int_Q s\mathcal{O}((sh)^2)\partial_t v\, A_h^2 v$. By virtue of Young's inequality we have

$$|Z_{12}^{(3)}| \leq \int_Q s^{-1}\mathcal{O}((sh)^2)|\partial_t v|^2 + \int_Q s^5\mathcal{O}((sh)^2)|A_h^2 v|^2.$$

Now, using (2.6) and a discrete integration by parts concerning the average operator, it follows that

$$|Z_{12}^{(3)}| \leq \int_{\overline{Q}} s^{-1}\mathcal{O}((sh)^2)|\partial_t v|^2 + \int_{\overline{Q}^*} s^5\mathcal{O}((sh)^2)|A_h v|^2,$$

where we have used that $t_r(A_h v) = 0$ on $\partial\overline{Q}$. Applying the previous step one more time we get

$$|Z_{12}^{(3)}| \leq \int_{\overline{Q}} s^{-1}\mathcal{O}((sh)^2)|\partial_t v|^2 + \int_{\overline{Q}} s^5\mathcal{O}((sh)^2)|v|^2.$$

Similar computations work for the remaining terms of Z_{12}, and the claim follows.

On the other hand, following the methodology of the estimate for Z_{21}, we obtain

$$\begin{aligned} |W_{21}| \leq & \int_{\mathcal{M}} \left(s^4(\partial_x\varphi)^4 + s^3\mathcal{O}(1) + s^4\mathcal{O}((sh)^2))\right)|v|^2\Big|_{t=0} \\ & + \int_{\mathcal{M}} \left(s^4(\partial_x\varphi)^4 + s^3\mathcal{O}(1) + s^4\mathcal{O}((sh)^2)\right)|v|^2\Big|_{t=T}. \end{aligned} \tag{4.42}$$

Thus, combining (4.40) with (4.41) and (4.42) we can assert that

$$I_{21} \geq \int_Q s^4\mathcal{O}(T)|v|^2 + X_{21} - Y_{21},$$

where

$$X_{21} := \int_Q \left(s^5\mathcal{O}((sh)^2) + s^2\mathcal{O}((sh)^2)\right)|v|^2 + \int_{Q^*} s^5\mathcal{O}((sh)^2)|D_h v|^2$$

and

$$\begin{aligned} Y_{21} := & \int_{\mathcal{M}} \left(s^4(\partial_x\varphi)^4 + s^2\mathcal{O}((sh)^2) + s^4\mathcal{O}((sh)^2)\right)|v|^2\Big|_{t=0} + \int_Q \mathcal{O}((sh)^2)s^{-1}|\partial_t v|^2 \\ & + \int_{\mathcal{M}} \left(s^4(\partial\varphi)^4 + s^2\mathcal{O}((sh)^2) + s^4\mathcal{O}((sh)^2)\right)|v|^2\Big|_{t=T}, \end{aligned}$$

which is the desired conclusion. □

4.3.6. Estimate of I_{22}

Lemma 4.10 *For $\lambda h(\delta T^2)^{-1} \leq 1$, we have*

$$I_{22} = -7\int_Q s^7(\partial_x\varphi)^6\partial_x^2\varphi|v|^2 + X_{22} + Y_{22}$$

where

$$\begin{aligned} X_{22} := & \int_Q \left(s^6\mathcal{O}(1) + s^7\mathcal{O}((sh)^2)\right)|v|^2 + \int_{\overline{Q}} s^3\mathcal{O}((sh)^4)|D_h^2 v|^2 \\ & - \int_{Q^*} s^5\mathcal{O}((sh)^2)\,|D_h v|^2 - \int_{Q^*} s\mathcal{O}((sh)^6)|D_h^3 v|^2 \end{aligned}$$

and

$$Y_{22} := \int_{\partial Q^*} s^2\mathcal{O}((sh)^5))t_r(|D_h^2 v|^2).$$

Proof. Let us set $q_{22} := \rho^2 D_h^4 r\, A_h D_h^3 r$, then $I_{22} := 4\int_Q q_{22} A_h^4 v\, D_h A_h^3 v$. By using Lemma 2.1 we have

$$A_h^4 v\, D_h A_h^3 v = \frac{1}{2} D_h A_h \left(|v + \frac{h^2}{4} D_h^2 v|^2 \right) - \frac{h^2}{8} D_h (|D_h v + \frac{h^2}{4} D_h^3 v|^2),$$

then I_{22} ca be written as

$$I_{22} = 2\int_Q q_{22} D_h A_h \left(|v + \frac{h^2}{4} D_h^2 v|^2 \right) - \frac{h^2}{2}\int_Q q_{22} D_h (|D_h v + \frac{h^2}{4} D_h^3 v|^2).$$

We use Corollary 2.2 for the first integral above and Proposition 2.1 for the second one to write

$$\begin{aligned} I_{22} =& -2\int_{\overline{Q}} D_h A_h(q_{22}) |v + \frac{h^2}{4} D_h^2 v|^2 + h\int_{\partial Q^*} D_h q_{22} t_r (|v + \frac{h^2}{4} D_h^2 v|^2) \\ &+ 2\int_{\partial Q} q_{22} t_r (A_h (|v + \frac{h^2}{4} D_h^2 v|^2)) n + \frac{h^2}{2}\int_{Q^*} D_h q_{22}\, |D_h v + \frac{h^2}{4} D_h^3 v|^2 \\ &- \frac{h^2}{2}\int_{\partial Q} q_{22} t_r (|D_h v + \frac{h^2}{4} D_h^3 v|^2) n \\ :=& Z_{22} + W_{22}, \end{aligned} \tag{4.43}$$

where

$$Z_{22} := -2\int_{\overline{Q}} D_h A_h(q_{22}) |v + \frac{h^2}{4} D_h^2 v|^2 + \frac{h^2}{2}\int_{Q^*} D_h q_{22}\, |D_h v + \frac{h^2}{4} D_h^3 v|^2$$

and

$$\begin{aligned} W_{22} :=& h\int_{\partial Q^*} D_h q_{22} t_r (|v + \frac{h^2}{4} D_h^2 v|^2) + 2\int_{\partial Q} q_{22} t_r (A_h (|v + \frac{h^2}{4} D_h^2 v|^2)) n \\ &- \frac{h^2}{2}\int_{\partial Q} q_{22} t_r (|D_h v + \frac{h^2}{4} D_h^3 v|^2) n. \end{aligned}$$

The task is now to find lower bounds for Z_{22} and W_{22}. First, we establish some estimate for discrete operator applying on our weight function. Thanks to Theorem 2.2 and Corollary 2.5 it follows that

$$\begin{aligned} q_{22} =& s^7 (\partial_x \varphi)^7 + s^6 \mathcal{O}(1) + s^7 \mathcal{O}((sh)^2) \\ D_h q_{22} =& 7 s^7 (\partial_x \varphi)^6 \partial_x^2 \varphi + s^6 \mathcal{O}(1) + s^7 \mathcal{O}((sh)^2) \text{ and } A_h D_h q_{22} = s^7 \mathcal{O}(1). \end{aligned}$$

Now, Z_{22} can be written as

$$\begin{aligned} Z_{22} =& -2\int_{\overline{Q}} \left(7 s^7 (\partial_x \varphi)^6 \partial_x^2 \varphi + s^6 \mathcal{O}(1) + s^7 \mathcal{O}((sh)^2) \right) |v + \frac{h^2}{4} D_h^2 v|^2 \\ &+ \frac{h^2}{2}\int_{Q^*} s^7 \mathcal{O}(1)\, |D_h v + \frac{h^2}{4} D_h^3 v|^2. \end{aligned}$$

Thus, by virtue of Young's inequality we obtain the following lower bound for Z_{22}

$$\begin{aligned} Z_{22} \geq& -\int_{\overline{Q}} \left(7 s^7 (\partial_x \varphi)^6 \partial_x^2 \varphi + s^6 \mathcal{O}(1) + s^7 \mathcal{O}((sh)^2) \right) |v|^2 + \int_{\overline{Q}} s^3 \mathcal{O}((sh)^4) |D_h^2 v|^2 \\ &- \int_{Q^*} s^5 \mathcal{O}((sh)^2)\, |D_h v|^2 - \int_{Q^*} s \mathcal{O}((sh)^6) |D_h^3 v|^2. \end{aligned} \tag{4.44}$$

Our next goal is to determinate a lower bound for W_{22}. We note that

$$t_r(|v + \frac{h^2}{4} D_h^2 v|^2) = \frac{h^4}{16} t_r(|D_h^2 v|^2),$$

on ∂Q^*, since $t_r(v) = 0$ on ∂Q^*. Besides

$$2 \int_{\partial Q} q_{22} t_r(A_h(|v + \frac{h^2}{4} D_h^2 v|^2))n - \frac{h^2}{2} \int_{\partial Q} q_{22} t_r(|D_h v + \frac{h^2}{4} D_h^3 v|^2)n = 2 \int_{\partial Q} q_{22} t_r(|A_h^3 v|^2)n.$$

We thus get

$$\begin{aligned} W_{22} &= \int_{\partial Q^*} s^2 \mathcal{O}((sh)^5)) t_r(|D_h^2 v|^2) + \int_{\partial Q} \left(2s^7 (\partial_x \varphi)^7 + s^6 \mathcal{O}(1) + s^7 \mathcal{O}((sh)^2)\right) t_r(|A_h^3 v|^2) \\ &\geq \int_{\partial Q^*} s^2 \mathcal{O}((sh)^5)) t_r(|D_h^2 v|^2). \end{aligned} \tag{4.45}$$

Combining (4.43) with (4.45) and (4.44) completes the proof. □

4.3.7. Estimate of I_{23}

Lemma 4.11 *For $\lambda h(\delta T^2)^{-1} \leq 1$, we have*

$$I_{23} \geq 30 \int_{Q^*} s^5 (\partial_x \varphi)^4 \partial_x^2 \varphi |D_h v|^2 - 10 \int_Q s^5 \partial_x^2 \left((\partial_x \varphi)^4 \partial_x^2 \varphi\right) |v|^2 + X_{23} + Y_{23},$$

where

$$\begin{aligned} X_{23} = &\int_{\overline{Q}} \left(s^4 \mathcal{O}(1) + s^5 \mathcal{O}((sh)^2)\right) |v|^2 + \int_{\overline{Q}^*} s^3 \mathcal{O}((sh)^2) |D_h v|^2 + \int_{Q^*} s \mathcal{O}((sh)^2) |D_h^3 v|^2 \\ &+ \int_{\overline{Q}} \left(s^4 \mathcal{O}(1) + s^3 \mathcal{O}((sh)^2) + s^5 \mathcal{O}((sh)^2)\right) |D_h^2 v|^2 \end{aligned}$$

and

$$\begin{aligned} Y_{23} := &\int_{\partial Q^*} s^3 \mathcal{O}((sh)^2) t_r\left(|D_h^2 v|^2\right) n + \int_{\partial Q} s^3 \mathcal{O}((sh)^2) |D_h^2 v|^2 n + \int_{\partial Q} s \mathcal{O}((sh)^4) t_r(|D_h^3 v|^2) n \\ &- \int_{\partial Q} s^3 \mathcal{O}((sh)^3) |D_h^2 v|^2 - \int_{\partial Q} s \mathcal{O}((sh)^3) t_r(|D_h^3 v|^2), \end{aligned}$$

Proof. Let us set $q_{23} := \rho^2 D_h^4 r \, A_h^3 D_h r$ to write $I_{23} := 4 \int_Q q_{23} \, A_h^4 v \, D_h^3 A_h v$. Using the identity $A_h^4 v = v + \frac{h^2}{4} D_h^2 v + \frac{h^2}{4} D_h^2 A_h^2 v$, due to Lemma 2.1, it follows that

$$\begin{aligned} I_{23} =& 4 \int_Q q_{23} v \, D_h^3 A_h v + h^2 \int_Q q_{23} D_h^2 v \, D_h^3 A_h v + h^2 \int_Q q_{23} D_h^2 A_h^2 v \, D_h^3 A_h v \\ =& I_{23}^{(a)} + I_{23}^{(b)} + I_{23}^{(c)}. \end{aligned}$$

Let us apply Proposition 2.1 to $I_{23}^{(a)}$, and then Lemma 2.1, to obtain

$$I_{23}^{(a)} = -4 \int_{Q^*} D_h(q_{23}) A_h v \, D_h^2 A_h v - 4 \int_{Q^*} A_h(q_{23}) D_h v \, D_h^2 A_h v,$$

where we have used that $v = 0$ on ∂Q. Repeated application of Proposition 2.1 and Lemma

2.1 enables us to write

$$I_{23}^{(a)} = 4\int_{\overline{Q}} D_h^2(q_{23})A_h^2 v\, D_hA_hv + 8\int_{\overline{Q}} D_hA_h(q_{23})|D_hA_hv|^2 + 4\int_{\overline{Q}} A_h^2(q_{23})D_h^2vD_hA_hv.$$

We note that the following identities $A_h^2vD_hA_hv = \frac{1}{2}D_h\left((A_hv)^2\right)$ and $D_h^2vD_hA_hv = \frac{1}{2}D_h\left((D_hv)^2\right)$ holds, thanks to Lemma 2.1. Then, $I_{23}^{(a)}$ can be written as follows

$$I_{23}^{(a)} = 2\int_{\overline{Q}} D_h^2(q_{23})D_h\left((A_hv)^2\right) + 8\int_{\overline{Q}} D_hA_h(q_{23})|D_hA_hv|^2 + 2\int_{\overline{Q}} A_h^2(q_{23})D_h\left((D_hv)^2\right).$$

Applying Proposition 2.1 to the first and third integral from $I_{23}^{(a)}$, and using that $t_r((D_hv)) = 0$ and $t_r(A_hv) = 0$ on $\partial\overline{Q}$, we get

$$I_{23}^{(a)} = -2\int_{\overline{Q}^*} D_h^3(q_{23})|A_hv|^2 + 8\int_{\overline{Q}} D_hA_h(q_{23})|D_hA_hv|^2 - 2\int_{\overline{Q}^*} D_hA_h^2(q_{23})|D_hv|^2.$$

Finally, using the identity $(A_hv)^2 = A_h(v^2) - \frac{h^2}{4}(D_hv)^2$, Proposition 2.1 one last time and that $v = t_r(D_hv) = 0$ on $\partial\overline{Q}$, we obtain

$$I_{23}^{(a)} = -2\int_{\overline{Q}} A_hD_h^3(q_{23})|v|^2 + \frac{h^2}{2}\int_{\overline{Q}^*} D_h^3(q_{23})|D_hv|^2 + 6\int_{\overline{Q}^*} D_hA_h^2(q_{23})|D_hv|^2.$$

We can proceed analogously for $I_{23}^{(b)}$ and $I_{23}^{(c)}$ to obtain $I_{23} = Z_{23} + W_{23}$ where

$$\begin{aligned} Z_{23} = &-2\int_{\overline{Q}} A_hD_h^3(q_{23})|v|^2 + \frac{h^2}{2}\int_{\overline{Q}^*} D_h^3(q_{23})|D_hv|^2 + 6\int_{\overline{Q}^*} D_hA_h^2(q_{23})|D_hv|^2 \\ &- 2h^2\int_{\overline{Q}} D_hA_h(q_{23})|D_h^2v|^2 + \frac{h^2}{2}\int_{\overline{Q}} D_hA_h(q_{23})|D_h^2v|^2 - \frac{3h^2}{2}\int_{\overline{Q}} D_hA_h(q_{23})|D_h^2v|^2 \\ &+ \frac{3h^4}{8}\int_{Q^*} D_hq_{23}|D_hv|^2 \end{aligned}$$

and

$$\begin{aligned} W_{23} = &-\frac{h^2}{2}\int_{\partial Q^*} A_h(q_{23})t_r\left(|D_h^2v|^2\right)n + \frac{h^2}{2}\int_{\partial Q} q_{23}t_r\left(|D_h^2A_hv|^2\right)n \\ &+ 2h^2\int_{\partial Q} q_{23}D_h^2vt_r(D_h^2A_hv)n + \frac{3h^3}{4}\int_{\partial Q^*} D_hq_{23}t_r(|D_h^2v|^2). \end{aligned}$$

According to Theorem 2.2 we have the following estimates

$$\begin{aligned} D_h^3A_h(q_{23}) =& 5s^5\partial_x^2\left((\partial_x\varphi)^4\partial_x^2\varphi\right) + s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2),\ D_h(q_{23}) = s^5\mathcal{O}(1) \\ D_hA_h^2(q_{23}) =& 5s^5(\partial_x\varphi)^4\partial_x^2\varphi + s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2),\ D_h^3(q_{23}) = s^5\mathcal{O}(1), \\ D_hA_h(q_{23}) =& s^5\mathcal{O}(1),\ q_{23} = s^5\mathcal{O}(1),\ A_h(q_{23}) = s^5\mathcal{O}(1). \end{aligned}$$

On account of the above estimates, we can estimate Z_{23} and W_{23} as

$$\begin{aligned} Z_{23} = & -10 \int_{\overline{Q}} (s^5 \partial_x^2 \left((\partial_x \varphi)^4 \partial_x^2 \varphi \right) + s^4 \mathcal{O}(1) + s^5 \mathcal{O}((sh)^2)) |v|^2 + \int_{\overline{Q}^*} s^3 \mathcal{O}((sh)^2) |D_h v|^2 \\ & - \int_{\overline{Q}} s^3 \mathcal{O}((sh)^2) |D_h^2 v|^2 + 30 \int_{\overline{Q}^*} (s^5 (\partial_x \varphi)^4 \partial_x^2 \varphi + s^4 \mathcal{O}(1) + s^5 \mathcal{O}((sh)^2)) |D_h v|^2 \\ & + \int_{\overline{Q}} s^3 \mathcal{O}((sh)^2 |D_h^2 v|^2 + \int_{Q^*} s \mathcal{O}((sh)^4) |D_h v|^2. \end{aligned} \tag{4.46}$$

and

$$\begin{aligned} W_{23} \geq & \int_{\partial Q^*} s^3 \mathcal{O}((sh)^2) t_r \left(|D_h^2 v|^2 \right) n + \int_{\partial Q} s^3 \mathcal{O}((sh)^2) t_r \left(|D_h^2 A_h v|^2 \right) n \\ & + \int_{\partial Q} s^3 \mathcal{O}((sh)^2) D_h^2 v \, t_r (D_h^2 A_h v) n + \int_{\partial Q^*} s^2 \mathcal{O}((sh)^2) t_r (|D_h^2 v|^2). \end{aligned}$$

In addition, for the boundary terms of the second integral from W_{23}, we have $t_r(D_h^2 A_h v)(x_0) = D_h^2 v(x_0) + \frac{h}{2} D_h^3 v(x_0)$ for $x_0 \in \partial Q^-$ and $t_r(D_h^2 A_h v)(x_0) = D_h^2 v(x_0) - \frac{h}{2} D_h^3 v(x_0)$ for $x_0 \in \partial Q^+$. This, and Young's inequality, yields

$$\begin{aligned} W_{23} \geq & \int_{\partial Q^*} s^3 \mathcal{O}((sh)^2) t_r \left(|D_h^2 v|^2 \right) n + \int_{\partial Q} s^3 \mathcal{O}((sh)^2) |D_h^2 v|^2 n + \int_{\partial Q} s \mathcal{O}((sh)^4) t_r (|D_h^3 v|^2) n \\ & - \int_{\partial Q} s^3 \mathcal{O}((sh)^3) |D_h^2 v|^2 - \int_{\partial Q} s \mathcal{O}((sh)^3) t_r (|D_h^3 v|^2). \end{aligned} \tag{4.47}$$

Therefore, for I_{23} we have

$$I_{23} \geq 30 \int_{\overline{Q}^*} s^5 (\partial_x \varphi)^4 \partial_x^2 \varphi |D_h v|^2 - 10 \int_{\overline{Q}} s^5 \partial_x^2 \left((\partial_x \varphi)^4 \partial_x^2 \varphi \right) |v|^2 + X_{23} + Y_{23},$$

where

$$\begin{aligned} X_{23} = & \int_{\overline{Q}} \left(s^4 \mathcal{O}(1) + s^5 \mathcal{O}((sh)^2) \right) |v|^2 + \int_{\overline{Q}^*} s^3 \mathcal{O}((sh)^2) |D_h v|^2 + \int_{Q^*} s \mathcal{O}((sh)^2) |D_h^3 v|^2 \\ & + \int_{\overline{Q}} \left(s^4 \mathcal{O}(1) + s^3 \mathcal{O}((sh)^2) + s^5 \mathcal{O}((sh)^2) \right) |D_h^2 v|^2 \end{aligned}$$

and

$$\begin{aligned} Y_{23} := & \int_{\partial Q^*} s^3 \mathcal{O}((sh)^2) t_r \left(|D_h^2 v|^2 \right) n + \int_{\partial Q} s^3 \mathcal{O}((sh)^2) |D_h^2 v|^2 n + \int_{\partial Q} s \mathcal{O}((sh)^4) t_r (|D_h^3 v|^2) n \\ & - \int_{\partial Q} s^3 \mathcal{O}((sh)^3) |D_h^2 v|^2 - \int_{\partial Q} s \mathcal{O}((sh)^3) t_r (|D_h^3 v|^2), \end{aligned}$$

which is due to (4.46) and (4.47), and this is the precisely assertion of the Lemma. □

4.3.8. Estimate of I_{24}

Lemma 4.12 *For $\lambda h (\delta T^2)^{-1} \leq 1$, we have*

$$I_{24} \geq 6 \int_Q s^7 (\partial_x \varphi)^6 \partial_x^2 \varphi |v|^2 + X_{24},$$

where

$$X_{24} := \int_Q \left(s^6\mathcal{O}(1) + s^7\mathcal{O}((sh)^2)\right)|v|^2 + \int_{Q^*} s^5\mathcal{O}((sh)^2)|D_h v|^2 - \int_Q s^5\mathcal{O}((sh)^4)|v|^2$$
$$- \int_{Q^*} s\mathcal{O}((sh)^4)|D_h^3 v|^2 - \int_{Q^*} s^5\mathcal{O}((sh)^4)|D_h v|^2 - \int_{Q^*} s\mathcal{O}((sh)^4)|D_h^3 v|^2.$$

Proof. Define $q_{24} := \rho D_h^4(r)\, A_h D_h(\rho A_h D_h^3(r))$. Let us evaluate $I_{24} := 2\int_Q q_{24}A_h^4(v)v$. Using the identity $A_h^4(v) = v + \frac{h^2}{4}D_h^2 v + \frac{h^2}{4}D_h^2 A_h^2 v$, which is due to Corollary 2.1, we have

$$I_{24} = 2\int_Q q_{24}|v|^2 + \frac{h^2}{2}\int_Q q_{24}D_h^2(v)\, v + \frac{h^2}{2}\int_Q q_{24}D_h^2 A_h^2(v)\, v$$
$$=: I_{24}^{(a)} + I_{24}^{(b)} + I_{24}^{(c)}.$$

We begin by analyzing $I_{24}^{(c)}$. A discrete integration by parts concerning the difference operator D_h, and the fact that $v = 0$ on ∂Q, yields

$$I_{24}^{(c)} = -\frac{h^2}{2}\int_{Q^*} D_h(q_{24}v)D_h A_h^2(v).$$

Now, using Lemma 2.1 and Corollary 2.1, $I_{24}^{(c)}$ can be written as

$$I_{24}^{(c)} = \int_{Q^*} -\frac{h^2}{4}D_h q_{24} D_h(v^2) - \frac{h^4}{8}D_h q_{24} A_h v D_h^3 v - \frac{h^2}{2}A_h q_{24}|D_h v|^2 - \frac{h^4}{8}A_h q_{24} D_h v D_h^3 v.$$

We apply Proposition 2.1 to the first term of the above expression, and using $v = 0$ on ∂Q, to get

$$I_{24}^{(c)} = \int_Q \frac{h^2}{4}D_h^2 q_{24}|v|^2 - \int_{Q^*} \frac{h^4}{8}D_h q_{24} A_h v D_h^3 v + \frac{h^2}{2}A_h q_{24}|D_h v|^2 + \frac{h^4}{8}A_h q_{24} D_h v D_h^3 v.$$

We can proceed analogously for $I_{24}^{(b)}$ to obtain

$$I_{24}^{(b)} = \int_Q \frac{h^2}{4}D_h^2 q_{24}|v|^2 - \int_{Q^*} \frac{h^2}{2}A_h q_{24}|D_h v|^2.$$

Therefore, we rewrite I_{24} as

$$I_{24} = 2\int_Q q_{24}|v|^2 + \int_Q \frac{h^2}{4}D_h^2 q_{24}|v|^2 - \int_{Q^*} \frac{h^4}{8}D_h q_{24} A_h v D_h^3 v$$
$$- \int_{Q^*} h^2 A_h q_{24}|D_h v|^2 - \int_{Q^*} \frac{h^4}{8}A_h q_{24} D_h v D_h^3 v.$$

Using the estimates

$$D_h(q_{24}) = s^7\mathcal{O}(1),\ D_h^2(q_{24}) = s^7\mathcal{O}(1),\ A_h(q_{24}) = s^7\mathcal{O}(1),$$
$$q_{24} = 3s^7(\partial_x\varphi)^6\partial_x^2\varphi + s^6\mathcal{O}(1) + s^7\mathcal{O}((sh)^2);$$

which are due to Theorem 2.2 and Corollary 2.5, it follows that

$$I_{24} = 6\int_Q s^7(\partial_x\varphi)^6\partial_x^2\varphi|v|^2 + Z_{24},$$

where

$$Z_{24} = \int_Q \left(s^6\mathcal{O}(1) + s^7\mathcal{O}((sh)^2)\right)|v|^2 + \int_{Q^*} s^5\mathcal{O}((sh)^2)|D_hv|^2 + \int_{Q^*} s^3\mathcal{O}((sh)^4)A_hvD_h^3v + \int_{Q^*} s^3\mathcal{O}((sh)^4)D_hvD_h^3v.$$

The proof is completed by showing that

$$\begin{aligned} Z_{24} \geq & \int_Q \left(s^6\mathcal{O}(1) + s^7\mathcal{O}((sh)^2)\right)|v|^2 + \int_{Q^*} s^5\mathcal{O}((sh)^2)|D_hv|^2 - \int_Q s^5\mathcal{O}((sh)^4)|v|^2 \\ & - \int_{Q^*} s\mathcal{O}((sh)^4)|D_h^3v|^2 - \int_{Q^*} s^5\mathcal{O}((sh)^4)|D_hv|^2 - \int_{Q^*} s\mathcal{O}((sh)^4)|D_h^3v|^2 \\ :=& X_{24}, \end{aligned}$$

which follows by Young's inequality and (2.6). □

4.3.9. Estimate of I_{31}

Lemma 4.13 *For $\lambda h(\delta T^2)^{-1} \leq 1$, we have*

$$I_{31} \geq -C \int_{\overline{\mathcal{M}}} |D_h^2v|^2\Big|_0^T = Y_{31}.$$

PROOF. Let us set $q_{31} := \rho A_h^4(r)$. By virtue of Theorem 2.2 we have $q_{31} = 1+\mathcal{O}(sh) = C > 0$. Then

$$|I_{31}| \leq C \left|\int_Q D_h^4v\,\partial_tv\right|. \tag{4.48}$$

Let us examine the right hand sided above. By using Corollary 2.2 we have

$$\int_Q D_h^4(v)\partial_tv = \int_Q D_h^2(\partial_tv)D_hv,$$

since $\partial_tv = 0$ on ∂Q and $D_h(\partial_tv) = 0$ on ∂Q^*. We note that $\frac{1}{2}\partial_t\left(|D_h^2v|^2\right) = D_h^2(\partial_tv)D_h^2v$, then

$$\int_Q D_h^4(v)\partial_tv = \frac{1}{2}\int_{\overline{Q}} \partial_t\left(|D_h^2v|^2\right) = \frac{1}{2}\int_{\overline{\mathcal{M}}} |D_h^2v|^2\Big|_0^T. \tag{4.49}$$

Combining (4.48) and (4.49) proves the Lemma. □

4.3.10. Estimate of I_{32}

Lemma 4.14 *For $\lambda h(\delta T^2)^{-1} \leq 1$, we have*

$$I_{32} = 18\int_{\overline{Q}} s^3(\partial_x\varphi)^2\partial_x^2\varphi|D_h^2v|^2 - 6\int_{Q^*} s^3\partial_x^2((\partial_x\varphi)^2\partial_x^2\varphi)|D_hv|^2 + X_{32} + Y_{32},$$

where

$$X_{32} := \int_Q \left(s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2)\right)|D_h^2 v|^2 + \int_{Q^*}\left(s^2\mathcal{O}(1) + s3\mathcal{O}((sh)^2)\right)|D_h v|^2$$
$$+ \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2$$

and

$$Y_{32} := -2\int_{\partial Q^*}\left(s^3(\partial_x\varphi)^3 + s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2)\right)t_r(|D_h^2 v|^2)n$$
$$+ \int_{\partial Q} s\mathcal{O}((sh)^2)t_r\left(|D_h^3 v|^2\right)n.$$

Proof. Setting $q_{32} := \rho^2 A_h r\, A_h D_h^3 r$, we write $I_{32} := \int_Q q_{32} D_h^4 v\, D_h A_h^3 v$. Using Lemma 2.1 we have the identity $D_h A_h^3 v = D_h A_h v + \frac{h^2}{4} A_h D_h^3 v$, which allows us to write

$$I_{32} = 4\int_Q q_{32} D_h^4 v\, D_h A_h v + h^2 \int_Q q_{32} D_h^4 v\, A_h D_h^3 v := I_{32}^{(a)} + I_{32}^{(b)}.$$

Let us examine $I_{32}^{(a)}$. Using the integration by parts (2.8) and the identity (2.2) we have

$$I_{32}^{(a)} = -4\int_{Q^*} D_h q_{32}\, D_h A_h^2 v\, D_h^3 v - 4\int_{Q^*} A_h q_{32}\, D_h^2 A_h v\, D_h^3 v + 4\int_{\partial Q} q_{32}\, D_h A_h v\, t_r\left(D_h^3 v\right) n.$$

Now, we use the identities $D_h A_h^2 v = D_h v + \frac{h^2}{4} D_h^3 v$ and $D_h^2 A_h v D_h^3 v = \frac{1}{2} D_h(|D_h^2 v|^2)$, due to (2.4) and (2.2) respectively, to obtain

$$I_{32}^{(a)} = -4\int_{Q^*} D_h q_{32}\, D_h v\, D_h^3(v) - h^2\int_{Q^*} D_h q_{32} |D_h^3 v|^2 - 2\int_{Q^*} A_h q_{32}\, D_h\left(|D_h^2 v|^2\right).$$

Applying the discrete integration by parts (2.8) on the first and third integral above, and then using the identity (2.2), it follows that

$$I_{32}^{(a)} = 4\int_{\overline{Q}} D_h^2 q_{32}\, A_h D_h v\, D_h^2 v + 6\int_{\overline{Q}} D_h A_h q_{32}\, |D_h^2 v|^2 - h^2\int_{Q^*} D_h q_{23}\, |D_h^3 v|^2$$
$$- 2\int_{\partial Q^*} A_h q_{32}\, t_r(|D_h^2 v|^2) n,$$

where we also have used that $D_h v = 0$ on ∂Q^*. Note that on the first integral above we can use the identity $A_h D_h v D_h^2 = \frac{1}{2} D_h(|D_h v|^2)$, use the discrete integral by parts (2.8), and the fact $t_r(D_h v) = 0$ on ∂Q to get

$$I_{32}^{(a)} = -2\int_{\overline{Q}^*} D_h^3 q_{32}\, |D_h v|^2 + 6\int_{\overline{Q}} D_h A_h q_{32}\, |D_h^2 v|^2 - h^2\int_{Q^*} D_h q_{23}\, |D_h^3 v|^2$$
$$- 2\int_{\partial Q^*} A_h q_{32}\, t_r(|D_h^2 v|^2) n,$$

This part of the proof finishes using Theorem 2.2, which enables us to write

$$\begin{aligned}
I_{32}^{(a)} =&18\int_{Q} s^3(\partial_x\varphi)^2\partial_x^2\varphi|D_h^2 v|^2 - 6\int_{Q^*} s^3\partial_x^2((\partial_x\varphi)^2\partial_x^2\varphi)|D_h v|^2 \\
&+\int_{Q}\left(s^2\mathcal{O}(1)+s^3\mathcal{O}((sh)^2)\right)|D_h^2 v|^2+\int_{Q^*}\left(s^2\mathcal{O}(1)+s3\mathcal{O}((sh)^2)\right)|D_h v|^2 \\
&+\int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2 \\
&-2\int_{\partial Q^*}\left(s^3(\partial_x\varphi)^3+s^2\mathcal{O}(1)+s^3\mathcal{O}((sh)^2)\right)t_r(|D_h^2 v|^2)n.
\end{aligned} \tag{4.50}$$

On the other hand, let us estimate $I_{32}^{(b)}$. Thanks to Lemma 2.1 we note that $D_h^4 v\, A_h D_h^3 v = \frac{1}{2}D_h\left(|D_h^3 v|^2\right)$, thus we can rewrite $I_{32}^{(b)}$ as

$$I_{32}^{(b)} = \frac{h^2}{2}\int_Q q_{32} D_h\left(|D_h^3 v|^2\right).$$

From the discrete integration by parts (2.8), for $I_{32}^{(b)}$ we get

$$I_{32}^{(b)} = -\frac{h^2}{2}\int_{Q^*} D_h q_{32}|D_h^3 v|^2 + \frac{h^2}{2}\int_{\partial Q} q_{32}\, t_r\left(|D_h^3 v|^2\right)n.$$

Hence, by virtue of Theorem 2.2, $I_{32}^{(b)}$ can be estimated as

$$I_{32}^{(b)} = \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2 + \int_{\partial Q} s\mathcal{O}((sh)^2)\, t_r\left(|D_h^3 v|^2\right)n. \tag{4.51}$$

The proof is completed by combining (4.50) and (4.51). □

4.3.11. Estimate of I_{33}

Lemma 4.15 *For $\lambda h(\delta T^2)^{-1} \le 1$, we have*

$$I_{33} := -2\int_{Q^*} s\partial_x^2\varphi|D_h^3 v|^2 + X_{33} + Y_{33},$$

where

$$X_{33} := -2\int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2$$

and

$$Y_{33} :=2\int_{\partial Q} s\partial_x\varphi t_r\left(|D_h^3 v|^2\right)n + 2\int_{\partial Q} s\mathcal{O}((sh)^2)t_r\left(|D_h^3 v|^2\right)n.$$

Proof. Setting $q_{33} := \rho^2 A_h^4 r\, A_h^3 D_h r$, we have $I_{33} := 4\int_Q q_{33} D_h^4 v\, D_h^3 A_h v$. We note that $D_h^4 v\, D_h^3 A_h v = \frac{1}{2}D_h\left(|D_h^3 v|^2\right)$, due to (2.2). Rewritten I_{33} with the previous identity and then applying the discrete integration by parts (2.8) we obtain

$$I_{33} = -2\int_{Q^*} D_h(q_{33})|D_h^3 v|^2 + 2\int_{\partial Q} q_{33} t_r\left(|D_h^3 v|^2\right)n.$$

Using Theorem 2.2 the Lemma follows. □

4.3.12. Estimate of I_{34}

Lemma 4.16 *For $\lambda h(\delta T^2)^{-1} \leq 1$, we have*

$$\begin{aligned} I_{34} =& 6 \int_{\overline{Q}} s^3 (\partial_x \varphi)^2 \partial_x^2 \varphi |D_h^2 v|^2 - 12 \int_{Q^*} s^3 \partial_x^2 \left((\partial_x \varphi)^2 \partial_x^2 \varphi \right) |D_h v|^2 \\ &+ 3 \int_Q s^3 \partial_x^4 \left((\partial_x \varphi)^2 \partial_x^2 \varphi \right) |v|^2 + X_{34}, \end{aligned}$$

where

$$\begin{aligned} X_{34} :=& \int_{\overline{Q}} \left(s\mathcal{O}((sh)^2) + s^2 \mathcal{O}(1) + s^3 \mathcal{O}((sh)^2) \right) |D_h^2 v|^2 + \int_Q \left(s^2 \mathcal{O}(1) + s^3 \mathcal{O}((sh)^2) \right) |v|^2 \\ &+ \int_{Q^*} \left(s^2 \mathcal{O}(1) + s^3 \mathcal{O}((sh)^2) \right) |D_h v|^2. \end{aligned}$$

Proof. Setting $q_{34} := \rho A_h^4 r \, A_h D_h (\rho A_h D_h^3 r)$, I_{34} ca be written as

$$I_{34} := 2 \int_Q q_{34} \, v \, D_h^4 v.$$

By virtue of Corollary 2.2 and Lemma 2.1 we write

$$I_{34} = 2 \int_{\overline{Q}} D_h^2 q_{34} \, A_h^2 v \, D_h^2 v + 4 \int_{\overline{Q}} D_h A_h q_{34} \, D_h A_h v \, D_h^2 v + 2 \int_{\overline{Q}} A_h^2 q_{34} \, |D_h^2 v|^2,$$

where we have used that $v = 0$ on ∂Q and $D_h v = A_h v = 0$ on ∂Q^*. We note that $D_h A_h v \, D_h^2 v = \frac{1}{2} D_h (|D_h v|^2)$ and $A_h^2 v = v + \frac{h^2}{4} D_h^2 v$, thanks to Lemma 2.1. This allows us to rewrite I_{34} as

$$I_{34} = 2 \int_{\overline{Q}} D_h^2 q_{34} v D_h^2 v + 2 \int_{\overline{Q}} D_h A_h q_{34} D_h (|D_h v|^2) + \frac{h^2}{2} \int_{\overline{Q}} D_h^2 q_{34} |D_h^2 v|^2 + 2 \int_{\overline{Q}} A_h^2 q_{34} |D_h^2 v|^2.$$

A discrete integration by parts involving the difference operator on the first two terms above yields

$$\begin{aligned} I_{34} =& - \int_{\overline{Q^*}} D_h^3 q_{34} \, D_h (|v|^2) - 2 \int_{\overline{Q^*}} D_h^2 A_h q_{34} \, |D_h v|^2 - 2 \int_{\overline{Q^*}} D_h^2 A_h q_{34} \, |D_h v|^2 \\ &+ \frac{h^2}{2} \int_{\overline{Q}} D_h^2 q_{34} \, |D_h^2 v|^2 + 2 \int_{\overline{Q}} A_h^2 q_{34} \, |D_h^2 v|^2, \end{aligned}$$

since $t_r(D_h v) = 0$ on $\partial \overline{Q}$ and $A_h v \, D_h v = \frac{1}{2} D_h (|v|^2)$. Repeating the previous steps on the first integral from above leads to

$$I_{34} = \int_Q D_h^4 q_{34} \, |v|^2 - 4 \int_{Q^*} D_h^2 A_h q_{34} \, |D_h v|^2 + \frac{h^2}{2} \int_Q D_h^2 q_{34} \, |D_h^2 v|^2 + 2 \int_{\overline{Q}} A_h^2 q_{34} \, |D_h^2 v|^2.$$

We shall have established the Lemma if we prove the following

$$\begin{aligned}
D_h^4(q_{34}) =& 3s^3\partial_x^4((\partial_x\varphi)^2\partial_x^2\varphi) + s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2),\\
A_h^2(q_{34}) =& 3s^3(\partial_x\varphi)^2\partial_x^2\varphi + s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2)\\
D_h^2A_h(q_{34}) =& 3s^3\partial_x^2((\partial_x\varphi)^2\partial_x^2\varphi) + s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2),\ D_h^2(q_{34}) = s^3\mathcal{O}(1),
\end{aligned}$$

which is clear from Theorem 2.2 and Corollary 2.5 □

4.3.13. Estimate of I_{42}

Lemma 4.17 *For $\lambda h(\delta T^2)^{-1} \leq 1$, we have*

$$I_{42} \geq 48\int_Q s^5\,(\partial_x\varphi)^4\,\partial_x^2\varphi|D_hv|^2 + X_{42},$$

where

$$X_{42} := \int_{Q^*}\left(s^4\mathcal{O}(1) + s^3\mathcal{O}((sh)^2) + s^5\mathcal{O}((sh)^2)\right)|D_hv|^2 + \int_Q s^3\mathcal{O}((sh)^2)|D_h^2v|^2$$

and

$$\begin{aligned}
Y_{42} :=& \int_{\partial Q} s^4\mathcal{O}((sh))\,t_r(|D_hv|^2) - \int_{\partial Q} s^2\mathcal{O}((sh)^3)\,t_r(|D_hv|^2) + \int_{\partial Q} s\mathcal{O}((sh)^2)\,|D_h^2v|^2n\\
&- \int_{\partial Q} s^2\mathcal{O}((sh)^2)\,|D_h^2v|^2 - \int_{\partial Q} s^4\mathcal{O}((sh)^2)\,t_r\left(|D_hv|^2\right) - \int_{\partial Q} s^5\mathcal{O}((sh)^2)\,t_r(|D_hv|^2)\\
&- \int_{\partial Q} s\mathcal{O}((sh)^2)|D_h^3v|^2 + \int_{\partial Q} s^2\mathcal{O}((sh)^3)\,|D_h^2v|^2n - \int_{\partial Q} s\mathcal{O}((sh)^4)\,|D_h^2v|^2\\
&- \int_{\partial Q} s\mathcal{O}((sh)^4)\,t_r(|D_h^3v|^2).
\end{aligned}$$

Proof. Let us set

$$q_{42} := A_hD_h\left(\rho A_h^2D_h^2r\right)\rho A_hD_h^3r.$$

Then, I_{42} is defined as

$$I_{42} := 24\int_Q q_{42}\,A_hD_hv\,D_hA_h^3v.$$

By virtue of Lemma 2.1 the following identity $D_hA_h^3 = A_hD_hv + \frac{h^2}{4}D_h^3A_hv$ holds. Thus, we can rewrite I_{42} as

$$I_{42} = 24\int_Q q_{42}|A_hD_hv|^2 + 6h^2\int_Q q_{42}\,A_hD_hv\,D_h^3A_hv.$$

We note that $|A_hD_h(v)|^2 = A_h\left(|D_hv|^2\right) - \frac{h^2}{4}|D_h^2v|^2$, thanks to Lemma 2.1. Then

$$I_{42} = 24\int_Q q_{42}\,A_h\left(|D_hv|^2\right) - 6h^2\int_Q q_{42}|D_h^2v|^2 + 6h^2\int_Q q_{42}\,D_hA_hv\,D_h^3A_hv.$$

A discrete integration by parts on the first integral above involving the average operator and with respect to the difference operator on the last one, and then using Lemma 2.1 yield

$$
\begin{aligned}
I_{42} =& 24\int_{Q^*} A_h q_{42}\,|D_h v|^2 - 12h\int_{\partial Q} q_{42}\, t_r(|D_h v|^2) - 6h^2\int_Q q_{42}|D_h^2 v|^2\\
&- 6h^2\int_{Q^*} D_h q_{42}\, D_h A_h^2 v\, D_h^2 A_h v - 6h^2\int_{Q^*} A_h q_{42}\,|D_h^2 A_h v|^2\\
&+ 6h^2\int_{\partial Q} q_{42}\, D_h A_h v\, t_r(D_h^2 A_h v) n.
\end{aligned}
$$

Now, using the identity $D_h^2 A_h v\, A_h^2 D_h v = \frac{1}{2} D_h(|D_h A_h v|^2)$, due to 2.1, we have

$$
\begin{aligned}
I_{42} =& 24\int_{Q^*} A_h q_{42}\,|D_h v|^2 - 12h\int_{\partial Q} q_{42}\, t_r(|D_h v|^2) - 6h^2\int_Q q_{42}|D_h^2 v|^2\\
&- 3h^2\int_{Q^*} D_h q_{42}\, D_h(|D_h A_h v|^2) - 6h^2\int_{Q^*} A_h q_{42}\,|D_h^2 A_h v|^2\\
&+ 6h^2\int_{\partial Q} q_{42}\, D_h A_h v\, t_r(D_h^2 A_h v) n.
\end{aligned}
$$

Finally, a discrete integration by parts with respect to the difference operator yields

$$
\begin{aligned}
I_{42} =& 24\int_{Q^*} A_h q_{42}\,|D_h v|^2 - 12h\int_{\partial Q} q_{42}\, t_r(|D_h v|^2) - 6h^2\int_Q q_{42}|D_h^2 v|^2\\
&3h^2\int_{\overline{Q}} D_h^2 q_{42}\,|D_h A_h v|^2 - 3h^2\int_{\partial Q^*} D_h q_{42}\, t_r(|D_h A_h|^2) n\\
&- 6h^2\int_{Q^*} A_h q_{42}\,|D_h^2 A_h v|^2 + 6h^2\int_{\partial Q} q_{42}\, D_h A_h v\, t_r(D_h^2 A_h v) n.
\end{aligned}
$$

From Theorem 2.2 and Corollary 2.5 it follows that

$$
\begin{aligned}
A_h(q_{42}) =& 2s^5(\partial_x\varphi)^4\partial_x^2\varphi + s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2),\ q_{42} = s^5\mathcal{O}(1),\\
D_h(q_{42}) =& s^5\mathcal{O}(1),\ D_h^2(q_{42}) = s^5\mathcal{O}(1).
\end{aligned}
$$

We thus can estimate I_{42} as

$$
I_{42} = 48\int_{Q^*} s^5\,(\partial_x\varphi)^4\,\partial_x^2\varphi|D_h v|^2 + Z_{42} + W_{42},
$$

where

$$
\begin{aligned}
Z_{42} :=& \int_{Q^*}\left(s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2)\right)|D_h v|^2 + \int_Q s^3\mathcal{O}((sh)^2)|D_h^2 v|^2\\
&+ \int_{Q^*} s^3\mathcal{O}((sh)^2)|D_h^2 A_h v|^2 + \int_Q s^3\mathcal{O}((sh)^2)|D_h A_h v|^2
\end{aligned}
$$

and

$$
\begin{aligned}
W_{42} :=& \int_{\partial Q} s^4\mathcal{O}((sh))\, t_r(|D_h v|^2) + \int_{\partial Q^*} s^3\mathcal{O}((sh)^2)\, t_r(|D_h A_h|^2) n\\
&+ \int_{\partial Q} s^3\mathcal{O}((sh)^2)\, D_h A_h v\, t_r(D_h^2 A_h v) n.
\end{aligned}
$$

The proof is completed by showing that

$$|Z_{42}| \le \int_{Q^*} \left(s^4\mathcal{O}(1) + s^3\mathcal{O}((sh)^2) + s^5\mathcal{O}((sh)^2)\right) |D_h v|^2 + \int_Q s^3\mathcal{O}((sh)^2)|D_h^2 v|^2 \\ := X_{42} \tag{4.52}$$

and

$$\begin{aligned} W_{42} \ge & \int_{\partial Q} s^4\mathcal{O}((sh))\, t_r(|D_h v|^2) - \int_{\partial Q} s^2\mathcal{O}((sh)^3)\, t_r(|D_h v|^2) + \int_{\partial Q} s\mathcal{O}((sh)^2)\, |D_h^2 v|^2 n \\ & - \int_{\partial Q} s^2\mathcal{O}((sh)^2)\, |D_h^2 v|^2 - \int_{\partial Q} s^4\mathcal{O}((sh)^2)\, t_r\left(|D_h v|^2\right) - \int_{\partial Q} s^5\mathcal{O}((sh)^2)\, t_r(|D_h v|^2) \\ & - \int_{\partial Q} s\mathcal{O}((sh)^2)|D_h^3 v|^2 + \int_{\partial Q} s^2\mathcal{O}((sh)^3)\, |D_h^2 v|^2 n - \int_{\partial Q} s\mathcal{O}((sh)^4)\, |D_h^2 v|^2 \\ & - \int_{\partial Q} s\mathcal{O}((sh)^4)\, t_r(|D_h^3 v|^2) \\ :=& Y_{42}. \end{aligned} \tag{4.53}$$

The inequality (4.52) follows using (2.6) and a discrete integration by parts respect to the average operator. To deal with W_{42}, we note that $D_h A_h v = t_r(D_h v) + \frac{h}{2} D_h^2 v\, n$ and $t_r(D_h^2 A_h v) = D_h^2 v - \frac{h}{2} t_r(D_h^3 v)\, n$ on ∂Q. Then

$$\begin{aligned} W_{42} =& \int_{\partial Q} s^4\mathcal{O}((sh))\, t_r(|D_h v|^2) + \int_{\partial Q} s^3\mathcal{O}((sh)^2)\, t_r(|D_h v|^2) n + \int_{\partial Q} s^3\mathcal{O}((sh)^2)\, t_r(|D_h v|^2) n \\ & + \frac{h}{2}\int_{\partial Q} s^3\mathcal{O}((sh)^2)\, t_r(|D_h v|)\, D_h^2 v + \frac{h^2}{4}\int_{\partial Q} s^3\mathcal{O}((sh)^2)\, |D_h^2 v|^2 n \\ & + \int_{\partial Q} s^3\mathcal{O}((sh)^2)\, D_h^2 v\, t_r\left(D_h v\right) + \int_{\partial Q} s^3\mathcal{O}((sh)^2)\, t_r\left(D_h v\, D_h^3 v\right) n \\ & + \int_{\partial Q} s^2\mathcal{O}((sh)^3)\, |D_h^2 v|^2 n + \int_{\partial Q} s\mathcal{O}((sh)^4)\, D_h^2 v\, t_r(D_h^3 v). \end{aligned}$$

The Young's inequality allows us to conclude (4.53), and the proof is complete. ☐

4.3.14. Estimate of I_{43}

Lemma 4.18 *For $\lambda h(\delta T^2)^{-1} \le 1$, we have*

$$I_{43} = -48\int_{\overline{Q}} s^3(\partial_x\varphi)^2\partial_x^2\varphi|D_h^2 v|^2 + 24\int_{\overline{Q}^*} s^3\partial_x^2\left((\partial_x\varphi)^2\partial_x^2\varphi\right)|D_h v|^2 + X_{43} + Y_{43},$$

where

$$\begin{aligned} X_{43} :=& \int_{\overline{Q}} \left(s^2\mathcal{O}(1) + s\mathcal{O}((sh)^2) + s^3\mathcal{O}((sh)^2)\right) |D_h^2 v|^2 \\ & + \int_{\overline{Q}^*} \left(s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2)\right) |D_h v|^2 + \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2. \end{aligned}$$

and

$$Y_{43} := \int_{\partial Q} s^2\mathcal{O}(1)\,|D_h^2 v|^2 + \int_{\partial Q} s^4\mathcal{O}((sh)^2)\,t_r\left(|D_h v|^2\right) - \int_{\partial Q} s\mathcal{O}((sh)^2)|D_h^2 v|^2$$
$$- \int_{\partial Q} s\mathcal{O}((sh)^2)\,t_r(|D_h^3 v|^2) + \int_{\partial Q^*} s^2\mathcal{O}(sh)\,t_r(|D_h^2 v|^2).$$

Proof. Let us set $q_{43} := A_h D_h\left(\rho A_h^2 D_h^2 r\right)\rho A_h^3 D_h r$. Let us compute

$$I_{43} = 24\int_Q q_{43}\,A_h D_h v\,D_h^3 A_h v.$$

A discrete integration by parts and Lemma 2.1 yield

$$\begin{aligned} I_{43} =& -24\int_{Q^*} D_h q_{43}\,A_h^2 D_h v\,D_h^2 A_h v - 24\int_{Q^*} A_h q_{43}\,|D_h^2 A_h v|^2 \\ & - 24\int_{\partial Q} q_{43}\,D_h A_h v\,t_r\left(D_h^2 A_h v\right)n. \\ =& I_{24}^{(a)} + I_{24}^{(b)} + I_{24}^{(c)}. \end{aligned}$$

Noting that $A_h^2 D_h v\,D_h^2 A_h v = \frac{1}{2}D_h(|D_h A_h v|^2)$, due to Lemma 2.1, and a discrete integration by parts related to the difference operator; for $I_{24}^{(a)}$ it follows that

$$I_{43}^{(a)} = 12\int_{\overline{Q}} D_h^2 q_{43}\,|D_h A_h v|^2 - 12\int_{\partial Q^*} D_h q_{43}\,t_r(|D_h A_h v|^2)n.$$

Now, by virtue of Lemma 2.1 we note $|D_h A_h v|^2 = A_h(|D_h v|^2) - \frac{h^2}{4}|D_h^3 v|^2$. Hence, we can rewrite $I_{43}^{(a)}$ as

$$I_{43}^{(a)} = 12\int_{\overline{Q}} D_h^2 q_{43}\,A_h(|D_h v|^2) - 3h^2\int_{\overline{Q}} D_h^2 q_{43}\,|D_h^3 v|^2 - 12\int_{\partial Q^*} D_h q_{43}\,t_r(|D_h A_h v|^2)n.$$

Integrating by parts conncerning the average operator leads to

$$I_{43}^{(a)} = 12\int_{\overline{Q}^*} A_h D_h^2 q_{43}\,|D_h v|^2 - 3h^2\int_{\overline{Q}} D_h^2 q_{43}\,|D_h^3 v|^2 - 12\int_{\partial Q^*} D_h q_{43}\,t_r(|D_h A_h v|^2)n,$$

since $t_r(D_h v) = 0$ on $\partial\overline{Q}$.

In the same manner we can see that

$$I_{43}^{(b)} = -24\int_{\overline{Q}} A_h^2 q_{43}\,|D_h^2 v|^2 + 12h\int_{\partial Q^*} A_h q_{43}\,t_r(|D_h^2 v|^2) + 6h^2\int_{Q^*} A_h q_{43}\,|D_h^3 v|^2.$$

We note that Theorem 2.2 and Corollary 2.5 enable us to write the following estimates

$$\begin{aligned} A_h D_h^2(q_{43}) =& 2s^3\partial_x((\partial_x\varphi)^2\partial_x^2\varphi) + s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2),\ D_h^2(q_{43}) = s^3\mathcal{O}(1),\ D_h(q_{43}) = s^3\mathcal{O}(1), \\ A_h^2(q_{43}) =& 2s^3(\partial_x\varphi)^2\partial_x^2\varphi + s^3\mathcal{O}((sh)^2),\ q_{43} = s^3\mathcal{O}(1),\ A_h(q_{43}) = s^3\mathcal{O}(1). \end{aligned}$$

The above implies that I_{43} can be estimated as

$$I_{43} = -48\int_{\overline{Q}} s^3(\partial_x\varphi)^2\partial_x^2\varphi|D_h^2 v|^2 + 24\int_{\overline{Q}^*} s^3\partial_x^2\left((\partial_x\varphi)^2\partial_x^2\varphi\right)|D_h v|^2 + X_{43},$$

where

$$X_{43} := \int_{\overline{Q}} \Big(s^2\mathcal{O}(1) + s\mathcal{O}((sh)^2) + s^3\mathcal{O}((sh)^2)\Big)\,|D_h^2 v|^2 + \int_{\overline{Q}^*} \Big(s^2\mathcal{O}(1) + s^3\mathcal{O}((sh)^2)\Big)\,|D_h v|^2 \\ + \int_{Q^*} s\mathcal{O}((sh)^2)|D_h^3 v|^2$$

and

$$W_{43} := -\int_{\partial Q^*} s^3\mathcal{O}(1)t_r(|D_hA_hv|^2)n - \int_{\partial Q} s^3\mathcal{O}(1)D_hA_hv\,t_r(D_h^2A_hv)n \\ + h\int_{\partial Q^*} s^3\mathcal{O}(1)\,t_r(|D_h^2v|^2).$$

The only point remaining concerns the behaviour of W_{43}. We note that $t_r(D_hA_hv) = D_hv - \frac{h}{2}t_r(D_h^2v)n = -\frac{h}{2}t_r(D_h^2v)n$ on ∂Q^*, since $D_hv = 0$ on ∂Q^*. Besides, $D_hA_hv = t_r(D_hv) + \frac{h}{2}D_h^2v\,n$ and $t_r(D_h^2A_hv) = D_h^2v - \frac{h}{2}t_r(D_h^3v)\,n$ on ∂Q. From this identities we see that

$$W_{43} = -\int_{\partial Q} s^3\mathcal{O}(1)\,D_h^2v\,t_r\,(D_hv) + \int_{\partial Q} s^2\mathcal{O}((sh)^2)\,t_r\,\Big(D_hv\,D_h^3v\Big)\,n + \int_{\partial Q} s^2\mathcal{O}(sh)|D_h^2v|^2 \\ + \int_{\partial Q} s\mathcal{O}((sh)^2)D_h^2v\,t_r(D_h^3v) + \int_{\partial Q^*} s^2\mathcal{O}(sh)\,t_r(|D_h^2v|^2) + \int_{\partial Q^*} s\mathcal{O}((sh)^2)\,t_r(|D_h^2v|^2)n.$$

Moreover, using the Young's inequality we obtain

$$W_{43} \geq \int_{\partial Q} s^2\mathcal{O}(1)\,|D_h^2v|^2 + \int_{\partial Q} s^4\mathcal{O}(1)\,t_r\,\Big(|D_hv|^2\Big) - \int_{\partial Q} s^3\mathcal{O}((sh)^2)\,t_r(|D_hv|^2) \\ - \int_{\partial Q} s\mathcal{O}((sh)^2)\,t_r(|D_h^3v|^2) + \int_{\partial Q} s^2\mathcal{O}(sh)|D_h^2v|^2 - \int_{\partial Q} s\mathcal{O}((sh)^2)|D_h^2v|^2 \\ - \int_{\partial Q} s\mathcal{O}((sh)^2)\,t_r(|D_h^3v|^2) + \int_{\partial Q^*} s^2\mathcal{O}(sh)\,t_r(|D_h^2v|^2) + \int_{\partial Q^*} s\mathcal{O}((sh)^2)\,t_r(|D_h^2v|^2)n \\ := Y_{43},$$

and the Lemma follows. □

4.3.15. Estimate of I_{44}

Lemma 4.19 *For $\lambda h(\delta T^2)^{-1} \leq 1$, we have*

$$I_{44} = -\,36\int_Q s^5\partial_x(\partial_x(\varphi^3)\partial_x^2(\varphi^2))|v|^2 + X_{44},$$

where

$$X_{44} := \int_Q (s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2))|v|^2 + \int_{Q^*} s^3\mathcal{O}((sh)^2)|D_hv|^2.$$

Proof. We define $q_{44} := A_hD_h\left(\rho A_h^2D_h^2r\right)A_hD_h\left(\rho A_hD_h^3r\right)$. Then, we write I_{44} as

$$I_{44} := 12\int_Q q_{44}\,v\;D_hA_hv.$$

A discrete integration by parts with respect to the average operator and Lemma 2.1 yield

$$I_{44} = 12\int_{Q^*} A_h(q_{44})\, D_h v\, A_h v - 3h^2 \int_{Q^*} D_h q_{44}\, |D_h v|^2,$$

where we have used that $v = 0$ on ∂Q. We note that $D_h v\, A_h v = \frac{1}{2} D_h(|v|^2)$, due to Lemma 2.1, thus

$$I_{44} = 6\int_{Q^*} A_h q_{44}\, D_h(|v|^2) - 3h^2 \int_{Q^*} D_h q_{44}\, |D_h v|^2.$$

We continue in this fashion to obtain

$$I_{44} = -6\int_{Q} D_h A_h q_{44}\, |v|^2 - 3h^2 \int_{Q^*} D_h q_{44}\, |D_h v|^2,$$

since $v = 0$ on ∂Q. By virtue of Theorem 2.2, we have the following estimates $A_h D_h q_{44} = 6s^5 \partial_x(\partial_x \varphi^3 \partial_x^2 \varphi^2) + s^4\mathcal{O}(1) + s^5\mathcal{O}((sh)^2)$ and $D_h(q_{44}) = s^5\mathcal{O}((sh)^2)$, which completes the proof. □

4.3.16. Proof of Lemma 4.2

We recall that P_h is defined by

$$P_h v := -\partial_t v + 4\rho A_h D_h^3(r)\, D_h A_h^3 v + 4\rho A_h^3 D_h(r)\, D_h^3 A_h v + 2A_h D_h(\rho A_h D_h^3(r))v. \tag{4.54}$$

We note that $\lambda \geq \lambda_1(T^2 + T)$ implies $s(t) \geq \lambda_1 > 0$ for any t. Hence, multiplying (4.54) by $s^{-1}(t)$, and then using the Cauchy-Schwartz and triangular inequality we obtain

$$\begin{aligned}
\left\| s^{-\frac{1}{2}} \partial_t v \right\|^2_{L^2_h(Q)} \leq & C\left(\left\| s^{-\frac{1}{2}} P_h v \right\|^2_{L^2_h(Q)} + \left\| s^{\frac{5}{2}} D_h A_h^3 v \right\|^2_{L^2_h(Q)} + \left\| s^{\frac{1}{2}} D_h^3 A_h v \right\|^2_{L^2_h(Q)} + \left\| s^{\frac{5}{2}} v \right\|^2_{L^2_h(Q)} \right) \\
\leq & C\left(\| P_h v\|^2_{L^2_h(Q)} + \left\| s^{\frac{5}{2}} D_h A_h^3 v \right\|^2_{L^2_h(Q)} + \left\| s^{\frac{1}{2}} D_h^3 A_h v \right\|^2_{L^2_h(Q)} + \left\| s^{\frac{7}{2}} v \right\|^2_{L^2_h(Q)} \right),
\end{aligned} \tag{4.55}$$

since from Theorem 2.2 we have

$$\begin{aligned}
|4\rho A_h D_h^3(r)| \leq & Cs^3, \\
|4\rho A_h^3 D_h(r)| \leq & Cs, \\
|2A_h D_h(\rho A_h D_h^3(r))| \leq & Cs^3.
\end{aligned}$$

Moreover, from (2.5) it follows that $D_h A_h^3 v = D_h A_h v + \dfrac{h^2}{4} D_h^3 A_h v$. Thus, by using Young's inequality and (2.6) we obtain

$$\begin{aligned}
\int_Q s^5 |D_h A_h^3 v|^2 \leq & C\left(\int_Q s^5 |D_h A_h v|^2 + \int_Q s\mathcal{O}_{\mathfrak{K}}((sh)^4) |D_h^3 A_h v|^2 \right) \\
\leq & C\left(\int_Q s^5 A_h(|D_h v|^2) + \int_Q s\mathcal{O}_{\mathfrak{K}}((sh)^4) A_h(|D_h^3 v|^2) \right) \\
\leq & C\left(\int_{Q^*} s^5 |D_h v|^2 + \int_{Q^*} s\mathcal{O}_{\mathfrak{K}}((sh)^4) |D_h^3 v|^2 \right).
\end{aligned} \tag{4.56}$$

Similarly, by using (2.6), we get

$$\int_Q s|D_h^3 A_h v|^2 \leq \int_Q s A_h \left(|D_h^3 v|^2\right) \leq \int_{Q^*} s|D_h^3 v|^2. \tag{4.57}$$

Therefore, combining (4.55) with (4.56) and (4.57) proves the Lemma.

Finally, we give the proof of the Lemmas that enables us to write the Carleman estimate in the original variable.

4.3.17. Proof of Lemma 4.3

We begin proving the inequality related with the operator D_h^3. Using Lemma 2.1 we have

$$D_h^3 w = D_h^3 r\, A_h^3 v + 3D_h^2 A_h r\, A_h^2 D_h v + 3D_h A_h^2 r\, A_h D_h^2 v + A_h^3 r\, D_h^3 v.$$

Then, by virtue of Theorem 2.2 it follows that

$$\int_{Q^*} s|\rho D_h^3 w|^2 \leq C\left(\int_{Q^*} s^7 |A_h^3 v|^2 + \int_{Q^*} s^5 |A_h^2 D_h v|^2 + \int_{Q^*} s^3 |A_h D_h^2 v|^2 + \int_{Q^*} s|D_h^3 v|^2\right).$$

Now, thanks to Lemma 2.1 we have the identities $A_h^3 v = A_h v + \frac{h^2}{4} A_h D_h^2 v$ and $A_h^2 D_h v = D_h v + \frac{h^2}{4} D_h^3$. This enables us to rewrite the first two integrals above to obtain

$$\begin{aligned}\int_{Q^*} s|\rho D_h^3 w|^2 \leq C \Bigg(& \int_{Q^*} s^7 |A_h v|^2 + \int_{Q^*} s^3 \mathcal{O}((sh)^2) |A_h D_h^2 v|^2 + \int_{Q^*} s^5 |D_h v|^2 \\ &+ \int_{Q^*} s\mathcal{O}((sh)^4)|D_h^3 v|^2 + \int_{Q^*} s^3 |A_h D_h^2 v|^2 + \int_{Q^*} s|D_h^3 v|^2\Bigg).\end{aligned}$$

Finally, applying (2.6) to those integrals that involve the average operator and then performing a discrete integral by parts respect to it we get

$$\int_{Q^*} s|\rho D_h^3 w| \leq C\left(\int_Q s^7 |v|^2 + \int_{Q^*} s^5 |D_h v|^2 + \int_{\overline{Q}} s^2 |D_h^2 v|^2 + \int_{Q^*} s|D_h^3 v|^2\right),$$

which concludes the proof of the inequality for the operator D_h^3. The proof of the other inequalities follow the same methodology.

4.3.18. Proof of Lemma 4.4

Let us prove the second inequality of the Lemma. Using Lemma 2.1 we note that

$$D_h^2 w := D_h^2(rv) = D_h^2 r\, A_h^2 v + 2D_h A_h r\, D_h A_h v + A_h^2 r\, D_h^2 v.$$

Thus we can write

$$\rho\, A_h^2 r\, D_h^2 v = \rho\, D_h^2 w - \rho\, D_h^2 r\, A_h^2 v - 2\rho, D_h A_h r\, D_h A_h v.$$

Moreover, thanks to Theorem 2.2 we have

$$|D_h^2 v|^2 \le C\left(|\rho\, D_h^2 w|^2 + s^4 |A_h^2 v|^2 + s^2 |D_h A_h v|^2\right).$$

Now, using $A_h^2 v = v + \frac{h^2}{4} D_h^2 v$ and Young's inequality it follows that

$$|D_h^2 v|^2 \le C\left(|\rho\, D_h^2 w|^2 + s^4 |v|^2 + \mathcal{O}((sh)^4)|D_h^2 v|^2 + s^2 |D_h A_h v|^2\right).$$

Hence, noting that $D_h A_h v(0,t) = \frac{h}{2} D_h^2 v(0,t)$ we obtain

$$\begin{aligned}
\int_0^T s^3 |D_h^2 v|^2\Big|_0 \le & C\left(\int_0^T s^3 |\rho\, D_h^2 w|^2\Big|_0 + \int_0^T s^7 |v|^2\Big|_0 + \int_0^T s^3 \mathcal{O}((sh)^4)|D_h^2 v|^2\Big|_0\right. \\
& \left. + \int_0^T s^3 |D_h v|^2\Big|_{h/2} + \int_0^T s^3 \mathcal{O}((sh)^2)|D_h^2 v|^2\Big|_0\right) \\
= & C\left(\int_0^T s^3 |\rho\, D_h^2 w|^2\Big|_0 + \int_0^T s^3 \mathcal{O}((sh)^4)|D_h^2 v|^2\Big|_0\right. \\
& \left. + \int_0^T s^3 \mathcal{O}((sh)^2)|D_h^2 v|^2\Big|_0\right),
\end{aligned}$$

since $v(0,t) = D_h v(h/2,t) = 0$. Therefore

$$\int_0^T s^3 |D_h^2 v|^2\Big|_0 \le C\left(\int_0^T s^3 |\rho\, D_h^2 w|^2\Big|_0\right), \tag{4.58}$$

which is the desired inequality.

We now turn to the proof of the first inequality of our Lemma. Since $D_h v(-h/2,t) = 0$ we write $D_h v(h/2,t) = D_h v(h/2) - D_h v(-h/2) = h D_h^2 v(0,t)$. Then

$$\int_0^T s^5 |D_h v|^2\Big|_{h/2} = \int_0^T s^3 \mathcal{O}((sh)^2)|D_h^2 v|^2\Big|_0. \tag{4.59}$$

Thus, combining (4.58) with (4.59) yields

$$\int_0^T s^5 |D_h v|^2\Big|_{h/2} \le C\left(\int_0^T s^3 \mathcal{O}((sh)^2)|\rho\, D_h^2 w|^2\Big|_0\right). \tag{4.60}$$

Finally, for the third inequality, we note that from (4.3.17) we have

$$\rho A_h^3 r\, D_h^3 v = \rho D_h^3 w - \rho D_h^3 r\, A_h^3 v - 3\rho D_h^2 A_h r\, A_h^2 D_h v - 3\rho D_h A_h^2 r\, A_h D_h^2 v.$$

Futhermore, thanks to Theorem 2.2 and the above expression it follows that

$$|D_h^3 v|^2 \le C|\rho D_h^3 w|^2 + s^6 |A_h^3 v|^2 + s^4 |A_h^2 D_h v|^2 + s^2 |A_h D_h^2 v|^2.$$

We note that $A_h^3 v = A_h v + \frac{h^2}{4} D_h^2 A_h v$ and $A_h^2 D_h v = D_h v + \frac{h^2}{4} D_h^3 v$, due to Lemma 2.1, then

$$\begin{aligned}|D_h^3 v|^2 &\leq C \Big(|\rho D_h^3 w|^2 + s^6 |A_h v|^2 + s^2 \mathcal{O}((sh)^2) |D_h^2 A_h v|^2 \\ &\qquad + s^4 |D_h v|^2 + \mathcal{O}((sh)^4) |D_h^3 v|^2 + s^2 |A_h D_h^2 v|^2 \Big) \\ &\leq C \left(|\rho D_h^3 w|^2 + s^6 |A_h v|^2 + s^4 |D_h v|^2 + s^2 |A_h D_h^2 v|^2 \right).\end{aligned}$$

Thus, using the identity $D_h^2 A_h v(h/2, t) = D_h^2 v(0, t) + \frac{h}{2} D_h^3 v(h/2, t)$ we obtain

$$\begin{aligned}\int_0^T s|D_h^3 v|^2\Big|_{h/2} &\leq C \left(\int_0^T s|\rho D_h^3 w|^2\Big|_{h/2} + \int_0^T s^7 |A_h v|^2\Big|_{h/2} + \int_0^T s^5 |D_h v|^2\Big|_{h/2} \right. \\ &\qquad \left. + \int_0^T s^3 |D_h^2 v|^2\Big|_0 + \int_0^T s\mathcal{O}((sh)^2) |D_h^3 v|^2\Big|_{h/2} \right) \\ &= C \left(\int_0^T s|\rho D_h^3 w|^2\Big|_{h/2} + \int_0^T s^5 \mathcal{O}((sh)^2) |D_h v|^2\Big|_{h/2} + \int_0^T s^5 |D_h v|^2\Big|_{h/2} \right. \\ &\qquad \left. + \int_0^T s^3 |D_h^2 v|^2\Big|_0 + \int_0^T s\mathcal{O}((sh)^2) |D_h^3 v|^2\Big|_{h/2} \right),\end{aligned} \tag{4.61}$$

since $A_h v(h/2, t) = \frac{h}{2} D_h v(h/2, t)$. Therefore, combining (4.61) with (4.59) and (4.58) we conclude

$$\int_0^T s|D_h^3 v|^2\Big|_{h/2} \leq C \left(\int_0^T s^3 |\rho\, D_h^2 w|^2\Big|_0 + \int_0^T s|\rho D_h^3 w|^2\Big|_{h/2} \right),$$

and the proof is complete.

Chapter 5

Discrete Carleman estimates for inverse problem with partial data

In this chapter we extend for higher dimension the concepts of meshes and discrete operator, and integration-by parts formulae. Then, we prove a Carleman estimate for Laplacian operator with boundary observation. This result allow us to state a stability estimate for the Calderón problem with partial data, by using limiting Carleman weight functions.

In 1980 Calderón [11] asked if it was possible to determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This inverse method is nowadays known as the Calderón problem. A great survey of the history of this problem and its developments can be found in [48]. The precise question can be described as follows: Let $\Omega \subset \mathbb{R}^d$, be a regular domain. Given σ a conductivity and q a potential it is possible to define the Dirichlet to Neumann (DtN) map $\Lambda[\sigma, q] : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ in the following way. For a prescribed voltage g on $\partial\Omega$,

$$\Lambda[\sigma, q] : g \mapsto \Lambda[\sigma, q](g) = \sigma \nabla u[\sigma, q] \cdot \nu|_{\partial\Omega}$$

where ν denotes the outer normal to $\partial\Omega$ and $u[\sigma, q]$ solves the problem

$$\operatorname{div}(\sigma \nabla u) + qu = 0 \text{ in } \Omega \text{ and } u = g \text{ on } \partial\Omega. \tag{5.1}$$

It is well known that the Liouville transform allows to rewrite (5.1) into

$$-\Delta v + q'v = 0 \text{ in } \Omega \text{ and } v = \sigma^{1/2} g \text{ on } \partial\Omega. \tag{5.2}$$

so Calderón problem can be reduced to the case $\sigma = 1$. The first question to be answered is the uniqueness of the DtN map, that is, the inyectivity of $\Lambda[q]$. The second question is the stability of the inverse, that is, trying to understand the modulus of continuity of Λ^{-1}. There is an extensive literature regarding inverse problems for different equations in the continuous case. For a complete survey see [48] and the references therein. However, up to our knowledge, there are few results in the discrete situation. The paper [20] treats the stability of the discrete Calderón problem. The aim of this Chapter is to extend the discrete Carleman estimates of Ervedoza and De Gournay [20] to the case of non-vanishing boundary terms. We give a stability result for the Calderón problem for a class of potentials in a cube in $\mathbb{R}^3$ when measurements are taken in all the faces of the cube but one. We show that in fact all the coefficients of the Fourier series expansion of the differences of the potentials are small excepts in the normal direction to the face where no measurements are made.

5.1. Some preliminaries

In this Section, we first introduce the notation of meshes and operators that will be used throughout this paper. Then, we establish discrete calculus formulas, product rule and integration by parts, for the discrete operators.

Let $d \geq 2$, and let us consider $N \in \mathbb{N}$, and $h := \frac{1}{N+1}$ small enough, which represent the size of our mesh. We define the Cartesian grid of $[0,1]^d$ as:

$$\mathcal{K}_h := \left\{ x \in [0,1]^d \mid \exists k \in \mathbb{Z}^d \text{ such that } x = hk \right\}.$$

We set $\Omega := (0,1)^d \cap \mathcal{K}_h$. For any set of points $\mathcal{W} \subseteq \Omega$, we define the meshes in the direction e_k, with $\{e_k\}_{k=1}^d$ the usual base of $\mathbb{R}^d$, as

$$\mathcal{W}_k^* := \tau_k(\mathcal{W}) \cup \tau_{-k}(\mathcal{W}) \text{ and } \mathcal{W}_k' := \tau_k(\mathcal{W}) \cap \tau_{-k}(\mathcal{W}),$$

where $\tau_{\pm k}(\mathcal{W}) := \left\{ x \pm \frac{h}{2} e_k \mid x \in \mathcal{W} \right\}$. Similarly, we define $\overline{\mathcal{W}}_{kj} := (\mathcal{W}_k^*)_j^*$ and $\mathring{\mathcal{W}}_{kj} := (\mathcal{W}_k')_j'$. We will write briefly $\overline{\mathcal{W}}_k$(resp. $\mathring{\mathcal{W}}_k$) when $k = j$. This enables us to consider the boundary points, in the e_k direction, as $\partial_k \mathcal{W} := \overline{\mathcal{W}}_k \setminus \mathcal{W}$. Moreover, we define the interior and the boundary of a set $\mathcal{W}$ as follows

$$\mathring{\mathcal{W}} := \bigcap_{k=1}^{d} \mathring{\mathcal{W}}_k \text{ and } \partial \mathcal{W} := \bigcup_{k=1}^{d} \overline{\mathcal{W}}_k \setminus \mathcal{W}.$$

In figure 5.1 we consider the set $\mathcal{W}$, in dimension two, and we indicate the sets $\mathcal{W}_1^*$ and $\overline{\mathcal{W}}$. We note that $\mathring{\overline{\mathcal{W}}} = \mathcal{W}$.

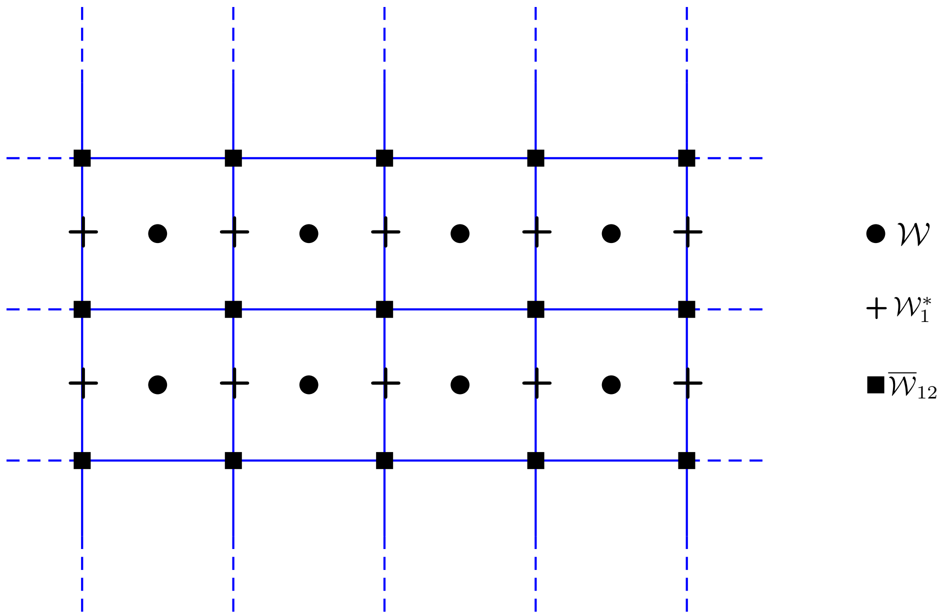


Figure 5.1: Representation of the primal and dual meshes in dimension two.

We denote by $C(\mathcal{W})$ the set of functions defined on a regular mesh $\mathcal{W}$ to $\mathbb{C}$. To obtain the discrete version of our system, using the finite difference method, we introduce the difference operator in the direction e_k as

$$(D_k u)(x) := \frac{\tau_k u(x) - \tau_{-k} u(x)}{h}.$$

We also define the average operator in the e_k direction as

$$(A_k u)(x) := \frac{\tau_k u(x) + \tau_{-k} u(x)}{2}.$$

To establish a discrete integration by parts formulas, for the difference and average operators, we define the outward normal of the set $\mathcal{W}$ in the direction e_k as $n_k \in C(\partial_k \mathcal{W})$ by

$$\forall x \in \partial_k \mathcal{W}, n_k(x) = \begin{cases} 1 & \text{if } \tau_{-k}(x) \in \mathcal{W}_k^* \text{ and } \tau_k(x) \notin \mathcal{W}_k^*, \\ -1 & \text{if } \tau_{-k}(x) \notin \mathcal{W}_k^* \text{ and } \tau_k(x) \in \mathcal{W}_k^*, \\ 0 & \text{otherwise.} \end{cases}$$

Additionally, to introduce the boundary condition we also define the trace operator in the direction k, denoted by t_r^k, for $u \in C(\mathcal{W}_k^*)$ as

$$\forall x \in \partial_i \mathcal{W},\ t_r^k(u) := \begin{cases} \tau_{-k} u(x) & n_k(x) = 1, \\ \tau_k u(x) & n_k(x) = -1, \\ \ 0 & n_k(x) = 0. \end{cases}$$

Now, the following Propositions gives us the product rule for the average and the difference operators

Proposition 5.1 *For $u, v \in C(\Omega)$, we have the following identities on Ω_k^*:*

$$D_k(u\,v) = D_k u\, A_k v + A_k u\, D_k v, \tag{5.3}$$

$$A_k(u\,v) = A_k u\, A_k v + \frac{h^2}{4} D_k u\, D_k v. \tag{5.4}$$

PROOF. By definition, we have

$$D_k(u\,v) = \frac{1}{h}\left(\tau_k(u\,v) - \tau_{-k}(u\,v)\right)$$

Then, adding the term $-\tau_k u\,\tau_{-k} v + \tau_k u\,\tau_{-k} v$ it follows that

$$D_k(u\,v) = \tau_k u\, D_k v + \tau_{-k} v\, D_k u.$$

Analogously, we also have $D_k(u\,v) = D_k u\,\tau_k v + \tau_{-k} u\, D_k v$. Averaging the last two equations we obtain the first identity.

We proceed similarly for the second identity. Considering the expression $\tau_k(uv) + \tau_{-k}(uv)$ we write

$$\begin{aligned} 4A_k u\, A_k v =& 2\tau_k(u\,v) + 2\tau_{-k}(u\,v) + \tau_k u\,(\tau_{-k} v - \tau_k v) + \tau_{-k} u\,(\tau_k v - \tau_{-k} v) \\ =& 4A_k(u\,v) - h^2 D_k u\, D_k v. \end{aligned}$$

□

To state the discrete integral by parts formulas, for a regular set $\mathcal{W} \subseteq \Omega$, we define the discrete integral for $u \in C(\mathcal{W})$ as

$$\int_{\mathcal{W}} u := h^d \sum_{x \in \mathcal{W}} u(x),$$

and the following L_h^2 inner product in $C(\mathcal{W})$

$$\langle u, v\rangle_{\mathcal{W}} := \int_{\mathcal{W}} u\,v, \quad u\,, v \in C(\mathcal{W}),$$

with the associated norm

$$\|u_h\|_{L_h^2(\mathcal{W})} := \sqrt{\langle u, u\rangle_{\mathcal{W}}}.$$

We also consider the following semi-norm for the discrete derivatives

$$\|u\|_{\mathring{H}_h^1(\mathcal{W})} = \left(\sum_{k=1}^{d} \int_{\mathcal{W}_k^*} |D_k u|^2 \right)^{1/2}.$$

For $u \in C(\mathcal{W})$, we define its $L_h^\infty(\mathcal{W})$ norm as

$$\|u\|_{L_h^\infty(\mathcal{W})} := \max_{x\in\mathcal{W}} \{|u(x)|\}.$$

Let us finally introduce the discrete integration on the boundary for $u \in C(\partial_k \mathcal{W})$ as

$$\int_{\partial_k \mathcal{W}} u := h^{d-1} \sum_{x\in\partial_k\mathcal{W}} u(x).$$

Thus, following the notation previously introduced we establish a discrete integral by parts formula for the discrete average and difference operator.

Proposition 5.2 *For any $v \in C(\mathcal{W}_k^*)$, $u \in C(\overline{\mathcal{W}}_k)$, we have*

$$\int_{\mathcal{W}} u\, D_k v = -\int_{\mathcal{W}_k^*} D_k u\, v + \int_{\partial_k\mathcal{W}} u\, t_r^k(v) n_k, \tag{5.5}$$

$$\int_{\mathcal{W}} u\, A_k v = \int_{\mathcal{W}_k^*} A_k u\, v - \frac{h}{2}\int_{\partial_k\mathcal{W}} u\, t_r^k(v). \tag{5.6}$$

Proof. We note that for the operator $\tau_\pm$ we write

$$\int_{\mathcal{W}} u\, \tau_k(v) = \int_{\mathcal{W}_h^*} \tau_{-k}(u)\, v - \int_{\tau_{-k}^2(\mathcal{W})\setminus\mathcal{W}} u\, \tau_k(v). \tag{5.7}$$

and

$$\int_{\mathcal{W}} u\, \tau_{-k}(v) = \int_{\mathcal{W}_k^*} \tau_k(u)\, v - \int_{\tau_k^2(\mathcal{W})\setminus\mathcal{W}} u\tau_{-k}(v). \tag{5.8}$$

Combining (5.7) with (5.8) we get (5.5). Similarly, averaging (5.7) and (5.8) we obtain (5.6), which concludes the proof. ☐

5.2. Carleman estimate with boundary observations

This section is devoted to prove the Carleman estimate (5.27). Let us mention that there are similar estimates like [8] or [20], but they do not fulfill our propose for the inverse problem. For instance, the weight function considered in [8] is not used to prove the existence of the CGO solution in the continuous setting. Moreover, the estimate proved in [20] does not consider boundary observation due to they perform the Carleman estimate for function supported in meshes that are at least two nodes from the boundary. Thus, our hypothesis on the meshes where the function u is defined is weaker than [20, Theorem 3.1]. For this reason, we obtain boundary observation in the estimate (number). The proof is similar in spirit to [20] although from our assumption in the proof of Theorem 5.2 there are some extra terms, and as we will see later, we can handle with it.

Let us begin considering the mesh regularity. We will consider the function σ^k beloging to $C(\overline{\Omega}_k^*)$. It represent

Definition 5.1 *We define*

$$\epsilon_d(h) := \sum_{k,j} \|D_j(\sigma^k)\|_{L_h^\infty(\overline{\Omega}_k)}, \quad \epsilon_a(h) := \sum_k \|A_k(\sigma^k) - 1\|_{L_h^\infty(\overline{\Omega}_k)},$$

and $M(h) := \sum_k \|\sigma^k\|_{L_h^\infty(\overline{\Omega}_k^*)}$.

In the rest of this Section we assume.
Assumption: We suppose that

$$M := \sup_{h\to 0} M(h) < \infty, \quad \epsilon_a, \epsilon_d < 1.$$

We observe that if $\epsilon_a < 1$, there exist $\underline{\sigma}$ and $\overline{\sigma}$, such that

$$0 < \underline{\sigma} \leq A_k(\sigma^k) \leq \overline{\sigma} \text{ in } \overline{\Omega}_k.$$

5.2.1. Carleman estimate

As usual for this kind of estimates, we consider the conjugate operator defined by $\Delta_{s,h}u :=$ For any $s \in \mathbb{R}^d$, we introduce the linear weight function $\phi_s(x) = s \cdot x$ and we define the operator $\Delta_{s,h}$ from $C(\overline{\Omega})$ to $C(\Omega)$ as

$$\Delta_{s,h}u := \sum_{k=1}^{d} e^{-\phi_s} D_k(\sigma^k D_k(e^{\phi_s}u)) \text{ in } \Omega.$$

Repeated application of (5.3) and (5.4) enables us to write

Lemma 5.1 *For any u in $C(\overline{\Omega})$, we have*

$$\Delta_{s,h}u = P_{s,h}^a u + P_{s,h}^b u \text{ in } \Omega,$$

where the operator $P_{s,h}^a$ and $P_{s,h}^b$ are defined by

$$P_{s,h}^a u = \sum_{k=1}^{d} A_k(\sigma^k)\left(\alpha_1^k D_k^2 u + \alpha_2^k u + 2\alpha_3^k A_k D_k u\right) \text{ in } \Omega,$$

and

$$P_{s,h}^b u = \sum_{k=1}^{d} D_k(\sigma^k)\left(\frac{h^2}{2}\alpha_3^k D_k^2 u + \alpha_3^k u + \alpha_1^k A_k D_k u\right) \text{ in } \Omega,$$

where α_1^k, α_2^k *and* α_3^k *are given by*

$$\alpha_1^k \;:=\; e^{-\phi_s} A_k^2 e^{\phi_s} + \frac{h^2}{4} e^{-\phi_s} D_k^2 e^{\phi_s} = \cosh\left(h\, s \cdot e_k\right),$$

$$\alpha_2^k \;:=\; e^{-\phi_s} D_k^2 e^{\phi_s} = \frac{4}{h^2} \sinh^2\left(\frac{h}{2} s \cdot e_k\right),$$

$$\alpha_3^k \;:=\; e^{-\phi_s} D_k A_k e^{\phi_s} = \frac{1}{h} \sinh\left(h\, s \cdot e_k\right).$$

Proof. Let us denote $F := e^{\phi_s} D_k(\sigma^k D_k(e^{\phi_s} u)) := e^{\phi_s} D_k(\sigma^k G)$. Using (5.3) and (5.4) on G we have

$$G = A_k(e^{\phi_s}) D_k u + D_k e^{\phi_s} A_k u.$$

Then, by virtue of Proposition 5.1 we obtain

$$D_k(G) = 2 D_k A_k e^{\phi_s} D_k A_k u + A_k^2 e^{\phi_s} D_k^2 u + D_k^2 e^{\phi_s} A_h^2 u$$
$$A_k(G) = \left(A_k^2 e^{\phi_s} + \frac{h^2}{4} D_k^2 e^{\phi_s}\right) D_k A_k u + \frac{h^2}{4} D_k A_k e^{\phi_s} D_k^2 u + A_k D_k e^{\phi_s} A_k^2 u$$

This, and (5.3), enables us to write

$$F = A_k(\sigma^k)\left(\alpha_1^k D_k^2 u + \alpha_2^k u + 2\alpha_3^k A_k D_k u\right) + D_k(\sigma^k)\left(\frac{h^2}{2}\alpha_3^k D_k^2 u + \alpha_3^k u + \alpha_1^k A_k D_k u\right).$$

It remains to prove that $\alpha_1^k = \cosh(h\, s \cdot e_k)$, $\alpha_2^k = \frac{4}{h^2} \sinh\left(\frac{h}{2} s \cdot e_k\right)$ and $\alpha_3^k = \frac{1}{h} \sinh h\, s \cdot e_k$, which follows from [20, Lemma 2.2]. □

Our task now is to estimate the operators $P_{s,h}^a$ and $P_{s,h}^d$ in order to estimate $\Delta_{s,h}$. Let us first focus on $P_{s,h}^a$. We can split $P_{s,h}^a$ into $P_{s,h}^a := A_{s,h} u + S_{s,h} u$, where

$$A_{s,h} u := \sum_{k=1}^d 2 A_k(\sigma^k) \alpha_3^k A_k D_k u \quad \text{and} \quad S_{s,h} u := \sum_{k=1}^d A_k(\sigma^k)\left(\alpha_1^k D_k^2 u + \alpha_2^k u\right), \text{ in } \Omega.$$

Thus, the inner product $\langle S_{s,h}, A_{s,h}\rangle$ can be bounded as

Proposition 5.3 *If* $h|s| \le 1$*, then there exists* $C > 0$*, such that for any* $u \in C_c(\overline{\Omega})$*, we have*

$$2\int_\Omega S_{s,h} u\, A_{s,h} u \ge -C\epsilon_d |s| \left(|s|^2 \|u\|^2_{L_h^2(\Omega)} + \|u\|^2_{\mathring{H}_h^1(\Omega)}\right) + \sum_{k=1}^d \int_{\partial_k \Omega} \frac{1}{h} \sinh(2hs \cdot e_k)\, |A_k(\sigma^k)|^2 t_r^k(|D_k u|^2)\, n_k$$

Proof. Setting $\beta_{jk} := 2A_j(\sigma^j) A_k(\sigma^k) \alpha_2^j \alpha_3^k$ and $\gamma_{jk} := 2A_j(\sigma^j) A_k(\sigma^k) \alpha_1^j \alpha_3^k$, we write

$$\langle A_{s,h} u, S_{s,h} u\rangle = \sum_{j,k=1}^d \int_\Omega \beta_{jk}\, u\, A_k D_k u + \gamma_{jk} A_k D_k u D_j^2 u := \sum_{j,k=1}^d I_{jk} + J_{jk}. \tag{5.9}$$

We split the proof into two steps. Firstly, let us estimate I_{jk}. Applying a discrete integration by parts with respect to the difference operator D_k we have

$$I_{jk} = -\int_{\Omega_k^*} D_k(\beta_{jk}\, u) A_k u,$$

where we have used that $u = 0$ on $\partial_k \Omega$. Now, using (5.3), I_{jk} can be rewritten as

$$I_{jk} = -\frac{1}{2}\int_{\Omega_k^*} A_k(\beta_{jk}) D_k(u^2) - \int_{\Omega_k^*} D_k(\beta_{jk}) |A_k u|^2.$$

By virtue of (5.5), and the fact that $u = 0$ on $\partial_k \Omega$, it follows that

$$I_{jk} = \frac{1}{2}\int_{\Omega} D_k A_k(\beta_{jk})\, |u|^2 - \int_{\Omega_k^*} D_k(\beta_{jk}) |A_k u|^2.$$

Therefore

$$|I_{jk}| \leq C\epsilon_d |s|^3 \|u\|^2_{L^2(\Omega)}, \tag{5.10}$$

since $|D_k A_k(\beta_{jk})| \leq C\epsilon_d |s|^3$ and $|D_k(\beta_{jk})| \leq C\epsilon_d |s|^3$.

Secondly, let us focus on J_{jk}. We identify two cases.

Case $k = j$ Thanks to (5.3) J_{kk} can be rewritten as

$$J_{kk} = \frac{1}{2}\int_{\Omega} \gamma_{kk}\, D_k(|D_k u|^2).$$

A discrete integration by parts yields

$$J_{kk} = -\frac{1}{2}\int_{\Omega_k^*} D_k(\gamma_{kk})\, |D_k u|^2 + \frac{1}{2}\int_{\partial_k \Omega} \gamma_{kk} t_r^k(|D_k u|^2)\, n_k.$$

Then, noting that $|D_k(\gamma_{kk})| \leq C\epsilon_d$ and $\gamma_{kk} := \frac{1}{h}\sinh(2hs \cdot e_k)|A_k(\sigma^k)|^2$, we obtain

$$J_{kk} \geq -C\epsilon_d |s| \|u\|^2_{\mathring{H}_h^1(\Omega)} + \frac{1}{2h}\int_{\partial_k \Omega} \sinh(2hs \cdot e_k)|A_k(\sigma^k)|^2 t_r^k(|D_k u|^2)\, n_k. \tag{5.11}$$

Case $k \neq j$ A discrete integration by parts with respect to the difference operator D_j gives

$$J_{jk} = -\int_{\Omega_j^*} D_j(\gamma_{jk} A_k D_k u)\, D_j u,$$

since $A_k D_k u = 0$ on $\partial_j \Omega$ when $k \neq j$. Moreover, J_{jk} can be rewritten as

$$J_{jk} = -\int_{\Omega_j^*} A_j \gamma_{j,k}\, D_j u\, A_k D_{jk}^2 u - \int_{\Omega_j^*} D_j \gamma_{jk}\, D_j u A_{jk}^2 D_k u := J_{jk}^{(1)} + J_{jk}^{(2)},$$

due to (5.3). Integrating by parts on $J_{jk}^{(1)}$ gives

$$J_{jk}^{(1)} = \frac{1}{2}\int_{\overline{\Omega}_{jk}} A_{jk}^2\gamma_{jk} D_k(|D_j u|^2) + \int_{\overline{\Omega}_{jk}} D_k A_j \gamma_{jk}\, |A_k D_j u|^2.$$

where we have used (5.3) and $t_r^k(D_j u) = 0$ on $\partial_k\Omega$ for $k \neq j$. Repeating this argument on the first integral above we get

$$J_{jk}^{(1)} = -\frac{1}{2}\int_{\Omega_{j^*}} D_k A_{jk}^2\gamma_{jk}|D_j u|^2 + \int_{\overline{\Omega}_{jk}} D_k A_j \gamma_{jk}\, |A_k D_j u|^2.$$

Then, using that $|A_k D_j u|^2 \leq A_k(|D_j u|^2)$, a discrete integration by parts, and that $D_j u = 0$ on $\partial_k\Omega$ we obtain

$$|J_{jk}^{(1)}| \leq C\epsilon_d|s| \int_{\Omega_j^*} |D_j u|^2, \tag{5.12}$$

where we have used that $|D_k A_{jk}^2\gamma_{jk}| \leq \epsilon|s|$. Similarly, using that $|D_j\gamma_{jk}| \leq C\epsilon_d|s|$ and the Young's inequality, for $J_{jk}^{(2)}$ we have

$$|J_{jk}^{(2)}| \leq C\epsilon_d|s| \left(\int_{\Omega_j^*} |D_j u|^2 + |A_{jk}^2 D_k u|^2\right).$$

We note that $|A_{jk}^2 D_j u| \leq A_j(|A_k D_k u|^2)$. This, and a discrete integration by parts with respect to the average operator yield

$$|J_{jk}^{(2)}| \leq C\epsilon_d|s| \left(\int_{\Omega_j^*} |D_j u|^2 + \int_{\Omega} |A_k D_k u|^2\right),$$

since $A_k D_k u = 0$ on $\partial_j\Omega$. We repeated one more time the previous argument to deal with the second integral from the right-hand side above to write

$$|J_{jk}^{(2)}| \leq C\epsilon_d|s| \left(\int_{\Omega_j^*} |D_j u|^2 + \int_{\Omega_k^*} |D_k u|^2\right) \tag{5.13}$$

Collecting (5.11), (5.12) and (5.13) we get

$$J_{jk} \geq -C\epsilon_d|s|\, \|u\|_{\mathring{H}_h^1(\Omega)}^2 + \frac{1}{2h}\int_{\partial_k\Omega} \sinh(2hs\cdot e_k) t_r^k(|D_k u|^2) n_k \tag{5.14}$$

Therefore, combining (5.9) with (5.10) and (5.14) completes the proof.

□

At this stage, since $P^a_{s,h}u := A_{s,h}u + S_{s,h}u$, Proposition 5.3 yields

$$\|P^a_{s,h}u\|^2_{L^2_h(\Omega)} + C\epsilon_d|s|\left(|s|^2\|u\|^2_{L^2_h(\Omega)} + \|u\|^2_{\mathring{H}^1_h(\Omega)}\right) \geq \sum_{k=1}^{d}\int_{\partial_k\Omega}\frac{1}{h}\sinh(2hs\cdot e_k)\,|A_k(\sigma^k)|^2 t^k_r(|D_k u|^2)\,n_k + \|S_{s,h}u\|^2_{L^2_h(\Omega)} + \|A_{s,h}u\|^2_{L^2_h(\Omega)} . \tag{5.15}$$

Our next task is to estimate the norm of the operators $S_{s,h}$ and $A_{s,h}$. Let us begin with $A_{s,h}$.

Lemma 5.2 *If $h|s| \leq 1$ then there exist $\epsilon_d < 1$ and $C > 0$, such that for all $s \in \mathbb{R}^d$,*

$$|s|^2\|u\|^2_{L^2_h(\Omega)} \leq C\left(\|A_{s,h}u\|^2_{L^2_h(\Omega)} + h^2|s|^2\|u\|^2_{\mathring{H}^1_h(\Omega)}\right), \quad \forall u \in C_c(\Omega).$$

Proof. Multiplying $A_{s,h}u$ by $(s\cdot x)u$ and integrating over Ω we have

$$\int_\Omega (s\cdot x)u\,A_{s,h}u = \sum_{k=1}^{d} 2\alpha_3^k \int_\Omega (s\cdot x)u\,A_k(\sigma^k)A_k D_k u := \sum_{k=1}^{d} 2\alpha_3^k I_k. \tag{5.16}$$

A discrete integration by parts yields

$$I_k = -\int_{\Omega_k^*} D_k\left((s\cdot x)u\,A_k(\sigma^k)\right)A_k u,$$

since $u = 0$ on $\partial_k\Omega$. Thanks to (5.3), I_k can be rewritten as

$$I_k = -\int_{\Omega_k^*} D_k((s\cdot x)u)\,A_k^2(\sigma^k)A_k u - \int_{\Omega_k^*} A_k((s\cdot x)u)\,D_k A_k(\sigma^k)A_k u := -I_k^{(1)} - I_k^{(2)}.$$

Using (5.4) on $I_k^{(1)}$, and noting that $D_k(s\cdot x) = s\cdot e_k$ and $A_k(s\cdot x) = s\cdot x$, it follows that

$$I_k^{(1)} = \int_{\Omega_k^*} A_k^2(\sigma^k)\,(s\cdot e_k)|A_k u|^2 + \int_{\Omega_k^*} A_k^2(\sigma^k)(s\cdot x)D_k u A_k u.$$

Now, due to Proposition 5.1 we have

$$I_k^{(1)} = \int_{\Omega_k^*} A_k^2(\sigma^k)\,(s\cdot e_k)A_k(|u|^2) - \frac{h^2}{4}\int_{\Omega_k^*} A_k^2(\sigma^k)\,(s\cdot e_k)|D_k u|^2 + \frac{1}{2}\int_{\Omega_k^*} A_k^2(\sigma^k)(s\cdot x)D_k(|u|^2).$$

Integrating by parts with respect to the average operator on the first integral above and with respect to the difference on the third one we get

$$I_k^{(1)} = \frac{1}{2}\int_\Omega A_k^3(\sigma^k)(s\cdot e_k)|u|^2 - \frac{1}{2}\int_\Omega D_k A_k^2(\sigma^k)(s\cdot x)|u|^2 - \frac{h^2}{4}\int_{\Omega_k^*} A_k^2(\sigma^k)(s\cdot e_k)|D_k u|^2, \tag{5.17}$$

where we have used (5.3) and $u = 0$ on $\partial_k\Omega$.

On the other hand, using (5.4), for $I_k^{(2)}$ we have

$$I_k^{(2)} = \int_{\Omega_k^*} D_k A_k(\sigma^k)(s\cdot x)|A_k u|^2 + \frac{h^2}{4}\int_{\Omega_k^*} D_k A_k(\sigma^k)(s\cdot e_k) D_k u A_k u.$$

We note that the following identities $D_k(|u|^2) = 2D_k u A_k u$ and $|A_k u|^2 = A_k(|u|^2) - \frac{h^2}{4}|D_k u|^2$, due to Proposition 5.1, enables us to rewrite $I_k^{(2)}$ as

$$I_k^{(2)} = \int_{\Omega_k^*} D_k A_k(\sigma^k)(s\cdot x) A_k(|u|^2) - \frac{h^2}{4}\int_{\Omega_k^*} D_k A_k(\sigma^k)(s\cdot x)|D_k u|^2 + \frac{h^2}{8}\int_{\Omega_k^*} D_k A_k(\sigma^k)(s\cdot e_k) D_k(|u|^2).$$

Using a discrete integral by parts for the average operator on the first integral above and with respect to the difference operator for the third one, and the applying Proposition 5.1, we obtain

$$I_k^{(2)} = \int_{\Omega} D_k A_k^2(\sigma^k)(s\cdot x)|u|^2 - \frac{h^2}{4}\int_{\Omega_k^*} D_k A_k(\sigma^k)(s\cdot x)|D_k u|^2 + \frac{h^2}{8}\int_{\Omega} D_k^2 A_k(\sigma^k)(s\cdot e_k)|u|^2. \tag{5.18}$$

From (5.17) and (5.18) it follows that

$$\begin{aligned} I_k = &-\frac{1}{2}\int_\Omega \left(A_k^3(\sigma^k) + \frac{h^2}{4} D_k^2 A_k(\sigma^k)\right)(s\cdot e_k)|u|^2 - \frac{1}{2}\int_\Omega D_k A_k^2(\sigma^k)(s\cdot x)|u|^2 \\ &+ \frac{h^2}{4}\int_{\Omega_k^*}\left(D_k A_k(\sigma^k)(s\cdot x) + A_k^2(\sigma^k)(s\cdot e_k)\right)|D_k u|^2. \end{aligned} \tag{5.19}$$

Combining (5.16) with (5.19) enables us to write

$$\begin{aligned} \sum_{k=1}^d \int_\Omega \alpha_3^k \left(A_k^3(\sigma^k) + \frac{h^2}{4} D_k^2 A_k(\sigma^k)\right)(s\cdot e_k)|u|^2 = &\frac{h^2}{2}\sum_{k=1}^d \int_{\Omega_k^*} \alpha_3^k \left(D_k A_k(\sigma^k)(s\cdot x) + A_k^2(\sigma^k)(s\cdot e_k)\right)|D_k u|^2 \\ &- \int_\Omega (s\cdot x)\, u A_{s,h} u + \sum_{k=1}^d \int_\Omega \alpha_3^k D_k A_k^2(\sigma^k)(s\cdot x)|u|^2. \end{aligned} \tag{5.20}$$

Thanks to the following assumption on σ^k given by $0 < \underline{\sigma} \leq A_k(\sigma^k) \leq \overline{\sigma}$, the left-hand side of (5.20) can be bounded as

$$C\underline{\sigma}|s|^2\|u\|^2_{L^2_h(\Omega)} \leq \int_\Omega \sum_{k=1}^d \alpha_3^k (s\cdot e_k)\left(A_k^3(\sigma^k) + \frac{h^2}{4} D_k^2 A_k(\sigma^k)\right)|u|^2.$$

Using Young's inequality on the right-hand side of (5.20) gives

$$\begin{aligned} C\underline{\sigma}|s|^2\|u\|^2_{L^2_h(\Omega)} &\leq -\int_\Omega (s\cdot x) u\, A_{s,h} u + Ch^2|s|^2(1+\varepsilon_d)\|u\|^2_{\mathring{H}^1_h(\Omega)} + C|s|^2\varepsilon_d\|u\|^2_{L^2_h(\Omega)} \\ &\leq \varepsilon|s|^2\|u\|^2_{L^2_h(\Omega)} + \frac{C}{\varepsilon}\|A_{s,h}u\|^2_{L^2_h(\Omega)} + Ch^2|s|^2(1+\epsilon_d)\|u\|^2_{\mathring{H}^1_h(\Omega)} + C|s|^2\varepsilon_d\|u\|^2_{L^2_h(\Omega)}. \end{aligned}$$

Taking ε and ϵ_d small enough, there exists $C > 0$ such that from the above estimate we obtain

$$|s|^2\|u\|^2_{L^2_h(\Omega)} \leq C\|A_{s,h}u\|^2_{L^2_h(\Omega)} + Ch^2|s|^2\|u\|^2_{\mathring{H}^1_h(\Omega)},$$

which conclude the proof. □

Remark: This previous result is similar to [20, Lemma 3.5]. The main difference is that we drop the condition that $u = 0$ on the boundary of a mesh being at least two nodes away from the original mesh where u is defined. The price to pay is the additional term on the right-hand side of (5.16).

The following Proposition state an upper bound to the operator $P^b_{s,h}$.

Proposition 5.4 *If $h|s| \leq 1$, then there exists $C > 0$, such that*

$$\|P^b_{s,h}u\|^2_{L^2_h(\Omega)} \leq \epsilon_d^2 C\left(|s|^2\|u\|^2_{L^2_h(\Omega)} + \|u\|^2_{\mathring{H}^1_h(\Omega)}\right), \quad \forall u \in C_c(\overline{\Omega}).$$

Proof. Let us begin recalling the definition of $P_{s,h}u$.

$$P^b_{s,h}u := \sum_{k=1}^{d} D_k(\sigma^k)\left(\frac{h^2}{2}\alpha_3^k D_k^2 u + \alpha_3^k u + \alpha_1^k A_k D_k u\right).$$

Then, thanks to Young's inequality we have

$$\left\|P^b_{s,h}u\right\|^2_{L^2_h(\Omega)} \leq \epsilon_d^2 C\left(|s|^2\|u\|^2_{L^2_h(\Omega)} + \|u\|^2_{\mathring{H}^1_h(\Omega)}\right),$$

where we have used that $|h^2D_k^2u|^2 \leq C\left(|u(x+he_k)|^2 + |u(x)|^2 + |u(x-he_k|^2)\right)$, a discrete integration by parts with respect to the average operator, and that $u = 0$ on $\partial\Omega$. □

Lemma 5.3 *If $h|s| \leq 1$ then, there exists $C, s_0 > 0$ such that, for all $|s| > s_0$, we have*

$$\|u\|^2_{\mathring{H}^1_h(\Omega)} \leq C(\|S_{s,h}u\|^2_{L^2_h(\Omega)} + |s|^2\|u\|^2_{L^2_h(\Omega)}), \quad \forall u \in C_c(\Omega).$$

Proof. Multiplying $S_{s,h}u$ by u and integrating over Ω we have

$$\int_\Omega S_{s,h}u\,u = \sum_{k=1}^{d}\alpha_1^k\int_\Omega A_k(\sigma^k)u\,D_k^2u + \alpha_2^k\int_\Omega A_k(\sigma^k)|u|^2 := \sum_{k=1}^{d}\alpha_1^kJ_k^{(1)} + \alpha_2^kJ_k^{(2)}. \tag{5.21}$$

Let us focus on $J_k^{(1)}$. A discrete integration by parts with respect the difference operator D_k yields

$$\begin{aligned} J_k^{(1)} &= -\int_{\Omega_k^*} D_k(A_k(\sigma^k)u)D_ku + \int_{\partial_k\Omega} A_k(\sigma^k)u\,t_r^k(D_ku)n_k \\ &= -\int_{\Omega_k^*} D_kA_k(\sigma^k)A_ku\,D_ku - \int_{\Omega_k^*} A_k^2(\sigma^k)|D_ku|^2, \end{aligned}$$

where we have used that $u = 0$ on $\partial_k\Omega$ and (5.3). Now, we use the identity $D_k(|u|^2) = 2D_kA_ku$

and a discrete integration by parts on the first integrate above to obtain

$$J_k^{(1)} = \frac{1}{2}\int_\Omega D_k^2 A_k(\sigma^k)|u|^2 - \int_{\Omega_k^*} A_k^2(\sigma^k)|D_k u|^2. \tag{5.22}$$

Combining (5.21) with (5.22) we have

$$\int_\Omega S_{s,h}u\, u = \sum_{k=1}^d \alpha_1^k \frac{1}{2}\int_\Omega D_k^2 A_k(\sigma^k)|u|^2 - \alpha_1^k \int_{\Omega_k^*} A_k^2(\sigma^k)|D_k u|^2 + \alpha_2^k \int_\Omega A_k(\sigma^k)|u|^2. \tag{5.23}$$

Applying the Young's inequality and using the assumption on σ^k give

$$\underline{\sigma}\|u\|^2_{\mathring{H}^1_h(\Omega)} \leq \frac{1}{2}\|S_{s,h}u\|^2_{L^2_h(\Omega)} + \frac{1}{2}\|u\|^2_{L^2_h(\Omega)} + C|s|^2\|u\|^2_{L^2_h(\Omega)}, \tag{5.24}$$

which establishes the result. □

Theorem 5.1 *There exist constants $C_1, C_2, s_0 > 0$, and $c > 0$, such that, for any $h, \epsilon_d \in (0, c)$ $\forall |s| \in [s_0, c\min\{h^{-1}, \epsilon_d^{-1}\}]$, we have, $\forall u \in C_c(\Omega)$*

$$C_1|s|^2\|u\|^2_{L^2(\Omega)} + C_1\|u\|^2_{\mathring{H}^1(\Omega)} + \sum_{k=1}^d \int_{\partial_k\Omega} \frac{1}{h}\sinh(2hs\cdot e_k)\,|A_k(\sigma^k)|^2 t_r^k |D_k u|^2\, n_k \leq C_2\|\Delta_{s,h}u\|^2_{L^2(\Omega)}.$$

Proof. Combining Lemma 5.2 with Lemma 5.3, there exist $C, s_0 > 0$ and $\epsilon_d < 1$ such that for all $|s| > s_0$ we have

$$|s|^2\|u\|^2_{L^2_h(\Omega)} + \|u\|^2_{\mathring{H}^1_h(\Omega)} \leq C\left(\|S_{s,h}u\|^2_{L^2_h(\Omega)} + \|A_{s,h}u\|^2_{L^2_h(\Omega)} + h^2|s|^2\|u\|^2_{\mathring{H}^1_h(\Omega)}\right), \tag{5.25}$$

provided $h|s| \leq 1$. Then, recalling that $P^a_{s,h}u := S_{s,h}u + A_{s,h}u$, combining Proposition 5.3 and (5.25) we have

$$\begin{aligned}|s|^2\|u\|^2_{L^2(\Omega)} + \|u\|^2_{\mathring{H}^1(\Omega)} + C\sum_{k=1}^d \int_{\partial_k\Omega} \frac{1}{h}\sinh(2hs\cdot e_k)\,|A_k(\sigma^k)|^2 t_r^k|D_k u|^2\, n_k \\ \leq C\left(\|P^a_{s,h}u\|^2_{L^2} + C\epsilon_d|s|\left(|s|^2\|u\|^2_{L^2} + \|u\|^2_{\mathring{H}^1}\right) + h^2|s|^2\|u\|^2_{\mathring{H}^1(\Omega)}\right).\end{aligned}$$

Additionally, from Proposition 5.4 the right-hand side above can bound as follows

$$\begin{aligned}|s|^2\|u\|^2_{L^2(\Omega)} + \|u\|^2_{\mathring{H}^1(\Omega)} + C\sum_{k=1}^d \int_{\partial_k\Omega} \frac{1}{h}\sinh(2hs\cdot e_k)\,|A_k(\sigma^k)|^2 t_r^k|D_k u|^2\, n_k \\ \leq C\left(\|\Delta_{s,h}u\|^2_{L^2} + (\epsilon_d|s| + \epsilon_d^2)\left(|s|^2\|u\|^2_{L^2} + \|u\|^2_{\mathring{H}^1}\right) + h^2|s|^2\|u\|^2_{\mathring{H}^1(\Omega)}\right),\end{aligned} \tag{5.26}$$

since $\Delta_{s,h}u := P^a_{s,h}u + P^b_{s,h}u$. Now, if we take $\varepsilon > 0$, such that

$$|s|\max\{h, \epsilon_d\} \leq \varepsilon,$$

and $|s| > \epsilon_d$, from (5.26) we obtain

$$|s|^2\|u\|^2_{L^2_h(\Omega)} + \|u\|^2_{\mathring{H}^1_h(\Omega)} + C\sum_{k=1}^{d}\int_{\partial_k\Omega}\frac{1}{h}\sinh(2hs\cdot e_k)\,|A_k(\sigma^k)|^2 t_r^k(|D_k u|^2)\,n_k \\ \leq C\left(\|\Delta_{s,h}u\|^2_{L^2_h} + \varepsilon\left(|s|^2\|u\|^2_{L^2_h} + \|u\|^2_{\mathring{H}^1_h}\right) + \varepsilon^2\|u\|^2_{\mathring{H}^1_h(\Omega)}\right).$$

Taking ε small enough the last two term from the right-hand side above can be absorb from its left-hand side, which completes the proof. □

Using Cauchy-Schwartz and Young's inequality, and decreasing the parameters h and ϵ_d if necessary, as a consequence of the previous result we state.

Lemma 5.4 *Given $q \in C(\mathring{\Omega})$, such that $\|q\|_{L^\infty_h(\Omega)} \leq m$. There exist constants $C_1, C_2, c > 0$, and $s_0 > 0$, such that, $\forall |s| \in [s_0, c\min\{h^{-1}, \epsilon_d^{-1}\}]$, we have that, $\forall u \in C_c(\Omega)$*

$$C_2\|\Delta_{s,h}u + qu\|^2_{L^2_h(\Omega)} \geq C_1|s|^2\|u\|^2_{L^2_h(\Omega)} + C_1\|u\|^2_{\mathring{H}^1_h(\Omega)} \\ + \sum_{k=1}^{d}\int_{\partial_k\Omega}\frac{1}{h}\sinh(2hs\cdot e_k)\,|A_k(\sigma^k)|^2 t_r^k(|D_k u|^2)\,n_k. \tag{5.27}$$

Theorem 5.2 *Given $q \in C(\Omega)$, such that $\|q\|_{L^\infty_h(\mathring{\Omega})} \leq m$. There exist constants $C_1, C_2, c > 0$, and $s_0 > 0$, such that, $\forall |s| \in [s_0, c\min\{h^{-1}, \epsilon_d^{-1}\}]$, we have, $\forall v \in C_c(\Omega)$*

$$C_2\|e^{-\phi_s}(\Delta_h + q)v\|^2_{L^2_h(\Omega)} \geq C_1|s|^2\|e^{-\phi_s}v\|^2_{L^2(\Omega)} + C_1\sum_{k=1}^{d}\int_{\Omega_k^*}|D_k v|^2 e^{-2\phi_s} \\ + \sum_{k=1}^{d}\int_{\partial_k\Omega} t_r^k(e^{-2t_r^k(\phi_s)})\,\frac{1}{h}\sinh(2hs\cdot e_k)\,|A_k(\sigma^k)|^2 t_r^k(|D_k v|^2)\,n_k.$$

Proof. Let us consider $v = ue^{\phi_s}$. Using (5.3) we have

$$D_k(v) = D_k(u)A_k(e^{\phi_s}) + A_k(u)D_k(e^{\phi_s}).$$

Thanks to Proposition 5.1 we write $A_k(e^{\phi_s}) = e^{\phi_s}\cosh\left(s\cdot e_k\frac{h}{2}\right)$ and $D_k(e^{\phi_s}) = \frac{2}{h}e^{\phi_s}\sinh\left(s\cdot e_k\frac{h}{2}\right)$. Thus

$$D_k v = D_k(u)e^{\phi_s}\cosh\left(s\cdot e_k\frac{h}{2}\right) + \frac{2}{h}A_k(u)e^{\phi_s}\sinh\left(s\cdot e_k\frac{h}{2}\right).$$

Therefore, using Young's inequality and that $|A_k u|^2 \leq A_k(|u|^2)$, there exists a constant $C > 0$ such that

$$|D_k v|^2 \leq C\left(|D_k u|^2 e^{2\phi_s} + |s|^2 A_k(|u|^2)e^{2\phi_s}\right).$$

Then a discrete integration by parts concerning the average operator yields

$$\int_{\Omega_k^*} |D_k v|^2 e^{-2\phi_s} \le C \left(\int_{\partial_k \Omega} |D_k u|^2 + |s|^2 \int_{\Omega} |u|^2 \right),$$

since $u = 0$ on $\partial_k \Omega$. From the above estimate we obtain

$$|s|^2 \|e^{-\phi_s} v\|^2_{L^2(\Omega)} + \sum_{k=1}^{d} \int_{\Omega_k^*} |D_k v|^2 e^{-2\phi_s} \le C_1 |s|^2 \|u\|^2_{L^2_h(\Omega)} + C_1 \|u\|^2_{\mathring{H}^1_h(\Omega)}.$$

We note that the proof is complete by showing that

$$t_r^k \left(|D_k u|^2 \right) = t_r^k(|D_k v|^2) t_r^k(e^{-2t_r^k(\phi_s)}) \text{ on } \partial_k \Omega.$$

Indeed, thanks to (5.3) and Proposition 5.1 it follows that

$$D_k u = D_k(v) e^{-\phi_s} \cosh\left(s \cdot e_k \frac{h}{2} \right) - \frac{2}{h} A_k(v) e^{-\phi_s} \sinh\left(s \cdot e_k \frac{h}{2} \right). \tag{5.28}$$

Moreover, for all $x \in \partial_k \Omega$ we have $\frac{2}{h} t_r^k \left(A_k v \right) = t_r^k(D_k v)\, n_k$ by virtue of $v = 0$ on $\partial_k \Omega$. Thus from (5.28) we write

$$t_r^k \left(D_k u \right) = t_r^k(D_k v) e^{-t_r^k(\phi_s) + s \cdot e_k \frac{h}{2} n_k}.$$

Therefore

$$t_r^k \left(|D_k u|^2 \right) = t_r^k(|D_k v|^2) t_r^k(e^{-2t_r^k(\phi_s)}),$$

since $-t_r^k(2\phi_s) + s \cdot e_k h n_k = -2(\phi_s \pm h e_k \cdot s)$, and the proof is complete. □

5.3. Application to an inverse problem with partial data

The aims of this Section is to analyze the discrete Calderón inverse problem with partial data. Firstly, we define the discrete normal derivative to introduce the Dirichlet to Neumann operator. To this end, let us consider the definition of the discrete normal derivative.

Definition 5.2 *We define the normal derivative in $\partial\Omega$, for any $u \in C(\overline{\Omega})$*

$$\partial_n u := \sum_{k=1}^{d} t_r^k(\sigma^k D_k u) n_k, \quad \text{on } \partial\Omega,$$

Moreover, via the discrete normal derivative, we consider the operator Dirichlet to Neumann.

Definition 5.3 *We define the Dirichlet to Neumann map:*

$$\Lambda_h[q](g) := \partial_n u, \quad \text{on } \partial\Omega,$$

where u is solution of

$$\begin{aligned} -\Delta_h u + qu &= 0, \quad \text{in } \Omega, \\ u &= g, \quad \text{on } \partial\Omega. \end{aligned}$$

Similar to the continuum case, the proof of the stability estimate use the existence of CGO solutions. The following result state their existence, and since the direction of our numerical scheme coincide with the space dimension we need the condition $h|s| \le c$ on the Carleman parameter s and the mesh size h instead of $h^{2/3}|h| \le c$ as it was mentioned in [20].

Theorem 5.3 (Theorem 4.4 in [20]) *Let $m \in \mathbb{R}_+$. For all $q \in C(\Omega)$ satisfying $\|q\|_{L^\infty(\Omega)} \le m$, there exists $s_0 > 0$ that depends on m such that $\forall \eta \in \mathbb{C}^d$ such that $\eta \cdot \eta = 0$, if $s := \mathcal{R}(\eta)$ verifies $s_0 \le |s| \le ch^{-1}$, there exists $u \in C(\overline{\Omega})$ a solution of*

$$-\Delta_h u + qu = 0, \quad \text{on } \Omega,$$

that satisfies

$$u(x) = e^{\eta \cdot x}(1 + r(x)), \quad \text{on } \overline{\Omega},$$

with

$$\|r\|_{\mathring{H}(\Omega)} + |s|\|r\|_{L^2(\overline{\Omega})} \le C(1 + |s|^4 h^2).$$

Let us finally introduce the boundary set where we will make the measurements.

Definition 5.4 *For a fixed vector $\mu \in \mathbb{R}^3$ we consider*

$$\Gamma_\mu^+ := \{x \in \partial\Omega \mid \vec{n}(x) \cdot \mu > 0\},$$

and

$$\Gamma_\mu^- := \{x \in \partial\Omega \mid \vec{n}(x) \cdot \mu \le 0\},$$

where $\vec{n}(x) \cdot \mu = \sum_{k=1}^d (\mu \cdot e_k) n_k(x)$, for $x \in \partial\Omega$. Thus, we consider $\partial\Omega = \Gamma_\mu^+ \cup \Gamma_\mu^-$.

Here, Γ_μ^- is the part of the boundary where we make our measure, where we assume is know the Derichlet-Neumann map. On the other hand, Γ_μ^+ is the part of the boundary where we do not know the map.

Let us now consider $m \in \mathbb{R}_+$ and $q_1, q_2 \in C(\Omega)$ satisfying $\|q_1\|_{L^\infty(\Omega)}, \|q_2\|_{L^\infty(\Omega)} \le m$. We take $\beta \in \mathbb{R}^d$, and the real vectors s and γ, such that $s \cdot \beta = \beta \cdot \gamma = \gamma \cdot s = 0$;

$$|s|^2 = |\gamma|^2 + |\beta|^2.$$

Finally, we assume that s satisfice
Assumption:

$$\Gamma_\mu^+ \subseteq \Gamma_s^+.$$

Then, we set η_1 and η_2 as

$$\eta_1 = s + i\beta + i\gamma, \quad \eta_2 = -s + i\beta - i\gamma,$$

using Theorem 5.3, we have

$$\begin{aligned} u_1 &= e^{\eta_1 \cdot x}(1 + r_1), \\ u_2 &= e^{\eta_2 \cdot x}(1 + r_2), \end{aligned}$$

where u_1, u_2 are solutions of

$$\begin{aligned} -\Delta_h u_1 + q_1 u_1 &= 0, \quad \text{in } \Omega, \\ -\Delta_h u_2 + q_2 u_2 &= 0, \quad \text{in } \Omega. \end{aligned}$$

We have

$$\int_\Omega (q_1 - q_2) u_1 u_2 = \int_{\partial\Omega} u_2(\Lambda_h[q_1](u_1) - \Lambda_h[q_2](u_1))$$
$$= \int_{\Gamma_\mu^+} u_2(\Lambda_h[q_1](u_1) - \Lambda_h[q_2](u_1)) + \int_{\Gamma_\mu^-} u_2(\Lambda_h[q_1](u_1) - \Lambda_h[q_2](u_1)).$$

If we define u, such that

$$\begin{aligned} -\Delta_h u + q_2 u &= (q_2 - q_1) u_1, \quad \text{in } \Omega \\ u &= 0, \qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{5.29}$$

Then

$$\partial_n u = \Lambda_h[q_1](u_1) - \Lambda_h[q_2](u_1), \quad \text{on } \partial\Omega,$$

and we have

$$\int_\Omega (q_1 - q_2) u_1 u_2 = \int_{\partial\Omega} u_2 \partial_n u$$
$$\int_\Omega (q_1 - q_2) e^{2i\beta}(1 + r_1 + r_2 + r_1 r_2) = \int_{\Gamma_\mu^+} u_2 \partial_n u + \int_{\Gamma_\mu^-} u_2 \partial_n u \tag{5.30}$$

We consider the following Lemma to estimate the integral over Γ_μ^+

Lemma 5.5 *If there exists $\delta \in (0,1)$, such that*

$$\min_{x \in \Gamma_\mu^+} |s \cdot \vec{n}| \geq \delta |s|,$$

and s, h satisfies the hypothesis of Theorem 5.2. Then, there exists a constant $C > 0$ such that

$$\left| \int_{\Gamma_\mu^+} u_2 \partial_n u \right|^2 \leq \frac{C}{\delta} \left(\frac{1}{|s|} + \int_{\Gamma_\mu^-} e^{-2\phi_s} |\partial_n u|^2 \right).$$

Proof. We note that

$$\left| \int_{\Gamma_\mu^+} u_2 \partial_n u \right| \leq \left(\int_{\Gamma_\mu^+} |u_2|^2 e^{2\phi_s} \right)^{1/2} \left(\int_{\Gamma_\mu^+} |\partial_n u|^2 e^{-2\phi_s} \right)^{1/2}. \tag{5.31}$$

Using the Carleman inequality on u, being solution of (5.29), we obtain

$$
\begin{aligned}
&C_2\|(-\Delta+q_2)u\|^2_{L^2(\Omega)}+\int_{\Gamma_s^-}\frac{1}{h}\sinh(2hs\cdot\vec{n})\sum_{k=1}^{3}t_r^kt_r^k(e^{-2\phi_s})\,t_r^k|D_ku|^2\,|n_k|\\
&\geq\int_{\Gamma_s^+}\frac{1}{h}\sinh(2hs\cdot\vec{n})\sum_{k=1}^{3}t_r^kt_r^k(e^{-2\phi_s})\,t_r^k|D_ku|^2\,|n_k|\\
&\geq 2\int_{\Gamma_\mu^+}|s\cdot\vec{n}|\sum_{k=1}^{3}t_r^kt_r^k(e^{-2\phi_s})\,t_r^k|D_ku|^2\,|n_k|\\
&\geq\frac{2}{\sigma}\min_{x\in\Gamma_\mu^+}|s\cdot\vec{n}|\int_{\Gamma_\mu^+}e^{-2\phi_s}\;|\partial_nu|^2
\end{aligned}
$$

Then, if we consider $\delta\in(0,1)$, such that

$$
\min_{x\in\Gamma_\mu^+}|s\cdot\vec{n}|\geq\delta|s|,
$$

and using the definition of u_1, we have

$$
\int_{\Gamma_\mu^+}e^{-2\phi_s}|\partial_nu|^2\leq\frac{C_2}{\delta|s|}\|e^{-\phi_s}(q_2-q_1)u_1\|^2_{L^2(\Omega)}+\frac{C}{\delta}\int_{\Gamma_s^-}e^{-2\phi_s}\;|\partial_nu|^2\leq\frac{C_2}{\delta|s|}+\frac{C}{\delta}\int_{\Gamma_\mu^-}e^{-2\phi_s}\;|\partial_nu|^2 \tag{5.32}
$$

On the other hand, using Theorem 5.3, we have

$$
\int_{\Gamma_\mu^+}|u_2|^2e^{2\phi_s}\leq\int_{\partial\Omega}|1+r_2|^2\leq C\|1+r_2\|^2_{H^1(\Omega)}\leq C\left(1+\|r_2\|^2_{L^2(\overline{\Omega})}+\|r_2\|^2_{\mathring{H}^1(\Omega)}\right)\leq C. \tag{5.33}
$$

Thus combine (5.31),(5.32) and (5.33) we conclude the proof. □

Therefore, using Lemma 5.5 in equation (5.30), we obtain

$$
\left|\int_\Omega(q_1-q_2)u_1u_2\right|^2\leq\frac{C}{\delta|s|}+\frac{C}{\delta}\int_{\Gamma_\mu^-}e^{-2\phi_s}\;|\partial_nu|^2+C\left|\int_{\Gamma_\mu^-}u_2\partial_nu\right|^2.
$$

Then using the definition of u, u_1, u_2 and (5.33), we have

$$
\begin{aligned}
\left|\int_\Omega(q_1-q_2)e^{2i\beta\cdot x}\right|^2\leq&\frac{C}{\delta}\left(\frac{1}{|s|}+\int_{\Gamma_\mu^-}e^{-2\phi_s}\;|\partial_nu|^2\right)\\
\leq&\frac{C}{\delta}\left(\frac{1}{|s|}+e^{c|s|}\int_{\Gamma_\mu^-}\;|\partial_nu|^2\right),
\end{aligned}
$$

for any β perpendicularly to s.

Theorem 5.4 *Given $\beta\in\mathbb{R}^3$, $|b|<\frac{c}{\sqrt{h}}$, such that exist vectors $s,\gamma\in\mathbb{R}^3$, wish satisfies, $s\cdot\beta=s\cdot\beta=\beta\cdot\gamma=0$,*

$$
|s|=|\gamma|+|\beta|,
$$

$\Gamma_\mu^+ \subseteq \Gamma_s^+$, and here exists $\delta \in (0,1)$, such that

$$\min_{x\in\Gamma_\mu^+} |s\cdot\vec{n}| \geq \delta|s|.$$

And s, h satisfies the hypothesis of Theorem 5.2. Then, there exist a constant $C > 0$, such that

$$\left|\int_\Omega (q_1-q_2)e^{2i\beta\cdot x}\right|^2 \leq \frac{C}{\delta}\max\left\{h^{\frac{1}{2}}, \left|\ln\left(\int_{\Gamma_\mu^-} |\partial_n u|^2\right)\right|^{-1}\right\}.$$

Theorem 5.5 *Given $\varepsilon \in (0,1)$, we consider a cone*

$$\mathcal{V}_{\varepsilon,h} = \{\beta\in\mathbb{R}^3 : |\beta\cdot e_j| \leq (1-\varepsilon)|\beta| \ \wedge |\beta| < c/\sqrt{h}\ \},$$

then, there exist $\delta \in (0,1)$ and a constant $C > 0$, such that for all $\beta \in \mathcal{V}_\varepsilon$, we have

$$\left|\int_\Omega (q_1-q_2)e^{2i\beta\cdot x}\right|^2 \leq \frac{C}{\delta}\max\left\{h^{\frac{1}{2}}, \left|\ln\left(\int_{\Gamma_{e_j}^-} |\partial_n u|^2\right)\right|^{-1}\right\}.$$

Chapter 6

Some conclusions and commentaries

In this chapter we present a brief discussion on some perspective about the main results presented in this thesis.

6.1. On Chapter 2

The results presented in Section 2.2, from Chapter 2, are of independent interest in view of its potential applications on problems related to semi-discrete or discrete Carleman estimates. For instance, it could be used to answer the challenge proposed by C. Zheng in [52], that is, to obtain a semi-discrete global Carleman estimates for fourth-order Schrödinger equation and establish a semi-discrete counterpart of the main results presented in that work. Even in the continuous setting, there are few papers about the stability of an inverse problem for higher-order equations, via Carleman estimates, due to tedious computation and the increased complexity. To our knowledge, the semi-discrete Carleman estimate presented in Chapter 4 is the first one for higher-order operators. Thus, the results from Section 2.2 can be a useful tool to obtain results in that direction.

An extension of the main results of this Chapter is to state Theorem 2.3 in its fully discrete version. This allows us to study controllability issues, via Carleman estimates, for higher full discrete systems. Let us mention that there are some results in that direction. For instance, in [30] the authors considered the null controllability problem for a fully discrete heat equation with Dirichlet boundary conditions. In that work, an estimate for the discrete (time and space) operators applied on the Carleman weight function is needed to obtain their main result.

6.2. On Chapter 3

A possible extension of the result from Section 3.4 could be to reformulate Theorem 3.2 for some families of non-uniform meshes. The Carleman estimate (3.10) is established for uniform mesh and could be adapted to some non-uniform meshes obtained as the smooth image of a uniform grid, following the methodology of [7]. Another interesting question is to consider the fully discrete case of our problem, particularly due to the term $\partial_x^2\partial_t$, which mixes time and space. Perhaps a first attempt is just consider the time-discrete case

$$\frac{u^{n+1}-u^n}{\Delta t} - \frac{\partial_x^2 u^{n+1} - \partial_x^2 u^n}{\Delta t} = p^{n+1}\partial_x u^{n+1} + q^{n+1}u^{n+1}, \quad n = 0, 1, ...,$$

which is a possible discretization in time.

6.3. On Chapter 4

The cornerstone of the controllability results presented in Chapter 4 is a semi-discrete Carleman estimate. We note that this estimate has been established in the one-dimensional setting and for uniform meshes. Following the methodology of [10], the Carleman estimate exhibited in Section 4.1 could be adapted to some non-uniform meshes obtained as the smooth image of a uniform grid. Moreover, it could be possible to extend our strategy to the multi-dimensional case. For instance, the discrete integration by parts concerning the difference operator from Proposition 2.1 could be written as

$$\int_Q u\, D_i v = -\int_{Q_i^*} D_i u\, v + \int_{\partial_i Q} u\, t_r^i(v)\, n_i,$$

where D_i is the difference operator acting in the direction i, defined as $(D_i u)(x) = \frac{1}{h}[u(x + he_i/2) - u(x - he_i/2)]$ in the case of uniform meshes, being $\{e_i\}_{i=1}^N$ a base of the space $\mathbb{R}^N$, for $N \geq 2$. Furthermore, the definition of the sets Q_i^* and $\partial_i Q$ are similar to the one-dimensional case presented in Section 5.1, as well as for the trace operator t_r^i.

In [29], it was proved by Guerrero and Kassab, in the continuous setting, the null controllability of a parabolic equation via a Carleman estimate for arbitrary dimension; which could be a guide to this purpose as it was the continuous Carleman estimate due to Cerpa and Mercado in [15] to obtain (4.11). Therefore, the controllabilty result that we have obtained in Section 4.2 is the first step to address a possible extension to arbitrary dimension. In any direction, we expect the notion of $\phi-$controllability.

The previous extension can be considered since Theorem 2.2 can be stated for arbitrary dimension in the semi-discrete setting. Indeed, the arguments presented in [37, Section 4] are straightforwardly extended for discrete difference or average operator defined in arbitrary dimension for uniform meshes. Moreover, this result is of independent interest given its potential applications on problems related to semi-discrete Carleman estimate as inverse problems.

Another natural extension could be to consider the fully-discrete case. Recently, there is some progress of controllability results for a fully discrete parabolic system through finite difference. In [30], González Casanova and Hernández-SantaMaría considered a fully-discrete approximation of the heat equation with Dirichlet boundary conditions. They obtain $\phi-$controllability results via a fully-discrete Carleman estimate. As expected, the time and space-discrete parameters are connected to the Carleman parameter. Thus, the first task to study the fully-discrete approximation of the system (4.1) would be to establish a similar result as Theorem 2.3 for the time-discrete difference operator.

References

[1] G. Alessandrini, L. Rondi, E. Rosset, and S. Vessella. The stability for the Cauchy problem for elliptic equations. *Inverse Problems*, 25(12):123004, 47, 2009.

[2] D. Allonsius, F. Boyer, and M. Morancey. Spectral analysis of discrete elliptic operators and applications in control theory. *Numer. Math.*, 140(4):857–911, 2018.

[3] L. Baudouin and S. Ervedoza. Convergence of an inverse problem for a 1-D discrete wave equation. *SIAM J. Control Optim.*, 51(1):556–598, 2013.

[4] L. Baudouin, S. Ervedoza, and A. Osses. Stability of an inverse problem for the discrete wave equation and convergence results. *J. Math. Pures Appl. (9)*, 103(6):1475–1522, 2015.

[5] T. B. Benjamin, J. L. Bona, and J. J. Mahony. Model equations for long waves in nonlinear dispersive systems. *Philos. Trans. Roy. Soc. London Ser. A*, 272(1220):47–78, 1972.

[6] F. Boyer. On the penalised HUM approach and its applications to the numerical approximation of null-controls for parabolic problems. In *CANUM 2012, 41e Congrès National d'Analyse Numérique*, volume 41 of *ESAIM Proc.*, pages 15–58. EDP Sci., Les Ulis, 2013.

[7] F. Boyer, F. Hubert, and J. Le Rousseau. Discrete Carleman estimates for elliptic operators and uniform controllability of semi-discretized parabolic equations. *J. Math. Pures Appl. (9)*, 93(3):240–276, 2010.

[8] F. Boyer, F. Hubert, and J. Le Rousseau. Discrete Carleman estimates for elliptic operators in arbitrary dimension and applications. *SIAM J. Control Optim.*, 48(8):5357–5397, 2010.

[9] F. Boyer, F. Hubert, and J. Le Rousseau. Uniform controllability properties for space/time-discretized parabolic equations. *Numer. Math.*, 118(4):601–661, 2011.

[10] F. Boyer and J. Le Rousseau. Carleman estimates for semi-discrete parabolic operators and application to the controllability of semi-linear semi-discrete parabolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(5):1035–1078, 2014.

[11] A.-P. Calderón. On an inverse boundary value problem. In *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)*, pages 65–73. Soc. Brasil. Mat., Rio de Janeiro, 1980.

[12] T. Carleman. Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes. *Ark. Mat., Astr. Fys.*, 26(17):9, 1939.

[13] C. Carthel, R. Glowinski, and J.-L. Lions. On exact and approximate boundary con-

trollabilities for the heat equation: a numerical approach. *J. Optim. Theory Appl.*, 82(3):429–484, 1994.

[14] E. Cerpa, R. Lecaros, T. N. T. Nguyen, and A. Pérez. Carleman estimates andcontrollability for a semi-discrete fourth-order parabolic equation. *Submitted*, 2021.

[15] E. Cerpa and A. Mercado. Local exact controllability to the trajectories of the 1-D Kuramoto-Sivashinsky equation. *J. Differential Equations*, 250(4):2024–2044, 2011.

[16] D. Chenais and E. Zuazua. Controllability of an elliptic equation and its finite difference approximation by the shape of the domain. *Numer. Math.*, 95(1):63–99, 2003.

[17] M. Choulli. *Applications of elliptic Carleman inequalities to Cauchy and inverse problems*. SpringerBriefs in Mathematics. Springer, [Cham]; BCAM Basque Center for Applied Mathematics, Bilbao, 2016. BCAM SpringerBriefs.

[18] P. L. da Silva and I. L. Freire. A geometrical demonstration for continuation of solutions of the generalised BBM equation. *Monatsh. Math.*, 194(3):495–502, 2021.

[19] L. de Teresa, R. Lecaros, J. H. Ortega, and A. Pérez. Discrete Calderón problem with partial data. *Work in progress*, 2021.

[20] S. Ervedoza and F. de Gournay. Uniform stability estimates for the discrete Calderón problems. *Inverse Problems*, 27(12):125012, 37, 2011.

[21] H. O. Fattorini and D. L. Russell. Exact controllability theorems for linear parabolic equations in one space dimension. *Arch. Rational Mech. Anal.*, 43:272–292, 1971.

[22] A. Fernández-Bertolin, L. Roncal, A. Ruland, and D. Stan. Discrete carleman estimates and three balls inequalities. *arXiv: Analysis of PDEs*, 2020.

[23] E. Fernández-Cara and A. Münch. Numerical null controllability of semi-linear 1-D heat equations: fixed point, least squares and Newton methods. *Math. Control Relat. Fields*, 2(3):217–246, 2012.

[24] E. Fernández-Cara and A. Münch. Numerical exact controllability of the 1D heat equation: duality and Carleman weights. *J. Optim. Theory Appl.*, 163(1):253–285, 2014.

[25] X. Fu, Q. Lü, and X. Zhang. *Carleman estimates for second order partial differential operators and applications*. SpringerBriefs in Mathematics. Springer, Cham, 2019. A unified approach, BCAM SpringerBriefs.

[26] A. V. Fursikov and O. Y. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[27] A. V. Fursikov and O. Y. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[28] M. Guadie and E. Malinnikova. On three balls theorem for discrete harmonic functions. *Comput. Methods Funct. Theory*, 14(4):721–734, 2014.

[29] S. Guerrero and K. Kassab. Carleman estimate and null controllability of a fourth order parabolic equation in dimension $N \geq 2$. *J. Math. Pures Appl. (9)*, 121:135–161, 2019.

[30] V. Hernández-SantaMaría and P. González Casanova. Carleman estimates and

controllability results for fully-discrete approximations of 1-d parabolic equations. *arXiv:2012.02156*, 2020.

[31] V. Isakov. *Inverse source problems*, volume 34 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[32] V. Isakov. On uniqueness in the inverse conductivity problem with local data. *Inverse Probl. Imaging*, 1(1):95–105, 2007.

[33] C. Kenig and M. Salo. Recent progress in the Calderón problem with partial data. In *Inverse problems and applications*, volume 615 of *Contemp. Math.*, pages 193–222. Amer. Math. Soc., Providence, RI, 2014.

[34] K. Knudsen and M. Salo. Determining nonsmooth first order terms from partial boundary measurements. *Inverse Probl. Imaging*, 1(2):349–369, 2007.

[35] S. Labbé and E. Trélat. Uniform controllability of semidiscrete approximations of parabolic control systems. *Systems Control Lett.*, 55(7):597–609, 2006.

[36] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20(1-2):335–356, 1995.

[37] R. Lecaros, J. H. Ortega, and A. Pérez. Stability estimate for the semi-discrete linearized Benjamin-Bona-Mahony equation. *ESAIM Control Optim. Calc. Var.*, 27:Paper No. 93, 30, 2021.

[38] J.-L. Lions. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, volume 8 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1988. Contrôlabilité exacte. [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

[39] A. Lopez and E. Zuazua. Some new results related to the null controllability of the 1-d heat equation. In *Séminaire sur les Équations aux Dérivées Partielles, 1997–1998*, pages Exp. No. VIII, 22. École Polytech., Palaiseau, 1998.

[40] S. Micu. On the controllability of the linearized Benjamin-Bona-Mahony equation. *SIAM J. Control Optim.*, 39(6):1677–1696, 2001.

[41] A. Münch and E. Zuazua. Numerical approximation of null controls for the heat equation: ill-posedness and remedies. *Inverse Problems*, 26(8):085018, 39, 2010.

[42] T. N. T. Nguyen. Carleman estimates for semi-discrete parabolic operators with a discontinuous diffusion coefficient and applications to controllability. *Math. Control Relat. Fields*, 4(2):203–259, 2014.

[43] T. N. T. Nguyen. Uniform controllability of semidiscrete approximations for parabolic systems in Banach spaces. *Discrete Contin. Dyn. Syst. Ser. B*, 20(2):613–640, 2015.

[44] L. Rosier and B.-Y. Zhang. Unique continuation property and control for the Benjamin-Bona-Mahony equation on a periodic domain. *J. Differential Equations*, 254(1):141–178, 2013.

[45] J. Sylvester and G. Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.

[46] D. Tataru. Unique continuation problems for partial differential equations. In *Geometric*

methods in inverse problems and PDE control, volume 137 of *IMA Vol. Math. Appl.*, pages 239–255. Springer, New York, 2004.

[47] G. Uhlmann. Electrical impedance tomography and Calderón's problem. *Inverse Problems*, 25(12):123011, 39, 2009.

[48] G. Uhlmann. 30 years of Calderón's problem. In *Séminaire Laurent Schwartz—Équations aux dérivées partielles et applications. Année 2012–2013*, Sémin. Équ. Dériv. Partielles, pages Exp. No. XIII, 25. École Polytech., Palaiseau, 2014.

[49] M. Yamamoto. One unique continuation for a linearized Benjamin-Bona-Mahony equation. *J. Inverse Ill-Posed Probl.*, 11(5):537–543, 2003.

[50] M. Yamamoto. Carleman estimates for parabolic equations and applications. *Inverse Problems*, 25(12):123013, 75, 2009.

[51] X. Zhang and E. Zuazua. Unique continuation for the linearized Benjamin-Bona-Mahony equation with space-dependent potential. *Math. Ann.*, 325(3):543–582, 2003.

[52] C. Zheng. Inverse problems for the fourth order Schrödinger equation on a finite domain. *Math. Control Relat. Fields*, 5(1):177–189, 2015.

[53] E. Zuazua. Propagation, observation, and control of waves approximated by finite difference methods. *SIAM Review*, 47(2):197–243, 2005.

[54] E. Zuazua. Control and numerical approximation of the wave and heat equations. In *International Congress of Mathematicians. Vol. III*, pages 1389–1417. Eur. Math. Soc., Zürich, 2006.