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OPTIMAL PRICING WITH TIME-DEPENDENT CONSUMER VALUATIONS

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN GESTIÓN DE OPERACIONES
MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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FIJACIÓN DE PRECIOS ÓPTIMA CON CONSUMIDORES CON VALORACIONES DEPENDIENTES DEL TIEMPO

En este trabajo se estudian tres problemas. Primero, se resuelve el problema de un vendedor dotado de infinitas unidades de cierto producto homogéneo para la venta en un horizonte finito y discreto con el fin de maximizar sus ingresos esperados. En segundo lugar, se estudian condiciones para garantizar que la política de precios que fija el *precio monopólico* para cada instante o período de tiempo sea no decreciente. Bajo esta condición de monotonía de los precios se muestra que una gran cantidad de problemas puede ser resuelta, aún considerando consumidores estratégicos. Finalmente se resuelve el problema del consumidor frente a una restricción de capacidad esperada y consumidores miopes. Para este problema se encuentran las soluciones óptimas de manera explícita y se estudia como cambian estas soluciones al variar la capacidad de inventario.

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In this work, three problems are explored. First, we solve the problem of a seller facing infinitely patient consumers. The seller owns infinitely many units of a certain homogeneous product for sale over a finite and discrete horizon in order to maximize his expected revenue. Secondly, conditions are studied to guarantee that the pricing policy that sets the *monopoly price* for each instant or period of time is non-decreasing. Under this condition of price monotony it is shown that a large number of problems can be solved, even considering strategic consumers. Finally, the problem of the consumer facing an expected capacity constraint and myopic consumers is addressed. For this problem the optimal solutions are found explicitly and it is studied how these solutions change when varying the inventory capacity.

Para mi madre, Oriana.

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Chapter 1

Introduction

Dynamic pricing and revenue management are a set of tactics and strategies that companies use to manage the allocation of their capacity to different fare classes over time in order to maximize revenues. Currently its applications are widely used by a variety of industries such as airlines, hotels, car rental companies, art performances, and sports events.

The main problem that will be studied could be described as follows. Suppose a seller has infinite items to sell, where sales take place over a finite selling horizon. An expected number of consumers arrive at fixed periods to purchase the object. The seller before the sales season posts a series of prices which will be fixed throughout the periods. Consumers, depending on the model, may wait and buy at a later instance of the period they arrive or be highly impatient and buy on arrival if their value for the item is higher than the posted price. The seller's problem is to maximize his expected revenue by trying to choose the best pricing policy considering consumer behavior. Although the literature on revenue management and dynamic pricing is vast, recent models that solve problems considering distinct variations of the above setting are [2], [4], [12]. For further reading in revenue management see [5].

This work makes two contributions to the area of revenue management and dynamic pricing. First, we find sufficient conditions for the pricing policy that charges monopolistically according to the consumers' valuation to be non-decreasing and therefore optimal in a variety of problems. Second, two finite horizon pricing problems are solved. The first problem tries to find the best pricing policy given infinitely patient consumers that arrive over T periods. The second one deals with the previous problem with myopic consumers and an *expected capacity* constraint. For both problems the optimal solutions (or pricing policies) are found explicitly.

Before starting, we need to introduce some notation. We denote $[T] := \{1, \dots, T\}$ the set of the first T natural numbers, $\mathbb{1}_A$ the indicator function of a set A and $\mathbb{R}_+$ the set of positive reals including zero. For a random variable V we will usually denote $F(p) = \mathbb{P}(V \leq p)$ its cumulative distribution function. The support of F is the smallest closed set S_F (also denoted as $\operatorname{supp} F$) such that $\mathbb{P}(V \in S_F) = 1$. Throughout this thesis, we will consider only positive random variables, i.e., with support in $\mathbb{R}_+$. Finally, all functions considered will be Lebesgue-measurable functions.

Chapter 2

Pricing with Infinitely Patient Buyers

In this chapter we will study optimal pricing policies in a setting in which a seller posts a price over T periods and sequentially arriving consumers strategically respond to these prices. In simple terms, a seller faces the problem of pricing for different periods as different types of consumers arrive. We will find the optimal solution of the problem constructively and provide a simple algorithm to find this optimal pricing policy.

The easiest way to visualize the model is to think of a seller with infinite homogeneous items trying to sell the objects to a continuum of consumers whose total mass of is normalized to 1 and value the item according to a different distribution depending on the time they arrive. With this in mind and thinking in only one period, if the seller posts the price p, $1 - F(p)$ is just how many consumers will buy the item at price p, at a normalized mass of 1. We will immediately formalize the model and consumer behavior over time.

2.1 Model definition

In its simpler form, we study the problem faced by a monopolist of setting prices prior to the consumer's arrival. The seller sells items over T periods, in which he has to set T prices for the different periods, where $T > 1$. After announcing a pricing path $(p_t)_{t=1}^T$ prior to the arrival of consumers, the seller is fully committed to this path and cannot change any of the prices. In each of the T periods $t = 1, 2, \dots, T$, a deterministic number of buyers α_t (and known to the seller) arrive with a private random valuation V_t for the item distributed according a distribution F_t with support contained in $\mathbb{R}_+$. Customers behave strategically and take the current and the future prices into consideration when deciding to buy early or late. Specifically, given a pricing path $(p_t)_{t=1}^T$, a buyer arriving in period τ with valuation v solves

$$\begin{aligned} \max_{t} \quad & v - p_t \\ \text{s.t.} \quad & t \in \{\tau, \dots, T\}, \end{aligned} \tag{BP}$$

provided that there is a price p_t such that $v > p_t$, otherwise the consumer will not purchase the item and will receive 0 utility. The objective of the seller is to set prices $p_1, \dots, p_T$ for each period in order to maximize its expected revenue. The monopolist has a production cost c per item that without loss of generality is normalized to zero. Given a pricing path $p = (p_t)_{t=1}^T$ and a consumer arriving in period τ, its demand in any period $t \geq \tau$ is given by

$$\mathrm{d}_\tau(p,t) = (1 - F_\tau(p_t))\mathbb{1}_{\{p_t < p_k, \forall k \geq \tau\}}.$$

The revenue this consumer generates to the seller is then $R_\tau(p) = \sum_{t=\tau}^T p_t \mathrm{d}_\tau(p,t)$. Note that this value is exactly $p_t(1 - F_\tau(p_t))$ for some $t \geq \tau$. Consequently, the formulation of the seller's problem is

$$\max_{p_1,\dots,p_n} \quad \sum_{t=1}^T \alpha_t \cdot R_t(p) \tag{SP}$$

The above problem can be relaxed under a non-decreasing pricing policy. Since the seller knows that a consumer will buy if and only if there will not be a lower price in the future, the problem that the seller solves under any non-decreasing pricing path is

$$\begin{aligned} \max_{p_1,\dots,p_n} \quad & \sum_{t=1}^T \alpha_t \cdot p_t(1 - F_t(p_t)) \\ \text{s.t.} \quad & p_1 \leq \dots \leq p_n. \end{aligned} \tag{RSP}$$

An important observation is that if the non-decreasing constraint were not in place, then the seller could clearly announce prices that maximize each summand independently. We will focus on studying solutions to SP. The first thing to consider is how SP relates to RSP.

2.2 Structure of the optimal solution

We start by defining general notions and hypotheses to solve the relaxed seller's problem. Define the *monopoly price* for a cumulative distribution function F_t as any price p_t^* that satisfies

$$p_t^* \in \arg\max_p R_t(p) = \arg\max_{p \geq 0} \alpha_t \cdot p(1 - F_t(p)),$$

which is motivated by the price that maximizes the seller's revenue when confronted with buyers with valuation F_t, when he can do make price discrimination. A naive approach for the seller to solve SP might be to announce the prices $p_1^*, \dots, p_T^*$, but nothing assures that it can be non-decreasing. The idea of making the previous pricing path non-decreasing motivates us to define the *monopoly price of an set $I \subseteq [T]$* as any price p_I^* such that

$$p_I^* \in \arg\max_p R_I(p) = \arg\max_{p \geq 0} \sum_{t \in I} \alpha_t \cdot p(1 - F_t(p)).$$

We will assume that the set $\arg\max_p R_I(p)$ consists of only one element for each $I \subseteq [T]$, and consequently denote the set as p_I^* and write without ambiguity $p_I^* = \arg\max_p R_I(p)$. Note that this also implies that $p_t^* = \arg\max_p R_t(p)$ for all $t \in [T]$. In order to solve SP, one strong condition that we could assume is that the revenue functions are concave, i.e. for each $t \in [T]$, $R_t(p) = \alpha_t \cdot p(1 - F_t(p))$ is concave. This hypothesis is quite restrictive and fails for most common distributions, such as exponential and normal distributions [10, 19]. For this reason, we can restrict ourselves to the family used for future proofs in this section. We will need two conditions over the family of distributions $\{F_t\}_{t=1}^T$:

(C1) For all $t \in [T]$, $R_t(p)$ is increasing in $(0, p_t^*)$ and decreasing in (p_t^*, ∞).

(C2) The family is closed under sums of revenue functions. This is, for all $I \subseteq [T]$, $R_I(p)$ is increasing in $(0, p_I^*)$ and decreasing in (p_I^*, ∞).

The above conditions are less restrictive than concavity. In fact, when all the distributions F_t have finite expectactation, condition (C1) and the set of maximizers being a singleton for each revenue function is equivalent to strict quasiconcavity of the revenue functions. Moreover, we will see that condition (C2) is only sufficient and can be relaxed. For now we will consider all families in the set $\mathcal{F}$, defined as

$$\mathcal{F} = \left\{ \{F_t\}_{t=1}^T \,:\, \{F_t\}_{t=1}^T \text{ is a family of distributions that satisfies (C1) and (C2)} \right\}.$$

Assumption 1 The family of distributions of consumers' valuation $\{F_t\}_{t=1}^T$ is an element of $\mathcal{F}$.

An important property about the set of families $\mathcal{F}$ is that it satisfies a property about the ordering of maximizers. Consider $R_1(p)$ and $R_2(p)$ revenue functions satisfying (C1) and (C2), with respective maximizers p_1 and p_2 and without loss of generality $p_1 \leq p_2$. Then the maximizer of $(R_1 + R_2)(p)$ belongs to the interval $[p_1, p_2]$. This is simple to check, since $(R_1 + R_2)(p)$ is increasing in $(0, p_1)$ and decreasing in (p_2, ∞). In addition, since $R_1(p)$ and $R_2(p)$ satisfy (C2), then this implies the maximizer of $(R_1 + R_2)(p)$ is necessarily in the interval $[p_1, p_2]$. As R_1 and R_2 were arbitrary revenue functions, then any family $\{F_{\mathrm{i}}\}_{t=1}^T \in \mathcal{F}$ satisfy the previous property on any arbitrary sum of their revenue functions. We have proven the following lemma.

Lemma 2.1 *Suppose Assumption 1 holds. Then, for any pair of disjoint intervals I_{i}, $I_j \subseteq [T]$ such that $p_{I_{\mathrm{i}}} \leq p_{I_j}$, $\arg\max_p (R_{I_{\mathrm{i}}} + R_{I_j})(p) \in [p_{I_{\mathrm{i}}}, p_{I_j}]$.*

The first proposition we will show will be to verify that in fact by solving RSP we also solve SP, since any feasible pricing path in RSP is feasible in SP and both problems have the same optimal value.

Proposition 2.2 *There exists an optimal pricing scheme $\{p_t\}_{t=1}^T$ that is non-decreasing, i.e., $p_1 \leq p_2 \leq \cdots \leq p_T$. Furthermore, SP is equivalent to RSP.*

The proof of the lemma is simple. It is based on choosing any optimal price path in [SP] and taking the higher increasing pricing policy that goes below the optimal price path (see Figure 2.2). Because of strategic consumer behaviour, both policies generate the same revenue and the lemma follows. In the proof of the lemma we show how to construct the above mentioned pricing policy.

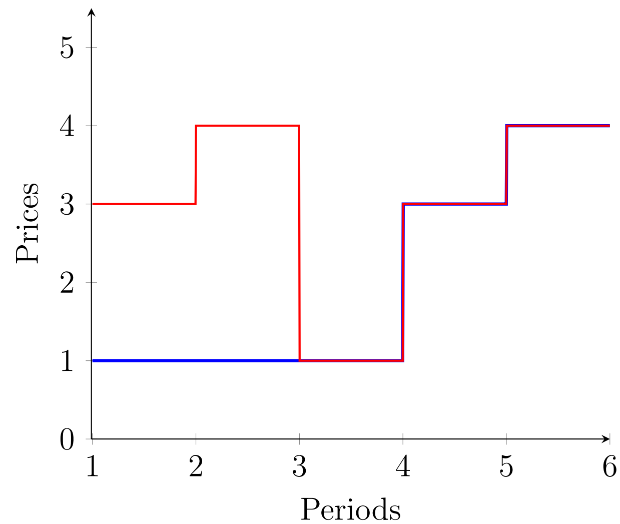

Figure 2.1: Two pricing paths that produce the same revenue for the seller.

Proof. Take any optimal pricing path $(\hat{p}_t)_{t=1}^T$ in SP. If it is non-decreasing, there is nothing to prove. Otherwise, let $\hat{p}_{\min} = \min\{\hat{p}_t \mid t \in [T]\}$ the lowest price in the T periods and $t_{\min}$ the lowest index on which this price is set. Since buyers are strategic, those who arrived on an earlier period will purchase in $t_{\min}$. Set $p_j = \hat{p}_{\min}$ for every index $j < t_{\min}$. Continue this procedure with the indices $\{t_{\min}+1, \ldots, T\}$. Clearly the resulting pricing policy is non-decreasing, and the revenue from the policies $(p_t)_{t=1}^T$ and $(\hat{p}_t)_{t=1}^T$ is identical by the behaviour of the buyers: both policies produce an expected revenue of $\sum_{j<t_{\min}} \hat{p}_{\min} \cdot (1 - F_j(\hat{p}_{\min}))$ before the period $t_{\min}$, and subsequent changes in prices by maintaining this procedure do not decrease the revenue in later periods as we have just mentioned.

The equivalence between the problems follows as any feasible pricing path in RSP is feasible in SP, and by the construction of the non-decreasing pricing policy given an optimal solution in SP that provides the same revenue. $\square$

Due to the previous proposition, from now on we will focus on finding the optimal solution to RSP. We will refer to a *interval* I as a subset of $[T]$ which has only consecutive integers. For instance, $\{1,2\}$ and $\{4,5,6\}$ are intervals, but $\{1,3\}$ is not. Similarly, we define a *partition* of $[T]$ through intervals (or just a partition) a collection of disjoint intervals $I_1, I_2, \ldots$ such that their union is $[T]$.

The following proposition shows that the optimal pricing policy has a partition structure. Moreover, the price announced in any interval I_j is set as p_{I_j} which is to be expected, since the prices in those periods respond to the aggregate demand $\mathrm{d}_{I_j}(p) = \sum_{\mathrm{i}\in I_j} \alpha_{\mathrm{i}} \cdot (1 - F_{\mathrm{i}}(p))$.

Proposition 2.3 *Suppose Assumption 1 holds. Then, the optimal pricing policy has a partition structure: there are $m \leq T$ disjoint intervals $I_1, \ldots, I_m$ in which in every period t such that $t \in I_j$ satisfies $p_t = p^*_{I_j}$.*

Proof. For any pricing policy $(\hat{p}_t)_{t=1}^T$ which is non-decreasing, define the m different intervals as the different sets of indices where a price is maintained. Formally, define $I_1 = \{t \in [T] \,|\, p_t =$

$\hat{p}_1\}$ which is clearly an interval since the policy $(\hat{p}_t)_{t=1}^T$ is non-decreasing. If this is not the only price fixed through all periods, let $\hat{p}_2$ the lowest price set different from $\hat{p}_1$ and define $I_2 = \{t \in [T] \,|\, p_t = \hat{p}_2\}$. Inductively we continue defining the intervals $I_j = \{t \in [T] \,|\, p_t = \hat{p}_j\}$ until there are no more periods left to cover. This procedure clearly ends as there are at most T different prices.

It remains to check that in the optimal policy, that every index t such that $t \in I_j$ satisfies $p_t = p^*_{I_j}$. To show this, take any optimal pricing policy $(\hat{p}_t)_{t=1}^T$ that is minimal in the number of intervals defined as before, and I_j any interval with $j \in [m]$. Let $\hat{p}_{I_j}$ the price set in the interval I_j. There are three cases to consider:

1. $j = 1$. Focusing only in this interval, by definition of the intervals we can think of maximizing $\hat{p}_{I_1}$ without changing consumer behavior, that is

$$\begin{aligned} \max_{\hat{p}_{I_1}} \quad & R_{I_1}(\hat{p}_{I_1}) \\ \text{s.t.} \quad & \hat{p}_{I_1} \leq \hat{p}_{I_2}. \end{aligned}$$

   If $m = 1$ (that is, there is only one interval and hence only one price set) then $\hat{p}_{I_2}$ is not defined but in that case we set $\hat{p}_{I_2} = \infty$. If the price set in this interval is not $p^*_{I_1}$, by Assumption 1 then by setting a price closer to $p^*_{I_1}$ increases the revenue. If the restriction is binding, then this pricing policy is setting the same price as another interval, which contradicts the minimality of the pricing path on intervals.

2. $j = m$. This case is analogous to the previous one.

3. $1 < j < m$. Focusing only in this period and given $\hat{p}_{I_{j-1}}$ and $\hat{p}_{I_{j+1}}$, we maximize

$$\begin{aligned} \max_{\hat{p}_{I_j}} \quad & R_{I_j}(\hat{p}_{I_j}) \\ \text{s.t.} \quad & \hat{p}_{I_{j-1}} \leq \hat{p}_{I_j} \leq \hat{p}_{I_{j+1}}. \end{aligned}$$

   Again, if the price set in this interval is not $p^*_{I_j}$, then by setting a price closer to this price increases the revenue. Furthermore, if some restriction is binding, say $\hat{p}_{I_{j-1}} = \hat{p}_{I_j}$ or $\hat{p}_{I_{j+1}} = \hat{p}_{I_j}$ then it contradicts the minimality of the chosen pricing path on intervals.

The proposition follows. $\square$

The previous proposition is fundamental to relaxing hypothesis (C2) in Assumption 1. This tells us that the optimal solution follows an interval structure, so it is not necessary to verify that the entire family is closed under sums of revenue functions, just closed over intervals. This reduces the number of conditions to check from 2^T to the number of partitions of T, which has no explicit solution but is significantly less. Take for example, the problem with 10 periods and distinct distributions $\{F_t\}_{t=1}^{10}$. Condition (C2) must check that $2^{10} = 1024$ functions are strictly quasiconcave while the number of partitions of [10] by intervals is only 42. We now set the relaxed condition on the families of valuations.

Assumption 1' The family of distributions $\{F_t\}_{t=1}^T$ is an element of $\mathcal{F}^*$, where

$$\mathcal{F}^* = \left\{ \{F_t\}_{t=1}^T \,:\, \{F_t\}_{t=1}^T \text{ is a family of distributions that satisfies (C1) and (C2')} \right\}.$$

(C1) For all $t \in [T]$, $R_t(p)$ is increasing in $(0, p_t^*)$ and decreasing in (p_t^*, ∞).

(C2') The family is closed under sums over intervals of revenue functions. This is, for all $I \subseteq [T]$ an interval, $R_I(p)$ is increasing in $(0, p_I^*)$ and decreasing in (p_I^*, ∞).

In what follows we will prove the key proposition to formulate an algorithm that finds the optimal price policy. Recall that is the monopoly prices are increasing, then the pricing path $(p_t^*)_{t=1}^T$ is optimal in SP. Consider the case in which there is only one decrease in the pricing policy (see Figure 2.2). The optimal pricing policy should include consecutive periods of decreasing prices in a single interval to make the optimal pricing policy non-decreasing. This turns out to be true.

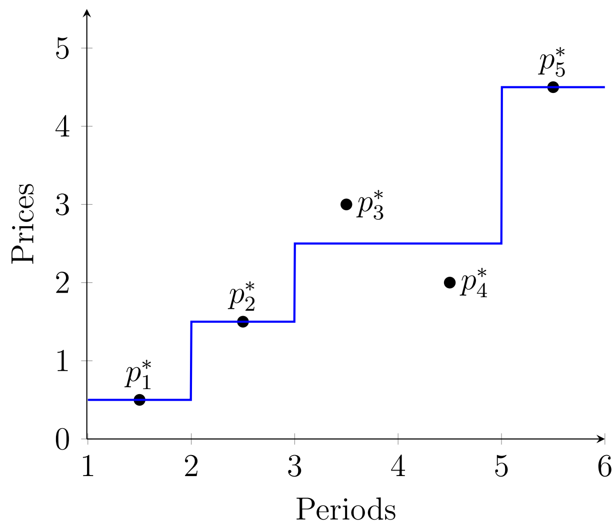

Figure 2.2: Monopoly prices $(p_t^*)_{t=1}^5$ and their respective optimal pricing policy (blue).

Proposition 2.4 *Suppose Assumption 1' holds. Let $(p_t^*)_{t=1}^T$ be the monopoly prices. If there exists a period j such that $p_j^* > p_{j+1}^*$ then $\{j, j+1\} \subseteq I$ for some interval I.*

Proof. Let $(\hat{p}_t)_{t=1}^T$ any optimal non-decreasing pricing policy minimal in the number of intervals. We proceed by contradiction. Let I and I' the different intervals in which j and $j+1$ belong, respectively. Since $(\hat{p}_t)_{t=1}^T$ is non-decreasing and j, $j+1$ are in different intervals, it must be the case that $p_I^* < p_{I'}^*$. There are two cases to consider in which we will find contradictions increasing or decreasing a price in a period in direction to the respective monopoly price and hence increasing the seller's revenue in virtue of Assumption 1 (see Figure 2.3) while all other prices in the pricing policy remain unchanged.

1. $p_{j+1}^* \geq p_{I'}^*$: Define a new pricing policy $(p_t)_{t=1}^T$ as

$$p_t = \begin{cases} \hat{p}_t & t \neq j, \\ p_{I'}^* & t = j. \end{cases}$$

   By construction $(p_t)_{t=1}^T$ is non-decreasing and produces a higher than $(\hat{p}_t)_{t=1}^T$, since in this case $\hat{p}_j = p_I^* < p_{I'}^* = p_j < p_j^*$.

2. $p^*_{j+1} < p^*_{I'}$: Define a new pricing policy $(p_t)_{t=1}^T$ as

$$p_t = \begin{cases} \hat{p}_t & t \neq j+1, \\ \max\{p^*_{j+1}, p^*_I\} & t = j+1. \end{cases}$$

Once again by construction $(p_t)_{t=1}^T$ is non-decreasing and produces a higher produces than the $(\hat{p}_t)_{t=1}^T$ policy, since in this case $\hat{p}_{j+1} = p^*_{I'} > \max\{p^*_{j+1}, p^*_I\} = p_{j+1} \geq p^*_{j+1}$.

□

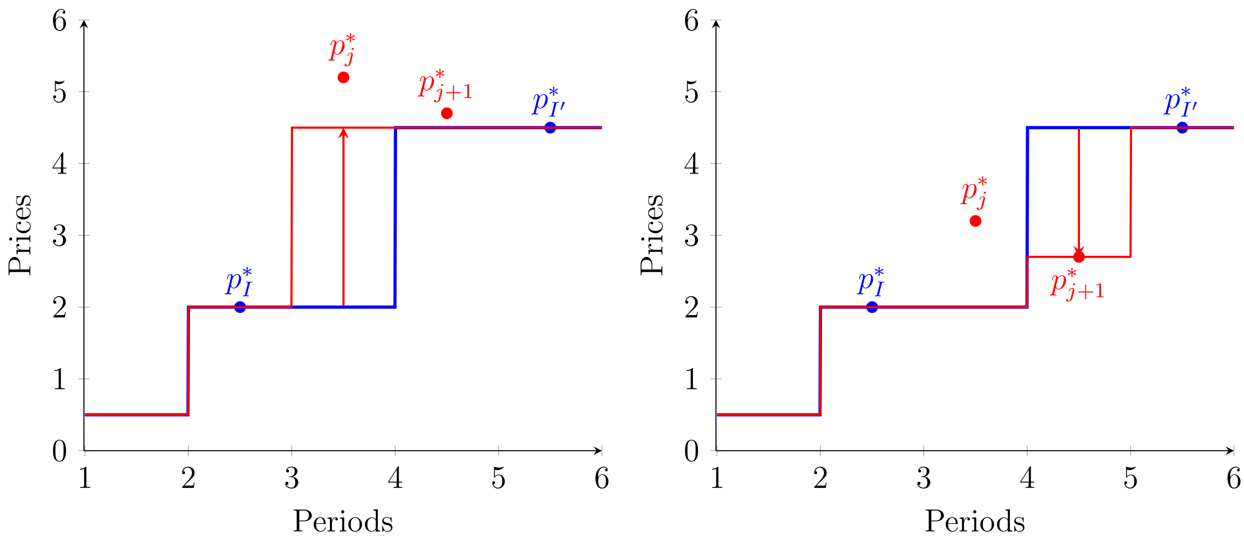

Figure 2.3: Contradictions during the proof of Proposition 2.4, case 1 (left) and case 2 (right). The pricing path $(\hat{p}_t)_{t=1}^T$ is represented as the blue line and the red line as the constructed policy $(p_t)_{t=1}^T$ which yields a contradiction.

An immediate corollary is that every decreasing sequence of monopoly prices is contained in an interval. This is because the interval I in the previous proposition will be fixed between decreasing pairs of monopoly prices.

Corollary 2.5 *Suppose Assumption 1' holds. Let $(p^*_t)_{t=1}^T$ the monopoly prices and $k > 1$. If there exists a sequence of periods such that $p^*_j \geq p^*_{j+1} \geq \ldots \geq p^*_{j+k}$ then $\{j, \ldots, j+k\} \subseteq I$ for some interval I.*

In particular, the previous corollary implies that every *maximal* decreasing subsequence of monopoly prices is in an interval.

2.2.1 Algorithm

We present an algorithm that computes the prices that solve SP under Assumption 1'. The construction is based on Corollary 2.5.

Algorithm 1: Optimal pricing policy of T periods.

Input: $F_1, \dots, F_T$ distributions satisfying Assumption 1' and $\alpha_1, \dots, \alpha_T$ masses of arrival.

Initialize $p_1 = 0$, $p_2 = 0$, $\dots$, $p_T = 0$, $p_{T+1} = \infty$, $J \leftarrow \emptyset$
for $t = 1, \dots T$ **do**
  Compute $p_t^* = \arg\max_p \alpha_t \cdot p(1 - F_t(p))$
  Set $p_t = p_t^*$
end
while *exists some* $j \in [T]$ *such that* $p_j > p_{j+1}$ **do**
  for $t = 1, \dots, T$ **do**
    $J \leftarrow J \cup \{t\}$
    if $p_t < p_{t+1}$ **then**
      Compute $p_J^* = \arg\max_p \sum_{\mathrm{i} \in J} \alpha_\mathrm{i} \cdot p(1 - F_\mathrm{i}(p))$
      Set $p_\mathrm{i} = p_J^*$ for every $\mathrm{i} \in J$
      $J \leftarrow \emptyset$
    end
  end
end
return $p_1, p_2, \dots, p_T$

The main issue with Corollary 2.5 arises after grouping the maximal decreasing sequences in an interval, since nothing implies that in a given interval a period outside a decreasing sequence is not in that interval in the optimum solution. We can solve this issue by doing a reduction technique on the number of periods. Consider $I_1, \dots I_m$ the m intervals containing maximal decreasing sequences. For any interval I_j, we have that

$$\sum_{\mathrm{i} \in I_j} \alpha_\mathrm{i} \cdot p(1 - F_\mathrm{i}(p)) = \left(\sum_{\mathrm{i} \in I_j} \alpha_\mathrm{i} \right) \cdot p \left(1 - \frac{\sum_{\mathrm{i} \in I_j} \alpha_\mathrm{i} \cdot F_\mathrm{i}(p)}{\sum_{\mathrm{i} \in I_j} \alpha_\mathrm{i}} \right). \tag{2.1}$$

Thus we can group periods we know to be in the same interval, and reduce the intervals I_1 through I_m as new and equivalent problem with m periods, with mass of arrival $\overline{\alpha}_j$ and distribution function $\overline{F}_j$ defined as

$$\overline{\alpha}_j = \sum_{\mathrm{i} \in I_j} \alpha_\mathrm{i}, \quad \overline{F}_j(p) = \frac{\sum_{\mathrm{i} \in I_j} \alpha_\mathrm{i} \cdot F_\mathrm{i}(p)}{\sum_{\mathrm{i} \in I_j} \alpha_\mathrm{i}}.$$

In the reduced (and equivalent) problem, there are m monopoly prices $(\overline{p}_t)_{t=1}^m$ which can contain decreasing sequences. Continue the previous procedure as in Corollary 2.5 until there are no more decreasing sequences. By construction, this is the optimum of the reduced problem and by identifying the respective indices of the solution we can recover the optimal solution to the original problem.

Although the proof of why the solution achieved by Algorithm 1 is optimal is based on the reduction of periods, the algorithm does not employ this reduction. It simply takes the information about the prices $(p_t)_{t=1}^T$ being stored and uses the aggregate demand to calculate the monopoly prices for each maximal decreasing interval as in equation 2.1.

Observation 2.6 A natural question that arises is how many iterations does the *while* loop do in Algorithm 1. In each iteration, at least one maximal decreasing sequence is eliminated. The worst case scenario for the cycle is that it eliminates exactly one sequence in each iteration, and then another decreasing sequence is created with the updated prices. This is the case of a very high price in period one and an increasing sequence of low prices thereafter. Hence, the while loop iterations is bounded by T. Moreover, assuming that there is a subroutine that computes a good approximation of monopoly prices in logarithmic time complexity (such as binary search), Algorithm 1 has a worst time complexity of $O(T^2 \log(N)R(N))$ where $R(N)$ is the cost of calculating $R(p)/R'(p)$ for some revenue function, with N-digit precision.

Chapter 3

Families of distributions and monopoly prices

In this chapter we will study different stochastic orders to study the optimality of setting the monopoly price at all times. We observe different conditions known in the literature that do not guarantee optimality, while the decreasing hazard rate order property (DHR) condition is sufficient to guarantee optimality under buyers with or without time discounting. Moreover, the DHR condition can be restated for random variables of all types (continuous, discrete and mixed).

The relevance behind the fact that the monopoly price policy is increasing is that all opportunities for buyers to strategize are nullified. That is, a buyer with valuation v observes an increasing price policy, so his only rational choice is to buy immediately. The fact that the seller can set monopoly prices is particularly interesting since it indicates that he is earning as much as possible from each consumer in expectation. We will formalize these notions in the following section.

3.1 Preliminaries

Consider the same setting as in Chapter 2 in which a seller interacts with a mass of buyers, in which the seller is trying to sell an item to various consumers arriving in enumerated periods. A natural modification of the previous model is where buyers can now arrive randomly over a set in $\mathbb{R}_+$. An ideal scenario for the seller would be to set the monopoly prices in each period, but this price path is not necessarily increasing, so consumers wait for the lowest price to make their purchase. If the monopoly prices are non-decreasing, then consumers have no choice but to buy when they arrive (a later purchase obtains less utility) and the seller obtains the maximum expected revenue.

In this way, we formalize this modifications of the model in Chapter 2 (specifically, consumer behavior). Denote $\mathcal{T}$ as a subset of the non-negative real numbers, which will indicate the times in which the buyer will be able to arrive. Specifically consumers arrive according to an arbitrary distribution over $\mathcal{T}$, and if he arrives at time $t \in \mathcal{T}$, he has a private random valuation V_t over the item with distribution F_t with its support contained in $\mathbb{R}_+$. The seller knowing all the distributions F_t, posts a price policy of his own choice p_t for all $t \in \mathcal{T}$. A mass of costumers perform their arrival and valuation through a realization of the associated random variables, and decides whether to buy in the period in which he arrived or later according to his utility function $u(v,t) = v - p_t$. In simpler words, a consumer (with arrival time and valuation τ and v respectively) chooses the period to purchase $t \in \mathcal{T}$ that maximizes his utility $u(v,t)$, subject to $t \geq \tau$. The seller's problem is to maximize his expected revenue. We will not focus for now on the seller's problem (and its different continuous-discrete variations depending on the arrival of consumers) but on conditions that make the monopoly prices p_t^* non-decreasing.

We will refer to the *monopolistic pricing* policy $(p_t^*)_{t\in\mathcal{T}}$ as the one that announces at any time $t \in \mathcal{T}$ the price p_t^*. The goal is to characterize when the monopolistic pricing policy is non-decreasing, through properties over the family of distributions $\{F_t\}_{t\in\mathcal{T}}$. The non-decreasing property over the monopoly prices is very important, since it implies that buyers will not deviate to any other period and will buy at the instant t they arrive if their valuation is above p_t^*.

3.2 Stochastic Orderings

We introduce different notions of order of random variables taking values in $\mathbb{R}$. In this manner, we will see the effect of providing the family of distributions $\{F_t\}_{t\in\mathcal{T}}$ with certain properties.

Definition 3.1 *A family of cumulative distribution functions $\{F_t\}_{t\in\mathcal{T}}$ is said to be increasing in terms of first-order stochastic dominance (SD) if*

$$F_{t_1}(p) \leq F_{t_0}(p) \text{ for all } p \in \mathbb{R} \text{ and } t_1 > t_0.$$

In particular for $t_1 > t_0$, we say that F_{t_1} dominates F_{t_0} in the sense of first-order stochastic dominance.

It is also found in the literature as the usual stochastic order [17]. In the case of two random variables X and Y, the condition that Y dominates X can be written as

$$\mathbb{P}(X > p) \leq \mathbb{P}(Y > p), \text{ for all } p \in \mathbb{R}. \tag{3.1}$$

In simple words the SD condition says that X is less likely to take values higher than Y, where higher means any value larger than p, for any $p \in \mathbb{R}$. An interesting property is noted by integrating on both sides of the inequality (3.1). Since we are dealing with positive random variables, it is straightforward to verify that $\mathbb{E}(X) \leq \mathbb{E}(Y)$. In particular, for a family $\{F_t\}_{t\in\mathcal{T}}$ the first-order stochastic dominance condition indicates that valuations are increasing in expectation. That is, for any choice of $t_1 \leq \cdots \leq t_n$ of times in $\mathcal{T}$,

$$\mathbb{E}(V_{t_1}) \leq \cdots \leq \mathbb{E}(V_{t_n}).$$

We introduce another notion of stochastic order. Consider a positive random variable V with an absolutely continuous distribution F, with density f. We define the hazard rate λ as follows.

$$\lambda(p) := \frac{f(p)}{1-F(p)}, \text{ for all } p \text{ in supp } F. \tag{3.2}$$

The hazard rate is strongly related to the *virtual value function*, which has much relevance in the theory of optimal auctions. Recall that the virtual value of the random variable $V \sim F$ is defined as

$$\phi(p) = p - \frac{1-F(p)}{f(p)} = p - \frac{1}{\lambda(p)}.$$

Note that the condition $\phi(p) = 0$ is exactly the first order conditions for the revenue function $\Pi(p) = p(1 - F(p))$. That is, $\phi(p)$ can be interpreted as the marginal revenue of the seller. Introduced by Myerson [15] in his seminal work, the following condition is sufficient for a second-price auction with reserve price $\phi^{-1}(0)$ to be optimal in the symmetric buyer case.

Definition 3.2 *An absolutely continuous distribution function F with density f has monotone hazard rate (MHR) property if its hazard rate λ satisfies that*

$$\lambda(p) = \frac{f(p)}{1-F(p)} \text{ is non-decreasing in } p.$$

We will denote $\lambda_F(p)$ to make explicit that the hazard rate of the cdf F is being used. A family of distribution functions $\{F_t\}_{t\in\mathcal{T}}$ is said to satisfy the MHR property if each F_t has the MHR property.

In practice, there are many distributions that have an increasing hazard rate, such as the uniform, exponential and normal distributions. In particular, the exponential distribution has a constant hazard rate equal to the parameter of the exponential distribution.

The next stochastic order we will introduce is widely used in several areas of statistical theory, such as in hypothesis testing to produce uniformly most powerful tests (UMP). Consider the following experiment. Suppose we have two absolutely continuous random variables X and Y with probability densities f and g respectively. Moreover, suppose that the quotient $f(p)/g(p)$ is increasing in p. Knowing the previous probability densities, a realization of X or Y is observed, and the goal is to decide from which distribution the realization came. The monotonicity condition provides us with information once the realization has been observed. High values suggest that the realization came from density f, while low values from g. In fact, there is a critical value to decide from which distribution the realization came from by the Karlin-Rubin theorem (see Theorem 8.3.17 in [3]). We define the generalization of the previous experiment for a family of distributions.

Definition 3.3 *A family of distribution functions $\{F_t\}_{t\in\mathcal{T}}$ with $\mathcal{T} \subseteq \mathbb{R}_+$ is said to satisfy the monotone likelihood ratio (MLRP) property if for every $t \in \mathcal{T}$, F_t is absolutely continuous and*

$$\frac{f_{t_1}(p_1)}{f_{t_0}(p_1)} \geq \frac{f_{t_1}(p_0)}{f_{t_0}(p_0)} \text{ for all } p_1 > p_0 \text{ and } t_1 > t_0, \quad p_1, p_0 \in \mathbb{R}, \quad t_1, t_0 \in \mathcal{T}.$$

For certain values of p, it may happen that $f_t(p) = 0$. To avoid these complications we can use a natural and equivalent definition as in [8]: a family of distribution functions $\{F_t\}_{t\in\mathcal{T}}$

satisfy MLRP if the following inequality holds

$$f_{t_1}(p_1)f_{t_0}(p_0) \geq f_{t_1}(p_0)f_{t_0}(p_1), \quad \text{for all } p_1 > p_0 \text{ and } t_1 > t_0, \quad p_1, p_0 \in \mathbb{R}, \quad t_1, t_0 \in \mathcal{T}.$$

The next proposition gives a relation between MLRP, SD and a connection to hazard rates.

Proposition 3.4 *Assume the family $\{F_t\}_{t\in\mathcal{T}}$ satisfies MLRP. Then, the following conditions hold:*

1. *The family $\{F_t\}_{t\in\mathcal{T}}$ satisfy SD.*
2. *The family has decreasing hazard rates in t, i.e. $h_{t_2}(p) \leq h_{t_1}(p)$ for all $p \in \mathbb{R}$ and for all $t_1, t_2 \in \mathcal{T}$ such that $t_1 < t_2$.*

Proof. Since the inequality $f_{t_1}(p_1)f_{t_0}(p_0) \geq f_{t_1}(p_0)f_{t_0}(p_1)$ holds for all $p_1 > p_0$ and $t_1 > t_0$, fixing p_0 and integrating from p_0 to ∞ with respect to p_1 we arrive at

$$\begin{aligned} f_{t_0}(p_0)\int_{p_0}^{\infty} f_{t_1}(p_1)\mathrm{d}p_1 &= f_{t_0}(p_0)(1 - F_{t_1}(p_0)) \\ &\geq f_{t_1}(p_0)(1 - F_{t_0}(p_0)) = f_{t_1}(p_0)\int_{p_0}^{\infty} f_{t_0}(p_1)\mathrm{d}p_1. \end{aligned}$$

Now fix p_1 and integrate from $-\infty$ to p_1 with respect to p_0,

$$f_{t_1}(p_1)\int_{-\infty}^{p_1} f_{t_0}(p_0)\mathrm{d}p_0 = f_{t_1}(p_1)F_{t_0}(p_1) \geq f_{t_0}(p_1)F_{t_1}(p_1) = f_{t_0}(p_1)\int_{-\infty}^{p_1} f_{t_1}(p_0)\mathrm{d}p_0.$$

Noting that p_0 as well as p_1 were arbitrary, rearranging terms we conclude that

$$\frac{1 - F_{t_1}(p)}{1 - F_{t_2}(p)} \leq \frac{f_{t_1}(p)}{f_{t_2}(p)} \leq \frac{F_{t_1}(p)}{F_{t_2}(p)}, \text{ for all } p \in \mathbb{R}.$$

In particular, the above inequalities imply that $F_{t_2}(p)(1 - F_{t_1}(p)) \leq F_{t_1}(p)(1 - F_{t_2}(p))$ for all $p \in \mathbb{R}$, and this implies $F_{t_2}(p) \leq F_{t_1}(p)$ for all $p \in \mathbb{R}$. It follows that the family $\{F_t\}_{t\in\Theta}$ is SD. Taking the left-hand side of the above inequality and rearranging we obtain that $h_{t_2}(p) \leq h_{t_1}(p)$ for all $p \in \mathbb{R}$. The second claim follows. □

The second point of the previous proposition will be key in the proof of the theorem in the next section. This motivates us to define the families that fulfill this condition.

Definition 3.5 *A family of cumulative distribution functions $\{F_t\}_{t\in\mathcal{T}}$ is said to have the decreasing hazard rate order property (DHR) if*

$$h_{t_2}(p) \leq h_{t_1}(p) \text{ for all } p \in \mathbb{R} \text{ and for all } t_1, t_2 \in \mathcal{T} \text{ such that } t_1 < t_2.$$

In the next section we will see how these notions of order for probability distributions relate to the monopolistic pricing policy.

3.3 Monopolistic Pricing

As mentioned previously, in this section we will focus on the monopolistic pricing policy, which sets the price $p_t^* = \arg\max_p p(1 - F_t(p))$ in every time $t \in \mathcal{T}$. In particular, we provide a series of examples in which it can be observed that certain conditions on their own fail to guarantee that p_t^* is non-decreasing in t. This implies that by adding a discount on the utility function of the buyer that heavily penalizes waiting between periods, in some instances a strategic consumer with high valuation over the item will deviate from buying in the period they arrive, to buy in a later period. As a result we obtain sufficient conditions on the family $\{F_t\}_{t\in\mathcal{T}}$ to ensure that p_t^* is non-decreasing in t and hence nullify the strategic behaviour of a consumer.

Focusing only on a single period, if the seller announces a non-decreasing price policy we can think about a posted price setting, where the seller sets a price p over an item and then the buyer buys the item if his valuation is higher than or equal to p. In the case where the distribution F is differentiable with density f and has finite expectation, the revenue function is also differentiable and we can find its maximum among its stationary points, i.e., those that satisfy the equation $\frac{\mathrm{d}}{\mathrm{d}p}\Pi(p) = 0$. Via Markov's inequality it is easy to notice that the seller's expected revenue is also bounded by the expectation of the buyer's valuation. The following lemma summarizes what have been stated.

Lemma 3.6 (Lemma 2.10 in [16]) *In a posted price setting, the seller revenue is given by $\Pi(p) = p(1 - F(p))$ where F is the distribution of the buyer's valuation (which is a positive random variable). Assume that F is differentiable and has finite expectation. Then,*

$$\lim_{p\to\infty} \Pi(p) = 0.$$

Furthermore, Π is upper bounded by $\mathbb{E}_{V\sim F}(V)$ and the monopoly price p^* is a solution of

$$f(p^*)\left(p^* - \frac{1 - F(p^*)}{f(p^*)}\right) = 0.$$

The above lemma states that in the differentiable case the first order conditions are sufficient to find the monopoly price.

3.3.1 Insufficient conditions for the monopoly prices to be non-decreasing

In all the following counterexamples we set $\mathcal{T} = \{0, 1\}$. That is to say, there are only two periods on which we will focus.

We start with only the SD condition. Let $\varepsilon_1, \varepsilon_2 > 0$ such that $1/3 > \varepsilon_1, \varepsilon_2$. We will start by defining the piecewise distribution functions and then connect the pieces by linear interpolation. Let $F_0(p) = 1 - \varepsilon_1$ for $p \in [0, 2/3]$, $F_0(p) = 1$ for $p \in [2/3 + \varepsilon_2, +\infty)$ and in the missing interval $(2/3, 2/3 + \varepsilon_2)$ we will consider a linear interpolation with the values of

$F_0(2/3)$ and $F_0(2/3+\varepsilon_2)$. Specifically, the function F_0 is defined as

$$F_0(p) = \begin{cases} 0 & p < 0, \\ 1-\varepsilon_1 & 0 \le p < 2/3, \\ \frac{\varepsilon_1}{\varepsilon_2}(p-2/3)+1-\varepsilon_1 & 2/3 \le p < 2/3+\varepsilon_2, \\ 1 & 2/3+\varepsilon_2 \le p. \end{cases}$$

Now let $F_1(x) = p$ for $p \in [0, 2/3]$, $F_1(p) = 1$ for $p \in [2/3+\varepsilon_2, +\infty)$ and linear interpolation in the missing interval. Once again we can explicitly define F_1 as follows

$$F_1(p) = \begin{cases} 0 & p < 0, \\ p & 0 \le p < 2/3, \\ \frac{1}{3\varepsilon_2}(p-2/3)+2/3 & 2/3 \le p < 2/3+\varepsilon_2, \\ 1 & 2/3+\varepsilon_2 \le p. \end{cases}$$

In Figure 3.1 we consider $\varepsilon_1 = 10^{-1}$ and $\varepsilon_2 = 10^{-3}$. By construction $F_1(p) \le F_0(p)$ for all $p \in \mathbb{R}$ so the stochastic dominance condition holds in this case. Furthermore, one can check that $p_0^* = 2/3 > 1/2 = p_1^*$. This counterexample is interesting because although valuations are increasing in expectation, counter intuitively the monopolistic pricing policy is decreasing.

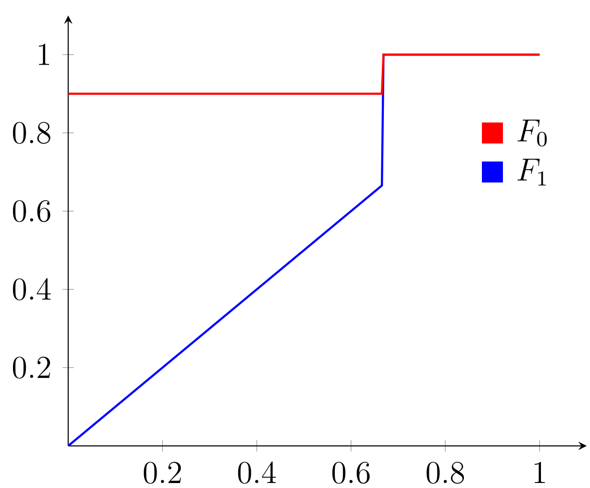

Figure 3.1: The monopolistic pricing policy values can be decreasing having only SD as hypothesis ($\varepsilon_1 = 10^{-1}, \varepsilon_2 = 10^{-3}$).

Now, if we only rely on the MHR property then we do not have that the monopolistic pricing policy is increasing either. Le F_0, F_1 the probability distributions of two exponential random variables with mean 1 and 2 respectively. That is, $F_0(p) = (1-\mathrm{e}^{-p})\mathbb{1}_{\{p\ge 0\}}$ and $F_1(p) = (1-\mathrm{e}^{-p/2})\mathbb{1}_{\{p\ge 0\}}$ (see Figure 3.2). Since the hazard rate of an exponential distribution is constant and equal to the exponential distribution parameter, both distributions verify non-decreasing hazard rates, but $p_0^* = 1 > 1/2 = p_1^*$. This is not surprising due to the fact that MHR is a property of the distribution and not of the family to which the function belongs.

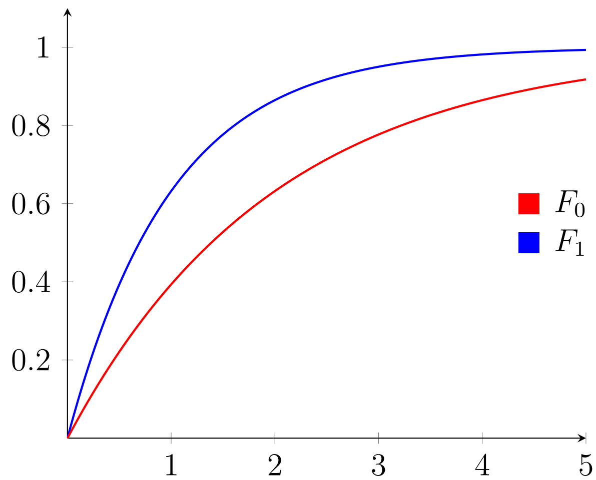

Figure 3.2: The monopolistic pricing policy values can be decreasing having only MHR as hypothesis.

Finally, if we consider both MHR and SD we cannot conclude monoticity of the monopolistic pricing policy either. Consider F_1 the cdf of an uniformly distributed random variable between $1/2$ and 1, and F_0 defined as follows

$$F_0(x) = \begin{cases} 0 & x < 0, \\ (\beta/\alpha)x & 0 \le x < \alpha, \\ F_1(x) & \alpha \le x. \end{cases}$$

where α and β are such that $2(\alpha - 1/2) = \beta$, for α, $\beta > 0$. This is simply a linear function with slope β/α in the interval $[0, \alpha]$, which then coincides with F_1 in a continuous fashion. For $\beta < 1$, it is straightforward to verify SD and MHR (F_0 is in fact convex, and thus f_0 is non-decreasing if we choose $f_0(\alpha) = f_1(\alpha)$, or any value in $[\beta/\alpha, f_1(\alpha)]$. Since $1 - F_0(p)$ is decreasing, it is ensured that h_{F_0} is non-decreasing). Letting $\beta = 1/10$, we obtain that $p_0^* = 11/20 > 1/2 = p_1^*$.

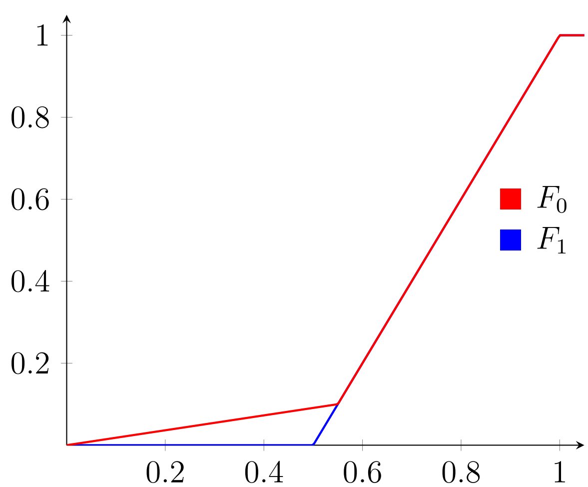

Figure 3.3: The monopolistic pricing policy values can be decreasing having SD and MHR as hypothesis ($\alpha = 11/20$, $\beta = 10^{-1}$).

This counterexample tells us that even if the variables are stochastically dominated and we have a notion of regularity (according to Myerson) we still cannot ensure the monoticity of the monopoly pricing policy.

3.3.2 Sufficient conditions for the monopoly prices to be non-decreasing

We now return to the usual setting, where $\mathcal{T}$ is any subset of $\mathbb{R}_+$. The goal of this section is to prove the following theorem.

Theorem 3.7 *Let $\{F_t\}_{t\in\mathcal{T}}$ be a family of distributions satisfying DHR. Then the monopolistic price policy p_t^* is non-decreasing.*

For completeness, we will write a result in which having the MHR and DHR hypotheses on the family of distributions gives us the desired monotonicity result using the first order conditions, providing an alternative simple proof.

Proposition 3.8 *If the family of cdfs $\{F_t\}_{t\in\mathcal{T}}$ satisfy DHR and MHR conditions, then the monopolistic price policy p_t^* is non-decreasing.*

Proof. In search of a contradiction, let us suppose that p_t^* is decreasing. Then, there exists t_1 and t_2 both in $\mathcal{T}$, $t_1 < t_2$ such that $p_{t_2}^* < p_{t_1}^*$. The first-order conditions associated with the maximization problem state that the optimum satisfies the following equation

$$1 - F_t(p_t^*) - f_t(p_t^*)p_t^* = 0,\ t = t_1, t_2.$$

Then for $t = t_1, t_2$, p_t^* satisfies $h_t(p_t^*) = 1/p_t^*$. Thus,

$$\frac{1}{p_{t_2}^*} = h_{t_2}(p_{t_2}^*) \le h_{t_1}(p_{t_2}^*) \le h_{t_1}(p_{t_1}^*) = \frac{1}{p_{t_1}^*}$$

where in the first inequality we used DHR and in the second one we used MHR by our assumption $p_{t_2}^* < p_{t_1}^*$. Therefore, rearranging terms we arrive at $p_{t_2}^* \ge p_{t_1}^*$, a contradiction. □

At first glance, it would appear that the MHR property was key in the above demonstration. We will now present a rather surprising result, in which we will not use MHR. In what follows, we will state two lemmas that will be key to the proof of the Theorem 3.7.

Lemma 3.9 *Let F and G be distributions such that $m_{F,G}(p) = \frac{1-F(p)}{1-G(p)}$ is non-decreasing in p. Then $p_G = \arg\max_p p(1 - G(p)) \le \arg\max_p p(1 - F(p)) = p_F$.*

Proof. Assume that $p_F < p_G$. By definition of p_F, for all $p \in \mathbb{R}$,

$$p(1 - F(p)) < p_F(1 - F(p_F)).$$

In particular we obtain $p_G(1 - F(p_G)) < p_F(1 - F(p_F))$. Since $p_F < p_G$ and $m_{F,G}(p)$ is non-decreasing

$$\frac{1 - F(p_F)}{1 - G(p_F)} \leq \frac{1 - F(p_G)}{1 - G(p_G)} \iff \frac{p_G(1 - G(p_G))}{p_F(1 - G(p_F))} \cdot p_F(1 - F(p_F)) \leq p_G(1 - F(p_G)).$$

By definition of p_G we know that $p_G(1 - G(p_G)) > p_F(1 - G(p_F))$, thus

$$p_F(1 - F(p_F)) < \frac{p_G(1 - G(p_G))}{p_F(1 - G(p_F))} \cdot p_F(1 - F(p_F))$$

and putting the last two inequalities together we obtain that $p_F(1-F(p_F)) < p_G(1-F(p_G))$, which contradicts the definition of p_F. □

Lemma 3.10 *Let F, G be absolutely continuous distributions, and let f, g be their respective probability density functions. Then $m_{F,G}(p)$ is non-decreasing if and only if $h_F(p) \leq h_G(p)$ for all $p \in \mathbb{R}$.*

Proof. It is straightforward to verify the following equivalences.

$$\begin{aligned} m_{F,G}(x) \text{ is non-decreasing in } p &\iff \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1 - F(p)}{1 - G(p)}\right) \geq 0 \\ &\iff -f(p)(1 - G(p)) + g(p)(1 - F(p)) \geq 0 \\ &\iff \frac{f(p)}{1 - F(p)} \leq \frac{g(p)}{1 - G(p)} \iff h_F(p) \leq h_G(p). \end{aligned}$$

□

With these lemmas, we are ready to see that MHR hypothesis was not necessary in the proof of Proposition 3.8.

Proof of Theorem 3.7. Let $t_1, t_2 \in \mathcal{T}$, such that $t_1 < t_2$. By definition of an DHR family, $h_{t_2}(p) \leq h_{t_1}(p)$ for all $p \in \mathbb{R}$. Using Lemma 3.10, $m_{F_{t_2},F_{t_1}}(p)$ is non-decreasing in p. Finally, by Lemma 3.9, $p_{t_1} < p_{t_2}$. □

Since we have already seen that MLRP implies DHR, we easily obtain the following corollary.

Corollary 3.11 *Let $\{F_t\}_{t \in \mathcal{T}}$ be a family of distributions satisfying MLRP. Then p_t^* is non-decreasing.*

Table 3.1 summarizes which properties are sufficient for monotony to be satisfied. Note that in the proof of the above theorem the key element was that $m_{F_{t_2},F_{t_1}}(p)$ is non-decreasing in p. Therefore, we can extend the result to any type of random variable (continuous, discrete and mixed) if the family of distributions $\{F_t\}_{t \in \mathcal{T}}$ satisfies that for any $t_1 < t_2$, the quotient $m_{F_{t_2},F_{t_1}}(p)$ is non-decreasing in p.

	MHR	No property
DHR	✓	✓
MLRP	✓	✓
SD	✗	✗
No property	✗	✗

Table 3.1: Properties under which the family $\{F_t\}_{t\in\mathcal{T}}$ make the monopolistic pricing policy non-decreasing.

It can be checked directly that $m_{F_{t_2},F_{t_1}}(p)$ being non-decreasing in p for all $t_1 < t_2$ is equivalent to $(1 - F_t(p_1))/(1 - F_t(p_2))$ is non-increasing in t, for all $p_1 > p_2$. This condition has already been studied by Zhao and Zheng [18] in their dynamic programming model. The interpretation of this last ratio is the conditional probability that a customer, given that he or she is willing to pay a lower price p_2, would buy at a higher price p_1. In other words, this condition holds if the probability that a customer is willing to pay a premium decreases over time.

3.3.3 Further generalizations

Theorem 3.7 can be used in a variety of models. An example of this can be for different utility functions for buyers, and adding intertemporal discounts for the buyers and the seller utility.

Consumer's utility functions The buyer's utility function $u(v, t) = v - p_t$ can be generalized to any positive utility function $\overline{u}$ increasing in $v - p_t$, If a consumer observes an increasing pricing policy, then a consumer that has a valuation v for the item and arrives on t, will purchase if and only if

$$u(v, t) = \overline{u}(v - p_t) \geq 0 \iff v - p_t \geq \overline{u}^{-1}(0), \tag{3.3}$$

where $\overline{u}^{-1}$ is the inverse of $\overline{u}$, which exists and is increasing since $\overline{u}$ is increasing. The seller's revenue of posting the price p_t at time t is $p_t \cdot \mathbb{1}_{\{v-p_t\geq\overline{u}^{-1}(0)\}}$. It follows that the expected revenue of the seller in the period t is $p_t(1 - F_t(p_t + \overline{u}^{-1}(0)))$. Thus, the goal of the seller is to solve

$$\max_p p(1 - F_t(p + \overline{u}^{-1}(0))).$$

Via the one-to-one change of variables $p = y - \overline{u}^{-1}(0)$, the previous problem is equivalent to solving

$$\max_y (y - \overline{u}^{-1}(0))(1 - F_t(y)) = \max_p (p - \overline{u}^{-1}(0))(1 - F_t(p)).$$

Hence, the reserve value $\overline{u}^{-1}(0)$ of the buyer produces an artificial valuation (or a cost of production per item) for the seller. Remember that we normalized the seller's valuation of the item to 0, and the proof of Theorem 3.7 is the same with the revenue functions $(p - c)(1 - F_t(p))$ for a constant $c \in \mathbb{R}_+$. Hence if $\{F_t\}_{t\in\mathcal{T}}$ has the DHR property the optimal pricing policy is exactly the monopolistic pricing policy, which is increasing. The explanation behind this behaviour is that the consumer has a reserve value $\overline{u}^{-1}(0)$ in the case of not buying the item, thus inducing a negative effect on the seller's revenue. We will see in Chapter 4 (Proposition 4.8) that a greater reserve value produces an increase in prices.

Intertemporal discounts The importance of nullifying strategic behavior allows us to incorporate any positive non-increasing intertemporal discounts $\alpha(t)$ and $\beta(t)$ for the consumer's and seller's utility function. Specifically, the following consumer function is considered as

$$u(v,t) = \alpha(t) \cdot \overline{u}(v - p_t)$$

where $\overline{u}$ is a positive and increasing function that without loss of generality $\overline{u}^{-1}(0) = 0$. The seller's objective function can be considered for random[1] consumer arrivals, but for simplicity let us consider the objective function of SP with $\beta(t)$ as a discount factor. For the case of an non-decreasing pricing policy, the objective functions becomes

$$\mathrm{Rev}(p) = \sum_{t=1}^{T} \beta(t) \cdot p_t(1 - F_t(p_t)).$$

The monopolistic pricing policy $(p_t^*)_{t=1}^T$, if feasible, it is clearly optimal because it maximizes each summand independent of the discount factor $\beta(t)$. Furthermore, a similar argument as in inequality 3.3 leads to a consumer with valuation v and arriving in period t, buying the item if and only if $v \geq \overline{u}^{-1}(0) + p_t$ and this condition is independent of the intertemporal discount $\alpha(t)$.

[1]'Random' here means any distribution of arrival G. To make this clear, see the objective function of GFCP(K) in Section 4.

Chapter 4

Finite Capacity Problem

In this section we will study the effect that capacity has with respect to prices and revenue for the seller in the problem of Chapter 2. There are two ways of approaching at this problem, viewing capacity as a decision variable or as a constraint either exogenously or endogenously derived. Our results focus on the latter variation of the problem. There are several examples where an interpretation can be given to the above problem, such as limited storage capacity and budget constraints.

First we will start by focusing on the difficulty of the rationing problem as a decision variable with strategic consumers since the price curves may be decreasing. After this, we will then study the seller's problem under a capacity constraint in depth, characterize its optimal solutions by means of the dual problem, and analyze the properties that they satisfy when varying the capacity.

4.1 Capacity as a decision variable

Consider the following setting: there are two periods, in the first period 1000 neutral to risk consumers arrive with a deterministic value of 100 for the object, and in the second period 3000 neutral to risk consumers arrive with a deterministic value of 50 for the object. The seller decides the prices p_1, p_2 in each of the two periods and how many items K he will sell. On one hand, it is simple to note that the optimal non-decreasing price policy (setting infinite capacity) is the constant price equal to 50, with optimal revenue equal to $\text{Rev}_{NC} = 2 \cdot 10^5$. On the other hand, if we consider capacity and decreasing sequences, we can obtain a higher revenue than in the previous case.

Let us take the pricing policy that sets $p_1 = 60$ in the first period and $p_2 = 50$ in the second. The seller also announces the public availability of only $K = 2000$ units for sale and completely commits to this availability. First-period consumers know that second-period consumers will buy all units left, so they anticipate and react to this behaviour so they have two options; the first is to buy on the period on they arrive and obtain 40 utils, or deviate and purchase on the second period. It is easy to see that the probability π of getting the item in the second period is upper bounded by 2/3, the best case for consumers in the second period (assuming that there are zero consumers in period 1). Thus, comparing expected utilities of a consumer of the first period of purchasing in period 2 and period 1 respectively:

$$50 \cdot \pi \leq \frac{100}{3} < 40.$$

Hence is always better for a consumer in the first period to buy when they arrive. With this we can compute the revenue of the seller. Let Rev_C the optimal revenue in the previous problem with capacity as a decision variable. Obviously, the pricing policy $(60, 40)$ may be not optimal but is sufficient to bound Rev_{NC}. More precisely,

$$\mathrm{Rev}_{NC} = 2 \cdot 10^5 < 60 \cdot 1000 + 50 \cdot 3000 = 2.1 \cdot 10^5 \leq \mathrm{Rev}_C.$$

Therefore capacity can be used to generate more revenue, as long as the capacity is public information for consumers. In general, the problem with more periods becomes harder and it is necessary to study the expected continuation payment of the game for consumers, who depend on the action of the rest of the consumers. This problem is particularly difficult to solve and go beyond the scope of this thesis. For further reading in this topic, see [11].

4.2 Capacity as a constraint

We will now study the finite capacity problem as a constraint imposed on the seller, which is only information available to the seller. For now, we will assume that consumers are myopic and therefore buy as soon as they arrive as long as the price they set in the period is above their valuation, otherwise leave and receive zero utility. An analogous way of thinking about myopic consumers is for the seller to have the power to do discriminate pricing. That is, for consumers arriving in period t he offers the product at price p_t as a take-it-or-leave-it offer (after the seller announces the price path $(p_t)_{t=1}^T$). The process of arrival of consumers is the same as in Chapter 2, except that the quantity of consumers that arrive in period t can be generalized to a random variable A_t with finite mean α_t independent of their valuation V_t. The good is infinitely divisible, and in each period a random mass of consumers A_t demand the object according to their distribution of valuation F_t and the price set. The seller wants to maximize his expected revenue subject to an *expected capacity/supply* constraint of K units. That is, if he announces a pricing policy $(p_t)_{t=1}^T$ the expected capacity constraint is (using independence of A_t and V_t)

$$\mathbb{E}\left[\sum_{t=1}^{T} A_t \cdot \mathbb{1}_{\{V_t \geq p_t\}}\right] \leq K \iff \sum_{t=1}^{T} \alpha_t \cdot [1 - F_t(p_t)] \leq K.$$

And his objective function is $\mathbb{E}[\sum_{t=1}^T A_t \cdot p_t \mathbb{1}_{\{V_t \geq p_t\}}] = \sum_{t=1}^T \alpha_t \cdot p_t[1 - F_t(p_t)]$. Thus, the problem of revenue maximization under the capacity constraint of K units for sale, and myopic buyers is stated as

$$\begin{aligned} \max_{p_1,\dots,p_T} \quad & \sum_{t=1}^{T} \alpha_t \cdot p_t[1 - F_t(p_t)] && \mathrm{FCP}(K) \\ \text{s.t.} \quad & \sum_{t=1}^{T} \alpha_t \cdot [1 - F_t(p_t)] \leq K. \end{aligned}$$

Certainly, if the value of K is high the constraint may not be active. Specifically, if the value of K is greater than $\sum_{t=1}^{T} \alpha_t \cdot [1 - F_t(p_t^*)]$ where p_t^* is the monopoly price under the distribution F_t then the constraint is non-binding and the optimal solution of FCP(K) is $(p_t^*)_{t=1}^T$.

Throughout this section we will use the term strategic consumers for consumers who maximize their utility given the price path announced by the seller. That is, a utility-maximizing consumer is who choose the optimal time $t \geq \tau$ at which to buy the item, given his arrival at time τ. Naturally, if there is no price at which they receive positive utility, consumers in this condition receive zero utility and do not make any purchase.

Without loss of generality, we normalize the sum of arrival masses to 1, i.e. $\sum_{t=1}^{T} \alpha_t = 1$ and assume that every α_t is strictly positive. In the following we will use lagrangian duality and monotone comparative statics to study the optimal solutions of FCP(K).

4.2.1 Dual problem and sufficient conditions

We introduce the dual problem of FCP(K). In this way, we introduce the lagrange multiplier $z \in \mathbb{R}_+$ associated with the expected capacity constraint. The Lagrangian function $\mathcal{L} : \mathbb{R}^T \times \mathbb{R} \to \mathbb{R}$ is given by

$$\begin{aligned}
\mathcal{L}(p, z) &= \sum_{t=1}^{T} \alpha_t \cdot p_t[1 - F_t(p_t)] + z\left(K - \sum_{t=1}^{T} \alpha_t \cdot [1 - F_t(p_t)]\right) \\
&= zK + \sum_{t=1}^{T} \alpha_t \cdot (p_t - z)[1 - F_t(p_t)].
\end{aligned}$$

Define the Lagrangian dual function $\psi_K : \mathbb{R} \to \mathbb{R}$ as

$$\psi_K(z) = \sup_{p_1,\ldots,p_T} \mathcal{L}(p, z) = zK + \sum_{t=1}^{T} \alpha_t \cdot \mathcal{R}_t(z)$$

where $\mathcal{R}_t(z) = \sup_p (p - z)[1 - F_t(p)]$. Note that, for $z \geq 0$, $(p_t)_{t=1}^T$ feasible in FCP(K) and by definition of supremum

$$\begin{aligned}
\psi_K(z) &\geq \sum_{t=1}^{T} \alpha_t \cdot p_t[1 - F_t(p_t)] + z\left(K - \sum_{t=1}^{T} \alpha_t \cdot [1 - F_t(p_t)]\right) \\
&\geq \sum_{t=1}^{T} \alpha_t \cdot p_t[1 - F_t(p_t)]
\end{aligned}$$

By the previous inequality, it follows that for any $z \geq 0$ the lagrangian dual function yields upper bounds on the optimal value val(FCP(K)). That is to say,

$$\text{val}(\text{FCP}(K)) \leq \psi_K(z), \text{ for all } z \geq 0.$$

Hence, the Lagrangian dual problem of FCP(K) is

$$\begin{aligned} \inf_{z} \quad & \psi_K(z) = zK + \sum_{t=1}^{T} \alpha_t \cdot \mathcal{R}_t(z) \\ \text{s.t.} \quad & z \geq 0. \end{aligned} \tag{$\Gamma(K)$}$$

We now present a sufficient condition under the family of distributions F_t, in order to solve $\Gamma(K)$. Moreover, with the solution of the dual problem we can find an explicit solution to the primal problem.

Assumption 2 For each $z \geq 0$ and $t \in [T]$, the function $\mathcal{R}_t(z) = \sup_p (p - z)(1 - F_t(p))$ is differentiable as a function of z. In addition, as least one of the following three conditions is satisfied for each $t \in [T]$:

1. The support of F_t is contained in $[0, v_t]$, for some constant v_t. This is, $\operatorname{supp} F_t \subseteq [0, v_t]$.
2. $\mathbb{E}_{V \sim F_t}[V] < \infty$.
3. The supremum of the revenue in period t is upperly bounded by a constant v_t. That is, $\sup_p p(1 - F_t(p)) \leq v_t$.

The goal of Assumption 2 is to bound $\psi_K(0)$ and to use the well known envelope theorem. Under this assumption, define $\overline{v} := \max_t \{\overline{v}_t \,|\, t \in [T]\}$. Clearly, the first point leads directly of bounding the expectation $\mathbb{E}_{V \sim F_t}[V]$ of each periods via an application of Hölder inequality. If $\mathbb{E}_{V \sim F_t}[V]$ is finite, we can bound the revenue of each period by a constant v_t and hence by $\overline{v}$ (See Lemma 3.6).

Now we prove under these conditions, the existence of an optimal solution to the lagrangian dual problem.

Lemma 4.1 *Suppose Assumption 2 holds. Then, for all $K > 0$, there exists $z(K) < \infty$ an optimal solution to the dual problem $\Gamma(K)$.*

Proof. Since for each t and $z \geq 0$ the function $\mathcal{R}_t(z)$ is differentiable, it is also continuous for each t and $z \geq 0$. Hence, by the algebra of continuous functions $\psi_K(z)$ is also a continuous function. We will prove that

$$\inf_{z \geq 0} \psi_K(z) = \inf_{0 \leq z \leq \overline{z}_K} \psi_K(z)$$

where $\overline{z}_K = \overline{v}/K$. In fact, consider $z' \geq \overline{z}_K$, then

$$\psi_K(z') \geq Kz' > K \cdot \frac{\overline{v}}{K} = \overline{v} \geq \psi_K(0),$$

where we used the positivity of $\mathcal{R}_t(z)$ and assumption 2 to bound $\psi_K(0)$:

$$\psi_K(0) = \sum_{t=1}^{T} \alpha_t \cdot \mathcal{R}_t(0) \leq \overline{v} \sum_{t=1}^{T} \alpha_t \leq \overline{v}.$$

The latter inequality follows from noting that the sum of masses of arrival is normalized to 1. We conclude that the infimum value of the continuous function ψ_K has to lie in the interval $[0, \overline{z}_K]$, which is a compact set. Finally, by Weierstrass extreme-value theorem ψ_K attains a minimum $z(K) \in [0, \overline{z}_K]$. $\square$

Theorem 4.2 *Suppose that Assumption 2 holds. Then, there is a pair of* $\big((p_t(K))_{t=1}^T, z(K)\big)$ *primal-dual optimal solutions to FCP(K) and* $\Gamma(K)$*.*

Proof. Take $z(K)$ as in Lemma 4.1, and define $p(K) = (p_t(K))_{t=1}^T$ as $p_t(K) = \arg\max_p (p - z(K))(1 - F_t(p))$ for each $t \in [T]$. By Proposition 5.1.5 in [1] we have to check four conditions to assure optimality of some primal-dual solutions. That is, we need to check primal and dual feasibility, Lagrangian optimality and complementary slackness.

1. *Primal and dual feasibility:* Dual feasibility follows because by construction $z(K) \geq 0$. By the envelope theorem (see, by e.g. Theorem 1 in [13]), we have that $\mathcal{R}_t'(z(K)) = -(1 - F_t(p_t(K)))$ and then

$$\psi_K'(z(K)) = K + \sum_{t=1}^T \alpha_t \mathcal{R}_t'(z(K)) = K - \sum_{t=1}^T \alpha_t (1 - F_t(p_t(K))).$$

Since $z(K)$ is dual optimal solution, and the feasible region is convex in the dual problem, by Proposition 2.1.2 in [1] the first order conditions of ψ_K can be written as

$$\psi_K'(z(K))(z - z(K)) \geq 0, \quad \forall z \geq 0. \tag{4.1}$$

Taking $z > z(K)$, it must be that $\psi_K'(z(K)) \geq 0$. Thus,

$$\psi_K'(z(K)) = K - \sum_{t=1}^T \alpha_t (1 - F_t(p_t(K))) \geq 0.$$

Primal feasibility follows.

2. *Lagrangian optimality:* By definition,

$$\begin{aligned}\arg\max_p \mathcal{L}(p, z(K)) &= \arg\max_{p_1,\ldots,p_T} z(K)K + \sum_{t=1}^T \alpha_t \cdot (p_t - z(K))[1 - F_t(p_t)] \\ &= \arg\max_{p_1,\ldots,p_T} \sum_{t=1}^T \alpha_t \cdot (p_t - z(K))[1 - F_t(p_t)]\end{aligned}$$

Since the masses of arrival α_t are fixed and positive, and noting that each p_t^* maximizes its respective summand, thus $\mathrm{argmax}_p\, \mathcal{L}(p, z(K)) = (p_t(K))_{t=1}^T$.

3. *Complementary slackness:* We must check that

$$z(K)\left(K - \sum_{t=1}^T \alpha_t \cdot [1 - F_t(p_t^*)]\right) = 0.$$

If $z(K) = 0$ complementary slackness follows. Otherwise, $z(K)$ must be positive. We can take $\varepsilon > 0$ small enough with $z(K)+\varepsilon$ and $z(K)-\varepsilon$ belonging to $\mathbb{R}^*_+$, and plugging in this values in equation (4.1) we obtain that $\psi'_K(z(K)) = 0$. This is

$$\sum_{t=1}^{T} \alpha_t \cdot [1 - F_t(p_t(K))] = K,$$

and complementary slackness follows, which completes the proof.

□

This result indicates an interesting connection between the capacity and prices in each period. A naive approach to the FCP(K) problem may be to raise the prices of the periods where less revenue is produced until the expected capacity constraint is met, maintaining the revenue in the periods where more revenue is earned. Theorem 4.2 implies that this is not optimal. Capacity becomes a production cost or item valuation for the seller, affecting all periods with this value given by the optimal dual variable $z(K)$.

The previous problem allows us to solve the period capacity problem easily; consider the same setting as FCP(K), except for the expected capacity constraint, which is replaced by T capacity constraints for each period. The formulation is as follows.

$$\begin{aligned} \max_{p_1,\ldots,p_T} \quad & \sum_{t=1}^{T} \alpha_t \cdot p_t[1 - F_t(p_t)] \qquad\qquad \text{FCP}(K_1,\ldots,K_T) \\ \text{s.t.} \quad & \alpha_t \cdot [1 - F_t(p_t)] \le K_t, \text{ for each } t \in [T]. \end{aligned}$$

One way to use Theorem 4.2 to solve the problem is as follows. First, let us note that

$$\begin{aligned} & \max_{p_1,\ldots,p_T} \left\{ \sum_{t=1}^{T} \alpha_t \cdot p_t[1 - F_t(p_t)] \,\middle|\, \alpha_t \cdot [1 - F_t(p_t)] \le K_t, \forall t \in [T] \right\} \\ \le \quad & \sum_{t=1}^{T} \max_{p_t} \left\{ \alpha_t \cdot p_t[1 - F_t(p_t)] \,\middle|\, \alpha_t \cdot [1 - F_t(p_t)] \le K_t \right\}. \end{aligned}$$

Focusing on the latter T problems, in each period t the solution is to set a price such that $p_t = \arg\max_p (p - z(K_t))(1 - F_t(p))$, for some dual optimal solutions $z(K_t)$ associated with each period capacity restriction. The pricing policy $(p_t)_{t=1}^T$ is feasible in FCP($K_1, \ldots, K_T$) and optimal because that pricing policy makes the previous inequality hold as an equality.

Observation 4.3 If we drop the hypothesis of consumers being myopic the FCP(K) problem changes drastically. The main issue is that the objective function does not capture strategic behaviour such as in SP. However, a non-decreasing pricing policy nullifies the strategic behaviour as seen in Chapter 2. In virtue of Theorem 3.7, if the family of distributions $\{F_t\}_{t=1}^T$ satisfies the DHR property, then

$$p_t(z(K)) = \arg\max_p \, (p - z(K))(1 - F_t(z(K)))$$

is non-decreasing in t, leading to an optimal non-decreasing pricing policy in the FCP(K) problem.

However, the pricing policy $p_t(z(K_t)) = \arg\max_p (p - z(K_t))(1 - F_t(p))$ is not necessarily non-decreasing since the dual optimal solutions $z(K_t)$ are not necessarily the same. Therefore, even under DHR the previous pricing policy may not be optimal in FCP($K_1, \ldots, K_T$).

4.2.2 Generalized Capacity Problem

The model introduced at the beginning of this section can be generalized considering continuous time, where buyers arrive according to a known distribution G with support over the interval $[0, T]$. The goal of this subsection, is to describe and to prove that, under mild conditions under the family of distributions $\{F_t\}_{t\in[0,T]}$, results stated in Theorem 4.4 still hold. Formally, we consider the measurable space $([0, T], \mathcal{B}([0, T]))$ equipped with the measure induced by G, dG. In this variation the seller commits to a price function $p : S_G \to \mathbb{R}_+$ knowing the arrival distribution G and the distributions of valuation F_t, where $S_G \subseteq [0, T]$ is the support of G. Consumers are still assumed myopic, and the value of the item for the seller is zero. In any time t, conditional on a consumer arrival and a borel measurable pricing policy $p(\cdot)$, the instantaneous expected revenue is given by $p(t)[1 - F_t(p(t))]$. Hence, the *generalized finite capacity problem* of capacity K can be stated as

$$\begin{aligned} \max_{p(\cdot)} \quad & \int_0^T p(t)[1 - F_t(p(t))] \, \mathrm{d}G(t) \qquad \text{(GFCP(K))} \\ \text{s.t.} \quad & \int_0^T [1 - F_t(p(t))] \, \mathrm{d}G(t) \leq K, \end{aligned}$$

where both integrals are in the Lebesgue–Stieltjes sense with respect to the induced measure dG. For instance, we can recover FCP(K) setting G as the distribution of a random variable Y such that $\mathbb{P}(Y = t) = \alpha_t$ for every $t \in [T]$ and $\sum_{t\in[T]} \alpha_t = 1$. Under assumptions on the family of distributions $\{F_t\}_{t\in S_G}$, we can derive analogous results in which the proofs are also analogous, exchanging the summation sign for an integral sign.

Assumption 3 For each $z \geq 0$, $\mathcal{R}_t(z) = \sup_p (p - z)(1 - F_t(p))$ is measurable as a function of t and for each $t \in S_G$, $\mathcal{R}_t(z)$ differentiable as a function of z. In addition, at least one of the following three conditions is satisfied:

1. For each $t \in S_G$, the support of F_t is contained in $[0, \overline{v}]$, for some constant $\overline{v}$ independent of t.
2. $\sup_{t\in S_G} \mathbb{E}_{V\sim F_t}[V] = \overline{v} < \infty$. This is, $\operatorname{supp} F_t \subseteq [0, \overline{v}]$ for all $t \in S_G$.
3. For each $t \in S_G$, the supremum of the revenue at time t is upperly bounded by a constant $\overline{v}$ independent of t. That is, $\sup_p p(1 - F_t(p)) \leq \overline{v}$, $t \in S_G$.

Under a continuity or measurability assumption, we can also derive the Lagrangian dual problem in a similar manner to the previous section:

$$\begin{aligned} \inf_{z} \quad & \psi_K(z) = zK + \int_0^T \mathcal{R}_t(z)\,\mathrm{d}G(t) \\ \text{s.t.} \quad & z \geq 0. \end{aligned} \tag{G$\Gamma(K)$}$$

For the dual problem to take the above form, it is necessary that

$$\sup_{p(\cdot)} \int_0^T (p(t) - z)[1 - F_t(p(t))]\,\mathrm{d}G(t) = \int_0^T \sup_{p(t)} \{(p(t) - z)[1 - F_t(p(t))]\}\,\mathrm{d}G(t),$$

where the first supremum is taken over all borel measurable functions. This equality holds if for every $t \in [0, T]$, the function $R_z(p) = (p - z)[1 - F_t(p)]$ is upper-semicontinuous (see Section 2 in [7]). Since the LHS is always less than or equal to the RHS, another condition for this inequality to hold is that the function $p(t) = \arg\max_p (p - z)[1 - F_t(p)]$ is borel measurable. From now on, we will assume this latter condition.

It is again verified that every point in Assumption 3 implies the following one, and that these conditions are to prove an analogue of Lemma 4.1, the existence of a solution $z(K)$ for G$\Gamma(K)$. Although the proof is similar, we need the continuity of ψ_K. After this, if we continue the steps to prove that $z(K)$ and $p_t(z(K))$ are optimal solutions of the dual and primal problem, to verify primal feasibility it is necessary to differentiate the function ψ_K, which depends on a Lebesgue-Stieltjes integral. Under Assumption 3, ψ_K is differentiable (and then continuous) as a function of z but the proof is technical and is based on the measure theory statement of the Leibniz integral rule (see Theorem 6.28 in [9]). Three hypotheses need to be verified:

1. For any $z \geq 0$, the map $t \to R_t(z)$ is integrable with respect to $\mathrm{d}G$: By Assumption 3 and the inequality

$$0 \leq R_t(z) \leq R_t(0) = \sup_p p(1 - F_t(p)) \leq \overline{v}, \qquad \text{for all } z \geq 0.$$

   Integrating yields the result.

2. For almost all $t \in [0, T]$, the map $t \to R_t(z)$ is differentiable: This comes from Assumption 3, since S_G is a set with measure 1 with respect to $\mathrm{d}G$.

3. There is an integrable map h with respect to $\mathrm{d}G$, such that $|\frac{\partial}{\partial z} R_t(z)| \leq h(z)$ almost surely for all $z \geq 0$: For each $t \in S_G$, by the envelope theorem

$$\left| \frac{\partial}{\partial z} R_t(z) \right| = |-[1 - F_t(p_t(z(K)))]| \leq 1.$$

   Then, the constant map equal to 1 bounds $|\frac{\partial}{\partial z} R_t(z)|$ and it is clearly integrable with respect to $\mathrm{d}G$.

Therefore ψ_K is differentiable, there is a dual optimal solution $z(K)$ and $\psi_K{}'(z(K))$ can be computed in the same way as in Theorem 4.2. Repeating the steps in the proof of Theorem 4.2, we arrive at the following theorem for the general case (proof is omitted).

Theorem 4.4 *Suppose that Assumption 3 holds. Then, there is a pair of* $((p_t(K))_{t\in S_G}, z(K))$ *primal-dual optimal solutions to GFCP(K) and* $G\Gamma(K)$*. Furthermore, for each* $t \in S_G$

$$p_t(z(K)) = \arg\max_p (p - z(K))(1 - F_t(p)).$$

Likewise, Observation 4.3 remains true for the general case if we endow families with the DHR property. That is, $p_t(z(K))$ as defined in Theorem 4.4 is increasing and therefore in a setting where consumers are strategic and there is an expected capacity restriction of K units, $p_t(z(K))$ is optimal.

4.2.3 Properties of the optimal solutions

As we constructed the optimal solutions of each problem, we can analyze its monotonicity properties. Milgrom and Shannon in 1997 [14] established the necessary and sufficient conditions for $\arg\max_{x\in S} f(x,t)$ to be non-decreasing in t given a set S. The usual theory considers X as a lattice and T as a partially ordered set. In our setting, X and T are subsets of $\mathbb{R}$.

Definition 4.5 *A function* $f : X \times T \to \mathbb{R}$ *satisfies the single-crossing condition in* (x,t) *if, for all* $x, x' \in X$ *and* $t, t' \in T$ *such that* $x' > x''$ *and* $t' > t''$

$$\begin{aligned} f(x', t'') > f(x'', t'') &\implies f(x', t') > f(x'', t'), \\ f(x', t'') \geq f(x'', t'') &\implies f(x', t') \geq f(x'', t'). \end{aligned}$$

We can simplify the above condition in the case of having only two functions. Consider $f, g : X \to \mathbb{R}$ such that

$$\begin{aligned} f(x') > f(x'') &\implies g(x') > g(x''), \\ f(x') \geq f(x'') &\implies g(x') \geq g(x''). \end{aligned}$$

Then we say that g dominates f by the single-crossing property and we denote $g \succeq_{sc} f$. This is useful to avoid overloading notation with many parameters to just focus on two functions, and the single crossing conditions reduces to proving $f(\cdot, t') \succeq_{sc} f(\cdot, t'')$ for every $t' > t''$. Since we are dealing only with subsets of $\mathbb{R}$, we can get rid of certain conditions that are necessary in any other case such as a property known as quasi-supermodularity. The only necessary and sufficient condition in our setting for the $\arg\max_{x\in S} f(x,t)$ to be non-decreasing in t is the so-called single-crossing property. Under these conditions, we state a special case of Theorem 4 in [14].

Theorem 4.6 (Milgrom-Shannon) *Suppose* $X \subseteq \mathbb{R}$*, and* f, g *real valued functions. Then* $\arg\max_{x\in S} g(x) \geq \arg\max_{x\in S} f(x)$ *for any* $S \subseteq X$ *if and only if* $g \succeq_{sc} f$.

Consider the dual optimal solution of the problem $\Gamma(K)$, given $K > 0$ (when it is unique). It is natural to expect that the more restrictive the capacity constraint, i.e. lower values of K, the dual variable will increase by the way it acts in the optimal pricing policy. For $K > 0$, we define the *dual application* $z : \mathbb{R}_+ \to \mathbb{R}_+$ as the function that takes a positive number K (a capacity) and returns the optimal solution of the dual problem $\Gamma(K)$. This application is well-defined if the solution of the dual problem is unique. This definition can also be extended to the generalized problem $\mathrm{G}\Gamma(K)$. We show the aforementioned monoticity property of $z(\cdot)$ in the following proposition.

Proposition 4.7 *If well-defined, the dual application is non-increasing in $\mathbb{R}_+$ for both problems $\Gamma(K)$ and $G\Gamma(K)$. Furthermore, supposing Assumption 2 holds, the dual application is continuous.*

Proof. The proof is the same for both problems, considering the respective dual function ψ_K. Consider $K_2 \geq K_1 \geq 0$. We must prove that $z(K_1) \geq z(K_2)$. Note that by definition of $z(\cdot)$, this is equivalent to

$$\begin{aligned} z(K_1) \geq z(K_2) &\iff \arg\min_{z\geq 0} \psi_{K_1}(z) \geq \arg\min_{z\geq 0} \psi_{K_2}(z) \\ &\iff \arg\max_{z\geq 0} -\psi_{K_1}(z) \geq \arg\max_{z\geq 0} -\psi_{K_2}(z) \end{aligned}$$

We prove the latter. By Theorem 4.6 it suffices to prove that $-\psi_{K_1} \succeq_{sc} -\psi_{K_2}$. Let $z' > z'' \geq 0$ and assume that $-\psi_{K_2}(z') + \psi_{K_2}(z'') \geq 0$. It is straightforward to verify equality $\psi_{K_1}(z) = \psi_{K_2}(z) - z(K_2 - K_1)$, for all $z \geq 0$. In view of this,

$$\begin{aligned} \psi_{K_1}(z'') &= \psi_{K_2}(z'') - z''(K_2 - K_1) \\ &\geq \psi_{K_2}(z') - z''(K_2 - K_1) \\ &\geq \psi_{K_2}(z') - z'(K_2 - K_1) \\ &= \psi_{K_1}(z'). \end{aligned}$$

Analogously, assuming that $-\psi_{K_2}(z') + \psi_{K_2}(z'') > 0$ leads to $-\psi_{K_1}(z') + \psi_{K_1}(z'') > 0$. This proves that $-\psi_{K_1} \succeq_{sc} -\psi_{K_2}$ and then $z(\cdot)$ is non-increasing.

The fact that $z(\cdot)$ is continuous is a simple application of Berge Maximum Theorem (Theorem 17.31 in [6]). We have already seen in Lemma 4.1 that $z(K)$ is a solution of

$$\min_{z\in[0,\overline{z}_K]} \psi_K(z) = -\max_{z\in[0,\overline{z}_K]} -\psi_K(z).$$

Since the correspondence $\varphi : \mathbb{R}_+ \rightrightarrows \mathbb{R}_+$ defined by $\varphi(K) = [0, \overline{z}_K]$ for all $K \in \mathbb{R}_+$ is clearly continuous with non-empty compact values, then the arg max correspondence is upper-hemicontinuous. If $z(\cdot)$ is well-defined, then the arg max correspondence is singleton-valued and thus continuous[1]. Noting that

$$\{z(K)\} = \arg\min_{z\in[0,\overline{z}_K]} \psi_K(z) = \arg\max_{z\in[0,\overline{z}_K]} -\psi_K(z),$$

it follows that the dual application is continuous in K. □

[1]This a simple fact of singleton-valued correspondences. See Lemma 17.6 in [6].

One related question comes with the dual application being decreasing. Let us denote as $p_t(z(K))$ the optimal price of period t of the FCP(K) problem (or GFCP(K) problem), given the capacity K. The question that arises is how optimal pricing policies are ordered while increasing capacity.

Proposition 4.8 *If $z(K) > 0$, then $p_t(z(K)) \geq p_t^*$ for all $t \in [T]$. Furthermore, if $K_1 \leq K_2$ then $p_t(z(K_2)) \leq p_t(z(K_1))$ for all $t \in [T]$. The proposition also holds in GFCP(K) exchanging $[T]$ with S_G.*

In other words, optimal price curves stack on top of each other if we consider more restrictive inventories.

Proof. Again, the proof is the same for both statements. Fix any t, and consider $R_{z(K)}(p) = (p - z(K))(1 - F_t(p))$ the revenue function that takes into account the artificial value for the item that produces the capacity K. Since $z(K)$ is decreasing in K, it suffices to prove that $R_{z(K_1)} \preceq_{sc} R_{z(K_2)}$. Let $p' > p''$ and assume that $R_{z(K_1)}(p') - R_{z(K_1)}(p'') \geq 0$. Note that

$$\begin{aligned}
R_{z(K_2)}(p') - R_{z(K_2)}(p'') &= (p' - z(K_2))(1 - F_t(p')) - (p'' - z(K_2))(1 - F_t(p'')) \\
&= (p' - z(K_1))(1 - F_t(p')) - (p'' - z(K_1))(1 - F_t(p'')) \\
&\quad + (z(K_2) - z(K_1))(F_t(p') - F_t(p'')) \\
&= R_{z(K_1)}(p') - R_{z(K_1)}(p'') + (z(K_2) - z(K_1))(F_t(p') - F_t(p'')) \geq 0.
\end{aligned}$$

Where in the last inequality we used that F_t is a cumulative distribution function and thus increasing, Proposition 4.7 and the hypothesis that $R_{z(K_1)}(p') - R_{z(K_1)}(p'') \geq 0$. Analogously, assuming that $R_{z(K_1)}(p') - R_{z(K_1)}(p'') > 0$ leads to $R_{z(K_2)}(p') - R_{z(K_2)}(p'') > 0$. This proves that $R_{z(K_1)} \preceq_{sc} R_{z(K_2)}$ and consequently that $p_t(z(K_2)) \leq p_t(z(K_1))$ for all $t \in [T]$. An identical procedure shows that $R_0 \preceq_{sc} R_{z(K)}$, proving that $p_t(z(K)) \geq p_t^*$ for all $t \in [T]$. □

4.3 Numerical experiments

In this small subsection we will illustrate the results of the previous sections using two families of distributions. Let us consider the families

$$F_t^{(1)}(p) = 1 - \mathrm{e}^{-p/t}, \quad F_t^{(2)}(p) = \frac{p}{t}\mathbb{1}_{[0,t)}(p) + \mathbb{1}_{[t,+\infty)}(p),$$

this is, $\{F_t^{(1)}\}_{t=1}^T$ is a family of exponential families indexed with a parameter $1/t$ and $\{F_t^{(2)}\}_{t=1}^T$ a family of uniform $[0,t]$ families. A direct computation shows that these families have the MLRP property and hence optimal pricing policies will be non-decreasing. We will consider discrete periods with $T = 8$, and $\alpha = (\alpha_t)_{t=1}^8 = (0.1, 0.2, 0.1, 0.1, 0.3, 0.05, 0.05, 0.1)$ although the arrival masses have almost no influence in the results if we vary the parameters slightly.

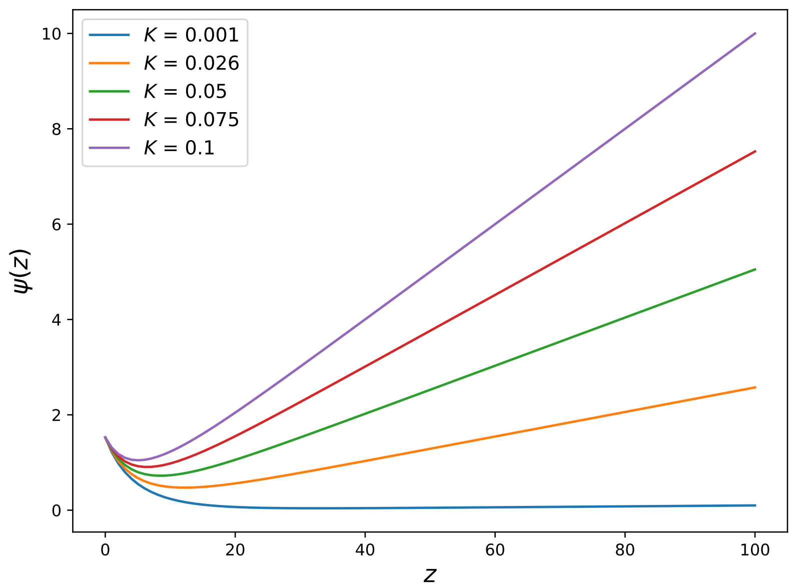

Figure 4.1: $\psi_K(z)$ for the family $\{F_t^{(1)}\}_{t=1}^8$.

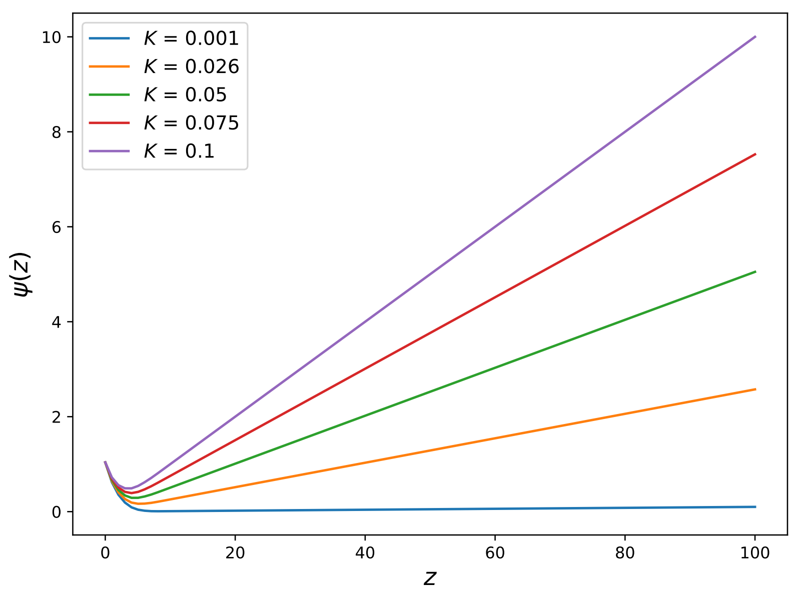

Figure 4.2: $\psi_K(z)$ for the family $\{F_t^{(2)}\}_{t=1}^8$.

Figures 4.1 and 4.2 show the ψ_K functions for families 1 and 2 respectively, for different values of K. It is straightforward to note from the definition that as K increases, the function increases its values. Moreover, the asymptotic behavior of the function is noticeable. This is because the term $\mathcal{R}_t^{(\mathrm{i})}(z)$ goes to 0 as $z \to \infty$, for $\mathrm{i} = 1, 2$. The reason for this is Lemma 3.6, as both families have finite expectation fixing a value of t (and hence the sum has finite expectation too). We proceed to compute the values for $z(K)$. The behavior of the dual application is clear; as K goes to 0 then $z(K) \to \infty$ and as K increases then $z(K) \to 0$. The explanation for this is as follows: for any fixed t, as $\mathcal{R}_t^{(\mathrm{i})}(z)$ can be made arbitrarily close to 0 then for low values of K the $\arg\min_{z\geq 0} \psi_K(z)$ can be made arbitrarily high, and vice-versa: for high values of K the term zK becomes more significant to minimize than the sum of the functions $\mathcal{R}_t^{(\mathrm{i})}(z)$.

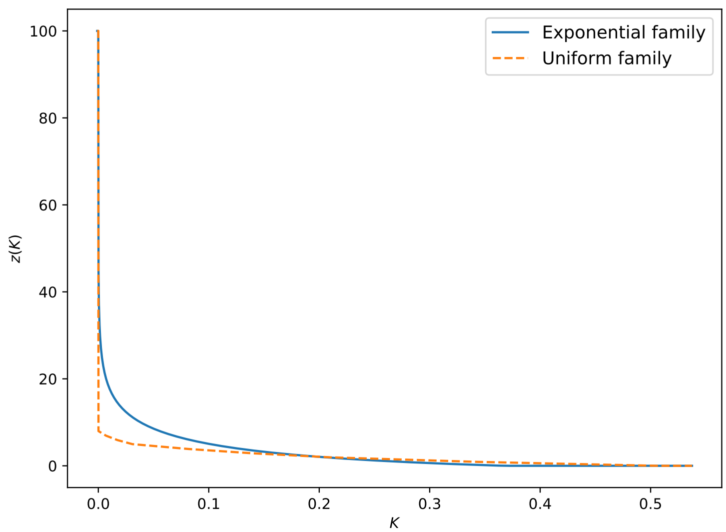

Figure 4.3: $z(K)$ for the exponential and uniform families.

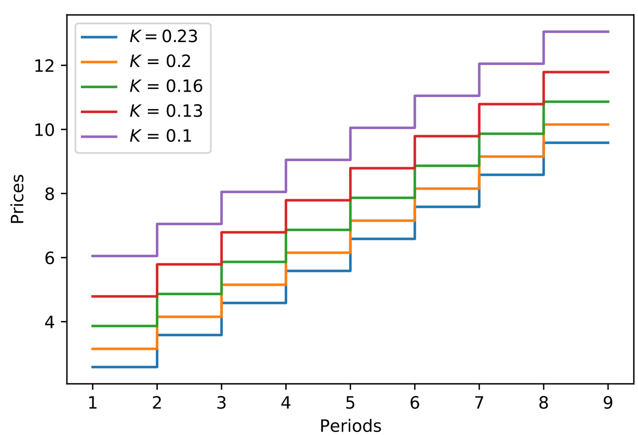

Figure 4.4: Optimal pricing policies varying K for the exponential family.

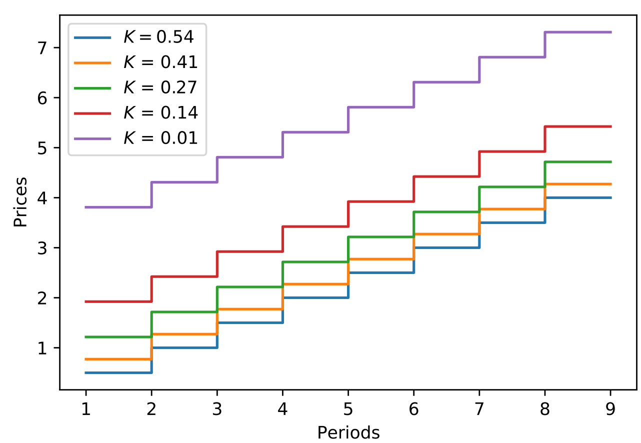

Figure 4.5: Optimal pricing policies varying K for the uniform family.

Figures 4.3, 4.4 and 4.5 empirically show Propositions 4.7 and 4.8. We note that as K decreases, prices do not rise regularly: although the distances between capacities are equispaced, large jumps are observed between the smallest inventory capacities (this is more notorious in Figure 4.5 and can be checked in the values of Table 4.1).

	Uniform family					Exponential family				
K	0.01	0.12	0.22	0.33	0.44	0.10	0.13	0.16	0.20	0.23
$z(K)$	6.62	3.24	1.84	0.98	0.36	5.05	3.79	2.87	2.15	1.58
$p_1(K)$	3.81	2.12	1.42	0.99	0.68	6.05	4.79	3.87	3.15	2.58
$p_2(K)$	4.31	2.62	1.92	1.49	1.18	7.05	5.79	4.87	4.15	3.58
$p_3(K)$	4.81	3.12	2.42	1.99	1.68	8.05	6.79	5.87	5.15	4.58
$p_4(K)$	5.31	3.62	2.92	2.49	2.18	9.05	7.79	6.87	6.15	5.58
$p_5(K)$	5.81	4.12	3.42	2.99	2.68	10.05	8.79	7.87	7.15	6.58
$p_6(K)$	6.31	4.62	3.92	3.49	3.18	11.05	9.79	8.87	8.15	7.58
$p_7(K)$	6.81	5.12	4.42	3.99	3.68	12.05	10.79	9.87	9.15	8.58
$p_8(K)$	7.31	5.62	4.92	4.49	4.18	13.05	11.79	10.87	10.15	9.58
Rev(K)	0.07	0.54	0.8	0.95	1.03	1.05	1.19	1.29	1.37	1.43

Table 4.1: Optimal pricing policies with their respective dual variable, for some values of K.

Finally, we can plot how the optimal revenue evolves as a function of K. Figure 4.6 shows how volatile the seller's revenue is as capacity declines. We notice that for small variations from the critical K value (the expected capacity setting the monopoly prices), the optimal revenue does not have much variation but as this value decreases even more, the changes are more abrupt and decrease in logarithmic fashion with respect to K.

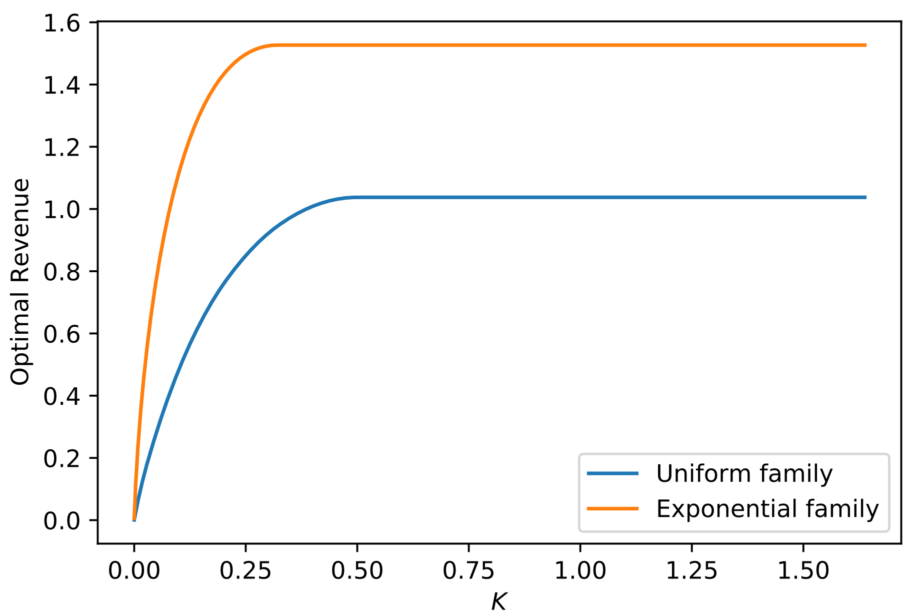

Figure 4.6: Optimal revenues in function of K.

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