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SOME CONTRIBUTIONS TO DEGENERATE STATE-DEPENDENT SWEEPING PROCESSES

TESIS PARA OPTAR AL GRADO DE
DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

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ALGUNAS CONTRIBUCIONES A LOS PROCESOS DE ARRASTRE DEGENERADOS DEPENDIENTES DEL ESTADO

Esta tesis tiene como objetivo estudiar las inclusiones diferenciales relacionadas con conos normales conducidos por conjuntos móviles no regulares (subsuaves y positivamente α-far) en espacios de Hilbert. En especial, nos interesa una variante particular del proceso de arrastre clásico conocido como proceso de arrastre degenerado dependiente del estado, que corresponde al caso en el que se incorpora un operador lineal/no lineal al proceso de arrastre clásico. Esta inclusión diferencial ha sido motivada por desigualdades cuasi-variacionales que surgen, por ejemplo, en problemas de evolución cuasi-estática con fricción, la evolución de pilas de arena, y modelos de daño micromecánico para materiales de hierro, entre otros. Cabe mencionar que el proceso de arrastre degenerado ha sido estudiado sólo en el marco de conjuntos regulares (convexos/prox-regulares) que varían de forma Lipschitz o absolutamente continua con respecto a la distancia Hausdorff.

Introducimos las definiciones fundamentales para el estudio de los procesos de arrastre degenerados dependientes del estado en el Capítulo 1. Nos centramos en el cono normal de Clarke, el subdiferencial de Clarke, el subdiferencial proximal, la distancia de Hausdorff truncada y los conjuntos ρ-uniformemente prox-regulares junto con algunas de las propiedades más útiles para demostrar la existencia y unicidad de soluciones a las variantes del proceso de arrastre clásico.

En el Capítulo 2, nos centramos en el estudio del problema del proceso de arrastre degenerado dependiente del estado. Para ello, utilizando la técnica de regularización de Moreau-Yosida, garantizamos la existencia de soluciones bajo el supuesto de que los conjuntos móviles varían de forma Lipschitz con respecto a la distancia Hausdorff truncada. Todos los resultados de esta sección se pueden encontrar en [56].

En el Capítulo 3 obtenemos resultados sobre la existencia de soluciones de los denominados procesos de arrastre degenerados perturbados dependientes del estado, este capítulo se basa en [57]. Utilizando un resultado de existencia apropiado para inclusiones diferenciales con perturbación monovaluada, y una adecuada adaptación de la técnica de regularización de Moreau-Yosida, mostramos la existencia de soluciones bajo el supuesto de que los conjuntos móviles varían de una manera Lipschitz con respecto a la distancia Hausdorff truncada. En consecuencia, se puede obtener la existencia de soluciones para los procesos de arrastre integralmente perturbados dependientes del estado. Nuestros resultados del Capítulo 3, pueden aplicarse al estudio del llamado método de descenso espejo en línea, el cual está relacionado con el problema de los k-servidores.

Este trabajo finaliza con algunas conclusiones y preguntas abiertas.

Palabras clave: Inclusión diferencial, Pertubado/Proceso de arrastre degenerado, Regularización de Moreau-Yosida, Distancia de Hausdorff truncada, método de descenso espejo.

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SOME CONTRIBUTIONS TO DEGENERATE STATE-DEPENDENT SWEEPING PROCESSES

This dissertation aims to study differential inclusions related to normal cones driven by nonregular moving sets (subsmooth and positively α-far) on Hilbert spaces. Specially, we are interested in a particular variant of the classical sweeping process known as the degenerate state-dependent sweeping process, which corresponds to the case where a linear/non-linear operator is incorporated into the classical sweeping process. This differential inclusion has been motivated by quasi-variational inequalities that arise, for instance, in problems of quasistatic evolution with friction, the evolution of sandpiles, and micromechanical damage models for iron materials, among others. It is worth to mention that degenerate sweeping process has been studied only in the framework of regular (convex/prox-regular) sets that vary in a Lipschitz or absolutely continuous way with respect to the Hausdorff distance.

We introduce the fundamental definitions for the study of the degenerate state-dependent sweeping processes in Chapter 1. We focus on the Clarke normal cone, Clarke subdifferential, proximal subdifferential, truncated Hausdorff distance, and ρ-uniformly prox-regular sets together with some of the most useful properties to show the existence and uniqueness of solutions to the variants of the classic sweeping process.

In Chapter 2, we focus on the study of the degenerate state-dependent sweeping processes problem. For this purpose, using the Moreau-Yosida regularization technique, we guarantee the existence of solutions under the assumption that moving sets varies of a Lipschitz way with respect to truncated Hausdorff distance. All the results from this section can be found in [56].

In Chapter 3, we obtain results on the existence of solutions of the so called perturbed degenerate state-dependent sweeping processes, which is based on [57]. Using an appropriate existence result for differential inclusions with single-valued perturbation, and together to a suitable adaptation of the Moreau-Yosida regularization technique, we show the existence of solutions under the assumption that moving sets varies of a Lipschitz way with respect to truncated Hausdorff distance. Consequently, the existence of solutions for the integrally perturbed degenerate state-dependent sweeping processes can be obtained. Our results from Chapter 3, can be applied to study of the so-called online mirror descent method, which is related to the k-server problem.

This work ends with some conclusions and open questions.

Keywords: Differential inclusion, Perturbed/degenerate state-dependent sweeping processes, Moreau-Yosida regularization, Truncated Hausdorff distance, Mirror descent method.

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Notations

Operations and symbols

$:=$	Equal by definition
$\equiv$	Identically equal
$\langle \cdot, \cdot \rangle$	Inner product on a Hilbert space $\mathcal{H}$
$\mathrm{haus}_r(A, B)$	r-truncated Hausdorff distance

Spaces

$\mathbb{R}$	Real line
$\mathcal{H}$	Hilbert space
$\mathrm{AC}\left([0, T] ; \mathcal{H}\right)$	$\mathcal{H}$-valued absolutely continuous functions
$L^1\left([0, T] ; \mathcal{H}\right)$	$\mathcal{H}$-valued Lebesgue integrable functions over $[0, T]$

Sets

$\mathbb{B}$	Closed unit ball
$\mathrm{Proj}_S\left(x\right)$	Projection of x to S
$T_S\left(x\right)$	Clarke tangent cone to S at x
$N_S\left(x\right)$	Clarke normal cone to S at x
${N_S}^P\left(x\right)$	Proximal normal cone to S at x
$\partial f\left(x\right)$	Clarke subdifferential of a function f in x
$\partial_L f\left(x\right)$	Limiting proximal subdifferential of f at x
$\partial^P f\left(x\right)$	Proximal subdifferential of f at x
$U_\rho\left(S\right)$	Open ρ-tube around a set S

General Introduction

The objective of this work is to contribute to the development of two kinds of variants of the so called *degenerate sweeping processes.* Specifically, we focused on the *perturbated/degenerate state-dependent sweeping processes*, i.e., differential inclusions of the form

$$\begin{cases} -\dot{x}(t) \in N_{C(t,x(t))}\left(\mathcal{A}(x(t))\right) & \text{a.e. } t \in [T_0, T]\,, \\ x(0) = x_0,\ \mathcal{A}(x_0) \in C(0, x_0), \end{cases} \tag{1}$$

where $C\colon\ [0,T]\times\mathcal{H} \rightrightarrows \mathcal{H}$ is a set-valued mapping with nonempty closed values of a separable Hilbert $\mathcal{H}$, $N_{C(t,x(t))}\left(\mathcal{A}\left(x\left(t\right)\right)\right)$ is the Clarke normal cone to $C(t,x(t))$ (state- dependent subset of constraints) at $\mathcal{A}(x(t)) \in C(t,x(t))$, and the operator $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ is a (possibly) nonlinear Lipschitz and strongly monotone operator.

The state-dependent sweeping process corresponds to the case where the moving set also depends on the state. This differential inclusion has been motivated by quasi-variational inequalities arising, e.g., in quasistatic evolution problems with friction, the evolution of sandpiles and micromechanical damage models for iron materials, among others (see [47] and the references therein). On the other hand, the degenerate sweeping process corresponds to the case where a linear/nonlinear operator is added "inside" the sweeping process. This dynamics was proposed by Kunze and Monteiro-Marques as a model for quasistatic elasto-plasticity (see [43]). Since then, the degenerate sweeping process has been studied by several authors in the framework of convex/prox-regular sets (see [1,42–45]), and when such sets vary in a Lipschitz or absolutely continuous way with respect to the Hausdorff distance, limiting the spectrum of possible applications to only bounded moving sets.

This thesis is based on [56,57], and rigorously develops one specific important topic in the formulation of any dynamical system: in both Chapter 2 and Chapter 3, we prove results of existence of solutions for the perturbed/degenerate state-dependent sweeping processes with nonregular sets, which we apply to *integro-differential sweeping processes*, as well as to the dynamics known as *online mirror descent method*, and *degenerate perturbed second-order sweeping process*, so as to obtain results of existence of a solution for these systems, but now in the framework of the nonregular sets of a separable Hilbert.

Chapter 1: Introduction

In this chapter, we establish the fundamental definitions and some lemmas that have become efficient tools to prove existence results for the perturbed/degenerate state-dependent sweeping processes.

The following result (see [14, Theorem 2]) is used in the chapter 2 to prove the existence of solutions for the Moreau-Yosida regularization scheme.

Lemma Let $F\colon [T_0,T]\times\mathcal{H}\rightrightarrows\mathcal{H}$ be with nonempty, closed and convex values satisfying:

(i) for every $x\in\mathcal{H}$, $F(\cdot,x)$ is measurable.

(ii) for every $t\in[T_0,T]$, $F(t,\cdot)$ is upper semicontinuous from $\mathcal{H}$ into $\mathcal{H}_w$.

(iii) for a.e. $t\in[T_0,T]$ and $A\subset\mathcal{H}$ bounded,

$$\gamma\left(F\left(t,A\right)\right)\ \le\ k\left(t\right)\gamma\left(A\right)$$

for some $k\in L^1\left(T_0,T\right)$ with $k\left(t\right)<+\infty$ for all $t\in[T_0,T]$, where $\gamma=\alpha$ or $\gamma=\beta$ is either the Kuratowski or the Hausdorff measure of noncompactness.

Then, the differential inclusion

$$\begin{cases}\dot{x}(t)\in F(t,x(t)) & \text{a.e. } t\in\left[T_0,T\right],\\ x(0)=x_0,\end{cases}$$

has at least one solution $x\in\mathrm{AC}\left(I;\mathcal{H}\right)$.

Now, let us consider the following result, which allows us shows that the set-valued map $(t,x)\rightrightarrows\frac{1}{2}\partial d^2_{C(t,x)}(\mathcal{A}(x))$ satisfies the conditions of last Lemma.

Proposition Assume that $(\mathcal{H}^2_{\mathcal{A}})$, $(\mathcal{H}^x_1)$, $(\mathcal{H}^x_3)$ and $(\mathcal{H}^x_4)$ hold. Then, the set-valued map $G\colon [0,T]\times\mathcal{H}\rightrightarrows\mathcal{H}$ defined by $G(t,x):=\frac{1}{2}\partial d^2_{C(t,x)}(\mathcal{A}(x))$ satisfies:

(i) for all $x\in\mathcal{H}$ and all $t\in[0,T]$, $G(t,x)=\mathcal{A}(x)-\operatorname{cl}\operatorname{co}\operatorname{Proj}_{C(t,x)}(\mathcal{A}(x))$.

(ii) for every $x\in\mathcal{H}$ the set-valued map $G(\cdot,x)$ is measurable.

(iii) for every $t\in[0,T]$, $G(t,\cdot)$ is upper semicontinuous from $\mathcal{H}$ into $\mathcal{H}_w$.

(iv) for every $t\in[0,T]$ and $B\subset\mathcal{H}$ bounded, $\gamma\left(G(t,B)\right)\le(1+k(t))M\gamma\left(B\right)$, where $\gamma=\alpha$ or $\gamma=\beta$ is the Kuratowski or the Hausdorff measure of non-compactness of B and $k\in L^1(0,T)$ is given by $(\mathcal{H}^x_4)$.

(v) Let $\mathcal{A}(x_0)\in C(0,x_0)$. Then, for all $t\in[0,T]$ and $x\in\mathcal{H}$,

$$\|G(t,x)\|:=\sup\left\{\|w\|\colon w\in G(t,x)\right\}\le(M+L)\|x-x_0\|+\kappa_r t,$$

where $r=\max\{\|\mathcal{A}(x)\|,\|\mathcal{A}(x_0)\|\}$ and κ_r is the constant given by $(\mathcal{H}^x_1)$.

The next result will be used to prove the existence of solutions for the Moreau-Yosida regularization scheme with single-valued perturbations.

Theorem Let $\mathcal{H}$ be a separable Hilbert space and $I = [0, T]$ for some $T > 0$. Assume that $(\mathcal{H}_1^F)$-$(\mathcal{H}_4^F)$ and $(\mathcal{H}_1^g)$-$(\mathcal{H}_3^g)$ hold. Then, the differential inclusion

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) + g(t, x(t)) & \text{a.e. } t \in [0, T], \\ x(0) = x_0 \in \mathcal{H}, \end{cases}$$

has at least one solution $x \in \mathrm{AC}([0, T]; \mathcal{H})$. Moreover, if $\mathcal{H}$ is finite-dimensional, then assumption $(\mathcal{H}_2^g)$ can be weakened to $(\mathcal{H}_4^g)$.

We emphasize that our main contribution in this chapter corresponds to the adaptation and extension of technical results from [41] so as to address the perturbed/degenerate state-dependent sweeping process governed by Lipschitz nonregular moving sets with respect to the truncated Hausdorff distance.

Chapter 2: Degenerate State-Dependent Sweeping Processes

In this chapter, we will follow a similar approach used by Vilches and Jourani in [41] to prove, under some compactness conditions, an existence result of solutions of the degenerate state-dependent sweeping processes with nonregular moving sets whose dynamic is given by (1). To do so, we use the Moreau-Yosida regularization technique, which consists in approaching a given differential inclusion by a penalized one, depending on a parameter, whose existence is easier to obtain (for example, by using the classical Cauchy-Lipschitz Theorem), and then to pass to the limit as the parameter goes to zero. More precisely, let $\lambda > 0$ and consider the following differential inclusion

$$\begin{cases} -\dot{x}_\lambda(t) \in \dfrac{1}{2\lambda} \partial d^2_{C(t, x_\lambda(t))} \left(\mathcal{A}(x_\lambda(t))\right) & \text{a.e. } t \in [T_0, T], \\ x_\lambda(0) = x_0, \end{cases} \tag{2}$$

where $\mathcal{A}(x_0) \in C(0, x_0)$. The first contribution of this chapter is given by the prove of the existence of solutions for (2) with nonregular sets, using the notion of truncated Hausdorff distance.

Proposition Assume that $(\mathcal{H}_{\mathcal{A}}^2)$, $(\mathcal{H}_1^x)$, $(\mathcal{H}_3^x)$ and $(\mathcal{H}_4^x)$ hold. Then, for every $\lambda > 0$ there exists at least one absolutely continuous solution x_λ of (2).

The truncated Hausdorff distance allows us to localize the trajectories of (2) in a ball of radius ρ. These trajectories stay uniformly close (with respect to λ) to the moving sets in a small interval of $I = [0, T]$. This has been established in the next Proposition.

Proposition Assume, in addition to the hypotheses of Proposition mentioned above, that $(\mathcal{H}_{\mathcal{A}}^1)$ holds. Then, for every $R > 0$ there exists $\tau_R \in]0, T]$ (independent of λ) such that if $\lambda < \frac{(m\alpha^2 - L)}{\tilde{\kappa}} \rho$,

$$\dot{\varphi}_\lambda(t) \leq \tilde{\kappa} + \frac{L - m\alpha^2}{\lambda} \varphi_\lambda(t) \quad \text{a.e. } t \in [0, \tau_R],$$

where $\alpha \in \left]\sqrt{\frac{L}{m}}, 1\right]$ and $\rho > 0$ are given by Remark 2.2 and $\tilde{\kappa} := \kappa_{\|\mathcal{A}(x_0)\|+MR}$ is the constant given by $(\mathcal{H}_1^x)$. Moreover,

$$\varphi_\lambda(t) \leq \frac{\tilde{\kappa}\lambda}{m\alpha^2 - L} \text{ for all } t \in [0, \tau_R].$$

Keeping in mind the aforementioned, and using some technical results of the chapter 1, the second contribution of this chapter is the following Theorem.

Theorem Assume that $(\mathcal{H}_\mathcal{A}^1)$, $(\mathcal{H}_\mathcal{A}^2)$, $(\mathcal{H}_1^x)$, $(\mathcal{H}_3^x)$ and $(\mathcal{H}_4^x)$ hold. Then, there exists at least one solution $x \in \mathrm{AC}\,(I; \mathcal{H})$ of $(\mathcal{DSSP})$.

We emphasize that the results and methods used in this chapter forms the first approach on the study of the degenerate state-dependent sweeping processes with nonregular moving sets.

Chapter 3: Perturbed Degenerate State-Dependent Sweeping Processes

The main object of this chapter is to prove, under some compactness conditions, an existence result of solution of the perturbed degenerate state-dependent sweeping processes with nonregular moving sets given by following differential inclusion

$$\begin{cases} \dot{x}(t) \in -N_{C(t,x(t))}\left(\mathcal{A}(x(t))\right) + g(t, x(t)) & \text{a.e. } t \in [0, T], \\ x(0) = x_0, \ \mathcal{A}(x_0) \in C(0, x_0). \end{cases} \tag{3}$$

Here, the set $N_{C(t,x(t))}\left(\mathcal{A}(x(t))\right)$ stands for the Clarke normal cone to $C(t, x(t))$ at $\mathcal{A}(x(t)) \in C(t, x(t))$ where $C\colon [0, T] \times \mathcal{H} \rightrightarrows \mathcal{H}$ is a set-valued mapping with nonempty closed values of a separable Hilbert $\mathcal{H}$. The operator $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ is (possibly) nonlinear Lipschitz and strongly monotone, and the perturbation is a single-valued function $g\colon [0, T] \times \mathcal{H} \to \mathcal{H}$ which is a locally μ-Lipshchitz with respect to the state variable, measurable in the time variable and satisfies a linear growth condition.

For this purpose, we use the Moreau-Yosida regularization technique, which consists in approaching a given differential inclusion by a penalized one, depending on a parameter, whose existence is easier to obtain (for example, by using the classical Cauchy-Lipschitz Theorem), and then to pass to the limit as the parameter goes to zero. More precisely, let $\lambda > 0$ and consider the following differential inclusion

$$-\dot{x}_\lambda(t) \in \frac{1}{2\lambda}\partial d^2_{C(t,x_\lambda(t))}\left(\mathcal{A}(x_\lambda(t))\right) + g(t, x_\lambda(t))$$

where $\mathcal{A}(x_0) \in C(0, x_0)$. The first contribution of this chapter is the following Theorem, which assures that a regularization process provides a family $(x_\lambda)_\lambda$ of solutions of the last differential inclusion converging (up to a subsequence) to a Lipschitz solution of the system (3).

Theorem Assume that $(\mathcal{H}_{\mathcal{A}}^1)$, $(\mathcal{H}_{\mathcal{A}}^2)$, $(\mathcal{H}_1^x)$, $(\mathcal{H}_3^x)$ and $(\mathcal{H}_4^x)$, that $(\mathcal{H}_1^g)$, $(\mathcal{H}_2^g)$ and $(\mathcal{H}_3^g)$ hold. Then, the problem (3) admits at least one absolutely continuous solution. Moreover, if $\mathcal{H}$ is finite-dimensional, then assumption $(\mathcal{H}_2^g)$ can be weakened to $(\mathcal{H}_4^g)$.

Although there are many works about the sweeping process, we are particularly interested in (3) since it allows us to study different differential inclusions, for instance, the integrally perturbed sweeping process [16, 17], projected dynamical systems [24] and complementarity dynamical systems (CDS), among others. In this chapter we concentrate on the study of the first two kinds of sweeping process.

Firstly, let us focus on the following dynamic system

$$\begin{cases} \dot{x}(t) \in -N\left(C(t,x(t)); \mathcal{A}(x(t))\right) + g_1(t,x(t)) + \displaystyle\int_0^t g_2(s,x(s))ds \quad \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in \mathcal{A}^{-1}(C(0,x_0)), \end{cases} \tag{4}$$

where $C\colon [0,T] \times \mathcal{H} \rightrightarrows \mathcal{H}$ is a set-valued map with nonempty and closed values, $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ is a nonlinear operator and $g_1, g_2\colon [0,T] \times \mathcal{H} \to \mathcal{H}$ are two single-valued functions.

Our second contribution is the following result where we give some general conditions under which the system (4) has at least one absolutely continuous solution.

Theorem Assume, in addition to $(\mathcal{H}_{\mathcal{A}}^1)$, $(\mathcal{H}_{\mathcal{A}}^2)$, $(\mathcal{H}_1^x)$, $(\mathcal{H}_3^x)$ and $(\mathcal{H}_4^x)$, that $(\mathcal{H}_1^g)$, $(\mathcal{H}_2^g)$ and $(\mathcal{H}_3^g)$ hold for g_1 and g_2. Then, the problem (4) admits at least one absolutely continuous solution $x\colon [0,T] \to \mathcal{H}$. Moreover, if $\mathcal{H}$ is finite-dimensional, then assumption $(\mathcal{H}_2^g)$ can be weakened to $(\mathcal{H}_4^g)$.

Secondly, let us focus on find $x\colon [0,T] \to \mathcal{H}$ such that $x(0) = x_0$ and

$$\nabla^2\phi(x(t))\dot{x}(t) \in -N_{\mathcal{K}}(x(t)) + g(t,x(t)) \text{ a.e. } t \in [0,T], \tag{5}$$

where $\phi\colon \mathcal{H} \to \mathbb{R}$ is a convex function which is twice differentiable over $\mathcal{H}$ and $\mathcal{K} \subset \mathcal{H}$ is a nonempty closed set. Now, because the dynamical system (5) is formally equivalent to the following perturbed degenerate sweeping process

$$\dot{z}(t) \in -N_{\mathcal{K}}(\nabla\phi^*(z(t))) + g(t, \nabla\phi^*(z(t))) \text{ a.e. } t \in [0,T],$$

where ϕ^* is the convex conjugate of ϕ, we obtain the following result.

Theorem Let $\mathcal{K}$ be a closed, ball-compact and positively α-far set (see Definition 1.19). Let $\phi\colon \mathcal{H} \to \mathbb{R}$ be a strongly convex function with Lipschitz continuous gradient. Assume that $g\colon [0,T] \times \mathcal{H} \to \mathcal{H}$ satisfies $(\mathcal{H}_1^g)$, $(\mathcal{H}_2^g)$ and $(\mathcal{H}_3^g)$. Then, for any $x_0 \in \mathcal{H}$, the problem (5) admits at least one absolutely continuous solution. Moreover, if $\mathcal{H}$ is finite-dimensional, then assumption $(\mathcal{H}_2^g)$ can be weakened to $(\mathcal{H}_4^g)$.

We emphasize that a similar theorem, under hypotheses of compactness and convexity on the moving set, was studied in [25, Theorem 2.2], so, the relaxed hypotheses used in the last theorem offer a significant improvement on this matter.

Chapter 4: Some ideas for future work

In this chapter, we present a brief discussion of some ideas to be developed as a future work.

Chapter 1

Introduction

In this chapter, we introduce the notation, the concepts, and preliminary results used throughout the thesis.

From now on $\mathcal{H}$ is a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm denoted by $\|\cdot\|$. The *closed ball* (resp. *open*) of $\mathcal{H}$ centered at $x \in \mathcal{H}$ of radius $r \in (0, +\infty]$ is defined by $\mathbb{B}[x,r] := \{y \in \mathcal{H} : \ \|x-y\| \ \leq \ r\}$ (resp. $\mathbb{B}(x,r) := \{y \in \mathcal{H} : \ \|x-y\| \ < \ r\}$). We will use the notation $\mathbb{B}$ for the *closed unit ball centered at zero*, that means, $\mathbb{B} = \mathbb{B}[0,1]$. Furthermore, if $r = +\infty$ by convention, we will set $r\mathbb{B} = \mathcal{H}$. The notation $\mathcal{H}_w$ stands for $\mathcal{H}$ equipped with the weak topology and $x_n \rightharpoonup x$ denotes the *weak of a sequence* $(x_n)_{n \in \mathbb{N}}$ *to* x.

Given a subset S of $\mathcal{H}$, we say that S is *ball compact* if, for any $r > 0$, the set $S \cap r\mathbb{B}$ is compact.

Let S be a closed subset of $\mathcal{H}$. As usual, d_S (or $d(\cdot, S)$) denotes the *distance function* from S, that is, $d_S(x) = d(x,S) := \inf_{y \in S} \|x-y\|$ for all $x \in \mathcal{H}$.

The *support function* of S is the function from $\mathcal{H}$ into $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ given by

$$\sigma_S(h) \ = \ \sigma(h,S) \ := \ \sup_{x \in S} \langle x, h \rangle \ \text{ for all } h \in \mathcal{H}.$$

The set-valued map $\text{Proj}_S \colon \mathcal{H} \rightrightarrows S$ of nearest points in S is defined by

$$\text{Proj}_S(x) \ := \ \{z \in S : \ \|x - z\| \ = \ d_S(x)\}, \ \text{ for all } x \in \mathcal{H}. \tag{1.1}$$

Whenever the latter set is a singleton for some $\bar{x} \in \mathcal{H}$, that is $\text{Proj}_S(\bar{x}) = \{\bar{y}\}$, the vector $\bar{y} \in S$ is denoted by $\text{proj}_S(\bar{x})$ or $P_S(\bar{x})$.

We define the *Clarke tangent cone* to S at $x \in S$ as

$$T_S(x) \ := \ \left\{ v \in \mathcal{H} : \ \forall \ x_n \underset{S}{\to} x \ \forall \ t_n \downarrow 0, \ \exists \ v_n \to v \ \forall \ n : x_n + t_n v_n \in S \right\}.$$

This cone is closed and convex, and its negative polar $N_S(x)$ is the *Clarke normal cone* to S at $x \in S$, i.e.,

$$N_S(x) := (T_S(x))^\circ = \{p \in \mathcal{H} : \langle p, v \rangle \leq 0 \quad \forall\, v \in T_S(x)\}.$$

As usual, $N_S(x) = \emptyset$ if $x \notin S$, and we also observe that

$$\text{if } x \in \operatorname{Int}(S), \text{ then } N_S(x) = \{0\}.$$

The *Clarke subdifferential* of a function $f \colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is defined through the normal cone by

$$\partial f(x) := \left\{v \in \mathcal{H} : (v, -1) \in N_{\operatorname{epi}(f)}(x, f(x))\right\},$$

where

$$\operatorname{epi}(f) := \{(x, t) \in \mathcal{H} \times \mathbb{R} : f(x) \leq t\}$$

is the *epigraph* of f. When the function f is finite and locally Lipschitz around x, the Clarke subdifferential is characterized (see [28]) in the following simple way

$$\partial f(x) := \{v \in \mathcal{H} : \langle v, h \rangle \leq f^\circ(x, h) \text{ for all } h \in \mathcal{H}\},$$

where

$$f^\circ(x, h) := \limsup_{(t,y) \to (0^+, x)} t^{-1}[f(y + th) - f(y)]$$

is the *generalized directional derivative* of the locally Lipschitz function f at x in the direction $h \in \mathcal{H}$. The function $f^\circ(x, \cdot)$ is in fact the support function of $\partial f(x)$. That characterization easily yields that the Clarke subdifferential of any locally Lipschitz function has the important property of upper semicontinuity from $\mathcal{H}$ into $\mathcal{H}_w$ (see definition below).

The following equality (see [28]) establishes the relation between the Clarke normal cone and the Clarke subdifferential of the distance function

$$N_S(x) = \operatorname{cl}^w(\mathbb{R}_+ \partial d_S(x)) \text{ for } x \in S. \tag{1.2}$$

As usual, it will be convenient to write $\partial d_S(x)$ in place of $\partial d(\cdot, S)(x)$.

Remark 1.1 In this thesis, we will compute the Clarke subdifferential of the distance function to a moving set. In doing so, the subdifferential will always be compute respect to the variable involved in the distance function by assuming that the set is fixed. Explicitly, $\partial d_{C(t,y)}(x)$ means the subdifferential of the function $d_{C(t,y)}(\cdot)$ (here $C(t, y)$ is fixed) calculated at the point x, i.e., $\partial\left(d_{C(t,y)}(\cdot)\right)(x)$.

The following formula gives a precise representation of the subdifferential of the distance from a closed set S in $\mathcal{H}$ (see [37]):

$$\partial d_S(x) = \bigcap_{\gamma>0} \overline{\text{co}} \left(\frac{x - \text{proj}_S^\gamma(x)}{d_S(x)} \right), \tag{1.3}$$

where $\text{proj}_S^\gamma(x) = \{z \in S : \|x - z\| < d_S(x) + \gamma\}$.

Let $f \colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $x \in \text{dom} f$. A vector $\zeta \in \mathcal{H}$ is a *proximal subgradient* of f at x (see [28, Chapter 1]) if there exist two positive numbers σ and η such that

$$f(y) \geq f(x) + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2 \quad \forall y \in \mathbb{B}(x, \eta).$$

The set $\partial^P f(x)$ of all proximal subgradients of f at x is the *proximal subdifferential* of f at x. Note that $\partial^P f(x) = \emptyset$ when f is not finite at x.

Given a closed subset S of $\mathcal{H}$, a vector $\zeta \in \mathcal{H}$ is a *proximal normal vector* to the set S at $x \in S$ when it is a proximal subgradient of the indicator function of S, that means, if there exist two positive numbers σ and η such that

$$\langle \zeta, y - x \rangle \leq \sigma \|y - x\|^2 \quad \forall y \in \mathbb{B}(x, \eta) \cap S.$$

Moreover, the following equivalence holds

$$\zeta \in \partial^P f(x) \quad \Leftrightarrow \quad (\zeta, -1) \in N^P_{\text{epi} f}(x, f(x)).$$

On the other hand, the *proximal normal cone* to S at x is also related to the proximal subdifferential of the distance function from S since the following relations hold true for all $x \in S$ (see for instance [22, 28])

$$\partial^P d_S(x) = N_S^P(x) \cap \mathbb{B} \quad \text{and } N_S{}^P(x) = \mathbb{R}_+ \partial^P d_S(x).$$

Another important property is the following: for each $v \in \mathcal{H}$ with $\text{Proj}_S(v) \neq \emptyset$ is

$$v - w \in N_S^P(w) \text{ for all } w \in \text{Proj}_S(v). \tag{1.4}$$

We will also consider the Limiting Proximal Subdifferential :

$$\partial_L f(x) := \left\{ w - \lim \zeta_i \colon \zeta_i \in \partial^P f(x_i),\ x_i \to x,\ f(x_i) \to f(x) \right\}.$$

Moreover, if f is locally Lipschitz around x, then the following equality holds

$$\partial f(x) = \text{cl co}\, \partial_L f(x).$$

We refer to [28, 51] for more details.

1.1 Function Spaces

We denote by $L^1([T_0,T];\mathcal{H})$ the space of $\mathcal{H}$-valued Lebesgue integrable functions defined over $[T_0,T]$. The space $L^1([T_0,T];\mathcal{H})$ endowed with the weak topology will be denoted by $L^1_w([T_0,T];\mathcal{H})$. We say that a set $\mathcal{K}\subseteq L^1([T_0,T];\mathcal{H})$ is *uniformly integrable* if

$$\lim_{\lambda\to+\infty}\left[\sup_{f\in K}\int_{\{\|f\|\geq\lambda\}}\|f(s)\|\,ds\right] = 0.$$

Furthermore, if there exists $\psi\in L^1(T_0,T)$ such that for all $f\in\mathcal{K}$

$$\|f(t)\|\leq\psi(t) \text{ a.e. } t\in[T_0,T],$$

then the set $\mathcal{K}$ is uniformly integrable.

We recall below the Dunford-Pettis theorem (see [34, Theorem 2.3.24]), which characterizes relatively weakly compact subsets of $L^1(\Omega)$.

Theorem 1.2 (Dunford-Pettis theorem) *Let $\mathcal{H}$ be a Hilbert space. A bounded set $\mathcal{K}\subseteq L^1([T_0,T];\mathcal{H})$ is relatively weakly compact in $L^1([T_0,T];\mathcal{H})$ if and only if it is uniformly integrable.*

Let us recall the following characterization of weak convergence in $C([T_0,T];\mathcal{H})$ (see [12, Theorem 4.2]).

Lemma 1.3 *$(x_n)_{n\in\mathbb{N}}\subseteq C([T_0,T];\mathcal{H})$ weakly converges in $C([T_0,T];\mathcal{H})$ to x if and only if $(x_n)_{n\in\mathbb{N}}$ is bounded in $C([T_0,T];\mathcal{H})$ and*

$$x_n(t)\rightharpoonup x(t) \quad \forall\, t\in[T_0,T].$$

We say that,

- $u\in W^{1,1}([T_0,T];\mathcal{H})$ if there exists $f\in L^1([T_0,T];\mathcal{H})$ and $u_0\in\mathcal{H}$ such that

$$u(t) = u_0+\int_0^t f(s)\,ds \quad \forall\, t\in[T_0,T].$$

- $u\in W^{2,1}([T_0,T];\mathcal{H})$ if $\dot{u}\in W^{1,1}([T_0,T];\mathcal{H})$.

Moreover, for $u:[T_0,T]\to\mathcal{H}$ we define

$$\mathrm{Lip}(u) := \sup_{t\neq s}\frac{\|u(t)-u(s)\|}{|t-s|},$$

and the space of *$\mathcal{H}$-valued Lipschitz functions* is defined by

$$\mathrm{Lip}\left([T_0, T]\,; \mathcal{H}\right) \;:=\; \{u : [T_0, T] \to \mathcal{H} :\; \mathrm{Lip}\,(u) \;<\; \infty\}\,.$$

Now, we recall the classical Arzelà-Ascoli theorem (see [34, Theorem 2.3.2]), which characterizes the relatively compact subsets of $C\left([T_0, T]\,; \mathcal{H}\right)$, equipped with the norm

$$\|f\|_{\infty} \;=\; \sup\left\{\|f\,(t)\| :\; t \in [T_0, T]\right\}.$$

Theorem 1.4 (Arzelà - Ascoli theorem) *A set $\mathcal{K} \subseteq \left(\mathcal{C}\left([T_0, T]\,; \mathcal{H}\right), \|\cdot\|_{\infty}\right)$ is relatively compact if and only if*

(i) *for every $t \in [T_0, T]$, the set*

$$\mathcal{K}\,(t) := \{u\,(t) :\; u \in \mathcal{K}\} \subseteq \mathcal{H}$$

is relatively compact in $\mathcal{H}$; and

(ii) *$\mathcal{K} \subseteq C\left([T_0, T]\,; \mathcal{H}\right)$ is uniformly equicontinuous, i.e., for every $\varepsilon \;>\; 0$, there is $\delta\,(\varepsilon) > 0$ such that, if $t,\, s \in [T_0, T]$ and $|t - s| \;<\; \delta$, then*

$$\|u\,(t) - u\,(s)\| \;<\; \varepsilon, \quad \forall\, u \in \mathcal{K}.$$

Definition 1.5 *We say that a function $x\colon\, [T_0, T] \to \mathcal{H}$ is* absolutely continuous *if for any $\varepsilon \;>\; 0$, there is $\delta \;>\; 0$ such that for any countable collection of disjoint subintervals $[\alpha_n, \beta_n]$ of $[T_0, T]$ such that*

$$\sum_n |\beta_n - \alpha_n| \;<\; \delta,$$

we have that

$$\sum_n \|x\,(\beta_n) - x\,(\alpha_n)\| \;<\; \varepsilon.$$

Theorem 1.6 *A function $x\colon\, [T_0, T] \;\to\; \mathcal{H}$ is absolutely continuous if there exists $f \in L^1\left([T_0, T]\,; \mathcal{H}\right)$ and $x_0 \in \mathcal{H}$ such that*

$$x\,(t) \;=\; x_0 + \int_{T_0}^{t} f\,(s)\,ds, \quad \forall\, t \in [T_0, T]\,.$$

In this case, we have that x is a.e. derivable with $\dot{x} = f$.

The following Lemma, proved in [41], is a sufficient condition for compactness of absolutely continuous functions.

Lemma 1.7 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of absolutely continuous functions from $[T_0, T]$ into $\mathcal{H}$ with $x_n(T_0) = x_0^n$. Assume that for all $n \in \mathbb{N}$*

$$\|\dot{x}_n(t)\| \leq \psi(t) \quad a.e. \quad t \in [T_0, T],$$

where $\psi \in L^1(T_0, T)$ and that $x_0^n \to x_0$ as $n \to \infty$. Then, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and an absolutely continuous function x such that

(i) $x_{n_k}(t) \rightharpoonup x(t)$ *in* $\mathcal{H}$ *as* $k \to +\infty$ *for all* $t \in [T_0, T]$.

(ii) $x_{n_k}(t) \rightharpoonup x(t)$ *in* $L^1([T_0, T]; \mathcal{H})$ *as* $k \to +\infty$.

(iii) $\dot{x}_{n_k}(t) \rightharpoonup \dot{x}(t)$ *in* $L^1([T_0, T]; \mathcal{H})$ *as* $k \to +\infty$.

(iv) $\|\dot{x}_n(t)\| \leq \psi(t)$ *a.e.* $t \in [T_0, T]$.

1.2 Measures of noncompactness

For a given bounded subset A of $\mathcal{H}$, the *Kuratowski measure* of noncompactness of A, $\alpha(A)$, is defined by

$$\alpha(A) = \{ d > 0 : A \text{ admits a finite cover by sets of diameter } \leq d \}$$

and the *Hausdorff measure* of noncompactness of A, $\beta(A)$, as

$$\beta(A) = \{ r > 0 : A \text{ can be covered by finitely many balls of radius } r \}.$$

The following result gives the main properties of the Kuratowski and Hausdorff measure of noncompactness (see [31, Proposition 9.1 from Section 9.2]) which will be used throughout the thesis.

Proposition 1.8 *Let $\mathcal{H}$ be a Hilbert space and B, B_1, B_2 be bounded subsets of $\mathcal{H}$. Let γ be the Kuratowski or the Hausdorff measure of noncompactness. Then,*

(i) $\gamma(B) = 0$ *if and only if* $\mathrm{cl}(B)$ *is compact.*

(ii) $\gamma(\lambda B) = |\lambda| \gamma(B)$ *for every* $\lambda \in \mathbb{R}$.

(iii) $\gamma(B_1 + B_2) \leq \gamma(B_1) + \gamma(B_2)$.

(iv) $B_1 \subseteq B_2$ *implies* $\gamma(B_1) \leq \gamma(B_2)$.

(iv) $\gamma(\mathrm{conv}(B)) = \gamma(B)$.

(v) $\gamma(\mathrm{cl}(B)) = \gamma(B)$.

(vi) *If* $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ *is a Lipschitz map of constant* $M \geq 0$, *then*

$$\gamma(\mathcal{A}(B)) \leq M\gamma(B).$$

The following lemma (see [31, Proposition 9.3]) provides a useful rule for the interchange of γ and integration.

Lemma 1.9 *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $v_n\colon [T_0,T]\to\mathcal{H}$ such that*

$$\sup_n \|v_n(t)\| \;\le\; \psi(t) \;\; a.e.\; t\in[T_0,T]$$

where $\psi\in L^1(T_0,T)$. Then

$$\gamma\left(\left\{\int_t^{t+h} v_n(s)\,ds : n\in\mathbb{N}\right\}\right) \;\le\; \int_t^{t+h}\gamma\left(\{v_n(s):n\in\mathbb{N}\}\right)ds,$$

for $T_0 \;\le\; t \;<\; t+h \;\le\; T$.

1.3 Set-valued maps

The following definitions related to set-valued mappings will be used in the sequel.

Definition 1.10 *Let $\mathcal{L}$ be the Lebesgue σ-field on $[T_0,T]$ and $\Phi\colon [T_0,T]\rightrightarrows\mathcal{H}$ a set-valued map with closed values. We say that Ψ is* measurable *if*

$$\Phi^{-1}(C) \;=\; \{t\in[T_0,T] : \Phi(t)\cap C \;\neq\; \emptyset\}\in\mathcal{L},$$

for every closed subset C of $\mathcal{H}$.

Moreover, if $\Phi\colon [T_0,T]\rightrightarrows\mathcal{H}$ has nonempty, closed, convex and bounded values, $\Phi\colon [T_0,T]\rightrightarrows\mathcal{H}$ is measurable if and only if its support function $v\mapsto\sigma(v,\Phi(t))$ is $\mathcal{L}$-measurable for all $v\in\mathcal{H}$. Furthermore, if $\mathcal{B}$ denotes the Lebesgue Borel σ-field on $\mathcal{H}$ and $\Psi\colon [T_0,T]\times\mathcal{H}\rightrightarrows\mathcal{H}$ is a set-valued map with nonempty, closed, convex and bounded values, Ψ is called $\mathcal{L}\otimes\mathcal{B}$-measurable if its support function $(t,x)\mapsto\sigma(v,\Psi(t,x))$ is $\mathcal{L}\otimes\mathcal{B}$-measurable for all $v\in\mathcal{H}$.

Definition 1.11 *A set-valued map $\Psi\colon\mathcal{H}\rightrightarrows\mathcal{H}$ is called* upper semicontinuous *from $\mathcal{H}$ into $\mathcal{H}_w$ if the set*

$$\Psi^{-1}(C) \;=\; \{x\in\mathcal{H} : \Psi(x)\cap C \;\neq\; \emptyset\}$$

is norm-closed for every C weakly closed set of $\mathcal{H}$.

Furthermore, if $\Psi\colon\mathcal{H}\rightrightarrows\mathcal{H}$ has nonempty, closed, convex and bounded values, Ψ is upper semicontinuous from $\mathcal{H}$ into $\mathcal{H}_w$ if and only if its support function $x\mapsto\sigma(v,\Psi(x))$ is upper semicontinuous for all $v\in\mathcal{H}$.

The following result (see [14, Theorem 2]) is used in the subsequent chapter to prove the existence of solutions for the Moreau-Yosida regularization scheme.

Lemma 1.12 *Let $F\colon [T_0, T]\times\mathcal{H} \rightrightarrows \mathcal{H}$ be with nonempty, closed and convex values satisfying:*

(i) *for every $x \in \mathcal{H}$, $F(\cdot, x)$ is measurable.*

(ii) *for every $t \in [T_0, T]$, $F(t, \cdot)$ is upper semicontinuous from $\mathcal{H}$ into $\mathcal{H}_w$.*

(iii) *for a.e. $t \in [T_0, T]$ and $A \subset \mathcal{H}$ bounded,*

$$\gamma(F(t, A)) \leq k(t)\gamma(A)$$

for some $k \in L^1(T_0, T)$ with $k(t) < +\infty$ for all $t \in [T_0, T]$, where $\gamma = \alpha$ or $\gamma = \beta$ is either the Kuratowski or the Hausdorff measure of noncompactness.

Then, the differential inclusion

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(0) = x_0, \end{cases}$$

has at least one solution $x \in \mathrm{AC}(I; \mathcal{H})$.

1.4 Truncated Hausdorff Distance

Let $r \in]0, +\infty]$ be a given extended real and let A and B be nonempty subsets of $\mathcal{H}$. We define the *r-truncated excess* of A over B as the extended real

$$\mathrm{exc}_r(A, B) := \sup_{x\in A\cap r\mathbb{B}} d(x, B).$$

It is clear that under the usual convention $r\mathbb{B} = \mathcal{H}$ for $r = +\infty$, the r-truncated excess of A over B is the usual *excess* of A over B, i.e.,

$$\mathrm{exc}_\infty(A, B) = \sup_{x\in A} d(x, B) =: \mathrm{exc}(A, B).$$

In this work, our results are based on the following basic notions of distance between sets.

- $$\mathrm{haus}_r(A, B) := \max\{\mathrm{exc}_r(A, B), \mathrm{exc}_r(B, A)\}$$

- $$\widehat{\mathrm{haus}}_r(A, B) := \sup_{z\in r\mathbb{B}} |d(z, A) - d(z, B)|.$$

The quantity $\mathrm{haus}_r(A, B)$ receives the name of *r-truncated Hausdorff distance*.

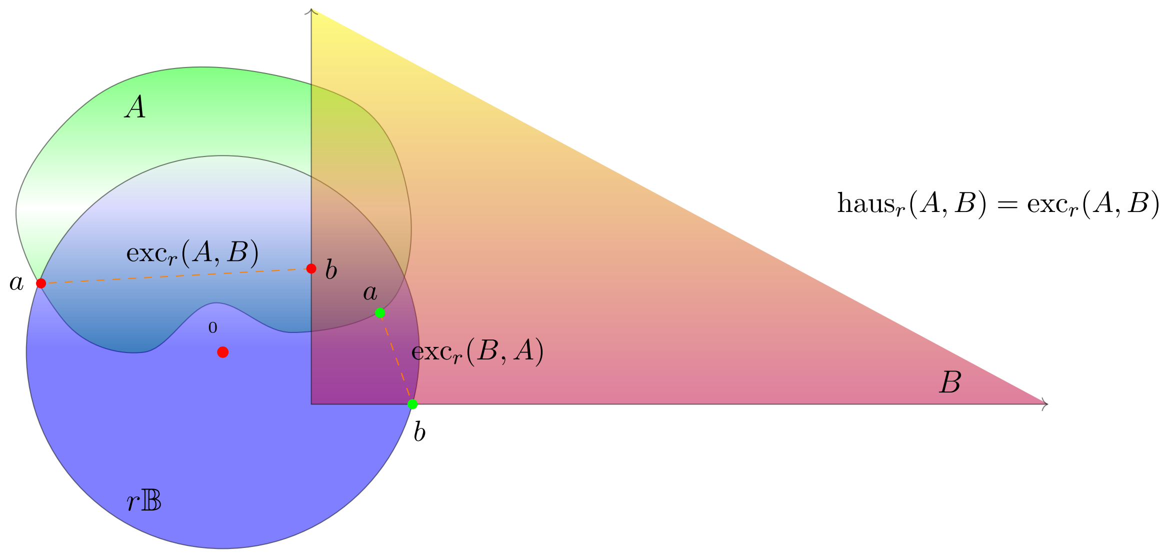

Figure 1.1: r-truncated Hausdorff distance

It is important to emphasize that

$$\mathrm{haus}_r(A,B) \leq \widehat{\mathrm{haus}}_r(A,B).$$

Moreover, for any extended real $r' \geq 2r + \max\{d_A(0), d_B(0)\}$, the following inequality holds

$$\widehat{\mathrm{haus}}_r(A,B) \leq \mathrm{haus}_{r'}(A,B).$$

As a consequence of the last two inequalities, we obtain the following result.

Proposition 1.13 *Let $C\colon [0,T] \rightrightarrows \mathcal{H}$ be a set-valued map with nonempty and closed sets. Then, the following assertions are equivalent.*

(a) *for all $r \geq 0$, there exists $\kappa_r \geq 0$ such that*

$$\widehat{\mathrm{haus}}_r(C(t), C(s)) \leq \kappa_r |t-s| \text{ for all } s,t \in [0,T]\,.$$

(b) *for all $r \geq 0$, there exists $\kappa_r \geq 0$ such that*

$$\mathrm{haus}_r(C(t), C(s)) \leq \kappa_r |t-s| \text{ for all } s,t \in [0,T]\,.$$

Therefore, we use (a) and (b) indistinctly.

Definition 1.14 *We say that a set-valued map $C\colon [0,T] \rightrightarrows \mathcal{H}$ is*

(i) Lipschitz with respect to the Hausdorff distance *if there exists $\kappa \geq 0$ such that*

$$\mathrm{haus}(C(t), C(s)) := \sup_{z\in\mathcal{H}} |d(z, C(t)) - d(z, C(s))| \leq \kappa |t-s| \text{ for all } s,t \in [0,T]\,.$$

(ii) Lipschitz with respect to the truncated Hausdorff distance *if for all* $r \geq 0$, *there exists* $\kappa_r \geq 0$ *such that*

$$\widehat{\text{haus}}_r(C(t), C(s)) \leq \kappa_r |t-s| \text{ for all } s, t \in [0, T].$$

It is clear that (i) implies (ii), but the reverse implication is not true. Moreover, as discussed in [55, 67], the Hausdorff distance can be too restrictive for practical applications. Indeed, let us consider the half-space moving set $C\colon [0,T] \times \mathcal{H} \rightrightarrows \mathcal{H}$ defined by

$$C(t,x) := \{z \in \mathcal{H} \colon \langle \zeta(t), z\rangle - \beta(t,x) \leq 0\} \text{ for all } (t,x) \in [0,T] \times \mathcal{H},$$

where $\zeta\colon [0,T] \to \mathcal{H}$ is a γ-Lipschitz mapping for some real $\gamma \geq 0$ and the map $\beta\colon [0,T] \times \mathcal{H} \to \mathcal{H}$ satisfies the following Lipschitz property:

$$|\beta(s,x) - \beta(t,y)| \leq \gamma|t-s| + L\|x-y\| \text{ for all } s,t \in [0,T] \text{ and } x,y \in \mathcal{H}.$$

Assume that the following normalization condition holds true: $\|\zeta(t)\| = 1$ for all $t \in [0,T]$. According to [32, Theorem 6.30], we have

$$d_{C(t,x)}(z) = (\langle \zeta(t), z\rangle - \beta(t,x))^+ \text{ for all } (t,x) \in [0,T] \times \mathcal{H},$$

where $r^+ := \max\{0, r\}$ for all $r \in \mathbb{R}$. Hence, we can check that for each $r > 0$ and any $s,t \in [0,T]$ with $s \neq t$ and $x,y \in \mathcal{H}$,

$$\widehat{\text{haus}}_r(C(s,x), C(t,y)) = \sup_{z \in r\mathbb{B}} |d_{C(s,x)}(z) - d_{C(t,y)}(z)| \leq \gamma|t-s|\,(r+1) + L\|x-y\|.$$

The situation mentioned above can be seen more clearly in Figure 1.4.

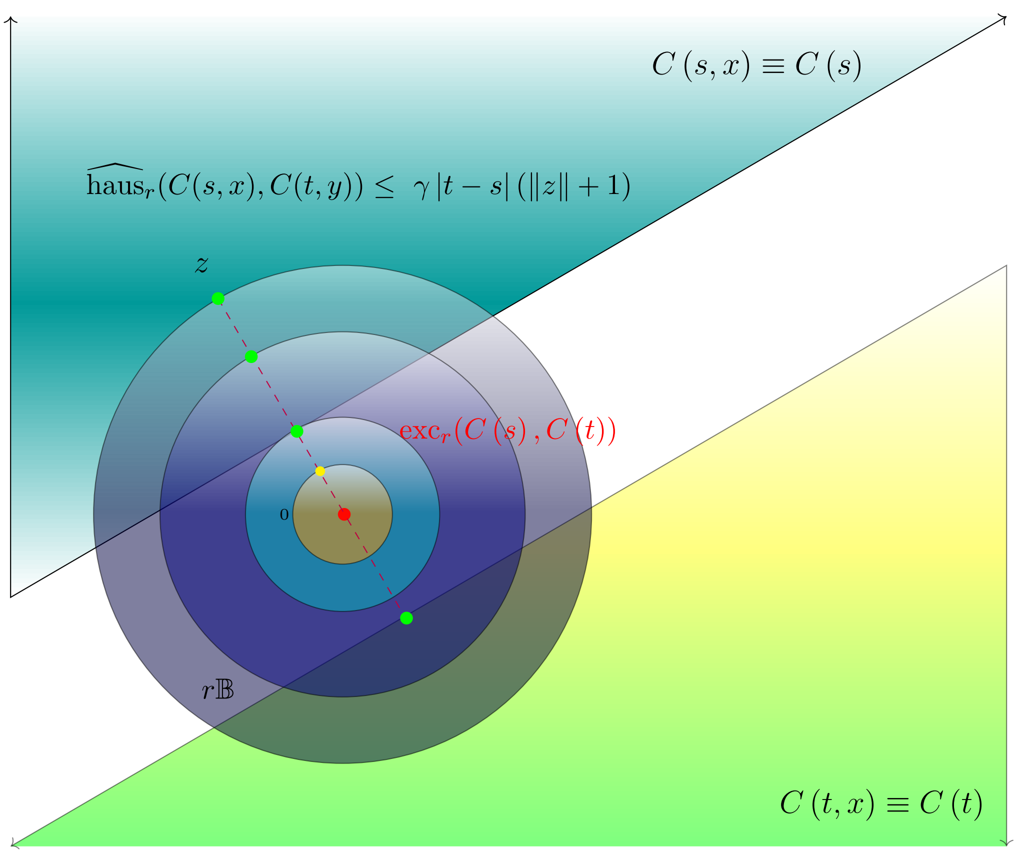

Figure 1.2: Truncated Hausdorff Distance

However, $\text{Haus}(C(t,x), C(s,y)) = +\infty$, which shows that C is Lipschitz with respect to the truncated Hausdorff distance but not Lipschitz with respect to Hausdorff distance.

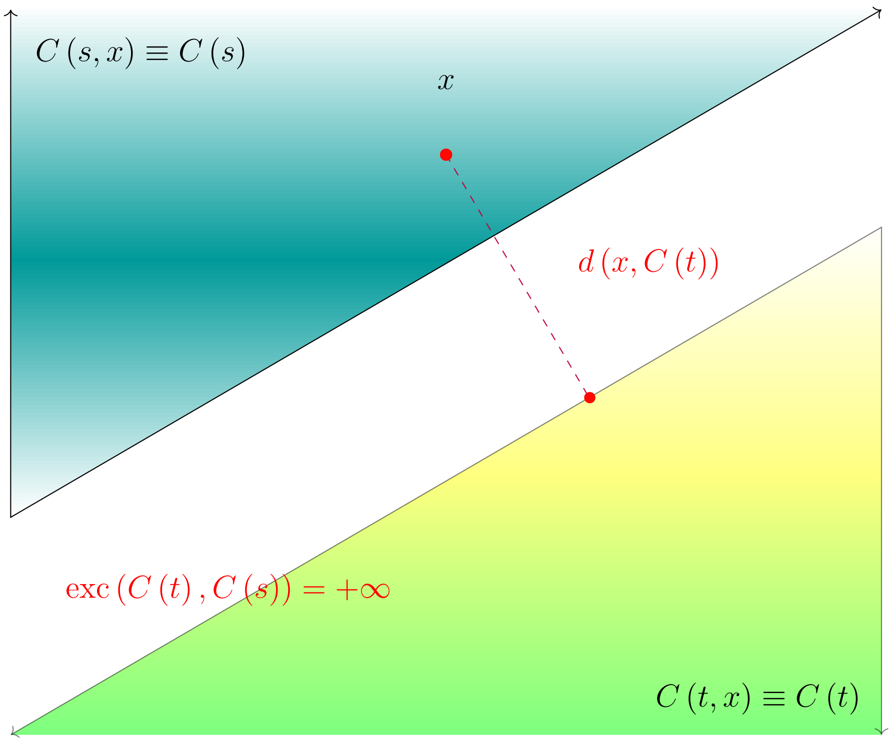

Figure 1.3: Hausdorff Distance

1.5 Some classes of sets

Here we define some classes of sets that generalize the class of convex sets satisfying

$$\text{Proj}_S(x) \neq \emptyset \text{ for all } x \in \mathcal{H}. \tag{1.5}$$

In the literature, the set S on $\mathcal{H}$ that satisfies (1.5) is called *proximinal* subset of $\mathcal{H}$ (see [11] for more details). Note that a proximinal set is necessarily closed. In particular, the case in which the set $\text{Proj}_S(x)$ is a singleton on a neighborhood of x was treated in the work [30], in which a detailed study is done over prox-regular sets, also known as sets with positive reach as defined by Federer in [33].

1.5.1 Uniformly prox-regular sets

Definition 1.15 ([59]) *For a fixed $\rho > 0$, the set S is said to be ρ-uniformly prox-regular if for any $x \in S$ and $\zeta \in N_S^P(x) \cap \mathbb{B}$, x is the unique nearest point of S to $x + \rho\zeta$, i.e.,*

$$x = \text{proj}_S(x + \rho\zeta).$$

It is known that S is ρ-uniformly prox-regular if and only if every nonzero proximal vector

$\zeta \in N_S^P(x)$ to S at any point $x \in S$ can be realized by an ρ-ball, that is,

$$S \cap \mathbb{B}\left(x + \rho\frac{\zeta}{\|\zeta\|}\right) \;=\; \emptyset.$$

The notion of uniformly prox-regular sets is related to the differentiability of the distance function. The following theorem recalls some useful characterizations and properties of prox-regular sets (see, e.g., [30]). Before stating it, define for any real $\rho \in (0, +\infty)$ the *open tubular neighborhood of radius* ρ of a set S by

$$U_\rho(S) \;:=\; \{x \in \mathcal{H} :\; 0 \;<\; d_S(x) \;<\; \rho\}.$$

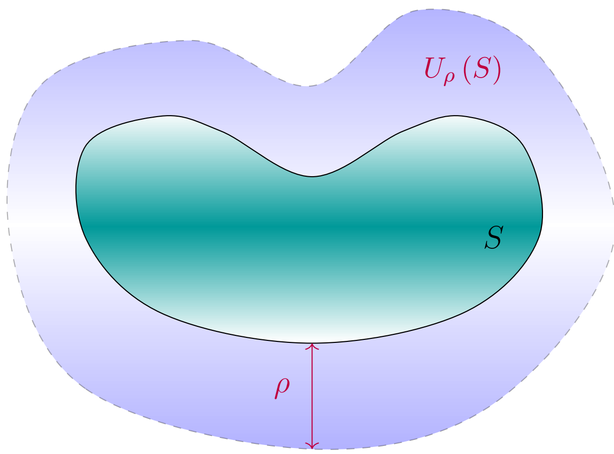

Figure 1.4: Open tubular neighborhood of radius ρ for a prox-regular set S.

Theorem 1.16 *Let S be a nonempty closed subset of $\mathcal{H}$. Consider the following assertions.*

(a) *The set S is ρ-uniformly prox-regular.*

(b) *For all x, $x' \in S$, for all $v \in N^P(S; x)$ we have that*

$$\langle v, x' - x\rangle \;\leq\; \frac{1}{2\rho}\|v\|\,\|x' - x\|^2.$$

(c) *For all $x_i \in S$, $v_i \in N^P(S; x_i) \cap \mathbb{B}_{\mathcal{H}}$ with $i = 1, 2$ we have*

$$\langle v_2 - v_1, x_2 - x_1\rangle \;\geq\; -\|x_2 - x_1\|^2.$$

with $\|v_1\| \;\leq\; \rho$ and $\|v_2\| \;\leq\; \rho$ (ρ-hypomonotonicity property).

(d) *For any real $\gamma \in (0, 1)$, for all x, $x' \in U_{\rho\gamma}(S)$*

$$\|\mathrm{proj}_S(x) - \mathrm{proj}_S(x')\| \;\leq\; (1 - \gamma)^{-1}\|x - x'\|.$$

(e) *For all* $u \in U_\rho(S) \setminus S$ *we have (with* $x = \mathrm{proj}_S(u)$*)*

$$x = \mathrm{proj}_S\left(x + t\frac{u-x}{\|u-x\|}\right) \text{ for all } t \in [0,\rho).$$

(f) *The function* d_S^2 *is* $\mathcal{C}^{1,1}$ *on* $U_\rho(S)$ *and*

$$\nabla d_S^2(x) = 2(x - \mathrm{proj}_S(x)) \text{ for all } x \in U_\rho(S).$$

(g) *The set* S *is normally regular in the sense that*

$$N^P(S;x) = N^L(S;x) = N(S;x) \text{ for all } x \in \mathcal{H},$$

further, the following equality holds:

$$\partial_P d_S(x) = \partial_L d_S(x) = \partial d_S(x) \text{ for all } x \in U_\rho(S).$$

Then, the assertions a), b), c), d), e) *and* f) *are pairwise equivalent and each one implies* g).

The next result, due to Balashov and Ivanov [9, Theorem 2], provides an estimation on the behaviour of the projection to prox-regular sets. We refer to [58, Theorem 2] and [55, Lemma 6.1] for more details.

Lemma 1.17 *Let* S_1, S_2 *be* ρ*-uniformly prox-regular subsets of* $\mathcal{H}$ *with* $\rho \in]0,+\infty[$, $\gamma \in]0,1[$, $r \in [0,+\infty[$, $x \in r\mathbb{B} \cap U_{\rho\gamma}(S_1) \cap U_{\rho\gamma}(S_2)$. *If* $\mathrm{haus}_{\rho\gamma+r}(S_1,S_2) \leq \rho$, *then*

$$\left\|\mathrm{proj}_{S_1}(x) - \mathrm{proj}_{S_2}(x)\right\| \leq \left(\frac{2\gamma\rho}{1-\gamma}\,\mathrm{haus}_{\rho\gamma+r}(S_1,S_2)\right)^{1/2}.$$

The above lemma can be improved when the sets are convex. We refer to [55] for the proof.

Lemma 1.18 *Let* $S_1, S_2 \subset \mathcal{H}$ *be two nonempty closed convex sets. Then, for all* $x, x' \in \mathcal{H}$ *we have*

$$\begin{aligned}\left\|\mathrm{proj}_{S_1}(x) - \mathrm{proj}_{S_2}(x')\right\|^2 &\leq \|x-x'\|^2 + 2d_{S_1}(x)\,\mathrm{exc}(S_2,S_1) + 2d_{S_2}(x')\,\mathrm{exc}(S_1,S_2)\\ &\leq \|x-x'\|^2 + 2\left(d_{S_1}(x) + d_{S_2}(x')\right)\mathrm{haus}(S_1,S_2).\end{aligned}$$

1.5.2 Nonregular sets

We study degenerate sweeping processes driven by regular and non-regular sets. We will consider two classes of non-regular sets: positively α-far sets and subsmooth sets.

The class of positively α-far sets was introduced in [35] and then broadly studied in [39]. This class includes several notions of sets as paraconvexity, uniformly prox-regularity, and uniformly subsmoothness (see [39]).

Before introducing the notion of α-far set, we notice that any ρ-uniformly prox-regular set $S \subset \mathcal{H}$ satisfies the following property:

$$d(0, \partial d(\cdot, S)(x)) = 1 \text{ for all } x \in U_\rho(S), \tag{1.6}$$

where $U_\rho(S) := \{x \in \mathcal{H} \colon 0 < d(x, S) < \rho\}$. The above property does not require the existence of the projection onto S. Thus, it can be used to define an extended class of nonregular sets.

Definition 1.19 *Let $\alpha \in]0, 1]$ and $\rho \in]0, +\infty]$. Let S be a nonempty closed subset of $\mathcal{H}$ with $S \neq \mathcal{H}$. We say that the Clarke subdifferential of the distance function $d(\cdot, S)$* keeps the origin α-far-off *on the open ρ-tube around S, $U_\rho(S) := \{x \in \mathcal{H} \colon 0 < d(x, S) < \rho\}$, provided*

$$0 < \alpha \leq \inf_{x \in U_\rho(S)} d(0, \partial d(\cdot, S)(x)). \tag{1.7}$$

Moreover, if E is a given nonempty set, we say that the family $(S(t))_{t \in E}$ is positively α-far *if every $S(t)$ satisfies* (1.7) *with the same $\alpha \in]0, 1]$ and $\rho > 0$.*

Definition 1.20 *Let S be a closed subset of $\mathcal{H}$. We say that S is* uniformly subsmooth*, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that*

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\| \tag{1.8}$$

holds for all $x_1, x_2 \in S$ satisfying $\|x_1 - x_2\| < \delta$ and all $x_i^* \in N(S; x_i) \cap \mathbb{B}$ for $i = 1, 2$. Also, if E is a given nonempty set, we say that the family $(S(t))_{t \in E}$ is equi-uniformly subsmooth*, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that* (1.8) *holds for each $t \in E$ and all $x_1, x_2 \in S(t)$ satisfying $\|x_1 - x_2\| < \delta$ and all $x_i^* \in N(S(t); x_i) \cap \mathbb{B}$ for $i = 1, 2$.*

Proposition 1.21 ([39]) *Assume that S is uniformly subsmooth. Then, for all $\varepsilon \in]0, 1[$ there exists $\rho \in]0, +\infty[$ such that*

$$\sqrt{1 - \varepsilon} \leq \inf_{y \in U_\rho(S)} d(0, \partial d(y, S)).$$

The class of positively α-far sets contains strictly that of uniformly subsmooth sets. Indeed, let us consider the set $S = \{(x, y) \in \mathbb{R}^2 \colon y \geq -|x|\}$. In [35] it is proved that S is positively $\frac{\sqrt{2}}{2}$-far but not uniformly prox-regular.

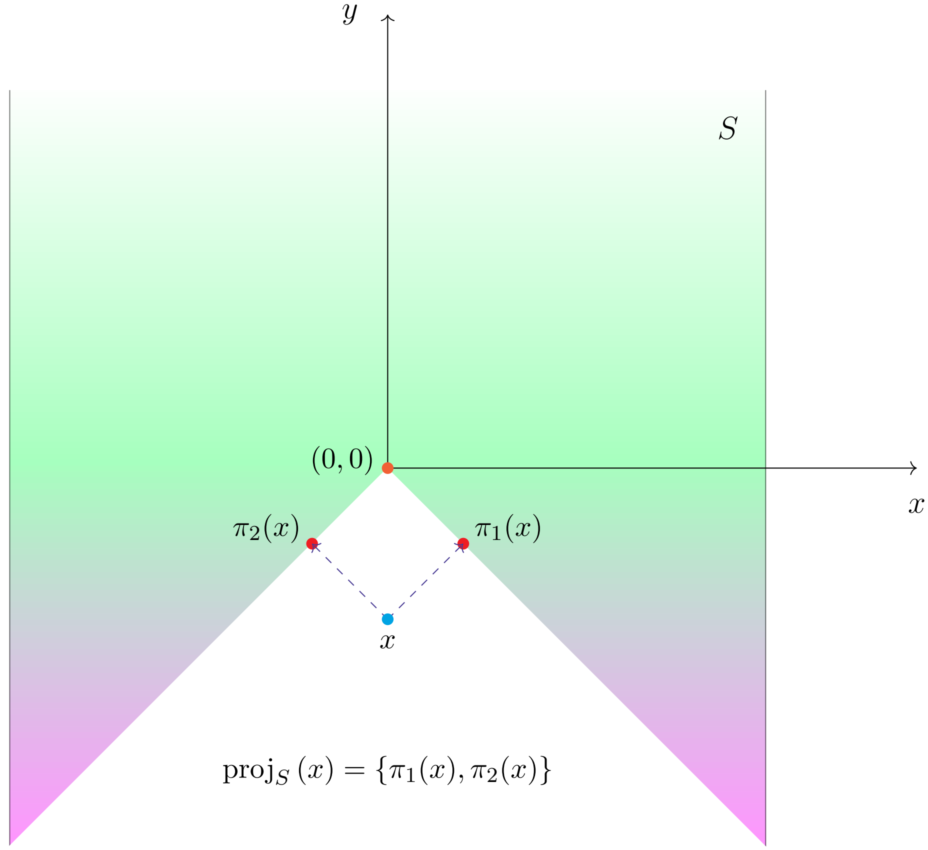

Figure 1.5: The set S is positively $\frac{\sqrt{2}}{2}$-far but not uniformly prox-regular.

1.6 Standing assumptions

This subsection gather the main assumptions that will be used throughout the thesis.

Hypotheses on the map $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$: $\mathcal{A}$ is a (possibly) nonlinear operator satisfying the following conditions:

$(\mathcal{H}_{\mathcal{A}}^1)$ There exists a constant $m > 0$ such that

$$\langle \mathcal{A}(x) - \mathcal{A}(y), x - y \rangle \geq m\|x - y\|^2 \text{ for all } x, y \in \mathcal{H},$$

i.e., $\mathcal{A}$ is strongly monotone.

$(\mathcal{H}_{\mathcal{A}}^2)$ There exists a constant $M > 0$ such that

$$\|\mathcal{A}(x) - \mathcal{A}(y)\| \leq M\|x - y\| \text{ for all } x, y \in \mathcal{H},$$

i.e., $\mathcal{A}$ satisfies a global Lipschitz condition.

Hypotheses on the set-valued map $C\colon [0, T] \times \mathcal{H} \rightrightarrows \mathcal{H}$: C is a set-valued map with closed and nonempty values. Moreover, the following hypotheses will be considered in this chapter.

($\mathcal{H}_1^x$) For all $r \geq 0$, there exist $\kappa_r \geq 0$ such that for $s, t \in [0,T]$ and $x, y \in \mathcal{H}$

$$\sup_{z \in r\mathbb{B}} |d(z, C(t,x)) - d(z, C(s,y))| \leq \kappa_r |t-s| + L\|x-y\|,$$

where $L \in [0, m[$ is independent of r and m is the constant given by ($\mathcal{H}_{\mathcal{A}}^1$).

($\mathcal{H}_2^x$) There exist constants $\alpha \in]0,1]$ and $\rho \in]0, +\infty]$ such that for every $y \in \mathcal{H}$

$$0 < \alpha \leq \inf_{x \in U_\rho(C(t,y))} d\left(0, \partial d(\cdot, C(t,y))(x)\right) \quad \text{a.e. } t \in [0,T],$$

where $U_\rho\left(C(t,y)\right) = \{x \in \mathcal{H} \colon 0 < d(x, C(t,y)) < \rho\}$.

($\mathcal{H}_3^x$) The family $\{C(t,v) \colon (t,v) \in [0,T] \times \mathcal{H}\}$ is equi-uniformly subsmooth.

($\mathcal{H}_4^x$) There exists $k \in L^1(0,T)$ such that for every $t \in [0,T]$, every $r > 0$ and every bounded set $B \subset \mathcal{H}$,

$$\gamma\left(C(t,B) \cap r\mathbb{B}\right) \leq k(t)\gamma(\mathcal{A}(B)),$$

where $\gamma = \alpha$ or $\gamma = \beta$ is either the Kuratowski or the Hausdorff measure of non-compactness (see Proposition 1.8), $k(t) < 1$ for all $t \in [0,T]$ and $\mathcal{A} \colon \mathcal{H} \to \mathcal{H}$ is an operator satisfying ($\mathcal{H}_{\mathcal{A}}^1$) and ($\mathcal{H}_{\mathcal{A}}^2$).

Remark 1.22 (i) *Let $L \in [0, m[$. Under ($\mathcal{H}_3^x$) for every $\alpha \in \left]\sqrt{\frac{L}{m}}, 1\right]$ there exists $\rho > 0$ such that ($\mathcal{H}_2^x$) holds. This follows from Proposition 1.21.*

(ii) *As shown in [44], the condition $L \in [0, m[$ in ($\mathcal{H}_1^x$) is sharp for the state-dependent sweeping processes, that is, it is possible to build an example of state-dependent sweeping processes without solutions for $L = m$.*

Hypotheses on $C \colon [0,T] \rightrightarrows \mathcal{H}$: C is a set-valued map with closed and nonempty values. Moreover, the following condition will be used in the chapter.

($\mathcal{H}_1$) For all $r \geq 0$, there exists $\kappa_r \geq 0$ such that for all $s, t \in [0,T]$

$$\sup_{z \in r\mathbb{B}} |d(z, C(t)) - d(z, C(s))| \leq \kappa_r |t-s|.$$

($\mathcal{H}_2$) There exist two constants $\alpha \in]0,1]$ and $\rho \in]0, +\infty]$ such that

$$0 < \alpha \leq \inf_{x \in U_\rho(C(t))} d\left(0, \partial d(x, C(t))\right) \quad \text{a.e. } t \in [0,T],$$

where $U_\rho\left(C(t)\right) = \{x \in \mathcal{H} \colon 0 < d(x, C(t)) < \rho\}$.

($\mathcal{H}_3$) For a.e. $t \in [0,T]$ the set $C(t)$ is ball-compact, that is, for every $r > 0$ the set $C(t) \cap r\mathbb{B}$ is compact in $\mathcal{H}$.

($\mathcal{H}_4$) For a.e. $t \in [0,T]$ the set $C(t)$ is r-uniformly prox-regular for some $r > 0$.

Hypotheses on $F\colon [0,T] \times \mathcal{H} \rightrightarrows \mathcal{H}$**:** is a set-valued map with nonempty closed and convex values satisfying:

($\mathcal{H}_1^F$) For every $x \in \mathcal{H}$, $F(\cdot, x)$ is measurable.

($\mathcal{H}_2^F$) For every $t \in [0,T]$, $F(t, \cdot)$ is upper semicontinuous from $\mathcal{H}$ into $\mathcal{H}_w$.

($\mathcal{H}_3^F$) There exist $\overline{c}$, $\overline{d} \in L^1(0,T)$ such that

$$d(0, F(t,v)) := \inf\{\|w\| : w \in F(t,v)\} \leq \overline{c}(t)\|v\| + \overline{d}(t),$$

for all $v \in \mathcal{H}$ and a.e. $t \in [0,T]$.

($\mathcal{H}_4^F$) For a.e. $t \in [0,T]$ and $A \subset \mathcal{H}$ bounded,

$$\gamma(F(t,A)) \leq k(t)\gamma(A)$$

for some $k \in L^1(0,T)$ with $k(t) < +\infty$ for all $t \in [0,T]$, where $\gamma = \alpha$ or $\gamma = \beta$ is either the Kuratowski or the Hausdorff measure of non-compactness.

Hypotheses on $g\colon [0,T] \times \mathcal{H} \to \mathcal{H}$**:** g is a function satisfying:

($\mathcal{H}_1^g$) For every $x \in \mathcal{H}$, the map $t \mapsto g(t,x)$ is measurable.

($\mathcal{H}_2^g$) For all $r > 0$, there exists $\mu_r \geq 0$ such that the map $x \mapsto g(t,x)$ is Lipschitz of constant μ_r on $r\mathbb{B}$ for a.e. $t \in [0,T]$.

($\mathcal{H}_3^g$) There are constants c, $d \geq 0$ such that

$$\|g(t,x)\| \leq c\|x\| + d$$

for all $x \in \mathcal{H}$ and a.e. $t \in [0,T]$.

($\mathcal{H}_4^g$) For a.e. $t \in [0,T]$, the map $x \mapsto g(t,x)$ is continuous.

1.7 Preliminary lemmas

In this section, we give some preliminary lemmas that will be used in the following sections. They are related to set-valued maps and properties of the distance function.

Recall that $-d(\cdot, S)$ has a directional derivative that coincides with the Clarke directional derivative of $-d(\cdot, S)$ whenever $x \notin S$ (see, e.g., [13]). Thus, we obtain the following lemma.

Lemma 1.23 *Let $S \subset \mathcal{H}$ be a closed set, $x \notin S$ and $v \in \mathcal{H}$. Then*

$$\lim_{h \downarrow 0} \frac{d(x+hv, S) - d(x,S)}{h} = \min_{y^* \in \partial d(x,S)} \langle y^*, v\rangle .$$

The following lemma characterizes the Clarke subdifferential of the distance function to moving sets depending on the state (see [41, Lemma 4.2]).

Lemma 1.24 ([41]) *Assume that* $(\mathcal{H}_4^x)$ *holds. Let* $t \in I$, $y \in \mathcal{H}$ *and* $x \notin C(t,y)$. *Then,*

$$\partial d_{C(t,y)}(x) = \frac{x - \operatorname{cl}\operatorname{co}\operatorname{Proj}_{C(t,y)}(x)}{d_{C(t,y)}(x)}.$$

The following result can be proved in the same way that [41, Lemma 4.3].

Lemma 1.25 *Let* $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ *be a Lipschitz operator. If* $(\mathcal{H}_1^x)$, $(\mathcal{H}_3^x)$ *and* $(\mathcal{H}_4^x)$ *hold then, for all* $t \in I$, *then the set-valued map* $x \rightrightarrows \partial d(\cdot, C(t,x))(\mathcal{A}(x))$ *is upper semicontinuous from* $\mathcal{H}$ *into* $\mathcal{H}_w$.

The following lemma provides some estimations for the distance function to a moving set depending on time and state.

Lemma 1.26 *Let* $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ *be a map satisfying* $(\mathcal{H}_{\mathcal{A}}^1)$ *and* $(\mathcal{H}_{\mathcal{A}}^2)$, $x, y\colon [0,T] \to \mathcal{H}$ *be two absolutely continuous functions, and* $C\colon [0,T] \times \mathcal{H} \rightrightarrows \mathcal{H}$ *be a set-valued map with nonempty closed values satisfying* $(\mathcal{H}_1^x)$. *Set* $z(t) := \mathcal{A}(x(t))$ *for all* $t \in [0,T]$. *Then,*

(i) *The function* $t \to d(z(t), C(t,y(t)))$ *is absolutely continuous over* $[0,T]$.

(ii) *For all* $t \in]0,T[$, *where* $\dot{y}(t)$ *exists,*

$$\begin{aligned}
&\limsup_{s\downarrow 0} \frac{d_{C(t+s,y(t+s))}(z(t+s)) - d_{C(t,y(t))}(z(t))}{s} \\
&\leq \kappa_r + L\|\dot{y}(t)\| + \limsup_{s\downarrow 0} \frac{d_{C(t,y(t))}(z(t+s)) - d_{C(t,y(t))}(z(t))}{s},
\end{aligned}$$

where κ_r *is the constant given by* $(\mathcal{H}_1^x)$ *with* $r := \sup_{t\in[0,T]} \|z(t)\| < +\infty$.

(iii) *For all* $t \in]0,T[$, *where* $\dot{z}(t)$ *exists,*

$$\limsup_{s\downarrow 0} \frac{d_{C(t,y(t))}(z(t+s)) - d_{C(t,y(t))}(z(t))}{s} \leq \max_{y^*\in\partial d(z(t),C(t,y(t)))} \langle y^*, \dot{z}(t)\rangle .$$

(iv) *For all* $t \in \{t \in]0,T[\colon z(t) \notin C(t,y(t))\}$, *where* $\dot{z}(t)$ *exists,*

$$\lim_{s\downarrow 0} \frac{d_{C(t,y(t))}(z(t+s)) - d_{C(t,y(t))}(z(t))}{s} = \min_{y^*\in\partial d(z(t),C(t,y(t)))} \langle y^*, \dot{z}(t)\rangle .$$

(v) *For every* $x \in \mathcal{H}$ *the set-valued map* $t \rightrightarrows \partial d(\cdot, C(t,y(t)))(\mathcal{A}(x))$ *is measurable.*

Proof. (iii) and (iv) follows from [41, Lemma 4.4]. To prove (i), let us consider $\psi\colon [0,T] \to \mathbb{R}$ be the function defined by

$$\psi(t) := d(z(t), C(t,y(t))),$$

where $z(t) := \mathcal{A}(x(t))$ for all $t \in [0,T]$. Fix $r = \sup_{t\in[0,T]} \|z(t)\| < +\infty$. Then, it follows from $(\mathcal{H}_1^x)$ the existence of $\kappa_r \geq 0$ and $L \in [0, m[$ such that

$$|\psi(t+h) - \psi(t)| \leq \kappa_r |t-s| + L\|z(t+h) - z(t)\|,$$

which implies the absolute continuity of $t \to d(z(t), C(t, y(t)))$.
To prove (ii), consider $t \in \operatorname{int}([0,T])$ where $\dot{y}(t)$ exists. Then, for $s > 0$ small enough,

$$\begin{aligned}\frac{\psi(t+s)-\psi(t)}{s} &= \frac{d(z(t+s), C(t+s, y(t+s))) - d(z(t+s), C(t, y(t)))}{s} \\ &+ \frac{d(z(t+s), C(t, y(t))) - d(z(t), C(t, y(t)))}{s} \\ &\leq \kappa_r + L\frac{\|y(t+s) - y(t)\|}{s} \\ &+ \frac{d(z(t+s), C(t, y(t))) - d(z(t), C(t, y(t)))}{s},\end{aligned}$$

and taking the superior limit, we get the desired inequality. □

The following result shows that the set-valued map $(t,x) \rightrightarrows \frac{1}{2}\partial d^2_{C(t,x)}(\mathcal{A}(x))$ satisfies the conditions of Lemma 1.12.

Proposition 1.27 *Assume that $(\mathcal{H}_{\mathcal{A}}^2)$, $(\mathcal{H}_1^x)$, $(\mathcal{H}_3^x)$ and $(\mathcal{H}_4^x)$ hold. Then, the set-valued map $G\colon [0,T] \times \mathcal{H} \rightrightarrows \mathcal{H}$ defined by $G(t,x) := \frac{1}{2}\partial d^2_{C(t,x)}(\mathcal{A}(x))$ satisfies:*

(i) *for all $x \in \mathcal{H}$ and all $t \in [0,T]$, $G(t,x) = \mathcal{A}(x) - \operatorname{cl}\operatorname{co}\operatorname{Proj}_{C(t,x)}(\mathcal{A}(x))$.*

(ii) *for every $x \in \mathcal{H}$ the set-valued map $G(\cdot, x)$ is measurable.*

(iii) *for every $t \in [0,T]$, $G(t,\cdot)$ is upper semicontinuous from $\mathcal{H}$ into $\mathcal{H}_w$.*

(iv) *for every $t \in [0,T]$ and $B \subset \mathcal{H}$ bounded, $\gamma(G(t,B)) \leq (1+k(t))M\gamma(B)$, where $\gamma = \alpha$ or $\gamma = \beta$ is the Kuratowski or the Hausdorff measure of non-compactness of B and $k \in L^1(0,T)$ is given by $(\mathcal{H}_4^x)$.*

(v) *Let $\mathcal{A}(x_0) \in C(0, x_0)$. Then, for all $t \in [0,T]$ and $x \in \mathcal{H}$,*

$$\|G(t,x)\| := \sup\{\|w\| \colon w \in G(t,x)\} \leq (M+L)\|x - x_0\| + \kappa_r t,$$

where $r = \max\{\|\mathcal{A}(x)\|, \|\mathcal{A}(x_0)\|\}$ and κ_r is the constant given by $(\mathcal{H}_1^x)$.

Proof. (i), (ii) and (iii) follow, respectively, from Lemma 1.24, (v) of Lemma 1.26 and Lemma 1.25. To prove (iv), let $B \subset \mathcal{H}$ be a bounded set included in the ball $r\mathbb{B}$, for some $r > 0$. Define the set-valued map $F(t,x) := \operatorname{Proj}_{C(t,x)}(\mathcal{A}(x))$. Then, for every $t \in I$,

$$\|F(t,B)\| := \sup\{\|w\| \colon w \in F(t,B)\} \leq \tilde{r}(t),$$

where $\tilde{r}_B(t) := \kappa_R t + Lr + L\|x_0\| + \|\mathcal{A}(x_0)\| + 2R$. Indeed, let $z \in F(t,B)$, then there exists $x \in B$ such that $z \in \operatorname{Proj}_{C(t,x)}(\mathcal{A}(x))$. Define

$$R := \max\left\{\sup_{x\in B}\|\mathcal{A}(x)\|, \|\mathcal{A}(x_0)\|\right\}.$$

Thus,

$$\begin{aligned}\|z\| &\leq d_{C(t,x)}(\mathcal{A}(x)) - d_{C(0,x_0)}(\mathcal{A}(x_0)) + \|\mathcal{A}(x)\| \\ &\leq \kappa_R t + L\|x - x_0\| + \|\mathcal{A}(x) - \mathcal{A}(x_0)\| + \|\mathcal{A}(x)\| \\ &\leq \kappa_R t + Lr + L\|x_0\| + \|\mathcal{A}(x_0)\| + 2R = \tilde{r}_B(t),\end{aligned}$$

where κ_R is the constant given by $(\mathcal{H}_1^x)$. Therefore,

$$\begin{aligned}\gamma\left(G(t,B)\right) &\leq \gamma(\mathcal{A}(B)) + \gamma\left(\operatorname{cl}\operatorname{co} F(t,B)\right) \\ &\leq M\gamma(B) + \gamma\left(F(t,B) \cap \tilde{r}_B(t)\mathbb{B}\right) \\ &\leq M\gamma(B) + \gamma\left(C(t,B) \cap \tilde{r}_B(t)\mathbb{B}\right) \\ &\leq (1 + k(t))M\gamma(B),\end{aligned}$$

where we have used $(\mathcal{H}_{\mathcal{A}}^2)$ and the last inequality is due to $(\mathcal{H}_4^x)$.

To prove (v), define $\tilde{G}(t,x) := \mathcal{A}(x) - \operatorname{Proj}_{C(t,x)}(\mathcal{A}(x))$. Then, due to $(\mathcal{H}_1^x)$,

$$\begin{aligned}\|\tilde{G}(t,x)\| &= d(\mathcal{A}(x), C(t,x)) - d(\mathcal{A}(x_0), C(0,x_0)) \\ &\leq (M + L)\|x - x_0\| + \kappa_r t,\end{aligned}$$

where κ_r is given by $(\mathcal{H}_1^x)$ with $r = \max\{\|\mathcal{A}(x)\|, \|\mathcal{A}(x_0)\|\}$. Then, by passing to the closed convex hull in the last inequality, we get the result. □

When the sets $C(t,x)$ are independent of x, the subsmoothness in Proposition 1.27 is no longer required. The following result follows in the same way as Proposition 1.27.

Proposition 1.28 *Assume that* $(\mathcal{H}_{\mathcal{A}}^2)$ $(\mathcal{H}_1)$ *and* $(\mathcal{H}_3)$ *hold. Then, the set-valued map* $G\colon [0,T] \times \mathcal{H} \rightrightarrows \mathcal{H}$ *defined by* $G(t,x) := \frac{1}{2}\partial d_{C(t)}^2(\mathcal{A}(x))$ *satisfies:*

(i) *for all* $x \in \mathcal{H}$ *and all* $t \in [0,T]$, $G(t,x) = \mathcal{A}(x) - \operatorname{cl}\operatorname{co}\operatorname{Proj}_{C(t)}(\mathcal{A}(x))$.

(ii) *for every* $x \in \mathcal{H}$ *the set-valued map* $G(\cdot,x)$ *is measurable.*

(iii) *for every* $t \in [0,T]$, $G(t,\cdot)$ *is upper semicontinuous from* $\mathcal{H}$ *into* $\mathcal{H}_w$.

(iv) *for every* $t \in [0,T]$ *and* $B \subset \mathcal{H}$ *bounded,* $\gamma\left(G(t,B)\right) \leq M\gamma\left(B\right)$, *where* $\gamma = \alpha$ *or* $\gamma = \beta$ *is the Kuratowski or the Hausdorff measure of non-compactness of* B.

(v) *Let* $\mathcal{A}(x_0) \in C(0)$. *Then, for all* $t \in [0,T]$ *and* $x \in \mathcal{H}$,

$$\|G(t,x)\| := \sup\left\{\|w\| \colon w \in G(t,x)\right\} \leq M\|x - x_0\| + \kappa_r t,$$

where $r = \max\{\|\mathcal{A}(x)\|, \|\mathcal{A}(x_0)\|\}$ *and* κ_r *is the constant given by* $(\mathcal{H}_1^x)$.

Let us recall the following consequence of Grönwall's Lemma for absolutely continuous functions, which we will need to get a priori bounds on the approximate solutions family of differential inclusions $(\mathcal{P}_\lambda)$ and $(\mathcal{P}'_\lambda)$.

Lemma 1.29 *Let a, b two positive numbers and $\zeta\colon [T_0, T] \to \mathbb{R}$ be an absolutely continuous function. If for almost every $t \in [T_0, T]$*

$$\dot{\zeta}(t) + b\zeta(t) \ \leq \ a,$$

then for all $t \in [T_0, T]$

$$\zeta(t) \ \leq \ \zeta(T_0)\exp(-b(t - T_0)) \ + \ \frac{a}{b}(1 - \exp(-b(t - T_0))).$$

The following result will be used to prove the existence of solutions for the Moreau-Yosida regularization scheme with single-valued perturbations.

Theorem 1.30 *Let $\mathcal{H}$ be a separable Hilbert space and $I = [0, T]$ for some $T > 0$. Assume that $(\mathcal{H}_1^F)$-$(\mathcal{H}_4^F)$ and $(\mathcal{H}_1^g)$-$(\mathcal{H}_3^g)$ hold. Then, the differential inclusion*

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) + g(t, x(t)) & \text{a.e. } t \in [0, T], \\ x(0) = x_0 \in \mathcal{H}, \end{cases}$$

has at least one solution $x \in \mathrm{AC}([0, T]; \mathcal{H})$. Moreover, if $\mathcal{H}$ is finite-dimensional, then assumption $(\mathcal{H}_2^g)$ can be weakened to $(\mathcal{H}_4^g)$.

Proof. Set

$$r(t) := \left(\|x_0\| + \int_0^t \tilde{d}(s)ds\right)\exp\left(\int_0^t \tilde{c}(s)ds\right),$$

where $\tilde{c}(t) := \bar{c}(t) + c$ and $\tilde{d}(t) := \bar{d}(t) + d$ (see $(\mathcal{H}_3^F)$ and $(\mathcal{H}_3^g)$). Let us consider the following differential inclusion

$$\begin{cases} \dot{x}(t) \in \tilde{F}(t, x) + \tilde{g}(t, x(t)) & \text{a.e. } t \in [0, T], \\ x(0) = x_0 \in \mathcal{H}, \end{cases} \tag{1.9}$$

where $\tilde{F}(t, x) := F(t, x) \cap (\bar{c}(t)\|x\| + \bar{d}(t))\mathbb{B}$, $\tilde{g}(t, x) = g(t, P_{r(t)}(x))$ and

$$P_{r(t)}(x) = \begin{cases} x & \text{if } \|x\| \leq r(t), \\ r(t)\frac{x}{\|x\|} & \text{elsewhere}, \end{cases}$$

is the retraction onto the ball $r(t)\mathbb{B}$. It is clear that $P_{r(t)}$ is 2-Lipschitz and, thus, by virtue of $(\mathcal{H}_2^g)$, for a.e. $t \in [0, T]$, the map $x \mapsto \tilde{g}(t, x)$ is Lipschitz continuous on $\mathcal{H}$. Indeed, let $x, y \in \mathcal{H}$, then

$$\|\tilde{g}(t, x) - \tilde{g}(t, y)\| \leq \mu_{r(T)}\|P_{r(t)}(x) - P_{r(t)}(y)\| \leq 2\mu_{r(T)}\|x - y\|,$$

where $\mu_{r(T)}$ is the Lipschitz constant given by $(\mathcal{H}_2^g)$. Therefore, the map

$$G(t,x) = \tilde{F}(t,x) + \tilde{g}(t,x)$$

satisfies the assumption of [14, Theorem 4]. Hence, the differential inclusion (1.9) admits at least one absolutely continuous solution $x\colon [0,T] \to \mathcal{H}$. Finally, by virtue of Gronwall's inequality, we can prove that $x(t) \in r(t)\mathbb{B}$, which proves that $x\colon [0,T] \to \mathcal{H}$ is a solution of (1.9). $\square$

The following result, known as the Convergence Theorem (see, e.g., [5, p.60]), will be used in Chapter 3.

Proposition 1.31 ([3]) *Let $F\colon I \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a set-valued map satisfying $(\mathcal{H}_1^F)$ and $(\mathcal{H}_2^F)$. Let (u_n), (f_n) be measurable functions such that*

(i) *(u_n) converges almost everywhere on I to a function $u\colon I \to \mathcal{H}$;*

(ii) *(f_n) converges weakly in $L^1(I,\mathcal{H})$ to $f\colon I \to \mathcal{H}$;*

(iii) *For all n, $f_n(t) \in F(t, u_n(t))$ a.e. $t \in I$.*

Then $f(t) \in F(t, u(t))$ a.e. on I.

Chapter 2

Moreau-Yosida Regularization of Degenerate State-Dependent Sweeping Processes

In this chapter, we are interested in the existence of solutions for *degenerate state-dependent sweeping processes* given by

$$\begin{cases} -\dot{x}(t) \in N_{C(t,x(t))}\left(\mathcal{A}(x(t))\right) & \text{a.e. } t \in [T_0, T], \\ x(0) = x_0, \ \mathcal{A}(x_0) \in C(0, x_0), \end{cases} \tag{$\mathcal{DSSP}$}$$

where for a set-valued mapping $C\colon [0,T] \times \mathcal{H} \rightrightarrows \mathcal{H}$ with nonempty closed values of a separable Hilbert $\mathcal{H}$, the set $N_{C(t,x(t))}\left(\mathcal{A}\left(x\left(t\right)\right)\right)$ denotes the Clarke normal cone to $C(t, x(t))$ at $\mathcal{A}(x(t)) \in C(t, x(t))$. The operator $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ is a (possibly) nonlinear Lipschitz and strongly monotone operator. This dynamic was proposed in [43] by Kunze and Monteiro-Marques as a model of quasistatic elastoplasticity for the case when the moving set $C(t)$ is convex and independent of the state.

We note that when $\mathcal{A} = I$ in ($\mathcal{DSSP}$) the dynamics can be seen as *state-dependent sweeping process*, which models quasi-variational inequalities arising, e.g., in the evolution of sandpiles, quasistatic evolution problems with friction, micromechanical damage models for iron materials, among others (see [47] and the references therein).

Another interesting case of ($\mathcal{DSSP}$) is when the sets $C\left(t, x\right)$ are independent of x. The obtained dynamical system is known as *degenerate sweeping process*

$$\begin{cases} -\dot{x}\left(t\right) \in N_{C(t)}\left(\mathcal{A}\left(x\left(t\right)\right)\right) & \text{a.e. } t \in [0, T], \\ x\left(0\right) = x_0, \ \mathcal{A}\left(x_0\right) \in C\left(0\right). \end{cases} \tag{$\mathcal{DSP}$}$$

The system ($\mathcal{DSP}$) was studied by Kunze and Monteiro-Marques in [43–45, 47]. Considering $\mathcal{A} = I$ in the previuos system we obatain the *classical sweeping process* that was introduced by J. J. Moreau in the seventies [52–54]. Typically, the degenerate sweeping process has been studied when the moving sets varies in a Lipschitz or absolutely continuous way with respect to the Hausdorff distance.

In [44], for the system $(\mathcal{DSP})$, the authors proved the existence and uniqueness of Lipschitz continuous solutions. In this case, the sets $C(t)$ are nonempty, closed, and convex of the Hilbert space $\mathcal{H}$ and they moving in a Lipschitz continuous way with respect to the Hausdorff distance and $\mathcal{A}$ is a strongly monotone operator. The method used in this work is a discretization technique based on a surjectivity of the sum of two maximal monotone operators one of these operators is the normal cone.

In the case where the sets $C(t)$ are uniformly prox-regular, Adly and Haddad studied the degenerate sweeping process problem through the *unconstrained differential inclusion*, see [1]. Using some ideas from [64] for the classical sweeping process, they proved that the sets of solutions of their unconstrained problem coincide with the solutions of the system $(\mathcal{DSP})$. The proof is based on the principle of reduction of degenerate sweeping process to unconstrained evolution problem which can be seen as penalization of the subdifferential of the distance function. By assuming that the moving set varies in an absolutely continuous way with respect to the Hausdorff distance, and $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is β-coercive operator an existence and uniqueness result of the degenerate sweeping process is proved when $\dim \mathcal{H} < \infty$.

Concerning the moving sets, they have been studied by considering continuity properties with respect to the Hausdorff distance in the case of the state-dependent sweeping process and the degenerate sweeping process. However, it has been clear that the Hausdorff distance is quite restrictive and thus limits the spectrum of possible applications (see, e.g., [55, 66]). For instance, the moving sets $C(t)$ whose values are hyperplanes or half-spaces do not satisfy the Lipschitzianity with respect to the Hausdorff distance, but the weaker notion of Lipschitzianity with respect to the truncated Hausdorff distance holds. For this reason, in this Chapter, we are interested in the case when the nonregular moving sets (subsmooth and positively α-far) satisfy a notion of Lipschitzianity with respect to the truncated Hausdorff distance for the system $(\mathcal{DSSP})$, and for the system $(\mathcal{DSSP})$ when $\mathcal{A} = I$. We note that these cases are a generalization of the systems studied in literature.

Our results are obtained through the Moreau-Yosida regularization technique. The Moreau-Yosida regularization technique consists of approaching a given differential inclusion by a penalized one, depending on a parameter, whose existence is easier to obtain (for example, by using the classical Cauchy-Lipschitz Theorem), and then to pass to the limit as the parameter goes to zero. This technique has been used several time to deal with sweeping processes (see [46, 48–50, 52, 63, 65] for more details). Recently, it was extended by Jourani and Vilches [41] to deal with state-dependent sweeping processes with Lipschitz nonregular moving sets with respect to the Hausdorff distance. We adapt and extend the techniques from [41] to deal with degenerate state-dependent sweeping processes driven by Lipschitz nonregular moving sets with respect to the truncated Hausdorff distance.

This Chapter is organized as follows. The main result of this chapter is Theorem 2.5, where we establish the convergence (up to a subsequence) of the Moreau-Yosida regularization for the degenerate state-dependent sweeping process under the Lipschitzianity of the moving sets with respect to the truncated Hausdorff distance.

2.1 Existence Results for Degenerate State-Dependent Sweeping Processes

In this section, we prove the existence of Lipschitz solutions for the degenerate state-dependent sweeping process $(\mathcal{DSSP})$ via Moreau-Yosida regularization.

Let $\lambda > 0$ and consider the following differential inclusion

$$\begin{cases} -\dot{x}_\lambda(t) \in \dfrac{1}{2\lambda}\partial d^2_{C(t,x_\lambda(t))}\left(\mathcal{A}(x_\lambda(t))\right) & \text{a.e. } t \in [T_0, T]\,, \\ x_\lambda(0) = x_0, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

where $\mathcal{A}(x_0) \in C(0, x_0)$. The following proposition follows from Lemma 1.12 and Proposition 1.27.

Proposition 2.1 *Assume that $(\mathcal{H}^2_{\mathcal{A}})$, $(\mathcal{H}^x_1)$, $(\mathcal{H}^x_3)$ and $(\mathcal{H}^x_4)$ hold. Then, for every $\lambda > 0$ there exists at least one absolutely continuous solution x_λ of $(\mathcal{P}_\lambda)$.*

Let us define $\varphi_\lambda(t) := d_{C(t,x_\lambda(t))}(\mathcal{A}(x_\lambda(t)))$ for $t \in [T_0, T]$.

Remark 2.2 *Recall that under $(\mathcal{H}^x_3)$, according to Proposition 1.21, for every $\alpha \in \left]\sqrt{\frac{L}{m}}, 1\right]$ there exists $\rho > 0$ such that $(\mathcal{H}^x_2)$ holds.*

The following proposition establishes that trajectories of $(\mathcal{P}_\lambda)$ stay uniformly close (with respect to λ) to the moving sets in a small interval of $I = [T_0, T]$.

Proposition 2.3 *Assume, in addition to the hypotheses of Proposition 2.1, that $(\mathcal{H}^1_{\mathcal{A}})$ holds. Then, for every $R > 0$ there exists $\tau_R \in]0, T]$ (independent of λ) such that if $\lambda < \frac{(m\alpha^2 - L)}{\tilde{\kappa}}\rho$,*

$$\dot{\varphi}_\lambda(t) \le \tilde{\kappa} + \frac{L - m\alpha^2}{\lambda}\varphi_\lambda(t) \quad a.e.\ t \in [0, \tau_R], \tag{2.1}$$

where $\alpha \in \left]\sqrt{\frac{L}{m}}, 1\right]$ and $\rho > 0$ are given by Remark 2.2 and $\tilde{\kappa} := \kappa_{\|\mathcal{A}(x_0)\| + MR}$ is the constant given by $(\mathcal{H}^x_1)$. Moreover,

$$\varphi_\lambda(t) \le \frac{\tilde{\kappa}\lambda}{m\alpha^2 - L} \text{ for all } t \in [0, \tau_R]. \tag{2.2}$$

Proof. Fix $R > 0$ and define the set

$$\Omega_\lambda = \{t \in I \colon \|x_\lambda(t) - x_0\| > R\}.$$

On the one hand, if $\Omega_\lambda = \emptyset$, then $x_\lambda(t) \in \mathbb{B}(x_0, R)$ for all $t \in I$. On the one hand, if $\Omega_\lambda \neq \emptyset$, then we can define

$$\tau_\lambda := \inf\{t \in I \colon \|x_\lambda(t) - x_0\| > R\} > 0.$$

Therefore, in what follows we can assume that

$$x_\lambda(t) \in \mathbb{B}(x_0, R) \text{ for all } t \in [0, \tau_\lambda], \tag{2.3}$$

where we have set $\tau_\lambda = T$ if $\Omega_\lambda = \emptyset$. Thus, by virtue of $(\mathcal{H}_\mathcal{A}^2)$ and (2.3), it follows that

$$\|\mathcal{A}(x_\lambda(t))\| \leq \|\mathcal{A}(x_0)\| + MR \text{ for all } t \in [0, \tau_\lambda]. \tag{2.4}$$

According to Proposition 2.1, the function x_λ is absolutely continuous. Moreover, due to $(\mathcal{H}_1^x)$ and (2.4), for every $t, s \in I$

$$|\varphi_\lambda(t) - \varphi_\lambda(s)| \leq (M + L)\|x_\lambda(t) - x_\lambda(s)\| + \tilde{\kappa}|t - s|,$$

where $\tilde{\kappa} := \kappa_{\|\mathcal{A}(x_0)\|+MR}$ is the constant given by $(\mathcal{H}_1^x)$. Hence φ_λ is absolute continuous. On the one hand, let $t \in [0, \tau_\lambda]$ where $\varphi_\lambda(t) \in]0, \rho[$ and $\dot{x}_\lambda(t)$ exists (recall that $\varphi_\lambda(0) = 0$). Then, by using (iii) from Lemma 1.26, we have

$$\begin{aligned}
\dot{\varphi}_\lambda(t) &\leq \tilde{\kappa} + L\|\dot{x}_\lambda(t)\| + \min_{w \in \partial d_{C(t,x_\lambda(t))}(z_\lambda(t))} \langle w, \dot{z}_\lambda(t)\rangle \\
&\leq \tilde{\kappa} + \frac{L}{\lambda}\varphi_\lambda(t) - \frac{m\alpha^2}{\lambda}\varphi_\lambda(t) \\
&= \tilde{\kappa} - \frac{m\alpha^2 - L}{\lambda}\varphi_\lambda(t),
\end{aligned}$$

where we have used $(\mathcal{H}_3^x)$ and Proposition 1.21.

On the other hand, let $t \in \varphi_\lambda^{-1}(\{0\}) \cap [0, \tau_\lambda]$ where $\dot{x}_\lambda(t)$ exists. Then, according to $(\mathcal{P}_\lambda)$, $\|\dot{x}_\lambda(t)\| = 0$. Indeed,

$$\|\dot{x}_\lambda(t)\| \leq \frac{1}{2\lambda}\sup\{\|z\| \colon z \in \partial d^2_{C(t,x_\lambda(t))}(\mathcal{A}(x_\lambda(t)))\} \leq \frac{\varphi_\lambda(t)}{\lambda} = 0,$$

where we have used the identity $\partial d_S^2(x) = 2d_S(x)\partial d_S(x)$. Then, due to $(\mathcal{H}_1^x)$,

$$\begin{aligned}
\dot{\varphi}_\lambda(t) &= \lim_{h\downarrow 0}\frac{1}{h}\big(d_{C(t+h,x_\lambda(t+h))}(z_\lambda(t+h)) - d_{C(t,x_\lambda(t))}(z_\lambda(t+h)) \\
&\qquad\qquad + d_{C(t,x_\lambda(t))}(z_\lambda(t+h)) - d_{C(t,x_\lambda(t))}(z_\lambda(t))\big) \\
&\leq \tilde{\kappa} + L\|\dot{x}_\lambda(t)\| + \lim_{h\downarrow 0}\frac{1}{h}\big(d_{C(t,x_\lambda(t))}(z_\lambda(t+h)) - d_{C(t,x_\lambda(t))}(z_\lambda(t))\big) \\
&\leq \tilde{\kappa} + (M + L)\|\dot{x}_\lambda(t)\| \\
&\leq \tilde{\kappa} + \frac{M + L}{\lambda}\varphi_\lambda(t) \\
&= \tilde{\kappa} - \frac{m\alpha^2 - L}{\lambda}\varphi_\lambda(t).
\end{aligned}$$

Also, we have that $\varphi_\lambda(t) < \rho$ for all $t \in [0, \tau_\lambda]$. Otherwise, since $\varphi_\lambda^{-1}(]-\infty, \rho[)$ is open and $0 \in \varphi_\lambda^{-1}(]-\infty, \rho[)$, there would exist $t^* \in]0, \tau_\lambda]$ such that $[0, t^*[\subset \varphi_\lambda^{-1}(]-\infty, \rho[)$ and $\varphi_\lambda(t^*) = \rho$. Then,

$$\dot{\varphi}_\lambda(t) \leq \tilde{\kappa} - \frac{m\alpha^2 - L}{\lambda}\varphi_\lambda(t) \quad \text{a.e. } t \in [0, t^*[,$$

which, by virtue of Grönwall's inequality, entrain that, for every $t \in [0, t^*[$

$$\varphi_\lambda(t) \leq \frac{\tilde{\kappa}\lambda}{m\alpha^2 - L}\left(1 - \exp\left(-\frac{m\alpha^2 - L}{\lambda}t\right)\right) \leq \frac{\tilde{\kappa}\lambda}{m\alpha^2 - L} < \rho,$$

that implies that $\varphi_\lambda(t^*) < \rho$, which is not possible. Thus, we have proved that φ_λ satisfies (2.1) and (2.2) in the interval $[0, \tau_\lambda]$.

It remains to prove that there exists $\tau_R \in]0, T]$ such that

$$\tau_R \leq \tau_\lambda \text{ for all } \lambda < \frac{(m\alpha^2 - L)}{\tilde{\kappa}}\rho. \tag{2.5}$$

Indeed, since x_λ satisfies $(\mathcal{P}_\lambda)$, we have

$$\|\dot{x}_\lambda(t)\| \leq \frac{1}{2\lambda}\sup\{\|z\| \colon z \in \partial d^2_{C(t,x_\lambda(t))}(\mathcal{A}(x_\lambda(t)))\} \leq \frac{\varphi_\lambda(t)}{\lambda} \leq \frac{\tilde{\kappa}}{m\alpha^2 - L},$$

where we have used the identity $\partial d_S^2(x) = 2d_S(x)\partial d_S(x)$. Thus, x_λ is $\frac{\tilde{\kappa}}{m\alpha^2 - L}$ Lipschitz on $[0, \tau_\lambda]$. Hence, by the continuity of x_λ and the definition of τ_λ, we obtain

$$R \leq \|x_\lambda(\tau_\lambda) - x_0\| \leq \int_0^{\tau_\lambda} \|\dot{x}_\lambda(s)\| ds \leq \frac{\tilde{\kappa}}{m\alpha^2 - L}\tau_\lambda,$$

which implies the existence of $\tau_R \in]0, T]$ such that (2.5) holds. Finally, we have prove that (2.1) and (2.2) holds for the interval $[0, \tau_R]$, which ends the proof. □

As a corollary of the last proposition, we obtain that x_λ is $\frac{\tilde{\kappa}}{m\alpha^2 - L}$-Lipschitz on $[0, \tau_R]$.

Corollary 2.4 *For every $\lambda < \frac{m\alpha^2 - L}{\tilde{\kappa}}\rho$ the function x_λ is $\frac{\tilde{\kappa}}{m\alpha^2 - L}$-Lipschitz on $[0, \tau_R]$.*

Let $(\lambda_n)_n$ be a sequence converging to 0. The next result shows the existence of a subsequence $(\lambda_{n_k})_k$ of $(\lambda_n)_n$ such that $\left(x_{\lambda_{n_k}}\right)_k$ converges (in the sense of Lemma 1.7) to a solution of $(\mathcal{DSSP})$ over all the interval I. We perform the analysis in the interval $[0, \tau_R]$ for some R and then we extend iteratively the solution to all the interval.

The next theorem extends all known results on State-Dependent Sweeping Processes and Degenerate Sweeping Processes.

Theorem 2.5 *Assume that $(\mathcal{H}^1_{\mathcal{A}})$, $(\mathcal{H}^2_{\mathcal{A}})$, $(\mathcal{H}^x_1)$, $(\mathcal{H}^x_3)$ and $(\mathcal{H}^x_4)$ hold. Then, there exists at least one solution $x \in \mathrm{AC}(I; \mathcal{H})$ of $(\mathcal{DSSP})$.*

Proof. Fix $R > 0$ and τ_R from Proposition 2.3. Then, by virtue of Proposition 2.3, the sequences $(x_{\lambda_n})_n$ and $(\mathcal{A}(x_{\lambda_n}))_n$ satisfy the hypotheses of Lemma 1.7 over the interval $[0, \tau_R]$ with

$$\psi(t) := \frac{\tilde{\kappa}}{m\alpha^2 - L} \text{ and } \tilde{\psi}(t) := \frac{\tilde{\kappa}M}{m\alpha^2 - L},$$

respectively. Therefore, there exist subsequences $(x_{\lambda_{n_k}})_k$ and $(\mathcal{A}(x_{\lambda_{n_k}}))_k$ of $(x_{\lambda_n})_n$ and $(\mathcal{A}(x_{\lambda_n}))_n$, respectively and functions $x\colon [0,\tau_R] \to \mathcal{H}$ and $\mathcal{A}x\colon [0,\tau_R] \to \mathcal{H}$ satisfying the hypotheses (i)-(iv) of Lemma 1.7. For simplicity, we write x_k and $\mathcal{A}(x_k)$ instead of $x_{\lambda_{n_k}}$ and $\mathcal{A}(x_{\lambda_{n_k}})$, respectively.

Claim 1 $(\mathcal{A}(x_k(t)))_k$ and $(x_k(t))_k$ are relatively compact in $\mathcal{H}$ for all $t \in [0,\tau_R]$.

Proof of Claim 1: Let $t \in [0,\tau_R]$. Let us consider $y_k(t) \in \mathrm{Proj}_{C(t,x_k(t))}(\mathcal{A}(x_k(t)))$. Then, $\|\mathcal{A}(x_k(t)) - y_k(t)\| = d_{C(t,x_k(t))}(\mathcal{A}(x_k(t)))$. Thus,

$$\begin{aligned}
\|y_k(t)\| &\le d_{C(t,x_k(t))}(\mathcal{A}(x_k(t))) + \|\mathcal{A}(x_k(t))\| \\
&\le \frac{\tilde{\kappa}\lambda_{n_k}}{m\alpha^2 - L} + M\|x_k(t) - x_0\| + \|\mathcal{A}(x_0)\| \\
&\le \tilde{r}(t) := \frac{\tilde{\kappa}}{m\alpha^2 - L}(\lambda_{n_k} + Mt) + \|\mathcal{A}(x_0)\|.
\end{aligned}$$

Also, since $(\mathcal{A}(x_k(t)) - y_k(t))$ converges to 0,

$$\gamma(\{\mathcal{A}(x_k(t))\colon k \in \mathbb{N}\}) = \gamma(\{y_k(t)\colon k \in \mathbb{N}\}).$$

Therefore, if $A := \{\mathcal{A}(x_k(t))\colon k \in \mathbb{N}\}$,

$$\gamma(A) = \gamma(\{y_k(t)\colon k \in \mathbb{N}\}) \le \gamma(C(t, \{x_k(t)\colon k \in \mathbb{N}\}) \cap \tilde{r}(t)\mathbb{B}) \le k(t)\gamma(A),$$

where we have used $(\mathcal{H}_4^x)$. Finally, since $k(t) < 1$, we obtain that $\gamma(A) = 0$, which shows that $(\mathcal{A}(x_k(t)))_k$ is relatively compact in $\mathcal{H}$. Finally, $(x_k(t))_k$ is relatively compact in $\mathcal{H}$ by virtue of $(\mathcal{H}_{\mathcal{A}}^2)$. □

Claim 2 $\mathcal{A}(x(t)) \in C(t, x(t))$ for all $t \in [0,\tau_R]$.

Proof of Claim 2: As a result of Claim 1 and the weak convergence $x_k(t) \rightharpoonup x(t)$ for all $t \in [0,\tau_R]$ (due to (i) of Lemma 1.7), we obtain the strong convergence of $(x_k(t))_k$ to $x(t)$ for all $t \in [0,\tau_R]$. Therefore, due to $(\mathcal{H}_1^x)$,

$$\begin{aligned}
d_{C(t,x(t))}(\mathcal{A}(x(t))) &\le \liminf_{k\to\infty} \left(d_{C(t,x_k(t))}(\mathcal{A}(x_k(t))) + (M+L)\|x_k(t) - x(t)\|\right) \\
&\le \liminf_{k\to\infty} \left(\frac{\tilde{\kappa}\lambda_{n_k}}{m\alpha^2 - L} + (M+L)\|x_k(t) - x(t)\|\right) = 0,
\end{aligned}$$

which proves the claim. □

Now, we prove that x is a solution of $(\mathcal{DSSP})$ over the interval $[0,\tau_R]$. Define

$$\tilde{F}(t,x) := \mathrm{cl\,co}\left(\frac{\tilde{\kappa}}{m\alpha^2 - L}\partial d_{C(t,x)}(\mathcal{A}(x)) \cup \{0\}\right) \text{ for } (t,x) \in [0,\tau_R] \times \mathcal{H}.$$

Then, for a.e. $t \in [0,\tau_R]$

$$-\dot{x}_k(t) \in \frac{1}{2\lambda}\partial d^2_{C(t,x_k(t))}(\mathcal{A}(x_k(t))) \subset \tilde{F}(t, x_k(t)),$$

where we have used Proposition 2.3.

Claim 3: $\tilde{F}$ has closed and convex values and satisfies:

(i) for each $x \in \mathcal{H}$, $\tilde{F}(\cdot, x)$ is measurable;

(ii) for all $t \in [0, \tau_R]$, $\tilde{F}(t, \cdot)$ is upper semicontinuous from $\mathcal{H}$ into $\mathcal{H}_w$;

(iii) if $\mathcal{A}(x) \in C(t, x)$ then $\tilde{F}(t, x) = \frac{\tilde{\kappa}}{m\alpha^2 - L} \partial d_{C(t,x)}(\mathcal{A}(x))$.

Proof of Claim 3: To prove Claim 3, we follow the ideas from [41, Theorem 6.1]. Define $G(t, x) := \frac{\tilde{\kappa}}{m\alpha^2 - L} \partial d_{C(t,x)}(\mathcal{A}(x)) \cup \{0\}$. We note that $G(\cdot, x)$ is measurable as the union of two measurable set-valued maps (see [4, Lemma 18.4]). Let us define $\Gamma(t) := \tilde{F}(t, x)$. Then, Γ takes weakly compact convex values. Fixing any $d \in \mathcal{H}$, by virtue of [38, Proposition 2.2.39], is enough to verify that the support function $t \mapsto \sigma(d, \Gamma(t)) := \sup\{\langle v, d\rangle : v \in \Gamma(t)\}$ is measurable. Thus,

$$\sigma(d, \Gamma(t)) := \sup\{\langle v, d\rangle : v \in \Gamma(t)\} = \sup\{\langle v, d\rangle : v \in G(t, x)\},$$

is measurable because $G(\cdot, x)$ is measurable. Thus (i) holds. Assertion (ii) follows directly from [4, Theorem 17.27 and 17.3]. Finally, if $\mathcal{A}(x) \in C(t, x)$ then $0 \in \partial d_{C(t,x)}(\mathcal{A}(x))$. Hence, using the fact that the subdifferential of a locally Lipschitz function is closed and convex,

$$\tilde{F}(t, x) = \operatorname{cl}\operatorname{co}\left(\frac{\tilde{\kappa}}{m\alpha^2 - L} \partial d_{C(t,x)}(\mathcal{A}(x))\right) = \frac{\tilde{\kappa}}{m\alpha^2 - L} \partial d_{C(t,x)}(\mathcal{A}(x)),$$

which shows (iii). □

In summary, we have

(i) for each $x \in \mathcal{H}$, $\tilde{F}(\cdot, x)$ is measurable.

(ii) for all $t \in [0, \tau_R]$, $\tilde{F}(t, \cdot)$ is upper semicontinuous from $\mathcal{H}$ into $\mathcal{H}_w$.

(iii) $\dot{x}_k \rightharpoonup \dot{x}$ in $L^1([0, \tau_R]; \mathcal{H})$ as $k \to +\infty$.

(iv) $x_k(t) \to x(t)$ as $k \to +\infty$ for all $t \in [0, \tau_R]$.

(v) $\mathcal{A}(x_k(t)) \to \mathcal{A}(x(t))$ as $k \to +\infty$ for all $t \in [0, \tau_R]$.

(vi) $-\dot{x}_k(t) \in \tilde{F}(t, x_k(t))$ for a.e. $t \in [0, \tau_R]$.

These conditions and Proposition 1.31 imply that x satisfies

$$\begin{cases} -\dot{x}(t) \in \tilde{F}(t, x(t)) & \text{a.e. } t \in [0, \tau_R], \\ x(0) = x_0 \in C(0, x_0), \end{cases}$$

which, according to Claim 3, implies that x is a solution of

$$\begin{cases} -\dot{x}(t) \in \dfrac{\tilde{\kappa}}{m\alpha^2 - L} \partial d_{C(t,x(t))}(\mathcal{A}(x(t))) & \text{a.e. } t \in [0, \tau_R], \\ x(0) = x_0 \in C(0, x_0). \end{cases}$$

Therefore, by virtue of (1.2) and Claim 2, x is a solution of $(\mathcal{DSSP})$ over $[0, \tau_R]$. Finally, to extend the solution over all the interval $[0, T]$, we can use [39, Lemma 5.7]. □

2.1.1 The Case of the Degenerate Sweeping Process

This subsection is devoted to the degenerate sweeping process:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); \mathcal{A}(x(t))\right) \quad \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in C(0). \end{cases} \tag{2.6}$$

This differential inclusion can be seen as a particular case of $(\mathcal{DSSP})$ when the sets $C(t,x)$ are state independent. We show that Theorem 2.5 is valid under the weaker hypothesis $(\mathcal{H}_2)$ instead of $(\mathcal{H}_3^x)$. The following theorem improves the results from [39, 40] regarding the existence for sweeping process with nonregular sets and also complements the results from [1, 42] regarding the existence of degenerate sweeping process with uniformly prox-regular sets.

Theorem 2.6 *Assume that $(\mathcal{H}_{\mathcal{A}}^1)$, $(\mathcal{H}_{\mathcal{A}}^2)$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ hold. Then, there exists at least one solution $x \in \mathrm{AC}\left(I;\mathcal{H}\right)$ of* (2.6).

Proof. According to the proof of Theorem 2.5, we observe that $(\mathcal{H}_3^x)$ was used to obtain $(\mathcal{H}_2^x)$ and the upper semicontinuity of $\partial d_{C(t,\cdot)}(\cdot)$ from $\mathcal{H}$ into $\mathcal{H}_w$ for all $t \in [0,T]$. Since in the present case these two properties hold under $(\mathcal{H}_2)$ (see Proposition 1.28), it is sufficient to adapt the proof of Theorem 2.5 to get the result. □

Chapter 3

Perturbed Degenerate State-Dependent Sweeping Processes

The degenerate sweeping process corresponds to the case where an operator is added "inside" the normal cone on the sweeping process. This dynamic was proposed by Kunze and Monteiro-Marques as a model for quasistatic elastoplasticity (see, e.g., [43]). Since then, the degenerate sweeping process has been studied by several authors in the context of convex and prox-regular sets (see, e.g., [1, 42–45]). However, there are not results about *perturbated degenerate state-dependent sweeping processes* with nonregular moving sets (subsmooth and positively α-far). This Chapter is intended to cover this gap by studying this issue when the right-hand side of the system ($\mathcal{DSSP}$) is perturbed by a locally μ-Lipshchitz function with respect to the state variable, measurable in the time variable which satisfies a linear growth condition. To be more precise, the perturbed systems takes the following form

$$\begin{cases} \dot{x}(t) \in -N_{C(t,x(t))}\left(\mathcal{A}(x(t))\right) + g(t,x(t)) & \text{a.e. } t \in [0,T], \\ x(0) = x_0, \ \mathcal{A}(x_0) \in C(0,x_0). \end{cases} \tag{$\mathcal{DSSP}_g$}$$

Here, the set $N_{C(t,x(t))}\left(\mathcal{A}(x(t))\right)$ stands for the Clarke normal cone to $C(t,x(t))$ at $\mathcal{A}(x(t)) \in C(t,x(t))$ where $C\colon [0,T]\times\mathcal{H} \rightrightarrows \mathcal{H}$ is a set-valued mapping with nonempty closed values of a separable Hilbert $\mathcal{H}$. The operator $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ is (possibly) nonlinear Lipschitz and strongly monotone, and the perturbation is a single-valued function $g\colon [0,T] \times \mathcal{H} \to \mathcal{H}$ defined as above.

We note that the case $\mathcal{A} = I$ in the system ($\mathcal{DSSP}_g$) have been well studied by Vilches in [68] and includes the perturbed classical sweeping process introduced and studied by Moreau [52, 54] to deal with contact problems in mechanical systems (see [47] for an introduction to the subject).

For the system ($\mathcal{DSSP}_g$) we prove the existence of solutions for regular (convex / prox-regular) and nonregular (subsmooth and positively α-far) moving sets which varies in a Lipschitz way with respect to the truncated Hausdorff distance. We prove our result by means of the Moreau-Yosida regularization (see, e.g., [41,46,48–50,52,63,65]) and by using an appropriate existence result for differential inclusions (see Theorem 1.30 below). The Moreau-Yosida regularization technique consists of approaching a given differential inclusion by a penalized one, depending on a parameter, whose existence is easier to obtain (for example, by using the classical Cauchy-Lipschitz Theorem), and then to pass to the limit as the parameter goes to zero. We refer to [55] for a detailed survey on the subject.

Let us finally prove the existence of solutions for integral perturbations of degenerate state-dependent sweeping processes:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t,x(t)); \mathcal{A}(x(t))\right) + g_1(t,x(t)) + \displaystyle\int_0^t g_2(s,x(s))ds \quad \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in \mathcal{A}^{-1}(C(0,x_0)), \end{cases} \tag{$\mathcal{IDSP}$}$$

where $C\colon [0,T] \times \mathcal{H} \rightrightarrows \mathcal{H}$ is a set-valued map with nonempty and closed values, $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ is a nonlinear operator and $g_1, g_2\colon [0,T] \times \mathcal{H} \to \mathcal{H}$ are two single-valued functions. Several recent works have been concerned with integrally perturbed versions of the sweeping process [15–17, 23, 29]. In these works, the authors considered convex or prox-regular moving sets that varies in a Lipschitz or absolutely continuous way with respect to the Hausdorff distance. Indeed, note that when $g_1 \equiv 0$, $\mathcal{A} = I$, and the moving sets are independent to the the state variable $C(t,x) \equiv C(t)$ for all $t \in [0,T]$ the system ($\mathcal{IDSP}$) takes the form of an *integral perturbation of Moreau's sweeping process*

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + \displaystyle\int_0^t g_2(s,x(s))ds \quad \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in C(0). \end{cases} \tag{$\mathcal{ISP}$}$$

Furthermore, considering $C(t) := \mathbb{K}^N$ the nonnegative orthant of $\mathbb{R}^N$, the system ($\mathcal{ISP}$) describes the flow of a compressible pressureless fluid when the fluid self-interacts through a force field generated by the fluid itself and a sticky particle dynamics is assumed (see [23] for details). On the other hand, in [29], Colombo and Kozaily considered the system ($\mathcal{ISP}$) for ρ-uniformly prox-regular moving sets. They established the existence and uniqueness of the solution of the system in the infinite-dimensional separable Hilbert space setting by means Moreau-Yosida regularization technique together with an appropriate Gronwall's inequality.

Let us also mention the work [16], where the type of perturbation is integro-differential which is known as *integro-differential sweeping process*. The methodology to prove the existence and uniqueness for this system is based on an approximation via sweeping process with the single-valued perturbation depending only on time, where the existence and uniqueness for this systems hold thanks to the ρ-hypomonotonicity property of the proximal normal cone.

The chapter is organized as follows. In Section 3.1, we present the central result of this work (Theorem 3.5), namely, the convergence (up to a subsequence) of the Moreau-Yosida regularization for perturbed degenerate state-dependent sweeping processes under the Lipschitzianity of the moving sets with respect to the truncated Hausdorff distance. Next, in Section 3.2 provides the existence of solutions for integro-differential sweeping processes. Later on, in Section 3.3, we apply our theoretical results to the existence of solutions for the online mirror descent method.

3.1 Main result

This section aims to extend the results from [41, 56, 68] to perturbed degenerate state-dependent sweeping processes. We use the Moreau-Yosida regularization technique from [41, 56, 68] to deal with $(\mathcal{DSSP}_g)$.

Let $\lambda > 0$ and consider the following differential inclusion: Find $x_\lambda\colon [0,T] \to \mathcal{H}$ with $x_\lambda(0) = x_0$ and such that for a.e. $t \in [0,T]$

$$-\dot{x}_\lambda(t) \in \frac{1}{2\lambda}\partial d^2_{C(t,x_\lambda(t))}(\mathcal{A}(x_\lambda(t))) + g(t, x_\lambda(t)) \qquad (\mathcal{P}'_\lambda)$$

where $\mathcal{A}(x_0) \in C(0, x_0)$. It follows from Theorem 1.30 and [56, Proposition 4.1] that the approached problem $(\mathcal{P}_\lambda)$ has an absolutely continuous solution.

Proposition 3.1 *Assume, in addition to $(\mathcal{H}^2_\mathcal{A})$, $(\mathcal{H}^x_1)$, $(\mathcal{H}^x_3)$, and $(\mathcal{H}^x_4)$, that $(\mathcal{H}^g_1)$-$(\mathcal{H}^g_3)$ hold. Then, for every $\lambda > 0$ there exists at least one absolutely continuous solution x_λ of $(\mathcal{P}'_\lambda)$.*

Let us define the function $\varphi_\lambda(t) := d_{C(t,x_\lambda(t))}(\mathcal{A}(x_\lambda(t)))$ for $t \in [0,T]$.

Remark 3.2 *Recall that under $(\mathcal{H}^x_3)$, according to Proposition 1.21, for every $\alpha \in \left]\sqrt{\frac{L}{m}}, 1\right]$ there exists $\rho > 0$ such that $(\mathcal{H}^x_2)$ holds.*

The following proposition establishes that trajectories of $(\mathcal{P}'_\lambda)$ stay uniformly close (with respect to λ) to the moving sets in a small interval of $[0,T]$. The proof is based on ideas from [56].

Proposition 3.3 *Assume, in addition to the hypotheses of Proposition 3.1, that $(\mathcal{H}^1_\mathcal{A})$ holds. Then, for every $R > 0$ there exists $\tau_R \in]0,T]$ (independent of λ) such that if $\lambda < \lambda_0 := \frac{(m\alpha^2 - L)}{\tilde{\kappa}_R + (L+M)(c(R+\|x_0\|)+d)}\rho$,*

$$\dot{\varphi}_\lambda(t) \leq \tilde{\kappa}_R + (L+M)(c(R+\|x_0\|)+d) - \frac{m\alpha^2 - L}{\lambda}\varphi_\lambda(t) \quad a.e.\ t \in [0,\tau_R], \qquad (3.1)$$

where $\alpha \in \left]\sqrt{\frac{L}{m}}, 1\right]$ and $\rho > 0$ are given by Remark 3.2 and $\tilde{\kappa}_R := \kappa_{\|\mathcal{A}(x_0)\|+MR}$ is the constant given by $(\mathcal{H}^x_1)$ and c, $d \geq 0$ are the constants given by $(\mathcal{H}^g_3)$. Moreover,

$$\varphi_\lambda(t) \leq \frac{\tilde{\kappa}_R + (L+M)(c(R+\|x_0\|)+d)}{m\alpha^2 - L}\lambda \quad \text{for all } t \in [0,\tau_R]. \qquad (3.2)$$

Proof. Fix $R > 0$ and define the set

$$\Omega_\lambda := \{t \in I\colon \|x_\lambda(t) - x_0\| > R\}.$$

On the hand, if $\Omega_\lambda = \emptyset$, then

$$x_\lambda(t) \in \mathbb{B}(x_0, R) \text{ for all } t \in I.$$

On the other hand, if $\Omega_\lambda \neq \emptyset$, then we can define

$$\tau_\lambda := \inf\{t \in I \colon \|x_\lambda(t) - x_0\| > R\} > 0.$$

Therefore, in what follows, we can assume that

$$x_\lambda(t) \in \mathbb{B}(x_0, R) \text{ for all } t \in [0, \tau_\lambda], \tag{3.3}$$

where we set $\tau_\lambda := T$ if $\Omega_\lambda = \emptyset$. Thus, by virtue of (3.3) and $(\mathcal{H}^2_{\mathcal{A}})$, it follows that

$$\|\mathcal{A}(x_\lambda(t))\| \leq \|\mathcal{A}(x_0)\| + MR \text{ for all } t \in [0, \tau_\lambda]. \tag{3.4}$$

Moreover, according to Proposition 2.1, x_λ is absolutely continuous and, due to (3.4) and $(\mathcal{H}^x_1)$, for every $s, t \in [0, \tau_\lambda]$,

$$|\varphi_\lambda(t) - \varphi_\lambda(s)| \leq (L + M)\|x_\lambda(t) - x_\lambda(s)\| + \tilde{\kappa}_R|t - s|,$$

where $\tilde{\kappa}_R := \kappa_{\|\mathcal{A}(x_0)\|+MR}$ is the constant given by $(\mathcal{H}^x_1)$. Hence, φ_λ is absolutely continuous.

Let $\lambda < \lambda_0$. Then, by using similar arguments to the given in [56, Proposition 5.2], it follows that

$$\dot{\varphi}_\lambda(t) \leq \tilde{\kappa}_R + \frac{L - m\alpha^2}{\lambda}\varphi_\lambda(t) + (L + M)\|g(t, x)\| \quad \text{ a.e. } t \in [0, \tau_\lambda],$$

which implies (3.2) in the interval $[0, \tau_\lambda]$.

It remains to prove that there exists $\tau_R \in]0, T]$ such that $\tau_R \leq \tau_\lambda$ for all $\lambda < \lambda_0$. Indeed, for a.e. $t \in [0, \tau_\lambda]$

$$\begin{aligned} \|\dot{x}_\lambda(t)\| &\leq \frac{1}{\lambda}\varphi_\lambda(t) + \|g(t, x_\lambda(t))\| \\ &\leq \frac{\tilde{\kappa}_R + (L + M)(c(R + \|x_0\|) + d)}{m\alpha^2 - L} + (c(R + \|x_0\|) + d). \end{aligned}$$

Hence, by continuity of x_λ and the definition of τ_λ,

$$\begin{aligned} R &\leq \|x_\lambda(\tau_\lambda) - x_0\| \\ &\leq \int_0^{\tau_\lambda} \|\dot{x}_\lambda(s)\| ds \\ &\leq \left(\frac{\tilde{\kappa}_R + (L + M)(c(R + \|x_0\|) + d)}{m\alpha^2 - L} + (c(R + \|x_0\|) + d)\right)\tau_\lambda, \end{aligned}$$

which implies the existence of $\tau_R \in]0, T]$ such that $\tau_R \leq \tau_\lambda$ for all $\lambda < \lambda_0$. Finally, we have proved that (3.1) and (3.2) hold for the interval $[0, \tau_R]$, which ends the proof. □

As a consequence of the last proposition, we obtain that x_λ is Lipschitz continuous on $[0, \tau_R]$ for $\lambda > 0$ small enough.

Corollary 3.4 *For $\lambda < \lambda_0$ the function x_λ is Lipschitz continuous of constant $C_R := \frac{\tilde{\kappa}_R+(L+M)(c(R+\|x_0\|)+d)}{m\alpha^2-L} + (c(R+\|x_0\|)+d)$ on $[0,\tau_R]$.*

Let $(\lambda_n)_n$ be a sequence converging to 0. The next result shows the existence of a subsequence $(\lambda_{n_k})_k$ of $(\lambda_n)_n$ such that $\left(x_{\lambda_{n_k}}\right)_k$ converges (in the sense of Lemma 1.7) to a solution of $(\mathcal{DSSP}_g)$ over all the interval $[0,T]$. We perform the analysis in the interval $[0,\tau_R]$ for some $R>0$ and then we extend iteratively the solution to all the interval.

The next theorem extends all known results on state-dependent sweeping processes and degenerate sweeping processes.

Theorem 3.5 *Assume that $(\mathcal{H}^1_{\mathcal{A}})$, $(\mathcal{H}^2_{\mathcal{A}})$, $(\mathcal{H}^x_1)$, $(\mathcal{H}^x_3)$ and $(\mathcal{H}^x_4)$, that $(\mathcal{H}^g_1)$, $(\mathcal{H}^g_2)$ and $(\mathcal{H}^g_3)$ hold. Then, the problem $(\mathcal{DSSP}_g)$ admits at least one absolutely continuous solution. Moreover, if $\mathcal{H}$ is finite-dimensional, then assumption $(\mathcal{H}^g_2)$ can be weakened to $(\mathcal{H}^g_4)$.*

Proof. Fix $R>0$ and τ_R from Proposition 3.3. Then, by virtue of Proposition 3.3, the sequences $(x_{\lambda_n})_n$ and $(\mathcal{A}(x_{\lambda_n}))_n$ satisfy the hypotheses of Lemma 1.7 over the interval $[0,\tau_R]$ with

$$\psi(t) := \frac{\tilde{\kappa}_R+(L+M)(c(R+\|x_0\|)+d)}{m\alpha^2-L} + (c(R+\|x_0\|)+d) \text{ and } \tilde{\psi}(t) := M\psi(t),$$

respectively. Therefore, there exist subsequences $(x_{\lambda_{n_k}})_k$ and $(\mathcal{A}(x_{\lambda_{n_k}}))_k$ of $(x_{\lambda_n})_n$ and $(\mathcal{A}(x_{\lambda_n}))_n$, respectively and functions $x\colon [0,\tau_R]\to\mathcal{H}$ and $\mathcal{A}x\colon [0,\tau_R]\to\mathcal{H}$ satisfying the hypotheses (i)-(iv) of Lemma 1.7. For simplicity, we write x_k and $\mathcal{A}(x_k)$ instead of $x_{\lambda_{n_k}}$ and $\mathcal{A}(x_{\lambda_{n_k}})$, respectively.

Claim 1: $(\mathcal{A}(x_k(t)))_k$ and $(x_k(t))_k$ are relatively compact in $\mathcal{H}$ for all $t\in[0,\tau_R]$.

Proof of Claim 1: Let $t\in[0,\tau_R]$. Let us consider $y_k(t)\in \operatorname{Proj}_{C(t,x_k(t))}(\mathcal{A}(x_k(t)))$. Then, $\|\mathcal{A}(x_k(t))-y_k(t)\| = d_{C(t,x_k(t))}(\mathcal{A}(x_k(t)))$. Thus,

$$\begin{aligned}
\|y_k(t)\| &\le d_{C(t,x_k(t))}(\mathcal{A}(x_k(t))) + \|\mathcal{A}(x_k(t))\| \\
&\le \left[\frac{\tilde{\kappa}_R+(L+M)(c(R+\|x_0\|)+d)}{m\alpha^2-L}\right]\lambda_{n_k} + M\|x_k(t)-x_0\| + \|\mathcal{A}(x_0)\| \\
&\le \tilde{r}(t) := \left[\frac{\tilde{\kappa}_R+(L+M)(c(R+\|x_0\|)+d)}{m\alpha^2-L}\right]\lambda_{n_k} + MC_Rt + \|\mathcal{A}(x_0)\|.
\end{aligned}$$

Also, since $(\mathcal{A}(x_k(t))-y_k(t))$ converges to 0,

$$\gamma\left(\{\mathcal{A}(x_k(t))\colon k\in\mathbb{N}\}\right) = \gamma\left(\{y_k(t)\colon k\in\mathbb{N}\}\right).$$

Therefore, if $A := \{\mathcal{A}(x_k(t))\colon k\in\mathbb{N}\}$,

$$\gamma(A) = \gamma(\{y_k(t)\colon k\in\mathbb{N}\}) \le \gamma\left(C\left(t,\{x_k(t)\colon k\in\mathbb{N}\}\right)\cap \tilde{r}(t)\mathbb{B}\right) \le k(t)\gamma(A),$$

where we have used $(\mathcal{H}^x_4)$. Finally, since $k(t)<1$, we obtain that $\gamma(A)=0$, which shows that $(\mathcal{A}(x_k(t)))_k$ is relatively compact in $\mathcal{H}$. Finally, $(x_k(t))_k$ is relatively compact in $\mathcal{H}$ by virtue of $(\mathcal{H}^2_{\mathcal{A}})$. ☐

Claim 2: $\mathcal{A}(x(t)) \in C(t, x(t))$ for all $t \in [0, \tau_R]$.

Proof of Claim 2: As a result of Claim 1 and the weak convergence $x_k(t) \rightharpoonup x(t)$ for all $t \in [0, \tau_R]$ (due to (i) of Lemma 1.7), we obtain the strong convergence of $(x_k(t))_k$ to $x(t)$ for all $t \in [0, \tau_R]$. Therefore, due to $(\mathcal{H}_1^x)$,

$$\begin{aligned} d_{C(t,x(t))}(\mathcal{A}(x(t))) &\leq \liminf_{k\to\infty} \left(d_{C(t,x_k(t))}\left(\mathcal{A}(x_k(t))\right) + (M+L)\|x_k(t) - x(t)\| \right) \\ &\leq \liminf_{k\to\infty} \left(\left[\frac{\tilde{\kappa}_R + (L+M)(c(R+\|x_0\|)+d)}{m\alpha^2 - L} \right] \lambda_{n_k} + (M+L)\|x_k(t) - x(t)\| \right) = 0, \end{aligned}$$

which proves the claim. □

Now, we will show that x is a solution of $(\mathcal{DSSP}_g)$ over the interval $[0, \tau_R]$.

Define
$$\tilde{F}(t, x) := -\overline{\text{co}}\left(\sigma \partial d_{C(t,x)}(\mathcal{A}(x)) \cup \{0\}\right) + g(t, x(t))$$
for $(t, x) \in [0, \tau_R] \times \mathcal{H}$, where

$$\sigma := \frac{\tilde{\kappa}_R + (L+M)(c(R+\|x_0\|)+d)}{m\alpha^2 - L}.$$

Then, for a.e. $t \in [0, \tau_R]$

$$\begin{aligned} \dot{x}_k(t) &\in -\frac{1}{2\lambda_k}\partial d^2_{C(t,x_k(t))}\left(\mathcal{A}(x_k(t))\right) + g(t, x_k(t)) \\ &\in -\frac{d_{C(t,x_k(t))}\left(\mathcal{A}(x_k(t))\right)}{\lambda_k}\partial d_{C(t,x_k(t))}\left(\mathcal{A}(x_k(t))\right) + g(t, x_k(t)) \\ &\subset \tilde{F}(t, x_k(t)), \end{aligned}$$

where we have used Proposition 3.3.

Claim 3: $\tilde{F}$ has closed and convex values and satisfies:

(i) for each $x \in \mathcal{H}$, $\tilde{F}(\cdot, x)$ is measurable;

(ii) for all $t \in [0, \tau_R]$, $\tilde{F}(t, \cdot)$ is upper semicontinuous from $\mathcal{H}$ into $\mathcal{H}_w$;

(iii) if $\mathcal{A}(x) \in C(t, x)$ then $\tilde{F}(t, x) = -\sigma\partial d_{C(t,x)}(\mathcal{A}(x)) + g(t, x(t))$.

Proof of Claim 3: It follows the same ideas given in the proof of [56, Theorem 5.1]. □

In summary, we have

(i) for each $x \in \mathcal{H}$, $\tilde{F}(\cdot, x)$ is measurable.

(ii) for all $t \in [0, \tau_R]$, $\tilde{F}(t, \cdot)$ is upper semicontinuous from $\mathcal{H}$ into $\mathcal{H}_w$.

(iii) $\dot{x}_k \rightharpoonup \dot{x}$ in $L^1([0, \tau_R]; \mathcal{H})$ as $k \to +\infty$.

(iv) $x_k(t) \to x(t)$ as $k \to +\infty$ for all $t \in [0, \tau_R]$.

(v) $\mathcal{A}(x_k(t)) \to \mathcal{A}(x(t))$ as $k \to +\infty$ for all $t \in [0, \tau_R]$.

(vi) $\dot{x}_k(t) \in \tilde{F}(t, x_k(t))$ for a.e. $t \in [0, \tau_R]$.

These conditions and Proposition 1.31 imply that x satisfies

$$\begin{cases} \dot{x}(t) \in \tilde{F}(t, x(t)) & \text{a.e. } t \in [0, \tau_R], \\ x(0) = x_0, \ \mathcal{A}(x_0) \in C(0, x_0), \end{cases}$$

which, according to Claim 3, implies that x is a solution of

$$\begin{cases} \dot{x}(t) \in -\sigma \partial d_{C(t,x(t))}(\mathcal{A}(x(t))) + g(t, x(t)) & \text{a.e. } t \in [0, \tau_R], \\ x(0) = x_0, \ \mathcal{A}(x_0) \in C(0, x_0), \end{cases}$$

Therefore, by virtue of (1.2) and Claim 2, x is a solution of $(\mathcal{DSSP}_g)$ over $[0, \tau_R]$. Finally, to extend the solution over all the interval $[0, T]$, we can use [39, Lemma 5.7]. □

3.1.1 Perturbed Degenerate Sweeping Processes

In this subsection, we provide existence results for the perturbed degenerate sweeping process:

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}\left(\mathcal{A}(x(t))\right) + g(t, x(t)) & \text{a.e. } t \in I := [0, T], \\ x(0) = x_0 \in \mathcal{A}^{-1}(C(0)). \end{cases} \tag{3.5}$$

The latter differential inclusion can be seen as a particular case of $(\mathcal{DSSP}_g)$ for state-independent moving sets.

The following theorem extends the result from [56, Theorem 5.2]. Moreover, it is well-known that $(\mathcal{H}_4)$ implies $(\mathcal{H}_2)$ with $\alpha = 1$. Thus, the following result includes the existence of solutions for regular (convex/prox-regular) sets.

Theorem 3.6 *Assume, in addition to $(\mathcal{H}_{\mathcal{A}}^1)$, $(\mathcal{H}_{\mathcal{A}}^2)$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, that $(\mathcal{H}_1^g)$, $(\mathcal{H}_2^g)$ and $(\mathcal{H}_3^g)$ hold. Then, the problem* (3.5) *admits at least one absolutely continuous solution. Moreover, if $\mathcal{H}$ is finite-dimensional, then assumption $(\mathcal{H}_2^g)$ can be weakened to $(\mathcal{H}_4^g)$.*

Proof. According to the proof of Theorem 3.5, we note that $(\mathcal{H}_3^x)$ was used to obtain $(\mathcal{H}_2^x)$ and the upper semicontinuity of $\partial d_{C(t,\cdot)}(\cdot)$ from $\mathcal{H}$ into $\mathcal{H}_w$ for all $t \in I$. Since in the state-independent case these two properties hold under $(\mathcal{H}_2)$ (see [56, Proposition 4.2]), it is sufficient to repeat the proof of Theorem 3.5 to get the result. □

3.2 Integro-differential sweeping processes

In this section, we study integrally perturbed sweeping processes. We first prove the existence of solutions for integrally perturbed degenerate state-dependent sweeping processes $(\mathcal{IDSP})$. Then, we obtain the existence for integrally perturbed degenerate sweeping processes:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); \mathcal{A}(x(t))\right) + g_1(t, x(t)) + \displaystyle\int_0^t g_2(s, x(s))ds & \text{a.e. } t \in I, \\ x(0) = x_0 \in \mathcal{A}^{-1}(C(0)). \end{cases} \tag{$\mathcal{IDSP}_g$}$$

The following result is a consequence of Theorem 3.5.

Theorem 3.7 *Assume, in addition to* $(\mathcal{H}_{\mathcal{A}}^1)$, $(\mathcal{H}_{\mathcal{A}}^2)$, $(\mathcal{H}_1^x)$, $(\mathcal{H}_3^x)$ *and* $(\mathcal{H}_4^x)$, *that* $(\mathcal{H}_1^g)$, $(\mathcal{H}_2^g)$ *and* $(\mathcal{H}_3^g)$ *hold for* g_1 *and* g_2. *Then, the problem* $(\mathcal{IDSP})$ *admits at least one absolutely continuous solution* $x\colon [0,T] \to \mathcal{H}$. *Moreover, if* $\mathcal{H}$ *is finite-dimensional, then assumption* $(\mathcal{H}_2^g)$ *can be weakened to* $(\mathcal{H}_4^g)$.

Proof. It is routine to prove that x solves $(\mathcal{IDSP})$ if and only if (x, z) solves

$$\begin{cases} \dot{x}(t) \in -N\left(C(t, x(t)); \mathcal{A}(x(t))\right) + g_1(t, x(t)) + z(t) & \text{a.e. } t \in I, \\ \dot{z}(t) = g_2(t, x(t)), \end{cases}$$

with initial conditions $(x(0), z(0)) = (x_0, 0)$. The above system can be regarded as a perturbed degenerate state-dependent sweeping process in the product space $\mathcal{H} \times \mathcal{H}$ and moving sets $C(t, x) \times \mathcal{H}$. Hence, Theorem 3.7 follows from Theorem 3.5. □

The next theorem follows by combining the arguments from the proof of Theorems 3.6 and 3.7.

Theorem 3.8 *Assume, in addition to* $(\mathcal{H}_{\mathcal{A}}^1)$, $(\mathcal{H}_{\mathcal{A}}^2)$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$, *that* $(\mathcal{H}_1^g)$-$(\mathcal{H}_3^g)$ *hold for* g_1 *and* g_2. *Then, the problem* $(\mathcal{IDSP}_g)$ *admits at least one absolutely continuous solution* $x\colon [0,T] \to \mathcal{H}$. *Moreover, if* $\mathcal{H}$ *is finite-dimensional, then assumption* $(\mathcal{H}_2^g)$ *can be weakened to* $(\mathcal{H}_4^g)$.

3.3 An application to online optimization

Let $\phi\colon \mathcal{H} \to \mathbb{R}$ be a convex smooth function. We say that ϕ is β-strongly convex if $\phi - (\beta/2)\|\cdot\|^2$ is convex (see, e.g., [10, Chapter 10]). It is well-known (see, e.g., [10, Exercise 17.5]) that ϕ is β-strongly convex if and only if

$$\langle \nabla\phi(x) - \nabla\phi(y), x - y\rangle \geq \beta\|x - y\|^2 \text{ for all } x, y \in \mathcal{H}.$$

We denote by ϕ^* the convex conjugate of ϕ. It is well-known that for strongly convex functions $\phi\colon \mathcal{H} \to \mathbb{R}$ the operator $(\nabla\phi)^{-1}\colon \mathcal{H} \to \mathcal{H}$ is well-defined and the following formula holds:

$$(\nabla\phi)^{-1}(x) = \nabla\phi^*(x) \text{ for all } x \in \mathcal{H}.$$

Moreover, ϕ^* is $1/\beta$-strongly convex if and only if $\nabla\phi$ is β-Lipschitz continuous (see, e.g., [10, Theorem 18.15]).

Assume that ϕ is twice differentiable over $\mathcal{H}$ and consider the problem: Find $x\colon [0,T] \to \mathcal{H}$ such that $x(0) = x_0$ and

$$\nabla^2\phi(x(t))\dot{x}(t) \in -N_{\mathcal{K}}(x(t)) + g(t, x(t)) \text{ a.e. } t \in [0,T], \tag{3.6}$$

where $\mathcal{K} \subset \mathcal{H}$ is a nonempty and closed set. The latter dynamic, called *online mirror descent*, was considered in [25] for convex and compact sets $\mathcal{K}$ to obtain a $O((\log k)^2)$-competitive randomized algorithm for the k-server problem on hierarchically separated trees (see [25, Theorem 2.2]). The system (3.6) also includes the so-called *oblique projected dynamical systems* considered in [36] as a way of preserving monotonicity in projected dynamical systems.

We observe that (3.6) is formally equivalent to the following perturbed degenerate sweeping process:

$$\dot{z}(t) \in -N_{\mathcal{K}}(\nabla\phi^*(z(t))) + g(t, \nabla\phi^*(z(t))) \text{ a.e. } t \in [0, T]. \tag{3.7}$$

Indeed, $x\colon [0,T] \to \mathcal{H}$ is a solution of (3.6) if and only if $z(t) = \nabla\phi(x(t))$ is a solution of (3.7). Hence, by using the above observation and Theorem 3.5, we obtain the following result that generalizes [25, Theorem 2.2].

Theorem 3.9 *Let $\mathcal{K}$ be a closed, ball-compact and positively α-far set (see Definition* 1.19*). Let $\phi\colon \mathcal{H} \to \mathbb{R}$ be a strongly convex function with Lipschitz continuous gradient. Assume that $g\colon [0,T] \times \mathcal{H} \to \mathcal{H}$ satisfies $(\mathcal{H}_1^g)$, $(\mathcal{H}_2^g)$ and $(\mathcal{H}_3^g)$. Then, for any $x_0 \in \mathcal{H}$, the problem* (3.6) *admits at least one absolutely continuous solution. Moreover, if $\mathcal{H}$ is finite-dimensional, then assumption $(\mathcal{H}_2^g)$ can be weakened to $(\mathcal{H}_4^g)$.*

The latter result offers a significant improvement to [25, Theorem 2.2] because we relax the compactness and the geometry of the set $\mathcal{K}$ (we recall that any convex or uniformly prox-regular set is positively α-far with $\alpha = 1$). Moreover, the problem (3.7) offers an equivalent formulation to (3.6), where no second differentiability of ϕ is involved. Finally, by means of Theorem 3.5, it is straightforward to consider the system (3.6) with moving sets, which can be useful for studying the k-server problem with time-dependent servers.

3.4 Perturbed second-order sweeping process

This section aims to apply our results to the so-called *degenerate perturbed second-order sweeping process* given by following differential inclusion

$$\begin{cases} \ddot{x}(t) \in -N_{C(t,x(t),\dot{x}(t))}\left(\mathcal{A}\left(\dot{x}(t)\right)\right) + g\left(t, x\left(t\right)\right) & \text{a.e. } t \in [0,T], \\ x(0) = x_0, \ \dot{x}(0) = v_0, \ \mathcal{A}\left(v_0\right) \in C(0, x_0, v_0). \end{cases} \tag{$\mathcal{SOSP}$}$$

Here, the set $N_{C(t,x(t),\dot{x}(t))}\left(\mathcal{A}\left(\dot{x}(t)\right)\right)$ stands for the Clarke normal cone to $C(t, x(t), \dot{x}(t))$ at $\mathcal{A}\left(\dot{x}(t)\right) \in C(t, x(t), \dot{x}(t))$, where $C\colon [0,T] \times \mathcal{H} \times \mathcal{H} \rightrightarrows \mathcal{H}$ is a set-valued mapping with nonempty closed values of a separable Hilbert space $\mathcal{H}$. The operator $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ is a (possibly) nonlinear Lipschitz and strongly monotone operator, and the perturbation is a single-valued function $g\colon [0,T] \times \mathcal{H} \to \mathcal{H}$ satisfying a linear growth condition.

When $\mathcal{A} = I$, the evolution $(\mathcal{SOSP})$ includes the *perturbed second-order sweeping process*, which was studied by Jourani and Vilches [40] when the moving sets are assumed to be nonempty, closed and subsmooth or positively α_0-far with absolutely continuous variation in time and Lipschitz variation in the state. On the other hand, this system has been studied by Castaing [26] for a moving set $C\left(t, x\right)$ under convex and compacts values assumption. Since then, several works have dealt with second-order sweeping processes with convex/prox-regular sets in Hilbert/Banach spaces (see [2, 6–8, 18–21, 27]).

The following result partially extends the results from [40].

Theorem 3.10 *Let $C\colon [0,T]\times\mathcal{H}\times\mathcal{H}\rightrightarrows\mathcal{H}$ be a set-valued mapping with nonempty closed values satisfying:*

- *For all $r\geq 0$, there exist $\kappa_r\geq 0$ such that for $s,t\in[0,T]$ and such that for all s, $t\in$ and all x, y, u, v, $z\in\mathcal{H}$*

$$\sup_{z\in r\mathbb{B}}|d(z,C(t,x,u))-d(z,C(s,y,v))|\leq\kappa_r|t-s|+L_1\|x-y\|+L_2\|u-v\|$$

  where $L_1\in[0,m[$ and $L_2\geq 0$ are independent of r. Here m is the constant given by $(\mathcal{H}_{\mathcal{A}}^1)$.

- *The family $\{C(t,u,v)\colon (t,u,v)\in[0,T]\times\mathcal{H}\times\mathcal{H}\}$ is equi-uniformly subsmooth.*

- *For every $t\in[0,T]$, every $r>0$ and every pair of bounded sets A, $B\subset\mathcal{H}$, the set $C(t,A,B)\cap r\mathbb{B}$ is relatively compact.*

Assume, in addition to $(\mathcal{H}_{\mathcal{A}}^1)$, $(\mathcal{H}_{\mathcal{A}}^2)$, that $(\mathcal{H}_1^g)$, $(\mathcal{H}_2^g)$ and $(\mathcal{H}_3^g)$ hold. Then, the problem $(\mathcal{SOSP})$ admits at least one absolutely continuous solution. Moreover, if $\mathcal{H}$ is finite-dimensional, then assumption $(\mathcal{H}_2^g)$ can be weakened to $(\mathcal{H}_4^g)$.

Proof. We observe that $(\mathcal{SOSP})$ is equivalent to the following system:

$$\begin{cases}\dot{x}(t)=y(t) & \text{a.e. } t\in[0,T],\\ \dot{y}(t)\in -N_{C(t,x(t),y(t))}(\mathcal{A}(y(t)))+g(t,x(t)) & \text{a.e. } t\in[0,T],\\ x(0)=x_0,\ y(0)=v_0,\ \mathcal{A}(v_0)\in C(0,x_0,v_0).\end{cases}\qquad(\mathcal{DFOSP}_g)$$

Hence, $(\mathcal{DFOSP}_g)$ can be rewritten as

$$\begin{pmatrix}\dot{x}(t)\\ \dot{y}(t)\end{pmatrix}\in -N_{\mathcal{H}\times C(t,x(t),y(t))}(x(t),\mathcal{A}(y(t)))+\begin{pmatrix}y(t)\\ g(t,x(t))\end{pmatrix},$$

with initial conditions $(x(0),y(0))=(x_0,v_0)$. The above differential inclusion is equivalent to

$$\begin{pmatrix}\dot{x}(t)\\ \dot{y}(t)\end{pmatrix}\in -N_{\mathcal{H}\times C(t,x(t),y(t))}\left(\tilde{\mathcal{A}}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\right)+\begin{pmatrix}y(t)\\ g(t,x(t))\end{pmatrix},$$

where

$$\tilde{\mathcal{A}}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}x(t)\\ \mathcal{A}(y(t))\end{pmatrix}.$$

The latter differential inclusion is a special case of the perturbated degenerate state-dependent sweeping processes with unbounded moving sets. Hence, Theorem 3.10 follows from Theorem 3.5. □

Chapter 4

Open questions and future work

In this thesis, we have introduced and studied the so-called perturbated/degenerate state-dependent sweeping processes. In Chapter 1 we presented the main definitions, standing assumptions, technical results that allowed us to obtain some estimates of the distance function to a moving set $C(t,x)$ and results of existence of solutions for differential inclusions that we apply to the Moreau-Yosida regularization schemes.

In Chapter 2, by means of a suitable adaptation of the arguments provided in [41], we proved an existence result of solutions for degenerate state-dependent sweeping processes ($\mathcal{DSSP}$) with nonregular moving sets that vary in a Lipschitz way with respect to the truncated Hausdorff distance. We emphasize that the study we have developed on the degenerate state-dependent sweeping processes corresponds to the first approach of this type of sweeping process under such general assumptions because, until now, the literature that addresses the problem of degenerate sweeping processes focuses on convex/prox-regular sets in Hilbert spaces that vary in a Lipschitz or absolutely continuous way with respect to the Hausdorff distance.

In Chapter 3, motivated by the interest of proving an existence result of solutions for the perturbed degenerate state-dependent sweeping processes, we adapted appropriately the demonstrative techniques used in the [68] to obtain the Theorem 3.5. The relevance of the previous result is due to the fact that it allows us to establish the Theorem 3.8 and Theorem 3.9: the first result guarantees the existence of solutions for the integro-differential degenerate state-dependent sweeping processes, in which case, once we make a suitable change of variable in ($\mathcal{IDSP}$), it can be seen as a perturbed degenerate state-dependent sweeping process for the unbounded moving sets, while the second result, framed in the optimization point of view, offers a significant improvement to [25, Theorem 2.2] due to the relaxation of the compactness and the geometry of the set $\mathcal{K}$.

Now, it is worth mentioning that this work leaves unanswered questions on the theory of the sweeping processes, which may become material for future work on this topic.

One interesting question is to consider the (BV) case. We are interested in the possibility of extending the results of [56] to the degenerate state-dependent sweeping process in the sense of differential measures. So, the study of the works of Jourani and Vilches [41], together with the works of Recupero [60–62], would configure a first approximation in the context of differential measures.

Because both existence and uniqueness of solutions are quite relevant to the formulation of any dynamical system, another natural problem corresponds to exploring under which assumptions we can provide results about the uniqueness of solutions for perturbated/degenerate state-dependent sweeping processes. It is important to emphasize that Theorem 2.5, presented in Section 2.1, and Theorem 3.5, presented in Section 3.1, are existence results.

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Annexed

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