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PARTIAL SYMMETRIES ON TOPOLOGICAL DYNAMICS AND SPECTRAL THEORY

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MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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RESUMEN DE LA MEMORIA PARA OPTAR AL TÍTULO DE: MAGÍSTER EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MATEMÁTICAS APLICADAS
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"SIMERÍAS PARCIALES EN DINÁMICA TOPOLÓGICA Y TEORÍA ESPECTRAL"

El enfoque principal de esta tesis es el estudio de los sistemas dinámicos gobernados por transformaciones parcialmente definidas. Estudiamos dos casos distintos: acciones grupoidales continuas sobre espacios topológicos y acciones parciales continuas de grupos sobre C^*-álgebras por *-automorfismos.

Los propósitos de estudio son los siguientes:

1. En el primer caso, estudiamos las propiedades dinámicas de la acción y los morfismos entre diferentes sistemas dinámicos. Mostramos que los morfismos respetan la gran mayoría de la estructura dinámica.
2. En el segundo caso, establecemos condiciones suficientes para que el álgebra de tipo L^1 asociada a la acción sea simétrica. Esta condición está fundamentalmente relacionada con la teoría espectral de estas álgebras.

PARTIAL SYMMETRIES ON TOPOLOGICAL DYNAMICS AND SPECTRAL THEORY

The main focus of this thesis is the study of dynamical systems governed by partially defined transformations. Two distinct cases are studied: continuous groupoid actions on topological spaces and continuous partial group actions on C^*-algebras by *-automorphisms.

The purposes of study are the following ones:

1. In the first case, we studied the dynamical properties of the action and the morphisms between different dynamical systems. We showed that the morphisms respect the vast majority of the dynamical structure.
2. In the second case, we gave sufficient conditions for the L^1-algebra associated to the action to be symmetric. This condition is fundamentally related to the spectral theory of these algebras.

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Content

Chapter 1

Introduction

In the classical setting, an invertible dynamical system is governed by a set of mutually compatible invertible transformations, what we typically call a 'group action'. Groups are fairly rigid objects that encapsulate -almost by definition- the modern notion of symmetry. However, there is a reasonable number of objects with imperfect or incomplete symmetry. This thesis aims to the study of dynamic systems where the transformations do not have a total compatibility, expressing a lack -but not absence- of symmetry; The symmetry that will interest us is not global but rather local. Such systems have acquired an increasing role in modern mathematics, and their impact in applications is already largely acknowledged. We explore two models for this lack of symmetry: groupoid actions on topological spaces and partial group actions on C^*-algebras.

Partial symmetry in topological dynamics is usually treated using concepts via *groupoids*, *partial group actions* or *inverse semigroup actions*. Several aspects are already very well developed. In fact our interest was raised by the applications to C^*-algebras, as in [24, 48, 51, 55]. However, up to our knowledge, the basic theory in the spirit of classical topological dynamics has not yet been developed systematically. In the literature, when notions and results are needed, they are briefly introduced and used in an ad hoc way. In addition, certain basic concepts barely appear in the partial symmetry setting. This led us to begin a systematic study of the basic notions (cf. [3, 19]) in the context of groupoid actions, with an special focus on recurrence phenomena.

On the other hand, one may consider directly a partial action of a group on a C^*-algebra, this is typically called a 'partial C^*-dynamical system'. Every such action generates a canonical Banach *-algebra which represents an invariant of the dynamical system. Naturally, one may study many aspects of this algebra in terms of the subyacent the action; This relation has been successfully studied for several years now and proven very fruitful, as some authors were able to express seemingly unrelated algebras in terms of a partial action of a group (cf. [22, 23]), revealing some insightful information about them.

Let us summarize the content of the thesis.

Chapter 2 is devoted to summarize the basics of two fairly classical theories: Banach algebras and Topological groups. It will include basic facts, mostly, but absolutely relevant for the developing of the crucial parts in this text. The **Section 2.1** introduces Banach *-alebras, the notion of spectrum, the basics of representations and the multiplier algebra $\mathcal{M}(A)$ of a C^*-algebra A. During this section we introduce useful concepts like the *minimal unitization*, *algebraically irreducible representations* or *approximate units*. We briefly discuss examples as $\mathbb{B}(\mathcal{H})$ and $\mathcal{C}_0(\Omega)$. This chapter includes very few proofs, as the results are very straightforward or too technical to be insightful. On the other hand, **Section 2.2** treats the subject of Topological groups, starting by a study of the topology of a topological group or its Hausdorffness. Then we introduce the Haar measure of a locally compact group and we use it to construct the L^1-algebra of a Fell bundle, which is the main object of study in Chapter 5.

Chapter 3 is based on the work [29], were we studied the basics of topological dynamics in the setting of groupoid actions. After an introduction of the subject (*groupoids*, *groupoids actions*, *recurrence sets*), we proceeded to illustrate several new examples belonging to this theory; namely *equivalence relations*, *the action groupoid associated to a group action*, *the Deaconu-Renault groupoid*, *the action groupoid associated to a group action* or *groupoid pull-backs*. To start the study, we break down the classical notion of *topological transitivity*, which transforms into four different notions for general groupoids, after proving that this notions differ by providing several counterexamples, we prove that these notions collapse for *open* groupoids. After introducing a new notion, *weak pointwise transitivity*, we proceed to study the classical notions of *fixed points*, *Periodic and almost periodic points* and *minimal sets*, both for open and general groupoids. Then we introduce factors -epimorphisms- and show that they preserve almost all properties, we end with an small subsection which proves that the notion of mixing is not relevant beyond groups.

Chapter 4 is extracted from the preprint [30]. In this chapter we study how to produce '*algebraic morphisms of dynamical systems*', starting from the notion of *algebraic morphisms of groupoids*. This is interesting (and distinct from factors) because it allows a change of groupoids and it works in a functorial way when one passes to the C^*-algebraic world. We define the notion of epimorphism here and proceed to show that this type of morphism preserves all the required properties, such as, all types of transitivity, minimality and recurrence.

Chapter 5 comes from the work [31]. This is the last chapter formed by original results and it focus on the cross-sectional L^1-algebra generated by a Fell bundle over a locally compact group. Specifically, the objective is to prove that the algebra $L^1(\mathsf{G}\,|\,\mathscr{C})$ is symmetric whenever G endowed with the discrete topology is rigidly symmetric. After an introduction to the terminology and history of the matter, we proceed to give the proof of the result, which is based on the *discretization* of an arbitrary algebraically irreducible representation of the algebra (a method presented in [49]). We present a particularization to the case of a Fell bundle representing a partial action, which was the main motivation for getting this result.

Finally, **Chapter 6** presents -as its own name suggests- a conclusion of the thesis and some possible lines for future work. We will not present spoilers for this very small ending chapter.

Chapter 2

Preliminaries

We will start this thesis by exposing basic facts on two topics, crucial to understand the results obtained and discussed in the following chapters. The two topics to be treated are Banach algebras and Topological groups.

We will avoid the complicated or technical proofs, as the presented results are objectively basic and well-known. In any case, and for the convenience of interested readers, we will include citations to the basic bibliography when needed. Exposure to classical analysis -topology, measure theory, functional analysis- and abstract algebra -group theory, ring theory- is assumed. Some basic knowledge of category theory will be useful.

2.1 Preliminaries on Banach algebras

2.1.1 *-Algebras and Banach *-algebras

Definition 2.1.1. An *-algebra is a complex vector space A equipped with an associative and bilinear multiplication $A \times A \to A : (a, b) \mapsto ab$ and an involution $a \mapsto a^*$, meaning an operation such that for all $\lambda, \mu \in \mathbb{C}$ and all $a, b \in A$, we have

$$(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$$
$$(ab)^* = b^* a^*.$$

Example 2.1.2. If Ω is a non-empty set, then $\mathbb{C}^\Omega$ denotes the vector space of complex-valued functions on Ω, with the pointwise operations. It becomes a *-algebra if the involution is defined as pointwise complex conjugation. Note that the *-algebra $\mathbb{C}^\Omega$ is commutative.

Example 2.1.3. If $\mathcal{H}$ is a complex Hilbert space, then the algebra $\mathbb{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ is a *-algebra under the operation of taking the adjoint of an operator. If $\mathcal{H}$ is of dimension at least two, then the *-algebra $\mathbb{B}(\mathcal{H})$ is not commutative.

Definition 2.1.4. Let A be a *-algebra. An element a of A is called Hermitian, or self-adjoint, if $a = a^*$. An element b of A is called normal, if $b^* b = b\, b^*$. Obviously, every Hermitian element is normal.

Definition 2.1.5. Let A be a *-algebra, and let B be a subset of A. One denotes by B_{sa} the set of Hermitian elements of B.

Remark 2.1.6. An arbitrary element c of a *-algebra A can be written uniquely as $c = a + \mathrm{i}b$ with a, b in A_{sa}, namely $a = (c + c^*)/2$ and $b = (c - c^*)/(2\mathrm{i})$. Furthermore the element c is normal if and only if a and b commute.

Definition 2.1.7. Let A be an algebra. A *unit* in A is a non-zero element e of A such that

$$ea = ae = a \quad \text{for all} \quad a \in A.$$

We say that A is a *unital* algebra if it has a unit.

Definition 2.1.8. Assume that A has no unit. One defines $\widetilde{A} := \mathbb{C} \oplus A$ (direct sum of vector spaces) and embeds A into $\widetilde{A}$ via $a \mapsto (0, a)$. One defines $e := (1, 0)$, so that $(\lambda, a) = \lambda e + a$ for $\lambda \in \mathbb{C}$, $a \in A$. In order for $\widetilde{A}$ to become a unital algebra with unit e, and with multiplication extending the one in A, the multiplication in $\widetilde{A}$ must be defined by

$$(\lambda e + a)(\mu e + b) = (\lambda\mu)e + (\lambda b + \mu a + ab) \qquad (\,\lambda, \mu \in \mathbb{C}, a, b \in A\,),$$

and this defn indeed satisfies the requirements. One says that $\widetilde{A}$ is the *unitization* of A. If A is a *-algebra, one makes $\widetilde{A}$ into a unital *-algebra by putting

$$(\lambda e + a)^* := \overline{\lambda} e + a^* \qquad (\,\lambda \in \mathbb{C}, a \in A\,).$$

Definition 2.1.9. Let A, B be algebras. An *homomorphism* from A to B is a linear mapping $\pi : A \to B$ such that $\pi(ab) = \pi(a)\pi(b)$ for all $a, b \in A$. If A, B are *-algebras, π is called a **-homomorphism* if it also satisfies $\pi(a^*) = \pi(a)^*$, for all a in A.

As always, we will say that an injective homomorphism is a *monomorphism* and a surjective homomorphism is a *epimorphism*.

Definition 2.1.10. Let A, B be unital algebras with units e_A, e_B respectively. An homomorphism π from A to B is called unital, if it satisfies $\pi(e_A) = e_B$.

Definition 2.1.11. An *algebra norm* is a norm $\|\cdot\|$ on an algebra A such that

$$\|ab\| \leq \|a\|\|b\| \quad \text{for all} \quad a, b \in A.$$

The pair $\big(A, \|\cdot\|\big)$ then is called a *normed algebra*. A normed algebra is called a *Banach algebra* if the underlying normed space is Banach. In the same fashion, $\big(A, \|\cdot\|\big)$ is called a Banach *-algebra if it is a Banach algebra with continuous involution. We will typically assume that the norm is understood and drop it off the notation.

Definition 2.1.12. A C^**-algebra* is a Banach *-algebra A, such that for all $a \in A$, one has

$$\|a\|^2 = \|a^*a\| \quad \text{as well as} \quad \|a\| = \|a^*\|.$$

The first equality is called the C^*-property.

Example 2.1.13. The *-algebra $\mathbb{B}(\mathcal{H})$ introduced in Example 2.1.3 is actually a unital C^*-algebra with the operator norm. This is the prototypical example in the theory: a famous theorem due to Gelfand, Neimark and Segal shows that every C^*-algebra can be realized as a closed *-subalgebra of some operator alegebra $\mathbb{B}(\mathcal{H})$.

Proposition 2.1.14. *Let A be a C^*-algebra. By defining*

$$\|a\| = \sup\{\|ax\| \mid x \in A, \|x\| \leq 1\} \qquad (a \in \widetilde{A}\,)$$

one makes $\widetilde{A}$ into a unital C^*-algebra. The above norm extends of course the norm in A.

Example 2.1.15. Suppose that Ω is a locally compact Hausdorff topological space and consider the *-subalgebra $\mathcal{C}_0(\Omega)$ of $\mathbb{C}^\Omega$ (see Example 2.1.2), composed of functions which vanish at infinity. If $\mathcal{C}_0(\Omega)$ is given the supremum norm

$$\|f\|_{\mathcal{C}_0(\Omega)} := \sup_{x\in\Omega} |f(x)|,$$

it becomes a commutative C^*-algebra.

This example is specially interesting because of a Theorem due to Gelfand: Every commutative C^*-algebra is isometrically *-isomorphic to some algebra of the form $\mathcal{C}_0(\Omega)$. This identification is functorial and leads to the so called *Gelfand duality*, which lets us think of C^*-algebras as '*non-commutative spaces*', a very actual and active trend of research in both mathematics and theoretical physics. This should be kept in mind during Chapter 5, were we will treat continuous group actions on C^*-algebras; If the algebra is commutative, this is the same as a continuous action on some locally compact Hausdorff topological space Ω.

Lets also note that $\mathcal{C}_0(\Omega)$ is not unital unless Ω is compact, in this case, we drop the 0 and simply denote it by $\mathcal{C}(\Omega)$. If Ω isn't compact, one can show that its unitization satisfies

$$\widetilde{\mathcal{C}_0(\Omega)} \cong \mathcal{C}(\Omega \cup \{\infty\}), \quad \text{where } \Omega \cup \{\infty\} \text{ is the Alexandroff compactification of } \Omega.$$

Definition 2.1.16. Let A be a Banach *-algebra without unit. If A is not a C^*-algebra, one makes $\widetilde{A}$ into a unital Banach *-algebra by putting

$$\|\lambda e + a\| := |\lambda| + \|a\| \qquad \big(\lambda \in \mathbb{C}, a \in A\big).$$

If A is a C^*-algebra, one makes $\widetilde{A}$ into a unital C^*-algebra with the norm defined in the preceding proposition.

Theorem 2.1.17. *If A is a Banach algebra, so is $\widetilde{A}$.*

Proof. This algebraic verifications are easy, the completeness follows from the fact that $\|\lambda e + a\| = |\lambda| + \|a\|$ is the product norm on $\mathbb{C} \oplus A$. □

Although many Banach *-algebras do not have a unit, some of them -as every Banach *-algebra that will be mentioned in this text- have a nice approximation of a unit.

Definition 2.1.18. A net $\{\Upsilon_i\}_{i\in I} \subset A$ is called an *approximate unit* if it bounded and for all $a \in A$ one has

$$\|a\Upsilon_i - a\| \to 0 \qquad \|\Upsilon_i a - a\| \to 0.$$

Theorem 2.1.19. *Every C^*-algebra A has an approximate unit.*

Proof. This theorem is standard, but giving its proof would require the development of techniques that are of little use for us. One can check this in [16, Thm. I.4.8] □

2.1.2 The Spectrum

During this subsection, A will be an algebra.

Definition 2.1.20. For $a \in A$ one defines

$$\mathrm{sp}_A(a) := \{\lambda \in \mathbb{C} \mid \lambda e - a \in \widetilde{A} \text{ is not invertible in } \widetilde{A}\}.$$

One says that $\mathrm{sp}_A(a)$ is the *spectrum* of the element a in the algebra A. We shall often abbreviate $\mathrm{sp}(a) := \mathrm{sp}_A(a)$.

The next result will be used tacitly in the sequel.

Proposition 2.1.21. *For $a \in A$ we have $\mathrm{sp}_{\widetilde{A}}(a) = \mathrm{sp}_A(a)$.*

Proof. The following statements are equivalent for $\lambda \in \mathbb{C}$.

$$\begin{aligned} \lambda \in \mathrm{sp}_{\widetilde{A}}(a) &\iff \lambda e - a \in \widetilde{\widetilde{A}} \text{ is not invertible in } \widetilde{\widetilde{A}} \\ &\iff \lambda e - a \in \widetilde{A} \text{ is not invertible in } \widetilde{A} \qquad \big(\text{because } \widetilde{\widetilde{A}} = \widetilde{A}\big), \\ &\iff \lambda \in \mathrm{sp}_A(a). \end{aligned}$$

□

Proposition 2.1.22. *If A has no unit, then $0 \in \mathrm{sp}_A(a)$ for all $a \in A$.*

Proof. Assume that $0 \notin \mathrm{sp}_A(a)$ for some a in A. We shall show that A is unital. a is invertible in $\widetilde{A}$, so let $a^{-1} = \mu e + b$ with $\mu \in \mathbb{C}$, $b \in A$. We obtain $e = (\mu e + b)a = \mu a + ba \in A$, so A is unital. □

Theorem 2.1.23. *Let B be another algebra, and let $\pi : A \to B$ be an algebra homomorphism. We then have*

$$\mathrm{sp}_B\big(\pi(a)\big) \setminus \{0\} \subset \mathrm{sp}_A(a) \setminus \{0\} \quad \text{for all} \quad a \in A.$$

If both A and B are unital algebras, and if π is unital as well, then

$$\mathrm{sp}_B\big(\pi(a)\big) \subset \mathrm{sp}_A(a) \quad \text{for all} \quad a \in A.$$

Proof. Let $a \in A$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Assume that $\lambda \notin \mathrm{sp}_A(a)$. We will show that $\lambda \notin \mathrm{sp}_B\big(\pi(a)\big)$. Let e_A be the unit in $\widetilde{A}$ and e_B the unit in $\widetilde{B}$. Extend π to an homomorphism $\widetilde{\pi} : \widetilde{A} \to \widetilde{B}$ by requiring that $\widetilde{\pi}(e_A) = e_B$ if A has no unit. Then the element $\lambda e_A - a$ has an inverse b in $\widetilde{A}$. It is not hard to see that $\widetilde{\pi}(b) + \lambda^{-1}\big(e_B - \widetilde{\pi}(e_A)\big)$ is the inverse of $\lambda e_B - \pi(a)$ in $\widetilde{B}$.

Assume now that A, B and π are unital, and consider the case $\lambda = 0$. If $0 \notin \mathrm{sp}_A(a)$, then a is invertible with inverse a^{-1}. But then $\pi(a)$ will be invertible with inverse $\pi(a^{-1})$. Hence $0 \notin \mathrm{sp}_B(\pi(a))$. □

Proposition 2.1.24. *If A is a *-algebra, then we have $\mathrm{sp}(a^*) = \overline{\mathrm{sp}(a)}$ for all $a \in A$.*

Proof. This follows from $(\lambda e - a)^* = \overline{\lambda} e - a^*$. □

It does not follow that the spectrum of a Hermitian element is real. In fact, a *-algebra, in which each Hermitian element has real spectrum, is called a symmetric *-algebra, cf. Definition 5.0.1.

2.1.3 Representations

This small subsection will be devoted to state a couple of definitions; we will introduce several types of representations. They will all appear at some point of Chapter 5.

Definition 2.1.25. Let $\mathcal{E}$ be a complex vector space and A an algebra. A *representation of A on the vector space $\mathcal{E}$* is an homomorphism $\Gamma : A \to \mathbb{B}(\mathcal{E})$.

Definition 2.1.26. Let $\mathcal{E}$ be a Banach space and A a Banach algebra. A *representation of A on the Banach space $\mathcal{E}$* is a continuous homomorphism $\Gamma : A \to \mathbb{B}(\mathcal{E})$, where $\mathbb{B}(\mathcal{E})$ is given the operator norm.

Definition 2.1.27. Let $\mathcal{H}$ be a complex Hilbert space and A a Banach *-algebra. A **-representation of A on $\mathcal{H}$* is a continuous *-homomorphism $\Gamma : A \to \mathbb{B}(\mathcal{H})$. Again, $\mathbb{B}(\mathcal{H})$ is given the operator norm.

Obviously, a *-representation is a particular case of a Banach space representation, which in turn is a particular case of a vector space representation. The following definitions are motivated by the different levels of compatibility between the available norm structure and the algebraic structure.

Definition 2.1.28. Let $\Gamma : A \to \mathbb{B}(\mathcal{E})$ be a representation of the Banach *-algebra A on the Banach space $\mathcal{E}$. We say that Γ is

1. *algebraically irreducible* if no non-trivial subspace of $\mathcal{E}$ is invariant for the action of A.
2. *topologically irreducible* if no non-trivial closed subspace of $\mathcal{E}$ is invariant for the action of A.
3. *algebraically non-degenerate* if $\{\pi(a)f \mid a \in A, f \in \mathcal{E}\} = \mathcal{E}$.
4. *topologically non-degenerate* if $\{\pi(a)f \mid a \in A, f \in \mathcal{E}\}$ is dense in $\mathcal{E}$.

2.1.4 Multipliers of a C^*-algebra

During this subsection, A will denote a C^*-algebra.

Definition 2.1.29. A *multiplier* of A is a pair (L, R) of functions $L, R : A \to A$ satisfying

$$R(x)y = xL(y), \quad \text{for all } x, y \in A.$$

The set of all multipliers is denoted $\mathcal{M}(A)$ and called the *multiplier algebra* of A.

The pair (L, R) is also called a *double centralizer*, with L being the *left centralizer* and R being the *right centralizer*.

Proposition 2.1.30. *If (L, R) is a double centralizer for A, then*

$$L(xy) = L(x)y, \quad R(xy) = xR(y), \quad \text{for all } x, y \in A$$

and both functions are linear and bounded with $\|L\| = \|R\|$ (operator norm).

Proof. Let $\{\Upsilon_i\}_{i\in I}$ be an approximate unit for A, then

$$\Upsilon_i L(xy) = R(\Upsilon_i)xy = \Upsilon_i L(x)y$$

and, for $\lambda \in \mathbb{C}$,

$$\Upsilon_i L(x+\lambda y) = R(\Upsilon_i)x + \lambda y = \Upsilon_i\big(L(x)+\lambda L(y)\big).$$

Passing to the limit yields $L(xy) = L(x)y$ and $L(x+\lambda y) = L(x)+\lambda L(y)$. Now we shall see that L is bounded by applying the Closed Graph Theorem. Let $x_n \in A$ be a sequence such that $x_n \to x$ and $L(x_n) \to y$. Then, for $a \in A$,

$$\begin{aligned}\|aL(x) - ay\| &\leq \|aL(x) - aL(x_n)\| + \|aL(x_n) - ay\| \\ &\leq \|R(a)\|\|x - x_n\| + \|aL(x_n) - ay\| \to 0.\end{aligned}$$

So $aL(x) - ay = 0$, implying that $y = L(x)$ and L is bounded. Obviously, the same applies to R, so the only remaining thing is showing the equality of norms:

$$\begin{aligned}\|L(x)\|^2 &= \|L(x)^*L(x)\| \\ &= \|R(L(x)^*)x\| \\ &\leq \|R\|\|L(x)\|\|x\| \\ &\leq \|R\|\|L\|\|x\|^2.\end{aligned}$$

So $\|L\| \leq \|R\|$, reversing the argument yields the desired result. $\square$

Proposition 2.1.31. *$\mathcal{M}(A)$ is a unital C^*-algebra containing A when the norm and operations are defined by*

$$(L_1,R_1)+(L_2,R_2) := (L_1+L_2, R_1+R_2) \tag{2.1.1}$$
$$(L_1,R_1)(L_2,R_2) := (L_1L_2, R_1R_2) \tag{2.1.2}$$
$$\lambda(L,R) := (\lambda L, \lambda R) \tag{2.1.3}$$
$$\|(L,R)\| := \|L\| = \|R\| \tag{2.1.4}$$
$$e_{\mathcal{M}(A)} := (\mathrm{id}_A, \mathrm{id}_A) \tag{2.1.5}$$
$$(L,R)^* = (R^*, L^*) \tag{2.1.6}$$

Where the operation * is defined by $f^*(a) = f(a^*)^*$.

Proof. The algebraic verifications are strightforward, the completeness is too technical and misses the point. Let us describe how A sits inside $\mathcal{M}(A)$. Given $a \in A$, define

$$L_a(x) = ax \quad \text{and} \quad R_a(x) = xa.$$

The identification $a \mapsto (L_a, R_a)$ is an isometric *-isomorphism. One obviously has $\|L_a\| \leq \|a\|$ and the converse follows by taking an approximate unit Υ_i:

$$\|L_a(\Upsilon_i)\| = \|a\Upsilon_i\| \to \|a\|$$

implies $\|L_a\| \geq \|a\|$. $\square$

2.2 Preliminaries on Topological groups

2.2.1 Topological groups

Definition 2.2.1. A topological group is a group G equipped with a topology with respect to which the group operations are continuous; that is, $(x, y) \mapsto xy$ is continuous from $\mathsf{G} \times \mathsf{G} \to \mathsf{G}$ and $x \mapsto x^{-1}$ is continuous from G to G. If G is a topological group, we shall denote the unit element of G by 1. If $A \subset \mathsf{G}$ and $x \in \mathsf{G}$, we define

$$Ax = \{ax \mid a \in A\}, \quad xA = \{xa \mid a \in A\}, \quad A^{-1} = \{a^{-1} \mid a \in A\},$$

and if also $B \subset \mathsf{G}$,

$$AB = \{ab \mid a \in A, b \in B\}.$$

We say that A is symmetric if $A = A^{-1}$

Remark 2.2.2. It is a useful observation that $A \cap B = \emptyset$ if and only if $1 \notin A^{-1}B$.

The following proposition lists some of the basic properties of topological groups that we will use.

Proposition 2.2.3. *Let* G *be a topological group.*

1. *The topology of* G *is invariant under translations and inversion; that is, if* U *is open then so are* xU*,* Ux*, and* U^{-1}*. Moreover, if* U *is open then so are* AU *and* UA *for any* $A \subset \mathsf{G}$*.*
2. *For every neighborhood* U *of* 1 *there is a symmetric neighborhood* V *of* 1 *such that* $VV \subset U$*.*
3. *If* H *is a subgroup of* G*, so is* $\overline{H}$*.*
4. *Every open subgroup of G is closed.*
5. *If* A *and* B *are compact sets in* G*, so is* AB*.*

Proof.

1. The first assertion is equivalent to the separate continuity of the map $(x, y) \mapsto xy$ and the continuity of the map $x \mapsto x^{-1}$. The second one follows since $AU = \bigcup_{x \in A} xU$ and $UA = \bigcup_{x \in A} Ux$.
2. Continuity of $(x, y) \mapsto xy$ at 1 means that for every neighborhood U of 1 there are neighborhoods W_1, W_2 of 1 with $W_1W_2 \subset U$. The desired set V can be taken to be $W_1 \cap W_2 \cap W_1^{-1} \cap W_2^{-1}$.
3. If $x, y \in H$ there are nets $x_\alpha, y_\beta \in H$ converging to x, y. Then $x_\alpha y_\beta \to xy$ and $x_\alpha^{-1} \to x^{-1}$, so xy and x^{-1} are in H.
4. If H is open, so are all its cosets xH; its complement $\mathsf{G} \setminus H$ is the union of all these cosets except H itself; hence $\mathsf{G} \setminus H$ is open and H is closed.
5. AB is the image of the compact set $A \times B$ under the continuous map $(x, y) \mapsto xy$, hence is compact.

$\square$

Suppose H is a subgroup of the topological group G. Let G/H be the space of left cosets of H, and let $q : \mathsf{G} \to \mathsf{G}/H$ be the canonical quotient map. We impose the quotient topology on G/H; that is, $U \subset \mathsf{G}/H$ is open if and only if $q^{-1}(U)$ is open in G. q maps open sets in G to open sets in G/H, for if V is open in G then $q^{-1}(q(V)) = VH$ is also open by Proposition 2.2.3; hence $q(V)$ is open.

Proposition 2.2.4. *Suppose H is a subgroup of the topological group G.*

1. *If H is closed, G/H is Hausdorff.*
2. *If G is locally compact, so is G/H.*
3. *If H is normal, G/H is a topological group.*

Proof. 1. Suppose $\widetilde{x} = q(x)$, $\widetilde{y} = q(y)$ are distinct points of G/H. If H is closed, xHy^{-1} is a closed set that does not contain 1, so by Proposition 2.2.3 there is a symmetric neighborhood U of 1 with $UU \cap xHy^{-1} = \emptyset$. Since $U = U^{-1}$ and $H = HH$, $1 \notin UxH(Uy)^{-1} = (UxH)(UyH)^{-1}$, so $(UxH) \cap (UyH) = \emptyset$. Thus $q(Ux)$ and $q(Uy)$ are disjoint neighborhoods of x and y.

2. If U is a compact neighborhood of 1 in G, $q(Ux)$ is a compact neighborhood of $q(x)$ in G/H.

3. If $x, y \in \mathsf{G}$ and U is a neighborhood of $q(xy)$ in G/H, the continuity of multiplication in G at (x, y) implies that there are neighborhoods V, W of x, y such that $VW \subset q^{-1}(U)$. Then $q(V)$ and $q(W)$ are neighborhoods of $q(x)$ and $q(y)$ such that $q(V)q(W) \subset U$, so multiplication is continuous on G/H. Similarly, inversion is continuous.

□

Corollary 2.2.5. *If G is T_1 then G is Hausdorff. If G is not T_1 then $\overline{\{1\}}$ is a closed normal subgroup, and $\mathsf{G}/\overline{\{1\}}$ is a Hausdorff topological group.*

Proof. The first assertion follows by taking $H = \{1\}$ in Proposition 2.2.4(1). $\overline{\{1\}}$ is a subgroup by Proposition 2.2.3; it is clearly the smallest closed subgroup of G. It is therefore normal, for otherwise one would obtain a smaller closed subgroup by intersecting it with one of its conjugates. Therefore the second assertion also follows from Proposition 2.2.4, by taking $H = \overline{\{1\}}$. □

In view of Corollary 2.2.5, it is essentially no restriction to assume that a topological group is Hausdorff (one could simply work with $\mathsf{G}/\overline{\{1\}}$ instead of G), and we do so henceforth. In particular, by a locally compact group we shall mean a topological group whose topology is locally compact and Hausdorff.

To end this subsection, let us include a classical, very useful lemma.

Definition 2.2.6. Given a metric space X, we say that the function $f : \mathsf{G} \to X$ is *left uniformly continuous* if the translated function $L_y f(x) := f(y^{-1}x)$ converges uniformly to f, as $y \to 1$.

Lemma 2.2.7. *Let (X, d_X) be a metric space. If $f : \mathsf{G} \to X$ is continuous and has compact support, then it is left uniformly continuous.*

Proof. Let $\varepsilon > 0$ and $K = \operatorname{supp}(f)$. For every $x \in K$, there exists a symmetric neighborhood U_x of 1 such that $d_X\big(f(y^{-1}x), f(x)\big) < \varepsilon/2$, for all $y \in U_x$. Moreover, let us pick a symmetric neighborhood V_x of 1, such that $V_xV_x \subset U_x$. The sets V_xx form a cover of K, so we are able to select finitely many such that $K \subset \bigcup_{i=1}^n V_{x_i}x_i$. Let $V = \bigcap_{i=1}^n V_{x_i}$; we claim that $\sup_{x\in K} d_X\big(f(y^{-1}x), f(x)\big) < \varepsilon$, for $y \in V$.

Indeed, for $x \in K$, there exists i such that $xx_i^{-1} \in V_{x_i}$, so $y^{-1}x = y^{-1}(xx_i^{-1})x_i \in U_{x_i}x_i$. Then

$$d_X\big(f(y^{-1}x), f(x)\big) \leq d_X\big(f(y^{-1}x), f(x_i)\big) + d_X\big(f(x_i), f(x)\big) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

If $y^{-1}x$ is in K, an analogous argument applies. □

2.2.2 The Haar measure

Definition 2.2.8. A (left) Haar measure on the group G is a non-zero Radon measure μ such that $\mu(xE) = \mu(E)$ for every borel set E. Right Haar measures are defined in an analogous fashion.

The following theorem is well-known, but its proof is too long to be stated here. We recommend the proof in [32, chapter 2.2], which is based on identifying Radon measures with linear functionals of $\mathcal{C}_c(\mathsf{G})$. For the most interested readers, we also recommend the extensive treatment of the subject done in [18].

Theorem 2.2.9. *Every locally compact group G possesses a left Haar measure μ. Moreover, every other left Haar measure is an scalar multiple of μ.*

Let us give some examples:

Example 2.2.10.

1. The Lebesgue measure in $\mathbb{R}^n$ is a Haar measure.
2. $\mathrm{d}x/|x|$ is a Haar measure in the multiplicative group $\mathbb{R}^{\times}$.
3. $\mathrm{d}x\mathrm{d}y/(x^2+y^2)$ is a Haar measure in the multiplicative group $\mathbb{C}^{\times}$.
4. $|\det(T)|^{-n}\mathrm{d}T$ is a left and right Haar measure on the matrix group $GL_n(\mathbb{R})$, where $\mathrm{d}T$ is the Lebesgue measure.
5. The affine group (also known as the $ax{+}b$-*group*) is the group of all the affine transformations $x \mapsto ax+b$ of $\mathbb{R}$. Here $\mathrm{d}a\mathrm{d}b/a^2$ is a left Haar measure and $\mathrm{d}a\mathrm{d}b/a$ is a right Haar measure.
6. One obtains a Haar measure on $\mathbb{Q}_p$, the group of p-adic numbers, by assigning the measure p^{-n} to the open ball $a + p^n\mathbb{Z}_p$ and extending.

If G is not abelian, a left Haar measure doesn't needs to be right-invariant. Anyhow, we can measure the extent to which μ fails to be right-invariant. For $x \in \mathsf{G}$, define $\mu_x(E) = \mu(Ex)$, which is again a left Haar measure, so by Theorem 2.2.9, there exists a positive real number $\Delta(x)$ such that $\mu_x = \Delta(x)\mu$.

Definition 2.2.11. The function $\Delta : \mathsf{G} \to \mathbb{R}^{>0}$ defined as above is called the *modular function*.

Proposition 2.2.12. *The modular function is a continuous group homomorphism. Moreover, it satisfies*

$$\int f(xy)\mathrm{d}\mu(x) = \Delta(y^{-1}) \int f(x)\mathrm{d}\mu(x),$$

for all $f \in L^1(\mathsf{G}, \mu)$.

Proof. We have

$$\Delta(xy)\mu(E) = \mu(Exy) = \Delta(y)\mu(Ex) = \Delta(y)\Delta(x)\mu(E)$$

and

$$\int \chi_E(xy)\mathrm{d}\mu(x) = \mu(Ey^{-1}) = \Delta(y^{-1})\mu(E) = \Delta(y^{-1}) \int \chi_E(x)\mathrm{d}\mu(x).$$

So the formula holds. The continuity follows, as $y \mapsto \int f(xy)\mathrm{d}\mu(x) \in \mathbb{C}$ is continuous for $f \in \mathcal{C}_\mathrm{c}(\mathsf{G})$. □

Remark 2.2.13. If we replace y^{-1} by y in the previous proposition, we obtain the formula

$$\Delta(y) \int f(x)\mathrm{d}\mu(x) = \int f(xy^{-1})\mathrm{d}\mu(x) = \int f(x)\mathrm{d}\mu(xy). \tag{2.2.1}$$

So we may write $\mathrm{d}\mu(xy) = \Delta(y)\mathrm{d}\mu(x)$.

We now proceed to introduce a large class of interesting groups.

Definition 2.2.14. G its called *unimodular* if $\Delta(x) \equiv 1$. Equivalently, G is unimodular if and only if every left Haar measure is a right Haar measure and vice versa.

Obviuosly, every abelian group is unimodular. A more interesting observation is that every compact group is unimodular:

Proposition 2.2.15. *If* $K \leq \mathsf{G}$ *is compact,* $\Delta(k) = 1$ *for all* $k \in K$.

Proof. $\Delta(K)$ is a compact subgroup of $\mathbb{R}^{>0}$, hence $\Delta(K) = \{1\}$. □

Corollary 2.2.16. *If* $\mathsf{G}/[\mathsf{G},\mathsf{G}]$ *is compact, then* G *is unimodular.*

Proof. As $\mathbb{R}^{>0}$ is abelian, the homomorphism Δ factorizes through $\mathsf{G}/[\mathsf{G},\mathsf{G}]$, which must have compact image. □

Observe that one can associate a right Haar measure to every left Haar measure by the formula $\widetilde{\mu}(E) = \mu(E^{-1})$. One has

Proposition 2.2.17. $\mathrm{d}\widetilde{\mu}(x) = \Delta(x^{-1})\mathrm{d}\mu(x)$.

Proof. Let $f \in \mathcal{C}_c(\mathsf{G})$, one has

$$\int f(xy)\Delta(x^{-1})\mathrm{d}\mu(x) = \Delta(y)\int f(xy)\Delta((xy)^{-1})\mathrm{d}\mu(x)$$
$$= \int f(x)\Delta(x^{-1})\mathrm{d}\mu(x).$$

Thus $\Delta(x^{-1})\mathrm{d}\mu(x)$ is a right Haar measure and so exists $c > 0$ such that $c\mathrm{d}\widetilde{\mu}(x) = \Delta(x^{-1})\mathrm{d}\mu(x)$. We will be ready if we manage to show that $c = 1$. Indeed, let us pick a compact, symmetric neighborhood U of 1 such that $|\Delta(x^{-1}) - 1| \leq \frac{1}{2}|c-1|$. As U is symmetric, $\mu(U) = \widetilde{\mu}(U)$ and

$$|c-1|\mu(U) = |c\widetilde{\mu}(U) - \mu(U)| = |\int_U \Delta(x^{-1}) - 1\mathrm{d}\mu(x)| \leq \frac{1}{2}|c-1|\mu(U),$$

which is absurd if $c \neq 1$. □

2.2.3 Fell bundles, partial group actions and the cross-sectional algebra

In this subsection we will introduce the notion of a Fell bundle. The idea is to generalize C^*-dynamical systems (i.e. continuous actions of groups on C^*-algebras) to a wider notion, namely *partial C^*-dynamical systems*. For the general theory of Fell bundles we followed [26, Chapter VIII], to which we refer for details. Nevertheless, the broad idea is the following: Each $x \in \mathsf{G}$ should define a transformation $\alpha_x : \mathrm{Dom}(\alpha_x) \subset \mathcal{A} \to \mathcal{A}$, subject to certain continuity conditions. The collection of all the domains $\{\mathrm{Dom}(\alpha_x)\}_{x\in\mathsf{G}}$ is a bundle (in the classical, topological sense) over G, where each fiber is a Banach space. This leads to the following definition.

Definition 2.2.18. Let X be a Hausdorff topological space. A *Banach bundle* over X is a topological space $\mathscr{C}$, equipped with a continuous open surjection $q : \mathscr{C} \to X$ such that $\mathfrak{C}_x := q^{-1}(x)$ is a Banach space with norm $\|\cdot\|_{\mathfrak{C}_x}$, satisfying the following conditions:

1. The function $a \mapsto \|a\|_{\mathfrak{C}_{q(a)}}$ is continuous.
2. The operation $+$ is continuous as a function from $\{(a,b) \in \mathscr{C} \times \mathscr{C} \mid q(a) = q(b)\}$ to $\mathscr{C}$.
3. For each complex number λ, $a \mapsto \lambda a$ is continuous.
4. If $x \in X$ and $\{a_i\}_{i\in I} \subset \mathscr{C}$ is a net such that $q(a_i) \to x$ and $\|a_i\|_{\mathfrak{C}_{q(a_i)}} \to 0$, then $a_i \to 0_{\mathfrak{C}_x}$.

Definition 2.2.19. A cross-section of the bundle $\mathscr{C}$ is a function $\Phi : X \to \mathscr{C}$, such that $\Phi(x) \in \mathfrak{C}_x$. This is, $q \circ \Phi = \mathrm{id}_X$. We say that Φ passes through $a \in \mathscr{C}$ if $a \in \mathrm{range}(\Phi)$. We also say that $\mathscr{C}$ has enough cross-sections if for every $a \in \mathscr{C}$, there exists a cross-section passing through a.

Cross-sections are the way that we have to pass from a (Fell) bundle to a Banach *-algebra, as we will see very soon. Let us now introduce the main object here:

Definition 2.2.20. Let G be a Hausdorff locally compact group. A *Fell bundle* over G is a Banach bundle $q : \mathscr{C} \to \mathsf{G}$, equipped with two operations, $\bullet : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and $^\bullet : \mathscr{C} \to \mathscr{C}$, satisfying

1. $q(a \bullet b) = q(a)q(b)$, or equivalently, $\mathfrak{C}_{q(a)} \bullet \mathfrak{C}_{q(b)} \subset \mathfrak{C}_{q(ab)}$.

2. For each pair of elements $x, y \in \mathsf{G}$, the restriction of $\bullet$ to $\mathfrak{C}_x \times \mathfrak{C}_y$ is bilinear.

3. The operation $\bullet$ is continuous and associative.

4. $\|a \bullet b\|_{\mathfrak{C}_{q(ab)}} \leq \|a\|_{\mathfrak{C}_{q(a)}} \|b\|_{\mathfrak{C}_{q(b)}}$.

5. $q(a^\bullet) = q(a)^{-1}$.

6. For each $x \in \mathsf{G}$, $^\bullet$ is conjugate-linear from $\mathfrak{C}_x$ to $\mathfrak{C}_{x^{-1}}$.

7. $(ab)^\bullet = b^\bullet a^\bullet$ and $(a^\bullet)^\bullet = a$.

8. $^\bullet$ is continuous, isometric (meaning that $\|a^\bullet\|_{\mathfrak{C}_{q(a)^{-1}}} = \|a\|_{\mathfrak{C}_{q(a)}}$) and satisfies the so called 'C^*-property' (see Definition 2.1.12):
$$\|a^\bullet a\|_{\mathfrak{C}_1} = \|a\|^2_{\mathfrak{C}_{q(a)}}.$$

Unlike the sum, the operations $\bullet$ and $^\bullet$ are globally defined on the bundle. They should be interpreted as a product and a conjugation, respectively.

Remark 2.2.21. Observe that, for every $x \in \mathsf{G}$, one has $\mathfrak{C}_1 \bullet \mathfrak{C}_x \subset \mathfrak{C}_x$ and $\mathfrak{C}_x \bullet \mathfrak{C}_1 \subset \mathfrak{C}_x$. This implies that $\mathfrak{C}_1$ is a C^*-algebra, called the *unit algebra* of $\mathscr{C}$.

Example 2.2.22. A C^*-dynamical $(\mathsf{G}, \mathcal{A}, \alpha)$ is simply a continuous action of G on the C^*-algebra $\mathcal{A}$ by *-automorphisms. Every such dynamical system can be easily encoded as a Fell bundle, as follows: Define $\mathscr{C} = \mathcal{A} \times \mathsf{G}$, with q being the obvious projection. The operations are given by
$$(a, x) \bullet (b, y) = (a\alpha_x(b), xy) \quad \text{and} \quad (a, x)^\bullet = (\alpha_{x^{-1}}(a^*), x^{-1}).$$
If the action α is trivial, one says that the resulting bundle is the *trivial bundle with fibre* $\mathcal{A}$.

We will now proceed to define (twisted) partial actions

Definition 2.2.23. A *twisted partial action* of G on the C^*-algebra $\mathcal{A}$ is a triple
$$\Theta = \big(\{\mathcal{A}_x\}_{x\in\mathsf{G}}, \{\theta_x\}_{x\in\mathsf{G}}, \{w(r,s)\}_{(r,s)\in\mathsf{G}\times\mathsf{G}}\big)$$
where for each x in G, $\mathcal{A}_x$ is a closed two sided ideal in $\mathcal{A}$, θ_x is a *-isomorphism from $\mathcal{A}_{x^{-1}}$ onto $\mathcal{A}_x$ and for each (r, s) in $\mathsf{G} \times \mathsf{G}$, $w(r, s)$ is a unitary multiplier of $\mathcal{A}_r \cap \mathcal{A}_{rs}$, satisfying the following postulates, for all r, s and x in G

1. $\mathcal{A}_1 = \mathcal{A}$ and θ_1 is the identity automorphism of $\mathcal{A}$.

2. $\theta_r(\mathcal{A}_{r^{-1}} \cap \mathcal{A}_s) = \mathcal{A}_r \cap \mathcal{A}_{rs}$

3. $\theta_r(\theta_s(a)) = w(r, s)\theta_{rs}(a)w(r, s)^*, \quad$ for $a \in \mathcal{A}_{s^{-1}} \cap \mathcal{A}_{s^{-1}r^{-1}}$

4. $w(1, t) = w(t, 1) = 1$

5. $\theta_r(aw(s, t))w(r, st) = \theta_r(a)w(r, s)w(rs, t), \quad$ for $a \in \mathcal{A}_{r^{-1}} \cap \mathcal{A}_s \cap \mathcal{A}_{st}$

Given a twisted partial action Θ, define $\mathscr{C}(\Theta)$ by
$$\mathscr{C}(\Theta) := \{(a, x) \mid a \in \mathcal{A}_x\}$$

Definition 2.2.24. $\Theta = \big(\{D_x\}_{x\in\mathsf{G}}, \{\theta_x\}_{x\in\mathsf{G}}, \{w(r,s)\}_{(r,s)\in\mathsf{G}\times\mathsf{G}}\big)$ is a twisted partial action of the locally compact group G on the *-algebra $\mathcal{A}$, we say that Θ is continuous if

1. The projection onto the second variable $q : \{(a,x) \mid a \in \mathcal{A}_x\} \to \mathsf{G}$ is an open map.
2. The map
$$(a,x) \in \{(a,x) \mid a \in \mathcal{A}_{x^{-1}}\} \mapsto (\theta_x(a), x) \in \{(a,x) \mid a \in \mathcal{A}_x\}$$
is continuous.
3. Let $\mathscr{D}$ be the Banach bundle over $\mathsf{G}\times\mathsf{G}$, where the fibre over (r,s) is $\mathcal{A}_{rs} \cap \mathcal{A}_r$. Then, for every cross-section $\gamma : \mathsf{G}\times\mathsf{G} \to \mathscr{D}$, the maps
$$(r,s) \mapsto \gamma(r,s)w(r,s) \quad \text{and} \quad (r,s) \mapsto w(r,s)\gamma(r,s)$$
are continuous.

We are interested in continuous twisted partial actions and the Fell bundles that they induce. This is a huge part of the motivation for the results obtained in [31]/Chapter 5, this is why we will postpone the rest of the discussion on this type of Fell bundles until reaching Subsection 5.4. To end this chapter, let us construct the cross-sectional algebra associated to a Fell bundle. μ will be denoting the Haar measure on G.

Definition 2.2.25. A cross-section $\Phi : \mathsf{G} \to \mathscr{C}$ is called locally μ-measurable if satisfies the following condition: For each compact subset $\mathrm{K} \subset \mathsf{G}$, there exists a sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ of continuous cross-sections such that $\Psi_n(x) \to \Phi(x)$ for μ-almost all $x \in \mathrm{K}$.

Definition 2.2.26. The L^p space associated to a Fell bundle $\mathscr{C}$ is the set of locally μ-measurable cross-sections $\Phi : \mathsf{G} \to \mathscr{C}$ which vanish outside of a σ-compact subset of G and
$$\int_{\mathsf{G}} \|\Phi(x)\|^p_{\mathfrak{C}_x} \mathrm{d}\mu(x) < \infty.$$
It is denoted by $L^p(\mathsf{G}\,|\,\mathscr{C})$ and becomes a Banach space with the norm
$$\|\Phi\|_{L^p(\mathsf{G}\,|\,\mathscr{C})} := \Big(\int_{\mathsf{G}} \|\Phi(x)\|^p_{\mathfrak{C}_x} \mathrm{d}\mu(x)\Big)^{1/p}.$$
Moreover, the L^1 space can be made a Banach *-algebra by considering the following operations:
$$\big(\Phi * \Psi\big)(x) = \int_{\mathsf{G}} \Phi(y) \bullet \Psi(y^{-1}x)\,\mathrm{d}\mu(y) \quad \text{and} \quad \Phi^*(x) = \Delta(x^{-1})\,\Phi(x^{-1})^{\bullet}\,,$$
were the integral sign is meant to be interpreted in the sense of Bochner (observe that $\Phi(y) \bullet \Psi(y^{-1}x)$ belongs to the Banach space $\mathfrak{C}_x$, for every $y \in \mathsf{G}$). The algebra obtained is called the *cross-sectional algebra*.

Example 2.2.27. The cross-sectional algebra obtained in the setting of the Example 2.2.22 is typically denoted in the literature by $L^1_\alpha(\mathsf{G},\mathcal{A})$. If the action is trivial and $\mathcal{A} = \mathbb{C}$, we obtain the group algebra $L^1(\mathsf{G})$.

Theorem 2.2.28. *Every cross-sectional algebra $L^1(\mathsf{G}\,|\,\mathscr{C})$ has an approximate unit.*

Proof. Again (as in Theorem 2.1.19), we shall not give the proof but the reference. The same reasons apply. This result follows from [26, Prop. 16.3] and [26, Thm. 5.11]. $\square$

Chapter 3

Topological Dynamics of Groupoid Actions

The present chapter is dedicated to a systematic study of the most elementary dynamical notions in the framework of *continuous groupoid actions on topological spaces*. We emphasize *recurrence phenomena*. As a general convention, all the topological spaces (including the groupoids) are Hausdorff. Local compactness is required only when needed. In certain cases we ask all the fibers of the groupoid to be non-compact, in order to avoid triviality.

In most cases (but not always), guessing the analog of the standard notions in the groupoid case is quite straightforward. Both in the statements and the proofs, one has to take into account the fibered nature of a groupoid action. When a groupoid Ξ with unit space X acts on a topological space Σ, the action θ is only defined on a subset of the product $\Xi\times\Sigma$. The element $\theta_\xi(\sigma)$ is defined only if the domain $\mathrm{d}(\xi)$ of $\xi\in\Xi$ coincides with the image $\rho(\sigma)$ of $\sigma\in\Sigma$ by the *anchor map* $\rho:\Sigma\to X$, which is part of the definition of the action. There are also some technical difficulties that we discuss briefly:

(a) When dealing with $\theta_\xi(\sigma)$, sometimes one needs to approximate ξ by a net $\{\xi_i\mid i\in I\}$. Usually $\theta_{\xi_i}(\sigma)$ makes no sense, and this requires more involved arguments.

(b) When a group G acts over a topological space Σ, if $\mathsf{A}\subset\mathsf{G}$ and $U\subset\Sigma$ are open, $\mathsf{A}U\subset\Sigma$ is also open. Rather often, this plays an important role in the proofs. For groupoid actions, this is true only in the special case when the domain (source) map $\mathrm{d}:\Xi\to\Xi$ is open; we say that the groupoid is *open* if this holds. This may fail even in simple examples. In such cases some expected properties do not hold, as counterexamples will show.

(c) Even if the map d is open, in most cases the translation $\theta_\xi(U)$ of an open set is no longer open. For group actions this phenomenon is absent, since the action is composed of (global) homeomorphisms. Therefore, extra care will be needed in some proofs.

3.1 Groupoids and groupoid actions

3.1.1 The framework

Let us define groupoids:

Definition 3.1.1. A groupoid consists of 7 elements: a set Ξ, the subset of units $\Xi^{(0)} \equiv X \subset \Xi$, the domain and range map $\mathrm{d}, \mathrm{r} : \Xi \to \Xi^{(0)}$, the family of composable pairs $\Xi^{(2)} := \{(\xi, \gamma) \in \Xi \times \Xi \mid \mathrm{d}(\xi) = \mathrm{r}(\gamma)\}$ and two operations $(\xi, \gamma) \mapsto \xi\gamma$, from $\Xi^{(2)}$ to Ξ and $\gamma \mapsto \gamma^{-1}$, from Ξ to Ξ, such that

1. (associativity) if $(\xi, \gamma) \in \Xi^{(2)}$ and $(\gamma, \eta) \in \Xi^{(2)}$, then $(\xi, \gamma\eta) \in \Xi^{(2)}$, $(\xi\gamma, \eta) \in \Xi^{(2)}$ and $(\xi\gamma)\eta = \xi(\gamma\eta)$
2. (involution) $(\xi^{-1})^{-1} = \xi$ for all $\xi \in \Xi$
3. (identity) for all $\xi \in \Xi$, $(\xi, \xi^{-1}) \in \Xi^{(2)}$ and $(\xi, \gamma) \in \Xi^{(2)}$ implies $\xi^{-1}(\xi\gamma) = \gamma$ and $(\xi\gamma)\gamma^{-1} = \xi$.

The idea is to generalize the concept of a group, by weakening the group operation. The operation in a groupoid is no longer globally defined, but partially. Ξ is a group if and only if X is a singleton. All of the above can be succintly -and somewhat pedantic- stated as 'a groupoid is simply an small category where every arrow is invertible'. We will give several examples of groupoids very soon. For $M, N \subset X$ one uses the standard notations

$$\Xi_M := \mathrm{d}^{-1}(M)\,, \quad \Xi^N := \mathrm{r}^{-1}(M)\,, \quad \Xi_M^N := \Xi_M \cap \Xi^N.$$

Particular cases are the d-fibre $\Xi_x \equiv \Xi_{\{x\}}$, the r-fibre $\Xi^x \equiv \Xi^{\{x\}}$ and the isotropy group $\Xi_x^x \equiv \Xi_{\{x\}}^{\{x\}}$ of a unit $x \in X$. Clearly $\Xi_x^x\, \Xi^x \subset \Xi^x$ and $\Xi_x\, \Xi_x^x \subset \Xi_x$.

The subset Δ of the topological groupoid Ξ is called a *subgroupoid* if for every $(\xi, \eta) \in (\Delta \times \Delta) \cap \Xi^{(2)}$ one has $\xi\eta \in \Delta$ and $\xi^{-1} \in \Delta$. This subgroupoid is *wide* if $\Delta^{(0)} = \Xi^{(0)}$.

A topological groupoid is a groupoid Ξ with a topology such that the inversion $\xi \mapsto \xi^{-1}$ and multiplication $(\xi, \eta) \mapsto \xi\eta$ are continuous. The topology in $\Xi^{(2)}$ is the topology induced by the product topology. Whenever the map $\mathrm{d} : \Xi \to X$ is open, we say that *the groupoid* Ξ *is open*. Equivalent conditions are: (i) $\mathrm{r} : \Xi \to X$ is open and (ii) the multiplication is an open map. It is known that a locally compact groupoid possessing a Haar system is open. In particular, étale groupoids and Lie groupoids are open.

An equivalence relation on X is defined by $x \approx y$ if $x = \mathrm{r}(\xi)$ and $y = \mathrm{d}(\xi)$ for some $\xi \in \Xi$. This leads to the usual notions of *orbit, invariant (saturated) set, saturation, transitivity*, etc. *The orbit of a point* x will be denoted by $\mathcal{O}_x = \mathrm{r}(\Xi_x)$. Its closure $\overline{\mathcal{O}}_x$ is called *an orbit closure*.

Definition 3.1.2. *A groupoid action* is a 4-tuple $(\Xi, \rho, \theta, \Sigma)$ consisting of a groupoid Ξ , a set Σ , a surjective map $\rho : \Sigma \to X$ (*the anchor*) and the action map

$$\theta : \Xi \bowtie \Sigma := \{(\xi, \sigma)\,|\, \mathrm{d}(\xi) = \rho(\sigma)\} \ni (\xi, \sigma) \mapsto \theta_\xi(\sigma) \equiv \xi \bullet_\theta \sigma \in \Sigma \tag{3.1.1}$$

satisfying the axioms:

1. $\rho(\sigma)\bullet_\theta\sigma = \sigma,\, \forall\, \sigma\in\Sigma\,$,
2. if $(\xi,\eta)\in\Xi^{(2)}$ and $(\eta,\sigma)\in\Xi\bowtie\Sigma$, then $(\xi,\eta\bullet_\theta\sigma)\in\Xi\bowtie\Sigma$ and $(\xi\eta)\bullet_\theta\sigma=\xi\bullet_\theta(\eta\bullet_\theta\sigma)\,$.

An action of a topological groupoid in a topological space is just an action (Ξ,ρ,θ,Σ) where Ξ is a topological groupoid, Σ is a Hausdorff topological space and the maps ρ,θ are continuous.

If the action θ is understood, we will write $\xi\bullet\sigma$ instead of $\xi\bullet_\theta\sigma$. If ρ is not supposed surjective, then $\rho(\Sigma)$ is an invariant subset of the unit space and only the reduction $\Xi^{\rho(\Sigma)}_{\rho(\Sigma)}$ really acts on Σ, so asking ρ to be onto seems convenient.

Example 3.1.3. Each topological groupoid acts continuously in a canonical way on its unit space. For this $\Sigma = X$ and $\rho=\mathrm{id}_X$, and then (note the special notation) $\xi\circ x := \xi x\xi^{-1}$ as soon as $\mathrm{d}(\xi)=x$. Putting this differently, ξ sends $\mathrm{d}(\xi)$ into $\mathrm{r}(\xi)$. One could also name this *the terminal action*; see Lemma 3.5.2.

Example 3.1.4. Let (Ξ,ρ,θ,Σ) be a continuous groupoid action and Δ a wide subgroupoid of Ξ. The restricted action is defined by keeping the same anchor ρ and just restricting the map θ to

$$\Delta\bowtie\Sigma=\{(\xi,\sigma)\in\Delta\times\Sigma\,|\,\mathrm{d}(\xi)=\rho(\sigma)\}=\{(\xi,\sigma)\in\Xi\bowtie\Sigma\,|\,\xi\in\Delta\}\,.$$

Example 3.1.5. The topological groupoid Ξ also acts on itself, with $\Sigma:=\Xi$, $\rho:=\mathrm{r}$ and $\xi\bullet\eta:=\xi\eta$.

For $\xi\in\Xi$, $\mathsf{A},\mathsf{B}\subset\Xi$, $M\subset\Sigma$ we use the notations

$$\mathsf{AB} := \big\{\xi\eta\,\big|\,\xi\in\mathsf{A}\,,\,\eta\in\mathsf{B}\,,\,\mathrm{d}(\xi)=\mathrm{r}(\eta)\big\}\,, \tag{3.1.2}$$

$$\xi\bullet M := \big\{\xi\bullet\sigma\,\big|\,\sigma\in M\cap\rho^{-1}[\mathrm{d}(\xi)]\big\}\,,$$

$$\mathsf{A}\bullet M := \big\{\xi\bullet\sigma\,\big|\,\xi\in\mathsf{A}\,,\sigma\in M\,,\mathrm{d}(\xi)=\rho(\sigma)\big\}=\bigcup_{\xi\in\mathsf{A}}\xi\bullet M\,.$$

These sets could be void in non-trivial situations.

Remark 3.1.6. In [55] it is shown that if the groupoid Ξ is open then $\mathsf{A}\bullet M\subset\Sigma$ is open whenever the sets $\mathsf{A}\subset\Xi$ and $M\subset\Sigma$ are open. For topological group actions the product $\mathsf{A}M$ is open provided that only the subset M is open. In addition, if A,B are subsets of the group, AB is open whenever at least one of the subsets is. Examples below will show that for groupoids this is not true, and this will require some special care in some of our proofs. Note that, even for open groupoids, the translation $\xi\bullet M$ of an open subset M of Σ could not be open.

Definition 3.1.7. We are going to use *orbits* $\mathfrak{O}_\sigma:=\Xi_{\rho(\sigma)}\bullet\sigma$ and *orbit closures* $\overline{\mathfrak{O}}_\sigma$. *The orbit equivalence relation* will be denoted by $\sim$. A subset $M\subset\Sigma$ is called *invariant* if $\xi\bullet M\subset M$, for every $\xi\in\Xi$. If $N\subset\Sigma$, its *saturation*

$$\mathsf{Sat}(N)=\Xi\bullet N=\bigcap_{\substack{N\subset M\\ M\text{ invariant}}} M$$

is the smallest invariant subset of Σ containing N.

Proposition 3.1.8. *Assume that Ξ is an open groupoid. The saturation of an open set is also open. The interior M°, the closure $\overline{M}$ and the boundary ∂M of an invariant subset M of Σ are also invariant.*

Proof. If N is an open set, $\mathsf{Sat}(N) = \Xi \bullet N$ is open, as d is an open map; see Remark 3.1.6.

One has $(M^\circ)^c = \overline{M^c}$ and $\partial M = \overline{M} \setminus M^\circ$. Since the difference of two invariant sets is clearly invariant, it is enough to show that M° is invariant. If $\sigma \in M$ is an interior point, there exists some open set $U \subset M$ containing σ, so we have

$$\xi \bullet \sigma \in \Xi \bullet U \subset \Xi \bullet M = M,$$

implying that $\xi \bullet \sigma$ is also an interior point of M, since $\Xi \bullet U$ is open. $\square$

Remark 3.2.1 and Example 3.3.2 will show that the openness assumption cannot be removed.

Lemma 3.1.9. *For every $\sigma \in \Sigma$ one has $\rho\big(\mathfrak{O}_\sigma\big) = \mathcal{O}_{\rho(\sigma)}$. The map ρ sends $\bullet$-invariant subsets of Σ into $\circ$-invariant subsets of the unit space X (see Example 3.1.3).*

Proof. If $y \in \rho\big(\mathfrak{O}_\sigma\big)$, for some $\xi \in \Xi_{\rho(\sigma)}$ one has

$$y = \rho(\xi \bullet \sigma) = \mathrm{r}(\xi) \in \mathcal{O}_{\rho(\sigma)}\,.$$

On the other hand, if $y \in \mathcal{O}_{\rho(\sigma)}$, there exists $\xi \in \Xi_{\rho(\sigma)}$ with $y = \mathrm{r}(\xi) = \rho(\xi \bullet \sigma) \in \rho\big(\mathfrak{O}_\sigma\big)$ and the equality is proven. From this, the last part is trivial. $\square$

3.1.2 Recurrence sets

Definition 3.1.10. Let us introduce the function

$$\Xi \bowtie \Sigma \ni (\xi, \sigma) \xrightarrow{\vartheta} (\xi \bullet \sigma, \sigma) \in \Sigma \times \Sigma$$

and denote by q the projection on the first variable $\Xi \times \Sigma \to \Xi$ and by $\mathfrak{q}$ its restriction to $\Xi \bowtie \Sigma$. For every $M, N \subset \Sigma$ one defines *the recurrence set*

$$\widetilde{\Xi}_M^N := \mathfrak{q}\big[\vartheta^{-1}(N \times M)\big]\,.$$

A simple inspection of the definitions reveals that

$$\widetilde{\Xi}_M^N = \{\xi \in \Xi \,|\, (\xi \bullet M) \cap N \neq \emptyset\}\,,$$

which explains the terminology (see also Example 3.2.3). The set $\widetilde{\Xi}_M^N$ is increasing in M and N. In the group case, one also uses the term "dwelling set".

Remark 3.1.11. Note that $\widetilde{\Xi}_M^N = \bigcup_{\sigma \in M} \widetilde{\Xi}_\sigma^N$, where

$$\widetilde{\Xi}_\sigma^N \equiv \widetilde{\Xi}_{\{\sigma\}}^N = \{\xi \in \Xi_{\rho(\sigma)} \,|\, \xi \bullet \sigma \in N\} \subset \Xi_{\rho(\sigma)}^{\rho(N)} \subset \Xi_{\rho(\sigma)}\,.$$

It follows immediately that $\widetilde{\Xi}_M^N \subset \Xi_{\rho(M)}^{\rho(N)}$. Actually, when ρ is also injective, one has $\widetilde{\Xi}_M^N = \Xi_{\rho(M)}^{\rho(N)}$. This applies, in particular, to Example 3.1.3. *The stabilizer* $\widetilde{\Xi}_\sigma^\sigma = \{\xi \in \Xi \,|\, \xi \bullet \sigma = \sigma\}$ is a closed subgroup of the isotropy group $\Xi_{\rho(\sigma)}^{\rho(\sigma)}$. They coincide whenever ρ is injective.

Example 3.1.12. In the setting of Example 3.1.5 one has

$$\widetilde{\Xi}_M^N = \left\{\xi \in \Xi \,\middle|\, \Xi^{\mathrm{d}(\xi)} \cap M \cap \xi^{-1} N \neq \emptyset\right\}.$$

Example 3.1.13. Given a topological space Σ, *the fundamental groupoid* Ξ, typically denoted by $\Pi_1(\Sigma)$, is just the set of homotopy classes of paths between pairs of points. The space of all paths is given the compact-open topology and this induces in Ξ the quotient topology. We observe that X is basically Σ and the action $\xi \circ \sigma$ moves the starting point σ through the path ξ. The recurrence sets are expressed in terms of the path-connectedness of Σ:

$$\widetilde{\Xi}_M^N = \Xi_M^N = \{\gamma : [0,1] \to \Sigma \mid \gamma \text{ continuous }, \gamma(0) \in M \text{ and } \gamma(1) \in N\}$$

modulo homotopy. The stabilizer of $\sigma \in X$ is the fundamental group $\Xi_\sigma^\sigma = \pi(\Sigma, \sigma)$ of Σ rooted at σ.

The next straightforward results will be useful in the next sections.

Lemma 3.1.14. *If $M, N \subset \Sigma$ and $\eta_1, \eta_2 \in \Xi$, then $\widetilde{\Xi}_{\eta_1 \bullet M}^{\eta_2 \bullet N} = \eta_2\, \widetilde{\Xi}_M^N \eta_1^{-1}$ and $\big(\widetilde{\Xi}_M^N\big)^{-1} = \widetilde{\Xi}_N^M$.*

Proof. We only show the first equality:

$$\begin{aligned}
\xi \in \widetilde{\Xi}_{\eta_1 \bullet M}^{\eta_2 \bullet N} &\Leftrightarrow \exists\, \sigma \in M\,, \tau \in N \text{ with } \xi \bullet (\eta_1 \bullet \sigma) = \eta_2 \bullet \tau \\
&\Leftrightarrow \exists\, \sigma \in M\,, \tau \in N \text{ with } \big(\eta_2^{-1} \xi\, \eta_1\big) \bullet \sigma = \tau \\
&\Leftrightarrow \eta_2^{-1} \xi\, \eta_1 \in \widetilde{\Xi}_M^N \\
&\Leftrightarrow \xi \in \eta_2\, \widetilde{\Xi}_M^N \eta_1^{-1}.
\end{aligned}$$

□

Lemma 3.1.15. *Let $M, N \subset \Sigma$. Then*

$$\mathsf{Sat}(M) \cap N \neq \emptyset \Leftrightarrow \mathsf{Sat}(M) \cap \mathsf{Sat}(N) \neq \emptyset \Leftrightarrow \widetilde{\Xi}_M^N \neq \emptyset.$$

Proof. One has $\mathsf{Sat}(M) \cap \mathsf{Sat}(N) \neq \emptyset$ if and only if there exist $\xi_1, \xi_2 \in \Xi$, $\sigma \in M$ and $\tau \in N$ such that $\xi_1 \bullet \sigma = \xi_2 \bullet \tau$, which is equivalent to $\big(\xi_2^{-1}\xi_1\big) \bullet \sigma = \tau \in N$, so we have $\xi_2^{-1}\xi_1 \in \widetilde{\Xi}_M^N$ and $\big(\xi_2^{-1}\xi_1\big) \bullet \sigma \in \mathsf{Sat}(M)$. We used the fact that

$$\mathrm{d}\big(\xi_2^{-1}\big) = \mathrm{r}(\xi_2) = \rho(\xi_2 \bullet \tau) = \rho(\xi_1 \bullet \sigma) = \mathrm{r}(\xi_1).$$

For the converse: If $\widetilde{\Xi}_M^N \neq \emptyset$, there exists $\xi \in \Xi$ such that $\xi \bullet \sigma = \tau$, with $\sigma \in M$ and $\tau \in N$. Then $\mathsf{Sat}(M) \cap N \neq \emptyset$, from which $\mathsf{Sat}(M) \cap \mathsf{Sat}(N) \neq \emptyset$ follows. □

Remark 3.1.16. Let $(\Xi, \rho, \theta, \Sigma)$ be a continuous groupoid action and Δ a wide subgroupoid of Ξ. In terms of the restricted action from Example 3.1.4, if $M, N \subset \Sigma$, the contention $\widetilde{\Delta}_M^N \subset \widetilde{\Xi}_M^N$ between the corresponding recurrence sets is obvious. It is also clear that the Δ-orbit of any point of Σ is contained in the Ξ-orbit of this point and that the invariant sets under Ξ are also invariant under Δ. From this one deduces many simple connections between dynamical properties of the two actions, that we will not write down.

3.2 Some examples

3.2.1 Equivalence relations

If $\Pi \subset X \times X$ is an equivalence relation on the Hausdorff topological space X, one can make Π into a topological groupoid by using the product topology in $X \times X$ and the operations

$$\mathrm{d}(x,y) = (y,y)\,, \quad \mathrm{r}(x,y) = (x,x)\,, \quad (x,y)(y,z) = (x,z)\,, \quad (x,y)^{-1} = (y,x)\,.$$

The unit space is $\mathsf{Diag}(X)$ and we identify it with X, via the homeomorphism $(x,x) \mapsto x$. As a particular case of Example 3.1.3, the groupoid $\Xi := \Pi$ acts in a canonical way on X by

$$(x,y) \circ y = x, \quad \forall\, x,y \in X.$$

For every $M, N \subset X$ we get

$$\widetilde{\Xi}_M^N = \Xi_M^N = \Pi \cap (N \times M)\,. \tag{3.2.1}$$

There are two extreme particular cases: (i) $\Pi = \mathsf{Diag}(X)$ (*the trivial groupoid*), for which Ξ_M^N may be identified with $N \cap M$, and (ii) $\Pi = X \times X$ (*the pair groupoid*), when $\Xi_M^N = N \times M$. Actually, an equivalence relation on X is a wide subgroupoid of the pair groupoid.

Remark 3.2.1. We keep the same notations. The first projection $X \times X \to X$ is always open. The restriction to a subset (Π in our case) may not be an open function in general. (However, when Π is an open subset in $X \times X$, it is also an open groupoid.) One special example [55, Ex. 6.2] consists of setting $X = \mathbb{R}$, the relation being $x\,\Pi\,y \Leftrightarrow x,y \in [-1,1]$ or $x = y$. This groupoid is Haussdorf and locally compact, but it is not open: observe that $\Xi \circ \big(-\frac{1}{2}, \frac{1}{2}\big) = [-1,1]$. This is relevant for Remark 3.1.6, Proposition 3.1.8 and their subsequent consequences.

Even if the equivalence relation is open, very often ”translations” of open sets are not open. For the pair groupoid, if $(x,y) \in X \times X$ and $y \in U \subset X$ is open, then $(x,y) \circ U = \{x\}$.

Non-open equivalence relations will be used repeatedly as counterexamples. It is true, however, that in some situations one considers on equivalence relations topologies which are different from the one induced from the Cartesian product.

3.2.2 Group actions

We indicate two ways to encode group actions by groupoids. The second one will be convenient for our purposes. We thought interesting to also mention the first one, since it seems natural.

Example 3.2.2. As a particular case of Example 3.1.3, *the transformation groupoid* $\Xi \equiv \mathsf{G} \ltimes_\gamma X$ associated to the continuous action γ of the topological group G on the topological space X naturally acts on X by $(a,x) \circ x := \gamma_a(x)$. We recall that, as a topological space, it is just $\mathsf{G} \times X$. The composition is $\big(b, \gamma_a(x)\big)(a,x) := (ba, x)$ and inversion reads $(a,x)^{-1} := \big(a^{-1}, \gamma_a(x)\big)$. If $M, N \subset \Sigma \equiv X$, then

$$\widetilde{\Xi}_M^N = \Xi_M^N = \big\{(a,x) \in \mathsf{G} \times M \;\big|\; \gamma_a(x) \in N\big\} \subset \mathsf{G} \times X.$$

The first projection $p:\mathsf{G}\times X\to\mathsf{G}$ restricts to a surjection

$$p:\widetilde{\Xi}_M^N\to\mathrm{Rec}_\gamma(M,N):=\big\{a\in\mathsf{G}\,\big|\,\gamma_a(M)\cap N\neq\emptyset\big\}\subset\mathsf{G}, \tag{3.2.2}$$

where $\mathrm{Rec}_\gamma(M,N)$ *is the usual recurrence set for dynamical systems* [3, 19, 21]. Injectivity fails in general: for instance, $\widetilde{\Xi}_X^X=\mathsf{G}\times X$ while $\mathrm{Rec}_\gamma(X,X)=\mathsf{G}$. On the other hand, if M or N are singletons, injectivity holds. Using slightly simplified notations, one has $\widetilde{\Xi}_{x_0}^N\cong\mathrm{Rec}_\gamma(x_0,N)$ and $\widetilde{\Xi}_M^{x_0}\cong\mathrm{Rec}_\gamma(M,x_0)$. Although the relation (3.2.2) is quite concrete, it will not be precise enough to make suitable connections between dynamical properties in the group and in the groupoid framework.

Example 3.2.3. So we implement differently the classical dynamical system $(\mathsf{G},\gamma,\Sigma)$ (for a better correspondence of notations, we set Σ for the space of the group action). The group is an open groupoid in the obvious way; so we have $\Xi:=\mathsf{G}$ and the unit space $X=\{\mathsf{e}\}$ is only composed of the unit of the group. The source and the domain maps are constant, the same being true for $\rho:\Sigma\to\{\mathsf{e}\}$; it follows that $\mathsf{G}\bowtie\Sigma=\mathsf{G}\times\Sigma$. One sets $a\bullet\sigma:=\gamma_a(\sigma)$ for every $a\in\mathsf{G},\sigma\in\Sigma$ (thus $\theta=\gamma$). Note that this is not covered by Examples 3.1.3 or 3.1.5. A simple inspection of the definitions shows that

$$\widetilde{\Xi}_S^T\equiv\widetilde{\mathsf{G}}_M^N=\mathrm{Rec}_\gamma(M,N)\,, \tag{3.2.3}$$

and this will be very convenient below.

3.2.3 The Deaconu-Renault groupoid

Let $\nu:X\to X$ be a local homeomorphism of the Hausdorff topological space X. To unify many constructions in groupoid C^*-algebras, one defines *the Deaconu-Renault groupoid* [17, 51]

$$\Xi(\nu):=\big\{(x,k-l,y)\,\big|\,x,y\in X\,,k,l\in\mathbb{N}\,,\nu^k(x)=\nu^l(y)\big\}\,,$$

with structure maps

$$(x,n,y)(y,m,z):=(x,n+m,z)\,,\quad (x,n,y)^{-1}:=(y,-n,x)\,,$$

$$\mathrm{d}(x,n,y):=y\,,\quad \mathrm{r}(x,n,y):=x\,.$$

With a suitable topology (not needed here), it turns into a Hausdorff étale groupoid over X. If ν is a (global) homeomorphism, this results in the transformation groupoid of Example 3.2.2 for the group $\mathsf{G}=\mathbb{Z}$.

An important tool is *the canonical cocycle* (a groupoid morphism)

$$c:\Xi(\nu)\to\mathbb{Z}\,,\quad c(x,n,y):=n\,,$$

which is, of course, the restriction to $\Xi(\nu)$ of the middle projection. For every $M,N\subset X$ set

$$\mathbb{Z}_\nu(N,M):=\big\{k-l\,\big|\,\nu^k(N)\cap\nu^l(M)\neq\emptyset\big\}\subset\mathbb{Z}\,.$$

The orbit of y is $\mathcal{O}_y=\{x\in X\mid\mathbb{Z}_\nu(x,y)\neq\emptyset\}$, where notationally we identify singletons to points. The canonical cocycle restricts to a surjection

$$c:\widetilde{\Xi(\nu)}_M^N\equiv\Xi(\nu)_M^N\to\mathbb{Z}_\nu(N,M)\,. \tag{3.2.4}$$

In contrast with the global case, if $M=\{y\}$, (3.2.4) could still fail to be injective. But clearly c restricts to a one to one map allowing to identify $\Xi(\nu)_y^x$ with $\mathbb{Z}_\nu(x,y)$ for every $x,y\in X$.

3.2.4 The action groupoid of a groupoid action

A continuous groupoid action $(\Xi, \rho, \bullet, \Sigma)$ being given, one constructs *the action (or transformation, or crossed product) groupoid* which, as a set, is the closed subspace $\Xi \bowtie \Sigma$ of $\Xi \times \Sigma$ introduced in (3.1.1) and the structure maps are

$$\big(\eta, \xi \bullet \sigma\big)(\xi, \sigma) := (\eta\,\xi, \sigma)\,, \quad (\xi, \sigma)^{-1} := \big(\xi^{-1}, \xi \bullet \sigma\big)\,,$$
$$\mathsf{d}(\xi, \sigma) := (\rho(\sigma), \sigma) \equiv \sigma\,, \quad \mathsf{r}(\xi, \sigma) := (\mathrm{r}(\xi), \xi \bullet \sigma) \equiv \xi \bullet \sigma. \tag{3.2.5}$$

To stress the origin of the construction, we are going to denote by $\Xi \ltimes_\theta \Sigma$ this groupoid, in analogy with the group case which is a particular example. The space of units $(\Xi \ltimes_\theta \Sigma)^{(0)}$ identifies with Σ. The canonical action of $\Xi \ltimes_\theta \Sigma$ on Σ (see Example 3.1.3)

$$(\xi, \sigma) \circ (\rho(\sigma), \sigma) := (\xi, \sigma)(\rho(\sigma), \sigma)(\xi, \sigma)^{-1} = (\mathrm{r}(\xi), \xi \bullet \sigma) = (\rho(\xi \bullet \sigma), \xi \bullet \sigma)$$

may also be written $(\xi, \sigma) \circ \sigma = \xi \bullet \sigma$, and thus reproduces in some way the initial action. The invariant subsets, in particular the orbits, are the same. This has a series of obvious consequences on the way dynamical properties are preserved when passing from $(\Xi, \rho, \bullet, \Sigma)$ to $\Xi \ltimes_\theta \Sigma$.

The connection between the relevant recurrent sets is similar with (and in fact generalizes) that of Example 3.2.2: if $M, N \subset \Sigma$ we get

$$(\Xi \ltimes_\theta \Sigma)_M^N = \{(\xi, \sigma) \in \Xi \ltimes_\theta \Sigma \,|\, \sigma \in M\,,\ \xi \bullet \sigma \in N\}\,.$$

Consequently

$$\widetilde{\Xi}_M^N = p\big[(\Xi \ltimes_\theta \Sigma)_M^N\big]\,, \tag{3.2.6}$$

where p is the restriction of the first projection of the product $\Xi \times \Sigma$. This restriction is not always injective, being constant on any set of the type $\{\xi\} \times \rho^{-1}[\mathrm{d}(\xi)]$. It follows that *for recurrence phenomena it is not always a good idea to replace the initial action by the action groupoid.*

3.2.5 Groupoid pull-backs

There is a powerful method to construct new more sophisticated groupoids from simpler ones, which, however, does not loose control over the orbit structure or the recurrence sets. Let $\mathrm{d}, \mathrm{r} : \Xi \to X$ be the domain and the range maps of a topological groupoid, Ω a topological space and $h : \Omega \to X$ an open continuous surjection. Let

$$h^{\downarrow\downarrow}(\Xi) := \{(\omega, \xi, \omega') \in \Omega \times \Xi \times \Omega \mid \mathrm{r}(\xi) = h(\omega), \mathrm{d}(\xi) = h(\omega')\}$$

be *the associated pull-back groupoid* [15, 43]. We recall its structural maps:

$$(\omega_1, \xi, \omega_2)(\omega_2, \eta, \omega_3) := (\omega_1, \xi\eta, \omega_3)\,, \quad (\omega, \xi, \omega')^{-1} := \big(\omega', \xi^{-1}, \omega\big)\,,$$
$$\mathfrak{d}(\omega, \xi, \omega') := \omega'\,, \quad \mathfrak{r}(\omega, \xi, \omega') := \omega\,.$$

Note the relations between orbits and orbit closures in the two groupoids (closures commute with open continuous surjections):

$$\mathcal{O}_\omega^{\downarrow\downarrow} = h^{-1}\big(\mathcal{O}_{h(\omega)}\big)\,, \quad \overline{\mathcal{O}_\omega^{\downarrow\downarrow}} = h^{-1}\big(\overline{\mathcal{O}_{h(\omega)}}\big)\,.$$

For $M, N \subset \Omega$, by inspection one gets

$$h^{\downarrow\downarrow}(\Xi)_M^N = N \times \Xi_{h(M)}^{h(N)} \times M. \tag{3.2.7}$$

3.3 Topological transitivity

3.3.1 The standard notions

Let us fix a continuous groupid action $(\Xi, \rho, \theta, \Sigma)$. We start with the simplest notions. If there is just one orbit, the action is *transitive*. This happens for the pair groupoid, for instance. (In [20, Prop. 3.18] it is shown that a compact transitive groupoid is open.) A point having a dense orbit is called *a transitive point*. If there is a dense orbit, i.e. if a transitive point does exist, one says that the action is *pointwise transitive*. For the more refined notions, one needs first to prove the next result:

Theorem 3.3.1. *Let us consider the following conditions:*

(i) *Σ is not the union of two proper invariant closed subsets.*

(i') *Any two open non-void invariant subsets of Σ have non-trivial intersection.*

(ii) *Each non-empty open invariant subset of Σ is dense.*

(iii) *For every $U, V \subset \Sigma$ open and non-void, $\widetilde{\Xi}_U^V \neq \emptyset$ holds (*recurrent transitivity*).*

(iv) *Each invariant subset of Σ is ether dense, or nowhere dense (*topological transitivity*).*

Then the following implications hold:

1. *Transitivity $\Rightarrow$ pointwise transitivity $\Rightarrow$ recurrent transitivity (iii).*
2. *One has $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i') \Leftrightarrow (i)$. None of the other implications holds in general. Pointwise transitivity does not imply topological transitivity.*
3. *If the groupoid Ξ is open, all the conditions in 2. are equivalent.*

Proof. 1. The first implication is trivial (and obviously it is not an equivalence). We verify now the second one. Assume that Σ has a dense orbit $\mathfrak{O}_\sigma$ and let $\emptyset \neq V_1, V_2$ be open sets. One has

$$\xi_1 \bullet \sigma \in \mathfrak{O}_\sigma \cap V_1 \neq \emptyset \quad \text{and} \quad \xi_2 \bullet \sigma \in \mathfrak{O}_\sigma \cap V_2 \neq \emptyset$$

for elements $\xi_1, \xi_2 \in \Xi$ with $\mathrm{r}\big(\xi_1^{-1}\big) = \mathrm{d}(\xi_1) = \rho(\sigma) = \mathrm{d}(\xi_2)$. Since $\big(\xi_2\, \xi_1^{-1}\big)(\xi_1 \bullet \sigma) = \xi_2 \bullet \sigma$ we infer that $\xi_2\, \xi_1^{-1} \in \widetilde{\Xi}_{V_1}^{V_2} \neq \emptyset$.

2. $(i) \Leftrightarrow (i')$ If (i) fails, i.e. $\Sigma = C \cup D$ with C and D proper closed invariant subsets, then $C^c \cap D^c = \emptyset$. This contradicts (i'), since C^c and D^c are open, non-void and invariant. On the other hand, if A, B are open non-empty invariant sets such that $A \cap B = \emptyset$, then $A^c \cup B^c = \Sigma$ with A^c, B^c proper closed invariant subsets, finishing the proof of the equivalence.

$(ii) \Rightarrow (i')$ If each non-void open invariant subset is dense, it meets every other (invariant) non-void open set.

$(iii) \Rightarrow (ii)$ Let $\emptyset \neq U \subset \Sigma$ open and invariant. By assumption, for every non-void open set $V \subset \Sigma$ there exists some $\xi \in \Xi$ making $(\xi \bullet U) \cap V$ non-void. But $\xi \bullet U \subset U$, implying $U \cap V \neq \emptyset$. Thus, U meets every other non-void open set and must be dense.

$(iv) \Rightarrow (iii)$ Suppose (iv) holds. Let $\emptyset \neq U, V \subset \Sigma$ open sets. $\mathsf{Sat}(U)$ is an invariant set containing U, so it cannot be nowhere dense, meaning that it is dense. Hence $\mathsf{Sat}(U) \cap V \neq \emptyset$ and we conclude by Lemma 3.1.15.

The fact that all the other implications fail without extra assumptions will be showed in a series of counterexamples below.

3. Provided that d is open, it is enough to prove that (i') implies (iv). So let us assume (i'), but let $A \subset \Sigma$ be invariant, neither dense, nor nowhere dense. Then $\big(\overline{A}\big)^\circ$ and $(A^c)^\circ = \big(\overline{A}\big)^c$ are both non-void open sets, which are invariant by Proposition 3.1.8. They should meet, by (i'), but this is obviously false. $\square$

Let us indicate now the necessary counterexamples for groupoids that are not open.

Example 3.3.2. This example shows that $(ii) \not\Rightarrow (iii), (iv)$ (redundantly but explicitly). Let

$$\Sigma = X \equiv \mathbb{R}^\times := (-\infty, 0) \cup (0, \infty).$$

Define the sets

$$X_1 = \{-1\} \cup \big((0,\infty) \setminus \{1\}\big) \quad \text{and} \quad X_2 = \big((-\infty,0) \setminus \{-1\}\big) \cup \{1\}.$$

Forming a partition of $\mathbb{R}^\times$, they induce an equivalence relation Π, viewed as a topological groupoid acting on $\mathbb{R}^\times$ as in subsection 3.2.1. It is easy to check that Π is not open. The orbits are precisely X_1 and X_2 (neither open nor closed), with closures

$$\overline{X}_1 = \{-1\} \cup (0,\infty) \quad \text{and} \quad \overline{X}_2 = (-\infty, 0) \cup \{1\}.$$

These orbit closures are not invariant, showing already that for non-open groupoids the conclusion of Proposition 3.1.8 does not necessarily hold. (And we see that Π is not an open groupoid in an indirect way.)

It is easy to see that $X = X_1 \cup X_2$ is the single (non-void) open invariant set, since any invariant set is a union of orbits, and X_1, X_2 are not open. Thus the condition (ii) from Theorem 3.3.1 holds. On the other hand the condition (iv) definitely fails, since the invariant sets X_1, X_2 are neither dense, nor nowhere dense. Of course recurrent transitivity also fails. To see this directly, take for instance $U = (2, \infty)$ and $V = (-\infty, -2)$, for which

$$\widetilde{\Xi}_U^V = \{(x,y) \in \Pi \mid y \in U, x \in V\} = \emptyset,$$

by (3.2.1) or by a simple computation.

Example 3.3.3. The previous example can be easily modified to show that $(i') \not\Rightarrow (ii)$. In $\mathbb{R}^\times$, define the equivalence relation Υ associated to the partition

$$Y_1 = \{-1\} \cup (0,1) \cup (1,2) \cup (3,\infty), \quad X_2 = \big((-\infty,0) \setminus \{-1\}\big) \cup \{1\}, \quad Y_3 = [2,3]$$

and consider the canonical action of $\Xi = \Upsilon$ on $\mathbb{R}^\times$. As invariant set are union of orbits, the only non-void invariant open sets are $\mathbb{R}^\times$ and

$$Y_1 \cup X_2 = (-\infty, 0) \cup (0, 2) \cup (3, \infty)\,,$$

which have non-empty intersection, so (i') holds. But the set $Y_1 \cup X_2$ is not dense, hence (ii) fails.

Example 3.3.4. Define on $\Sigma = X = \mathbb{R}$ the equivalence relation

$$x \,\Pi\, y \;\Leftrightarrow\; x, y \in \mathbb{Q} \;\text{ or }\; x = y\,.$$

This provides a non-open groupoid acting on $\mathbb{R}$, and this action is pointwise transitive ($\mathbb{Q}$ is a dense orbit), hence recurrently transitive. The invariant set $(0,1) \setminus \mathbb{Q}$ is neither nowhere dense nor dense, so this action is not topologically transitive. This example shows that *pointwise transitivity* $\not\Rightarrow (iv)$ and (consequently) $(iii) \not\Rightarrow (iv)$.

Proposition 3.3.5. *If Ξ is open and Σ is a Baire second-countable space, then topological transitivity, recurrent transitivity and pointwise transitivity are equivalent (and also equivalent to the properties (i) and (ii)).*

Proof. Having in view Theorem 3.3.1, what remains is to show that (ii) implies pointwise transitivity. Since Σ is second-countable, its topology has a countable basis $\{V_n \neq \emptyset\}_{n\in\mathbb{N}}$. By defining $U_n = \Xi \bullet V_n$ we get countably many dense open subsets of a Baire space, so $U = \cap_n U_n$ is also a dense (invariant) set. Let $W \neq \emptyset$ be an open subset of Σ. By the definition of a basis, there exists some $V_n \subset W$. Hence we have $U \subset U_n = \Xi \bullet V_n \subset \Xi \bullet W$. Therefore, if $\sigma \in U$ then $\xi^{-1} \bullet \sigma \in W$ for some $\xi \in \Xi$. Hence σ has a dense orbit and the action is pointwise transitive. $\square$

We recall that Hausdorff locally compact spaces and complete metric spaces are Baire.

Remark 3.3.6. What we used in the proof of Proposition 3.3.5 is the fact that the intersection U is non-void. Actually, one could improve: *under the given requirements, the set of points with dense orbit is a dense G_δ-set.*

Example 3.3.7. In the classical dynamical system case of Example 3.2.3 we recover known results [3, 19, 33]. Since the source map $\mathsf{G} \to \{\mathsf{e}\}$ is clearly open, Theorem 3.3.1 simplifies a lot. Even in this particular case, without second countability the full equivalence from Proposition 3.3.5 fails.

Example 3.3.8. The Deaconu-Renault groupoid is recurrently transitive if and only if for every $\emptyset \neq U, V \subset X$ open sets, there exist $x \in V,\ y \in U,\ k, l \in \mathbb{N}$ such that $\nu^k(x) = \nu^l(y)$.

Example 3.3.9. In the setting of subsection 3.2.4, *the groupoid action $(\Xi, \rho, \bullet, \Sigma)$ is recurrently transitive if and only if the action groupoid $\Xi \ltimes_\theta \Sigma$ is so* (p is surjective, hence $p(B)$ is void if and only if B is void).

Example 3.3.10. Although the pullback groupoid of subsection 3.2.5 might be much more complicated than the initial one, $h^{\downarrow\downarrow}(\Xi)$ and Ξ are simultaneously recurrently transitive. This follows from (3.2.7).

3.3.2 Weak pointwise transitivity

Definition 3.3.11. Define *the invariant closure of* $A \subset \Sigma$ by

$$\mathfrak{C}(A) := \bigcap_{\substack{A \subset M \\ M \text{ closed, invariant}}} M\,.$$

This is, the smallest closed and invariant set including A. One very special case (deserving its own notation) is $\mathfrak{C}_\sigma = \mathfrak{C}(\{\sigma\})$, *the invariant orbit closure of* σ.

Proposition 3.3.12. *The invariant closure verifies*

$$\mathfrak{C}(A) = \mathfrak{C}(\mathsf{Sat}(A)) = \mathfrak{C}\big(\,\overline{\mathsf{Sat}(A)}\,\big)\,.$$

In particular

$$\mathfrak{C}_\sigma = \mathfrak{C}(\mathfrak{O}_\sigma) = \mathfrak{C}(\overline{\mathfrak{O}}_\sigma) \supset \overline{\mathfrak{O}}_\sigma\,.$$

If Ξ *is open,* $\mathfrak{C}(A) = \overline{\mathsf{Sat}(A)}$ *and* $\mathfrak{C}_\sigma = \overline{\mathfrak{O}}_\sigma$.

Proof. Since $A \subset \mathsf{Sat}(A) \subset \overline{\mathsf{Sat}(A)}$, one readily gets

$$\mathfrak{C}(A) \subset \mathfrak{C}(\mathsf{Sat}(A)) \subset \mathfrak{C}\big(\,\overline{\mathsf{Sat}(A)}\,\big)\,.$$

The set $\mathfrak{C}(A)$ is closed and invariant, so it contains $\overline{\mathsf{Sat}(A)}$, implying that $\mathfrak{C}\big(\,\overline{\mathsf{Sat}(A)}\,\big) \subset \mathfrak{C}(A)$.

If Ξ is open, by Proposition 3.1.8, $\overline{\mathsf{Sat}(A)}$ is a closed and invariant set containing A, so $\mathfrak{C}(A) = \overline{\mathsf{Sat}(A)}$. Then use the fact that $\mathfrak{O}_\sigma = \mathsf{Sat}(\{\sigma\})$. ◻

It is not difficult to see that $\mathfrak{C}(A)$ contains the sets

$$\overline{\mathsf{Sat}(A)}\,, \overline{\mathsf{Sat}(\overline{\mathsf{Sat}(A)})}\,, \overline{\mathsf{Sat}(\overline{\mathsf{Sat}(\overline{\mathsf{Sat}(A)})})}\,, \dots$$

Whenever the groupoid is open, this ascending chain of sets collapses in the first step. The relevance of the set $\mathfrak{C}(A)$ in groupoids that are not open is a new phenomenon.

Definition 3.3.13. An action $(\Xi, \rho, \theta, \Sigma)$ is called *weakly pointwise transitive* (wpt) if exists a point $\sigma \in \Sigma$ such that $\mathfrak{C}_\sigma = \Sigma$. In this case, we say that σ is a *weakly transitive point*.

Remark 3.3.14. Since $\overline{\mathfrak{O}}_\sigma \subset \mathfrak{C}_\sigma$, it is clear that *pointwise transitive* $\Rightarrow$ *weakly pointwise transitive*. When Ξ is an open groupoid, *weakly pointwise transitive* $\Leftrightarrow$ *pointwise transitive*.

Example 3.3.15. We compute the invariant closures in the previous examples:

- In example 3.3.2, one has $\mathfrak{C}_\sigma = \mathbb{R}^\times$ for all $\sigma \in \Sigma = \mathbb{R}^\times$. So *every point is weakly transitive*. We recall that this example is not topologically transitive or recurrently transitive.
- In example 3.3.3,

$$\mathfrak{C}_\sigma = \begin{cases} [2,3] & \text{if } \sigma \in [2,3]\,, \\ \mathbb{R}^\times & \text{if } \sigma \in \mathbb{R}^\times \setminus [2,3]\,. \end{cases}$$

- In example 3.3.4,

$$\mathfrak{C}_\sigma = \begin{cases} \mathbb{R} & \text{if } \sigma \in \mathbb{Q}\,, \\ \{\sigma\} & \text{if } \sigma \notin \mathbb{Q}\,. \end{cases}$$

All of these are *weakly pointwise transitive* systems. This examples also show that the sets $\{\mathfrak{C}_\sigma \mid \sigma \in \Sigma\}$ doesn't need to be disjoint: some could intersect or even (strictly) contain others. Of course, this often happens for orbit closures, but for invariant closures the overlaps tend to be larger.

Proposition 3.3.16. *Weak pointwise transitivity implies the condition (i) of Theorem 3.3.1.*

Proof. Suppose that $\Sigma = N_1 \cup N_2$, being N_1, N_2 closed and invariant sets. Without loss of generality, there exists $\sigma \in N_1$ such that $\mathfrak{C}_\sigma = \Sigma$. As N_1 is closed, invariant and contains σ, the relation $\Sigma = \mathfrak{C}_\sigma \subset N_1$ follows. We conclude that (i) holds. $\square$

Remark 3.3.17. Example 3.3.3 shows that *weakly pointwise transitivity does not imply the condition* (ii) of Theorem 3.3.1, in general. We recall that pointwise transitivity does imply (iii) , which is stronger than (ii), so none of these properties is implied by weak topological transitivity. We summarize most of the implications in the following diagram, in which the arrows indicate implications:

$$\begin{tikzcd}
(iv) \arrow[r] & (iii) \arrow[r] & (ii) \arrow[r] & (i) \arrow[r, leftrightarrow] & (i') \\
(t) \arrow[r] \arrow[u] & (\mathrm{pt}) \arrow[rr] \arrow[u] & & (\mathrm{wpt}) \arrow[u] &
\end{tikzcd}$$

3.4 Minimality and almost periodicity

3.4.1 Fixed points

Let $(\Xi, \rho, \theta, \Sigma)$ be a continuous groupoid action.

Definition 3.4.1. *A fixed point* is a point $\sigma \in \Sigma$ such that $\xi \bullet \sigma = \sigma$ for every $\xi \in \Xi_{\rho(\sigma)}$. This is equivalent with $\widetilde{\Xi}_\sigma^\sigma = \Xi_{\rho(\sigma)}$. We write $\sigma \in \Sigma_{\text{fix}}^\theta \equiv \Sigma_{\text{fix}}$.

Example 3.4.2. For group actions, converted into groupoid actions as in Exemple 3.2.3, the notion of fixed point boils down to the usual one. The same can be said if the model is that of Example 3.2.2.

Example 3.4.3. Let $\Xi(\nu)$ be the Deaconu-Renault groupoid attached to the local homeomorphism $\nu : X \to X$. The unit y is a fixed point if and only if for every $x \neq y$ there are no positive integers k, l with $\nu^k(x) = \nu^l(y)$.

Example 3.4.4. An element $\omega \in \Omega$ is a fixed point of the pull-back groupoid $h^{\downarrow\downarrow}(\Xi)$ introduced in subsection 3.2.5 if and only if $h(\omega)$ is a fixed point of the canonical action of Ξ on its unit space.

Proposition 3.4.5. *The set Σ_{fix} is invariant. If Ξ is open, Σ_{fix} is also closed.*

Proof. If $\sigma \in \Sigma_{\text{fix}}$, then $\mathfrak{O}_\sigma = \{\sigma\}$, which makes Σ_{fix} trivially invariant.

If Ξ is open and $\sigma \in \overline{\Sigma}_{\text{fix}}$, then exists a net $(\sigma_i)_{i\in I}$ of fixed points converging to σ. Let $\xi \in \Xi_{\rho(\sigma)}$. By Fell's criterion [55, Prop. 1.1], applied to the open map d and the net $\rho(\sigma_i) \in X$, there exists a net (ξ_j) and a subnet (σ_{i_j}) such that $\xi_j \to \xi$ and $\rho(\sigma_{i_j}) = \mathrm{d}(\xi_j)$. By continuity, we have

$$\xi \bullet \sigma = (\lim_j \xi_j) \bullet (\lim_j \sigma_{i_j}) = \lim_j (\xi_j \bullet \sigma_{i_j}) = \lim_j \sigma_{i_j} = \sigma.$$

So σ is a fixed point for the action. □

Examples 3.4.7 and 3.5.8 will show that openness of Ξ is important.

3.4.2 Periodic and almost periodic points

Although it is not always necessary, in this subsection we prefer to assume that in the continuous action $(\Xi, \rho, \theta, \Sigma)$ the groupoid Ξ is strongly non-compact (all the d-fibres are non-compact).

Definition 3.4.6. (a) Let $x \in X$. The subset A of Ξ_x is called *syndetic* if $\mathsf{KA} = \Xi_x$ for a compact subset K of Ξ .

(b) We say that $\sigma \in \Sigma$ is *periodic*, and we write $\sigma \in \Sigma_{\text{per}}$, if $\widetilde{\Xi}_\sigma^\sigma$ is syndetic in $\Xi_{\rho(\sigma)}$.

(c) The point σ is called *weakly periodic* (we write $\sigma \in \Sigma_{\text{wper}}$) if the subgroup $\widetilde{\Xi}_\sigma^\sigma$ is not compact.

(d) The point $\sigma \in \Sigma$ is said to be *almost periodic* if $\widetilde{\Xi}_\sigma^U$ is syndetic in $\Xi_{\rho(\sigma)}$ for every neighborhood U of σ in Σ . We denote by Σ_{alper} the set of all the almost periodic points. If $\Sigma_{\text{alper}} = \Sigma$, the action is *pointwise almost periodic*.

Example 3.4.7. Consider the equivalence relation Π introduced in Remark 3.2.1, and form the groupoid product $\Xi = \Pi \times \mathbb{Z}$. By considering the action of Ξ in $\Sigma = \mathbb{R}$, we see that

$$\Sigma_{\text{fix}} = (-\infty, -1) \cup (1, \infty)\,, \quad \Sigma_{\text{per}} = \Sigma_{\text{wper}} = \Sigma_{\text{alper}} = \mathbb{R}\,,$$

which illustrates that, in general, the set Σ_{fix} is not closed. This action is pointwise almost periodic.

Example 3.4.8. We fix a unit $y \in X$ of the Deaconu-Renault groupoid of the local homeomorphism $\nu : X \to X$. We remarked that (3.2.4) becomes a bijection for singletons. Then y is periodic if and only if it is weakly periodic, and this happens exactly when $\mathbb{Z}_\nu(x, x)$ (a subgroup of $\mathbb{Z}$) does not coincide with $\{0\}$ (then it will be both infinite and syndetic). This means that for some positive integers $k \neq l$ one has $\nu^k(x) = \nu^l(x)$.

Example 3.4.9. The situation is particularly simple for equivalence relations, outlined in subsection 3.2.1, especially because of equation (3.2.1). In particular, for $y \in X \equiv \Sigma$ one gets

$$\widetilde{\Xi}_y \cong \{x \in X \,|\, x\,\Pi\, y\} \quad \text{and} \quad \widetilde{\Xi}_y^y = \{(y, y)\} \cong \{y\}\,.$$

To insure that the associated groupoid is strongly non-compact, we require that for every $y \in X$ the set $\{x \in X \mid x\,\Pi\, y\}$ is non-compact. Then $\Sigma_{\text{wper}} = \emptyset$ (so there are no fixed points or periodic points). With some abuse of notation and interpretation, y will be almost periodic if and only if $\{x \in U \mid x\,\Pi\, y\}$ is syndetic in $\{x \in X \mid x\,\Pi\, y\}$ for every neighborhood U of y . If X is locally compact, one could choose a relatively compact neighborhood and syndeticity contradicts the fact that the fiber in y is non-compact. Therefore, in the locally compact strongly non-compact case, equivalent relations do not exhibit almost periodic points.

Example 3.4.10. Denoting by β any of the properties "periodic", "weakly periodic" and "almost periodic", it is easy to check that $\omega \in \Omega$ has β in the pull-back $h^{\downarrow\downarrow}(\Xi)$ if and only if $h(\omega) \in X$ has β in Ξ. This happens mostly because of equality 3.2.7.

Proposition 3.4.11. *The sets $\Sigma_{\rm per}, \Sigma_{\rm wper}$ are invariant.*

Proof. By Lemma 3.1.14, one has $\widetilde{\Xi}^{\xi\bullet\sigma}_{\xi\bullet\sigma} = \xi\, \widetilde{\Xi}^{\sigma}_{\sigma}\xi^{-1}$, so $\Sigma_{\rm wper}$ is invariant.

We focus now on $\Sigma_{\rm per}$; let $\sigma \in \Sigma_{\rm per}$ and $\xi \in \Xi_{\rho(\sigma)}$. For some compact set K , we have

$$\mathsf{K}\,\xi^{-1}\widetilde{\Xi}^{\xi\bullet\sigma}_{\xi\bullet\sigma}\,\xi = \mathsf{K}\,\widetilde{\Xi}^{\sigma}_{\sigma} = \Xi_{\rho(\sigma)} = \Xi_{\rho(\xi\bullet\sigma)}\xi\,.$$

Hence

$$(\mathsf{K}\xi^{-1})\widetilde{\Xi}^{\xi\bullet\sigma}_{\xi\bullet\sigma} = \Xi_{\rho(\xi\bullet\sigma)}\,,$$

meaning that $\widetilde{\Xi}^{\xi\bullet\sigma}_{\xi\bullet\sigma}$ is syndetic in $\Xi_{\rho(\xi\bullet\sigma)}$, and thus $\xi \bullet \sigma \in \Sigma_{\rm per}$ holds. □

The sets of periodic, weakly periodic or almost periodic points might fail to be closed, even for group actions.

3.4.3 Compact orbits

We connect now periodicity with the notion of a periodicoid point, introduced in [7, 3.1] in a more restricted context.

Proposition 3.4.12. *Every periodic point σ has a compact orbit.*

Proof. The following construction should be kept (briefly) in mind, as it will be useful for Proposition 3.4.14. For $\sigma \in \Sigma$, let us define the continuous surjective function

$$\alpha^{\sigma} : \Xi_{\rho(\sigma)} \to \mathfrak{O}_{\sigma} \subset \Sigma\,, \quad \alpha^{\sigma}(\xi) := \theta_{\xi}(\sigma) \equiv \xi \bullet \sigma.$$

If $\sigma \in \Sigma_{\rm per}$, then $\mathsf{K}\widetilde{\Xi}^{\sigma}_{\sigma} = \Xi_{\rho(\sigma)}$ for some compact set $\mathsf{K} \subset \Xi$. By using the definition of $\widetilde{\Xi}^{\sigma}_{\sigma}$, one gets

$$\mathfrak{O}_{\sigma} = \alpha^{\sigma}\big(\Xi_{\rho(\sigma)}\big) = \alpha^{\sigma}(\mathsf{K})\,,$$

which is compact, as a direct continuous image of a compact set. □

For deriving a converse proposition, we will use a well known lemma (with proof, for the convenience of the reader):

Lemma 3.4.13. *Let $\Xi = \mathsf{G}$ be a locally compact, second countable group acting on a topological space Σ. If $\sigma \in \Sigma$ has a compact orbit, then $\widetilde{\mathsf{G}}^{\sigma}_{\sigma} \equiv {\rm Rec}(\sigma, \sigma)$ (see subsection 3.2.2) is syndetic in G.*

Proof. Let $\mathsf{N}_g \subset \mathsf{G}$ be a relatively compact, open neighborhood of g, for every $g \in \mathsf{G}$. Observe that

$$\mathsf{G} = \mathsf{G}\,\widetilde{\mathsf{G}}^{\sigma}_{\sigma} = \Big(\bigcup_{g\in\mathsf{G}} \mathsf{N}_g\Big)\,\widetilde{\mathsf{G}}^{\sigma}_{\sigma}.$$

So

$$\mathfrak{O}_\sigma = \mathsf{G} \bullet \sigma = \Big(\bigcup_{g\in\mathsf{G}} \mathsf{N}_g\Big)\widetilde{\mathsf{G}}^{\sigma}_{\sigma} \bullet \sigma = \bigcup_{g\in\mathsf{G}} \mathsf{N}_g \bullet \sigma.$$

Let us, for the sake of the argument, use (and prove it later) that the set $\mathsf{N}_g \bullet \sigma$ is a neighborhood of σ. By compactness we can extract a finite index set $\mathsf{F} = \{g_1, \dots, g_n\}$ such that

$$\mathfrak{O}_\sigma = \bigcup_{i=1}^{n} \mathsf{N}_{g_i} \bullet \sigma \subset \Big(\bigcup_{i=1}^{n} \overline{\mathsf{N}}_{g_i}\Big) \bullet \sigma.$$

Define $\mathsf{K} = \bigcup_{i=1}^{n} \overline{\mathsf{N}}_{g_i}$ and notice that for every $g \bullet \sigma \in \mathfrak{O}_\sigma$, there exists $h \in \mathsf{K}$ such that

$$g \bullet \sigma = h \bullet \sigma \Rightarrow (h^{-1}g) \bullet \sigma = \sigma,$$

meaning that $h^{-1}g \in \widetilde{\mathsf{G}}^{\sigma}_{\sigma}$ and $g \in \mathsf{K}\widetilde{\mathsf{G}}^{\sigma}_{\sigma}$. As K is compact, we conclude that $\widetilde{\mathsf{G}}^{\sigma}_{\sigma}$ is syndetic in G.

Now, we fill the remaining gap: Pick $W \subset \mathsf{G}$, another relatively compact, open neighborhood of g but satisfying $\overline{W} \subset \mathsf{N}_{\mathrm{g}}$. As G is second countable, it has the Lindelöf property, so exists a countable subset $\mathsf{C} \subset \mathsf{G}$ making $\mathsf{G} = \mathsf{C}W$ true. If $\mathsf{N}_g \bullet \sigma$ had void interior, so would $c \bullet \big[W \bullet \sigma\big] = \big[cW\big] \bullet \sigma$. But then we could decompose $\mathfrak{O}_\sigma$ as a countable union of closed nowhere dense sets:

$$\mathfrak{O}_\sigma = \mathsf{G} \bullet \sigma = \mathsf{C}\mathsf{W} \bullet \sigma = \bigcup_{\mathsf{c}\in\mathsf{C}} \big[\mathsf{c}\overline{\mathsf{W}}\big] \bullet \sigma,$$

contradicting Baire category theorem. $\square$

Proposition 3.4.14. *Assume that Ξ is locally compact, second countable and open. If $\sigma \in \Sigma$ has a compact orbit, then it is a periodic point.*

Proof. Let us first treat the case of the canonical action of Ξ in its unit space X (and notice that, being X closed in Ξ, is a locally compact, second countable space by its own). We will use a small amount of information from [55, Sect. 22], treating the Mackey-Glimm-Ramsay Dichotomy for groupoids; see also [50]. For $x \in X$, let us define the continuous surjective function

$$\alpha^x : \Xi_x \to \mathcal{O}_x \subset X\,, \quad \alpha^x(\xi) := \xi \circ x\,.$$

One has $\alpha^x(\xi) = \alpha^x(\eta)$ if and only if $\xi^{-1}\eta \in \Xi^x_x$. This leads to a continuous bijection

$$\widetilde{\alpha}^x : \Xi_x / \,\Xi^x_x \to \mathcal{O}_x\,. \tag{3.4.1}$$

The quotient map $p : \Xi_x \to \Xi_x/\,\Xi^x_x$ is (surjective, continuous and) open, cf. [55, Ex. 2.2.1]. Then $\widetilde{\alpha}^x$ is a homeomorphism if and only if α^x is an open function. By [55] (see the non-trivial implication $(2) \Rightarrow (\mathrm{e})$ on pages 41-42), this happens if the orbit $\mathcal{O}_x$ is Baire. In our case this is insured, since it is (Hausdorff and) compact. So we conclude that (3.4.1) is a homeomorphism and thus $\Xi_x/\,\widetilde{\Xi}^x_x$ is compact.

To finish this part of the proof, we show now that this implies that Ξ_x^x is syndetic in Ξ_x. Let $\{\mathsf{V}_i \mid i \in I\}$ be a covering of the locally compact space Ξ_x by open subsets with compact closures. Since p is open and surjective, $\{W_i := p(\mathsf{V}_i) \mid i \in I\}$ will be an open covering of the compact space Ξ_x / Ξ_x^x, from which we extract a finite subcovering $\{W_i := p(\mathsf{V}_i) \mid i \in I_0\}$. Then $\mathsf{V} := \bigcup_{i \in I_0} \mathsf{V}_i$ has a compact closure K such that $p(\mathsf{K}) = \Xi_x / \Xi_x^x$. To check that $\mathsf{K}\,\Xi_x^x = \Xi_x$, pick $\xi \in \Xi_x$. Then $p(\xi) = p(\eta)$ for some $\eta \in \mathsf{K}$. By the definition of p, this means $\xi \in \eta\,\Xi_x^x$ and we are done.

Now, we will derive the full result: Suppose that Ξ acts on a very general space Σ and $\mathfrak{O}_\sigma \subset \Sigma$ is compact, for some $\sigma \in \Sigma$. As the anchor map is continuous, $\rho(\mathfrak{O}_\sigma) = \mathcal{O}_{\rho(\sigma)} \subset X$ (Lemma 3.1.9) is compact and by the previous discussion, the decomposition $\Xi_{\rho(\sigma)} = \mathsf{K}_1\,\Xi_{\rho(\sigma)}^{\rho(\sigma)}$ holds for some compact set $\mathsf{K}_1 \subset \Xi$. Remark that $\mathsf{G} = \Xi_{\rho(\sigma)}^{\rho(\sigma)}$ is a (locally compact) group acting continuously on the compact space

$$\mathfrak{O}_\sigma \cap \{\tau \in \Sigma \mid \rho(\tau) = \rho(\sigma)\}.$$

By applying Lemma 3.4.13 (with a change of notations), we obtain another compact set $\mathsf{K}_2 \subset \Xi$ such that

$$\Xi_{\rho(\sigma)} = \mathsf{K}_1 \Xi_{\rho(\sigma)}^{\rho(\sigma)} = \mathsf{K}_1\big(\mathsf{K}_2 \widetilde{\Xi}_\sigma^\sigma\big) = \big(\mathsf{K}_1\mathsf{K}_2\big)\widetilde{\Xi}_\sigma^\sigma\,,$$

finishing the proof. □

3.4.4 Minimal sets

Minimality is a very important property in classical topological dynamics; it extends straightforwardly to groupoid actions, denoted below by $(\Xi, \rho, \theta, \Sigma)$. During this subsection, both Ξ and Σ are assumed to be locally compact.

Definition 3.4.15. A closed invariant subset $M \subset \Sigma$ is called *minimal* if it does not contain proper non-void closed invariant subsets. Equivalently, M is *minimal* if all the orbits contained in M are dense in M. *The action is minimal* if Σ itself is minimal.

The minimal sets are the closed invariant non-empty subsets of Σ which are minimal under such requirements. Two minimal sets either coincide or are disjoint. Clearly *transitivity* $\Rightarrow$ *minimality* $\Rightarrow$ *pointwise transitivity) and the implications are strict in general (even for group actions).*

Remark 3.4.16. For open groupoids, minimal sets are either clopen or nowhere dense. This follows from Proposition 3.1.8: the boundary of an invariant set is invariant. So if Ξ is open and M is minimal, the boundary $\partial M \subset M$ is closed and invariant, therefore it should be void (i.e. M is open) or coincide with M (meaning that M is nowhere dense). But in general this is not the case: in Remark 3.2.1, $[-1,1] \subset \mathbb{R}$ is a minimal set.

The main result of this section is

Theorem 3.4.17. *If the point $\sigma \in \Sigma$ is almost periodic, its orbit closure $\overline{\mathfrak{O}}_\sigma$ is minimal and compact. If, in addition, the groupoid Ξ is open, σ is almost periodic if and only if $\overline{\mathfrak{O}}_\sigma$ is minimal and compact.*

Proof. Suppose that σ is almost periodic; we show first that its orbit closure $\overline{\mathfrak{O}}_\sigma$ is compact. Let U_0 be a compact neighborhood of σ. Using the assumptions, for some compact set K one has

$$\mathfrak{O}_\sigma = \Xi_{\rho(\sigma)} \bullet \sigma = \big(\mathsf{K}\,\widetilde{\Xi}_\sigma^{U_0}\big) \bullet \sigma = \mathsf{K} \bullet \big(\widetilde{\Xi}_\sigma^{U_0} \bullet \sigma\big) \subset \mathsf{K} \bullet U_0 = \text{compact}\,,$$

so $\overline{\mathfrak{O}}_\sigma$ is compact.

If $\overline{\mathfrak{O}}_\sigma$ is not minimal, it strictly contains a minimal (and compact) set M. The point σ does not belong to M, so there are disjoint open sets $U, V \subset \Sigma$ such that $\sigma \in U$ and $M \subset V$. For an arbitrary compact set $\mathsf{K} \subset \Xi$ we will now show that $\mathsf{K}\,\widetilde{\Xi}_\sigma^U \neq \Xi_{\rho(\sigma)}$, implying that in fact σ is not almost periodic.

The set M being invariant, $\mathsf{K}^{-1}\bullet M \subset M$ holds. Let W be a neighborhood of M with $\mathsf{K}^{-1}\bullet W \subset V$. Since $M \subset \overline{\mathfrak{O}}_\sigma = \overline{\Xi_{\rho(\sigma)}\bullet \sigma}$, there exists $\eta \in \Xi_{\rho(\sigma)}$ such that $\eta \bullet \sigma \in W$. Then

$$\big(\mathsf{K}^{-1}\eta\big) \bullet \sigma = \mathsf{K}^{-1} \bullet (\eta \bullet \sigma) \subset \mathsf{K}^{-1}\bullet W \subset V,$$

and thus $\big(\mathsf{K}^{-1}\eta\big) \bullet \sigma$ is disjoint from U, meaning that $\mathsf{K}^{-1}\eta$ is disjoint from $\widetilde{\Xi}_\sigma^U$. This shows that

$$\Xi_{\rho(\sigma)} \ni \eta \notin \mathsf{K}\,\widetilde{\Xi}_\sigma^U \neq \Xi_{\rho(\sigma)}\,,$$

finishing the proof.

For the converse, suppose now that d is open and $\overline{\mathfrak{O}}_\sigma = \overline{\Xi_{\rho(\sigma)}\bullet \sigma}$ is minimal and compact. Let U be an open neighborhood of σ. For each $\xi \in \Xi$, choose an open neighborhood N_ξ of ξ with compact closure. The sets $\Xi \bullet U$ and $\mathsf{N}_\xi \bullet U$ are open in Σ, cf. [55, Ex. 2.1.11]. By minimality one has

$$\overline{\mathfrak{O}}_\sigma \subset \Xi \bullet U = \bigcup_{\xi\in\Xi} \xi \bullet U \subset \bigcup_{\xi\in\Xi} \mathsf{N}_\xi \bullet U.$$

By compactness of $\overline{\mathfrak{O}}_\sigma$ applied to the open cover above, for a finite set $\mathsf{F} = \{\xi_1, \dots, \xi_k\} \subset \Xi$ we get

$$\overline{\mathfrak{O}}_\sigma \subset \bigcup_{i=1}^k \mathsf{N}_{\xi_i} \bullet U.$$

If $\eta \in \Xi_{\rho(\sigma)}$ then $\eta \bullet \sigma \in \mathsf{N}_{\xi_j}\bullet U$ for some j, and then one has $\eta \bullet \sigma \in \eta_j \bullet U$ for some $\eta_j \in \mathsf{N}_{\xi_j}$. It follows immediately that $\mathrm{r}(\eta) = \mathrm{r}(\eta_j) = \mathrm{d}\big(\eta_j^{-1}\big)$ and

$$\eta_j^{-1}\bullet (\eta \bullet \sigma) = \big(\eta_j^{-1}\eta\big) \bullet \sigma \in U.$$

This means that $\eta_j^{-1}\eta \in \widetilde{\Xi}_\sigma^U$ or, equivalently, that

$$\eta \in \eta_j\,\widetilde{\Xi}_\sigma^U \subset \mathsf{N}_{\xi_j}\widetilde{\Xi}_\sigma^U \subset \bigcup_{i=1}^k \Big(\overline{\mathsf{N}_{\xi_i}}\,\widetilde{\Xi}_\sigma^U\Big) = \Big(\bigcup_{i=1}^k \overline{\mathsf{N}_{\xi_i}}\Big)\,\widetilde{\Xi}_\sigma^U.$$

Since η is arbitrary one gets $\Xi_{\rho(\sigma)} \subset \mathsf{K}\,\widetilde{\Xi}_\sigma^U$, where $\mathsf{K} := \bigcup_{i=1}^k \overline{\mathsf{N}_{\xi_i}}$ is compact. We checked that $\widetilde{\Xi}_\sigma^U$ is syndetic in $\Xi_{\rho(\sigma)}$, so σ is almost periodic. $\square$

Example 3.4.18. In the case of the transformation groupoid associated to a topological dynamical system (G,γ,X), one recovers the classical result ([3, pag.11] and [21, pag. 28, 38, 39]). For this, we use Example 3.2.3. First of all, it is clear that minimality of the group action coincides with minimality in the sense of groupoids, since the orbits are the same. The relevant recurrence sets also coincide: set $S=\{\sigma\}$ and $T=U$ in (3.2.3). Finally, syndeticity has the same meaning in the two cases.

Example 3.4.19. Consider the case of the pair groupoid $\Xi:=X\times X$ acting on its unit space X, taken to be compact. The action is transitive (only one orbit), so every $x\in X$ should be almost periodic. And it is, since $\Xi_x=X\times\{x\}$ and $\widetilde{\Xi}_x^U=U\times\{x\}$. The set $\mathsf{K}:=X\times\{x\}$ itself is compact, and

$$\mathsf{K}\,\widetilde{\Xi}_x^U=(X\times\{x\})(U\times\{x\})=(X\times\{x\})\{(x,x)\}=X\times\{x\}=\Xi_x\,.$$

This example also shows another difference between the group and the groupoid case. For groups, the compact set K in the definition of syndeticity can always be taken finite (see for example [19, pag.271], where this property is called 'discrete syndeticity'). In this groupoid no finite set $\mathsf{K}\subset X\times\{x\}$ makes the equality $\mathsf{K}\,\widetilde{\Xi}_x^U=\Xi_x$ true if X itself is infinite.

If Σ is a compact space, Zorn's Lemma implies that it has a minimal subset $M\subset\Sigma$. If in addition Ξ is open, every $x\in M$ is almost periodic, by Theorem 3.4.17. Thus, in this setting, almost periodic points always exist.

Corollary 3.4.20. *If Ξ is open (and locally compact), the set $\Sigma_{\rm alper}$ is invariant.*

Proof. The second part of Theorem 3.4.17 guarantees that, if $\sigma\in\Sigma_{\rm alper}$, then $\mathfrak{O}_\sigma\subset\Sigma_{\rm alper}$. □

Corollary 3.4.21. *Suppose that Σ is compact, Ξ is open and the action is minimal. Then $\widetilde{\Xi}_\sigma^V$ is syndetic for every $\sigma\in\Sigma$ and every open non-void subset V of Σ (and not just for neighborhoods of σ).*

Proof. By minimality, there exists $\zeta\in\Xi_{\rho(\sigma)}$ such that V is an open neighborhood of $\zeta\bullet\sigma$. Hence $\widetilde{\Xi}_{\zeta\bullet\sigma}^V$ is syndetic by Theorem 3.4.17; it can be written as $\mathsf{K}\,\widetilde{\Xi}_{\zeta\bullet\sigma}^V=\Xi_{\rho(\zeta\bullet\sigma)}$ for some compact subset K of Ξ. Then

$$\Xi_{\rho(\sigma)}=\Xi_{{\rm d}(\zeta)}=\Xi_{{\rm r}(\zeta)}\,\zeta=\Xi_{\rho(\zeta\bullet\sigma)}\,\zeta=\mathsf{K}\,\widetilde{\Xi}_{\zeta\bullet\sigma}^V\,\zeta=\mathsf{K}\,\widetilde{\Xi}_\sigma^V,$$

meaning that $\widetilde{\Xi}_\sigma^V$ is also syndetic. □

We say that the action is *semisimple* if all the orbit closures are minimal (equivalently: the orbit closures form a partition of Σ).

Corollary 3.4.22. *If all the orbits are closed, the action is semisimple. A pointwise almost periodic action is semisimple. If Ξ is open and all the orbits are compact, the action is pointwise almost periodic.*

Proof. The statements are obvious or they follow easily from Theorem 3.4.17. □

Remark 3.4.23. In Example 3.1.5 the orbits are precisely the d-fibers, automatically closed: $\mathfrak{O}_\eta=\overline{\mathfrak{O}}_\eta=\Xi_{{\rm d}(\eta)}$. In particular this action of Ξ on itself to the left is semisimple. Pointwise almost periodicity hangs on compactness of the fibers.

3.5 Factors

We indicate in this section the fate of some of the properties above under epimorphisms. Some of the results do not use surjectivity; we leave this to the reader.

Definition 3.5.1. An *homomorphism* of the groupoid actions $(\Xi, \rho, \theta, \Sigma)$, $(\Xi, \rho', \theta', \Sigma')$ is a continuous function $f : \Sigma \to \Sigma'$ such that for all $\sigma \in \Sigma\,, \xi \in \Xi_{\rho(\sigma)}$ one has

$$\rho'\big(f(\sigma)\big) = \rho(\sigma) \quad \text{and} \quad f\big(\theta(\xi,\sigma)\big) = \theta'\big(\xi, f(\sigma)\big)\,. \tag{3.5.1}$$

An *epimorphism* is a surjective homomorphism; in such a case we say that $(\Xi, \rho', \theta', \Sigma')$ is *a factor* of $(\Xi, \rho, \theta, \Sigma)$ and that $(\Xi, \rho, \theta, \Sigma)$ is *an extension* of $(\Xi, \rho', \theta', \Sigma')$.

Writing $\bullet$ instead of θ and $\bullet'$ instead of θ', the second requirement in (3.5.1) is

$$f(\xi \bullet \sigma) = \xi \bullet' f(\sigma)\,, \quad \forall\, (\xi, \sigma) \in \Xi \bowtie \Sigma\,. \tag{3.5.2}$$

Lemma 3.5.2. *The canonical action* $(\Xi, \mathrm{id}_X, \circ, X)$ *from Example 3.1.3 is a factor of any other continuous action* $(\Xi, \rho, \bullet, \Sigma)$.

Proof. The continuous surjection $f := \rho : \Sigma \to X$ is an epimorphism, since it satisfies

$$\rho(\xi \bullet \sigma) = \mathrm{r}(\xi) = \xi \circ \mathrm{d}(\xi) = \xi \circ \rho(\sigma)\,, \quad \forall\, (\xi, \sigma) \in \Xi \bowtie \Sigma\,.$$

□

Lemma 3.5.3. *If f is an epimorphism between the groupoid dynamical systems $(\Xi, \rho, \theta, \Sigma)$ and $(\Xi, \rho', \theta', \Sigma')$ and $\sigma \in \Sigma$, then $f(\mathfrak{O}_\sigma) = \mathfrak{O}'_{f(\sigma)}$ and $f\big(\overline{\mathfrak{O}}_\sigma\big) \subset \overline{\mathfrak{O}}'_{f(\sigma)}$. Direct images of invariant sets are also invariant.*

Proof. The elementary proof relies on (3.5.2) and on the properties of continuous functions. We recall that for continuous surjections the direct image may not commute with the closure. □

Let us see what happens with recurrence sets under epimorphisms.

Lemma 3.5.4. *If $M, N \subset \Sigma$ then $\widetilde{\Xi}_M^N \subset \widetilde{\Xi}\,'^{f(N)}_{f(M)}$, where the later set is computed with respect to the factor $(\Xi, \rho', \theta', \Sigma')$ of the groupoid dynamical system $(\Xi, \rho, \theta, \Sigma)$.*

Proof. One verifies easily that the next sequence of equivalences and implications is rigorous:

$$\begin{aligned}\xi \in \widetilde{\Xi}_M^N &\Leftrightarrow (\xi \bullet M) \cap N \neq \emptyset \Leftrightarrow f\big[(\xi \bullet M) \cap N\big] \neq \emptyset \\ &\Rightarrow f(\xi \bullet M) \cap f(N) \neq \emptyset \Leftrightarrow \big(\xi \bullet' f(M)\big) \cap f(N) \neq \emptyset \\ &\Leftrightarrow \xi \in \widetilde{\Xi}\,'^{f(N)}_{f(M)}\,.\end{aligned}$$

The second equivalence is true because f is onto. In general one has $f(A \cap B) \subset f(A) \cap f(B)$ and the inclusion could be strict; this shows why (and when) there is no equality in the statement. □

To see the usefulness of this Lemma, we hurry to apply it. On many occasions we are going to use the equality $f\big[f^{-1}(B')\big]=B'$ for $B'\subset\Sigma'$, valid by surjectivity.

Theorem 3.5.5. *Let f be an epimorphism between the actions $(\Xi,\rho,\bullet,\Sigma)$ and $\big(\Xi,\rho',\bullet',\Sigma'\big)$. For every index $\alpha\in\{\mathrm{fix},\mathrm{per},\mathrm{wper},\mathrm{alper}\}$ one has $f\big(\Sigma_\alpha\big)\subset\Sigma'_\alpha$. In particular, $\rho\big(\Sigma_\alpha\big)\subset X_\alpha$, where X_α indicates the set of units of X having the property α with respect to the canonical action.*

Proof. The proof is easy. For instance, the statement about almost periodicity follows easily from the definitions and from Lemma 3.5.4; a set containing a syndetic subset is obviously syndetic. □

Proposition 3.5.6. *Let f be an epimorphism between the actions $(\Xi,\rho,\bullet,\Sigma)$ and $\big(\Xi,\rho',\bullet',\Sigma'\big)$. Suppose that the action $(\Xi,\rho,\bullet,\Sigma)$ has one of the properties*

$$\mathcal{P}\in\{\textit{transitivity},\ \textit{pointwise transitivity},\textit{weak pointwise transitivity},(i),(ii),(iii)\}$$

from Theorem 3.3.1. Then the action $\big(\Xi,\rho',\bullet',\Sigma'\big)$ also has $\mathcal{P}$.

Proof. $\mathcal{P}=$ *transitivity*. By Lemma 3.5.3, the epimorphism f transforms the single orbit $\mathfrak{O}_\sigma=\Sigma$ into an orbit $\mathfrak{O}'_{f(\sigma)}=f(\mathfrak{O}_\sigma)=\Sigma'$.

$\mathcal{P}=$ *pointwise transitivity*. This also follows immediately from Lemma 3.5.3, the part referring to orbit closures.

$\mathcal{P}=$ *weak pointwise transitivity*. If $C'\subset\Xi'$ is a closed and invariant set containing $f(\sigma)$, then $f^{-1}(C')\subset\Xi$ is a closed and invariant set containing σ, hence $\Sigma=\mathfrak{C}_\sigma\subset f^{-1}(C')$ and $\Sigma'=C'$.

$\mathcal{P}=(i)\Leftrightarrow(i')$. Let $\emptyset\neq U',V'\subset\Sigma'$ be invariant open sets and let $f^{-1}(U')\,,f^{-1}(V')$ be their open non-void invariant inverse images. Since $(\Xi,\rho,\bullet,\Sigma)$ satisfies (i'), one has $f^{-1}(U')\cap f^{-1}(V')\neq\emptyset$. Consequently

$$U'\cap V'=f\big[f^{-1}(U')\big]\cap f\big[f^{-1}(V')\big]\supset f\big[f^{-1}(U')\cap f^{-1}(V')\big]\neq\emptyset$$

and $\big(\Xi,\rho',\bullet',\Sigma'\big)$ also satisfies (i'), which is equivalent with (i).

$\mathcal{P}=(ii)$. Let $U'\subset\Sigma'$ be open and invariant. Then $f^{-1}(U')\subset\Sigma$ is open and invariant, so it is dense. Since f is surjective it follows that $U'=f\big(f^{-1}(U')\big)$, and this one is dense in Σ'.

$\mathcal{P}=(iii)$. Let $\emptyset\neq U',V'\subset\Sigma'$ be open subsets. By recurrent transitivity of the initial action, one has $\widetilde{\Xi}_{f^{-1}(U')}^{f^{-1}(V')}\neq\emptyset$. Then, by Lemma 3.5.4, we get

$$\emptyset\neq\widetilde{\Xi}_{f^{-1}(U')}^{f^{-1}(V')}\subset\widetilde{\Xi}'^{f[f^{-1}(V')]}_{f[f^{-1}(U')]}=\widetilde{\Xi}'^{V'}_{U'}$$

and the proof is finished. □

Corollary 3.5.7. *We say that* the groupoid Ξ has the property $\mathcal{P}$ *if its canonical action on its unit space has this property. Suppose that the topological groupoid Ξ admits a continuous action (ρ,θ,Σ) having one of the properties $\mathcal{P}$ mentioned in Proposition 3.5.6. Then Ξ itself has this property.*

Proof. This follows from Proposition 3.5.6 and Lemma 3.5.2. □

If we require Ξ to be open, there is a direct proof that the property (iv) from Theorem 3.3.1 (called topological transitivity) also transfer to factors; it uses Lemma 3.1.8. This also follows joining Theorem 3.3.1 and Proposition 3.5.6. But see the next example for a non-topologically transitive groupoid, which however possesses a topologically transitive action. It follows that topological transitivity is not preserved by factors (which is rather surprising).

Example 3.5.8. Form the product groupoid $\Xi = \Pi \times \mathbb{R}$, with the relation

$$x \,\Pi\, y \;\Leftrightarrow\; x, y \in \mathbb{Q} \text{ or } x = y$$

over $X = \mathbb{R}$, and consider the wide subgroupoid

$$\Delta = \{(x,y,g) \in \Xi \mid x, y \notin \mathbb{Q} \Rightarrow g = 0\} = \big(\mathbb{Q}\times\mathbb{Q}\times\mathbb{R}\big) \cup \big(\mathrm{Diag}(\mathbb{R}\times\mathbb{R})\times\{0\}\big)\,.$$

By analogy with Example 3.3.4, its easy to see that the canonical action of Δ on $X = \mathbb{R}$ is not topologically transitive. Actually, the orbits of rational points all coincide with $\mathbb{Q}$, but each irrational point is a fixed point, so $(s,t)\backslash\mathbb{Q}$ is invariant for every $s < t$, without being dense or nowhere dense. Now we will exhibit a topologically transitive action of Ξ: Let

$$\Sigma = \{(y,h) \in \mathbb{R}^2 \mid h = 0 \text{ or } y \in \mathbb{Q}\} = (\mathbb{R}\times\{0\}) \cup (\mathbb{Q}\times\mathbb{R})$$

with the topology inherited from $\mathbb{R}^2$ and define the continuous action

$$\rho(y,h) = y \quad \text{and} \quad (x,y,g) \bullet (y,h) = (x, g+h)\,.$$

Notice that the orbits of $\bullet$ are $\mathbb{Q}\times\mathbb{R}$ and (the fixed points) $\{(y,0) \mid y \in \mathbb{R}\setminus\mathbb{Q}\}$, implying that all of the invariant sets are either dense or nowhere dense, since $\Sigma\backslash(\mathbb{Q}\times\mathbb{R}) = (\mathbb{R}\backslash\mathbb{Q})\times\{0\}$ is already nowhere dense.

We finish this subsection with a result on the behavior of minimality under epimorphisms, in both directions.

Proposition 3.5.9. *Let $f : \Sigma \to \Sigma'$ be an epimorphism between the actions $(\Xi, \rho, \theta, \Sigma)$ and $\big(\Xi, \rho', \theta', \Sigma'\big)$.*

(i) If $M \subset \Sigma$ is minimal and $f(M)$ is closed in Σ', then $f(M)$ is minimal.

(ii) Suppose that Σ is compact (hence Σ' is also compact). If $M' \subset \Sigma'$ is minimal, there exists $M \subset \Sigma$ minimal such that $f(M) = M'$. If Ξ is open and locally compact and $\sigma' \in \Sigma'$ is almost periodic, then $\sigma' = f(\sigma)$ for some almost periodic point σ of Σ.

Proof. (i) This is dealt with easily by Lemma 3.5.3: if $\sigma \in M$ then

$$f(M) = f\big(\overline{\mathfrak{O}}_\sigma\big) \subset \overline{\mathfrak{O}}'_{f(\sigma)} \subset \overline{f(M)} = f(M)\,,$$

so the orbit of $f(\sigma)$ is dense in the closed set $f(M)$.

(ii) The inverse image $f^{-1}(M')$ is non-void closed and $\bullet$-invariant. By Zorn's Lemma, it contains a minimal (and compact) subsystem M. The direct image $f(M) \subset M'$ is non-void closed and $\bullet'$-invariant, so it must coincide with M'. For the second part: As σ' is almost periodic, $M' := \overline{\mathfrak{O}}'_{\sigma'}$ is minimal, so we can find some (compact) minimal subset $M \subset \Sigma$, such that $f(M) = M'$ and $\sigma' = f(\sigma)$ for some $\sigma \in M$. By Theorem 3.4.17, any $\sigma \in M$ is almost periodic. □

3.6 A disappointing notion: mixing

The next definition seems a legitimate generalization of the classical one:

Definition 3.6.1. The action $(\Xi, \rho, \theta, \Sigma)$ is called *weakly mixing* whenever for every $U, U', V, V' \subset \Sigma$ non-empty open sets, one has $\widetilde{\Xi}_U^V \cap \widetilde{\Xi}_{U'}^{V'} \neq \emptyset$. It is called *strongly mixing* if the complement of $\widetilde{\Xi}_U^V$ is relatively compact for every open sets $U, V \neq \emptyset$.

If Ξ itself is not compact, strongly mixing implies weakly mixing. This follows from the equality

$$\big(\widetilde{\Xi}_U^V \cap \widetilde{\Xi}_{U'}^{V'}\big)^c = \big(\widetilde{\Xi}_U^V\big)^c \cup \big(\widetilde{\Xi}_{U'}^{V'}\big)^c$$

and the fact that the union of two relatively compact sets is relatively compact. On the other hand, *weakly mixing always implies recurrent transitivity*: just take $U = U'$ and $V = V'$.

Example 3.6.2. For classical group actions one recovers the usual concepts; see Example 3.2.3.

The next result shows that there in no point in exploring besides Example 3.6.2, which is already extensively treated in all the standard textbooks in Topological Dynamics. Recall that we assumed ρ surjective and X Hausdorff.

Proposition 3.6.3. *Suppose that $(\Xi, \rho, \theta, \Sigma)$ is weakly mixing. Then $\rho(\Sigma) = X$ consists of a single point, so the groupoid is a group.*

Proof. Suppose that exists two distinct points $x, y \in \rho(\Sigma)$, and let $U_0, V_0 \subset X$ be disjoint open sets separating them. Form the non-void open sets $U = \rho^{-1}(U_0)$, $V = \rho^{-1}(V_0)$. By weakly mixing, there exists some $\xi \in \widetilde{\Xi}_U^U \cap \widetilde{\Xi}_U^V$. Thus, there are points $\sigma_1, \sigma_2, \sigma_3 \in U, \tau \in V$ such that

$$\xi \bullet \sigma_1 = \sigma_2 \quad \text{and} \quad \xi \bullet \sigma_3 = \tau,$$

implying that

$$\rho(\sigma_2) = \rho(\xi \bullet \sigma_1) = \mathrm{r}(\xi) = \rho(\xi \bullet \sigma_3) = \rho(\tau)\,.$$

But $\rho(\sigma_2) \in U_0$ and $\rho(\tau) \in V_0$. This contradiction shows that $\rho(\Sigma)$ consist of a single point. $\square$

Chapter 4

Morphisms of Groupoid Actions and Recurrence

In the previous chaper, epimorphism (factor) meant *one* topological groupoid acting on two topological spaces and a continuous equivariant surjection connecting them, we extend here the study of morphisms in a new direction, the main purpose being to *allow a change of groupoids*. There are several ways to make this change, depending on what "groupoid morphism" means.

As groupoids are special classes of categories, the first (natural) notion of groupoid morphism is, of course, a continuous functor. This is what is taken into account in most of the references. We used this type of morphism to introduce, in Subsection 3.5, morphisms of actions, consisting of pairs of maps

$$f:(\Xi,\Sigma)\to\big(\Xi',\Sigma'\big)\,,$$

where the subyacent functor was the identity $\Xi\to\Xi$ and $f:\Sigma\to\Sigma'$ is a continuous map between the topological spaces. This leads naturally to the following definition

Definition 4.0.1. Let $(\Xi,\rho,\theta,\Sigma)\,,(\Xi',\rho',\theta',\Sigma')$ be two groupoid dynamical systems. *A morphism of groupoid actions* is a pair $(\Psi,f)\,,$ where $\Psi:\Xi\to\Xi'$ a continuous groupoid morphism (or continuous functor, if one prefers the category point of view) and $f:\Sigma\to\Sigma'$ is a continuous function, such that the next diagram commutes

$$\begin{tikzcd}
\Xi \arrow[r, "\mathrm{r}", "\mathrm{d}"'] \arrow[d, "\Psi"'] & X \arrow[d, "\psi"'] & \Sigma \arrow[l, "\rho"'] \arrow[d, "f"'] \\
\Xi' \arrow[r, "\mathrm{r}'", "\mathrm{d}'"'] & X' & \Sigma' \arrow[l, "\rho'"']
\end{tikzcd}$$

and such that

$$f\big[\theta_\xi(\sigma)\big]=\theta'_{\Psi(\xi)}\big[f(\sigma)\big]\,,\quad\text{if }\ \mathrm{d}(\xi)=\rho(\sigma)\ (\text{which implies } \mathrm{d}'[\Psi(\xi)]=\rho'\big[f(\sigma)\big])\,. \qquad (4.0.1)$$

The object map ψ is the restriction of the arrow map Ψ. If both Ψ and f are surjective, we say that (Ψ,f) is *an epimorphism*, $(\Xi',\rho',\theta',\Sigma')$ is *a factor* of (Ξ,ρ,θ,Σ) and (Ξ,ρ,θ,Σ) is *an extension* of $(\Xi',\rho',\theta',\Sigma')\,.$

The relation (4.0.1) may be written as

$$f \circ \theta_\xi = \theta'_{\Psi(\xi)} \circ f\,, \quad \text{on } \Sigma_{\mathrm{d}(\xi)} := \rho^{-1}\big(\{\mathrm{d}(\xi)\}\big)\,,$$

or even as

$$f(\xi \bullet \sigma) = \Psi(\xi) \bullet' f(\sigma)\,, \quad \text{if } \mathrm{d}(\xi) = \rho(\sigma)\,. \tag{4.0.2}$$

Nevertheless, the type of morphism we will treat in this work is based on *algebraic morphisms*. When studyind groupoid C^*-algebras, experts remarked that usual groupoid morphisms do not fit well, and this can be seen at the level of very simple extreme particular cases, groups and spaces. Group algebras $C^*(\mathsf{G}), C^*(\mathsf{G}')$ behave well with respect to (direct) group morphisms $\mathsf{G} \to \mathsf{G}'$, while the Abelian C^*-algebras $C_0(X), C_0(X')$ prefer (proper continuous) maps $X \leftarrow X'$ acting in the opposite direction. Recall that the Gelfand functor is contravariant!

References as [56, 57, 12, 11, 13, 45] contributed to a solution, introducing *algebraic morphisms* [13] (also called *actors* in [45]), insuring a functorial groupoid C^*-algebra construction. Instead of a mapping, such an algebraic morphism $\Phi : \Xi \rightsquigarrow \Xi'$ is a continuous action of Ξ on the topological space Ξ', commuting with the right action of the groupoid Ξ' on itself. Buss, Exel and Meyer argued that "algebraic morphisms are exactly the same as functors between the categories of actions that do not change the underlying space". We refer to [13, Th.4.12] for a precise statement.

4.1 Algebraic morphisms

4.1.1 Algebraic morphisms of groupoid actions

Definition 4.1.1. Let $\big(\Xi_1, \rho_1, \bullet_1, \Sigma\big)$ be a left action and $\big(\Xi_2, \rho_2, \bullet_2, \Sigma\big)$ a right action. We say that *the two actions commute* if

(i) $\rho_1\big(\sigma \bullet_2 \xi_2\big) = \rho_1(\sigma)$ if $\mathrm{r}_2(\xi_2) = \rho_2(\sigma)$,

(ii) $\rho_2\big(\xi_1 \bullet_1 \sigma\big) = \rho_2(\sigma)$ if $\mathrm{d}_1(\xi_1) = \rho_1(\sigma)$,

(iii) $\xi_1 \bullet_1 \big(\sigma \bullet_2 \xi_2\big) = \big(\xi_1 \bullet_1 \sigma\big) \bullet_2 \xi_2$ if $\mathrm{d}_1(\xi_1) = \rho_1(\sigma)$ and $\mathrm{r}_2(\xi_2) = \rho_2(\sigma)$.

The following notion was taken from [11, 12, 13]:

Definition 4.1.2. Let $\Xi\,, \Xi'$ be two groupoids, with unit spaces X and X', respectively. *An algebraic morphism* (also called *an actor*) $\Xi \rightsquigarrow \Xi'$ is an action $\big(\Xi, \mu, \diamond, \Xi'\big)$ of Ξ on the topological space Ξ', which commutes with the right action of Ξ' on itself by right multiplications.

Remark 4.1.3. In this case

$$\Xi_1 = \Xi\,, \quad \Xi_2 = \Sigma = \Xi', \quad \rho_1 = \mu : \Xi' \to X, \quad \bullet_1 = \diamond\,, \quad \rho_2 = \mathrm{d}' : \Xi' \to X'$$

and commutativity means that

$$\mu(\eta'\xi') = \mu(\eta')\,, \quad \text{if } \mathrm{d}'(\eta') = \mathrm{r}'(\xi')\,, \tag{4.1.1}$$

$$\mathrm{d}'(\xi\diamond\eta')=\mathrm{d}'(\eta')\,,\quad \text{if }\ \mathrm{d}(\xi)=\mu(\eta')\,, \tag{4.1.2}$$
$$\xi\diamond(\eta'\xi')=(\xi\diamond\eta')\xi',\quad \text{if }\ \mathrm{d}(\xi)=\mu(\eta')\,,\ \mathrm{d}'(\eta')=\mathrm{r}'(\xi')\,.$$

We are going to denote by ν the restriction of μ to $X'=\Xi'^{(0)}$. From (4.1.1) one infers immediately that $\mu=\nu\circ\mathrm{r}'$; it follows that here μ is determined by ν.

Example 4.1.4. For two topological groups G,G', algebraic morphisms are exactly topological group morphisms $\beta:\mathsf{G}\to\mathsf{G}'$. In this case $X=\{\mathsf{e}\}$ and $X'=\{\mathsf{e}'\}$, so $\mu(\xi')=\mathsf{e}$ for every $\xi'\in\mathsf{G}'$, and then $\xi\diamond\xi':=\beta(\xi)\xi'$ defines the action (note that $\beta(\xi)=\xi\diamond\mathsf{e}'$).

Example 4.1.5. More generally, let

$$\Xi=\bigsqcup_{x\in X}\mathsf{G}_x\,,\quad \Xi'=\bigsqcup_{x'\in X'}\mathsf{G}'_{x'}$$

two group bundles and $(\Xi,\mu,\diamond,\Xi')$ an algebraic morphism. We recall that $\mu=\nu\circ\mathrm{r}'$ is determined by $\nu:X'\to X$. The elements $\xi\in\Xi_x$ and $\xi'\in\Xi'_{x'}$ may be composed by $\diamond$ if and only if $x=\nu(x')$. The action $\diamond$ reduces to a family of actions

$$\diamond_{x'}:\mathsf{G}_{\nu(x')}\times\mathsf{G}'_{x'}\to\mathsf{G}'_{x'}\,,\quad x'\in X'$$

and finally, by Example 4.1.4, to a family of topological group morphisms

$$\beta_{x'}:\mathsf{G}_{\nu(x')}\to\mathsf{G}'_{x'}\,,\quad x'\in X'$$

tied together by some continuity condition.

Example 4.1.6. If Ξ' is a trivial groupoid (this is $\Xi'=X'$), the only possible algebraic morphism $\Xi\rightsquigarrow\Xi'$ is the one induced by the trivial action $\xi\diamond x'=x'$, $\forall\,\xi\in\Xi_{\mu(x')}$ (and this can only happen if $\mathrm{d}(\xi)=\mathrm{r}(\xi)$, for every $\xi\in\Xi_{\mu(X')}$). So it may be identified with the anchor map $\mu:\Xi'\to X$. If, in addition, the groupoid Ξ is also trivial, the algebraic morphism reduces to a continuous map $\Xi'=X'\xrightarrow{\mu}X=\Xi$ (note the inverse direction) and the outer multiplication is

$$\mu(x')\diamond x':=x',\quad \forall\,x'\in X'.$$

There is a way to multiply algebraic morphisms $\Xi_1\overset{\Phi_{12}}{\rightsquigarrow}\Xi_2\overset{\Phi_{23}}{\rightsquigarrow}\Xi_3$, resulting in the new algebraic morphism $\Xi_1\overset{\Phi_{13}}{\rightsquigarrow}\Xi_3$, where $\Phi_{13}:=\Phi_{23}\circ\Phi_{12}$: Let $\Phi_{12}:=\big(\Xi_1,\mu_{12},\diamond_{12},\Xi_2\big)$ and $\Phi_{23}:=\big(\Xi_2,\mu_{23},\diamond_{23},\Xi_3\big)$ be the two algebraic morphisms. We set

$$\Phi_{13}:=\Phi_{23}\circ\Phi_{12}\equiv\big(\Xi_1,\mu_{13},\diamond_{13},\Xi_3\big)\,, \tag{4.1.3}$$
$$\mu_{13}:=\mu_{12}\circ\mu_{23}:\Xi_3\to X_1\subset\Xi_1\,, \tag{4.1.4}$$
$$\xi_1\diamond_{13}\xi_3:=\big(\xi_1\diamond_{12}\mu_{23}(\xi_3)\big)\diamond_{23}\xi_3\quad \text{if}\quad \mathrm{d}_1(\xi_1)=\mu_{13}(\xi_3)=\mu_{12}\big[\mu_{23}(\xi_3)\big]\,. \tag{4.1.5}$$

Both compositions in the r.h.s. of (4.1.5) are possible; in particular, by (4.1.2), one has

$$\mathrm{d}_2\big(\xi_1\diamond_{12}\mu_{23}(\xi_3)\big)=\mathrm{d}_2\big(\mu_{23}(\xi_3)\big)=\mu_{23}(\xi_3)\,.$$

Definition 4.1.7. Let $\Theta:=(\Xi,\rho,\theta=\bullet,\Sigma)$ and $\Theta':=(\Xi',\rho',\theta'=\bullet',\Sigma')$ be two groupoid actions, with unit spaces X and X', respectively. *An algebraic morphism of actions* $\Theta\rightsquigarrow\Theta'$ is a pair (Φ,g), where $\Phi\equiv(\Xi,\mu,\diamond,\Xi'):\Xi\rightsquigarrow\Xi'$ is an algebraic morphism, $g:\Sigma\to\Sigma'$ is a continuous function, such that

$$\rho=\nu\circ\rho'\circ g\,, \tag{4.1.6}$$
$$g(\xi\bullet\sigma)=\big[\xi\diamond\rho'(g(\sigma))\big]\bullet' g(\sigma)\quad \text{whenever}\quad \mathrm{d}(\xi)=\rho(\sigma)=\nu\big[\rho'(g(\sigma))\big]\,. \tag{4.1.7}$$

The composability condition $\mathrm{d}'\big[\xi \diamond \rho'(g(\sigma))\big] = \rho'(g(\sigma))$ is automatic, by (4.1.2). Part of the story is told in the diagram

$$\begin{tikzcd} \Xi \arrow[r, "\mathrm{d}"] & X & \Sigma \arrow[l, "\rho"'] \arrow[d, "g"] \\ \Xi' \arrow[r, "\mathrm{d}'"] \arrow[ur, "\mu"] & X' \arrow[u, "\nu"] & \Sigma' \arrow[l, "\rho'"'] \end{tikzcd}$$

Example 4.1.8. Assume that both of the groupoids are groups: $\Xi = \mathsf{G}$, $\Xi' = \mathsf{G}'$, acting in the topological spaces Σ, Σ', respectively. Recalling the setting of Example 4.1.4 and the fact that $X = \{\mathsf{e}\}$, $X' = \{\mathsf{e}'\}$, condition (4.1.7) translates into

$$g(\xi \bullet \sigma) = \big[\xi \diamond \rho'(g(\sigma))\big] \bullet' g(\sigma) = \big[\beta(\xi)\rho'(g(\sigma))\big] \bullet' g(\sigma) = \beta(\xi) \bullet' g(\sigma)\,,$$

so algebraic morphisms become ordinary morphisms of (group) actions. (Note that the condition (4.1.6) gets trivialized)

Example 4.1.9. If both action Θ and Θ' are canonical (terminal) actions, as in Example 3.1.3, then the above conditions translate into

$$\mathrm{id}_X = \nu \circ g\,,$$

$$\mathrm{r}'\big[\xi \diamond g(\mathrm{d}(\xi))\big] = g(\mathrm{r}(\xi))\,.$$

Maybe it is interesting to note that $\mathrm{d}'\big[\xi \diamond g(\mathrm{d}(\xi))\big] = g(\mathrm{d}(\xi))$ also holds, because of condition (4.1.2).

Definition 4.1.10. *The composition of two algebraic morphisms of actions*

$$\big(\Xi_1, \rho_1, \bullet_1, \Sigma_1\big) \underset{g_{12}}{\overset{\Phi_{12}}{\rightsquigarrow}} \big(\Xi_2, \rho_2, \bullet_2, \Sigma_2\big) \underset{g_{23}}{\overset{\Phi_{23}}{\rightsquigarrow}} \big(\Xi_3, \rho_3, \bullet_3, \Sigma_3\big)\,,$$

where

$$\Phi_{12} \equiv \big(\Xi_1, \mu_{12}, \diamond_{12}, \Xi_2\big)\,, \quad \Phi_{23} \equiv \big(\Xi_2, \mu_{23}, \diamond_{23}, \Xi_3\big)$$

is naturally defined as

$$(\Phi_{23}, g_{23}) \circ (\Phi_{12}, g_{12}) := (\Phi_{23} \circ \Phi_{12}, g_{23} \circ g_{12})\,,$$

where $\Phi_{23} \circ \Phi_{12}$ comes from (4.1.3), (4.1.4), (4.1.5) and $g_{23} \circ g_{12}$ is the usual composition of functions.

Proposition 4.1.11. *The composition of two algebraic morphisms of actions is also an algebraic morphism of actions.*

Proof. We are going to use the notations of Definition 4.1.10. At the level of anchor maps one has

$$\rho_1 = \nu_{12} \circ \rho_2 \circ g_{12} = \nu_{12} \circ \big(\nu_{23} \circ \rho_3 \circ g_{23}\big) \circ g_{12} = \nu_{13} \circ \rho_3 \circ g_{13}.$$

We must also verify that, if $\mathrm{d}_1(\xi_1) = \rho_1(\sigma_1)$, then

$$g_{23}\big(g_{12}(\xi_1 \bullet_1 \sigma_1)\big) = \big[\xi_1 \diamond_{13} \rho_3(g_{23}(g_{12}(\sigma_1)))\big] \bullet_3 g_{23}\big(g_{12}(\sigma_1)\big)$$

holds. Indeed, we will prove this, starting from the left hand side:

$$\begin{aligned}
g_{23}\big(g_{12}(\xi_1 \bullet_1 \sigma_1)\big) &\overset{(4.1.7)}{=} g_{23}\big(\big[\xi_1 \diamond_{12} \rho_2(g_{12}(\sigma_1))\big] \bullet_2 g_{12}(\sigma_1)\big) \\
&\overset{(4.1.7)}{=} \Big[\big[\xi_1 \diamond_{12} \rho_2(g_{12}(\sigma_1))\big] \diamond_{23} \rho_3(g_{23}(g_{12}(\sigma_1)))\Big] \bullet_3 g_{23}\big(g_{12}(\sigma_1)\big) \\
&\overset{(4.1.6)}{=} \Big[\big[\xi_1 \diamond_{12} \nu_{23}\big(\rho_3(g_{23}(g_{12}(\sigma_1)))\big)\big] \diamond_{23} \rho_3\big(g_{23}(g_{12}(\sigma_1))\big)\Big] \bullet_3 g_{23}\big(g_{12}(\sigma_1)\big) \\
&\overset{(4.1.5)}{=} \big[\xi_1 \diamond_{13} \rho_3\big(g_{23}(g_{12}(\sigma_1))\big)\big] \bullet_3 g_{23}\big(g_{12}(\sigma_1)\big)\,.
\end{aligned}$$

□

It is not hard to see that the composition of algebraic morphisms is associative and has an identity morphism for every action: If $\Theta = (\Xi, \rho, \bullet, \Sigma)$ is an action, the algebraic morphism of actions consisting of the action of Ξ on itself by left multiplication plus the identity function $g = \mathrm{id}_\Sigma$ is the identity morphism associated to Θ. So we are in presence of a category whose objects are groupoid actions and the arrows are algebraic morphisms.

Let us finish this subsection with a lemma which briefly explores some of the dynamics of the action $\diamond$; it will be used below. We denote by $\mathfrak{O}^{\diamond}_{\xi'}$ the orbit of $\xi' \in \Xi'$ under this action. $\mathrm{Sat}^{\diamond}(\cdot)$ represents the saturation (i.e. the smallest invariant set containing $\cdot$) with respect to the action $\diamond$.

Lemma 4.1.12. *The function* $\mathrm{d}' : \Xi' \to X'$ *is constant on the orbits of* $\diamond$. *That is, if* $\xi_1' \overset{\diamond}{\sim} \xi_2'$, *then* $\mathrm{d}'(\xi_1') = \mathrm{d}'(\xi_2')$. *One may write this as*

$$\mathfrak{O}^{\diamond}_{\xi'} \subset \big(\Xi'\big)_{\mathrm{d}'(\xi')}\,, \quad \forall\, \xi' \in \Xi'.$$

Moreover, if $\mathrm{Sat}^{\diamond}\big[\rho'(g(\Sigma))\big] = \Xi'$, *the equality is achieved.*

Proof. If ξ_1' and ξ_2' are related by $\xi_1' = \xi \diamond \xi_2'$, then

$$\mathrm{d}'(\xi_1') = \mathrm{d}'\big(\xi \diamond \xi_2'\big) = \mathrm{d}'(\xi_2')\,.$$

If now $\mathrm{Sat}^{\diamond}\big[\rho'(g(\Sigma))\big] = \Xi'$, for every pair of elements $\xi_1', \xi_2' \in \big(\Xi'\big)_{\mathrm{d}'(\xi')}$ we have

$$\mathrm{r}'\big((\xi_2')^{-1}\big) = \mathrm{d}'(\xi_2') = \mathrm{d}'(\xi_1')\,,$$

so there exists some $\xi \in \Xi$ and some $x' = \rho'(g(\sigma)) \in \rho'(g(\Sigma))$ such that

$$\xi \diamond x' = \xi_1'(\xi_2')^{-1}.$$

Observe that

$$\mathrm{r}'(\xi_2') = \mathrm{d}'(\xi_1'(\xi_2')^{-1}) = \mathrm{d}'(\xi \diamond x') = x',$$

so, in fact we have $\xi \diamond \mathrm{r}'(\xi_2') = \xi_1'(\xi_2')^{-1}$. This implies $\xi \diamond \xi_2' = \xi_1'$, so ξ_1' and ξ_2' belong to the same $\diamond$-orbit. □

In general, we will use the assumption $\mathrm{Sat}^{\diamond}\big[\rho'(g(\Sigma))\big] = \Xi'$ as being the analogous of surjectivity. A justification for this will be given in Remark 4.2.2. This assumption is stronger than $\mathrm{Sat}^{\diamond}(X') = \Xi'$, but weaker than $\mathrm{Sat}^{\diamond}(X') = \Xi$ and g surjective. It will play an important role in the next subsection.

Remark 4.1.13. If $\mathrm{Sat}^{\diamond}\big[\rho'(g(\Sigma))\big]=\Xi'$, then for every $x'\in X'$, there exists $\sigma\in\Sigma$ and $\xi\in\Xi$ such that $\xi\diamond\rho'(g(\sigma))=x'$. This implies that

$$x'=\mathrm{d}'(x')=\mathrm{d}'\big(\xi\diamond\rho'(g(\sigma))\big)=\mathrm{d}'\big(\rho'(g(\sigma))\big)=\rho'(g(\sigma))\,,$$

which means that $\rho'\circ g$ must be surjective.

4.1.2 Dynamical properties under algebraic morphisms of groupoid actions

Let us fix an algebraic morphism of groupoid actions

$$(\Phi,g):(\Xi,\rho,\bullet,\Sigma)\rightsquigarrow\big(\Xi',\rho',\bullet',\Sigma'\big)\,,\ \ \text{with}\ \ \Phi=(\mu,\diamond)\,.$$

Recall that the anchor maps ρ,ρ' are assumed surjective. If necessary, we indicate the action by an upper label.

Let us start with the relationship between recurrence sets of the two actions.

Theorem 4.1.14. *For $M,N\subset\Sigma$ we have*

$$\widetilde{\Xi}_M^N\diamond\rho'\big(g(M)\big)\subset\big(\widetilde{\Xi}'\big)_{g(M)}^{g(N)}\,. \tag{4.1.8}$$

If $\mathrm{Sat}^{\diamond}\big[\rho'(g(\Sigma))\big]=\Xi'$ and g is injective, equality is achieved.

Proof. Let $\xi\diamond\rho'\big(g(\sigma)\big)$ be an element of $\widetilde{\Xi}_M^N\diamond\rho'\big(g(M)\big)$, with $\xi\in\widetilde{\Xi}_M^N$ and $\sigma\in M$. This implies that

$$\mathrm{d}(\xi)=\nu\big[\rho'(g(\sigma))\big]\overset{(4.1.6)}{=}\rho(\sigma)\,.$$

So one has

$$\big[\xi\diamond\rho'(g(\sigma))\big]\bullet' g(\sigma)=g(\xi\bullet\sigma)\in g(N)\,.$$

It follows that $\xi\diamond\rho'(g(\sigma))\in\big(\widetilde{\Xi}'\big)_{g(M)}^{g(N)}$.

If $\mathrm{Sat}^{\diamond}\big[\rho'(g(\Sigma))\big]=\Xi'$, then every $\xi'\in\big(\widetilde{\Xi}'\big)_{g(M)}^{g(N)}$ can be written as

$$\xi'=\xi\diamond\rho'(g(\sigma))\,.$$

Without lose of generality we may assume that σ is in M and satisfies $\xi'\bullet' g(\sigma)\in g(N)$ (if not, select some $\sigma_0\in M$ such that $\xi'\bullet' g(\sigma_0)\in g(N)$ and observe that $\mathrm{d}'(\xi')=\rho'(g(\sigma))=\rho'(g(\sigma_0))$). In this case, one has $\mathrm{d}(\xi)=\nu\big[\rho'(g(\sigma))\big]=\rho(\sigma)$, so

$$\xi'\bullet' g(\sigma)=\big(\xi\diamond\rho'(g(\sigma))\big)\bullet' g(\sigma)=g(\xi\bullet\sigma)\in g(N)\,.$$

Using the injectivity of g one gets $\xi\bullet\sigma\in N$, meaning that $\xi\in\Xi_M^N$, and the equality in (4.1.8) follows from the way ξ' was defined. $\square$

Proposition 4.1.15. *(i) Let $\sigma,\tau\in\Sigma$ such that $\sigma\overset{\bullet}{\sim}\tau$. Then $g(\sigma)\overset{\bullet'}{\sim}g(\tau)$. Consequently*

$$g\big(\mathfrak{O}_\sigma\big)\subset\mathfrak{O}'_{g(\sigma)}\quad\text{and}\quad g\Big(\overline{\mathfrak{O}}_\sigma\Big)\subset\overline{\mathfrak{O}}'_{g(\sigma)}\,. \tag{4.1.9}$$

(ii) If $\mathrm{Sat}^{\diamond}\big[\rho'(g(\Sigma))\big] = \Xi'$*, then*

$$g\big(\mathfrak{O}_\sigma\big) = \mathfrak{O}'_{g(\sigma)} \,.$$

Hence the function g maps $\bullet$-invariant subsets of Σ to $\bullet'$-invariant subsets of Σ'.

(iii) Inverse images through g of $\bullet'$-invariant subsets of Σ' are $\bullet$-invariant subsets of Σ.

Proof. (i) For $\sigma, \tau \in \Sigma$, if $\sigma \overset{\bullet}{\sim} \tau$ then

$$\emptyset \neq \widetilde{\Xi}_\sigma^\tau \diamond \rho'(g(\sigma)) \overset{(4.1.8)}{\subset} \big(\widetilde{\Xi}'\big)_{g(\sigma)}^{g(\tau)} ,$$

which implies that $\big(\widetilde{\Xi}'\big)_{g(\sigma)}^{g(\tau)} \neq \emptyset$, i.e. $g(\sigma) \overset{\bullet'}{\sim} g(\tau)$. In its turn, this implies (4.1.9).

(ii) Pick $\tau' \overset{\bullet'}{\sim} g(\sigma)$, related through $\tau' = \xi' \bullet' g(\sigma)$. By the assumption $\mathrm{Sat}^{\diamond}\big[\rho'(g(\Sigma))\big] = \Xi'$, write

$$\xi' = \xi \diamond \rho'(g(\sigma_0)) \,.$$

We have

$$\mathrm{d}'(\xi') = \rho'(g(\sigma)) = \rho'(g(\sigma_0))$$

and thus we get

$$\tau' = \xi' \bullet' g(\sigma) = \big[\xi \diamond \rho'(g(\sigma))\big] \bullet' g(\sigma) = g(\xi \bullet \sigma) \in g\big(\mathfrak{O}_\sigma\big) \,.$$

(iii) This follows easily from (4.1.7). Let $B' \subset \Sigma'$ be $\bullet'$-invariant. If

$$\tau = \xi \bullet \sigma \overset{\bullet}{\sim} \sigma \in g^{-1}(B') \,,$$

then

$$g(\tau) = g(\xi \bullet \sigma) = \big[\xi \diamond \rho'(g(\sigma))\big] \bullet' g(\sigma) \in B'.$$

□

Clearly, what happens on $\Sigma' \setminus g\big(\Sigma\big)$ cannot be related to the action of Ξ' on Σ'; this is particularly relevant for relating the different notions of transitivity.

Corollary 4.1.16. *Suppose that the map g is surjective.*

(i) If Θ has any form of transitivity of the ones stated in Theorem 3.3.1, then Θ' also has it.

(ii) If $M \subset \Sigma$ is minimal and $g(M) \subset \Sigma'$ is closed, $g(M)$ is also minimal.

Proof. The proof only relies on Proposition 4.1.15. □

Corollary 4.1.17. *Assume that the map g is surjective. If Θ is recurrently transitive, Θ' is also recurrently transitive.*

Proof. Suppose that Θ is recurrently transitive and let $\emptyset \neq U, V \subset \Sigma'$ two open sets. Then $g^{-1}(U), g^{-1}(V) \subset \Sigma$ are non-void open sets, so $\widetilde{\Xi}_{g^{-1}(U)}^{g^{-1}(V)} \neq \emptyset$ which implies

$$\emptyset \neq \widetilde{\Xi}_{g^{-1}(U)}^{g^{-1}(V)} \diamond \rho'(U) \subset \big(\widetilde{\Xi}'\big)_U^V .$$

So Θ' is recurrently transitive. $\square$

Corollary 4.1.18. *Let $(\Phi, g) : (\Xi, \rho, \bullet, \Sigma) \rightsquigarrow \big(\Xi', \rho', \bullet', \Sigma'\big)$ be an algebraic morphism of groupoid actions. Assume that $\mathrm{Sat}^\diamond\big[\rho'(g(\Sigma))\big] = \Xi'$. If σ is periodic, $g(\sigma)$ is also periodic.*

Proof. If σ is periodic, then $\widetilde{\Xi}_\sigma^\sigma$ is syndetic in $\Xi_{\rho(\sigma)}$ with compact set K. By (4.1.6) and Theorem 4.1.14, one has

$$\mathfrak{O}^\diamond_{\rho'(g(\sigma))} = \Xi_{\rho(\sigma)} \diamond \rho'(g(\sigma)) = \big(\mathsf{K}\widetilde{\Xi}_\sigma^\sigma\big) \diamond \rho'(g(\sigma)) = \mathsf{K} \diamond \Big[\widetilde{\Xi}_\sigma^\sigma \diamond \rho'(g(\sigma))\Big] \subset \mathsf{K} \diamond \big(\widetilde{\Xi}'\big)_{g(\sigma)}^{g(\sigma)} .$$

Because of the Lemma 4.1.12, we see that

$$\big(\Xi'\big)_{\rho'(g(\sigma))} = \mathfrak{O}^\diamond_{\rho'(g(\sigma))} \subset \mathsf{K} \diamond \big(\widetilde{\Xi}'\big)_{g(\sigma)}^{g(\sigma)} = \mathsf{K} \diamond \Big[\rho'(g(\sigma))\big(\widetilde{\Xi}'\big)_{g(\sigma)}^{g(\sigma)}\Big] = \big[\mathsf{K} \diamond \rho'(g(\sigma))\big]\big(\widetilde{\Xi}'\big)_{g(\sigma)}^{g(\sigma)} .$$

Hence $\big(\widetilde{\Xi}'\big)_{g(\sigma)}^{g(\sigma)}$ is syndetic in $\big(\Xi'\big)_{\rho'(g(\sigma))}$, with compact set $\mathsf{K}' = \mathsf{K} \diamond \rho'(g(\sigma))$. $\square$

Corollary 4.1.19. *Let $(\Phi, g) : (\Xi, \rho, \bullet, \Sigma) \rightsquigarrow \big(\Xi', \rho', \bullet', \Sigma'\big)$ be an algebraic morphism of groupoid actions, with Σ' a locally compact space. Assume that $\mathrm{Sat}^\diamond\big[\rho'(g(\Sigma))\big] = \Xi'$. If σ is almost periodic, $g(\sigma)$ is also almost periodic.*

Proof. Assume that σ is almost periodic, let $U' \subset \Sigma'$ be a neighborhood of $g(\sigma)$ and, without lose of generality, assume that U' has a compact closure. Set $U = g^{-1}(U')$. Since σ is almost periodic, we have $\Xi_{\rho(\sigma)} = \mathsf{K}\widetilde{\Xi}_\sigma^U$ for some compact set $\mathsf{K} \subset \Xi$. We apply Theorem 4.1.14 just as in Corollary 4.1.18 to find that

$$\mathfrak{O}^\diamond_{\rho'(g(\sigma))} = \mathsf{K} \diamond \Big[\widetilde{\Xi}_\sigma^U \diamond \rho'(g(\sigma))\Big] \subset \mathsf{K} \diamond \big(\widetilde{\Xi}'\big)_{g(\sigma)}^{U'} .$$

Because of Lemma 4.1.12, we see that

$$\big(\Xi'\big)_{\rho'(g(\sigma))} = \mathfrak{O}^\diamond_{\rho'(g(\sigma))} \subset \mathsf{K} \diamond \big(\widetilde{\Xi}'\big)_{g(\sigma)}^{U'} = \big(\mathsf{K} \diamond \rho'(U')\big)\big(\widetilde{\Xi}'\big)_{g(\sigma)}^{U'} .$$

So we have $\mathsf{K}'\big(\widetilde{\Xi}'\big)_{g(\sigma)}^{U'} = \big(\Xi'\big)_{\rho'(g(\sigma))}$, with $\mathsf{K}' = \mathsf{K} \diamond \rho'(\overline{U'})$ compact. The result follows. $\square$

4.2 Some extra remarks on algebraic morphisms

A particular case of the next proposition appears in [11, Remark 2.4, Prop. 2.12]. Our present result shows that, in a (very) special case, algebraic morphisms of groupoid actions can be reinterpreted as usual morphisms of actions. This also (greatly) extends Example 4.1.8.

Proposition 4.2.1. *Let $\Theta := (\Xi, \rho, \bullet, \Sigma)$ and $\Theta' := \big(\Xi', \rho', \bullet', \Sigma'\big)$ two continuous groupoid actions. There is a one-to-one correspondence between*

- *algebraic morphisms of groupoid actions* $\big(\Phi = (\Xi, \mu, \diamond, \Xi'), g\big)$ *with* $\nu := \mu|_{X'} : X' \to X$ *a homeomorphism,*
- *morphisms of groupoid actions* $(\Psi, f) : \Theta \to \Theta'$ *(as in Definition 4.0.1) with* $\psi := \Psi|_X : X \to X'$ *a homeomorphism.*

Proof. Given (Ψ, f), we set

$$g := f, \quad \nu := \psi^{-1}, \quad \mu := \nu \circ \mathrm{r}' = \psi^{-1} \circ \mathrm{r}', \quad \xi \diamond \xi' := \Psi(\xi)\xi'. \tag{4.2.1}$$

The definition of the action $\diamond$ is justified by

$$\mathrm{d}(\xi) = \mu(\xi') = \psi^{-1}\big[r'(\xi')\big] \Longrightarrow \psi[\mathrm{d}(\xi)] = \mathrm{d}'[\Psi(\xi)] = \mathrm{r}'(\xi').$$

Let us use (4.2.1) to check (4.1.7):

$$\begin{aligned} g(\xi \bullet \sigma) &= f(\xi \bullet \sigma) \\ &\overset{(4.0.2)}{=} \Psi(\xi) \bullet' f(\sigma) \\ &= \Psi(\xi) \bullet' \big[\rho'(f(\sigma)) \bullet' f(\sigma)\big] \\ &= \big[\Psi(\xi)\rho'(f(\sigma))\big] \bullet' f(\sigma) \\ &\overset{(4.2.1)}{=} \big[\xi \diamond \rho'(f(\sigma))\big] \bullet' f(\sigma) \\ &= \big[\xi \diamond \rho'(g(\sigma))\big] \bullet' g(\sigma). \end{aligned}$$

In the opposite direction, if (Φ, g) is given, set

$$f := g, \quad \psi := \nu^{-1}, \quad \Psi(\xi) := \xi \diamond \nu^{-1}[\mathrm{d}(\xi)]. \tag{4.2.2}$$

This is possible, as $\mu\big(\nu^{-1}[\mathrm{d}(\xi)]\big) = \mathrm{d}(\xi)$. We now check (4.0.2), using (4.2.2):

$$\begin{aligned} f(\xi \bullet \sigma) &= g(\xi \bullet \sigma) \\ &\overset{(4.1.7)}{=} \big[\xi \diamond \rho'(g(\sigma))\big] \bullet' g(\sigma) \\ &\overset{(4.1.6)}{=} \big[\xi \diamond \nu^{-1}(\rho(\sigma))\big] \bullet' g(\sigma) \\ &= \big[\xi \diamond \nu^{-1}(\mathrm{d}(\xi))\big] \bullet' g(\sigma) \\ &= \Psi(\xi) \bullet' g(\sigma) \\ &= \Psi(\xi) \bullet' f(\sigma). \end{aligned}$$

□

Remark 4.2.2. Under the situation of the previous Proposition, one finds out that

$$\Psi(\Xi) = \mathrm{Sat}^{\diamond}(X').$$

Since ρ' is assumed surjective, it follows that (Ψ, f) is an epimorphism of actions if and only if g is surjective and $\mathrm{Sat}^{\diamond}(X') = \mathrm{Sat}^{\diamond}\big[\rho'(g(\Sigma))\big] = \Xi'$. We may think of the latter condition as being the equivalent for surjectivity in a generalized way.

We finish this chapter with two results on stabilizations under the action $\diamond$ resulting from an algebraic morphism of groupoids.

Proposition 4.2.3. *For every subset* $Y' \subset X'$, $\mathrm{Sat}^\diamond(Y')$ *is a subsemigroupoid of* Ξ' *(stable for the multiplication). If* Y' *is invariant* ($\mathrm{d}'(\xi') \in Y' \Leftrightarrow \mathrm{r}'(\xi') \in Y'$), $\mathrm{Sat}^\diamond(Y')$ *is a subgroupoid. We have* $\mathrm{Sat}^\diamond(Y') = \mathrm{Sat}^\diamond(Z')$ *if and only if* $Y' = Z'$.

Proof. If $\xi_1 \diamond x_1', \xi_2 \diamond x_2' \in \mathrm{Sat}^\diamond(Y')$ are elements such that $x_1' = \mathrm{d}'\big(\xi_1 \diamond x_1'\big) = \mathrm{r}'\big(\xi_2 \diamond x_2'\big)$, then

$$\big(\xi_1 \diamond x_1'\big)\big(\xi_2 \diamond x_2'\big) = \xi_1 \diamond \Big[x_1'\big(\xi_2 \diamond x_2'\big)\Big] = \xi_1 \diamond \big(\xi_2 \diamond x_2'\big) = (\xi_1\xi_2) \diamond x_2' \in \mathrm{Sat}^\diamond(Y')\,.$$

Therefore, $\mathrm{Sat}^\diamond(Y')$ is closed under multiplication. Actually, the same computation proves the (apparently) stronger property

$$\mathrm{Sat}^\diamond(X')\mathrm{Sat}^\diamond(Y') \subset \mathrm{Sat}^\diamond(Y')\,.$$

Let us now assume that Y' is invariant and prove that $\mathrm{Sat}^\diamond(Y')$ is closed by inversion: Let

$$\xi' = \xi \diamond \mathrm{d}'(\xi') \in \mathrm{Sat}^\diamond(Y')\,.$$

We claim that

$$(\xi')^{-1} := \xi^{-1} \diamond \mathrm{r}'(\xi') \in \mathrm{Sat}^\diamond(Y')\,.$$

Indeed $\mathrm{r}'(\xi') \in Y'$, by invariance, and

$$\mu(\mathrm{r}'(\xi')) = \mu(\xi') = \mu\big(\xi \diamond \mathrm{d}'(\xi')\big) = \mathrm{r}(\xi) = \mathrm{d}(\xi^{-1})$$

proves that ξ^{-1} may be applied to $\mathrm{r}'(\xi')$. Straightforward computations show that $\xi^{-1} \diamond \mathrm{r}'(\xi')$ is indeed the inverse of ξ'. For example

$$\big(\xi^{-1} \diamond \mathrm{r}'(\xi')\big)\xi' = \xi^{-1} \diamond \Big[\mathrm{r}'(\xi')\xi'\Big] = \xi^{-1} \diamond \xi' = \xi^{-1} \diamond \big[\xi \diamond \mathrm{d}'(\xi')\big] = \mathrm{d}(\xi) \diamond \mathrm{d}'(\xi') = \mathrm{d}'(\xi')\,.$$

Finally, if $\mathrm{Sat}^\diamond(Y') = \mathrm{Sat}^\diamond(Z')$, then, for any $z' \in Z'$ there exists $y' \in Y'$ such that $z' = \xi \diamond y'$. But z' is a unit. Apply (4.1.2) to get $z' = y' \in Y'$, hence $Z' \subset Y'$. Then repeat the argument to conclude that $Y' = Z'$. ☐

If our algebraic morphism comes from an algebraic morphism of actions, we not only have that $\mathrm{Sat}^\diamond(X')$ is a subgroupoid of Ξ' but we can also show that $\mathrm{Sat}^\diamond\big[\rho'(g(\Sigma))\big]$ is a subgroupoid. This holds because we don't actually need the full strength of invariance. If $\mathrm{r}'(\xi') \in Y'$ holds for every $\xi' \in \Xi'_{Y'}$ of the form $\xi' = \xi \diamond y'$, the conclusion still follows. (And this occurs in $\rho'(g(\Sigma))$ because of (4.1.7))

In the previous chapter, the notion of *openness* played a crucial role. It seems interesting to explore the fate of openness under algebraic morphisms. In the proof of the next result we will use Fell's criterion [55, Prop. 1.1] for open functions twice.

Proposition 4.2.4. *If* Ξ *is an open groupoid and* $\mathrm{Sat}^\diamond(X') = \Xi'$, *then* Ξ' *is an open groupoid.*

Proof. Let $x' = \mathrm{d}'(\xi') \in X'$ and pick some net $x'_i \in X'$ converging to x'. By the assumptions, we must have $\xi' = \xi \diamond x'$, for some $\xi \in \Xi$. Remark that $\mu\big(x'_i\big) \to \mu(x') = \mathrm{d}(\xi')$ so, by Fell's criterion, there exists a subnet x'_{i_j} and a net $\xi_j \in \Xi$ such that $\mu\big(x'_{i_j}\big) = \mathrm{d}(\xi_j)$ and $\xi_j \to \xi$. Hence $\xi_j \diamond x'_{i_j}$ is well defined and

$$\xi'_j := \xi_j \diamond x'_{i_j} \to \xi \diamond x' = \xi'.$$

So we found a subnet $x'_{i_j} = \mathrm{d}'(\xi'_j)$ with $\xi'_j \in \Xi$ converging to ξ'. We conclude, by applying Fell's criterion, that d' is an open map. $\square$

Chapter 5

On symmetry of L^1-Algebras Associated to Fell Bundles

This chapter treats the symmetry of certain Banach *-algebras connected with Fell bundles over locally compact groups (always supposed to be Hausdorff). The main concept is

Definition 5.0.1. A Banach *-algebra $\mathfrak{B}$ is called *symmetric* if the spectrum of b^*b is positive for every $b \in \mathfrak{B}$ (this happens if and only if the spectrum of any self-adjoint element is real.)

The symmetry of a Banach *-algebra admits many reformulations and has many useful and interesting consequences [47, 34], which will not be discussed here. A very useful and readable presentation may be found in [34].

A great deal of effort has been dedicated to Banach algebras associated to a locally compact group G. The basic example is the convolution algebra $L^1(\mathsf{G})$; actually the interest in the symmetry property arouse around the Banach algebra interpretation and treatment of the classical result [54] of Wiener on Fourier series. But there are increasingly general other classes, as global crossed products, partial crossed products (both twisted by 2-cocycles or not) and L^1-types algebras associated to Fell bundles over G. All of these played an important theoretical role and lead to many applications. When looking for results and examples, one aims to enlarge the collection of groups that can be treated, as well as the class of symmetric Banach *-algebras assigned to them. The present chapter is concerned with the second purpose.

Until very recently, the largest class for which general results have been found was that of crossed products attached to a global action of the group over a C^*-algebra. A cohomological twist has also been permitted. The simplest case of a trivial action leads to the projective tensor product between $L^1(\mathsf{G})$ and a C^*-algebra. Some references are: [42, 6, 40, 49, 47, 35, 36, 5, 28, 34, 8, 44, 27].

It is convenient to use the following terminology; the points (i) are (ii) are classical notions, the third was introduced recently by D. Jauré and M. Măntoiu in [38].

Definition 5.0.2. (i) The locally compact group G is called *symmetric* if the convolution Banach *-algebra $L^1(\mathsf{G})$ is symmetric.

(ii) The locally compact group G is called *rigidly symmetric* if given any C^*-algebra $\mathcal{A}$, the algebra $L^1(\mathsf{G},\mathcal{A}) \cong L^1(\mathsf{G}) \otimes \mathcal{A}$ is symmetric.

(iii) The locally compact group G is called *hypersymmetric* if for every Fell bundle $\mathscr{C}$ over G, the Banach *-algebra $L^1(\mathsf{G}\,|\,\mathscr{C})$ is symmetric.

A diagram with implications and equivalences is supposed to systematize the actual state of art. If G is a locally compact group, we will denote by $\mathsf{G}^{\rm dis}$ the same group with the discrete topology.

$$\begin{tikzcd}
\mathsf{G} \text{ symmetric} & \mathsf{G} \text{ rigidly symmetric} \arrow[l, Rightarrow, "2"'] & \mathsf{G} \text{ hypersymmetric} \arrow[l, Rightarrow, "7"'] \\
\mathsf{G}^{\rm dis} \text{ symmetric} \arrow[u, Rightarrow, "3"] & \mathsf{G}^{\rm dis} \text{ rigidly symmetric} \arrow[l, Rightarrow, "1"'] \arrow[u, Rightarrow, "4"] \arrow[ur, Rightarrow, "6"] \arrow[r, Leftrightarrow, "5"'] & \mathsf{G}^{\rm dis} \text{ hypersymmetric} \arrow[u, Rightarrow, "8"']
\end{tikzcd}$$

Clearly, a rigidly symmetric group is symmetric, which is expressed by the two horizontal arrows 1 and 2. It is still not known if the two notions are really different.

The implications 3 and 4 are due to Poguntke [49].

In 5 one direction is trivial, since projective tensor products of the form $\ell^1(\mathsf{G},\mathcal{A})$ are easily written as L^1-algebras of some Fell bundle, as we know. The fact that, for discrete groups, rigid symmetry implies hypersymmetry is the main result of [38].

The main result of the present chapter are the (equivalent) implications 6 *and* 8.

Replacing the implication 7 by an equivalence would certainly be considered a nice result. Neither arrow 3 nor arrow 4 can be reversed: Any compact connected semisimple real Lie group is rigidly symmetric, but in [10] it is shown that it contains a (dense) free group on two elements. The discretization will not be amenable; by [52] it cannot be symmetric.

In [27] there is shown a global action of the 'ax+b' group, which associated L^1-algebra is not symmetric. This provides an example of a symmetric but not hypersymmetric group.

There are large classes of examples known to be symmetric or rigidly symmetric. Classes of rigidly symmetric discrete groups are (cf. [42]): (a) Abelian, (b) finite, (c) finite extensions of discrete nilpotent. This last class includes all the finitely generated groups with polynomial growth. A central extension of a rigidly symmetric group is rigidly symmetric, by [42, Thm. 7]. In [44, Cor. 2.16] it is shown that the quotient of a discrete rigidly symmetric group by a normal subgroup is rigidly symmetric.

5.1 The main results

All over this section, G will be a (Hausdorff) locally compact group with unit 1 and left Haar measure $d\mu(x) \equiv dx$. We are also given a Fell bundle $\mathscr{C} = \bigsqcup_{x\in\mathsf{G}} \mathfrak{C}_x$ over G. Its sectional $L^1(\mathsf{G}\,|\,\mathscr{C})$ Banach *-algebra is the completion of the space $C_{\mathsf{c}}(\mathsf{G}\,|\,\mathscr{C})$ of continuous sections with compact support. Its (universal) C^*-algebra its denoted by $\mathrm{C}^*(\mathsf{G}\,|\,\mathscr{C})$. The case of a discrete group and associated graded C^*-algebras is developed in [25]. We only recall the product on $L^1(\mathsf{G}\,|\,\mathscr{C})$

$$\big(\Phi * \Psi\big)(x) = \int_{\mathsf{G}} \Phi(y) \bullet \Psi(y^{-1}x)\,\mathrm{d}y \tag{5.1.1}$$

and its involution

$$\Phi^*(x) = \Delta(x^{-1})\,\Phi(x^{-1})^{\bullet}\,, \tag{5.1.2}$$

in terms of the operations $\big(\bullet,^{\bullet}\big)$ on the Fell bundle. (see Definition 2.2.26)

We are going to make use both of $\mathscr{C}$ and its discrete version; the following result is obvious.

Lemma 5.1.1. *Let $\mathscr{C} = \bigsqcup_{x\in\mathsf{G}} \mathfrak{C}_x$ be a Fell bundle over the locally compact group G and $\mathsf{G}^{\rm dis}$ the same group with the discrete topology. We denote by $\mathscr{C}^{\rm dis}$ the space $\mathscr{C}$ with the disjoint union topology defined by the fibres $\big\{\mathfrak{C}_x \,\big|\, x \in \mathsf{G}\big\}$ (a subset is open if and only if its intersection with every $\mathfrak{C}_x$ is open). Then $\mathscr{C}^{\rm dis}$ is a Fell bundle over $\mathsf{G}^{\rm dis}$, called* the discretization of the initial one. *One has*

$$C_{\rm c}\big(\mathsf{G}^{\rm dis}\,|\,\mathscr{C}^{\rm dis}\big) = \big\{\Phi : \mathsf{G} \to \mathscr{C} \,\big|\, \Phi(x) \in \mathfrak{C}_x\,,\,\forall\, x \in \mathsf{G} \text{ and } \operatorname{supp}(\Phi) \text{ is finite}\big\}\,,$$

$$L^1\big(\mathsf{G}^{\rm dis}\,|\,\mathscr{C}^{\rm dis}\big) \equiv \ell^1(\mathsf{G}\,|\,\mathscr{C}) = \Big\{\Phi : \mathsf{G} \to \mathscr{C} \,\Big|\, \Phi(x) \in \mathfrak{C}_x\,,\,\forall\, x \in \mathsf{G} \text{ and } \sum_{x\in\mathsf{G}} \|\,\Phi(x)\,\| < \infty\Big\}\,,$$

$$C^*\big(\mathsf{G}^{\rm dis}\,|\,\mathscr{C}^{\rm dis}\big) = \text{the enveloping } C^*\!-\text{algebra of } \ell^1(\mathsf{G}\,|\,\mathscr{C})\,.$$

The critical technical result is the next theorem. It will be proven in Section 5.3, on the lines of [49, 27], where only particular cases of Fell bundles have been treated. Some technical complications which have to be overcome are due to the fact that our L^1-sections do not take values in a single Banach space.

Theorem 5.1.2. *Let $\mathscr{C}$ be a Fell bundle over the locally compact group G. If $\ell^1(\mathsf{G}\,|\,\mathscr{C})$ is symmetric, then $L^1(\mathsf{G}\,|\,\mathscr{C})$ is also symmetric.*

The discrete case has been treated in [38], using the setting of graded C^*-algebras. We rephrase here the result, in the language of Fell bundles.

Theorem 5.1.3. *Let $\mathscr{D}$ be a Fell bundle over the discrete group H. Assume any of the hypothesis:*

- H *is rigidly symmetric,*
- H *is symmetric and $C^*\big(\mathsf{H}\,|\,\mathscr{D}\big)$ is a type I C^*-algebra.*

Then $\ell^1(\mathsf{H}\,|\,\mathscr{D})$ is a symmetric Banach *-algebra.

We state now the main results of the chapter. They follow immediately from the two theorems above, where $\mathsf{H} = \mathsf{G}^{\rm dis}$ and $\mathscr{D}$ is the discretization of $\mathscr{C}$.

Theorem 5.1.4. *Let $\mathscr{C}$ be a Fell bundle over the locally compact group G for which the discrete group $\mathsf{G}^{\rm dis}$ is rigidly symmetric. Then $L^1(\mathsf{G}\,|\,\mathscr{C})$ is a symmetric Banach *-algebra.*

If one only knows that $\mathsf{G}^{\mathrm{dis}}$ is symmetric, there is still a result if we ask more on the C^*-algebraic side.

Theorem 5.1.5. *Let $\mathscr{C}$ be a Fell bundle over the locally compact group G for which the discrete group $\mathsf{G}^{\mathrm{dis}}$ is symmetric. Suppose that $C^*\big(\mathsf{G}^{\mathrm{dis}}|\mathscr{C}^{\mathrm{dis}}\big)$ is a type I C^*-algebra. Then $L^1(\mathsf{G}\,|\,\mathscr{C})$ is symmetric.*

5.2 Discretization of representations

Our proof of Theorem 5.1.2, to be found in the next section, relies on a discretization procedure in the setting of Fell bundles, that we now present.

Let $\Gamma : L^1(\mathsf{G}\,|\,\mathscr{C}) \to \mathbb{B}(\mathcal{E})$ be an algebraically irreducible representation on the vector space $\mathcal{E}$. Even for general Banach algebras, it is known how to turn Γ into a Banach space representation. In our case, for $\xi_0 \in \mathcal{E}$, the norm

$$\|\xi\|_{\mathcal{E}} := \inf\big\{\|\Phi\|_{L^1(\mathsf{G}\,|\,\mathscr{C})}\,|\,\Gamma(\Phi)\xi_0 = \xi\big\}$$

makes $\mathcal{E}$ a Banach space and Γ a contractive homomorphism. If $\mathcal{E}$ was already a Banach space, this new norm is equivalent to the previous one, so we lose nothing by assume that this is the norm of $\mathcal{E}$.

We will define a representation Γ^{dis} of $\ell^1\big(\mathsf{G}\,|\,\mathscr{C}\big) \equiv L^1\big(\mathsf{G}^{\mathrm{dis}}\,|\,\mathscr{C}^{\mathrm{dis}}\big)$ on $\mathcal{E}$, called its *discretization*. The procedure consists in disintegrating the initial representation, reinterpreting it in the discrete setting and integrating it back. The new one will act in the same space, but it will be very different.

By algebraic irreducibility, we may write every $\xi \in \mathcal{E}$ as $\xi = \Gamma(\Psi)\xi_0$. Using this form, the representation Γ induces bounded operators $\{\gamma(a)\}_{a\in\mathscr{C}}$, given by

$$\gamma(a)\Big[\Gamma(\Psi)\xi_0\Big] := \Gamma(a\cdot\Psi)\xi_0$$

where, in terms of the bundle projection $q : \mathscr{C} \to \mathsf{G}$, we set

$$(a\cdot\Psi)(x) := a \bullet \Psi\big(q(a)^{-1}x\big) \in \mathfrak{C}_x\,, \quad \forall\, x \in \mathsf{G}\,.$$

We are thinking of $a \in \mathscr{C}$ as a multiplier of $L^1(\mathsf{G}\,|\,\mathscr{C})$. We do not insist on the multiplier techniques and interpretation, for simplicity, and because we do not have here Hilbert space *-representations for which it is easy to cite the relevant facts. So we provide a direct proof of the following result:

Lemma 5.2.1. *The map $\gamma : \mathscr{C} \to \mathbb{B}(\mathcal{E})$ is well-defined, strongly continuous, contractive and does not depend on the choices.*

Proof. Let $\xi = \Gamma(\Psi)\xi_0 = \Gamma(\Phi)\xi_1$ and let $\big\{\Upsilon_i\,\big|\,i \in I\big\} \subset L^1(\mathsf{G}\,|\,\mathscr{C})$ be some approximate unit. Then one has

$$\begin{aligned} 0 &= \Gamma(a\cdot\Upsilon_i)(\xi-\xi) \\ &= \Gamma(a\cdot\Upsilon_i)\big(\Gamma(\Psi)\xi_0 - \Gamma(\Phi)\xi_1\big) \\ &= \Gamma(a\cdot\big[\Upsilon_i * \Psi\big])\xi_0 - \Gamma(a\cdot\big[\Upsilon_i * \Phi\big])\xi_1 \end{aligned}$$

So by passing to the limit we get the equality

$$\gamma(a)\xi = \Gamma(a\cdot\Psi)\xi_0 = \Gamma(a\cdot\Phi)\xi_1.$$

We now show that $\|\gamma(a)\|_{\mathbb{B}(\mathcal{E})} \leq \|a\|_{\mathfrak{C}_{q(a)}}$. By the definition of $\|\cdot\|_{\mathcal{E}}$, there exists $\{\Phi_n\}_{n\in\mathbb{N}} \subset L^1(\mathsf{G}\,|\,\mathscr{C})$ for which $\Gamma(\Phi_n)\xi_0 = \xi$ and $\|\Phi_n\|_{L^1(\mathsf{G}\,|\,\mathscr{C})} \to \|\xi\|_{\mathcal{E}}$. Then

$$\gamma(a)\xi = \gamma(a)\Gamma(\Phi_n)\xi_0 = \Gamma\big(a\cdot\Phi_n\big)\xi_0\,,$$

which yields $\|\gamma(a)\xi\|_{\mathcal{E}} \leq \|a\cdot\Phi_n\|_{L^1(\mathsf{G}\,|\,\mathscr{C})}$. But

$$\begin{aligned}
\|a\cdot\Phi_n\|_{L^1(\mathsf{G}\,|\,\mathscr{C})} &= \int_{\mathsf{G}} \|a\bullet\Phi_n(q(a)^{-1}x)\|_{\mathfrak{C}_x}\mathrm{d}x\\
&\leq \int_{\mathsf{G}} \|a\|_{\mathfrak{C}_{q(a)}}\|\Phi_n(q(a)^{-1}x)\|_{\mathfrak{C}_{q(a)^{-1}x}}\mathrm{d}x\\
&= \|a\|_{\mathfrak{C}_{q(a)}}\int_{\mathsf{G}} \|\Phi_n(x)\|_{\mathfrak{C}_x}\mathrm{d}x\\
&= \|a\|_{\mathfrak{C}_{q(a)}}\|\Phi_n\|_{L^1(\mathsf{G}\,|\,\mathscr{C})}.
\end{aligned}$$

So

$$\|a\cdot\Phi_n\|_{L^1(\mathsf{G}\,|\,\mathscr{C})} \leq \|a\|_{\mathfrak{C}_{q(a)}}\|\Phi_n\|_{L^1(\mathsf{G}\,|\,\mathscr{C})} \to \|a\|_{\mathfrak{C}_{q(a)}}\|\xi\|_{\mathcal{E}}\,,$$

hence $\|\gamma(a)\xi\|_{\mathcal{E}} \leq \|a\|_{\mathfrak{C}_{q(a)}}\|\xi\|_{\mathcal{E}}$ and $\|\gamma(a)\|_{\mathbb{B}(\mathcal{E})} \leq \|a\|_{\mathfrak{C}_{q(a)}}$.

For the continuity of $a\mapsto\gamma(a)\xi$, write (again) $\xi = \Gamma(\Psi)\xi_0$ and observe that

$$\begin{aligned}
\gamma(b)\xi - \gamma(a)\xi &= \gamma(b)\Gamma(\Psi)\xi_0 - \gamma(a)\Gamma(\Psi)\xi_0\\
&= \Gamma\big(b\bullet\Psi(q(b)^{-1}\,\cdot\,) - a\bullet\Psi(q(a)^{-1}\,\cdot\,)\big)\xi_0\\
&= \Gamma\big(b\cdot\Psi - a\cdot\Psi\big)\xi_0\,.
\end{aligned}$$

In consequence, the continuity of $a\mapsto\gamma(a)\xi$ follows from the continuity of $a\mapsto a\cdot\Psi\in L^1(\mathsf{G}\,|\,\mathscr{C})$. □

Remark 5.2.2. The initial representation Γ can be recovered from the operators $\{\gamma(a)\}_{a\in\mathscr{C}}$ by

$$\Gamma(\Phi)\xi = \int_{\mathsf{G}} \gamma\big(\Phi(x)\big)\xi\,\mathrm{d}x\,. \tag{5.2.1}$$

Indeed, writing $\xi\in\mathcal{E}$ as $\xi = \Gamma(\Psi)\xi_0$ we get

$$\begin{aligned}
\int_{\mathsf{G}} \gamma\big(\Phi(x)\big)\xi\,\mathrm{d}x &= \int_{\mathsf{G}} \gamma\big(\Phi(x)\big)\big[\Gamma(\Psi)\xi_0\big]\,\mathrm{d}x\\
&= \int_{\mathsf{G}} \Gamma\big(\Phi(x)\bullet\Psi(x^{-1}\,\cdot\,)\big)\xi_0\,\mathrm{d}x\\
&= \Gamma\Big(\int_{\mathsf{G}} \Phi(x)\bullet\Psi\big(x^{-1}\,\cdot\,\big)\,\mathrm{d}x\Big)\xi_0\\
&= \Gamma(\Phi*\Psi)\xi_0 = \Gamma(\Phi)\big[\Gamma(\Psi)\xi_0\big] = \Gamma(\Phi)\xi\,.
\end{aligned}$$

This is part of the so called 'integration/desintegration' theory for Fell Bundles.

Definition 5.2.3. With the assumptions given above, the *discretization* of the representation $\Gamma : L^1(\mathsf{G}\,|\,\mathscr{C}) \to \mathbb{B}(\mathcal{E})$ is $\Gamma^{\rm dis} : \ell^1\big(\mathsf{G}\,|\,\mathscr{C}\big) \to \mathbb{B}(\mathcal{E})$ given by

$$\Gamma^{\rm dis}(\varphi)\xi := \sum_{x\in\mathsf{G}} \gamma\big(\varphi(x)\big)\xi\,, \quad \text{ for } \varphi \in \ell^1\big(\mathsf{G}\,|\,\mathscr{C}\big)\,. \tag{5.2.2}$$

Remark 5.2.4. $\Gamma^{\rm dis}$ is an algebraically irreducible representation of a cross-sectional algebra, so it also induces operators $\{\gamma^{\rm dis}(a)\}_{a\in\mathscr{C}}$. We point out that $\gamma^{\rm dis}(a) = \gamma(a)$, which follows from the existing definitions by a straightforward computation. On the other hand, the discrete setting allows a simpler treatment (multipliers are no longer needed). The Fell bundle $\mathscr{C}^{\rm dis}$ injects isometrically in $\ell^1\big(\mathsf{G}\,|\,\mathscr{C}\big)$ by

$$\big[\mu(a)\big](x) := \delta_{q(a),x}\, a\,, \quad \forall\, a \in \mathscr{C}, x \in \mathsf{G} \tag{5.2.3}$$

(which may be written simply $\mu(a) = a\delta_{q(a)}$), and then we set $\gamma^{\rm dis} := \Gamma^{\rm dis} \circ \mu$. To check that this is the same object, one proves immediately that $\mu(a) * \psi = a \cdot \psi$ and then apply $\Gamma^{\rm dis}$:

$$\Gamma^{\rm dis}\big[\mu(a)\big]\Gamma^{\rm dis}(\psi)\xi_0 = \Gamma^{\rm dis}\big[\mu(a) * \psi\big]\xi_0 = \Gamma^{\rm dis}(a\cdot\psi)\xi_0 = \gamma(a)\Gamma^{\rm dis}(\psi)\xi_0\,.$$

To obtain the main result of this section, which is Proposition 5.2.7, let us state a couple of lemmas:

Lemma 5.2.5. *Let $T : \mathcal{X} \to \mathcal{Y}$ be a bounded operator between Banach spaces. If T satisfies the condition*

$$\forall\, y \in \mathcal{Y}, \forall\, \varepsilon > 0\,, \exists\, x \in \mathcal{X} \text{ such that } \max\{\|x\| - \|y\|, \|Tx - y\|\} \le \varepsilon\,,$$

then it must be surjective.

Proof. This is Lemma A2 from [9]; see also [49, pag. 193]. ☐

Lemma 5.2.6. *For every $\eta \in \mathcal{E}$ and $\varepsilon > 0$ there exists $\varphi \in \ell^1(\mathsf{G}\,|\,\mathscr{C})$ such that*

$$\|\,\varphi\,\|_{\ell^1(\mathsf{G}\,|\,\mathscr{C})} \le \|\,\eta\,\|_{\mathcal{E}} + \varepsilon\,, \quad \|\,\Gamma^{\rm dis}(\varphi)\xi_0 - \eta\,\|_{\mathcal{E}} \le \varepsilon\,. \tag{5.2.4}$$

Proof. Let $0 < \delta$, to be fixed later. Due to the definition of the norm in $\mathcal{E}$ and the density of $C_{\rm c}(\mathsf{G}\,|\,\mathscr{C})$ in $L^1(\mathsf{G}\,|\,\mathscr{C})$, there is an element $\Phi \in C_{\rm c}(\mathsf{G}\,|\,\mathscr{C})$ such that

$$\|\,\Phi\,\|_{L^1(\mathsf{G}\,|\,\mathscr{C})} \le \|\,\eta\,\|_{\mathcal{E}} + \delta\,, \quad \|\,\Gamma(\Phi)\xi_0 - \eta\,\|_{\mathcal{E}} \le \delta\,\|\,\xi_0\,\|_{\mathcal{E}}\,. \tag{5.2.5}$$

Recall that, for such a section, the maps $x \mapsto \|\,\Phi(x)\,\|_{\mathfrak{C}_x}$ and $x \mapsto \gamma\big(\Phi(x)\big)\xi_0$ are uniformly continuous. The support of Φ will be denoted by Σ. Let us fix an open relatively compact neighborhood V of $1 \in \mathsf{G}$ such that

$$\|\,\gamma\big(\Phi(xu)\big)\xi - \gamma\big(\Phi(x)\big)\xi\,\|_{\mathcal{E}} \le \delta/\mu(\Sigma)\,, \quad \forall\, x \in \mathsf{G}\,, u \in V \tag{5.2.6}$$

and

$$\big|\,\|\,\Phi(yu)\,\|_{\mathfrak{C}_{yu}} - \|\,\Phi(y)\,\|_{\mathfrak{C}_y}\big| \le \delta/\mu(\Sigma)\,, \quad \forall\, x \in \mathsf{G}\,, u \in V. \tag{5.2.7}$$

(see Lemma 2.2.7). Choose $x_1, \dots, x_m \in \mathsf{G}$ such that $\Sigma \subset \bigcup_{j=1}^m x_jV$. Setting $M_1 := x_1V \cap \Sigma$, and then inductively

$$M_k := \big(x_kV \cap \Sigma\big) \setminus \bigcup_{j<k} M_j\,, \quad k = 2, \dots, m\,,$$

one gets a measurable partition of Σ. The solution to our problem is expected to be the finitely supported function

$$\varphi := \sum_{j=1}^{m} \mu(M_j)\Phi(x_j)\delta_{x_j}\,.$$

Its single non-null values are $\varphi(x_j) = \mu(M_j)\Phi(x_j) \in \mathfrak{C}_{x_j}$, $j = 1, \dots, m$, so $\varphi \in \ell^1(\mathsf{G}\,|\,\mathscr{C})$. One has

$$\begin{aligned}
\|\,\Phi\,\|_{L^1(\mathsf{G}\,|\,\mathscr{C})} &= \sum_{j=1}^{m} \int_{M_j} \|\,\Phi(x)\,\|_{\mathfrak{C}_x}\, d\mu(x) \\
&\geq \sum_{j=1}^{m} \big[\, \|\,\Phi(x_j)\,\|_{\mathfrak{C}_{x_j}} - \delta/\mu(\Sigma)\,\big] \int_{M_j} d\mu(x) \\
&= \sum_{j=1}^{m} \|\,\Phi(x_j)\,\|_{\mathfrak{C}_{x_j}}\ \mu(M_j) - \delta \\
&= \|\,\varphi\,\|_{\ell^1(\mathsf{G}\,|\,\mathscr{C})} - \delta\,.
\end{aligned}$$

For the inequality we used (5.2.7) with $y = x_k$ and the fact that $M_j \subset x_j V$. Combining this with the first identity in (5.2.5), we get

$$\|\,\varphi\,\|_{\ell^1(\mathsf{G}\,|\,\mathscr{C})} \leq \|\,\eta\,\|_{\mathcal{E}} + 2\delta\,.$$

To reach the second condition in (5.2.4), by (5.2.5), we only need to control the difference $\Gamma(\Phi)\xi_0 - \Gamma^{\mathrm{dis}}(\varphi)\xi_0$, using the formulae (5.2.1) and (5.2.2), which in our case become

$$\Gamma(\Phi)\xi_0 = \sum_{j=1}^{m} \int_{M_j} \gamma\big(\Phi(x)\big)\xi_0 \,\mathrm{d}x$$

and

$$\Gamma^{\mathrm{dis}}(\varphi)\xi_0 = \sum_{j=1}^{m} \mu(M_j)\gamma\big(\Phi(x_j)\big)\xi_0 = \sum_{j=1}^{m} \int_{M_j} \gamma\big(\Phi(x_j)\big)\xi_0 \,\mathrm{d}x.$$

Therefore, by (5.2.6)

$$\begin{aligned}
\|\Gamma(\Phi)\xi_0 - \Gamma^{\mathrm{dis}}(\varphi)\xi_0\|_{\mathcal{E}} &= \|\sum_{j=1}^{m} \int_{M_j} \gamma\big(\Phi(x)\big)\xi_0 - \gamma\big(\Phi(x_j)\big)\xi_0 \,\mathrm{d}x\|_{\mathcal{E}} \\
&\leq \sum_{j=1}^{m} \int_{M_j} \|\gamma\big(\Phi(x)\big)\xi_0 - \gamma\big(\Phi(x_j)\big)\xi_0\|_{\mathcal{E}} \,\mathrm{d}x \\
&\leq \sum_{j=1}^{m} \delta\mu(M_j)/\mu(\Sigma) \\
&= \delta\,.
\end{aligned}$$

So we have

$$\begin{aligned}
\|\,\Gamma^{\mathrm{dis}}(\varphi)\xi_0 - \eta\,\|_{\mathcal{E}} &\leq \|\,\Gamma(\Phi)\xi_0 - \Gamma^{\mathrm{dis}}(\varphi)\xi_0\,\|_{\mathcal{E}} + \|\,\Gamma(\Phi)\xi_0 - \eta\,\|_{\mathcal{E}} \\
&\leq \delta(1 + \|\xi_0\|_{\mathcal{E}})\,.
\end{aligned}$$

We finish by taking $\delta < \min\big\{\frac{\varepsilon}{1+\|\xi_0\|_{\mathcal{E}}}, \frac{\varepsilon}{2}\big\}$. $\square$

Proposition 5.2.7. *The discretization* $\Gamma^{\rm dis}$ *is an algebraically irreducible representation.*

Proof. We need to show that every non-null vector ξ_0 is cyclic. This can be restated as the surjectivity of the operator

$$T_{\xi_0}:\ell^1\big(\mathsf{G}\,|\,\mathscr{C}\big)\to\mathcal{E},\quad T_{\xi_0}(\varphi):=\Gamma^{\rm dis}(\varphi)\xi_0\,,$$

which is a consequence of Lemmas 5.2.5 and 5.2.6. □

5.3 Proof of Theorem 5.1.2

Definition 5.3.1. Let $\Gamma:\mathfrak{B}\to\mathbb{B}(\mathcal{E})$ be a representation of the Banach *-algebra $\mathfrak{B}$ in the Banach space $\mathcal{E}$. The representation is called *preunitary* if there exists a Hilbert space $\mathcal{H}$, a topologically irreducible *-representation $\Pi:\mathfrak{B}\to\mathbb{B}(\mathcal{H})$ and an injective linear and bounded operator $V:\mathcal{E}\to\mathcal{H}$ such that

$$V\Gamma(\phi)=\Pi(\phi)V,\quad\forall\,\phi\in\mathfrak{B}\,.$$

Our interest in this notion lies in the next characterization, taken from [41]:

Lemma 5.3.2. *The Banach* **-algebra* $\mathfrak{B}$ *is symmetric if and only if all its non-trivial algebraically irreducible representations are preunitary.*

Let us now prove Theorem 5.1.2.

Proof. We start with a non-trivial algebraically irreducible representation $\Gamma:L^1(\mathsf{G}\,|\,\mathscr{C})\to\mathbb{B}(\mathcal{E})$ and, as above, we denote by $\Gamma^{\rm dis}$ its discretization. By Proposition 5.2.7, it is also algebraically irreducible. Since $\ell^1\big(\mathsf{G}\,|\,\mathscr{C}\big)$ is symmetric, Lemma 5.3.2 tells us that $\Gamma^{\rm dis}$ is preunitary, hence there exists a topologically irreducible *-representation $\Pi^{\rm dis}:\ell^1\big(\mathsf{G}\,|\,\mathscr{C}\big)\to\mathbb{B}(\mathcal{H})$ and an injective bounded linear operator $V:\mathcal{E}\to\mathcal{H}$ such that

$$V\,\Gamma^{\rm dis}(\varphi)=\Pi^{\rm dis}(\varphi)V,\quad\forall\,\varphi\in\ell^1\big(\mathsf{G}\,|\,\mathscr{C}\big)\,. \tag{5.3.1}$$

The representation $\Pi^{\rm dis}$ induces a representation $\pi^{\rm dis}$ of the Fell bundle $\mathscr{C}^{\rm dis}$ in the same Hilbert space $\mathcal{H}$, which also satisfies $\pi^{\rm dis}(a^*)=\pi^{\rm dis}(a)^*$ for every $a\in\mathscr{C}$. Similarly to Remark 5.2.4, it is enough to set $\pi^{\rm dis}:=\Pi^{\rm dis}\circ\mu$, where μ has been defined in (5.2.3); one easily checks that

$$\Pi^{\rm dis}(\varphi)=\sum_{y\in G}\pi^{\rm dis}\big[\varphi(y)\big]\,,\quad\forall\,\varphi\in\ell^1\big(\mathsf{G}\,|\,\mathscr{C}\big)\,.$$

For each $a\in\mathscr{C}$, using (5.3.1) and Remark 5.2.4, we obtain

$$\pi^{\rm dis}(a)V=\Pi^{\rm dis}\big[\mu(a)\big]V=V\Gamma^{\rm dis}\big[\mu(a)\big]=V\gamma^{\rm dis}(a)\,. \tag{5.3.2}$$

Let us define the *-representation $\Pi:L^1(\mathsf{G}\,|\,\mathscr{C})\to\mathbb{B}(\mathcal{H})$ by

$$\Pi(\Phi)\xi:=\int_{\mathsf{G}}\pi^{\rm dis}\big(\Phi(x)\big)\xi\,{\rm d}x\,. \tag{5.3.3}$$

Recall that $\gamma^{\rm dis}=\gamma$ (see Remark 5.2.2). We have

$$V\Gamma(\Phi)\xi \overset{(5.2.1)}{=} V\int_{\mathsf G}\gamma\big(\Phi(x)\big)\xi\,{\rm d}x \overset{(5.3.2)}{=} \int_{\mathsf G}\pi^{\rm dis}\big(\Phi(x)\big)V\xi\,{\rm d}x \overset{(5.3.3)}{=} \Pi(\Phi)V\xi.$$

This means that $\Pi(\Phi)V=V\Gamma(\Phi)$ for every $\Phi\in L^1(\mathsf G\,|\,\mathscr C)$. So the representation Γ is preunitary, hence $L^1(\mathsf G\,|\,\mathscr C)$ is symmetric. ☐

5.4 Twisted partial group actions and crossed products

Let $\Theta:=(\mathsf G,\theta,\omega,\mathcal A)$ be *a continuous twisted partial action* of the locally compact group $\mathsf G$ on the C^*-algebra $\mathcal A$. We refer to [23, 1] for an exposure of the general theory. In any case, the action is implemented by isomorphisms between ideals

$$\theta_x:\mathcal A_{x^{-1}}\to\mathcal A_x\,,\quad x\in\mathsf G$$

and unitary multipliers

$$\omega(y,z)\in\mathbb{UM}\big(\mathcal A_y\cap\mathcal A_{yz}\big)\,,\quad y,z\in\mathsf G$$

satisfying suitable algebraic and topological axioms. (See [23, Def. 2.1])

In [23], Exel associates to $(\mathsf G,\theta,\omega,\mathcal A)$ *the twisted partial semidirect product Fell bundle* in the following way: The total space is

$$\mathscr C(\Theta):=\big\{(a,x)\,\big|\,a\in\mathcal A_x\big\}$$

with the topology inherited from $\mathcal A\times\mathsf G$ and the obvious bundle projection $p(a,x):=x$. One has

$$\mathfrak C_x(\Theta)=\mathcal A_x\times\{x\}\,,\quad\forall\,x\in\mathsf G\,,\tag{5.4.1}$$

with the Banach space structure transported from $\mathcal A_x$ in the obvious way. The algebraic laws of the bundle are

$$(a,x)\bullet_\Theta(b,y):=\big(\theta_x\big[\theta_x^{-1}(a)b\big]\omega(x,y),xy\big)\,,\quad\forall\,x,y\in\mathsf G\,,\,a\in\mathcal A_x\,,\,b\in\mathcal A_y\,,\tag{5.4.2}$$

$$(a,x)^{\bullet_\Theta}:=\big(\theta_x^{-1}(a^*)\omega(x^{-1},x)^*,x^{-1}\big)\,,\quad\forall\,x\in\mathsf G\,,\,a\in\mathcal A_x\,.\tag{5.4.3}$$

Remark 5.4.1. In [23] it is shown that twisted partial semidirect product Fell bundles are very general: every separable Fell bundle with stable unit fiber $\mathfrak C_1$ is of such a type.

For a (continuous) section Φ and for any $x\in\mathsf G$ one has $\Phi(x)=\big(\tilde\Phi(x),x\big)$, with $\tilde\Phi(x)\in\mathcal A_x\subset\mathcal A$. This allows identifying Φ with a function $\tilde\Phi:\mathsf G\to\mathcal A$ such that $\tilde\Phi(x)\in\mathcal A_x\subset\mathcal A$ for each $x\in\mathsf G$. By this identification $L^1\big(\mathsf G\,|\,\mathscr C(\Theta)\big)$ can be seen as $L^1_\Theta\big(\mathsf G,\mathcal A\big)$, the completion of

$$C_{\rm c}\big(\mathsf G,\{\mathcal A_x\}_x\big):=\big\{\tilde\Phi\in C_{\rm c}(\mathsf G,\mathcal A)\,\big|\,\tilde\Phi(x)\in\mathcal A_x\,,\forall\,x\in\mathsf G\big\}$$

in the norm

$$\|\,\tilde\Phi\,\|_{L^1_\Theta(\mathsf G,\mathcal A)}:=\int_{\mathsf G}\|\,\tilde\Phi(x)\,\|_{\mathcal A}\,dx\equiv\|\,\tilde\Phi\,\|_{L^1(\mathsf G,\mathcal A)}\,,$$

so it sits as a closed (Banach) subspace of $L^1(\mathsf{G},\mathcal{A})$. The substantial difference is the semidirect product Fell bundle algebraic structure of $C_{\rm c}\big(\mathsf{G},\{\mathcal{A}_x\}_x\big)$, consequence of (5.1.1), (5.1.2), (5.4.2) and (5.4.3):

$$\big(\tilde\Phi *_\Theta \tilde\Psi\big)(x)=\int_{\mathsf{G}}\theta_y\big[\theta_y^{-1}\big(\tilde\Phi(y)\big)\tilde\Psi(y^{-1}x)\big]\omega\big(y,y^{-1}x\big){\rm d}y$$

and

$$\tilde\Phi^{*_\Theta}(x)=\Delta(x^{-1})\theta_x\big[\tilde\Phi(x^{-1})^*\big]\omega(x^{-1},x)\,,$$

which extends to $L^1_\Theta(\mathsf{G},\mathcal{A})$. The C^*-envelope of $L^1_\Theta(\mathsf{G},\mathcal{A})$ is $\mathcal{A}\rtimes_\Theta\mathsf{G}:=C^*\big(\mathsf{G}\,|\,\mathscr{C}(\Theta)\big)$ and it is called *the (partial, twisted) crossed product of* G *and* $\mathcal{A}$. The translation of Theorems 5.1.4 and 5.1.5 to this setting implies the following corollary.

Corollary 5.4.2. *Let* $\Theta:=(\mathsf{G},\theta,\omega,\mathcal{A})$ *be a continuous twisted partial action of the locally compact group* G *for which ether the discrete group* $\mathsf{G}^{\rm dis}$ *is rigidly symmetric, or it is symmetric and* $\mathcal{A}\rtimes_\Theta\mathsf{G}^{\rm dis}$ *is type* I *. Then* $L^1_\Theta(\mathsf{G},\mathcal{A})$ *is a symmetric Banach* **-algebra.*

Remark 5.4.3. A (very) particular case it the one of *global* twisted actions, basically characterized by $\mathcal{A}_x=\mathcal{A}$ for every $x\in\mathsf{G}$. Even more particularly, one may take $\mathcal{A}=\mathbb{C}$, with the trivial action, and then $\omega:\mathsf{G}\times\mathsf{G}\to\mathbb{T}$ is *a multiplier*. In [4] Austad shows for such a case similar results, but assuming that the extension G_ω of $\mathbb{T}$ by G associated to ω is symmetric. These are good assumptions, since rigid symmetry or discretization are not required. On the other hand, the extension could be much more complicated than the group itself.

Chapter 6

Conclusions and future work

The study of partial symmetries is notably more complicated than the global case. Nevertheless, its new found generality allows us to greatly extend a huge variety of results. This is particularly important for the program of large endeavours such as the classification of C^*-algebras, to which this subject has contributed largely in the recent years.

With respect to topological dynamics, the results obtained seem to vary a lot. Some of the classic notions got dissected into various pieces, other were trivialized outside the group setting. This leads to very interesting phenomena. The study of algebraic morphisms is notably more technical than the study of the naive morphisms. Open groupoids seem to behave just as groups.

On the other hand, the results obtained in Chapter 5 largely generalize the ones available in the actual literature, which reduce to global actions, basically. So we have proven the symmetry of a large, new, class of algebras associated to the groups involved. This enlights the spectral theory of these algebras and hopefully it will help to understand them better.

Finally, we propose a couple of lines of future work related to each of the treated topics:

1. An expansion on the study of topological dynamics of groupoid actions; Many topics were left aside, for instance, one could try to develop notions of *equicontinuity*, *distality*, *disjointness* or an analog of *Furstenberg's structure theorem*.
2. The class of groupoid dynamical systems with the algebraic morphisms form a category. A future line of work could be the study of this category. My particular interests involve *existence of products*, *coproducts*, *equalizers*, *completeness*, etc.
3. To prove or disprove that every symmetric group is rigidly symmetric. My personal belief is that a counterexample exists, but I do not have one.
4. To prove or disprove that every rigidly symmetric group is hypersymmetric. This seems hard, the technique used for discrete groups no longer works and this failure seems to enlighten a possible negative answer.

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