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STABILITY AND APPLICATIONS OF MULTI-ORDER FRACTIONAL SYSTEMS

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Abstract. This paper establishes conditions for global/local robust asymptotic stability for a class of multi-order nonlinear fractional systems consisting of a linear part plus a global/local Lipschitz nonlinear term. The derivation order can be different in each coordinate and take values in $(0, 2)$. As a consequence, a linearized stability theorem for multi-order systems is also obtained. The stability conditions are order-dependent, reducing the conservatism of order-independent ones. Detailed examples in robust control and population dynamics show the applicability of our results. Simulations are attached, showing the distinctive features that justify multi-order modelling.

1. **Introduction.** A multi-order (also called mixed-order [11], or incommensurate) fractional system is a set of fractional differential equations (i.e. involving fractional derivatives) where each equation is allowed to have its own differential order. Since fractional systems have been mainly used to get empirical models of complex processes [1], the importance of the qualitative study of multi-order fractional systems becomes evident as they represent the more general structure. Moreover, in applications of fractional operators to classic integer-order problems, these systems naturally appear.

Lyapunov stability is one of the main research topics in the qualitative study of nonlinear differential equations due to its capability to characterize asymptotic properties. Although most of the research has been focused on systems where each equation has the same differentiation order, some partial results have been obtained for the multi-order case. The triangular linear case was studied in [8] and BIBO (bounded-input, bounded-output) stability, through Laplace domain arguments, has been studied in [2] and subsequent papers. Attractiveness for a class of local Lipschitz nonlinear systems was studied in [21]. In [15, 21], a comparison lemma and the knowledge of asymptotically stable multi-order fractional systems were used to study multi-order systems.

In this paper, we establish conditions for Lyapunov stability of multi-order nonlinear systems in which the nonlinear part is a Lipschitz function. The proposed

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results establish a kind of small-gain conditions on the non-linearity to ensure stability. These conditions are order-dependent and reduce the conservatism of order-independent ones. By contrast, the Lyapunov-like method yields conditions that do not depend on the derivation order (e.g. [12]). However, in systems defined through fractional-order derivatives, order-independent conditions for stability are conservative, which can be easily seen in linear time-invariant systems. Moreover, serious limitations of the Lyapunov technique have been found in comparison with the integer-derivative counterpart, mainly due to the lack of monotony [20].

Our method relies on a fixed-point technique, devised in [5] to prove the local stability of single-order systems, that we have extended to prove the global and local stability of multi-order ones. Moreover, robustness properties, i.e. the preservation of the convergence or the boundedness under uncertainty, are also provided. By contrast, the Laplace method devised in [19] cannot assert the stability in the Lyapunov sense and contains some gaps when establishing the asymptotic convergence (essentially, the same problems that have been indicated in [20]). Finally, we show, by applications of the main results, how the multi-order approach enhances the representation capabilities by providing qualitatively different behaviour from single-order systems.

The rest of the paper is organized as follows. In Section 2 we provide some definitions and notations. In Section 3 the main results are established. These results are discussed and extended in Section 4, meanwhile, in section 5 we provide illustrations of the main results.

2. **Preliminaries.** This section is devoted to review notation and basic results adapted to the paper context.

We start with some notation and definitions. $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. $\|\cdot\|$ denotes a norm in $\mathbb{R}^n$ and also a compatible matrix norm in $\mathbb{R}^{n\times n}$ so that $\|Ax\| \leq \|A\|\|x\|$ (e.g. the induced norm). $B(x,r) \subset \mathbb{R}^n$ denotes the closed ball around $x \in \mathbb{R}^n$ of radius r. $'$ or $^\top$ denote the transpose operation for vectors in $\mathbb{R}^n$. I_n (or simply I) denotes the identity matrix in $\mathbb{R}^{n\times n}$. $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ are the minimal and maximal eigenvalue functions. $C([0,\infty),\mathbb{R}^n)$ is the space of continuous functions from $[0,\infty)$ to $\mathbb{R}^n$ endowed with the infinite norm $\|f\|_\infty = \sup_{t\geq 0}\|f(t)\|$.

Definition 2.1. The two parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad z \in \mathbb{C}.$$

The convention $E_\alpha(z) := E_{\alpha,\alpha}(z)$ is used.

Definition 2.2. For a function $x : [a,b] \to \mathbb{R}$, the Caputo derivative operator of order α is defined by

$$D^{\alpha}_{0^+}x(t) = \frac{1}{\Gamma(\alpha-m)}\int_0^t (t-s)^{\alpha+1-m}x^{(m)}(s)ds, \quad t \in [a,b], \tag{1}$$

where Γ is the gamma function and m is an integer satisfying $m-1 < \alpha \leq m$.

Our main results concern to nonlinear systems with Lipschitz nonlinearities. So, we recall this concept:

Definition 2.3. A local Lipschitz continuous function $g : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$, uniformly in its second argument, is a function satisfying for any $t > 0$

$$\|g(x,t) - g(\hat{x},t)\| \le L(r)\|x - \hat{x}\|, \quad \forall x, \hat{x} \in \{x \in \mathbb{R}^n : \|x\| < r\} \tag{2}$$

where $L(r) > 0$. A global Lipschitz continuous function $g : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ in the first argument with Lipschitz constant L uniformly in t is a local Lipschitz function such that $L(r)$ is a constant.

2.1. **Laplace domain.** For a function $f : [0, \infty) \to \mathbb{R}^{n\times m}$, $\hat{f}(s) := \mathcal{L}[f(t)](s)$ is its Laplace transform, which is computed component-wise so that its component $\hat{f}_{ij}$ corresponds to the Laplace transform –in its usual definition [16]– of the scalar component f_{ij} of f for each $1 \le i \le n, 1 \le j \le m$ and $s \in \mathbb{C}$. Similarly, let $\mathcal{L}^{-1}[\hat{f}(s)](t)$ be the Laplace anti-transform, also defined component-wise. A pole of a matrix function $\hat{f}(z)$ is a complex number z such that it is a pole of some scalar component $\hat{f}_{ij}$ of $\hat{f}$.

We will need the following version of the Final Value Theorem.

Theorem 2.4. *Let $y : [0, \infty) \to \mathbb{R}^{n\times m}$ be a function such that y and $\frac{dy}{dt}$ have Laplace transforms, with $\frac{dy}{dt}$ piecewise continuous. Assume that every pole of $s\hat{y}(s)$ lies in the open left-hand side of the complex plane. Then, $\lim_{t\to\infty} y(t)$ exists and is given by*

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s), \tag{3}$$

where the limit is taken component-wise, i.e. $\lim_{t\to\infty} y_{ij}(t) = \lim_{s\to 0} s\hat{y}_{ij}(s)$ for each $1 \le i \le n, 1 \le j \le m$.

Proof. Let $\hat{x}_{ij} := s\hat{y}_{ij}$ any component of $s\hat{y}$. From the hypothesis, $\hat{x}$ has all its poles in the open left-hand side of the complex plane. Then, x_{ij} is dominated by an exponential (see [4, p. 98 at the bottom]), and in particular, it is absolutely integrable. Since $\hat{y}_{ij} = s^{-1}\hat{x}_{ij}$, we have that $y_{ij}(t) = \int_0^t x_{ij}(\tau)d\tau$. Since $\int_0^t x_{ij}(\tau)d\tau = \int_0^t x_{ij}^+(\tau)d\tau - \int_0^t x_{ij}^-(\tau)d\tau$, where $x_{ij}^+(t) = \max(x(t), 0)$ and $x_{ij}^-(t) = \max(-x(t), 0)$, and x is absolutely integrable, we have $\lim_{t\to\infty} y_{ij}(t)$ exists and is denoted by $y_{ij}(\infty)$. Then, $\lim_{s\to 0} \mathcal{L}[\frac{d}{dt}y_{ij}(t)](s) = \lim_{s\to 0} \int_0^\infty \frac{d}{dt}y_{ij}(t)e^{-st}dt = \int_0^\infty \frac{d}{dt}y_{ij}(t)ds = y_{ij}(\infty) - y_{ij}(0)$, where we pass the limit inside the integral in the second equality due to the uniform convergence [16, p. 315]. On the other hand, using a classical property, $\mathcal{L}[\frac{d}{dt}y_{ij}(t)](s) = s\hat{y}_{ij}(s) - y_{ij}(0)$, which implies $\lim_{t\to\infty} y_{ij}(t) = \lim_{s\to 0} s\hat{y}_{ij}(s)$. □

Notice that the limit on the right-hand side of (3) is implicitly assumed to exist.

2.2. **Operator norms.** For a function $f : (0, \infty) \to \mathbb{R}^n$, its Lebesgue $p-$norm is defined by $\|f\|_p^p := \int_0^\infty \|f(t)\|^p dt$; L^p is the set of functions with finite $p-$norm. For an operator G on L^p, its induced $p-$norm is given by $\|G\|_{L^p} := \sup_{u\neq 0, u\in L^p} \frac{\|Gu\|_p}{\|u\|_p}$.

Recall that the convolution $y = h * u$ may represent the zero state response of a linear system of input u and output y. Such linear systems are often specified by the Laplace transform H of h because in this domain one has $\hat{y} = H\hat{u}$. Thus, the notation $\|H(s)\|_1$ means $\|h\|_1$.

Since $\|H(s)\|_1 = \|H\|_{L^\infty}$ when H is a linear operator (see e.g. [7, p.23]), a linear system $H(s)$ is BIBO stable (i.e. bounded input u implies bounded output y) if and only if $\|H(s)\|_1 < \infty$. The following result establishes that the boundedness of $\|H(s)\|_1$ is determined by the sign of the real part of the poles of $H(s)$.

Lemma 2.5. *Let $H(s)$ be a complex matrix function. Then $\|H(s)\|_1 < \infty$ implies $H(s)$ has no poles in $\{Re\ s \geq 0\}$. Conversely, suppose that $H(s)$ has no poles in $\{Re\ s \geq 0\}$ then $\|H(s)\|_1 < \infty$ when each component of $H(s)$ has the form $p(s)/q(s)$, where $p(s) = \sum_{k=1}^{m} b_k s^{\beta_k}$, $q(s) = \sum_{k=1}^{n} a_k s^{\alpha_k}$ and $\max_{k=1,\ldots,m}\{\beta_k\} < \max_{k=1,\ldots,n}\{\alpha_k\}$.*

Proof. Since $\|H(s)\|_1 < \infty$, each component of H satisfies $\|H_{ij}(s)\|_1 < \infty$ for $1 \leq i \leq n, 1 \leq j \leq m$. This means that $\int_0^\infty |h_{ij}| d\tau < \infty$. By the definition of the Laplace transform, it follows that $H_{ij}(s) = \int_{t=0}^{\infty} h_{ij}(t) e^{-st} dt$ is bounded for any $Re\ s \geq 0$, implying that $H_{ij}(s)$ and, hence, H have no pole on $\{Re\ s \geq 0\}$. On the other hand, when $H(s)$ has no poles in $\{Res \geq 0\}$, then each component has no poles in $\{Re\ s \geq 0\}$. Applying [2, Theorem 3.1] (considered without delayed terms), it follows that each component satisfies $\|H_{ij}(s)\|_1 < \infty$ and hence $\|H(s)\|_1 < \infty$. □

3. **Order-dependent conditions for stability.** In this section, we study the stability problem for the class of multi-order systems described by the following set of equations

$$D_{0^+}^{\alpha_i} x_i(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + g_i(x_1, \ldots, x_n, t), \quad i = 1, \ldots, n, \tag{4}$$

which will be compactly written as

$$D_{0^+}^{\hat{\alpha}} x(t) = Ax(t) + g(x, t), \tag{5}$$

where $D^{\hat{\alpha}} x(t)$ is a short notation for the vector of components $D^{\alpha_i} x_i(t)$ with $\alpha_i \in (0, 2)$ for $i = 1, \ldots, n$. The vectors $x : [0, \infty) \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$, and the matrix $A \in \mathbb{R}^{n \times n}$ are similarly defined. The results of this paper are stated for the continuous solutions to (5) (see [11] for sufficient conditions on g).

To work with a compact notation, we denote $s^{\hat{\alpha}} I$ the diagonal matrix whose entries are s^{α_i}, i.e.

$$s^{\hat{\alpha}} I := diag(s^{\alpha_1}, \ldots, s^{\alpha_n}). \tag{6}$$

The following result is obtained by generalizing to the multi-order case the method proposed in [5], which is concerned with the stability analysis for the single-order (commensurate) instance of (4).

Theorem 3.1. *Let g be a global Lipschitz function in its first argument, uniformly in its second argument, with Lipschitz constant L and such that $g(0, t) \equiv 0$. If*

$$L < \frac{1}{\|(s^{\hat{\alpha}} I - A)^{-1}\|_1} < \infty, \tag{7}$$

then $x = 0$ is a globally asymptotically stable point for system (4).

Proof. Consider first the case $0 < \alpha_i \leq 1$ for any $i = 1, \ldots, n$. For any $x_0 \in \mathbb{R}^n$, define the operator $\mathcal{T}_{x_0}$ on $C([0, \infty), \mathbb{R}^n)$ by

$$\mathcal{T}_{x_0}(\xi)(t) := \psi_{x_0}(t, \hat{\alpha}, A) + \int_0^t \Psi_E((t - \tau), \hat{\alpha}, A) g(\xi(\tau), \tau) d\tau. \tag{8}$$

where $\psi_{x_0}(t, \hat{\alpha}, A) := \mathcal{L}^{-1}((s^{\hat{\alpha}} I - A)^{-1} s^{\hat{\alpha}-1} I x_0)(t) =: \psi(t, \hat{\alpha}, A) x_0$, $\Psi_E(t, \hat{\alpha}, A) = \mathcal{L}^{-1}((s^{\hat{\alpha}} I - A)^{-1})(t)$, and $s^{\hat{\alpha}-1} I := diag(s^{\alpha_1 - 1}, \ldots, s^{\alpha_n - 1})$

The form of $\mathcal{T}_{x_0}$ is taken from what can be called the variation of constants formula for (4) which is obtained by solving (4) (with initial condition $x(0) = x_0$)

in the Laplace domain and then applying the inverse transform. This procedure yields[1]

$$x(t) = \psi_{x_0}(t, \hat{\alpha}, A) + \int_0^t \Psi_E((t-\tau), \hat{\alpha}, A) g(x(\tau), \tau) d\tau. \tag{9}$$

This solution is unique in $C([0,\infty), \mathbb{R}^n)$ because the existence of another continuous solution y for the same initial condition implies that the difference $z = x - y$ has a Laplace transform $\hat{z} \equiv 0$, and its inverse Laplace transform gives $z(t) = 0$ almost everywhere, implying $z(t) \equiv 0$ when $x, y \in C([0,\infty), \mathbb{R}^n)$. Comparing (8) with (9), it follows that any fixed point of $\mathcal{T}_{x_0}$ on $C([0,\infty), \mathbb{R}^n)$ only can be equals to the unique continuous solution to (4) satisfying $x(0) = x_0$. Thus, we can prove the theorem by studying the properties of $\mathcal{T}_{x_0}$ and its fixed point.

We first note that $\mathcal{T}_{x_0}(\xi)$ defines, for each fixed ξ, a system that is better appreciated by making explicit the dependence on $u(t) := g(\xi(t), t)$, which plays the role of input, namely $\mathcal{T}_{x_0,u}(\xi)$. One then can ask if this system is BIBO stable, which concerns to the operator $\int_0^t \Psi_E((t-\tau), \hat{\alpha}, A) g(\xi(\tau), \tau) d\tau$ since the BIBO stability is studied for null initial condition ($x_0 = 0$, in our case). By the identification $u(t) = g(\xi(t), t)$, one realizes that this is exactly as ask the BIBO stability of the linear system $D_{0+}^{\hat{\alpha}} x(t) = Ax(t) + u(t)$ for which the results of Section 2.2 can be applied, after applying the Laplace transform. In particular, condition (7) implies that in fact this system is BIBO stable. By the first claim of Lemma 2.5, this implies that $(s^{\hat{\alpha}} I - A)^{-1}$ has no poles in $\{s \in \mathbb{C} : Re(s) \geq 0\}$. Then, we can apply[2] Theorem 2.4 to obtain that ψ_{x_0}, whose Laplace transform is $(s^{\hat{\alpha}} I - A)^{-1} s^{\hat{\alpha}-1} I x_0$ and hence it satisfies the hypotheses of Theorem 2.4, converges to zero. Being the solution to (4) with $g \equiv 0$, ψ_{x_0} is continuous[3] [11], and hence $\|\psi_{x_0}\|_\infty < \infty$. Since $\|(s^{\hat{\alpha}} I - A)^{-1}\|_1 < \infty$ and g is Lipschitz uniformly on t, the second term in the right hand of (9) is continuous when evaluated on $C([0,\infty), \mathbb{R}^n)$. Then, $\mathcal{T}_{x_0}$ maps $C([0,\infty), \mathbb{R}^n)$ to $C([0,\infty), \mathbb{R}^n)$.

From the Lipschitz continuity of g and using (7), we obtain $\|\mathcal{T}_{x_0}\xi - \mathcal{T}_{x_0}\hat{\xi}\|_\infty \leq q\|\xi - \hat{\xi}\|_\infty$ for any $\|\xi\|_\infty, \|\hat{\xi}\|_\infty < \infty$, where $q := L\|(s^{\hat{\alpha}} I - A)^{-1}\|_1 < 1$. Since $g(0,t) \equiv 0$, we also have

$$\|\mathcal{T}_{x_0}\xi\|_\infty \leq \|\psi_{x_0}\|_\infty + q\epsilon. \tag{10}$$

for any $\epsilon > 0$ and any $\|\xi\|_\infty < \epsilon$. Since $\|\psi_{x_0}\|_\infty < \infty$, it follows $\|\mathcal{T}_{x_0}\xi\|_\infty < \infty$, and hence $\mathcal{T}_{x_0}$ is a contractive map in the Banach subspace B_∞ of $C([0,\infty), \mathbb{R}^n)$ defined by all the bounded functions for the $\|\cdot\|_\infty$ norm. Using the Contraction Mapping Principle, $\mathcal{T}_{x_0}$ has a unique fixed point $\xi_0 \in B_\infty \subset C([0,\infty), \mathbb{R}^)$, which is then a bounded function. From the reasoning in the third paragraph of this proof, the continuous solution x of (4) with $x(0) = x_0$ is also a fixed point of $\mathcal{T}_{x_0}$. From

[1]Notice that each component of the vector $(s^{\hat{\alpha}} I - A)^{-1}\hat{g}$ is a sum of convolutions in the time domain, which can be regrouped to form the integral in (9).

[2]Using the Lipschitz assumption, each x_i is dominated by an exponential as $t \to \infty$ implying that the Laplace transform of x exists. This is due to the fact that one can find a linear bound to the nonlinear equation by employing the Lipschitz inequality. Then, one can bound the solution by choosing the largest $\alpha_i \leq 1$, which yields the solutions that can grow faster asymptotically, obtaining a Mittag-Leffler upper bounding function dominated by an exponential. Since $\dot{x}$ satisfies a similar equation as x but with the addition of a term proportional to $t^{\alpha_i - 1}$ for each i when x continuous (see [11, eqn. (3.5)]), the same holds for $\dot{x}$.

[3]More rigorously, this can be seen from the fact that each component of ψ_{x_0} is the sum of fractional integrals of integrable functions according to the anti-transform of $(s^{\hat{\alpha}} I - A)^{-1} s^{\hat{\alpha}-1} I x_0$, whereby they are continuous

the uniqueness of the fixed point in $C([0,\infty),\mathbb{R}^n)$, we conclude that $x\equiv\xi_0$, and hence, the continuous solutions of (5) are bounded functions.

Pick any $\epsilon>0$ and choose $x_0\in\mathbb{R}^n$ such that $\|x_0\|<\delta:=\epsilon(1-q)/\|\psi(t,A)\|_\infty$. From (10), it follows that $\mathcal{T}_{x_0}$ maps the ball $B(0,\epsilon)$ of $C([0,\infty),\mathbb{R}^n)$ to $B(0,\epsilon)$. Applying the Contraction Mapping Principle again, it follows that the fixed point of $\mathcal{T}_{x_0}$ must be contained in $B(0,\epsilon)$. On the other hand, we know that the unique continuous solution x such that $x(0)=x_0$ is a fixed point of $\mathcal{T}_{x_0}$. Therefore, $\|x\|_\infty<\epsilon$, and the Lyapunov stability of $x=0$ follows.

Now pick any $x_0\in\mathbb{R}^n$. Using in the following order (9) the triangular inequality, properties of $\limsup$, the fact that $\|\psi_{x_0}(t,\hat\alpha,A)\|$ converges to zero as $t\to\infty$, and the Lipschitz assumption on g together with the fact that $g(0,t)\equiv 0$ (which implies that $\|g(x,t)\|\le L\|x(t)\|$), we have respectively

$$\begin{aligned}
&\limsup_{t\to\infty}\|x(t)\|\\
&\le \limsup_{t\to\infty}\big(\|\psi_{x_0}(t,\hat\alpha,A)\|+\|\int_0^t \Psi_E((t-\tau),\hat\alpha,A)g(x(\tau),\tau)d\tau\|\big),\\
&\le \limsup_{t\to\infty}\|\psi_{x_0}(t,\hat\alpha,A)\|+\limsup_{t\to\infty}\|\int_0^t \Psi_E((t-\tau),\hat\alpha,A)g(x(\tau),\tau)d\tau\|,\\
&= \limsup_{t\to\infty}\|\int_0^t \Psi_E((t-\tau),\hat\alpha,A)g(x(\tau),\tau)d\tau\|\\
&\le L\limsup_{t\to\infty}\int_0^t \|\Psi_E((t-\tau),\hat\alpha,A)\|\|x(\tau)\|d\tau.
\end{aligned}$$

From the already established boundedness of the solutions to equation (5), we have $\limsup_{t\to\infty}\|x(t)\|<\infty$. Then for any $\epsilon>0$ there exists $T(\epsilon)$ such that $\|x(t)\|\le \limsup_{t\to\infty}\|x(t)\|+\epsilon$ for any $t>T(\epsilon)$. Hence,

$$\begin{aligned}
\limsup_{t\to\infty}\|x(t)\|\le L(\limsup_{t\to\infty}\|x(t)\|+\epsilon)\limsup_{t\to\infty}\Big(&\int_{T(\epsilon)}^t\|\Psi_E((t-\tau),\hat\alpha,A)\|d\tau\\
&+\|x\|_\infty\int_{t-T(\epsilon)}^t\|\Psi_E((t-\tau),\hat\alpha,A)\|d\tau\Big).
\end{aligned}$$

Using (7), which implies, in particular, the integrability of Ψ_E, the integral term in the second line above converges to zero and the first integral is bounded by $\|(s^{\hat\alpha}I-A)^{-1}\|_1$. This implies $\limsup_{t\to\infty}\|x(t)\|\le L\|(s^{\hat\alpha}I-A)^{-1}\|_1(\limsup_{t\to\infty}\|x(t)\|+\epsilon)$. Sending $\epsilon\to 0^+$, we obtain

$$\limsup_{t\to\infty}\|x(t)\|\le L\|(s^{\hat\alpha}I-A)^{-1}\|_1\limsup_{t\to\infty}\|x(t)\|.$$

Using (7), $L\|(s^{\hat\alpha}I-A)^{-1}\|_1<1$ and the above inequality only makes sense if $\limsup_{t\to\infty}\|x(t)\|=0$. Therefore, $0\le\lim_{t\to\infty}\|x(t)\|\le\limsup_{t\to\infty}\|x(t)\|=0$, completing the proof of the asymptotic stability of $x=0$.

Consider now the case where some derivation orders satisfy $1\le\alpha_i<2$. Without loss of generality, suppose that $\alpha_i>1$ for $i=1,\dots,m$ with $m\le n$, and $\alpha_i\le 1$ otherwise. Instead of (8), we define

$$\mathcal{T}_{x_0,\dot x_0}(\xi)(t):=\psi_{x_0,\dot x_0}(t,\hat\alpha,A)+\int_0^t\Psi_E((t-\tau),\hat\alpha,A)g(\xi(\tau),\tau)d\tau,$$

where

$$\psi_{x_0,\dot{x}_0}(t,\hat{\alpha},A) = \mathcal{L}^{-1}\{(s^{\hat{\alpha}}I - A)^{-1}s^{\hat{\alpha}-1}Ix_0 \\ + (s^{\hat{\alpha}}I - A)^{-1}(s^{\alpha_1-2}\dot{x}_{01},\ldots,s^{\alpha_m-2}\dot{x}_{0m},0,\ldots,0)'\}(t).$$

That is, the only difference with the precedent case occurs every time that the function $\psi_{x_0,\dot{x}_0}(t,\hat{\alpha},A)$, associated to the initial condition, is used. We have $\lim_{s\to 0}(s^{\hat{\alpha}}I - A)^{-1}s^{\hat{\alpha}}Ix_0 = 0$ and $\lim_{s\to 0}(s^{\hat{\alpha}}I - A)^{-1}s(s^{\alpha_1-2}\dot{x}_{01},\ldots,s^{\alpha_m-2}\dot{x}_{0m},0,\ldots,0)' = 0$, as in the latter case $\alpha_i - 2 \in (0,-1)$ for $i = 1,\ldots,m$. Since Lemma 2.5 and the continuity results in [11] hold for arbitrary orders in $(0,2)$, the boundedness claim on $\psi_{x_0,\dot{x}_0}(t,\hat{\alpha},A)$ can be obtained as above. The explicit form of the initial condition term is used to assert the stability. To reproduce the arguments above, we must consider the augmented vector $(x_0',\dot{x}_0')$ and define the diagonal matrix $\psi(t,A)$ of elements, written in the s-domain, $(s^{\hat{\alpha}}I - A)^{-1}s^{\hat{\alpha}-1}I$ and $(s^{\hat{\alpha}}I - A)^{-1}diag(s^{\alpha_1-2},\ldots,s^{\alpha_m-2},0,\ldots,0)$. □

Remark 3.2. Despite its concision, condition (7) looks hard to compute since it involves the L^1-norm of the Laplace inverse of $(s^{\hat{\alpha}}I - A)^{-1}$. In the next section, we will see a more practical condition, which can be verified by simple calculations.

Remark 3.3. Seen g as a disturbance of the linear part, Theorem 3.1 ensures the preservation of the asymptotic stability under Lipschitz disturbances whenever the order-dependent condition (7) is fulfilled.

The proof of Theorem 3.1 relies on the Lipschitz characteristic of the nonlinearity. According to Definition 2.3, similar reasoning can be developed when the nonlinearity is locally Lipschitz. Indeed, the next result shows that the (local) asymptotic stability can be established without the condition (7).

Theorem 3.4. *Consider system* (4) *with g a local Lipschitz function in the first argument, uniformly in t, such that $\lim_{r\to 0} L(r) = 0$ and with $g(0,t) \equiv 0$. If $(s^{\hat{\alpha}}I - A)^{-1}$ has no poles in $\{s \in \mathbb{C} : Re(s) \geq 0\}$, then the origin of* (4)*, $x = 0$, is asymptotically stable.*

Proof. Since $(s^{\hat{\alpha}}I - A)^{-1}$ has no poles in $\{s \in \mathbb{C} : Re(s) \geq 0\}$, we use Lemma 2.5 to obtain $\|(s^{\hat{\alpha}}I - A)^{-1}\|_1 < \infty$. From the hypotheses, we have that $L(r) \to 0$ as $r \to 0^+$, and hence, there exists a small enough r_0 such that $\|(s^{\hat{\alpha}}I - A)^{-1}\|_1 < 1/L(r)$ for any $r < r_0$. The rest of the proof follows along the same lines as the proof of Theorem 3.1, by restricting to the Banach subspace of $C([0,\infty),\mathbb{R}^n)$ such that for any ξ in that subset, we have $\|\xi\|_\infty \leq r_0 - \epsilon$, where $\epsilon > 0$ is a small enough constant independent of ξ. □

Remark 3.5. In the linear case, i.e. system (4) with $g \equiv 0$, the asymptotic stability is global. In [6], authors obtain the same condition of Theorem 3.4 to assert just the attractiveness of $x = 0$ for this particular case, while Theorem 3.4 asserts the Lyapunov stability of $x = 0$.

Remark 3.6. In [21, Theorem 5], the attractiveness of $x = 0$ for system (5) with rational multi-orders belonging to $(0,1)$ is asserted. In addition to the local Lipschitz of g and the pole conditions, they impose a nonnegative definiteness condition on A, restricting the result to Hurwitz matrices. Therefore, this result is strictly included as a particular case in Theorem 3.4.

From Theorem 3.4, a generalization of the Lyapunov indirect method can be obtained for the following nonlinear multi-order system

$$D_{0^+}^{\alpha_i} x_i(t) = f_i(x_1, \dots, x_n), \quad i = 1, \dots, n, \tag{11}$$

or, in compact notation,

$$D^{\hat{\alpha}} x(t) = f(x), \tag{12}$$

where $D^{\hat{\alpha}} x(t)$ is the vector of components $D^{\alpha_i} x_i(t)$ for $\alpha_i \in (0,2)$, $i = 1, \dots, n$, $x : [0,\infty) \to \mathbb{R}^n$, and $f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ is a continuously differentiable function with $f(0) \equiv 0$. Although rather a corollary, we state the next result as a theorem due to its applicability.

Theorem 3.7. *Let $A := J_f(x = 0)$ be the Jacobian of the function f evaluated at $x = 0$. If $(s^{\hat{\alpha}} I - A)^{-1}$ has no poles in $\{s \in \mathbb{C} : Re(s) \geq 0\}$, then the origin of system (11), $x = 0$, is asymptotically stable.*

Proof. By making a Taylor expansion, we can write $f(x) = Ax + g(x)$, where $g(x)$ is the Taylor polynomial. Since g is locally Lipschitz with $\lim_{r\to 0} L(r) = 0$, the result follows from the application of Theorem 3.4. □

Remark 3.8. Theorem 3.7, which is a local result and has the intuitive motivation of the linear approximation of a nonlinear system, has been informally used in previous works.

Finally, we study the robustness under a more general disturbance. The next result generalizes to the multi-order case the results for the single-order in [9].

Corollary 3.9. Consider system (5) where $(s^{\hat{\alpha}} I - A)^{-1}$ has no poles in $\{s \in \mathbb{C} : Re(s) \geq 0\}$. Let $g(x,t) = g_0(x,t) + g_1(t)$ with g_0 locally Lipschitz with $\lim_{r\to 0} L(r) = 0$ or Lipschitz of parameter $L < \frac{1}{\|(s^{\hat{\alpha}} I - A)^{-1}\|_1}$, uniformly on its second argument. If g_1 is a bounded function, then the solutions of (5) are bounded, and if, in addition, g_1 converges to zero, then $x = 0$ is locally or globally attractive, respectively.

Proof. For brevity's sake, we only prove the case $\alpha_i \leq 1$ for any i and g_0 being Lipschitz with $L < \frac{1}{\|(s^{\hat{\alpha}} I - A)^{-1}\|_1}$. Firstly, we state the boundedness. Let

$$\begin{aligned} \mathcal{T}_{x_0}(\xi)(t) := \psi_{x_0}(t, \hat{\alpha}, A) &+ \int_0^t \Psi_E((t-\tau), \hat{\alpha}, A) g_0(\xi(\tau), \tau) d\tau \\ &+ \int_0^t \Psi_E((t-\tau), \hat{\alpha}, A) g_1(\tau) d\tau. \end{aligned}$$

From the hypothesis that $(s^{\hat{\alpha}} I - A)^{-1}$ has no poles in $\{s \in \mathbb{C} : Re(s) \geq 0\}$ and using the same arguments of the proof Theorem 3.1, we have that $\Psi_E \in L^1$ and ψ_{x_0} is bounded. Since g_1 is bounded, the convolution of g_1 with an L^1 function is bounded by, say, $C < \infty$. Thus, instead of (10), we have

$$\|\mathcal{T}_{x_0}\xi\|_\infty \leq \|\psi_{x_0}\|_\infty + C + q\epsilon, \tag{13}$$

that is to say, $\mathcal{T}_{x_0}$ maps the subspace B_∞ of $C([0,\infty), \mathbb{R}^n)$ defined by all the bounded functions for the $\|\cdot\|_\infty$ norm to itself. Furthermore, we still have $\|\mathcal{T}_{x_0}\xi - \mathcal{T}_{x_0}\hat{\xi}\|_\infty \leq q\|\xi - \hat{\xi}\|_\infty$ for any $\|\xi\|_\infty, \|\hat{\xi}\|_\infty < \infty$. Therefore, the boundedness follows by noting that the unique fixed point of $\mathcal{T}_{x_0}$ in B_∞ is also the solution to (5) with $g = g_0 + g_1$. Secondly, we prove the convergence when g_1 converges to zero. Since $\Psi_E \in L^1$

and g_1 is bounded and converges to zero, the convolution of them, i.e. the second integral in $\mathcal{T}_{x_0}$, converges to zero. By redefining $\psi_{x_0}(t,\hat{\alpha},A)$ as $\psi_{x_0}(t,\hat{\alpha},A)$ plus this integral, yielding a term that converges to zero, the same arguments of the proof Theorem 3.1 can be carried out to prove the final claim. $\square$

4. **Discussion.** In this section, we discuss the results of Section 3.

4.1. **A more tractable hypothesis.** The condition (7) in Theorem 3.1, i.e. $L < \|(s^{\hat{\alpha}}I - A)^{-1}\|_1^{-1}$, has the advantage of concision, but it is hardly useful to solve practical problems. In the following result, we provide a necessary condition for (7), which is computationally tractable.

Proposition 4.1. Consider $0 < \alpha_i < 2$ for any $i = 1,\dots,n$, $\hat{\alpha}$ the vector of components α_i and the notation in (6). If $(s^{\hat{\alpha}}I - A)^{-1}$ has no poles in the closed right-hand complex plane and A is invertible, then

$$\|(s^{\hat{\alpha}}I - A)^{-1}\|_1 \geq \|A^{-1}\|, \tag{14}$$

and the condition

$$L \leq \|A^{-1}\|^{-1} \tag{15}$$

implies (7).

Proof. Recall that $\|(s^{\hat{\alpha}}I - A)^{-1}\|_1$ is a short notation for $\|\mathcal{L}^{-1}[(s^{\hat{\alpha}}I - A)^{-1}](t)\|_1$. If $(s^{\hat{\alpha}}I - A)^{-1}$ has no poles in the complex right-hand plane, then according to the proof of Theorem 2.4, $\lim_{t\to\infty}\int_0^t \mathcal{L}^{-1}[(s^{\hat{\alpha}}I - A)^{-1}](\tau)d\tau$ as $t \to \infty$ exists and can be obtained from the asymptotic value of $ss^{-1}(s^{\hat{\alpha}}I - A)^{-1} = (s^{\hat{\alpha}}I - A)^{-1}$ as $s \to 0$. Thus, we have

$$\begin{aligned}\|(s^{\hat{\alpha}}I - A)^{-1}\|_1 &= \lim_{t\to\infty}\int_0^t \|\mathcal{L}^{-1}[(s^{\hat{\alpha}}I - A)^{-1}](\tau)\|d\tau \\ &\geq \|\lim_{t\to\infty}\int_0^t \mathcal{L}^{-1}[(s^{\hat{\alpha}}I - A)^{-1}](\tau)d\tau\| \\ &= \|\lim_{s\to 0}(s^{\hat{\alpha}}I - A)^{-1}\| = \|A^{-1}\|,\end{aligned}$$

which proves the first claim. Since $L \leq \|A^{-1}\|^{-1} \leq \|(s^{\hat{\alpha}}I - A)^{-1}\|_1^{-1}$, the second claim is also proved. The restriction to $0 < \alpha_i < 2$ is only to stress that when $\alpha_i = \alpha_0$ for any $i = 1,\dots,n$, $(s^{\hat{\alpha}}I - A)^{-1}$ can only satisfy the pole's hypothesis on this range. $\square$

Remark 4.1. Inequality (14) is order-independent, but it is a sharp one, as it becomes an equality in the scalar case with $\alpha \leq 1$. Indeed, when $A = \lambda < 0$, $(s^\alpha - \lambda)^{-1}$ has anti-transform given by $t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$. From [13, equation (1.10.7), p.50], we have

$$\frac{d}{dt}t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha), \quad \forall \alpha > 0.$$

Thus,

$$\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(\lambda\tau^\alpha)d\tau = t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha).$$

From [13, equation (1.8.28), p.43], $E_{\alpha,\alpha+1}(\lambda t^\alpha) = \frac{-1}{\lambda t^\alpha} + \mathcal{O}(\frac{1}{(\lambda t^\alpha)^2})$ for any $0 < \alpha < 2$, $t \to \infty$ and any λ such that $\pi\alpha/2 < |arg(\lambda t)| \le \pi$. Since $E_{\alpha,\alpha}(-x) \ge 0$ for $0 < \alpha \le 1$, we conclude

$$\|\tau^{\alpha-1} E_{\alpha,\alpha}(\lambda\tau^\alpha)\|_1 = \frac{1}{|\lambda|}.$$

Although order-independent conditions are conservative, they provide robustness under uncertainty in the derivation order as shown in the following result.

Theorem 4.2. *Let system (4) with $\alpha_i = 1$ for any $i = 1, \dots, n$ and $g \equiv 0$ be the nominal case. If condition (15) holds for A invertible, then $x = 0$ is robustly stable in the sense that for any disturbance in the derivation order such that $0 < \alpha_i < 2$ for any $i = 1, \dots, n$ and $(s^{\hat{\alpha}} I - A)^{-1}$ has no poles in the closed right hand side of the complex plane, and for any nonlinear disturbance with Lipschitz constant lesser or equal than L, $x = 0$ is asymptotically stable.*

Proof. The claim follows from Theorem 3.1 and Proposition 4.1 by recalling that a Lipschitz function with parameter L_g is also a Lipschitz function with parameter L when $L \ge L_g$. □

4.2. **Commensurate case.** The (so-called) commensurate case for system (5) occurs when $\alpha_i = \alpha$ for all $i = 1, \dots, n$. In this case, $\mathcal{L}^{-1}((s^\alpha I - A)^{-1})(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha)$. Also, the condition of A having all its eigenvalues in $\{\lambda \in \mathbb{C} - \{0\} : |arg(\lambda)| > \frac{\alpha\pi}{2}\}$ implies $\|t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha)\|_1 < \infty$ (e.g. [2]). So, from Theorem 3.1, we need in addition the order-dependent condition $L < \|t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha)\|_1^{-1}$ (or, from Proposition 4.1, $L < \|A\|^{-1}$), to ensure the stability of $x = 0$. In turn, Remark 4.1 suggests carrying out the problem in a scalar formulation. Consider thus the function $V(x) = x'Px$ with $P > 0$ a constant matrix. After some manipulations, we obtain

$$D^\alpha V \le \frac{\lambda_m(Q)}{\lambda_M(P)} V + \bar{g}(V),$$

where $\bar{g}(V) = \frac{\eta}{\lambda_m(P)} V$, $\eta = L^2 + \|P\|^2$, and $Q := A'P + PA < 0$ is an arbitrary matrix. Then, (15) is satisfied if $|\lambda_M(Q)|\eta < \lambda_m(P)\lambda_M(P)$. Since Q is arbitrary, we can have for instance $L^2 < \lambda_M(P) - \|P\|^2$.

Notice that the order-dependent condition on Theorem 1 includes the case of A having all its eigenvalues in $\{\lambda \in \mathbb{C} - \{0\} : |arg(\lambda)| > \frac{\alpha\pi}{2}\}$, as seen above. Meanwhile, the condition using $Q := A'P + PA < 0$ implies that the eigenvalues of A must belong to $\{\lambda \in \mathbb{C} - \{0\} : |arg(\lambda)| > \frac{\pi}{2}\}$, reflecting the conservatism of order-independent conditions.

4.3. **Parametric uncertainty.** Consider now the case where the matrix A in (4) is partly unknown due to uncertainties in its entries. The approach here is different from that of [18] because, on the one hand, we study the multi-order fractionalization of the state equation rather than the single-order input-output one, and on the other, we consider time-varying perturbations.

Corollary 4.3. Consider system (4) with $g \equiv 0$ and $A = A_0 + \Delta A(t)$, where $A_0 \in \mathbb{R}^{n \times n}$ is a known constant matrix, and $\Delta A \in \mathbb{R}^{n \times n}$ is an unknown perturbation. Then, $x = 0$ is asymptotically stable if $\|\Delta A(t)\| \le \|(s^{\hat{\alpha}} I - A_0)^{-1}\|_1^{-1} < \infty$ for any $t > 0$.

Proof. It is a direct consequence of Theorem 3.1 by defining $\bar{g}(x,t) := \Delta A(t)x$, which is Lipschitz uniformly on t, and noting that the perturbed system becomes $D^{\hat{\alpha}}x = A_0 x + \Delta A x$. □

5. **Applications.** We explore some practical application of the obtained results. The performed simulations are accomplished using a filter approximation of the fractional integral.

5.1. **Robust control.** For the nominal system

$$\dot{x} = Ax + Bu, \tag{16}$$

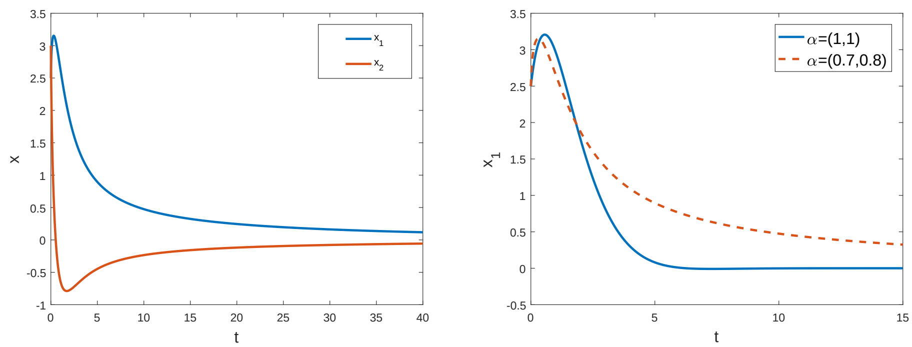
consider the robust control problem consisting in synthesizing a signal u such that the state $x(\cdot)$ converges to the origin $x = 0$ despite uncertainties. The traditional way in that uncertainties are accounted for is by adding a term $g(x,t)$, which can express parametric (see Corollary 4.3) and/or unmodelled uncertainty. Theorem 4.2 enables us to consider a more general instance by including uncertainties in the derivation order. The practical relevance of this kind of uncertainty is supported by the fact that fractional-order derivatives have been mainly employed to match observed responses in complex dynamical systems, taking inspiration from a model written originally in integer-order derivatives [1], what is called fractionalization.

Assuming that the pair (A, B) is controllable, we can find a matrix K such that $A_c := A + BK$ has arbitrary poles in the complex open left-half plane. This implies that A_c^{-1} exists and that its norm can be dominated, using the relationship between the induced norm and the spectral radius, in order to satisfy (15). To apply Theorem 4.2, what remain is to verify the pole condition on $(s^{\hat{\alpha}}I - A_c)^{-1}$ for each $\hat{\alpha}$ in the allowed range of derivation orders. For numerical detail, consider the pair in canonical controllable form $A = [0, 1; -1, 2]$, $B = [0; 1]$. To deal with a Lipschitz disturbance g with parameter $L < 1/4$ in the Euclidean norm, we can set the eigenvalues of A_c in -1 by choosing $u = k_1x_1 + k_2x_2 = 4x_2$, so that $|A_c^{-1}|^{-1} = 0.41$. Considering a derivation order disturbance $\hat{\alpha}$, the poles of $(s^{\hat{\alpha}}I - A_c)^{-1}$ are determined by the solutions to $s^{\alpha_1+\alpha_2} + 2s^{\alpha_1} + 1 = 0$. Using [3, Proposition 1, 3(a)], it follows that all the poles belongs to the open left-half plane when $\alpha_1, \alpha_2 \in (0, 1)$[4].

A simulation is made for the disturbances $g(x,t) = 1/4(0, \sin(x_2))'$ and $\hat{\alpha} = (0.7, 0.8)'$, while the initial condition was set at $x(0) = (2.5, 3)$. In the spirit of the problem, g and $\hat{\alpha}$ are unknown. Fig. 1a shows the effectiveness of the robust control. Fig. 1b reveals that a slow convergence is an indication that a derivation order disturbance can be present, as the speed of convergence is notoriously affected when compared for the same g using the nominal instance $\hat{\alpha} = (1, 1)'$. In this sense, Fig. 1a also shows that the motivation for using multiple order of derivation as disturbance, as opposed to the *commensurate case*, is to capture different speeds of convergence as x_1 and x_2 converges in different orders.

5.2. **Population dynamic.** Due to their capability to incorporate long-memory effects, fractional systems seem suited to model population dynamics in which learned habits or instincts affect the present behaviour of individuals [17]. Moreover, since the derivation order of an equation determines restrictively the convergence order of its solution [10], this order can represent qualitatively different adaptation capabilities of species, which can evolve with a speed lower than exponential as shown in

[4] We conjecture that this is always the case, i.e. that for any $\alpha_i \in (0, 1)$ the poles of $(s^{\hat{\alpha}}I - A_c)^{-1}$ have negative real value when A_c is Hurwitz.



(A) (B)

FIGURE 1. Robust performance

virus propagation. This provides the motivation for the introduction of multi-order models in population dynamics.

Consider thus the fractionalization of the classic competition model given by the Verhulst-Pearl logistic equation (see e.g., [14, Eqn. 4.4.5, p.154]),

$$D_{0^+}^{\alpha_i} x_i = x_i\big(a_i + f_i(x)\big), \quad x_i(0) > 0, i = 1, \ldots, n, \tag{17}$$

where x_i is the population of the $i-$specie, $i = 1, \ldots, n$, $x = (x_1, \ldots, x_n)'$, $\alpha_i \in (0, 1]$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ is a smooth function codifying the interaction (competition/cooperation) of the $i-$specie with the other species.

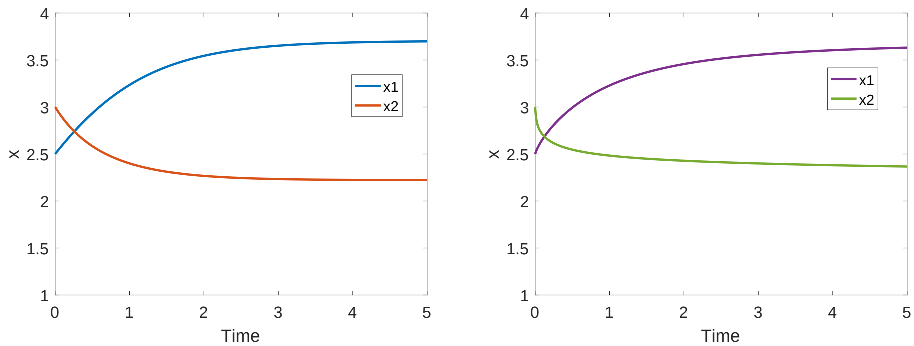
Equilibrium points of (17), i.e. the zeros of its right-hand side, are relevant as they characterize the asymptotic distribution of the population. Theorem 3.7 provides a sufficient condition for the asymptotic stability of an equilibrium point. Around or close enough to non-trivial stable equilibrium points, we are assured that (17) is a meaningful population model, i.e. $x_i \geq 0$, which is a nontrivial fact for fractional systems.

For numerical visualization, we consider the following dynamic

$$D^{\alpha_1} x_1 = x_1\big(a_1 + b_1 x_1 + b_{12} x_2\big),$$
$$D^{\alpha_2} x_2 = x_2\big(a_2 + b_2 x_2 + b_{21} x_1\big),$$

where $a_1 = a_2 = 1$, $b_1 = -0.3, b_2 = -0.5$, $b_{12} = 0.03$ and $b_{21} = 0.05$. The equilibrium points are $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (0, 2)$, $(x_1, x_2) = (10/3, 0)$, and $(x_1, x_2) = (1060/297, 700/297) \approx (3.56, 2.35)$. The latter equilibrium point is asymptotically stable since the Jacobian of the corresponding function f associated to (17) evaluated in this point is given by $[-1.06, 0.10; 0.11, -1.17]$, and hence, the characteristic polynomial has the form $s^{\alpha_1+\alpha_2} + as^{\alpha_1} + bs^{\alpha_2} + c$ where $a = 1.175$, $b = 1.068$, $c = 0.106 * 0.1175$, for which [3, Proposition 1, 3(a)] provides the satisfaction of the hypothesis of Theorem 3.7.

We study the case in which a change in the dominant species occurs due to their interactions. The initial condition $x(0) = (2.5, 3)$ avoids the crossing with the other equilibrium points in the case that the attraction region of $(x_1, x_2) = (1060/297, 700/297)$, whose existence follows by Theorem 3.7, includes $(2.5, 3)$. Figure 2 shows that this is the case and also how the choice of the derivation order determines the behaviour in the population dynamic while the population equilibrium



(A) $\alpha_1 = \alpha_2 = 1$

(B) $\alpha_1 = 0.8, \alpha_2 = 0.5$

FIGURE 2. Population dynamics depending on the derivation order

is preserved. On the one hand, the speed of convergence decreases as the derivation order decreases, as the bottom curve in Figure 2b is still descending meanwhile the bottom curve in Figure 2a is nearly flat for the same simulation time. On the other hand, for t near zero, the fastest response corresponds to the smallest derivation order, which explains the decreasing of the crossing time in Figure 2b.

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