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EMBEDDING DE ANTI-ÁRBOLES EN GRAFOS ORIENTADOS

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MATEMÁTICAS APLICADAS

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERA CIVIL MATEMÁTICA

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EMBEDDING DE ANTI-ÁRBOLES EN GRAFOS ORIENTADOS

La pregunta que dio inicio a esta tesis fue decidir si semigrado mínimo mayor a $\frac{k}{2}$ en un grafo orientado garantiza tener como subgrafo a cualquier camino orientado de k aristas. Este enunciado correspondería a la extensión natural de un resultado clásico para grafos simples, que en vez de semigrado pide grado mínimo $\frac{k}{2}$.

El resultado obtenido responde la pregunta para los caminos de largo k cuyas aristas alternan dirección, llamados *anticaminos*, en grafos suficientemente grandes con semigrado mínimo mayor a $\frac{k}{2}$, para todo $k \in \mathbb{N}$. Aún mejor, también funciona para todo antiárbol balanceado con k aristas y grado máximo acotado.

Para llegar al resultado, se introduce el concepto de *antimatching conexo* y se utiliza el Lema de Regularidad, en su versión para grafos orientados. El proceso de embedding consiste en dividir el antiárbol en unos antiárboles más pequeños y distribuirlos en las aristas del antimatching encontrado en el grafo reducido orientado y hacer las conexiones.

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EMBEDDING ANTI-TREES IN ORIENTED GRAPHS

The question that marked the beginning of this thesis was to decide if minimum semidegree greater than $\frac{k}{2}$ in an oriented graph guarantees having any oriented path with k edges as a subgraph. This statement would correspond to the natural extension for digraphs of a classic result for simple graphs, that instead of minimum semidegree asks for minimum degree $\frac{k}{2}$.

The obtained result answers the question for the paths of length k with edges that alternate directions, called *antipaths*, for sufficiently large graphs with minimum semidegree greater than $\frac{k}{2}$, for every $k \in \mathbb{N}$. And, even better, it also works for every balanced antitree with k edges and bounded maximum degree.

In order to prove the result, we introduce the concept of *connected antimatching* and use the Regularity Lemma in its version for oriented graphs. For the embedding, we split our antitree in smaller antitrees, distribute them in the antimatching edges of our reduced oriented graph and make the connections.

"And it was the most fun I've had in my life."
- Kozume Kenma

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Chapter 1

Introduction

The purpose of this chapter is to present the context necessary to understand the work done in this thesis. We will go through results, questions and conjectures about Extremal Graph Theory in Section 1.1, delving into oriented graphs in Section 1.2, to finalize with a sketch of the main proof done in the present work Section 1.3. The formal definitions for the concepts included in this chapter will be found in Chapter 2.

1.1 Extremal Graph Theory

Extremal Graph Theory studies how global graph properties influence its local substructures. An interesting question in the area is knowing how many edges are enough to force a certain subgraph H. A graph G on n vertices with the largest possible numbers of edges such that $H \nsubseteq G$ is called *extremal* for n and H, and its number of edges is denoted by $\text{ex}(n, H)$.

One of the first approaches on this matter can be attributed to Mantel [16], who proved in 1907 that a graph on n vertices without triangles has, at most, $\frac{n^2}{4}$ edges. In the 1940's, Turán analyzed the same question but for any complete subgraph:

Theorem 1.1 *(Turán [20]) For all integers r, n with $r > 1$, every graph G with $K^r \nsubseteq G$ on n vertices and $\text{ex}(n, K^r)$ edges is a $T^{r-1}(n)$, this means, an $(r-1)$-partite graph on $n \geq r-1$ vertices whose partition sets differ in size by at most 1.*

A few years later, Erdős and Stone [10] generalised this result for any r-partite graph with exactly s vertices in each class, K_s^r.

Theorem 1.2 *(Erdős-Stone [10]) For all integers $r \geq 2$ and $s \geq 1$, and every $\varepsilon > 0$, there exists an integer n_0 such that every graph with $n \geq n_0$ vertices and at least*

$$e(T^{r-1}(n)) + \varepsilon n^2$$

edges contains K_s^r as a subgraph.

This theorem led to an interesting asymptotic result for any graph H, in function of its chromatic number $\chi(H)$.

Corollary 1.3 *(Erdős-Simonovits [9]) For every graph H with at least one edge,*

$$\lim_{n\to\infty} \mathrm{ex}(n, H)\binom{n}{2}^{-1} = \frac{\chi(H)-2}{\chi(H)-1}.$$

Conditioning on the number of edges is equivalent to conditioning on the average degree. In fact, statements asking for requirements on the degree are very typical, not only on the average degree, but also on the minimum degree, maximum degree or a combination of them as well. To illustrate this, a widely known conjecture on the average degree is the one made by Erdős and Sós,

Conjecture 1.4 *(Erdős-Sós [8]) Let $k \in \mathbb{N}$. Every graph with average degree greater than $k-1$ contains every tree with k edges as a subgraph.*

On the other hand, there exists a variety of results with minimum degree as a condition. A classic one is Dirac's theorem [7], which states that any graph on $n \geq 3$ vertices and minimum degree at least $\frac{n}{2}$ has a cycle on n vertices. An easy observation in a similar direction is that, for every integer k, a minimum degree of k implies having every tree on k edges as a subgraph. For paths, this result can be improved, using a condition similar to Dirac's condition scaled down to the size of our path.

Theorem 1.5 *(Erdős-Gallai [11]) If $\delta(G) \geq \frac{k}{2}$, G is connected and $|V(G)| \geq k+1$, then G contains a path of length k.*

1.2 Oriented Graphs

A *digraph* consists on a set of vertices V and edges E, which are ordered pairs of distinct vertices. An *oriented graph* is a digraph that allows at most one edge between a pair of vertices. If an edge in a digraph G is directed from u to v, we write $uv \in E(G)$, denoting v as an *out-neighbour* of u and u an *in-neighbour* of v. The set of out-neighbours of a vertex v is denoted by $N^+(v)$ and the set of in-neighbours, $N^-(v)$. We call a complete oriented graph a *tournament*. The *underlying graph* of an oriented graph is the corresponding graph without the orientations.

The questions in extremal graph theory we presented in the previous section have their analogous statements for graphs with directed edges. Let us start analyzing Theorem 1.5. Observe that in this theorem we ask for a path. If we are looking for its analogue for oriented graphs, the first thing to notice is that there are lots of orientations of a path with k edges. For example, orienting every edge in the same direction gives us the *directed path*, and alternating directions is what we call an *antidirected path*. So, we shall ask for every graph that has a path as its underlying graph.

Another important question is: what do we do with the degree condition? One could replace $\delta(G)$ with the minimum degree of the underlying graph of G. But is easy to show that such a condition would not be sufficient: it suffices to consider an oriented graph G with two tournaments on $\frac{k}{2}$ vertices joined to a cutvertex v such that for every vertex $u \neq v$, $uv \in E(G)$. In that case, this oriented graph does not contain a directed path of length k. One could think that this example fails because of the non existent connectivity (in the digraph sense). So, consider now an oriented graph D consisting of four sets $\{V_1, V_2, V_3, V_4\}$ of $k/4$ vertices each and a central vertex v such that:

- for every $v_1 \in V_1$, $N^+(v_1) = V_2 \cup V_4$ and $N^-(v_1) = \{v\}$,
- for every $v_2 \in V_2$, $N^+(v_2) = V_3$ and $N^-(v_2) = V_1 \cup \{v\}$,
- for every $v_3 \in V_3$, $N^+(v_3) = \{v\}$ and $N^-(v_3) = V_2 \cup V_4$, and
- for every $v_4 \in V_4$, $N^+(v_4) = V_3 \cup \{v\}$ and $N^-(v_4) = V_1$.

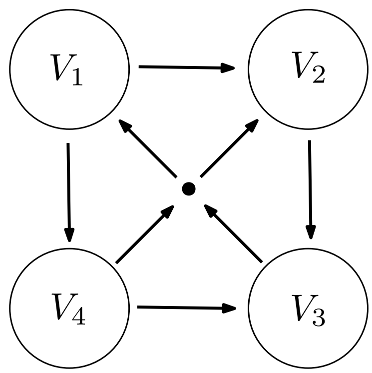

Figure 1.1: Oriented graph D from the counterexample with connectivity.

Again, is not possible to form a directed path on k edges if $k > 4$, as we would need to pass repeated times through the central vertex. To avoid this kind of complications, we will use a condition on the *minimum semidegree*, which is defined as $\delta^0(G) := \min\{\delta^+(G), \delta^-(G)\}$, with $\delta^+(G)$ the minimum $|N^+(v)|$ and $\delta^-(G)$ the minimum $|N^-(v)|$, for $v \in V(G)$. Observe that in Figure 1.1, $\delta^0(D) = 1$. This leads to the following conjecture:

Conjecture 1.6 *[17] Every oriented graph D with $\delta^0(D) > \frac{k}{2}$ contains every oriented path on k edges.*

If true, the conjecture would be tight: it is enough to observe that an antidirected path of length k cannot be found in the digraph obtained by a blow-up of the directed triangle, replacing each vertex with an independent set of size $\frac{k}{2}$ [17]. Before the formulation of the conjecture, Jackson [12] was already studying directed paths and his results give that the conjecture is true for this orientation. Recently, Stein and Klimošová [18] showed that Conjecture 1.6 is true for antipaths if the condition on the minimum semidegree is changed from $\frac{k}{2}$ to $\frac{3k}{4}$.

As Conjecture 1.6 considers every orientation of a path, others authors have also considered every length of a path. Kelly, Kühn and Osthus proved a result using a condition on the minimum semidegree that gives us every orientation and every length of an oriented cycle. Observe that the condition used is strong, as it is based on the number of vertices of the graph:

Theorem 1.7 *(Kelly, Kühn, Osthus [14]) For any $\alpha > 0$ there exists $n_0 = n_0(\alpha)$ such that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq (3/8 + \alpha)n$ contains every possible orientation of an l-cycle, for all $3 \leq l \leq n$.*

From now on, we will focus on the antidirected orientation. This orientation has been studied as well, not only for paths as Klimošová and Stein did, but also for trees. For instance, Addario-Berry, Havet, Linhares Sales, Reed and Thomassé have a well known conjecture for antitrees.

Conjecture 1.8 *(Addario-Berry, Havet, Linhares Sales, Reed, Thomassé [1]) Every digraph D with more than $(k-1)|V(D)|$ edges contains each antitree on k edges.*

The same authors proved their conjecture for antitrees of diameter at most 3. They also show that it does not hold for other orientations of trees, as we could consider a bipartite graph $G = (A, B)$ with every edge oriented from A to B. They also note that restricted to *symmetric digraphs*, this is a digraph D such that $uv \in E(D) \Leftrightarrow vu \in E(D)$, Conjecture 1.8 is equivalent to Conjecture 1.4.

So, we could ask ourselves if it is possible to extend Conjecture 1.6 to include every oriented tree on k edges, maintaining the condition on the degree. In this thesis we answer this question for a specific class of oriented trees: the ones that are *antidirected* (every vertex v has $d^+(v) = 0$ or $d^-(v) = 0$), *balanced* (the number of vertices with $d^+(v) = 0$ is equal to the number of vertices with $d^-(v) = 0$) and have bounded maximum degree $\Delta(T) := \max_{v \in V(T)}\{|N^+(v)|, |N^-(v)|\}$.

Theorem 1.9 *For all $\eta \in (0,1), \Delta \in \mathbb{N}$, there exists n_0 such that for every oriented graph D on $n \geq n_0$ vertices and for every $k \geq \eta n$, if $\delta^0(D) > (1+\eta)\frac{k}{2}$ then D contains an embedding of every balanced rooted antidirected tree T with k edges and $\Delta(T) \leq \Delta$.*

Observe that antipaths of odd length are included in this type of trees. For an antipath of even length l, we could add a vertex and an edge to obtain an antipath of odd length. This new antipath is balanced and, because the semidegree is an integer, using $\frac{l}{2}$ or $\frac{l+1}{2}$ would be equivalent. Hence, for sufficiently large graphs, Theorem 1.9 improves the advances made for antipaths on Conjecture 1.6 in [18]. In the following section we present an outline of the proof for Theorem 1.9, which will help to understand the full proof, which can be found in Chapter 3.

1.3 Sketch of Proof

Before proving Theorem 1.9, we introduce a new concept: *connected antimatchings*. Roughly, a connected antimatching in an oriented graph D is defined as a set M of disjoint edges in

D such that for every pair of edges in M, there exists an antiwalk in D that contains them. A more detailed and formal definition is given in Section 3.1.

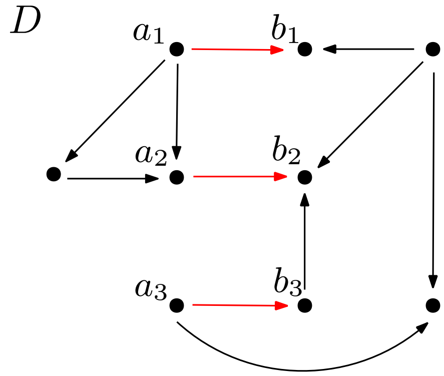

Figure 1.2: The set of edges $\{a_1b_1, a_2b_2, a_3b_3\}$ forms a connected antimatching in D.

Now, we can start the proof. First, we apply the Diregularity Lemma in our oriented graph D, which gives us an oriented graph D' partitioned in r clusters. Each cluster C_i is further divided into two sections: a small one, C_i^1, and a big one, C_i^2. This division will be useful in the embedding. Lastly, we obtain the corresponding reduced oriented graph R, with a minimum semidegree greater than $t := (1 + \frac{\eta}{2})\frac{k}{2n}r$.

In Section 3.1, we prove that graphs with minimum semidegree at least d have a connected antimatching of size d and that for every pair of edges in this antimatching there is an antiwalk of length at most $16d$ that contains them. Therefore, we have a connected antimatching in R of size $\lceil t \rceil$.

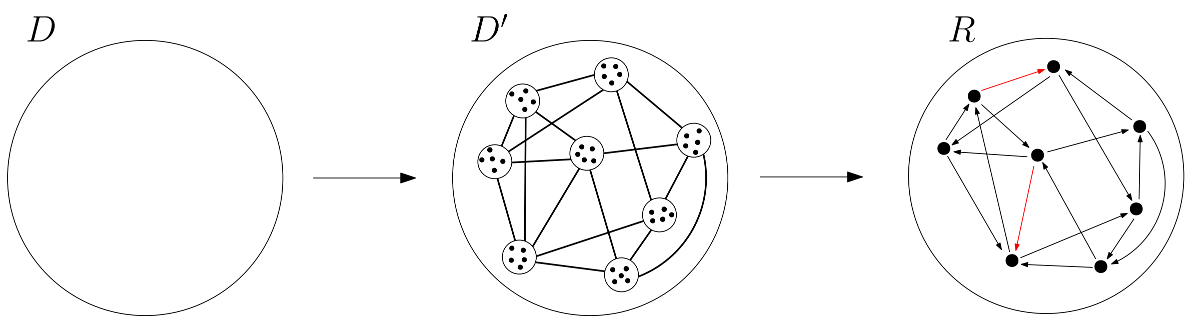

Figure 1.3: Starting with D, we obtain D' and lastly R, with a connected antimatching marked in red.

Now we work with our antitree T. A *β-decomposition* is a partition of our tree into a set W of vertices (often called *seeds*), of size at most $\frac{1}{\beta} + 2$, and a family $\mathcal{T}$ of disjoint subtrees of size at most βk, such that the root of each $S \in \mathcal{T}$ has a $w \in W$ as a parent.

After obtaining such a decomposition, we consider the trees in $\mathcal{T}$ but without their first $16t$ levels. Those new antiforests are almost balanced, thanks to the original balance of T. Thus we prove in Section 3.1 that it is possible to distribute these antiforests into $\lceil t \rceil$ sets such that none of them will have more than $|C_1^2|$ in-vertices (vertices v with $d^+(v) = 0$) or more than $|C_1^2|$ out-vertices (vertices v with $d^-(v) = 0$). This means that there will be enough space in the big slices of a pair of clusters to embed one of these set of antiforests. Since there are also $\lceil t \rceil$ antimatching edges, we will associate each one of the $\lceil t \rceil$ sets of antiforests to an antimatching edge.

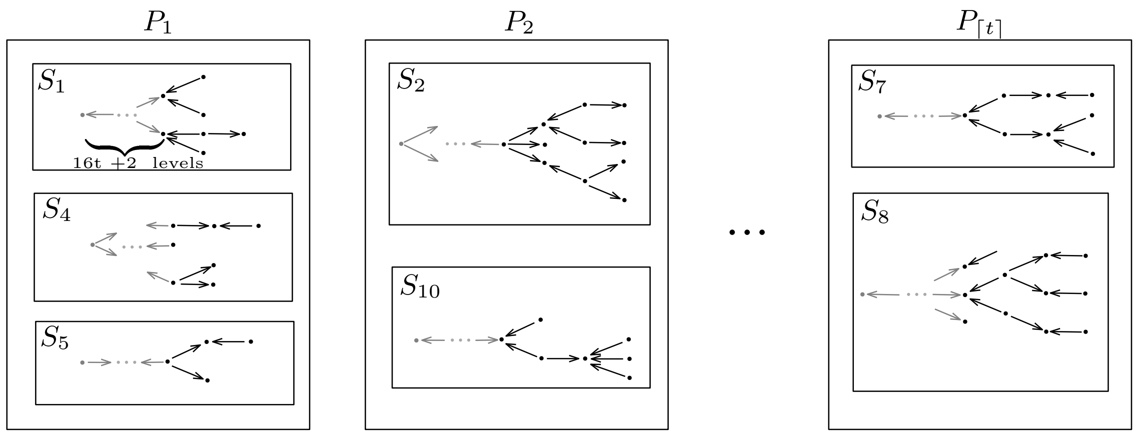

Figure 1.4: Example of distributing the antitrees $\{S_i\}$ in $\mathcal{T}$ without their first $16t + 2$ levels in the $\lceil t \rceil$ sets.

Having this, the idea of the embedding is the following: let C_1C_2 be the first edge of the antimatching. We embed the root of T, $r(T)$, in a vertex from C_1^1 that has more than $\sqrt{\varepsilon}|C_2^1|$ neighbours in C_2^1. Then, there exists $S \in \mathcal{T}$ that has $r(T)$ as its parent. We also know that S has a corresponding antimatching edge C_iC_j, because of the partition in $\lceil t \rceil$ parts. Define S' as S plus the seeds $w \in W$ that have its parent in S. So, we embed the first $16t + 2$ levels of S' in an antiwalk of length at most $16t + 2$ using only the small slices of the clusters, and arrive at C_i, where we embed the rest of S' in C_i^2 and C_j^2. And then we repeat the process tree by tree until we have embedded all T.

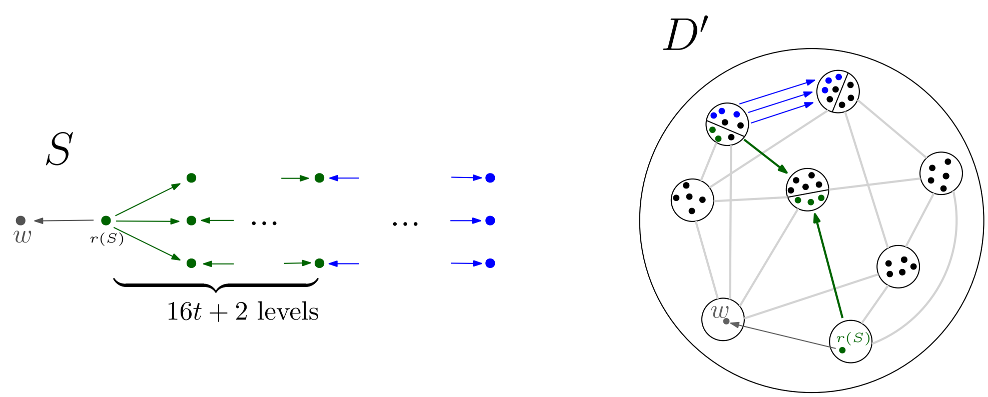

Figure 1.5: Example of the process of embedding an antitree S in D'.

The basic concepts and notations about graphs and regularity can be found in Section 2, and the details of the proof and the auxiliary lemmas in Section 3.

Chapter 2

Preliminaries

We dedicate this chapter to introduce the concepts, notations and previous results necessary for this thesis. The concepts presented in this section are all standard, and more details can be found in [4] and [6].

2.1 The Basics

A *graph* G is a pair (V, E) such that V is a set and $E \subseteq \binom{V}{2}$. The elements of V are the *vertices* and the elements of E are the *edges*. If H is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we say H is a *subgraph* of G. A subgraph H of G is *induced* if it contains all the edges $\{x, y\} \in E(G)$ for every $x, y \in V(H)$.

An edge $\{u, v\}$ will be simply denoted uv (or vu). If $uv \in E(G)$, then we say that u and v are *neighbours*, and the set that contains every neighbour of v is called *neighbourhood*, denoted by $N_G(v)$, or simply $N(v)$ if is clear by the context. The *degree* of a vertex v, $d_G(v)$ or $d(v)$, is the number of neighbours of v in G. We define the *minimum degree* of a graph G as $\delta(G) := \min\{d(v) : v \in V(G)\}$ and the *maximum degree* as $\Delta(G) := \max\{d(v) : v \in V(G)\}$.

A graph D such that its edges are ordered pairs in $V(D) \times V(D)$ instead of sets is called a *digraph*. The edge (u, v) will be simply denoted uv, and therefore in digraphs, contrary to non oriented graphs, the edges uv and vu are different. An *oriented graph* is a digraph such that there are no edges of the form (u, u) and for every pair of vertices there is at most one edge between them. We define the *underlying graph* of an oriented graph as the graph without the orientations.

The *in-neighbourhood* of a vertex v, $N_D^-(v)$ or $N^-(v)$, is the set of all vertices in $V(D)$ such that $uv \in E(D)$, and the *out-neighbourhood*, $N_D^+(v)$ or $N^+(v)$, is defined in a similar manner, but with $vu \in E(D)$. The *in-degree*, $d^-(v)$, counts the number of vertices in $N^-(v)$. The *minimum in-degree* corresponds to $\delta^-(D) := \min\{d^-(v) : v \in V(D)\}$ and the *maximum in-degree* to $\Delta^-(D) := \max\{d^-(v) : v \in V(D)\}$. All these notions have their analogous *out*-version. We define the *minimum semidegree* of a digraph as $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$.

We also need to define a few special graphs that will be useful in the next chapters: a *walk* of length k is a sequence of vertices $v_0v_1v_2...v_k$ joined by edges of the form v_iv_{i+1}. A *path* of length k, P_k, is a *walk* of length k such that all vertices (and thus also all edges) are distinct. If we add the edge $v_{k+1}v_1$ to P_k, we obtain the *cycle* of size k, C_k. In an oriented graph, a *directed path* of length k is an oriented graph such that its underlying graph is a path of length k and every edge is oriented from v_i to v_{i+1}. An *antipath* of length k is an oriented graph such that its underlying graph is a path of length k and the edges alternate directions. An *antiwalk* of length k is a sequence of vertices $v_0v_1v_2...v_k$ joined by edges of the form v_iv_{i+1} if i is even and $v_{i+1}v_i$ if i is odd, or of the form v_iv_{i+1} if i is odd and $v_{i+1}v_i$ if i is even. In the same manner we define *directed cycle* and *anticycle* (which exists only for even lengths).

Figure 2.1: On the left, a directed path of length 3. On the right, an antipath of length 3.

A graph is called a *tree* if it does not have cycles and is *connected*, this means that for every pair of vertices u, v there exists a path that starts in u and ends in v. An *antitree* is an oriented graph such that its underlying graph is a tree and it does not have a directed path of length 2 as a subgraph.

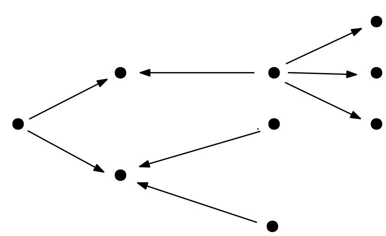

Figure 2.2: Example of an antitree.

Let T be an antitree. We define $V_{\text{in}}(T)$ as the set of vertices of T that do not have out-neighbours, and we define $V_{\text{out}}(T)$ analogously. Observe that $\{V_{\text{in}}(T), V_{\text{out}}(T)\}$ partitions T if T is an antitree. An antitree is *balanced* if $|V_{\text{in}}(T)| = |V_{\text{out}}(T)|$. When T is rooted we denote its root by $r(T)$.

Let T be a rooted oriented tree. We define the i-th level of T as the set of vertices $t \in V(T)$ such that there exists an oriented path from t to $r(T)$ of length i.

2.2 Diregularity

Let G be a graph, let $\varepsilon > 0$ and let A, B be two disjoint subsets of $V(G)$. We define the *density* of the pair (A, B) as $d(A, B) = \frac{|E(A,B)|}{|A|\cdot|B|}$, with $|E(A, B)|$ denoting the number of edges between A and B. The pair (A, B) is ε-*regular* if $|d(X, Y) - d(A, B)| < \varepsilon$ holds for every $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$.

Let (A, B) be an ε-regular pair with density d and let $Y \subseteq B$ be such that $|Y| > \varepsilon|B|$. A vertex $x \in A$ is called *ε-typical* (or simply *typical*) with respect to Y if it has more than $(d-\varepsilon)|Y|$ neighbours in Y. A useful lemma about typical vertices is the following:

Lemma 2.1 *[6] Let (A, B) be an ε-regular pair of density d. Let $Y \subseteq B$ with $|Y| \geq \varepsilon|B|$. Then A has at most $\varepsilon|A|$ vertices that are not typical with respect to Y.*

Let $\{V_0, ..., V_k\}$ be a partition of V. This partition is called an *ε-regular* partition of G if:

(i) $|V_0| \leq \varepsilon|V|$,

(ii) $|V_1| = ... = |V_k|$,

(iii) all but at most εk^2 of the pairs (V_i, V_j), with $1 \leq i < j \leq k$, are ε-regular.

Lemma 2.2 *(Regularity Lemma, Szemerédi [19])*
For every $\varepsilon > 0$ and every integer $M' \geq 1$, there exist two integers M and n_0 such that every graph G with $n \geq n_0$ vertices has an ε-regular partition $\{V_0, V_1, ..., V_k\}$ with $M' \leq k \leq M$.

From this lemma it is possible to obtain what is known as the "degree form" of the statement:

Lemma 2.3 *(Degree Form of the Regularity Lemma) For every $\varepsilon \in (0, 1)$ and every integer M', there are integers M and n_0 such that if G is a graph on $n \geq n_0$ vertices and $d \in [0, 1]$ is any real number, then there is a partition of G into $V_0, ..., V_k$ and a spanning subdigraph G' of G, called regularized graph, such that the following holds:*

- $M' \leq k \leq M$,
- $|V_0| \leq \varepsilon n$,
- $|V_1| = \ldots = |V_k| =: m$,
- $d_{G'}(x) > d_G(x) - (d+\varepsilon)n$ *for all vertices* $x \in V(G)$,
- *for all $1 \leq i < j \leq k$, the bipartite graph $(V_i, V_j)_{G'}$ whose vertex classes are V_i and V_j and whose edge set consists of all the V_i-V_j edges in G' is ε-regular and has density either 0 or at least d,*
- *for all $1 \leq i \leq k$ the digraph $G'[V_i]$ is empty.*

Let D be a digraph and let $A, B \subseteq V(D)$ disjoint. We denote by (A, B) the oriented subgraph of D with vertex set $A \cup B$ and every edge directed from A to B in D. In this case, we say the pair (A, B) is *ε-regular* if the underlying graph is ε-regular. Having these notions and the Regularity Lemma for graphs without orientation, it is possible to extend it to digraphs:

Lemma 2.4 *(Degree form of the Diregularity Lemma, Alon and Shapira [3])*
For every $\varepsilon \in (0,1)$ and every integer M', there are integers M and n_0 such that if D is a digraph on $n \geq n_0$ vertices and $d \in [0,1]$ is any real number, then there is a partition of the vertices of D into $V_0, V_1, ..., V_k$ and a spanning subdigraph D' of D, called regularized digraph, such that the following holds:

- $M' \leq k \leq M$,
- $|V_0| \leq \varepsilon n$,
- $|V_1| = \ldots = |V_k| =: m$,
- $d^+_{D'}(x) > d^+_D(x) - (d+\varepsilon)n$ *for all vertices* $x \in D$,
- $d^-_{D'}(x) > d^-_D(x) - (d+\varepsilon)n$ *for all vertices* $x \in V(D)$,
- *for every ordered pair V_iV_j with $1 \leq i,j \leq k$ and $i \neq j$, the bipartite graph $(V_i, V_j)_{D'}$ whose vertex classes are V_i and V_j and whose edge set consists of all the $V_i - V_j$ edges in D' is ε-regular and has density either 0 or at least d,*
- *for all $1 \leq i \leq k$ the digraph $D'[V_i]$ is empty.*

Given a regularized digraph D' with clusters $V_1, ..., V_t$, the *reduced digraph* R is a digraph with vertices $V_1, ..., V_t$ such that the edge V_iV_j exists only if D' contains a V_iV_j edge. Observe that, even if D' is only oriented, R might have two edges between some vertices.

In [13], it is proved that discarding appropiate edges from the reduced digraph we can preserve the semidegree and obtain the *reduced oriented graph*. This is formalized in the following lemma.

Lemma 2.5 *For every $\varepsilon \in (0,1)$ there exist integers $M' = M'(\varepsilon)$ and $n_0 = n_0(\varepsilon)$ such that the following holds. Let $d \in [0,1]$, let D be an oriented graph of order at least n_0 and let R' be a reduced digraph and D' the regularized digraph obtained by applying the Diregularity Lemma to D with parameters ε, d and M'. Then, R' has a spanning oriented subgraph R with*

(a) $\delta^+(R) \geq (\delta^+(D)/|D| - (3\varepsilon + d))|R|$,

(b) $\delta^-(R) \geq (\delta^-(D)/|D| - (3\varepsilon + d))|R|$.

This oriented graph R is called the (ε, d)-reduced oriented graph.

Chapter 3

Proofs

In the first section of this chapter we define new concepts and prove auxiliary lemmas necessary to prove the main result of this thesis, which is shown in the second section. We finalize the chapter with comments and conclusions regarding the results obtained in this work.

3.1 Auxiliary Lemmas

3.1.1 Connected antimatchings

Let P be an antiwalk. A vertex $v \in V(P)$ is called an *out-vertex* if P has one or more edges of the form vx. If P has one or more edges of the form xv, v will be called *in-vertex*. Every vertex in an antiwalk has to be an in- or an out-vertex, and it also can be both at the same time. In the case that P has only one vertex v, we will say v is an in- and an out-vertex of P.

Let C be a non-empty oriented graph and let $a \in V(C)$. We say (C, a) is *anticonnected* if for every $v \in V(C) \setminus \{a\}$ there exists a non-trivial antiwalk P_v that starts in an edge of the form ax and ends in v. This type of walk will be called an *out-walk from a to v*. We define $\mathrm{In}(C, a)$ as the set of vertices $v \in V(C)$ such that there exists an out-walk from a to v in which v is an in-vertex. We define $\mathrm{Out}(C, a)$ as the set of vertices $v \in V(C)$ such that $v = a$ or there exists an out-walk from a to v in which v is an out-vertex. Observe that $\mathrm{In}(C, a)$ and $\mathrm{Out}(C, a)$ are not necessarily disjoint, and their union covers $V(C)$ if (C, a) is anticonnected.

Let D be an oriented graph. We say $M = \{a_i b_i\}_{1 \leq i \leq m}$ is a *connected antimatching* of size m in D if $a_i \neq b_j$, $a_i \neq a_j$, $b_i \neq b_j$ for every $1 \leq i \neq j \leq m$, and if there is a subdigraph C of D such that $M \subseteq E(C)$, (C, a_1) is anticonnected and $a_i \in \mathrm{Out}(C, a_1)$ for every $1 \leq i \leq m$.

Our first lemma links together this new concept, connected antimatching, with the minimum semidegree of an oriented graph. And the next one, Lemma 3.2, gives us a bound on the distance between the edges of a connected antimatching in this context. Both proofs can be done in an algorithmic way, as you may observe next.

Lemma 3.1 *Let d be a positive integer and let D be an oriented graph of minimum semidegree $\delta^0(D) \geq d$. Then D has a connected antimatching of size d.*

Proof. Let $M = \{a_i b_i\}_{1 \leq i \leq m}$ be a connected antimatching of maximum size in D. For contradiction, we assume that $|M| < d$. Note that $|M| \geq 1$, because any single edge is a connected antimatching. Define C as the largest induced subdigraph such that $M \subseteq E(C)$, (C, a_1) is anticonnected and $a_i \in \mathrm{Out}(C, a_1)$ for every $1 \leq i \leq m$. We claim that:

$$\text{If } v \in \mathrm{In}(C, a_1) \text{ (resp. } v \in \mathrm{Out}(C, a_1)\text{), then } N^-(v) \subseteq V(C) \text{ (resp. } N^+(v) \subseteq V(C)\text{).} \quad (3.1)$$

Indeed, let $v \in \mathrm{In}(C, a_1)$ and suppose that there exists a vertex $x \in N^-(v) \setminus V(C)$. By definition, v is in-vertex in an antiwalk P_v from a_1 to v. So, there exists an out-walk from a_1 to x formed by P_v and the edge xv. This means that if we add the vertex x and the edge xv to C, then we obtain a new subdigraph C' such that $M \in E(C')$, (C', a_1) is anticonnected and C' is larger than C, a contradiction. The proof for $v \in \mathrm{Out}(C, a_1)$ is analogous. This proves (3.1).

Next, we show that:

$$\text{If } v \in \mathrm{In}(C, a_1) \setminus V(M) \text{ (resp. } v \in \mathrm{Out}(C, a_1) \setminus V(M)\text{), then } N^-(v) \subseteq V(M) \text{ (resp. } N^+(v) \subseteq V(M)\text{).} \quad (3.2)$$

In order to see this, take $v \in \mathrm{In}(C, a_1) \setminus V(M)$. By (3.1), we know $N^-(v) \subseteq V(C)$, so suppose v has an in-neighbour x such that $x \in V(C) \setminus V(M)$.

Define $a_{m+1} := x$, $b_{m+1} := v$ and let $M' := \{a_i b_i\}_{1 \leq i \leq m+1}$. Then, $M' \subseteq E(C)$ and (C, a_1) is anticonnected. Take an out-walk from a_1 to v and add the edge xv at the end. The resulting walk is an out-walk from a_1 to x such that x is an out-vertex. Therefore, M' is a connected antimatching of size $m + 1$, a contradiction. We can proceed similarly for $v \in \mathrm{Out}(C, a_1) \setminus V(M)$. This proves (3.2).

We claim that $|V(C) \setminus V(M)| \leq 1$. To prove this, suppose we have two vertices u, v in $V(C) \setminus V(M)$. Because of our assumption on the minimum semidegree of D and by (3.2), u has at least d in-neighbours in $V(M)$ or has at least d out-neighbours in $V(M)$. And the same holds for v. Since we assume that $|M| < d$, there is an edge $xy \in M$ such that:

- one of xu, ux is in $E(C)$, and
- one of yv, vy is in $E(C)$.

Adding these edges that are in $E(C)$ to M and removing xy gives us a larger connected antimatching. This can be proved similarly as in the proof for (3.2). Thus, we conclude that $|V(C) \setminus V(M)| \leq 1$. In particular, and since we assume M to have less than d edges,

$$|V(C)| < 2d. \quad (3.3)$$

Now, by definition of $\mathrm{In}(C, a_1)$ and $\mathrm{Out}(C, a_1)$, the out-neighbourhood of a_1 is contained in $\mathrm{In}(C, a_1)$, and the in-neighbourhood of b_1 is in $\mathrm{Out}(C, a_1)$. Considering the minimum semidegree of D and (3.1), we have that $|\mathrm{In}(C, a_1)|, |\mathrm{Out}(C, a_1)| \geq d$. By (3.3), we conclude $\mathrm{In}(C, a_1) \cap \mathrm{Out}(C, a_1) \neq \varnothing$.

Take a vertex $v \in \mathrm{In}(C, a_1) \cap \mathrm{Out}(C, a_1)$. By (3.1), v has its in-neighbourhood and out-neighbourhood contained in C. But this means that C has at least $2d$ vertices, since D is an oriented graph and thus $N^+(v) \cap N^-(v) = \varnothing$, which contradicts (3.3). $\square$

In a digraph D, we define the *distance* between two edges $e, e' \in E(D)$ as the length of the shortest antiwalk that begins in any order of the edge e and ends with any order of the edge e'. We denote it as $\mathrm{dist}(e, e')$. Observe that if M is a connected antimatching, then $\mathrm{dist}(e, e')$ is well defined for $e, e' \in M$, since that antiwalk exists by definition.

Lemma 3.2 *Let $d \in \mathbb{N}$ and let D be an oriented graph of minimum semidegree $\delta^0(D) \geq d$. Then there exists a connected antimatching $M = \{a_i b_i\}_{1 \leq i \leq d}$ such that $\mathrm{dist}(a_1 b_1, a_i b_i) \leq 8d$ for every $1 \leq i \leq d$.*

Proof. By Lemma 3.1, there exists a connected antimatching of size d in D. Let $M := \{a_i b_i\}_{1 \leq i \leq d}$ be one such that $\sum_{i=2}^{d} \mathrm{dist}(a_1 b_1, a_i b_i)$ is minimum.

For the sake of contradiction, suppose there exists k such that the shortest out-walk P from $a_1 b_1$ to $a_k b_k$ has length greater than $8d$. We claim the following:

Every vertex in P appears at most once as an in-vertex and at most once as an out-vertex. (3.4)

To see this, take a vertex $v \in V(P)$ and suppose it appears at least twice in P as in-vertex. Let $xv, yv \in E(P)$. If $v \neq b_1$, then P has the form $P_1 xv P_2 vy P_3$, with $a_1 b_1$ the first edge of P_1 and $a_k b_k$ the last edge of P_3. If $v = b_1$, then P has the form $xv P_2 vy P_3$, with $xv = a_1 b_1$. Observe that in both cases $|P_2| \geq 2$, as we need to start and end in v as an in-vertex. Since both proofs are similar, we only show the proof for the case $v \neq b_1$. If we delete P_2, we are left with the subdigraph $P' = P_1 xvy P_3$, which is an antiwalk, because P_1 and P_3 are antiwalks and $xv, yv \in E(P)$. But this is a contradiction, because P' would be an out-walk from $a_1 b_1$ to $a_k b_k$ shorter than P. An analogous argument holds when v appears twice as an out-vertex. This proves (3.4).

From (3.4) follows that a shortest out-walk does not repeat edges. Therefore, and since we assumed that P has length greater than $8d$,

$$|E(P)| > 8d. \tag{3.5}$$

By (3.4) we obtain that every a_i is incident to, at most, 4 edges in P. The same holds for every b_i. This gives us at most $8d$ edges having one of its extremes on M, and so, by (3.5), there exists an edge $f = xy$ such that $x, y \notin V(M)$.

Then we could replace $a_k b_k$ with f in M, with x taking the role of a_k and y the role of b_k. This gives us a connected antimatching with the sum of the distances to $a_1 b_1$ strictly less than the sum for the original M, a contradiction. $\square$

3.1.2 Tree decomposition

Let T be a tree. In this work we use the term *tree decomposition* not as it is used to define treewidth, but as a partition of $E(T)$ in subtrees T_i. A type of tree decomposition is proved by Ajtai, Komlós and Szemerédi [2]. The version we use can be found in [5]:

Lemma 3.3 *(Besomi, Pavez-Signé, Stein [5]) Let $\beta \in (0,1)$, and let T be a rooted tree on $k+1$ vertices. Then there exists a set $W \subseteq V(T)$ and a family $\mathcal{T}$ of disjoint rooted trees such that*

1. *$r(T) \in W$;*
2. *$\mathcal{T}$ consists of the components of $T - W$, and each $S \in \mathcal{T}$ is rooted at the vertex closest to the root of T;*
3. *$|S| \leq \beta k$ for each $S \in \mathcal{T}$; and*
4. *$|W| < \frac{1}{\beta} + 2$.*

The pair $(W, \mathcal{T})$ will be called a β-decomposition of T.

Let T be an oriented rooted tree. We say $(W, \mathcal{T})$ is a β-decomposition of T if is a β-decomposition for the underlying graph of T. Let $(W, \mathcal{T})$ be a β-decomposition of T. We define $L_m(T)$ as the union of the first m levels of every $S \in \mathcal{T}$. We now prove that after removing $L_m(T)$ from T, the remainder of T still maintains a certain balance.

Lemma 3.4 *Let $k, m, \Delta \in \mathbb{N}^+$, $\alpha, \beta \in (0, \frac{1}{2})$, such that $\frac{4}{\alpha}(\frac{1}{\beta} + 2)\Delta^{m+1} \leq k$. Let T be a balanced rooted antitree on $k+1$ vertices, with $\Delta(T) \leq \Delta$. Then, for any β-decomposition $(W, \mathcal{T})$ of T it is true that*

$$1 - \alpha \leq \frac{|V_{in}(T) - L_m(T) - W|}{|V_{out}(T) - L_m(T) - W|} \leq 1 + \alpha.$$

Proof. Let T be a balanced rooted antitree on $k+1$ vertices with $\Delta(T) \leq \Delta$ and let $(W, \mathcal{T})$ be a β-decomposition of T.

First, we are going to bound the size of $L_m(T) + W$:

$$|L_m(T) + W| < \tfrac{\alpha k}{4}. \tag{3.6}$$

To see this, for every $w \in W$ take the tree τ_w rooted in w that contains every $S \in \mathcal{T}$ such that w is the parent of $r(S)$, with $r(S)$ a descendant of w in T. Observe that the first m levels of any $S \in \mathcal{T}$ are included in the first $m+1$ levels of some τ_w, and therefore the size of $L_m(T) + W$ is at most the total size of the first $m+1$ levels of every τ_w.

Because $\Delta(T) \leq \Delta$, the first $m+1$ levels of any subtree of T contains at most Δ^{m+1} vertices. This implies that $|L_m(T) + W| \leq (\frac{1}{\beta} + 2)\Delta^{m+1} < \frac{\alpha k}{4}$, where the last inequality holds by our condition on k, proving (3.6).

Then, using the fact that $|V_{\text{in}}(T)| = |V_{\text{out}}(T)| = \frac{k+1}{2}$, due to the balance of T, and applying (3.6) we obtain the following:

$$\frac{|V_{\text{in}}(T) - L_m(T) - W|}{|V_{\text{out}}(T) - L_m(T) - W|} \geq \frac{\frac{k+1}{2} - |L_m(T) + W|}{\frac{k+1}{2}} \geq 1 - \frac{\alpha k}{2k} \geq 1 - \alpha.$$

We calculate similarly for the upper bound:

$$\frac{|V_{\text{in}}(T) - L_m(T) - W|}{|V_{\text{out}}(T) - L_m(T) - W|} \leq \frac{\frac{k+1}{2}}{\frac{k+1}{2} - |L_m(T) + W|} \leq 1 + \frac{\alpha k}{(2-\alpha)k} \leq 1 + \alpha,$$

which concludes the proof. □

Let $P = v_0 v_1 ... v_m$ be an antiwalk and T a rooted antitree. We say P and T are *consistent* if at least one of the following holds:

i) $|V(T)| = 1$,

ii) $r(T) \in V_{\text{out}}(T)$ and $v_0 v_1$ is the first edge of P, or

iii) $r(T) \in V_{\text{in}}(T)$ and $v_1 v_0$ is the first edge of P.

Next, we prove a lemma that is helpful for embedding antitrees in antiwalks found in the reduced oriented graph.

Lemma 3.5 *Let $\varepsilon \in (0, \frac{1}{4})$, $m, h, l \in \mathbb{N}$ with $1 \leq h \leq l$. Let D be an oriented graph and $C = C_0 ... C_{h-1} C_h$ be an antiwalk in an $(\varepsilon, 2\sqrt{\varepsilon})$-reduced oriented graph R of D. Let $Z_0 \subseteq C_0$ be a set with $|Z_0| > \varepsilon m$. For every $1 \leq i \leq h$, let $Z_i \subseteq C_i$ be such that $|Z_i| \geq 3\sqrt{\varepsilon} m$, with the convention $Z_{h+j} = Z_{h-1}$, if j is odd, and $Z_{h+j} = Z_h$, if j is even. Let $X_{h-1} \subseteq C_{h-1} \setminus Z_{h-1}$ and $X_h \subseteq C_h \setminus Z_h$ with $|X_{h-1}|, |X_h| > \varepsilon m$. Let S be a rooted antitree with $|S| < 2\varepsilon m$ such that C and S are consistent.*

Then it is possible to embed the first l levels of S in $\cup_{i=0}^{h} Z_i$ such that the i-th level of S goes in Z_i. Also, if the l-th level of S is not empty, then

a) if $l - h$ is odd, every vertex in the level l is embedded in Z_{h-1} and is typical with respect to X_h and to Z_h, and

b) if $l - h$ is even, every vertex in the level l is embedded in Z_h and is typical with respect to X_{h-1} and to Z_{h-1}.

Proof. Let $r(S)$ be the root of S and denote the embedding by $\phi : V(S) \longrightarrow \cup_{i=0}^{h} C_i$. Without loss of generality, suppose that $C_0 C_1 \in E(R)$, and $r(S) \in V_{\text{out}}(S)$.

First, we embed $r(S)$ in a typical vertex of Z_0 with respect to Z_1. Such vertex exists since $|Z_0| > \varepsilon m$ and there are, at most, εm vertices that are not typical with respect to Z_1. Next, we will embed the first $k = \min\{h, l-1\}$ levels of S such that the i-th level goes to typical vertices in Z_i.

Suppose we are embedding the i-th level of S. Let v be a vertex in the i-th level and denote its father by p_v. Because p_v is in the $(i-1)$-th level, $\phi(p_v)$ is in Z_{i-1} and is typical with respect to Z_i. Observe that we use at most $2\varepsilon m$ vertices in total in the embedding and $|Z_i| \geq 3\sqrt{\varepsilon}m$, thus at any point of the process $|Z_i| > 2\sqrt{\varepsilon}m$. Also note that $\phi(p_v)$ has more than εm neighbours in Z_i and at most εm of them are not typical with respect to Z_{i+1}. Therefore there exists a vertex u in $Z_i \cap N(\phi(p_v))$ that is typical with respect to Z_{i+1}. Fix $\phi(p_v) = u$ and delete u from Z_i. Repeat this process for every vertex in the i-th level. We do this process until the first k levels are completely embedded.

For the remaining levels, we will start by embedding the levels $h+i$, with $1 \leq i < l-k$, if there are any. Take a vertex v in the $(h+1)$-th level. Because p_v is in level h, we know that $\phi(p_v)$ is typical with respect to Z_{h-1}. In the same manner we chose the images for the first k levels, it is possible to find a vertex u in $Z_{h-1} \cap N(\phi(p_v))$ that is typical with respect to Z_h. Then, fix $\phi(v) = u$ and delete u from Z_h. Repeat for every vertex in the $(h+1)$-th level. For v in the $(h+2)$-th level the process is analogous, but observing that since $\phi(p_v)$ is in Z_h and typical with respect to Z_{h-1}, the image of v will be in Z_{h-1} and will be typical with respect to Z_h.

Take a vertex v in the $(h+i)$-th level with $2 < i < l-h$. If i is odd, we repeat the process made for $i = 1$. Then, the image of v will be in Z_h and will be typical with respect to Z_{h-1}. On the other hand, if i is even, we repeat the process made for $i = 2$, with the image of v in Z_{h-1} and typical with respect to Z_h. We repeat this process for every $1 \leq i < l-h$.

Finally, we embed the l-th level of S, if it is not empty. To do this, take v in the level l of S. Suppose $l-h$ is even and thus $\phi(p_v) \in Z_{h-1}$. The case of $l-h$ and $\phi(p_v) \in Z_h$ odd is analogous. Then, choose a vertex $u \in N(\phi(p_v)) \cap Z_h$ that is typical with respect to X_{h-1} and to Z_{h-1}. Note that at most $2\varepsilon m$ vertices are not typical with respect to both X_{h-1}, Z_{h-1} at the same time. Since $|Z_i| > 2\sqrt{\varepsilon}m$ at any point of the process, then $|N(\phi(p_v)) \cap Z_h| > 2\varepsilon m$ and thus there exists such vertex u. Fix $\phi(v) = u$ and update $Z_h = Z_h \setminus u$. Repeating this process for every vertex in the l-th level completes the embedding of S. □

3.1.3 Partition

The following lemma shows that it is possible to partition a family in $\mathbb{N}^2$ under certain conditions. This will be useful to partition the trees in $\mathcal{T}$ without their first levels, with $(W, \mathcal{T})$ a β-decomposition.

Lemma 3.6 *Let $m, t \in \mathbb{N}$, $\alpha > 0$ and let $(p_i, q_i)_{i \in I} \subseteq \mathbb{N}^2$ be a family such that:*

a) $(1-\alpha)\sum_{i \in I} p_i \leq \sum_{i \in I} q_i \leq (1+\alpha)\sum_{i \in I} p_i$,

b) $p_i + q_i \leq \alpha m$, *for all* $i \in I$, *and*

c) $\sum_{i \in I} p_i, \sum_{i \in I} q_i < (1-10\alpha)mt$.

Then, there exists a partition $\mathcal{J}$ of I of size t such that

$$\sum_{j\in J} p_j \le (1-7\alpha)m, \quad \sum_{j\in J} q_j \le (1-7\alpha)m$$

for every $J \in \mathcal{J}$.

Proof. For every $i \in I$, we define δ_i as the difference $p_i - q_i$ and, similarly for every set $S \subseteq I$, we define δ_S as $\sum_{i\in S}(p_i - q_i) = \sum_{i\in S}\delta_i$.

Take a partition $\{A_1, ..., A_t, R\}$ of I, allowing the sets to be empty, such that:

i) $|R|$ is minimal,

ii) for every $j \in [t]$, $\sum_{i\in A_j} p_i \le (1-9\alpha)m$ and $\delta_{A_j} \in [-\alpha m, \alpha m]$.

Observe that such partition exists, since $A_1 = ... = A_t = \varnothing$ satisfies ii). Note that the second condition implies that $\sum_{i\in A_j} q_i < (1-7\alpha)m$:

$$\sum_{i\in A_j} q_j = \sum_{i\in A_j}(p_j - \delta_{A_j}) \le (1-9\alpha)m + \alpha m < (1-7\alpha)m.$$

Therefore, if R was empty, we could conclude the proof taking $\mathcal{J} = \{A_1, ..., A_t\}$. So, assume otherwise. We claim that

$$\delta_i\delta_j \ge 0 \text{ for every } i, j \in R. \tag{3.7}$$

To see this, suppose there exists $a, b \in R$ such that $\delta_a \cdot \delta_b < 0$. Without loss of generality, suppose that $p_a \ge p_b$. Because $|R|$ is minimal and $a \in R$, we know that adding a to any A_k, $k \in [t]$, the thus obtained partition would not satisfy ii):

$$p_a + \sum_{i\in A_k} p_i > (1-9\alpha)m \text{ or } \delta_{A_k} + \delta_a \notin [-\alpha m, \alpha m].$$

If the first of these two conditions holds for each $k \in [t]$, then we have

$$\sum_{i\in A_k} p_i > (1-9\alpha)m - p_a > (1-9\alpha)m - \alpha m \ge (1-10\alpha)m,$$

which is a contradiction, because we would have $\sum_{i\in I} p_i = \sum_{k\in[t]}\sum_{i\in A_k} p_i \ge (1-10\alpha)mt$ and, at the same time by c), $\sum_{i\in I} p_i < (1-10\alpha)mt$. Therefore,

$$\text{there exists } k \text{ such that } p_a \textit{ fits} \text{ in } A_k\text{, this is, such that } \sum_{i\in A_k} p_i + p_a \le (1-9\alpha)m. \tag{3.8}$$

So $\delta_{A_k} + \delta_a \notin [-\alpha m, \alpha m]$. Suppose that $\delta_a < 0$. The case $\delta_a > 0$ is proved similarly. We claim that

$$\delta_{A_k} \in [-\alpha m, 0). \tag{3.9}$$

To see this, suppose that $\delta_{A_k} \notin [-\alpha m, 0)$ and therefore $\delta_{A_k} \in [0, \alpha m]$ by ii). Then, adding δ_a to δ_{A_k} we would obtain that

$$-\alpha m \leq \delta_{A_k} + \delta_a \leq \alpha m,$$

with the first inequality coming from *b*) and $\delta_{A_k} \in [0, \alpha m]$. The second inequality comes from ii) and the assumption that $\delta_a < 0$. This would contradict that $\delta_{A_k} + \delta_a \notin [-\alpha m, \alpha m]$, proving (3.9).

Since we supposed that $p_a \geq p_b$, by (3.8) p_b fits in A_k. So $\delta_{A_k} + \delta_b \notin [-\alpha m, \alpha m]$. Because we assumed $\delta_a \cdot \delta_b < 0$ and $\delta_a < 0$, $\delta_b > 0$. Also, $\delta_b \leq \alpha m$ by *b*). Thus, by (3.9), $\delta_{A_k} + \delta_b \in [-\alpha m, \alpha m]$, a contradiction to i) since we could add b to A_k and reduce the size of R. This proves (3.7).

Next, we will prove that it is possible to assign every index in R to one of the sets A_i:

There exists a partition $\{R_1, ..., R_t\}$ of R such that $\sum_{i \in A_j \cup R_j} p_i \leq (1 - 7\alpha)m$ and $\sum_{i \in A_j \cup R_j} q_i \leq (1 - 7\alpha)m$, for each $j \in [t]$. (3.10)

Observe that if we prove this claim, then the proof of the Lemma is complete, taking $\mathcal{J}$ as $\{A_1 \cup R_1, ..., A_t \cup R_t\}$, with $\{R_i\}_{i=1}^t$ as in (3.10).

A collection $\{R_1, ..., R_t\}$ of subsets of R will be called *good* if the sets $\{R_1, ..., R_t\}$ are pairwise disjoint, $|R \setminus (R_1 \cup ... \cup R_t)|$ is minimal and, for each $j \in [t]$, the conditions $\sum_{i \in A_j \cup R_j} p_i \leq (1 - 7\alpha)m$, $\sum_{i \in A_j \cup R_j} q_i \leq (1 - 7\alpha)m$ are satisfied.

To see (3.10), take a good collection $\{R_1, ..., R_t\}$ of R. If $R \setminus (R_1 \cup ... \cup R_t)$ is empty, then the proof is complete. If not, take $k \in R \setminus (R_1 \cup ... \cup R_t)$. Because $|R \setminus (R_1 \cup ... \cup R_t)|$ is minimal, we have that

(1) $p_k + \sum_{i \in A_j \cup R_j} p_i > (1 - 7\alpha)m$, or

(2) $q_k + \sum_{i \in A_j \cup R_j} q_i > (1 - 7\alpha)m$.

From now on we assume that $\delta_i \geq 0$ for every $i \in R$, since by (3.7), there are only two cases: $\delta_i \geq 0$ for every $i \in R$ or $\delta_i \leq 0$ for every $i \in R$. The proof for the other case (that is, $\delta_i \leq 0$ for every $i \in R$) is analogous and can be done inverting the roles of p and q in the next part of the proof.

We know that

$$\sum_{i \in A_j \cup R_j} p_i = \sum_{i \in A_j \cup R_j} (q_i + \delta_i) \geq \sum_{i \in A_j \cup R_j} q_i + \delta_{A_j} \geq \sum_{i \in A_j \cup R_j} q_i - \alpha m, \tag{3.11}$$

with the inequalities coming from $\delta_{R_j} \geq 0$ and ii) for δ_{A_j}.

We claim that

$$\sum_{i \in A_j \cup R_j} p_i > (1 - 10\alpha)m. \tag{3.12}$$

Indeed, this is easy to see if (1) holds. If (2) holds, note that

$$
\begin{aligned}
\sum_{i\in A_j\cup R_j} p_i \geq \sum_{i\in A_j\cup R_j} q_i - \alpha m \\
&> (1-7\alpha)m - q_k - \alpha m \\
&> (1-10\alpha)m,
\end{aligned}
$$

by (3.11) and b), proving (3.12).

Using (3.12) we calculate

$$
\sum_{i\in I} p_i > \sum_{j\in[t]} \sum_{i\in A_j\cup R_j} p_i > (1-10\alpha)mt
$$

which contradicts c), completing the proof of (3.10). □

3.2 Proving Theorem 1.9

PROOF. **Defining the constants.**

Let $\varepsilon := \frac{\eta^4}{10^6}$. Using Lemma 2.4 with $(\varepsilon, 2\sqrt{\varepsilon})$ playing the role of (ε, d), we obtain constants m_0 and M_0, such that we can apply the lemma to digraphs on $n \geq m_0$ vertices and the resulting regularized digraph will be partitioned into, at most, M_0 clusters.

Let $\beta := \frac{\varepsilon}{2M_0}$. Suppose $\Delta \geq 3$ and set

$$
n_0 := \max\left\{m_0,\ \frac{1}{\beta\eta},\ \frac{12M_0^2}{\varepsilon^2}\left(\frac{8M_0(1+\eta/2)}{1+\eta}+2\right),\ \frac{12M_0^2}{\varepsilon^2}\Delta(\Delta-1)^{8(1+\eta/2)\cdot\frac{M_0}{1+\eta}+1}\right\}
$$

In conclusion, we have $\frac{1}{n_0} \ll \beta \ll \varepsilon \ll \eta < 1$.

Let $n \geq n_0$ and $k \geq \eta n$ given.

Preparing the graph.

Let D be a digraph on n vertices. We apply Lemma 2.4 with $(\varepsilon, 2\sqrt{\varepsilon})$ to obtain D', a digraph with $r \leq M_0$ clusters $C_1, ..., C_r$ of size m and $\delta^0(D') > (1+\frac{\eta}{2})\frac{k}{2}$. We divide each cluster of D' into two slices: C_i^1 of size $4\sqrt{\varepsilon}m$ and C_i^2 of size $(1-4\sqrt{\varepsilon})m$.

We define R as the reduced oriented graph of D', which consists of vertices $C_1, ..., C_r$, and has minimum semidegree greater than $\lceil(1+\frac{\eta}{2})\frac{k}{2n}r\rceil =: t$, as stated in Lemma 2.5. By Lemma 3.2, R has a connected antimatching $M = \{a_ib_i\}_{i=1}^t$ such that

$$
\operatorname{dist}(a_1b_1, a_jb_j) \leq 8t, \text{ for } j \in [t]. \tag{3.13}
$$

Preparing the tree.

Let T be a given balanced rooted antitree with k edges and $\Delta(T) \leq \Delta$. By Lemma 3.3, we obtain a β-decomposition $(W, \mathcal{T})$ of T, with $|W| < \frac{1}{\beta} + 2$. For each $S \in \mathcal{T}$, we define $(p_S, q_S) := (|V_{\text{in}}(S) - L_{16t+2}(T)|, |V_{\text{out}}(S) - L_{16t+2}(T)|)$.

We claim that the family $(p_S, q_S)_{S \in \mathcal{T}}$ satisfies a), b) and c) of Lemma 3.6, with I as $\mathcal{T}$: to see this, we choose $\alpha = \sqrt{\varepsilon}$ and, by Lemma 3.4, this gives us a) of Lemma 3.6.

Because we have a β-decomposition, we know that $p_S + q_S \leq |S| \leq \beta k$. So, we have

$$\beta k \leq \beta n \leq \frac{\sqrt{\varepsilon} n}{2M_0} \leq \sqrt{\varepsilon} m$$

which implies b): $p_S + q_S < \sqrt{\varepsilon} m$ for all $S \in \mathcal{T}$, and thus, condition b) of Lemma 3.6 holds.

Lastly, to see c), we observe that by the way we chose our constants the following holds:

$$\begin{aligned}\sum_{S \in \mathcal{T}} p_S \leq \frac{k+1}{2} &\leq \frac{1}{2}(1-\varepsilon)(1-10\sqrt{\varepsilon})(1+\frac{\eta}{2})\frac{k}{2} \\ &\leq (1-10\sqrt{\varepsilon}) \cdot \frac{(1-\varepsilon)n}{r} \cdot \lceil (1+\frac{\eta}{2})\frac{k}{2n} r \rceil \\ &\leq (1-10\sqrt{\varepsilon}) mt,\end{aligned}$$

with the first inequality coming from the fact that T is balanced. Observe that the same holds for $\sum_{S \in \mathcal{T}} q_S$, completing the proof of c).

Therefore, we can apply Lemma 3.6, obtaining a partition $\{P_j\}_{j=1}^{t}$ of $\mathcal{T}$, such that

$$\text{For every } j \in [t],\ \textstyle\sum_{i \in P_j} p_i, \sum_{i \in P_j} q_i \leq (1-7\sqrt{\varepsilon}) m. \tag{3.14}$$

The embedding.

The idea of the embedding procedure is to execute an iterative process. To simplify the explanation, we will refer as C_{a_i} (resp. C_{b_i}) to the cluster corresponding to a_i (resp. b_i) in our reduced oriented graph R, for every antimatching edge $a_i b_i$. The process starts by embedding the root of T in $C^1_{a_1}$. After that, in every step of the process, we will embed, for some $S \in \mathcal{T}$, the antitree $T[S \cup W_S]$, where $W_S \subseteq W$ is the set of all seeds having their parent in S. We choose S such that $p_{r(S)}$, the parent of $r(S)$, has already an image $\phi(p_{r(S)})$ in some cluster $\bar{C}$. Suppose $S \in P_j$. By (3.13), we know that there is an antiwalk $\bar{C} C_0 ... C_h C_{a_j}$ in R with $h < 16t$, as $\bar{C}$ is in an edge of M or antiwalk in R. Then, thanks to the bounded degree of T and the way we chose $\phi(p_{r(S)})$ earlier, we can assure that $\phi(p_{r(S)})$ has enough neighbours in C_0 to embed the root of S. So, we embed the first $16t+2$ levels of $T[S \cup W_S]$ in $C^1_0, ..., C^1_h, C^1_{a_j}$, and the rest in $\{C^2_{a_j}, C^2_{b_j}\}$, repeating the procedure until there are no more antitrees left in $\mathcal{T}$ to embed.

Let us make this sketch more precise. Let $V_1 = \bigcup_{i=1}^{r} C^1_i$ and $V_2 = \bigcup_{i=1}^{r} C^2_i$. We embed $r(T)$ in a vertex of $C^1_{a_1}$ that is typical towards $C^1_{b_1}$. If $N(r(T)) \subseteq W$, we embed $N(r(T))$ in vertices of $C^1_{b_1}$ that are typical towards $C^1_{a_1}$. We repeat this process every time until there exists an $S \in \mathcal{T}$ with its root already embedded. We will show that at every step of the process, this means, every time we embed a new antitree $T[S \cup W_S]$, the following conditions are satisfied:

(A) for each $S \in \mathcal{T}$, the first $16t+2$ levels of $T[S \cup W_S]$ are embedded into V_1, and the rest into V_2,

(B) for each $S \in P_j$, $V_{\text{out}}(T[S \cup W_S]) \setminus L_{16t+2}(T)$ is embedded in $C^2_{a_j}$ and $V_{\text{in}}(T[S \cup W_S]) \setminus L_{16t+2}(T)$ is embedded in $C^2_{b_j}$,

(C) every vertex in $V_{\text{in}}(T[S \cup W_S])$ (resp. $V_{\text{out}}(T[S \cup W_S])$) is embedded in a cluster corresponding to an in-vertex (resp. out-vertex) of an antiwalk of length at most $8t$ starting in a_1b_1 and ending in another edge of M, and

(D) for every $w \in W$, the image of w is typical with respect to C^1_i, with C_i a cluster in an antiwalk of length at most $8t$ starting in a_1b_1 and ending in another edge of M.

Start by assuming we are in a step of the process and about to embed an antitree $T[S \cup W_S]$, with $S \in \mathcal{T}$, such that the parent of $r(S)$ is already embedded in a typical vertex of a cluster $\bar{C}$, by (D). Since $S \in P_j$, for some $j \in [t]$, we are looking for an antiwalk of length at most $16t+2$ from $\bar{C}$ to C_{a_j} to embed the first $16t+2$ levels of $T[S \cup W_S]$. Because of (C), (3.13) ensures the existence of a walk from $\bar{C}$ to C_{a_1} and another from C_{a_1} to C_{a_j}, with total length less than $16t+2$.

Suppose $r(S) \in V_{\text{in}}(S)$. To see that the first $16t+2$ levels of $T[S \cup W_S]$ can be embedded in that sequence of clusters, it suffices to apply Lemma 3.5. Let us see that the hypotheses of the lemma are satisfied: we already have the desired path $C_0 ... C_h$, with C_1C_0 and $C_{a_j}C_{b_j}$ its first and last edges, and $C_{h-1} \cup C_h = C_{a_j} \cup C_{b_j}$.

Define X_{h-1} as the unoccupied vertices of C^2_{h-1} and X_h as the unoccupied vertices of C^2_h. By (A), the only vertices in C^2_{h-1}, C^2_h are seeds and antitrees in P_j without their first $16t+2$ levels. By (3.14) and the size of C^2_{h-1}, C^2_h, we conclude that $|X_{h-1}|, |X_h| > 2\sqrt{\varepsilon}m > \varepsilon m$.

Let $p_{r(S)}$ be the parent of $r(S)$. Define Z_0 as $N(\phi(p_{r(S)})) \cap C^1_0$ minus the already occupied vertices of C^1_0. For $0 < i \leq h$, define Z_i as C^1_i minus its already occupied vertices. So, the size of Z_i is $4\sqrt{\varepsilon}m$ minus the number of occupied vertices in C^1_i. We will see that the number of occupied vertices in any C^1_i is at most $\sqrt{\varepsilon}m$: because the tree has bounded maximum degree, the first $16t+2$ levels of S' have at most $\Delta(T) \cdot (\Delta(T)-1)^{16t+1}$ vertices (or $16t+2$ in the case $\Delta < 3$). By (A), we know that C^1_i has only been occupied with vertices from $L_{16t+2}(T)$

and W. Hence, we can compute the amount of vertices used for embedding $L_{16t+2}(T)$:

$$\begin{aligned}|W|\cdot\Delta(T)\cdot(\Delta(T)-1)^{16t+1} &\leq \frac{3}{\beta}\cdot\Delta(\Delta-1)^{16t+1}\\ &= \frac{3\cdot 2M_0}{\varepsilon}\Delta(\Delta-1)^{16(1+\eta/2)\cdot\frac{k}{2n}\cdot r+1}\\ &\leq \frac{3\cdot 2M_0}{\varepsilon}\Delta(\Delta-1)^{8(1+\eta/2)\cdot\frac{M_0}{1+\eta}+1}\\ &\leq \varepsilon\cdot\frac{n_0}{2M_0}\\ &\leq \varepsilon m\end{aligned}$$

Using in the third line that $r\leq M_0$ and that $\frac{k}{n}\leq\frac{1}{1+\eta}$, which comes from the fact that $n=|V(D)|>2\delta^0(D)>(1+\eta)k$.

Then the number of occupied vertices for the first $16t+2$ levels of every tree in $\mathcal{T}$ is, at most, εm. Since the seeds are also bounded by εm, we have that the number of occupied vertices in any C_i^1 is less than $\sqrt{\varepsilon}m$.

Because $\phi(p_{r(S)})$ was chosen typical with respect to the unoccupied vertices of C_0^1, $|Z_0|\geq 3\varepsilon m$. For $0<i\leq h$, we would have that $|Z_i|>3\sqrt{\varepsilon}m$. Thus, we can apply Lemma 3.5 and we conclude the embedding of the first $16t+2$ levels of $T[S\cup W_S]$. This construction proves (A), (C) and (D) for the vertices in the first levels.

Now it is left to show that the remaining levels of $T[S\cup W_S]$, if there are any, can be embedded. Let v be a vertex in the level $16t+3$ of $T[S\cup W_S]$. Its parent, p_v, is embedded in Z_h or Z_{h-1}, depending on the orientation of vp_v in T. Suppose that $\phi(p_v)\in Z_{h-1}$ and that $C_{h-1}=C_{a_j}$. Because of Lemma 3.5, $\phi(p_v)$ is typical with respect to X_h. Observe that, at any point of the process, $|X_h|\geq 2\sqrt{\varepsilon}m$. This is true because $|C_h^2|=(1-4\sqrt{\varepsilon})m$ and the number of occupied vertices in C_h^2 is, at most, $(1-6\sqrt{\varepsilon})m$. This comes from (A) and (B), both of which state that the occupied vertices in C_h^2 are only seeds, that are less than $\sqrt{\varepsilon}m$, and the in-vertices of P_j, that add less than $(1-7\sqrt{\varepsilon}m)$ by 3.14. This implies that $|N(\phi(p_v))\cap X_h|\geq 2\varepsilon m$.

If $v\in S$, we choose its image u in X_h typical with respect to X_{h-1}. If $v\in W_S$, we choose its image u in X_h and it has to be typical with respect to Z_{h-1} by (D). This is possible in both cases since there are, at most, εm vertices in X_h that are not typical with respect to Z_{h-1} or X_{h-1} and $|N(\phi(p_v))\cap X_h|\geq 2\varepsilon m$. Fix $\phi(v)=u$ and delete u from X_h. Repeat this process for every vertex in the $(16t+3)$-th level of $T[S\cup W_S]$. This completes the embedding of the whole $(16t+3)$-th level.

The process for the remaining levels of $T[S\cup W_S]$ is the same as the one we did for the $(16t+3)$-th level, so we repeat it until there are no more levels to embed. By construction, (A), (B), (C) and (D) hold for $T[S\cup W_S]$. This completes the embedding of $T[S\cup W_S]$. $\square$

3.3 Conclusions

As usual in research, there are many questions and paths open to keep studying. For instance our main result, Theorem 1.9, leaves place for questions and opportunities to be improved. An example of these questions is:

Question 3.7 *Does Theorem 1.9 remain true if the antitree is not balanced?*

For simple graphs, there are results on embedding trees in which the maximum degree of the tree is not bounded by a constant, but by a function in n, the order of the host graph:

Theorem 3.8 *(Komlós, Sárközy and Szemerédi [15]) For all $\delta > 0$, there are n_0 and c such that every graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1+\delta)\frac{n}{2}$ contains each n vertex tree T with $\Delta(T) \leq \frac{cn}{\log(n)}$.*

So, instead of removing a condition as we asked in Question 3.7, improving one is what inspires the following question.

Question 3.9 *Could the condition on the maximum degree of the antitree in Theorem 1.9 be improved?*

Although, as there exist conditions to be improved in Theorem 1.9, one that is tight is the condition on the minimum semidegree. The example of the C_3 blow-up mentioned in the introduction for Conjecture 1.6 works, since the antipath is a balanced antitree. The example also works for other antitrees, since the problem is the same: we would need vertices from the three clusters, and is impossible to have that without going through a directed path of length 2.

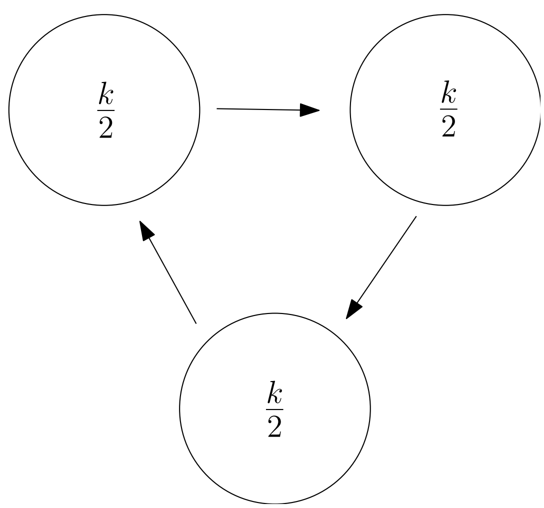

Figure 3.1: Blow-up of a C_3.

Going back to Conjecture 1.6, this conjecture was stated for every orientation of an oriented path and we only studied the antidirected one. Thus, the natural question to ask given our result would be:

Question 3.10 *Does Theorem 1.9 work for oriented trees that are not antidirected?*

The approach we followed in this thesis may not be useful for these unanswered questions. In this thesis the Regularity Lemma and the connected antimatchings were very helpful tools, since we benefited from the non existent directed paths in the tree. But, for example, for other orientations of a tree, using antimatchings may not be the best approach. Though the same idea elaborated on this thesis using a different structure depending on the orientation we are looking for might work. So, innovating in methods and techniques could be a way to answer the mentioned questions.

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