Non-propagation for zero order pseudodifferential operators

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El hombre tiene mil planes para sí mismo. El azar, sólo uno para cada uno.

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Abstract

We show phase space localization at suitable energies for zero order pseudodifferential operators, implying non-propagation properties for the associated evolution groups. This extends previous results which only treated configuration space anisotropic behavior. The proofs rely on Rieffel's strict deformation quantization of C^*-algebras acted by a vector group and on a quasiorbit analysis of some connected locally compact dynamical systems.

1 Introduction

The main purpose of this thesis is to prove some phase-space localization results for the functional calculus and for the evolution group of certain Weyl pseudodifferential operators $H = \mathfrak{Op}(f)$ acting in the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^n)$ with symbols presenting full phase-space anisotropy. Very roughly, a symbol

$$\mathbb{R}^n \times (\mathbb{R}^n)^* \ni (x, \xi) \mapsto f(x, \xi) \in \mathbb{R}$$

has full phase-space anisotropy if it has non-trivial behavior both for $|x| \to \infty$ and $|\xi| \to \infty$; of course more assumptions will be needed to make our theory work. For us, the trivial behavior would be convergence to either zero or infinity.

To describe the localization issues let us consider a (maybe unbounded) self-adjoint operator H in the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^n)$. We think it to be the quantum Hamiltonian of a physical system moving in $\mathbb{R}^n$, so its evolution group $\{e^{itH} \mid t \in \mathbb{R}\}$ describes the time evolution of the quantum system. Thus, if at the initial moment the system is in a state modellized by the normalized vector $v \in \mathcal{H}$, at time t it will be in the state associated to $v_t := e^{itH}v$.

By general principles of Quantum Mechanics, the probability at time t for the system to be localized within the Borel subset U of $\mathbb{R}^n$ is given by the number

$$\|\chi_U e^{itH} v\|^2 = \int_U dx\, |v_t(x)|^2 .$$

Very often one is interested in the behavior of this quantity when the initial state v has a certain localization in energy. If E is a Borel subset of $\mathbb{R}$, an interval for instance, we say that the state has energy belonging to E if $v = \chi_E(H)v$, where the characteristic function of E is applied to the self-adjoint operator H via the usual Borel functional calculus. For technical reasons we also consider as interesting vectors satisfying the condition $v = \rho(H)v$, where $\rho : \mathbb{R} \to \mathbb{R}_+$ is a continuous (or a smooth) function; it can be, for example, a continuous approximation of the characteristic function χ_E . Anyhow, we are motivated to consider the dependence of the quantity $\|\chi_U e^{itH}\rho(H)v\|^2$ on the parameters U, ρ and t . The normalization of v is not essential, so we shall replace it by an arbitrary vector u .

The type of result we are looking for would say that, under certain assumptions on H and ρ and for a given family $\mathcal{U}$ of non-void Borel subsets of $\mathbb{R}^n$, for every $\varepsilon > 0$ there is an element U of the family $\mathcal{U}$ such that

$$\|\chi_U e^{itH}\rho(H)u\|^2 \le \varepsilon^2 \|u\|^2 \quad \text{for all } t \in \mathbb{R} \text{ and } u \in \mathcal{H} . \tag{1.1}$$

Admitting that in some sense the family $\mathcal{U}$ converges to some region F (eventually situated "at infinity"), this means roughly that states with energies contained in the support of the function ρ cannot propagate towards F .

All these being said, let us notice however that (1.1), although dynamically significant, does not really have a dynamical nature. It is perfectly equivalent to the estimate

$$\|\chi_U \rho(H)\|_{\mathbb{B}(\mathcal{H})} \leq \varepsilon \,, \tag{1.2}$$

written in terms of the operator norm of $\mathbb{B}(\mathcal{H})$, the C^*-algebra of all linear bounded operators in the Hilbert space $\mathcal{H}$. It is obvious that such an estimate needs some tuning between the energy-localization function ρ and the family $\mathcal{U}$; without it one can only write

$$\|\chi_U \rho(H)\|_{\mathbb{B}(\mathcal{H})} \leq \|\chi_U\|_{\mathbb{B}(\mathcal{H})} \|\rho(H)\|_{\mathbb{B}(\mathcal{H})} = \sup_{\lambda \in \mathrm{sp}(H)} \rho(\lambda) \,,$$

and clearly we are interested in the case in which the support of ρ has a non-trivial intersection with the spectrum $\mathrm{sp}(H)$ of the Hamiltonian H.

A simple-minded relevant situation is as follows: If the support of the function ρ is disjoint from the essential spectrum $\mathrm{sp}_{\mathrm{ess}}(H)$ of H, it is known that the operator $\rho(H)$ is compact (finite-rank actually). If, in addition, this support contains points of the discrete spectrum $\mathrm{sp}_{\mathrm{dis}}(H) := \mathrm{sp}(H) \setminus \mathrm{sp}_{\mathrm{ess}}(H)$, then $\rho(H) \neq 0$. Let $\mathcal{U}$ be the filter formed by the complements of all the relatively compact subsets of $\mathbb{R}^n$. Then the family of operators of multiplication by χ_U converges strongly to zero. Multiplication with a compact operator improves this to norm convergence, so for each $\varepsilon > 0$ there is a sufficiently large (relatively) compact set $K \in \mathbb{R}^n$ such that $\|\chi_{K^c} \rho(H)\|_{\mathbb{B}(\mathcal{H})} \leq \varepsilon$. In dynamical terms, this would mean that states localized in the discrete spectrum cannot propagate to infinity.

For less trivial situations we consider the case of generalized Schrödinger operators $H = \mathfrak{Op}(f)$ in $L^2(\mathbb{R}^n)$ defined by the Weyl quantization of the symbol

$$f(x,\xi) = h(\xi) + V(x) \,,$$

where

$$V : \mathbb{R}^n \to \mathbb{R} \,, \quad h : (\mathbb{R}^n)^* \to \mathbb{R}$$

are convenient functions. Then

$$H = \mathfrak{Op}(f) = h(D) + V(Q) \,,$$

where Q is the position operator, $D := -i\nabla$ is the momentum and $h(D)$, $V(Q)$ can also be constructed by the usual functional calculus associated to (families of commuting) self-adjoint operators. Of course $h(D)$ is a convolution operator and even a constant coefficient differential operator if h is a polynomial, while $V(Q)$ is the operator of multiplication with the function V.

Assume now that $n = 1$, that V is continuous and

$$\lim_{x \to \pm\infty} V(x) = V_\pm \in \mathbb{R} \ \text{ with } \ V_- < V_+$$

and take for simplicity $h(\xi) := \xi^2$, so

$$H = -\Delta + V(Q)$$

is a one-dimensional Schrödinger Hamiltonian with configuration space anisotropy. Below V_- the spectrum of H is discrete, so one can apply the discussion above. But it is more interesting to take ρ supported in the interval (V_-, V_+). If the convergence of V towards the limits $V_\pm$ is fast enough, propagation towards infinity is possible in this region. But for physical reasons one expects this to happen only "to the left". This is not difficult to prove rigorously: for every $\varepsilon > 0$ there exists a real number a such that

$$\|\chi_{(a,+\infty)}(Q)\rho(H)\|_{\mathbb{B}(\mathcal{H})} \leq \varepsilon\,.$$

Thus "propagation to the right is forbidden" at energies smaller than V_+. In this example we make use of the filter base $\mathcal{U} := \{(a,\infty)\,|\,a \in \mathbb{R}\}$ formed of neighborhoods of the point $+\infty$ in the two-point compactification $[-\infty, +\infty]$ of the real axis.

A more complicated version is less easy to guess just by physical grounds. We consider the same Hamiltonian $H = -\Delta + V(Q)$ for $n = 1$ but now

$$\lim_{x\to\pm\infty} [V(x) - V_\pm(x)] = 0\,,$$

where $V_\pm$ are two periodic functions, with periods $T_\pm > 0$. In this case

$$\mathrm{sp}_{\mathrm{ess}}(H) = \mathrm{sp}(H_-) \cup \mathrm{sp}(H_+)\,,$$

where the asymptotic Hamiltonians

$$H_\pm := -\Delta + V_\pm(Q)\,,$$

being periodic, have a band structure for the spectrum. We don't know if this intuitive enough, but it can be shown however, that if the support of ρ does not meet $\mathrm{sp}(H_+)$, then propagation to the right is impossible in the same precise meaning as above. It is not difficult to construct a two-tori compactification of $\mathbb{R}$ of the form

$$\Omega := (\mathbb{R}/T_-\mathbb{Z}) \cup \mathbb{R} \cup (\mathbb{R}/T_+\mathbb{Z})$$

such that V satisfies the stated conditions if and only if it extends to a continuous function on this compactification. Then the two asymptotic Hamiltonians are fabricated from the restrictions of this extension to the two tori and the regions of non-propagation can be once again described in terms of neighborhoods of these tori in the compactification.

To illustrate the different types of anisotropy on the simple example of generalized Schrödinger operators, assume again that $n = 1$ and

$$f(x,\xi) = h(\xi) + V(x)$$

(the case $f(x,\xi) = h(\xi)V(x)$ can also be discussed along the same lines), where $V : \mathbb{R}^n \to \mathbb{R}$ and $h : (\mathbb{R}^n)^* \to \mathbb{R}$ are continuous functions. Let us assume for simplicity that

$$\lim_{\xi\to\pm\infty} h(\xi) = h_\pm \;\text{ and }\; \lim_{x\to\pm\infty} V(x) = V_\pm\,;$$

the limits are elements of the extended real axis. If $h_\pm = \infty$ (or if $h_\pm = 0$) and $V_\pm \in \mathbb{R}$, the operator is said to possess *configuration space anisotropy* (especially if $V_- \neq V_+$). But if $h_\pm \in \mathbb{R}$ and $V_\pm \in \mathbb{R}$, we are in the presence of *a full phase-space anisotropic problem.* Let us keep this classification incomplete. For larger dimension n and with more complicated types of behavior at infinity it might not be so easy to say exactly what kind of anisotropy we are dealing with, so the reader should take the present discussion at a heuristic level.

For a given self-adjoint operator L we denote by $\mathrm{sp}(L)$ the spectrum and by $\mathrm{sp}_{\mathrm{ess}}(L)$ the essential spectrum. In the example above, if $h_\pm = \infty$ (anisotropy in configuration space), denoting $\min\{g(y)\}$ by g_m and $\max\{g(y)\}$ by g_M, one has

$$\mathrm{sp}_{\mathrm{ess}}(H) = [h_m + \min(V_-, V_+), \infty) = \mathrm{sp}[h(D) + V_-] \cup \mathrm{sp}[h(D) + V_+]. \qquad (1.3)$$

It is easy to generalize a result above to this case and show that if $\mathrm{supp}(\rho)$ does not meet

$$\mathrm{sp}[h(D) + V_+] = [h_m + V_+, \infty),$$

then for every $\varepsilon > 0$ there exists $a \geq 0$ such that

$$\|\chi_{(a,+\infty)}(Q)\rho(H)\|_{\mathbb{B}(\mathcal{H})} \leq \varepsilon.$$

A similar result leading to "non-propagation to the left" is available by replacing + by − and $(a, +\infty)$ with $(-\infty, -a)$.

On the other hand, for full phase-space anisotropy ($h_\pm \in \mathbb{R}$ and $V_\pm \in \mathbb{R}$), the essential spectrum is given by four contributions

$$\begin{aligned}\mathrm{sp}_{\mathrm{ess}}(H) &= \mathrm{sp}[h(D) + V_-] \cup \mathrm{sp}[h(D) + V_+] \cup \mathrm{sp}[V(Q) + h_-] \cup \mathrm{sp}[V(Q) + h_+] \\ &= [h_m + V_-, h_M + V_-] \cup [h_m + V_+, h_M + V_+] \\ &\cup [h_- + V_m, h_- + V_M] \cup [h_+ + V_m, h_+ + V_M].\end{aligned} \qquad (1.4)$$

In this case one can show once again that $\|\chi_{(a,+\infty)}(Q)\rho(H)\|_{\mathbb{B}(\mathcal{H})}$ can be made arbitrary small for big $a \in \mathbb{R}_+$ if

$$\mathrm{supp}(\rho) \cap \mathrm{sp}[h(D) + V_+] = \emptyset$$

and that $\|\chi_{(-\infty,-a)}(Q)\rho(H)\|_{\mathbb{B}(\mathcal{H})}$ can be made arbitrary small for big $a \in \mathbb{R}_+$ if

$$\mathrm{supp}(\rho) \cap \mathrm{sp}[h(D) + V_-] = \emptyset.$$

But a new phenomenon appears, connected to the presence of the two other components in the essential spectrum of H: Suppose that the support of ρ does not meet $\mathrm{sp}[V(Q)+h_+]$. Then it can be shown that for every $\varepsilon > 0$ there exists $b \in \mathbb{R}_+$ such that

$$\|\chi_{(b,+\infty)}(D)\rho(H)\|_{\mathbb{B}(\mathcal{H})} \leq \varepsilon$$

(and a similar result for + replaced by −). This can be converted in an estimate of the form

$$\|\chi_{(b,+\infty)}(D)e^{itH}\rho(H)u\| \leq \varepsilon \, \|u\|$$

which is uniform in $t \in \mathbb{R}$ and $u \in L^2(\mathbb{R})$. It is no longer a statement about the probability of spatial localisation, but one about the probability of the system to have momentum larger than the number b.

In both cases the essential spectrum of the Hamiltonian $H = \mathfrak{O}\mathfrak{p}(f)$ can be written as union of spectra of "asymptotic Hamiltonians" that can be in some way obtained by extending the symbol $f(x, \xi) = h(\xi) + V(x)$ to a compactification of the phase space $\Xi := \mathbb{R} \times \mathbb{R}^*$ having the form of a square and then restricting it to the four edges situated "at infinity" (some simple reinterpretations are needed). Notice that the partial (configuration space) anisotropy is simpler: the restrictions to two of the edges do not contribute. In some sense the two corresponding asymptotic Hamiltonians are infinite and their spectrum is void. The reader is asked to imagine what would happen both at the level of the essential spectrum and at the level of localization estimates in the case of a pure momentum space anisotropy, when

$$\lim_{\xi \mapsto \pm\infty} h(\xi) = h_{\pm} \in \mathbb{R} \quad \text{and} \quad \lim_{x \mapsto \pm\infty} V(x) = \infty .$$

In n dimensions and for more general types of anisotropy (recall the periodic limits) one expects more sophisticated things to happen. Suppose that our Hamiltonian H is obtained via Weyl quantization from a convenient real function f defined in phase-space $\Xi := \mathbb{R}^n \times (\mathbb{R}^n)^*$. If its behaviour at infinity in both variables (x, ξ) is sophisticaded enough (corresponding to what could be called "phase-space anisotropy") then one could expect the following picture:

1. The essential spectrum is the (closure of the) union of spectra of a family of "asymptotic Hamiltonians" $H(F)$ associated to remote regions F of phase-space. There are several ways to express this. One would be to say that the behavior of f at infinity in Ξ can be described by a compactification $\Sigma = \Xi \sqcup \partial\Sigma$ of Ξ and that F is a conveniently defined subset of "the boundary at infinity" $\partial\Sigma$.

2. If a bounded continuos function ρ is supported away from one of the components $\mathrm{sp}[H(F)]$, then "propagation towards F is forbidden" at energies belonging to the support of ρ. This would be deduced from an estimate of the form

$$\| \mathfrak{O}\mathfrak{p}(\chi_W^\infty)\rho(H) \|_{\mathbb{B}(\mathcal{H})} \leq \varepsilon$$

written in terms of the Weyl quantization

$$\mathfrak{O}\mathfrak{p}(\chi_W^\infty) \equiv \chi_W^\infty(Q, D)$$

of a smooth regularization of the characteristic function χ_W of a subset W of Ξ. For small ε, the set W should be very close to the set F; for example it can be the intersection with Ξ of a small neighborhood $\mathcal{W}$ of F in the compactification Σ.

Until recently, there have been few general results for the essential spectrum of phase-space anisotropic pseudodifferential operators and this was the main obstacle to getting localization estimates. Techniques involving crossed products, very efficient for configurational anisotropy [12, 13, 14, 24, 2], are not available in such a case. In [26, 27] this problem was solved in a rather general setting, by using the good functorial properties of Rieffel's pseudodifferential calculus [38, 39], developed in the context of strict deformation quantization [40]. Roughly, if the symbol presents full phase-space anisotropy, the essential spectrum of the corresponding pseudodifferential operator can be written as the closed union of spectra of a family of "asymptotic" pseudodifferential operators. To obtain the symbols of these asymptotic operators one constructs a compactification of the phase space, which is naturally a dynamical system, and then determines the quasi-orbits of this dynamical system which are disjoint from the phase space itself. The extensions of the initial symbol to these quasi-orbits define the required asymptotic operators that contribute to the essential spectrum.

In the present thesis we are going to show that Rieffel's calculus can also be used to get the localization estimates, leading in their turn to non-propagation results for the evolution group; this extends the treatment in [2, 30, 21] of purely configurational anisotropic systems.

Let us describe briefly the content of this work.

First, in the next section, we give a brief description of some previous results. This will hopefully motivate our approach to cover the full anisotropy.

Section 3 will review some properties of the Rieffel quantization, one of our main tools. It has as basic data the action Θ of the vector space Ξ on a C^*-algebra $\mathcal{A}$ (for our purposes it is enough to take it commutative). The canonic symplectic form on Ξ is used to twist the product on $\mathcal{A}$. This twisting is done first on the set of smooth elements of $\mathcal{A}$ under the action. Then a C^*-norm is found on the resulting non-commutative *-algebra. The outcome will be a new C^*-algebra $\mathfrak{A}$ (the quantization of $\mathcal{A}$, composed of pseudodifferential symbols) also endowed with an action of the vector space Ξ.

In Section 4 we prove our first abstract result; it refers to the algebra of symbols. The proof has three steps: the first is basically topological, involving Abelian C^*-algebras; the second replaces the pointwise commutative composition with the relevant pseudodifferential-theoretical one and the third corrects the C^*-algebraic norm.

To get familiar statements, refering to pseudodifferential operators, one applies Hilbert space representations to this abstract result; this is done in Sections 5, 6 and 7 at various levels of generality.

The final section 8 is dedicated to some examples.

Finally, let us mention that in [5] localization estimates and the structure of the essential spectrum have been obtained for finite-difference operators on certain graphs, connected to the 1-dimensional Heisenberg model of ferromagnetism. Besides a certain C^*-algebraic background, the connections with the present work are rather limited. Results on the essential spectrum of natural families of discrete operators on rooted trees can be found in [15, 11].

2 A short review of previous results

As we said in the Introduction, we are interested in estimates of the form (1.2). After some preliminary previous results contained in [7], such estimates have been obtained in [2] for Schrödinger operators $H := -\Delta + V$, where Δ is the Laplace operator and V is the potential (the operator of multiplication by a real continuous function on $\mathbb{R}^n$). Thus in suitable units H is the Hamiltonian of a non-relativistic particle moving in $\mathbb{R}^n$ in the presence of the potential V and "localization" or "non-propagation" refers to this physical system. In [30] and [21] the results were significantly extended to certain pseudodifferential operators with variable magnetic fields, using the magnetic version of the Weyl calculus [28, 29, 17].

Leaving the magnetic fields apart, for simplicity, the Hamiltonians have now the form

$$H = \mathfrak{Op}(f),$$

being defined as the Weyl quantization of some real symbol f defined in phase-space

$$\Xi := \mathbb{R}^n \times (\mathbb{R}^n)^*.$$

The order of the elliptic symbol f (in Hörmander sense) is strictly positive, so one has

$$\lim_{\xi \to \infty} f(x, \xi) = \infty$$

and the behavior in $x \in \mathbb{R}^n$ is modelled by a C^*-algebra of bounded, uniformly continuous functions on $\mathbb{R}^n$. So the symbols defining the operators are still confined to the restricted configuration space anisotropy.

To be more precise, to suitable functions h defined on the phase space Ξ, one assigns operators acting on functions $u : \mathscr{X} := \mathbb{R}^n \to \mathbb{C}$ by

$$[\mathfrak{Op}(h)u](x) := (2\pi)^{-n} \int_{\mathscr{X}} \int_{\mathscr{X}^*} dy\, d\xi\, e^{i(x-y)\cdot\xi}\, h\left(\frac{x+y}{2}, \xi\right) u(y). \tag{2.1}$$

This is basically the Weyl quantization and, under convenient assumptions on h, (2.1) makes sense and has nice properties in the Hilbert space $\mathcal{H} := \mathsf{L}^2(\mathscr{X})$ or in the Schwartz space $\mathcal{S}(\mathscr{X})$.

Let $h : \Xi \to \mathbb{R}$ be an elliptic symbol of strictly positive order m. This means that h is smooth and satisfies estimates of the form

$$\left|\left(\partial_x^\alpha \partial_\xi^\beta h\right)(x, \xi)\right| \le C_{\alpha,\beta}(1 + |\xi|)^{m-|\beta|}, \quad \forall \alpha, \beta \in \mathbb{N}^n, \quad \forall (x, \xi) \in \Xi \tag{2.2}$$

and

$$|h(x, \xi)| \ge C(1 + |\xi|)^m, \quad \forall (x, \xi) \in \Xi, \ |\xi| \text{ large enough.} \tag{2.3}$$

It is well-known that under these assumptions $\mathfrak{Op}(h)$ makes sense as an unbounded self-adjoint operator in $\mathcal{H}$, defined on the m'th order Sobolev space. The problem is to evaluate the essential spectrum of this operator and to derive estimates for its functional calculus.

It comes out that the relevant information is contained in the behavior at infinity of h in the x variable. This one is conveniently taken into account through an Abelian algebra $\mathscr{A}$ composed of uniformly continuous functions un $\mathscr{X}$, which is invariant under translations (if $\varphi \in \mathscr{A}$ and $y \in \mathscr{X}$ then $\theta_y(\varphi) := \varphi(\cdot + y) \in \mathscr{A}$). Let us also assume (for simplicity) that $\mathscr{A}$ is unital and contains the ideal $C_0(\mathscr{X})$ of all complex continuous functions on $\mathscr{X}$ which converge to zero at infinity. We ask that the elliptic symbol h of strictly positive order m also satisfy

$$\left(\partial_x^\alpha \partial_\xi^\beta h\right)(\cdot,\xi) \in \mathscr{A}, \quad \forall\, \alpha,\beta \in \mathbb{N}^n,\ \forall\, \xi \in \mathscr{X}^*. \tag{2.4}$$

Then the function h extends continuously on $\Omega \times \mathscr{X}^*$, where Ω is the Gelfand spectrum of the C^*-algebra $\mathscr{A}$; this space Ω is a compactification of the locally compact space $\mathscr{X}$. By translational invariance of $\mathscr{A}$, it is a compact dynamical system under an action of the group $\mathscr{X}$. After removing the orbit $\mathscr{X}$, one gets a $\mathscr{X}$-dynamical system $\Omega_\infty := \Omega \setminus \mathscr{X}$; its quasi-orbits (closure of orbits) contain the relevant information about the essential spectrum of the operator $H := \mathfrak{Op}(h)$. For each quasi-orbit Q, one constructs a self adjoint operator H_Q. It is actually the Weyl quantization of the restriction of h to $Q \times \mathscr{X}^*$, suitably reinterpreted. Using the notations $\mathrm{sp}(T)$ and $\mathrm{sp}_{\mathrm{ess}}(T)$, respectively, for the spectrum and the essential spectrum of an operator T, one gets finally

$$\mathrm{sp}_{\mathrm{ess}}(H) = \overline{\bigcup_Q \mathrm{sp}(H_Q)}\,. \tag{2.5}$$

Many related results exist in the literature, some of them for special type of functions h, but with less regularity required, others including anisotropic magnetic fields, others formulated in a more geometrical framework or referring to Fredholm properties. We only cite [1, 4, 6, 10, 12, 13, 14, 16, 19, 18, 20, 21, 22, 23, 24, 30, 32, 33, 34, 35, 36, 37]; see also references therein. As V. Georgescu remarked [12, 13], when the function h does not diverge for $\xi \to \infty$, the approach is more difficult and should also take into account the asymptotic values taken by h in "directions contained in $\mathscr{X}^*$".

Now, in the framework above, we indicate the localization results. Let $H = \mathfrak{Op}(h)$ be a Weyl pseudodifferential operator with elliptic symbol of order $m > 0$. For some unital translation-invariant Abelian C^*-algebra $\mathscr{A}$ composed of uniformly continuous functions on $\mathscr{X}$ and containing $C_0(\mathscr{X})$, assume that $h(x,\xi)$ is $\mathscr{A}$-isotropic in the variable x, i.e. (2.4) holds. Choose a quasi-orbit Q in the boundary $\Omega_\infty := \Omega \setminus \mathscr{X}$ of the Gelfand spectrum of $\mathscr{A}$. As said above, one associates to Q a self-adjoint operator $H(Q)$; its spectrum is contained (very often strictly) in the essential spectrum of H. We also fix a bounded continuous function $\rho : \mathbb{R} \to [0,\infty)$ whose support is disjoint from $\mathrm{sp}[H(Q)]$. Then for every $\epsilon > 0$ there exists a neighborhood $\mathcal{U}$ of Q in Ω such that, setting $U := \mathcal{U} \cap \mathscr{X}$,

$$\|\chi_U(Q)\rho(H)\|_{\mathbb{B}(\mathcal{H})} \le \varepsilon \tag{2.6}$$

and

$$\|\chi_U(Q)e^{itH}\rho(H)u\|_{\mathcal{H}} \le \varepsilon\, \|u\|_{\mathcal{H}}, \quad \forall\, t \in \mathbb{R},\ u \in \mathcal{H}. \tag{2.7}$$

We recall that $\chi_U(Q)$ is, by definition, the operator of multiplication by the function χ_U in the Hilbert space $\mathcal{H} = L^2(\mathscr{X})$. Concrete examples has been indicated in [2, 30].

As remarked by V. Georgescu, a very efficient tool for obtaining some of the results cited above was the crossed product, associated to C^*-dynamical systems. In the setting presented before, one uses the action θ of $\mathscr{X}$ by translations on the C^*-algebra $\mathscr{A}$ to construct a larger, non-commutative C^*-algebra $\mathscr{A} \rtimes_\theta \mathscr{X}$. After a partial Fourier transform, this one can be seen to be generated by pseudodifferential operators of strictly negative order, with coefficients in $\mathscr{A}$. So it will contain resolvent families of elliptic strictly positive order Weyl operators satisfying (2.4) and the structure of the crossed product will rather easily imply spectral results. A basic fact is that the crossed product is a functor, also acting on equivariant morphisms, and that it behaves nicely with respect to quotients and direct sums. One drawback is, however, that ξ-anisotropy cannot be treated easily. The symbols of order 0 are not efficiently connected to the crossed products (treating them as multiplier would not be enough for our purposes).

As a substitute for crossed products, in [26, 27] Rieffel's version of the Weyl pseudodifferential calculus has been used to investigate the essential spectrum of full phase-space anisotropic Hamiltonians. Since it will also be needed for our study of localization results, we dedicate the next section to a recall.

3 Rieffel's pseudodifferential calculus

We shall recall briefly Rieffel's deformation procedure, sending to [38] for proofs and more details. Some convention will be different. Rieffel's main purpose was to provide a unified framework for a large class of examples in deformation quantization (cf. also [40]) and to study their convergence to some corresponding Poisson algebras, but this will not be important here.

Let us denote by $\mathscr{X}$ the vector space $\mathbb{R}^n$ on which, when necessary, the canonical base $(e_1, \dots, e_n)$ will be used. Its dual is denoted by $\mathscr{X}^*$ with the dual base $(e_{n+1}, \dots, e_{2n})$. Then "the phase space" $\Xi = \mathscr{X} \times \mathscr{X}^*$ with points generically denoted by

$$X = (x, \xi), Y = (y, \eta), Z = (z, \zeta)$$

is canonically a symplectic space with the symplectic form

$$[\![X, Y]\!] := x \cdot \eta - y \cdot \xi . \tag{3.1}$$

The duality between $x \in \mathscr{X}$ and $\xi \in \mathscr{X}^*$ has been denoted by a dot; by abuse we could think of it as a scalar product on $\mathbb{R}^n$, identified with its own dual.

We start with *a classical data*, which is by definition a quadruplet

$$(\mathcal{A}, \Theta, \Xi, [\![\cdot, \cdot]\!]) ,$$

where $\mathcal{A}$ is a C^*-algebra and a continuous action Θ of Ξ by automorphisms of $\mathcal{A}$ is also given. For $(f, X) \in \mathcal{A} \times \Xi$ we are going to use the notations

$$\Theta(f, X) = \Theta_X(f) \in \mathcal{A}$$

for the X-transformed of the element f. The function Θ is assumed to be continuous and the automorphisms Θ_X, Θ_Y satisfy

$$\Theta_X \circ \Theta_Y = \Theta_{X+Y} , \quad \forall X, Y \in \Xi .$$

Let us denote by $\mathcal{A}^\infty$ the vector space of all smooth elements f under Θ, those for which the mapping $\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{A}$ is C^∞ in norm; it is a dense *-algebra of $\mathcal{A}$. It is also a Fréchet *-algebra for the family of semi-norms

$$\| f \|^{(k)}_{\mathcal{A}} := \sum_{|\mu| \le k} \frac{1}{\mu!} \left\| \partial^\mu_X (\Theta_X(f))\big|_{X=0} \right\|_{\mathcal{A}} , \qquad k \in \mathbb{N} .$$

In the sequel we are going to use the abbreviations $\mathcal{D}^\mu f := \partial^\mu_X(\Theta_X(f))\big|_{X=0}$ for all the multi-indices $\mu \in \mathbb{N}^{2n}$. All the operators $\mathcal{D}^\mu$ are well-defined, linear and continuous on the Fréchet *-algebra $\mathcal{A}^\infty$.

Then one introduces on $\mathcal{A}^\infty$ the product

$$f \# g := \pi^{-2n} \int_\Xi \int_\Xi dY dZ\, e^{2i[\![Y,Z]\!]}\, \Theta_Y(f)\, \Theta_Z(g)\,, \tag{3.2}$$

rigorously defined as an oscillatory integral. There are three equivalent ways to give a meaning to this kind of expression; we review them briefly, sending to [38] for details.

1. **Integration by parts.** For $f, g \in \mathcal{A}^\infty$ we set

$$F(Y,Z) := \Theta_Y(f)\, \Theta_Z(g) \in \mathcal{A}^\infty\,,$$

but the arguments below are valid for many other functions F. We can define a scalar valued function by

$$K(Y,Z) = \left(1 + |(Y,Z)|^2\right)^{-1}$$

and M_K as the operator of pointwise multiplication by K. Using integration by parts it can be shown for every positive integer m that

$$\int_\Xi \int_\Xi dY dZ\, e^{2i[\![Y,Z]\!]} F(Y,Z) = \int_\Xi \int_\Xi dY dZ\, e^{2i[\![Y,Z]\!]} \left[\left((1+\widetilde{\Delta})M_K\right)^m F\right](Y,Z)\,,$$

where $\widetilde{\Delta}$ denotes a constant coefficient operator of second order in all $4n$ dimensions. Since the first-order derivatives of K are of the form KB, where B is a bounded rational function, using a induction argument we can express the product as

$$f \# g = \int_\Xi \int_\Xi dY dZ\, e^{2i[\![Y,Z]\!]} K^m(Y,Z) \sum_{|\mu| \le 2m} B_\mu(Y,Z)(\partial^\mu F)(Y,Z) \tag{3.3}$$

for a family $(B_\mu)_\mu$ of bounded functions. This, the fact that f, g are smooth vectors of the action Θ and the decay of K^m show that (3.3) makes sense for m large enough.

2. **Partition of unity.** One can use a representation of the composition $f \# g$ given by [38, Lemma 1.6, Cor. 1.7], in terms of a regular partition of unity of Ξ. Let $\mathfrak{L}$ be a lattice in Ξ; for example one could take

$$\mathfrak{L} := \left\{ \sum_{i=1}^{2n} \alpha_i e_i \mid \alpha_i \in \mathbb{Z} \right\}.$$

Pick then a non-trivial positive, smooth, compactly supported function ψ on Ξ such that

$$\Psi(X) := \sum_{P \in \mathfrak{L}} \psi(X - P) > 0\,, \qquad \forall\, X \in \Xi$$

and set $\psi_0 := \psi / \Psi$ and

$$\psi_P(\cdot) := \psi_0(\cdot - P)\,, \qquad \forall\, P \in \mathfrak{L}\,.$$

Then $\{\psi_P \mid P \in \mathfrak{L}\}$ will be a locally finite partition of unity on Ξ. It can be shown that the infinite sum

$$f\#g = \pi^{-2n} \sum_{P,Q\in\mathfrak{L}} \int_\Xi \int_\Xi dY dZ\, e^{2i[\![Y,Z]\!]}\, \psi_P(Y)\, \psi_Q(Z)\, \Theta_Y(f)\, \Theta_Z(g) \qquad (3.4)$$

converges absolutely.

3. **Cut-off.** For every $k \in \mathbb{N}$ let $\nu_k \in C_c^\infty(\Xi\times\Xi)$, with uniform bounds on the derivatives and such that $\nu_k(Y,Z)$ is equal to 1 in ball $B(0,r_k)$ of radius r_k that diverges to ∞. Then we have

$$f\#g = \lim_{k\to\infty} \pi^{-2n} \int_\Xi \int_\Xi dY dZ\, e^{2i[\![Y,Z]\!]}\, \nu_k(Y,Z)\, \Theta_Y(f)\, \Theta_Z(g)\,. \qquad (3.5)$$

Since the three expressions (3.3), (3.4) and (3.5) coincide, they are also independent of the various choices $\psi, \mathfrak{L}, \nu_m$.

To complete the algebraical structure, we keep the same involution *; one gets a *-algebra $(\mathcal{A}^\infty, \#, ^*)$. This *-algebra admits a C^*-completion $\mathfrak{A}$ in a C^*-norm $\|\cdot\|_{\mathfrak{A}}$ which is defined by Hilbert module techniques. Since the construction is rather involved and it will not play an explicit role for us, we only refer to [38, Ch. 4] for the details and justifications.

The deformation can be extended to Ξ-morphisms, giving rise to a covariant functor. Let

$$\left(\mathcal{A}_j, \Theta_j, \Xi, [\![\cdot,\cdot]\!]\right),\ j = 1,2$$

be two classical data and let

$$\mathcal{R} : \mathcal{A}_1 \to \mathcal{A}_2$$

be a Ξ-morphism, *i.e.* a (C^*-)morphism intertwining the two actions:

$$\mathcal{R} \circ \Theta_{1,X} = \Theta_{2,X} \circ \mathcal{R}, \qquad \forall X \in \Xi\,.$$

Then $\mathcal{R}$ sends $\mathcal{A}_1^\infty$ into $\mathcal{A}_2^\infty$ and extends to a morphism

$$\mathfrak{R} : \mathfrak{A}_1 \to \mathfrak{A}_2$$

that also intertwines the corresponding actions.

For us, the main property of this functor is that it preserves short exact sequences of Ξ-morphisms. Let $\mathcal{J}$ be a (closed, self-adjoint, two-sided) invariant ideal in $\mathcal{A}$ and denote by $\mathfrak{J}$ its deformation, using the procedure indicated above. Then $\mathfrak{J}$ is isomorphic (and will be identified) with an invariant ideal in $\mathfrak{A}$. In addition, on the quotient $\mathcal{A}/\mathcal{J}$ there is a natural quotient action of Ξ, so we can perform its Rieffel deformation. This one is canonically isomorphic to the quotient $\mathfrak{A}/\mathfrak{J}$.

If $h \in \mathfrak{A}$, the spectrum of its canonical image in the quotient C^*-algebra $\mathfrak{A}/\mathfrak{J}$ will be denoted by $\mathrm{sp}_{\mathfrak{J}}(h)$. Later we are going to need

Lemma 3.1. *1. Let $\rho : \mathbb{R} \to \mathbb{R}_+$ a bounded continuous function.*

1. *If $h \in \mathfrak{A}$ and $\mathrm{supp}(\rho) \cap \mathrm{sp}_{\mathfrak{J}}(h) = \emptyset$, then $\rho(h) \in \mathfrak{J}$.*
2. *If $h \in \mathfrak{A}^\infty$ and $\mathrm{supp}(\rho) \cap \mathrm{sp}_{\mathfrak{J}}(h) = \emptyset$, then $\rho(h) \in \mathfrak{J}^\infty$.*

Proof. 1. This is a minor variation of [2, Lemma 1]. It holds for every closed bi-sided self-adjoint ideal of a C^*-algebra.

2. The second assertion follows from the first one; $\mathfrak{J}$ is invariant under the action Θ and clearly $\mathfrak{J}^\infty = \mathfrak{A}^\infty \cap \mathfrak{J}$. □

Actually we are interested in deforming Abelian C^*-algebras. Let (Σ, Θ, Ξ) be a topological dynamical system with group $\Xi = \mathbb{R}^{2n}$. This means that Σ is a locally compact space, $\Theta : \Sigma \times \Xi \to \Sigma$ is a continuous map and, using notations as

$$\Theta(\sigma, X) =: \Theta_X(\sigma) = \Theta_\sigma(X), \qquad \forall\, X \in \Xi,\ \sigma \in \Sigma,$$

each $\Theta_X : \Sigma \to \Sigma$ is a homeomorphism and one has

$$\Theta_X \circ \Theta_Y = \Theta_{X+Y}, \qquad \forall\, X, Y \in \Xi.$$

One denotes by $\mathcal{B}(\Sigma)$ the C^*-algebra of all bounded complex functions on Σ with pointwise multiplication, complex conjugation and the obvious norm

$$\| f \|_\infty := \sup_\sigma |f(\sigma)|.$$

The action Θ of Ξ on Σ induces an action of Ξ on $\mathcal{B}(\Sigma)$ (also denoted by Θ) given by

$$\Theta_X(f) := f \circ \Theta_X. \tag{3.6}$$

In general this action fails to have good continuity or smoothness properties, so we introduce

$$\mathcal{B}_\Theta(\Sigma) := \{ f \in \mathcal{B}(\Sigma) \mid \Xi \ni X \mapsto \Theta_X(f) \in \mathcal{B}(\Sigma) \text{ is norm} - \text{continuous} \} \tag{3.7}$$

and

$$\mathcal{B}^\infty_\Theta(\Sigma) := \{ f \in \mathcal{B}(\Sigma) \mid \Xi \ni X \mapsto \Theta_X(f) \in \mathcal{B}(\Sigma) \text{ is } C^\infty \text{ in norm} \}. \tag{3.8}$$

We also denote by $C_0(\Sigma)$ the C^*-algebra of all complex continuous functions f on Σ such that for any $\varepsilon > 0$ there is a compact subset K of Σ such that

$$|f(\sigma)| \le \varepsilon \text{ if } \sigma \notin K.$$

Notice that $C_0(\Sigma)$ is a C^*-subalgebra of $\mathcal{B}_\Theta(\Sigma)$, but not an ideal in general. When Σ is compact, $C(\Sigma)$ is unital. The action Θ of Ξ on Σ induces an action on $C_0(\Sigma)$; we denote by $C^\infty_0(\Sigma)$ the set of smooth elements. The Rieffel deformations of $\mathcal{B}_\Theta(\Sigma)$ and $C_0(\Sigma)$ will be denoted, respectively, by $\mathfrak{B}_\Theta(\Sigma)$ and $\mathfrak{C}_0(\Sigma)$. Clearly, the deformation procedure can be applied to any C^*-subalgebra of $\mathcal{B}_\Theta(\Sigma)$ that is invariant under the action Θ.

Later on we shall need the following smoothing procedure. For $\varphi \in C_c^\infty(\Xi)$ and $g \in \mathcal{B}(\Sigma)$ one sets

$$g^\varphi \equiv \varphi *_\Theta g := \int_\Xi dY \varphi(Y)\, \Theta_{-Y}(g)\,. \tag{3.9}$$

If the action Θ consists in translations: $[\Theta_Y(f)]\,(X) := f(X+Y)$, then $*_\Theta$ coincides with the usual convolution. In this case $g^\varphi \in \mathrm{BC}^\infty(\Sigma) = \mathrm{BC_u}(\Sigma)^\infty$ and $\mathrm{supp}(g^\varphi) \subset \mathrm{supp}(g) + \mathrm{supp}(\varphi)$. We are going to need the next more general statement.

Lemma 3.2. *1. One has* $g^\varphi \in \mathcal{B}^\infty_\Theta(\Sigma)$.

2. *For every multi-index* $\alpha \in \mathbb{N}^{2n}$ *one has* $\mathcal{D}^\alpha g^\varphi = g^{\partial^\alpha \varphi}$.

3. *One has* $\mathrm{supp}(g^\varphi) \subset \Theta_{\mathrm{supp}(\varphi)}[\mathrm{supp}(g)]$.

Proof. By a change of variables one easily gets

$$\Theta_X(g^\varphi) = g^{\mathcal{T}_X \varphi}\,, \qquad \text{where} \quad (\mathcal{T}_X\varphi)(Y) := \varphi(Y+X)\,.$$

This and a standard application of the Dominated Convergence Theorem lead easily to the statements 1. and 2.

Now we show 3. Since Θ is continuous, $\mathrm{supp}(\varphi)$ is compact in Ξ and $\mathrm{supp}(g)$ is closed in Σ, it follows easily that $\Theta_{\mathrm{supp}(\varphi)}[\mathrm{supp}(g)]$ is closed in Σ. Let $\sigma \notin \Theta_{\mathrm{supp}(\varphi)}[\mathrm{supp}(g)]$; then there exists a neighborhood V of σ such that

$$V \cap \Theta_{\mathrm{supp}(\varphi)}[\mathrm{supp}(g)] = \emptyset\,.$$

For each $\sigma' \in V$ one has

$$[\varphi *_\Theta g](\sigma') = \int_{\mathrm{supp}(\varphi)} dY \varphi(Y) g[\Theta_{-Y}(\sigma')]$$

and if $Y \in \mathrm{supp}(\varphi)$ then $\Theta_{-Y}(\sigma') \notin \mathrm{supp}(g)$. This shows that V is disjoint from $\mathrm{supp}(g^\varphi)$. □

An important example to which Rieffel deformation apply is given by Ξ-*algebras, i.e.* C^*-algebras $\mathcal{B}$ composed of bounded, uniformly continuous function on Ξ, under the additional assumption that the action $\mathcal{T}$ of Ξ on itself by translations, raised to functions as in (3.6), leaves $\mathcal{B}$ invariant. Let us denote by Σ the Gelfand spectrum of $\mathcal{B}$. By Gelfand theory, there exists a continuous function : $\Xi \mapsto \Sigma$ with dense image, which is equivariant with respect to the actions $\mathcal{T}$ on Ξ, respectively Θ on Σ. The function is injective if and only if $C_0(\Xi) \subset \mathcal{B}$.

The largest such C^*-algebra $\mathcal{B}$ is $\mathrm{BC_u}(\Xi)$, consisting of all the bounded uniformly continuous functions : $\Xi \mapsto \mathbb{C}$. It coincides with the family of functions $g \in \mathrm{BC}(\Xi)$ (just bounded and continuous) such that

$$\Xi \ni X \mapsto g \circ \mathcal{T}_X = g(\cdot + X) \in \mathrm{BC}(\Xi)$$

is continuous. Then the Fréchet *-algebra of C^∞-vectors is

$$\mathsf{BC_u}(\Xi)^\infty \equiv \mathsf{BC}^\infty(\Xi) := \{f \in C^\infty(\Xi) \mid |(\partial^\alpha f)(X)| \le C_\alpha\,,\ \forall\, \alpha, X\}\,.$$

Another important particular case is $\mathcal{B} = C_0(\Xi)$ (just put $\Sigma = \Xi$ in the general construction). It is shown in [38] that at the quantized level one gets the usual Weyl calculus and the emerging non-commutative C^*-algebra $\mathfrak{C}_0(\Xi)$ is isomorphic to the ideal of all compact operators on an infinite-dimensional separable Hilbert space.

4 Localization in the symbolic calculus

We are given a topological dynamical system (Σ, Θ, Ξ) to which we associate, as in section 3, the Abelian C^*-algebras $\mathcal{B}_\Theta(\Sigma)$ and $C_0(\Sigma)$ as well as their Rieffel deformations $\mathfrak{B}_\Theta(\Sigma)$ and $\mathfrak{C}_0(\Sigma)$. Recall that, with respect to the canonical basis $(e_1, .., e_{2n})$ of Ξ, one defines the higher-order partial derivatives

$$\mathcal{D}^\mu f := \partial_X^\mu [\Theta_X(f)]_{X=0}\,,$$

where μ is a multi-index and $f \in \mathcal{B}_\Theta^\infty(\Sigma)$. Recall also the form of the seminorms of the Fréchet space $\mathcal{B}_\Theta^\infty(\Sigma)$

$$\| f \|^{(j)}_{\mathcal{B}_\Theta(\Sigma)} := \sum_{|\mu| \le j} \frac{1}{\mu!} \| \mathcal{D}^\mu f \|_{\mathcal{B}_\Theta(\Sigma)} \equiv \sum_{|\mu| \le j} \frac{1}{\mu!} \| \mathcal{D}^\mu f \|_\infty\,, \tag{4.1}$$

and of the Fréchet space $\mathfrak{B}_\Theta^\infty(\Sigma)$

$$\| f \|^{(j)}_{\mathfrak{B}_\Theta(\Sigma)} := \sum_{|\mu| \le j} \frac{1}{\mu!} \| \mathcal{D}^\mu f \|_{\mathfrak{B}_\Theta(\Sigma)}\,. \tag{4.2}$$

We fix a closed invariant set $F \subset \Sigma$; invariance means that $\Theta_X(F) \subset F$ for every $X \in \Xi$. Then

$$C_0(\Sigma)^F := \{ f \in C_0(\Sigma) \mid f|_F = 0 \}$$

is an invariant ideal of $C_0(\Sigma)$; its Rieffel quantization $\mathfrak{C}_0(\Sigma)^F$ is identified to an ideal of $\mathfrak{C}_0(\Sigma)$. As explained above, the quotient $\mathfrak{C}_0(\Sigma)/\mathfrak{C}_0(\Sigma)^F$ can be regarded as the deformation of the Abelian quotient $C_0(\Sigma)/C_0(\Sigma)^F$, which in its turn can be identified with $C(F)$, the C^*-algebra of all continuous functions on the compact space F. Along these lines, we identify $\mathfrak{C}_0(\Sigma)/\mathfrak{C}_0(\Sigma)^F$ with the Rieffel quantization $\mathfrak{C}(F)$ of $C(F)$.

Let us denote by $\varphi \in C_c^\infty(\Xi)_+^{\mathfrak{n}}$ the family of all positive functions $\varphi \in C_c^\infty(\Xi)$ which satify the normalization condition $\int_\Xi \varphi = 1$. If $\mathcal{W} \subset \Sigma$ is an open (or closed) set, the function

$$\chi_{\mathcal{W}}^\varphi = \varphi *_\Theta \chi_{\mathcal{W}} = \int_\Xi dY\, \varphi(Y) \chi_{\Theta_Y(\mathcal{W})}$$

belongs to $\mathcal{B}_\Theta^\infty(\Sigma)$ by Lemma 3.2 and one has

$$\operatorname{supp}\big(\chi_{\mathcal{W}}^\varphi\big) \subset \Theta_{\operatorname{supp}(\varphi)}[\operatorname{supp}(\chi_{\mathcal{W}})] = \Theta_{\operatorname{supp}(\varphi)}(\mathcal{W})\,. \tag{4.3}$$

Notice that in general the characteristic function $\chi_{\mathcal{W}}$ is not an element of $\mathcal{B}_\Theta(\Sigma)$.

Let us also fix a basis of open neighborhoods $\mathcal{N}_F$ of F in the space Σ.

Theorem 4.1. *Let $h \in \mathfrak{C}_0^\infty(\Sigma)$ and $\rho : \mathbb{R} \to [0, \infty)$ a continuous function with support disjoint from the spectrum of $h^F := h|_F$ computed in the non-commutative C^*-algebra $\mathfrak{C}(F)$. For any $\varphi \in C_c^\infty(\Xi)_+^{\mathfrak{n}}$, $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $\mathcal{W} \in \mathcal{N}_F$ such that*

$$\big\| \chi_{\mathcal{W}}^\varphi \# \rho(h) \big\|^{(k)}_{\mathfrak{B}_\Theta(\Sigma)} \le \varepsilon\,. \tag{4.4}$$

Remark 4.2. The Theorem is our main abstract localization result, expressed in terms of the symbolic calculus defined by Rieffel's deformation. Note that it contains a rich amount of information, involving all the seminorms $\|\cdot\|^{(k)}_{\mathfrak{B}_\Theta(\Sigma)}$; for $k=0$ one gets the norm of the C^*-algebra $\mathfrak{B}_\Theta(\Sigma)$. It will be turned into an assertion about pseudodifferential operators in the next sections.

Remark 4.3. It is clear that $1-\chi^\varphi_{\mathcal{W}}=\chi^\varphi_{\mathcal{W}^c}$, whose support is included in $\Theta_{\mathrm{supp}(\varphi)}(\mathcal{W}^c)$. Therefore $\chi^\varphi_{\mathcal{W}}=1$ on the complement of $\Theta_{\mathrm{supp}(\varphi)}(\mathcal{W}^c)$. Taking $\mathcal{W}$ open, $\mathcal{W}^c$ will be closed and included in $\Sigma\setminus F$, which is Θ-invariant. Then $\Theta_{\mathrm{supp}(\varphi)}(\mathcal{W}^c)$ will also be closed and disjoint from F, so $\chi^\varphi_{\mathcal{W}}=1$ on an open neighborhood of F.

Remark 4.4. As an example of closed invariant subset one can consider a quasi-orbit, i.e. the closure of an orbit. Any closed invariant set $F\subset\Sigma$ is the union of all the quasi-orbits it contains. Note that the spectrum of $h^F:=h|_F$ computed in $\mathfrak{C}(F)$ is an increasing function of F. So for small closed invariant subsets F (as quasi-orbits, for instance), the support of the localization ρ will probably allowed to be large. The interesting case is, of course, that in which $\mathrm{supp}(\rho)$ has a large intersection with the spectrum in $\mathfrak{C}^\infty_0(\Sigma)$ of the initial symbol h (which is obtained formally setting $F=\emptyset$).

We are going to prove Theorem 4.1 in several steps.

Proposition 4.5. *For every $f\in C^\infty_0(\Sigma)^F$, $\varepsilon>0$, $j\in\mathbb{N}$ and $\varphi\in C^\infty_c(\Xi)^{\mathfrak{n}}_+$ there exists $\mathcal{U}\in\mathcal{N}_F$ such that*

$$\left\|\chi^\varphi_{\mathcal{U}}f\right\|^{(j)}_{\mathcal{B}_\Theta(\Sigma)}\le\varepsilon. \tag{4.5}$$

Proof. One has

$$\left\|\chi^\varphi_{\mathcal{U}}f\right\|^{(j)}_{\mathcal{B}_\Theta(\Sigma)}=\sum_{|\mu|\le j}\frac{1}{\mu!}\left\|\mathcal{D}^\mu\left(\chi^\varphi_{\mathcal{U}}f\right)\right\|_{\mathcal{B}_\Theta(\Sigma)}\le\sum_{|\mu|\le j}\frac{1}{\mu!}\sum_{\nu\le\mu}C^\mu_\nu\left\|\mathcal{D}^{\mu-\nu}\chi^\varphi_{\mathcal{U}}\,\mathcal{D}^\nu f\right\|_{\mathcal{B}_\Theta(\Sigma)},$$

so one must estimate $\left\|\mathcal{D}^\alpha\chi^\varphi_{\mathcal{U}}\,\mathcal{D}^\beta f\right\|_{\mathcal{B}_\Theta(\Sigma)}$ for a finite number of multi-indices α,β.

We know from Lemma 3.2 that $\mathcal{D}^\alpha\chi^\varphi_{\mathcal{U}}=\chi^{\partial^\alpha\varphi}_{\mathcal{U}}$. Then

$$\begin{aligned}
\left\|\mathcal{D}^\alpha\chi^\varphi_{\mathcal{U}}\,\mathcal{D}^\beta f\right\|_{\mathcal{B}_\Theta(\Sigma)} &= \left\|\int_\Xi dY(\partial^\alpha\varphi)(Y)\,\Theta_{-Y}(\chi_{\mathcal{U}})\,\mathcal{D}^\beta f\right\|_{\mathcal{B}_\Theta(\Sigma)}\\
&\le \|\partial^\alpha\varphi\|_{L^1(\Xi)}\sup_{Y\in\mathrm{supp}(\varphi)}\left\|\Theta_{-Y}(\chi_{\mathcal{U}})\,\mathcal{D}^\beta f\right\|_{\mathcal{B}_\Theta(\Sigma)}\\
&= \|\partial^\alpha\varphi\|_{L^1(\Xi)}\sup_{Y\in\mathrm{supp}(\varphi)}\left\|\Theta_{-Y}\left[\chi_{\mathcal{U}}\,\Theta_Y(\mathcal{D}^\beta f)\right]\right\|_\infty\\
&= \|\partial^\alpha\varphi\|_{L^1(\Xi)}\sup_{Y\in\mathrm{supp}(\varphi)}\sup_{\sigma\in\mathcal{U}}\left|\left[\Theta_Y(\mathcal{D}^\beta f)\right](\sigma)\right|\\
&= \|\partial^\alpha\varphi\|_{L^1(\Xi)}\sup_{Y\in\mathrm{supp}(\varphi)}\sup_{\sigma\in\mathcal{U}}\left|(\mathcal{D}^\beta f)\left[\Theta_Y(\sigma)\right]\right|.
\end{aligned}$$

Therefore one can write

$$\begin{aligned}\left\|\chi^{\varphi}_{\mathcal{U}}f\right\|^{(j)}_{\mathcal{B}_\Theta(\Sigma)} &\le \sum_{|\mu|\le j}\frac{1}{\mu!}\sum_{\nu\le\mu}C^{\mu}_{\nu}\,\|\partial^{\mu-\nu}\varphi\|_{L^1(\Xi)}\sup\left\{\left|(\mathcal{D}^{\nu}f)(\tau)\right| \mid \tau\in\Theta_{\mathrm{supp}(\varphi)}(\mathcal{U})\right\}\\ &\le C(j,\varphi)\max_{|\nu|\le j}\sup\left\{\left|(\mathcal{D}^{\nu}f)(\tau)\right| \mid \tau\in\Theta_{\mathrm{supp}(\varphi)}(\mathcal{U})\right\},\end{aligned} \tag{4.6}$$

where $C(j,\varphi)$ is a finite constant depending on j and φ.

Given $\varepsilon>0$ we now find $\mathcal{U}$. Since the action Θ is strongly continuous, for every $Z\in\mathrm{supp}(\varphi)$ there exists a ball $\mathbf{B}(Z,\delta_Z)$ centered in Z such that if $Y\in\mathbf{B}(Z,\delta_Z)$ one has for all $|\nu|\le j$

$$\left\|\Theta_Y(\mathcal{D}^{\nu}f)-\Theta_Z(\mathcal{D}^{\nu}f)\right\|_{\mathcal{B}_\Theta(\Sigma)}\le\frac{\varepsilon}{2\,C(j,\varphi)}. \tag{4.7}$$

The balls $\mathbf{B}(Z,\delta_Z)$ form a covering of the compact set $\mathrm{supp}(\varphi)$, from which we extract a finite subcovering indexed by $\{Z_i\mid i\in I\}$. Since

$$\Theta_{Z_i}(\mathcal{D}^{\nu}f)\in C_0(\Sigma)^F,\quad \forall\, i,\nu,$$

there exists $\mathcal{U}^{\nu}_i\in\mathcal{N}_F$ such that

$$\left|[\Theta_{Z_i}(\mathcal{D}^{\nu}f)](\sigma)\right|\le\frac{\varepsilon}{2\,C(j,\varphi)},\quad \forall\,\sigma\in\mathcal{U}^{\nu}_i. \tag{4.8}$$

Setting

$$\mathcal{U}:=\bigcap\left\{\mathcal{U}^{\nu}_i\,\middle|\, i\in I,|\nu|\le j\right\}\in\mathcal{N}_F$$

one gets from (4.7) and (4.8)

$$\left|[\Theta_Y(\mathcal{D}^{\nu}f)](\sigma)\right|\le\frac{\varepsilon}{C(j,\varphi)},\quad \forall\,\sigma\in\mathcal{U},\ \forall\,Y\in\mathrm{supp}(\varphi),\ \forall\,|\nu|\le j.$$

Inserting this into (4.6) finishes the proof. □

Now we prove an estimation as (4.5), but with the pointwise product $\cdot$ replaced by the deformed product $\#$.

Proposition 4.6. *For any $f\in\mathfrak{C}^{\infty}_0(\Sigma)^F=C^{\infty}_0(\Sigma)^F$, $\varphi\in C^{\infty}_{\mathrm{c}}(\Xi)^{\mathrm{n}}_+$, $\varepsilon>0$ and $j\in\mathbb{N}$, there exists $\mathcal{V}\in\mathcal{N}_F$ such that*

$$\left\|\chi^{\varphi}_{\mathcal{V}}\#f\right\|^{(j)}_{\mathcal{B}_\Theta(\Sigma)}\le\varepsilon. \tag{4.9}$$

Proof. For the composition $\chi^{\varphi}_{\mathcal{V}}\#f$ we are going to use the representation (3.4).

For $G\in BC^{\infty}(\Xi\times\Xi;\mathcal{B}^{\infty}_{\Theta}(\Sigma))$ and $j,m\in\mathbb{N}$ we set

$$\begin{aligned}\|G\|^{(j,m)}_{\mathcal{B}_\Theta(\Sigma)} &:= \max_{i\le j}\sum_{|(\mu,\nu)|\le m}\frac{1}{\mu!\nu!}\sup_{Y,Z\in\Xi}\left\|(\partial^{\mu}_Y\partial^{\nu}_Z G)(Y,Z)\right\|^{(i)}_{\mathcal{B}_\Theta(\Sigma)}\\ &= \max_{i\le j}\sum_{|(\mu,\nu)|\le m}\frac{1}{\mu!\nu!}\sup_{Y,Z\in\Xi}\sum_{|\alpha|\le i}\frac{1}{\alpha!}\left\|\mathcal{D}^{\alpha}\left[(\partial^{\mu}_Y\partial^{\nu}_Z G)(Y,Z)\right]\right\|_{\mathcal{B}_\Theta(\Sigma)}.\end{aligned} \tag{4.10}$$

By [38, Prop. 1.6], for every $k > 2n$ we have estimates given by

$$\Big\| \int_\Xi \int_\Xi dY dZ\, e^{2i[\![Y,Z]\!]} \psi_P(Y)\psi_Q(Z) G(Y,Z) \Big\|^{(j)}_{\mathcal{B}_\Theta(\Sigma)} \le C_{PQ}(k) \| G \|^{(j,2k)}_{\mathcal{B}_\Theta(\Sigma)}, \tag{4.11}$$

where $\sum_{P,Q} C_{PQ}(k) < \infty$. Applying this to

$$G^\varphi_{\mathcal{V}}(Y,Z) := \Theta_Y\big[\chi^\varphi_{\mathcal{V}}\, \Theta_{Z-Y}(f)\big]$$

and relying on the representation (3.4), one gets

$$\Big\| \int_\Xi \int_\Xi dY dZ\, e^{2i[\![Y,Z]\!]} \psi_P(Y)\psi_Q(Z) \Theta_Y\big[\chi^\varphi_{\mathcal{V}}\, \Theta_{Z-Y}(f)\big] \Big\|^{(j)}_{\mathcal{B}_\Theta(\Sigma)} \le C_{PQ}(k) \big\| G^\varphi_{\mathcal{V}} \big\|^{(j,2k)}_{\mathcal{B}_\Theta(\Sigma)}.$$

A direct computation shows that the quantity $\big\| G^\varphi_{\mathcal{V}} \big\|^{(j,2k)}_{\mathcal{B}_\Theta(\Sigma)}$ is bounded uniformly in $\mathcal{V}$, because it is dominated by a finite linear combination of terms of the form

$$\big\| \chi^{\partial^\beta \varphi}_{\mathcal{V}} \big\|_\infty \big\| \mathcal{D}^\gamma f \big\|_\infty \le \big\| \partial^\beta \varphi \big\|_{L^1(\Xi)} \big\| \mathcal{D}^\gamma f \big\|_\infty .$$

Thus, for any $\varepsilon > 0$, there exists $m_j \in \mathbb{N}$ such that for every $\mathcal{V} \in \mathcal{N}_F$ one has

$$\sum_{|P|+|Q|>m_j} \Big\| \int_\Xi \int_\Xi dY dZ\, e^{2i[\![Y,Z]\!]} \psi_P(Y)\psi_Q(Z) \Theta_Y\big[\chi^\varphi_{\mathcal{V}}\, \Theta_{Z-Y}(f)\big] \Big\|^{(j)}_{\mathcal{B}_\Theta(\Sigma)} \le \varepsilon/2 .$$

We still have to bound by $\varepsilon/2$ the remaining finite family of terms, this time for some special neighborhood $\mathcal{V}$ of F. Using the continuity of the action Θ and the compacity of the support of ψ_P, ψ_Q, there exists a finite family of balls $\{\mathbf{B}(Y_i,\delta_i) \times \mathbf{B}(Z_i,\delta'_i)\}_{i\in I}$ which covers the suport of $\psi_P \otimes \psi_Q$, such that for $(Y,Z) \in \mathbf{B}(Y_i,\delta_i) \times \mathbf{B}(Z_i,\delta'_i)$ one has

$$\big\| \chi^\varphi_{\mathcal{V}} \left(f_{Z_i - Y_i} - f_{Z-Y}\right) \big\|^{(j)}_{\mathcal{B}_\Theta(\Sigma)} \le \varepsilon/2M, \quad \forall\, j \le k, \tag{4.12}$$

where

$$f_X := \Theta_{-X}(f) \in C^\infty_0(\Sigma)^F$$

and M is some positive number. In addition, by Proposition 4.5, for every $i \in I$ there is some $\mathcal{V}_i \in \mathcal{N}_F$ such that

$$\big\| \chi^\varphi_{\mathcal{V}_i} f_{Z_i - Y_i} \big\|^{(j)}_{\mathcal{B}_\Theta(\Sigma)} \le \varepsilon/2M, \quad \forall\, j \le k. \tag{4.13}$$

One takes the finite intersection $\mathcal{V} = \bigcap_{i\in I} \mathcal{V}_i$ and then, by (4.12), (4.13) and the fact that the action Θ is isometric with respect to all the semi-norms, we can estimate the compactly supported integral

$$\begin{aligned}
&\Big\| \int_\Xi \int_\Xi dY dZ\, e^{2i[\![Y,Z]\!]} \psi_P(Y)\psi_Q(Z) \Theta_Y\big[\chi^\varphi_{\mathcal{V}}\, \Theta_{Z-Y}(f)\big] \Big\|^{(j)}_{\mathcal{B}_\Theta(\Sigma)} \\
&\le M_{P,Q} \sup \Big\{ \big\| \chi^\varphi_{\mathcal{V}} f_{Z-Y} \big\|^{(j)}_{\mathcal{B}_\Theta(\Sigma)} \;\big|\; Y \in \operatorname{supp}(\psi_P), Z \in \operatorname{supp}(\psi_Q) \Big\} \\
&\le M_{P,Q} \sup_{i\in I} \big\| \chi^\varphi_{\mathcal{V}} f_{Z_i - Y_i} \big\|^{(j)}_{\mathcal{B}_\Theta(\Sigma)} \\
&+ M_{P,Q} \sup_{i\in I} \sup \Big\{ \big\| \chi^\varphi_{\mathcal{V}} (f_{Z_i - Y_i} - f_{Z-Y}) \big\|^{(j)}_{\mathcal{B}_\Theta(\Sigma)} \;\big|\; Y \in \mathbf{B}(Y_i,\delta_i), Z \in \mathbf{B}(Z_i,\delta'_i) \Big\} \\
&\le M_{P,Q} \left(\frac{\varepsilon}{2M} + \frac{\varepsilon}{2M} \right) = \frac{M_{P,Q}}{M}\, \varepsilon .
\end{aligned}$$

Then, choosing $M := \sum_{|P|+|Q|\le m_j} C_{P,Q} M_{P,Q}$, one gets the estimation. □

Now we change the semi-norms.

Proposition 4.7. *For any* $f \in \mathfrak{C}_0^\infty(\Sigma)^F$, $\varphi \in C_c^\infty(\Xi)_+^{\mathfrak{n}}$, $\varepsilon > 0$ *and* $k \in \mathbb{N}$, *there exists* $\mathcal{W} \in \mathcal{N}_F$ *such that*

$$\left\| \chi_{\mathcal{W}}^{\varphi} \# f \right\|_{\mathfrak{B}_\Theta(\Sigma)}^{(k)} \le \varepsilon . \tag{4.14}$$

Proof. This follows from our Proposion 4.6 and the equivalence [38, Ch. 7] of the families of seminorms (4.1) and (4.2), which is a rather deep result. □

End of the proof of Theorem 4.1.

To finish the proof of Theorem 4.1, one uses Lemma 3.1 with $\mathfrak{A} := \mathfrak{C}_0(\Sigma)$ and $\mathfrak{J} := \mathfrak{C}_0(\Sigma)^F$. Notice the identification

$$\mathrm{sp}_{\mathfrak{C}_0(\Sigma)^F}(h) = \mathrm{sp}\Big(h^F \Big| \mathfrak{C}(F)\Big) .$$

This allows us to take $f = \rho(h) \in \mathfrak{C}_0^\infty(\Sigma)^F$ in Proposition 4.7 and to get

$$\left\| \chi_{\mathcal{W}}^{\varphi} \# \rho(h) \right\|_{\mathfrak{B}_\Theta(\Sigma)}^{(k)} \le \varepsilon \tag{4.15}$$

under the stated conditions. The case $k = 0$ is enough for our purposes.

5 The simplest main represented result

We start with the simplest situation. We take Σ to be a locally compact space containing $\Xi = \mathbb{R}^{2n}$ densely. If Σ is even compact, it will be a compactification of Ξ. One denotes by $\mathcal{A}_\Sigma(\Xi)$ the C^*-algebra composed of restrictions to Ξ of all the elements of $C_0(\Sigma)$. Then $\mathcal{A}_\Sigma(\Xi)$ is a C^*-subalgebra of $\mathsf{BC}(\Xi)$ which is canonically isomorphic to $C_0(\Sigma)$ by the extension/restriction isomorphism. Thus the Gelfand spectrum of $\mathcal{A}_\Sigma(\Xi)$ is homeomorphic to Σ.

Let us also assume that $\mathcal{A}_\Sigma(\Xi)$ is contained in $\mathsf{BC}_{\mathrm{u}}(\Xi)$ and it is invariant under translations. It follows easily that the action of Ξ on itself by translations extends to a continuous action Θ of Ξ by homeomorphisms of Σ. This action is topologically transitive: Ξ is an open dense orbit. Let us set $\Sigma_\infty := \Sigma \setminus \Xi$ for the boundary.

Since $(\mathcal{A}_\Sigma(\Xi), \Theta, \Xi)$ is a (commutative) C^*-dynamical system, one can perform Rieffel's procedure to turn it in the (non-commutative) C^*-dynamical system $(\mathfrak{A}_\Sigma(\Xi), \Theta, \Xi)$. The common set of smooth vectors $\mathfrak{A}_\Sigma^\infty(\Xi) = \mathcal{A}_\Sigma^\infty(\Xi)$ is contained in $\mathsf{BC}^\infty(\Xi)$.

It is known that $\mathsf{BC}^\infty(\Xi)$ is the family of smooth vectors of the Ξ-algebra $\mathsf{BC}_{\mathrm{u}}(\Xi)$, whose Rieffel quantization will be denoted by $\mathfrak{BC}_{\mathrm{u}}(\Xi)$. But on $\mathsf{BC}^\infty(\Xi)$, by the Calderón-Vaillancourt Theorem [8], one can apply the Schrödinger representation in $\mathcal{H} := \mathsf{L}^2(\mathscr{X})$

$$\mathfrak{Op} : \mathsf{BC}^\infty(\Xi) \to \mathbb{B}(\mathcal{H}) \tag{5.1}$$

given in the sense of oscillatory integrals by

$$[\mathfrak{Op}(f)u](x) = (2\pi)^{-n} \int_{\mathscr{X}} dy \int_{\mathscr{X}^*} d\xi \, e^{i(x-y)\cdot\xi} f\left(\frac{x+y}{2}, \xi\right) u(y). \tag{5.2}$$

In particular, this works for $f \in \mathfrak{A}_\Sigma^\infty(\Xi)$.

We also fix a closed Θ-invariant subset F of Σ_∞; it can be a quasiorbit for instance. As in Section 4, we also consider a neighborhood basis $\mathcal{N}_F$ of the set F in Σ. For every $\mathcal{W} \in \mathcal{N}_F$ we set $W := \mathcal{W} \cap \Xi$. Then the function χ_W^φ is the restriction of $\chi_{\mathcal{W}}^\varphi$ to Ξ and it belongs to $\mathsf{BC}^\infty(\Xi)$, hence $\mathfrak{Op}(\chi_W^\varphi)$ makes sense as a bounded operator in $L^2(\mathscr{X})$.

Finally let $h \in C_0^\infty(\Sigma) = \mathfrak{C}_0^\infty(\Sigma)$ be a real function and set $H := \mathfrak{Op}(h) = H^*$, a bounded operator in $\mathcal{H} := L^2(\mathscr{X})$ (we use the same notation h for the restriction of $h : \Sigma \to \mathbb{R}$ to Ξ). Relying on [26], we give an operator interpretation for the set $\mathrm{sp}(h^F)$, the spectrum of $h^F := h|_F$ computed in the non-commutative C^*-algebra $\mathfrak{C}(F)$. Let us write $\mathfrak{Q}(F)$ for the set of all quasi-orbits of the closed invariant set F and denote by $\mathfrak{Q}_0(F)$ a subset of $\mathfrak{Q}(F)$ such that $F = \bigcup_{Q \in \mathfrak{Q}_0(F)} Q$. In each quasi-orbit Q pick a point σ_Q such that the orbit of this point is dense in Q. Then

$$h^{\sigma_Q} : \Xi \to \mathbb{R}, \qquad h^{\sigma_Q}(X) := h[\Theta_X(\sigma_Q)] \tag{5.3}$$

is an element of $\mathsf{BC}^\infty(\Xi)$, to which one can apply $\mathfrak{Op}$; let us set $H^{\sigma_Q} := \mathfrak{Op}(h^{\sigma_Q})$. It can be shown [26] that:

- The spectrum $S^{\mathcal{Q}}$ of the bounded self-adjoint operator $H^{\sigma_{\mathcal{Q}}}$ depends only of the quasi-orbit Q and not of the generating point σ_Q.
- One has

$$S^F = \mathrm{sp}(h^F) = \overline{\bigcup_{Q\in\mathfrak{Q}(F)} S^{\mathcal{Q}}} = \overline{\bigcup_{Q\in\mathfrak{Q}_0(F)} S^{\mathcal{Q}}}. \tag{5.4}$$

  Of course, if F is itself a quasi-orbit one can take $\mathfrak{Q}_0(F) = \{F\}$ and the statements simplify a lot.

- The set $\mathrm{sp}(h^F)$ is contained in the essential spectrum $\mathrm{sp}_{\mathrm{ess}}(H)$ of the initial operator H.
- Actually, if we cover Σ_∞ by closed Θ-invariant sets F, one has

$$\mathrm{sp}_{\mathrm{ess}}(H) = \overline{\bigcup_F S^F} = \overline{\bigcup_{Q\in\mathfrak{Q}(\Sigma_\infty)} S^{\mathcal{Q}}}. \tag{5.5}$$

Now we can state and prove

Theorem 5.1. *Let $h \in \mathfrak{A}^\infty_\Sigma(\Xi) = \mathcal{A}^\infty_\Sigma(\Xi)$ be a real function and set $H := \mathfrak{Op}(h)$. Let $\rho : \mathbb{R} \to \mathbb{R}_+$ be a bounded continuous function such that $\mathrm{supp}(\rho) \cap S^F = \emptyset$. For every $\varepsilon > 0$ and for every positive function $\varphi \in C^\infty_c(\Xi)$ with $\int_\Xi \varphi = 1$ there exists $\mathcal{W} \in \mathcal{N}_F$ such that*

$$\left\| \mathfrak{Op}\left(\chi^\varphi_W\right) \rho(H) \right\|_{\mathbb{B}(\mathcal{H})} \leq \varepsilon. \tag{5.6}$$

In particular, one has uniformly in $t \in \mathbb{R}$ and $u \in \mathcal{H}$

$$\left\| \mathfrak{Op}\left(\chi^\varphi_W\right) e^{itH} \rho(H) u \right\| \leq \varepsilon \|u\|. \tag{5.7}$$

Proof. It is known (cf. [3, Lemma 3.1] or [26, Prop. 2.6]) that the mapping $\mathfrak{Op}$ extends to a faithful (therefore isometric) representation of $\mathfrak{BC}_{\mathrm{u}}(\Xi)$ in $\mathcal{H}$. We can use its restriction to our algebra $\mathfrak{A}_\Sigma(\Xi)$ and apply it to the element $\rho(h)$. Note however that χ^φ_W, element of $\mathsf{BC}^\infty(\Xi) \subset \mathfrak{BC}_{\mathrm{u}}(\Xi)$, has a priori no reason to belong to $\mathfrak{A}_\Sigma(\Xi)$.

For the first estimate we use the fact that, being a representation, $\mathfrak{Op}$ is multiplicative and commutes with the functional calculus:

$$\mathfrak{Op}\left(\chi^\varphi_W\right)\rho(H) = \mathfrak{Op}\left(\chi^\varphi_W\right)\rho[\mathfrak{Op}(h)] = \mathfrak{Op}\left(\chi^\varphi_W\right)\mathfrak{Op}[\rho(h)] = \mathfrak{Op}\left[\chi^\varphi_W \sharp \rho(h)\right].$$

We denoted by $\sharp$ the Weyl composition law of symbols [8], corresponding isomorphically to the composition $\#$. Then we use Theorem 4.1, the isomorphisms

$$\mathfrak{A}_\Sigma(\Xi) \cong \mathfrak{C}_0(\Sigma) \quad \text{and} \quad \mathfrak{BC}_{\mathrm{u}}(\Xi) \cong \mathfrak{B}_\Theta(\Sigma)$$

and the fact that $\mathfrak{Op}$ is an isometry to write

$$\left\| \mathfrak{Op}\left(\chi^\varphi_W\right)\rho(H) \right\|_{\mathbb{B}(\mathcal{H})} = \left\| \mathfrak{Op}\left[\chi^\varphi_W \sharp \rho(h)\right] \right\|_{\mathbb{B}(\mathcal{H})} = \left\| \chi^\varphi_{\mathcal{W}} \# \rho(h) \right\|_{\mathfrak{B}_\Theta(\Sigma)} \leq \varepsilon. \tag{5.8}$$

As it has been said repeatedly, the second estimate (5.7) follows from (5.6). □

6 Other represented result

Here we treat a more general case. The action Θ will no longer be composed of translations.

Our framework starts with a continuous action Θ of Ξ by homeomorphims of the locally compact space Σ. Θ is continuous and the homeomorphims Θ_X,Θ_Y satisfy $\Theta_X \circ \Theta_Y = \Theta_{X+Y}$ for every $X, Y \in \Xi$. The action Θ of Ξ on Σ induces a continuous action of Ξ on $C_0(\Sigma)$ given by $\Theta_X(f) = f \circ \Theta_X$. We want to realize the algebra $(C_0(\Sigma), \Theta, \Xi)$ as a subalgebra of $(\mathsf{BC_u}(\Xi), \mathcal{T}, \Xi)$ by a Ξ-monomorphim (we denoted by $\mathcal{T}$ the usual translations in Ξ). For this purpose, it is convenient to have a closer look at the quasi-orbit structure of the dynamical system (Σ, Θ, Ξ) in connection with C^*-algebras and Hilbert space representations.

Let us use the convenient notation $\Theta_\sigma(X) := \Theta_X(\sigma)$. For each $\sigma \in \Sigma$, we denote by $E_\sigma := \overline{\Theta_\sigma(\Xi)}$ *the quasi-orbit generated by* σ and set

$$\mathcal{P}_\sigma : C_0(\Sigma) \to \mathsf{BC_u}(\Xi), \quad \mathcal{P}_\sigma(f) := f \circ \Theta_\sigma .$$

The range of the Ξ-morphism $\mathcal{P}_\sigma$ is called $\mathcal{B}_\sigma$ and it is a Ξ-algebra. Defining analogously $\mathcal{P}'_\sigma : C_0(E_\sigma) \to \mathsf{BC_u}(\Xi)$ one gets a Ξ-monomorphism with the same range $\mathcal{B}_\sigma$, which shows that the Gelfand spectrum of $\mathcal{B}_\sigma$ can be identified with the quasi-orbit E_σ.

For each quasi-orbit E, one has the natural restriction map

$$\mathcal{R}_E : C_0(\Sigma) \to C_0(E), \quad \mathcal{R}_E(f) := f|_E,$$

which is a Ξ-epimorphism. Actually one has $\mathcal{P}_\sigma = \mathcal{P}'_\sigma \circ \mathcal{R}_{E_\sigma}$.

Being respectively invariant under the actions Θ and $\mathcal{T}$, the C^*-algebras $C_0(E)$ and $\mathcal{B}_\sigma$ are also subject to Rieffel deformation. By quantization, one gets C^*-algebras and morphisms

$$\mathfrak{R}_E : \mathfrak{C}_0(\Sigma) \to \mathfrak{C}_0(E), \qquad \mathfrak{P}_\sigma : \mathfrak{C}_0(\Sigma) \to \mathfrak{B}_\sigma, \qquad \mathfrak{P}'_\sigma : \mathfrak{C}_0(E_\sigma) \to \mathfrak{B}_\sigma ,$$

satisfying $\mathfrak{P}_\sigma = \mathfrak{P}'_\sigma \circ \mathfrak{R}_{E_\sigma}$. While $\mathfrak{R}_E$ and $\mathfrak{P}_\sigma$ are epimorphisms, $\mathfrak{P}'_\sigma$ is an isomorphism.

We also need Hilbert space representations. For each Ξ-algebra $\mathcal{B}$, we restrict $\mathfrak{Op}$ from $\mathsf{BC}^\infty(\Xi)$ to $\mathcal{B}^\infty = \mathfrak{B}^\infty$ (the dense *-algebra of smooth vectors of $\mathcal{B}$) and then we extend it to a faithful representation in $\mathcal{H} = L^2(\mathscr{X})$ of the C^*-algebra $\mathfrak{B}$. We can apply the construction to the C^*-algebras $\mathfrak{B}_\sigma$. By composing, we get a family

$$\{\, \mathfrak{Op}_\sigma := \mathfrak{Op} \circ \mathfrak{P}_\sigma \mid \sigma \in \Sigma \}$$

of representations of $\mathfrak{C}_0(\Sigma)$ in $\mathcal{H}$, indexed by the points of Σ. For $f \in \mathfrak{C}_0^\infty(\Sigma)$ one has $\mathfrak{P}_\sigma(f) \in \mathfrak{B}_\sigma^\infty = \mathcal{B}_\sigma^\infty$, and the action on $\mathcal{H}$ is given by

$$[\mathfrak{Op}_\sigma(f)u](x) = (2\pi)^{-n} \int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} d\xi \, e^{i(x-y)\cdot\xi} f\Big[\Theta_{(\frac{x+y}{2},\xi)}(\sigma)\Big] u(y) \tag{6.1}$$

in the sense of oscillatory integrals. If the function f is real, all the operators $\mathfrak{Op}_\sigma(f)$ will be self-adjoint. To conclude, a single element $f \in \mathfrak{C}_0(\Sigma)$ leads to a family

$$\{ H_\sigma := \mathfrak{Op}_\sigma(f) \mid \sigma \in \Sigma \}$$

of bounded operators in $L^2(\mathscr{X})$. Note that, seen as a quantization of the symbol f, (6.1) can be quite different from a Weyl operator.

Remark 6.1. The point σ will be called *of the first kind* when $C_0(\Xi) \subset \mathcal{B}_\sigma$. It has been shown in [27, Prop. 3.1] that in such a case $\mathfrak{Op}(\mathfrak{B}_\sigma)$ contains all the compact operators in $\mathcal{H}$, thus $\mathfrak{Op}_\sigma$ is irreducible.

Remark 6.2. Notice that $\mathfrak{Op}_\sigma$ is faithful exactly when $\mathfrak{P}_\sigma$ is injective, i.e. when $\mathcal{P}_\sigma$ is injective, which is obviously equivalent to $E_\sigma = \Sigma$. Consequently, if the dynamical system is not *topologically transitive* (i.e. no orbit is dense in Σ), none of the Schrödinger-type representations $\mathfrak{Op}_\sigma$ will be faithful. In such a case, we are not able to transform the abstract algebraic Theorem 4.1 into an assertion involving operators. There is always an injective representation, the so-called *regular representation*, but spectral analysis and localization results in such a setting seem to have a limited interest.

So we restrict now to the case of a topologically transitive dynamical system and study the operators $\mathfrak{Op}_\sigma(h)$ associated to a suitable symbol h and a *generic point* $\sigma \in \Sigma$, *i.e.* a point generating a dense orbit. The non-generic points τ will define subsets S_τ of the essential spectrum of $\mathfrak{Op}_\sigma(h)$ as well as regions of non-propagation for its evolution group.

Theorem 6.3. *Let (Σ, Θ, Ξ) a topologically transitive dynamical system, $(C_0(\Sigma), \Theta, \Xi)$ its associated C^*-dynamical system and $\sigma \in \Sigma$ a generic point.*

For a fixed real function $h \in C_0^\infty(\Sigma) = \mathfrak{C}_0^\infty(\Sigma)$ set $H_\sigma := \mathfrak{Op}_\sigma(h)$. Choose a non-generic point $\tau \in \Sigma$, denote by E_τ its quasi-orbit (strictly contained in Σ) and set $H_\tau := \mathfrak{Op}_\tau(h)$. Let $\rho : \mathbb{R} \to \mathbb{R}_+$ be a continuous function such that $\operatorname{supp}(\rho) \cap \operatorname{sp}(H_\tau) = \emptyset$.

For every $\varepsilon > 0$ and for every positive function $\varphi \in C_c^\infty(\Xi)$ with $\int_\Xi \varphi = 1$ there exists a neighborhood $\mathcal{W}$ of E_τ in Σ such that

$$\left\| \mathfrak{Op}_\sigma\big(\chi_{\mathcal{W}}^\varphi\big) \rho(H_\sigma) \right\|_{\mathbb{B}(\mathcal{H})} \le \varepsilon . \tag{6.2}$$

In particular, one has uniformly in $t \in \mathbb{R}$ and $u \in L^2(\mathscr{X})$

$$\left\| \mathfrak{Op}_\sigma\big(\chi_{\mathcal{W}}^\varphi\big)\, e^{itH_\sigma} \rho(H_\sigma) u \right\| \le \varepsilon \| u \| . \tag{6.3}$$

Proof. Since (Σ, Θ, Ξ) is topologically transitive and σ is generic, the mapping $\mathfrak{Op}_\sigma$ is a faithful representation of our algebra $\mathfrak{C}_0(\Sigma)$.

Then the present result follows easily from Theorem 4.1, along the lines of the proof of Theorem 5.1. The role of the closed invariant subset F is played here by the non-generic quasi-orbit E_τ. □

Remark 6.4. Recall that we set $\mathfrak{Op}_\sigma\left(\chi^\varphi_{\mathcal{W}}\right) := \mathfrak{Op}\left(\chi^\varphi_{\mathcal{W}} \circ \Theta_\sigma\right)$. The function $\chi^\varphi_{\mathcal{W}} \circ \Theta_\sigma$ is a mollified version of $\chi_{\mathcal{W}} \circ \Theta_\sigma$, which in its turn is the characteristic function of the subset $\Theta_\sigma^{-1}(\mathcal{W})$ of the phase-space Ξ. By our choice of the points σ, τ, one has $O_\tau \cap O_\sigma = \emptyset$ and

$$E_\tau \subset \mathcal{W} \subset E_\sigma = \Sigma,$$

where the inclusions are strict.

Remark 6.5. For a better understanding of the dependence on the points σ and τ, we recall some results from [26]. If σ_1, σ_2 belong to the same orbit ($O_{\sigma_1} = O_{\sigma_2}$), the two operators H_{σ_1} and H_{σ_2} are unitarily equivalent. If the two points only generate the same quasi-orbit

$$E_{\sigma_1} := \overline{O_{\sigma_1}} = \overline{O_{\sigma_1}} =: E_{\sigma_2}$$

they may not be unitarily equivalent, but they still have the same spectrum and the same essential spectrum. In applications, very often, there is a privileged generic point σ_0 defining an interesting Hamiltonian H_{σ_0} as in (6.1) and the remaining objects are auxiliary constructions. Their usefulness comes from the fact that the behavior of the symbol requires a topological dynamical system encoding spectral information.

7 An extension

The Weyl system $\pi : \Xi \to \mathbb{U}(\mathcal{H})$ defined for all $X \in \Xi$ and $u \in \mathcal{H} := L^2(\mathscr{X})$ by

$$[\pi(X)u](y) := e^{i(y-x/2)\cdot\xi}u(y-x). \tag{7.1}$$

It is a projective unitary representation with multiplier given by the imaginary exponential of the symplectic form:

$$\pi(X)\pi(Y) = \exp\left(\frac{i}{2}[\![X, Y]\!]\right)\pi(X+Y), \qquad \forall X, Y \in \Xi. \tag{7.2}$$

We denote by

$$\Pi : \Xi \to \mathfrak{Aut}[\mathbb{B}(\mathcal{H})], \qquad \Pi(X)T := \pi(X)T\pi(-X) \tag{7.3}$$

the automorphism group associated to π. The C^0-vectors of this automorphism group form a C^*-subalgebra

$$\mathbb{B}^0(\mathcal{H}) := \{S \in \mathbb{B}(\mathcal{H}) \mid \Xi \ni X \mapsto \Pi(X)S \in \mathbb{B}(\mathcal{H}) \ \ \|\cdot\| \text{-continuous}\}, \tag{7.4}$$

while the C^∞-vectors

$$\mathbb{B}^\infty(\mathcal{H}) := \{S \in \mathbb{B}(\mathcal{H}) \mid \Xi \ni X \mapsto \Pi(X)S \in \mathbb{B}(\mathcal{H}) \ \text{ is } \mathrm{C}^\infty \text{ in norm}\} \tag{7.5}$$

form a dense *-subalgebra.

We also denote by $\delta(\Xi)$ the family of all ultrafilters on Ξ that are finer than the Fréchet filter. Recall from [27] that the essential spectrum of any self-adjoint operator H belonging to $\mathbb{B}^0(\Xi)$ is given by

$$\mathrm{sp}_{\mathrm{ess}}(H) = \overline{\cup_{X\in\delta(\Xi)}\mathrm{sp}(H_X)}, \tag{7.6}$$

where the limits

$$H_X := \lim_{X\to X}\Pi(X)H \tag{7.7}$$

are shown to exist in the strong sense.

The next result is a variant of Theorem 5.1.

Theorem 7.1. *Let H be a self-adjoint operator in $\mathcal{H} = L^2(\mathscr{X})$ belonging to $\mathbb{B}^\infty(\mathcal{H})$. Let us fix an ultrafilter X on Ξ finer than the Fréchet filter and choose a bounded continuous function $\rho : \mathbb{R} \to \mathbb{R}_+$ such that* $\mathrm{supp}(\rho) \cap \mathrm{sp}(H_X) = \emptyset$.

For every $\epsilon > 0$ and $\varphi \in C_c^\infty(\Xi)_+^n$ there exists $W \in X$ such that

$$\left\| \mathfrak{Op}(\chi_W^\varphi)\rho(H) \right\|_{\mathbb{B}(\mathcal{H})} \le \epsilon. \tag{7.8}$$

Proof. In [27, Prop. 3.1] it has been shown that $\mathbb{B}^0(\mathcal{H}) = \mathfrak{Op}\,[\mathfrak{BC}_u(\Xi)]$, where $\mathfrak{BC}_u(\Xi)$ is the Rieffel quantization of the C^*-algebra $\mathsf{BC}_u(\Xi)$ of all bounded uniformly continuous functions on Ξ. As we said above, this one is the largest one on which Ξ acts continuously by translations (that were denoted by $\mathcal{T}$). It is well-known that

$$\pi(X)\mathfrak{Op}(f)\pi(-X) = \mathfrak{Op}[\mathcal{T}_X(f)], \qquad \forall\, X \in \Xi .$$

Then, clearly, one also has

$$\mathbb{B}^\infty(\mathcal{H}) = \mathfrak{Op}\,[\mathfrak{BC}^\infty(\Xi)] = \mathfrak{Op}\,[\mathsf{BC}^\infty(\Xi)] .$$

Then the methods of the previous sections became available and the proof is very similar to the proof of Theorem 5.1. $\square$

8 Some examples

Example 8.1. Let us first exploit the fact that the phase space Ξ is the Cartesian product of the configuration space $\mathscr{X}$ and the momentum space $\mathscr{X}^*$ and that the translations in Ξ admit the decomposition $\mathcal{T}_{(x,\xi)} = \tau_x \otimes \tau^*_\xi$ in terms of translations in $\mathscr{X}$ and in $\mathscr{X}^*$, respectively. We consider unital C^*-algebras $\mathcal{A}(\mathscr{X}) \subset \mathsf{BC_u}(\mathscr{X})$ and $\mathcal{A}(\mathscr{X}^*) \subset \mathsf{BC_u}(\mathscr{X}^*)$. We assume that $\mathcal{A}(\mathscr{X})$ is τ-invariant and contains $C_0(\mathscr{X})$ and similarly about $\mathcal{A}(\mathscr{X}^*)$. The relevant algebra will be $\mathcal{A}(\Xi) := \mathcal{A}(\mathscr{X}) \otimes \mathcal{A}(\mathscr{X}^*)$, which is a $\mathcal{T}$-invariant C^*-subalgebra of $\mathsf{BC_u}(\Xi)$ containing the ideal $C_0(\Xi)$. Its Gelfand spectrum can be naturally regarded as a compactification $\Sigma = \Omega \times \Omega^*$ of Ξ, which is the product between a compactification $\Omega := \mathscr{X} \sqcup \Omega_\infty$ of $\mathscr{X}$ and a compactification $\Omega^* := \mathscr{X}^* \sqcup \Omega^*_\infty$ of $\mathscr{X}^*$.

Very simple elements of $\mathcal{A}^\infty(\Xi)$ (smooth symbols with $\mathcal{A}(\Sigma)$-anisotropy) are finite sums

$$h(x,\xi) = \sum_{j=1}^{m} \phi_j(x)\psi_j(\xi) \tag{8.1}$$

with $\phi_j \in \mathcal{A}^\infty(\mathscr{X})$ and $\psi_j \in \mathcal{A}^\infty(\mathscr{X}^*)$.

We denote by Θ the action of Ξ on Σ which extends the translations. Then

$$\Sigma = (\mathscr{X} \sqcup \Omega_\infty) \times (\mathscr{X}^* \sqcup \Omega^*_\infty) = \Xi \sqcup \Sigma_\infty , \tag{8.2}$$

where we have the invariant decomposition

$$\Sigma_\infty = (\mathscr{X} \times \Omega^*_\infty) \sqcup (\Omega_\infty \times \mathscr{X}^*) \sqcup (\Omega_\infty \times \Omega^*_\infty) . \tag{8.3}$$

Let us fix an $\mathscr{X}$-quasiorbit $E \subset \Omega_\infty$ and a basis of neighborhoods $\mathcal{N}_E$ of E in Ω. Similarly, fix an $\mathscr{X}^*$-quasiorbit $E^* \subset \Omega_\infty$ and a basis of neighborhoods $\mathcal{N}_{E^*}$ of E^* in Ω^*. We will be interested in the quasiorbits $\Omega \times E^*$, $E \times \Omega^*$ and $E \times E^*$ of Σ_∞.

Assume that $\mathcal{U} \subset \Omega$ is a neighborhood of E and set $U := \mathcal{U} \cap \mathscr{X}$. Then $\mathcal{W} := \mathcal{U} \times \Omega^*$ is a neighborhood of $F := E \times \Omega^*_\infty$ in Σ and one has $W := \mathcal{W} \cap \Xi = U \times \mathscr{X}^*$. Therefore $\chi_W = \chi_U \otimes 1$. It is convenient to choose the regularizing function φ of the form $\alpha \otimes \beta$, where $\alpha \in C^\infty_c(\mathscr{X})^n_+$ and $\beta \in C^\infty_c(\mathscr{X}^*)^n_+$; then obviously $\chi^\varphi_W = \chi^\alpha_U \otimes 1$ and thus $\mathfrak{Op}(\chi^\varphi_W) = \chi^\alpha_U(Q) \otimes 1$, where we recall that Q denotes the position operator in $\mathcal{H} = L^2(\mathscr{X})$ and $\chi^\alpha_U(Q)$ is constructed via the usual functional calculus.

Coming now to the Hamiltonian, for simplicity, let us consider only the case

$$h := \phi \otimes 1 + 1 \otimes \psi, \quad h(x,\xi) = \phi(x) + \psi(\xi) , \tag{8.4}$$

with $\phi \in \mathcal{A}^\infty(\mathscr{X})$ and $\psi \in \mathcal{A}^\infty(\mathscr{X}^*)$ real functions. By Weyl quantization one gets the bounded self-adjoint operator

$$H := \mathfrak{Op}(h) = \psi(P) + \phi(Q) , \tag{8.5}$$

where $P := -i\nabla$ is the momentum operator.

To construct the asymptotic Hamiltonian H^F corresponding to the quasiorbit $F = E \times \Omega^*$ one has to choose a generic point $\omega \in E$; then $(\omega, 0_{\mathscr{X}^*})$ will generate F. With the function

$$\phi^\omega : \mathscr{X} \to \mathbb{R}, \quad \phi^\omega(x) := \phi[\tau_x(\omega)]$$

one constructs the multiplication operator $\phi^\omega(Q)$ and the Hamiltonian

$$H^\omega := \mathfrak{Op}\,(1 \otimes \psi + \phi^\omega \otimes 1) = \psi(P) + \phi^\omega(Q). \tag{8.6}$$

As said before, the spectrum S^E of this Hamiltonian only depends on the quasiorbit E.

All these being said, we can now make a statement, deduced from Theorem 5.1. Fix a positive bounded continuous function ρ defined on $\mathbb{R}$ having a support disjoint of S^E. For every $\varepsilon > 0$ and every $\alpha \in C_c^\infty(\mathscr{X})_+^n$ there exist a neighborhood $\mathcal{U}$ of E in Ω such that

$$\left\| \chi_U^\alpha(Q)\rho(H) \right\|_{\mathbb{B}(\mathcal{H})} \leq \varepsilon. \tag{8.7}$$

The corresponding estimate on the evolution group is also available. Note that, although the symbol of the Hamiltonian does have full phase-space anisotropy, due to the particular form of the quasiorbit $E \times \Omega^*$, one gets a configuration-space localization result.

Similarly, if we consider the quasi-orbit $\Omega \times E^*$, one obtains easily momentum-space estimates describing forbidden momenta for suitable energies.

Finally, let us take into account the quasiorbit $F' := E \times E^*$. With choices of generic points $\omega \in E$ and $\omega^* \in E^*$ we construct the asymptotic Hamiltonian

$$H^{\omega,\omega^*} := \mathfrak{Op}\left(1 \otimes \psi^{\omega^*} + \phi^\omega \otimes 1\right) = \psi^{\omega^*}(P) + \phi^\omega(Q) \tag{8.8}$$

with spectrum $S^{F'}$ independent of the pair (ω, ω^*) (and contained in the previous set S^E, which in its turn is contained in the essential spectrum of H). If the support of ρ is disjoint from $S^{F'}$, very much as above, for every $\varepsilon > 0$ we get the phase-space localization estimate

$$\left\| \mathfrak{Op}\left[\chi_{U\times U^*}^{\varphi}\right]\rho(H) \right\|_{\mathbb{B}(\mathcal{H})} \leq \delta, \tag{8.9}$$

valid for $U := \mathcal{U} \cap \mathscr{X}$ and $U^* := \mathcal{U}^* \cap \mathscr{X}^*$, where $\mathcal{U}^*$ is a small neighborhood of $E^* \subset \Omega^*$.

Of course, if the C^*-algebras $\mathcal{A}(\mathscr{X})$ and $\mathcal{A}(\mathscr{X}^*)$ are explicit and easy to understand, the results of this Example can be made more concrete; sources of inspiration can be found in [2], in which however $\mathscr{X}$ or $\mathscr{X}^*$ is replaced by Ξ. Some of the results described in the Introduction can be obtained easily taking $\mathscr{X} = \mathbb{R}$ and $\mathcal{A}(\mathscr{X}), \mathcal{A}(\mathscr{X}^*)$ be the C^*-algebras of continuous functions (on $\mathscr{X}$ and $\mathscr{X}^*$, respectively) having (a priori different) limits at $\pm\infty$.

Example 8.2. Assume that (Σ^0, Θ, Ξ) is a topologically transitive dynamical system. To have a concrete example in mind not consisting of translations, we could take Ξ to be $\mathbb{R} \times \mathbb{R}$ acting on the closed quarter plane $[0, \infty) \times [0, \infty)$ by dilations:

$$\Theta_{(x,\xi)}(y, \eta) := (e^x y, e^\xi \eta). \tag{8.10}$$

For bounded continuous functions $f : \Sigma^0 \to \mathbb{C}$ and subsets L of Ξ we define *the oscillation of f on L along the action Θ* by

$$\operatorname{osc}_L^\Theta(f) : \Sigma^0 \to \mathbb{R}_+, \quad \left[\operatorname{osc}_L^\Theta(f)\right](\sigma) := \sup_{X\in L}\left|f[\Theta_X(\sigma)] - f(\sigma)\right|. \tag{8.11}$$

Note the simple properties

$$\left[\operatorname{osc}_L^\Theta(f+g)\right](\sigma) \le \left[\operatorname{osc}_L^\Theta(f)\right](\sigma) + \left[\operatorname{osc}_L^\Theta(g)\right](\sigma), \tag{8.12}$$

$$\left[\operatorname{osc}_L^\Theta(af)\right](\sigma) = |a|\left[\operatorname{osc}_L^\Theta(f)\right](\sigma), \tag{8.13}$$

$$\left[\operatorname{osc}_L^\Theta(\overline{f})\right](\sigma) = \left[\operatorname{osc}_L^\Theta(f)\right](\sigma), \tag{8.14}$$

$$\left[\operatorname{osc}_L^\Theta(fg)\right](\sigma) \le \| g\|_\infty\left[\operatorname{osc}_L^\Theta(f)\right](\sigma) + \| f\|_\infty\left[\operatorname{osc}_L^\Theta(g)\right](\sigma). \tag{8.15}$$

The family $\mathrm{VO}^\Theta(\Sigma^0)$ of *vanishing oscillations functions along the action* Θ is defined by requiring that $\operatorname{osc}_K^\Theta(f) \in C_0(\Sigma^0)$ for every compact subset K of Ξ. To make it interesting, we assume that the locally compact space Σ^0 is not compact. Using (8.12), (8.13) (8.14) and (8.15) we see that $\mathrm{VO}^\Theta(\Sigma^0)$ is a unital *-algebra. The easy to prove inequality

$$\| \operatorname{osc}_L^\Theta(f) - \operatorname{osc}_L^\Theta(g)\|_\infty \le 2\, \| f - g\|_\infty \tag{8.16}$$

even shows that it is a C^*-algebra. For $Y \in \Xi$ and $K \subset \Xi$ compact, one has

$$\operatorname{osc}_K^\Theta[\Theta_Y(f)] \le \operatorname{osc}_{Y+K}^\Theta(f) + \operatorname{osc}_{\{Y\}}^\Theta(f), \tag{8.17}$$

so $\mathrm{VO}^\Theta(\Sigma^0)$ is Θ-invariant. Thus Rieffel deformation apply, yielding a non-commutative C^*-algebra $\mathfrak{VO}^\Theta(\Sigma^0)$.

We need a certain understanding of the Gelfand spectrum of $\mathrm{VO}^\Theta(\Sigma^0)$, denoted by Σ. Since $\Theta : \Xi\times\Sigma^0 \to \Sigma^0$ is continuous, the set $\Theta_K(\Upsilon)$ is compact in Σ^0 if $K \subset \Xi$ and $\Upsilon \subset \Sigma^0$ are compact; this leads easily to a proof of the inclusion $C_0(\Sigma^0) \subset \mathrm{VO}^\Theta(\Sigma^0)$. It follows that Σ is a compactification of Σ^0 and $\Gamma := \Sigma \setminus \Sigma^0$ is a compact dynamical system under an extension of the action Θ. We are going to show that its elements γ are fixed points for this extension, also denoted by Θ. The point γ is a character of $\mathrm{VO}^\Theta(\Sigma^0)$ which is trivial on $C_0(\Sigma^0)$. Thus for every $X \in \Xi$ and every $f \in \mathrm{VO}^\Theta(\Sigma^0)$ one has

$$[\Theta_X(\gamma) - \gamma](f) = \gamma[\Theta_X(f) - f] = \gamma[f\circ\Theta_X - f] = 0,$$

because $f\circ\Theta_X - f \in C_0(\Sigma^0)$. The proof also shows that $\mathrm{VO}^\Theta(\Sigma^0)$ is the largest C^*-algebra having this property.

Let now h be a smooth element of the C^*-algebra $\mathfrak{VO}^\Theta(\Sigma_0)$. Recall that Σ^0 has been supposed topologically transitive; then clearly Σ is also topologically transitive, because Σ^0 is dense in Σ. Choosing a point σ belonging to a dense orbit one constructs in the Hilbert space $\mathcal{H} := L^2(\mathscr{X})$ the generalized Weyl operator $H_\sigma := \mathfrak{Op}(h\circ\Theta_\sigma)$ (see Section 6, especially (6.1), for its definition).

By general principles [26], the essential spectrum of H_σ is the closed union of spectra of generalized Weyl operators defined by asymptotic symbols, which actually are restrictions to non-generic quasiorbits of the extension of h to Σ (also denoted by h). In Σ^0 there could already exist such quasiorbits that we do not know. So we concentrate on those that we do know, the fixed points belonging to Γ, that will only describe a subset of the essential spectrum. The asymptotic symbol corresponding to such a point γ is just given by a real number $h(\gamma)$ which, after Weyl quantization, will yield the constant operator $h(\gamma)1_{\mathcal{H}}$ with singleton spectrum $\{h(\gamma)\}$. Consequently, one gets $h(\Gamma) \subset \mathrm{sp}_{\mathrm{ess}}(H_\sigma)$ and this inclusion is an equality if there are no non-generic quasiorbits in Σ^0.

Since the "boundary at infinity" Γ is difficult to understand, it helps to express $h(\Gamma)$ only in terms of the values taken by h on Σ^0. Let us choose $\tau \in \Sigma^0$ generic (belonging to a dense orbit). By mimiking arguments from [25] it can be shown that

$$h(\Gamma) = \bigcap \left\{ \overline{h[\Theta_{\Xi\setminus K}(\tau)]} \mid K \subset \Xi \text{ is compact} \right\}.$$

Of course this set is τ-independent. It could be called *the asymptotic range of the symbol h along the action Θ*. Another description of the points of $h(\Gamma)$ are in terms of *δ-level sets*. For $\lambda \in \mathbb{R}$ and $\delta > 0$ one sets

$$M_\delta^{\Theta,\tau}(h;\lambda) := \{ X \in \Xi \mid |h[\Theta_X(\tau)] - \lambda| < \varepsilon \}.$$

Then λ belongs to $h(\Gamma)$ if and only if $M_\delta^{\Theta,\tau}(h;\lambda) \subset \Xi$ is not relatively compact for any $\delta > 0$. Note the relation

$$M_\delta^{\Theta,\Theta_Y(\tau)}(h;\lambda) = M_\delta^{\Theta,\tau}(h;\lambda) - Y,$$

valid for two points τ and $\Theta_Y(\tau)$ placed on the same (dense) orbit, that certifies once again the independence of the condition on the generic point $\tau \in \Sigma^0$.

So let us now fix a bounded continuous function $\rho : \mathbb{R} \to [0,\infty)$ which is zero in a neighborhood of the point $\lambda = h(\gamma)$ (which is the spectrum of the asymptotic Hamiltonian $h(\gamma)1_{\mathcal{H}}$). By Theorem 6.3, for any $\varepsilon > 0$ there exists a neighborhood $\mathcal{W}$ of γ in Σ such that

$$\left\| \mathfrak{Op}_\sigma\big(\chi_{\mathcal{W}}^\varphi\big)\rho(H_\sigma) \right\|_{\mathbb{B}(\mathcal{H})} \le \varepsilon, \tag{8.18}$$

with the usual consequence for the evolution group generated by H_σ.

It is difficult in general to describe properly the neighborhood $\mathcal{W}$. For the case of translations acting on $\Sigma^0 = \Xi$, one can adapt the analysis from [2] . This also can be done quite easily for the action by dilations of $\mathbb{R}\times\mathbb{R}$ on $[0,\infty)\times[0,\infty)$. In this situation, however, one has the extra three non-generic quasiorbits

$$\{0\}\times\{0\}, \quad \overline{[0,\infty)\times\{0\}} \quad \text{and} \quad \overline{\{0\}\times[0,\infty)}$$

to be taken into account. It is easier to write the localization and the non-propagation results for these quasiorbits. We indicate the nice simple form of the corresponding three asymptotic Hamiltonians, following from (6.1):

$$H_{(0,0)} = h(0,0)1_{\mathcal{H}},$$

$$(H_{(1,0)}u)(x) = h(e^x, 0)u(x)\,,$$
$$(H_{(0.1)}u)(x) = \int_{\mathscr{X}} dy\, \widehat{k}(x-y)u(y)\,,$$

where $\widehat{k}$ is the Fourier transform of the function

$$k : \mathscr{X}^* \to \mathbb{R}, \qquad k(\eta) := h(0, e^\eta)\,.$$

The contribution to the essential spectrum coming from these quasiorbits consists of the two real segments $h([0,\infty) \times \{0\})$ and $h(\{0\} \times [0,\infty))$. To be relevant and interesting, the localization function ρ must now be supported away from these two segments but it should be non-trivial at least on part of the set $h(\Gamma)$ described above. At energies belonging to the support of ρ propagation towards the two edges $[0,\infty) \times \{0\}$ and $\{0\} \times [0,\infty)$ is forbidden.

Remark 8.3. As a particular case of Example 8.2 one can take $\Sigma^0 := \Xi$ on which Θ is the action by translations. At the level of the essential spectrum this has been treated in [26]. The localization results can easily be infered from the discussion above; no extra non-generic quasiorbits are present here.

Remark 8.4. On the other hand, vanishing oscillation functions can be considered in the setting of Example 8.1 for the C^*-algebras $\mathcal{A}(\mathscr{X})$ and/or $\mathcal{A}(\mathscr{X}^*)$. Notice that $\mathrm{VO}(\Xi)$ and $\mathrm{VO}(\mathscr{X}) \otimes \mathrm{VO}(\mathscr{X}^*)$ are quite different, thus leading to different types of phase-space localization results.

Remark 8.5. In [2] (see also [24, 25]) it was shown how to extend localization and propagation results to C^*-algebras generated by a mixture of vanishing oscillation and minimal (in particular almost periodic) functions on a vector space. This can be further extended to actions of vector groups on locally compact spaces, thus generalizing Example 8.2. Since this is quite straightforward, we are not going to do this here.

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