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Coloraciones de aristas con restricciones en subgrafos

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por

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Se informa a la escuela de Postgrado de la Facultad de Ciencias que la Tesis de Doctorado presentada por la candidata

Natacha Astromujoff

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A mi mamá, que leía a Cortazar en las clases de matemática y me enseñó a ver la magia de los hilos de colores en la historia de un Cronopio, aunque de a ratos parezca una Esperanza o me porte como un Fama.

Biografía

Nací el 6 de mayo de 1973, en Buenos Aires. A los cinco años me fui con mi familia a vivir a Barcelona, pero volví a Buenos Aires un año después, con mi mamá y mi hermana. Allí todavía vive mi papá, y desde hace veinte años Soledad, mi hermana menor. En el año 91 mi mamá se mudó a Santiago de Chile. Yo me fuí un año a Barcelona, y volví a Buenos Aires. Estudié en la UBA Lic. en letras algunos años, hasta que descubrí la lingüística y decidí estudiar matemáticas al mismo tiempo. Todo esto mientras trabajaba en producción de cine, con lo que solo hice el primer año de la Lic. en matemáticas. En el año 2002, por diversos motivos llegué a Chile, y comencé a estudiar en la facultad de ciencias de la Universidad de Chile la licenciatura en matemáticas. Luego el doctorado en la misma facultad.

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Resumen:

En esta tesis se estudian tres tipos de coloraciones de aristas de grafos. En el primer capítulo, la introducción, damos una breve historia de las coloraciones de grafos. Luego hacemos una descripción de coloraciones de aristas en las que dos colores cualesquiera de la misma forman un ciclo hamiltoniano, *coloraciones perfectas*, o *coloraciones acíclicas*, donde dos colores forman un subgrafo acíclico. También introducimos un nuevo tipo de coloración, que hemos llamado *coloración aritmética de aristas*. En cada caso se describen los resultados existentes y los que se han obtenido en esta tesis.

En el segundo capítulo nos concentramos en los resultados que obtuvimos en coloraciones perfectas. Estos corresponden a un análisis de la cantidad de *pares perfectos*, pares de colores que inducen un ciclo hamiltoniano, que puede tener una coloración del grafo bipartito balanceado completo con $2n$ vértices, dando el valor exacto en el caso de que n sea par y una cota inferior si n es impar.

En el tercer capítulo utilizamos resultados del capítulo 2 para construir una coloración acíclica del grafo bipartito completo balanceado con $2n$ vértices, usando $2n + 4$ colores, para n un número primo impar.

En el último capítulo, se define la noción de coloración aritmética y se demuestran diferentes cotas superiores e inferiores para la cantidad mínima de colores necesaria para obtenerla, en términos de parámetros como el grado máximo del grafo y el *número cromático inyectivo*. También se presentan resultados sobre la complejidad computacional del cálculo de una coloración aritmética óptima, i.e. el cálculo del *índice aritmético*.

En el apéndice damos algunas definiciones básicas de Teoría de Grafos.

Abstract:

In this thesis three types of edge colorings of graphs are studied. In the first chapter, the introduction, we give a brief history of graph colorings. Then we make a description of edge colorings in which any two colors form a special subgraph. An edge coloring is a *perfect coloring* when the subgraph that these colors determine is a Hamiltonian cycle, while it is an *acyclic coloring* when this subgraph is acyclic. We also introduce a new type of edge coloring, which we call *arithmetic coloring*. In each case the previous results and those obtained in this thesis are described.

In the second chapter we focus on results we have obtained about perfect colorings in balanced complete bipartite graphs. These correspond to a quantitative analysis of the number of pairs of colors inducing a Hamiltonian cycle. When the graph has $2n$ vertices, we determine the exact value if n is even, and a lower bound if n is odd.

In the third chapter we use results of Chapter 2 to construct an acyclic coloring of the complete bipartite balanced graph with $2n$ vertices, using $2n + 4$ colors, when n is an odd prime number.

In the last chapter, the notion of arithmetic coloring is defined. We obtain upper and lower bounds for the minimum number of colors required to obtain an optimal arithmetic coloring. These bounds are given in terms of the maximum degree of the graph and the injective chromatic number. We also present some results on the computational complexity of computing an optimal arithmetic coloring.

In the appendix we give some basic definitions of Graph Theory.

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Capítulo 1

Introducción

Grafos y coloración: El hilo conductor de esta tesis está dado por las coloraciones de aristas en *grafos*. Un grafo, informalmente, es una colección de puntos llamados *vértices*, que pueden o no estar unidos por líneas, llamadas *aristas*. Formalmente, un grafo es un conjunto V, cuyos elementos se llaman vértices, junto con una familia, E, de subconjuntos de tamaño 2 de V, cuyos elementos se llaman aristas. Cuando $e = \{u, v\}$ es una arista, se dice que u y v son *vecinos* o vértices *adyacentes* y que la arista e es incidente en u y en v. También diremos en este caso que u y v son los *extremos* de e.

La teoría de grafos se remonta a la primera mitad del siglo XVIII, cuando Leonhard Euler modela el problema conocido como "los puentes de Königsberg", que pregunta si es posible dar un paseo por la ciudad cruzando cada puente exactamente una vez [24].

Por su parte, los problemas de *coloración* en grafos tienen su origen a mediados del siglo XIX. El primero puede rastrearse a una carta entre August De Morgan y William Hamilton, donde De Morgan pregunta si es posible colorear las regiones de un mapa con cuatro colores evitando que las regiones que comparten frontera tengan el mismo color.

Este problema puede formularse mediante grafos de la siguiente forma: Si cada región es un vértice y tenemos aristas entre los vértices cuando las regiones que éstos representan son vecinas, lo que estamos buscando es una coloración de los vértices del grafo con cuatro colores, es decir, una asignación de colores a cada vértice tal que dos vértices que están unidos por una arista no reciban el mismo color. En general, el problema de coloración consiste en saber cuál es la mínima cantidad de colores con la que se puede conseguir una coloración como la descrita para cada grafo.

Nuestro trabajo está centrado en *coloraciones de aristas*. Expresada directamente en el grafo, una coloración de aristas es una función que a cada arista del mismo le asigna un color, de forma que dos aristas que inciden en un mismo vértice reciben colores diferentes. Al igual que con las coloraciones de vértices, la primera pregunta que surge es con cuántos colores como mínimo puede realizarse una coloración de aristas. A este número mínimo se lo llama *índice cromático*, y se nota $\chi'(G)$. Vizing demostró en 1964 que este número puede acotarse linealmente con respecto al *grado máximo* del grafo, $\Delta(G)$, que es el máximo número de aristas que inciden en un vértice del grafo. La cota que demostró es $\chi'(G) \leq \Delta(G)+1$ [56].

Cuando para un grafo G cada vértice v tiene exactamente Δ aristas incidentes, es decir G es Δ-*regular*, y tenemos que $\chi'(G) = \Delta(G)$, entonces cada color debe ser usado en una arista incidente en v, para cada vértice v . Cuando esto ocurra diremos informalmente que el vértice v *ve* todos los colores o que todos los colores *aparecen* en v.

Un conjunto de aristas M es un *emparejamiento* si para cada vértice v del grafo, a lo sumo una arista de M incide en v. Una manera equivalente de definir una coloración de aristas es como una partición de las aristas del grafo en emparejamientos; cada clase de la partición corresponde a las aristas de un color dado. Decimos que un emparejamiento M es *perfecto* cuando todos los vértices del grafo son el extremo de una arista de M.

A pesar de que sabemos que $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ y que es posible determinar eficientemente el valor $\Delta(G)$, la determinación algorítmica del índice cromático $\chi'(G)$ es difícil. Más precisamente, en [36] Hoyler demuestra que es NP-completo decidir cuándo un *grafo cúbico* tiene índice cromático 3 o 4. Sin embargo, para *grafos bipartitos*, aquellos que tienen una coloración de vértices con dos colores, se sabe que $\chi'(G) = \Delta(G)$ y que una coloración de aristas óptima puede encontrase en tiempo polinomial [38].

En este trabajo se estudian tres tipos de coloraciones de aristas diferentes. Comenzaremos por las *coloraciones perfectas* de aristas, luego *coloraciones acíclicas* de aristas y finalmente *coloraciones aritméticas* de aristas. Estas tres variantes agregan distintos tipos de restricciones a la coloración de aristas que hemos descrito. En la historia de las coloraciones de grafos pueden encontrarse distintos tipos de coloraciones, tanto de aristas como de vértices, que agregan restricciones a las originales, en muchos casos con el objeto de modelar un problema particular. Para mayor conocimiento del tema puede verse [38].

1.1 Preliminares

1.1.1 Coloración perfecta de aristas

Diremos que una coloración de aristas es perfecta si el conjunto de aristas de cada color es un emparejamiento perfecto y al tomar el subgrafo inducido por cualquier par de colores obtenemos un *ciclo hamiltoniano*, ciclo que cubre todos los vértices del grafo. Por lo antes dicho, para que una coloración como esta sea posible en un grafo G, necesitamos como mínimo que el grafo sea Δ-regular, con cantidad par de vértices y que tenga índice cromático $\chi'(G) = \Delta$. Esto se debe a que, para obtener un ciclo hamiltoniano con cualquier par de colores, cada color debe aparecer en cada vértice del grafo. Así esta coloración se relaciona naturalmente con los parámetros $\chi'(G)$ y $\Delta(G)$.

Con respecto a este tipo de coloración existe una conjetura para grafos completos con cantidad par de vértices. Diremos que G es un *grafo completo*, y lo notaremos $G = K_n$ si el conjunto de vértices V tiene tamaño n y todos los pares de vértices son adyacentes.

Conjetura: 1 (Coloración perfecta [40]). *Existe una coloración perfecta del grafo completo con* $2n$ *vértices,* K_{2n}.

Sobre la conjetura 1 Anderson demostró que el grafo completo K_{2p} tiene una coloración perfecta si p es un número primo impar [4]. El grafo completo K_{p+1}, tiene una coloración perfecta si p es un número primo impar [5]. Éstas son las dos únicas familias infinitas para las que se ha demostrado la conjetura. También se sabe que si $2n \in \{16, 28, 36, 40, 50, 126, 170, 244, 344, 730, 1332, 1370, 1850, 2198, 3126, 6860\}$ entonces K_{2n} satisface la conjetura (ver [47]).

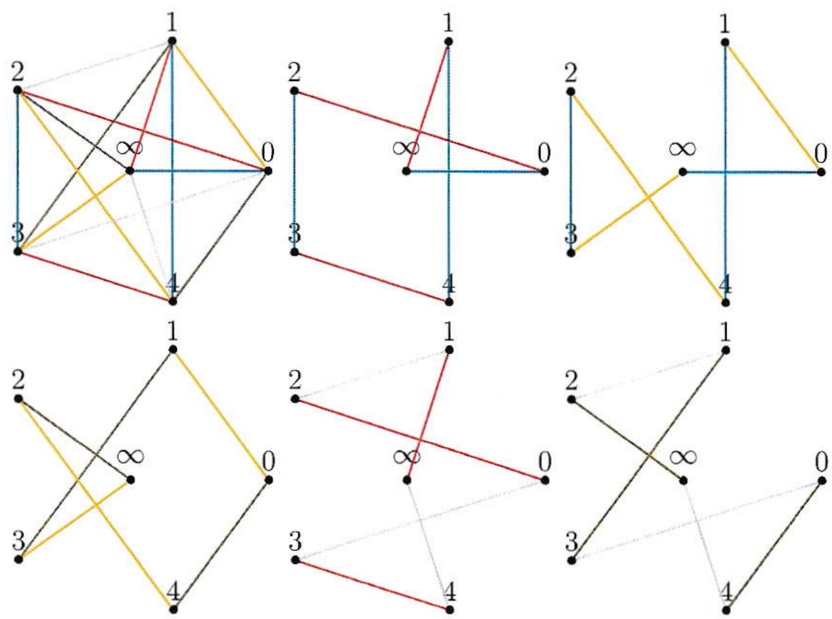

Figura 1.1: Coloración perfecta de $K_{p+1}, p = 5$

La conjetura 1 es sobre coloraciones perfectas en grafos completos y de ella se desprende una nueva conjetura sobre coloraciones perfectas de *grafos bipartitos completos balanceados*. Un grafo G es bipartito completo balanceado si su conjunto de vértices V se particiona en dos conjuntos X e Y del mismo tamaño, y sus aristas son todos los subconjuntos $\{u, v\}$ con $u \in X, v \in Y$. Los conjuntos X e Y se llaman las *partes* del grafo o sus *conjuntos independientes*. La notación utilizada para este grafo es $K_{n,n}$ donde n es el tamaño de X y de Y. Läufer [41] demuestra que si K_{n+1} admite una coloración perfecta, entonces $K_{n,n}$ tiene una coloración perfecta. Como hemos dicho, una coloración perfecta de K_n puede existir sólo si n es par. En cambio, para el grafo bipartito completo una coloración perfecta de aristas puede existir sólo si el grafo es balanceado y además sus partes son de tamaño 2 o impar mayor que dos [11]. Cabe decir que una coloración perfecta de $K_{n,n}$ no implica una de K_{n+1}, i.e. que el recíproco no es cierto.

De este resultado, y de los antes mencionados para K_n se puede inferir que tanto $K_{p,p}$ como $K_{2p-1,2p-1}$ tienen coloraciones perfectas, para p un primo impar. En 2001, Briant, Maenhaut y Wanless demuestran que el grafo bipartito completo K_{p^2,p^2} admite una coloración perfecta cuando p es un número primo impar [17].

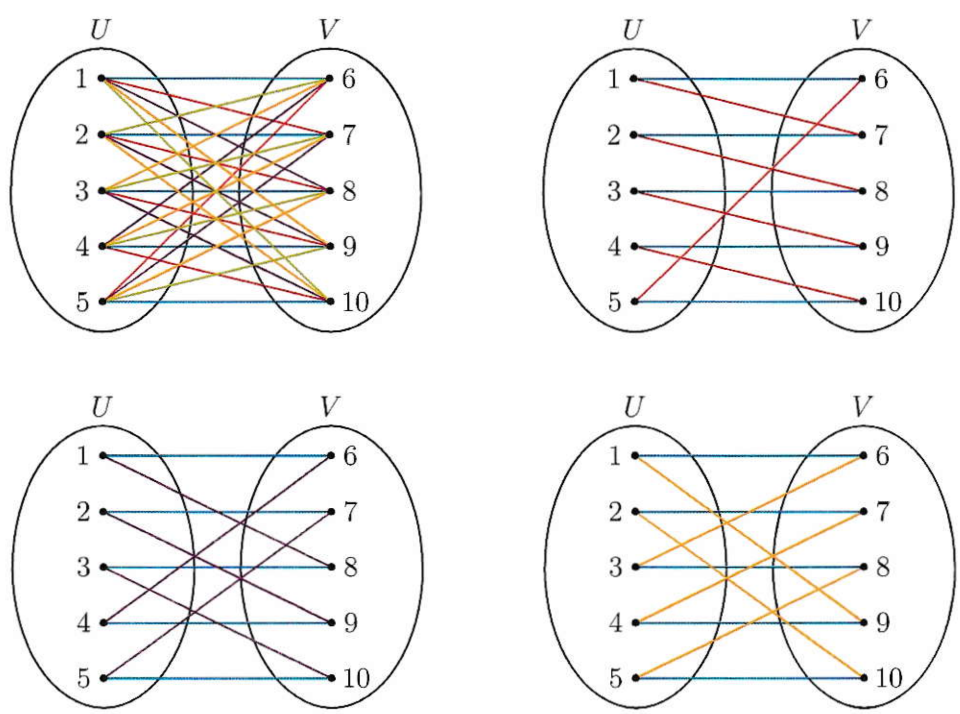

Figura 1.2: Coloración perfecta de $K_{p,p}, p = 5$

Usando el resultado de Läufer la conjetura 1 tiene una versión en grafos bipartitos completos:

Conjetura: 2 (Coloración perfecta del grafo bipartito completo [60]). *Todo grafo bipartito completo $K_{2n-1,2n-1}$ para n un entero positivo, admite una coloración perfecta.*

Hasta aquí los resultados que hemos descrito se enfocan solamente en demostrar la conjetura 1 para grafos completos, o la conjetura 2 para grafos bipartitos completos en ciertas subfamilias de los mismos. Un enfoque diferente a la conjetura fue dado por Wagner [57]. Éste corresponde a una versión cuantitativa de la misma, referida a grafos completos. En esta tesis realizamos un análisis similar al hecho por Wagner para grafos completos, pero en grafos bipartitos completos. El detalle de los resultados obtenidos se dará en la sección 1.2.1.

1.1.2 Coloraciones acíclicas de aristas

El segundo problema que abordamos, también sobre coloraciones de aristas, es el de las coloraciones acíclicas. Una coloración acíclica del grafo G es una coloración de vértices o aristas en la que al tomar el subgrafo inducido por dos colores cualquiera, éste es acíclico. Originalmente fueron definidas por Grünbaum como coloraciones de vértices en [30]. Allí él demuestra que un *grafo planar* siempre es acíclicamente coloreable con 9 colores. Esta cota es mejorada en diversas ocasiones, hasta que en 1979 Borodin demuestra que estos grafos son acíclicamente coloreables con 5 colores [14], este resultado es óptimo [39].

Algunos años más tarde aparece la noción de coloración acíclica de aristas. La notación usada para designar el índice acíclico es $a'(G)$, i.e. la menor cantidad de colores con la cual se puede obtener una coloración de las aristas de un grafo G tal que no existan *ciclos bicromáticos*, i.e. ciclos que están coloreados con dos colores. En la literatura puede también encontrarse como χ'_a, pero evitamos esta notación, pues la utilizaremos más adelante cuando definamos la coloración aritmética. Con respecto al índice acíclico se han demostrado algunas cotas inferiores. Para grafos Δ-regulares, se sabe que $a'(G) \geq \Delta + 1$. Esto se debe a que si coloreamos el grafo con Δ colores, el subgrafo inducido por dos colores es un 2-*factor*, un subgrafo con todos sus vértices de grado 2, y es un resultado conocido que un 2-factor es la unión disjunta de ciclos. Para grafos Δ-regulares con $2n$ vértices y $\Delta > n$, se ha demostrado que $a'(G) \geq \Delta + 2$ [10]. En el caso de grafos bipartitos completos balanceados, se ha demostrado que $a'(K_{n,n}) \geq n + 2$ cuando n es un número entero impar [11].

En la Figura 1.3 puede verse un pequeño ejemplo de este tipo de coloración.

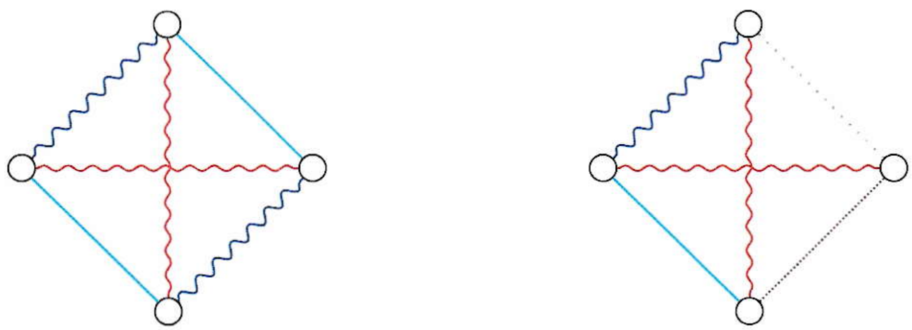

Figura 1.3: Izq.: Coloración de aristas de K_4, con 3 colores. Der.: Coloración acíclica de aristas de K_4 con 5 colores.

Respecto del índice acíclico se conjetura que $\Delta(G) + 2$ es una cota superior. Esta conjetura fue hecha independientemente [27] por Fiamčik y [1] por Alon, Mc Diarmid y Reed. Pese a que existen trece años de diferencia entre estos trabajos, los de Fiamčik no fueron traducidos del ruso hasta hace pocos años.

Conjetura: 3 (Coloración acíclica [27]). *Para cada grafo G, $a'(G) \leq \Delta(G) + 2$.*

Con respecto a esta conjetura se sabe que los siguientes grafos la cumplen:

Los grafos planares G, que no contienen ciclos de largo 5 como subgrafo, tienen índice acíclico $a'(G) \leq \Delta(G) + 2$ [52]. Si G es tal que $\mathsf{mad}\ (G) \leq 4$, donde mad es el máximo sobre el promedio del grado máximo de los subgrafos de G, se tiene

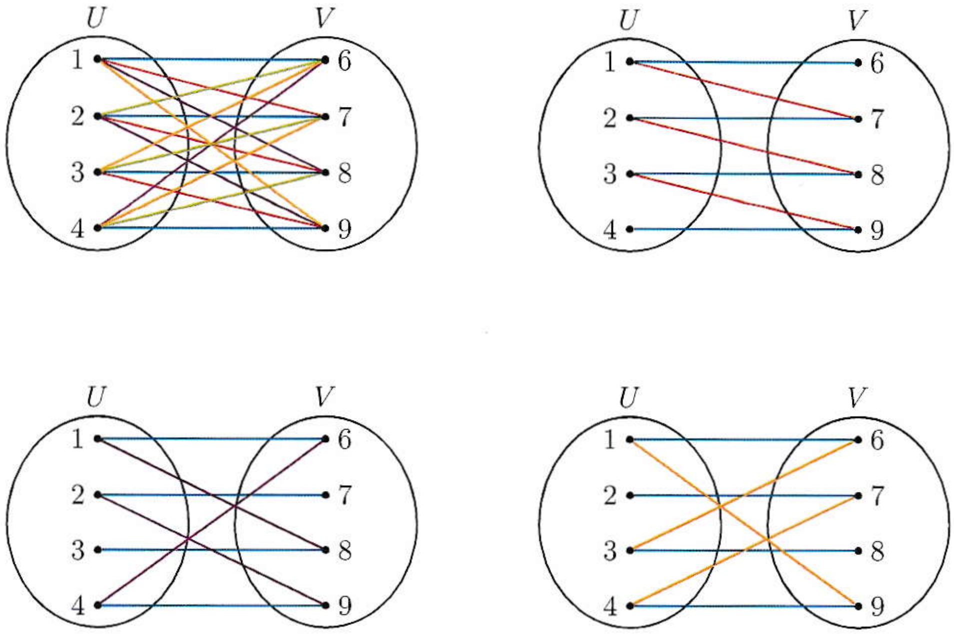

Figura 1.4: Coloración acíclica de $K_{p-1,p-1}$ derivada de coloración perfecta de $K_{p,p}, p = 5$

que $a'(G) \leq \Delta(G)+2$ [59]. Para grafos planares exteriores, i.e. grafos que pueden ser dibujados en el plano sin que sus aristas se crucen, y con todos sus vérices en el borde de la cara exterior $a'(G) \leq \Delta(G)+1$, e incluso se tiene que $a'(G) = \Delta(G)$ si $\Delta(G) \geq 5$ [37]. También los grafos 2-degenerados cumplen $a'(G) \leq \Delta(G) + 1$ [12]. Los grafos 4-regulares tienen una coloración acíclica con 5 colores [18]. Los grafos subcúbicos cumplen la conjetura, y si el grafo no es 3-regular se necesita un color menos [9]. Casi todo grafo Δ-regular cumple la conjetura [2]. En [46] prueban que si G es un grafo Δ-regular aleatorio, $a'(G) \leq \Delta(G) + 1$.

Las mejores cotas superiores para un grafo general se deben a Molloy y Reed [43] quienes demuestran mediante métodos probabilísticos que $a'(G) \leq 16\Delta(G)$, mejorando $a'(G) \leq 64\Delta(G)$, previamente demostrado por Alon et al.: [1]. Con las mismas herramientas se demuestra en [44] que $a'(G) \leq 4{,}52\Delta(G)$, si G tiene cintura (*girth* en inglés) de al menos 220. Además, en [2] se demuestra que existe una constante k, tal que si la cintura de G es al menos $k\Delta(G) \log \Delta(G)$, entonces G cumple la conjetura. Por otra parte, la mejor cota *constructiva* para grafos genéricos es $a'(G) \leq 5\Delta(G)(\log \Delta(G) + 2)$, demostrada por Subramanian [54].

1.1.3 Coloraciones acíclicas derivadas de coloraciones perfectas

Si bien las nociones de coloración perfecta y coloración acíclica de aristas parecen opuestas, en varios trabajos se ha explorado la posibilidad de construir coloraciones acíclicas a partir de coloraciones perfectas. Además de los resultados antes mencionados, se demostraron cotas superiores e inferiores para grafos completos y bipartitos completos, explotando la relación existente con las coloraciones perfectas. Por una parte, una coloración perfecta con k colores de un grafo G induce una coloración acíclica, con el mismo número de colores, en cualquier subgrafo inducido propio. Por ejemplo, para grafos completos se sabe que si una coloración perfecta de K_{2n+2} existe, entonces el subgrafo inducido K_{2n+1} tiene una coloración acíclica con $2n+1$ colores, es decir con $\Delta(K_{2n+1}) + 1$ colores. Esto es lo mínimo para colorear acíclicamente un grafo Δ-regular, como fue notado por Alon et. al. [2]. Esta construcción toma una coloración perfecta de K_{2n+2} y construye lo que llamaremos una coloración *casi-perfecta* de K_{2n+1}. La misma se obtiene retirando un vértice de K_{2n+2}. Así al mirar la coloración en el subgrafo $(K_{2n+2} \setminus v) = K_{2n+1}$ y tomar el subgrafo inducido por dos colores, lo que se obtiene es un camino hamiltoniano, i.e. un camino que pasa por todos los vértices del grafo. A partir de esta relación, Alon et al. demostraron [2] que son equivalentes:

- El grafo completo K_{2n+2} tiene una coloración perfecta.
- El grafo completo K_{2n+1} tiene una coloración casi-perfecta.
- El índice acíclico de K_{2n+1} cumple $a'(K_{2n+1}) = 2n+1$.

Un resultado similar, para grafos bipartitos balanceados completos, es mencionado por Basavaraju y Chandran [10] donde aseveran que tomando una coloración perfecta de $K_{n+2,n+2}$, y retirando un vértice de cada conjunto de la bipartición se obtiene una coloración de $K_{n+1,n+1}$ como subgrafo inducido. De la misma forma se puede obtener una coloración de $K_{n,n}$. En ambos casos la coloración inducida es acíclica, lo que nos da una cota superior para estos grafos $a'(K_{n,n}) \leq a'(K_{n+1,n+1}) \leq n+2$. Por otro lado, ya vimos que $a'(K_{n+1,n+1}) \geq n+2$, pues este grafo es regular, y que $a'(K_{n,n}) \geq n+2$ para n impar, [10]. Por lo tanto, cuando $K_{n+2,n+2}$ tiene una coloración perfecta, se cumple que $a'(K_{n,n}) = a'(K_{n+1,n+1}) = n+2$. Se conoce una coloración perfecta de $K_{n+2,n+2}$ si $n+2 \in \{p, 2p-1, p^2\}$ [4, 5, 17], donde los dos primeros se deben a una coloración perfecta de K_{n+3}. Por lo tanto, se tiene que $a'(K_{n,n}) = n+2, n \in \{p-2, 2p-3, p^2-2\}$, y que $a'(K_{n+1,n+1}) = n+2$, para $n+1 \in \{p-1, 2p-2, p^2-1\}$, en los tres conjuntos anteriores p es un número primo impar.

Por otra parte, dada una coloración perfecta de aristas, basta cambiar el color en una de ellas en cada uno de los ciclos hamiltonianos, para obtener una coloración acíclica. El punto crítico es hacerlo con la menor cantidad de colores, teniendo la precaución de no generar nuevos ciclos con los colores que agregamos. Usando esta idea, se ha demostrado que $a'(K_{p,p}) = p+2$ para p un primo impar [10].

1.1.4 Coloraciones aritméticas

En la tercera parte de esta tesis trabajamos un nuevo tipo de coloración de aristas que llamamos coloración aritmética, este tipo de coloración fue definida por Matamala y Zamora. Dado un grafo $G = (V, E)$, consideraremos una función $\varphi : V \to \mathbb{N}$ que asigna a cada vértice un número natural. Decimos que φ es un *potencial* para G si la función que asigna a cada arista la distancia entre los números dados por φ a sus extremos, es una coloración de aristas. Esta coloración se llamará *la coloración inducida por* φ. El origen de este tipo de coloraciones es la llamada conjetura de los árboles elegantes (*the graceful tree conjeture*) [48], donde se pregunta si todo árbol con n vértices, tiene un potencial φ inyectivo, con imagen en $\{1, \ldots, n\}$, de modo que su coloración inducida sea también inyectiva. En nuestro estudio se relaja la condición de inyectividad, tanto en el potencial como en su coloración inducida. De este modo, decimos que una coloración de las aristas de un grafo es *aritmética* si es una coloración inducida por un potencial del grafo.

Así, dado un grafo G, tiene sentido buscar el menor entero positivo k para el cual existe un potencial φ con imagen $\Im(\varphi) \subseteq \{1, 2, \ldots k\}$. A este número k lo denotamos $\chi'_a(G)$. Es claro que, como la coloracion inducida usa a lo más $\chi'_a(G)$ colores en las aristas, se tiene que $\chi'(G) \leq \chi'_a(G)$.

Una noción similar, estudiada en la literatura, es la llamada coloración inyectiva de vértices.[1] La misma pide que vértices con un vecino común tengan colores distintos. El parámetro asociado se llama *número cromático inyectivo* y se denota $\chi_i(G)$ [31]. Este número nos dice cuál es el menor número de colores con que es posible una coloración inyectiva del grafo G. Claramente el potencial de una coloración aritmética define una coloración inyectiva de los vértices del grafo y por ende $\Delta(G) \leq \chi_i(G) \leq \chi'_a(G)$. El recíproco es falso y puede verse un ejemplo de coloración inyectiva que no es aritmética en la figura 1.5, donde se muestra que para obtener una coloración aritmética, los números que el potencial φ asigna a dos vértices vecinos de otro dado, junto con el valor del vértice, no pueden formar una progresión aritmética de largo 3.

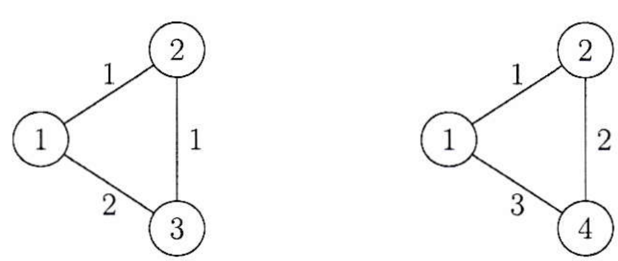

Figura 1.5: Izq.: Coloración inyectiva que no es aritmética de $K_3 = C_3$. Der.: Coloración aritmética de $K_3 = C_3$. Vértices: imagen del potencial φ. Aristas: coloración aritmética.

[1]A pesar del nombre, ésta no es una coloración como las descritas en esta tesis.

1.2 Resultados

1.2.1 Coloración perfecta, resultados cuantitativos

Como brevemente comentamos en la sección 1.1.1, Wagner realiza un análisis cuantitativo de las coloraciones perfectas de K_n [57]. Allí define la función $c(n)$ como el máximo número de pares de colores perfectos, sobre todas las coloraciones de K_n, n un número entero par. Un par perfecto es un par de colores que induce un ciclo hamiltoniano en el grafo. Así, si K_n tiene una coloración perfecta, entonces $c(n)$ toma el máximo valor posible, $\binom{n-1}{2}$. Su trabajo se enfoca a buscar cotas inferiores para la función $c(n)$.

También extiende esta función para números enteros impares, definiendo en este caso un par perfecto de colores como dos colores que inducen un camino hamiltoniano en el grafo. Como vimos antes, K_{2n} tiene una coloración perfecta si y sólo si K_{2n-1} tiene una coloración casi-perfecta, por lo que inmediatamente se tiene que $c(2n-1) = c(2n)$.

En [57] Wagner demuestra que para dos números enteros, n, m ambos impares y coprimos entre sí, $c(nm) \geq 2c(n)c(m)$. También demuestra que $c(n) \geq \frac{n}{2}\varphi(n)$, donde $\varphi(n)$ es la función de Euler.

En este trabajo hacemos un análisis similar para grafos bipartitos completos $K_{n,n}$. Para un entero positivo n definimos la función $\mathsf{pf}(n)$ como el máximo número de pares perfectos sobre todas las coloraciones de $K_{n,n}$. Se sabe, por los resultados de Wanless en [61] que para n un entero par, $\mathsf{pf}(n) \leq \frac{1}{4}n^2$ y que esta cota se alcanza cuando $n = 2p$, para p un número primo impar.

Nosotros demostramos que $\mathsf{pf}(n) \geq \frac{1}{4}n^2$, para todo n [7]. En el caso de n par, este resultado implica que $\mathsf{pf}(n) = \frac{1}{4}n^2$. Esto lo demostramos dando una coloración explícita de $K_{n,n}$ con $\frac{1}{4}n^2$ pares perfectos.

Los otros resultados concernientes a esta función están enfocados a encontrar cotas inferiores para $\mathsf{pf}(n)$ cuando n es un número compuesto. Para esto debemos extender la noción de coloración perfecta al caso de familias de emparejamientos perfectos, disjuntos dos a dos, de tamaño $m \leq n$. Así se define $\mathsf{pf}(m, n)$ como el máximo número de pares perfectos en una familia de tamaño m, de uno-factores de tamaño n, con $m \leq n$. De esta forma, la familia es perfecta si $\mathsf{pf}(m, n) = \binom{m}{2}$. Los resultados que obtenemos en esta dirección son que para todo número impar y compuesto n, con p siendo su menor divisor primo, $\mathsf{pf}(n, n) \geq \mathsf{pf}(p, \frac{n}{p})(\frac{n}{p})^2$, y que para dos números enteros impares n y m, tales que $3 \leq m \leq n$, se cumple que $\mathsf{pf}(m, n) > \frac{1}{4}m^2$. En comparación a lo hecho por Wagner, nuestro trabajo logra mejorar la cota $\frac{n}{2}\varphi(n)$, que para ciertas subsucesiones infinitas de enteros n es $o(n^2)$, por la cota $\frac{1}{4}n^2$[2].

En la figura 1.6 puede verse la construcción para $n = 5$. Para leer el cuadrado latino debemos tener en cuenta que las columnas pueden numerarse con los vértices de uno de los conjuntos de la bipartición del grafo y los símbolos en cada celda como los vértices del otro conjunto de la bipartición. De esta forma cada fila representa un uno-factor en el grafo. Dadas dos filas, los uno-factores representados forman un par perfecto si el ciclo que se describe tiene largo $4p$, y no forman un par perfecto si el ciclo se cierra antes. En la figura están marcados dos ciclos entre las filas del cuadrado, las filas de la primera mitad del cuadrado con respecto a las de la segunda mitad forman pares perfectos, mientras que las filas de una misma mitad cierran sus ciclos antes de recorrer ambas filas. (En la figura 1.7 la biyección utilizada entre el cuadrado latino y el grafo bipartito es la misma que acabamos de describir, y a esto se le agrega que el color de cada celda representa el color de la arista que une al vértice que da nombre en la columna con el vértice que representa el símbolo de la celda).

[2]Los resultados cuantitativos sobre coloraciones perfectas comprenden un artículo sometido

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00	01	02	03	04	05	06	10	13	16	12	15	11	14
01	02	03	04	05	06	00	14	10	13	16	12	15	11
02	03	04	05	06	00	01	11	14	10	13	16	12	15
03	04	05	06	00	01	02	15	11	14	10	13	16	12
04	05	06	00	01	02	03	12	15	11	14	10	13	16
05	06	00	01	02	03	04	16	12	15	11	14	10	13
06	00	01	02	03	04	05	13	16	12	15	11	14	10
10	13	16	12	15	11	14	06	00	01	02	03	04	05
13	16	12	15	11	14	10	05	06	00	01	02	03	04
16	12	15	11	14	10	13	04	05	06	00	01	02	03
12	15	11	14	10	13	16	03	04	05	06	00	01	02
15	11	14	10	13	16	12	02	03	04	05	06	00	01
11	14	10	13	16	12	15	01	02	03	04	05	06	00
14	10	13	16	12	15	11	00	01	02	03	04	05	06

Figura 1.6: Cuadrado Latino de orden $2p, p = 5$, donde se muestran los ciclos Hamiltonianos y los de largo $2p$.

1.2.2 Coloración acíclica

En la subsección anterior se mencionó que probamos que $\mathsf{pf}(2n, 2n) = \frac{1}{4}(2n)^2$. Este resultado se obtuvo mediante la construcción de una coloración de $K_{2n,2n}$ con n^2 pares perfectos. Cuando n es un número primo, esta coloración tiene una característica adicional: los pares de colores que no son perfectos inducen en el grafo exactamente dos ciclos, cada uno de largo $2n$. Esta propiedad nos permite construir una coloración acíclica de $K_{2n,2n}$, para n un número primo, que mejora los resultados conocidos. Pese a que esta cota no llega a ser la propuesta por la conjetura 3, hasta ahora no existía ninguna cota cercana para la coloración acíclica de este grafo.

La coloración acíclica que contruimos para $K_{2p,2p}$ tiene $2p + 4$ colores. Esto implica que $a'(K_{2p,2p}) \leq 2p + 4$ [8]. Para demostrar este resultado nos apoyamos en algunas ideas de la construcción propuesta en [10] para obtener una coloración acíclica de $K_{p,p}$, con p un primo impar. Allí toman una coloración perfecta de $K_{p,p}$, con los colores $1, \ldots, p$ y un emparejamiento perfecto M con una arista de cada color. Se recolorean las aristas de M con el color $p+1$, con excepción de dos: una de ellas mantiene su color original y la otra se pinta con un nuevo color $p+2$. Nuestra construcción aplica estas ideas a cuatro copias de $K_{p,p}$ que particionan las aristas de $K_{2p,2p}$.

En la figura 1.7, se puede ver la construcción en un cuadrado latino de orden $2p$, para p un número primo impar, cuando $p = 5$, (ver explicación sobre la biyección entre el cuadrado latino y el grafo dada en el último párrafo de 1.2.1).

00	01	02	03	04	10	12	14	11	13
01	02	03	04	00	13	10	12	14	11
02	03	04	00	01	11	13	10	12	14
03	04	00	01	02	14	11	13	10	12
04	00	01	02	03	12	14	11	13	10
10	12	14	11	13	04	00	01	02	03
12	14	11	13	10	03	04	00	01	02
14	11	13	10	12	02	03	04	00	01
11	13	10	12	14	01	02	03	04	00
13	10	12	14	11	00	01	02	03	04

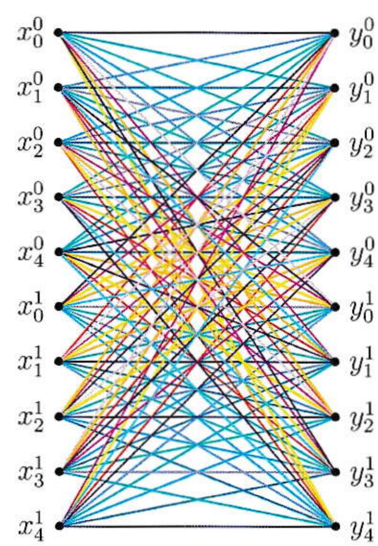

Figura 1.7: Coloración acíclica de $K_{2p,2p}$ vista en un cuadrado Latino de orden $2p, p = 5$.

1.2.3 Coloraciones aritméticas

Por las relaciones explicadas en 1.1.4, el índice aritmético $\chi'_a(K_n)$ del grafo completo de tamaño n es el menor número entero m tal que existe un conjunto de n números naturales sin progresiones aritméticas de largo 3 con máximo elemento m. La determinación de este número corresponde a un problema antiguo de la teoría de números que recibió mucha atención durante gran parte del siglo pasado, formulado inicialmente por Erdös y Turán [23]. Utilizando la relación entre el número cromático inyectivo y el índice aritmético, demostramos que para un grafo general G se cumple que $\chi'_a(G) \leq \chi'_a(K_{\chi_i(G)})$. Se demuestra también una cota súper lineal para un grafo general, en términos del número cromático inyectivo y de las cotas conocidas para conjuntos de números enteros sin progresiones aritméticas de largo tres. Esta cota es: $\chi'_a(G) \leq \chi_i(G)2^{\sqrt{2\log(\chi_i(G))}}$. Se puede ver también que los grafos completos son extremales para esta cota súper lineal.

Para ciertas familias de grafos, los *árboles* y los grafos bipartitos probamos una cota superior lineal para el valor de $\chi'_a(G)$ en términos del parámetro $\chi_i(G)$. Más precisamente, probamos que $\chi'_a(G) \leq 2\chi_i(G)-1$ para grafos bipartitos. En la figura 1.8 se ve la coloración aritmética de $K_{7,7}$. Los grafos bipartitos balanceados completos, son extremales para esta cota. Dado que en un árbol T, $\chi_i(T) = \Delta(T)$, este resultado implica que $\chi'_a(T) \leq 2\Delta(T) - 1$, pues los árboles son bipartitos. Para los árboles, conseguimos mejorar esta cota: demostramos que para un árbol T de grado máximo Δ, el índice aritmético cumple $\chi'_a(T) \leq \lceil 5\Delta/3 \rceil - 1$.

Para el índice inyectivo se sabe que en función del grado máximo existe una cota superior cuadrática para todo grafo G, $\chi_i(G) \leq \Delta^2 - \Delta + 1$ [31]. Si relacionamos la cota obtenida en función del índice inyectivo para grafos generales con el grado máximo del grafo, lo que obtenemos en nuestro caso es una cota súper cuadrática. Trabajando en esta dirección, lo primero que demostramos es un resultado para grafos bipartitos que son el grafo de incidencia de un plano proyectivo de orden Δ, P_Δ. En este caso probamos que $\chi'_a(P_\Delta) = \Delta^2 - \Delta + 1$. (Un ejemplo puede verse en 1.9.) Luego para un grafo genérico G demostramos una cota cuadrática en función de $\Delta(G)$, probando que se cumple que $\chi'_a(G) \leq 2\Delta(\Delta - 1) + 1$.

Un segundo aspecto de las coloraciones aritméticas desarrollado en esta tesis es el estudio de la complejidad computacional de problemas relacionados al cálculo de $\chi'_a(G)$. En este aspecto demostramos que para encontrar la coloración

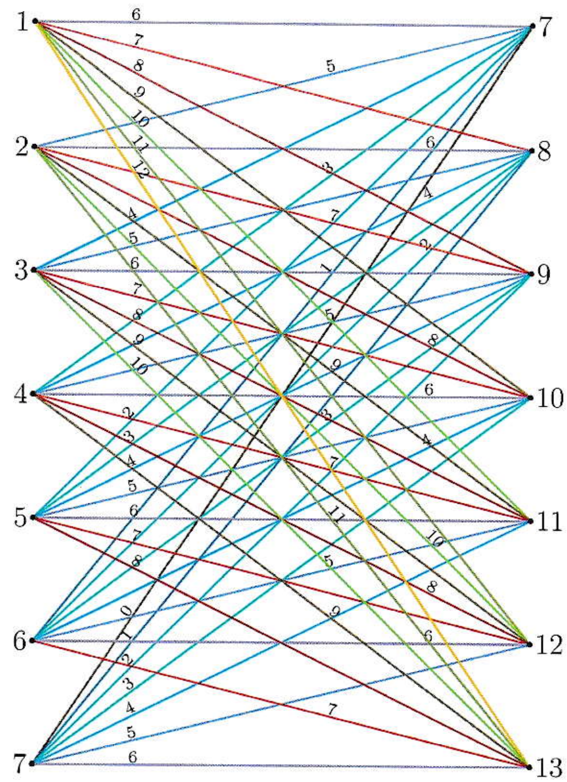

Figura 1.8: Coloración aritmética de $K_{n,n}, n = 7$

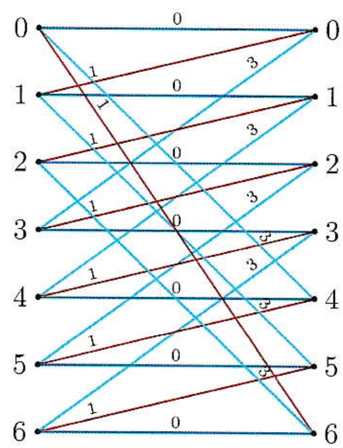

Figura 1.9: Coloración aritmética del Plano proyectivo $p = 2, S = \{0, 1, 3\}$

aritmética óptima de ciertas familias de grafos existen algoritmos en tiempo polinomial, por ejemplo en la familia de los árboles. Pero incluso para grafos regulares de grado 3 es un problema NP-completo decidir cuándo $\chi'_a(G) \leq 4$. Pese a esto, para algunas familias de grafos es posible encontrar un algoritmo de aproximación en tiempo polinomial, por ejemplo grafos *split*. Sin embargo probamos que esto no es posible para la clase de grafos *chordal*[3].

Adicionalmente hemos determinado el índice aritmético para productos cartesianos de ciclos y caminos. Para los casos trabajados se cumple que $\chi'_a \leq 6$, pero su valor exacto depende de los largos de los caminos y ciclos considerados, lo que se resume en la figura 1.10.

	P_2	P_3	P_4	P_5	P_6	P_7
C_3	4	6	6	6	6	6
C_4	4	5	5	5	5	5
C_5	4	5	5	5	5	6
C_6	4	5	6	6	6	6
C_7	4	5	6	6	6	6

Figura 1.10: Índice aritmético de producto de ciclos por caminos.

Un ejemplo completo de una coloración aritmética se ve en la figure 1.11.

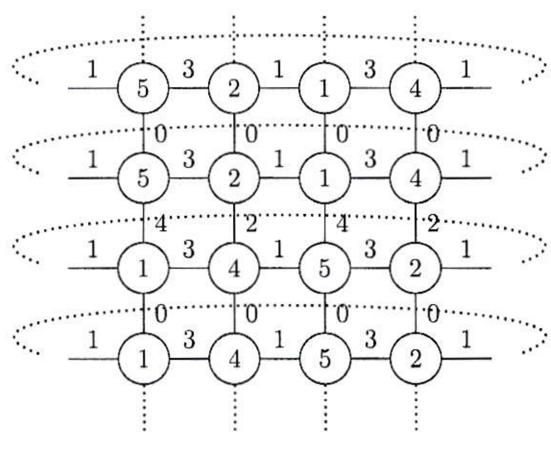

Figura 1.11: Coloración aritmética de $C_{4m} \Box P_n$.

[3]Los resultados mencionados en esta subsección coinciden con el artículo sometido [6], y fueron obtenidos en trabajo conjunto con M. Chapell, M. Matamala, I. Todinca, J. Zamora.

Chapter 2

A quantitative approach to perfect one-factorization of complete bipartite graphs[1]

2.1 Abstract

In this work we prove that $\mathsf{pf}(n) \geq \frac{1}{4}n^2$ for all $n \geq 2$, where $\mathsf{pf}(n)$ is the maximum over all one-factorizations $\mathcal{F}$ of the complete bipartite graph $K_{n,n}$ of the number of pairs of perfect matchings in $\mathcal{F}$ inducing a Hamiltonian cycle in $K_{n,n}$. For n even this lower bound is known to be an upper bound which proves that $\mathsf{pf}(n) = \frac{1}{4}n^2$, for even n. For odd n we can improve this lower bound to $\mathsf{pf}(n) > \frac{1}{4}n^2$.

2.2 Perfect one-factorizations

A *one-factorization* of a graph G is a partition of its set of edges into perfect matchings (see Wallis [58]). The union of two distinct perfect matchings (or 1-factors) A and B in any one-factorization of a graph G induces a spanning subgraph $G_{A,B}$ which is the vertex disjoint union of cycles. When this graph $G_{A,B}$ is a Hamiltonian cycle the pair A, B is called a *perfect pair.* A one-factorization $\mathcal{F}$ is *perfect* if the graph induced by any two distinct perfect matchings in $\mathcal{F}$ is a Hamiltonian cycle (see Seah [42]).

[1]The results in this chapter are part of a joint work with Martín Matamala, [7].

The *perfect one-factorization conjecture* states that for any positive integer n, the complete graph K_{2n} admits a perfect one-factorization. The conjecture seems to be first explicitly mentioned in [21]. There are only two infinite families for which it is known that K_{2n} has a perfect one-factorization: $n \in \{\frac{p+1}{2}, p\}$, for any prime p [4, 40].

A quantitative version of this conjecture was pursued by Wagner [57]. For even n, Wagner defined $c(n)$ as the maximum over all one-factorization $\mathcal{F}$ of K_n of the number of perfect pairs in $\mathcal{F}$. For odd n, $c(n) = c(n+1)$. Then for any even n such that K_n has a perfect one-factorization, $c(n) = \binom{n-1}{2}$. Wagner showed that $c(nm) \geq 2c(n) \cdot c(m)$ whenever m and n are odd and coprimes and $c(n) \geq n\phi(n)/2$, where n is odd and ϕ is the Euler totient function.

In this work we make a similar analysis for complete bipartite graphs. For a positive integer n we define $\mathsf{pf}(n)$ as the maximum over all one-factorizations $\mathcal{F}$ of $K_{n,n}$ of the number of perfect pairs in $\mathcal{F}$. As before, if the complete bipartite graph $K_{n,n}$ has a perfect one-factorization, then $\mathsf{pf}(n) = \binom{n}{2}$. It is known that if K_n has a perfect one factorization then $K_{n-1,n-1}$ has a perfect one-factorization (see Wanless and Ihrig [62]). Hence, the perfect one-factorization conjecture leads to the following version for complete bipartite graphs first stated by Wanless in [60].

Conjecture: 1. *For every odd $n \geq 3$, $\mathsf{pf}(n) = \frac{n(n-1)}{2} = \binom{n}{2}$.*

In contrast for each even n it is known that $\mathsf{pf}(n) \leq 1/4n^2$ and that this upper bound is achieved for each $n = 2p$, where p is an odd prime [61]. Our first result given in Theorem 2.2.1, is a construction that shows that $\mathsf{pf}(n) = 1/4n^2$ for each even n.

Our presentation is easier to understand in the language of *Latin rectangles.* Given two positive integers m and n with $m \leq n$, a *Latin rectangle* L of size $m \times n$ is a matrix with m rows and n columns filled with symbols in an alphabet Σ_L of size n, such that each row contains each symbol in Σ_L once, and each column has each symbol in Σ_L at most once. When $m = n$ a Latin rectangle is called a *Latin square of order n*. Two rows i and j of a Latin rectangle L of size $m \times n$ form a *perfect pair* if $i \neq j$ and the permutation $L_{i,j}$ which assigns to the symbol x in row i and column k the symbol y in row j and column k is a cyclic permutation. We denote by $\mathsf{pf}(R)$ the number of perfect pairs in a Latin rectangle R and by $\mathsf{pf}(m,n)$ the maximum of $\mathsf{pf}(R)$ over all Latin rectangles of size $m \times n$. Then $\mathsf{pf}(m,n) \leq \binom{m}{2}$. When a Latin rectangle of size $m \times n$ achieves this upper bound, it is called a *perfect or pan-Hamiltonian* Latin rectangle ([35],[60]). It is known that there is a one-to-one correspondence between perfect Latin squares of order n and perfect one-factorizations of $K_{n,n}$ ([62]). Hence, $\mathsf{pf}(n,n)$ is in fact the same as $\mathsf{pf}(n)$.

We label rows and colums of Latin rectangles with symbols from their alphabet usually implicitely.

Theorem 2.2.1. *For each even n, pf$(n) = \frac{1}{4}n^2$.*

Proof: We only need to prove that for each even n, there is a Latin square of order n having $\frac{1}{4}n^2$ perfect pairs.

In the rest of this proof all additions are made using arithmetic in $\mathbb{Z}_{\frac{n}{2}}$. Let A, B and C the following Latin squares of orders $\frac{n}{2}$. For each $i, j \in \mathbb{Z}_{\frac{n}{2}}$,

- $A(i,j) = j + i$.
- $B(i,j) = j + i + 1$.
- $C(i,j) = j - i$.

Let D be the Latin square of order n with $\Sigma_D = \mathbb{Z}_{\frac{n}{2}} \times \{0,1\}$ and defined as follows. For $a, c \in \mathbb{Z}_{\frac{n}{2}}$ and $b, d \in \{0,1\}$,

- $D((a,b),(c,d)) = (C(a,c),0)$ when $b = d = 0$,
- $D((a,b),(c,d)) = (C(a,c),1)$ when $b = 1 = 1 - d$,

- $D((a,b),(c,d)) = (A(a,c),1)$ when $1-b=d=1$, and
- $D((a,b),(c,d)) = (B(a,c),0)$ when $b=d=1$.

In the definition of D, rows and columns are indexed by symbols in Σ_D.

It is easy to see that for each pair $(i,0)$ and $(l,1)$, the permutation $D_{(i,0),(l,1)}$ satisfies $D^2_{(i,0),(l,1)}((-t,0)) = (-t+1,0)$, for each $i,l,t \in \mathbb{Z}_{\frac{n}{2}}$. Therefore, the permutation $D_{(i,0),(l,1)}$ is a cyclic permutation for each pair $i,l \in \mathbb{Z}_{\frac{n}{2}}$. This finishes the proof. □

The Latin squares A, B and C given in the proof of Theorem 2.2.1 all have order $m = \frac{n}{2}$. In the sequel, these Latin squares will be called *Cyclic Latin squares of order m*: They are obtained by cyclically rotating a fixed permutation of a set of m symbols. It is clear that for m prime, they are perfect Latin squares.

By using arithmetic in $\mathbb{Z}_m$, given two rows a and a' of C the permutation $C_{a,a'}(x)$ is given by $C_{a,a'}(x) = x + a - a'$. Hence, if $C^l_{a,a'}(x) = x$ for some x, then $a = a'$ or l and m are not coprimes. We shall use this property in the proof of the Theorem 2.2.2.

Let R be a Latin rectangle of size $m \times n$ and let r be a row of R which we use as a reference. We denote by R_a the permutation $R_{a,r}$, for each row a of R. With this notation, for every two rows a and a' of R we have the following relations: $R_a^{-1} = R_{r,a}$ and $R_{a,a'} = R_{a'}^{-1} \circ R_a$.

In Theorem 2.2.2, we describe a way to construct a Latin square A of order n based on a perfect Latin squares P of order p, where p divides n, a Cyclic Latin square C of order $\frac{n}{p}$ and a Latin rectangle K of size $p \times \frac{n}{p}$ such that $\mathsf{pf}(K) = \mathsf{pf}(p, \frac{n}{p})$. Roughly speaking we use the perfect Latin square P to glue cycles of the row permutations of K into cycles of length n. This defines a Latin rectangle K' of size $p \times n$ such that $\mathsf{pf}(K') = \mathsf{pf}(K)$. The final Latin square A is built by taking $\frac{n}{p}$ copies of the Latin rectangle K' glued appropriately with the help of the Cyclic Latin square C.

Theorem 2.2.2. *Let n be an odd composite integer and let p be the smallest prime divisor of n. Then,*

$$\mathsf{pf}(n,n) \geq \mathsf{pf}(p,\frac{n}{p})(\frac{n}{p})^2.$$

Proof: Let P, K be Latin rectangles of sizes $p \times p$ and $p \times \frac{n}{p}$, respectively, such that $\mathsf{pf}(P) = \binom{p}{2}$ and $\mathsf{pf}(K) = \mathsf{pf}(p, \frac{n}{p})$. Let C be a Cyclic Latin square of order $\frac{n}{p}$. We assume that the set of symbols of P, K and C satisfy $\Sigma_P \subseteq \Sigma_C = \Sigma_K$.

Let $\Sigma_T := \Sigma_C \times \Sigma_P$ and let T be the Cartesian product of C and P given, for each $(a,b),(c,d) \in \Sigma_T$, by

$$T((a,b),(c,d)) = (C(a,c), P(b,d)).$$

Notice that we are implicitly labeling rows and columns of T with symbols in Σ_T. It is not hard to see that for every $(a,b),(a',b') \in \Sigma_T$, the permutation $T_{(a,b),(a',b')}$ is the Cartesian product of the permutation $C_{a,a'}$ and $P_{b,b'}$. Hence, symbols appearing in rows (a,b) and (a',b') of T can be ordered (by permuting columns if necessary) as

$$\begin{array}{llll} T((a,b),\cdot): & (x,y) & (C_{a,a'}(x), P_{b,b'}(y)) \cdots (C^{p-1}_{a,a'}(x), P^{p-1}_{b,b'}(y)) \cdots \\ T((a',b'),\cdot): & (C_{a,a'}(x), P_{b,b'}(y)) & (C^2_{a,a'}(x), P^2_{b,b'}(y)) \cdots (C^p_{a,a'}(x), y) \cdots \end{array}$$

Given a fixed symbol $o_p \in \Sigma_P$, let T' be the $n \times n$ matrix obtained from T by changing in each row (a,b) of T the symbol (x, o_p) by $(K_b^{-1} \circ C_a^p(x), o_p)$, for each $x \in \Sigma_C = \Sigma_K$.

In order to prove the result, it is enough to show that T' is a Latin square and that for each pair (b,b') which is perfect in K and for every a and a' in Σ_C, the permutation $T'_{(a,b),(a',b')}$ is a cyclic permutation.

As T' is obtained by modifying only symbols of the form (x, o_p) we get that rows (a, b) and (a', b') in T' are given by:

$$T'((a,b),\cdot): \quad (K_b^{-1} \circ C_a^p(x), o_p) \quad (C_{a,a'}(x), P_{b,b'}(o_p)) \cdots (C_{a',a}(x), P_{b',b}(o_p)) \cdots$$

$$T'((a',b'),\cdot): \quad (C_{a,a'}(x), P_{b,b'}(o_p)) \quad (C_{a,a'}^2(x), P_{b,b'}^2(o_p)) \cdots (K_{b'}^{-1} \circ C_{a'}^p(C_{a,a'}^p(x)), o_p) \cdots$$

From the definition of C_a we get that

$$C_{a'}^p(C_{a,a'}^p(x)) = (C_{a'} \circ C_{a,a'})^p(x) = C_a^p(x).$$

If z denotes $K_b^{-1} \circ C_a^p(x)$, then we have that

$$K_{b'}^{-1} \circ C_{a'}^p(C_{a,a'}^p(x)) = K_{b'}^{-1}(C_a^p(x)) = K_{b'}^{-1} \circ K_b(z) = K_{b,b'}(z).$$

Therefore, after p iterations of T' the symbol (z, o_p) is transformed in $(K_{b,b'}(z), o_p)$, for each a and a' in Σ_C. Hence, if (b, b') is a perfect pair in K, then the permutation $T'_{(a,b),(a'b')}$ is cyclic, for every a and a' in Σ_C.

It remains to show that T' is a Latin square. It is clear that the Cartesian product T is a Latin square. By the definition of T' we only need to check that the modification $(x, o_p) \to (K_b^{-1} \circ C_a^p(x), o_p)$ induces a permutation of Σ_T in each row and each column. By definition $K_b^{-1} \circ C_a^p$ is an injective function. Hence, the first components of the symbols in each row of T' forms a permutation of Σ_T. Therefore, each row of T' is a permutation of Σ_T. Let us assume that for some rows (a, b), (a', b') and some column (c, d) of T' we have that

$$K_b^{-1} \circ C_a^p(C(a,c)) = K_{b'}^{-1} \circ C_{a'}^p(C(a',c))$$

and $P(b,d) = o_p = P(b',d)$. Since P is a Latin square we have that $b = b'$ which implies that

$$C_a^p(C(a,c)) = C_{a'}^p(C(a',c)).$$

From its definition $C_a^{-p} \circ C_{a'}^p = C_{a',a}^p$ and $C(a',c) = C_{a,a'}(C(a,c))$. Therefore,

$$C(a,c) = C_{a',a}^p \circ C_{a,a'}(C(a,c)) = C_{a',a}^{p-1}(C(a,c)).$$

Since C is a Cyclic Latin square of order $\frac{n}{p}$ and $p-1$ does not divides $\frac{n}{p}$ we get that $a = a'$. Hence each column of T' is a permutation of Σ_T. Therefore T' is a Latin square. This finishes the proof.

□

When proving Theorem 2.2.4, we use induction on the size of Latin rectangles. The induction hypothesis will give us a Latin rectangle K of size $p \times \frac{n}{p}$ such that $\mathsf{pf}(K) > \frac{1}{4}p^2$. By plugging this into Theorem 2.2.2 we obtain the result for Latin square: $\mathsf{pf}(n,n) > \frac{1}{4}n^2$. In order to get the result for Latin rectangle, we have to choose some rows of the Latin square so as the resulting Latin rectangle has the desired number of perfect pairs. This last step can be achieved by using Lemma 2.2.3 which proves that a Latin square of order n with more than $\frac{1}{4}n^2$ perfect pairs has m rows such that the subrectangles induced by these rows has more than $\frac{1}{4}m^2$ perfect pairs. In fact, the property is slightly more general since it corresponds to a *density* property of subgraph of a *dense* graph. For a graph $G = (V,E)$ we denote by $e(G)$ the cardinality of the set of edges E and by $v(G)$ the cardinality of the set of vertices V.

Lemma 2.2.3. *Let G be a graph on n vertices having $e(G) > \frac{1}{4}n^2$. Then, for every $3 \le m \le n$, there is a subgraph H of G with $v(H) = m$ and $e(H) > \frac{1}{4}m^2$.*

Proof: The case $3 = m = n$ being obvious we can assume that $n \ge 4$. We proceed by induction on n. By deleting edges we can assume that $\frac{1}{4}n^2 < e(G) \le \frac{1}{4}n^2 + 1$.

We know that the minimum degree $\delta(G)$ of G satisfies $\delta(G)n \leq 2e(G)$. Hence, $\delta(G) \leq \frac{n}{2} + \frac{2}{n}$. Let v_0 be a vertex with $d_G(v_0) = \delta(G)$. Then, $e(G - v_0) = e(G) - \delta(G)$. For even n, we have that

$$e(G - v_0) \geq \frac{n^2}{4} + 1 - \frac{n}{2} - \frac{2}{n} = \frac{(n-1)^2}{4} + \frac{3}{4} - \frac{2}{n}.$$

For $n \geq 4$ we have that $\frac{3}{4} - \frac{2}{n} > 0$ which implies $e(G - v_0) > \frac{(n-1)^2}{4}$. For odd $n \geq 5$, we have that $\frac{2}{n} < \frac{1}{2}$, Hence, $\delta(G) \leq \frac{n-1}{2}$ and the following inequalities hold.

$$e(G - v_0) > \frac{n^2}{4} - \frac{n-1}{2} = \frac{(n-1)^2}{4} + \frac{1}{4} > \frac{(n-1)^2}{4}.$$

Therefore, the subgraph $G - v_0$ has $n - 1$ vertices and more than $\frac{1}{4}(n-1)^2$ edges. By the induction hypothesis, for each m, $3 \leq m \leq n - 1$ there is a subgraph H of $G - v_0$ such that $v(H) = m$ and $4e(H) > m^2$. As H is also a subgraph of G, we get the conclusion.

□

The result is tight in the sense that we can not replace the strict inequality in the hyphotesis by inequality even if we relax the conclusion in the same manner. In fact, the complete bipartite graph $K_{n,n}$ has n^2 edges and $2n$ vertices but any subgraph with $2n - 1$ vertices has $n(n-1)$ edges.

Theorem 2.2.4. *For each odd integer n and each m, with $3 \leq m \leq n$, we have that $\mathsf{pf}(m, n) > \frac{1}{4}m^2$.*

Proof: We proceed by induction on n. The basis case, $n = 3$, is obvious. If n is a prime number, then we know that a Cyclic Latin square C of order n satisfies $\mathsf{pf}(C) = \binom{n}{2}$. Given m, with $3 \leq m \leq n$, we can choose any Latin rectangle R from C of size $m \times n$ and we have $\mathsf{pf}(R) = \binom{m}{2}$. Then $\mathsf{pf}(m, n) = \binom{m}{2} > \frac{1}{4}m^2$, when $m \geq 3$.

If n is a composite number, then let p be the smallest prime divisor of n. From Theorem 2.2.2 we know that $\mathsf{pf}(n,n) \geq \mathsf{pf}(p, \frac{n}{p})(\frac{n}{p})^2$. As $\frac{n}{p}$ is odd, less than n and at least p, we can apply the induction hypothesis to get that $\mathsf{pf}(p, \frac{n}{p}) > \frac{1}{4}p^2$ which shows $\mathsf{pf}(n,n) > \frac{1}{4}n^2$.

Let N be a Latin square of order n such that $\mathsf{pf}(N) > \frac{1}{4}n^2$. Let G be the graph on Σ_N such that ab is an edge of G if and only if $N_{a,b}$ is a cyclic permutation. Then $e(G) > \frac{1}{4}n^2$. From Lemma 2.2.3 we know that for each m, with $3 \leq m \leq n$, there is a subgraph H with $e(H) > \frac{1}{4}m^2$ and $v(H) = m$. Therefore, the Latin rectangle N' obtained from N by choosing rows whose label is in the set of vertices of H satisfies $\mathsf{pf}(N') > \frac{1}{4}m^2$. This shows that $\mathsf{pf}(m,n) > \frac{1}{4}m^2$.

□

2.3 Conclusion

In this work we have considered an approximative version of Conjecture 1 focusing into uniform lower bounds for the function $\mathsf{rp}(n) := \mathsf{pf}(n)/n^2$: we have proved that $\mathsf{rp}(n) > \frac{1}{4}$, for each odd n.

Proof of Theorem 2.2.4 suggests the following weakening of Conjecture 1.

Conjecture: 2. *For every two odd integers $n \geq p \geq 3$, $\mathsf{rp}(n) \geq \mathsf{rp}(p)$.*

If true, then our lower bound $\frac{1}{4}$ might be replaced by $\frac{1}{3}$ which is best possible since $\mathsf{rp}(3) = \frac{1}{3}$. From the proof of Theorem 2.2.2 we can prove that $\mathsf{rp}(n) = \mathsf{pf}(n,n)/n^2 \geq \mathsf{pf}(p, n/p)/p^2$ when p is a prime that divides n and $n \geq p^2$. Hence, for p a prime divisor of n such that $p^2 \leq n$ we have $\mathsf{rp}(n) \geq \mathsf{rp}(p)$ if the following is true.

Conjecture: 3. *For every two odd integers p and n with $p \leq n$, it holds $\mathsf{pf}(p,n) = \mathsf{pf}(p,p) = \mathsf{pf}(p)$.*

We have confirmed Conjecture 3 when p and n are odd, $p \leq n$ and $p \leq 25$.

Chapter 3

Acyclic edge coloring of the complete bipartite graph $K_{2p,2p}$[1]

3.1 Abstract

In this work we prove that there is an acyclic edge coloring of the bipartite graph $K_{2p,2p}$ with $2p+4$ colors, for p prime. For this purpose we use a partition $\mathcal{F}$ of the set of edges of $K_{2p,2p}$ into $2p$ perfect matchings with p^2 pairs of perfect matching in $\mathcal{F}$ inducing a Hamiltonian cycle in $K_{2p,2p}$.

3.2 Introduction.

An *acyclic edge coloring* of a graph is a proper coloring of its edges in which the subgraph induced by any two colors has no cycles. The *acyclic chromatic index* of a graph G is the smallest integer k such that there is an acyclic edge coloring of G using k colors; it is denoted by $a'(G)$. It has been conjectured that $a'(G) \leq \Delta + 2$, for any G [1, 27]. This conjecture is still open even for balanced complete bipartite graphs. It is known that for an even number n $a'(K_{n,n}) \geq n+1$, and for an odd n $a'(K_{n,n}) \geq n + 2$ [11]. Moreover, it is known that if $K_{n,n}$ has a perfect factorization, then n is odd, and $a'(K_{n-2,n-2}) \leq a'(K_{n-1,n-1}) \leq n$ [10]. Aditionally, if K_{n+1} has a perfect factorization then $K_{n,n}$ also does [41].

[1]The results in this chapter are part of a joint work with Martín Matamala, [8].

The only values of n for which it is known that $a'(K_{n,n}) \leq n+2$ is when $n \in \{p, p-1, p-2, 2p-2, 2p-1, p^2-2, p^2-1\}$, for a prime p [10, 1].

All these results are obtained from perfect factorizations of auxiliary graphs but, with two different techniques. On the one hand, all but the first one are obtained from perfect factorizations of K_{p+1}, K_{2p} and K_{p^2,p^2} with $p, 2p-1$ and p^2 colors -one for each one-factor-, respectively, by removing one or two vertices. Clearly, any perfect factorization of a graph G defines in any proper induced subgraph an acyclic coloring. On the other hand, the first one is obtained from a perfect factorization $\{M_1, \ldots, M_p\}$ of $K_{p,p}$ and a perfect matching M that intersects M_i in exactly one edge, for each $i \in \{1, \ldots, p\}$. The acyclic coloring is defined as follows. Each edge in $M \cap M_i$, with $1 < i < p$ is colored with color $p+1$,the edge in $M \cap M_1$ gets color $p+2$ and each remaining edge get color i if it belongs to $M_i \setminus M$, for each $i \in \{1, \ldots, p-1\}$ and all edges in M_p get color p [10].

In this work we improve on the best known constructive upper bound $a'(K_{n,n}) \leq 5n(\log n + 2)$ proved in [54], when $n = 2p$ and p is prime. We prove that $a'(K_{2p,2p}) \leq 2p + 4$. The construction used to color the graph is based on a one-factorization of the $K_{2p,2p}$ with p^2 pairs of perfect matchings inducing a Hamiltonian cycle, similar to that done in [10].

3.3 Some special matchings and perfect matchings of $K_{2p,2p}$.

Let $K_{2p,2p} = (X \cup Y, E)$ be a balanced bipartite graph with independent sets X and Y each of size $2p$. Our construction is easier to describe by using arithmetic in $\mathbb{Z}_p$. Hence, from now on all the arithmetic operations will be carried out in $\mathbb{Z}_p$. Let us assume that $X = \{x_j^i : i \in \{0,1\}, j \in \mathbb{Z}_p\}$ and $Y = \{y_j^i, i \in \{0,1\}, j \in \mathbb{Z}_p\}$.

The following sets of edges will be the fundamental parts of our construction.

For $b \in \mathbb{Z}_p$, we define

$$P_b = \{x_j^0 y_{j+b}^0 | j \in \mathbb{Z}_p\}, \quad S_b = \{x_j^1 y_{j-(b+1)}^0 | j \in \mathbb{Z}_p\},$$

$$Q' = \{x_j^1 y_j^1 | j \in \mathbb{Z}_p\} \text{ and } R' = \{x_j^0 y_j^1 | j \in \mathbb{Z}_p\}.$$

For a generator a of $\mathbb{Z}_p^*$, we also define

$$Q_b = \{x_j^1 y_{aj-ab}^1 | j \in \mathbb{Z}_p\}, \quad R_b = \{x_j^0 y_{aj+ab}^1 | j \in \mathbb{Z}_p\},$$

$$P' = \{x_j^1 y_{aj}^1 | j \in \mathbb{Z}_p\} \text{ and } S' = \{x_j^0 y_{aj-1}^1 | j \in \mathbb{Z}_p\}.$$

One can see that for each $b, b' \in \mathbb{Z}_p$ the following sets are perfect matchings of $K_{2p,2p}$: $P_b \cup Q_{b'}$, $R_b \cup S_{b'}$, and also $P' \cup Q'$ and $S' \cup R'$. Moreover, the set $\mathcal{F} = \{P_b \cup Q_b, R_b \cup S_b | b \in \mathbb{Z}_p\}$ is a partition of the set of edges of the graph $K_{2p,2p}$ into perfect matchings, *i.e.* $\mathcal{F}$ is a *one-factorization* of $K_{2p,2p}$ (see Figure 3.1).

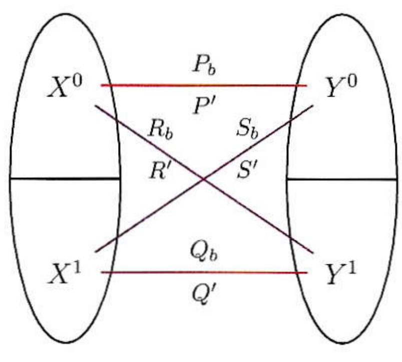

Figure 3.1: Structure of the graph

Among the sets $P_b, Q_b, R_b, S_b, P', Q', S'$ and R' there are only a few non-empty intersections. In fact, the only non-empty intersections are $F_b \cap F$, for $F \in \{P, Q, R, S\}$. It follows that $F \cap F_b = \{e_b^F\}$, where $\bar{a} = (a-1)^{-1}$,

$$e_b^P = x_{\bar{a}b}^0 y_{(1+\bar{a})b}^0, \quad e_b^Q = x_{(1+\bar{a})b}^1 y_{(1+\bar{a})b}^1,$$

$$e_b^R = x^0_{-(1+\bar{a})b} y^1_{-(1+\bar{a})b} \text{ and } e_b^S = x^1_{-\bar{a}b} y^0_{-(1+\bar{a})b-1}.$$

All the sets described above use a function $\sigma_{\alpha,\beta}(z) = \alpha z + \beta$, with $\alpha, \beta \in \mathbb{Z}_p$ and $\alpha \neq 0$. In the sequel we shall use the following properties of $\sigma_{\alpha,\beta}$

- For each $\beta \in \mathbb{Z}_p$, $\sigma_{\alpha,\beta}$ is a bijective function whose inverse is $\sigma^{-1}_{\alpha,\beta}(z) = \frac{(z-\beta)}{\alpha}$.
- $\sigma^i_{\alpha,\beta}(z) = \alpha^i z + \beta\frac{(\alpha^i-1)}{(\alpha-1)}$, for $\alpha \neq 1$ and $\sigma^i_{1,\beta}(z) = z + ib$.
- For $\alpha \neq 1$, $\sigma^i_{\alpha,\beta}(z) = z$ if and only if $\alpha^i = 1$, or $z = \frac{\beta}{1-\alpha}$. When α is a generator of the multiplicative group, the smallest i such that $\alpha^i = 1$ is $i = p-1$.
- For $\beta \neq 0, \sigma^i_{1,\beta}(z) = z$ if and only if i is a multiple of p.

3.4 Cycle structure induced by two matchings.

We start by considering the cycles that can appear when we take the union of two of our initial sets.

Lemma 3.4.1. *For each $b, b' \in \mathbb{Z}_p$ and any $F \in \{P, Q, R, S\}$, we have that $F_b \cup F'_b$ is a cycle of length $2p$.*

Proof: We use the scheme given in Figure 3.2 to describe the graph induced by edges in $P_b \cup P_{b'}$. By starting at x^0_i we continue to y^0_{i+b} by an edge in P_b and then we go to $x^0_{i+b-b'}$ by using an edge of $P_{b'}$.

$$x^0_i \overset{P_b}{\text{———}} y^0_{i+b} \overset{P_{b'}}{\text{———}} x^0_{i+(b-b')}$$

$$x^0_{i+(b-b')} \overset{P_b}{\text{——}} y^0_{i+(2b-b')} \overset{P_{b'}}{\text{——}} x^0_{i+2(b-b')} \cdots y^0_{i+(tb-(t-1)b')} \overset{P'_b}{\text{——}} x^0_{i+t(b-b')}$$

Figure 3.2: Scheme of the graph induced by P_b and $P_{b'}$.

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Since $b \neq b'$, this path continue until we get x_i^0 again. This happens only after $2p$ steps. The analysis of the other cases is similar and it is depicted in Figure 3.3

$$
\begin{array}{ccccccccc}
x_i^1 & \overset{Q_b}{\text{———}} & y^1_{ai-ab} & \overset{Q_{b'}}{\text{——}} & x^1_{i-(b-b')} & \cdot\cdot & y^1_{ai-a(tb-(t-1)b')} & \overset{Q'_b}{\text{——}} & x^1_{i-t(b-b')} \\
x_i^0 & \overset{R_b}{\text{———}} & y^1_{ai+ab} & \overset{R_{b'}}{\text{——}} & x^0_{i+(b-b')} & \cdot\cdot & y^1_{ai+a(tb-(t-1)b')} & \overset{R'_b}{\text{——}} & x^0_{i+t(b-b')} \\
x_i^1 & \overset{S_b}{\text{———}} & y^0_{i-(b+1)} & \overset{S_{b'}}{\text{——}} & x^1_{i-(b-b'} & \cdot\cdot & y^0_{i-(tb-1-(t-1)b')} & \overset{S_{b'}}{\text{——}} & x^1_{i+t(b-b')}
\end{array}
$$

Figure 3.3: Scheme for the cycles induced by $Q_b \cup Q_{b'}$, $R_b \cup R_{b'}$ and $S_b \cup S_{b'}$.

□

Lemma 3.4.2. *The following sets induce Hamiltonian cycles.*

- *For each $b, b' \in \mathbb{Z}_p$, the set $P_b \cup Q_b \cup R_{b'} \cup S_{b'}$.*
- *For each $b \neq \bar{a}$, the sets $S_b \cup R_b \cup P' \cup Q'$ and $P_b \cup Q_b \cup S' \cup R'$.*
- *The set $P' \cup Q' \cup S' \cup R'$.*

Proof:

We again describe the situation by using a diagram given in Figure 3.4.

$$
\begin{array}{ccccccccc}
x_i^0 & \overset{P_b}{\text{———}} & y^0_{i+b} & \overset{S_{b'}}{\text{——}} & x^1_{i+(b+b'+1)} & \overset{Q_b}{\text{—}} & y^1_{ai+a(b'+1)} & \overset{R_{b'}}{\text{——}} & x^0_{i+1} \\
x_i^0 & \overset{P'}{\text{———}} & y^0_{ai} & \overset{R'}{\text{———}} & x^1_{i+\frac{1}{a}} & \overset{Q'}{\text{———}} & y^1_{i+\frac{1}{a}} & \overset{S'}{\text{———}} & x^0_{i+\frac{1}{a}}
\end{array}
$$

$$
\begin{array}{ccccccccc}
x_i^0 & \overset{P'}{\text{———}} & y^0_{ai} & \overset{S_b}{\text{———}} & x^1_{ai+(b+1)} & \overset{Q'}{\text{——}} & y^1_{ai+(b+1)} & \overset{R_b}{\text{—}} & x^0_{i+\frac{b(1-a)+1}{a}} \\
x_i^0 & \overset{P_b}{\text{———}} & y^0_{i+b} & \overset{R'}{\text{———}} & x^1_{\frac{i+b+1}{a}} & \overset{Q_b}{\text{——}} & y^1_{i+b(1-a)+1} & \overset{S'}{\text{—}} & x^0_{i+b(1-a)+1}
\end{array}
$$

Figure 3.4: Sets defining Hamiltonian cycles.

In the first two cases, after following 4 edges of the cycle we go from x_i^0 to x_{i+1}^0 or $x_{i+\frac{1}{a}}^0$. If we go on $4p$ edges the vertex will be x_{i+p}^0 and $x_{i+\frac{p}{a}}^0$, which is x_i^0 since $\frac{p}{a} = 0$. In the other two cases, as $b \neq \bar{a}$ we have that $b(1-a)+1 \neq 0$. Then, the cycle will have $4p$ edges. □

3.5 Non-acyclic colorings.

If we color the edges of the set $P_b \cup Q_b$ with color b and the edges of the set $R_b \cup S_b$ with color $p+b$, for each $b \in \mathbb{Z}_p$, then we obtain a (proper) edge coloring of $K_{2p,2p}$ with $2p$ colors. Let us call this coloring C_1.

From previous results, given any two colors c and c' of C_1, the subgraph induced by the edges colored with these two colors is a 2-regular graph which consists of one cycle when $0 \leq c < p \leq c' < 2p$, and two cycles when $0 \leq c, c' < p$ or $p \leq c, c' < 2p$. In the first case, the cycles is induced by a set $P_c \cup Q_b \cup R_{c'-p} \cup S_{b'}$. In the second case, if $c, c' < p$, then one cycle is induced by the set $P_c \cup P_{c'}$ and the other by the set $Q_c \cup Q_{c'}$, while when $c, c' \geq p$, one cycle is induced by the set $S_{c-p} \cup S_{c'-p}$ and the other by the set $R_{c-p} \cup R_{c'-p}$.

The role that the sets P', Q', S', R' will play is to determine a set of edges that will change its colors. Since, for each $b \in \mathbb{Z}_p$ and for each $F \in \{P, Q, R, S\}$, $F_b \cap F'$ has exactly one edge e_b^F, by recoloring these edges with colors greater than $2p-1$, bi-chromatic cycles of C_1 using colors smaller than $2p$ disappear.

Let us try the following recoloring of C_1. We assign colors $2p$, $2p+1$, $2p+2$ and $2p+3$ to edges in P', Q', R' and S', respectively. Let us call this new coloring C_2. Clearly it is an edge coloring. Since the set $P' \cup Q' \cup S' \cup R'$ induces a Hamiltonian cycle, it is clear that no two colors of C_2 larger than $2p-1$ can induce a cycle. We know that if $b \neq \bar{a}$, then $P_b \cup Q_b \cup R' \cup S'$ and $S_b \cup R_b \cup P' \cup S'$ are Hamiltonian cycles. Hence, in C_2, if c' is such that $2p+2 \leq c'$, then the subgraph induced by colors b and c' is acyclic. Moreover, if $2p \leq c' \leq 2p+1$, then the subgraph induced by colors $b+p$ and c' is also acyclic.

In the next lemma we show that $P' \cup P_b$ is the vertex disjoint union of an edge, e_b^P, and a cycle of length $2(p-1)$. Hence C_2 is not an acyclic coloring: it has several bi-chromatic cycles of length $2(p-1)$.

Lemma 3.5.1. *For each $b \in \mathbb{Z}_p$ and each $F \in \{P, Q, R, S\}$, the subgraph induced by the set $F_b \cup F'$ has two connected components, an edge and a cycle of length $2(p-1)$.*

Proof: It is clear that for each $b \in \mathbb{Z}_p$ and each $F \in \{P, Q, R, S\}$, the graph induced by the set $F_b \cup F'$ has maximum degree at most two. Hence, each connected component is either a cycle or a path. In the scheme given in Figure 3.5 we show the structure of this subgraph.

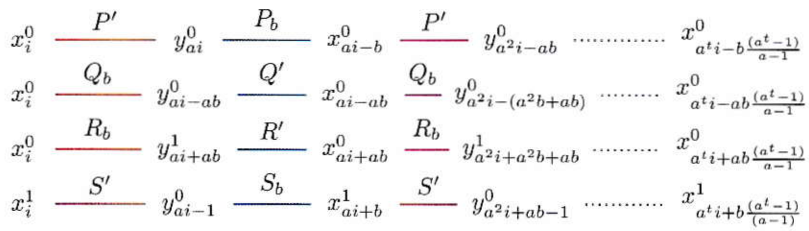

Figure 3.5: Structure of the subgraph induced by $F_b \cup F'$

Hence, starting at x_i^j and taking alternately edges from F_b and from F' we reach the vertex $x^j_{a^ti \pm \gamma\bar{a}(a^t-1)}$ after $2t$ steps, where $\gamma \in \{b, ab\}$. In order to have $a^ti \pm \gamma\bar{a}(a^t-1) = i$, there are two possibilities. The first one is that $t = p-1$, because this is the order of a; and the second one is that $i = \mp\gamma\bar{a}$. In the first case we obtain a cycle of length $2(p-1)$ while in the second case we obtain the edge e_b^F. Hence, the subgraph induced by the set $F_b \cup F'$ is the vertex disjoint union of an edge and a cycle of length $2(p-1)$. □

In order to eliminate the bi-chromatic cycles of C_2, we are forced to assign at least two of the new colors to each set P', Q', R' and S'. In the next lemma we show that we still have another difficulty to consider.

The graphs G' and G'' induced by the sets $P_{\bar{a}} \cup Q_{\bar{a}} \cup S' \cup R'$ and $R_{\bar{a}} \cup S_{\bar{a}} \cup P' \cup Q'$, respectively, have a lot of cycles!

Lemma 3.5.2. *The subgraphs G' and G'' defined above are the vertex disjoint union of p cycles of length* 4.

Proof: In Figure 3.6, we consider the cycles that can appear in the considered subgraphs.

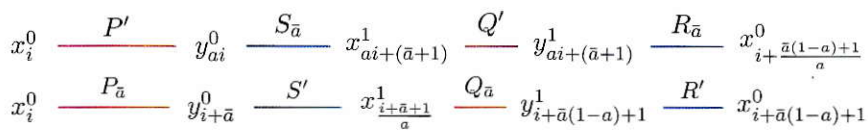

Figure 3.6: Structure of the subgraphs G' and G''.

It is easy to see that $\bar{a}(1-a)+1 = \frac{1}{a-1}(1-a)+1 = 0$. That is to say, that after going over 4 edges the cycle is closed. Hence, G' and G'' are the vertex disjoint union of p cycles of length four. □

Notice that in the subgraph G' the two edges e^P_{-1} and e^Q_0 are in the same cycle. The same holds for e^P_0 and e^Q_1. Similarly, in the subgraph G'' the two edges e^S_{-1} and e^R_0 are in the same cycle; the same holds for e^S_0 and e^R_1.

3.6 The acyclic coloring of $K_{2p,2p}$.

In order to obtain an acyclic coloring of $K_{2p,2p}$ we modify coloring C_2 by giving to edges e^P_0 and e^P_{-1} the color $2p+1$, to egdes e^Q_0 and e^Q_1 the color $2p$, to e^S_0, e^S_{-1} the color $2p+2$ and to edges e^R_0, e^R_1 the color $2p+3$. Therefore the final coloring $C : E \longrightarrow \{0, 1, \ldots 2p+3\}$ is given as follows.

$$C(e) = b, \text{if } e \in (P_b \cup Q_b) \setminus (P' \cup Q').$$

$$C(e) = b + p, \text{if } e \in (R_b \cup S_b) \setminus (R' \cup S').$$

$$C(e) = 2p, \text{if } e \in (P' \setminus \{e_0^P, e_{-1}^P\}) \cup \{e_0^Q, e_1^Q\}.$$

$$C(e) = 2p + 1, \text{if } e \in (Q' \setminus \{e_0^Q, e_1^Q\}) \cup \{e_0^P, e_{-1}^P\}.$$

$$C(e) = 2p + 2, \text{if } e \in (R' \setminus \{e_0^R, e_1^R\}) \cup \{e_0^S, e_{-1}^S\}.$$

$$C(e) = 2p + 3, \text{if } e \in (S' \setminus \{e_0^S, e_1^S\}) \cup \{e_0^R, e_1^R\}.$$

It is clear that C is an edge coloring. We now show that C is an acyclic coloring.

Theorem 3.6.1. *The coloring C is an acyclic coloring of $K_{2p,2p}$.*

Proof: We only need to prove that after the last recoloring there is no further bi-chromatic cycles.

As we recolor two edges from each of the sets P', Q', R', S', we have that each cycle of length $2(p-1)$ appearing in $F' \cup F_b$, for $F \in \{P, Q, R, S\}$ has three colors: the colors $b, 2p, 2p+1$ if $F \in \{P, Q\}$ or colors $b+p, 2p+2, 2p+3$ if $F \in \{R, S\}$ (see figures 3.7 and 3.8). The remaining case is when $c \in \{\bar{a}, \bar{a}+p\}$ and $c' \geq 2p$. More precisely, $c = \bar{a}$ and $c' \in \{2p+2, 2p+3\}$, and $c = \bar{a}+p$ and $c' \in \{2p, 2p+1\}$.

From Lemma 3.5.2, we know that the subgraphs G' and G'' are the vertex disjoint union of p cycles of length four. In figures 3.9 and 3.10, we consider the colors that C assigns to G' and G'', respectively.

We first consider the case $c = \bar{a}$. Let $\mathcal{C}$ be one cycle of length four in G'. Then $\mathcal{C}$ contains exactly one edge of each of the sets $P_{\bar{a}}, Q_{\bar{a}}, R'$ and S'. Let $e_{S'}, e_{R'}, e_P$ and e_Q be these edges and let us assume that they belong to the sets S', R', P and Q, respectively. Hence, $\mathcal{C}$ has three colors unless $C(e_{R'}) = C(e_{S'})$. In this

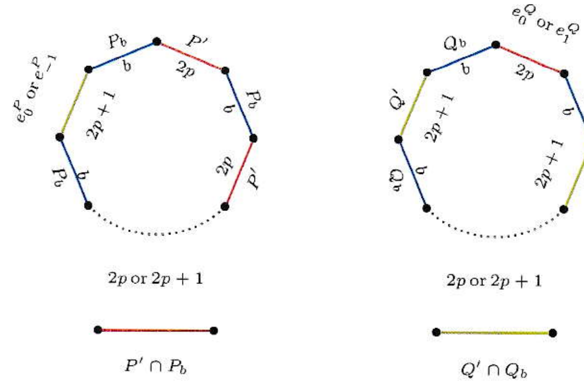

Figure 3.7: $P_b \cup Q_b \cup P' \cup Q'$

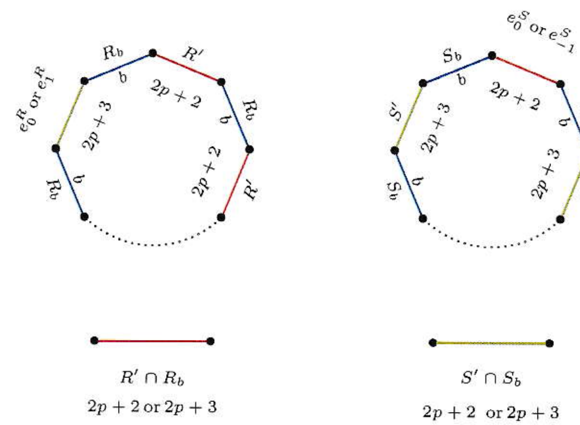

Figure 3.8: $R_b \cup S_b \cup R' \cup S'$

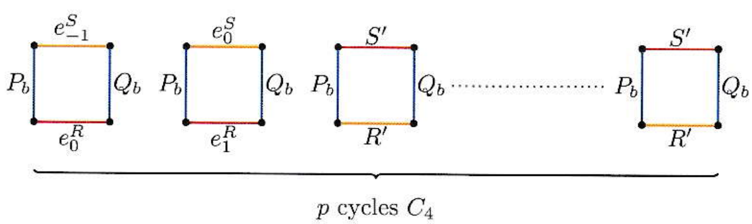

Figure 3.9: $S' \cup R' \cup P_b \cup Q_b$, for $b = \bar{a}$

case, $C(e_{R'}) = 2p+2$ and $e_{S'} \in \{e_0^S, e_{-1}^S\}$ or $C(e_{S'}) = 2p+3$ and $e_{R'}$ belongs to $\{e_0^R, e_1^R\}$. But e_0^S and e_1^R are in the same cycle of G' as well as e_{-1}^S and e_0^R. Hence, $C(e_{R'}) \neq C(e_{S'})$. A similar argument can be made for the cycles in the subgraph

G''.

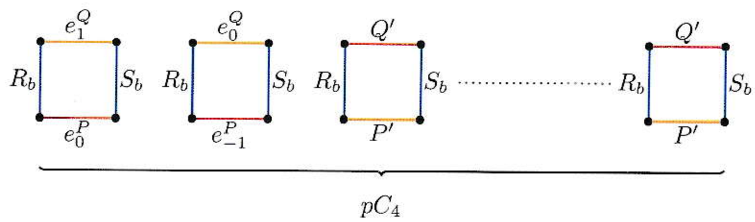

Figure 3.10: $P' \cup Q' \cup S_b \cup R_b$, for $b = \bar{a}$.

□

Chapter 4

Injective colorings with arithmetic constraints[1]

4.1 Abstract

An injective coloring of a graph is a vertex labeling such that two vertices sharing a common neighbor get different labels. In this work we introduce and study what we call *arithmetic colorings*. These are injective colorings using positive integers such that for every pair of different vertices, the average of the colors assigned to them is not the color assigned to any of their common neighbors. The smallest integer k such that an injective (resp. arithmetic) coloring of a given graph G exists with k colors (resp. colors in $\{1, \ldots, k\}$) is called the *injective (resp. arithmetic) chromatic number (resp. index)*. They are denoted by $\chi_i(G)$ and $\chi'_a(G)$, respectively.

In the first part of this work, we present several upper bounds for the arithmetic chromatic index. On the one hand, we prove a super linear upper bound in terms of the injective chromatic number for arbitrary graphs, as well as a linear upper bound for bipartite graphs and trees. Complete graphs are extremal graphs for the super linear bound, while complete balanced bipartite graphs are extremal graphs for the linear bound. On the other hand, we prove a quadratic

[1]The results in this chapter are part of a joint work with M. Chapelle,M. Matamala, I. Todinca, J. Zamora [8].

upper bound in terms of the maximum degree.

In the second part, we study the computational complexity of computing $\chi'_a(G)$. We prove that it can be computed in polynomial time for trees. We also prove that for bounded treewidth graphs, to decide whether $\chi'_a(G) \leq k$, for a fixed k, can be done in polynomial time. On the other hand, we show that for cubic graphs it is NP-complete to decide whether $\chi'_a(G) \leq 4$. We also prove that for every $\epsilon > 0$ there is a polynomial time approximation algorithm with approximation factor $n^{1/3+\epsilon}$ for $\chi'_a(G)$, when restricted to split graphs. However, unless $\mathsf{P} = \mathsf{NP}$, for every $\epsilon > 0$ there is no polynomial time approximation algorithm with approximation factor $n^{1/3-\epsilon}$ for $\chi'_a(G)$, even when restricted to split graphs.

4.2 Introduction

An *injective coloring* of a graph G is an assignment of colors to the vertices of G such that two vertices sharing a common neighbor get different colors. Injective colorings were introduced in [31]. The smallest integer k such that an injective coloring with k colors exists is called the injective chromatic number of G and it is denoted by $\chi_i(G)$.

In this work, we introduce another concept which we call *arithmetic coloring.* An arithmetic coloring c of a graph $G = (V, E)$ is a function assigning positive integers to its vertices such that by assigning to each edge uv the value $|c(u)-c(v)|$, we obtain a *proper* edge coloring $\tilde{c}$ of G. In order to ease the presentation we set

$$[k] := \{1, \dots, k\}.$$

The smallest integer k such that an arithmetic colorings exists with colors in the set $[k]$ is called the *arithmetic chromatic index* of G. We denote it by $\chi'_a(G)$. If an arithmetic coloring c uses colors in the set $[k]$, then the associated proper edge coloring $\tilde{c}$ uses colors in the set $[k-1] \cup \{0\}$. Hence, we have that $\chi'(G) \leq \chi'_a(G)$, where $\chi'(G)$ is the chromatic index of G.

Notice that c is an arithmetic coloring if and only if for every three distinct vertices x, y, z with $xy, yz \in E$, the following two properties hold: $c(x) \neq c(z)$, and $c(x)+c(z) \neq 2c(y)$. Hence, arithmetic colorings are injective colorings. Then, $\Delta(G) \leq \chi_i(G) \leq \chi'_a(G)$, as any arithmetic coloring using colors in the set $[k]$ is an injective coloring with at most k colors. We shall see throughout this work that these two parameters are closely related and that the additive structure of integers is closely related with the existence of arithmetic colorings.

4.3 Upper bounds

We start by showing a general upper bound of $\chi'_a(G)$ in terms of $\chi_i(G)$.

Proposition 1. . *Let G be a graph. Then*

$$\chi'_a(G) \leq \chi'_a(K_{\chi_i(G)}),$$

where K_m denotes the complete graph on m vertices.

Proof: Let $m = \chi_i(G)$. We assume that the set of vertices of the complete graph K_m is the set $[m]$. Let c be an arithmetic coloring of K_m with maximum value l. Then, given any three distinct vertices i, j, k in $[m]$, colors $c(i), c(j)$ and $c(k)$ are distinct, and $c(i) + c(j) \neq 2c(k)$.

Let c' be an injective coloring of G with m colors. Then, given any three distinct vertices u, v, w in G, with u and w neighbors of v, we have that $c'(u) \neq c'(w)$.

If $c'(v) \in \{c'(u), c'(w)\}$, then $c(c'(u)) + c(c'(w)) \neq 2c(c'(v))$. Otherwise, $c'(u), c'(v), c'(w)$ are distinct and $c(c'(u)) + c(c'(w)) \neq 2c(c'(v))$. Hence, by defining $c''(u) := c(c'(u))$ for every vertex u, we obtain an arithmetic coloring of G

m	$\chi'_a(K_m)$	m	$\chi'_a(K_m)$	m	$\chi'_a(K_m)$	m	$\chi'_a(K_m)$	m	$\chi'_a(K_m)$
1	1	2	2	3	4	4	5	5	9
6	11	7	13	8	14	9	20	10	24
11	26	12	30	13	32	14	36	15	40
16	41	17	51	18	54	19	58	20	63
21	71	22	74	23	82	24	84	25	92
26	95	27	100	28	104	29	111	30	114
31	121	32	122	33	137	34	145	35	150
36	157	37	163	38	165	39	169	40	174

Table 4.1: $\chi'_a(K_m)$ for $m \leq 40$, [28].

with maximum value l. □

Obviously, this upper bound is tight for complete graphs as $\chi_i(K_m) = m$. We remark that a set of integers defines an arithmetic coloring of K_m if and only if it does not contain arithmetic progressions of length three. Hence, in order to effectively apply the upper bound given in Proposition 1, we need some information about the smallest integer l such that there is a set of m integers without arithmetic progression of length three and contained in $[l]$. The determination of this value, in our terms $\chi'_a(K_m)$, has been the focus of research for more than 70 years, initiated in [23] where the first upper of m in terms of l was given, which was later improved in [49, 50], [55], and [33]. Currently, the best upper bound appears in [15]. On the other hand, the first lower bound was proved in [13] and it was later improved in [22]. These best bounds can be stated as follows: there are constants c_1 and c_2 such that

$$c_1 l \sqrt{\frac{\log^{\frac{1}{8}} l}{2^{4\sqrt{2\log l}}}} \leq m \leq c_2 l \sqrt{\frac{\log\log l}{\log l}}. \tag{4.1}$$

In [28] the exact value of $\chi'_a(K_m)$ [2], for $m \leq 41$, was computed, and lower and upper bounds, for $m \leq 100$, were given (see Table 1).

By doing some standard calculus manipulations, from Proposition 1 and In-

[2]In fact, they studied the dependency of m in terms of l, which they called the Szemerédi number $sz(l)$.

equality 4.1 we get the following super linear upper bound for $\chi'_a(G)$. This bound will be applied later to obtain some (in)approximability results for the computation of the arithmetic chromatic index.

Theorem 2. . *Let G be a graph. Then,*

$$\chi'_a(G) \leq \chi_i(G) g(\chi_i(G)),$$

where $g(x) = 2^{\sqrt{2\log(x)}}$.

One can see that the injective chromatic number of a graph G is the chromatic number of the graph $G^{(2)}$, obtained from G by adding edges between vertices at distance two and by removing all the original edges. When G is a bipartite graph with independent sets U and W, the graph $G^{(2)}$ is the disjoint union of the graphs $G^U = G^2[U]$ and $G^W = G^2[W]$ induced by the independent sets U and W of G in $G^{(2)}$, respectively. Then, for bipartite graphs we have the following ([31]).

$$\chi_i(G) = \max\{\chi(G^U), \chi(G^W)\}. \tag{4.2}$$

A similar result holds for the arithmetic chromatic index which allow us to get a linear upper bound for $\chi'_a(G)$ in terms of $\chi_i(G)$, for bipartite graphs. In the sequel, $N_G(u)$ denotes the set of neighbors of a vertex u in a graph G.

Theorem 3. . *Let G be a bipartite graph. Then,*

$$\chi'_a(G) \leq 2\chi_i(G) - 1.$$

Proof: Let G be a bipartite graph with independent sets U and W. Let G^U and G^W defined as above. Let $k_U := \chi(G^U)$ and $k_W := \chi(G^W)$. Let c and c' be proper vertex colorings of G^U and G^W, respectively such that $c(u) \in [k_U]$, for each $u \in G^U$, and $c'(w) \in [k_W]$, for each $w \in G^W$. We define $\varphi : U \cup W \to \mathbb{N}$

as follows. For each $u \in U$, $\varphi(u) = c(u)$ and $\varphi(v) = c'(v) + k_U - 1$, for each $v \in W$. Then $\max \varphi = k_U + k_W - 1 \leq 2\chi_i(G) - 1$ and it is easy to see that φ is an injective coloring. Hence, in order to prove that φ is an arithmetic coloring we prove that $\varphi(u) + \varphi(v) \neq 2\varphi(w)$ for every u, v, w and $w \in N_G(u) \cap N_G(v)$. When $u, v \in U$, as c is a proper coloring of G^U, $\varphi(u) \neq \varphi(v)$ and $\varphi(u), \varphi(v) \leq k_U$. Moreover, $\varphi(w) \geq k_U$. Therefore, $\varphi(u) + \varphi(v) \neq 2\varphi(w)$. We now consider the situation for vertices $u, v \in W$. As c' is a proper coloring of G^W, $\varphi(u) \neq \varphi(v)$ and $\varphi(u), \varphi(v) \geq k_U$. Moreover, $\varphi(w) \leq k_U$. Therefore, $\varphi(u) + \varphi(v) \neq 2\varphi(w)$.

□

In the next result we prove that the previous upper bound is tight.

Proposition 4. *Let n be odd, with $n \geq 9$. Then, $\chi'_a(K_{n,n}) = 2n - 1$.*

Proof: Let U and W be the two independent sets of $K_{n,n}$ each of size n. It is clear that both $(K_{n,n})^U$ and $(K_{n,n})^W$ are complete graphs of size n. Hence, $\chi_i(K_{n,n}) = n$.

By coloring U with colors in the set $[n]$ and W with colors in the set $[2n - 1] \setminus [n-1]$, we get an arithmetic coloring of $K_{n,n}$. Then, $\chi'_a(K_{n,n}) \leq 2n - 1$.

We now prove the lower bound $\chi'_a(K_{n,n}) \geq 2n - 1$. Let φ be an arithmetic coloring of $K_{n,n}$, and let $A = \varphi(U)$ and $B = \varphi(W)$. Since every vertex in U is adjacent to every vertex in W, the function φ must be injective when restricted to U. Similarly, it must be injective when restricted to W. Hence, $|A| = |B| = n$.

In this situation, φ is an arithmetic coloring if and only if $av(A) \cap B = av(B) \cap A = \emptyset$, where $av(C) := \left\{\frac{x+y}{2} \in \mathbb{N} : x, y \in C, x \neq y\right\}$. Therefore,

$$\max\{|av(A)|, |av(B)|\} + n \leq \chi'_a(K_{n,n}).$$

For the sake of contradiction let us assume that $\chi'_a(K_{n,n}) \leq 2n - 2$. Then, $|av(A)| \leq n - 2$ and $|av(B)| \leq n - 2$. We first show that in this situation, $av(B) = \{a, a+1, \ldots, a + (n-3)\}$, for some integer a. To this purpose we need the following property.

Claim 4.3.1. *Let D be a set with all its elements with the same parity. If $|D| \geq 2$, then $|av(D)| \geq 2|D| - 3$. Moreover, for $|D| \geq 5$, $|av(D)| = 2|D| - 3$ if and only if D is an arithmetic progression.*

Proof: Without loss of generality we can assume that all elements of D are even integers. The case $|D| = 2$ is direct. Let $D = \{a_1 < a_2 < \ldots < a_k\}$ be the elements of D and let us assume that $k \geq 3$. Let $b_{i,j} := a_i + a_j$. Then, the following $2k - 3$ integers are all distinct and even,

$$b_{1,2} < b_{1,3} < b_{2,3} < \cdots < b_{k-2,k} < b_{k-1,k}.$$

This shows that $|av(D)| \geq 2|D| - 3$. Moreover, for each $i \geq 4$ the following two sequences have $2i - 3$ integers, all distinct and even.

$$b_{1,2} < \cdots < b_{1,i-1} < b_{1,i} < b_{2,i} < \cdots < b_{i-1,i}$$

and

$$b_{1,2} < \cdots < b_{1,i-1} < b_{2,i-1} < b_{2,i} < \cdots < b_{i-1,i}.$$

Since we can extend each of the above two sequences with $2k-3-(2i-3)$ distinct terms, we get that if $|av(D)| = 2|D| - 3$ then $b_{1,i} = b_{2,i-1}$ for each $i \geq 4$. From this equality we get $a_1 + a_i = a_2 + a_{i-1}$ and then $a_2 - a_1 = a_i - a_{i-1}$, for $i \geq 4$. It is easy to see that the equality $a_2 + a_5 = a_3 + a_4$ holds, when $k \geq 5$. Therefore, D is an arithmetic progression. □

We now prove that $av(B) = \{a, a+1, \ldots, a+(n-3)\}$. As $|B| = n$ is odd, it has a subset D with all its elements with the same parity, and such that $2|D| \geq n+1$. From the claim it follows that $|av(D)| \geq 2|D| - 3$. Since $av(D) \subseteq av(B)$ and $|av(B)| \leq n-2$ we conclude that $av(D) = av(B)$ and $|av(D)| = 2|D| - 3 = n - 2$. Since $n \geq 9$, we get that $|D| \geq 5$ which again from the claim implies that D is an arithmetic progression. It is clear that in this case $av(D)$, and then $av(B)$, are arithmetic progressions as well.

Let $a, b \geq 1$ integers such that $av(B) = \{a, a+b, \ldots, a+b(n-3)\}$. Then, $a \geq 2$ and $a+b(n-3) \leq 2n-3$. Since $n \geq 9$ we obtain that $b \leq 2+1/(n-3) < 3$ which implies $b \leq 2$. If $b = 2$, then elements in $av(B)$ have the same parity. As we are assuming that $av(B) \cap A = \emptyset$ and that $A \subseteq [2n-2]$, A must contain a set D' of size at least $|av(B)|+1 = n-1$ whose elements have the same parity. Again we apply the claim and we get that $av(A) \geq 2(n-1)-3 = 2n-5 > n-2$ which, for $n \geq 9$, contradicts $\chi'_a(K_{n,n}) \leq 2n-2$. Therefore, we conclude that $b = 1$. That is, $av(B) = \{a, a+1, \ldots, a+(n-3)\}$. Then

$$A = \{1, \ldots, a-1\} \cup \{a+n-2, \ldots, n+n-2\}.$$

Hence, the set $av(A)$ contains the sets $\{2, \ldots, a-2\}$, $\{a+n-1, \ldots, 2n-3\}$, and $\mathbb{N} \cap \{\frac{a+n-1+i}{2} : i = 0, \ldots, n-3\}$. Therefore, $av(A)$ has at least $n-4+\lfloor (n-2)/2 \rfloor$ elements which is larger than $n-2$, for $n \geq 9$ which is a contradiction with $\chi'_a(K_{n,n}) \leq 2n-2$.

□

We can still improve our previous upper bounds when we consider trees. It is easy to see that for trees the injective chromatic number equals the maximum degree. So Theorem 3 applied to a tree T implies that $\chi'_a(T) \leq 2\Delta(T)-1$. It is also clear that this upper bound reduces to $\chi_i(T) = \Delta(T) = \chi'_a(T)$, when T has radius 1. Similarly, when the radius of T is two, we have that $\chi'_a(T) \leq \lceil 3/2\Delta \rceil - 1$. More generally we have the following.

Proposition 5. *Let T be a tree of maximum degree Δ. If T has radius at least three, then*

$$\chi'_a(T) \leq \lceil 5\Delta/3 \rceil - 1.$$

Proof: By induction we prove something slightly stronger. T has an arithmetic coloring using colors in the set

$$\Omega := \{1, \dots, \Delta + \lceil 2\Delta/3 \rceil - 1\} \setminus \{\lceil 2\Delta/3 \rceil + 1, \dots, \Delta - 1\}.$$

Let d be the radius of T and let v_0 be a vertex such that the distance of each leaf of T to v_0 is at most d. If we remove all leaves of T at distance d of v_0, we get a tree T' with radius $d-1$. Notice that each vertex in T' having a neighbor in $T - T'$ must be a leaf of T'. By induction hypothesis there is an arithmetic coloring of T' with maximum value at most $\Delta + \lceil 2\Delta/3 \rceil - 1$ and not using colors in $\{\lceil 2\Delta/3 \rceil + 1, \dots \Delta - 1\}$.

Let v be a leaf of T' with color i. When $i \leq \lceil 2\Delta/3 \rceil$, we color its neighbors in $T - T'$ with colors in $\{1, \dots, i\} \cup \{\Delta, \dots, \Delta + \lceil 2\Delta/3 \rceil - 1\}$, if $2i \geq \lceil 2\Delta/3 \rceil$, and with colors in $\{i, \dots, \lceil 2\Delta/3 \rceil\} \cup \{\Delta, \dots, \Delta + \lceil 2\Delta/3 \rceil - 1\}$, otherwise. When $i \geq \Delta$, we color its neighbors in $T - T'$ with colors in $\{\Delta, \dots, i\} \cup \{1, \dots, \lceil 2\Delta/3 \rceil\}$, if $2(i - \Delta) \geq \lceil 2\Delta/3 \rceil$, and with colors in $\{i, \dots, \Delta + \lceil 2\Delta/3 \rceil - 1\} \cup \{1, \dots, \lceil 2\Delta/3 \rceil\}$, otherwise. It is clear in this way we can obtain an arithmetic coloring of T which uses colors in Ω. □

Previous result leads us to seek an upper bound for $\chi'_a(G)$ in terms of the maximum degree in arbitrary graph. In [31] it was shown that $\chi_i(G) \leq \Delta^2 - \Delta + 1$ and that this upper bound is attained by the incidence graph of the projective plane of order Δ (P_Δ). We prove a similar result for the arithmetic chromatic index. We first prove that the incidence graph of P_Δ has arithmetic chromatic index $\Delta(\Delta - 1) + 1$. To this purpose we shall use as set of colors a *perfect difference set S*. It is a set of Δ integers, $s_1, \dots, s_\Delta$, having the property that their $\Delta(\Delta - 1)$ differences, $s_i - s_j, i \neq j; i, j = 1, \dots \Delta$, are congruent modulo $\Delta(\Delta - 1) + 1$, to the integers $1, 2, \dots \Delta(\Delta - 1)$, in some order. In [53] it was shown that for each $\Delta - 1$ which is a power of a prime number, there is a *perfect difference set* of size Δ.

Notice that if S is a perfect difference set so is the set $S' = \{s - m : s \in S\}$, where m is the minimum element in S. Hence, we shall assume in the sequel that $0 \in S$. Under this assumption it follows that for every two elements $s, s' \in S$ we have $s + s' \neq 0 \mod n$, where $n = \Delta(\Delta - 1) + 1$. Otherwise, the two differences $0 - s'$ and $s - 0$ coincide modulo n.

For each perfect difference set S, with $0 \in S$, and having Δ elements, we define the following representation of the incidence graph of P_Δ. Let $n = \Delta(\Delta - 1) + 1$ and let $G(S) = (U \cup W, E(S))$ be a bipartite graph where U and W are copies of $[n-1] \cup \{0\}$ and $E(S) = \{xy | x \in U, y \in W; \exists s \in S : x + s = y \bmod n\}$.

Lemma 6. *The graph $G(S)$ corresponds to the incidence graph of P_Δ. Moreover, $\chi'_a(G(S)) = \Delta(\Delta - 1) + 1$.*

Proof:

By its definition, the set of neighbors of a vertex x in U (resp. W) is $\{x + s \bmod n : s \in S\}$ (resp. $\{x - s \bmod n : s \in S\}$), where $n = \Delta(\Delta - 1) + 1$. Hence $G(S)$ is a $\Delta-$regular graph.

By a counting argument one can see that two vertices x and x' in U have exactly one common neighbor in W, if they have at least one.

As S is a perfect difference set, for each $z = x - x' \bmod n \in U$, there always exist s and s' such that $x - x' = s' - s \bmod n$. Hence, $y := x + s = x' + s$ is a common neighbor of x and x'. A similar argument can be applied to prove that two vertices in W have exactly one common neighbor in U. Therefore, $G(S)$ corresponds to the incidence graph of P_Δ.

Moreover, previous analysis shows that $n = \chi(G^U) \leq \chi'_a(G(S))$.

We prove that $\chi'_a(G(S)) = n$, by showing that the coloring obtained by assigning to each vertex $i \in U \cup W$ the value i, is an arithmetic coloring. It is clear that this coloring is an injective coloring as $x + s = x + s' \bmod n$ implies $s = s' \bmod n$. By the choice of S this implies that $s = s'$. On the other hand, if there are $x, x' \in U$, $s, s' \in S$ such that $x+s = x'+s' =: y \bmod n$ and $x+x' = 2y$, then we obtain the contradiction $s + s' = 0 \bmod n$. □

Our previous construction shows that the following upper bound is tight up to a constant factor.

Theorem 7. *Let G be a graph of maximum degree Δ. Then*

$$\chi'_a(G) \leq 2\Delta(\Delta - 1) + 1.$$

Proof: Our first proof of this upper bound was rather elaborated and only gives the result asymptotically. We present here a simpler proof given by B. Reed [3]. Let $v_1, \ldots, v_n$ be an ordering of the vertices. We construct an arithmetic coloring greedily by following this ordering. When coloring a vertex v_i we have already colored at most $\Delta(\Delta - 1)$ vertices at distance two of v_i. Each such vertex forbids two colors to be used at vertex v. Hence, with $2\Delta(\Delta - 1) + 1$ colors this greedy strategy produces an arithmetic coloring of G with maximum value $2\Delta(\Delta - 1) + 1$.

□

4.4 Lower bounds

We now show some non-trivial lower bounds for the arithmetic chromatic index in terms of the minimum degree.

Theorem 8. *Let $G = (V, E)$ be a graph with minimum degree δ. Then $\chi'_a(G) \geq 5(\delta - 1)/3$.*

Proof: Let $\alpha := \chi'_a(G)$, $l =: \lfloor(\alpha - \delta)/2\rfloor + 1$, and $c := \alpha - \delta - 2(l - 1)$. We shall prove that $2(l - 1) + c \geq (2\delta - 5)/3$, and hence that $\alpha \geq 5(\delta - 1)/3$.

Let φ be an arithmetic coloring of G, with $\varphi : V \to [\alpha]$. Let u be a vertex such that $\varphi(u) < \delta$. Then, at most one color between $\varphi(u) - a$ and $\varphi(u) + a$

[3] Personal Communication

can be used to color neighbors of u, for each $a = 0, \ldots, \varphi(u) - 1$. Hence, at least $\delta - \varphi(u)$ colors in $\{2\varphi(u), \ldots, \alpha\}$ are needed to color neighbors of u. This gives $\varphi(u) \leq \alpha - \delta + 1$. Given the definition of c and l, we have that the set of colors that can be used to color neighbors of u is contained in

$$\{1, \ldots, 2l - 1 + c\} \cup \{\delta, \ldots, \delta + 2(l - 1) + c\}.$$

We define four parameters $\theta^-, \theta^+, \rho^-, \rho^+$ as follows.

$$\begin{aligned}
\theta^- &:= \min\{l + c - \varphi(u) : \varphi(u) \leq l + c\}, \\
\theta^+ &:= \min\{\varphi(u) - l - c : l + c \leq \varphi(u) \leq 2l - 1 + c\}, \\
\rho^- &:= \min\{\delta - 1 + l + c - \varphi(u) : \delta \leq \varphi(u) \leq \delta - 1 + l + c\}, \\
\rho^+ &:= \min\{\varphi(u) - l - c - \delta + 1 : \delta - 1 + l + c \leq \varphi(u) \leq \delta - 1 + 2l - 1 + c\}.
\end{aligned}$$

Let u be a vertex such that $\varphi(u) = l + c - \theta^-$. Then, the neighbors of u can use at most $l + c - \theta^-$ colors in $\{1, \ldots, 2(l + c - \theta^-) - 1\}$, at most $2\theta^- - c$ colors in $\{2(l+c-\theta^-), \ldots, 2l+c-1\}$, and at most $2l+c-(\rho^-+\rho^+)$ in $\{\delta, \ldots, \delta+2(l-1)+c\}$. Hence, at most

$$3l + c + \theta^- - \rho^- - \rho^+ \tag{4.3}$$

in total.

By considering a vertex u such that $\varphi(u) = l + c + \theta^+$, we deduce that there are at most

$$3l + 2c + \theta^+ - \rho^- - \rho^+ \tag{4.4}$$

colors available to color neighbors of u.

Similarly, for a vertex u with $\varphi(u) = \delta - 1 + l + c - \rho^-$, there are at most

$$3l + c + \rho^- - \theta^+ - \theta^- \tag{4.5}$$

colors that can be used to color neighbors of u.

Finally, for a vertex with $\varphi(u) = \delta - 1 + l + c + \rho^+$, there are no more than

$$3l + 2c + \rho^+ - \theta^+ - \theta^- - 2 \tag{4.6}$$

colors for coloring neighbors of u.

By adding Equations (4.3), (4.4), (4.5), and (4.6), we get

$$12l + 6c - \theta^- - \theta^+ - \rho^- - \rho^+ - 2 \geq 4\delta,$$

which implies

$$2(l-1) + c \geq (2\delta - 5)/3.$$

□

When we apply previous bound to Δ-regular graphs we get the following result.

Corollary 9. *Let $G = (V, E)$ be a Δ-regular graph. Then $\chi'_a(G) \geq 5(\Delta - 1)/3$.*

4.5 Computational complexity of computing $\chi'_a(G)$

We have seen so far that arithmetic colorings are injective colorings with additional constraint on the color allowed in the neighborhood of each vertex. A similar restriction applies to $L(p,q)$-labeling, a related concept introduced in [29]. A vertex coloring c of a graph G with colors in $[k] \cup \{0\}$ such that $|c(u) - c(v)| \geq p$ when u and v are adjacent and $|c(u) - c(v)| \geq q$ when they are at distance two in G is called a $L(p,q)$-labeling of G. Let $\lambda(p,q)(G)$ be the smallest integer k such that there is a $L(p,q)$-labeling with colors in $[k-1] \cup \{0\}$. From the computational complexity point of view it is known that for each $p \geq q$, such that q does not divide p, the problem of computing $\lambda_{p,q}(G)$ is NP-hard, even when restricted to trees [26]. On the other hand, in [19], it is shown that $\lambda_{2,1}(G)$ can be computed in polynomial time on trees. More recently, a linear time algorithm for computing $\lambda_{2,1}(G)$ for trees was proved in [32]. The method presented in [19] can be extended to compute $\lambda_{p,1}(G)$ in polynomial time.

Here we show how a similar idea can be used to compute the arithmetic chromatic index of trees. For a tree of maximum degree Δ, we can reduce the computation of its arithmetic chromatic index to the problem of deciding whether it has an arithmetic coloring with maximum value l, for each value $l \in \{\Delta, \dots, 5\Delta/3\}$. We present the result in a slightly more general framework which includes previous notions.

Let us consider injective colorings with colors in a given set $\mathcal{C}$. We associate to each color $a \in \mathcal{C}$ a graph $H_a = (C_a, R_a)$, where C_a is a subset of $\mathcal{C}$. Let $T = (V, E)$ be a tree and $c : V \to \mathcal{C}$ an injective coloring. We say that c is *feasible* (for $(H_a)_{a \in \mathcal{C}}$) if for each vertex $u \in V$, the set $c(N_T(u))$ is contained in $C_{c(u)}$ and it is an independent set in the graph $H_{c(u)}$.

Let $\mathcal{C} = [k]$, for some positive integer k and for each $a \in [k]$, let $H_a = ([k], \emptyset)$. In this situation, a feasible injective coloring is just an injective coloring using colors in $[k]$. On the other hand, given positive integers p, q, $p \geq q$, if for each $a \in [k] \cup \{0\}$ the graph H_a has vertex set $([k] \cup \{0\}) \setminus \{a-p+1, \dots, a+p-1\}$ and edge set $\{ij : i, j \in [k] \cup \{0\}, i \neq j, |i-j| < q\}$, then a feasible injective coloring is, in fact, a $L(p,q)$-labeling. Finally, if for each $a \in [k]$, the vertex set of H_a is $[k]$ and its edge set is $\{ij : i + j = 2a\}$, then we have that a feasible injective coloring is an arithmetic coloring. We notice that each family $(H_a)_{a \in C}$ previously associated to injective colorings, arithmetic colorings or $L(p,1)$-labelings satisfies the following property: for each $a \in \mathcal{C}$, each connected component of H_a is a complete graph. In fact, in these situations each connected component is either an isolated vertex or consists of two adjacent vertices.

In order to determine the existence of feasible injective coloring we use the natural extension of the dynamic programming algorithm given in [19] to our framework.

Given a leaf r of T and a vertex $u \neq r$, we denote by T^u the subtree of T which contains the neighbor of u in the path in T between r and u, which we call the *father* of u in T, and every vertex v such that the path in T between v and r contains u. By instance if u is the neighbor of r in T, then r is the father of u and $T^u = T$. Let $N'_T(u)$ denote the set $N_T(u) \setminus \{f(u)\}$ which is the set of all neighbors of u in the tree T excluding its father.

For each vertex u and each color $a \in \mathcal{C}$ we say color $b \in C_a$ is *compatible* with u and a, if T^u has a feasible injective coloring c such that $c(f(u)) = a$ and $c(u) = b$. Let $C(u, a)$ denote the set of colors which are compatible with u and a. It is clear that if there are colors a and b such that $b \in C(u, a)$, where u is the (unique) neighbor of r in T, then $T^u = T$ has a feasible injective coloring.

The following dynamic programming algorithm computes the compatible sets.

Compatible Sets

Input: A tree $T = (V, E)$ and r a leaf of T; a set of color $\mathcal{C}$ and a family of graphs $(H_a = (C_a, R_a))_{a \in \mathcal{C}}$, where $C_a \subseteq \mathcal{C}$, for each $a \in \mathcal{C}$.

Output: For each vertex u, $u \neq r$, and each color $a \in \mathcal{C}$, the set of compatible colors $C(u, a)$.

1. For each leaf u, $u \neq r$, for each color a, set $C(u, a) = C_a$; mark vertex u as processed.

2. Iteratively, take a vertex u in T, $u \neq r$, not yet processed and such that for every vertex in $N'_T(u)$ all compatible sets have been determined. For each color a, compute $C(u, a)$ using the following equivalence: $b \in C(u, a)$ if and only if for each $v \in N'_T(u)$ there exists a color $d_v \in C(v, b)$, such that the

set $\{d_v : v \in N'_T(u)\} \cup \{a\}$ is an independent set of H_b. At the end, mark vertex u as processed and continue with unprocessed vertices.

This strategy can be implemented in time $O(n|\mathcal{C}|^2K)$, where K is the time needed to determine whether $b \in C(u,a)$, for a given vertex u, and given colors a and b.

When for each $a \in \mathcal{C}$, each connected component of the graph H_a is a complete graph, a feasible injective coloring c of T must assign to each vertex $v \in N_T(u)$ a color in a different connected component of $H_{c(u)}$. In this situation, given colors a and b and a vertex u, the problem of determining whether $b \in C(u,a)$ can be formulated as a maximum matching problem in the auxiliary bipartite graph $G = (N'_T(u) \cup B, E)$, where B is the set of connected components of H_b which do not contain color a and $vs \in E$ whenever $v \in N'_T(u)$, $s \in B$ and $V(s) \cap C(v,b) \neq \emptyset$, where $V(s)$ is the set of colors in s.

We have that T^u admits a feasible injective coloring c with $c(u) = b$ and $c(f(u)) = a$ if and only if G has a matching M such that $N'_T(u)$ is contained in the set $V(M) := \{v : \exists e \in M, v \in e\}$. In fact, if G has a matching M with $N'_T(u) \subseteq V(M)$, then for each $v \in N'_T(u)$ there is a connected component s of H_b such that $C(v,b) \cap V(s) \neq \emptyset$. Hence, for each $v \in N'_T(u)$, the subtree T^v has a feasible injective coloring such that $c_v(v) \in V(s)$ and $c_v(u) = b$. These feasible injective colorings can be extended to an injective coloring c of T^u by defining $c(w) = c_v(w)$ for each $w \in T^v$ and $c(f(u)) = a$. As each $c(v)$ belongs to a different connected component of H_b, the set $\{c(v) : v \in N'_T(u)\} \cup \{a\}$ is an independent set of H_b. Therefore, c is a feasible injective coloring of T^u. Conversely, if T^u has a feasible injective coloring c, then c must assign to each vertex in $N'_T(u)$ a color in a different connected component of H_b. As $c(f(u)) = a$, no color in the connected component of H_b containing a can be used to color vertices in $N'_T(u)$. Moreover, for every two distinct vertices v and w in $N'_T(u)$, $c(v)$ and $c(w)$ belong to different connected components in H_b. Then, $M := \{vs : c(v) \in V(s) \in B\}$ is a matching of G such that $N'_T(u)$ is contained in $V(M)$.

The time needed to compute a maximum matching in a bipartite graph whose independent sets are of size $|N(u)| - 1 \leq \Delta$ and $|C_a| \leq |\mathcal{C}|$ is $O(\Delta^{1.5}|\mathcal{C}|)$.

Theorem 10. *In time $O(\Delta^{1.5}n|\mathcal{C}|^3)$ we can decide whether a tree with n vertices and of maximum degree Δ admits a feasible injective coloring for $(H_a)_{a\in\mathcal{C}}$, when for each $a \in \mathcal{C}$, each connected component of H_a is a complete graph.*

From Theorem 10 applied to the family $(H_a)_{a\in\mathcal{C}}$ associated to arithmetic colorings we get that in time $O(\Delta^{1.5}nl^3)$ we can decide whether a tree T with n vertices and of maximum degree Δ admits an arithmetic coloring with colors in $[l]$. From Proposition 5 we get the following corollary.

Corollary 11. *The arithmetic chromatic index can be computed in time $O(\Delta^{4.5}n \log \Delta)$ in a tree with n vertices and of maximum degree Δ.*

We now consider the computation of the arithmetic chromatic index in larger classes of graphs. A natural class to consider is the class of bounded treewidth graphs. For this class it is known that any decision problem admitting a Monadic Second Order Logic formula has a polynomial time algorithm [20]. Let k-ARITHMETIC COLORING denote the problem of deciding whether a graph G has arithmetic chromatic index at most k. For each fixed k, we have the following.

Lemma 12. *The problem k-ARITHMETIC COLORING can be expressed by a Monadic Second Order Logic formula of size only depending on k.*

Proof:

In the following formula, the sets $X_1, \ldots, X_k$ will correspond to the color sets, and the three vertices x, y, z to any path of length 2. Part (4.8) of the formula states that vertices x, y, z receive colors in $[k]$, part (4.9) states that vertices x and z receive different colors, and part (4.10) states that the colors given to vertices x, y, z do not form an arithmetic progression.

$$\exists X_1, \ldots, X_k \subseteq V(G) \text{ s.t. } \forall x, y, z \in V(G) : \big(\{x,y\} \in E(G) \wedge \{y,z\} \in E(G)\big) \tag{4.7}$$

$$\Big[\big(\textstyle\bigvee_{i\in[k]} x \in X_i\big) \wedge \big(\textstyle\bigvee_{i\in[k]} y \in X_i\big) \wedge \big(\textstyle\bigvee_{i\in[k]} z \in X_r\big)\Big] \tag{4.8}$$

$$\wedge \qquad \neg\Big[\textstyle\bigvee_{i\in[k]} \big(x \in X_i \wedge z \in X_i\big)\Big] \tag{4.9}$$

$$\wedge \qquad \neg\Big[\textstyle\bigvee_{i,j\in[k]} \big(x \in X_i \wedge y \in X_{i+j} \wedge z \in X_{i+2j}\big)\Big] \tag{4.10}$$

□

From the result in [20] we immediately get the following.

Theorem 13. *The problem k-*Arithmetic Coloring*, for any fixed k, has a polynomial time algorithm when restricted to classes of graphs of bounded treewidth.*

We do not know whether the problem remains polynomially solvable when k is part of the input, even for serie-parallel graphs, i.e. graphs of treewidth at most two. On the other hand, we prove that the problem is hard for $k = 4$, even when restricted to 3-regular graphs. To this end we show that 4-Arithmetic Coloring reduces the problem of deciding whether a graph has a proper 3-edge coloring. This latter problem was proved to be NP-complete in [36], even when restricted to 3-regular graphs.

Theorem 14. *The problem k-*Arithmetic Coloring*, for $k = 4$, is NP-complete, even when restricted to 3-regular graphs.*

Proof: To see that k-Arithmetic Coloring belongs to NP, we notice that the arithmetic chromatic index is monotone under subgraphs. Hence an upper bound for its value on the complete graph K_n is an upper bound for its value in any graph on n vertices. As the right hand side of Equation 4.1 is $O(n^2)$, an arithmetic coloring for a graph on n vertices has a description of length polynomial in n. This shows that the problem belongs to NP.

In order to prove the statement, for each 3-regular graph G we build a 3-regular graph G' such that G is 3-edge-colorable if and only if G' has an arithmetic coloring with colors in $\{1, a, 4\}$, with $a = 2$ or $a = 3$.

The graph G' is obtained from G by replacing each vertex v of G by a copy of the complete graph K_3, which we denote $A(v)$, and we replace each edge $e = uv \in G$ by an edge $a_e b_e$ in G', where $a_e \in A_u$, $b_e \in A_v$ and so as each vertex in A_u finishes with degree 3 in G'. The vertices a_e and b_e in the above construction are called the vertices of G' *associated* to the edge e in G. It is clear that the construction of G' can be done in polynomial time and that G' is 3-regular.

Any proper edge coloring of G with three colors can be transformed in an arithmetic coloring of G' which uses colors in the set $\{1, 2, 4\}$ as follows. In G' we assign color i to the two vertices a_e and b_e associated to an edge e of G with color i, for $i = 1, 2$, and we assign color 4 to those vertices of G' associated with edges of G with color 3.

Conversely, an arithmetic coloring with colors in $\{1, 2, 3, 4\}$ uses either colors $1, 2, 4$ or $1, 3, 4$ in each set A_v. Therefore, colors 1 and 4 are used in each set A_v. Moreover, if a is the neighbor of a vertex $b \in A_v$ not in A_v, then it has the same color as b. This allows us to color each edge e with color 1 (resp. 3), when its associated vertices a_e and b_e are colored with color 1 (resp. 4) in G'. The remaining edges are colored with color 2. □

In view of previous results it is interesting to consider whether we can approximate the arithmetic chromatic number in polynomial time. We can use Theorem 2 to obtain (in)approximability results for the arithmetic chromatic index based on previous results obtained for the injective chromatic number. In [34], it was proved that there is a polynomial time approximation algorithm for $\chi_i(G)$ with an approximation factor $n^{1/3}$ when restricted to split graphs. Moreover, they proved that this result is tight in the following sense. They showed that unless

ZPP = NP, for each $\epsilon > 0$ there is no polynomial time approximation algorithm for $\chi_i(G)$ with a factor $n^{1/3-\epsilon}$, even for the class of split graphs. This result was based on an inapproximability result for the chromatic number obtained in [25]. From a result obtained in [63], we now know that the condition ZPP = NP can be strengthened to P = NP.

Let us assume that there are $\epsilon > 0$ and a polynomial time approximation algorithm which on input G computes an approximation α for the arithmetic chromatic index laying between $\chi'_a(G)$ and $n^{1/3-\epsilon}\chi'_a(G)$.

From Theorem 2 we have that $\chi'_a(G) \leq \chi_i(G)g(\chi_i(G))$, where $g(x) = 2^{\sqrt{2\log(x)}}$ is a non negative, non decreasing function, and $g(n) \leq n^{\epsilon/2}$, for n large enough. Then, we have

$$\chi_i(G) \leq \alpha \leq n^{1/3-\epsilon}\chi_i(G)g(n) \leq n^{1/3-\epsilon/2}\chi_i(G),$$

for each graph G with n vertices, and n large enough. This implies the following result.

Theorem 15. *For each $\epsilon > 0$, unless P = NP, there is no polynomial time approximation algorithm with approximation factor $n^{1/3-\epsilon}$ for the arithmetic chromatic index, even when restricted to split graphs.*

On the other hand, if for a graph G on n vertices we can compute in polynomial time a value v such that $\chi_i(G) \leq v \leq \chi_i(G)n^{1/3}$, then we can use $vg(v)$ to get an approximated value for $\chi'_a(G)$. In fact, we know from Theorem 2 that $\chi'_a(G) \leq \chi_i(G)g(\chi_i(G))$. Hence $\chi'_a(G) \leq vg(v)$, as $xg(x)$ is a non-decreasing function.

Applying the function $xg(x)$ to $\chi_i(G)n^{1/3}$ we get $\chi_i(G)n^{1/3}g(\chi_i(G)n^{1/3})$. But, $g(\chi_i(G)n^{1/3}) \leq g(n^{4/3})$ and for each value $\epsilon > 0$, $g(n^{4/3})$ is smaller than n^{ϵ}, for n large enough. Therefore, for n large enough we have

$$\chi'_a(G) \leq vg(v) \leq \chi'_a(G)n^{1/3+\epsilon}.$$

As we previously mentioned, in [34], it was proved that there is a polynomial time approximation algorithm for $\chi_i(G)$ with an approximation factor $n^{1/3}$ when restricted to split graphs. Hence, our previous analysis immediately implies the following result.

Theorem 16. *For each $\epsilon > 0$, there is a polynomial time approximation algorithm with approximation factor $n^{1/3+\epsilon}$ for the arithmetic chromatic index, when restricted to split graphs.*

It is clear that based upon Theorem 3 we can obtain similar approximation results between the arithmetic chromatic index and the injective chromatic number for bipartite graphs. This linear relation between these two parameters makes interesting the following result.

Theorem 17. *For each fixed $k \geq 3$, the problem of deciding whether an input graph has injective chromatic number at most k, is NP-complete, even restricted to bipartite graphs of maximum degree k.*

Proof: Let $G = (V, E)$ be a k-regular graph and let $I(G)$ be the *incident graph* of G defined as the bipartite graph with independent sets V and E such that ve is an edge of $I(G)$, whenever e is incident with v in G. Notice that if G has maximum degree k, then so has $I(G)$.

It is not hard to see that for the incidence graph $I(G)$, it holds that $(I(G))^V$ is the original graph G and that $(I(G))^E$ is the line graph of G. Therefore, from Equation 4.2 we get

$$\chi_i(I(G)) = \max\{\chi(G), \chi'(G)\}.$$

From Brooks' Theorem ([16]) we get that $\chi_i(I(G)) = \chi'(G)$, unless G is a complete graph. Therefore, computing the injective chromatic number of $I(G)$, a bipartite graph of maximum degree k, is as hard as computing the chromatic index of G, a k-regular graph. As it is known that computing the chromatic index of k regular graphs is NP-hard we obtain the statement of the theorem. □

4.6 Conclusion

We have seen that computing the arithmetic chromatic number on complete graphs as well as in balanced complete bipartite graphs depends on non-trivial arithmetic properties of integers. We think that it is worth to consider the problem in others classes of well structured graphs as products of paths and/or cycles, balanced complete 3-partite graphs, and more generally, balanced complete k-partite graphs. We think that this study may require more sophisticated arithmetic combinatorics tools in order to obtain lower bounds.

Appendix A

Basic concepts in graphs

A.1 Basics

Definition A.1.1. *A* graph *is a pair $G = (V, E)$ of sets where E is a subset of the set of unorderer pairs of V. We call $V = V(G)$ the set of vertices of G and $E = E(G)$ the set of edges ofG.*

An edge *of E will be denoted indistinctly by $e = uv = \{u, v\}$. In this case, we say that e is incident in u and v, or that these two vertices are the endpoints of e.*

In this thesis we assume that the set $V(G)$ is *finite*. There are no loops in the graph. That means that there is no element $xy \in E(G)$ such that $x = y$. We also asume that there is at most one edge between two vertices. These two restrictions define a *simple graph*. We say that two vertices are **neighbors** or **adjacent** one to the other when there is an edge between them. The set of neighbors of a vertex v is denoted $N_G(v)$ or $N(v)$.

Definition A.1.2. *Given a graph $G = (V, E)$ we say that the* order *of the graph is n, if the cardinality of V is n. The notation is $|V| = n$. The* size *of the graph, denoted by $m(G)$ is the size of the edge set, $|E(G)|$.*

Definition A.1.3. *The degree of a vertex v, denoted by $d(v)$ is the number of edges incident to that vertex. We use $\delta(G) and \Delta(G)$ to refer to the minimum and maximum degree over all vertices in $V(G)$.*

Definition A.1.4. *When $\delta(G) = \Delta(G)$, we say that G is a regular graph, or a $\Delta-$regular graph. The 3-regular graphs are called cubic graphs.*

Definition A.1.5. *A graph $H = (V, E)$ is said to be a* subgraph *of $G = (V, E)$ if and only if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.*

There is a special kind of subgraph that will appear throughout this work. A subgraph H of G is said to be an *induced subgraph* of G if each edge of G has its endpoints in $V(H)$ is an edge of H.

In this context we use the notation G_A to refer to the induced graph by A with $A \subseteq V(G)$. We say that $G' = (V', E')$ is a *spanning subgraph* of $G = (V, E)$ if $G_{V'}$ induces G, i.e. $V' = V$.

Definition A.1.6. *The* girth *of a graph is the minimum length a cycle contained as a subgraph in G.*

A.2 Graph families

In this thesis we have worked with some special families of graphs.

Definition A.2.1. *A simple graph is said to be a* ***complete graph*** *if every pair of vertices in G are adjacent. It is denoted K_n, where n is the order of the graph.*

Definition A.2.2. *A* ***bipartite graph****, is a graph in which its set of vertices can be partitioned into two nonempty subsets, A and B such that all the edges in $E(G)$ have one endpoint in A and the other in B. The bipartite graph G with bipartition (A, B) is denoted $G(A, B)$. We say that $G(A, B)$ is a* ***bipartite complete graph*** *if every vertex in A is adjacent to every vertex in B. If $|A| = n$ and $|B| = m$ we denote the complete bipartite graph $G(A, B) = K_{n,m}$. If the two sets of vertices have the same size, we call this graph* ***balanced****.*

Definition A.2.3. *A* ***path*** *P is a nonempty graph, $P = (V, E)$ with vertex set $V = \{v_0, v_1, \ldots v_n\}$ and edge set $E = \{v_0v_1, v_1v_2, \ldots v_{n-1}v_n\}$ with $v_i \neq v_j$ for $i \neq j$. The length of a path is the number of edges in it, and the path of length n is denoted P_n.*

Definition A.2.4. *A* ***cycle*** *is a graph obtained from adding an edge between the endpoints of a path. The length of a cycle is again the number of edges on it, which is the same of the number of vertices, we denote C_n the cycle of length n.*

Definition A.2.5. *A* ***tree*** *T is a nonempty graph, $T = (V, E)$ with no cycles contained as a subgraph.*

Definition A.2.6 (Cartesian product)**.** *Given two graphs $G = (V, E)$ and $H = (U, F)$ The Cartesian product of G and H, denoted by $G \square H$, is the graph on the vertex set $V \times H$, and edge set $[(v, x), (u, y)] \in E(G \square H)$ if either $x = y$ and $\{v, u\} \in E$, or if $v = u$ and $\{x, y\} \in F$.*

A.3 Special subgraphs

Definition A.3.1. *A subgraph H of a graph G is said to be* induced *if, for any pair of vertices v and w of H, vw is an edge of H if and only if vw is an edge of G. In other words, H is an induced subgraph of G if it has exactly the edges that appear in G over the same vertex set. If the vertex set of H is the subset S of $V(G)$, then H can be written as $G[S]$ and is said to be induced by S.*

Definition A.3.2. *A* matching *M in a graph $G = (V, E)$ is a subset of the edges of G such that every vertex of G is at most on one edge in M.*

Definition A.3.3. *A **factor** of a graph G is a spanning subgraph of G. A k-factor of G is a factor of G that is k-regular. A one-factor of G is a matching M which includes all the vertices of G. We also call this special factor **perfect matching**.*

Definition A.3.4. *A* Hamiltonian cycle *in G is a cycle that contains every vertex of $V(G)$ exactly once. If G has a Hamiltonian cycle we say that G is Hamiltonian.*

Definition A.3.5. *A **factorization** of a graph G is a set of factors of G which are pairwise edge disjoint, and whose union is all G. A **one-factorization** of a graph is a decomposition of the set of edges into one-factors, or perfect matchings.*

Definition A.3.6. *A* one-factorization *of a graph G, $\mathcal{F} = \{F_0, F_1, \ldots F_n\}$ is said to be perfect if $F_i \cup F_j$ form a Hamiltonian cycle in G for every $i \neq j$.*

A.4 Edge coloring

Definition A.4.1. *A proper* edge coloring *of a graph $G = (V, E)$ is a map $\phi : E(G) \longrightarrow C$, with C a set of colors, with the restriction that two edges sharing an end point must receive different colors. The smallest integer n such that a proper coloring of G is possible by using n colors is call the **chromatic index** of G, denoted by $\chi'(G)$.*

Another way to see the chromatic index of a graph, which will be useful in this thesis, is to think of the coloring as a partition of the edges of the graph. Then each color class defines a matching in G, and the chromatic index, $\chi'(G)$, is the smallest number of matchings into which the set of edges can be partitioned.

Definition A.4.2. *A* perfect edge coloring *is an edge coloring of the graph G, in which every pair of colors induces a Hamiltonian cycle.*

Definition A.4.3. *An* acyclic edge coloring *is an edge coloring of the graph G, in which every pair of colors induces an acyclic subgraph, i.e. a set of trees, called forest.*

Definition A.4.4. *An* arithmetic edge coloring *c of a graph $G = (V, E)$ is a function assigning positive integers to its vertices such that by assigning to each edge uv the value $|c(u) - c(v)|$, we obtain a proper edge coloring. The smallest integer k such that an arithmetic colorings exists with colors in the set $[k]$ is called the* arithmetic chromatic index *of G. We denote it by $\chi'_a(G)$.*

Definition A.4.5. *An* injective coloring *of a graph is a vertex labeling such that two vertices sharing a common neighbor get different labels. The smallest integer k such that an injectve coloring exists with colors in the set $[k]$ is called the* injective chromatic number *of G. We denote it by $\chi_i(G)$. This coloring is not a proper coloring of the vertex of G.*

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