EQUIVARIANT FAMILIES OF PSEUDODIFFERENTIAL OPERATORS COMING FROM DEFORMATION QUANTIZATION AND COVARIANT FIELDS OF RIEFFEL C^*-ALGEBRAS

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Resumen:

El propósito de esta tesis es el estudio y desarrollo de dos nuevos tipos de teorías pseudodiferenciales basadas en el cálculo de Weyl, y también el estudio de ciertas técnicas C^*-algebraicas relacionadas con estas. Estos cálculos consisten en asociar a cada elemento de una cierta clase de funciones (a quienes llamamos símbolos) una familia de operadores en $L^2(\mathbb{R}^n)$ indexada por los puntos de un espacio en el que $\mathbb{R}^n$ o $\mathbb{R}^{2n}$ actúa, de modo que estas definen familias equivariantes de operadores. Por equivariantes me refiero a que los operadores indexados por puntos en la misma órbita, correspondientes a un símbolo fijo, son unitariamente equivalentes (la equivalencia unitaria será implementada por un operador unitario independiente del símbolo). Los símbolos son elementos de un álgebra de Poisson de funciones sobre un espacio que puede ser visto como un encolado continuo de varios espacios de fase usuales.

Desde el punto de vista C^*-algebraico, para cada uno de estos cálculos obtendremos una C^*-álgebra que tiene al álgebra de símbolos como una *-subálgebra densa. Las familias de operadores provienen de restringir una cierta familia de representaciones de la C^*-álgebra al álgebra de Poisson.

El primer cálculo fue introducido por M. Lein, M. Măntoiu y yo. Este cálculo fue construido de modo que los operadores que emergen a través de este puedan ser interpretados como Hamiltonianos magnéticos. También logramos hacer al formalismo dependiente de un parámetro real $\hbar$ (interpretado como la constante de Planck) y obtuvimos varios resultados de tipo semiclásico. Además apartir de cierta subálgebra de Poisson de símbolos logramos construir lo que se conoce como una cuantización por deformación estricta.

El segundo cálculo está basado en un proceso de deformación de C^*-álgebras. Probaremos que si el álgebra no deformada es el álgebra de secciones de un campo continuo de C^*-álgebras entonces el álgebra deformada también lo es y, de hecho, las correspondientes fibras son el resultado de deformar las fibras del álgebra de secciones inicial. Esto será usado para probar resultados espectrales para los operadores determinados por el segundo cálculo.

Abstract:

The aim of this thesis is the study and development of two recent types of pseudodifferential theories rooted in the usual Weyl calculus, and also the study of certain C^*-algebraic techniques related to them. These calculi consist in to associate to each element of certain class of functions (called the symbols of the calculi) a family of operators on $L^2(\mathscr{X})$, indexed by the points of a space on which $\mathbb{R}^n$ or $\mathbb{R}^{2n}$ acts, such that these define equivariant families of operators. By a equivariant family we mean that the operators indexed by points on the same orbit, corresponding to a fixed symbol, are unitary equivalent (the unitary equivalence will be implemented by an unitary operator independent of the symbol). The symbols are elements of a Poisson algebra of functions on a space that can be seen as a continuous gluing of several standard phase spaces.

From the C^*-algebraic point of view, for each of these calculi we will obtain a C^*-algebra which has the Poisson algebra of symbols as a dense *-subalgebra. The families of operators will be the result of restricting a certain family of representations of the C^*-algebra to the Poisson algebra.

The first calculus was introduced by M. Lein, M. Măntoiu and I. This calculus was meant to generate operators which can be considered as magnetic Hamiltonian. We made the formalism dependent of a real parameter $\hbar$ (which must be interpreted as Planck's constant) and we obtained several semiclassical results. We also constructed, from certain Poisson subalgebra of symbols, what is known as a strict deformation quantization.

The second calculus is based on a deformation procedure of C^*-algebras. We will prove that if the undeformed algebra is the section algebra of a continuous field of C^*-algebras then the deformed algebra will also be a section algebra, in fact, the corresponding fibers will be the deformation of the fibers of the initial section algebra. This will be applied to obtain spectral results about the operators given by the second calculus.

Contents

Chapter 1

Overview

1.1 Quantization: basic examples

By general principles, in Classical Mechanics the observables of a physical system are real valued C^∞-functions on its phase space (which is assumed to be a Poisson manifold or just a symplectic manifold). On the other hand, in Quantum Mechanics the observables of the physical system should be self-adjoint operators on some Hilbert space. We understand by *quantization* of a physical system, a systematic way to convert suitable classical observables of the system into quantum observables. An admissible classical observable for such procedure is usually called a symbol for the quantization.

The Weyl pseudodifferential calculus is the best understood of such procedures; it can be regarded as a quantization of the physical system consisting of a non-relativistic spinless particle moving in the configuration space $\mathscr{X} := \mathbb{R}^n$, in the absence of any magnetic field. The phase space of this system is the cotangent bundle $T^*\mathscr{X} \cong \mathscr{X} \times \mathscr{X}^* =: \Xi$ of the configuration space, where $\mathscr{X}^*$ is the dual of $\mathscr{X}$. Explicitly, it is given by

$$[\mathfrak{Op}(f)u](x) := (2\pi)^{-n} \int_{\mathscr{X}} \int_{\mathscr{X}^*} e^{i(x-y)\cdot\xi} f\left(\frac{x+y}{2}, \xi\right) u(y)\, \mathrm{d}\xi \mathrm{d}y,$$

where $u \in L^2(\mathscr{X})$. This expression makes sense for suitable class of functions f (symbols) and defines a (not necessarily bounded) operator.

Recall that the canonical (constant) symplectic form

$$\sum_{j=1}^{n} \mathrm{d}\xi_j \wedge \mathrm{d}x_j$$

expressed in canonical coordinates $\{x_j, \xi_j\}_{j=1}^n$ transforms $T^*\mathscr{X}$ into a symplectic manifold. The bracket associated with it is given by

$$\{f, g\} := \sum_{j=1}^{n} (\partial_{\xi_j} f\, \partial_{x_j} g - \partial_{x_j} f\, \partial_{\xi_j} g),$$

where $f, g \in C^\infty(\Xi)$. It endows Ξ with the structure of a Poisson Manifold, in other words, the triple $(C^\infty(\Xi)\,,\{\cdot\,,\cdot\}\,,\cdot)$ where $\cdot$ denote the pointwise product forms a Poisson Algebra. The explicit definition can be found in [24] together with a treatment of Poisson and symplectic manifolds from the classical mechanics point of view.

Now we briefly expose what happens when a (non-necessarily constant) continuous magnetic field B (a continuous closed 2-form of $\mathscr{X}$) is turned on. Recall that the vector bundle structure of the space $T^*\mathscr{X} \cong \mathscr{X} \times \mathscr{X}^* = \Xi$ is given just by the projection on the first component. Let $\tilde{\pi}_1 B$ be the pullback of B by this projection, and let us consider the closed 2-form given by the sum of the canonical symplectic form and $\tilde{\pi}_1 B$. This gives a new symplectic structure on $T^*\mathscr{X}$ and the bracket associated to it is given by

$$\{f, g\}_B := \sum_{j=1}^{n} (\partial_{\xi_j} f\, \partial_{x_j} g - \partial_{x_j} f\, \partial_{\xi_j} g) + \sum_{j,k} B^{jk}\, \partial_{\xi_j} f\, \partial_{\xi_k} g. \tag{1.1.1}$$

In the literature it has been considered another change in the Poisson structure of Ξ, based on an application of the minimal coupling principle: Choose a vector potential A for the magnetic field ($B = \mathrm{d}A$) and define

$$\{f, g\}^A := \{f_A, g_A\},$$

where $f, g \in C^\infty(\Xi)$ and $f_A(x, \xi) := f(x, \xi - A(x))$. These Poisson brackets are related by the following formula:

$$\{f, g\}^A = (\{f, g\}_B)_A.$$

By physical considerations, for each vector potential A one should have a quantization procedure $\mathfrak{Op}^A$ of this (magnetic) physical system. They should have the gauge covariance property, meaning that if $A' = A + \nabla\rho$ (i.e. A y A' are potentials for the same magnetic field) then $e^{i\rho}\mathfrak{Op}^A(f)e^{-i\rho} = \mathfrak{Op}^{A'}(f)$. In [22] and [30] a solution for this problem was given; it is called the magnetic Weyl calculus and is formally given by:

$$\left[\mathfrak{Op}^A(f)u\right](x) = (2\pi)^{-n} \int_{\mathscr{X}} \int_{\mathscr{X}^*} e^{i(x-y)\cdot\xi} f\left(\frac{x+y}{2}, \xi\right) e^{-i\Gamma^A<x,y>} u(y) \mathrm{d}\xi \mathrm{d}y,$$

where $\Gamma^A < x, y >$ is the circulation of the potential A through the segment leading from x to y.

Each continuous magnetic field B has a potential called the transversal gauge potential. It is given by

$$A_j(x) := \sum_{k=1}^{n} \int_0^1 B_{jk}(sx) s x_k \mathrm{d}s.$$

This potential satisfies that $\Gamma^A < x, y >= \Gamma^B < 0, x, y >$, where $\Gamma^B < 0, x, y >$ is the flux of B through the triangle with vertices at 0, x and y. The calculus associated to this potential will be denoted by $\mathfrak{Op}^B$.

It appears from time to time in the literature the proposition $\mathfrak{Op}_A(f) = \mathfrak{Op}(f_A)$. In general this fails to be a physically admissible quantization because it does not satisfies the covariance property; however if A is linear then $\mathfrak{Op}^A = \mathfrak{Op}_A$.

1.2 Pseudodifferential calculi and gluing basic physical systems

The aim of this thesis is the study and development of two recent types of pseudodifferential theories rooted in the usual Weyl calculus, and also the study of certain C^*-algebraic techniques related to them. These calculi are also meant to be interpreted as a quantization procedure. The purpose is associating to each symbol a family of operators on $L^2(\mathscr{X})$, indexed by the points of a space on which $\mathscr{X}$ or Ξ acts, such that these define equivariant families of operators. By a equivariant family we mean that the operators indexed by points on the same orbit, corresponding to a fixed symbol, are unitary equivalent (the unitary equivalence will be implemented by an unitary operator independent of the symbol). The symbols are elements of a Poisson algebra of functions on a space that can be seen as a continuous gluing of several standard phase spaces.

From the C^*-algebraic point of view, for each of these calculi we will obtain a C^*-algebra which has the Poisson algebra of symbols as a dense *-subalgebra. The families of operators will be the result of restricting a certain family of representations of the C^*-algebra to the Poisson algebra.

The first calculus is meant to be a generalization of the magnetic Weyl calculus. Let us describe its setting and interpretation briefly.

Let θ be a jointly continuous action of $\mathscr{X}$ on a Hausdorff locally compact space Ω. We start by showing that Ω can be regarded as a continuous gluing of usual configuration spaces; then θ will allow us to define a notion of differentiability for functions on Ω. So Ω can be considered as a new kind of (global) configuration space. This will lead us to define naturally a phase space for our setting and the symbols for our first calculus.

Recall that the orbit $\mathcal{O}_\omega$ generated by a point $\omega \in \Omega$ is homeomorphic to the quotient of $\mathscr{X}$ by the closed stability subgroup $\mathscr{X}_\omega := \{x \in \mathscr{X} \mid \theta_x(\omega) = \omega\}$. It is well known that every closed subgroup of $\mathscr{X}$ is basically of the form $\mathbb{R}^k \times L \times \{0\}^m$, where L is a lattice of dimension l, and $k + l + m = n$. Then each orbit is homeomorphic to a simple configuration space (a product of an euclidean space, some tori and points).

Although Ω is just a locally compact space, the given action allows us to define a notion of differentiability. To motivate this definition we appeal to an essential idea of our construction. For a function $\varphi : \Omega \to S$ and a point $\omega \in \Omega$, we define $\varphi_\omega : \mathscr{X} \to S$ by $\varphi_\omega(x) := \varphi(\theta_x(\omega))$. Note that if ω and ω' are in the same orbit then φ_ω and $\varphi_{\omega'}$ differ just by a translation and φ_ω is well defined over $\mathscr{X}/\mathscr{X}_\omega$. Moreover, if φ is continuous, then each φ_ω is continuous.

We could declare that $\varphi \in C^\infty(\Omega)$ if $\varphi_\omega \in C^\infty(\mathscr{X})$ for each $\omega \in \Omega$. However, this first attempt does not consider the global topology of Ω. The following definition can be interpreted as a way to verify uniformly that each φ_ω is C^∞. We define the space of bounded smooth vectors by

$$BC^\infty(\Omega) := \{\varphi \in BC(\Omega) \mid \mathscr{X} \ni x \to \theta_x(\varphi) \in BC(\Omega) \text{ is } C^\infty\},$$

where $BC(\Omega)$ is the C^*-algebra of complex bounded continuous functions over Ω, $\theta_x(\varphi)$ is defined by $\theta_x(\varphi)(\omega) := \varphi(\theta_x(\omega))$ and C^∞ is meant in norm-sense. For example, if we consider $\Omega := \mathscr{X}$ and θ the action by translation, the resulting space is the usual space $BC^\infty(\mathscr{X})$ of infinitely differentiable functions which together with all its partial derivatives are bounded. Clearly, if $\varphi \in BC^\infty(\Omega)$ then $\varphi_\omega \in BC^\infty(\mathscr{X})$, for each $\omega \in \Omega$.

Since Ω is meant to be a global configuration space, the natural phase space for our setting is $\Omega \times \mathscr{X}^*$. The action $\theta \otimes \tau^\star$ of $\mathscr{X} \times \mathscr{X}^* = \Xi$ on $\Omega \times \mathscr{X}^*$ given by

$$(\theta \otimes \tau^\star)\left((x,\eta),(\omega,\xi)\right) := (\theta_x(\omega), \eta + \xi) \tag{1.2.1}$$

allows us to define analogously a new space of bounded smooth vectors $BC^\infty(\Omega \times \mathscr{X}^*)$.

The last ingredient is the appropriate definition of a magnetic field for our setting. We call a magnetic field on Ω a continuous function $B : \Omega \to \bigwedge^2 \mathscr{X}$ such that B_ω is a usual magnetic field for any ω. Inspired by the usual magnetic case, we define

$$\{f,g\}_B := \sum_{j=1}^{n} \left(\partial_{\xi_j} f\, \delta_j g - \delta_j f\, \partial_{\xi_j} g\right) + \sum_{j,k} B^{jk}\, \partial_{\xi_j} f\, \partial_{\xi_k} g, \tag{1.2.2}$$

where $f, g \in BC^\infty(\Omega \times \mathscr{X}^*)$ and

$$\delta_i : BC^\infty(\Omega) \to BC^\infty(\Omega), \qquad \delta_i \varphi := \frac{\partial \theta_x(\varphi)}{\partial_{x_i}}|_{x=0}. \tag{1.2.3}$$

Clearly $BC^\infty(\Omega \times \mathscr{X}^*)$ together the pointwise product and $\{\cdot,\cdot\}_B$ forms a Poisson algebra. This will be the Poisson algebra of symbols for the general magnetic calculus.

For comparison, let us leave for a while the first calculus and start to describe the second one.

The second calculus is based on M. Rieffel's article [43]. In this article a quantization in the sense of strict deformation was defined (see [24], [43] or [44] for details and motivation of this notion). On the other hand, in [29] it was showed how to obtain Schrödinger type representations for Rieffel's quantization; we call all the formalism Rieffel's pseudodifferential calculus.

The initial data for Rieffel's calculus is a quadruplet (Σ, Θ, Ξ, J), where Σ is a Hausdorff locally compact topological space, Θ is a jointly continuous action of Ξ on Σ and J is a $2n \times 2n$ skew-symmetric matrix. As before, each orbit is homeomorphic to a product of an euclidean space, some tori and points; we interpret each $\mathcal{O}_\sigma$ as a standard phase space. Moreover, using the action Θ as before, we define the space of bounded smooth vectors $BC^\infty(\Sigma)$; this is the space of symbols for Rieffel's calculus. So again we can interpret Σ as a gluing of standard phase spaces and also as a global phase space. This point of view is very fruitful; it allows us to consider global phase spaces which are not the cotangent bundle of a global configuration space. This could seem to be just heuristical, but there are some very interesting examples (chapter 12 at [43]) where the global phase space Σ is actually a Poisson manifold which is not symplectic (in particular, it isn't a cotangent bundle of a manifold) and the standard phase space components $\mathcal{O}_\sigma$ are its symplectic leafs.

As before for a function $\varphi : \Sigma \to S$ and a point $\sigma \in \Sigma$, we define $\varphi_\sigma : \Xi \to S$ by $\varphi_\sigma(X) := \varphi(\Theta_X(\sigma))$. We also define the derivations $\{\delta_j\}_{j=1}^{2n}$ replacing in (1.2.3) Ω by Σ, θ by Θ and $x \in \mathscr{X}$ by $X \in \Xi$.

Note that although the phase space considered before $\Omega \times \mathscr{X}^*$ is a particular case, we didn't consider in the present general case magnetic fields.

We can also define a Poisson bracket on $BC^\infty(\Sigma)$:

$$\{f,g\}_J := \sum_{j,k} J_{jk} \delta_j f\, \delta_k g. \tag{1.2.4}$$

We will usually consider the standard skew-symmetric matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$; in this case the bracket is given by

$$\{f, g\} := \sum_{j=1}^{n} \left(\delta_{n+j} f \, \delta_j g - \delta_j f \, \delta_{n+j} g\right).$$

Note also that the skew-symmetric bilinear form

$$[[X, Y]]_J = JY \cdot X \tag{1.2.5}$$

transform Ξ into a symplectic vector space (so into a symplectic manifold with a constant symplectic form), which are the most standard objets used to describe classical mechanics. Choosing the standard symplectic matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, we obtain the standard symplectic form which we simple denote by $[[\cdot \cdot]]$.

$\{\cdot, \cdot\}_J$ can be obtained formally by

$$\{f, g\}_J := [[\nabla f, \nabla g]]_J$$

where $\nabla f := (\delta_1 f, \delta_2 f, ..., \delta_{2n} f)$.

Finally, the calculi are respectively given by

$$H_\omega^B(f) := \mathfrak{Op}^{B_\omega}(f_{(\omega,0)}) \text{ y } H_\sigma(f) := \mathfrak{Op}(f_\sigma), \tag{1.2.6}$$

where f belongs to $BC^\infty(\Omega \times \mathscr{X}^*)$ in the first case, and to $BC^\infty(\Sigma)$ in the second case.

Note that, since each $f_{(\omega,0)}$ and f_σ belong to $BC^\infty(\Xi)$, the above expressions have sense and they define families of bounded standard pseudodifferential operators, thanks to the Calderon-Vaillancourt Theorem ([15]) and his magnetic version ([20]). Note also that each symbol $f_{(\omega,0)}$ or f_σ could be considered as a symbol on the corresponding standard phase space. For example, if the orbit $\mathcal{O}_\sigma$ of σ is homeomorphic to $\{0\} \times \mathbb{T} \times \mathbb{R}^{2n-2}$ (recall that each orbit is homeomorphic to the product of points, tori an euclidean spaces) then f_σ will be a symbol on Ξ independent of the first variable and 2π-periodic in the second one, or it could just be considered as a symbol on $\{0\} \times \mathbb{T} \times \mathbb{R}^{2n-2}$.

In [29] it was proved that if σ and σ' belong to the same orbit then $H_\sigma(f)$ and $H_{\sigma'}(f)$ are unitary equivalent for each $f \in BC^\infty(\Sigma)$. If $\sigma' = \Theta_Z(\sigma)$ then the unitary equivalence is implemented by the unitary operator $\mathfrak{Op}(\mathfrak{e}_Z)$, where $\mathfrak{e}_Z(X) = e^{-i[\![X,Z]\!]}$. The corresponding result for the general magnetic calculus will be proved in the second chapter (2.3.9). This properties motivate us to think that these calculi could give a convenient setting to use pseudodifferential techniques in some continuous models of random operators; obviously, for this, we need to endow Ω (respectively Σ) with a measure invariant by θ (respectively Θ).

In all the formalism above we can introduce dependence of a real parameter $\hbar > 0$ interpreted as Planck's constant. For the Weyl calculus this dependence is given by

$$[\mathfrak{Op}_\hbar(f)u](x) := (2\pi\hbar)^{-n} \int_{\mathscr{X}} \int_{\mathscr{X}^*} f\left(\frac{x+y}{2}, \xi\right) e^{i/\hbar(x-y)\cdot\xi} u(y) \, d\xi dy,$$

and it is transmitted to Rieffel's pseudodifferential operators through (1.2.6). The dependence for the general magnetic Weyl calculus will be given explicitly in chapter 2 (see (2.3.7)).

1.3 C^*-algebraic techniques

The development and study of the calculi described above have in common the use of C^*-algebraic techniques. This is motivated principally by the relation already present for the Weyl theory with certain C^*-algebras. The core of this relationship comes from the following well known fact: Given two suitable symbols f and g (for example $f, g \in BC^\infty(\Xi)$), there is a symbol h such that $\mathfrak{Op}(h) = \mathfrak{Op}(f)\mathfrak{Op}(g)$. The corresponding h is denoted by $f\#g$, and $\#$ is called the Moyal product. It is given by

$$(f \# g)(x,\xi) := \pi^{-2n} \int_\Xi \int_\Xi e^{2i[\![Y,Z]\!]} f((x,\xi)+Y)\, g((x,\xi)+Z)\, \mathrm{d}Y\mathrm{d}Z,$$

where the above integral is usually defined by oscillatory integral techniques.

There are certain sets of symbols which together with the pointwise sum, the Moyal product, and pointwise complex conjugation form a $*$-algebra, so one can ask for a natural C^*-norm. For example, since $\mathfrak{Op}$ is faithful and $\mathfrak{Op}(\bar{f}) = \mathfrak{Op}(f)^*$, over those *-algebra whose image by $\mathfrak{Op}$ consists of bounded operators, we can define $||f|| := ||\mathfrak{Op}(f)||_{\mathbb{B}(L^2(\mathscr{X}))}$. The Schwartz class $\mathcal{S}(\Xi)$ and $BC^\infty(\Xi)$ are important examples of such *-algebras; note that, with the usual Poisson bracket each of these spaces becomes also a Poisson subalgebra of $C^\infty(\Xi)$.

If we consider the $\hbar$-dependent Weyl calculus $\mathfrak{Op}_\hbar$, we can define $\#_\hbar$ by imposing $\mathfrak{Op}_\hbar(f\#_\hbar g)$ $= \mathfrak{Op}_\hbar(f)\mathfrak{Op}_\hbar(g)$. For $f \in BC^\infty(\Xi)$, we can also define $||f||_\hbar := ||\mathfrak{Op}_\hbar(f)||_{\mathbb{B}(L^2(\mathscr{X}))}$.

It is well known that if f and g belong to $BC^\infty(\Xi)$, then

$$f\#_\hbar g = fg + \frac{i\hbar}{2}\{f,g\} + \hbar^2 R_\hbar(f,g) \tag{1.3.1}$$

and $||R_\hbar(f,g)||_\hbar \le C$ uniformly in $\hbar$. In particular

$$\lim_{\hbar\to 0} ||\mathfrak{Op}_\hbar(fg) - \left(\frac{\mathfrak{Op}_\hbar(f)\mathfrak{Op}_\hbar(g) + \mathfrak{Op}_\hbar(g)\mathfrak{Op}_\hbar(f)}{2}\right)||_{\mathbb{B}(L^2(\mathscr{X}))} = 0 \tag{1.3.2}$$

and

$$\lim_{\hbar\to 0} ||\mathfrak{Op}_\hbar(\{f,g\}) - \frac{1}{i\hbar}[\mathfrak{Op}_\hbar(f), \mathfrak{Op}_\hbar(g)]||_{\mathcal{B}(L^2(\mathscr{X}))} = 0, \tag{1.3.3}$$

where $[\cdot,\cdot]$ is the usual commutator of $\mathbb{B}\left(L^2(\mathscr{X})\right)$.

For the magnetic Weyl calculus the situation is similar. First a $\hbar$-dependent magnetic Moyal product was defined by imposing $\mathfrak{Op}_\hbar^A(f\#_\hbar^B g) = \mathfrak{Op}_\hbar^A(f)\mathfrak{Op}_\hbar^A(g)$; remarkably the $\hbar$-dependent magnetic Moyal product does not depend of the vector potential A. From theorem 2.11. at [27] it is easy to obtain the analogue of (1.3.1) for the magnetic Moyal product, with the usual bracket $\{\cdot,\cdot\}$ replaced by the magnetic bracket $\{\cdot,\cdot\}_B$ given in (1.1.1).

One of the purposes of this thesis is to define, for the general magnetic calculus, a corresponding Moyal product, a suitable $\hbar$-dependent norm and to prove the analogue of (1.3.1). For Rieffel's pseudodifferential calculus this was already settled in [43]. From these results we will also obtain an analogue of (1.3.3) and (1.3.2) for both calculi.

1.3.1 Twisted crossed products and the general magnetic calculus

For simplicity, we initially define a general magnetic Moyal product on a space which can be regarded as a kind of Schwartz class. Later, this product will be extended to a much more general class of symbols. Let $\mathcal{S}(\mathscr{X}^*; C_0^\infty(\Omega))$ denote the space of elements of $BC^\infty(\Omega \times \mathscr{X}^*)$ such that their partial derivatives of any order evaluated on any point of $\mathscr{X}^*$ belongs to $C_0(\Omega)$ and its product with any polynomial on $\mathscr{X}^*$ is uniformly bounded on Ω (see (2.2.8) for the full definition). For $f, g \in \mathcal{S}(\mathscr{X}^*; C_0^\infty(\Omega))$ define

$$(f\sharp^B g)(\omega, \xi) = \pi^{-2n} \int_{\mathscr{X}} \int_{\mathscr{X}^*} \int_{\mathscr{X}} \int_{\mathscr{X}^*} e^{2i[\![(y,\eta),(z,\zeta)]\!]}\, e^{-i\Gamma^{B_\omega}<y-z,y+z,z-y>} \cdot$$
$$\cdot f\big(\theta_y[\omega], \xi + \eta\big)\, g\big(\theta_z[\omega], \xi + \zeta\big)\, \mathrm{d}\zeta\, \mathrm{d}z\, \mathrm{d}\eta\, \mathrm{d}y.$$

Note that $(f\#^B g)_{(\omega,0)} = f_{(\omega,0)} \#^{B_\omega} g_{(\omega,0)}$, where in the right hand side we consider the standard magnetic Moyal product and in the left hand side the new one. Therefore

$$H^B_\omega(f\#^B g) = H^B_\omega(f) H^B_\omega(g).$$

The above definition of the general magnetic Moyal product is motivated by the C^*-algebraic techniques that were used in the study and development of the magnetic Weyl calculus. To explain this point, let us consider the Banach space $L^1(\mathscr{X}, C_0(\Omega))$ of Bochner integrable functions on $\mathscr{X}$ with values in $C_0(\Omega)$. Let us also define

$$1 \otimes \mathcal{F} : L^1(\mathscr{X}, C_0(\Omega)) \to C_0(\mathscr{X}^*, C_0(\Omega)), \quad 1 \otimes \mathcal{F}^{-1} : L^1(\mathscr{X}^*, C_0(\Omega)) \to C_0(\mathscr{X}, C_0(\Omega))$$

by

$$[(1 \otimes \mathcal{F})\Phi](\zeta) = \int_{\mathscr{X}} \Phi(x) e^{-i\xi\cdot x}\, \mathrm{d}x, \quad [(1 \otimes \mathcal{F}^{-1}) f](x) = \int_{\mathscr{X}^*} f(\zeta) e^{i\xi\cdot x}\, \mathrm{d}\zeta.$$

It is easy to check, as in the standard case, that $\mathcal{S}(\mathscr{X}; C_0^\infty(\Omega)) \subset L^1(\mathscr{X}, C_0(\Omega))$ and that $1 \otimes \mathcal{F}$ restricts to an isomorphism between $\mathcal{S}(\mathscr{X}; C_0^\infty(\Omega))$ and $\mathcal{S}(\mathscr{X}^*; C_0^\infty(\Omega))$, with the corresponding restriction of $1 \otimes \mathcal{F}^{-1}$ as its inverse.

For $\Phi, \Psi \in \mathcal{S}(\mathscr{X}; C_0^\infty(\Omega))$ we define

$$\Phi \diamond^B \Psi := (1 \otimes \mathcal{F}^{-1}) \left[(1 \otimes \mathcal{F})\Phi \,\#^B\, (1 \otimes \mathcal{F})\Psi\right].$$

Then

$$(\Phi \diamond^B \Psi)(\omega, x) := \int_{\mathscr{X}} \left[\Phi(\theta_{\frac{y-x}{2}}(\omega), y)\right] \left[\Psi(\theta_{\frac{y}{2}}(\omega), x - y)\right] \left[\kappa(\theta_{-\frac{x}{2}}(\omega); y, x - y)\right]\, \mathrm{d}y,$$

where

$$\kappa^B(\omega; x, y) = e^{-i\Gamma^{B_\omega}\langle 0, x, x+y\rangle}, \quad \forall\, x, y \in \mathscr{X},\ \omega \in \Omega.$$

We can recover the above integral by evaluating in ω the function given by the following Bochner integral:

$$(\Phi \diamond^B \Psi)(x) := \int_{\mathscr{X}} \theta_{\frac{y-x}{2}}[\Phi(y)]\, \theta_{\frac{y}{2}}[\Psi(x - y)]\, \theta_{-\frac{x}{2}}\left[\kappa^B(y, x - y)\right]\, \mathrm{d}y.$$

The above product belongs to the theory of twisted crossed product C^*-algebras. In general, the definition of such C^*-algebras has as ingredients an initial C^*-algebra $\mathcal{A}$, a locally compact group G, a strongly continuous action θ of G on $\mathcal{A}$ and a so called normalized 2-cocycle for the triple $(\mathcal{A}, G, \theta)$.

When $\mathcal{A} = C_0(\Omega)$, a normalized 2-cocycle of $(C_0(\Omega), G, \theta)$ is a continuous map $\kappa : G \times G \to C(\Omega, \mathbb{T})$ satisfying for all $x, y, z \in G$:

$$\kappa(x+y, z)\, \kappa(x, y) = \theta_x[\kappa(y, z)]\, \kappa(x, y+z)$$

and $\kappa(x, 0) = \kappa(0, x) = 1$, where $C(\Omega, \mathbb{T})$ is the algebra of continuous functions on Ω with values in the circle endowed with the open-compact topology. This is the definition of a 2-cocycle of G with coefficients in $C(\Omega, \mathbb{T})$ given in group cohomology theory. $(C_0(\Omega), G, \theta, \kappa)$ is called a twisted C^*-dynamical system (if the 2-cocycle is trivial, $(C_0(\Omega), G, \theta)$ is just a C^*-dynamical system).

Since the setting for the general magnetic calculus requires an action of the vector group $\mathscr{X}$, we will give the definition of a twisted crossed product just for $G = \mathscr{X}$ (although the definition below also work for G abelian).

If a twisted C^*-dynamical system $(C_0(\Omega), \mathscr{X}, \theta, \kappa)$ is given, then we can endow $L^1(\mathscr{X}; C_0(\Omega))$ with a product $\diamond^\kappa$, given by (1.3.1), and with the involution $\star$ given by

$$\Phi^{\diamond^\kappa}(x) = \overline{\Phi(-x)}.$$

Then $(L^1(\mathscr{X}; C_0(\Omega)), \diamond^\kappa, {}^{\diamond^\kappa}, \|\cdot\|_1)$ is a *-Banach algebra but it is not a C^*-algebra. However there is a C^*-norm $\|\cdot\|$ on $L^1(\mathscr{X}; \mathcal{A})$ (see [39] and [40] for the pioneering works and [34] for a treatment related to magnetics fields). The completion with respect to this norm is called the twisted crossed product C^*-algebra associated to $(\mathcal{A}, \mathscr{X}, \theta, \kappa)$, and is denoted by $\mathcal{A} \rtimes_\theta^\kappa \mathscr{X}$. If κ is trivial, then the resulting C^*-algebra is just called a crossed product C^*-algebra.

We will prove later that κ^B define a 2-cocycle for $(C_0(\Omega), \mathscr{X}, \theta)$; for this Stokes' theorem will be the main tool. For simplicity, we will denote the emerging C^*-algebra $C_0(\Omega) \rtimes_\theta^{\kappa^B} \mathscr{X}$ by $\mathfrak{C}^B$. We will also denote by $\mathfrak{B}^B$ the completion of $(1 \otimes \mathcal{F})[L^1(\mathscr{X}, C_0(\Omega))]$ with respect to the norm $||1 \otimes \mathcal{F}(\Phi)||_{\mathfrak{B}^B} = ||\Phi||_{\mathfrak{C}^B}$. Since $L^1(\mathscr{X}, C_0(\Omega))$ is dense in $\mathfrak{C}^B$, the partial Fourier transform can be extended to an isomorphism between $\mathfrak{C}^B$ and $\mathfrak{B}^B$. By definition, $\mathfrak{B}^B$ has as product an extension of the general magnetic Moyal product.

The whole description given above was already settled for the usual magnetic Moyal product and this is the main motivation for our present treatment; the principal reference is [34]. Let us explain briefly which is the corresponding setting in this case. Initially, it was considered the action τ of $\mathscr{X}$ on itself by translation; this gives an action of $\mathscr{X}$ in $C_0(\mathscr{X})$, as in our general case. However, this action makes sense in any C^*-algebra of bounded, uniformly continuous functions on $\mathscr{X}$ and stable by translations; we called such C^*-algebra a standard $\mathscr{X}$-algebra. It follows from Gelfand's theory that for any standard $\mathscr{X}$-algebra $\mathcal{A}$ there is a locally compact space $S_\mathcal{A}$ such that $\mathcal{A} \cong C_0(S_\mathcal{A})$ and τ also comes from an action of $\mathscr{X}$ on $S_\mathcal{A}$. If B is an usual continuous magnetic field, then κ^B defined by

$$[\kappa^B(x, y)](z) = e^{-i\Gamma^B\langle z, x+z, x+y+z\rangle}, \quad \forall x, y, z \in \mathscr{X},$$

is a 2-cocycle for $(C_0(\mathscr{X}), \mathscr{X}, \tau)$. It was proved in [34] that if we suppose that $C_0(\mathscr{X}) \subseteq \mathcal{A}$, then $\mathscr{X} \subseteq S_{\mathcal{A}}$ and if in addition the components of $B = (B_{ij})$ are extendible from $\mathscr{X}$ to a continuous function on $S_{\mathcal{A}}$ then $[\kappa^B(x,y)]$ can also be defined on $S_{\mathcal{A}}$ and κ^B becomes a 2 cocycle of $(\mathcal{A}, \mathscr{X}, \tau)$.

From the cohomological point of view, what we have been showing in the last paragraph is a way to associate to each closed 2-form of $\mathscr{X}$ a 2-cocycle of $\mathscr{X}$ with coefficients in $C(\mathscr{X}, \mathbb{T})$. Moreover, in [34] it was shown that this is also true for 1-forms and 1-cocycles, and this association respect the respective coboundary maps, i.e. if A is a vector potential and we denote by λ^A the corresponding 1-cocycle and by ρ the coboundary map at the level of group cohomology, then $\kappa^{\mathrm{d}A} = \rho(\lambda^A)$.

We also must recall that the standard magnetic Moyal product was obtained by imposing the equality $\mathfrak{Op}^B(f \#^B g) = \mathfrak{Op}^B(f)\mathfrak{Op}^B(g)$ (as in the nonmagnetic case). In the other hand, the definition of our general magnetic Moyal product is motivated by the standard magnetic Moyal product and the C^*-algebraic techniques described above, but we did not figure it out from some faithful representation of our space of symbols as operators in some Hilbert space (as in the standard case); mainly because there isn't a Hilbert space naturally associated to the setting. However, if Ω is endowed with a θ-invariant measure μ, then we will give a faithful representation $\mathfrak{OP}^B$ of $\mathfrak{B}^B$ on $\mathbf{B}\big[L^2(\mathscr{X};\mathcal{H}')\big]$, where $\mathcal{H}' := L^2(\Omega,\mu)$; moreover we will also show that for each $f \in \mathfrak{B}^B$, $\mathfrak{OP}^B(f) = \int_\Omega^\oplus H_\omega^B(f)\mathrm{d}\mu(\omega)$ (see the last paragraphs of the subsection (2.3.3)).

Let us come back to the description of twisted crossed product C^*-algebras. One of the most important property of these C^*-algebras is that there is a one to one correspondence between their non-degenerate representations and certain triples $(\mathscr{H}, r, T)$ called covariant representations. More precisely, if $(C_0(\Omega), \mathscr{X}, \theta, \kappa)$ is a twisted $*$-dynamical system, we call covariant representation $(\mathcal{H}, r, T)$ a Hilbert space $\mathcal{H}$ together with two maps $r : C_0(\Omega) \to \mathbf{B}(\mathcal{H})$ and $T : \mathscr{X} \to \mathcal{U}(\mathcal{H})$ ($\mathbf{B}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ denote the C^*-algebra of bounded operator on $\mathcal{H}$ and the group of unitary operators on $\mathcal{H}$, respectively) satisfying

(i) r is a non-degenerate representation,

(ii) T is strongly continuous and $T(x)T(y) = r[\kappa(x,y)]T(x+y), \quad \forall x, y \in \mathscr{X}$,

(iii) $T(x)r(\varphi)T(x)^* = r[\theta_x(\varphi)], \quad \forall x \in \mathscr{X},\ \varphi \in \mathcal{A}$.

If $(\mathcal{H}, r, T)$ is a covariant representation of $(C_0(\Omega), \mathscr{X}, \theta)$, then $\mathfrak{Rep}_r^T$ defined on $L^1(\mathscr{X}; C_0(\Omega))$ by

$$\mathfrak{Rep}_r^T(\Phi) := \int_{\mathscr{X}} r\left[\theta_{y/2}(\Phi(y))\right] T(y)\, \mathrm{d}y$$

extends to a representation of $C_0(\Omega) \rtimes_\theta^\kappa \mathscr{X}$.

In our setting an important example is the following: let $\omega \in \Omega$, $\mathcal{H} = L^2(\mathscr{X})$ and r_ω be the representation of $C_0(\Omega)$ in $\mathbf{B}(\mathcal{H})$ given for any $\varphi \in C_0(\Omega)$, $u \in \mathcal{H}$ and $x \in \mathscr{X}$ by

$$[r_\omega(\varphi)u](x) = [\theta_x(\varphi)](\omega)\ u(x) \equiv \varphi\big(\theta_x[\omega]\big)\ u(x).$$

Let also T_ω be the map from $\mathscr{X}$ into the set of unitary operators on $\mathcal{H}$ given by

$$[T_\omega^B(y)u](x) := \kappa^B(\omega; x, y)\ u(x+y) = e^{-i\Gamma^{B_\omega}\langle 0,x,x+y\rangle}u(x+y).$$

The general magnetic calculus can be lifted from the family of representations $\mathfrak{Rep}_{r_\omega}^{T_\omega^B}$ through the partial Fourier transform defined at the beginning of this section. In other words we have that

$$H_\omega^B([1 \otimes \mathcal{F}](\Phi)) = \mathfrak{Rep}_{r_\omega}^{T_\omega^B}(\Phi) =: \mathfrak{Rep}_\omega^B(\Phi),\ \forall \omega \in \Omega\,, \Phi \in \mathfrak{C}^B.$$

Now we want to introduce $\hbar$-dependence in our formalism. It is easy to check that $(C_0(\Omega), \mathscr{X}, \theta^\hbar, \kappa^{B,\hbar})$, where

$$\theta_x^\hbar := \theta_{\hbar x} \text{ and } \kappa^{B,\hbar}(x,y) = \kappa^{\frac{B}{\hbar}}(\hbar x, \hbar y)$$

also form a twisted C^*-dynamical system. So we can consider the twisted crossed product C^*-algebras $C_0(\Omega) \rtimes_{\theta^\hbar}^{\kappa^{B,\hbar}} \mathscr{X}$ which will be denoted simply by $\mathfrak{C}_\hbar^B$. The algebra of symbols isomorphic to $\mathfrak{C}_\hbar^B$ via the partial Fourier transform will be denoted by $\mathfrak{B}_\hbar^B$; its product is the $\hbar$-dependent magnetic Moyal product for our calculus.

Again, the $\hbar$-dependent general magnetic calculus is lifted from a family of representations of $\mathfrak{C}_\hbar^B$ through the partial Fourier transform; the required covariant representations is obtained by replacing T_ω^B by $T_\omega^{B;\hbar}$, which is given by

$$T_\omega^{B,\hbar}(y) := T_\omega^B(\hbar y).$$

Let us come back to our space of symbols $\mathcal{S}\big(\mathscr{X}^*; C_0^\infty(\Omega)\big)$. It is easy to check that $\mathcal{S}\big(\mathscr{X}^*; C_0^\infty(\Omega)\big)$ is a Poisson subalgebra of $BC^\infty(\Omega \times \mathscr{X}^*)$. We will prove that it is a dense $*$-subalgebra of $\mathfrak{B}_\hbar^B$ for each $\hbar \in \mathbb{R}$ and get an analogue of (1.3.1) for it. Indeed we will prove that, for each $f, g \in \mathcal{S}\big(\mathscr{X}^*; C_0^\infty(\Omega)\big)$ we have

$$f \#_\hbar^B g = fg + \frac{i\hbar}{2}\{f,g\}_B + \hbar^2 R_\hbar(f,g), \tag{1.3.4}$$

where $\{\cdot,\cdot\}_B$ is our general magnetic Poisson bracket given by (1.2.2), each term belongs to $\mathcal{S}\big(\mathscr{X}^*; C_0^\infty(\Omega)\big)$ and $||R_\hbar(f,g)||_{\mathfrak{B}_\hbar^B}$ is uniformly bounded in $\hbar$.

To prove this results we will transport both the Poisson bracket and the pointwise product from
$\mathcal{S}\big(\mathscr{X}^*; C_0^\infty(\Omega)\big)$ to $\mathcal{S}\big(\mathscr{X}; C_0^\infty(\Omega)\big)$ through the inverse partial Fourier transform; then we will obtain the correspondent result for $\mathcal{S}\big(\mathscr{X}; C_0^\infty(\Omega)\big)$ by using some properties of twisted crossed products C^*-algebras, and finally we will come to back to $\mathcal{S}\big(\mathscr{X}^*; C_0^\infty(\Omega)\big)$ now through the partial Fourier transform.

Since for each $\omega \in \Omega$ $H_{\omega,\hbar}^B$ is a representation of $\mathfrak{B}_\hbar^B$ we have that

$$||H_{\omega,\hbar}^B(fg) - \frac{H_{\omega,\hbar}^B(f)H_{\omega,\hbar}^B(g) + H_{\omega,\hbar}^B(g)H_{\omega,\hbar}^B(f)}{2}|| \le ||fg - \left(\frac{f\#_\hbar^B g + g\#_\hbar^B f}{2}\right)||_{\mathfrak{B}_\hbar^B} \underset{\hbar\to 0}{\rightarrow} 0 \tag{1.3.5}$$

and

$$||H_{\omega,\hbar}^B(\{f,g\}) - \frac{1}{i\hbar}[H_{\omega,\hbar}^B(f), H_{\omega,\hbar}^B(g)]|| \le ||\{f,g\} - \frac{1}{i\hbar}[f,g]_{\mathfrak{B}_\hbar^B}||_{\mathfrak{B}_\hbar^B} \underset{\hbar\to 0}{\rightarrow} 0, \tag{1.3.6}$$

where $[\cdot,\cdot]_{\mathfrak{B}_\hbar^B}$ is the commutator in $\mathfrak{B}_\hbar^B$.

We will also prove that, for each $f \in \mathcal{S}\big(\mathscr{X}^*; C_0^\infty(\Omega)\big)$, the map

$$[0,1] \ni \hbar \to ||f||_{\mathfrak{B}_\hbar^B} \in [0,\infty) \quad \text{is continuous.} \tag{1.3.7}$$

This result will be a direct consequence of some strong property of twisted crossed product C^*-algebras. We will discuss this point in the next section.

Putting together the right side of (1.3.6), the right side of (1.3.5), (1.3.7) and the fact that $\mathcal{S}\big(\mathscr{X}^*; C_0^\infty(\Omega)\big)$ is a dense *-subalgebra of $\mathfrak{B}_\hbar^B$ for each $\hbar \in \mathbb{R}$, we get that the family of inclusions

$$\left\{\mathcal{S}\big(\mathscr{X}^*, C_0^\infty(\Omega)\big)_{\mathbb{R}} \hookrightarrow \mathfrak{B}_{\hbar,\mathbb{R}}^B\right\}_{\hbar\in[0,1]}$$

($\mathcal{S}\big(\mathscr{X}^*, C_0^\infty(\Omega)\big)_{\mathbb{R}}$ and $\mathfrak{B}_{\hbar,\mathbb{R}}^B$ are the real valued elements of $\mathcal{S}\big(\mathscr{X}^*, C_0^\infty(\Omega)\big)$ and the self adjoint part of $\mathfrak{B}_\hbar^B$, respectively) is what Marc Rieffel called a strict deformation quantization (see [24], [43] or [44] for details and motivation or 2.5.1 for the explicit definition inside this thesis). For us, this is the appropriate way to condense the basic properties that a quantization procedure must satisfy.

1.3.2 Rieffel's deformation quantization and covariant fields of C^*-algebras.

Recall that for Rieffel's calculus we have as initial data a quadruplet (Σ, Θ, Ξ, J), where Σ is a Hausdorff locally compact topological space, Θ is a jointly continuous action of Ξ on Σ and J is a $2n \times 2n$ skew-symmetric matrix. Recall also that we defined $BC^\infty(\Sigma)$ using Θ and we endowed it with the Poisson bracket $\{\cdot,\cdot\}_J$ given by (1.2.4), which is related to the skew-symmetric bilinear form $[\![\cdot,\cdot]\!]_J$ given by (1.2.5). As we explained before, it is meant to cover global phase spaces which are not necessarily of the form $\Omega \times \mathscr{X}^*$. So, we cannot use the same C^*-algebraic techniques as before. However, if we look more closely the definition of the usual Moyal product, we can note that $(f\#g)(x,\xi)$ can be recovered from evaluating at (x,ξ) the following Bochner integral

$$\pi^{-2n}\int_\Xi\int_\Xi e^{2i[\![Y,Z]\!]}\,(\tau\otimes\tau^\star)_Y(f)\,(\tau\otimes\tau^\star)_Z(g)\,\mathrm{d}Y\mathrm{d}Z,$$

where $[(\tau\otimes\tau^\star)_{(y,\eta)}(f)](x,\xi) := f(x+y,\xi+\eta)$. Motivated by this expression, for each $f,g \in BC^\infty(\Sigma)$ we can define

$$f\#g = \pi^{-2n}\int_\Xi\int_\Xi e^{2i[\![Y,Z]\!]}\,\Theta_Y(f)\,\Theta_Z(g)\,\mathrm{d}Y\mathrm{d}Z, \tag{1.3.8}$$

where the above integral is defined by oscillatory integral techniques. This is the starting point of [43] (up to a change of variable). More precisely, in [43] it was show how from a C^*-dynamical system of the form $(\mathcal{A},\Theta,\Xi)$ and a $2n\times 2n$ skew-symmetric matrix J a new C^*-algebra $\mathfrak{A}_J$ can be constructed, which can be regarded as a deformation of the initial C^*-algebra. We will call this procedure Rieffel deformation quantization and the emerging C^*-algebra $\mathfrak{A}_J$ the Rieffel deformed C^*-algebra.

For the construction of this C^*-algebra, let us consider first the space of smooth vectors

$$\mathcal{A}^\infty = \{f \in \mathcal{A} \,|\, \Xi \ni X \to \Theta_X(f) \in \mathcal{A} \text{ is } C^\infty\}.$$

It is well known that $\mathcal{A}^\infty$ is a dense *-subalgebra of $\mathcal{A}$. We endow $\mathcal{A}^\infty$ with the same involution of $\mathcal{A}$ and with the product

$$f \#_J g = \pi^{-2n} \int_\Xi \int_\Xi e^{2i[\![Y,Z]\!]_J}\, \Theta_Y(f)\, \Theta_Z(g)\, \mathrm{d}Y \mathrm{d}Z,$$

suitably defined by oscillatory integral techniques. We can recover (1.3.8) as a particular case if J is the standard symplectic matrix.

In [43] it was proved that the *-algebra $(\mathcal{A}^\infty, \#_J, ^*)$ admits a C^*-completion $\mathfrak{A}$ in a C^*-norm $\|\cdot\|_{\mathfrak{A}}$.

We will give a more detailed discussion about the whole construction and some of its properties in chapter 3.

Many interesting examples of Rieffel's deformed C^*-algebras were given in [43], among them we highlight the noncommutative tori, the quantum closed disk, the quantum quadrant, the Podles' quantum spheres, and the Woronowicz's quantum group $SU_\mu(2)$. It is remarkable that if $(C_0(\Omega), \mathscr{X}, \theta)$ is a C^*-dynamical system then the crossed product C^*-algebra $C_0(\Omega) \rtimes_\theta \mathscr{X}$ is isomorphic to the Rieffel deformed C^*-algebra associated to the C^*-dynamical system $(C_0(\Omega \times \mathscr{X}^*), \theta \otimes \tau^*, \Xi)$ together with the standard symplectic matrix, where $\theta \otimes \tau^*$ is given by (1.2.1).

In [43] $\hbar$-dependence was introduced by replacing the skew-symmetric matrix J by $\hbar J$, leading to Rieffel's deformed algebras $\mathfrak{A}_{\hbar J}$. We will denote by $\|\cdot\|_\hbar$ the corresponding C^*-norm.

Recall that $\mathcal{A}^\infty$ was equipped with the Poisson bracket $\{\cdot,\cdot\}_J$ given by (1.2.4). Fortunately, it was already proved in [43] an analogue of (1.3.1); explicitly it was proved that, for each $f, g \in \mathcal{A}^\infty$ we have

$$f \#_{\hbar J} g = fg + \frac{i\hbar}{2}\{f,g\}_J + \hbar^2 R_\hbar(f,g)$$

and $\|R_\hbar(f,g)\|_{\mathfrak{B}^B_\hbar}$ is uniformly bounded in $\hbar$. So, as for the $\hbar$-dependent general magnetic Moyal product, we also get the analogue of (1.3.6) and (1.3.5). In [43] it was also proved that for each $f \in \mathcal{A}^\infty$ the function

$$[0,1] \ni \hbar \to \|f\|_\hbar \in [0,\infty) \text{ is continuous.} \tag{1.3.9}$$

Thus the family of inclusions

$$\left\{\mathcal{A}^\infty \hookrightarrow \mathcal{A}_{\hbar J}\right\}_{\hbar \in [0,1]}$$

is a strict deformation quantization.

Our next purpose can be considered as a step in the study of Rieffel's deformed C^*-algebras, but it will also lead us to some spectral results concerning the operators given by Rieffel's calculus.

Recall that a (upper semi-)continuous field of C^*-algebras is a family of epimorphisms of C^*-algebras $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ indexed by the locally compact topological space T and satisfying:

1. For every $b \in \mathcal{B}$ one has $\| b \|_{\mathcal{B}} = \sup_{t\in T} \| \mathcal{P}(t)b \|_{\mathcal{B}(t)}$.

2. For every $b \in \mathcal{B}$ the map $T \ni t \mapsto \| \mathcal{P}(t)b \|_{\mathcal{B}(t)}$ is (upper semi-)continuous and decays at infinity.

3. There is a multiplication $\mathcal{C}(T) \times \mathcal{B} \ni (\varphi, b) \to \varphi * b \in \mathcal{B}$ such that

$$\mathcal{P}(t)[\varphi * b] = \varphi(t)\,\mathcal{P}(t)b, \quad \forall t \in T\,,\ \varphi \in \mathcal{C}(T)\,,\ b \in \mathcal{B}.$$

If in addition $\mathcal{B}$ is endowed with a strongly continuous action α of a locally compact group G and each $Ker\mathcal{P}(t)$ is invariant under α, then $\left\{ \mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T \right\}$ is called a covariant (upper semi-)continuous field of C^*-algebras. $\mathcal{B}$ is usually called the algebra of sections of the field.

Let $\left\{ \mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T \right\}$ be a covariant (upper semi-) continuous field of C^*-algebras. Let us denote $\mathrm{Ker}\mathcal{P}(t)$ by $\mathcal{I}(t)$. Since each ideal $\mathcal{I}(t)$ is invariant, the restriction of α to $\mathcal{I}(t)$ $\alpha_t(g) := \alpha(g)|_{\mathcal{I}(t)}$, define a strongly continuous action on $\mathcal{I}(t)$. For each $t \in T$ we can also define a strongly continuous action α^t on $\mathcal{B}(t)$ by

$$\alpha^t_g(\mathcal{P}(t)b) := \mathcal{P}(t)[\alpha_g(b)].$$

So, if $G = \mathbb{R}^{2n}$, we can consider the Rieffel deformed algebras of each member of the short exact sequence

$$0 \to \mathcal{I}(t) \to \mathcal{B} \to \mathcal{B}(t) \to 0,$$

which we denote by $\mathfrak{I}(t)$, $\mathfrak{B}$ and $\mathfrak{B}(t)$, respectively.

Fortunately, the resulting C^*-algebras also form a short exact sequence (theorem 7.7 in [43]). In other words

$$\mathfrak{B}/\mathfrak{I}(t) \cong \mathfrak{B}(t).$$

So, one can ask if the resulting family of epimorphism is a (upper semi-)continuous field of C^*-algebras. One of the principal purposes of this thesis is to prove this, i.e. we will prove that the Rieffel deformation of a covariant continuous field of C^*-algebras is a continuous field of Rieffel's deformed algebras (theorem (3.0.1)).

For twisted crossed product C^*-algebras a similar program was already settled. More precisely, if $(\mathcal{B}, G, \alpha, K)$ is a twisted C^*-dynamical system and $\left\{ \mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T \right\}$ is a covariant (upper semi-)continuous field of C^*-algebras, then we can transfer α and K to the ideals $\mathcal{I}(t)$ and the algebras $\mathcal{B}(t)$, so it also follows that the twisted crossed product of a covariant short exact sequence is a short exact sequence of twisted crossed products. Moreover, the resulting family of epimorphisms also form a (upper semi-)continuous field of C^*-algebras; in other words, twisted crossed products of a covariant (upper semi-)continuous field of C^*-algebras is a (upper semi-)continuous field of twisted crossed products C^*-algebras: see theorem 5.1 and corollary 5.3 in [37] for details and the proof of this fact. We can apply this result to prove (1.3.7). Indeed, take $\mathcal{B} := C([0,1]; C_0(\Omega)) \cong C_0([0,1] \times \Omega)$, for each $\hbar \in \mathbb{R}$ define $\mathcal{P}(\hbar) : \mathcal{B} \to C_0(\Omega)$ by $\mathcal{P}(\hbar)(\varphi) = \varphi(\hbar)$, and also define the action α on $\mathcal{B}$ and a 2-cocycle K for $(\mathcal{B}, \mathscr{X}, \alpha)$ by

$$[\alpha_x(\varphi)](\hbar) := \theta^\hbar_x[\varphi(\hbar)] \quad \text{and} \quad [K(x,y)](\hbar) := \kappa^\hbar(x,y).$$

(1.3.7) follows after considering each element of $\mathcal{S}(\mathscr{X};C_0^\infty(\Omega))$ as an $\hbar$-independent element of $L^1(\mathscr{X};\mathcal{B})$
$\subset \mathcal{B}\rtimes_\alpha^K \mathscr{X}$ and noticing that $\mathcal{B}(\hbar)\rtimes_{\alpha^\hbar}^{K^\hbar}\mathscr{X} = C_0(\Omega)\rtimes_{\theta^\hbar}^{\kappa^\hbar}\mathscr{X} =: \mathfrak{C}_\hbar^B$.

Although our theorem (3.0.1) wasn't available until now, in [43] (1.3.9) was obtained following the same above strategy. More precisely, in [43] (3.0.1) was checked for a specific covariant continuous field of C^*-algebras (similar to the one used above, at the beginning of chapter 8 in [43]), and as before, this implied (1.3.9).

We must mention that (1.3.9) is just one example of an application of our theorem (3.0.1). In fact, we will give many other examples. Most of these examples have as initial undeformed section algebra an abelian algebra $\mathcal{A}$. In this case, it is easy to characterize the notion of covariant continuous field of C^*-algebras. Recall that, by Gelfand's theory, every abelian C^*-algebra is of the form $C_0(\Sigma)$, and every strongly continuous action Θ of Ξ on $\mathcal{A} := C_0(\Sigma)$ comes from a jointly continuous action of Ξ on Σ, which we also denote by Θ, and which is given by the equation $[\Theta_X(f)](\sigma) = f(\Theta_X(\sigma))$. So we are in the setting of Rieffel's pseudodifferential calculus. Let T be a locally compact Hausdorff space, and let $q:\Sigma\to T$ be an open and continuous surjection such that $q(\Theta_X(\sigma)) = q(\sigma)$, $\forall X\in\Xi$. Consider $\Sigma_t = q^{-1}(\{t\})$, $\mathcal{A}(t) := C_0(\Sigma_t)$, and

$$\mathcal{R}(t) : C_0(\Sigma)\to C_0(\Sigma_t)\,, \qquad \mathcal{R}(t)f := f|_{\Sigma_t}\,,\ \ \forall t\in T\,.$$

Then $\left\{\mathcal{C}(\Sigma)\xrightarrow{\mathcal{R}(t)}\mathcal{C}(\Sigma_t)\mid t\in T\right\}$ is a covariant continuous field of commutative C^*-algebras, and this is the only possible way to regard $C_0(\Sigma)$ as a section algebra. So, given such $q:\Sigma\to T$, we will apply (3.0.1) to show that the family of epimorphism

$$\left\{\mathfrak{C}(\Sigma)\xrightarrow{\mathfrak{R}(t)}\mathfrak{C}(\Sigma_t)\mid t\in T\right\}\quad \text{is a continuous field of } C^*\text{-algebras,} \tag{1.3.10}$$

where $\mathfrak{C}(\Sigma)$ and $\mathfrak{C}(\Sigma_t)$ are the Rieffel quantization of $C_0(\Sigma)$ and $C_0(\Sigma_t)$, respectively, and $\mathfrak{R}(t)$ denote the epimorphism induced from $\mathcal{R}(t)$ by the Rieffel's quantization.

To explain how this results will be applied to the spectral analysis of the families of operators given by Rieffel calculus, first we need to introduce some notions. We say that a real set valued function S is inner continuous at t_0 if each real open set which meets $S(t_0)$ also meets $S(t)$ if t is close enough to t_0; so, intuitively, S is inner continuous at t_0 if $S(t)$ doesn't suddenly shrink close to t_0. The notion of outer continuity follows by replacing above "open" by "compact" and "meets" by "doesn't meet"; so, S is inner continuous at t_0 if $S(t)$ doesn't suddenly expand close to t_0. Let us also assume that Σ is compact (just for simplicity) and that each Σ_t is a quasi-orbit (the closure of a single orbit). It was proved in [29] that if $\sigma,\sigma'\in\Sigma_t$, then the spectra of $H_\sigma(f)$ and $H_{\sigma'}(f)$ coincide for every real function $f\in BC^\infty(\Sigma)$. So one can define, choosing $\sigma\in\Sigma_t$, $S(t) := \mathrm{sp}[H_\sigma(f)]$. We will show that (1.3.10) will imply both outer and inner continuity of S. Note that $S\circ q$ is also (outer and inner) continuous, so although σ and σ' leave in different quasi-orbits, the spectra of $H_\sigma(f)$ and $H_{\sigma'}(f)$ can be "compared" using the definition of outer and inner continuity. For example, a gap in the spectrum of $H_\sigma(f)$ will still be a gap in the spectrum of $H_{\sigma'}(f)$, if σ and σ' are close enough. If we apply this result to the example used by Rieffel to prove (1.3.9) (at the beginning of chapter 8 in [43]), we will get that for each $\sigma\in\Sigma$ and each real $f\in BC^\infty(\Sigma)$, $\mathrm{sp}[H_\sigma^\hbar(f)]\underset{\hbar\to 0}{\to}\mathrm{sp}[H_\sigma^0(f)] = \overline{f(\Sigma)}$ in the sense described above.

Finally we will also obtain that, fixing $\sigma \in \Sigma_t$, the set valued function $S^{\text{ess}}(t) := \text{sp}_{\text{ess}}[H_\sigma(f)]$ (the essential spectrum of the operator $H_\sigma(f)$) is outer continuous, for every real $f \in BC^\infty(\Sigma)$. So, gaps of $\text{sp}_{\text{ess}}[H_\sigma(f)]$ remains as gaps if we replace σ by σ' close enough.

The second chapter of this thesis consist of the article "Magnetic twisted actions on general abelian C^*-algebras" which is joint work with Max Lein and Marius Măntoiu. This article is mainly devoted to the study of the general magnetic calculus, the proof of (1.3.4) and to obtaining the strict deformation quantization described above. It will appear in Journal of Operator Theory.

The third chapter consists of the article "Covariant Fields of C^*-Algebras and Continuity of Spectra in Rieffel's Pseudodifferential Calculus" which is joint work with Marius Măntoiu. This article is devoted to the proof of the theorem concerning covariant continuous field of C^*-algebras described above (3.0.1), to some examples and to applying the main theorem to spectral analysis. It has been submitted for publication.

Chapter 2

Magnetic twisted actions on general abelian C^*-algebras

2.1 Introduction

The usual pseudodifferential calculus in phase space $\Xi := T^*\mathcal{R}^n$ is connected to crossed product C^*-algebras $\mathcal{A} \rtimes_\theta \mathscr{X}$ associated to the action by translations θ of the group $\mathscr{X} := \mathcal{R}^n$ on an abelian C^*-algebra $\mathcal{A}$ composed of functions defined on $\mathscr{X}$. Such a formalism has been used in the quantization of a physical system composed of a spin-less particle moving in $\mathscr{X}$, where the operators acting on $L^2(\mathscr{X})$ can be decomposed into the building block observables position and momentum which are associated to $\mathscr{X}$ and its dual $\mathscr{X}^*$. When dealing with Hamiltonian operators, the algebra $\mathcal{A}$ encapsulates properties of electric potentials, for instance.

During the last decade, it was shown how to incorporate correctly a variable magnetic field in the picture, cf. [36, 22, 23, 30, 32, 34, 33, 20, 21] (see also [7, 8, 9] for extensions involving nilpotent groups). This relies on twisting both the pseudodifferential calculus and the crossed product algebras by a 2-cocycle defined on the group $\mathscr{X}$ and taking values in the (Polish, non-locally compact group) $\mathcal{U}(\mathcal{A})$ of unitary elements of the algebra $\mathcal{A}$. This 2-cocycle is given by imaginary exponentials of the magnetic flux through triangles. The resulting gauge-covariant formalism has position and *kinetic* momentum as its basic observables. The latter no longer commute amongst each other due to the presence of the magnetic field. It was shown in [31] that the family of twisted crossed products indexed by $\hbar \in (0, 1]$ can be understood as a strict deformation quantization (in the sense of Marc Rieffel) of a natural Poisson algebra defined by a symplectic form which is the sum of the canonical symplectic form in Ξ and a magnetic contribution.

A natural question is what happens when the algebra $\mathcal{A}$ (composed of functions defined on $\mathscr{X}$) is replaced by a general abelian C^*-algebra. By Gelfand theory this one is isomorphic to $C_0(\Omega)$, the C^*-algebra of all the complex continuous functions vanishing at infinity defined on the locally compact space Ω. To define crossed products and pseudodifferential operators we also need a continuous action θ of $\mathscr{X}$ on Ω by homeomorphisms. $C_0(\Omega)$ can be seen as a C^*-algebra of functions on $\mathscr{X}$ exactly when Ω happens to have a distinguished dense orbit. In the general case, the twisting ingredient will be "a general magnetic field", i.e. a continuous family B of magnetic fields indexed by the points of Ω and satisfying an equivariance condition with respect to the action θ.

The purpose of this article is to investigate the emerging formalism, both classical and quantal.

To the quadruplet $(\Omega, \theta, B, \mathscr{X})$ described above we first assign in Section 2.2 a Poisson algebra that is the setting for classical mechanics. The Poisson bracket is written with derivatives defined by the abstract action θ and it also contains the magnetic field B. Since Ω does not have the structure of a manifold, this Poisson algebra does not live on a Poisson manifold, let alone a symplectic manifold (as it is the case when a dense orbit exists). But it admits symplectic representations and, at least in the free action case, $\Omega \times \mathscr{X}^*$ is a Poisson space [24] in which symplectic manifolds (the orbits of the action raised to the phase-space Ξ) are only glued together continuously.

Twisted crossed product C^*-algebras are available in a great generality [39, 40]. We use them in Section 2.3 to define algebras of quantum observables with magnetic fields. By a partial Fourier transformation they can be rewritten as algebras of generalized magnetic pseudodifferential symbols. The outcome has some common points with Rieffel's pseudodifferential calculus [43], which starts from an action of $\mathcal{R}^N$ on a C^*-algebra. In our case this algebra is abelian and the action has a somehow restricted form; on the other hand the magnetic twisting cannot be covered by Rieffel's formalism. We also study Hilbert-space representations of the algebras of symbols. Their interpretation as equivariant families of usual magnetic pseudodifferential operators with anisotropic coefficients [27] is available. This will be developed in a forthcoming article and applied to spectral analysis of deterministic and random magnetic quantum Hamiltonians.

Section 4 is dedicated to a development of the magnetic composition law involving Planck's constant. The first and second terms are written using the classical Poisson algebra conterpart. We insist on reminder estimates valid in the relevant C^*-norms.

All these are used in Section 5 to show that the quantum formalism converges to the classical one when Planck's constant $\hbar$ converges to zero, in the sense of strict deformation quantization [43, 44, 24, 25]. The semiclassical limit of dynamics [24, 45] generated by generalized magnetic Hamiltonians will be studied elsewhere.

An appendix is devoted to some technical results about the behavior of the magnetic flux through triangles. These results are used in the main body of the text.

2.2 Classical

2.2.1 Actions

Let $\mathcal{A}$ denote an abelian C^*-algebra. By Gelfand theory, this algebra is isomorphic to the algebra $C_0(\Omega)$ of continuous functions vanishing at infinity on some locally compact (Hausdorff) topological space Ω, and we shall treat this isomorphism as an identification. Furthermore, we shall always assume that $\mathcal{A}$ is endowed with a continuous action θ of the group $\mathscr{X} := \mathcal{R}^n$ by automorphisms: For any $x, y \in \mathscr{X}$ and $\varphi \in \mathcal{A}$,

$$\theta_0[\varphi] = \varphi, \qquad \theta_x\left[\theta_y[\varphi]\right] = \theta_{x+y}[\varphi]$$

and the map $\mathscr{X} \ni x \mapsto \theta_x[\varphi] \in \mathcal{A}$ is continuous for any $\varphi \in \mathcal{A}$. The triple $(\mathcal{A}, \theta, \mathscr{X})$ is usually called an (abelian) *$\mathscr{X}$-algebra*.

Equivalently, we can assume that the spectrum Ω of $\mathcal{A}$ is endowed with a continuous action of $\mathscr{X}$ by homeomorphisms, which with abuse of notation will also be denoted by θ. In other words, $(\Omega, \theta, \mathscr{X})$ is a locally compact dynamical system. We shall use all of the notations $\theta(\omega, x) = \theta_x[\omega] = \theta_\omega(x)$ for $(\omega, x) \in \Omega \times \mathscr{X}$ and choose the convention $\big(\theta_x[\varphi]\big)(\omega) = \varphi\big(\theta_x[\omega]\big)$ to connect the two actions.

An important, but very particular family of examples of $\mathscr{X}$-algebras is constructed using functions on $\mathscr{X}$. We denote by $BC(\mathscr{X})$ the C^*-algebra of all bounded, continuous functions $\phi : \mathscr{X} \longrightarrow \mathcal{C}$. Let τ denote the action of the locally compact group $\mathscr{X} = \mathcal{R}^n$ on itself, *i.e.* for any $x, y \in \mathscr{X}$ we set $\tau(x, y) = \tau_x[y] := y + x$. This notation is also used for the action of $\mathscr{X}$ on $BC(\mathscr{X})$ given by $\tau_x[\varphi](y) := \varphi(y + x)$. The action is continuous only on $BC_{\mathrm{u}}(\mathscr{X})$, the C^*-subalgebra composed of bounded and uniformly continuous functions. Any C^*-subalgebra of $BC_{\mathrm{u}}(\mathscr{X})$ which is invariant under translations is an $\mathscr{X}$-algebra. Motivated by the above examples, we define $BC(\Omega) := \{\varphi : \Omega \to \mathbb{C} \mid f \text{ is bounded and continuous}\}$ and

$$\mathcal{B} \equiv BC_{\mathrm{u}}(\Omega) := \big\{\varphi \in BC(\Omega) \mid \mathscr{X} \ni x \mapsto \theta_x[\varphi] \in BC(\Omega) \text{ is continuous}\big\}.$$

By a *$\mathscr{X}$-morphism* we denote either a continuous map between the underlying spaces of two dynamical systems which intertwines the respective actions, or a morphism between two $\mathscr{X}$-algebras which also intertwines their respective actions.

Let us recall some definitions related to the dynamical system $(\Omega, \theta, \mathscr{X})$. For any $\omega \in \Omega$ we set $\mathcal{O}_\omega := \big\{\theta_x[\omega] \mid x \in \mathscr{X}\big\}$ for *the orbit of* ω and $\mathcal{Q}_\omega := \overline{\mathcal{O}_\omega}$ for *the quasi-orbit of* ω, which is the closure of $\mathcal{O}_\omega$ in Ω. We shall denote by $\mathbf{O}(\Omega) \equiv \mathbf{O}(\Omega, \theta, \mathscr{X})$ the set of orbits of $(\Omega, \theta, \mathscr{X})$ and by $\mathfrak{Q}(\Omega) \equiv \mathfrak{Q}(\Omega, \theta, \mathscr{X})$ the set of quasi-orbits of $(\Omega, \theta, \mathscr{X})$. For fixed $\omega \in \Omega$, $\varphi \in C_0(\Omega)$ and $x \in \mathscr{X}$, we set $\varphi_\omega(x) := \varphi(\theta_x[\omega]) \equiv \varphi\big(\theta_\omega(x)\big)$. It is easily seen that $\varphi_\omega : \mathscr{X} \to \mathcal{C}$ belongs to $BC_u(\mathscr{X})$. Furthermore, the C^*-algebra

$$\mathcal{A}_\omega := \big\{\varphi_\omega \mid \varphi \in C_0(\Omega)\big\} = \theta_\omega[C_0(\Omega)]$$

is isomorphic to the C^*-algebra $C_0(\mathcal{Q}_\omega)$ obtained by restricting the elements of $C_0(\Omega)$ to the closed invariant subset $\mathcal{Q}_\omega$. Then, one clearly obtains that

$$\theta_\omega : C_0(\Omega) \ni \varphi \mapsto \varphi_\omega = \varphi \circ \theta_\omega \in BC_u(\mathscr{X}) \tag{2.2.1}$$

is a $\mathscr{X}$-morphism between $\big(C_0(\Omega), \theta, \mathscr{X}\big)$ and $\big(BC_u(\mathscr{X}), \tau, \mathscr{X}\big)$ which induces a $\mathscr{X}$-isomorphism between $\big(C_0(\mathcal{Q}_\omega), \theta, \mathscr{X}\big)$ and $\big(\mathcal{A}_\omega, \tau, \mathscr{X}\big)$.

We recall that the dynamical system is *topologically transitive* if an orbit is dense, or equivalently if $\Omega \in \mathfrak{Q}(\Omega)$. This happens exactly when the morphism (2.2.1) is injective for some ω. The dynamical system $(\Omega, \theta, \mathscr{X})$ is *minimal* if all the orbits are dense, *i.e.* $\mathfrak{Q}(\Omega) = \{\Omega\}$. This property is also equivalent to the fact that the only closed invariant subsets are $\emptyset$ and Ω.

Definition 2.2.1. *Let $(\mathcal{A}, \theta, \mathscr{X})$ be an $\mathscr{X}$-algebra. We define the spaces of smooth vectors*

$$\mathcal{A}^\infty := \{\varphi \in \mathcal{A} \mid \mathscr{X} \ni x \mapsto \theta_x(\varphi) \in \mathcal{A} \text{ is } C^\infty\}.$$

For the $\mathscr{X}$-algebras $C_0(\Omega)$ and $BC_u(\Omega)$ we will often use the notations $C_0^\infty(\Omega)$, respectively. Despite these notations, we stress that in general Ω is not a manifold; the notion of differentiability is defined only along orbits. By setting for any $\alpha \in \mathbb{N}^n$

$$\delta^\alpha : C_0^\infty(\Omega) \to C_0^\infty(\Omega), \qquad \delta^\alpha \varphi := \partial_x^\alpha (\varphi \circ \theta_x)\,|_{x=0},$$

one defines a Fréchet structure on $C_0^\infty(\Omega)$ by the semi-norms

$$s^\alpha(\varphi) := \left\|\delta^\alpha \varphi\right\|_{C_0(\Omega)} = \sup_{\omega\in\Omega}\left|(\delta^\alpha\varphi)(\omega)\right|.$$

Each of the two spaces, $C_0^\infty(\Omega)$ and $\mathcal{A}_\omega^\infty$, is a dense Fréchet *-subalgebra of the corresponding C^*-algebra.

Lemma 2.2.2. *(i) For each $\omega \in \Omega$ one has*

$$\mathcal{A}_\omega^\infty = \left\{\phi \in C^\infty(\mathscr{X}) \mid \partial^\beta \phi \in \mathcal{A}_\omega,\ \forall \beta \in \mathbb{N}^n\right\}.$$

In particular $\mathcal{A}_\omega^\infty \subset BC^\infty(\mathscr{X}) := \left\{\phi \in C^\infty(\mathscr{X}) \mid \partial^\beta\phi \text{ is bounded } \forall\beta \in \mathbb{N}^n\right\}$.

(ii) Let $\varphi \in C_0(\Omega)$. Then

$$\varphi \in C_0^\infty(\Omega) \iff \varphi \circ \theta_\omega \in \mathcal{A}_\omega^\infty,\ \forall \omega \in \Omega.$$

Proof. The proof consists in some routine manipulations of the definitions. The only slightly non-trivial fact is to show that point-wise derivations are equivalent to the uniform ones, required by the uniform norms. This follows from the Fundamental Theorem of Calculus, using the higher order derivatives, which are assumed to be bounded. A model for such a standard argument is the proof of Lemma 2.7 in [27]. □

Remark 2.2.3. In the following, we will use repeatedly and without further comment the identification of point-wise and uniform derivatives under the assumption that higher-order point-wise derivatives exist and are bounded.

Although in our setting the classical observables are functions defined on $\Omega \times \mathscr{X}^*$, we are going to relate them to functions on *phase space* $\Xi := \mathscr{X} \times \mathscr{X}^*$ whose points are denoted by capital letters $X = (x,\xi)$, $Y = (y,\eta)$, $Z = (z,\zeta)$. The dual space $\mathscr{X}^*$ also acts on itself by translations: $\tau_\eta^*(\xi) := \xi+\eta$, and this action is raised to various function spaces as above. Similarly, phase space Ξ can also be regarded as a group acting on itself by translations, $(\tau\otimes\tau^*)_{(y,\eta)}(x,\xi) := (x+y, \xi+\eta)$. Phase space Ξ acts on $\Omega \times \mathscr{X}^*$ as well, via the action $\theta \otimes \tau^*$, and this defines naturally function spaces on $\Omega \times \mathscr{X}^*$ as above; they will be used without further comment.

2.2.2 Cocycles and magnetic fields

We first recall the definition of a 2-cocycle κ on the abelian algebra $\mathcal{A} = C_0(\Omega)$ endowed with an action θ of $\mathscr{X}$. We mention that the group $\mathcal{U}(\mathcal{A})$ of unitary elements of the unital C^*-algebra $BC(\Omega)$ coincides with $C(\Omega;\mathbb{T}) := \left\{\varphi \in C(\Omega) \mid |\varphi(\omega)| = 1,\ \forall\omega \in \Omega\right\}$, on which we consider the topology of uniform convergence on compact sets.

Definition 2.2.4. A normalized 2-cocycle on $\mathcal{A}$ *is a continuous map* $\kappa : \mathscr{X} \times \mathscr{X} \to \mathcal{U}(\mathcal{A})$ *satisfying for all* $x, y, z \in \mathscr{X}$*:*

$$\kappa(x+y, z)\,\kappa(x, y) = \theta_x[\kappa(y, z)]\,\kappa(x, y+z) \tag{2.2.2}$$

and $\kappa(x, 0) = \kappa(0, x) = 1$.

Proposition 2.2.5. *If* $\kappa : \mathscr{X} \times \mathscr{X} \to C(\Omega; \mathbb{T})$ *is a 2-cocycle of* $C_0(\Omega)$ *then for any* $\omega \in \Omega$, $\kappa_\omega(\cdot, \cdot) := \kappa(\cdot, \cdot) \circ \theta_\omega$ *is a 2-cocycle of* $\mathcal{A}_\omega$ *with respect to the action* τ.

Proof. Everything is straightforward. To check the 2-cocycle property, one needs the identity

$$\theta_x \circ \theta_\omega = \theta_\omega \circ \tau_x, \qquad x \in \mathscr{X},\ \omega \in \Omega.$$

It is easy to show that $\kappa : \mathscr{X} \times \mathscr{X} \to C(\Omega, \mathbb{T})$ is continuous iff the function

$$\Omega \times \mathscr{X} \times \mathscr{X} \ni (\omega, x, y) \mapsto \kappa(\omega; x, y) := \big(\kappa(x, y)\big)(\omega) \in \mathbb{T}$$

is continuous. Recalling the isomorphism $\mathcal{A}_\omega \cong C(\mathcal{Q}_\omega)$ one easily finishes the proof. □

We shall be interested in *magnetic 2-cocycles.*

Definition 2.2.6. *We call* magnetic field on Ω *a continuous function* $B : \Omega \to \bigwedge^2 \mathscr{X}$ *such that* $B_\omega := B \circ \theta_\omega$ *is a magnetic field (continuous closed 2-form on* $\mathscr{X}$*) for any* ω.

Using coordinates, B can be seen as an anti-symmetric matrix $\left(B^{jk}\right)_{j,k=1,\dots,n}$ where the entries are continuous functions $B^{jk} : \Omega \to \mathcal{R}$ satisfying (in the distributional sense)

$$\partial_j B^{kl}_\omega + \partial_k B^{lj}_\omega + \partial_l B^{jk}_\omega = 0, \qquad \forall\, \omega \in \Omega,\ \forall\, j, k, l = 1, \dots, n.$$

Proposition 2.2.7. *Let* B *a magnetic field on* Ω*. Set*

$$\big(\kappa^B(x, y)\big)(\omega) \equiv \kappa^B(\omega; x, y) := \exp\big(-i\Gamma^{B_\omega}\langle 0, x, x+y\rangle\big),$$

where $\Gamma^{B_\omega}\langle a, b, c\rangle := \int_{\langle a,b,c\rangle} B_\omega$ *is the integral (flux) of the 2-form* B_ω *through the triangle* $\langle a, b, c\rangle$ *with corners* $a, b, c \in \mathscr{X}$*. Then* κ^B *is a 2-cocycle on the* $\mathscr{X}$*-algebra* $C_0(\Omega)$.

Proof. The algebraic properties follow from the properties of the integration of 2-forms. For example, (2.2.2) is a consequence of the identity

$$\Gamma^{B_\omega}\langle 0, x, x+y\rangle + \Gamma^{B_\omega}\langle 0, x+y, x+y+z\rangle = \Gamma^{B_{\theta_x[\omega]}}\langle 0, y, y+z\rangle + \Gamma^{B_\omega}\langle 0, x, x+y+z\rangle.$$

This one follows from Stokes' Theorem, after noticing that

$$\Gamma^{B_{\theta_x[\omega]}}\langle 0, y, y+z\rangle = \Gamma^{B_\omega}\langle x, x+y, x+y+z\rangle. \tag{2.2.3}$$

One still has to check that $\kappa^B \in C(\Omega \times \mathscr{X} \times \mathscr{X})$. This reduces to the obvious continuity of

$$(\omega, x, y) \mapsto \Gamma^{B_\omega}\langle 0, x, x+y\rangle = \sum_{j,k=1}^{n} x_j y_k \int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}s\, s\, \theta_{sx+sty}[B^{jk}](\omega),$$

where we have used a parametrization of the flux involving the components of the magnetic field in the canonical basis of $\mathscr{X} = \mathcal{R}^n$. □

By (2.2.3) one easily sees that $\left(\kappa^B\right)_\omega = \kappa^{B_\omega}$, where the l.h.s. was defined in Proposition 2.2.5, while

$$\kappa^{B_\omega}(z;x,y) := \exp\left(-i\Gamma^{B_\omega}\langle z, z+x, z+x+y\rangle\right).$$

2.2.3 Poisson algebras

We intend now to define a Poisson structure (cf. [24, 35]) on spaces of functions that are smooth under the action $\theta \times \tau^*$ of Ξ on $\Omega \times \mathscr{X}^*$. This Poisson algebras can be represented by families of subalgebras of $BC^\infty(\Xi)$, indexed essentially by the orbits of Ω, each one endowed with the Poisson structure induced by a magnetic symplectic form [31]. For simplicity, we shall concentrate on a Poisson subalgebra consisting of functions which have Schwartz-type behavior in the variable $\xi \in \mathscr{X}^*$. For this smaller algebra of functions, we will prove strict deformation quantization in section 2.5. One can also define $C^\infty(\Omega \times \mathscr{X}^*)$ in terms of the action $\theta \otimes \tau^*$; this one is also a Poisson algebra, but we will not need it here.

When necessary, we shall use $f(\xi)$ as short-hand notation for $f(\cdot,\xi)$, i. e. $f(\omega,\xi) = \big(f(\xi)\big)(\omega)$ for $(\omega,\xi) \in \Omega \times \mathscr{X}^*$, and we will think of $f(\cdot,\xi)$ as an element of some algebra of functions on Ω. Note that

$$BC^\infty(\Omega \times \mathscr{X}^*) =$$
$$\left\{f \in BC(\Omega \times \mathscr{X}^*) \mid f(\cdot,\xi) \in BC^\infty(\Omega) \text{ and } f(\omega,\cdot) \in BC^\infty(\mathscr{X}^*),\ \forall \omega \in \Omega, \xi \in \mathscr{X}^*\right\}.$$

Definition 2.2.8. *We say that $f \in BC^\infty(\Omega \times \mathscr{X}^*)$ belongs to $\mathcal{S}\big(\mathscr{X}^*; C_0^\infty(\Omega)\big)$ if and only if*

(i) $\partial_\xi^\beta f(\xi) \in C_0^\infty(\Omega)$, $\forall \xi \in \mathscr{X}^*$ *and*

(ii) $\|f\|_{a\alpha\beta} := \sup_{\xi \in \mathscr{X}^*} \left\|\xi^a \delta^\alpha \partial_\xi^\beta f(\xi)\right\|_{C_0(\Omega)} < \infty$ *for all* $a, \alpha, \beta \in \mathbb{N}^n$.

Proposition 2.2.9. *We assume from now on that $B^{jk} \in BC^\infty(\Omega)$ for any $j,k = 1,\dots,n$.*

(i) $BC^\infty(\Omega \times \mathscr{X}^*)$ is a Poisson algebra under point-wise multiplication and the Poisson bracket

$$\{f,g\}_B := \sum_{j=1}^n \left(\partial_{\xi_j} f\, \delta_j g - \delta_j f\, \partial_{\xi_j} g\right) - \sum_{j,k} B^{jk}\, \partial_{\xi_j} f\, \partial_{\xi_k} g. \tag{2.2.4}$$

(ii) $\mathcal{S}\big(\mathscr{X}^*; C_0^\infty(\Omega)\big)$ is a Poisson subalgebra of $BC^\infty(\Omega \times \mathscr{X}^*)$.

Proof. The two vector spaces are stable under point-wise multiplication and derivations with respect to ξ and along orbits in Ω via ∂_ξ and δ, respectively. They are also stable under multiplication with elements of $BC^\infty(\Omega)$. The axioms of a Poisson algebra are verified by direct computation. $\square$

To analyze the quantum calculus which is to be defined below, a change of realization is useful. Defining $\mathcal{S}\big(\mathscr{X}; C_0^\infty(\Omega)\big)$ as in Definition 2.2.8, but with $\mathscr{X}^*$ replaced with $\mathscr{X}$, we transport by the partial Fourier transformation the Poisson structure from $\mathcal{S}\big(\mathscr{X}^*; C_0^\infty(\Omega)\big)$ to $\mathcal{S}\big(\mathscr{X}; C_0^\infty(\Omega)\big)$ setting

$$(\Phi \diamond_0 \Psi)(\omega; x) := (1 \otimes \mathcal{F})^{-1}\big((1 \otimes \mathcal{F})\Phi \cdot (1 \otimes \mathcal{F})\Psi\big)(\omega; x) = \int_{\mathscr{X}} \mathrm{d}y\, \Phi(\omega; y)\, \Psi(\omega; x - y) \quad (2.2.5)$$

and

$$\begin{aligned} \{\Phi, \Psi\}^B &:= (1 \otimes \mathcal{F})^{-1}\big\{(1 \otimes \mathcal{F})\Phi, (1 \otimes \mathcal{F})\Psi\big\}_B \\ &= -i \sum_{j=1}^n \big(Q_j \Phi \diamond_0 \delta_j \Psi - \delta_j \Phi \diamond_0 Q_j \Psi\big) + \sum_{j,k=1}^n B^{jk}\,\big(Q_j \Phi \diamond_0 Q_k \Psi\big), \end{aligned} \quad (2.2.6)$$

where $(Q_j\Phi)(x) = x_j \Phi(x)$ defines the multiplication operator by x_j. Obviously this also makes sense on larger spaces.

To get a better idea of the Poisson structure of $BC^\infty(\Omega \times \mathscr{X}^*)$, we will exploit the orbit structure of the dynamical system $\big(\Omega \times \mathscr{X}^*, \theta \otimes \tau^*, \mathscr{X} \times \mathscr{X}^*\big)$ and relate this big Poisson algebra to a family of smaller, symplectic-type ones. For each $\omega \in \Omega$, we can endow $\Xi = \mathscr{X} \times \mathscr{X}^*$ with a symplectic form

$$\big[\sigma_\omega^B\big]_Z(X, Y) := y \cdot \xi - x \cdot \eta + B_\omega(z)(x, y) = \sum_{j=1}^n (y_j\, \xi_j - x_j\, \eta_j) + \sum_{j,k=1}^n B^{jk}\big(\theta_z[\omega]\big)\, x_j\, y_k,$$

which makes the pair (Ξ, σ_ω^B) into a symplectic space. This canonically defines a Poisson bracket

$$\{f, g\}_{B_\omega} := \sum_{j=1}^n \big(\partial_{\xi_j} f\, \partial_{x_j} g - \partial_{x_j} f\, \partial_{\xi_j} g\big) - \sum_{j,k=1}^n B_\omega^{jk}\, \partial_{\xi_j} f\, \partial_{\xi_k} g. \quad (2.2.7)$$

Proposition 2.2.10. *(i) For each $\omega \in \Omega$, the map*

$$\pi_\omega := \theta_\omega \otimes 1 : \big(BC^\infty(\Omega \times \mathscr{X}^*), \cdot, \{\cdot, \cdot\}_B\big) \to \big(BC^\infty(\Xi), \cdot, \{\cdot, \cdot\}_{B_\omega}\big)$$

is a Poisson map, i. e. for all $f, g \in BC^\infty(\Omega \times \mathscr{X}^*)$

$$\pi_\omega(f \cdot g) = \pi_\omega(f) \cdot \pi_\omega(g), \qquad \pi_\omega\big(\{f, g\}_B\big) = \big\{\pi_\omega(f), \pi_\omega(g)\big\}_{B_\omega}.$$

(ii) If $\omega, \omega' \in \Omega$ belong to the same orbit, the corresponding Poisson maps are connected by a symplectomorphism (they may be called equivalent representations of the Poisson algebra*).*

Proof. To simplify notation, we use the shorthand $f_\omega := \pi_\omega(f)$ for $f \in BC^\infty(\Omega \times \mathscr{X}^*)$ and $\omega \in \Omega$.

(i) For any $\omega \in \Omega$, $f, g \in BC^\infty(\Omega \times \mathscr{X}^*)$, we have

$$(f\, g)_\omega(x, \xi) = \big((f\, g) \circ (\theta_\omega \otimes 1)\big)(x, \xi) = f\big(\theta_\omega(x), \xi\big)\, g\big(\theta_\omega(x), \xi\big) = (f_\omega\, g_\omega)(x, \xi).$$

Similarly, $\big(\{f, g\}_B\big)_\omega = \big\{f_\omega, g_\omega\big\}_{B_\omega}$ follows from direct computation, using

$$\partial_{x_j} f_\omega = \partial_{x_j}\big(f \circ (\theta_\omega \otimes 1)\big) = (\delta_j f) \circ (\theta_\omega \otimes 1) = (\delta_j f)_\omega.$$

(ii) If there exists $z \in \mathscr{X}$ such that $\theta_z[\omega] = \omega'$, then

$$\theta_{\omega'} \otimes 1 = (\theta_\omega \otimes 1) \circ (\tau_z \otimes 1),$$

where $\tau_z \otimes 1 : (\Xi, \sigma^B_\omega) \to (\Xi, \sigma^B_{\omega'})$ is a symplectomorphism.

□

Remark 2.2.11. It is easy to see that the mapping

$$\pi_\omega := \theta_\omega \otimes 1 : \mathcal{S}(\mathscr{X}^*, C_0^\infty(\Omega)) \longrightarrow \mathcal{S}(\mathscr{X}^*, \mathcal{A}_\omega^\infty)$$

is a surjective morphism of Poisson algebras, for any $\omega \in \Omega$. On the second space we consider the Poisson structure defined by the magnetic field B_ω, as in [31].

For any $\omega \in \Omega$ we define the stabilizer $\mathscr{X}_\omega := \{x \in \mathscr{X} \mid \theta_x[\omega] = \omega\}$. This is a closed subgroup of $\mathscr{X}$, the same for all ω belonging to a given orbit. We define the subspace of Ω on which the action θ is free:

$$\Omega_0 := \{\omega \in \Omega \mid \mathscr{X}_\omega = \{0\}\}.$$

Obviously Ω_0 is invariant under θ and $\Omega_0 \times \mathscr{X}^*$ is invariant under the free action $\theta \otimes \tau^*$, so we can consider the Poisson algebra $BC^\infty(\Omega_0 \times \mathscr{X}^*)$ with point-wise multiplication and Poisson bracket (2.2.4).

For any $\mathcal{O} \in \mathbf{O}(\Omega_0)$ (the family of all the orbits of the space Ω_0) we choose a point $\omega(\mathcal{O}) \in \mathcal{O}$. Then

$$\theta_{\omega(\mathcal{O})} \otimes 1 : \mathscr{X} \times \mathscr{X}^* \longrightarrow \Omega_0 \times \mathscr{X}^*$$

is a continuous injection with range $\mathcal{O} \times \mathscr{X}^*$ (which is one of the orbits of $\Omega_0 \times \mathscr{X}^*$ under the action $\theta \times \tau^*$). Of course, one has (disjoint union)

$$\Omega_0 \times \mathscr{X}^* = \bigsqcup_{\mathcal{O} \in \mathbf{O}(\Omega_0)} \mathcal{O} \times \mathscr{X}^*.$$

In addition, $\theta_{\omega(\mathcal{O})} \otimes 1$ is a Poisson mapping on $\Xi = \mathscr{X} \times \mathscr{X}^*$ if one considers the Poisson structure induced by the symplectic form $\sigma^B_{\omega(\mathcal{O})}$.

Referring to Definition I.2.6.2 in [24], we notice that actually $\Omega_0 \times \mathscr{X}^*$ *is a Poisson space.*

2.3 Quantum

2.3.1 Magnetic twisted crossed products

Definition 2.3.1. *We call* twisted C^*-dynamical system *a quadruplet* $(\mathcal{A}, \theta, \kappa, \mathscr{X})$*, where* θ *is an action of* $\mathscr{X} = \mathcal{R}^n$ *on the (abelian)* C^**-algebra* $\mathcal{A}$ *and* κ *is a normalized 2-cocycle on* $\mathcal{A}$ *with respect to* θ*.*

Starting from a twisted C^*-dynamical system, one can construct twisted crossed product C^*-algebras [39, 40, 34] (see also references therein). Let $L^1(\mathscr{X};\mathcal{A})$ be the complex vector space of $\mathcal{A}$-valued Bochner integrable functions on $\mathscr{X}$ and L^1-norm

$$\|\Phi\|_{L^1} := \int_{\mathscr{X}} \mathrm{d}x \, \|\Phi(x)\|_{\mathcal{A}} \, .$$

For any $\Phi, \Psi \in L^1(\mathscr{X};\mathcal{A})$ and $x \in \mathscr{X}$, we define the product

$$(\Phi \diamond^\kappa \Psi)(x) := \int_{\mathscr{X}} \mathrm{d}y \, \theta_{\frac{y-x}{2}}[\Phi(y)] \, \theta_{\frac{y}{2}}[\Psi(x-y)] \, \theta_{-\frac{x}{2}}[\kappa(y, x-y)]$$

and the involution $\Phi^{\diamond^\kappa}(x) := \overline{\Phi(-x)}$. With these two operations, $\big(L^1(\mathscr{X};\mathcal{A}), \diamond^\kappa, {}^{\diamond^\kappa}\big)$ forms a Banach-$*$-algebra.

Definition 2.3.2. *The enveloping C^*-algebra of $L^1(\mathscr{X};\mathcal{A})$ is called* the twisted crossed product $\mathcal{A} \rtimes_\theta^\kappa \mathscr{X}$.

We are going to indicate now the relevant twisted crossed products, also introducing Planck's constant $\hbar$ in the formalism. We define

$$\theta_x^\hbar := \theta_{\hbar x} \text{ and } \kappa^{B,\hbar}(x,y) = \kappa^{\frac{B}{\hbar}}(\hbar x, \hbar y),$$

which means

$$\kappa^{B,\hbar}(\omega; x, y) = e^{-\frac{i}{\hbar}\Gamma^{B_\omega}\langle 0, \hbar x, \hbar x + \hbar y\rangle}, \quad \forall x, y \in \mathscr{X}, \; \omega \in \Omega,$$

and check easily that $\big(C_0(\Omega), \theta^\hbar, \kappa^{B,\hbar}, \mathscr{X}\big)$ is a twisted C^*-dynamical system for any $\hbar \in (0,1]$. It will be useful to introduce $\Lambda_\hbar^B(x,y)$ via

$$\theta_{-\frac{\hbar}{2}x}[\kappa^{B,\hbar}(\omega; x, y)] = e^{-\frac{i}{\hbar}\Gamma^{B_\omega}\langle -\frac{\hbar}{2}x, \hbar y - \frac{\hbar}{2}x, \frac{\hbar}{2}x\rangle} =: e^{-i\hbar\Lambda_\hbar^{B_\omega}(x,y)},$$

as short-hand notation for the phase factor. This scaled magnetic flux can be parametrized explicitly as

$$\Lambda_\hbar^B(x,y) = \sum_{j,k=1}^n y_j \, (x_k - y_k) \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s \, \theta_{\hbar(s-1/2)x + \hbar(t-s)y}[B^{jk}]. \tag{2.3.1}$$

Plugging this particular choice of 2-cocycle and $\mathscr{X}$ action into the general form of the product, one gets

$$(\Phi \diamond_\hbar^B \Psi)(x) = \int_{\mathscr{X}} \mathrm{d}y \, \theta_{\frac{\hbar}{2}(y-x)}[\Phi(y)] \, \theta_{\frac{\hbar}{2}y}[\Psi(x-y)] \, e^{-i\hbar\Lambda_\hbar^B(x,y)}.$$

The twisted crossed product C^*-algebra $\mathcal{A} \rtimes_{\theta^\hbar}^{\kappa^{B,\hbar}} \mathscr{X}$ will be denoted simply by $\mathfrak{C}_\hbar^B$ with self-adjoint part $\mathfrak{C}_{\hbar,\mathcal{R}}^B$ and norm $\|\cdot\|_\hbar^B$. We also call $\mathfrak{C}_0$ the enveloping C^*-algebra of $L^1(\mathscr{X};\mathcal{A})$ with the commutative product $\diamond_0$; it is isomorphic with $C_0(\mathscr{X}^*;\mathcal{A}) \cong C(\mathscr{X}^*) \otimes \mathcal{A}$.

A quick computation shows that $\pi^\hbar_\omega := \theta^\hbar_\omega \otimes 1$ intertwines the involutions associated to the C^*-algebras $\mathfrak{C}^B_\hbar$ and $\mathcal{A}_\omega \rtimes^{\kappa^{B_\omega,\hbar}}_{\tau^\hbar} \mathscr{X}$, i. e. $\pi^\hbar_\omega(\Phi^{\diamond^B_\hbar}) = \pi^\hbar_\omega(\Phi)^{\diamond^{B_\omega}_\hbar}$ is satisfied for every $\Phi \in \mathfrak{C}^B_\hbar$. A slightly more cumbersome task is the verification of $\pi^\hbar_\omega(\Phi \diamond^B_\hbar \Psi) = \pi^\hbar_\omega(\Phi) \diamond^{B_\omega}_\hbar \pi^\hbar_\omega(\Psi)$. For any $\Phi, \Psi \in L^1(\mathscr{X};\mathcal{A})$ and $z, x \in \mathscr{X}$, we have

$$
\begin{aligned}
&\big[\pi^\hbar_\omega(\Phi \diamond^B_\hbar \Psi)\big](z;x) \\
&= \int_{\mathscr{X}} \mathrm{d}y \, \Big(\theta_{\frac{\hbar}{2}(y-\cdot)}[\Phi(y)] \, \theta_{\frac{\hbar}{2}y}[\Psi(\cdot - y)] \, e^{-\frac{i}{\hbar}\Gamma^B \langle \cdot - \frac{\hbar}{2}\cdot, -\frac{\hbar}{2}\cdot + \hbar y, \frac{\hbar}{2}\cdot \rangle} \Big) \circ (\theta^\hbar_\omega \otimes 1)(z;x) \\
&= \int_{\mathscr{X}} \mathrm{d}y \, \Phi\big(\theta_{\hbar z + \frac{\hbar}{2}(y-x)}[\omega], y\big) \, \Psi\big(\theta_{\hbar z + \frac{\hbar}{2}y}[\omega], x-y\big) \, e^{-\frac{i}{\hbar}\Gamma^{B_{\theta_{\hbar z}[\omega]}} \langle -\frac{\hbar}{2}x, -\frac{\hbar}{2}x + \hbar y, \frac{\hbar}{2}x \rangle} \\
&= \int_{\mathscr{X}} \mathrm{d}y \, \big(\tau_{\frac{\hbar}{2}(y-x)}[\pi^\hbar_\omega(\Phi)(y)]\big)(z) \, \big(\tau_{\frac{\hbar}{2}y}[\pi^\hbar_\omega(\Psi)(x-y)]\big)(z) \, e^{-\frac{i}{\hbar}\Gamma^{B_\omega} \langle \hbar z - \frac{\hbar}{2}x, \hbar z - \frac{\hbar}{2}x + \hbar y, \hbar z + \frac{\hbar}{2}x \rangle} \\
&= \big[\pi^\hbar_\omega(\Phi) \diamond^{B_\omega}_\hbar \pi^\hbar_\omega(\Psi)\big](z;x).
\end{aligned}
$$

It follows easily that $\{\pi^\hbar_\omega\}_{\omega \in \Omega}$ defines by extension a family of epimorphisms

$$
\pi^\hbar_\omega : \mathfrak{C}^B_\hbar \longrightarrow \mathcal{A}_\omega \rtimes^{\kappa^{B_\omega,\hbar}}_{\tau^\hbar} \mathscr{X}
$$

that map a twisted crossed product defined in terms of $C_0(\Omega)$ onto more concrete C^*-algebras defined in terms of subalgebras $\mathcal{A}_\omega$ of $BC_u(\mathscr{X})$.

As we have seen, $\mathcal{S}\big(\mathscr{X}^*; C^\infty_0(\Omega)\big)$ is a Poisson subalgebra of $BC^\infty(\Omega \times \mathscr{X}^*)$. For strict deformation quantization we also need that it is a *-subalgebra of each of the C^*-algebras $\mathfrak{C}^B_\hbar$. Since $\mathcal{S}\big(\mathscr{X}^*; C^\infty_0(\Omega)\big)$ is obviously stable under involution, this will follow from

Proposition 2.3.3. *If $B^{jk} \in BC^\infty(\Omega)$, then $\mathcal{S}\left(\mathscr{X}; C^\infty_0(\Omega)\right)$ is a subalgebra of $\big(L^1\big(\mathscr{X}; C_0(\Omega)\big), \diamond^B_\hbar\big)$, i.e.*

$$
\mathcal{S}\big(\mathscr{X}; C^\infty_0(\Omega)\big) \diamond^B_\hbar \mathcal{S}\big(\mathscr{X}; C^\infty_0(\Omega)\big) \subset \mathcal{S}\big(\mathscr{X}; C^\infty_0(\Omega)\big).
$$

Proof. Let $\Phi, \Psi \in \mathcal{S}\big(\mathscr{X}; C^\infty_0(\Omega)\big)$. As $\mathcal{S}\big(\mathscr{X}; C^\infty_0(\Omega)\big)$ is a subspace of $L^1\big(\mathscr{X}; C_0(\Omega)\big)$, $\Phi \diamond^B_\hbar \Psi$ exists in $L^1\big(\mathscr{X}; C_0(\Omega)\big)$. To prove the product $\Phi \diamond^B_\hbar \Psi$ is also in $\mathcal{S}\big(\mathscr{X}; C^\infty_0(\Omega)\big)$, we need to estimate all semi-norms: let $a, \alpha, \beta \in \mathbb{N}^n$. First, we show that we can exchange differentiation with respect to x and along orbits with integration with respect to y via Dominated Convergence, i. e. that for all x and ω

$$
\begin{aligned}
\big(x^a \partial^\alpha_x \delta^\beta (\Phi \diamond^B_\hbar \Psi)\big)(\omega;x) &= \int_{\mathscr{X}} \mathrm{d}y \, x^a \partial^\alpha_x \delta^\beta \Big(\Phi\big(\theta_{\frac{\hbar}{2}(y-x)}[\omega], y\big) \, \Psi\big(\theta_{\frac{\hbar}{2}y}[\omega], x-y\big) \, e^{-i\hbar\Lambda^{B_\omega}_\hbar(x,y)} \Big) \\
&=: \int_{\mathscr{X}} \mathrm{d}y \, I^a_{\alpha\beta}(\omega; x, y)
\end{aligned}
$$

holds. Hence, we need to estimate the absolute value of $I^a_{\alpha\beta}$ uniformly in x and ω by an integrable function. To do that, we write out the derivatives involved in $I^a_{\alpha\beta}$,

$$I^a_{\alpha\beta}(x,y) = x^a \partial_x^\alpha \delta^\beta \Big(\theta_{\frac{\hbar}{2}(y-x)}[\Phi(y)]\, \theta_{\frac{\hbar}{2}y}[\Psi(x-y)]\, e^{-i\hbar\Lambda_\hbar^B(x,y)} \Big)$$
$$= x^a \sum_{\substack{\alpha'+\alpha''+\alpha'''=\alpha \\ \beta'+\beta''+\beta'''=\beta}} \left(-\frac{\hbar}{2}\right)^{|\alpha'|} \theta_{\frac{\hbar}{2}(y-x)}\big[\delta^{\alpha'+\beta'}\Phi(y)\big]\, \theta_{\frac{\hbar}{2}y}\big[\partial_x^{\alpha''}\delta^{\beta''}\Psi(x-y)\big]\, \partial_x^{\alpha'''}\delta^{\beta'''} e^{-i\hbar\Lambda_\hbar^B(x,y)}.$$

Taking the $C_0(\Omega)$ norm of the above expression and using the triangle inequality, $\hbar \leq 1$, the fact that θ_z is an isometry as well as the estimates on the exponential of the magnetic flux from Lemma 2.5.4 (ii), we get

$$\big\|I^a_{\alpha\beta}(x,y)\big\|_{\mathcal{A}} \leq |x^a| \sum_{\alpha'+\alpha''+\alpha'''=\alpha\beta'+\beta''+\beta'''=\beta} \left(\frac{\hbar}{2}\right)^{|\alpha'|} \big\|\theta_{\frac{\hbar}{2}(y-x)}\big[\delta^{\alpha'+\beta'}\Phi(y)\big]\big\|_{\mathcal{A}}$$
$$\cdot \big\|\theta_{\frac{\hbar}{2}y}\big[\partial_x^{\alpha''}\delta^{\beta''}\Psi(x-y)\big]\big\|_{\mathcal{A}} \big\|\partial_x^{\alpha'''}\delta^{\beta'''} e^{-i\hbar\Lambda_\hbar^B(x,y)}\big\|_{\mathcal{A}}$$
$$\leq \left(\prod_{j=1}^n \big(|y_j| + |x_j - y_j|\big)^{a_j}\right) \sum_{\substack{\alpha'+\alpha''+\alpha'''=\alpha \\ \beta'+\beta''+\beta'''=\beta}} \big\|\delta^{\alpha'+\beta'}\Phi(y)\big\|_{\mathcal{A}} \big\|\partial_x^{\alpha''}\delta^{\beta''}\Psi(x-y)\big\|_{\mathcal{A}} \cdot$$
$$\cdot \sum_{|b|+|c|=2(|\alpha'''|+|\beta'''|)} K_{bc}\, |y^b|\, |(x-y)^c|$$
$$= \sum_{\substack{\alpha'+\alpha''+\alpha'''=\alpha \\ \beta'+\beta''+\beta'''=\beta}} \sum_{|b|+|c|\leq|a|+2|\alpha'''|+2|\beta'''|} \tilde{K}_{bc} \big\|(Q^b\delta^{\alpha'+\beta'}\Phi)(y)\big\|_{\mathcal{A}} \big\|(Q^c\partial_x^{\alpha''}\delta^{\beta''}\Psi)(x-y)\big\|_{\mathcal{A}}.$$

The polynomial with coefficients $\tilde{K}_{bc}$ comes from multiplying the other two polynomials in the $|y_j|$ and $|x_j - y_j|$. Taking the supremum in x only yields a function in y (independent of x and ω) which is integrable and dominates $\big|I^a_{\alpha\beta}(\omega;x,y)\big|$ since the right-hand side is a finite sum of Schwartz functions in y,

$$\sup_{x\in\mathscr{X}} \big\|I^a_{\alpha\beta}(x,y)\big\|_{C(\Omega)} \leq \sum_{\substack{\alpha'+\alpha''+\alpha'''=\alpha \\ \beta'+\beta''+\beta'''=\beta \\ |b|+|c|\leq|a|+2|\alpha'''|+2|\beta'''|}} \tilde{K}_{bc} \big\|(Q^b\delta^{\alpha'+\beta'}\Phi)(y)\big\|_{C(\Omega)} \big\|Q^c\partial_x^{\alpha''}\delta^{\beta''}\Psi\big\|_{000}$$
$$= \sum_{\substack{\alpha'+\alpha''+\alpha'''=\alpha \\ \beta'+\beta''+\beta'''=\beta \\ |b|+|c|\leq|a|+2|\alpha'''|+2|\beta'''|}} \tilde{K}_{bc} \big\|(Q^b\delta^{\alpha'+\beta'}\Phi)(y)\big\|_{C(\Omega)} \big\|\Psi\big\|_{c\alpha''\beta''}.$$

Hence, by Dominated Convergence, it is permissible to interchange differentiation and integration. To estimate the semi-norm of the product, we write for an integer N such that $2N \geq n+1$

$$\big\|\Phi \circ_\hbar^B \Psi\big\|_{a\alpha\beta} = \sup_{\substack{x\in\mathscr{X} \\ \omega\in\Omega}} \left| \int_{\mathscr{X}} \mathrm{d}y\, I^a_{\alpha\beta}(\omega;x,y) \right| \leq \int_{\mathscr{X}} \frac{\mathrm{d}y}{\langle y\rangle^{2N}} \langle y\rangle^{2N} \sup_{x\in\mathscr{X}} \big\|I^a_{\alpha\beta}(x,y)\big\|_{C(\Omega)}$$
$$\leq C_1(N) \sup_{x,y\in\mathscr{X}} \Big(\langle y\rangle^{2N} \big\|I^a_{\alpha\beta}(x,y)\big\|_{C(\Omega)}\Big) \leq C_2(N) \max_{|b|\leq 2N} \sup_{x,y\in\mathscr{X}} \big\|y^b I^a_{\alpha\beta}(x,y)\big\|_{C(\Omega)}. \quad (2.3.2)$$

The right-hand side involves semi-norms associated to $\mathcal{S}(\mathscr{X} \times \mathscr{X}; C^\infty(\Omega))$ which we will estimate in terms of the semi-norms of Φ and Ψ, by arguments similar to those leading to the domination of $\left\|I^a_{\alpha\beta}(x,y)\right\|_{C(\Omega)}$.

Thus, we have estimated $\left\|\Phi \diamond^B_\hbar \Psi\right\|_{a\alpha\beta}$ from above by a finite number of semi-norms of Φ and Ψ and $\Phi \diamond^B_\hbar \Psi \in \mathcal{S}(\mathscr{X}; C^\infty(\Omega))$. □

2.3.2 Twisted symbolic calculus

It is useful to transport the composition law $\diamond^B_\hbar$ by partial Fourier transform

$$1 \otimes \mathcal{F} : \mathcal{S}(\mathscr{X}; C^\infty_0(\Omega)) \longrightarrow \mathcal{S}(\mathscr{X}^*; C^\infty_0(\Omega)),$$

setting

$$f \sharp^B_\hbar g := (1 \otimes \mathcal{F}) \left[(1 \otimes \mathcal{F})^{-1} f \diamond^B_\hbar (1 \otimes \mathcal{F})^{-1} g\right]. \tag{2.3.3}$$

In this way one gets a multiplication on $\mathcal{S}(\mathscr{X}^*; C^\infty_0(\Omega))$ which generalizes the magnetic Weyl composition of symbols of [30, 31, 20] (and to which it reduces, actually, if Ω is just a compactification of the configuration space $\mathscr{X}$). Together with complex conjugation, they endow $\mathcal{S}(\mathscr{X}^*; C^\infty_0(\Omega))$ with the structure of a *-algebra. After a short computation one gets

$$\begin{aligned}
(f \sharp^B_\hbar g)(\omega,\xi) &= (\pi\hbar)^{-2n} \int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\eta \int_{\mathscr{X}} \mathrm{d}z \int_{\mathscr{X}^*} \mathrm{d}\zeta \, e^{i\frac{2}{\hbar}(z\cdot\eta - y\cdot\zeta)} \, e^{-\frac{i}{\hbar}\Gamma^{B_\omega}\langle \hbar y - \hbar z, \hbar y + \hbar z, \hbar z - \hbar y\rangle} \cdot \\
&\qquad \cdot f\big(\theta_y[\omega], \xi + \eta\big) \, g\big(\theta_z[\omega], \xi + \zeta\big) \\
&= (\pi\hbar)^{-2n} \int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\eta \int_{\mathscr{X}} \mathrm{d}z \int_{\mathscr{X}^*} \mathrm{d}\zeta \, e^{i\frac{2}{\hbar}\sigma[(y,\eta),(z,\zeta)]} \, e^{-\frac{i}{\hbar}\Gamma^{B_\omega}\langle \hbar y - \hbar z, \hbar y + \hbar z, \hbar z - \hbar y\rangle} \cdot \\
&\qquad \cdot \big(\Theta_{(y,\eta)}[f]\big)(\omega, x) \, \big(\Theta_{(z,\zeta)}[g]\big)(\omega, x),
\end{aligned} \tag{2.3.4}$$

where $\sigma[(y,\eta),(z,\zeta)] := z \cdot \eta - y \cdot \zeta$ is the canonical symplectic form on $\Xi := \mathscr{X} \times \mathscr{X}^*$ and

$$\big(\Theta_{(y,\eta)}[f]\big)(\omega,\xi) \equiv \big((\theta_y \otimes \tau^*_\eta)[f]\big)(\omega,\xi) = f\big(\theta_y[\omega], \xi + \eta\big).$$

This formula should be compared with the product giving Rieffel's quantization [43].

We note that $1 \otimes \mathcal{F}$ can be extended to $L^1(\mathscr{X}; C_0(\Omega))$ and then to $\mathfrak{C}^B_\hbar$. So we get a C^*-algebra $\mathfrak{B}^B_\hbar$, isomorphic to $\mathfrak{C}^B_\hbar$, on which the product is an extension of the twisted composition law (2.3.4). From the bijectivity of the partial Fourier transform and Proposition 2.3.3 we get the following

Corollary 2.3.4. *If the components of the magnetic field B are of class $BC^\infty(\Omega)$, then $\mathcal{S}(\mathscr{X}^*; C^\infty_0(\Omega))$ is a Fréchet *-subalgebra of $\mathfrak{B}^B_\hbar$, i. e. it is stable under complex conjugation and*

$$\mathcal{S}(\mathscr{X}^*; C^\infty_0(\Omega)) \sharp^B_\hbar \mathcal{S}(\mathscr{X}^*; C^\infty_0(\Omega)) \subset \mathcal{S}(\mathscr{X}^*; C^\infty_0(\Omega))$$

holds.

2.3.3 Representations

We first recall the definition of covariant representations of a magnetic C^*-dynamical system and the way they are used to construct representations of the corresponding C^*-algebras. We denote by $\mathcal{U}(\mathcal{H})$ the group of unitary operators in the Hilbert space $\mathcal{H}$ and by $\mathbf{B}(\mathcal{H})$ the C^*-algebra of all the linear bounded operators on $\mathcal{H}$.

Definition 2.3.5. *Given a magnetic C^*-dynamical system $(\mathcal{A}, \theta^\hbar, \kappa^{B,\hbar}, \mathscr{X})$, we call* covariant representation *$(\mathcal{H}, r, T)$ a Hilbert space $\mathcal{H}$ together with two maps $r : \mathcal{A} \to \mathbf{B}(\mathcal{H})$ and $T : \mathscr{X} \to \mathcal{U}(\mathcal{H})$ satisfying*

(i) r is a non-degenerate representation,

(ii) T is strongly continuous and $T(x)T(y) = r[\kappa^{B,\hbar}(x,y)]T(x+y), \quad \forall x, y \in \mathscr{X}$,

(iii) $T(x)r(\varphi)T(x)^* = r[\theta^\hbar_x(\varphi)], \quad \forall x \in \mathscr{X}, \ \varphi \in \mathcal{A}$.

Lemma 2.3.6. *If $(\mathcal{H}, r, T)$ is a covariant representation of $(\mathcal{A}, \theta^\hbar, \kappa^{B,\hbar}, \mathscr{X})$, then $\mathfrak{Rep}^T_r$ defined on $L^1(\mathscr{X}; \mathcal{A})$ by*

$$\mathfrak{Rep}^T_r(\Phi) := \int_{\mathscr{X}} \mathrm{d}y \, r\left[\theta^\hbar_{y/2}(\Phi(y))\right] T(y)$$

extends to a representation of $\mathfrak{C}^B_\hbar$.

By composing with the partial Fourier transformation, one gets representations of the pseudodifferential C^*-algebra $\mathfrak{B}^B_\hbar$, denoted by

$$\mathfrak{Op}^T_r : \mathfrak{B}^B_\hbar \to \mathbf{B}(\mathcal{H}), \quad \mathfrak{Op}^T_r(f) := \mathfrak{Rep}^T_r\left[(1 \otimes \mathcal{F})^{-1}(f)\right]. \tag{2.3.5}$$

Given any $\omega \in \Omega$, we shall now construct a concrete representation of $\mathfrak{C}^B_\hbar$ in $\mathcal{H} = L^2(\mathscr{X})$. Let r_ω be the representation of $\mathcal{A}$ in $\mathbf{B}(\mathcal{H})$ given for any $\varphi \in \mathcal{A}$, $u \in \mathcal{H}$ and $x \in \mathscr{X}$ by

$$[r_\omega(\varphi)u](x) = [\theta_x(\varphi)](\omega)\, u(x) \equiv \varphi(\theta_x[\omega])\, u(x).$$

Let also $T^\hbar_\omega$ be the map from $\mathscr{X}$ into the set of unitary operators on $\mathcal{H}$ given by

$$[T^\hbar_\omega(y)u](x) := \kappa^{B,\hbar}(\omega; x/\hbar, y)\, u(x + \hbar y) = e^{-\frac{i}{\hbar}\Gamma^{B_\omega}\langle 0, x, x+\hbar y\rangle} u(x + \hbar y).$$

Proposition 2.3.7. *$(\mathcal{H}, r_\omega, T^\hbar_\omega)$ is a covariant representation of the magnetic twisted C^*-dynamical system.*

Proof. Just use the definitions, Stokes Theorem for the magnetic field B_ω and the identities

$$\Gamma^{B_\omega}\langle x, x + \hbar y, x + \hbar y + \hbar z\rangle = \Gamma^{B_{\theta_x[\omega]}}\langle 0, \hbar y, \hbar y + \hbar z\rangle$$

and

$$\Gamma^{B_\omega}\langle 0, x + \hbar y, x\rangle = -\Gamma^{B_\omega}\langle 0, x, x + \hbar y\rangle,$$

valid for all $x, y, z \in \mathscr{X}$ and $\omega \in \Omega$. □

The integrated form $\mathfrak{Rep}^{\hbar}_{\omega} := \mathfrak{Rep}^{T^{\hbar}_{\omega}}_{r_\omega}$ has the following action on $L^1(\mathscr{X};\mathcal{A})$:

$$\begin{aligned}\left[\mathfrak{Rep}^{\hbar}_{\omega}(\Phi)u\right](x) &= \int_{\mathscr{X}} \mathrm{d}z\ \Phi\left(\theta_{x+\frac{\hbar z}{2}}[\omega];z\right)\ \kappa^{B,\hbar}(\omega;x/\hbar,z)\ u(x+\hbar z) \\ &= \hbar^{-n}\int_{\mathscr{X}} \mathrm{d}y\ \Phi\left(\theta_{\frac{x+y}{2}}[\omega];\frac{1}{\hbar}(y-x)\right)\ e^{-\frac{i}{\hbar}\Gamma^{B_\omega}\langle 0,x,y\rangle}\ u(y), \quad (2.3.6)\end{aligned}$$

and the corresponding representation $\mathfrak{Op}^{\hbar}_{\omega}$ of the C^*-algebra $\mathfrak{B}^B_{\hbar}$ has the following form on suitable $f\in\mathfrak{B}^B_{\hbar}$:

$$\left[\mathfrak{Op}^{\hbar}_{\omega}(f)u\right](x) = (2\pi\hbar)^{-n}\int_{\mathscr{X}}\mathrm{d}y\int_{\mathscr{X}^*}\mathrm{d}\xi\ e^{\frac{i}{\hbar}(x-y)\cdot\xi} f\left(\theta_{\frac{x+y}{2}}[\omega];\xi\right)\ e^{-\frac{i}{\hbar}\Gamma^{B_\omega}\langle 0,x,y\rangle}\ u(y). \tag{2.3.7}$$

It is clear that $\mathfrak{Op}^{\hbar}_{\omega}$ is *not* a faithful representation, since (2.3.7) only involves the values taken by f on $\mathcal{O}_\omega\times\mathscr{X}^*$, where $\mathcal{O}_\omega$ is the orbit passing through ω. It is rather easy to show that the kernel of $\mathfrak{Op}^{\hbar}_{\omega}$ can be identified with the twisted crossed product $C_0(\mathcal{Q}_\omega)\rtimes^{\kappa^B,\hbar}_{\theta^\hbar}\mathscr{X}$, constructed as explained above, with Ω replaced by $\mathcal{Q}_\omega := \overline{\mathcal{O}_\omega}$, the quasi-orbit generated by the point ω.

Remark 2.3.8. The expert in the theory of quantum magnetic fields might recognize in (2.3.7) the expression of a magnetic pseudodifferential operator with symbol $f\circ(\theta_\omega\otimes 1)$, written in the transverse gauge for the magnetic field B_ω. Then it will be a simple exercise to write down analogous representations associated to continuous (fields of) vector potentials $A:\Omega\to\wedge^1\mathscr{X}$ generating the magnetic field (i.e. $B_\omega = dA_\omega$, $\forall\omega\in\Omega$) and to check an obvious principle of gauge-covariance.

We show now that the family of representations $\{\mathfrak{Op}^{\hbar}_{\omega}\mid\omega\in\Omega\}$ actually has as a natural index set the orbit space of the dynamical system, up to unitary equivalence.

Proposition 2.3.9. *Let ω,ω' be two elements of Ω, belonging to the same orbit under the action θ. Then, for any $\hbar\in(0,1]$, one has $\mathfrak{Rep}^{\hbar}_{\omega}\cong\mathfrak{Rep}^{\hbar}_{\omega'}$ and $\mathfrak{Op}^{\hbar}_{\omega}\cong\mathfrak{Op}^{\hbar}_{\omega'}$ (unitary equivalence of representations).*

Proof. By assumption, there exists an element x_0 of $\mathscr{X}$ such that $\theta_{x_0}[\omega'] = \omega$. For $u\in\mathcal{H}$ and $x\in\mathscr{X}$ we define the unitary operator

$$\left(U^{\hbar}_{\omega,\omega'}u\right)(x) := e^{-\frac{i}{\hbar}\Gamma^{B_{\omega'}}\langle 0,x_0,x_0+x\rangle}\ u(x+x_0)\ .$$

To show unitary equivalence of the two representations, it is enough to show that for all $\varphi\in\mathcal{A}$ and $y\in\mathscr{X}$

$$U^{\hbar}_{\omega,\omega'}\, r_{\omega'}(\varphi) = r_\omega(\varphi)\, U^{\hbar}_{\omega,\omega'} \quad \text{and} \quad U^{\hbar}_{\omega,\omega'}\, T^{\hbar}_{\omega'}(y) = T^{\hbar}_{\omega}(y)\, U^{\hbar}_{\omega,\omega'}.$$

The first one is obvious. The second one reduces to

$$\begin{aligned}&\Gamma^{B_{\omega'}}\langle 0,x_0,x_0+x\rangle + \Gamma^{B_{\omega'}}\langle 0,x_0+x,x_0+x+\hbar y\rangle = \\ &\qquad = \Gamma^{B_{\omega'}}\langle x_0,x_0+x,x_0+x+\hbar y\rangle + \Gamma^{B_{\omega'}}\langle 0,x_0,x_0+x+\hbar y\rangle\,,\end{aligned}$$

which is true by Stokes Theorem. □

Remark 2.3.10. The Proposition reveals what we consider to be the main practical interest of the formalism we develop in the present article. To a fixed real symbol f and to a fixed value $\hbar$ of Planck's constant one associates a family

$$\{H^\hbar_\omega := \mathfrak{Op}^\hbar_\omega(f) \mid \omega \in \Omega\} \tag{2.3.8}$$

of self-adjoint magnetic pseudodifferential operators on the Hilbert space $\mathcal{H} := L^2(\mathscr{X})$, indexed by the points of a dynamical system $(\Omega, \theta, \mathscr{X})$ and satisfying the equivariance condition

$$H^\hbar_{\theta_x[\omega]} = \left(U^\hbar_{\omega,\theta_x[\omega]}\right)^{-1} H^\hbar_\omega \, U^\hbar_{\omega,\theta_x[\omega]}, \quad \forall (\omega, x) \in \Omega \times \mathscr{X}. \tag{2.3.9}$$

In concrete situations, such equivariance conditions usually carry some physical meaning. In a future publication we are going to extend the formalism to unbounded symbols f, getting realistic magnetic Quantum Hamiltonians organized in equivariant families, which will be studied in the framework of spectral theory.

To define other types of representations, we consider now Ω endowed with a θ-invariant measure μ. Such measures always exist, since $\mathscr{X}$ is abelian hence amenable. We set $\mathcal{H}'$ for the Hilbert space $L^2(\Omega, \mu)$ and consider first the faithful representation: $\tilde{r} : \mathcal{A} \to \mathbf{B}(\mathcal{H}')$ with $[\tilde{r}(\varphi)v](\omega) := \varphi(\omega)\ v(\omega)$ for all $v \in \mathcal{H}'$ and $\omega \in \Omega$. Then, (by a standard construction in the theory of twisted crossed products) the *regular representation of the magnetic C^*-dynamical system* $(\mathcal{A}, \theta^\hbar, \kappa^{B,\hbar}, \mathscr{X})$ *induced by* $\tilde{r}$ is the covariant representation $\left(L^2(\mathscr{X}; \mathcal{H}'), r, T^\hbar\right)$:

$$r : \mathcal{A} \to \mathbf{B}\big[L^2(\mathscr{X}; \mathcal{H}')\big] \quad \text{with} \quad [r(\varphi)w](\omega; x) := \left(\tilde{r}(\theta_x(\varphi))[w(x)]\right)(\omega) = \varphi\big(\theta_x(\omega)\big)\, w(\omega; x)\,,$$

$$T^\hbar : \mathscr{X} \to \mathcal{U}[L^2(\mathscr{X}; \mathcal{H}')] \quad \text{with} \quad [T^\hbar(y)w](\omega; x) := \kappa^{B,\hbar}(\omega; x/\hbar, y)\, w(\omega; x + \hbar y).$$

We identify freely $L^2(\mathscr{X}; \mathcal{H}')$ with $L^2(\Omega \times \mathscr{X})$ with the obvious product measure, so $r(\varphi)$ is the operator of multiplication by $\varphi \circ \theta$ in $L^2(\Omega \times \mathscr{X})$. Due to Stokes' Theorem, this is again a covariant representation.

The integrated form $\mathfrak{REP}^\hbar := \mathfrak{Rep}^{T^\hbar}_r$ associated to $(r, T^\hbar)$ is given on $L^1(\mathscr{X}; \mathcal{A})$ by

$$\left[\mathfrak{REP}^\hbar(\Phi)w\right](\omega; x) = \hbar^{-n} \int_{\mathscr{X}} \mathrm{d}y \; \Phi\left(\theta_{\frac{x+y}{2}}[\omega]; \frac{y-x}{\hbar}\right) \, e^{-\frac{i}{\hbar}\Gamma^{B_\omega}\langle 0,x,y\rangle} \; w(\omega; y)$$

and it admits the direct integral decomposition

$$\mathfrak{REP}^\hbar(\Phi) = \int_\Omega^\oplus \mathrm{d}\mu(\omega)\, \mathfrak{Rep}^\hbar_\omega(\Phi). \tag{2.3.10}$$

The group $\mathscr{X}$, being abelian, is amenable, and thus the regular representation $\mathfrak{REP}^\hbar$ is faithful. The corresponding representation $\mathfrak{OP}^\hbar : \mathfrak{B}^B_\hbar \to \mathbf{B}\left[L^2(\mathscr{X}; \mathcal{H}')\right]$ is given for f with partial Fourier transform in $L^1(\mathscr{X}; \mathcal{A})$ by

$$\left[\mathfrak{OP}^\hbar(f)w\right](\omega; x) = (2\pi\hbar)^{-n} \int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\eta \; e^{\frac{i}{\hbar}(x-y)\cdot\eta} \; f\left(\theta_{\frac{x+y}{2}}[\omega], \eta\right) \; e^{-\frac{i}{\hbar}\Gamma^{B_\omega}\langle 0,x,y\rangle} \; w(\omega; y)\,.$$

2.4 Asymptotic expansion of the product

The proof of strict deformation quantization hinges on the following Theorem:

Theorem 2.4.1 (Asymptotic expansion of the product). *Assume the components of B are in $BC^\infty(\Omega)$. Let $\Phi, \Psi \in \mathcal{S}(\mathscr{X}; C_0^\infty(\Omega))$ and $\hbar \in (0,1]$. Then the product $\Phi \diamond_\hbar^B \Psi$ can be expanded in powers of $\hbar$,*

$$\Phi \diamond_\hbar^B \Psi = \Phi \diamond_0 \Psi - \hbar \tfrac{i}{2} \{\Phi, \Psi\}^B + \hbar^2 R_\hbar^{\diamond,2}(\Phi, \Psi), \tag{2.4.1}$$

where $\{\Phi, \Psi\}^B$ is defined as in equation (2.2.6). All terms are in $\mathcal{S}(\mathscr{X}; C_0^\infty(\Omega))$ and $R_\hbar^{\diamond,2}(\Phi, \Psi)$ is bounded uniformly in $\hbar$, $\| R_\hbar^{\diamond,2}(\Phi, \Psi) \|_\hbar^B \leq C$.

Proof. We are going to use Einstein's summation convention, i. e. repeated indices in a product are summed over. Two types of terms in the product formula need to be expanded in $\hbar$, the group action of $\mathcal{X}$ on Ω,

$$\begin{aligned}
\left(\theta_{\frac{\hbar}{2}y}[\Phi(x)]\right)(\omega) &= \Phi\left(\theta_{\frac{\hbar}{2}y}[\omega]; x\right) = \Phi(\omega; x) + \hbar \int_0^1 \mathrm{d}\tau \, \tfrac{1}{2} y_j \, \theta_{\tau \frac{\hbar}{2} y}\left[(\delta_j \Phi)(\omega; x)\right] \\
&=: \Phi(\omega; x) + \hbar \left(R_{\hbar,y}^{\theta,1}(\Phi)\right)(\omega; x) \\
&= \Phi(\omega; x) + \tfrac{\hbar}{2} y_j \, (\delta_j \Phi)(\omega; x) + \hbar^2 \int_0^1 \mathrm{d}\tau \, \tfrac{1}{4}(1-\tau) \, y_j y_k \, \theta_{\tau \frac{\hbar}{2} y}\left[(\delta_j \delta_k \Phi)(\omega; x)\right] \\
&=: \Phi(\omega; x) + \tfrac{\hbar}{2} y_j \, (\delta_j \Phi)(\omega; x) + \hbar^2 \left(R_{\hbar,y}^{\theta,2}(\Phi)\right)(\omega; x),
\end{aligned}$$

and the exponential of the magnetic flux,

$$\begin{aligned}
e^{-i\hbar \Lambda_\hbar^B(x,y)} &= 1 + \hbar \int_0^1 \mathrm{d}\tau \, \frac{\mathrm{d}}{\mathrm{d}\epsilon} \left(e^{-i\epsilon \Lambda_\epsilon^B(x,y)}\right)\bigg|_{\epsilon = \tau\hbar} =: 1 + \hbar \, R_\hbar^{\kappa,1}(x, y) \\
&= 1 - \hbar i \Lambda_0^B(x, y) + \hbar^2 \int_0^1 \mathrm{d}\tau \, (1-\tau) \, \frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2} \left(e^{-i\epsilon \Lambda_\epsilon^B(x,y)}\right)\bigg|_{\epsilon = \tau\hbar} \\
&=: 1 - \hbar \tfrac{i}{2} B^{jk} \, y_j \, (x_k - y_k) + \hbar^2 \, R_\hbar^{\kappa,2}(x, y).
\end{aligned}$$

We will successively plug these expansions into the product formula, keeping only terms of $\mathcal{O}(\hbar^2)$:

$$\begin{aligned}
&(\Phi \diamond_\hbar^B \Psi)(x) \\
&= \int_{\mathcal{X}} \mathrm{d}y \, \Big(\Phi(y) + \tfrac{\hbar}{2}(y_j - x_j)\,(\delta_j \Phi)(y) + \hbar^2 \big(R^{\theta,2}_{\hbar,y-x}(\Phi)\big)(y)\Big)\, \theta_{\frac{\hbar}{2}y}[\Psi(x-y)]\, e^{-i\hbar\Lambda^B_\hbar(x,y)} \\
&= \int_{\mathcal{X}} \mathrm{d}y \, \Phi(y) \Big(\Psi(x-y) + \tfrac{\hbar}{2} y_j \,(\delta_j \Psi)(x-y) + \hbar^2 \big(R^{\theta,2}_{\hbar,y}(\Psi)\big)(x-y)\Big)\, e^{-i\hbar\Lambda^B_\hbar(x,y)} + \\
&\quad + \frac{\hbar}{2} \int_{\mathcal{X}} \mathrm{d}y \,(y_j - x_j)\,(\delta_j\Phi)(y) \Big(\Psi(x-y) + \hbar \big(R^{\theta,1}_{\hbar,y}(\Psi)\big)(x-y)\Big)\, e^{-i\hbar\Lambda^B_\hbar(x,y)} + \\
&\quad + \hbar^2 \int_{\mathcal{X}} \mathrm{d}y \, \big(R^{\theta,2}_{\hbar,y-x}(\Phi)\big)(y)\, \theta_{\frac{\hbar}{2}y}[\Psi(x-y)]\, e^{-i\hbar\Lambda^B_\hbar(x,y)} \\
&= \int_{\mathcal{X}} \mathrm{d}y \, \Phi(y)\, \Psi(x-y) \Big(1 - \hbar i \Lambda^B_0(x,y) + \hbar^2 R^{\kappa,2}_\hbar(x,y)\Big) + \\
&\quad + \frac{\hbar}{2} \int_{\mathcal{X}} \mathrm{d}y \, \Big(-(\delta_j\Phi)(y)\,(Q_j\Psi)(x-y) + (Q_j\Phi)(y)\,(\delta_j\Psi)(x-y)\Big) \Big(1 + \hbar R^{\kappa,1}_\hbar(x,y)\Big) + \\
&\quad + \hbar^2 \int_{\mathcal{X}} \mathrm{d}y \, \Big[\big(R^{\theta,2}_{\hbar,y-x}(\Phi)\big)(y)\, \theta_{\frac{\hbar}{2}y}[\Psi(x-y)] + \Phi(y)\, \big(R^{\theta,2}_{\hbar,y}(\Psi)\big)(x-y) - \\
&\qquad\qquad - \tfrac{1}{2}(\delta_j\Phi)(y)\, \big(Q_j R^{\theta,1}_{\hbar,y}(\Psi)\big)(x-y)\Big]\, e^{-i\hbar\Lambda^B_\hbar(x,y)} \\
&= \int_{\mathcal{X}} \mathrm{d}y \, \Phi(y)\, \Psi(x-y) + \frac{\hbar}{2} \int_{\mathcal{X}} \mathrm{d}y \, \Big((Q_j\Phi)(y)\,(\delta_j\Psi)(x-y) - (\delta_j\Phi)(y)\,(Q_j\Psi)(x-y) \\
&\qquad\qquad - i B^{jk}\,(Q_j\Phi)(y)\,(Q_k\Psi)(x-y)\Big) + \\
&\quad + \hbar^2 \int_{\mathcal{X}} \mathrm{d}y \, \Big[\Big(\big(R^{\theta,2}_{\hbar,y-x}(\Phi)\big)(y)\, \theta_{\frac{\hbar}{2}y}[\Psi(x-y)] + \Phi(y)\, \big(R^{\theta,2}_{\hbar,y}(\Psi)\big)(x-y)\Big)\, e^{-i\hbar\Lambda^B_\hbar(x,y)} - \\
&\qquad\qquad - \tfrac{1}{2}(\delta_j\Phi)(y)\, \big(Q_j R^{\theta,1}_{\hbar,y}(\Psi)\big)(x-y)\, e^{-i\hbar\Lambda^B_\hbar(x,y)} + \\
&\qquad\qquad + \tfrac{1}{2}\big((Q_j\Phi)(y)\,(\delta_j\Psi)(x-y) - (\delta_j\Phi)(y)\,(Q_j\Psi)(x-y)\big)\, R^{\kappa,1}_\hbar(x,y) + \\
&\qquad\qquad + \Phi(y)\, \Psi(x-y)\, R^{\kappa,2}_\hbar(x,y)\Big] \\
&=: (\Phi \diamond_0 \Psi)(x) - \hbar \tfrac{i}{2} \{\Phi, \Psi\}^B(x) + \hbar^2 \big(R^{\diamond,2}_\hbar(\Phi,\Psi)\big)(x).
\end{aligned}$$

In the above, we have used $(y_j - x_j)\,\Psi(x-y) = -(Q_j\Psi)(x-y)$, $y_j\,\Phi(y) = (Q_j\Phi)(y)$ and the explicit expression for $\Lambda^B_0(x,y)$.

Clearly, the leading-order and sub-leading-order terms are again in $\mathcal{S}(\mathscr{X}; C^\infty_0(\Omega))$. Thus also $R^{\diamond,2}_\hbar(\Phi,\Psi) = \hbar^{-2}\big(\Phi \diamond^B_\hbar \Psi - \Phi \diamond_0 \Psi + \hbar \tfrac{i}{2}\{\Phi,\Psi\}^B\big)$ is an element of $\mathcal{S}(\mathscr{X}; C^\infty_0(\Omega))$ for all $\hbar \in (0,1]$.

The most difficult part of the proof is to show that the $\hbar$-dependent C^*-norm of the remainder $R^{\diamond,2}_\hbar(\Phi,\Psi)$ can be uniformly bounded in $\hbar$. The first ingredient is the fact that the $\hbar$-dependent C^*-norm of the twisted crossed product is dominated by the $L^1(\mathcal{X};\mathcal{A})$-norm for all values of $\hbar \in (0,1]$,

$$\|\Phi\|^B_\hbar \leq \|\Phi\|_{L^1}, \quad \forall \Phi \in \mathcal{S}(\mathscr{X}; C^\infty_0(\Omega)) \subset L^1(\mathscr{X}; C_0(\Omega)) \subset \mathfrak{C}^B_\hbar.$$

Hence, if we can find $\hbar$-independent L^1 bounds on each term of the remainder, we have also estimated the $\hbar$-dependent C^*-norm uniformly in $\hbar$.

There are four distinct types of terms in the remainder. Let us start with the first: we define

$$\big(R^{\diamond,2}_{\hbar,1}(\Phi,\Psi)\big)(x) := \int_{\mathcal{X}} \mathrm{d}y\, \big(R^{\theta,2}_{\hbar,y-x}(\Phi)\big)(y)\, \theta_{\frac{\hbar}{2}y}[\Psi(x-y)]\, e^{-i\hbar\Lambda^B_\hbar(x,y)}.$$

Then we have

$$\begin{aligned}
\big\|R^{\diamond,2}_{\hbar,1}(\Phi,\Psi)\big\|^B_\hbar &\leq \big\|R^{\diamond,2}_{\hbar,1}(\Phi,\Psi)\big\|_{L^1(\mathcal{X};C_0(\Omega))} \\
&\leq \int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \mathrm{d}y\, \big\|\big(R^{\theta,2}_{\hbar,y-x}(\Phi)\big)(y)\big\|_{C_0(\Omega)}\, \big\|\theta_{\frac{\hbar}{2}y}[\Psi(x-y)]\big\|_{C_0(\Omega)}\, \big\|e^{-i\hbar\Lambda^B_\hbar(x,y)}\big\|_{C_0(\Omega)} \\
&= \int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \mathrm{d}y\, \big\|\big(R^{\theta,2}_{\hbar,-x}(\Phi)\big)(y)\big\|_{C_0(\Omega)}\, \big\|\Psi(x)\big\|_{C_0(\Omega)}.
\end{aligned}$$

We inspect $\big\|\big(R^{\theta,2}_{\hbar,-x}(\Phi)\big)(y)\big\|_{C_0(\Omega)}$ more closely:

$$\begin{aligned}
\big\|\big(R^{\theta,2}_{\hbar,-x}(\Phi)\big)(y)\big\|_{C_0(\Omega)} &\leq \frac{1}{4}\int_0^1 \mathrm{d}\tau\, \big|(-x_j)(-x_k)\big|\, \big\|\theta_{-\tau\frac{\hbar}{2}x}[(\delta_j\delta_k\Phi)(y)]\big\|_{C_0(\Omega)} \\
&= \frac{1}{8}|x_j\, x_k|\, \big\|(\delta_j\delta_k\Phi)(y)\big\|_{C_0(\Omega)}.
\end{aligned}$$

If we plug that back into the estimate of the L^1 norm, we get

$$\begin{aligned}
\big\|R^{\diamond,2}_{\hbar,1}(\Phi,\Psi)\big\|^B_\hbar &\leq \frac{1}{8}\int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \mathrm{d}y\, \big\|(\delta_j\delta_k\Phi)(y)\big\|_{C_0(\Omega)}\, \big\|(Q_jQ_k\Psi)(x)\big\|_{C_0(\Omega)} \\
&= \frac{1}{8}\big\|\delta_j\delta_k\Phi\big\|_{L^1}\big\|Q_jQ_k\Psi\big\|_{L^1}.
\end{aligned}$$

The right-hand side is finite by the definition of $\mathcal{S}\left(\mathscr{X};C^\infty_0(\Omega)\right)$. Similarly, the second term can be estimated, just the roles of Φ and Ψ are reversed.

Now to the second type of term: we define

$$\big(R^{\diamond,2}_{\hbar,3}(\Phi,\Psi)\big)(x) := -\frac{1}{2}\int_{\mathcal{X}} \mathrm{d}y\, (\delta_j\Phi)(y)\, \big(Q_jR^{\theta,1}_{\hbar,y}(\Psi)\big)(x-y)\, e^{-i\hbar\Lambda^B_\hbar(x,y)}$$

and estimate

$$\big\|R^{\diamond,2}_{\hbar,3}(\Phi,\Psi)\big\|^B_\hbar \leq \big\|R^{\diamond,2}_{\hbar,3}(\Phi,\Psi)\big\|_{L^1} \leq \frac{1}{2}\int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \mathrm{d}y\, \big\|(\delta_j\Phi)(y)\big\|_{C_0(\Omega)}\, \big\|\big(Q_jR^{\theta,1}_{\hbar,y}(\Psi)\big)(x)\big\|_{C_0(\Omega)}.$$

The last factor needs to be estimated by hand:

$$\begin{aligned}
\big\|\big(Q_jR^{\theta,1}_{\hbar,y}(\Psi)\big)(x)\big\|_{C_0(\Omega)} &\leq \frac{1}{2}\int_0^1 \mathrm{d}\tau\, |x_j\, y_k|\, \big\|\theta_{\tau\frac{\hbar}{2}y}[(\delta_k\Psi)(x)]\big\|_{C_0(\Omega)} \\
&= \frac{1}{2}|x_j\, y_k|\, \big\|(\delta_k\Psi)(x)\big\|_{C_0(\Omega)}.
\end{aligned}$$

This leads to the bound

$$\begin{aligned}\big\|R^{\diamond,2}_{\hbar,3}(\Phi,\Psi)\big\|^B_\hbar &\leq \frac{1}{4}\int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \mathrm{d}y \, \big\|(\delta_j\Phi)(y)\big\|_{C_0(\Omega)} \, |x_j \, y_k| \, \big\|(\delta_k\Psi)(x)\big\|_{C_0(\Omega)} \\ &= \frac{1}{4}\big\|Q_k\delta_j\Phi\big\|_{L^1} \, \big\|Q_j\delta_k\Psi\big\|_{L^1}.\end{aligned}$$

The right-hand side is again finite since $\Phi,\Psi \in \mathcal{S}\left(\mathscr{X};C^\infty_0(\Omega)\right)$ and does not depend on $\hbar$.

Estimating the two magnetic terms is indeed a bit more involved: we define

$$\big(R^{\diamond,2}_{\hbar,4}(\Phi,\Psi)\big)(x) := \frac{1}{2}\int_{\mathcal{X}} \mathrm{d}y \, (Q_j\Phi)(y) \, (\delta_j\Psi)(x-y) \, R^{\kappa,1}_\hbar(x,y).$$

The usual arguments show the C^*-norm can be estimated by

$$\big\|R^{\diamond,2}_{\hbar,4}(\Phi,\Psi)\big\|^B_\hbar \leq \int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \mathrm{d}y \, \big\|(Q_j\Phi)(y)\big\|_{C_0(\Omega)} \, \big\|(\delta_j\Psi)(x-y)\big\|_{C_0(\Omega)} \, \big\|R^{\kappa,1}_\hbar(x,y)\big\|_{C_0(\Omega)}$$

which warrants a closer inspection of the last term: first of all, we note that

$$\begin{aligned}R^{\kappa,1}_\hbar(x,y) &= \int_0^1 \mathrm{d}\tau \, \frac{\mathrm{d}}{\mathrm{d}\epsilon}\big(e^{-i\epsilon\Lambda^B_\epsilon(x,y)}\big)\Big|_{\epsilon=\tau\hbar} \\ &= \int_0^1 \mathrm{d}\tau \, \Big(-i\Lambda^B_\epsilon(x,y) - i\epsilon \, \tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Lambda^B_\epsilon(x,y)\Big) \, e^{-i\epsilon\Lambda^B_\epsilon(x,y)}\Big|_{\epsilon=\tau\hbar}.\end{aligned}$$

If we use Lemma 2.5.4 and $\hbar \leq 1$, then this leads to the following $C_0(\Omega)$ norm estimate of $R^{\kappa,1}_\hbar(x,y)$:

$$\begin{aligned}&\big\|R^{\kappa,1}_\hbar(x,y)\big\|_{C_0(\Omega)} \\ &\quad\leq \int_0^1 \mathrm{d}\tau \, \Big(\big\|\Lambda^B_{\tau\hbar}(x,y)\big\|_{C_0(\Omega)} + \hbar\tau \, \Big\|\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Lambda^B_\epsilon(x,y)\Big|_{\epsilon=\tau\hbar}\Big\|_{C_0(\Omega)}\Big) \, \big\|e^{-i\epsilon\Lambda^B_{\tau\hbar}(x,y)}\big\|_{C_0(\Omega)} \\ &\quad\leq \big\|B^{jk}\big\|_{C_0(\Omega)} \, |y_j| \, |x_k - y_k| + \tfrac{1}{2}\big\|\delta_l B^{jk}\big\|_{C_0(\Omega)} \, |y_j| \, |x_k - y_k| \, \big(|x_l - y_l| + |y_l|\big).\end{aligned}$$

Put together, this allows us to estimate the norm of $R^{\diamond,2}_{\hbar,4}$ by

$$\begin{aligned}\big\|R^{\diamond,2}_{\hbar,4}(\Phi,\Psi)\big\|^B_\hbar &\leq \big\|B^{mk}\big\|_{C_0(\Omega)} \, \|Q_jQ_m\Phi\|_{L^1} \, \|Q_k\delta_j\Psi\|_{L^1} + \\ &\quad + \tfrac{1}{2}\big\|\delta_l B^{mk}\big\|_{C_0(\Omega)} \Big(\|Q_jQ_mQ_l\Phi\|_{L^1} \, \|Q_k\delta_j\Psi\|_{L^1} + \|Q_jQ_m\Phi\|_{L^1} \, \|Q_kQ_l\delta_j\Psi\|_{L^1}\Big).\end{aligned}$$

Now on to the last term,

$$\big(R^{\diamond,2}_{\hbar,6}(\Phi,\Psi)\big)(x) := \int_{\mathcal{X}} \mathrm{d}y \, \Phi(y) \, \Psi(x-y) \, R^{\kappa,2}_\hbar(x,y).$$

Using the explicit form of $R^{\kappa,2}_{\hbar}(x,y)$,

$$\begin{aligned}R^{\kappa,2}_{\hbar}(x,y) &= \int_0^1 \mathrm{d}\tau\,(1-\tau)\,\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\big(e^{-i\epsilon\Lambda^B_\epsilon(x,y)}\big)\Big|_{\epsilon=\tau\hbar} \\ &= \int_0^1 \mathrm{d}\tau\,(1-\tau)\,\Big[-i2\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Lambda^B_\epsilon(x,y) - i\epsilon\tfrac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\Lambda^B_\epsilon(x,y) - \\ &\quad - \Big(\Lambda^B_\epsilon(x,y) + \epsilon\,\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Lambda^B_\epsilon(x,y)\Big)^2\Big]\Big|_{\epsilon=\tau\hbar} e^{-i\tau\hbar\Lambda^B_{\tau\hbar}(x,y)},\end{aligned}$$

in conjunction with the estimates derived in Lemma 2.5.4 (which are uniform in τ), we get

$$\begin{aligned}\big\|R^{\kappa,2}_{\hbar}(x,y)\big\|_{C_0(\Omega)} &\leq \int_0^1 \mathrm{d}\tau\,(1-\tau)\,\Big[2\big\|\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Lambda^B_\epsilon(x,y)\big|_{\epsilon=\tau\hbar}\big\|_{C_0(\Omega)} + \tau\big\|\tfrac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\Lambda^B_\epsilon(x,y)\big|_{\epsilon=\tau\hbar}\big\|_{C_0(\Omega)} + \\ &\quad + \Big(\big\|\Lambda^B_{\hbar\tau}(x,y)\big\|_{C_0(\Omega)} + \tau\,\big\|\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Lambda^B_\epsilon(x,y)\big|_{\epsilon=\tau\hbar}\big\|\Big)^2\Big]\,\big\|e^{-i\tau\hbar\Lambda^B_{\tau\hbar}(x,y)}\big\|_{C_0(\Omega)} \\ &= \int_0^1 \mathrm{d}\tau\,(1-\tau)\,\Big[2\big\|\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Lambda^B_\epsilon(x,y)\big|_{\epsilon=\tau\hbar}\big\|_{C_0(\Omega)} + \tau\big\|\tfrac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\Lambda^B_\epsilon(x,y)\big|_{\epsilon=\tau\hbar}\big\|_{C_0(\Omega)} + \\ &\quad + \big\|\Lambda^B_{\hbar\tau}(x,y)\big\|^2_{C_0(\Omega)} + 2\tau\,\big\|\Lambda^B_{\hbar\tau}(x,y)\big\|_{C_0(\Omega)}\,\big\|\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Lambda^B_\epsilon(x,y)\big|_{\epsilon=\tau\hbar}\big\|_{C_0(\Omega)} + \\ &\quad + \tau^2\big\|\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Lambda^B_\epsilon(x,y)\big|_{\epsilon=\tau\hbar}\big\|^2_{C_0(\Omega)}\Big].\end{aligned}$$

Hence, we can bound the $\hbar$-dependent C^*-norm of $R^{\diamond,2}_{\hbar,6}$ by

$$\begin{aligned}\big\|R^{\diamond,2}_{\hbar,6}\big\|^B_\hbar &\leq \big\|\delta_l B^{jk}\big\|_{C_0(\Omega)}\Big(\big\|Q_j\Phi\big\|_{L^1}\,\big\|Q_kQ_l\Psi\big\|_{L^1} + \big\|Q_jQ_l\Phi\big\|_{L^1}\,\big\|Q_k\Psi\big\|_{L^1}\Big) + \\ &\quad + \tfrac{1}{6}\big\|\delta_l\delta_m B^{jk}\big\|_{C_0(\Omega)}\Big(\big\|Q_j\Phi\big\|_{L^1}\,\big\|Q_kQ_lQ_m\Psi\big\|_{L^1} + \big\|Q_jQ_m\Phi\big\|_{L^1}\,\big\|Q_kQ_l\Psi\big\|_{L^1} + \\ &\qquad + \big\|Q_jQ_lQ_m\Phi\big\|_{L^1}\,\big\|Q_k\Psi\big\|_{L^1}\Big) + \\ &\quad + \tfrac{1}{2}\big\|B^{jk}\big\|_{C_0(\Omega)}\,\big\|B^{j'k'}\big\|_{C_0(\Omega)}\,\big\|Q_jQ_{j'}\Phi\big\|_{L^1}\,\big\|Q_kQ_{k'}\Psi\big\|_{L^1} + \\ &\quad + \tfrac{1}{3}\big\|B^{jk}\big\|_{C_0(\Omega)}\,\big\|\delta_{l'}B^{j'k'}\big\|_{C_0(\Omega)}\Big(\big\|Q_jQ_{j'}\Phi\big\|_{L^1}\,\big\|Q_kQ_{k'}Q_{l'}\Psi\big\|_{L^1} + \\ &\qquad + \big\|Q_jQ_{j'}Q_{l'}\Phi\big\|_{L^1}\,\big\|Q_kQ_{k'}\Psi\big\|_{L^1}\Big) + \\ &\quad + \tfrac{1}{12}\big\|\delta_l B^{jk}\big\|_{C_0(\Omega)}\,\big\|\delta_{l'}B^{j'k'}\big\|_{C_0(\Omega)}\Big(\big\|Q_jQ_{j'}\Phi\big\|_{L^1}\,\big\|Q_kQ_{k'}Q_lQ_{l'}\Psi\big\|_{L^1} + \\ &\quad + 2\big\|Q_jQ_{j'}Q_l\Phi\big\|_{L^1}\,\big\|Q_kQ_{k'}Q_{l'}\Psi\big\|_{L^1} + \big\|Q_jQ_{j'}Q_lQ_{l'}\Phi\big\|_{L^1}\,\big\|Q_kQ_{k'}\Psi\big\|_{L^1}\Big).\end{aligned}$$

Putting all these individual estimates together yields a bound on $\big\|R^{\diamond,2}_{\hbar}(\Phi,\Psi)\big\|^B_\hbar$ which is uniform in $\hbar$ and the proof of the Theorem is finished. □

Corollary 2.4.2. *Assume the components of B are in $BC^\infty(\Omega)$. Let $f,g \in \mathcal{S}\big(\mathscr{X}^*;C^\infty_0(\Omega)\big)$ and $\hbar \in (0,1]$. Then the product $f\sharp^B_\hbar g$ can be expanded in powers of $\hbar$,*

$$f\sharp^B_\hbar g = f\,g - \hbar\tfrac{i}{2}\{f,g\}_B + \hbar^2 R^{\sharp,2}_\hbar(f,g), \tag{2.4.2}$$

where fg is the pointwise product and $\{f,g\}_B$ is the magnetic Poisson bracket defined as in equation (2.2.4). All terms are in $\mathcal{S}(\mathscr{X}^*; C_0^\infty(\Omega))$ and the remainder satisfies $\| R_\hbar^{\sharp;2}(f,g) \|_{\mathfrak{B}_\hbar^B}$ $\leq C$ uniformly in $\hbar$.

Proof. The proof follows from equations (2.3.3), (2.2.5), (2.2.6) and Theorem 2.4.1, keeping in mind that the partial Fourier transforms are isomorphisms $\mathcal{S}(\mathscr{X}^*; C^\infty(\Omega)) \rightleftarrows \mathcal{S}(\mathscr{X}; C^\infty(\Omega))$ that extend to automorphisms between the C^*-algebras $\mathfrak{B}_\hbar^B$ and $\mathfrak{C}_\hbar^B$. □

2.5 Strict deformation quantization

To make this precise, we repeat an already standard concept. For more details and motivation, the reader could see [43, 44, 24] and references therein.

Definition 2.5.1. *Let $(\mathcal{S}, \circ, \{\cdot,\cdot\})$ be a real Poisson algebra which is densely contained on the selfadjoint part $\mathfrak{C}_{0,\mathcal{R}}$ of an abelian C^*-algebra $\mathfrak{C}_0$.* A strict deformation quantization of the Poisson algebra $\mathcal{S}$ *is a family of $\mathcal{R}$-linear injections $(\mathfrak{Q}_\hbar : \mathcal{S} \to \mathfrak{C}_{\hbar,\mathcal{R}})_{\hbar\in I}$, where $I \subset \mathcal{R}$ contains 0 as an accumulation point, $\mathfrak{C}_{\hbar,\mathcal{R}}$ is the selfadjoint part of the C^*-algebra $\mathfrak{C}_\hbar$, with products and norms denoted by $\diamond_\hbar$ and $\|\cdot\|_\hbar$, $\mathfrak{Q}_0$ is just the inclusion map and $\mathfrak{Q}_\hbar(\mathcal{S})$ is a subalgebra of $\mathfrak{C}_{\hbar,\mathcal{R}}$.*

The following conditions are required for each $\Phi, \Psi \in \mathcal{S}$

(i) Rieffel axiom*: the mapping $I \ni \hbar \mapsto \left\|\mathfrak{Q}_\hbar(\Phi)\right\|_\hbar$ is continuous.*

(ii) Von Neumann axiom*:*

$$\lim_{\hbar\to 0}\left\|\tfrac{1}{2}\left[\mathfrak{Q}_\hbar(\Phi)\diamond_\hbar \mathfrak{Q}_\hbar(\Psi) + \mathfrak{Q}_\hbar(\Psi)\diamond_\hbar \mathfrak{Q}_\hbar(\Phi)\right] - \mathfrak{Q}_\hbar(\Phi\circ\Psi)\right\|_\hbar = 0.$$

(iii) Dirac axiom*:*

$$\lim_{\hbar\to 0}\left\|\tfrac{i}{\hbar}\left[\mathfrak{Q}_\hbar(\Phi)\diamond_\hbar \mathfrak{Q}_\hbar(\Psi) - \mathfrak{Q}_\hbar(\Psi)\diamond_\hbar \mathcal{Q}_\hbar(\Phi)\right] - \mathfrak{Q}_\hbar(\{\Phi,\Psi\})\right\|_\hbar = 0.$$

Putting this into the present context, we have

Theorem 2.5.2. *Assume that $B^{jk} \in BC^\infty(\Omega)$ and $I = [0,1]$. Then the family of injections*

$$\left(\mathcal{S}(\mathscr{X}, C_0^\infty(\Omega))_\mathcal{R} \hookrightarrow \mathfrak{C}^B_{\hbar,\mathcal{R}}\right)_{\hbar\in I}$$

defines a strict deformation quantization.

Proof. By Proposition 2.2.9 and Proposition 2.3.3, $\mathcal{S}(\mathscr{X}, C_0^\infty(\Omega))_\mathcal{R}$ can be seen a Poisson algebra with respect to $\diamond_0$ and $\{\cdot,\cdot\}^B$ as well as a subalgebra of the real part of each of the twisted crossed product $\mathfrak{C}_\hbar^B$.

Von Neumann and Dirac axioms are direct consequences of Theorem 2.4.1.

The Rieffel axiom can be checked exactly as in [31], which builds on results from [37, 42]. The fact that the algebra $\mathcal{A}$ in [31] consisted of continuous functions defined on the group $\mathscr{X}$ itself does not play any role here. □

A partial Fourier transform transfers these results directly to $\mathcal{S}\big(\mathscr{X}^*, C_0^\infty(\Omega)\big)$ and $\mathfrak{B}_\hbar^B$, objects which are more natural in the context of Weyl calculus. In this way we extend the main result of [31] to magnetic twisted actions on *general* abelian C^*-algebras.

Corollary 2.5.3. *Assume that* $B^{jk} \in C^\infty(\Omega)$. *Let* $I = [0,1]$. *Then the family of injections*

$$\Big(\mathcal{S}\big(\mathscr{X}^*, C_0^\infty(\Omega)\big)_\mathcal{R} \hookrightarrow \mathfrak{B}_{\hbar,\mathcal{R}}^B\Big)_{\hbar\in I}$$

defines a strict deformation quantization, where the Poisson algebra structure in $\mathcal{S}\big(\mathscr{X}^*, C_0^\infty(\Omega)\big)_\mathcal{R}$ *is given by point-wise multiplication and the Poisson bracket* $\{\cdot,\cdot\}_B$.

Proof. The proof is straightforward from the Corollary 2.4.2 and the above theorem, after noticing that the partial Fourier transform is an isomorphism between the Poisson algebras $\mathcal{S}\big(\mathscr{X}^*; C_0^\infty(\Omega)\big)$ and $\mathcal{S}\big(\mathscr{X}; C_0^\infty(\Omega)\big)$, and it extends to an isomorphisms between the C^*-algebras $\mathfrak{B}_\hbar^B$ and $\mathfrak{C}_\hbar^B$. $\square$

Appendix: Estimates on the magnetic flux

In the next lemma we gather some useful estimates on the scaled magnetic flux and its exponential, that are used in the proofs of Propositions 2.3.3 and 2.4.1.

Lemma 2.5.4. *Assume the components of* B *are in* $BC^\infty(\Omega)$ *and* $\hbar \in (0,1]$.

(i) For all multiindices $a, \alpha \in \mathbb{N}^n$ *there exist constants* $C^j > 0$, $C^{jk} > 0$, $j,k \in \{1,\dots,n\}$, *depending on* B^{jk} *and its* δ*-derivatives up to* $(|a|+|\alpha|)$*th order, such that*

$$\big\|\partial_x^a \delta^\alpha \Lambda_\hbar^B(x,y)\big\|_{C_0(\Omega)} \leq \sum_{j=1}^n C_1^j\,|y_j| + \sum_{j,k=1}^n C_2^{jk}\,|y_j|\,|x_k - y_k|.$$

(ii) For all $a, \alpha \in \mathbb{N}^n$ *there exists a polynomial* $p_{a\alpha}$ *in* $2n$ *variables, with coefficients* $K_{bc} \geq 0$, *such that*

$$\begin{aligned}\big\|\partial_x^a \delta^\alpha e^{-i\hbar\Lambda_\hbar^B(x,y)}\big\|_{C_0(\Omega)} &\leq p_{a\alpha}\big(|y_1|,\dots,|y_n|,|x_1-y_1|,\dots,|x_n-y_n|\big)\\ &= \sum_{|b|+|c|\leq 2(|a|+|\alpha|)} K_{bc}\,|y^b|\,|(x-y)^c|.\end{aligned}$$

(iii) The following estimates which are uniform in $\hbar$ *and* τ *hold for the magnetic flux and its derivatives:*

$$\begin{aligned}\big\|\Lambda_{\hbar\tau}^B(x,y)\big\|_{C_0(\Omega)} &\leq \sum_{jk} \big\|B^{jk}\big\|_{C_0(\Omega)}\,|y_j|\,|x_k-y_k|,\\ \Big\|\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Lambda_\epsilon^B(x,y)\big|_{\epsilon=\tau\hbar}\Big\|_{C_0(\Omega)} &\leq \sum_{jkl} \big\|\delta_l B^{jk}\big\|_{C_0(\Omega)}\,|y_j|\,|x_k-y_k|\,\big(|x_l-y_l|+|y_l|\big),\\ \Big\|\tfrac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\Lambda_\epsilon^B(x,y)\big|_{\epsilon=\tau\hbar}\Big\|_{C_0(\Omega)} &\leq \sum_{jklm} \big\|\delta_l\delta_m B^{jk}\big\|_{C_0(\Omega)}\,|y_j|\,|x_k-y_k|\,\big(|x_l-y_l|\,|x_m-y_m|+\\ &\qquad +|y_l|\,|x_m-y_m|+|y_l|\,|y_m|\big).\end{aligned}$$

Proof. (i) and (ii) follow directly from the explicit parametrization of the magnetic flux.

(iii) Throughout the proof we are going to use Einstein's summation convention, i. e. repeated indices in a product are summed over from 1 to $\dim(\mathscr{X})$. From the explicit parametrization (2.3.1)

$$\Lambda^B_\epsilon(x,y) = y_j\,(x_k - y_k) \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s\, \theta_{\epsilon(s-1/2)x+\epsilon(t-s)y}[B^{jk}],$$

we compute first and second derivative of $\Lambda^B_\epsilon(x,y)$ with respect to ϵ, using dominated convergence to interchange differentiation with respect to the parameter ϵ and integration with respect to t and s,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Lambda^B_\epsilon(x,y) &= y_j\,(x_k - y_k) \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s\, \big(s(x_l - y_l) + t y_l - \tfrac{1}{2}x_l\big)\, \theta_{\epsilon(s-1/2)x+\epsilon(t-s)y}[\delta_l B^{jk}], \\
\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\Lambda^B_\epsilon(x,y) &= y_j\,(x_k - y_k) \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s\, \big(s(x_l - y_l) + t y_l - \tfrac{1}{2}x_l\big)\, \big(s(x_m - y_m) + \\
&\qquad\qquad t y_m - \tfrac{1}{2}x_m\big) \cdot \theta_{\epsilon(s-1/2)x+\epsilon(t-s)y}[\delta_l \delta_m B^{jk}].
\end{aligned}$$

The estimate on the flux itself follows from the fact that all the automorphisms θ_z are isometric in $C_0(\Omega)$:

$$\begin{aligned}
\big\|\Lambda^B_{\tau\hbar}(x,y)\big\|_{C_0(\Omega)} &\leq |y_j|\,|x_k - y_k| \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s\, \big\|\theta_{\epsilon(s-1/2)x+\epsilon(t-s)y}[B^{jk}]\big\|_{C_0(\Omega)} \\
&\leq \big\|B^{jk}\big\|_{C_0(\Omega)}\, |y_j|\,|x_k - y_k|.
\end{aligned}$$

Using the triangle inequality to estimate $|x_l|$ from above by $|x_l - y_l| + |y_l|$, we get

$$\begin{aligned}
\left\| \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Lambda^B_\epsilon(x,y)\bigg|_{\epsilon=\tau\hbar} \right\|_{C_0(\Omega)} &\leq |y_j|\,|x_k - y_k| \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s\, \big(s|x_l - y_l| + t|y_l| + \tfrac{1}{2}|x_l|\big) \cdot \\
&\qquad \cdot \big\|\theta_{\tau\hbar(s-1/2)x+\tau\hbar(t-s)y}[\delta_l B^{jk}]\big\|_{C_0(\Omega)} \\
&= \big\|\delta_l B^{jk}\big\|_{C_0(\Omega)}\, |y_j|\,|x_k - y_k| \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s\, \big(s|x_l - y_l| + t|y_l| + \tfrac{1}{2}|x_l|\big) \\
&\leq \big\|\delta_l B^{jk}\big\|_{C_0(\Omega)}\, |y_j|\,|x_k - y_k|\, \big(|x_l - y_l| + |y_l|\big).
\end{aligned}$$

In a similar fashion, we obtain the estimate for the second-order derivative,

$$\begin{aligned}
&\left\| \frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2} \Lambda^B_\epsilon(x,y) \bigg|_{\epsilon=\tau\hbar} \right\|_{C_0(\Omega)} \leq \\
&\leq |y_j|\,|x_k - y_k| \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s \left| \left(s(x_l - y_l) + ty_l - \tfrac{1}{2}x_l \right) \left(s(x_m - y_m) + ty_m - \tfrac{1}{2}x_m \right) \right| \cdot \\
&\qquad\qquad \cdot \left\| \theta_{\tau\hbar(s-1/2)x+\tau\hbar(t-s)y}[\delta_l \delta_m B^{jk}] \right\|_{C_0(\Omega)} \\
&\leq \left\| \delta_l \delta_m B^{jk} \right\|_{C_0(\Omega)} |y_j|\,|x_k - y_k| \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s \left(s^2 |x_l - y_l|\,|x_m - y_m| + 2st|y_l|\,|x_m - y_m| + \right. \\
&\qquad\qquad \left. + s|x_l - y_l|\,|x_m| + t|y_l|\,|x_m| + t^2|y_l|\,|y_m| + \tfrac{1}{4}|x_l|\,|x_m| \right) \\
&\leq \left\| \delta_l \delta_m B^{jk} \right\|_{C_0(\Omega)} |y_j|\,|x_k - y_k| \left(|x_l - y_l|\,|x_m - y_m| + |y_l|\,|x_m - y_m| + |y_l|\,|y_m| \right).
\end{aligned}$$

This finishes the proof. □

Chapter 3

Covariant Fields of C^*-Algebras and Continuity of Spectra in Rieffel's Pseudodifferential Calculus

Introduction

Let T be a locally compact topological space, always assumed to be Hausdorff. We denote by $\mathcal{C}(T)$ the Abelian C^*-algebra of all complex continuous functions on T that are arbitrarily small outside large compact subsets. A $\mathcal{C}(T)$-algebra [10, 37, 49] is a C^*-algebra $\mathcal{B}$ together with a non-degenerated injective morphism from $\mathcal{C}(T)$ to the center of $\mathcal{B}$ (multipliers are used if $\mathcal{B}$ is not unital). The main role of the concept of $\mathcal{C}(T)$-algebra consists in codifying in a simple and efficient way the idea that $\mathcal{B}$ is fibered in the sense of C^*-algebras over the base T [14, 47]. Actually $\mathcal{C}(T)$-algebras can be seen as upper semi-continuous fields of C^*-algebras over the base T; lower semi-continuity can also be put in this setting if one also uses the space of all primitive ideals [26, 37, 42, 46, 49]. We intend to put these concepts in the perspective of Rieffel quantization.

Rieffel's calculus [43, 44] is a machine that transforms functorially "simpler" C^*-algebras and morphisms into more complicated ones. The ingredients to do this are an action of the vector group $\Xi := \mathbb{R}^d$ by automorphisms of the "simple" algebra as well as a skew symmetric linear operator of Ξ. When morphisms are involved, they are always assumed to intertwine the existing actions.

Rieffel's machine is actually meant to be a quantization. The initial data are naturally defining a Poisson structure, regarded as a mathematical modelization of the observables of a classical physical system. After applying the machine to this classical data one gets a C^*-algebra seen as the family of observables of the same system, but written in the language of Quantum Mechanics. By varying a convenient parameter (Planck's constant $\hbar$) one can recover the Poisson structure (at $\hbar = 0$) from the C^*-algebras defined at $\hbar \neq 0$ in a way that satisfies certain natural axioms [24, 43, 44].

The spirit of this quantization procedure is that of a pseudodifferential theory [15]. At least in simple situations the multiplication in the initial C^*-algebra is just point-wise multiplication of functions defined on some locally compact topological space, on which Ξ acts by homeomorphisms. The non-commutative product in the quantized algebra can be interpreted as a symbol composition of a pseudodifferential type. Actually the concrete formulae generalize and are motivated by the usual Weyl calculus.

In a setting where all the relevant concepts make sense, we prove in Theorem 3.3.3 and Proposition 3.3.4 their compatibility: By Rieffel quantization an upper semi-continuous fields of C^*-algebras is turned into an upper semi-continuous fields of C^*-algebras with fibers which are easy to identify; the proof uses $C(T)$-algebras. Finally, using primitive ideals techniques, we show the analog of this result for lower semi-continuity; the key technical result is Proposition 3.4.1. Putting everything together one gets

Theorem 3.0.1. *Rieffel quantization transforms covariant continuous fields of C^*-algebras into covariant continuous fields of C^*-algebras.*

Maybe the most interesting cases, anyhow those which are closer to the spirit of Weyl quantization, involve Abelian initial algebras $\mathcal{A}$. In this situation the information is encoded in a topological dynamical system with locally compact space Σ and the upper semi-continuous field property can be read in the existence of a continuous covariant surjection $q : \Sigma \to T$; if this one is open, then lower semi-continuity also holds. If the orbit space of the dynamical system is Hausdorff, it serves as a good space T over which the Rieffel deformed algebra can be decomposed, with easily identified fibers. We treat the Abelian case in section 3.5.

We illustrate the results by some examples in section 3.6. Among others, the techniques we develop can be used to show that the C^*-algebras of some compact quantum groups constructed in [45] can be written as continuous fields, some of the fibers being isomorphic to certain non-commutative tori.

One naturally expects that topics or tools coming from the standard pseudodifferential theory could make sense and even work in the more general setting of Rieffel's calculus. In [29], some C^*-algebraic techniques of spectral analysis ([4, 5, 16, 28, 33] and references therein) were tuned with Rieffel quantization, getting results on spectra and essential spectra of certain self-adjoint operators that seemed to be out of reach by other methods. In the present article we continue the project by studying *spectral continuity*. Pioneering work on applying C^*-algebraic techniques to spectral continuity problems and applications to discrete physical systems may be found in [4, 6, 13]. Results on continuity of spectra for unbounded Schrödinger-like Hamiltonians (especially with magnetic fields) appear in [2, 3, 18, 38] and references therein.

Roughly, our problem can be stated as follows: For each point t of the locally compact space T we are given a self-adjoint element (a classical observable) $f(t)$ of a C^*-algebra $\mathcal{A}(t)$, which is Abelian for most of the applications, and we assume some simple-minded continuity property in the variable t for this family. By quantization, $f(t)$ is turned into a quantum observable $\mathfrak{f}(t)$ belonging to a new, non-commutative C^*-algebra $\mathfrak{A}(t)$. We inquire if the family $S(t) := \mathrm{sp}\,[\mathfrak{f}(t)]$ of spectra computed in these new algebras vary continuously with t. Intuitively, outer continuity says that the family cannot suddenly expand: if for some t_0 there is a gap in the spectrum of $\mathfrak{f}(t_0)$ around a point $\lambda_0 \in \mathbb{R}$, then for t close to t_0 all the spectra $S(t)$ will have gaps around λ_0. On the other hand, inner continuity insures that if $\mathfrak{f}(t_0)$ has some spectrum in a non-trivial interval of $\mathbb{R}$, this interval will contain spectral points of all the elements $\mathfrak{f}(t)$ for t close to t_0.

Although traditionally $\mathfrak{A}(t)$ is thought to be a C^*-algebra of bounded operators in some Hilbert space, the abstract situation is both natural and fruitful. One can work with abstract C^*-algebras $\mathfrak{A}(t)$ and then, if necessarily, they are represented faithfully in Hilbert spaces; the spectrum will be preserved under representation.

It comes out that spectral continuity can be obtained from corresponding continuity properties of resolvent families of the elements $\mathfrak{f}(t)$ and this involves both inversion and norm in each complicated C^*-algebra $\mathfrak{A}(t)$. Things are smoothed out if the family $\{\mathfrak{A}(t) \mid t \in T\}$ has a priori continuity properties, that may be connected to concepts as $C(T)$-algebras or (upper or lower semi)-continuous C^*-bundles.

In a final section, using the results of the article, we are going to investigate what happens when the quantization mapping $\mathcal{A}(t) \mapsto \mathfrak{A}(t)$ is Rieffel's quantization. For our situation, which has a rather small overlap with the references above, we also include an outer continuity result for *essential spectra* of Rieffel pseudodifferential operators. Continuity in Planck's constant $\hbar$, treated in [43] and in [29], is a very special case. The full strength of these spectral techniques would require an extension of Rieffel's calculus to suitable families of unbounded elements. Hopefully this will be achieved in the future, and this would be the right opportunity to present detailed examples.

3.1 Rieffel's pseudodifferential calculus; a short review

We start by describing briefly Rieffel quantization [43, 44]. The initial object, containing *the classical data*, is a quadruplet $(\mathcal{A}, \Theta, \Xi, [\![\cdot,\cdot]\!])$. The pair $(\Xi, [\![\cdot,\cdot]\!])$ will usually be taken to be a $2n$-dimensional symplectic vector space, but the skew-symmetric bilinear form $[\![\cdot,\cdot]\!]$ may be degenerate in most situations. On the other hand $(\mathcal{A}, \Theta, \Xi)$ is a C^**-dynamical system*, meaning that the vector group acts strongly continuously by automorphisms of the (maybe non-commutative) C^*-algebra $\mathcal{A}$. Let us denote by $\mathcal{A}^\infty$ the family of elements f such that the mapping $\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{A}$ is C^∞. It is a dense *-algebra of $\mathcal{A}$ and also a Fréchet algebra with the family of semi-norms

$$\| f \|_{\mathcal{A}}^{(k)} := \sum_{|\alpha|\le k} \frac{1}{|\alpha|!} \| \partial_X^\alpha [\Theta_X(f)]_{X=0} \|_{\mathcal{A}} \equiv \sum_{|\alpha|\le k} \frac{1}{|\alpha|!} \| \delta^\alpha(f) \|_{\mathcal{A}}, \quad k \in \mathbb{N}. \tag{3.1.1}$$

To quantize the above structure, one keeps the involution unchanged but introduce on $\mathcal{A}^\infty$ the product

$$f \# g := \pi^{-2n} \int_\Xi \int_\Xi dY\, dZ\, e^{2i[\![Y,Z]\!]}\, \Theta_Y(f)\, \Theta_Z(g)\,, \tag{3.1.2}$$

suitably defined by oscillatory integral techniques. One gets a *-algebra $(\mathcal{A}^\infty, \#, ^*)$, which admits a C^*-completion $\mathfrak{A}$ in a C^*-norm $\|\cdot\|_{\mathfrak{A}}$ defined by Hilbert module techniques [43]. The action Θ leaves $\mathcal{A}^\infty$ invariant and extends to a strongly continuous action of the C^*-algebra $\mathfrak{A}$, that will also be denoted by Θ. The space $\mathfrak{A}^\infty$ of C^∞-vectors coincide with $\mathcal{A}^\infty$ and it is a Fréchet space with the family of semi-norms

$$\| f \|_{\mathfrak{A}}^{(k)} := \sum_{|\alpha|\le k} \frac{1}{|\alpha|!} \| \partial_X^\alpha [\Theta_X(f)]_{X=0} \|_{\mathfrak{A}} \equiv \sum_{|\alpha|\le k} \frac{1}{|\alpha|!} \| \delta^\alpha(f) \|_{\mathfrak{A}}, \quad k \in \mathbb{N}. \tag{3.1.3}$$

By Proposition 4.10 in [43], there exist $k \in \mathbb{N}$ and $C_k > 0$ such that

$$\| f \|_{\mathfrak{A}} \leq C_k \| f \|_{\mathcal{A}}^{(k)}, \quad \forall f \in \mathcal{A}^\infty = \mathfrak{A}^\infty .$$

Replacing here f by $\delta^\alpha f$ for every multi-index α, it follows that on $\mathcal{A}^\infty$ the topology given by the semi-norms (3.1.1) is finer than the one given by the semi-norms (3.1.3). As a consequence of Theorem 7.5 in [43], the role of the C^*-algebras $\mathcal{A}$ and $\mathfrak{A}$ can be reversed: one obtains $\mathcal{A}$ as the quantization of $\mathfrak{A}$ by replacing the skew-symmetric form $[\![\cdot,\cdot]\!]$ by $-[\![\cdot,\cdot]\!]$. Thus $\mathcal{A}^\infty$ *and* $\mathfrak{A}^\infty$ *coincide as Fréchet spaces.*

The quantization transfers to Ξ-morphisms. Let $(\mathcal{A}_j, \Theta_j, \Xi, [\![\cdot,\cdot]\!])$, $j = 1, 2$, be two classical data and let $\mathcal{R} : \mathcal{A}_1 \to \mathcal{A}_2$ be a Ξ-morphism, i. e.a (C^*-)morphism intertwining the two actions Θ_1, Θ_2. Then $\mathcal{R}$ sends $\mathcal{A}_1^\infty$ into $\mathcal{A}_2^\infty$ and extends to a morphism $\mathfrak{R} : \mathfrak{A}_1 \to \mathfrak{A}_2$ that also intertwines the corresponding actions. In this way, one obtains a covariant functor. The functor is exact: it preserves short exact sequences of Ξ-morphisms. Namely, if $\mathcal{J}$ is a (closed, self-adjoint, two-sided) ideal in $\mathcal{A}$ that is invariant under Θ, then its quantization $\mathfrak{J}$ can be identified with an invariant ideal in $\mathfrak{A}$ and the quotient $\mathfrak{A}/\mathfrak{J}$ is canonically isomorphic to the quantization of the quotient $\mathcal{A}/\mathcal{J}$ under the natural quotient action.

We will refer to *the Abelian case* under the following circumstances: A continuous action Θ of Ξ by homeomorphisms of the locally compact Hausdorff space Σ is given. For $(\sigma, X) \in \Sigma \times \Xi$ we are going to use all the notations

$$\Theta(\sigma, X) = \Theta_X(\sigma) = \Theta_\sigma(X) \in \Sigma \tag{3.1.4}$$

for the X-transformed of the point σ. The function Θ is continuous and the homeomorphisms Θ_X, Θ_Y satisfy $\Theta_X \circ \Theta_Y = \Theta_{X+Y}$ for every $X, Y \in \Xi$.

We denote by $\mathcal{C}(\Sigma)$ the Abelian C^*-algebra of all complex continuous functions on Σ that are arbitrarily small outside large compact subsets of Σ. When Σ is compact, $\mathcal{C}(\Sigma)$ is unital. The action Θ of Ξ on Σ induces an action of Ξ on $\mathcal{C}(\Sigma)$ (also denoted by Θ) given by $\Theta_X(f) := f \circ \Theta_X$. This action is strongly continuous, i. e.for any $f \in \mathcal{C}(\Sigma)$ the mapping

$$\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{C}(\Sigma) \tag{3.1.5}$$

is continuous; thus we are placed in the setting presented above. We denote by $\mathcal{C}(\Sigma)^\infty \equiv \mathcal{C}^\infty(\Sigma)$ the set of elements $f \in \mathcal{C}(\Sigma)$ such that the mapping (3.1.5) is C^∞; it is a dense *-algebra of $\mathcal{C}(\Sigma)$. The general theory supplies a non-commutative C^*-algebra $\mathfrak{A} \equiv \mathfrak{C}(\Sigma)$, acted continuously by the group Ξ, with smooth vectors $\mathfrak{C}^\infty(\Sigma) = \mathcal{C}^\infty(\Sigma)$.

3.2 Families of C^*-algebras

Now we give a short review of $\mathcal{C}(T)$-algebras and semi-continuous fields of C^*-algebras (see [10, 24, 26, 37, 42, 49] and references therein), outlining the connection between the two notions.

If $\mathcal{B}$ is a C^*-algebra, we denote by $\mathcal{M}(\mathcal{B})$ its multiplier algebra and by $\mathcal{ZM}(\mathcal{B})$ its center. If $\mathcal{B}_1, \mathcal{B}_2$ are two vector subspaces of $\mathcal{M}(\mathcal{B})$, we set $\mathcal{B}_1 \cdot \mathcal{B}_2$ for the vector subspace generated by $\{b_1 b_2 \mid b_1 \in \mathcal{B}_1, b_2 \in \mathcal{B}_2\}$. We are going to denote by $\mathcal{C}(T)$ the C^*-algebra of all complex continuous functions on the (Hausdorff) locally compact space T that decay at infinity.

Definition 3.2.1. *We say that $\mathcal{B}$ is a $\mathcal{C}(T)$-*algebra *if a non-degenerate monomorphism $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{B})$ is given.*

We recall that non-degeneracy means that the ideal $\mathcal{Q}[\mathcal{C}(T)] \cdot \mathcal{B}$ is dense in $\mathcal{B}$.

Definition 3.2.2. *By* upper semi-continuous field of C^*-algebras *we mean a family of epimorphisms of C^*-algebras $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ indexed by the locally compact topological space T and satisfying:*

1. *For every $b \in \mathcal{B}$ one has $\| b \|_{\mathcal{B}} = \sup_{t\in T} \| \mathcal{P}(t)b \|_{\mathcal{B}(t)}$.*
2. *For every $b \in \mathcal{B}$ the map $T \ni t \mapsto \| \mathcal{P}(t)b \|_{\mathcal{B}(t)}$ is upper semi-continuous and decays at infinity.*
3. *There is a multiplication $\mathcal{C}(T) \times \mathcal{B} \ni (\varphi, b) \to \varphi * b \in \mathcal{B}$ such that*

$$\mathcal{P}(t)[\varphi * b] = \varphi(t)\,\mathcal{P}(t)b\,, \quad \forall t \in T\,,\ \varphi \in \mathcal{C}(T)\,,\ b \in \mathcal{B}\,.$$

If, in addition, the map $t \mapsto \| \mathcal{P}(t)b \|$ is continuous for every $b \in \mathcal{B}$, we say that $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ is a continuous field of C^*-algebras.

The requirement 2 is clearly equivalent with the condition that for every $b \in \mathcal{B}$ and every $\epsilon > 0$ the subset $\{t \in T \mid \| \mathcal{P}(t)b \|_{\mathcal{B}(t)} \geq \epsilon\}$ is compact. One can rephrase 1 as $\cap_t \ker[\mathcal{P}(t)] = \{0\}$, so one can identify $\mathcal{B}$ with a C^*-algebra of sections of the field; this make the connection with other approaches, as that of [37] for example. It will always be assumed that $\mathcal{B}(t) \neq \{0\}$ for all $t \in T$.

We are going to describe briefly in which way the two definitions above are actually equivalent.

First let us assume that $\mathcal{B}$ is a $\mathcal{C}(T)$-algebra and denote by $\mathcal{C}_t(T)$ the ideal of all the functions in $\mathcal{C}(T)$ vanishing at the point $t \in T$. We get ideals $\mathcal{I}(t) := \overline{\mathcal{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathcal{B}}$ in $\mathcal{B}$, quotients $\mathcal{B}(t) := \mathcal{B}/\mathcal{I}(t)$ as well as canonical epimorphisms $\mathcal{P}(t) : \mathcal{B} \to \mathcal{B}(t)$. One also sets

$$\varphi * b := \mathcal{Q}(\varphi)b\,, \quad \forall \varphi \in \mathcal{C}(T),\ b \in \mathcal{B}\,. \tag{3.2.1}$$

Then $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ is an upper semi-continuous field of C^*-algebras with multiplication $*$.

Conversely, if an upper semi-continuous field $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ is given, also involving the multiplication $*$, we set

$$\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{B})\,, \quad \mathcal{Q}(\varphi)b := \varphi * b\,. \tag{3.2.2}$$

In this way one gets a $\mathcal{C}(T)$-algebra and each of the quotients $\mathcal{B}/\mathcal{I}(t)$ is isomorphic to the fiber $\mathcal{B}(t)$.

To discuss lower semi-continuity we need $\mathrm{Prim}(\mathcal{B})$, the space of all the primitive ideals (kernels of irreducible representations) of $\mathcal{B}$. The hull-kernel topology turns $\mathrm{Prim}(\mathcal{B})$ into a locally compact (non necessarily Hausdorff) topological space. We recall that *the hull application*

$\mathcal{J} \mapsto h(\mathcal{J}) := \{\mathcal{K} \in \mathrm{Prim}(\mathcal{B}) \mid \mathcal{J} \subset \mathcal{K}\}$ realizes a decreasing bijection between the family of ideals of $\mathcal{B}$ and the family of closed subsets of $\mathrm{Prim}(\mathcal{B})$. Its inverse is *the kernel map* $\Omega \mapsto k(\Omega) := \cap_{\mathcal{K}\in\Omega}\mathcal{K}$, which is also decreasing.

The Dauns-Hoffman Theorem establishes the existence of a unique isomorphism $\Gamma : BC[\mathrm{Prim}(\mathcal{B})] \to \mathcal{ZM}(\mathcal{B})$, where $BC[\mathrm{Prim}(\mathcal{B})]$ is the C^*-algebra of bounded and continuous functions over $\mathrm{Prim}(\mathcal{B})$, such that for each $\mathcal{K} \in \mathrm{Prim}(\mathcal{B})$, $\Psi \in BC[\mathrm{Prim}(\mathcal{B})]$ and $b \in \mathcal{B}$ we have $\Gamma(\Psi)b + \mathcal{K} = \Psi(\mathcal{K})b + \mathcal{K}$. For a detailed study of the space $\mathrm{Prim}(\mathcal{B})$ and a proof of the Dauns-Hoffman Theorem, cf. sections A.2 and A.3 in [46]. Let us suppose that there is a continuous surjective function $q : \mathrm{Prim}(\mathcal{B}) \to T$. Then we can define $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{B})$ by $\mathcal{Q}(\varphi) = \Gamma(\varphi \circ q)$ and one can check that $\mathcal{Q}$ endows $\mathcal{B}$ with the structure of a $\mathcal{C}(T)$-algebra.

On the other hand, if we have a non-degenerate monomorphism $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{B})$, we can define canonically a continuous map $q : \mathrm{Prim}(\mathcal{B}) \to T$. One has $q(\mathcal{K}) = t$ if and only if $\mathcal{I}(t) \subset \mathcal{K}$, and we can recover $\mathcal{Q}$ from the above construction. Moreover *the map* $T \ni t \to \| b(t) \|_{\mathcal{B}(t)} \in \mathbb{R}_+$ *is continuous for every* $b \in \mathcal{B}$ *(so we have a continuous field of* C^**-algebras) if and only if* q *is open.* For the proof of this facts see propositions C.5 and C.10 in [49].

3.3 Covariant $\mathcal{C}(T)$-algebras and upper semi-continuity under Rieffel quantization

Let T be a locally compact Hausdorff space and $(\mathcal{A}, \Theta, \Xi, [\![\cdot,\cdot]\!])$ a classical data. The canonical C^*-dynamical system defined by Rieffel quantization is $(\mathfrak{A}, \Theta, \Xi)$.

Definition 3.3.1. *We say that* $\mathcal{A}$ *is a* covariant $\mathcal{C}(T)$-algebra *with respect to the action* Θ *if a non-degenerate monomorphism* $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{A})$ *is given (so it is a* $\mathcal{C}(T)$*-algebra) and in addition one has*

$$\Theta_X[\mathcal{Q}(\varphi)f] = \mathcal{Q}(\varphi)\left[\Theta_X(f)\right], \quad \forall f \in \mathcal{A},\, X \in \Xi,\, \varphi \in \mathcal{C}(T). \tag{3.3.1}$$

We intend to prove that the Rieffel quantization transforms covariant $\mathcal{C}(T)$-algebras into covariant $\mathcal{C}(T)$-algebras. For this and for a further result identifying the emerging quotient algebras, we are going to need

Lemma 3.3.2. *Let* I *be an ideal of* $\mathcal{C}(T)$ *and denote by* $\overline{\mathcal{Q}(I)\cdot\mathcal{A}}$ *the closure of* $\mathcal{Q}(I)\cdot\mathcal{A}$ *in the* C^**-algebra* $\mathcal{A}$*. Then* $\mathcal{Q}(I)\cdot\mathcal{A}^\infty$ *is dense in* $\left(\overline{\mathcal{Q}(I)\cdot\mathcal{A}}\right)^\infty \equiv \left(\overline{\mathcal{Q}(I)\cdot\mathcal{A}}\right)\cap\mathcal{A}^\infty$ *for the Fréchet topology inherited from* $\mathcal{A}^\infty$.

Proof. By the covariance condition $\overline{\mathcal{Q}(I)\cdot\mathcal{A}}$ is an invariant ideal of $\mathcal{A}$.

The proof uses regularization. Consider *the integrated form of* Θ, i.e. for each $\Phi \in C_c^\infty(\Xi)$ (compactly supported smooth function) and $g \in \mathcal{A}$ define

$$\Theta_\Phi(g) = \int_\Xi dY\,\Phi(Y)\Theta_Y(g).$$

Note that for every $X \in \Xi$ one has

$$\Theta_X\left[\Theta_\Phi(g)\right] = \int_\Xi dY\,\Phi(Y - X)\Theta_Y(g).$$

Then $\Theta_\Phi(g) \in \mathcal{A}^\infty$ and for each multi-index μ we have

$$\delta^\mu [\Theta_\Phi(g)] = (-1)^{|\mu|}\Theta_{\partial^\mu\Phi}(g) \quad \text{and} \quad \| \delta^\mu [\Theta_\Phi(g)] \|_{\mathcal{A}} \leq \| \partial^\mu \Phi \|_{L^1(\Xi)} \| g \|_{\mathcal{A}} .$$

One of the deepest theorems about smooth algebras, the Dixmier-Malliavin Theorem [11], say that $\mathcal{A}^\infty$ is generated (algebraically) by the set of all the elements of the form $\Theta_\Phi(g)$ with $\Phi \in C_c^\infty(\Xi)$ and $g \in \mathcal{A}$. Replacing $\mathcal{A}$ with $\overline{\mathcal{Q}(I) \cdot \mathcal{A}}$, for $f \in \left(\overline{\mathcal{Q}(I) \cdot \mathcal{A}}\right)^\infty$ there exist $\Phi_1, \dots, \Phi_m \in C_c^\infty(\Xi)$ and $f_1, \dots, f_m \in \overline{\mathcal{Q}(I) \cdot \mathcal{A}}$ such that $f = \sum_{i=1}^m \Theta_{\Phi_i}(f_i)$. Let $\epsilon > 0$ and fix a multi-index α. Choose $g_1, \dots, g_m \in \mathcal{Q}(I) \cdot \mathcal{A}$ such that for each i

$$\| f_i - g_i \|_{\mathcal{A}} \leq \frac{\epsilon}{m \| \partial^\alpha \Phi_i \|_{L^1(\Xi)}} .$$

Then

$$\left\| \delta^\alpha \left(f - \sum_{i=1}^m \Theta_{\Phi_i}(g_i) \right) \right\|_{\mathcal{A}} = \left\| \sum_{i=1}^m \Theta_{\partial^\alpha \Phi_i}(f_i - g_i) \right\|_{\mathcal{A}} \leq \sum_{i=1}^m \| \partial^\alpha \Phi_i \|_{L^1(\Xi)} \| f_i - g_i \|_{\mathcal{A}} \leq \epsilon .$$

Thus we only need to prove that for each $\Phi \in C_c^\infty(\Xi)$ and $g \in \mathcal{Q}(I) \cdot \mathcal{A}$ the element $\Theta_\Phi(g)$ belongs to $\mathcal{Q}(I) \cdot \mathcal{A}^\infty$. Let $\varphi_1, \dots, \varphi_j \in I$ and $h_1, \dots, h_j \in \mathcal{A}$ such that $g = \sum_{i=1}^j \mathcal{Q}(\varphi_i) h_i$. Then

$$\Theta_\Phi(g) = \sum_{i=1}^j \Theta_\Phi [\mathcal{Q}(\varphi_i) h_i] ,$$

and by covariance, for each index i one has

$$\Theta_\Phi [\mathcal{Q}(\varphi_i) h_i] = \int_\Xi dY \, \Phi(Y) \mathcal{Q}(\varphi_i) \Theta_Y(h_i) = \mathcal{Q}(\varphi_i) [\Theta_\Phi(h_i)] \in \mathcal{Q}(I) \cdot \mathcal{A}^\infty .$$

□

Theorem 3.3.3. *Rieffel quantization transforms covariant $C(T)$-algebras into covariant $C(T)$-algebras: there exists a non-degenerate monomorphism $\mathfrak{Q} : C(T) \to \mathcal{ZM}(\mathfrak{A})$ satisfying for all $\varphi \in C(T)$, $f \in \mathcal{A}$ and $X \in \Xi$ the covariance relation $\Theta_X[\mathfrak{Q}(\varphi) f] = \mathfrak{Q}(\varphi) [\Theta_X(f)]$.*

Proof. The action Θ of Ξ on $\mathcal{A}$ extends canonically to an action by automorphisms of the multiplier algebra $\mathcal{M}(\mathcal{A})$, also denoted by Θ, which is not strongly continuous in general. But, tautologically, it restricts to a strongly continuous action $\Theta : \Xi \to \mathrm{Aut}[\mathcal{M}_0(\mathcal{A})]$ on the C^*-subalgebra

$$\mathcal{M}_0(\mathcal{A}) := \{ m \in \mathcal{M}(\mathcal{A}) \mid \Xi \ni X \mapsto \Theta_X(m) \in \mathcal{M}(\mathcal{A}) \text{ is norm continuous} \} . \tag{3.3.2}$$

In these terms, the covariance condition on $\mathcal{Q}$ says simply that for any $\varphi \in C(T)$ the multiplier $\mathcal{Q}(\varphi)$ is a fixed point for all the automorphisms Θ_X (take $f = 1$ in (3.3.1)). As a very weak consequence one has $\mathcal{Q}[C(T)] \subset \mathcal{M}_0(\mathcal{A})^\infty$, with an obvious notation for the smooth vectors.

Proposition 5.10 from [43] applied to the unital C^*-algebra $\mathcal{M}_0(\mathcal{A})$ says that the Rieffel quantization of $\mathcal{M}_0(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{M}(\mathfrak{A})$. Consequently one has $\mathcal{Q}[C(T)] \subset \mathcal{M}_0(\mathcal{A})^\infty \subset$

$\mathcal{M}(\mathfrak{A})$ and this supplies a candidate $\mathfrak{Q} : \mathcal{C}(T) \to \mathcal{M}(\mathfrak{A})$. This is obviously an injective map and the range is only composed of fixed points, which insures covariance.

Let us set for a moment $\mathcal{M} := \mathcal{M}_0(\mathcal{A})$, with multiplication $\cdot$, and denote by $\mathfrak{M} \subset \mathcal{M}(\mathfrak{A})$ its Rieffel quantization, with multiplication legitimately denoted by $\#$. For smooth elements $m, n \in \mathcal{M}^\infty = \mathfrak{M}^\infty$, one of them being a fixed point central in $\mathcal{M}$, one has $m\#n = m \cdot n = n \cdot m = n\#m$ (Corollary 2.13 in [43]). This implies easily that $\mathfrak{Q}$ is again a morphism and its range is contained in $\mathcal{Z}\mathfrak{M}$. A density argument with respect to the strict topology implies that every $\mathfrak{Q}(\varphi)$ commutes with all the elements of $\mathcal{M}(\mathfrak{A})$, thus $\mathfrak{Q}[\mathcal{C}(T)] \subset \mathcal{Z}\mathcal{M}(\mathfrak{A})$ as required.

Now we only need to show non-degeneracy, i.e. the fact that $\mathfrak{Q}[\mathcal{C}(T)] \cdot \mathfrak{A}$ is dense in $\mathfrak{A}$. We show the even stronger assertion that $\mathcal{Q}[\mathcal{C}(T)] \cdot \mathcal{A}^\infty = \mathfrak{Q}[\mathcal{C}(T)] \cdot \mathfrak{A}^\infty$ is dense in $\mathfrak{A}$. This would follow if we knew that $\mathcal{Q}[\mathcal{C}(T)] \cdot \mathcal{A}^\infty$ is dense in $\mathfrak{A}^\infty$ with respect to its Fréchet topology given by the semi-norms (3.1.3); then we use denseness of $\mathfrak{A}^\infty$ in the weaker C^*-norm topology of $\mathfrak{A}$.

We recall from section 3.1 that $\mathcal{A}^\infty$ and $\mathfrak{A}^\infty$ coincide even as Fréchet spaces. Therefore one is reduced to showing that $\mathcal{Q}[\mathcal{C}(T)] \cdot \mathcal{A}^\infty$ is dense in $\mathcal{A}^\infty$ for its Fréchet topology. Taking $\mathcal{I} = \mathcal{C}(T)$ in Lemma 3.3.2, we find out that $\mathcal{Q}[\mathcal{C}(T)] \cdot \mathcal{A}^\infty$ is dense in $\left(\overline{\mathcal{Q}[\mathcal{C}(T)] \cdot \mathcal{A}}\right) \cap \mathcal{A}^\infty$, which equals $\mathcal{A}^\infty$ since $\mathcal{Q}$ has been assumed non-degenerate. This finishes the proof. □

If $\mathcal{A}$ is a covariant $\mathcal{C}(T)$-algebra, then $\mathcal{I}(t) := \overline{\mathcal{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathcal{A}}$ is an invariant ideal of $\mathcal{A}$. We can apply Rieffel quantization to $\mathcal{I}(t)$, to $\mathcal{A}(t) := \mathcal{A}/\mathcal{I}(t)$ (with the obvious actions of Ξ) and to the projection $\mathcal{P}(t) : \mathcal{A} \to \mathcal{A}(t)$. One gets C^*-algebras $\mathfrak{I}_t$, $\mathfrak{A}_t$ as well as the morphism $\mathfrak{P}_t : \mathfrak{A} \to \mathfrak{A}_t$. By [43, Th.7.7] the kernel of $\mathfrak{P}_t$ is $\mathfrak{I}_t$, so $\mathfrak{A}_t$ can be identified to the quotient $\mathfrak{A}/\mathfrak{I}_t$.

On the other hand, by using the $\mathcal{C}(T)$-structure of the C^*-algebra $\mathfrak{A}$ given by Theorem 3.3.3, we have ideals $\mathfrak{I}(t) := \overline{\mathfrak{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathfrak{A}}$ in $\mathfrak{A}$ as well as quotients $\mathfrak{A}(t) := \mathfrak{A}/\mathfrak{I}(t)$ to which we associates projections $\mathfrak{A} \xrightarrow{\mathfrak{P}(t)} \mathfrak{A}(t)$. However, one gets

Proposition 3.3.4. *With notation as above, for each $t \in T$ we have $\mathfrak{I}(t) = \mathfrak{I}_t$.*

In particular, the fibers $\mathfrak{A}(t) = \mathfrak{A}/\mathfrak{I}(t)$ of the $\mathcal{C}(T)$-algebra $\mathfrak{A}$ are isomorphic to the Rieffel quantization $\mathfrak{A}_t$ of the fibers $\mathcal{A}(t) = \mathcal{A}/\mathcal{I}(t)$ of $\mathcal{A}$ and for each $f \in \mathfrak{A}$ the mapping $t \mapsto \| \mathfrak{P}(t)f \|_{\mathfrak{A}(t)} = \| \mathfrak{P}_t f \|_{\mathfrak{A}_t}$ is upper semi-continuous.

Proof. We recall that $\mathcal{I}(t)^\infty$ and $\mathfrak{I}(t)^\infty$ coincide as Fréchet spaces. By Lemma 3.3.2, $\mathfrak{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathfrak{A}^\infty$ is dense in $\mathfrak{I}(t)^\infty$, thus in $\mathfrak{I}(t)$, and $\mathcal{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathcal{A}^\infty$ is dense in $\mathcal{I}(t)^\infty = \mathfrak{I}(t)^\infty$, thus also dense in $\mathfrak{I}_t$.

By construction one has $\mathfrak{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathfrak{A}^\infty = \mathcal{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathcal{A}^\infty$; consequently $\mathfrak{I}(t) = \mathfrak{I}_t$ for every $t \in T$ and the proof is finished. □

Remark 3.3.5. For obvious reasons, we are going to say that $\left\{\mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T\right\}$ and $\left\{\mathfrak{A} \xrightarrow{\mathfrak{P}(t)} \mathfrak{A}(t) \mid t \in T\right\}$ are *covariant upper semi-continuous fields of C^*-algebras*. The intrinsic definition, in the first case for instance, would be the following: $\left\{\mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T\right\}$ is required to be an upper semi-continuous field of C^*-algebras and we also ask the action Θ to leave invariant all the ideals $\mathcal{I}(t) = \ker[\mathcal{P}(t)]$. It is easily seen that this is equivalent to require the covariance of the associated $\mathcal{C}(T)$-structure. This makes the connection with Definition 3.1 in [42].

For section C^*-algebras of an upper semi-continuous field it is known [49] that each irreducible representation factorizes through one of the fibers. Therefore we get

Corollary 3.3.6. *Let $(\mathcal{A}, \Theta, \Xi, [\![\cdot,\cdot]\!])$ be a classical data and assume that $\mathcal{A}$ is a Θ-covariant $\mathcal{C}(T)$-algebra with respect to a Hausdorff locally compact space T, with fibers $\{\mathcal{A}(t) \mid t \in T\}$. Denote, respectively, by $\mathfrak{A}$ and $\mathfrak{A}(t)$ the corresponding quantized C^*-algebras. Then any irreducible representation of $\mathfrak{A}$ factorizes through one of the algebras $\mathfrak{A}(t)$.*

The $\mathcal{C}(T)$-structure $\mathfrak{Q}$ of $\mathfrak{A}$, given by Theorem 3.3.3, defines canonically the map $\mathfrak{q} : \mathrm{Prim}(\mathfrak{A}) \to T$, as explained at the end of section 3.2. If $\pi : \mathfrak{A} \to \mathbb{B}(\mathcal{H})$ is the irreducible Hilbert space representation of $\mathfrak{A}$, then the point t in Corollary 3.3.6 is $\mathfrak{q}[\ker(\pi)]$.

3.4 Lower semi-continuity under Rieffel quantization

We keep the previous setting and inquire now if lower semi-continuity of the mappings $t \mapsto \| \mathcal{P}(t)f \|_{\mathcal{A}(t)}$ for all $f \in \mathcal{A}$ implies lower semi-continuity of the mappings $t \mapsto \| \mathfrak{P}(t)f \|_{\mathfrak{A}(t)}$ for all $f \in \mathfrak{A}$. We start by noticing that $\mathrm{Prim}(\mathcal{A})$ and $\mathrm{Prim}(\mathfrak{A})$ are canonically endowed with continuous actions of the group Ξ; once again these actions will be denoted by Θ. By the discussion at the end of section 3.2 we are left with proving

Proposition 3.4.1. *Suppose that $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{A})$ is a covariant $\mathcal{C}(T)$-algebra structure on $\mathcal{A}$ and that the associated function $q : \mathrm{Prim}(\mathcal{A}) \to T$ is open. Then the function $\mathfrak{q} : \mathrm{Prim}(\mathfrak{A}) \to T$ associated to $\mathfrak{Q} : \mathcal{C}(T) \to \mathcal{Z}(\mathfrak{A})$ is also open.*

Proof. We remark first that q is Θ-covariant (Lemma 8.1 in [49]), i.e. one has $q \circ \Theta_X = q$ for every $X \in \Xi$. Consequently, if $\mathcal{O} \subset \mathrm{Prim}(\mathcal{A})$ is an open set, then $\Theta_\Xi(\mathcal{O}) := \{\Theta_X(\mathcal{K}) \mid X \in \Xi, \mathcal{K} \in \mathcal{O}\}$ will also be an open set and $q(\mathcal{O}) = q\left[\Theta_\Xi(\mathcal{O})\right]$. So q will be open iff it sends open *invariant* subsets of $\mathrm{Prim}(\mathcal{A})$ into open subsets of T. The same is true for $\mathfrak{q} : \mathrm{Prim}(\mathfrak{A}) \to T$. But the most general open subset of $\mathrm{Prim}(\mathcal{A})$ has the form

$$\mathcal{O}_\mathcal{J} := \{\mathcal{K} \in \mathrm{Prim}(\mathcal{A}) \mid \mathcal{J} \not\subset \mathcal{K}\} = h(\mathcal{J})^c$$

for some ideal $\mathcal{J}$ of $\mathcal{A}$, being the complement of the hull $h(\mathcal{J})$ of this ideal. In addition, $\mathcal{O}_\mathcal{J}$ is Θ-invariant iff $\mathcal{J}$ is an invariant ideal. We also recall that Rieffel quantization establishes a one-to-one correspondence between invariant ideals of $\mathcal{A}$ and invariant ideals of $\mathfrak{A}$.

So let $\mathcal{J}$ be an invariant ideal in $\mathcal{A}$ and $\mathfrak{J}$ its quantization (an invariant ideal in $\mathfrak{A}$). We would like to show that $q\left(\mathcal{O}_\mathcal{J}\right) = \mathfrak{q}\left(\mathcal{O}_\mathfrak{J}\right)$; by the discussion above this would imply that q and $\mathfrak{q}$ are simultaneously open. Using the fact that $q(\mathcal{K}) = t$ if and only if $\mathcal{I}(t) \subseteq \mathcal{K}$ and similarly for $\mathfrak{q}$, one gets

$$q\left(\mathcal{O}_\mathcal{J}\right) = \{t \in T \mid \exists \mathcal{K} \in \mathrm{Prim}(\mathcal{A}),\ \mathcal{J} \not\subset \mathcal{K},\ \mathcal{I}(t) \subset \mathcal{K}\}$$

and

$$\mathfrak{q}\left(\mathcal{O}_\mathfrak{J}\right) = \{t \in T \mid \exists \mathfrak{K} \in \mathrm{Prim}(\mathfrak{A}),\ \mathfrak{J} \not\subset \mathfrak{K},\ \mathfrak{I}(t) \subset \mathfrak{K}\}.$$

Using the hull application and the fact that both the hull and the kernel are decreasing, one can write

$$t \notin q(\mathcal{O}_{\mathcal{J}}) \iff h[\mathcal{I}(t)] \cap h[\mathcal{J}]^c = \emptyset \iff h[\mathcal{I}(t)] \subset h[\mathcal{J}] \iff \mathcal{I}(t) \supset \mathcal{J}$$

and

$$t \notin q(\mathcal{O}_{\mathfrak{J}}) \iff h[\mathfrak{I}(t)] \cap h[\mathfrak{J}]^c = \emptyset \iff h[\mathfrak{I}(t)] \subset h[\mathfrak{J}] \iff \mathfrak{I}(t) \supset \mathfrak{J}.$$

To finish the proof one only needs to notice that the Rieffel quantization of invariant ideals preserves inclusions. □

Remark 3.4.2. The definition of *a covariant continuous field of C^*-algebras* is naturally obtained by adding the lower semi-continuity condition to the definition of an upper semi-continuous field of C^*-algebras contained in Remark 3.3.5. Using this notion, *Theorem 3.0.1 is now fully justified.*

The C^*-dynamical system $(\mathcal{A}, \Theta, \Xi)$ being given, one could try one of the choices $T = \mathrm{Orb}[\mathrm{Prim}(\mathcal{A})]$ (*the orbit space*) or $T = \mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$ (*the quasi-orbit space*), both associated to the natural action of Ξ on the space $\mathrm{Prim}(\mathcal{A})$. We recall that, by definition, *a quasi-orbit* is the closure of an orbit and we refer to [49] for all the fairly standard assertions we are going to make about these spaces. The two spaces are quotients of $\mathrm{Prim}(\mathcal{A})$ with respect to obvious equivalence relations. Endowed with the quotient topology they are locally compact, but they may fail to possess the Hausdorff property. On the positive side, both *the orbit map* $p : \mathrm{Prim}(\mathcal{A}) \to \mathrm{Orb}[\mathrm{Prim}(\mathcal{A})]$ and *the quasi-orbit map* $q : \mathrm{Prim}(\mathcal{A}) \to \mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$ are continuous open surjections. So one can state:

Corollary 3.4.3. *If the quasi-orbit space of the dynamical system* $(\mathrm{Prim}(\mathcal{A}), \Theta, \Xi)$ *is Hausdorff, then the deformed C^*-algebra $\mathfrak{A}$ can be expressed as a continuous field of C^*-algebras over the base* $\mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$.

A similar statement holds with "quasi-orbit" replaced by "orbit" and with $\mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$ *replaced by* $\mathrm{Orb}[\mathrm{Prim}(\mathcal{A})]$.

Notice that, when $\mathrm{Orb}[\mathrm{Prim}(\mathcal{A})]$ happens to be Hausdorff, the orbits will be automatically closed (as inverse images by p of points); so one would actually have $\mathrm{Orb}[\mathrm{Prim}(\mathcal{A})] = \mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$.

3.5 The Abelian case

The most important is the Abelian case, that has been described at the end of section 3.1.

We assume given a continuous surjection $q : \Sigma \to T$. Then we have the disjoint decomposition of Σ in closed subsets

$$\Sigma = \sqcup_{t \in T} \Sigma_t, \quad \Sigma_t := q^{-1}(\{t\}). \tag{3.5.1}$$

Associated to the canonical injections $j_t : \Sigma_t \to \Sigma$, we have associated restriction epimorphisms

$$\mathcal{R}(t) : C(\Sigma) \to C(\Sigma_t), \quad \mathcal{R}(t)f := f|_{\Sigma_t} = f \circ j_t, \ \forall t \in T. \tag{3.5.2}$$

We give conditions on the topological data (Σ, q, T) in order to get a continuous field of Abelian C^*-algebras.

Proposition 3.5.1. *If q is continuous,* $\left\{\mathcal{C}(\Sigma) \xrightarrow{\mathcal{R}(t)} \mathcal{C}(\Sigma_t) \mid t \in T\right\}$ *is an upper semi-continuous field of commutative C^*-algebras. If q is also open, the field is continuous.*

Proof. Obviously $\cap_{t\in T} \ker[\mathcal{R}(t)] = \{0\}$, since $f|_{\Sigma_t} = 0$, $\forall t \in T$ implies $f = 0$. On the other hand, setting

$$\varphi * f := (\varphi \circ q)f\,, \qquad \forall \varphi \in \mathcal{C}(T)\,,\ f \in \mathcal{C}(\Sigma)\,, \tag{3.5.3}$$

we get immediately $\mathcal{R}(t)(\varphi * f) = \varphi(t)\,\mathcal{R}(t)f$, $\forall t \in T$.

We need to study continuity properties of the mapping

$$T \ni t \mapsto n_f(t) := \|\mathcal{R}(t)f\|_{\mathcal{C}(\Sigma_t)} = \sup_{\sigma\in\Sigma_t}|f(\sigma)| = \inf\left\{\|f+h\|_{\mathcal{C}(\Sigma)} \mid h \in \mathcal{C}(\Sigma),\ h|_{\Sigma_t} = 0\right\} \in \mathbb{R}_+.$$

The last expression for the norm can be justified directly easily, but it also follows from the canonical isomorphism $\mathcal{C}(\Sigma_t) \cong \mathcal{C}(\Sigma)/\mathcal{C}_{\Sigma_t}(\Sigma)$, where $\mathcal{C}_{\Sigma_t}(\Sigma)$ is the ideal of functions $h \in \mathcal{C}(\Sigma)$ such that $h|_{\Sigma_t} = 0$.

We first assume that q is only continuous. For every $S \subset T$ we set $\Sigma_S := q^{-1}(S)$. It is easy to see by Stone-Weierstrass Theorem that

$$\mathcal{C}_{(t)}(\Sigma) := \{h \in \mathcal{C}(\Sigma) \mid \exists \text{ an open neighborhood } U \text{ of } t \text{ such that } h|_{\Sigma_{\overline{U}}} = 0\}$$

is a self-adjoint 2-sided ideal dense in $\mathcal{C}_{\Sigma_t}(\Sigma)$. Let $t_0 \in T$ and $\varepsilon > 0$; by density and the definition of $\inf$

$$\exists\, h \in \mathcal{C}_{(t_0)}(\Sigma) \ \text{ such that } \ n_f(t_0) + \varepsilon \geq \|f+h\|_{\mathcal{C}(\Sigma)}\,.$$

Let U be the open neighborhood of t_0 for which $h|_{\Sigma_{\overline{U}}} = 0$. For any $t \in U$ one also has $h \in \mathcal{C}_{(t)}(\Sigma)$, so

$$n_f(t) = \inf\left\{\|f+g\|_{\mathcal{C}(\Sigma)} \mid g \in \mathcal{C}_{(t)}(\Sigma)\right\} \leq \|f+h\|_{\mathcal{C}(\Sigma)} \leq n_f(t_0) + \varepsilon$$

and this is upper semi-continuity.

Let us also suppose q open, let $t_0 \in T$ and $\varepsilon > 0$. By the definition of sup, there exists $\sigma_0 \in \Sigma_{t_0}$ such that $|f(\sigma_0)| \geq n_f(t_0) - \varepsilon/2$. Since f is continuous, there is a neighborhood V of σ_0 in Σ such that

$$|f(\sigma)| \geq |f(\sigma_0)| - \varepsilon/2 \geq n_f(t_0) - \varepsilon, \qquad \forall \sigma \in V.$$

Since q is open, $U := q(V)$ is a neighborhood of t_0. For every $t \in U$ we have $\Sigma_t \cap V \neq \emptyset$, so for such t

$$n_f(t) \geq \sup\{|f(\sigma)| \mid \sigma \in \Sigma_t \cap V\} \geq n_f(t_0) - \varepsilon$$

and this is lower semi-continuity. □

Remark 3.5.2. The result also follows from the fact that $\mathcal{C}(\Sigma)$ is a $\mathcal{C}(T)$-algebra for the injective morphism

$$\mathcal{Q} : \mathcal{C}(T) \to BC(\Sigma) \cong \mathcal{M}[\mathcal{C}(\Sigma)]\,, \qquad \mathcal{Q}(\varphi) := \varphi \circ q\,.$$

We have identified the multiplier algebra of $\mathcal{C}(T)$ with the unital C^*-algebra of all bounded continuous complex functions defined on Σ. The direct topological proof of Proposition 3.5.1 seemed to us more suitable.

We recall now that an action Θ of Ξ on Σ by homeomorphisms is given.

Definition 3.5.3. *We say that the continuous surjection* q is Θ-covariant *if it satisfies the equivalent conditions:*

1. *Each Σ_t is Θ-invariant.*
2. *For each $X \in \Xi$ one has $q \circ \Theta_X = q$.*
3. *For all $X \in \Xi$ and $\varphi \in C(T)$ one has $\Theta_X[\mathcal{Q}(\varphi)] = \mathcal{Q}(\varphi)$.*

The equivalence of the three conditions is straightforward. We conclude that $C(\Sigma)$ is a covariant $C(T)$-algebra (cf. Definition 3.3.1). If one wants to avoid the language of $C(T)$-algebras, by Remark 3.3.5, it should be noticed that all the ideals $\mathcal{I}(T) := \ker[\mathcal{R}(t)] = \{f \in C(T) \mid f|_{\Sigma_t} = 0\}$ are left invariant by the action Θ.

The Rieffel-quantized C^*-algebras $\mathfrak{C}(\Sigma)$ and $\mathfrak{C}(\Sigma_t)$ as well as the epimorphisms $\mathfrak{R}(t)$: $\mathfrak{C}(\Sigma) \to \mathfrak{C}(\Sigma_t)$ were introduced in Section 3.1. Applying now Proposition 3.5.1 and the results obtained in sections 3.3 and 3.4, one gets

Corollary 3.5.4. *Assume that the mapping $q : \Sigma \to T$ is a Θ-covariant continuous surjection. Then the family $\left\{\mathfrak{C}(\Sigma) \xrightarrow{\mathfrak{R}(t)} \mathfrak{C}(\Sigma_t) \mid t \in T\right\}$ forms a covariant upper semi-continuous field of non-commutative C^*-algebras.*

If q is also open, then the field is continuous.

Let us assume now that the orbit space $\mathrm{Orb}(\Sigma)$ is Hausdorff. Any orbit, being the inverse image of a point in $\mathrm{Orb}(\Sigma)$, will be closed in Σ and invariant; it will also be homeomorphic to the quotient of Ξ by the corresponding stability group. As a precise particular case of Corollary 3.4.3 one can state:

Corollary 3.5.5. *If the orbit space of the dynamical system (Σ, Θ, Ξ) is Hausdorff, then the deformed C^*-algebra $\mathfrak{C}(\Sigma)$ can be expressed as a continuous field of C^*-algebras over the base space $\mathrm{Orb}(\Sigma)$. The fiber over $\mathcal{O} \in \mathrm{Orb}(\Sigma)$ is the deformation of the Abelian algebra $C(\mathcal{O}) \cong C(\Xi/\Xi_{\mathcal{O}})$.*

Remark 3.5.6. It is known that the orbit space is Hausdorff if the action Θ is proper, meaning that the map $\Xi \times \Sigma \ni (X, \sigma) \mapsto (\Theta_X(\sigma), \sigma) \in \Sigma \times \Sigma$ is proper in the usual topological sense [49]. This happens for instance if Σ is a Hausdorff locally compact group on which the closed subgroup Ξ acts by left translations. More generally, assume that the action Θ factorizes through a compact group $\widehat{\Xi}$, i.e. the kernel of Θ contains a closed co-compact subgroup Z of Ξ (with $\widehat{\Xi} = \Xi/Z$). Then the orbit space under the initial action is the same as the orbit space of the action of the compact quotient. But the action of a compact group is proper and Corollary 3.5.5 applies.

3.6 Some examples

Example 3.6.1. Let $\mathscr{A}$ be a C^*-algebra and T a locally compact space. On

$$\mathcal{A} \equiv C(T; \mathscr{A}) := \{f : T \to \mathscr{A} \mid f \text{ is continuous and small at infinity}\} \tag{3.6.1}$$

we consider the natural structure of C^*-algebra. It clearly defines a continuous field of C^*-algebras

$$\left\{\mathcal{C}(T;\mathscr{A}) \xrightarrow{\delta(t)} \mathscr{A} \mid t \in T\right\}, \quad \delta(t)f := f(t)\,.$$

The associated $\mathcal{C}(T)$-structure is given by $[\mathcal{Q}(\varphi)f]\,(t) := \varphi(t)f(t)$ for $\varphi \in \mathcal{C}(T)$, $f \in \mathcal{A}$, $t \in T$. For each $t \in T$ an action θ^t of Ξ on $\mathscr{A}$ is given; we require for each $f \in \mathcal{A}$ the condition

$$\sup_{t\in T} \| \theta^t_X[f(t)] - f(t) \|_{\mathscr{A}} \underset{X\to 0}{\longrightarrow} 0\,. \tag{3.6.2}$$

Then obviously

$$\Theta : \Xi \to \mathrm{Aut}(\mathcal{A})\,, \qquad [\Theta_X(f)](t) := \theta^t_X[f(t)] \tag{3.6.3}$$

defines a continuous action of the vector group Ξ on $\mathcal{A}$. Each of the kernels

$$\mathcal{I}(t) := \ker[\delta(t)] = \{f \in \mathcal{C}(T;\mathscr{A}) \mid f(t) = 0\}$$

is Θ-invariant, so one actually has a covariant continuous field of C^*-algebras (see Remarks 3.3.5 and 3.4.2). It makes sense to apply Rieffel quantization, getting C^*-algebras (respectively) $\mathfrak{A} \equiv \mathfrak{C}(T;\mathscr{A})$ from the dynamical system $(\mathcal{A} \equiv \mathcal{C}(T;\mathscr{A}), \Theta)$ and $\mathfrak{A}(t)$ from the dynamical system $(\mathscr{A}, \theta^t)$ for all $t \in T$. From the results above one concludes that $\left\{\mathfrak{A} \xrightarrow{\Delta(t)} \mathfrak{A}(t) \mid t \in T\right\}$ *is also a covariant field of C^*-algebras*. For each t we denoted by $\Delta(t)$ the Rieffel quantization of the morphism $\delta(t)$.

Example 3.6.2. A particular case, considered in [43, Ch.8], consists in taking $T := \mathrm{End}(\Xi)$ the space of all linear maps $t : \Xi \to \Xi$; it is a locally compact (finite-dimensional vector) space with the obvious operator norm. If an initial action θ of Ξ on $\mathscr{A}$ is fixed, the choice $\theta^t_X := \theta_{tX}$ verify all the requirements above. Therefore one gets a covariant continuous field of C^*-algebras indexed by $\mathrm{End}(\Xi)$. This is basically [43, Th.8.3]; we think that our treatment gives a simpler and more unified proof of this result, especially concerning the lower semi-continuous part. In particular, for any $f \in \mathcal{C}[\mathrm{End}(\Xi);\mathscr{A}]$, one has $\lim_{t\to 0} \| f(t) \|_{\mathfrak{A}(t)} = \| f(0) \|_{\mathscr{A}}$. An interesting particular case is obtained restricting the arguments to the compact subspace $T_0 := \{t = \sqrt{\hbar}\,\mathrm{id}_\Xi \mid \hbar \in [0,1]\} \subset T$. The number $\hbar$ corresponds to the Plank constant and, even for constant $f : [0,1] \to \mathscr{A}$, the relation $\lim_{\hbar\to 0} \| f \|_{\mathfrak{A}(\hbar)} = \| f \|_{\mathscr{A}}$ is non-trivial and has an important physical interpretation concerning the semiclassical behavior of the Quantum Mechanical formalism. We refer to [24, 43, 44] for much more on this topic.

Remark 3.6.3. A way to convert Example 3.6.1 in a more sophisticated one is as follows:

For every $t \in T$ pick $\mathcal{B}(t)$ to be a C^*-subalgebra of $\mathscr{A}$ which is invariant under the action θ^t. Construct the C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ defined as $\mathcal{B} := \{f \in \mathcal{C}(T;\mathscr{A}) \mid f(t) \in \mathcal{B}(t),\ \forall t \in T\}$, which is obviously invariant under the action Θ. One gets a covariant continuous field of C^*-algebras $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$, where $\mathcal{P}(t)$ is a restriction of the epimorphism $\delta(t)$. The general theory developed in sections 3.3 and 3.4 supplies another covariant continuous field of C^*-algebras $\left\{\mathfrak{B} \xrightarrow{\mathfrak{P}(t)} \mathfrak{B}(t) \mid t \in T\right\}$, where $\mathfrak{B}(t)$ is the quantization of $\mathcal{B}(t)$ and can be identified with an invariant C^*-subalgebra of $\mathfrak{A}(t)$.

Example 3.6.4. Crossed products associated to actions of $\mathscr{X} := \mathbb{R}^n$ on C^*-algebras can be obtained from Rieffel's quantization procedure, as it is explained in [43, Ex.10.5]. From the results of the present article one could infer rather easily, as a particular case, that (informally) *the crossed product by a continuous field of C^*-algebras is a continuous field of crossed products.* Such results exist in a much greater generality, including (twisted) actions of amenable locally compact groups [37, 40, 42, 49], so we are not going to give details.

Example 3.6.5. Let $T \subset [0,\infty)$ with the relative topology, set $\Omega_T := \{z \in \mathbb{C} \mid |z| \in T\}$ and $\Sigma := \Omega_T \times \mathbb{R}$. We consider the action Θ of $\Xi := \mathbb{R}^2$ on Σ given by

$$\Theta_{(x,y)}(z,a) := \left(e^{2\pi i x}z, a+y\right) .$$

It is easily checked that $q : \Sigma \to T$ given by $q(z,a) = |z|$ is continuous, open and Θ-covariant. So, applying the theory to $\mathcal{A} := \mathcal{C}(\Sigma)$, we can construct the covariant continuous field of non-commutative C^*-algebras $\{\mathfrak{C}(\Sigma) \longrightarrow \mathfrak{C}(\Sigma_t) \mid t \in T\}$. Note that for any $t \in T$ one has $\Sigma_t := q^{-1}(\{t\}) = S_t \times \mathbb{R}$, where S_t is the circle of radius t. It follows easily that, up to isomorphisms, if $t \neq 0$ the quantized C^*-algebra $\mathfrak{C}(\Sigma_t)$ is the quantum cylinder [43, Ex.10.6], while for $t = 0$ it is the Abelian C^*-algebra $\mathcal{C}(\mathbb{R})$.

For a related version pick $T_1, T_2 \subset [0,\infty)$, set $\Sigma = \Omega_{T_1} \times \Omega_{T_2}$ and $q : \Sigma \to T := T_1 \times T_2$ given by $q(z,w) = (|z|,|w|)$. Introduce the action $\Theta_{(x,y)}(z,w) := \left(e^{2\pi i x}z, e^{2\pi i y}w\right)$ and replace the usual symplectic form $[\![\cdot,\cdot]\!]$ by $\beta/2[\![\cdot,\cdot]\!]$, where β is some real number. In this case $\Sigma_{(t_1,t_2)} = S_{t_1} \times S_{t_2}$; thus if $t_1 t_2 \neq 0$ then $\mathfrak{C}(\Sigma_{(t_1,t_2)})$ is the quantum torus A_β [43, Ex.10.2], if $t_1 t_2 = 0$, $t_1 + t_2 \neq 0$ it is $\mathcal{C}(\mathbb{T})$, and if $t_1 = t_2 = 0$ it is $\mathbb{C}^2$.

Example 3.6.6. In [45] one constructs C^*-algebras which can be considered quantum versions of a certain class of compact connected Lie groups. We will have nothing to say about the extra structure making them quantum groups; we are only going to apply the results above to present these C^*-algebras as continuous fields.

Let Σ be a compact connected Lie group, containing *a toral subgroup*, i.e. a connected closed Abelian subgroup H. Such a toral group is isomorphic to an n-dimensional torus $\mathbb{T}^n$. Assume given a continuous group epimorphism $\eta : \mathbb{R}^n \to H$ (for example the exponential map defined on the Lie algebra $\mathfrak{H} \equiv \mathbb{R}^n$). We use η to define an action of $\Xi := \mathbb{R}^n \times \mathbb{R}^n$ on Σ by $\Theta_{(x,y)}(\sigma) := \eta(-x)\sigma\eta(y)$. Then, by applying Rieffel deformation to $\mathcal{A} := \mathcal{C}(\Sigma)$ using the action Θ (and a certain type of skew-symmetric operator on Ξ), one gets the C^*-algebra $\mathfrak{A} := \mathfrak{C}(\Sigma)$ which, endowed with suitable extra structure, is regarded as a quantum group corresponding to Σ.

It is obvious that the action factorizes through the compact group $H \times H$. Thus the orbit space $\mathrm{Orb}(\Sigma)$ is Hausdorff and Remark 3.5.6 and Corollary 3.5.5 serve to express $\mathfrak{C}(\Sigma)$ as a continuous field of C^*-algebras. For the stability group of any orbit $\mathcal{O}$ one can write $\Xi_{\mathcal{O}} \supset \ker(\Theta) \supset \ker(\eta) \times \ker(\eta)$, thus $\mathcal{O} \cong \Xi/\Xi_{\mathcal{O}}$ is a continuous image of $H \times H$.

Example 3.6.7. An interesting particular case, taken from [45], involves the construction of a quantum version of the compact Lie group $\Sigma := \mathbb{T} \times SU(2)$. Here $\mathbb{T}$ is the 1-torus, the group $SU(2)$ contains diagonally a second copy of $\mathbb{T}$ and can be parametrised by the 3-sphere $S^3 := \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$, and so Σ contains a 2-torus. Initially $\Xi = \mathbb{R}^4$ acts on Σ in the given way, but it is shown in [45] (using results from [43]) that the same deformed algebra is obtained by the action

$$\Theta' : \Xi' := \mathbb{R}^2 \to \text{Homeo}\left(\mathbb{T} \times S^3\right), \qquad \Theta'_{(x,y)}(\tau; z, w) := \left(e^{-2\pi i x}\tau; z, e^{4\pi i y} w\right).$$

The orbit space is homeomorphic with the closed unit disk $T := \{z \in \mathbb{C} \mid |z| \leq 1\}$. The orbits corresponding to $|z| < 1$ are 2-tori, while the orbits corresponding to $|z| = 1$ (implying $w = 0$) are 1-tori. If we set $\mathcal{A} := C(\mathbb{T} \times SU(2))$, then the quantized C^*-algebra $\mathfrak{A} \cong \mathfrak{C}(\mathbb{T} \times SU(2))$ deserves to be called *a quantum* $\mathbb{T} \times SU(2)$. The deformation of the continuous functions on any of the 2-tori leads to a quantum tori. By multiplying the initial skew-symmetric form $[\![\cdot,\cdot]\!]$ with an irrational number β one can make this non-commutative torus $\mathfrak{C}_\beta(\mathbb{T}^2)$ irrational, which serves to show that the corresponding quantum $\mathbb{T} \times SU(2)$ (obtained for such a β) is not of type I. But applying the results obtained here one also gets the detailed information: *The algebra $\mathfrak{C}(\mathbb{T} \times SU(2))$ can be written over the closed unit disk T as a continuous field of non-commutative 2-tori and Abelian C^*-algebras (corresponding to the one-dimensional orbits).*

Many other particular cases can be worked out in detail. We propose to the reader the example $\Sigma := SU(2) \times SU(2)$.

3.7 Spectral continuity

Let us introduced the concept of continuity for families of sets that will be useful below.

Definition 3.7.1. *Let T be a Hausdorff locally compact topological space and $\{S(t) \mid t \in T\}$ a family of compact subsets of $\mathbb{R}$.*

1. *The family is called* outer continuous *if for any $t_0 \in T$ and any compact subset K of $\mathbb{R}$ such that $K \cap S(t_0) = \emptyset$, there exists a neighborhood V of t_0 with $K \cap S(t) = \emptyset$, $\forall t \in V$.*

2. *The family $\{S(t) \mid t \in T\}$ is called* inner continuous *if for any $t_0 \in T$ and any open subset A of $\mathbb{R}$ such that $A \cap S(t_0) \neq \emptyset$, there exists a neighborhood W of t_0 with $A \cap S(t) \neq \emptyset$, $\forall t \in W$.*

3. *If the family is both inner and outer continuous, we say simply that it is* continuous.

In applications the sets $S(t)$ are spectra of some self-adjoint elements $f(t)$ of (non-commutative) C^*-algebras $\mathfrak{A}(t)$. The next result states technical conditions under which one gets continuity of such families of spectra. It is taken from [2] and it has been inspired by the treatment in [4]. We include the proof for the convenience of the reader.

Proposition 3.7.2. *For any $t \in T$ let $f(t)$ be a self-adjoint element in a C^*-algebra $\mathfrak{A}(t)$ with norm $\|\cdot\|_{\mathfrak{A}(t)}$ and inversion $g \mapsto g^{(-1)_{\mathfrak{A}(t)}}$. We denote by $S(t) \subset \mathbb{R}$ the spectrum of $f(t)$ in $\mathfrak{A}(t)$.*

1. *Assume that for any $z \in \mathbb{C} \setminus \mathbb{R}$ the mapping*

$$T \ni t \mapsto \left\| (f(t) - z)^{(-1)_{\mathfrak{A}(t)}} \right\|_{\mathfrak{A}(t)} \in \mathbb{R}_+ \tag{3.7.1}$$

is upper semi-continuous. Then the family $\{S(t) \mid t \in T\}$ is outer continuous.

2. *Assume that for any $z \in \mathbb{C} \setminus \mathbb{R}$ the mapping (3.7.1) is lower semi-continuous. Then the family $\{S(t) \mid t \in T\}$ is inner continuous.*

Proof. We use the functional calculus for self-adjoint elements in the C^*-algebra $\mathfrak{A}(t)$ to define $\chi[f(t)]$ for every continuous function $\chi : \mathbb{R} \to \mathbb{C}$ decaying at infinity. Notice that

$$(f(t) - z)^{(-1)_{\mathfrak{A}(t)}} = \chi_z[f(t)] \,, \quad \text{with} \ \ \chi_z(\lambda) := (\lambda - z)^{-1} \,.$$

By a standard argument relying on Stone-Weierstrass Theorem, one deduces that the map $t \mapsto \|\chi[f(t)]\|_{\mathfrak{A}(t)}$ has the same continuity properties (upper or lower semi-continuity, respectively) as (3.7.1).

Let us suppose now upper semi-continuity in t_0 and assume that $S(t_0) \cap K = \emptyset$ for some compact set K. By Urysohn's Lemma, there exists $\chi \in C_0(\mathbb{R})_+$ with $\chi|_K = 1$ and $\chi|_{S(t_0)} = 0$, so $\chi\,[f(t_0)] = 0$. Choose a neighborhood V of t_0 such that for $t \in V$

$$\| \, \chi[f(t)] \, \|_{\mathfrak{A}(t)} \leq \| \, \chi[f(t_0)] \, \|_{\mathfrak{A}(t)} + \frac{1}{2} = \frac{1}{2} \,.$$

If for some $t \in V$ there exists $\lambda \in K \cap S(t)$, then

$$1 = \chi(\lambda) \leq \sup_{\mu \in S(t)} \chi(\mu) = \| \, \chi[f(t)] \, \|_{\mathfrak{A}(t)} \leq \frac{1}{2} \,,$$

which is absurd.

Let us assume now lower semi-continuity in t_0. Pick an open set $A \subset \mathbb{R}$ such that $S(t_0) \cap A \neq \emptyset$ and let $\lambda \in S(t) \cap A$. By Urysohn's Lemma there exist a positive function $\chi \in C_0(\mathbb{R})$ with $\chi(\lambda) = 1$ and $\mathrm{supp}(\chi) \subset A$; thus $\| \, \chi\,[f(t_0)] \, \| \geq 1$. Suppose moreover that for any neighborhood $W \subset I$ of t_0 there exists $t \in W$ such that $S(t) \cap A = \emptyset$ and thus $\chi\,[f(t)] = 0$. This clearly contradicts the lower semi-continuity of $t \mapsto \| \, \chi\,[f(t)] \, \|_{\mathfrak{A}(t)}$. We conclude thus the inner continuity condition for the family $S(t)$. □

Proving these properties of the resolvents is a priory a difficult task, since this involves working both with norms and composition laws that depend on t. But putting together the information obtained until now, we get our abstract result concerning spectral continuity:

Theorem 3.7.3. *Let $\left\{ \mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T \right\}$ be a covariant upper semi-continuous field of C^*-algebras indexed by a Hausdorff locally compact space T and let f be a smooth self-adjoint element of $\mathcal{A}$. For any $t \in T$ we denote by $\mathfrak{A}(t)$ the Rieffel quantization of $\mathcal{A}(t)$ and consider $f(t) := \mathcal{P}(t) f$ as an element of $\mathcal{A}(t)^\infty = \mathfrak{A}(t)^\infty \subset \mathfrak{A}(t)$, with spectrum $S(t)$ computed in $\mathfrak{A}(t)$. Then the family $\{S(t) \mid t \in T\}$ is outer continuous.*

If the field is continuous, the family of subsets will also be continuous.

Proof. The results of the first chapter allow us to conclude that the quantized field $\left\{ \mathfrak{A} \xrightarrow{\mathfrak{P}(t)} \mathfrak{A}(t) \mid t \in T \right\}$ has the same continuity properties as the original one.

For any $z \in \mathbb{C} \setminus \mathbb{R}$ one has $(f - z)^{(-1)_{\mathfrak{A}}} \in \mathfrak{A}$ and $(f(t) - z)^{(-1)_{\mathfrak{A}(t)}} = \mathfrak{P}(t)\left[(f - z)^{(-1)_{\mathfrak{A}}}\right]$. Therefore the assumptions of Proposition 3.7.2 are fulfilled both in the upper semi-continuous and in the lower semi-continuous case, so we obtain the desired continuity properties for the family $\{S(t) \mid t \in T\}$. □

Of course, the conclusion also holds for non-smooth self-adjoint elements $f \in \mathfrak{A}$. Very often they are much less "accessible" than the smooth elements, being obtained by an abstract completion procedure, so we only make the statements for C^∞ vectors.

Specializing to the Abelian case and using the notations of section 3.5 , one gets

Corollary 3.7.4. *Let $f \in \mathcal{C}^\infty(\Sigma)$ a real function and for each $t \in T$ denote by $S(t)$ the spectrum of $f(t) := f|_{\Sigma_t} \in \mathcal{C}^\infty(\Sigma_t) = \mathfrak{C}^\infty(\Sigma_t)$ seen as an element of the non-commutative C^*-algebra $\mathfrak{C}(\Sigma_t)$. Then the family $\{S(t) \mid t \in T\}$ of compact subsets of $\mathbb{R}$ is outer continuous.*

If q is also open, the family of subsets is continuous.

Remark 3.7.5. One can use [43, Ex.10.2] to identify quantum tori as Rieffel-type quantizations of usual tori. One is naturally placed in the setting above and can reproduce some known spectral continuity results [13, 4] on generalized Harper operators.

The standard approach of Quantum Mechanics asks for Hilbert space operators. This can be achieved by representing faithfully the C^*-algebras $\mathfrak{A}(t)$ in a Hilbert space of L^2-functions in a way that generalizes the Schrödinger representation. We are going to get continuity results for both spectra and essential spectra of the emerging self-adjoint operators. We work in the following

Framework.

1. $(\mathcal{C}(\Sigma), \Theta, \Xi, [\![\cdot,\cdot]\!])$ is an Abelian classical data, with Σ compact.

2. Ξ is symplectic, given in a Lagrangean decomposition $\Xi = \mathscr{X} \times \mathscr{X}^* \ni X = (x, \xi)$, $Y = (y, \eta)$, where $\mathscr{X}$ is a n-dimensional real vector space, $\mathscr{X}^*$ is its dual and the symplectic form on Ξ is given in terms of the duality between $\mathscr{X}$ and $\mathscr{X}^*$ by $[\![(x,\xi),(y,\eta)]\!] := y \cdot \xi - x \cdot \eta$.

3. $q : \Sigma \to T$ is a Θ-covariant continuous surjection. We also assume that each $\Sigma_t := q^{-1}(\{t\})$ is *a quasi-orbit*, i.e. there is a point $\sigma \in \Sigma_t$ such that the orbit $\mathcal{O}_\sigma := \Theta_\Xi(\sigma)$ is dense in Σ_t (we say that σ *generates the quasi-orbit* Σ_t).

4. We fix a real element $f \in \mathcal{C}^\infty(\Sigma)$. For each $t \in T$ and for any point σ generating the quasi-orbit Σ_t we define $f(t) := f|_{\Sigma_t}$ and $f_\sigma(t) := f(t) \circ \Theta_\sigma : \Xi \to \mathbb{R}$.

5. We set $H_\sigma(t) := \mathfrak{Op}\left[f_\sigma(t)\right]$ (self-adjoint operator in the Hilbert space $\mathcal{H} := L^2(\mathscr{X})$), by applying to $f_\sigma(t)$ the usual Weyl pseudodifferential calculus. We denote by $S(t)$ the spectrum of $H_\sigma(t)$.

Some explanations are needed. It is easy to see that each $f_\sigma(t)$ belongs to $BC^\infty(\Xi)$, i.e. it is a smooth function with bounded derivatives of any order. Therefore, using oscillatory integrals, one can define the self-adjoint operator in $L^2(\mathscr{X})$

$$[H_\sigma(t)u](x) \equiv \big[\mathfrak{Op}(f_\sigma(t))u\big](x) := (2\pi)^{-n}\int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\xi\, e^{i(x-y)\cdot\xi}\, [f_\sigma(t)]\left(\frac{x+y}{2},\xi\right) u(y)\,. \tag{3.7.2}$$

This operator is bounded by the Calderón-Vaillancourt Theorem [15]. Using the notation (3.1.4), we see that for every $X\in\Xi$ one has $[f_\sigma(t)](X) := f[\Theta_X(\sigma)]$; this depends on $t\in T$ through σ and only involves the values of f on the dense subset $\mathcal{O}_\sigma$ of Σ_t. The same is true about $H_\sigma(t)$, which can be written

$$[H_\sigma(t)u](x) = (2\pi)^{-n}\int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\xi\, e^{i(x-y)\cdot\xi} f\left[\Theta_{(\frac{x+y}{2},\xi)}(\sigma)\right] u(y)\,. \tag{3.7.3}$$

It is shown in [29] that if σ and σ' are both generating the same quasi-orbit Σ_t, then the operators $H_\sigma(t)$ and $H_{\sigma'}(t)$ are isospectral (but not unitarily equivalent in general). Thus the compact set $S(t)$ only depend on t and not on the choice of the generating element σ.

Theorem 3.7.6. *Assume the Framework above. Then the family $\{S(t)\mid t\in T\}$ is outer continuous.*

If q is also open, than the family is continuous.

Proof. By Corollary 3.7.4, it would be enough to show for every t that $S(t)$ coincides with the spectrum of $f(t)\in\mathfrak{C}(\Sigma_t)$. For this we define

$$\mathcal{N}_\sigma : C^\infty(\Sigma_t)\to BC^\infty(\Xi),\qquad \mathcal{N}_\sigma(g) := g\circ\Theta_\sigma$$

and then set

$$\mathfrak{Op}_\sigma := \mathfrak{Op}\circ\mathcal{N}_\sigma : C^\infty(\Sigma_t)\to\mathbb{B}(\mathcal{H})\,.$$

Then one has $H_\sigma(t) := \mathfrak{Op}[f_\sigma(t)] = \mathfrak{Op}_\sigma[f(t)]$. It is not quite trivial, but it has been shown in [29], that $\mathfrak{Op}_\sigma$ extends to a faithful representation of the Rieffel quantized C^*-algebra $\mathfrak{C}(\Sigma_t)$ in $\mathcal{H}$. Faithfulness is implied by the fact that σ generates the quasi-orbit Σ_t, which results in the injectivity of $\mathcal{N}_\sigma$, conveniently extended to $\mathfrak{C}(\Sigma_t)$. It follows then that $\mathrm{sp}[H_\sigma(t)] = \mathrm{sp}[f(t)]$, as required, so the family $\{S(t)\mid t\in T\}$ has the desired continuity properties. □

We recall that *the essential spectrum* of an operator is the part of the spectrum composed of accumulation points or infinitely-degenerated eigenvalues. Let us denote by $S^{\mathrm{ess}}(t)$ the essential spectrum of $H_\sigma(t)$; once again this only depends on t. To discuss the continuity properties of this family of sets we are going to need some preparations relying mainly on results from [29].

First we write each Σ_t as a disjoint Θ-invariant union $\Sigma_t = \Sigma_t^{\mathrm{g}}\sqcup\Sigma_t^{\mathrm{n}}$. The elements σ_1 of Σ_t^{g} are *generic points for* Σ_t, meaning that each of them is generating Σ_t. The points $\sigma_2\in\Sigma_t^{\mathrm{n}}$ are *non-generic*, i.e. the closure of the orbit $\mathcal{O}_{\sigma_2}$ is strictly contained in Σ_t.

Let us now fix a point $t\in T$ and a generating element $\sigma\in\Sigma_t$. The monomorphism $\mathcal{N}_\sigma$ extends to an isomorphism between $\mathcal{C}(\Sigma_t)$ and a C^*-subalgebra $\mathcal{B}_\sigma(t)$ of the C^*-algebra $BC_{\mathrm{u}}(\Xi)$ of all the bounded uniformly continuous complex functions on Ξ. It is shown in Lemma 2.2 from [29] that only two possibilities can occur, and this is independent of σ: either $\mathcal{C}(\Xi)\subset\mathcal{B}_\sigma(t)$ (and then t is called *of the first type*), or $\mathcal{C}(\Xi)\cap\mathcal{B}_\sigma(t) = \{0\}$ (and then we say that t is *of the second type*). Correspondingly, one has the disjoint decomposition $T = T_I\sqcup T_{II}$.

Theorem 3.7.7. *Assume the Framework above. Then the family* $\{S^{\mathrm{ess}}(t) \mid t \in T\}$ *is outer continuous.*

Proof. One must rephrase the essential spectrum $S^{\mathrm{ess}}(t) := \mathrm{sp}_{\mathrm{ess}}[H_\sigma(t)]$ in convenient C^*-algebraic terms. Assume first that t is of the second type. By [29, Prop. 3.4], the discrete spectrum of $H_\sigma(t)$ is void, thus one has $S^{\mathrm{ess}}(t) = S(t)$. If t is of the first type, the subset Σ_t^{n} is invariant under the action Θ and it is also closed by [29, Prop. 2.5]. Denoting by $f^{\mathrm{n}}(t)$ the restriction of $f(t)$ to Σ_t^{n}, one gets an element of $\mathcal{C}^\infty(\Sigma_t^{\mathrm{n}}) \subset \mathfrak{C}(\Sigma_t^{\mathrm{n}})$ with spectrum $S^{\mathrm{n}}(t)$. But [29, Th. 3.7] states among others that $S^{\mathrm{n}}(t)$ coincides with $S^{\mathrm{ess}}(t)$.

We need to construct now a suitable restricted dynamical system. Let us consider the decomposition

$$\Sigma = \Big(\bigsqcup_{t\in T_I}\Sigma_t\Big) \sqcup \Big(\bigsqcup_{t\in T_{II}}\Sigma_t\Big) = \Big(\bigsqcup_{t\in T_I}\Sigma_t^{\mathrm{g}}\Big) \sqcup \Big\{\Big(\bigsqcup_{t\in T_I}\Sigma_t^{\mathrm{n}}\Big) \sqcup \Big(\bigsqcup_{t\in T_{II}}\Sigma_t\Big)\Big\} =: \Sigma^{\mathrm{d}} \sqcup \Sigma^{\mathrm{ess}}.$$

One might set $\Sigma_t^{\mathrm{ess}} := \Sigma_t^{\mathrm{n}}$ if $t \in T_I$ and $\Sigma_t^{\mathrm{ess}} := \Sigma_t$ if $t \in T_{II}$. Notice that each Σ_t^{ess} is not void. This is clear for $t \in T_{II}$, since q has been supposed surjective. If $t \in T_I$ and $\Sigma_t^{\mathrm{n}} = \emptyset$, then $\Sigma_t = \Sigma_t^{\mathrm{g}}$ is minimal and compact, so $t \in T_{II}$ by Lemma 2.3 in [29], which is absurd. The disjoint union $\Sigma^{\mathrm{ess}} := \sqcup_{t\in T}\Sigma_t^{\mathrm{ess}}$ (with the topology induced from Σ) is a compact dynamical system under the restriction of the action Θ of Ξ and $q^{\mathrm{ess}} := q|_{\Sigma^{\mathrm{ess}}} : \Sigma^{\mathrm{ess}} \to T$ is a covariant continuous surjection. Thus we can apply the previous results and conclude that $\{\mathfrak{C}(\Sigma^{\mathrm{ess}}) \to \mathfrak{C}(\Sigma_t^{\mathrm{ess}}) \mid t \in T\}$ is an upper semi-continuous field of C^*-algebras; the arrows are Rieffel quantizations of obvious restriction maps.

From all these applied to $f|_{\Sigma^{\mathrm{ess}}} \in \mathfrak{C}^\infty(\Sigma^{\mathrm{ess}})$ it follows that $\{S^{\mathrm{ess}}(t) = \mathrm{sp}\,[f(t)|_{\Sigma^{\mathrm{ess}}(t)}] \mid t \in T\}$ is outer continuous. $\square$

Remark 3.7.8. Even in simple situations, the surjective restriction of a continuous open surjection may not be open. So q^{ess} may fail to be open and in general we don't obtain inner continuity for the family of essential spectra. On the other hand, if openness of the restriction q^{ess} is required explicitly, one clearly gets the inner continuity. Since only the dynamical system $(\Sigma^{\mathrm{ess}}, \Theta, \Xi)$ is involved in controlling the family of essential spectra, some assumptions weaker than those above would suffice.

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