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EXISTENCE AND STABILITY OF STEADY STATES SOLUTIONS OF FLAT VLASOV-POISSON SYSTEM WITH A CENTRAL MASS DENSITY

TESIS PARA OPTAR AL GRADO DE MAGISTER EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MATEMÁTICAS APLICADAS

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RESUMEN DE LA TESIS PARA OPTAR AL GRADO DE MAGISTER
EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MATEMÁTICAS APLICADAS
Y DE LA MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO
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EXISTENCIA Y ESTABILIDAD DE ESTADOS ESTACIONARIOS DEL SISTEMA DE VLASOV-POISSON CON UNA DENSIDAD DE MASA CENTRAL

Estudiamos un modelo Newtoniano que nos permite describir algunos fenómenos en dinámica estelar. Este modelo es descrito por una ecuación en derivadas parciales conocida como la *ecuación de Vlasov* o la *ecuación de Liouville*, cuyas soluciones describen la evolución temporal de un sistema de partículas libre de colisiones en el plano de fase, sujeto a un potencial gravitacional autointeractuante.

Este trabajo está dividido en dos partes. En la primera parte, estudiamos el sistema de Vlasov-Poisson Plano con un potencial gravitacional externo inducido por una densidad de masa fija. Este modelo describe algunos objetos extremadamente planos en dinámica de galaxias. El objetivo de esta parte es el estudio de la existencia y estabilidad de estados estacionarios del sistema de Vlasov-Poisson Plano en este caso. Resolvemos un problema variacional para encontrar minimizadores del *funcional de Energía-Casimir* en un conjunto de funciones adecuado. El problema de minimización es resuelto a través de una reducción del problema de optimización original (ver [22]), pero en lugar de utilizar un argumento de concentración-compacidad, usamos un argumento de simetrización, tomando el reordenamiento de una sucesión minimizante, y probamos que converge débil a un minimizador en un espacio L^p adecuado, con $p > 1$. Probamos que este minimizador induce una solución para el problema de minimización original. El problema de minimización nos entrega un resultado de estabilidad no lineal para el estado estacionario en el espacio L^p mencionado antes.

En la segunda parte, mostramos los resultados publicados en [13]. Probamos el *mixing* en el plano de fase para soluciones de la ecuación de Vlasov en sistemas integrables. Bajo una condición natural de no-armonicidad, obtenemos convergencia débil para la función de distribución con ratio t^{-1}. En una dimensión, también estudiamos el caso donde esta condición falla en cierta energía, probando que el mixing aún se mantiene pero con un ratio más lento. Cuando ocurre esta condición y las funciones tienen mayor regularidad, la convergencia puede ir más rápido.

Abstract

We study a Newtonian model, that allows us to describe some phenomena in stellar dynamics. This model is described by a partial differential equation known as the *Vlasov equation* or the *Liouville equation*, whose solutions describe the temporal evolution of a collisionless particle system in the phase space, subject to a self-interacting gravitational potential.

This work is divided in two parts. In the first part, we treat the Flat Vlasov-Poisson system with an external gravitational potential induced by a fixed mass density. This model describes some extremely flat objects in galactic dynamics. The aim of this part is the study of the existence and stability of steady states solutions of the Flat Vlasov-Poisson system in this case. We solve a variational problem to find minimizers for the *Casimir-Energy functional* in a suitable set of functions. The minimization problem is solved through a reduction of the original optimization problem (see [22]), but instead of a concentration-compactness argument, we use a symmetrization argument, taking the rearrangement of a minimizing sequence and we prove that it converges weakly to a minimizer in a suitable L^p space with $p > 1$. We prove that this minimizer induces a solution for the original minimization problem. The minimization problem give us a non-linear stability result for the steady state solution in the L^p space.

In the second part, we show the results published in [13]. We prove phase-space mixing for solutions of the Vlasov equation for integrable systems. Under a natural non-harmonicity condition, we obtain weak convergence of the distribution function with rate t^{-1}. In one dimension, we also study the case where this condition fails at a certain energy, showing that mixing still holds but with a slower rate. When the condition holds and functions have higher regularity, the rate can be faster.

Viva el sushi vegano.

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Table of Content

List of Figures

Introduction

In classical mechanics, if we want to describe the evolution of N particles self-interacting through some kind of force (e.g. the gravitational force or the electric force), we could model the dynamics of the particles with a system of N differential equations given by the *Newton's second law*. The trajectory of each particle satisfies the equation

$$m_i \ddot{\mathbf{x}}_i = \mathbf{F_i}(\mathbf{x_1}, ..., \mathbf{x_N}), \tag{1}$$

where $i = 1, ..., N$, and $m_i, \mathbf{x_i}$ are the mass and position of the particle i, respectively, and $\mathbf{F_i}$ is the net force acting over the particle i, induced by the presence of the other $N - 1$ particles. For example, if the force F_i acting on the particle i, is given by the gravitational interaction between the particles, equation (1) results in

$$m_i \ddot{\mathbf{x}}_i = -G \sum_{j \neq i}^{N} m_i m_j \frac{\mathbf{x_i} - \mathbf{x_j}}{|\mathbf{x_i} - \mathbf{x_j}|^{\mathbf{3}}}, \tag{2}$$

where $G \approx 6.674 \cdot 10^{-11} [\frac{\mathrm{N \cdot m^2}}{\mathrm{km^2}}]$ is the universal gravitational constant. Given an initial datum (position and velocity) for each particle, the system described above has a solution. The biggest problem with this model lies in the fact that in some contexts, the number of particles N, and therefore the number of the equations in the system is large. Hence, the computational cost increases until the system of equations is impossible to solve. Since it is impossible to study adequately the problem for values of N extremely larges through classical mechanics, we use statistical mechanics as a convenient way to model the particle system, studying the global behavior instead the dynamics of each particle. Hence, we describe the physical system through a one-particle distribution function f on the phase space $U \times V$, where $U, V \subseteq \mathbb{R}^d$, such that

$$(t, x, v) \in \mathbb{R} \times U \times V \mapsto f(t, x, v) \in \mathbb{R}_0^+ \tag{3}$$

where the particle distribution function envolves in time. Therefore, in the same context, assuming that the mass of the system is finite, and it is a fixed value $M > 0$, by the mass conservation we have that in all time t

$$\iint_W f(t, x, v) dv dx, \tag{4}$$

is the total mass in the section $W \subset U \times V$ of the phase space, at time t, and in general

$$\iint f(t, x, v) dv dx = M. \tag{5}$$

If we asume there are no collisions between the particles in the system, then the particle distribution function satisfies the hypothesis of *Liouville's theorem* (see [1, 9]) and therefore, it is constant along the particles trajectories. By the Newton's second law

$$\begin{aligned}\dot{\mathbf{x}}(s,t,x,v) &= \mathbf{v}(s,t,x,v)\\ \dot{\mathbf{v}}(s,t,x,v) &= \mathbf{F}(s,\mathbf{x}(s,t,x,v))\\ \mathbf{x}(t,t,x,v) &= x\\ \mathbf{v}(t,t,x,v) &= v,\end{aligned}$$

where

$$\mathbf{F} = -\nabla_x U, \tag{6}$$

is a force field induced by the potential U due to the self-interaction. Hence, by Liouville's theorem, the total derivative is zero:

$$\frac{d}{ds} f(s,\mathbf{x}(s),\mathbf{v}(s)) = 0,$$

which implies that

$$\partial_t f(s,\mathbf{x}(s),\mathbf{v}(s)) + \mathbf{v}(s)\cdot\nabla_x f(s,\mathbf{x}(s),\mathbf{v}(s)) + \mathbf{F}\cdot\nabla_v f(s,\mathbf{x}(s),\mathbf{v}(s)), \tag{7}$$

where for convenience, we denote $\mathbf{x}(s) := \mathbf{x}(s,t,x,v)$ and $\mathbf{v}(s) := \mathbf{v}(s,t,x,v)$. Therefore, we obtain the following partial differential equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_x U\cdot\nabla_v f = 0. \tag{8}$$

Equation (8) is called the *Vlasov equation*, also known as the *Liouville equation* or the *collisionless Boltzmann equation*. This thesis focuses on the analysis of the existence and stability of steady states solutions of the Vlasov equation in some contexts, where the self-interaction between the particles is given by the gravitational potential, and furthermore, these particles are under the influence of an external potential that we will describe in next chapters on this thesis. It is of general interest to study the stability of these steady states, because in many physical contexts, the estimation of a initial datum could have an error which could give us radically different solutions for the same problem. The stability guarantees for an initial datum, in some sense the solutions do not change for another initial datum nearby the original. We will prove the stability results for the steady states that it can be obtained as minimizers for a suitable energy functional.

The Vlasov-Poisson system

If we assume that the force field described in (6) comes from a gravitational potential generated by the particles system itself, then the potential U satisfies the Poisson equation

$$\Delta U(t,x) = 4\pi\rho_f(t,x) \tag{9}$$

where we denote by $\rho_f(t,x)$, the *spatial density* at point x and time t

$$\rho_f(t,x) = \int f(t,x,v)dv. \tag{10}$$

We request that the gravitational potential vanishes at infinity, in the sense of

$$\lim_{|x|\to\infty} U(t,x) = 0. \tag{11}$$

The gravitational potential U is a solution for the equations (9) and (11). This solution (the potential) can be written as a convolution of mass density against the fundamental solution of the Laplace equation (see [6, Pag 22, Definition])

$$U_f = \Upsilon_0 * \rho_f. \tag{12}$$

In particular, for dimension $d = 3$ the fundamental solution is given by

$$\Upsilon_0 = -\frac{1}{|\cdot|},$$

and the potential U_f can be written as

$$U_f(t,x) = -\int \frac{\rho_f(t,y)}{|x-y|} dy. \tag{13}$$

The equations (5), (10) and (11), together with the equations (8) and (13), give us the system known as the *Vlasov-Poisson system*. In [16], the existence and uniqueness of a classical solution for the Vlasov-Poisson system, given an initial datum $f_0 \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ is proved. We can prove with a brief calculation, that for a non-time depending potential, the local energy per particle $E : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined as

$$E(x,v) = \frac{1}{2}|v|^2 + U(x), \tag{14}$$

is a steady state solution of the Vlasov equation, because $\nabla_x E(x,v) = \nabla_x U(x)$ and $\nabla_v E(x,v) = v$, and hence

$$v \cdot \nabla_x U(x) - \nabla_x U(x) \cdot v = 0.$$

It is customary (see [8, Pag 3]), to search steady state solutions with the form

$$f = \phi(E), \tag{15}$$

for a suitable function ϕ, and therefore, we only need to solve (9) to obtain a solution for the system. A classical example of solutions in the form (15), are the *isotropic polytropes*

$$f(x,v) = (E_0 - E(x,v))_+^k, \tag{16}$$

where $E_0 < 0$, with $-1 < k < 7/2$, and $a \mapsto a_+$ is the positive part of a function. These functions are spherically symmetric solutions for the Vlasov equation in three dimensions, with compact support and finite mass. Existence and stability of those solutions was proved in [27].

It can be shown that if f solves the Vlasov equation, then the total energy of the system

$$t \mapsto \mathcal{E}(f(t,\cdot,\cdot)) := E_{kin}(f(t,\cdot,\cdot)) + E_{pot}(f(t,\cdot,\cdot)), \tag{17}$$

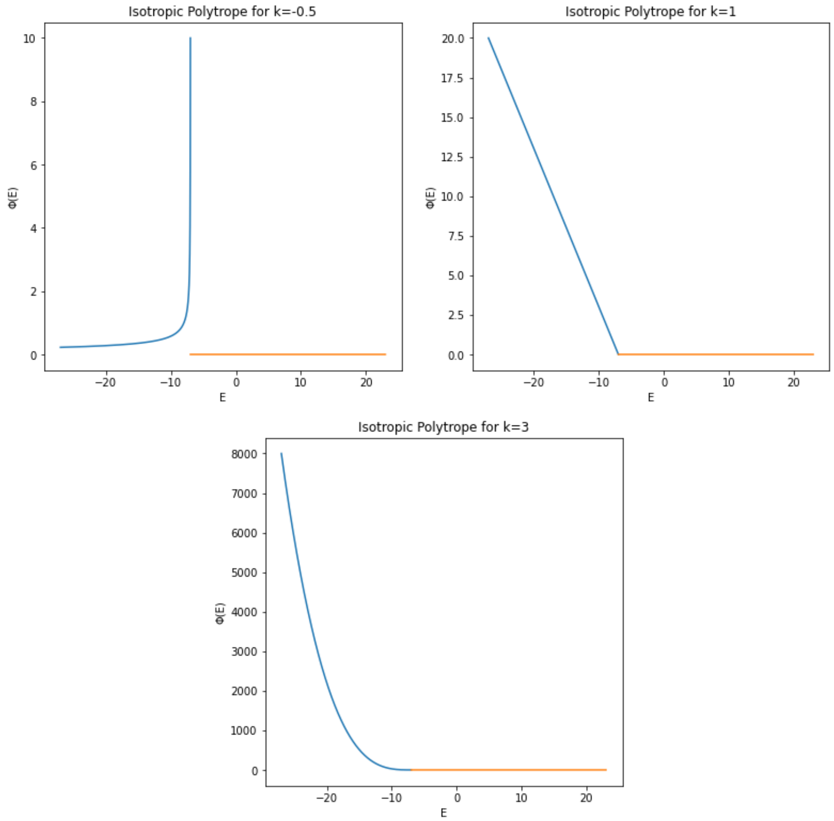

Figure 1: Isotropic polytropes as a function of energy E, for values $k = -0.5$, $k = 1$, $k = 3$. In blue, the values for which $\phi(E) = (E_0 - E)_+^k$ is not zero.

is conserved in time. That means

$$\frac{d}{dt}\mathcal{E}(f(t,x,v)) = 0, \tag{18}$$

for all $t \in \mathbb{R}$, $x, v \in \mathbb{R}^3$. Here, E_{kin} and E_{pot} are the kinetic and potential energy of the system, respectively. That is

$$E_{kin}(f(t,\cdot,\cdot)) := \iint \frac{|v|^2}{2} f(t,x,v)dvdx, \tag{19}$$

and

$$E_{pot}(f(t,\cdot,\cdot)) := \frac{1}{2}\int U_f(t,x)\rho_f(t,x)dx = -\frac{1}{2}\iint \frac{\rho_f(t,x)\rho_f(t,y)}{|x-y|}dxdy. \tag{20}$$

However, it can be shown that if f is a steady state solution of the Vlasov equation, then it cannot be a critical point of the total energy functional $\mathcal{E}$. Indeed, if f_0 is a steady state solution of Vlasov equation that induces a potential U_0, then we have

$$\mathcal{E}(f) - \mathcal{E}(f_0) = \iint \left(\frac{|v|^2}{2} + U_0(x)\right)(f - f_0)dvdx - \frac{1}{8\pi}\|\nabla U_f - \nabla U_0\|_2^2$$

whose linear part does not vanish, hence f_0 cannot be a critial point. It is customary to considerer the functionals in the form

$$\mathcal{C}(f(t,\cdot,\cdot)) := \iint \Phi(f(t,x,v))dvdx, \tag{21}$$

which are called *Casimir functionals*, are used to have a chance to find stationary solutions of the Vlasov equation as minimizers of the combined energy functional

$$\mathcal{E}_{\mathcal{C}}(f) = \mathcal{E}(f) + \mathcal{C}(f). \tag{22}$$

Here, $\Phi : [0, \infty) \to [0, \infty)$ will be a sufficiently differentiable function. These functionals are called *Casimir-Energy functionals.* In the same way as with the total energy, we can prove that the Casimir-Energy functional is a conserved quantity. The existence of steady state solutions of Vlasov-Poisson system which minimizes the Casimir-Energy functional is a well studied problem (see [10, 19, 21]).

The Flat Vlasov-Poisson system

Next, we will describe the flat Vlasov-Poisson system defined in [8, 19], which enables to model extremely flat objects in stellar dynamics. For simplicity, the first assumption that is made to model a flat stellar object with collisionless kinetic particles, is to suppose that all galactic matter is concentrated in an infinitesimal thin layer. If $x = (x_1, x_2, x_3)$ and $v = (v_1, v_2, v_3)$ are the coordinates for position and velocity of particles, we request that the particles stay concentrated in the (x_1, x_2)-plane. In order to keep the matter concentrated there, we need that the particles do not escape outside, then we request that velocities are restricted to move on the (v_1, v_2)-plane. Formally, we can write the particle distribution function f with this restrictions through a Dirac delta, in the sense of distribution as

$$f(t, x, v) = g(t, x_1, x_2, v_1, v_2)\delta(x_3)\delta(v_3) \tag{23}$$

$$\rho_f(t, x) = \rho_g(t, x_1, x_2)\delta(x_3). \tag{24}$$

In [8], it was shown that f is a solution of the Vlasov equation in $\mathbb{R}^3$ in the distribution sense if and only if g solves the Vlasov equation in $\mathbb{R}^2$, with the modified force term

$$\tilde{\mathbf{F}}(t, \tilde{x}) := -\int \frac{\tilde{x} - \tilde{y}}{|\tilde{x} - \tilde{y}|^3} \rho_g(t, \tilde{y}) d\tilde{y}. \tag{25}$$

This motives to study the following partial differential equations, named the *Flat Vlasov-Poisson system*

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \tag{26}$$

$$U(t, x) = -\int \frac{\rho_f(y)}{|x - y|} dy, \tag{27}$$

$$\lim_{|x| \to \infty} U(t, x) = 0. \tag{28}$$

Remark Note that the Vlasov-Poisson system in dimension two, is not the same that the Flat Vlasov-Poisson system, because the fundamental solution of Poisson equation in two dimensions is $\Upsilon(x) = C \ln(|x|)$ for a suitable constant $C > 0$, and therefore, the potential formula of U in (27) differs of $\Upsilon_0 * \rho_f$. The problem with the Flat Vlasov-Poisson system resides in the behavior of $|\cdot|^{-1}$ in $\mathbb{R}^2$ instead of $\mathbb{R}^3$.

In [8], it was shown that for suitable conditions for the function Φ, the Casimir-Energy functional $\mathcal{E}_{\mathcal{C}}$ has a minimizer in some feasible functional space, which is a stationary state with non-linear stability properties of the Flat Vlasov-Poisson system. In the first chapter of this thesis, we will describe the same system with an external potential which comes from a central object described by a fixed spatial density, and we will extend those ideas for this case, proving the existence of a minimizer for a Casimir-Energy functional which is an stationary state of the Flat Vlasov-Poisson system with the external potential, and which preserves the non-linear stability properties of the original case. For this, we will prove the existence using a rearrangement argument, instead of a Concentration-compactness argument (see [15, 22, 8]).

Phase-space mixing

Another interesting thing that we can study describing matter as a collisionless kinetic gas, is to study some macroscopic *observables* properties, through its microscopic average behavior over the phase space. As we mentioned in the introduction of the present text, the idea is to study the asymptotic average of these observables, and to find conditions to reach an equilibrium state. It is easy to prove that the Vlasov equation can be written as

$$\partial_t f = \nabla_x U \cdot \nabla_v f - v \cdot \nabla_x f = \{\mathcal{H}, f\}, \tag{29}$$

where $\mathcal{H}$ is the Hamiltonian of the system, which it is given by

$$\mathcal{H}(x,v) = \frac{1}{2}|v|^2 + U(x),$$

and $\{\cdot,\cdot\}$ is the Poisson's bracket. In some contexts, we can rewrite the equation (29) changing the canonical coordinates system conveniently. As the Poisson's bracket definition is the same for every canonical coordinates (q,k), then we can write

$$\partial_t f = \sum_{k=1}^{d} \left(\frac{\partial \mathcal{H}}{\partial q_i}\frac{\partial f}{\partial k_i} - \frac{\partial \mathcal{H}}{\partial k_i}\frac{\partial f}{\partial q_i} \right). \tag{30}$$

If the motion of a single particle in this system is integrable, we can write (30) using action-angle variables (see Appendix A.49), taking $q = (q_1, ..., q_d) \in \mathbb{T}^d$, and $k = (k_1, ..., k_d) \in K \subseteq \mathbb{R}^d$, where K is a suitable open set, obtaining an equation in the form

$$\partial_t f(t,q,k) + \omega(k) \cdot \nabla_q f(t,q,k) = 0. \tag{31}$$

In [26, 25], we can find some examples where we can write the dynamic of the system into the form (31). In classical mechanics, this occurs for instance in a potential well in one space-dimension or for spherically symmetric potentials in dimensions 2 and 3. But also in a relativistic context, geodesic motion in the Kerr family of space-times is integrable and Liouville's equation (or the collisionless Boltzmann equation) can be written in the form (31). If the system is anharmonic, in the sense that points with nearby energies move at different angular speeds $\omega(k)$, regular initial distributions will eventually stretch out to thin filaments that cover the region of phase space allowed by the conservation laws. This phenomenon is called *phase-space mixing*.

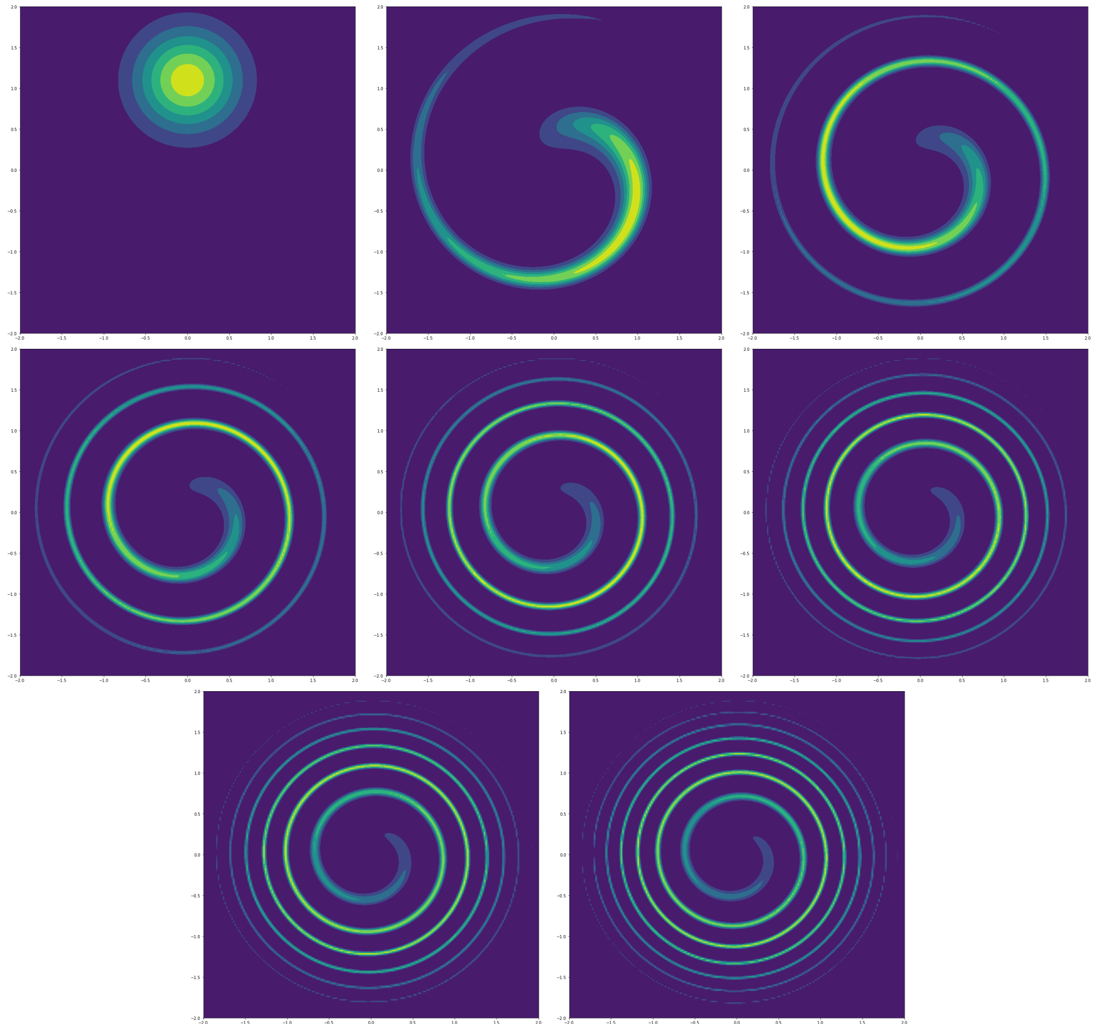

Figure 2: Snapshots at times $t = 10k\pi$ and $t = 2k\pi$, for k from 0 to 7, of the evolution of a Gaussian initial condition, in a perturbed harmonic oscillator with Hamiltonian $\mathcal{H} = p^2/2 + x^2/2 + \varepsilon x^4$ with $\varepsilon = 0.2$, approximate at first order in perturbation theory

The main problem here is the study of the asymptotic behavior of the phase space averages. In a physical context, macroscopic observables are obtained averaging the microscopic observable over each particle in the phase space, so we can study if the macroscopic observable reach an equilibrium state, studying the following limit

$$\lim_{t\to\infty} \iint \phi(x,v) f(t,x,v) dv dx = \lim_{t\to\infty} \iint (\phi \circ \gamma)(q,k) f(t,q,k) dq dk, \tag{32}$$

where γ is the action-angle coordinates transformation (see Appendix A.49). Phase-space mixing say us that the limit above exists and it is the average of the observable quantity over a non-time depending particle distribution function, and it occurs for all observable ϕ. In other words, it is a kind of *weak* convergence of $t \mapsto f(t,\cdot,\cdot)$ over all *test functions* ϕ. The key of this part consist of writing the equation in *action-angle* coordinates, and study the mixing over $\mathbb{T}^d \times K$ as the limit of the second equality on (32). For integrable systems, this change of coordinates is possible and transforms Vlasov equation in a transport equation in the form of (31).

The second part of this thesis are the results published in [13]. Using a *vector field method*, we proved the phase-space mixing for solutions of Vlasov equation for integrable systems, and we studied the rate of convergence for the limit above.

Chapter 1

Flat Vlasov-Poisson system with central mass density

We study the flat Vlasov-Poisson system as described in the introduction, with the addition of an external gravitational potential created by a fixed mass density. The main problem of this chapter is to search steady states solutions for this system as minimizers of Casimir-Energy functional, and to study non-linear stability properties.

1.1 Equations with the external potential

In this context, as the same way in the original case, we consider the Vlasov-Poisson system in $\mathbb{R}^3$ with the following potential

$$U(t,x) = -\int \frac{\rho_f(t,y)}{|x-y|}dy - \int \frac{\rho_{ext}(x)}{|x-y|}dy. \tag{1.1}$$

Here, the second term in the equality (1.1) is the external potential of the system which comes from the central mass density. Similarly to Flat Vlasov-Poisson system, to keep the matter concentrated in a flat thin layer, we have that the distribution f is written as

$$f(t,x,v) = g(t,x_1,x_2,v_1,v_2)\delta(x_3)\delta(v_3), \tag{1.2}$$

$$\rho_f(t,x) = \rho_g(t,x_1,x_2)\delta(x_3), \tag{1.3}$$

$$\rho_{ext}(x) = \nu_{ext}(x_1,x_2)\delta(x_3), \tag{1.4}$$

where ρ_{ext} is the spatial mass density from the external potential. Therefore, as in [8], the distribution f defined as in (1.2) is solution of the Vlasov equation in the sense of distributions if and only if g is solution of the Vlasov equation with the force term

$$\tilde{\mathbf{F}}(t,\tilde{x}) := -\int \frac{\tilde{x}-\tilde{y}}{|\tilde{x}-\tilde{y}|^3}\rho_g(t,\tilde{y})d\tilde{y} - \int \frac{\tilde{x}-\tilde{y}}{|\tilde{x}-\tilde{y}|^3}\nu_{ext}(\tilde{y})d\tilde{y}, \tag{1.5}$$

where $\tilde{x}$ and the integrals are over $\mathbb{R}^2$. Therefore, we define the flat Vlasov-Poisson system with the external potential as follows.

Definition 1.1 (Flat Vlasov-Poisson system with central mass density) *Let $\rho_{ext} \in L^1_+(\mathbb{R}^2)$ be a function with compact support. We define the Flat Vlasov-Poisson system with a central mass density, as the following system of partial differential equations*

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \tag{1.6}$$

$$U(t,x) = -\int \frac{\rho_f(t,y)}{|x-y|} dy - \int \frac{\rho_{ext}(y)}{|x-y|} dy, \tag{1.7}$$

$$\lim_{|x|\to\infty} U(t,x) = 0, \tag{1.8}$$

$$\iint f(t,x,v) dv dx = M, \tag{1.9}$$

where $M > 0$ is the total mass of the system, $x, v \in \mathbb{R}^2$ and $t \in [0, \infty)$.

Observe that the central mass density described by ρ_{ext}, generates a gravitational field that induces potential energy over the particle system described by the solutions f of the system above. Therefore, we have that

Definition 1.2 (Potential energy induced by central mass density) *Let ρ_{ext} the spatial mass density defined in 1.1 and f the particle function distribution of the system. We define the potential energy induced by ρ_{ext} over the flat particle system as*

$$E^{\varepsilon}_{pot}(f(t,\cdot,\cdot)) := \int U_{ext}(x)\rho_f(t,x) dx = -\iint \frac{\rho_{ext}(x)\rho_f(t,y)}{|x-y|} dx dy, \tag{1.10}$$

where

$$U_{ext}(x) := -\int \frac{\rho_{ext}(y)}{|x-y|} dy, \tag{1.11}$$

is the gravitational potential induced by ρ_{ext}. Hence, we define the potential energy of this system as

$$t \mapsto E_{pot}(f(t,\cdot,\cdot)) := E^1_{pot}(f(t,\cdot,\cdot)) + E^{\varepsilon}_{pot}(f(t,\cdot,\cdot)), \tag{1.12}$$

where E^1_{pot} is the potential energy associated to self-interaction defined in 20.

Therefore, the energy system is defined as

$$\mathcal{E}(f) := E_{kin}(f) + E_{pot}(f), \tag{1.13}$$

where the potential energy of the system is defined above. The main goal of this chapter is to prove analogous results of [8] about existence and stability, for the Flat Vlasov-Poisson system with a central mass density. Hence, we will assume some suitable properties for the function Φ, such that the Energy-Casimir functional

$$\mathcal{E}_{\mathcal{C}}(f) = \mathcal{E}(f) + \mathcal{C}(f),$$

has a minimizer over the following set of functions

$$\mathcal{F}_M := \left\{ f \in L^1_+(\mathbb{R}^4) \mid E_{kin}(f) + \mathcal{C}(f) < \infty, \iint f = M \right\}, \tag{1.14}$$

where $M > 0$ is the total mass of the system, which is fixed. Then we make the following assumptions for the function Φ:

(a) $\Phi \in C^1([0,\infty))$ is strictly convex,

(b) $\Phi'(0) = \Phi(0) = 0$,

(c) $\Phi(f) \gtrsim f^{1+1/k}$ for $f \geq 0$ big,

where $k \in (0,1)$ is a fixed parameter. Here for convenience, we adopt the next notation: if $x, y \in \mathbb{R}$, we denote

$$x \lesssim y,$$

when there is a constant $C > 0$ such that $x \leq Cy$, and we denote by $x \simeq y$ whenever $x \lesssim y$ and $y \lesssim x$. Under these assumptions, we have the following assertion:

Proposition 1.3 *If the function Φ satisfies (a), (b) y (c), then it is a non-negative function and its derivative Φ' is a bijection from $[0,\infty)$ to $[0,\infty)$.*

Proof. As Φ is a C^1 strictly convex function on $[0,\infty)$, we have that

$$\Phi(x) > \Phi(y) + \Phi'(y)(x-y), \tag{1.15}$$

for all $x, y \in [0,\infty)$ with $x \neq y$. Hence, for $x > 0$, by the inequality 1.15 we have that $\Phi(x) > 0$, and as $\Phi(0) = 0$, then Φ is a non-negative function. In the other way, inverting roles of x and y we conclude that

$$0 > (\Phi'(y) - \Phi'(x))(x-y), \tag{1.16}$$

so if $x \neq y$, then $\Phi'(y) - \Phi'(x)$ cannot be 0, then Φ is an injective map. Finally, if $y > 0$, for the inequality 1.15 we have that

$$\Phi'(y) > \Phi(y)/y. \tag{1.17}$$

Property (c) implies that $\Phi(y)/y \to \infty$ when $y \to \infty$, and therefore $\Phi'(y) \to \infty$ when $y \to \infty$. As Φ is C^1, we have that if $q > 0$, there exists $y_q \in [0,\infty)$ such that $\Phi'(y_q) = q$. As $\Phi'(0) = 0$, we have that Φ' is a surjective function. Therefore Φ' is a bijective function, as we wanted to show. □

We will prove that we can construct a minimizer of the Casimir-Energy functional over the feasible set 1.24, which is a steady state of the Flat Vlasov-Poisson system with a central mass density, and we will give some non-linear stability properties. The following theorems are the main results of this chapter.

Theorem 1.4 *Let $\Phi : [0,\infty) \to [0,\infty)$ be a function which satisfies the properties (a), (b), and (c), mentioned above, and suppose that $\rho_{ext} \in L^{4/3}(\mathbb{R}^2)$ is stricly symmetric decreasing. Let $n = k+1 \in (1,2)$. Then there exists some $E_0 < 0$ such that*

$$f_0 = (\Phi')^{-1}(E_0 - E)\chi_{E_0 > E} \tag{1.18}$$

is a minimizer of $\mathcal{E}_\mathcal{C}$ in $\mathcal{F}_M$. Moreover, if $\Phi \in C^2([0,\infty))$, $\Phi'' > 0$, and $\rho_{ext} \in L^{4/n}(\mathbb{R}^2)$, then f_0 is a stationary solution of Flat Vlasov-Poisson system with a central mass density. Here

$$E(x,v) = \frac{1}{2}|v|^2 + U_0 + U_{ext}.$$

is the local energy per particle.

Theorem 1.5 *Let f_0 be a minimizer for $\mathcal{E}_{\mathcal{C}}$ in $\mathcal{F}_M$ obtained from Theorem 1.4 and we suppose that it is unique, and let $\rho_0 := \rho_{f_0}$. Define*

$$d(f_1, f_2) := \mathcal{E}_{\mathcal{C}}(f_1) - \mathcal{E}_{\mathcal{C}}(f_2) - E^1_{pot}(\rho_{f_1} - \rho_{f_2}), \tag{1.19}$$

and let some arbitrary $\varepsilon > 0$. Then there exists some $\delta > 0$ such that for every solution of the Flat Vlasov-Poisson system with central mass density $t \mapsto f(t)$, with $f(0) \in C^1_c(\mathbb{R}^4) \cap \mathcal{F}_M$, if

$$d(f(0), f_0) + E^1_{pot}(\rho_{f(0)} - \rho_0) < \delta, \tag{1.20}$$

then

$$d(f(t), f_0) + E^1_{pot}(\rho_{f(t)} - \rho_0) < \varepsilon, \tag{1.21}$$

for every $t \geq 0$.

1.2 A reduction for the variational problem

In the same way as in [8], we solve a reduction of the original optimization problem. This reduction is defined over a suitable space of densities ρ, and then we will connect the solution for this reduced optimization problem with the original, constructing a solution using the Euler-Lagrange equation for the reduced solution. Since we search steady state solutions of the system, we have that

$$t \mapsto f(t, x, v),$$

and

$$t \mapsto \rho_f(t, x),$$

are time-independent flows, so henceforth we omit the time dependence in the notation. The idea is to prove that for $M > 0$, the Casimir-Energy functional $\mathcal{E}_{\mathcal{C}}$ has a minimizer over the following feasible set

$$\mathcal{F}_M := \left\{ f \in L^1_+(\mathbb{R}^4) \mid E_{kin}(f) + \mathcal{C}(f) < \infty, \iint f = M \right\}. \tag{1.22}$$

For this, in the same way as [8], we study the following reduced variational problem, consisting in to search minimizers for the functional

$$\mathcal{E}^r_{\mathcal{C}}(\rho) := \int \Psi(\rho(x))dx + E_{pot}(\rho), \tag{1.23}$$

over the set

$$\mathcal{F}^r_M := \left\{ \rho \in L^{4/3}(\mathbb{R}^2) \cap L^1_+(\mathbb{R}^2) \mid \int \Psi(\rho(x))dx < \infty, \int \rho(x)dx = M \right\}, \tag{1.24}$$

where the function Ψ is defined through the following variational problem

$$\Psi(r) := \inf_{g \in \mathcal{G}_r} \mathcal{I}(g), \tag{1.25}$$

where the functional $\mathcal{I}$ is given by

$$\mathcal{I}(g) := \int \frac{|v|^2}{2} g(v) + \Phi(g(v))dv, \tag{1.26}$$

over the feasible set

$$\mathcal{G}_r := \left\{ g \in L^1_+(\mathbb{R}^2) \mid \mathcal{I}(g) < \infty, \int g(v)dv = r \right\}. \tag{1.27}$$

The main goal is to minimize the Casimir-Energy functional over all functions $f(x, v)$ such that its spatial density is some fixed ρ, and after that, to minimize it over ρ. We state the following lemma, whose demonstration can be found in [8]. The main idea for proving this lemma comes from the *Legendre Transform* of a function f:

$$f^*(\lambda) = \sup_{x \in \mathbb{R}} (\lambda x + f(x)). \tag{1.28}$$

Lemma 1.6 *Let Φ and Ψ be defined as above, and extending both functions to $+\infty$ on $(-\infty, 0)$. Then we have the following assertions:*

(a) $\Psi \in C^1([0, \infty))$, is strictly convex and $\Psi(0) = \Psi'(0) = 0$.

(b) Let $k > 0$ and $n = k + 1$. Then

(i) If $\Phi(f) \simeq f^{1+1/k}$ for all $f \geq 0$, then $\Psi(\rho) \simeq \rho^{1+1/n}$ for $\rho \geq 0$

(ii) If $\Phi(f) \gtrsim f^{1+1/k}$ for big $f \geq 0$, then $\Psi(\rho) \gtrsim \rho^{1+1/n}$ for big $\rho \geq 0$.

1.2.1 The Euler-Lagrange equation for the original and reduced variational problem

Solving the reduced variational problem for $\mathcal{E}^r_{\mathcal{C}}$ has a motivation which lies in the following result. The next theorem makes a connection between the solution of the reduction and the original problem. This theorem is a modified version of [8, Teo 2.3] for the potential defined in 1.7

Theorem 1.7 *Let ρ_{ext} be the spatial density defined in (10). We have the following assertions:*

(a) For all functions $f \in \mathcal{F}_M$, we have that

$$\mathcal{E}_{\mathcal{C}}(f) \geq \mathcal{E}^r_{\mathcal{C}}(\rho_f) \tag{1.29}$$

with equality if f is a minimizer of $\mathcal{E}_{\mathcal{C}}$ over $\mathcal{F}_M$.

(b) Let ρ_0 be a minimizer of $\mathcal{E}^r_{\mathcal{C}}$ over $\mathcal{F}^r_M$ and let $U := U_0 + U_{ext}$, where U_0 is the gravitational potential induced by ρ_0 and U_{ext} is the gravitational potential induced by ρ_{ext}. Suppose also that ρ_0 is spherically symmetric and nonincreasing. Then there exist a Lagrange multiplier $E_0 < 0$ such that almost everywhere we have

$$\rho_0 = (\Psi')^{-1}(E_0 - U)\chi_{E_0 > U}, \tag{1.30}$$

and the function f_0 defined as

$$f_0 := (\Phi')^{-1}(E_0 - E)\chi_{E_0 > E}, \tag{1.31}$$

where $E(x,v) = \frac{1}{2}|v|^2 + U(x)$, is a minimizer of $\mathcal{E}_{\mathcal{C}}$ en $\mathcal{F}_M$.

First, we will prove the next result, which allow us to conclude that if f is a minimizer, then we have the equality in 1.29.

Proposition 1.8 *If $f \in \mathcal{F}_M$ is such that if $\Phi'(f) = E_0 - E$ almost everywhere over the points such that $f > 0$, and $E_0 - E \leq 0$ when $f = 0$, then $\mathcal{E}_{\mathcal{C}}(f) = \mathcal{E}^r_{\mathcal{C}}(\rho_f)$.*

Proof. Since Φ is a convex function, for $g \in \mathcal{G}_{\rho_f(x)}$ we have that almost everywhere in $x \in \mathbb{R}^2$

$$\mathcal{I}(g) \geq \mathcal{I}(f(x,\cdot)) + \int \left(\frac{1}{2}|v|^2 + \Phi'(f(x,v))\right)(g(v) - f(x,v))dv.$$

We separate the range of integration as

$$\int \left(\frac{1}{2}|v|^2 + \Phi'(f(x,v))\right)(g(v) - f(x,v))dv = I_1 + I_2,$$

where

$$I_1 = \int_{\{f>0\}} \left(\frac{1}{2}|v|^2 + \Phi'(f(x,v))\right)(g(v) - f(x,v))dv,$$

and

$$I_2 = \int_{\{f=0\}} \left(\frac{1}{2}|v|^2 + \Phi'(f(x,v))\right)(g(v) - f(x,v))dv.$$

For the first one, since $\Phi'(f(x,v)) = E_0 - E(x,v)$ a.e. whenever $f > 0$,

$$\begin{aligned} I_1 &= (E_0 - U_f(x)) \int_{\{f>0\}} (g(v) - f(x,v))dv, \\ &= (E_0 - U_f(x)) \left(\int (g(v) - f(x,v))dv - \int_{\{f=0\}} (g(v) - f(x,v))dv\right) \\ &= (U_f(x) - E_0) \int_{\{f=0\}} g(v)dv, \end{aligned}$$

where the last equality comes from the fact that $g \in \mathcal{G}_{\rho_f(x)}$. For the second integral, using the fact that $\Phi'(0) = 0$, we have that

$$I_2 = \int_{\{f=0\}} \frac{1}{2}|v|^2 g(v)dv,$$

and therefore

$$I_1 + I_2 = \int_{\{f=0\}} (E(x,v) - E_0) g(v) dv,$$

and as $g \in L^1_+(\mathbb{R}^2)$, and $E_0 - E \leq 0$ when $f = 0$ a.e., then $I_1 + I_2 \geq 0$. Hence $\mathcal{I}(g) \geq \mathcal{I}(f(x,\cdot))$ and as $v \mapsto f(x,v) \in \mathcal{G}_{\rho_f(x)}$, we have that

$$\Psi(\rho_f(x)) = \inf_{g \in \mathcal{G}_{\rho_f(x)}} \mathcal{I}(g) \geq \mathcal{I}(f(x,\cdot)) \geq \inf_{g \in \mathcal{G}_{\rho_f(x)}} \mathcal{I}(g) = \Psi(\rho_f(x)),$$

and therefore

$$\begin{aligned}
\mathcal{E}^r_{\mathcal{C}}(\rho_f) &= E_{pot}(\rho_f) + \int \Psi(\rho_f(x)) dx \\
&= E_{pot}(\rho_f) + \int \mathcal{I}(f(x,\cdot)) dx \\
&= E_{pot}(\rho_f) + E_{kin}(f) + \mathcal{C}(f) \\
&= \mathcal{E}_{\mathcal{C}}(f)
\end{aligned}$$

as we wanted to prove. □

Proof. (a) Let $f \in \mathcal{F}_M$, then for all $x \in \mathbb{R}^2$, the function $v \mapsto f(x,v)$ belongs to $L^1_+(\mathbb{R}^2)$. Moreover, if we had that

$$\int \frac{|v|^2}{2} f(x,v) + \Phi(f(x,v)) dv = \infty,$$

then

$$\begin{aligned}
E_{kin}(f) + \mathcal{C}(f) &= \int\int \frac{|v|^2}{2} f(x,v) dv dx + \int\int \Phi(f(x,v)) dv dx \\
&= \int \left(\int \frac{|v|^2}{2} f(x,v) + \Phi(f(x,v)) dv \right) dx \\
&= \infty
\end{aligned}$$

and it contradicts that $f \in \mathcal{F}_M$. Hence $f(x,\cdot) \in \mathcal{G}_{\rho_f(x)}$, and therefore

$$\Psi(\rho_f(x)) \leq \mathcal{I}(f(x,v)).$$

Thus

$$\begin{aligned}
\mathcal{E}^r_{\mathcal{C}}(\rho_f) &= \int \Psi(\rho_f(x)) dx + E_{pot}(\rho_f) \\
&\leq \int \mathcal{I}(f(x,v)) dx + E_{pot}(\rho_f) \\
&= \mathcal{E}_{\mathcal{C}}(f).
\end{aligned}$$

We can prove that if f_0 is a minimizer of the functional $\mathcal{E}_{\mathcal{C}}$, then f_0 satisfies the hypothesis of 1.8, and therefore we have the equality on 1.29.

(b) Let ρ_0 a minimizer of $\mathcal{E}_{\mathcal{C}}^r$ in $\mathcal{F}_M^r$ and let $\varphi \in C_c^\infty(\mathbb{R}^2)$ such that $\operatorname{supp}(\varphi)$ is strictly contained in $\operatorname{supp}(\rho_0)$. If we define

$$\delta := \inf_{x \in \operatorname{supp}(\varphi)} \rho_0(x),$$

then $\delta > 0$. As we have that ρ_0 is spherically symmetric and nonincreasing, then we can find $\lambda_\delta > 0$ such that

$$\lambda_\delta \cdot \sup_{x \in \operatorname{supp}(\varphi)} |\varphi(x)| < \frac{\delta}{2}.$$

Thus we have that $\rho_0 + \lambda\varphi \geq 0$ on $\operatorname{supp}(\rho_0)$, for every $\lambda \in I_{\lambda_\delta} := (-\lambda_\delta, \lambda_\delta)$. This allow us to construct a function $r : I_{\lambda_\delta} \to \mathbb{R}$ as

$$\xi(\lambda) = \mathcal{E}_{\mathcal{C}}^r(\rho_0 + \lambda\varphi),$$

which is well defined. Note that since ρ_0 is a minimizer of $\mathcal{E}_{\mathcal{C}}^r$, ξ has a minimum in $\lambda = 0$, therefore exists a Lagrange multiplier $E_0 \in \mathbb{R}$ such that

$$\xi'(0) = E_0 \cdot \frac{d}{d\lambda}\left(\int (\rho_0(x) + \lambda\varphi(x))dx - M\right) = \int E_0 \cdot \varphi(x)dx.$$

For the right-hand-side, we compute

$$\xi'(0) = \int (\Psi'(\rho_0(x)) + U(x))\varphi(x)dx,$$

which implies that

$$\int (\Psi'(\rho_0(x)) + U(x) - E_0)\varphi(x)dx = 0,$$

where φ is an arbitrary element of $C_c^\infty(\mathbb{R}^2)$ with $\operatorname{supp}(\varphi) \subset \operatorname{supp}(\rho_0)$. Hence we have almost everywhere on $\operatorname{supp}(\rho_0)$, that

$$\Psi'(\rho_0) = E_0 - U,$$

and $E_0 \leq U$ almost everywhere on $\mathbb{R}^2 \setminus \operatorname{supp}(\rho_0)$. Since Ψ' is non negative and a bijective map, we have the Euler-Lagrange equation for the minimizer:

$$\rho_0 = (\Psi')^{-1}(E_0 - U)\chi_{E_0 > U},$$

where $U = U_0 + U_{ext}$. On the other hand, we can prove that if f_0 is defined as in 1.31, then $\rho_0 = \rho_{f_0}$, therefore for an arbitrary $f \in \mathcal{F}_M$, we have that

$$\mathcal{E}_{\mathcal{C}}(f) \geq \mathcal{E}_{\mathcal{C}}^r(\rho_f) \geq \mathcal{E}_{\mathcal{C}}^r(\rho_0) = \mathcal{E}_{\mathcal{C}}^r(\rho_{f_0}) = \mathcal{E}_{\mathcal{C}}(f_0)$$

i.e. f_0 minimizes $\mathcal{E}_{\mathcal{C}}$ over $\mathcal{F}_M$. A brief calculation gives us that when $E_0 > U$

$$\begin{aligned}\int f_0(x,v)dv &= \int_{E_0 > E} (\Phi')^{-1}(E_0 - E)dv \\ &= (\Psi^*)'(E_0 - U) = (\Psi')^{-1}(E_0 - U) \\ &= \rho_0,\end{aligned}$$

and both sides are zero where $E_0 \leq U$, and since $U(x) \to 0$ when $|x| \to \infty$ we conclude that $E_0 < 0$. Here we denoted by Ψ^* the Legendre transform of the function Ψ.

□

1.3 The reduced variational problem

As we saw in the last section, Theorem 1.7 allows us to build a minimizer for the original problem, by solving the reduced problem and writing the Euler-Lagrange equation for the reduction. In this section we will prove the existence of a minimizer for reduced problem, using an argument based on rearrangements of minimizing sequences. As we saw in Lemma 1.6, the following assertions for the function Ψ hold

(a) $\Psi \in C^1([0,\infty))$ is strictly convex.

(b) $\Psi'(0) = \Psi(0) = 0$.

(c) $\Psi(\rho) \gtrsim \rho^{1+1/n}$ for big $\rho \geq 0$.

Here, we have that $n, n' \in (0,2)$ are fixed suitable parameters. Hence, we want to solve the following variational problem

$$I_M := \inf_{\rho \in \mathcal{F}_M^r} \mathcal{E}_{\mathcal{C}}^r(\rho) = \inf_{\rho \in \mathcal{F}_M^r} \left(E_{pot}(\rho) + \int \Psi(\rho(x)) dx \right) \tag{1.32}$$

over the feasible set defined in (1.24).

The following theorem that we will prove, give conditions which implies directly the result of Theorem 1.4.

Theorem 1.9 *Assume that $\rho_{ext} \in L^{4/3}(\mathbb{R}^2)$ is strictly symmetric decreasing. Under the assumptions (a),(b), and (c) on the function Ψ mentioned above, there is $\rho_0 \in \mathcal{F}_M^r$ such that is a solution for the reduced variational problem 1.32, and therefore, the function*

$$f_0 := (\Phi')^{-1}(E_0 - E)\chi_{E_0 > E}, \tag{1.33}$$

is a minimizer of $\mathcal{E}_{\mathcal{C}}$ in $\mathcal{F}_M$, where $E(x,v) = \frac{1}{2}|v|^2 + U(x)$ and $E_0 < 0$ is the Lagrange multiplier associated to Euler-Lagrange equation of ρ_0.

1.3.1 Weak convergence of minimizing sequences and a candidate for minimizer

The main idea for the proof is to take a minimizing sequence for the variational problem, and proving that it converges weakly in a suitable $L^p(\mathbb{R}^2)$ space, hoping that the weak limit is a minimizer and an element of the feasible set. In [13], the reduced problem for Flat Vlasov-Poisson system was solved using a concentration compactness argument (see [15]) and the general ideas for this method can be reviewed in [22]. In this thesis we used a symmetrization argument, taking the rearrangement of the minimizing sequence to obtain nonincreasing spherically symmetries, to keep the density concentrated in a finite region, avoiding the spatial translations and splitting.

Before proceeding with the proof of Theorem 1.9, we will prove some useful previous results.

Lemma 1.10 *If $\rho \in L^{4/3}(\mathbb{R}^2)$, then the gravitational potential U_ρ is an element of $L^4(\mathbb{R}^2)$ which is the dual space of $L^{4/3}(\mathbb{R}^2)$.*

Proof. By the weak version of Young's inequality (see A.48) with $p = 4/3$, $q = 2$ and $r = 4$ we have that

$$\|U_\rho\|_4 = \left\| \rho * \frac{1}{|\cdot|} \right\|_4 \lesssim \left\| \frac{1}{|\cdot|} \right\|_{w,2} \|\rho\|_{4/3} \, .$$

By Theorem A.24, is easy to see that $1/|\cdot| \in L^2_w(\mathbb{R}^2)$, where the result is direct, and by the fact that $3/4 + 1/4 = 1$, we have that $L^4(\mathbb{R}^2)$ can be identified with the dual space of $L^{4/3}(\mathbb{R}^2)$. In particular, the *Coulomb energy* defined as

$$\mathcal{D}(\rho, \sigma) := \frac{1}{2} \iint \frac{\rho(x)\sigma(y)}{|x-y|} dxdy, \tag{1.34}$$

is an inner product in $L^{4/3}(\mathbb{R}^2)$. □

The next is to prove that the reduced variational problem is well defined, in the sense that the Casimir-Energy functional is bounded below over the feasible set, and therefore the infimum does not *"escape"*. We have the following lemma:

Lemma 1.11 *Under the assumptions (a), (b), and (c) for the function Ψ, we have that $I_M > -\infty$.*

Proof. Let $\rho \in \mathcal{F}^r_M$. We have that

$$\mathcal{E}^r_{\mathcal{C}}(\rho) = \int \Psi(\rho(x))dx + E^1_{pot}(\rho) + E^\varepsilon_{pot}(\rho).$$

We must prove that this expression is bounded from below, uniformly in ρ. For this, we will find bounds for both terms separately. By Hardy-Littlewood-Sobolev inequality (see A.47) with $\lambda = 1$, $n = 2$, $p = r = 4/3$, as $\rho, \rho_{ext} \in L^{4/3}(\mathbb{R}^2)$ we have that

$$-E^1_{pot}(\rho) = \frac{1}{2} \int\int \frac{\rho(x)\rho(y)}{|x-y|} dxdy \lesssim \|\rho\|^2_{4/3}, \tag{1.35}$$

$$-E^\varepsilon_{pot}(\rho) = \frac{1}{2} \int\int \frac{\rho_{ext}(x)\rho(y)}{|x-y|} dxdy \lesssim \|\rho_{ext}\|_{4/3}\|\rho\|_{4/3} \lesssim \|\rho\|_{4/3}. \tag{1.36}$$

By *Riesz-Thorin Interpolation Lemma* (see Appendix A.45) with $p_\theta = 4/3$, $p_0 = 1$ y $p_1 = 1 + 1/n$, we have that

$$\|\rho\|_{4/3} \leq \|\rho\|_1^{\frac{3-n}{4}} \|\rho\|_{1+1/n}^{\frac{n+1}{4}} \lesssim \|\rho\|_{1+1/n}^{\frac{n+1}{4}}. \tag{1.37}$$

By the assumption (c) for the function Ψ, we have that there exist $\delta > 0$ such that for every $\rho > \delta$, we have $\Psi(\rho) \geq C\rho^{1+1/n}$ for some suitable constant $C > 0$. Hence, if we define

$\{\rho > \delta\} := \{x \in \mathbb{R}^2 : \rho(x) > \delta\}$, we have that

$$\int \rho(x)^{1+1/n} dx = \int_{\{\rho>\delta\}} \rho(x)^{1+1/n} dx + \int_{\mathbb{R}^2 \setminus \{\rho>\delta\}} \rho(x)^{1+1/n} dx \tag{1.38}$$

$$\leq C \int \Psi(\rho(x)) dx + \delta^{1/n} \int \rho(x) dx \tag{1.39}$$

$$\lesssim \int \Psi(\rho(x)) dx + 1. \tag{1.40}$$

Therefore, by the last bound we have

$$\|\rho\|_{1+1/n}^{\frac{n+1}{4}} = \left(\int \rho(x)^{1+1/n} dx \right)^{n/4} \lesssim \left(\int \Psi(\rho(x)) dx \right)^{n/2} + 1, \tag{1.41}$$

and moreover

$$\|\rho\|_{1+1/n}^{\frac{n+1}{2}} = \left(\int \rho(x)^{1+1/n} dx \right)^{n/2} \lesssim \left(\int \Psi(\rho(x)) dx \right)^{n/2} + 1. \tag{1.42}$$

Here we used the fact that for all $a \geq 0$ and $0 < n < 2$, we have the inequalities

$$(1+a)^{n/4} \leq (1+a)^{n/2} \leq 1 + a^{n/2}.$$

The first of them is directly true, while the second one comes from the fact that the function $a \mapsto 1 + a^{n/2} - (1+a)^{n/2}$ is increasing in $[0, \infty)$ with $0 \mapsto 0$. Therefore, combining the inequalities above, we have the following bounds for the potential energies

$$-E^1_{pot}(\rho) \lesssim \|\rho\|^2_{4/3} \lesssim \|\rho\|_{1+1/n}^{\frac{n+1}{2}} \lesssim \left(\int \Psi(\rho(x)) dx \right)^{n/2} + 1, \tag{1.43}$$

$$-E^\varepsilon_{pot}(\rho) \leq C\|\rho\|_{4/3} \lesssim \|\rho\|_{1+1/n}^{\frac{n+1}{4}} \lesssim \left(\int \Psi(\rho(x)) dx \right)^{n/2} + 1. \tag{1.44}$$

Hence, we have the next inequality for the reduced energy functional

$$\mathcal{E}^r_{\mathcal{C}}(\rho) \geq \int \Psi(\rho(x)) dx - C \left(\int \Psi(\rho(x)) dx \right)^{n/2} - C, \tag{1.45}$$

where $C > 0$ is a suitable constant. If we consider the function $g : \mathbb{R} \to \mathbb{R}$ defined as $g(x) = -Cx^{n/2} + x - C$, since $0 < n < 2$, it is easy to verify that $g''(x) \geq 0$ and the equation $g'(x) = 0$ has a solution, and therefore g has a global minimum over $\mathbb{R}^+$. Hence, we have in (1.45) that

$$\mathcal{E}^r_{\mathcal{C}}(\rho) = g(x_\Psi) \geq \inf_{x \in \mathbb{R}} g(x) > -\infty,$$

where

$$x_\Psi := \int \Psi(\rho(x)) dx.$$

Thus we have inmediatly that $I_M > -\infty$ as we wanted to prove. □

A corollary of above lemma is the following, which allow us to find a candidate to a minimizer of reduced variational problem

Corollary 1.12 *All minimizing sequences $(\rho_i)_{i\in\mathbb{N}}$ in $\mathcal{F}_M^r$ of $\mathcal{E}_\mathcal{C}^r$, and their rearrangements (see Appendix A.28), are bounded in $L^{1+1/n}(\mathbb{R}^2)$ and $L^{4/3}(\mathbb{R}^2)$. In particular, each of them has a subsequence that converges weakly in these spaces.*

Remark We will see later that it will be convenient to take the rearrangement of a minimizing sequence, to obtain symmetry properties of the elements of the sequence.

Proof. Since $L^{1+1/n}(\mathbb{R}^2)$ and $L^{4/3}(\mathbb{R}^2)$ are reflexive spaces, the unit ball in each spaces is weak-sequentially compact (see Appendix A.11), thus it is sufficient to prove that every minimizing sequence is bounded in these spaces, to extract some weakly convergent subsequences. Let $(\rho_i)_{i\in\mathbb{N}}$ be a minimizing sequence of reduced variational problem, i.e.

$$\lim_{i\to\infty} \mathcal{E}_\mathcal{C}^r(\rho_i) = I_M.$$

In particular, $\mathcal{E}_\mathcal{C}^r(\rho_i)$ is bounded in $\mathbb{R}$. By (1.45), we have that

$$\mathcal{E}_\mathcal{C}^r(\rho_i) \geq \int \Psi(\rho_i(x))dx \left(1 - C\left(\int \Psi(\rho_i(x))dx\right)^{\frac{n}{2}-1}\right) - C, \tag{1.46}$$

for a suitable constant $C > 0$. If $\int \psi(\rho_i)dx$ is not bounded, as $0 < n < 2$, the right side of (1.46) goes to ∞ when $i \to \infty$, and it contradicting that $\mathcal{E}_\mathcal{C}^r(\rho_i)$ is bounded. Hence $\int \psi(\rho_i)dx$ is bounded, and since

$$\int \rho_i(x)^{1+1/n}dx \lesssim 1 + \int \Psi(\rho_i(x))dx,$$

the boundedness result in $L^{1+1/n}(\mathbb{R}^2)$ is direct, and as $\|\rho_i\|_{4/3} \lesssim \|\rho_i\|_{1+1/n}^{\frac{n+1}{4}}$ we have the sequence is also bounded in $L^{4/3}(\mathbb{R}^2)$. In particular, as the rearrangement preserves the norm (see Appendix A.28), we have that $\|\rho_i\|_{1+1/n} = \|\rho_i^*\|_{1+1/n}$ and $\|\rho_i\|_{4/3} = \|\rho_i^*\|_{4/3}$, and therefore we conclude that the rearrangement sequence is bounded in each spaces and thus, it has a weakly convergent subsequence of the rearrangements of initial sequence which converges in each spaces.

Now, we just need to prove that the weak limit is the same. For this, let ρ_0 and ρ_1 the weak limits of $(\rho_i)_{i\in\mathbb{N}}$ in $L^{1+1/n}(\mathbb{R}^2)$ and $L^{4/3}(\mathbb{R}^2)$, respectively, and take any Lebesgue measurable set A. Then we have that $\rho_0\chi_{A\cap B(0,R)}$ and $\rho_1\chi_{A\cap B(0,R)}$ converge pointwise to $\rho_0\chi_A$ and $\rho_1\chi_A$ whenever $R \to \infty$, respectively, and both are dominated by ρ_0 and ρ_1 which are integrable. Since $\chi_{A\cap B(0,R)}$ is an element of every $L^p(\mathbb{R}^2)$ with $1 \leq p \leq \infty$, by weak convergence we have that if $i \to \infty$, then

$$\int \rho_i(x)\chi_{A\cap B(0,R)}(x)dx \longrightarrow \int \rho_0(x)\chi_{A\cap B(0,R)}(x)dx = \int \rho_1\chi_{A\cap B(0,R)}(x)dx$$

and taking $R \to \infty$, by the *Dominated Convergence Theorem* we have that

$$\int \rho_0(x)\chi_A(x)dx = \int \rho_1(x)\chi_A(x)dx,$$

for each Lebesgue measurable set A, and therefore $\rho_0 - \rho_1 = 0$ a.e. and thus

$$\|\rho_0 - \rho_1\|_{1+1/n} = 0.$$

$\square$

Now we will prove that the rearrangement of the minimizing sequence is also a minimizing sequence of the functional $\mathcal{E}_{\mathcal{C}}^r$ in $\mathcal{F}_M^r$.

Lemma 1.13 *If $(\rho_i)_{i\in\mathbb{R}}$ is a minimizing sequence of $\mathcal{E}_{\mathcal{C}}^r$ in $\mathcal{F}_M^r$, then $(\rho_i^*)_{i\in\mathbb{N}}$ is a minimizing sequence as well.*

Proof. Since the rearrangement preserves norms, $\|\rho_i\|_1 = \|\rho_i^*\|_1$, which implies that

$$\int \rho_i^*(x)dx = M, \tag{1.47}$$

and as Ψ is nonnegative and convex such that $\Psi(0) = 0$, by the Nonnexpansivity of Rearrangment (see Appendix A.29), we have that

$$\int \Psi(\rho_i^*(x))dx \leq \int \Psi(\rho_i(x))dx < \infty. \tag{1.48}$$

By (1.47) and (1.48) we have that $\rho_i^* \in \mathcal{F}_M^r$, for all $i \in \mathbb{N}$, and thus

$$I_M \leq \mathcal{E}_{\mathcal{C}}^r(\rho_i^*) = \int \Psi(\rho_i(x))dx + E_{pot}(\rho_i^*).$$

By *Riesz Rearrangement Inequality* (see Appendix A.30), we have that

$$E_{pot}^1(\rho_i) = -\frac{1}{2}\iint \frac{\rho_i(x)\rho_i(y)}{|x-y|}dxdy \geq -\frac{1}{2}\iint \frac{\rho_i^*(x)\rho_i^*(y)}{|x-y|}dxdy = E_{pot}^1(\rho_i^*),$$

moreover, as ρ_{ext} is strictly symmetric decreasing, by the simplest rearrangement inequality (see Appendix A.28) and as $\|\rho_{ext}\|_2 = \|\rho_\varepsilon^*\|_2$, we have that $\rho_{ext} = \rho_{ext}^*$ and therefore

$$-E_{pot}^\varepsilon(\rho_i) = \iint \frac{\rho_{ext}(x)\rho_i(y)}{|x-y|}dxdy = \int -U_{ext}(x)\rho_i^*(x)dx = -E_{pot}^\varepsilon(\rho_i^*),$$

and hence

$$E_{pot}^\varepsilon(\rho_i^*) + E_{pot}^1(\rho_i^*) \leq E_{pot}^\varepsilon(\rho_i) + E_{pot}^1(\rho_i).$$

Thus, when $i \to \infty$ we have that

$$I_M \leq \mathcal{E}_{\mathcal{C}}^r(\rho_i^*) \leq \mathcal{E}_{\mathcal{C}}^r(\rho_i) \to I_M$$

which implies that

$$\lim_{i\to\infty} \mathcal{E}_{\mathcal{C}}(\rho_i^*) = I_M$$

as we wanted. □

Remark Based on Corollary 1.12 and Lemma 1.13, we can assume that the minimizing sequence also is spherically symmetric nonincreasing, thus we will assume it without loss of generality. As $(\rho_i)_{i\in\mathbb{N}}$ is weakly-sequentially compact, passing through subsequences we can assume there is some ρ_0 such that $\rho_i \rightharpoonup \rho_0$ on $L^{1+1/n}(\mathbb{R}^2)$ and $L^{4/3}(\mathbb{R}^2)$.

Although the weak convergence usually could not give us enough information about the weak limit ρ_0, in this case we have the following result, which tell us that the weak convergence implies strong convergence of the potential energy.

Lemma 1.14 *Let $(\rho_i)_{i\in\mathbb{N}}$ be the rearrangement of a minimizing sequence. Then*

$$E_{pot}(\rho_i) \to E_{pot}(\rho_0), \tag{1.49}$$

where ρ_0 is the weak limit of the minimizing sequence on $L^{1+1/n}(\mathbb{R}^2)$.

PROOF. Recall that

$$E_{pot}(\rho_i) = E^1_{pot}(\rho_i) + E^{\varepsilon}_{pot}(\rho_i), \tag{1.50}$$

and

$$E^{\varepsilon}_{pot}(\rho_i) = \int U_{ext}(x)\rho_i(x)dx. \tag{1.51}$$

Since $U_{ext} = U_{\rho_{ext}}$ with $\rho_{ext} \in L^{4/3}(\mathbb{R}^2)$, by Lemma 1.10 we have that $U_{ext} \in L^4(\mathbb{R}^2) = (L^{4/3}(\mathbb{R}^2))^*$ and by weak convergence of the minimizing sequence in $L^{4/3}(\mathbb{R}^2)$ (see 1.12) we have that $E^{\varepsilon}_{pot}(\rho_i) \to E^{\varepsilon}_{pot}(\rho_0)$. It is enough to show that

$$E^1_{pot}(\rho_i) \to E^1_{pot}(\rho_0). \tag{1.52}$$

Define $\sigma_i := \rho_i - \rho_0 \rightharpoonup 0$ in $L^{1+1/n}(\mathbb{R}^2)$ and note that if $\mathcal{D}$ is the Coulomb energy defined in 1.34, then

$$\begin{aligned}
E^1_{pot}(\sigma_i) &= \mathcal{D}(\rho_i - \rho_0, \rho_i - \rho_0)\\
&= \mathcal{D}(\rho_i, \rho_i) - 2\mathcal{D}(\rho_i, \rho_0) + \mathcal{D}(\rho_0, \rho_0)\\
&= \mathcal{D}(\rho_i, \rho_i) - \mathcal{D}(\rho_0, \rho_0) - 2\mathcal{D}(\rho_i, \rho_0) + 2\mathcal{D}(\rho_0, \rho_0)\\
&= E^1_{pot}(\rho_i) - E^1_{pot}(\rho_0) - 2\mathcal{D}(\sigma_i, \rho_0),
\end{aligned}$$

then, it is enough to prove that $E^1_{pot}(\sigma_i) \to 0$, because by weak convergence we have that $\mathcal{D}(\sigma_i, \rho_0) \to 0$. Let $R_1 > 0$ and separate the integral in $E_{pot}(\sigma_i)$ in two parts, one of them inside, and the other outside of the strip $|x-y| < R_1$. Thus

$$\iint \frac{\sigma_i(x)\sigma_i(y)}{|x-y|}dxdy = \iint_{|x-y|<R_1} \frac{\sigma_i(x)\sigma_i(y)}{|x-y|}dxdy + \iint_{|x-y|\geq R_1} \frac{\sigma_i(x)\sigma_i(y)}{|x-y|}dxdy. \tag{1.53}$$

We denote by I_1, I_2 these two integrals, and we will try to find small bounds for them. First, as $\sigma_i \in L^{1+1/n}(\mathbb{R}^2)$ and $1/|\cdot| \in L^{(n+1)/2}(B(0,R_1))$, by Hölder's inequality (see Appendix A.44) and Young's inequality (see Appendix A.46), and the fact that $(\sigma_i)_{i\in\mathbb{N}}$ is bounded in $L^{1+1/n}(\mathbb{R}^2)$, we have that

$$\begin{aligned}
I_1 &= \iint \sigma_i(x)\sigma_i(y)\frac{\chi_{B(0,R_1)}(x-y)}{|x-y|}dxdy\\
&= \int \sigma_i(x)\cdot\left(\sigma_i * \frac{\chi_{B(0,R_1)}}{|\cdot|}\right)(x)dx\\
&\leq \|\sigma_i\|_{1+1/n}\left\|\sigma_i * \frac{\chi_{B(0,R_1)}}{|\cdot|}\right\|_{n+1}\\
&\leq \|\sigma_i\|^2_{1+1/n}\left\|\frac{\chi_{B(0,R_1)}}{|\cdot|}\right\|_{(n+1)/2}\\
&\lesssim \left\|\frac{\chi_{B(0,R_1)}}{|\cdot|}\right\|_{(n+1)/2}.
\end{aligned}$$

Therefore, we note that

$$\|\frac{\chi_{B(0,R_1)}}{|\cdot|}\|_{(n+1)/2} = \left(\int_0^{2\pi}\int_0^{R_1} \frac{1}{r^{(n+1)/2}} r dr d\theta\right)^{2/(n+1)} = CR_1^{(3-n)/2},$$

for a suitable constant $C > 0$. Here the last term goes to 0 when $R_1 \to 0$ because $0 < n < 2$, and thus for R_1 small enough, $I_1 < \varepsilon$, for an arbitrary $\varepsilon > 0$. For the second integral, we define

$$U_{R_2} = \{(x, y) \in \mathbb{R}^4 : |x| \geq R_2 \vee |y| \geq R_2\}.$$

Hence, we have that

$$I_2 = \iint_{|x-y|\geq R_1 \cap U_{R_2}} \frac{\sigma_i(x)\sigma_i(y)}{|x-y|} dxdy + \iint_{|x-y|\geq R_1 \cap U_{R_2}^c} \frac{\sigma_i(x)\sigma_i(y)}{|x-y|} dxdy. \tag{1.54}$$

If we call $I_{2,1}$, $I_{2,2}$ these two integrals, by Hardy-Littlewood-Sobolev inequality (see Appendix A.47) we have

$$\begin{aligned}
|I_{2,1}| &\leq \iint \frac{|\sigma_i(x)\sigma_i(y)|}{|x-y|}(\chi_{B(0,R_2)^c}(x) + \chi_{B(0,R_2)^c}(y))dxdy \\
&= \iint \frac{|\sigma_i(x)||\sigma_i(y)|}{|x-y|}\chi_{B(0,R_2)^c}(x)dxdy + \iint \frac{|\sigma_i(x)||\sigma_i(y)|}{|x-y|}\chi_{B(0,R_2)^c}(y)dxdy \\
&\lesssim \|\sigma_i\chi_{B(0,R_2)^c}\|_{4/3}\|\sigma_i\|_{4/3}.
\end{aligned}$$

As in the proof of Lemma 1.11, we have that $\|\sigma_i\|_{4/3} \lesssim \|\sigma_i\|_{1+1/n}^{\frac{n+1}{4}}$ and as σ_i converges weakly to 0 in $L^{1+1/n}(\mathbb{R}^2)$ and therefore it is bounded in that space, then this inequality implies that σ_i is also bounded in $L^{4/3}(\mathbb{R}^2)$. Hence, by Minkowski's inequality we have

$$|I_{2,1}| \lesssim \|\rho_i\chi_{B(0,R_2)^c}\|_{4/3} + \|\rho_0\chi_{B(0,R_2)^c}\|_{4/3}.$$

As ρ_i is symmetric nonincreasing, then pointwise on $x \in \mathbb{R}^2$, the function ρ_i is dominated by the average over a ball centered in the origin and radius $|x|$, i.e.

$$\rho_i(x) \leq \frac{1}{|B(0,1)| \cdot |x|^2}\int_{B(0,|x|)} \rho_i(y)dy \lesssim \frac{1}{|x|^2}.$$

Therefore,

$$\begin{aligned}
\|\rho_i\chi_{B(0,R_2)^c}\|_{4/3} &\lesssim \left(\int_{B(0,R_2)^c} \frac{1}{|x|^{8/3}}dx\right)^{3/4} \\
&\lesssim \left(\int_{R_2}^{\infty} r^{1-8/3}dr\right)^{3/4} \\
&\lesssim R_2^{-\frac{1}{2}}.
\end{aligned}$$

On the other hand, in the same way as before, we have that

$$\|\rho_0\chi_{B(0,R_2)^c}\|_{4/3} \lesssim \|\rho_0\chi_{B(0,R_2)^c}\|_{1+1/n}^{\frac{n+1}{4}},$$

and as $\rho_i \rightharpoonup \rho_0$ in $L^{1+1/n}(\mathbb{R}^2)$, then by weakly lower semicontinuity of norm (see Appendix A.8), we have that

$$\|\rho_0 \chi_{B(0,R_2)^c}\|_{1+1/n} \le \liminf_{i\to\infty} \|\rho_i \chi_{B(0,R_2)^c}\|_{1+1/n},$$

in a similar way as before, we can calculate

$$\begin{aligned}\|\rho_i \chi_{B(0,R_2)^c}\|_{1+1/n} &\lesssim \left(\int_{B(0,R_2)^c} \frac{1}{|x|^{2+2/n}} dx\right)^{\frac{n}{n+1}} \\ &\lesssim \left(\int_{R_2}^{\infty} r^{-1-2/n} dr\right)^{\frac{n}{n+1}} \\ &\lesssim R_2^{-\frac{2}{n+1}},\end{aligned}$$

and therefore

$$\|\rho_0 \chi_{B(0,R_2)^c}\|_{1+1/n} \le \liminf_{i\to\infty} \|\rho_i \chi_{B(0,R_2)^c}\|_{1+1/n} \lesssim R_2^{-\frac{2}{n+1}},$$

and thus

$$\|\rho_0 \chi_{B(0,R_2)^c}\|_{4/3} \lesssim \|\rho_0 \chi_{B(0,R_2)^c}\|_{1+1/n}^{\frac{n+1}{4}} \lesssim R_2^{-\frac{1}{2}}.$$

Hence, we have that

$$|I_{2,1}| \lesssim R_2^{-\frac{1}{2}} \to 0,$$

when $R_2 \to \infty$. Finally, we have that

$$\begin{aligned}I_{2,2} &= \iint_{|x-y|\ge R_1} \frac{\sigma_i(x)\sigma_i(y)}{|x-y|} \chi_{B(0,R_2)}(x)\chi_{B(0,R_2)}(y) dx dy \\ &= \int \sigma_i(x) h_i(x) dx,\end{aligned}$$

where we defined

$$h_i(x) := \chi_{B(0,R_2)}(x) \int_{|x-y|\ge R_1} \frac{\sigma_i(y)}{|x-y|} \chi_{B(0,R_2)}(y) dy.$$

Now, for each $x \in \mathbb{R}^2$ we denote by φ_x to the function defined as

$$\varphi_x(y) := \frac{\chi_{B(0,R_2)}(y)}{|x-y|} \chi_{\mathbb{R}^2\setminus B(0,R_1)}(x-y),$$

so that

$$h_i(x) = \chi_{B(0,R_2)}(x) \int \sigma_i(y)\varphi_x(y) dy.$$

We will prove that $\varphi_x \in L^{n+1}(\mathbb{R}^2)$. Indeed, we have that

$$\|\varphi_x\|_{n+1}^{n+1} = \int \frac{\chi_{B(0,R_2)}(x)}{|x-y|^{n+1}} \chi_{\mathbb{R}^2\setminus B(0,R_1)}(x-y) dy \le \frac{1}{R_1^{n+1}} |B(0,R_2)| < \infty.$$

Thus, as $\sigma_i \rightharpoonup 0$ in $L^{1+1/n}(\mathbb{R}^2)$, we have that $h_i(x) \to 0$, for all $x \in \mathbb{R}^2$, and as $|h_i(x)| \lesssim \chi_{B(0,R_2)}(x) \in L^{n+1}(\mathbb{R}^2)$, then by the *Dominated Convergence Theorem*, we have that $h_i \to 0$ en $L^{n+1}(\mathbb{R}^2)$. Then, by Hölder's inequality we have that

$$I_{2,2} \le \|\sigma_i\|_{1+1/n} \|h_i\|_{n+1} \lesssim \|h_i\|_{n+1} < \varepsilon,$$

for all i big enough. Therefore, when $R_2 \to \infty$, for R_1 and i big enough

$$\iint \frac{\sigma_i(x)\sigma_i(y)}{|x-y|} dxdy = I_1 + I_{2,1} + I_{2,2} < 2\varepsilon,$$

which implies that $E^1_{pot}(\sigma_i) \to 0$. Thus

$$E^1_{pot}(\rho_i) - E^1_{pot}(\rho_0) = E^1_{pot}(\sigma_i) + 2\mathcal{D}(\sigma_i, \rho_0) \to 0,$$

as we wanted to prove.

□

1.3.2 Existence of a minimizer for the reduced variational problem

The convergence of potential energies proved above, give us a strong evidence that the weak limit ρ_0 in $L^{1+1/n}(\mathbb{R}^2)$ of rearranged minimizing sequence is our best candidate to a minimizer of the functional $\mathcal{E}^r_{\mathcal{C}}$. In fact, we will prove this assertion, and for this, we must prove that $\mathcal{E}^r_{\mathcal{C}}(\rho_0)$ is at most the infimum over the feasible space, and ρ_0 is an element of this set. We will proceed with the proof of Theorem 1.9, which gives us the existence for a solution of variational reduced problem, and in consequence, a solution for the original variational problem, as we wanted.

Proof of Theorem 1.9. Let $(\rho_i)_{i\in\mathbb{N}} \subseteq \mathcal{F}^r_M$ be the rearranged minimizing sequence for $\mathcal{E}^r_{\mathcal{C}}$. By Corollary 1.12, there exists ρ_0 such that $\rho_i \rightharpoonup \rho_0$ in $L^{1+1/n}(\mathbb{R}^2)$. For $R > 0$ we have that $\chi_{B(0,R)} \in L^{n+1}(\mathbb{R}^2)$, therefore by weak convergence we have that

$$M = \int \rho_i(x)dx \geq \int_{B(0,R)} \rho_i(x) = \int \chi_{B(0,R)}\rho_i(x)dx \longrightarrow \int_{B(0,R)} \rho_0(x)dx$$

thus, when $R \to \infty$, we have that

$$\int \rho_0(x)dx \leq M,$$

and interpolating in the same way as in (1.37), and along with the above, we can prove that $\rho_0 \in L^1_+(\mathbb{R}^2) \cap L^{4/3}(\mathbb{R}^2)$, and by Lemma 1.14 we have that $E_{pot}(\rho_i) \to E_{pot}(\rho_0)$. By Mazur's Lemma (see Appendix A.9), there is a family of nonnegative and finite sequences $(B_n)_{n\in\mathbb{N}}$, $B_n := \{\alpha^n_k : k = n, ..., N_n\}$ such that

$$\sum_{k=n}^{N_n} \alpha^n_k = 1,$$

therefore, the sequence $(\hat{\rho}_n)_{n\in\mathbb{N}}$ defined as

$$\hat{\rho}_n = \sum_{k=n}^{N_n} \alpha^n_k \rho_k,$$

converges strongly to ρ_0 in $L^{1+1/n}(\mathbb{R}^2)$, and thus, there is a subsequence $(\hat{\rho}_{n_j})_{j\in\mathbb{N}}$ which converges pointwise almost everywhere to ρ_0. As the map $\rho \mapsto \Psi(\rho)$ is strictly convex, we have that the map $\rho \mapsto \int \Psi(\rho)dx$ is convex, and therefore

$$\int \Psi(\hat{\rho}_{n_j})dx = \sum_{k=n_j}^{N_{n_j}} \alpha_k^{n_j} \int \Psi(\rho_k)dx \leq \sup_{k\geq n_j} \int \Psi(\rho_k)dx,$$

else, by Fatou's Lemma and using the fact that $\Psi \in C^1([0,\infty))$, we have that

$$\begin{aligned}\int \Psi(\rho_0)dx &= \int \liminf_{j\to\infty} \Psi(\hat{\rho}_{n_j})dx \\ &\leq \liminf_{j\to\infty} \int \Psi(\hat{\rho}_{n_j})dx \\ &\leq \lim_{j\to\infty} \int \Psi(\hat{\rho}_{n_j})dx \\ &\leq \limsup_{k\to\infty} \int \Psi(\rho_k)dx,\end{aligned}$$

and therefore, as $E_{pot}(\rho_0) = \limsup_{i\to\infty} E_{pot}(\rho_i)$, and as $(\rho_i)_{i\in\mathbb{N}}$ is a minimizing sequence of $\mathcal{E}_{\mathcal{C}}^r$ in $\mathcal{F}_M^r$, we have that

$$\mathcal{E}_{\mathcal{C}}^r(\rho_0) \leq \limsup_{i\to\infty} \int \Psi(\rho_i)dx + \limsup_{i\to\infty} E_{pot}(\rho_i) = I_M.$$

Hence $\mathcal{E}_{\mathcal{C}}^r(\rho_0) \leq I_M$ and thus we only must prove that ρ_0 is an element of the feasible set, i.e. $\rho_0 \in \mathcal{F}_M^r$. For this, recall that

$$\mathcal{E}_{\mathcal{C}}^r(\rho_0) \geq \int \Psi(\rho_0(x))dx \left(1 - C\left(\int \Psi(\rho_0(x))dx\right)^{\frac{n}{2}-1}\right) - C. \tag{1.55}$$

If $\int \Psi(\rho_0(x))dx = \infty$, this implies in the inequality above, that

$$\mathcal{E}_{\mathcal{C}}^r(\rho_0) = \infty,$$

which is a contradiction. Thus, it is enough to prove that

$$\int \rho_0(x)dx = M.$$

We will proceed by contradiction. Assume that

$$0 < M'' = \int \rho_0(x)dx < M,$$

and consider $\bar{\rho}_R := \chi_{B(0,R)}\rho_0$, where $R > 0$ is such that

$$M' := \int \bar{\rho}_R(x)dx < M'' < M, \tag{1.56}$$

and define

$$I_M^0 := \inf_{\rho\in\mathcal{F}_M^r} \mathcal{E}_{\mathcal{C}}^{r,0}(\rho) = \inf_{\rho\in\mathcal{F}_M^r} \left(\int \Psi(\rho(x))dx + E_{pot}^1(\rho)\right).$$

In [8, Lema 3.4. (a)] it was proved that $I_M^0 < 0$ for every $M > 0$, and the assertion [8, Lema 3.4. (b)] of this lemma implies that $M \mapsto I_M^0$ is nonincreasing in M. Since $\bar{\rho}_R(x) \to \rho_0(x)$ pointwise when $R \to \infty$ and is a function dominated by $\rho_0 \in L^{1+1/n}(\mathbb{R}^2)$, by the *Dominated Convergence Theorem*, $\bar{\rho}_R \to \rho_0$ in $L^{1+1/n}(\mathbb{R}^2)$ and in particular $\bar{\rho}_R \rightharpoonup \rho_0$ in this space. As Ψ is strictly convex and Ψ' is a bijection over $[0,\infty)$, then is strictly increasing and thus

$$\int \Psi(\bar{\rho}_R(x))dx \leq \int \Psi(\rho_0(x))dx.$$

If we take the rearrangement of $\bar{\rho}_R$, as Ψ is convex and $\Psi(0) = 0$, by nonexpansivity of rearrangements (see Appendix A.28) together with the inequality above we have that

$$\int \Psi(\bar{\rho}_R^*(x))dx \leq \int \Psi(\rho_0(x))dx.$$

Hence, by Lemma 1.14, we have that $E_{pot}(\bar{\rho}_R^*) \to E_{pot}(\rho_0)$. Let $\varepsilon > 0$ such that

$$I_{M-M''}^0 < -\varepsilon \tag{1.57}$$

and take R big enough such that $E_{pot}(\bar{\rho}_R^*) - E_{pot}(\rho_0) \leq \frac{\varepsilon}{2}$. Then

$$\mathcal{E}_{\mathcal{C}}^r(\bar{\rho}_R^*) = \int \Psi(\bar{\rho}_R^*(x))dx + E_{pot}(\bar{\rho}_R^*) \tag{1.58}$$

$$\leq \int \Psi(\rho_0(x))dx + E_{pot}(\rho_0) + \frac{\varepsilon}{2} \tag{1.59}$$

$$= \mathcal{E}_{\mathcal{C}}^r(\rho_0) + \frac{\varepsilon}{2}. \tag{1.60}$$

By (1.56), we have that $\delta := M - M' \geq M - M'' > 0$ and we can take $\varphi \in \mathcal{F}_\delta^r$ such that $supp(\varphi) \subseteq B(0, R')$ with $R' > 0$ and such that

$$\mathcal{E}_{\mathcal{C}}^{r,0}(\varphi) < \frac{I_\delta^0}{2} \leq \frac{I_{M-M''}^0}{2}. \tag{1.61}$$

In this way, if $a \in \mathbb{R}^2$ is such that $|a| = R + R'$, then

$$\begin{aligned}\int \Psi(\bar{\rho}_R^*(x) + \varphi(x-a))dx &\leq \int \Psi(\bar{\rho}_R^*(x))dx + \int \Psi(\varphi(x-a))dx\\ &= \int \Psi(\bar{\rho}_R^*(x))dx + \int \Psi(\varphi(x))dx,\end{aligned}$$

and in the other hand

$$-\mathcal{D}(\bar{\rho}_R^* + \varphi(\cdot - a), \bar{\rho}_R^* + \varphi(\cdot - a)) \leq -\mathcal{D}(\bar{\rho}_R^*, \bar{\rho}_R^*) - \mathcal{D}(\varphi, \varphi),$$

and

$$-\mathcal{D}(\rho_{ext}, \bar{\rho}_R^* + \varphi(\cdot - a)) \leq -\mathcal{D}(\rho_{ext}, \bar{\rho}_R^*).$$

Thus, by (1.57), (1.60) and (1.61) we have that

$$\begin{aligned}\mathcal{E}_{\mathcal{C}}^r(\bar{\rho}_R^* + \varphi(\cdot - a)) &\leq \mathcal{E}_{\mathcal{C}}^r(\bar{\rho}_R^*) + \mathcal{E}_{\mathcal{C}}^{r,0}(\varphi) \\ &< \mathcal{E}_{\mathcal{C}}^r(\rho_0) + \frac{\varepsilon}{2} + \frac{I_{M-M''}^0}{2} \\ &< \mathcal{E}_{\mathcal{C}}^r(\rho_0) \\ &= I_M,\end{aligned}$$

and as the rearrangement preserves the $L^1(\mathbb{R}^2)$ norm, then $\bar{\rho}_R^* \in \mathcal{F}_{M'}^r$ and $\varphi \in \mathcal{F}_\delta^r$, then $\bar{\rho}_R^* + \varphi(\cdot - a) \in \mathcal{F}_M^r$, which is a contradiction. Hence

$$\int \rho_0(x)dx = M,$$

and therefore $\rho_0 \in \mathcal{F}_M^r$ which implies that is a minimizer of $\mathcal{E}_{\mathcal{C}}^r$ over the feasible set, i.e. a solution of the reduced variational problem. Hence, by Theorem 1.7, we have that the function f_0 which satisfies the following identity

$$f_0 = (\Phi')^{-1}(E_0 - E)\chi_{E_0 > E},$$

is a minimizer of the Casimir-Energy functional. □

Remark We have the following observations.

a) Note that the rearrangement argument also works for the original case of Flat Vlasov-Poisson system, putting $\rho_{ext} = 0 \in L^{4/3}(\mathbb{R}^2)$.

b) As in the proof of Lemma 1.13, we can prove that if ρ_0 is the solution of the reduced variational problem, then the rearrangement of ρ_0 is also a solution. Hence, we can assume that ρ_0 is spherically symmetric and nonincreasing.

c) In the proof of $\rho_0 \in \mathcal{F}_M^r$, we assumed that ρ_0 is almost everywhere zero. This assertion is true. Indeed, as $I_M^0 < 0$, then there exist $\rho \in \mathcal{F}_M^r$ such that $\mathcal{E}_{\mathcal{C}}^{r,0}(\rho) < 0$, then

$$\mathcal{E}_{\mathcal{C}}^r(\rho_0) \leq \mathcal{E}_{\mathcal{C}}^{r,0}(\rho) + E_{pot}^{\varepsilon}(\rho) < 0,$$

thus ρ_0 cannot be almost everywhere zero.

1.4 Regularity

In the same way as in the Vlasov-Poisson system, the idea is to construct solutions which are functions of the energy E. The problem for the Flat Vlasov-Poisson system, is that even E is not directly a steady state of the Vlasov equation. That is, for $f = E$ to be a solution of

$$v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \tag{1.62}$$

$U = U_0 + U_{ext}$ needs to be sufficiently regular to make sense of the second term. First of all, we have the following result

Lemma 1.15 *Assume also that $\rho_{ext} \in L^q(\mathbb{R}^2)$, where $q = \frac{p}{p-1}$, $1 < p < 2$. Let ρ_0 the solution of the reduced variational problem, then U_0 and therefore U are continuous.*

Proof. Since $\rho_0, \rho_{ext} \in L^{4/3}(\mathbb{R}^2)$, $U = U_0 + U_{ext} \in L^4(\mathbb{R}^2)$, and by the equation $\Psi'(\rho_0) = E_0 - U$ a.e. on $\mathrm{supp}(\rho_0) = \{E_0 - U > 0\}$ we can conclude that $\Psi'(\rho_0) \in L^4(\mathbb{R}^2)$. Recall that

$$\Psi(\rho) \gtrsim \rho^{1+1/n},$$

for big $\rho \geq 0$. Thus, by the convexity of Ψ we have that

$$\Psi'(\rho) > \frac{\Psi(\rho)}{\rho} \gtrsim \rho^{1/n},$$

where the last inequality holds for all $\rho \geq \delta$, for some $\delta \geq 0$ fixed. Thus we have that

$$\begin{aligned}\int \rho_0(x)^{4/n}dx &= \int_{\{\rho_0\geq\delta\}} \rho_0(x)^{4/n}dx + \int_{\{\rho_0<\delta\}} \rho_0(x)^{4/n}dx \\ &\lesssim \int_{\{\rho_0\geq\delta\}} \Psi'(\rho_0(x))^4 dx + \int_{\{\rho_0<\delta\}} \rho_0(x)^{4/n}dx \\ &\lesssim \int \Psi'(\rho_0(x))^4 dx + 1,\end{aligned}$$

and therefore $\rho_0 \in L^{4/n}(\mathbb{R}^2)$. It is easy to prove that if $n \in (1,2)$, then $1/|\cdot|\chi_{B(0,R)} \in L^{4/n}(\mathbb{R}^2)^* = L^{\frac{4}{4-n}}(\mathbb{R}^2)$ and $1/|\cdot|\chi_{B(0,R)^c} \in L^{4/3}(\mathbb{R}^2)^* = L^4(\mathbb{R}^2)$ for every $R > 0$. Hence, by A.21 we have that

$$-U_0 = \frac{1}{|\cdot|} * \rho_0 = \frac{1}{|\cdot|}\chi_{B(0,R)} * \rho_0 + \frac{1}{|\cdot|}\chi_{B(0,R)^c} * \rho_0,$$

is continuous. Finally, as $\rho_{ext} \in L^{4/3}(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$, we have that

$$-U_{ext} = \frac{1}{|\cdot|} * \rho_{ext} = \frac{1}{|\cdot|}\chi_{B(0,R)} * \rho_{ext} + \frac{1}{|\cdot|}\chi_{B(0,R)^c} * \rho_{ext},$$

and the argument is the same as above, just note that $1/|\cdot|\chi_{B(0,R)} \in L^p(\mathbb{R}^2)$. □

For the solution of variational problem

$$f_0 = (\Phi')^{-1}(E_0 - E)\chi_{E_0>E},$$

we need more regularity for Φ. In particular, we need to differentiate the inverse of the first derivative. For this, it is enough to have $\Phi \in C^2([0,\infty))$ and $\Phi'' > 0$, and thus, if we could prove more regularity for the potential U, then f_0 will be a steady state solution of Flat Vlasov-Poisson with the external potential. Another way to investigate the regularity of the potential comes from to the study of its *Fourier Transform* (see Appendix **??**). Note that since $d = 2$, then

$$\mathcal{F}(U) = \mathcal{F}\left(\rho_{0,\varepsilon} * \frac{1}{|\cdot|}\right) \simeq \mathcal{F}(\rho_{0,\varepsilon}) \cdot \frac{1}{|\cdot|}. \tag{1.63}$$

Since $\rho_0 \in L^{4/n}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, interpolating we can conclude that $\rho_0 \in L^2(\mathbb{R}^2)$ and therefore by *Plancherel's Theorem* (see Appendix A.39), we have that $\mathcal{F}(\rho_0) \in L^2(\mathbb{R}^2)$. If $\rho_{ext} \in L^2(\mathbb{R}^2)$, as before we have $\mathcal{F}(\rho_{ext})$ and hence $\mathcal{F}(\rho_{0,\varepsilon}) \in L^2(\mathbb{R}^2)$. Then we have that $|\cdot|\mathcal{F}(U) \in L^2(\mathbb{R}^2)$, and in particular we have that $(1+|\cdot|)\mathcal{F}(U) \in L^2_{loc}(\mathbb{R}^2)$, which implies that $U \in H^1_{loc}(\mathbb{R}^2)$. Since we are in two dimensions, this is not enough to embed this space to obtain more regularity. The main idea to overcome this problem comes from the *Riesz Transform* (see Appendix A.41), and more general, from the following result.

Theorem 1.16 *The operator T defined as*

$$Tf := -\frac{1}{|\cdot|} * f, \tag{1.64}$$

maps elements of $W^{k,p}(\mathbb{R}^2)$ into $W^{k+1,p}(\mathbb{R}^2)$.

Proof. If $f \in W^{k,p}(\mathbb{R}^2)$, then we need to prove that $D^\alpha(Tf) \in L^p(\mathbb{R}^2)$ for every multiindex α with $|\alpha| = k+1$. Note that $D^\alpha(Tf) = D^\beta(\partial_j(Tf))$ where β is a multiindex with $|\beta| = k$ and $D^\beta f \in L^p(\mathbb{R}^2)$. Then

$$\mathcal{F}(R_j(D^\beta f))(x) = -i\frac{x_j}{|x|}\mathcal{F}(D^\beta f)(x).$$

Recall that

$$\mathcal{F}(D^\beta f)(x) = i^{|\beta|} x_k^{\beta_k} x_j^{\beta_j} \mathcal{F}(f)(x),$$

where β_j and β_k are the components of the multiindex β. Thus

$$\begin{aligned}
\mathcal{F}(R_j(D^\beta f))(x) &= -i^{|\beta|} x_k^{\beta_k} x_j^{\beta_j}\left(ix_j\mathcal{F}(f)(x)\frac{1}{|x|}\right) \\
&\simeq i^{|\beta|} x_k^{\beta_k} x_j^{\beta_j}\left(ix_j\mathcal{F}(Tf)(x)\right) \\
&\simeq i^{|\beta|} x_k^{\beta_k} x_j^{\beta_j}\mathcal{F}(\partial_j(Tf))(x) \\
&\simeq \mathcal{F}(D^\beta(\partial_j Tf))(x).
\end{aligned}$$

Therefore, by Theorem A.43, we have

$$\|D^\alpha(Tf)\|_p = \|D^\beta(\partial_j Tf)\|_p \lesssim \|D^\beta f\|_p,$$

which implies the desires result. □

For our main problem, we need the following lemma.

Lemma 1.17 *If $\Phi \in C^2([0,\infty))$ and $\Phi'' > 0$ in $[0,\infty)$, then $(\Psi')^{-1} \in C^1([0,\infty))$.*

Proof. Note that since $\Phi(r) = +\infty$ on $(-\infty, 0)$, for $\lambda > 0$

$$\Psi^*(\lambda) = \int_{|v|<\sqrt{2\lambda}} \Phi^*\left(\lambda - \frac{|v|^2}{2}\right) dv.$$

Recall that

$$\frac{d}{d\lambda}[(\Psi')^{-1}](\lambda) = \frac{d^2}{d\lambda^2}[\Psi^*](\lambda) = \frac{d^2}{d\lambda^2}\int_{|v|<\sqrt{2\lambda}} \Phi^*\left(\lambda - \frac{|v|^2}{2}\right) dv.$$

Since

$$\begin{aligned}\frac{d}{d\lambda}\int_{|v|<\sqrt{2\lambda}} \Phi^*\left(\lambda - \frac{|v|^2}{2}\right) dv &= 2\pi \cdot \frac{d}{d\lambda}\int_0^{\sqrt{2\lambda}} \Phi^*\left(\lambda - \frac{r^2}{2}\right) rdr \\ &\simeq \frac{1}{\sqrt{2\lambda}}\Phi^*(0) + \int_0^{\sqrt{2\lambda}} (\Phi')^{-1}\left(\lambda - \frac{r^2}{2}\right) rdr \\ &\simeq \int_0^{\sqrt{2\lambda}} (\Phi')^{-1}\left(\lambda - \frac{r^2}{2}\right) rdr,\end{aligned}$$

we have

$$\begin{aligned}\frac{d}{d\lambda}[(\Psi')^{-1}](\lambda) &\simeq \frac{1}{\sqrt{2\lambda}}(\Phi')^{-1}(0) + \int_0^{\sqrt{2\lambda}} ((\Phi')^{-1})'\left(\lambda - \frac{r^2}{2}\right) rdr \\ &\simeq \int_0^{\sqrt{2\lambda}} ((\Phi')^{-1})'\left(\lambda - \frac{r^2}{2}\right) rdr \\ &\simeq \int_0^{\sqrt{2\lambda}} \frac{rdr}{\Phi''\left((\Phi')^{-1}\left(\lambda - \frac{r^2}{2}\right)\right)},\end{aligned}$$

which implies the desired result. □

Next we prove the following theorem, which allows us to say that f_0 (from (1.31)) inside on the support of ρ_0 is locally an steady state solution of Flat Vlasov-Poisson with a central mass density.

Theorem 1.18 *Suppose that $\Phi \in C^2([0,\infty))$, $\Phi'' > 0$, and $\rho_{ext} \in L^{4/n}(\mathbb{R}^2)$. Then $U \in C^{1,1-\frac{n}{2}}(\Omega)$, for every bounded set $\Omega \subset \mathbb{R}^2$ with C^1 boundary, and therefore*

$$f_0 = (\Phi')^{-1}(E_0 - E)\chi_{E_0 > E},$$

is an steady state solution of Flat Vlasov-Poisson system with a central mass density in Ω.

Proof. By Theorem 1.16, since $\rho_{0,ext} = \rho_0 + \rho_{ext} \in L^{4/n}(\mathbb{R}^2) = W^{0,4/n}(\mathbb{R}^2)$, we have

$$T\rho_{0,ext} = U \in W^{1,4/n}(\mathbb{R}^2).$$

Denote by ∇ the weak gradient. Thus from (1.30) we have

$$\nabla\rho_0 = -((\Psi')^{-1})'((E_0 - U)_+)\nabla U_+.$$

By Lemma 1.15 and Lemma 1.17, $((\Psi')^{-1})'((E_0 - U)_+)$ is bounded a.e., and therefore we have $\rho_0 \in W^{1,4/n}(\mathbb{R}^2)$. Again by Theorem 1.16 we can conclude that $U \in W^{2,4/n}(\mathbb{R}^2)$. Hence, by Sobolev Inequalities (see Appendix A.37), we can conclude that

$$U \in C^{1,1-\frac{n}{2}}(\bar{\Omega}).$$

for every bounded set $\Omega \subset \mathbb{R}^2$ with C^1 boundary. Provided of the regularity of U in every bounded set Ω of S with C^1 boundary, and the regularity of Φ, f_0 is an steady state solution of the Vlasov equation in Ω, with is the required result. □

Remark We observe that since ρ_0 is continuous, spherically symmetric and nonincreasing, the support of ρ_0 is a closed ball $\overline{B}(0, R_0)$ for some $R_0 > 0$. Then we can take $\Omega = B(0, R) \subset \mathbb{R}^2$ with $R > R_0$, where Theorem 1.18 holds. Outside of $B(0, R) \subset \mathbb{R}^2 \setminus \text{supp}(\rho_0)$ the function $1/|\cdot|$ is C^1, and therefore U preserves this regularity. Hence, we can replace open and bounded sets in the hypotesis of Theorem 1.18 with $\mathbb{R}^2$.

1.5 Stability of the minimizer

With the existence of a minimizer for the variational problem proved, in the same way as [8], we will prove that a similar result of stability, now for the case of flat Vlasov-Poisson with central mass density. Before that, we will prove some useful results. We expanded over the minimizer f_0 given by 1.9, and we have that

$$\mathcal{E}_{\mathcal{C}}(f) - \mathcal{E}_{\mathcal{C}}(f_0) = d(f, f_0) - E^1_{pot}(\rho_f - \rho_{f_0}), \tag{1.65}$$

where

$$d(f, f_0) = \iint \left[\Phi(f) - \Phi(f_0) + E(f - f_0)\right] dvdx, \tag{1.66}$$

and in this case

$$E(x, v) = \frac{1}{2}|v|^2 + U(x) = \frac{1}{2}|v|^2 + U_0(x) + U_{ext}(x)$$

is the energy defined in 1.7. Thus, as Φ is strictly convex, we have that

$$\begin{aligned} d(f, f_0) &\geq \iint \left[\Phi'(f_0)(f - f_0) + E(f - f_0)\right] dvdx \\ &= \iint \left[\Phi'(f_0) + (E - E_0)\right](f - f_0) dvdx \\ &\geq 0 \end{aligned}$$

with $d(f, f_0) = 0$ if and only if $f = f_0$. We have the following lemma.

Lemma 1.19 *Let $(f_i)_{i\in\mathbb{N}}$ a minimizing sequence for $\mathcal{E}_{\mathcal{C}}$ en $\mathcal{F}_M$. Then ρ_{f_i} is a minimizing sequence for $\mathcal{E}^r_{\mathcal{C}}$ in $\mathcal{F}^r_M$.*

Proof. It is clear from 1.7 that

$$\mathcal{E}_{\mathcal{C}}(f_i) \geq \mathcal{E}^r_{\mathcal{C}}(\rho_{f_i}) \geq \inf_{\rho\in\mathcal{F}^r_M} \mathcal{E}^r_{\mathcal{C}}(\rho).$$

If ρ_0 is the minimizer for $\mathcal{E}^r_{\mathcal{C}}$ obtained from reduced problem and f_0 is the minimizer for $\mathcal{E}_{\mathcal{C}}$ induced by ρ_0 from Theorem 1.9, as $\rho_0 = \rho_{f_0}$ we have that $\mathcal{E}^r_{\mathcal{C}}(\rho_{f_0}) = \mathcal{E}^r_{\mathcal{C}}(\rho_0)$. By Theorem 1.7, we have that

$$\mathcal{E}_{\mathcal{C}}(f_i) \to \inf_{f\in\mathcal{F}_M} \mathcal{E}_{\mathcal{C}}(f) = \mathcal{E}_{\mathcal{C}}(f_0) = \mathcal{E}^r_{\mathcal{C}}(\rho_0) = \inf_{\rho\in\mathcal{F}^r_M} \mathcal{E}^r_{\mathcal{C}}(\rho),$$

as we wanted to prove. □

Lemma 1.20 *Let $(f_i)_{i\in\mathbb{N}}$ a minimizing sequence for $\mathcal{E}_\mathcal{C}$ in $\mathcal{F}_M$. Then it is bounded in $L^{1+1/k}(\mathbb{R}^4)$. In particular the sequence is weakly-sequentially compact in that space.*

Proof. In the same way as (1.40), using the fact that $f^{1+1/k} \lesssim \Phi(f)$ for every big enough f, we can prove that

$$\iint f_i(x,v)^{1+1/k} dvdx \lesssim \mathcal{C}(f_i) + 1.$$

where we recall that $\mathcal{C}$ is the Casimir functional defined in (21)

$$\mathcal{C}(f_i) = \iint \Phi(f_i(x,v))dvdx.$$

In the same way as in the proof of the bounds (1.43) and (1.44) in 1.12, we can prove that

$$-E_{pot}(f_i) = -E_{pot}(\rho_{f_i}) \lesssim \mathcal{C}(f_i)^{k/2} + 1,$$

and hence we have that

$$\begin{aligned}\mathcal{E}_\mathcal{C}(f_i) &= E_{kin}(f_i) + E_{pot}(f_i) + \mathcal{C}(f_i)\\ &\geq \mathcal{C}(f_i) - C\cdot\mathcal{C}(f_i)^{k/2} - C\\ &= \mathcal{C}(f_i)\left(1 - C\cdot\mathcal{C}(f_i)^{\frac{k}{2}-1}\right) - C,\end{aligned}$$

for a suitable constant $C > 0$. Therefore, if the Casimir functional $\mathcal{C}(f_i)$ is not bounded, then $\mathcal{E}_\mathcal{C}(f_i) \to \infty$, which is a contradiction, because $(f_i)_{i\in\mathbb{N}}$ is minimizing sequence for $\mathcal{E}_\mathcal{C}$. Hence $\mathcal{C}(f_i)$ is bounded, and therefore, so is $(f_i)_{i\in\mathbb{N}}$ in $L^{1+1/k}(\mathbb{R}^4)$. As $L^{1+1/k}(\mathbb{R}^4)$ is a reflexive space, by *Banach-Aloglu Theorem* (see Appendix A.11), we can find a subsequence which is weakly-sequentially compact in that space. □

We know that by Lemma 1.19 proved above, if $(f_i)_{i\in\mathbb{N}}$ is a minimizing sequence for $\mathcal{E}_\mathcal{C}$ in $\mathcal{F}_M$, then ρ_{f_i} is minimizing sequence for $\mathcal{E}^r_\mathcal{C}$ in $\mathcal{F}^r_M$, and passing through subsequence, we already saw that converges weakly to a minimizer $\tilde{\rho}_0$ for the reduced functional. We have the following result:

Lemma 1.21 *Let $(f_i)_{i\in\mathbb{N}}$ a minimizing sequence for $\mathcal{E}_\mathcal{C}$ in $\mathcal{F}_M$ and let f_0 the minimizer obtained from theorem 1.9, induced by ρ_0, and we suppose that is unique. Then passing through subsequence, we have that $\rho_{f_i} \rightharpoonup \rho_0$ in $L^{1+1/n}(\mathbb{R}^2)$, and passing through subsequence, $f_i \rightharpoonup f_0$ en $L^{1+1/k}(\mathbb{R}^4)$.*

Proof. By Lemma 1.19, we know that $(\rho_{f_i})_{i\in\mathbb{N}}$ is a minimizing sequence for $\mathcal{E}^r_\mathcal{C}$ in $\mathcal{F}^r_M$, and by Corollary 1.12 and uniqueness of f_0, we have that passing through subsequence, $\rho_{f_i} \rightharpoonup \rho_0$ in $L^{1+1/n}(\mathbb{R}^2)$, where $\rho_0 = \rho_{f_0}$. By Lemma 1.20, we have that, passing by subsequence again, $f_i \rightharpoonup \tilde{f}_0$ en $L^{1+1/k}(\mathbb{R}^4)$. We will prove that $\rho_0 = \rho_{\tilde{f}_0}$ almost everywhere. For them, let A an arbitraty Lebesgue measurable set and let $R_1, R_2 > 0$ arbitrary positive numbers. We have that

$$\int_A \chi_{B(0,R_1)}(x)\rho_{\tilde{f}_0} = \iint_A \chi_{B(0,R_1)}(x)\chi_{B(0,R_2)}(v)\tilde{f}_0 + \iint_A \chi_{B(0,R_1)}(x)\chi_{B(0,R_2)^c}(v)\tilde{f}_0.$$

It is obvious that $(x,v) \mapsto \chi_{B(0,R_1)}(x)\chi_{B(0,R_2)}(v)$ is an element of $L^{1+1/k}(\mathbb{R}^4)^* = L^{k+1}(\mathbb{R}^4)$, and therefore

$$\begin{aligned}\iint_A \chi_{B(0,R_1)}(x)\chi_{B(0,R_2)}(v)\tilde{f}_0 &= \lim_{i\to\infty}\iint_A \chi_{B(0,R_1)}(x)\chi_{B(0,R_2)}(v) f_i \\ &\le \lim_{i\to\infty}\iint_A \chi_{B(0,R_1)}(x) f_i \\ &= \lim_{i\to\infty}\int_A \chi_{B(0,R_1)}(x)\rho_{f_i} \\ &= \int_A \chi_{B(0,R_1)}(x)\rho_0,\end{aligned}$$

where the last equality comes from the fact that $x \mapsto \chi_{B(0,R_1)}(x)$ is an element of $L^{1+1/n}(\mathbb{R}^2)^* = L^{n+1}(\mathbb{R}^2)$. In the other hand, we have that

$$\begin{aligned}\iint_A \chi_{B(0,R_1)}(x)\chi_{B(0,R_2)^c}(v)\tilde{f}_0 &\le \frac{2}{R_2^2}\iint \frac{|v|^2}{2} f_0(x,v)dvdx \\ &= \frac{2}{R_2^2}E_{kin}(\tilde{f}_0).\end{aligned}$$

Therefore

$$\int_A \chi_{B(0,R_1)}(x)\rho_{\tilde{f}_0} \le \int_A \chi_{B(0,R_1)}(x)\rho_0 + \frac{2}{R_2^2}E_{kin}(\tilde{f}_0),$$

and thus if $R_1, R_2 \to +\infty$ we have found

$$\int_A \rho_{\tilde{f}_0} \le \int_A \rho_0. \tag{1.67}$$

In the other hand, by weak convergence again we have that

$$\begin{aligned}\int_A \chi_{B(0,R_1)}(x)\rho_0 &= \int_A \chi_{B(0,R_1)}(x)\rho_0 \\ &= \liminf_{i\to\infty}\int_A \chi_{B(0,R_1)}(x)\rho_i \\ &= \liminf_{i\to\infty}\left(\iint_A \chi_{B(0,R_1)}(x)\chi_{B(0,R_2)}(v) f_i + \iint_A \chi_{B(0,R_1)}(x)\chi_{B(0,R_2)^c}(v) f_i\right).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\liminf_{i\to\infty}\iint_A \chi_{B(0,R_1)}(x)\chi_{B(0,R_2)}(v) f_i &= \iint_A \chi_{B(0,R_1)}(x)\chi_{B(0,R_2)}(v)\tilde{f}_0 \\ &\le \iint_A \chi_{B(0,R_1)}(x)\tilde{f}_0 \\ &= \int_A \chi_{B(0,R_1)}(x)\rho_{\tilde{f}_0}.\end{aligned}$$

It is enough to prove that the second term goes to 0. Note that

$$\begin{aligned}\liminf_{i\to\infty}\iint_A \chi_{B(0,R_1)}(x)\chi_{B(0,R_2)^c}(v) f_i &\le \frac{2}{R_2^2}\liminf_{i\to\infty} E_{kin}(f_i) \\ &\le \frac{C}{R_2^2}.\end{aligned}$$

Thus, if $R_1, R_2 \to \infty$, we have that

$$\int_A \rho_0 \leq \int_A \rho_{\tilde{f}_0}, \tag{1.68}$$

and therefore

$$\int_A \rho_0 = \int_A \rho_{\tilde{f}_0},$$

and this is for every Lebesgue measurable set A. Hence $\rho_0 = \rho_{\tilde{f}_0}$ almost everywhere. Therefore $E_{pot}(\tilde{f}_0) = E_{pot}(\rho_{\tilde{f}_0}) = E_{pot}(\rho_0) = E_{pot}(f_0)$, and thus

$$\begin{aligned}
\mathcal{E}_{\mathcal{C}}(\tilde{f}_0) &\leq E_{kin}(\tilde{f}_0) + E_{pot}(\tilde{f}_0) + \mathcal{C}(\tilde{f}_0) \\
&\leq \liminf_{i\to\infty} E_{kin}(f_i) + E_{pot}(\rho_0) + \limsup_{i\to\infty} \mathcal{C}(f_i) \\
&\leq \limsup_{i\to\infty} E_{kin}(f_i) + \limsup_{i\to\infty} E_{pot}(\rho_{f_i}) + \limsup_{i\to\infty} \mathcal{C}(f_i) \\
&= \limsup_{i\to\infty} E_{kin}(f_i) + \limsup_{i\to\infty} E_{pot}(f_i) + \limsup_{i\to\infty} \mathcal{C}(f_i) \\
&= \inf_{f\in\mathcal{F}_M} \mathcal{E}_{\mathcal{C}}(f).
\end{aligned}$$

Since $\rho_{\tilde{f}_0} = \rho_0$ almost everywhere, we have $\tilde{f}_0$ integrates M, and thus is an element of the feasible set $\mathcal{F}_M$. Therefore is a minimizer of Casimir-Energy functional, and by the uniqueness of minimizer we have that $\tilde{f}_0 = f_0$, as we wanted to prove. □

Next, we will prove the main result from this section, which gives us a notion of stability for the minimizer found, analogous to the stability result from [8].

Theorem 1.22 *Let f_0 a minimizer for $\mathcal{E}_{\mathcal{C}}$ in $\mathcal{F}_M$ and we suppose that is unique, and let $\rho_0 := \rho_{f_0}$. Let $\varepsilon > 0$, then there is some $\delta > 0$ such that for every solution of flat Vlasov-Poisson system with central mass density $t \mapsto f(t)$, with $f(0) \in C_c^1(\mathbb{R}^4) \cap \mathcal{F}_M$, if*

$$d(f(0), f_0) - E^1_{pot}(\rho_{f(0)} - \rho_0) < \delta, \tag{1.69}$$

then

$$d(f(t), f_0) - E^1_{pot}(\rho_{f(t)} - \rho_0) < \varepsilon, \tag{1.70}$$

for every $t \geq 0$.

Proof. We will proceed by contradiction. If the assertion is not true, then there exists some $\varepsilon_0 > 0$, such that for every $i \in \mathbb{N}$, there is some $t_i > 0$, and a solution f_i from flat Vlasov-Poisson system with central mass density, such that $f_i(0) \in C_c^1(\mathbb{R}^4) \cap \mathcal{F}_M$ with

$$d(f_i(0), f_0) - E^1_{pot}(\rho_{f_i(0)} - \rho_0) < \frac{1}{i}, \tag{1.71}$$

and

$$d(f_i(t_i), f_0) - E^1_{pot}(\rho_{f_i(t_i)} - \rho_0) \geq \varepsilon_0. \tag{1.72}$$

By (1.71), we have that $\mathcal{E}_{\mathcal{C}}(f_i(0)) - \mathcal{E}_{\mathcal{C}}(f_0) = d(f_i(0), f_0) - E^1_{pot}(\rho_{f_i(0)} - \rho_0) \to 0$. As the Casimir-Energy functional $\mathcal{E}_{\mathcal{C}}$ is a conserved quantity, then $\mathcal{E}_{\mathcal{C}}(f_i(0)) = \mathcal{E}_{\mathcal{C}}(f_i(t_i)) \to \mathcal{E}_{\mathcal{C}}(f_0)$.

Then we have that $(f_i(t_i))_{i\in\mathbb{N}}$ is a minimizing sequence for $\mathcal{E}_{\mathcal{C}}$ in $\mathcal{F}_M$, and therefore passing through subsequence, we have that $(\rho_{f_i(t_i)})_{i\in\mathbb{N}}$ is a minimizing sequence for $\mathcal{E}^r_{\mathcal{C}}$ in $\mathcal{F}^r_M$, and as f_0 is unique, by Lemma 1.20, we have that $\rho_{f_i(t_i)} \rightharpoonup \rho_0$ in $L^{1+1/n}(\mathbb{R}^2)$. By Corollary 1.12, we have that $E^1_{pot}(\rho_{f_i(t_i)} - \rho_0) \to 0$, and thus as

$$\mathcal{E}_{\mathcal{C}}(f_i(t_i)) - \mathcal{E}_{\mathcal{C}}(f_0) = d(f_i(t_i), f_0) - E^1_{pot}(\rho_{f_i(t_i)} - \rho_0),$$

we have that $d(f_i(t_i), f_0) \to 0$, which contradicts 1.72.

□

Remark As we mentioned in 1.34, the Coulomb energy $\mathcal{D}$ is an inner product over $L^{4/3}(\mathbb{R}^2)$ which induces a norm in that space, given by

$$\|\rho\|_{pot} := \mathcal{D}(\rho,\rho)^{1/2} = (-E^1_{pot}(\rho))^{1/2}, \tag{1.73}$$

and therefore we can replace $-E_{pot}(\cdot)$ by $\|\cdot\|_{pot}$ in Theorem 1.22.

Corollary 1.23 *Let $\varepsilon > 0$. Under the assumptions from 1.22, and supposing that $\|f(0)\|_{1+1/k} = \|f_0\|_{1+1/k}$, then there is some $\delta > 0$ such that if* (1.69) *holds, then*

$$\|f(t) - f_0\|_{1+1/k} < \varepsilon. \tag{1.74}$$

Proof. As the same way as the proof from the above theorem, if we assume the opposite, we can build a minimizing sequence $(f_i(t_i))_{i\in\mathbb{N}}$ such that

$$\|f_i(t_i)\|_{1+1/k} = \|f_i(0)\|_{1+1/k} = \|f_0\|_{1+1/k}.$$

By Lemma 1.21, we have that $f_i(t_i) \rightharpoonup f_0$ and also $\|f_i(t_i)\|_{1+1/k} \to \|f_0\|_{1+1/k}$. This implies, using the fact that $L^{1+1/k}(\mathbb{R}^4)$ is uniformly convex, that $\|f_i(t_i) - f_0\|_{1+1/k} \to 0$, which contradicts $\|f_i(t_i) - f_0\|_{1+1/k} \geq \varepsilon_0$. □

Remark We have the following observations.

a) Interpolating (see Appendix A.45), we have that the result of Corollary 1.23 is true in $L^p(\mathbb{R}^4)$ norm, for every $1 < p \leq 1 + 1/k$.

b) Remains pending to study the uniqueness of the minimizer for the variational problem, which is an important hypothesis in the non-linear stability properties proved above.

c) Global existence of classical solutions of Flat Vlasov-Poisson system given an initial datum $C^1_c(\mathbb{R}^2)$ have not been proved yet. The results of non-linear stability are conditional to have the existence of this suitable solutions.

Chapter 2

Mixing in anharmonic potential well

In this chapter, we will show the results which were obtained in a joint work with Hanne Van Den Bosch and Paola Rioseco [13]. We will give a proof of phase-space mixing phenomenon and an estimation for the rate of convergence to equilibrium for integrable systems. Since we are studying essentially a transport equation in $\mathbb{T}^d$ rather than $\mathbb{R}^d$ (equation (31)), the rate of decay does not improve with dimension. We state our main theorem of this chapter, where we denote $(D\omega)_{ij} = \partial_i \omega_j$ the Jacobian matrix of ω and

$$\bar{f}_0(k) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f_0(q,k) dq \tag{2.1}$$

the average of f_0 over $\mathbb{T}^d$.

Theorem 2.1 *Let $f(t,q,k)$ be a solution of (31) with initial datum $f_0 \in C^1(\mathbb{T}^d \times K)$. Assume that $\phi \in C^1_c(\mathbb{T}^d \times K)$ is bounded, and that*

$$\omega \in C^2(K) \quad \text{and } \det D\omega(k) \neq 0, \text{ for all } k \in K, \tag{2.2}$$

then there exists C, depending on ω, f_0 and ϕ such that

$$\left| \int_K \int_{\mathbb{T}^d} (f(t,q,k) - \bar{f}_0(q,k)) \phi(q,k) dq dk \right| \leq \frac{C}{1+|t|}.$$

Remark The constant C depends on the initial datum, on the test function ϕ, and on the inverse of the Jacobian matrix $D\omega$. In Propositions 2.4 and 2.8 below, we give more precise statements that allow to relax the hypotheses and estimate the constant for concrete cases.

2.1 The vector field method

The main tool that we will use throughout this chapter and allows us to facilitate some calculations, are the d-vector fields

$$W := tD\omega(k)\nabla_q + \nabla_k, \tag{2.3}$$

which we can write by components as

$$W_j = t\sum_{i=1}^{d}(\partial_{k_j}\omega_i(k))\partial_{q_i} + \partial_{k_j}, \tag{2.4}$$

where $j \in \{1, ..., d\}$. The usefulness of to define these vector field lies in the fact that the Liouville operator (29) commutes with each vector field defined in (2.4).

Proposition 2.2 *Let W be the operator defined in* (2.3)*, and denote by $\mathcal{L}$ the Liouville operator in action-angle variables, defined as*

$$\mathcal{L} = \partial_t + \omega(k)\nabla_q.$$

Then we have $\mathcal{L}W_j = W_j\mathcal{L}$, for all $j \in \{1, ..., d\}$.

Proof. A brief calculation give us the following equations (in Einstein notation)

$$W_j\mathcal{L} = t\partial_{k_j}\omega_i(k)\partial^2_{q_it} + t\omega_l(k)\partial_{k_j}\omega_i(k)\partial^2_{q_iq_l} + \partial^2_{k_jt} + \partial_{k_j}\omega_l(k)\partial_{q_l} + \omega_l(k)\partial^2_{k_jq_l},$$

and

$$\mathcal{L}W_j = \partial_{k_j}\omega_i(k)\partial_{q_i} + t\partial_{k_j}\omega_i(k)\partial^2_{tq_i} + \partial^2_{tk_j} + (\omega_i\partial_{q_i})(t\partial_{k_j}\omega_l(k)\partial_{q_l}) + \omega_i(k)\partial^2_{q_ik_j}.$$

Then it is easy to see that

$$t\omega_l(k)\partial_{k_j}\omega_i(k)\partial^2_{q_iq_l} = (\omega_i\partial_{q_i})(t\partial_{k_j}\omega_l(k)\partial_{q_l}),$$

which implies the desired result. □

One direct but usefull consequence for this property is the fact that if f is a solution of the Liouville equation, then W_j^nf it is too, for all $n \in \mathbb{N}$, and therefore we have the following property:

Proposition 2.3 *Let f be a solution of* (31)*. Then for sufficiently regular functions f and g, we have*

$$\iint |W_j^nf|(t, q, k)g(k)dkdq = \iint |W_j^nf_0|(q, k)g(k)dkdq.$$

Proof. We can calculate explicitly the solution of (31) as $f(t, q, k) = f_0(q - \omega(k)t, k)$. In this way, we have

$$\begin{aligned}\iint |f|(t, q, k)g(k)dkdq &= \iint |f_0|(q - \omega(k)t, k)g(k)dkdq \\ &= \iint |f_0|(q, k)g(k)dkdq,\end{aligned}$$

which is the result for $n = 0$. Using proposition 2.2 we have that W_j^nf is also a solution of the Liouville equation (31), for all $n \in \mathbb{N}$, and this implies directly the required result. □

2.2 The one-dimensional case

In the one-dimensional case, the operator W defined above takes the form

$$W = \omega'(k) t \partial_q + \partial_k, \tag{2.5}$$

and the Liouville's operator takes the form

$$\mathcal{L} = \partial_t + \omega(k)\partial_q. \tag{2.6}$$

We will use propositions 2.2 and 2.3 to obtain time-indepent bounds.

Proposition 2.4 *Let f denote the solution to (29) with initial datum $f_0 \in L^1$ and fix $\phi \in L^\infty$. Assume that either f or ϕ have compact support in $\mathbb{T}^d \times K$. Then, provided all terms on the right-hand-side are finite*

$$\left| \int_K \int_{\mathbb{T}} (f(t,q,k) - \bar{f}_0(q,k))\phi(q,k) dq dk \right| \leq \frac{2\pi}{t} \left(\left\| \frac{\bar{\phi}}{\omega'} \partial_k f_0 \right\|_1 + \left\| \bar{f}_0 \partial_k \frac{\phi}{\omega'} \right\|_1 \right).$$

Remark We have the following observations.

a) The hypotheses on f, ϕ and ω of Theorem 2.1 imply directly that the terms in the upper bound are indeed finite. Since it is sufficient to prove the decay for large values of t, this proposition implies the one-dimensional case of Theorem 2.1.

b) The hypothesis on compact support is only needed to ensure the absence of boundary terms when integrating by parts. It can be weakened by adding the value(s) of $\frac{\bar{f}_0 \bar{\phi}}{\omega'}$ at ∂K to the right-hand-side, provided these values are well-defined.

PROOF. Since the proof of 2.3, we have that

$$\begin{aligned} \bar{f}_0(k) &:= \frac{1}{2\pi} \int_{\mathbb{T}} f_0(q,k) dq \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} f_0(q - \omega(k)t, k) dq \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} f(t,q,k) dq. \end{aligned}$$

We insert this in the expression that we need to estimate and use the Fundamental Theorem of Calculus to write

$$\begin{aligned} \left| \int_K \int_{\mathbb{T}} (f(t,q,k) - \bar{f}_0(q,k))\phi(q,k) dq dk \right| &= \left| \frac{1}{2\pi} \int_K \int_{\mathbb{T}} \int_{\mathbb{T}} (f(t,q,k) - f(t,q',k))\phi(q,k) dq' dq dk \right| \\ &= \left| \frac{1}{2\pi} \int_K \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{q'}^{q} \partial_q f(t,\tilde{q},k)\phi(q,k) d\tilde{q} dq' dq dk \right| \\ &\leq \frac{1}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \int_K \partial_q f(t,\tilde{q},k)\phi(q,k) dk \right| d\tilde{q} dq' dq \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \int_K \partial_q f(t,\tilde{q},k)\phi(q,k) dk \right| d\tilde{q} dq. \end{aligned}$$

To obtain this inequality, we first use Fubini's theorem to perform the k-integral before the others, and then extended the range of $\tilde{q}$ (which requires inserting the absolute value). The last line is just the observation that the q'-dependence has disappeared from the integrand.

We now use W defined in 2.5 to write $\partial_q = (\omega'(k)t)^{-1}(W - \partial_k)$. The first term will have the required form to apply 2.3, and we can integrate by parts (the boundary terms disappear due to the assumptions on f and ϕ) to bring the second term in this form as well. This gives

$$\int_K \partial_q f(t,\tilde{q},k)\phi(q,k)dk = t^{-1}\int_K [Wf](t,\tilde{q},k)\frac{\phi(q,k)}{\omega'(k)}dk + t^{-1}\int_K f(t,\tilde{q},k)\partial_k\left[\frac{\phi(q,k)}{\omega'(k)}\right]dk.$$

Inserting this in the bound, we found

$$t^{-1}\int_K\int_{\mathbb{T}}\frac{|\partial_k f_0|(\tilde{q},k)}{|\omega'(k)|}\int_{\mathbb{T}}|\phi(q,k)|dqd\tilde{q}dk + t^{-1}\int_K\int_{\mathbb{T}}\tilde{f}_0(\tilde{q},k)\int_{\mathbb{T}}\left|\partial_k\left[\frac{\phi(q,k)}{\omega'(k)}\right]\right|dqd\tilde{q}dk.$$

This can be rewritten in terms of the averages over $\mathbb{T}$ to give the required result. □

2.2.1 Localization argument

Even if the condition det $D\omega \neq 0$ fails at some points, mixing may still hold. For simplicity, we state this result in the one-dimensional case and for a linearly vanishing ω'. We use the explicit rate of decay and the expression for the upper bound obtained in Proposition 2.4 allows for extensions when $\omega'(k)$ vanishes at some *energies* in the support of ϕ. We use a simple localization argument to treat the case where $\omega'(k)$ vanishes linearly.

Theorem 2.5 *Fix f_0 and ϕ of class C^1, with compact support, and let f denote the corresponding solution to Liouville's equation. Assume that $\omega \in C^2(K)$, and $\omega'(k) \neq 0$ except for k in the finite set $\{k_1, \cdots, k_N\}$, and that $\omega''(k_i) \neq 0$. Then, there is $C > 0$ such that*

$$\left|\int_K\int_{\mathbb{T}}(f(t,q,k) - \bar{f}_0(k))\phi(q,k)\right|dqdk \leq \frac{C}{1+|t|^{1/3}}.$$

Proof. Let $0 < \varepsilon < 1$ to be fixed later. We fix a smooth cutoff function χ with support in $(-1,1)$, values in $[0,1]$, and such that $\chi \equiv 1$ in $[-1/2,1/2]$. We define

$$\chi_{i,\varepsilon} := \chi\left(\frac{k-k_i}{\varepsilon}\right),$$

and

$$\eta_\varepsilon := \prod_{i=1}^N (1-\chi_{i,\varepsilon}).$$

Then, we write $\phi(q,k) = \eta_\varepsilon(k)\phi(q,k) + (1-\eta_\varepsilon(k))\phi(q,k)$. Note that $\eta_\varepsilon(k)\phi(q,k)$ satisfies the hypotheses of Proposition 2.4. Thus we have

$$\begin{aligned}&\left|\int_K \int_{\mathcal{T}} (f(t,q,k) - \bar{f}_0(k))\eta_\varepsilon(k)\phi(q,k)dqdk\right| \\ &\leq \frac{2\pi}{t} \left\|\frac{\eta_\varepsilon}{\omega'}\right\|_\infty \left(\left\|\bar{\phi}\partial_k f_0\right\|_1 + \left\|\bar{f}_0\partial_k\phi\right\|_1\right) \\ &\qquad + \frac{2\pi}{t}\left\|\partial_k\left(\frac{\eta_\varepsilon}{\omega'}\right)\right\|_\infty \left\|\phi\bar{f}_0\right\|_1.\end{aligned}$$

Now, we need to extract the ε-dependence from the L^∞-norms. Since $\omega''(k_i) \neq 0$, for some $C > 0$ and all $\varepsilon < 1$, we have

$$\inf_{\text{supp }(\eta_\varepsilon)\cap\text{supp }(f)} |\omega'(k)| \geq \frac{\varepsilon}{C}.$$

This gives us the bounds

$$\left\|\frac{\eta_\varepsilon}{\omega'}\right\|_\infty \leq \frac{C}{\varepsilon}, \qquad \left\|\partial_k\left(\frac{\eta_\varepsilon}{\omega'}\right)\right\|_\infty \leq \frac{C}{\varepsilon^2}.$$

We have obtained

$$\left|\int_K \int_{\mathbb{T}} (f(t,q,k) - \bar{f}_0(k))\eta_\varepsilon(k)\phi(q,k)dqdk\right| \leq \frac{C}{t\varepsilon^2}. \tag{2.7}$$

On the other hand,

$$\begin{aligned}\left|\int_K \int_{\mathbb{T}} (f(t,q,k) - \bar{f}_0(k))(1-\eta_\varepsilon(k))\phi(q,k)dqdk\right| &\leq 2\int_K (1-\eta_\varepsilon(k))\int_{\mathbb{T}} f(t,q,k)dqdk \\ &\leq C\varepsilon\left\|\bar{f}_0\right\|_\infty.\end{aligned}$$

We sum with (2.7), evaluate at some $T > 1$ and pick $\varepsilon = T^{-1/3}$, to obtain

$$\left|\int_K \int_{\mathbb{T}} (f(T,q,k) - \bar{f}_0(k))\phi(q,k)dqdk\right| \leq CT^{-1/3},$$

since for small T, both terms are bounded, which implies the result. □

2.2.2 Improved decay

If the initial condition is more regular, we can improve the estimate on the decay. To this end, we use the following L^1-version of Poincaré's inequality.

Lemma 2.6 (Poincaré's inequality) *Assume that $g : \mathbb{T} \mapsto \mathbb{R}$ is a periodic function of class C^l and $g(x) = 0$ for some $x \in [0, 2\pi)$. Then, for all $l \in \mathbb{N}$*

$$\int_{\mathbb{T}} |g(q)|dq \leq (\pi)^l \int_{\mathbb{T}} |g^{(l)}(q)|dq.$$

Proof. Without loss of generality, we may assume that $x = 0$. Then,

$$\begin{aligned}\int_0^\pi |g(q)|dq &= \int_0^\pi \left|\int_0^s g'(r)dr\right| ds\\ &\le \int_0^\pi |g'(r)| \int_r^\pi dsdr\\ &\le \pi \int_0^\pi |g'(r)|dr.\end{aligned}$$

Treating the contribution to the L^1-norm of the interval $[\pi, 2\pi]$ analogously, we find that

$$\int_{\mathbb{T}} |g(q)|dq \le \pi \int_{\mathbb{T}} |g'(q)|dq.$$

For the case $l \ge 2$, we proceed by induction. By periodicity we have that

$$\int_{\mathcal{T}} g^{(l-1)} = 0,$$

and hence $g^{(l-1)}(x) = 0$ for some $x \in \mathbb{T}$, and we can iterate the argument. □

As a consequence, we can obtain a faster rate of decay for more regular initial data and observables. For the sake of readability, we assume that the support of ϕ is compact (bounded away from the boundary of K), though it is possible to relax this to suitable decay of the functions and their derivatives.

Theorem 2.7 *For $d = 1$ and under the hypotheses of Theorem 2.1, assume that additionally, $\omega'(k)^{-1} \in C^l(K)$, $f_0, \phi \in C^l(\mathbb{T} \times K)$ for some $l \ge 2$. Then there exists $C > 0$ depending on ω, f_0 and ϕ such that*

$$\left|\int_K \int_{\mathbb{T}} (f(t,q,k) - \bar{f}_0(k))\phi(q,k)dqdk\right| \le \frac{C}{1+|t|^l}.$$

Remark A striking consequence is that mixing is actually super-polynomial when ω, f_0 and ϕ are of class C^∞.

Proof. As in the proof of Proposition 2.4, we bound

$$\begin{aligned}&\left|\int_K \int_{\mathbb{T}} (f(t,q,k) - \bar{f}_0(k))\phi(q,k)dqdk\right|\\ &\qquad\qquad \le \int_{\mathbb{T}} \int_{\mathbb{T}} \left|\partial_{\tilde{q}} \int_K f(t,\tilde{q},k)\phi(q,k)dk\right| d\tilde{q}dq.\end{aligned}$$

We then use Lemma 2.6 to insert $l - 1$ additional derivatives:

$$\begin{aligned}\left|\int_K \int_{\mathbb{T}} (f(t,q,k) - \bar{f}_0(k))\phi(q,k)dqdk\right| &\le \pi^{l-1} \int_{\mathbb{T}} \int_{\mathbb{T}} \left|\partial_{\tilde{q}}^l \int_K f(t,\tilde{q},k)\phi(q,k)dk\right| d\tilde{q}dq\\ &\le t^{-l}\pi^{l-1} \int_{\mathbb{T}} \int_{\mathbb{T}} \left|\int_K \left(\frac{W - \partial_k}{\omega'(k)}\right)^l f(t,\tilde{q},k)\phi(q,k)dk\right| d\tilde{q}dq.\end{aligned}$$

In the previous expression, we keep in mind that the operator W only affects the variables denoted by k and $\tilde{q}$, not q. Expanding the product makes appear 2^l terms. In order to integrate by parts, we iterate the identities

$$[\partial_K, W] = \omega''(k)t\partial_q = \frac{\omega''(k)}{\omega'(k)}(W - \partial_k)$$

and for any sufficiently regular function $g(k)$,

$$[W, g(k)] = [\partial_K, g(k)] = g'(k).$$

This allows to obtain an identity of the form

$$\left(\frac{W - \partial_k}{\omega'(k)}\right)^l = \sum_{j=0}^{l}\sum_{m=0}^{l} g_{j,m}(k)\partial_k^j W^m,$$

where each of the functions $g_{n,m}(k)$ is a complicated expression containing powers of $(\omega')^{-1}$ and its derivatives up to order $l-(m+j)$. In each term, we integrate by parts in K to obtain

$$\begin{aligned}\left|\int_K \left(\frac{W - \partial_k}{\omega'(k)}\right)^l f(t,\tilde{q},k)\phi(q,k)dk\right| &\leq \sum_{j=0}^{l}\sum_{m=0}^{l}\left|\int_K W^m f(t,\tilde{q},k)\partial_k^j(g_{j,m}(k)\phi(q,k))dk\right|. \\ &\leq \sum_{j=0}^{l}\sum_{m=0}^{l}\left\|\partial_k^j(g_{j,m}\phi)\right\|_\infty \int_K |W^m f(t,\tilde{q},k)|\, dk.\end{aligned}$$

Finally, we apply 2.3 to bound

$$\begin{aligned}\left|\int_K\int_{\mathbb{T}}(f(t,q,k) - \bar{f}_0(k))\phi(q,k)dqdk\right| &\leq \frac{\pi^l}{t^l}\sum_{j=0}^{l}\sum_{m=0}^{l}\left\|\partial_k^j(g_{j,m}\phi)\right\|_\infty \int_{\mathbb{T}}\int_K |W^m f(t,\tilde{q},k)|\, d\tilde{q} \\ &= \frac{\pi^l}{t^l}\sum_{j=0}^{l}\sum_{m=0}^{l}\left\|\partial_k^j(g_{j,m}\phi)\right\|_\infty \|\partial_k^m f_0\|_1\end{aligned}$$

□

2.3 Higher dimensions

In this section we will show the Phase-Space Mixing decay for the d-dimensional case, with $d \geq 2$. As we defined in 2.3, we have that the operators W_j commutes with the Liouville's operator, for all $j \in \{1, ..., d\}$. If $D\omega$ is an invertible matrix, most of the proof goes through as before. For the sake of completeness, we state Theorem 2.1 with an explicit bound on the right-hand-side. To this end, we define the matrix norm

$$|M|_\infty = \max_{i,j}|M_{i,j}|.$$

In this way, we have the following proposition

Proposition 2.8 *Let $f(t,q,k)$ be the solution to 31 with initial datum $f_0 \in C^1(\mathbb{T}^d \times K)$. Assume that $\phi \in C_c^1(\mathbb{T}^d \times K)$, and that*

$$\det D\omega(k) \neq 0, \qquad \text{for all } k \in K.$$

Then, $M := (D\omega(k))^{-1}$ is well-defined and

$$\left| \int_K \int_{\mathbb{T}^d} (f(t,q,k) - \bar{f}_0(k))\phi(q,k)dqdk \right|$$
$$\leq \frac{2\pi d}{t}\left(\| |M|_\infty \phi\|_\infty \sum_{i=1}^d \left\|\partial_{k_i}\bar{f}_0\right\|_1 + \|\nabla_k \cdot M\phi\hat{e}_j\|_\infty \left\|\bar{f}_0\right\|_1 \right).$$

Proof. As before, we express the left-hand-side as

$$\left| \int_K \int_{\mathbb{T}^d} (f(t,q,k) - \bar{f}_0(k))\phi(q,k)dqdk \right|$$
$$= \frac{1}{(2\pi)^d}\left| \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\int_K (f(t,q,k) - f(t,q',k))\phi(q,k)dkdqdq' \right|.$$

Then we write

$$f(t,q,k) - f(t,q',k) = \sum_{j=1}^d \int_{q_j'}^{q_j} \partial_{q_j} f(t, q_1, \cdots q_{j-1}, s, q_{j+1}', \cdots, q_d', k)ds$$

and we obtain the bound

$$\left| \int_K \int_{\mathbb{T}^d} (f(t,q,k) - \bar{f}_0(k))\phi(q,k)dqdk \right|$$
$$\leq \frac{1}{(2\pi)^d}\sum_{j=1}^d \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\int_0^{2\pi} \left| \int_K \partial_{q_j} f(t, q_1, \cdots q_{j-1}, s, q_{j+1}', \cdots, q_d', k)\phi(q,k)dk \right| dsdqdq'.$$

We now write $\partial_{q_j} = t^{-1}[M(W - \nabla_k)]_j$. Thus, by using the divergence theorem for the second term (and using the compact support of ϕ to conclude the absence of boundary terms), we obtain

$$\left| \int_K \partial_{q_j} f(t, q_1, \cdots q_{j-1}, s, q_{j+1}', \cdots, q_d', k)\phi(q,k)dk \right|$$
$$\leq \frac{1}{t}\int_K \left| [M(k)W]_j f(t, q_1, \cdots q_{j-1}, s, q_{j+1}', \cdots, q_d', k)\phi(q,k) \right| dk$$
$$+ \frac{1}{t}\int_K \left| f(t, q_1, \cdots q_{j-1}, s, q_{j+1}', \cdots, q_d', k)(\nabla_k \cdot M(k)^\top \phi(q,k)\hat{e}_j) \right| dk$$
$$\leq \frac{\| |M|_\infty \phi\|_\infty}{t} \sum_{i=1}^d \int_{\mathbb{R}_+^d} \left| W_i f(t, q_1, \cdots q_{j-1}, s, q_{j+1}', \cdots, q_d', k) \right| dk$$
$$+ \frac{\left\|\nabla_k \cdot M^\top \phi \hat{e}_j\right\|_\infty}{t} \int_K \left| f(t, q_1, \cdots q_{j-1}, s, q_{j+1}', \cdots, q_d', k) \right| dk.$$

Inserting this in the previous bound and using 2.3, we finally obtain

$$\left| \int_K \int_{\mathcal{T}^d} (f(t,q,k) - \bar{f}_0(k))\phi(q,k) dq dk \right|$$
$$\leq \frac{2\pi d}{t} \left(\| |M|_\infty \phi \|_\infty \sum_{i=1}^{d} \left\| \partial_{k_i} \bar{f}_0 \right\|_1 + \left\| \nabla_k \cdot M^\top \phi \hat{e}_j \right\|_\infty \left\| \bar{f}_0 \right\|_1 \right).$$

□

2.4 Hamiltonian with Coulomb potential

Finally, we study the Coulomb potential generated by a particle density F. We will use the notation F for the density in the physical phase space $\mathbb{R}^d \times \mathbb{R}^d$ and $f = F \circ N$ for the density in action-angle coordinates, where $N : \mathbb{T}^d \times K \mapsto G \subset \mathbb{R}^d \times \mathbb{R}^d$ denotes the transformation from action-angle variables to the position and momentum, where G is the open set of values of position and momenta for which this transformation is well-defined and invertible. The motivation to consider the Coulomb potential in particular, is to take into account the gravitational self-interaction (the Vlasov-Poisson system). As in [**?**], the results that we prove remain insufficient to treat the nonlinear equation. This is natural, since we don't expect in general that $\bar{f}_0$ is a steady state of the Vlasov-Poisson system.

The Coulomb potential (as the gravitational potential defined in Chapter 2) can be written as the integral of F against a *test function* with a singularity, which can be compensated by requiring some extra regularity of F. For a given F defined in Euclidean space, we define the Coulomb potential generated by its particle density as the unique solution to

$$\Delta U_F(x) = \int_{\mathbb{R}^d} F(x,v) dv,$$

with $U_F(0) = 0$ when $d = 1$, and $\lim_{|x|\to\infty} U_F(x) = 0$ for $d \geq 2$. We will assume that the system with Hamiltonian $\mathcal{H}(x,v) = |v|^2/2 + U(x)$ is integrable. Then, we will prove the next result.

Corollary 2.9 *Assume that N is a C^1-diffeomorphism, and that the frequencies $\omega(k)$ satisfy* (2.2)*. Let $F_0 \in C_c^1(G) \cap L^1(G)$. Denote by F the solution to Liouville's equation* (31)*, then*

$$\left\| U_F - U_{\widetilde{F}_0} \right\|_\infty \leq \frac{C}{1+|t|},$$

where

$$\widetilde{F}_0 := \overline{(F_0 \circ N)} \circ N^{-1}.$$

Proof. For fixed $x_0 \in \mathbb{R}^d$, we write Υ_{x_0} for the fundamental solution to Poisson's equation in dimension d. In particular,

$$\Upsilon_{x_0}(x) = \begin{cases} (x - x_0)\chi_{[-\infty,x_0]} + \max(0,x_0) & \text{if } d = 1 \\ -(2\pi)^{-1}\ln(|x-x_0|) & \text{if } d = 2 \\ \kappa_d |x-x_0|^{-d+2} & \text{if } d = 3, \end{cases}$$

for a suitable constant κ_d. Therefore, we can write

$$U_F(x_0) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Upsilon_{x_0}(x) F(t,x,v) dx dv = \int_{\mathbb{T}^d} \int_K \varphi_{x_0}(q,k) f(t,q,k) dk dq,$$

where $\varphi_{x_0} = \Upsilon_{x_0} \circ N$. Now, the integral is in a suitable form to apply the arguments in the proofs of Theorems 2.1 and 2.8, provided that φ_{x_0} is sufficiently regular. The Coulomb kernel ϕ_{x_0} belongs to the Sobolev space $W^{1,1}_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$, since the integral of its derivative in a ball is finite. Outside a sufficiently large ball, the function and its derivatives are bounded. Since we assume that N is of class C^1, φ_{x_0} inherits these properties. Thus, $\nabla_k \varphi_{x_0} \in L^1 + L^\infty$. For the L^∞-part, we can apply Proposition 2.8 directly, and for the L^1 part we switch the roles of f and ϕ in the proof of Proposition 2.8. □

Conclusion

In the first chapter of this thesis we proved the existence of a minimizer of Casimir-Energy functional described in 21 and provided of some regularity conditions for the function Φ, we proved that the minimizer is a steady state solution of Flat Vlasov-Poisson system with a central mass density, system described in 1.1. Instead of the ideas based in concentration-compactness, we used a *symmetrization* argument, taking the rearrangement of minimizing sequences for the reduced variational problem, which allow to control terms which appears from the gravitational potential. Also, it was proved an analogous result of [8] about non-linear stability for the steady state, in suitable L^p-norms, provided of uniqueness of the minimizer of Casimir-Energy functional. It remains an open problem to show the uniqueness of the minimizer, and the study of the existence of classical solutions of the Flat Vlasov-Poisson system provided of an initial datum. In the second part we talked about the results of [13], where we proved the phase-space mixing for solutions to the Liouville equation for integrable systems, obtaining a rate of convergence in time. In one dimension, we proved that when the non-harmonicity condition fails at a certain energy, the phase space mixing still holds but with a slower rate.

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Annexed

The following appendix presents some usefull results, definitions and theorems, which were used throughout this thesis. More details can be found in [1, 3, 6, 9, 12].

Functional Analysis

Uniformly convex spaces

Definition A.1 (Uniformly convex space) *Let X be a normed linear space. We say that X is **uniformly convex** if there exists a positive function $r \mapsto \varepsilon(r)$ with $\varepsilon(r) > 0$ and $\lim_{r\to 0} \varepsilon(r) = 0$, such that for all $x, y \in \bar{B}(0,1)$*

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \varepsilon(\|x-y\|)$$

Dual space and reflexive spaces

Definition A.2 (Dual space) *Let X a normed linear space. We define the **dual space** X^* of X, as the linear space of all continuous linear functionals $L : X \to \mathbb{R}$.*

Theorem A.3 *We have that $L \in X^*$ if and only if, there exists a constant $K > 0$ such that $|Lx| \leq K\|x\|$. We have that X^* is a Banach space, with the norm:*

$$\|L\| := \sup_{x\neq 0} \frac{|Lx|}{\|x\|} \tag{A.1}$$

Definition A.4 (Reflexive space) *A normed linear space X is called **reflexive**, if X es isomorphic to its bi-dual space $(X^*)^*$.*

Theorem A.5 *Every uniformly convex Banach space is reflexive.*

Weak convergence

Definition A.6 (Weak convergence) *Let X a normed linear space, and $(x_n)_{n\in\mathbb{N}}$ some sequence of elements in X. We have that x_n* ***converges weakly*** *to $x \in X$, if for every linear functional $L \in X^*$,*

$$Lx_n \to Lx. \tag{A.2}$$

It is denoted by $x_n \rightharpoonup x$.

Theorem A.7 (Uniform boundedness principle) *Every sequence weakly convergent in a normed linear space X is uniformly bounded in norm.*

Theorem A.8 (Lower semi-continuity of norm) *Let $(X, \|\cdot\|)$ a normed linear space, and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ a sequence weakly convergent to $x \in X$. Then*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\| \tag{A.3}$$

Theorem A.9 (Mazur's lemma) *Let $(X, \|\cdot\|)$ a Banach space, and $(x_n)_{n\in\mathbb{N}} \subseteq X$ a sequence weakly convergent to $x \in X$. Then there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in the convex hull of $(x_n)_{n\in\mathbb{N}}$, which converges strongly to x.*

Definition A.10 (Weak sequentially compact set) *A set $Y \subset X$ is said* ***weakly sequentially compact*** *if every sequence in Y has a subsequence weakly convergent in Y.*

Theorem A.11 (Banach-Alaoglu) *The closed unit ball in a Banach space X is weak sequentially compact if and only if X is a reflexive space.*

Theorem A.12 *Let X an uniformly convex Banach space and $(x_n)_{n\in\mathbb{N}}$ a sequence weakly convergent to $x \in X$ and such that*

$$\limsup_{n\to\infty} \|x_n\| \leq \|x\|.$$

Then x_n converges strongly to x in X.

L^p and L^p_w spaces

L^p spaces

Let $(\Omega, \mathcal{T}, \mu)$ a measurable set, and let $p \in [0, \infty]$. We have the following definitions.

Definition A.13 (Almost everywhere) *We say that some property P in a measurable space ocurrs almost everywhere, if the set of points for which P does not occur has measure zero.*

Definition A.14 (Almost everywhere equality) *Let $f, g \in \mathcal{M}$. We define the almost everywhere equality class of equivalence $(\sim)$ as $f \sim g$ if and only if $f = g$ a.e. Hence, the class of* *equivalence is denoted by*

$$[f]_\sim := \{g \in \mathcal{M} \mid g \sim f\} \tag{A.4}$$

Definition A.15 (L^p spaces, $1 \le p < \infty$) *We define the space $L^p(\Omega, \mathcal{T}, \mu)$, which is denoted for simplicity by $L^p(\Omega)$, as the quotient space of p-integrable functions quotiented with $\sim$. In other words*

$$L^p(\Omega) := \left\{ [f]_\sim \mid \int_\Omega |f|^p d\mu < \infty \right\}. \tag{A.5}$$

The L^p spaces are normed linear spaces with the norm

$$\|[f]_\sim\|_p = \|f\|_p := \left(\int_\Omega |f|^p d\mu \right)^{1/p} \tag{A.6}$$

Definition A.16 (L^∞ space) *We define the $L^\infty(\Omega, \mathcal{T}, \mu)$ space, which is denoted for simplicity by $L^\infty(\Omega)$, as the quotient space of almost everywhere bounded functions quotiented with $\sim$. In other words*

$$L^\infty(\Omega) := \{[f]_\sim \mid \exists a \in \mathbb{R}_+ \ \textit{tal que} \ |f| \le a \ \textit{a.e.}\}. \tag{A.7}$$

The L^∞ space is a normed linear space with the essential supremum norm

$$\|[f]_\sim\|_\infty = \|f\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} f(x) \tag{A.8}$$

Theorem A.17 *If $1 \le p \le \infty$, then $L^p(\Omega)$ is a Banach space, and for $1 < p < \infty$, $L^p(\Omega)$ is uniformly convex. In particular $L^2(\Omega)$ is a Hilbert space.*

Theorem A.18 *We have that $L^q(\Omega)$ is isomorphic to $L^p(\Omega)^*$, where $1/p + 1/q = 1$. In particular, if $1 < p < \infty$, then $L^p(\Omega)$ is reflexive.*

Theorem A.19 (Fatou's lemma) *Let $(f_n)_{n\in\mathbb{N}}$ a sequence of positive measurable functions. Then*

$$\int \liminf_n f_n dx \le \liminf \int f_n dx.$$

Moreover, if every f_n is integrable and g is another integrable function, then

a) If $\liminf_n f_n$ is integrable and $f_n \ge g$ for every $n \in \mathbb{N}$, then

$$\int \liminf_n f_n dx \le \liminf \int f_n dx.$$

b) If $\limsup_n f_n$ is integrable and $f_n \le g$ for every $n \in \mathbb{N}$, then

$$\int \limsup_n f_n dx \ge \limsup \int f_n dx.$$

Theorem A.20 (Dominated Convergence Theorem) *Let $(f_n)_{n\in\mathbb{N}}$ a sequence of integrable functions such that $f_n \to f$ pointwise. Suppose that there exists an integrable function g such that $(f_n)_{n\in\mathbb{N}}$ is dominated by g, i.e. $|f_n| \le g$. Then f is integrable and*

$$\int f_n dx \to \int f dx$$

Theorem A.21 (Continuity of convolution) *Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, with $\frac{1}{p}+\frac{1}{q}=1$. Then the convolution is a continuous function on $\mathbb{R}^d$, and for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that*

$$\sup_{|x|>\delta_\varepsilon} |(f * g)(x)| < \varepsilon \tag{A.9}$$

L^p_w spaces

Definition A.22 (L^p_w spaces) *We define $L^p_w(\Omega, \mathcal{T}, \mu)$, which is denoted for simplicity by $L^p_w(\Omega)$, as the quotient space of all measurable functions such that*

$$\sup_{\alpha>0} \alpha |\{x \in \Omega : |f(x)| > \alpha\}|^{1/p} < \infty, \tag{A.10}$$

that is

$$L^p_w(\Omega) := \{[f]_\sim : \sup_{\alpha>0} \alpha |\{x \in \Omega : |f(x)| > \alpha\}|^{1/p} < \infty\}. \tag{A.11}$$

If $p > 1$, then for $q > 1$ such that $1/p + 1/q = 1$, we have that

$$\|f\|_{p,w} := \sup_A |A|^{1/q} \int_A |f(x)| d\mu \tag{A.12}$$

induces a norm in the space $L^p_w(\Omega)$.

Theorem A.23 *We have that $L^p(\Omega) \subset L^p_w(\Omega)$.*

Theorem A.24 *Let $0 < \lambda < d$ and let $p = d/\lambda$. If $f := |\cdot|^{-\lambda}$, then*

$$\|f\|_{p,w} = \frac{d}{d-\lambda} (|\mathbb{S}^{d-1}|/d)^{1/p}. \tag{A.13}$$

Rearrangements

Definitions

Definition A.25 (Vanishing at infinity) *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function. It is said that f **vanishes at infinity** if the sets*

$$\{x \in \mathbb{R}^d : |f(x)| > t\} \tag{A.14}$$

have finite Lebesgue measure, for all $t > 0$.

Definition A.26 (Symmetric rearrangement of a set) *Let $A \subset \mathbb{R}^d$ a Borel set with finite Lebesgue measure. The **symmetric rearrangement** of A is defined as the open ball centered at the origin which volume is the same of A. That is*

$$A^* = B(0, r), \tag{A.15}$$

where r is such that $|A| = |B(0,r)| = |\mathbb{S}^{d-1}| r^d / d$.

Definition A.27 (Symmetric rearrangement of a function) *Let $f : \mathbb{R}^d \to \mathbb{R}$ a Borel-measurable function which vanishes at infinite. The **symmetric rearrangement** or simply the **rearrangement** of f as follows: if $f = \chi_A$, then $f^* = \chi_{A^*}$, and otherwise*

$$f^*(x) := \int_0^\infty \chi^*_{\{|f|>t\}}(x)dt. \tag{A.16}$$

Proposition A.28 *The rearrangements have the following properties*

1. *f^* is a nonnegative function.*
2. *f^* is radially symmetric and nonincreasing. That is*
$$f^*(x) \leq f^*(y), \ \ si \ |x| \geq |y|, \tag{A.17}$$
with equality if $|x| = |y|$.
3. *If $f \in L^p(\mathbb{R}^d)$, then $f^* \in L^p(\mathbb{R}^d)$ and also it preserves the norm. That is*
$$\|f\|_p = \|f^*\|_p, \tag{A.18}$$
with $1 \leq p \leq \infty$.

Useful theorems

Theorem A.29 (Non-expansivity of rearrangement) *Let $Q : \mathbb{R} \to \mathbb{R}$ a nonnegative convex function such that $Q(0) = 0$. Let f and g nonnegative functions in $\mathbb{R}^d$ which vanishes at infinity. Then we have that*

$$\int_{\mathbb{R}^d} Q(f^*(x) - g^*(x))dx \leq \int_{\mathbb{R}^d} Q(f(x) - g(x))dx. \tag{A.19}$$

If also we assume that Q is strictly convex, $f = f^*$ and f is strictly nonincreasing, then the equality in A.19 implies that $g = g^*$.

Theorem A.30 (Riesz's rearrangement inequality) *Let f, g and h three nonnegative functions in $\mathbb{R}^d$. Then*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(x-y)h(y)dxdy \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^*(x)g^*(x-y)h^*(y)dxdy. \tag{A.20}$$

Sobolev Spaces

Definitions

Definition A.31 (Hölder continuous functions) *Let Ω be an open subset of $\mathbb{R}^d$. A function $u : \Omega \to \mathbb{R}$ is said **Hölder continuous of exponent** $0 < \gamma \leq 1$, if there exists $C > 0$ such that*

$$|u(x) - u(y)| \leq C|x - y|^\alpha, \tag{A.21}$$

for every $x, y \in \Omega$.

Definition A.32 (Hölder space) *The **Hölder space** $C^{k,\gamma}(\bar{\Omega})$ consist of all functions $u \in C^k(\bar{\Omega})$ which are Hölder continuous with exponent $0 < \gamma \leq 1$.*

Theorem A.33 *The Hölder space $C^{k,\gamma}(\bar{\Omega})$ is a Banach space, provided with the norm*

$$\|u\|_{C^{k,\gamma}(\bar{\Omega})} := \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C(\bar{\Omega})} + \sup_{x\neq y} \frac{|u(x) - u(y)|}{|x-y|^\gamma}. \tag{A.22}$$

Definition A.34 (Weak Derivative) *Suppose $u, v \in L^1_{loc}(\Omega)$ and α is a multiindex. We say that v is the $\alpha^{th}-$**weak partial derivative of** u, written*

$$D^\alpha u = v, \tag{A.23}$$

provided

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v\phi dx, \tag{A.24}$$

for all test functions $\phi \in C_c^\infty(\Omega)$.

Definition A.35 (Sobolev Space) *Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer. We define the **Sobolev space** as*

$$W^{k,p}(\Omega) := \{u \in L^1_{loc}(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for each multiindex } |\alpha| \leq k\}. \tag{A.25}$$

Theorem A.36 *For every k, $W^{k,p}(\Omega)$ is a Banach space, provided the norm*

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p dx\right)^{1/p}, & \text{if } 1 \leq p < \infty \\ \sum_{|\alpha|\leq k} \operatorname{ess\,sup}_\Omega |D^\alpha u|, & \text{if } p = \infty. \end{cases} \tag{A.26}$$

If $p = 2$, we write $H^k(\Omega) = W^{k,2}(\Omega)$ which is a Hilbert space.

Theorem A.37 (General Sobolev Inequalities) *Let Ω be a bounded open subset of $\mathbb{R}^d$, with a C^1 boundary. Assume that $u \in W^{k,p}(\Omega)$.*

i) If

$$k < \frac{d}{p},$$

then $u \in L^q(\Omega)$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{d}.$$

We have in addition the estimate

$$\|u\|_q \lesssim \|u\|_{W^{k,p}(\Omega)}.$$

ii) If

$$k > \frac{d}{p},$$

then $u \in C^{k-\left[\frac{d}{p}\right]-1,\gamma}(\bar{\Omega})$, *where*

$$\gamma = \begin{cases} \left[\frac{d}{p}\right] + 1 - \frac{d}{p}, & \text{if } \frac{d}{p} \text{ is not an integer,} \\ \text{any positive number } < 1, & \text{if } \frac{d}{p} \text{ is an integer.} \end{cases}$$

We have in addition the estimate

$$\|u\|_{C^{k-\left[\frac{d}{p}\right]-1,\gamma}(\bar{\Omega})} \lesssim \|u\|_{W^{k,p}(\bar{\Omega})}.$$

Fourier Transform and Riesz Transform

Definition A.38 (Fourier Transform of $L^1(\mathbb{R}^d)$) *If* $u \in L^1(\mathbb{R}^d)$, *we define its* ***Fourier Transform*** $\mathcal{F}u$ *by*

$$(\mathcal{F}u)(y) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot y} u(x)dx, \tag{A.27}$$

and its ***inverse Fourier Transform*** $\mathcal{F}^{-1}u$ *by*

$$(\mathcal{F}^{-1}u)(y) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot y} u(x)dx. \tag{A.28}$$

Theorem A.39 (Plancharel's Theorem) *Assume* $u \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^2)$. *Then* $\mathcal{F}u, \mathcal{F}^{-1}u \in L^2(\mathbb{R}^2)$ *and*

$$\|u\|_2 = \|\mathcal{F}u\|_2 = \|\mathcal{F}^{-1}u\|_2.$$

Proposition A.40 (Properties of Fourier Transform) *Assume that* $u, v \in L^2(\mathbb{R}^d)$. *Then we have the following properties*

i) $\int_{\mathbb{R}^d} u\bar{v}dx = \int_{\mathbb{R}^d} \mathcal{F}u\bar{\mathcal{F}}v\,dx$.

ii) $\mathcal{F}(D^\alpha u)(y) = i^{|\alpha|} y_1^{\alpha_1} \cdot \ldots \cdot y_d^{\alpha_d} \mathcal{F}u(y)$ *for each multiindex* $\alpha = (\alpha_1, ..., \alpha_d)$ *such that* $D^\alpha u \in L^2(\mathbb{R}^d)$.

iii) If $u, v \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, *then* $\mathcal{F}(u * v) = (2\pi)^{d/2}\mathcal{F}(u)\mathcal{F}(v)$.

iv) Furthermore $u = \mathcal{F}^{-1}(\mathcal{F}(u))$

Definition A.41 (Riesz Transform) *The* ***Riesz Transform*** Rf *of a function* $f \in L^1(\mathbb{R}^2)$ *is defined by* $Rf = (R_1 f, ..., R_d f)$, *where*

$$R_j f(x) := \frac{\Gamma((d+1)/2}{\pi^{(d+1)/2}} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d \setminus B(x,\varepsilon)} \frac{(x_j - y_j)f(y)}{|x-y|^{d+1}} dy, \tag{A.29}$$

for every $j \in \{1, ..., d\}$.

Proposition A.42 *The Fourier Transform of* $R_j f$ *is given by*

$$\mathcal{F}(R_j f)(x) = -i\frac{x_j}{|x|}(\mathcal{F}f)(x).$$

Theorem A.43 *If $1 < p < \infty$, there are constants $C_p, C'_p > 0$ such that for all $f \in L^p(\mathbb{R}^d)$*

$$\frac{1}{C'_p}\|f\|_p \leq \|Rf\|_p \leq C_p\|f\|_p. \tag{A.30}$$

Useful inequalities

Theorem A.44 (Hölder's inequality) *Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $f \in L^p, g \in L^q$, then $fg \in L^r$ and we have that*

$$\|fg\|_r \leq \|f\|_p\|g\|_q.$$

Theorem A.45 (Riesz-Thorin interpolation lemma) *Let $0 < p_0 < p_1 \leq \infty$. For $\theta \in (0, 1)$, it is defined*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

If $f \in L^{p_0} \cap L^{p_1}$, then $f \in L^{p_\theta}$ and we have that

$$\|f\|_{p_\theta} \leq \|f\|_{p_0}^{1-\theta}\|f\|_{p_1}^{\theta}. \tag{A.31}$$

Theorem A.46 (Young's inequality, strong version) *Let $p, q, r \in [1, \infty]$ with $p, q \leq r$, such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1. \tag{A.32}$$

Let $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ and we have that

$$\|f * g\|_r \leq \|f\|_p\|g\|_q \tag{A.33}$$

Theorem A.47 (Hardy-Littlewood-Sobolev inequality) *Let $p, r > 1$ y and $0 < \lambda < d$ with $1/p + \lambda/d + 1/r = 2$. Let $f \in L^p$, $g \in L^r$. Then there exists a constant $C > 0$, independent of f, g, such that*

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)|x-y|^{-\lambda}g(y)dxdy\right| \lesssim \|f\|_p\|g\|_r \tag{A.34}$$

Theorem A.48 (Young's inequality, weak version) *Let $p, q, r \in [1, \infty]$ with $p, q \leq r$, such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1. \tag{A.35}$$

Let $f \in L^p$ and $g \in L^q_w$, then $f * g \in L^r$ and we have that there exists a constant $C > 0$ such that

$$\|f * g\|_r \lesssim \|f\|_p\|g\|_{w,q} \tag{A.36}$$

Action-angle variables in $\mathbb{R}^d \times \mathbb{R}^d$

Under hypothesis of Liouville's theorem, over a integrable system of $d-$degrees of freedom and d conserved quantities, when the energy level sets $C(E) := \{(x, v) \in \mathbb{R}^d \times \mathbb{R}^d \mid \mathcal{H}(x, p) = E\}$ are compact, it can be defined some kind of canonical coordinates

Definition A.49 (Action-angle variables) *Let $x = (x_1, ..., x_d)$ and $p = (p_1, ..., p_2)$ the canonical coordinates of position and momenta in the phase space. We define the* ***action variable*** *$k = (k_1, ..., k_d) \in K$ as*

$$k_i = \frac{A(\mathcal{H}(x,p))}{2\pi}, \quad A(\mathcal{H}(x,p)) = \oint x_i dp_i,$$

where K is a suitable open set of $\mathbb{R}^d$ and A is the ***area function****. The canonical variable of the action variable k is the* ***angle variable*** *$q = (q_1, ..., q_d) \in \mathbb{T}^d$.*

Proposition A.50 *The Hamiltonian $\mathcal{H}$ depends only on the action coordinates k, i.e.*

$$\mathcal{H}(x, p) = \mathcal{H}(k, q) = \mathcal{H}(k).$$

The angle variable satisfies the equation

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial k_i} = \omega_i(k) = \frac{2\pi}{T(\mathcal{H}(k))},$$

where ω_i are the ***frequencies*** *of the periodic motion with* ***period*** *T. The equation implies that q_i is a linear function of time.*