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Pseudo-Differential Operators on General Type I Locally Compact Groups

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Mathematics is the art
of giving the same name to different things.

HENRI POINCARÉ.

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Agradezco enormemente a mis padres por su apoyo, que sin ellos no estaría aquí. Gracias a todos mis profesores durante la licenciatura.

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Resumen

In a recent paper by M. Măntoiu and M. Ruzhansky a global pseudo-differential calculus has been developed for unimodular groups of type I. In the present thesis we generalize the main results to arbitrary locally compact groups of type I. Our methods involve defining suitable Weyl systems, Wigner transforms and the use of Plancherel's theorem for non-unimodular groups. We also give explicit constructions for the group of affine transformations of the real line and Grélaud's group.

Contents

Introduction

M. Măntoiu and M. Ruzhansky developed [33] a global pseudo-differential calculus, or quantization, for the class of second countable locally compact unimodular type I groups. Our aim is to generalize their main results to the more general class of non-unimodular groups. There are many important examples of non-unimodular groups, the simplest one is perhaps the affine group consisting of all affine transformations of the real line, the only non-unimodular Lie group in dimension two. In dimension three there are many infinite families of non-isomorphic non-unimodular Lie groups. Many other examples arise in the study of parabolic subgroups of semisimple Lie groups that are used to study irreducible representations through extensions of Mackey's machine [19–23].

Let G be a locally compact group. It will be assumed that G is second countable and of type I. Let $\widehat{G}$ be its unitary dual, that is, the space of all classes of unitary equivalence of (strongly continuous) irreducible unitary representations. The formula (cf. eq. (1.1) [33] for the simpler unimodular case)

$$[\mathrm{Op}(a)u]\,(x) = \int_G \int_{\widehat{G}} \mathrm{Tr}\left(a(x,\xi) D_\xi^{\frac{1}{2}} \pi_\xi(xy^{-1})^*\right) \Delta(y)^{-\frac{1}{2}} u(y)\, d\xi\, dy, \qquad (0.1)$$

is the starting point for a global pseudo-differential calculus on G. It involves special measures on G and $\widehat{G}$, namely, the Haar and Plancherel measures, operator-valued symbols defined on $G \times \widehat{G}$, the modular function Δ of the group and a family of unbounded operators; the formal dimension operators D_ξ introduced by Duflo and Moore in [8, §3]. Formula (0.1) also makes use of the Haar and Plancherel measures on G and $\widehat{G}$ respectively. In order to make sense of formula (0.1), we also have to fix a measurable field of irreducible representations $(\pi_\xi)_{\xi \in \widehat{G}}$ such that π_ξ is a representations in the class of ξ that acts on a Hilbert space $\mathcal{H}_\xi$. For the moment the symbols are essentially chosen so that the compositions $a(x,\xi)D_\xi^{1/2}$ are trace-class operators on the Hilbert space $\mathcal{H}_\xi$ almost everywhere. We also require the map sending ξ to the trace-class norm of $a(x,\xi)D_\xi^{1/2}$ to be absolutely integrable for almost all $x \in G$.

The notion of quantization comes from the passage from classical mechanics to quantum mechanics. It is a rigorous formalism in which one passes from abelian C^*-algebras of observables, as in Hamiltonian mechanics, to non-abelian ones, as in quantum mechanics where the observables are operators on an infinite-dimensional Hilbert space. Quantizations have been proven useful in the study of partial differential equations, quantum optics and signal processing. It also has many connections to Lie theory, as it is directly connected [11] to the Heisenberg group and to the metaplectic group, which is the double covering of the symplectic group.

Formula (0.1) is a generalization of the one derived in [33, eq. (1.1)] to the class of unimodular groups, but our formula has a difference in the order of the factors that has to do with the choice of a convention for the Fourier transform (cf. Remark 1.4) our quantization will give rise to right-invariant operators whereas the one in [33] gives rise to left-invariant operators.

Particular cases of compact Lie groups have been extensively studied in [35, 37] for example, and in the references cited therein. The class of nilpotent Lie groups is treated in [10] and in other references. For a general treatment of pseudo-differential operators in a group-theoretic setting see [10, 36]. The idea of using the irreducible representations of a group to define such calculus seems to come from [42, §1.2], but it was not developed in this abstract setting. All the articles cited above contain historical background and references to the existing literature treating pseudo-differential operators and quantization in a group-theoretic context. For a historical survey on harmonic analysis see [32], for example.

One of the advantages of using operator-valued symbols is that one gets a global approach. Even for compact Lie groups there is no notion of full scalar-valued symbols for a pseudo-differential operator using local coordinates. For a more detailed discussion of the advantages of this approach see [33].

When our group is $G = \mathbb{R}^n$, formula (0.1) boils down to the extensively studied Kohn-Nirenberg quantization rule. In that case there is a much bigger class of symbols, namely the Hörmander symbol classes $S^m_{\rho,\delta}(\mathbb{R}^n)$. More general classes of symbols have been studied, but definitively the Hörmander classes are the most important ones, they are extensively studied in –but not only– [25, 41, 44], and the spectral theory of pseudo-differential operators is studied in [38]. For $G = \mathbb{R}^n$ there are also τ-quantizations given by

$$[\mathrm{Op}^\tau(a)u](x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a\left((1-\tau)x + \tau y, \xi\right) e^{2\pi i\langle \xi, x-y\rangle} u(y)\, d\xi\, dy, \qquad (0.2)$$

with $\tau \in [0, 1]$ mostly related to ordering issues. The Kohn-Nirenberg case amounts to take $\tau = 0$. Another interesting case is $\tau = 1/2$, the so-called Weyl quantization, which is a more symmetric quantization that has the desirable property $\mathrm{Op}(\bar{a}) = \mathrm{Op}(a)^*$. It is possible to extend the idea of τ-quantizations to our pseudo-differential calculus on type I groups with a fixed measurable function $\tau : G \to G$ instead of a real number $\tau \in [0,1]$. For a τ-quantization the right modification of formula (0.1) turns out to be

$$[\mathrm{Op}^\tau(a)u](x) = \int_G \int_{\widehat{G}} \mathrm{Tr}\left(a(\tau(yx^{-1})x,\xi) D_\xi^{\frac{1}{2}} \pi_\xi(xy^{-1})^*\right) \Delta(y)^{-\frac{1}{2}} u(y)\, d\xi\, dy, \qquad (0.3)$$

from which one recovers (0.1) after setting $\tau(\cdot) = e$ to be the constant function, where e is the identity element of the group. Another interesting example is when $\tau(x) = x$. In the simplest case of $G = \mathbb{R}^n$ this amounts to taking $\tau = 1$ and one gets the quantization in which derivatives are composed to the left and position operators to the right. Anyhow, the formalisms corresponding to different choices of τ are actually isomorphic when we restrict to the classes of symbols we are considering in the present thesis.

If $G = \mathbb{R}^n$, then one can write the quantization as

$$\mathrm{Op}(a) = \int_G \widehat{a}(\xi, x) W(\xi, x)\, dx\, d\xi,$$

where the Weyl system is the family

$$\{W(\xi, x) = V(\xi)U(x) \mid (\xi, x) \in \mathbb{R}^{2n}\},$$

of unitary operators in $L^2(\mathbb{R}^n)$ obtained by putting together the operators of translation and modulation. The Weyl system offers a precise way to codify the canonical commutation relations between position and momentum operators from quantum mechanics, and the quantization Op can be seen as a non-commutative functional calculus on these operators. Besides the physical interest, this opens the way to some new topics or tools such as the Bargmann transform, the anti-Wick quantization, co-orbit spaces, and others. In Section 2.4 we show how to carry this point of view to the general category of locally compact type I groups, but one of the drawbacks of non-unimodular groups is that the resulting operators are only defined in a dense subspace.

Weyl systems were one of the first examples of projective representations. The study of projective representations of $\mathbb{R}^{2n}$ was one of the most important problems in the 20th century and had its roots deep in the foundation of quantum mechanics. One can see that the projective representations of $\mathbb{R}^{2n}$ can be seen as unitary representations of the Heinsenberg group with n degrees of freedom [1] and the representations of the latter were settled down with the Stone-von Neumann theorem, which says that under some hypothesis, and up to equivalence, there is only one possible representation that satisfies the canonical commutation relations.

Another approach to a quantization consists of using the formalism of C^*-algebras. Given a locally compact group G there is an action by left (or right) translations on various C^*-algebras of functions on G. In such situations there are natural crossed products associated to them (cf. Chapter 4): Among the non-degenerate representations of these C^*-algebras we mention the Schrödinger representation, acting on the Hilbert space $L^2(G)$. This formalism allows us to take full advantage of the theory of C^*-algebras, extending to the bigger class of compact operators on $L^2(G)$.

In the present thesis we are not going to rely on properties such as compactness, semisimplicity, nilpotency or smoothness. Almost all hypotheses shall be on the measure-theoretic side. The category of second-countable type I locally compact groups has a nice integration theory and their unitary duals have an amenable integration theory. This framework allows for a general form of Plancherel's Theorem, which is all that is needed to develop the basic features of a quantization, even for non-unimodular groups. The non-unimodular Plancherel theorem is originally due to [40] and had many contributions by Duflo, Moore [8], Lipsman and Kleppner [27], To apply the theorem one needs to know the complete unitary dual of a group, including its Plancherel measure. Later H. Führ [14] found the exact domain in which the inversion formula holds in the non-unimodular case. For an introduction to abstract harmonic analysis we refer to [12].

We now summarize the present thesis.

- In Chapter 1 we introduce notation used throughout the thesis and the general theory and tools required to properly develop a quantization, including tools from abstract harmonic analysis and functional analysis, the main tool being the non-unimodular Plancherel transform.

- In Chapter 2 we make a preliminary construction of the quantization Op on a densely defined subspace using formula (0.1). We include a discussion on the differences between the left and right quantizations which comes from the non-commutativity of the group, and we study how to recover the families of convolution and multiplication operators using our quantization.

- In Section 2.4 we introduce the notion of a Weyl system, a measurable family of densely defined closed operators. Then we define a τ-quantization for an arbitrary measurable function $\tau : G \to G$ that has to do with ordering issues of the operators. In Section 2.5 we introduce a more general τ-quantization, and prove that it is a unitary map from our class of symbols onto the Hilbert-Schmidt operators on $L^2(G)$.

- In Sections 3.1 and 3.2 we work out the explicit formulas of the quantization for the group of affine transformations of $\mathbb{R}$ and Grélaud's group, two examples of non-unimodular solvable Lie groups. We compute the unitary dual of Grélaud's group using the Mackey machine reviewed in § 1.4.

- In Chapter 4 we review the basic theory of crossed products of C^*-algebras and we show how it relates to our theory. This formalism is also used to cover a bigger class of compact operators using the Schrödinger representation associated to a natural crossed product.

In the future it is our goal to extend formula (0.1) for more general classes of densely defined operator-valued symbols, to cover for example differential operators on Lie groups, or even the class of bounded operators. Many developments have been done in this direction for connected nilpotent Lie groups [10] and compact groups [37].

Chapter 1

Framework

In this Chapter we set up the general framework of this thesis,and also recall some known results in the Fourier theory of non-unimodular groups of type I. We also briefly discuss the theory of square-integrable representations, for which we have many explicit constructions. We also review the basic aspects of the Mackey machine that will be used to compute the unitary dual of Grélaud's group in Section 3.1.

1.1 General remarks from functional analysis and measure theory

We denote Hilbert spaces, over the field of complex numbers, with the letter $\mathcal{H}$, using the convention that their scalar product, denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, will be linear in the first variable and anti-linear in its second. In the following we assume all Hilbert spaces to be separable. If $\mathcal{H}$ is a Hilbert space we denote its conjugate Hilbert space by $\mathcal{H}^{\dagger}$, whose elements are the same as those of $\mathcal{H}$ but the scalar product is defined as $\alpha \cdot u = \bar{\alpha}u$, and its inner product is conjugate to the one from $\mathcal{H}$, i.e. $\langle u, v\rangle_{\mathcal{H}^\dagger} = \langle v, u\rangle_{\mathcal{H}}$. $\mathcal{B}(\mathcal{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathcal{H}$, and $\mathcal{B}_0(\mathcal{H})$ stands for the two sided *-ideal of compact operators on $\mathcal{H}$. We also make use of the Schatten-von Neumann classes $\mathcal{B}_p(\mathcal{H})$ for $p \geq 1$; these are Banach *-algebras with the norm

$$\|T\|_{\mathcal{B}_p} = \mathrm{Tr}\left((T^*T)^{p/2}\right)^{1/p}.$$

The most important cases are $\mathcal{B}_1(\mathcal{H})$, the space of trace-class operators, and $\mathcal{B}_2(\mathcal{H})$, the space of Hilbert-Schmidt operators. The latter when endowed with the inner product

$$\langle T, S\rangle_{\mathcal{B}_2} = \mathrm{Tr}\,(TS^*)\,,$$

becomes a H^*-algebra. As a Hilbert space it is unitarily isomorphic with the Hilbert tensor product $\mathcal{H} \otimes \mathcal{H}^\dagger$ in a natural way. Both $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{B}_2(\mathcal{H})$ are two-sided *-ideals in $\mathcal{B}(\mathcal{H})$ whose closure in the operator norm is $\mathcal{B}_0(\mathcal{H})$.

Let $\mathcal{A}$ be a set of bounded operators on a Hilbert space $\mathcal{H}$. By $\mathcal{A}'$ we denote the family of all bounded operators on $\mathcal{H}$ that commute with all the elements of $\mathcal{A}$. This set is a von Neumann algebra, i.e. a C^*-algebra which is closed in the strong operator topology. There is a celebrated theorem due to von Neumann [4, Chapter IX, Theorem 6.4] that says that if $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity, then $\mathcal{A}''$ coincides with the closure of $\mathcal{A}$ in the weak operator topology.

A **Borel space** is a set X endowed with a σ-algebra of subsets of X, called Borel sets or measurable sets. We are going to refer interchangeably to Borel space as measurable spaces.

Definition 1.1. A measurable space X is called **countably separated** if there is a countable family $\{\Omega_i\}_{i\in\mathbb{N}}$ of measurable sets such that for all $x \in X$ one has

$\{x\} = \bigcap_{x\in\Omega_i}\Omega_i$. The space X is called a **standard Borel space** if there is a measurable isomorphism $X \to Y$, where Y is a complete second countable metric space, endowed with the Borel σ-algebra generated by its topology.

Clearly being a standard Borel space implies being countably generated, and the former is a much stronger hypothesis that one may initially think. In a famous classification due to Kuratowsky it is shown that every standard Borel space is Borel isomorphic either to the interval $[0, 1]$ or to a countable discrete set [34, Theorem 2.14]. We say that a measure υ is **countably separated (or standard)** if there is a measurable υ-conull subset $\Omega \subseteq X$ that is countably separated (standard).

Much of the interest of studying these properties comes from the study of quotients of measurable spaces. Let $\sim$ be an equivalence relation on a measurable space X. The **quotient Borel structure** on $X/\sim$ is the finest σ-algebra making the natural projection $X \to X/\sim$ a measurable map. Sometimes we shall need **measurable transversals** $\Omega \subseteq X$, that is a measurable set that contains exactly one element of each equivalence class in $X/\sim$. Another construction comes as follows. Let X, Y be Borel spaces, and let $f : X \to Y$ be a Borel map, let υ be a given measure in X. We say that a measure $\bar{\upsilon}$ on Y is a **pseudo-image** of υ, if $\bar{\upsilon}$ is equivalent to the measure υ_f given on measurable sets by $\upsilon_f(\Omega) = \upsilon(f^{-1}(\Omega))$, this means that for any measurable set $\Omega \subseteq Y$ one has that $\bar{\upsilon}(\Omega) = 0$ if and only if $\upsilon(f^{-1}(\Omega)) = 0$.

Definition 1.2. Two measures υ_1, υ_2 on a Borel space X are called **strongly equivalent** if there is a continuous function $g : X \to (0, \infty)$ such that for each compactly supported continuous function $f : X \to \mathbb{C}$ one has

$$\int_X f(x)\, d\upsilon_1(x) = \int_X f(x)g(x)\, d\upsilon_2(x).$$

This does not implies equivalent, since the function g does not need to be integrable.

1.2 Direct integrals of Hilbert spaces

Let X be a Borel space. A **field of Hilbert spaces** over X is just a family $(\mathcal{H}_x)_{x\in X}$ of separable Hilbert spaces $\mathcal{H}_x$. A function $s : X \to \coprod_{x\in X} \mathcal{H}_x$ is called a **section over** X, or a **vector field** if $s_x \in \mathcal{H}_x$ for all $x \in X$. A **measurable field of Hilbert spaces** over a field X is a field of Hilbert spaces $(\mathcal{H}_x)_{x\in X}$ together with a countable set $\{e^j\}_{j=1}^{\infty}$ of sections over X with the following properties:

(i) The functions $x \mapsto \langle e_x^i, e_x^j\rangle$ are measurable for all $i, j \in \mathbb{N}$.

(ii) for each $x \in X$, the set $\{e_x^i\}_{i\in\mathbb{N}}$ is a total subset of $\mathcal{H}_x$.

We say that a section s over X is a **measurable section** if the map $x \mapsto \langle s_x, e_x^i\rangle$ is measurable for all $i \in \mathbb{N}$.

Definition 1.3. Let X be a Borel space and let $(\mathcal{H}_x)_{x\in X}$ be a measurable field of Hilbert spaces over X. Suppose that v is a measure on X. The **direct integral** of the spaces $\mathcal{H}_x$, denoted by

$$\int_X^{\oplus} \mathcal{H}_x \, dv(x),$$

is the Hilbert space consisting the all the measurable sections s over X such that

$$\|s\|^2 = \int_X \|s_x\|_{\mathcal{H}_x}^2 \, dv(x) < \infty,$$

in where two sections are identified if they coincide in a v-conull set. The inner product of two sections s, s' is given by

$$\langle s, s'\rangle = \int_X \langle s_x, s'_x\rangle_{\mathcal{H}_x} \, dv(x).$$

Example 1.1. Let (X, ν) be a measure space. Then, if $\mathcal{H}_x = \mathcal{H}$ is a fixed Hilbert space for all $x \in X$, then

$$\int_X^{\oplus} \mathcal{H}_x \, dv(x) \cong L^2(X, v; \mathcal{H}).$$

This space is also naturally isomorphic to the Hilbert tensor product $\mathcal{H} \otimes L^2(X, v)$ under the identification $(\eta \otimes u)(x) = u(x)\,\eta$.

Definition 1.4. A **measurable field of operators** $(T_x)_{x\in X}$ over X is a family of (not necessarily bounded) operators $T_x : \mathcal{H}_x \to \mathcal{H}_x$, where $(\mathcal{H}_x)_{x\in X}$ is a measurable field of Hilbert spaces, such that the section $(T_x s_x)_{x\in X}$ is measurable whenever s is a measurable section over X.

Suppose that v is a measure on a Borel space X, then if $T = (T_x)_{x\in X}$ is a measurable field of operators such that $\operatorname{ess\,sup}_{x\in X} \|T_x\|$ is finite, then T defines a bounded operator on the direct integral $\int_X^{\oplus} \mathcal{H}_x \, dv(x)$ with operator norm bounded by the essential supremum of the norms $\|T_x\|$; we denote this operator by

$$\left(\left[\int_X^{\oplus} T_y \, dv(y) \right] s \right)_x = T_x s_x.$$

1.3 Basic representation theory

Let G be a Hausdorff second countable locally compact group. For the most part this means that the product and inversion laws of the group are continuous maps, and as a topological space G is both second countable and locally compact. We denote its unit by $e \in G$.

Remark 1.1. Recall that a second countable group is separable, Hausdorff, σ-compact and completely metrizable. In particular, as a measurable space it will be standard. Since every irreducible representation is cyclic, it must act on a separable Hilbert space.

Let H be a closed subgroup of G. Then the quotient G/H, endowed with the quotient topology, is a locally compact topological space. It is even a locally compact group when H is a normal subgroup. The spaces G and G/H are completely metrizable. When they are endowed with the Borel σ-algebra generated by their topology they become standard Borel spaces. The following proposition (cf. [28] Lemma 1.1) is due to G. Mackey.

Proposition 1.1. *Let G be a second countable group and let H be a closed subgroup. Then there exist a measurable transversal of G/H.*

Definition 1.5. A **representation** of a group G on a Hilbert space $\mathcal{H}_\pi$ is a function $\pi : G \to \mathcal{B}(\mathcal{H}_\pi)$, such that for each pair of elements $x, y \in G$

$$\pi(xy) = \pi(x)\pi(y).$$

The representation is called **unitary** if $\pi(x)^* = \pi(x^{-1})$ for all $x \in G$. It is called a **strongly continuous representation** if for each $u \in \mathcal{H}_\pi$ the map $x \mapsto \pi(x)u$ is continuous. Every unitary representation considered in this thesis is strongly continuous, even if it is not explicitly mentioned.

Definition 1.6. A **projective representation** on a Hilbert space $\mathcal{H}_\pi$ is a map $\pi : G \to \mathcal{H}_\pi$ such that there exist a measurable map $\omega : G \times G \to S^1$ into the unit circle satisfying

$$\pi(x)\pi(y) = \omega(x, y)\pi(xy),$$

and that for each pair of vectors $u, v \in \mathcal{H}_\pi$, the map $x \mapsto \langle u, \pi(x)v\rangle$ is measurable. We say that π is a projective representation with **multiplier** ω, or that it is a ω-projective representation.

Definition 1.7. A representation π on a Hilbert space $\mathcal{H}_\pi$ is called **irreducible** if there are no proper closed linear subspaces $\mathcal{E} \subsetneq \mathcal{H}_\pi$ such that $\pi(x)\mathcal{E} \subseteq \mathcal{E}$ for all $x \in G$.

Definition 1.8. Let π, σ be two representations. An **intertwining operator** $T : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ is a bounded operator such that for each $x \in G$ one has

$$T\,\pi(x) = \sigma(x)\,T.$$

We say that two representations are **unitarily equivalent** if there exist an unitary intertwining operator between them.

The most basic, and perhaps, the most important result about irreducible representations is the following.

Proposition 1.2 (Schur's lemma). *A unitary (projective) representation π is irreducible if and only if $\pi(G)' = \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_\pi}$. Suppose that π_1, π_2 are irreducible unitary (projective) representations of G. If they are equivalent, then there exists a unique (up to multiplication by a constant) intertwining operator. Otherwise there are no non-trivial intertwining operators.*

Let π be a strongly continuous unitary representation of G on a Hilbert space $\mathcal{H}_\pi$. For any such representation we also have its **contragradient** $\pi^\dagger$ acting on ${\mathcal{H}_\pi}^\dagger$ by $\pi^\dagger(x)u = \pi(x)u$. Note that in general π is not equivalent to $\pi^\dagger$.

We say that π' is a **subrepresentation** of π if it is unitarily equivalent to the restriction $\pi|_\mathcal{E}$, where $\mathcal{E}$ is a closed invariant subspace of $\mathcal{H}_\pi$.

Definition 1.9. We say that two representations π, ρ are **quasi-equivalent** if for every subrepresentation π' of π there is a non-trivial operator intertwining π' and ρ, and for every subrepresentation ρ' of ρ there is a non-trivial operator intertwining ρ' and π. This is an equivalence relation among representations. Two equivalent representations are clearly quasi-equivalent and the notions coincide when we refer to irreducible representations.

Definition 1.10. A unitary representation π is called **primary**, or a **factor representation**, if the center of the von Neumann algebra generated by $\pi(G) \subseteq \mathcal{B}(\mathcal{H}_\pi)$ consist only of multiples of the identity i.e.

$$\pi(G)' \cap \pi(G)'' = \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_\pi}.$$

Definition 1.11. A unitary representation π of G is called **multiplicity-free** if the von Neummann algebra $\pi(G)'$ is commutative. It is called a **type I** representation if it is quasi-equivalent to a multiplicity-free representation.

By Schur's lemma one has that any irreducible representation is a multiplicity-free primary representation, but the converse is not necessarily true, this is the content of the following definitions.

Definition 1.12. We say that a topological group G is **type I** if every primary representation is quasi-equivalent to an irreducible representation, or equivalently, if it is a direct sum of copies of some irreducible representation. If G is second countable, another characterization is that the group is type I if and only if every representation is type I [14, Theorem 3.23].

Example 1.2. Some examples of type I groups are:

- Compact groups.
- Connected semisimple Lie groups [20].
- Abelian groups.
- Exponentially solvable Lie groups [2], in particular connected simply connected nilpotent Lie groups.

– Real algebraic groups [6].

It is known that a discrete group is of type I if and only if it possesses an abelian normal subgroup of finite index [43].

Fix a left Haar measure μ on G, that is, a Radon measure μ such that

$$\int_G f(xy)\, d\mu(y) = \int_G f(y)\, d\mu(y),$$

for all $f \in C_c(G)$ and $x \in G$. We will denote this choice of left Haar measure by $d\mu(x) = dx$; Every locally compact groups admits a left Haar measure, and it is unique up to a positive constant. Once μ is fixed we get a right Haar measure μ^r defined by the formula $\mu^r(\Omega) = \mu(\Omega^{-1})$.

Let $\Delta : G \to (0, \infty)$ be the **modular function** of G, defined by the formula $\mu(\Omega x) = \Delta(x)\mu(\Omega)$ for all measurable sets $\Omega \subseteq G$ and $x \in G$. This implies in particular that $d\mu^r = \Delta^{-1}\, d\mu$. Hence the left and right Haar measures are strongly equivalent.

The modular function is a continuous (smooth if G is a Lie group) homomorphism into the multiplicative group $(0, \infty)$. We say that a group is **unimodular** if the modular function is a constant function. If G is a connected Lie group then

$$\Delta(x) = |\det \mathrm{Ad}(x^{-1})|,$$

where Ad is the adjoint representation of G on its Lie algebra [24, Chapter 10, Lemma 1.2].

Remark 1.2. The the following classes of groups are unimodular:

– Connected semisimple Lie groups.

– Abelian groups.

- Connected nilpotent Lie groups.
- Compact groups.
- Discrete groups.

We also note that G is unimodular provided that the abelianization of G is a compact group.

Let N be the kernel of the modular function; it is a closed normal subgroup of G, which is itself a unimodular group [12, Theorem 2.49]. We also note that since our groups are locally compact the image of the modular function is a subgroup of $\mathbb{R}_{>0}$. It may be either dense in $(0, \infty)$ or a closed discrete subset.

The spaces $L^p(G) = L^p(G, \mu)$ of p-integrable complex-valued functions on G will always refer to the left Haar measure. These are separable Banach spaces for $p \in [1, \infty)$. By $C_c(G)$ we denote the space of continuous complex-valued functions on G with compact support, a dense subspace of $L^p(G)$. The space $C_0(G)$ denotes the C^*-algebra of all continuous complex-valued functions defined on G that vanish at infinity.

1.3.1 The left and right regular representations

Every group naturally comes with a pair of unitary representations, namely, the left and right regular representations, defined on $L^2(G)$ by

$$\lambda_y f(x) = f(y^{-1}x)$$
$$\rho_y f(x) = \Delta(y)^{\frac{1}{2}} f(xy).$$

These are unitary strongly continuous representations of G [12, §2.5]. It is a deep fact [39] that one has the following equality of von Neumann algebras

$$\lambda(G)'' = \rho(G)', \qquad \rho(G)'' = \lambda(G)'.$$

There is also the **two-sided regular representation** $\lambda \otimes \rho$ of $G \times G$ given by

$$\lambda \otimes \rho\,(x, y)f = \lambda_x \rho_y f \qquad f \in L^2(G).$$

For the convenience of the reader we recall that the modular function plays the following role in integration by substitution of variables

$$\int_G f(y)\,dy = \int_G \Delta(x) f(yx)\,dy = \int_G \Delta(y)^{-1} f(y^{-1})\,dy. \tag{1.1}$$

The modular function implements a Banach *-algebra structure on $L^1(G)$: the convolution of two functions defined by the integral

$$(f * g)(x) = \int_G f(y) g(y^{-1}x)\,dy,$$

and the involution is given by

$$f^*(x) = \Delta(x)^{-1}\overline{f(x^{-1})}.$$

In general, one has a p-dependent involution on $L^p(G)$ given by

$$f^*(x) = \Delta(x)^{-\frac{1}{p}}\overline{f(x^{-1})}. \tag{1.2}$$

In the following we reserve the notation f^* for functions in the Hilbert space $L^2(G)$.

Definition 1.13. Let X be a Borel space. A **measurable field of representations** of G over X is a measurable field of operators $(\pi_y)_{y \in X}$ such that each π_y is a unitary representation of G. A measurable field of representations induces a unitary representation on the direct integral $\int_X^{\oplus} \mathcal{H}_y \, dv(y)$ given on sections s over X by

$$\left(\left[\int_X^{\oplus} \pi_y(x)\, dv(y)\right] s\right)_p = \pi_p(x) s_p.$$

An important result due to G. Mackey [30, Theorem 10.2] is that for each measurable subset $\Omega \subseteq \widehat{G}$ on which the Mackey Borel structure is standard, there exist a measurable field of irreducible representations $(\pi_\xi)_{\xi\in\Omega}$ over Ω, acting on canonical Hilbert spaces $\mathcal{H}_\xi$, such that $\pi_\xi \in \xi$ for each $\xi \in \Omega$.

Consider the Banach *-algebra $L^1(G)$ endowed with the **universal norm**

$$\|f\|_{C^*(G)} = \sup_\rho \|\rho(f)\|,$$

where the supremum is taken over the set of all non-degenerate *-representations of $L^1(G)$. The completion of $L^1(G)$ with respect to the universal norm is called the **group algebra** of G. We denote this C^*-algebra as $C^*(G)$. It is a standard fact that the irreducible unitary representations of G are in one-to-one correspondence with the non-degenerate *-representations of $C^*(G)$, the correspondence being given by

$$\pi(f) = \int_G f(y)\pi(y)\, dy$$

for functions in the dense subspace $L^1(G)$.

Remark 1.3. For a type I group G, its C^*-enveloping algebra is postliminal. An equivalent formulation of the type I hypothesis is that for all irreducible representations π one has that $\pi(C^*(G))$ contains all the compact operators on $\mathcal{H}_\pi$.

Given a locally compact group G, its **unitary dual** $\widehat{G}$ is the collection of all of its irreducible unitary representation modulo unitary equivalence. For a representative $\pi_\xi \in \xi$ of an element of the unitary dual of G, we denote the Hilbert space on which it acts by $\mathcal{H}_\xi = \mathcal{H}_{\pi_\xi}$. The unitary dual is known completely for various classes of groups, including abelian, compact, and connected nilpotent Lie groups. It is known up to a set of measure zero in the case of connected semisimple Lie groups. In the case of abelian groups, $\widehat{G}$ is also a second countable locally compact group, but as soon one leaves the abelian world, there does not seem to be a natural way in which this space is a group, even for the case of compact groups.

We endow $\widehat{G}$ with the Mackey Borel structure introduced in [29, §9]. This is done as follows: Let $\mathrm{Irr}_n(G)$ be the set of all irreducible unitary representation acting on a fixed Hilbert space $\mathcal{H}_n$ of dimension $n \in \mathbb{N} \cup \{\infty\}$ and let $\mathrm{Irr}(G) = \bigcup_{n=0}^{\infty} \mathrm{Irr}_n(G)$. We endow each $\mathrm{Irr}_n(G)$ with the weakest σ-algebra such that the maps $\pi \mapsto \langle u, \pi(x)v \rangle$ are measurable for all $u, v \in \mathcal{H}_n, x \in G$. Then a set $\Omega \subseteq \mathrm{Irr}(G)$ is said to be measurable if and only if $\Omega \cap \mathrm{Irr}_n(G)$ is measurable for all $n \in \mathbb{N} \cup \{\infty\}$. The **Mackey Borel structure** is the quotient Borel structure induced by the map $\mathrm{Irr}(G) \to \widehat{G}$ that sends each representation to its equivalence class. The next proposition (cf. [14] Lemma 3.15) sheds some light on the Mackey Borel structure.

Proposition 1.3. *Assume that X is a standard Borel space and let $(\pi_x)_{x \in X}$ be a measurable field of irreducible representations of G. Then the map $X \ni x \mapsto [\pi_x]$, where $[\pi]$ denotes the equivalence class of π, is a measurable map into $\widehat{G}$.*

A consequence of G being of type I is that $\widehat{G}$ is a standard Borel space [18]. It is known [7] that being of type I is equivalent to $\widehat{G}$ being countably separated and is also equivalent to being a standard Borel space. Since spaces which does not satisfy the former hypothesis are badly behaved, it is very hard to expect to have a reasonable integration theory for non-type I groups.

One also provides $\widehat{G}$ with the Fell topology. This topology is T_0 provided that G

is of type I, and then the Mackey Borel structure coincides with the Borel σ-algebra generated by this topology. For semisimple and nilpotent connected Lie groups this topology is T_1, but in general this space is not Hausdorff. For a proof of these assertions see [7, 18], and for more information on the Fell topology see [9, Chapter VII §1].

Definition 1.14. We say that a standard measure ν on $\widehat{G}$ is a **Plancherel measure** (cf. [14] Definition 3.30 and Theorem 3.24) if it yields a direct integral central decomposition of the left regular representations into irreducible representations. That is to say, ν is a Plancherel measure if the following conditions hold

(i) There is a measurable map $m : \widehat{G} \to \mathbb{N} \cup \{0, \infty\}$, a measurable field of irreducible representations $(\pi_\xi)_{\xi \in \widehat{G}}$ with $\pi_\xi \in \xi$, and a unitary isomorphism $U : L^2(G) \to \int_{\widehat{G}}^{\oplus} m(\xi) \cdot \mathcal{H}_\xi \, d\nu(\xi)$ such that such that for each element $x \in G$ one has

$$U\lambda_x = \int_{\widehat{G}}^{\oplus} m(\xi) \cdot \pi_\xi(x) \, d\nu(\xi) \, U.$$

More precisely, let I_n be a set with n elements endowed with the counting measure, then $m(\xi) \cdot \mathcal{H}_\xi = L^2(I_{m(\xi)}) \otimes \mathcal{H}_\xi$ and $m(\xi) \cdot \pi_\xi(x) = \mathrm{Id}_{L^2(I_{m(\xi)})} \otimes \pi_\xi(x)$.

(ii) U implements an equivalence of von Neumann algebras

$$\lambda(G)' \cap \lambda(G)'' \cong \int_{\widehat{G}}^{\oplus} \mathbb{C} \cdot \mathrm{Id}_{m(\xi) \cdot \mathcal{H}_\xi} \, d\nu(\xi).$$

Plancherel measures do exist for separable locally compact groups of type I and in fact they are all mutually equivalent (cf. Theorem 1.3 below). From now on we adopt the notation $d\nu(\xi) = d\xi$ for a Plancherel measure ν. If necessary, we will denote by ν_G the Plancherel measure of a group G.

There are various cases in which the Plancherel measure can be given explicitly.

For abelian groups their unitary dual is also an abelian group in a canonical way and the Plancherel measure coincides with a multiple of its Haar measure. For connected simply connected nilpotent Lie groups it corresponds to a measure on the space of coadjoint orbits arising from the Lebesgue measure on $\mathfrak{g}^*$ [26, Chapter 3 §2.7]. For compact groups the Peter-Weyl theorem says that the irreducible representations form a discrete set and that the Plancherel measure of an irreducible representation is equal to 1. Note that this is only valid using our convention of the Plancherel transform in which the Duflo-Moore operators are taken into account (cf. Theorem 1.2). For a proof see [12, Theorem 5.12].

1.4 Induced representations and the Mackey machine

The Mackey machine is one of the most important tools for computing the unitary dual of a general locally compact group, it consist of inducing representations from a closed normal subgroup and a family of "small subgroups" that appear in the quotient group. This method is developed mainly in [30] and is the basis of a more geometric study of the unitary dual. This point has been proven to be very fruitful, it is directly connected to Kirillov's orbit method, which is the tool of choice to compute the unitary dual of a connected simply connected nilpotent Lie group. This method can be even extended to exponentially solvable Lie groups without many changes. It is even possible to extend this method for the class of connected semisimple Lie groups, since they do not have proper closed normal subgroups, one has to study the parabolic subgroups instead, and there is a similar result in this case allowing one to compute at least the support of the Plancherel measure.

1.4.1 Induced representations

Let υ be a Borel measure on a locally compact Hausdorff G-space X, define the measures $\upsilon_x(\Omega) = \upsilon(x^{-1}\Omega)$ for each Borel subset $\Omega \subseteq X$, and all $x \in G$. Note that for all integrable functions f one has

$$\int_X f(p)\, d\upsilon_x(p) = \int_X f(xp)\, d\upsilon(p). \tag{1.3}$$

Definition 1.15. A measure υ on a G-space X is said to be **invariant** if the measures υ_x coincide. It is called **quasi-invariant** if the measures υ_x are all mutually absolutely continuous. We will call υ **strongly quasi-invariant** if the Radon-Nikodym derivative $(x, p) \mapsto (d\upsilon_x/d\upsilon)(p)$ is a continuous function defined on $G \times X$. This is a stronger assumption than the hypothesis of having all the measures υ_x mutually strongly equivalent, since the Radon-Nikodym derivative must be jointly continuous.

It is a standard fact [12, §2.6] that any transitive locally compact G-space admits a strongly quasi-invariant measure and in fact it is unique up to strong equivalence. Moreover if $X = G/H$ is a transitive G-space, there is a G-invariant Radon measure υ on X if and only if the modular functions on G and H satisfy the relation $\Delta_G\big|_H = \Delta_H$. In such a case, υ unique up to a positive factor.

Let H be a closed subgroup of G, and assume that $X = G/H$ admits a strongly quasi-equivalent measure υ. Let σ be a representation of H on a Hilbert space $\mathcal{H}_\sigma$. The **induced representation** $\pi = \mathrm{Ind}_H^G(\sigma)$ acts on the Hilbert space $\mathcal{H}_\pi = L^2(G, H, \sigma, \upsilon)$ consisting of classes of equivalence of functions $f : G \to \mathcal{H}_\sigma$ such that

(i) The maps $x \mapsto \langle f(x), u\rangle$ are measurable for all $u \in \mathcal{H}_\sigma$.

(ii) $f(xh) = \sigma(h)^* f(x)$ for all $x \in G, h \in H$, except for a possible set of pairs (x, h) such that the corresponding products xh's belong to a υ-null set in G/H.

(iii) The following quantity is finite

$$\|f\|^2 = \int_{G/H} \|f(x)\|^2 \, d\nu(xH).$$

As usual, we impose the equivalence relation $f \cong g$ if and only if $\|f - g\| = 0$ (cf. [16] §4, Theorem 9). The inner product of $\mathcal{H}_\pi$ is given by

$$\langle f, g \rangle = \int_{G/H} \langle f(x), g(x) \rangle_{\mathcal{H}_\sigma} \, d\nu(xH).$$

Thanks to condition *(ii)* the quantities $\langle f(x), g(x) \rangle_{\mathcal{H}_\sigma}$ depends only on the right H-coset of x, so the formulas above are well defined. The induced representation π is given by formula

$$(\pi(x)f)(y) = \sqrt{\frac{d\nu_x}{d\nu}(yH)} \, f(x^{-1}y).$$

Thanks to formula (1.3), and the fact that the Radon-Nikodym are jointly continuous, π is a strongly continuous unitary representation of G.

1.4.2 The Mackey Machine

Let N be a closed normal subgroup of G. The **dual action** of G on the unitary dual of N is given by

$$x.\sigma(n) = \sigma(x^{-1}nx) \quad x \in G, n \in N.$$

With this action $\widehat{N}$ becomes a Borel G-space. We denote by G_σ the stabilizer in G of $\sigma \in \widehat{N}$. Let $H = G/N$, since N acts trivially on $\widehat{N}$, the dual action gives rise to an action of H. We denote by H_σ the stabilizer in H of $\sigma \in \widehat{N}$. These groups are usually called "small groups" in the literature. We denote the orbit of an element $\sigma \in \widehat{N}$ as $\mathcal{O}_\sigma$, the type I hypothesis implies that $\mathcal{O}_\sigma \cong \widehat{N}/H_\sigma$ as Borel spaces [14, §3.2].

Definition 1.16. Let N be a closed normal subgroup of G and let $H = G/N$, endowed with the Borel quotient structure. We say that N is **regularly embedded** in G if the orbit space $\widehat{N}/H$ is a countably separated Borel space.

Let $N = \ker(\Delta)$ and let $H = G/N$. It is proved in [8, Theorem 6] that the left regular representation of G is type I if and only if the left regular representation of N is type I and the orbit space $\widehat{N}/H$ is a standard Borel space. In particular since our groups are type I, N is automatically regularly embedded in G.

Definition 1.17. We say that a representation $\sigma \in \widehat{N}$ has **trivial Mackey obstruction** if it can be extended to a unitary representation of G_σ.

There are two very simple cases in which a representation $\sigma \in \widehat{N}$ has trivial Mackey obstruction. Namely when $G_\sigma = N$ or when $G = N \rtimes H$ is a semi-direct product and N is abelian. Nevertheless, these cases cover a great number of examples.

In general it is possible to extend σ to a projective representation of G_σ in the following way. If $x \in H_\sigma$ stabilizes σ, there is a unitary operator U_x such that

$$U_x \sigma(n) U_x^* = \sigma(xnx^{-1}).$$

This choice is unique up to a factor of norm 1. Without loss of generality we choose them so that U_e is the identity and $x \mapsto U_x$ is a measurable map, that is to say, $x \mapsto \langle u, U_x v\rangle$ is a measurable map for all $u, v \in \mathcal{H}_\sigma$. Let Ω be a measurable transversal of G_σ/N. An element $y \in G_\sigma$ can be written in a unique way as $y = sn$, where $s \in \Omega$ and $n \in N$. We define $\bar{\sigma}(y) = U_{sN}\sigma(n)$. Then one sees that for all $x, y \in G_\sigma$

$$U_x U_y \sigma(n) U_y^* U_x^* = \sigma(xyn(xy)^{-1}).$$

Hence, by the uniqueness of the operators U_x's, there is a constant $\omega(x, y)$ of norm 1 such that $U_x U_y = \omega(x, y) U_{xy}$ for all $x, y \in G_\sigma$. Since U is measurable, one sees that ω is measurable. Hence $\bar{\sigma}$ is a projective representation that extends σ to G_σ with multiplier ω. The Mackey obstruction is trivial precisely when we can chose $\bar{\sigma}$ so that $\omega = 1$.

We cite the celebrated theorem of G. Mackey [30] which is the main tool for computing the unitary dual of a group in terms of a normal subgroup N and a family of "small groups" that depend on the dual action. For simplicity we restrict ourselves to unitary representations. The hypothesis of having trivial Mackey obstruction can be removed, but the resulting induced representations turn out to be projective representations instead of unitary.

Theorem 1.1 (The Mackey machine). *Let $N \subseteq G$ be a regularly embedded closed normal subgroup and suppose that each $\sigma \in \widehat{N}$ has trivial Mackey obstruction. Given a representation $\sigma \in \widehat{N}$, denote a fixed extension to G_σ by $\tilde{\sigma}$. Given a unitary representation $\rho \in \widehat{H_\sigma}$, we define the unitary representation $\sigma \times \rho$ of G_σ, which acts on the Hilbert tensor product $\mathcal{H}_\sigma \otimes \mathcal{H}_\rho$ by*

$$(\sigma \times \rho)(x) = \tilde{\sigma}(x) \otimes \rho(xN).$$

Then one has that

(i) The induced representation $\mathrm{Ind}_{G_\sigma}^{G}(\sigma \times \rho)$ is a unitary irreducible representation of G.

(ii) Suppose that N is of type I. Then every irreducible unitary representation of G is of the above form.

(iii) Moreover, if we fix a measurable transversal $\Omega \subseteq \widehat{N}$ i.e. a set that contains exactly one representative for each orbit in $\widehat{N}/H$, then the map

$$\{(\sigma, \rho) \mid \sigma \in \widehat{N}, \rho \in \widehat{H_\sigma}\} \to \widehat{G} \text{ given by } (\sigma, \rho) \mapsto \mathrm{Ind}_{G_\sigma}^{G}(\sigma \times \rho),$$

is a bijection.

We also cite an extension of the theorem due to Kleppner and Lipsman [27] that allows us to compute the Plancherel measure of the unitary dual. And only requires the Mackey obstruction to be trivial ν_N-almost everywhere.

Proposition 1.4. *Let G be a locally compact group and let N be a closed normal type I unimodular subgroup. Suppose that N is regularly embedded and that there is a G-invariant measurable ν_N-conull subset $\Omega \subseteq \widehat{N}$ such that all $\sigma \in \Omega$ have trivial Mackey obstruction. Then the set*

$$\bigcup_{\mathcal{O}_\sigma \in \Omega/H} \{\mathrm{Ind}_{G_\sigma}^{G}(\sigma \times \rho) \mid \rho \in \widehat{H_\sigma}\},$$

is a ν-conull subset of $\widehat{G}$ and the Plancherel measure may be obtained as follows: Pick a pseudo-image $\bar{\nu}_N$ of the Plancherel measure of $\widehat{N}$ on Ω/H. Then there is for $\bar{\nu}_N$-almost all $\mathcal{O}_\sigma \in \Omega/H$, a normalized Plancherel measure ν_{H_σ} such that if we identify $\mathrm{Ind}_{G_\sigma}^{G}(\sigma \times \rho)$ as the point (σ, ρ) one has that the Plancherel measure of $\widehat{G}$ is given by

$$d\nu(\sigma, \rho) = d\nu_{H_\sigma}(\rho)\, d\bar{\nu}_N(\mathcal{O}_\sigma).$$

1.5 Square-integrable representations

Given a representation π on a Hilbert space $\mathcal{H}_\pi$, and two vectors $u, v \in \mathcal{H}_\pi$, we define the **matrix coefficient** of π at the pair (u, v) as

$$C_{u,v}(x) = \langle u, \pi(x)v \rangle.$$

Note that $C_{u,v}$ is a uniformly continuous bounded function and the algebra generated by all the matrix coefficients of a given representation depends only on its equivalence class. If a matrix coefficient $C_{u,v}$ is square-integrable for some non-zero $u, v \in \mathcal{H}_\pi$ we say that π is a **square-integrable** representation. If π is an irreducible representation it is known [7] that the representation is square integrable if and only if $C_{u,v} \in L^2(G)$ holds for all $u, v \in \mathcal{H}_\pi$.

Theorem 1.2. *Suppose π is a square integrable irreducible unitary representation. There is a densely defined positive self-adjoint operator $D_\pi : \mathrm{Dom}(D_\pi) \to \mathcal{H}_\pi$ with dense image, called the **Duflo-Moore operator**, satisfying:*

(i) For all vectors $u, u' \in \mathcal{H}_\pi, v, v' \in \mathrm{Dom}(D_\pi^{1/2})$

$$\left\langle C_{u,D_\pi^{1/2}v}, C_{u',D_\pi^{1/2}v'} \right\rangle_{L^2(G)} = \langle u, u' \rangle \langle v', v \rangle.$$

In particular this implements an isometric linear map $C : \mathcal{H}_\pi \otimes \mathcal{H}_\pi^\dagger \to L^2(G)$.

(ii) Up to normalization by a positive constant, D_π is uniquely determined by the relation

$$\pi(x) D_\pi \pi(x)^* = \Delta(x)^{-1} D_\pi. \tag{1.4}$$

We presented this result as it is stated in [14, p. 97], in which an explicit construction of the operators D_π is made for square integrable representations. When the group is unimodular the operators D_π are just multiplication by a positive scalar d_π that coincides with the dimension of $\mathcal{H}_\pi$ when the latter is finite. An operator satisfying (1.4) is called **semi-invariant** with weight Δ^{-1}. In general not all irreducible representations admit a semi-invariant operator, they only exist ν-a.e. (cf. Theorem 1.3).

1.6 The Fourier and Plancherel transformations

Suppose we have fixed a Plancherel measure ν in $\widehat{G}$, a measurable field of representations $(\pi_\xi)_{\xi\in\widehat{G}}$ and there is a family of densely defined self-adjoint positive operators $D_\xi : \mathcal{H}_\xi \to \mathcal{H}_\xi$ satisfying relation (1.4) for ν-almost all $\xi \in \widehat{G}$. We define (in the weak sense) the **operator-valued Fourier transform** of a function $f \in L^1(G)$ as

$$\mathcal{F}(f)(\xi) = \int_G f(y)\pi_\xi(y)\,dy.$$

This is the unique operator $\mathcal{F}(f)(\xi)$ such that for all $u, v \in L^2(G)$ one has

$$\langle \mathcal{F}(f)(\xi)u, v\rangle = \int_G f(y)\langle \pi_\xi(y)u, v\rangle\,dy.$$

The Fourier transform is a non-degenerate *-representation of $L^1(G)$, but in the non-unimodular case it fails to intertwine the two-sided regular representation of G with $\int_{\widehat{G}}^{\oplus} \xi \otimes \xi^\dagger\,d\xi$ and it also fails to be a unitary map. So we also introduce the **Plancherel transform** of $f \in L^1(G) \cap L^2(G)$ as the operator

$$\mathcal{P}(f)(\xi) = \mathcal{F}(f)(\xi)D_\xi^{\frac{1}{2}}.$$

In the following we denote by $\mathcal{P}(f) = \widehat{f}$, the Plancherel transform of a function f.

The Plancherel transform satisfies the following identities for functions $f, g \in L^2(G) \cap L^1(G)$,

$$\begin{aligned}
\pi_\xi(x)\widehat{f}(\xi)\pi_\xi(y)^* &= \widehat{\lambda_x\rho_y f}(\xi),\\
\widehat{f}(\xi)^* &= \widehat{f^*}(\xi),\\
\widehat{f * g}\,(\xi) &= \pi_\xi(f)\widehat{g}(\xi).
\end{aligned}$$

When π_ξ is square integrable, $\widehat{f}(\xi)$ extends to a Hilbert-Schmidt operator on $\mathcal{H}_\xi$. It turns out that this will hold for ν-almost every $\xi \in \widehat{G}$ and not only for the representations which are square-integrable (cf. Theorem 1.3 below); for a proof see [8, Theorem 5.1].

We are going to present a formulation of the Plancherel Theorem for non-unimodular groups. For a proof in the unimodular case we refer to [7]. The non-unimodular Plancherel Theorem was developed by N. Tatsuuma [40] and later an extension of his theory, including a clarification of the role of the hypothesis required, was obtained by Duflo and Moore [8]. Similar results were obtained by Kleppner and Lipsman in [27]. We state the following theorem as it was derived in the article of Duflo and Moore and in the spirit of [14, Theorem 3.48].

Theorem 1.3. *Let G be a type I second countable locally compact group. Then there exists a σ-finite Plancherel measure ν on $\widehat{G}$, a measurable field of Hilbert spaces $(\mathcal{H}_\xi)_{\xi\in\widehat{G}}$, a measurable field of irreducible representations $(\pi_\xi)_{\xi\in\widehat{G}}$ with $\pi_\xi \in \xi$, and a measurable field $(D_\xi)_{\xi\in\widehat{G}}$ of densely defined self-adjoint positive operators on $\mathcal{H}_\xi$ with dense image satisfying* (1.4) *for ν-almost every $\xi \in \widehat{G}$, which have the following properties:*

(i) Let $f \in L^1(G) \cap L^2(G)$. For ν-almost all $\xi \in \widehat{G}$, the operator $\widehat{f}(\xi)$ extends to a Hilbert-Schmidt operator on $\mathcal{H}_\xi$ and

$$\|f\|_2^2 = \int_{\widehat{G}} \|\widehat{f}(\xi)\|_{\mathcal{B}_2}^2 \, d\xi. \tag{1.5}$$

(ii) The Plancherel transformation extends in a unique way to a unitary operator

$$\mathcal{P} : L^2(G) \to \int_{\widehat{G}}^{\oplus} \mathcal{B}_2(\mathcal{H}_\xi) \, d\xi. \tag{1.6}$$

(iii) $\mathcal{P}$ implements the following unitary equivalences of representations and von

Neumann algebras

$$\lambda_x \cong \int_{\widehat{G}}^{\oplus} \pi_\xi(x) \otimes \mathrm{Id}_{\mathcal{H}_\xi^\dagger}\, d\xi, \tag{1.7}$$

$$\rho_x \cong \int_{\widehat{G}}^{\oplus} \mathrm{Id}_{\mathcal{H}_\xi} \otimes \pi_\xi^\dagger(x)\, d\xi, \tag{1.8}$$

$$\lambda(G)' \cong \int_{\widehat{G}}^{\oplus} \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_\xi} \otimes \mathcal{B}(\mathcal{H}_\xi^\dagger)\, d\xi, \tag{1.9}$$

$$\lambda(G)'' \cong \int_{\widehat{G}}^{\oplus} \mathcal{B}(\mathcal{H}_\xi) \otimes \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_\xi^\dagger}\, d\xi. \tag{1.10}$$

In particular these relations show that ν satisfies the axioms required by Definition 1.14 to be a Plancherel measure.

(iv) *The Plancherel measure and the operator field may be chosen to satisfy the inversion formula*

$$f(x) = \int_{\widehat{G}} \mathrm{Tr}\left(\widehat{f}(\xi) D_\xi^{\frac{1}{2}} \pi_\xi(x)^*\right) d\xi, \tag{1.11}$$

for all f in the Fourier algebra of G (cf. Section 1.6.1 below). The integral in the inversion formula converges absolutely in the sense that $\widehat{f}(\xi) D_\xi^{\frac{1}{2}}$ extends to a trace-class operator ν-a.e. and the integral over $\widehat{G}$ of the trace-class norms is finite.

(v) *The choice of $\left(\nu, (D_\xi)_{\xi \in \widehat{G}}\right)$ is essentially unique: The semi-invariance relation* (1.4) *fixes each D_ξ up to a multiplicative constant, and once we fix these, ν is fixed by* (1.11). *On the other hand if we fix ν, (which is unique up to equivalence) the operators D_ξ are completely determined by ν.*

(vi) *G is unimodular if and only if there exist positive constants d_ξ such that $D_\xi = d_\xi \, \mathrm{Id}_{\mathcal{H}_\xi}$ for ν-almost all ξ. If G is non-unimodular, D_ξ is an unbounded operator for ν-almost all ξ (this can be seen from equation* (1.4)*).*

(vii) *Suppose there is another Plancherel measure ν' on $\widehat{G}$ and measurable fields*

$(\pi_\xi', D_\xi')_{\xi\in\widehat{G}}$ *that share the properties i)-iii). Then* ν *and* ν' *are equivalent measures, and there is a measurable field of unitary operators* $(U_\xi)_{\xi\in\widehat{G}}$*, intertwining* π_ξ *and* π_ξ'*, such that for* ν*-almost all* $\xi \in \widehat{G}$ *the Radon-Nikodym derivative of* ν' *with respect to* ν *satisfies*

$$\frac{d\nu'}{d\nu}(\xi)\, D_\xi' = U_\xi\, D_\xi\, U_\xi^*. \tag{1.12}$$

The operators D_ξ are called the **formal dimension operators**, or the **Duflo-Moore operators.** When π is induced from a subgroup H on which the modular function is trivial, the Hilbert space $\mathcal{H}_\pi$ is then formed by vector-valued functions on G, and the Duflo-Moore operators have the very simple expression (cf. [12] Theorem 7.42)

$$(D_\pi f)(x) = \Delta(x) f(x).$$

Moreover, if we require for N to be type I, then ν-a.e. the representation $\xi \in \widehat{G}$ is induced from a representation of N (cf. [40]).

In the unimodular case, if one replaces the Plancherel transform with the usual Fourier transform, then formula (1.11) reads

$$f(x) = \int_{\widehat{G}} d_\xi \cdot \operatorname{Tr}\left(\widehat{f}(\xi)\pi_\xi(x)^*\right)\, d\xi.$$

Most books that treat the unimodular case refer to $d_\xi \cdot \nu$ as the Plancherel measure. This explains how to recover the unimodular theory using the non-unimodular Plancherel theorem.

Remark 1.4. Alternatively, we could also define the Plancherel transform of a integrable function f as

$$\dot{\mathcal{P}}(f)(\xi) = D_\xi^{\frac{1}{2}} \int_G f(x)\pi_\xi(x)^*\, dx.$$

Using the semi-invariance relation of D_ξ and the involution given in (1.2), we get the following relation

$$\begin{aligned}\dot{\mathcal{P}}(f)(\xi) &= \int_G f(x)\pi_\xi(x)^* D_\xi^{\frac{1}{2}} \Delta(x)^{-\frac{1}{2}}\, dx \\ &= \int_G f(x^{-1})\Delta(x)^{-\frac{1}{2}} \pi_\xi(x) D_\xi^{\frac{1}{2}}\, dx \\ &= \mathcal{P}(\bar{f}^*)(\xi),\end{aligned}$$

so the two definitions differ by an automorphism of $L^2(G)$. Another thing to have in mind is that the inversion formula (1.11) takes the form

$$\begin{aligned}f(x) &= \bar{f}^*(x^{-1})\Delta(x)^{-\frac{1}{2}} \\ &= \int_{\widehat{G}} \mathrm{Tr}\left(\mathcal{P}(\bar{f}^*)(\xi) D_\xi^{\frac{1}{2}} \pi_\xi(x^{-1})^*\right) \Delta(x)^{-\frac{1}{2}}\, d\xi \\ &= \int_{\widehat{G}} \mathrm{Tr}\left(D_\xi^{\frac{1}{2}} \dot{\mathcal{P}}(f)(\xi)\pi_\xi(x)\right) d\xi.\end{aligned}$$

For simplicity we make use of the following notation

$$\mathcal{B}_2^{\oplus}(G) = \int_{\widehat{G}}^{\oplus} \mathcal{B}_2(\mathcal{H}_\xi)\, d\xi, \qquad \mathcal{B}_1^{\oplus}(G) = \int_{\widehat{G}}^{\oplus} \mathcal{B}_1(\mathcal{H}_\xi) D_\xi^{-\frac{1}{2}}\, d\xi,$$

$$\mathcal{S}(G) = L^2(G) \otimes \mathcal{B}_2^{\oplus}(G), \qquad \mathcal{S}(\widehat{G}) = \mathcal{B}_2^{\oplus}(G) \otimes L^2(G).$$

$\mathcal{S}(G)$ will be the natural space for our symbols. It comes with a natural inner product given by

$$\langle a, b\rangle_{\mathcal{S}(G)} = \int_G \int_{\widehat{G}} \mathrm{Tr}\left(a(x,\xi)b(x,\xi)^*\right) d\xi\, dx.$$

Remark 1.5. By formula (1.7), a representation $\pi_\xi \in \xi$ is subrepresentation of the left regular representation if and only if the singleton $\{\xi\}$ has positive Plancherel measure. It is known that a representation appears as a summand in the decomposition of λ only if it is square-integrable. In addition one checks that for unimodular groups, all the square-integrable representations satisfy $\nu(\{\xi\}) = 1$ [14, pp. 84], for some normalization. When the Hilbert space has finite dimension d_π, the normalization is given by $D_\pi = d_\pi \cdot \mathrm{Id}$.

1.6.1 The Fourier algebra and Plancherel inversion

Most of the results of this section are presented in the works [13, 14] of H. Führ. In order to shed some light on the trace-class hypothesis imposed on our symbols, we elaborate a little on the natural domain of the Plancherel transform in such a way that formula (1.11) holds. We also give the natural domain on the Plancherel side for the inversion formula (1.11).

Definition 1.18. The **Fourier algebra** $A(G)$ of a locally compact group G is defined as the closure of the linear span of

$$\{f * g^\flat \mid f, g \in L^2(G)\},$$

where $g^\flat(x) = \overline{g(x^{-1})}$, with the norm

$$\|u\|_{A(G)} = \inf\{\|f\|_2 \|g\|_2 \mid u = f * g^\flat\}.$$

It becomes a Banach *-algebra with convolution as the product law and $^\flat$ as the involution. This is the space of matrix coefficient functions of the left regular representation of G (cf. (1.13) below).

We record some calculations for further use in the following lemma (cf. [14] Lemma 4.14).

Lemma 1.4. *Let f, g be two square integrable functions on G. Suppose that $g^\flat \in L^1(G)$, then if*

$$h(x) = \langle f, \lambda_x g\rangle = (f * g^\flat)(x). \tag{1.13}$$

Is a matrix coefficient of λ, we have that

$$\widehat{h}(\xi) = \widehat{f}(\xi)\,\widehat{g}(\xi)^* D_\xi^{-\frac{1}{2}}. \tag{1.14}$$

Hence $\widehat{h}(\xi)\,D_\xi^{\frac{1}{2}}$ extends to a trace-class operator ν-almost everywhere.

For a function in $L^2(G) \cap A(G)$, by the previous lemma, its Plancherel transform is in $\mathcal{B}_1^{\oplus}(G) \cap \mathcal{B}_2^{\oplus}(G)$ for ν-almost every $\xi \in \widehat{G}$. A straightforward calculation shows that the inversion formula holds for such a function.

In [14, Theorem 4.12] it is shown that the Plancherel transform induces an isomorphism between the Banach spaces $A(G)$ and $\mathcal{B}_1^{\oplus}(G)$. This induces an isomorphism of Hilbert spaces

$$\mathcal{P} : A(G) \cap L^2(G) \to \mathcal{B}_1^{\oplus}(G) \cap \mathcal{B}_2^{\oplus}(G).$$

The next proposition (cf. [14] Theorem 4.15) shows that $A(G) \cap L^2(G)$ is the natural domain of the Plancherel transform in such a way that the inversion formula holds. Considering the preceding paragraph it also shows that, on the Plancherel side, the natural domain for the inversion formula is $\mathcal{B}_1^{\oplus}(G) \cap \mathcal{B}_2^{\oplus}(G)$.

Proposition 1.5. *Let $F \in \mathcal{B}_2^{\oplus}(G)$ and suppose that for ν-almost everywhere the operator $F(\xi)D_\xi^{\frac{1}{2}}$ extends to a trace-class operator. Suppose moreover that*

$$\int_{\widehat{G}} \|F(\xi)D_\xi^{\frac{1}{2}}\|_{\mathcal{B}_1}\, d\xi < \infty.$$

If f is the inverse Plancherel transform of F, then we have for μ-almost everywhere

$$f(x) = \int_{\widehat{G}} \operatorname{Tr}\left(F(\xi)D_\xi^{\frac{1}{2}}\pi_\xi(x)^*\right) d\xi. \tag{1.15}$$

Chapter 2

Quantization on locally compact groups of type I

In this chapter we introduce a quantization leading to a pseudo-differential calculus for operator-valued symbols defined on the whole group times its dual. In order to do so we take advantage of the irreducible representations of the group. In section 2.4 we develop the notion of a Weyl system which will be used to make a sense of the formulas in a rigorous way for more general symbols. This will also makes things clearer when dealing with a general τ-quantization.

In the following we fix a Plancherel measure and choice of a measurable field of irreducible representations $(\pi_\xi)_{\xi\in\widehat{G}}$ and formal dimension operators $(D_\xi)_{\xi\in\widehat{G}}$ so that Theorem 1.3 holds. By *(vii)* of Theorem 1.3, different choices of the measurable fields of representation or formal dimension operators will lead to isomorphic formulations.

2.1 The quantization

Given a symbol $a \in L^2(G) \otimes (\mathcal{B}_1^{\oplus}(G) \cap \mathcal{B}_2^{\oplus}(G))$ we define the operator $\mathrm{Op}(a)$: $L^2(G) \to L^2(G)$ with symbol a to be

$$[\mathrm{Op}(a)u]\,(x) = \int_G \int_{\widehat{G}} \mathrm{Tr}\left(a(x,\xi) D_\xi^{\frac{1}{2}} \pi_\xi(xy^{-1})^*\right) \Delta(y)^{-\frac{1}{2}} u(y)\, d\xi\, dy,$$

where u is a square integrable function. The operator $\mathrm{Op}(a)$ is called the **pseudo-differential operator** with symbol a. Let

$$\ker_a(x,y) = \Delta(y)^{-\frac{1}{2}} \int_{\widehat{G}} \mathrm{Tr}\left(a(x,\xi) D_\xi^{\frac{1}{2}} \pi_\xi(xy^{-1})^*\right) d\xi.$$

Since a is in the domain of the inverse Plancherel transformation in its second variable $\mathcal{P}_2$, the above integral converges absolutely and

$$\ker_a(x,y) = [\mathcal{P}_2^{-1} a](x, xy^{-1}) \Delta(y)^{-\frac{1}{2}}.$$

By Plancherel's theorem and the change of variables given by equation (1.1), we conclude that $\ker_a$ is a square integrable function on $G \times G$. Hence $\mathrm{Op}(a)$ is a Hilbert-Schmidt operator with kernel $\ker_a$ and Hilbert-Schmidt norm

$$\|\mathrm{Op}(a)\|_{\mathcal{B}_2} = \|\ker_a\|_{L^2(G\times G)} = \|a\|_{\mathcal{S}(G)}.$$

Now we are ready to extend the definition of $\mathrm{Op}(a)$ for an arbitrary symbol $a \in \mathcal{S}(G)$ using the previous formula and the fact that $L^2(G) \otimes (\mathcal{B}_1^{\oplus}(G) \cap \mathcal{B}_2^{\oplus}(G))$ is a dense subspace of $\mathcal{S}(G)$. Hence Op extends to a unitary map $\mathrm{Op} : \mathcal{S}(G) \to \mathcal{B}_2(L^2(G))$ in a unique way.

2.2 Left and right quantizations

Having in mind the familiar Kohn-Nirenberg quantization for $G = \mathbb{R}^n$ (cf. (0.2) with $\tau = 0$), one notes that for non-abelian groups there are at least two possible generalizations; a left quantization Op_L (the one used so far in this thesis)and a right quantization given by

$$[\mathrm{Op}_R(a)u](x) = \int_G \int_{\widehat{G}} \mathrm{Tr}\left(a(x,\xi) D_\xi^{\frac{1}{2}} \pi_\xi(x^{-1}y)^* \right) u(y)\, d\xi\, dy.$$

Actually, these two quantizations are related in the following sense: let a be a symbol, and consider the symbol defined by

$$\tilde{a}(x,\xi) = \pi_\xi(x)\dot{\mathcal{P}}_2\mathcal{P}_2^{-1}a(x,\xi)\pi_\xi(x)^*.$$

It is an easy exercise to check that $\mathrm{Op}_L(\tilde{a}) = \mathrm{Op}_R(a)$ using symbols of the form $a = f \otimes \widehat{g}$. For unimodular groups the assignment $a \mapsto \tilde{a}$ is isometric, but for general non-unimodular groups this is not the case since the modular function is not bounded.

2.3 Some operators arrising from the calculus

One of the most important families of operators in $L^2(G)$ is the one given by convolution operators. In this section we show how to recover the usual convolution and multiplication operators using pseudo-differential calculus.

For $f, g \in L^2(G)$ define the operators

$$[\mathrm{Mult}_f\, u](x) = f(x)u(x),$$

$$[\mathrm{Conv}_g^L\, u](x) = \int_G g(y) f(y^{-1}x)\, dy.$$

In general Mult_f and Conv_g^L are not bounded operators. In fact, Mult_f is bounded if and only if f is essentially bounded. Similarly, Conv_g^L is bounded if and only if $\operatorname{ess\,sup}_{\xi\in\widehat{G}}\|\widehat{g}(\xi)\| < \infty$ nevertheless, for general non-unimodular groups the composition $\mathrm{Mult}_f\,\mathrm{Conv}_g^L$ extends to a Hilbert-Schmidt operator.

Suppose now that G is unimodular, and let $f, g \in L^2(G)$. Define the symbol a by

$$a(x,\xi) = f(x)\,\widehat{g}(\xi).$$

Using the Plancherel inversion formula one gets that the quantizations give us a very simple way to recover these families of operators. Namely,

$$\mathrm{Op}_L(a) = \mathrm{Mult}_f\,\mathrm{Conv}_g^L, \qquad \mathrm{Op}_R(a)u = \mathrm{Mult}_f\,\mathrm{Conv}_g^R,$$

where Conv_g^R is the operator given by $\mathrm{Conv}_g^R(u) = u * g$.

For non-unimodular groups the picture changes dramatically, the main reason being that the compositions $\mathrm{Mult}_f\,\mathrm{Conv}_g^L$ are no longer Hilbert-Schmidt operators under the assumption that $f, g \in L^2(G)$. In fact one has that

$$\|\mathrm{Mult}_f\,\mathrm{Conv}_g^L\|_{\mathcal{B}_2} = \|\Delta^{-\frac{1}{2}}f\|_2\|\Delta^{\frac{1}{2}}g\|_2.$$

For general non-unimodular groups $\mathrm{Mult}_f\,\mathrm{Conv}_g$ is not even a bounded operator if f and g are not chosen in a suitable manner.

One way to fix this is to take functions in an appropriate dense subspace. Chose $f, g \in L^2(G)$ such that the functions $\Delta^{-\frac{1}{2}}f, \Delta^{\frac{1}{2}}g$ are square integrable, and set

$$a(x,\xi) = \Delta^{-\frac{1}{2}}(x)f(x)\widehat{[\Delta^{\frac{1}{2}}g]}(\xi).$$

Then for $u \in L^2(G)$ we have that $\mathrm{Op}(a)u = f \cdot (g * u)$ for every $u \in L^2(G)$. Indeed,

$$\begin{aligned}[\mathrm{Op}(a)u](x) &= \int_G \Delta(x)^{-\frac{1}{2}} f(x) \Delta(xy^{-1})^{\frac{1}{2}} g(xy^{-1}) \Delta(y)^{-\frac{1}{2}} u(y)\, dy \\ &= f(x) \int_G \Delta(y)^{-1} g(xy^{-1}) u(y)\, dy \\ &= f(x)(g * u)(x).\end{aligned}$$

Another way to express the relation between symbols of the form $a = f \otimes \widehat{g}$ and operators of multiplication and convolution is given in the formulas

$$\begin{aligned}\mathrm{Op}_L(f \otimes \widehat{g}) &= \mathrm{Mult}_f \, \mathrm{Conv}_g^L \, \mathrm{Mult}_{\Delta^{1/2}} && (2.1)\\ &= \mathrm{Mult}_{\Delta^{1/2} f} \, \mathrm{Conv}^L_{\Delta^{-1/2} g}, \\ \mathrm{Op}_R(f \otimes \widehat{g}) &= \mathrm{Mult}_f \, \mathrm{Conv}^R_{g^\sharp}, && (2.2)\end{aligned}$$

here $g^\sharp(x) = g(x^{-1})$. We also note that the left regular representation of G induces a representation acting on $\mathcal{S}(G)$: let a be a symbol and $y \in G$, then $\lambda_y \, \mathrm{Op}(a)$ is a Hilbert-Schmidt operator. Hence by Proposition 2.3 below there is some symbol $y.a$ such that

$$\lambda_y \, \mathrm{Op}(a) = \mathrm{Op}(y.a).$$

It is easy to see that this defines an action of G and that

$$y.a(x, \xi) = \pi_\xi(y) a(y^{-1}x, \xi).$$

2.3.1 Other convolution operators that appear in the literature

In [5, §1.2] the author introduces a family of convolution (to the right) operators given by

$$\check{\mathrm{Conv}}^{R}_{g} u(x) = \int_G u(xy)\Delta(y)^{\frac{1}{2}} g(y)\, dy = (u * \bar{g}^*)(x).$$

These operators are then used to study the space of left-invariant operators. If we want to follow this path there are two ways to study these operators. One consists of defining

$$h(x) = \Delta(x)^{-1} g(x^{-1}).$$

Note that h is absolutely integrable if and only if g is. Then, formally one has

$$\mathrm{Op}_R(f \otimes \widehat{h}) = \mathrm{Mult}_{\Delta^{-\frac{1}{2}} f} \check{\mathrm{Conv}}^{R}_{g}.$$

The other possibility is to put

$$h(x) = \Delta(x)^{-\frac{1}{2}} g(x^{-1}) = \bar{g}^*(x).$$

In this case, by the definition of h and equation (2.2) one gets

$$\mathrm{Op}_R(f \otimes \widehat{h}) = \mathrm{Mult}_f \check{\mathrm{Conv}}^{R}_{g} \mathrm{Mult}_{\Delta^{-1/2}}.$$

2.4 Quantization by a Weyl system

In this section we introduce the notion of a Weyl system for a general locally compact group. This is then used to define pesudo-differential operators through τ-quantization for an arbitrary measurable function $\tau : G \to G$. The goal in this section is to make sense of formulas (0.1) and (0.3) in a rigorous way and to clarify the role of the required hypothesis. Throughout this section we fix a Plancherel measure ν and a measurable field $(\pi_\xi, D_\xi)_{\xi\in\widehat{G}}$ as in Theorem 1.3.

We start by defining a family of integral kernels that will turn out to be very useful for the rest of this section.

Definition 2.1. Given two square integrable functions u, v, we define the **Weyl kernel** associated to u, v by

$$K_{u,v}(x,y) = \Delta(y^{-1}x)^{\frac{1}{2}}\overline{u(y^{-1}x)}v(x) = v(x)u^*(x^{-1}y). \tag{2.3}$$

We also introduce the τ-**Weyl kernels** associated to the pair (u, v) as

$$K^\tau_{u,v}(x,y) = K_{u,v}(\tau(y^{-1})^{-1}x, y), \tag{2.4}$$

which just amounts to a left translation in the first variable of K.

Remark 2.1. Fubini's theorem yields

$$\begin{aligned}\int_G\int_G |K^\tau_{u,v}(x,y)|^2\,dx\,dy &= \int_G\int_G |K_{u,v}(x,y)|^2\,dx\,dy \\ &= \int_G\int_G |v(x)u^*(x^{-1}y)|^2\,dx\,dy \\ &= \|u\|_2^2\|v\|_2^2.\end{aligned}$$

Hence $K^\tau_{u,v}$ is square integrable. On the other hand if we identify L^2-functions of two variables with tensors of the form $u\otimes v \in L^2(G)^\dagger \otimes L^2(G)$ under the identification $(u\otimes v)(x,y) = u^*(y)v(x)$, then the adjoint of K^τ may be identified with the operator

$$[(K^\tau)^*S](x,y) = S(\tau((xy)^{-1})x, xy)$$

acting on $L^2(G\times G)$. This operator is injective, hence K^τ defines a unitary isomorphism

$$K^\tau : L^2(G)^\dagger \otimes L^2(G) \to L^2(G\times G),$$

that we will denote by the same letter.

Remark 2.2. Note that u^* being integrable is equivalent to $\Delta^{-\frac{1}{2}}u$ being integrable, moreover $\|\Delta^{-\frac{1}{2}}u\|_1 = \|u^*\|_1$. By equation (2.3), $\int_G K^\tau_{u,v}(x,y)\,dx = (v * u^*)(y)$ so, if v, u^* are integrable, then

$$\begin{aligned}\int_G\int_G |K^\tau_{u,v}(x,y)|\,dx\,dy &= \int_G\int_G \Delta(y)^{-\frac{1}{2}}|u(y^{-1})v(x)|\,dy\,dx\\ &= \int_G\int_G \Delta(y)^{-\frac{1}{2}}|u(y)v(x)|\,dy\,dx\\ &= \|\Delta^{-\frac{1}{2}}u\|_1\,\|v\|_1.\end{aligned}$$

Also note that for an arbitrary $y \in G$, by Hölder's inequality

$$\int_G |K^\tau_{u,v}(x,y)|\,dx \leq \|\Delta^{\frac{1}{2}}u\|_2\|v\|_2.$$

By the previous remarks we see that if u, v belong to an appropriate dense subspace, like $C_c(G)$ for example, then it makes sense to take the Plancherel transform of $K^\tau_{u,v}$ in both its first and second variables.

Definition 2.2. Let $\tau : G \to G$ be given a measurable function, $x \in G$ and a representation π_ξ in the class of $\xi \in \widehat{G}$. For nice enough $\Theta \in L^2(G;\mathcal{H}_\xi)$, in a sense we specify below, define the **τ-Weyl System** by

$$[W^\tau(\pi_\xi, y)\Theta](x) = \Delta(y^{-1}x)^{\frac{1}{2}}\pi_\xi(\tau(y^{-1})x)D_\xi^{\frac{1}{2}}(\Theta(y^{-1}x)).$$

We drop the index τ in the notation when the choice $\tau(\cdot) = e$ is made. It is important to note that these operators are only defined for elements in

$$\{\Theta \in L^2(G;\mathcal{H}_\xi) \mid \Theta(x) \in \mathrm{Dom}(D_\xi^{\frac{1}{2}})\ \mu\text{-a.e. and } \Delta^{\frac{1}{2}}D_\xi^{\frac{1}{2}}\Theta \in L^2(G;\mathcal{H}_\xi)\}.$$

The domain of $W^\tau(\pi_\xi, y)$ contains the space of vectors of the form $\eta \otimes u$, where $\eta \in \mathrm{Dom}(D_\xi^{\frac{1}{2}})$ and $u \in \mathrm{Dom}(\mathrm{Mult}_{\Delta^{1/2}})$. Here Mult_f denotes the operator of multiplication by f, so in particular $W^\tau(\pi_\xi, y)$ is a densely defined operator.

Proposition 2.1. *Let ν be a fixed Plancherel measure and let $(\pi_\xi, D_\xi^{\frac{1}{2}})_{\xi\in\widehat{G}}$ be a measurable field as in Theorem 1.3. The operators $(W^\tau(\pi_\xi, y))_{(\xi,x)\in\widehat{G}\times G}$ form a measurable field of densely defined closed operators on $\int_{\widehat{G}}^{\oplus} \mathcal{H}_\xi \, d\xi \otimes L^2(G)$. If π_ξ and π'_ξ are unitarily equivalent representations with intertwining operator U, then*

$$W^\tau(\pi_\xi{}', y) = (U \otimes \mathrm{Id}_{\mathcal{H}_\xi})\, W^\tau(\pi_\xi, y)\, (U^* \otimes \mathrm{Id}_{\mathcal{H}_\xi}).$$

Because of this, once we fix a measurable field of representations and Duflo-Moore operators, we will just write $W^\tau(\xi, y)$ instead of $W^\tau(\pi_\xi, y)$.

Proof. Let $(\eta_\xi)_{\xi\in\widehat{G}}$ be a measurable section of $\int_{\widehat{G}}^{\oplus} \mathcal{H}_\xi \, d\xi$ such that $\eta_\xi \in \mathrm{Dom}(D_\xi^{\frac{1}{2}})$ for ν-almost everywhere, and let u be a square integrable function such that $\Delta^{1/2}u$ is also square integrable, then

$$W^\tau(\pi_\xi, y)(\eta_\xi \otimes u) = \pi_\xi(\tau(y^{-1}) \cdot\,) D_\xi^{\frac{1}{2}} \eta_\xi \otimes [\lambda_y \, \mathrm{Mult}_{\Delta^{\frac{1}{2}}} \, u], \tag{2.5}$$

is clearly a measurable section of $\mathcal{H}_\xi \otimes L^2(G) \cong L^2(G; \mathcal{H}_\xi)$. In particular $\eta_\xi \otimes u \in \mathrm{Dom}(W^\tau(\pi_\xi, y))$. Let $\{(\eta_\xi^i)_{\xi\in\widehat{G}}\}_{i\in\mathbb{N}}$ be a total subset of $\int_{\widehat{G}}^{\oplus} \mathcal{H}_\xi \, d\xi$ such that ν-a.e. each η_ξ^i is in the domain of $D_\xi^{\frac{1}{2}}$, and let $\{u^j\}_{j\in\mathbb{N}} \subseteq C_c(G)$ be a total subset of $L^2(G)$. Then $\{(\eta_\xi^i \otimes u^j)_{\xi\in\widehat{G}}\}_{i,j\in\mathbb{N}}$ is a total subset of $\int_{\widehat{G}}^{\oplus} \mathcal{H}_\xi \, d\xi \otimes L^2(G)$ contained in the domain of $(W^\tau(\pi_\xi, y))_{(\xi,x)\in\widehat{G}\times G}$; and the map

$$(\xi, y) \mapsto \langle W^\tau(\pi_\xi, y)(\eta_\xi^i \otimes u^j), (\eta_\xi^{i'} \otimes u^{j'}) \rangle$$

is measurable for each $i, j, i', j' \in \mathbb{N}$. Hence the Weyl system forms a measurable field of densely defined operators. By equation (2.5), one sees that it is a composition of closed operators, hence closed for each pair $(\xi, y) \in \widehat{G} \times G$. Let U be a unitary operator such that .

$$\pi_\xi{}'(x) = U \pi_\xi(x) U^*$$

for each $x \in G$. Let $D'_\xi = UD_\xi U^*$. By the semi-invariance relation and equation (1.12), this is the Duflo-Moore operator associated to π'_ξ. Then one has

$$\begin{aligned} W^\tau(\pi'_\xi, y)(\eta_\xi \otimes u) &= [U\pi_\xi(\tau(y^{-1})\cdot)U^* D'^{\frac{1}{2}}_\xi \eta_\xi] \otimes [\lambda_y \operatorname{Mult}_{\Delta^{\frac{1}{2}}} u] \\ &= [U\pi_\xi(\tau(y^{-1})\cdot)D^{\frac{1}{2}}_\xi U^* \eta_\xi] \otimes [\lambda_y \operatorname{Mult}_{\Delta^{\frac{1}{2}}} u] \\ &= (U \otimes \mathrm{Id}_{\mathcal{H}_\xi})\, W^\tau(\pi_\xi, y)(U^*\eta_\xi \otimes u) \end{aligned}$$

and therefore the required relation

$$W^\tau({\pi_\xi}', x) = (U \otimes \mathrm{Id}_{\mathcal{H}_\xi})\, W^\tau(\pi_\xi, x)\, (U^* \otimes \mathrm{Id}_{\mathcal{H}_\xi}).$$

□

Given two square integrable functions u, v such that $\Delta^{\frac{1}{2}} u \in L^2(G)$, and a given vector $\phi \in \mathrm{Dom}(D_\xi^{1/2})$, define the operator $\mathcal{W}^\tau_{u,v} : L^2(G) \to L^2(G)$ by

$$\mathcal{W}^\tau_{u,v}(\xi, y)\phi = \int_G (W^\tau(\xi, y)\, \bar{u} \otimes \phi)\, (x) v(x)\, dx.$$

We defined this operator on a dense subspace, but it can be extended to a Hilbert-Schmidt operator (cf. Proposition 2.2 below). Note that

$$\begin{aligned} \mathcal{W}^\tau_{u,v}(\xi, y)\, \phi &= \pi_\xi(\tau(y^{-1})) \int_G \left(\Delta^{\frac{1}{2}} \bar{u}\right) (y^{-1}x)\, v(x) \pi_\xi(x) D^{\frac{1}{2}}_\xi\, \phi\, dx \\ &= \pi_\xi(\tau(y^{-1}))\, \mathcal{P}_1(K_{u,v})(\xi, y)\, \phi \\ &= \left(\mathcal{P}_1 K^\tau_{u,v}\right)(\xi, y)\, \phi. \end{aligned}$$

Here $\mathcal{P}_1$ denotes the Plancherel transform in the first variable. This short remark leads to the following proposition.

Proposition 2.2. *The assignment* $u \otimes v \mapsto \mathcal{W}^\tau_{u,v}$ *extends to a unique unitary map* $\mathcal{W}^\tau : L^2(G)^\dagger \otimes L^2(G) \to \mathcal{S}(\widehat{G})$ *called the* ***Fourier-Wigner*** τ***-transformation.***

Proof. Given two square integrable functions u, v such that $\Delta^{1/2} u$ is also square integrable, applying Plancherel theorem one gets

$$\begin{aligned}\|\mathcal{W}^{\tau}_{u,v}\|^2_{\mathcal{S}(\widehat{G})} &= \int_G \int_{\widehat{G}} \|\mathcal{W}^{\tau}_{u,v}(\xi, y)\|^2_{\mathcal{B}_2}\, d\xi\, dy \\ &= \int_G \int_{\widehat{G}} \|(\mathcal{P}_1 K^{\tau}_{u,v})(\xi, y)\|^2_{\mathcal{B}_2}\, d\xi\, dy \\ &= \int_G \int_G |K^{\tau}_{u,v}(x, y)|^2\, dx\, dy \\ &= \|u\|_2^2 \|v\|_2^2,\end{aligned}$$

which shows that $\mathcal{W}$ extends to a unitary isomorphism. □

We introduce the **Wigner τ-transformation** of two functions $u, v \in L^2(G)$ as

$$\mathcal{V}^{\tau}_{u,v} = \mathcal{P}_2 \mathcal{P}_1^{-1} \mathcal{W}^{\tau}_{u,v} = \mathcal{P}_2 K^{\tau}_{u,v} \in \mathcal{S}(G).$$

More explicitly,

$$\mathcal{V}^{\tau}_{u,v}(x, \xi) = \int_G \Delta(y^{-1}\tau(y^{-1})^{-1}x)^{\frac{1}{2}} \overline{u(y^{-1}\tau(y^{-1})^{-1}x)}\, v(\tau(y^{-1})^{-1}x)\, \pi_\xi(y) D_\xi^{\frac{1}{2}}\, dy.$$

We record for further use the orthogonality relations, valid for $u, u', v, v' \in L^2(G)$

$$\langle \mathcal{W}_{u,v}, \mathcal{W}_{u',v'} \rangle_{\mathcal{S}(\widehat{G})} = \langle v, v' \rangle \langle u', u \rangle = \langle \mathcal{V}_{u,v}, \mathcal{V}_{u',v'} \rangle_{\mathcal{S}(G)}. \tag{2.6}$$

2.5 Pseudo-differential operators

As before, fix a measurable map $\tau : G \to G$. In the next definition we formalize the τ-quantization $\mathrm{Op}^{\tau}(a)$ introduced in equation (0.1). When τ is the constant function $\tau(x) = e$ we drop the superscript in the notation.

Definition 2.3. Let $a \in \mathcal{S}(G)$ be a symbol with Plancherel transform in both variables $\widehat{a} = \mathcal{P}_1 \mathcal{P}_2^{-1} a \in \mathcal{S}(\widehat{G})$. Define $\mathrm{Op}^{\tau}(a)$ to be the unique bounded linear operator in $L^2(G)$ defined by the relation

$$\langle \mathrm{Op}^\tau(a)u, v\rangle = \langle \widehat{a}, \mathcal{W}^\tau_{u,v}\rangle_{\mathcal{S}(\widehat{G})}.$$

Or equivalently,

$$\langle \mathrm{Op}^\tau(a)u, v\rangle = \langle a, \mathcal{V}^\tau_{u,v}\rangle_{\mathcal{S}(G)}.$$

$\mathrm{Op}^\tau(a)$ is then called the **τ-pseudo-differential operator** with symbol a, while the map $a \mapsto \mathrm{Op}^\tau(a)$ will be called the **τ-pseudo-differential calculus** or **τ-quantization.**

Note that

$$|\langle \mathrm{Op}^\tau(a)u, v\rangle| \leq \|a\|_{\mathcal{S}(G)}\|\mathcal{W}_{u,v}\|_{\mathcal{S}(\widehat{G})} = \|a\|_{\mathcal{S}(G)}\|u\|_2\|v\|_2.$$

So in particular $\|\mathrm{Op}^\tau(a)\| \leq \|a\|_{\mathcal{S}(G)}$. By Theorem 1.3, a different choice of Plancherel measure and tuple $(\pi_\xi, D_\xi)_{\xi\in\widehat{G}}$ gives rise to an equivalent calculus.

We have the following proposition. With only small its proof follows that of [33, Theorem 3.8].

Proposition 2.3. *Let us define*

$$\Lambda_{u,v}(w) = \langle w, u\rangle v, \quad \forall\, w \in L^2(G),$$

the rank-one operator associated to the pair (u, v). Then one has

$$\Lambda_{u,v} = \mathrm{Op}^\tau(\mathcal{V}^\tau_{u,v}), \quad \forall\, u, v \in L^2(G).$$

In particular, the mapping Op^τ sends $\mathcal{S}(G)$ unitarily onto the Hilbert space of all Hilbert-Schmidt operators in $L^2(G)$.

Proof. Note that relation (2.6) gives

$$\begin{aligned}\langle \mathrm{Op}^\tau(\mathcal{V}^\tau_{u,v})u', v'\rangle &= \langle \mathcal{V}^\tau_{u,v}, \mathcal{V}^\tau_{u',v'}\rangle \\ &= \langle v, v'\rangle\langle u', u\rangle \\ &= \langle \Lambda_{u,v}(u'), v'\rangle,\end{aligned}$$

for all square integrable functions. Hence $\Lambda_{u,v} = \mathrm{Op}^\tau(\mathcal{V}^\tau_{u,v})$. Since the rank-one operators are dense in the space of Hilbert-Schmidt operators, the desired conclusion holds. □

After working out the formulas and assuming that u, v belong to an appropriate dense subset, and that for μ-almost all $x \in G$ the operator $a(x,\xi)$ satisfies the hypothesis of Proposition 1.5, one has

$$\begin{aligned}\langle \mathrm{Op}^\tau(a)u, v\rangle &= \langle \mathcal{P}_2^{-1}a, K^\tau_{u,v}\rangle_{L^2(G\times G)} \\ &= \int_G\int_G\int_{\widehat{G}} \mathrm{Tr}\left(a(x,\xi)D_\xi^{\frac{1}{2}}\pi_\xi(y)^*\right)\overline{K^\tau_{u,v}(x,y)}\,d\xi\,dx\,dy.\end{aligned}$$

Thus by making the substitution $y \mapsto yx$ and using (2.4)

$$\begin{aligned}[\mathrm{Op}^\tau(a)u](x) &= \int_G\int_{\widehat{G}} \mathrm{Tr}\Big(a(\tau(y^{-1})x,\xi)D_\xi^{\frac{1}{2}}\pi_\xi(y)^*\Big)\Delta(y^{-1}x)^{\frac{1}{2}}u(y^{-1}x)\,d\xi\,dy \\ &= \int_G\int_{\widehat{G}} \mathrm{Tr}\Big(a(\tau((xy)^{-1})x,\xi)D_\xi^{\frac{1}{2}}\pi_\xi(xy)^*\Big)\Delta(y)^{-\frac{1}{2}}u(y^{-1})\,d\xi\,dy \\ &= \int_G\int_{\widehat{G}} \mathrm{Tr}\Big(a(\tau(yx^{-1})x,\xi)D_\xi^{\frac{1}{2}}\pi_\xi(yx^{-1})\Big)\Delta(y)^{-\frac{1}{2}}u(y)\,d\xi\,dy.\end{aligned}$$

Thus, the kernel of $\mathrm{Op}^\tau(a)$ is the square integrable function

$$\mathrm{ker}^\tau_a(x,y) = \Delta(y)^{-\frac{1}{2}}\int_{\widehat{G}} \mathrm{Tr}\left(a(\tau(yx^{-1})x,\xi)D_\xi^{\frac{1}{2}}\pi_\xi(xy^{-1})^*\right)\,d\xi \tag{2.7}$$

$$= \Delta(y)^{-\frac{1}{2}}[\mathcal{P}_2^{-1}a](\tau(yx^{-1})x, xy^{-1}). \tag{2.8}$$

Formula (2.8) shows in another way that Op^τ is unitary. Indeed by consecutive use of Fubini's theorem,

$$\begin{aligned}\int_G\int_G \left|\ker_a^\tau(x,y)\right|^2\,dx\,dy &= \int_G\int_G \Delta(y)^{-1}\left|\mathcal{P}_2^{-1}a(\tau(xy^{-1})x,xy^{-1})\right|^2\,dx\,dy\\ &= \int_G\int_G \left|\mathcal{P}_2^{-1}a(\tau(y)x,y)\right|^2\,dx\,dy\\ &= \|a\|^2_{\mathcal{S}(G)}.\end{aligned}$$

We summarize the most important properties of Op^τ in the following theorem (cf. Section 2.6 for the definition of an H^*-algebra).

Theorem 2.1. *The τ-quantization $\mathrm{Op}^\tau : \mathcal{S}(G) \to \mathcal{B}_2\left(L^2(G)\right)$ is a unitary isomorphism of Hilbert spaces. Additionally Op^τ has the following properties*

(i) If $\tau(x) = e$, then $\mathrm{Op}^\tau(f \otimes \widehat{g}) = \mathrm{Mult}_f\,\mathrm{Conv}_g\,\mathrm{Mult}_{\Delta^{1/2}}$.

(ii) The integral kernel of $\mathrm{Op}^\tau(a)$ is given by

$$\ker_a(x,y) = \Delta(y)^{-\frac{1}{2}}[\mathcal{P}_2^{-1}a](\tau(yx^{-1})x, xy^{-1}).$$

Remark 2.3. Suppose G is unimodular. If $\tau(x) = e$ for all $x \in G$, then formula (2.7) reads

$$\ker_a(x,y) = \int_{\widehat{G}} d_\xi^{1/2}\,\mathrm{Tr}\left(a(x,\xi)\pi_\xi(xy^{-1})^*\right)\,d\xi = [\mathcal{P}_2^{-1}a](x,xy^{-1}).$$

Remark 2.4. When $G = \mathbb{R}^n$, under the identification of $\widehat{G}$ with $\mathbb{R}^n$ defined by $\xi(x) = e^{-2\pi i\langle\xi,x\rangle}$, with Haar measure as the Plancherel measure, and setting $D_\xi = \mathrm{Id}_{L^2(\mathbb{R})}$, we recover the Kohn-Nirenberg calculus (cf. eq. (0.2)).

2.5.1 Relations between different τ-quantizations

The choice of measurable function $\tau : G \to G$ has to do with ordering issues in the quantization arising from the non-commutativity of the operators involved. One may ask for is the exact relation between the quantizations given by Op and Op^τ. Let $a \in \mathcal{S}(G)$ be a symbol defined on the group, and consider the unitary map $\Omega^\tau : L^2(G \times G) \to L^2(G \times G)$ given by

$$\Omega^\tau S(x, y) = S(\tau(y)^{-1}x, y).$$

Then if we consider the symbol

$$a^\tau = \mathcal{P}_2\,(\Omega^\tau)^*\,\mathcal{P}_2^{-1}a,$$

after successive uses of the changes of variables shown in equation (1.1) we arrive at the relation between the quantizations

$$\mathrm{Op}(a^\tau) = \mathrm{Op}^\tau(a).$$

This relation shows how to pass from the quantization where $\tau(x) = e$ to an arbitrary τ-quantization. Note that $a \mapsto a^\tau$ is a unitary isomorphism.

Example 2.1. Consider $\tau(x) = x$. Then if $a = f \otimes \widehat{g}$ we have

$$\mathrm{Op}^\tau(a) = \mathrm{Conv}^L_g\,\mathrm{Mult}_f\,\mathrm{Mult}_{\Delta^{1/2}}\,.$$

With formula (2.1) in mind one sees that this choice of τ changes the order in which the operators of multiplication and convolution are composed.

Example 2.2. We implement a right τ-quantization via the formula

$$[\mathrm{Op}^\tau_R(a)u](x) = \int_G\int_{\widehat{G}} \mathrm{Tr}\left(a(\tau(yx^{-1})x, \xi)D_\xi^{\frac{1}{2}}\pi_\xi(x^{-1}y)^*\right)u(y)\,d\xi\,dy.$$

As in the previous example, if we take $\tau(x) = x$ the operator with symbol $a = f \otimes \widehat{g}$ is

$$\mathrm{Op}^\tau_R(a) = \mathrm{Conv}^R_{g^\natural}\,\mathrm{Mult}_f\,.$$

2.6 Involutive algebras of symbols

The fact that Op^{τ} is an isomorphisms allows us to define a product $*_{\tau}$, which we will call the **Moyal product**, and an involution ${}^{*_{\tau}}$ on $\mathcal{S}(G)$ by the formula

$$\mathrm{Op}^{\tau}(a *_{\tau} b) = \mathrm{Op}^{\tau}(a)\,\mathrm{Op}^{\tau}(b),$$
$$\mathrm{Op}^{\tau}(a^{*_{\tau}}) = \mathrm{Op}^{\tau}(a)^{*}.$$

With this extra structure $\mathcal{S}(G)$ becomes an H^***-algebra** [7, Appendix A], i.e. a complete Hilbert algebra, which we will denote by $H^*(G)$. Being an H^*-algebra means that the following relations hold,

$$\langle a *_{\tau} b, c\rangle_{H^*(G)} = \langle b, a^{*_{\tau}} *_{\tau} c\rangle_{H^*(G)},$$
$$\langle a, b\rangle_{H^*(G)} = \langle b^{*_{\tau}}, a^{*_{\tau}}\rangle_{H^*(G)},$$

for all $a, b, c \in H^*(G)$. These relations follow from Proposition 2.3 and the fact that the Hilbert-Schmidt operators are a H^*-algebra with the usual composition and involution.

Chapter 3

Some concrete examples of non-unimodular groups

In this chapter we work out in detail the representation theory of the affine group and of Grélaud's group, two simple examples of non-unimodular groups. Then we show how quantization works in theses examples.

3.1 A pseudo-differential calculus for the affine group

In this section we develop a pseudo-differential calculus on the affine group of the real line. The theory of unitary representations of the affine group is worked out in [17] or in [12, §6.7]. In this section

$$G = \{(b, a) \in \mathbb{R}^2 \mid a \neq 0\},$$

denotes the Affine group, with product law

$$(b, a) \cdot (b', a') = (ab' + b, aa').$$

The group G is a Lie group and the connected component of the identity is a simply connected Lie group; it is also the semi-direct product $\mathbb{R} \rtimes \mathbb{R}^{\times}$, where $\mathbb{R}^{\times} = \mathbb{R} \setminus \{0\}$ is the multiplicative group of $\mathbb{R}$ acting on $\mathbb{R}$ by multiplication. Since it is also a real algebraic group, it is type I.

Let $\mathfrak{g} = \mathbb{R}^2$ be the Lie algebra of G with bracket defined by

$$[(\beta, \alpha), (\beta', \alpha')] = (\alpha\beta' - \alpha'\beta, 0).$$

The left Haar measure is $|a|^{-2}\, da\, db$, and its right Haar measure is given by $|a|^{-1} da\, db$. Hence the modular function is

$$\Delta(b, a) = |a|^{-1}.$$

The Haar measures of G are a product of a continuous function on G and the Lebesgue measure of $\mathbb{R}^2$. Hence they are strongly equivalent measures. The connected component of the affine group is the only connected simply connected (non-unimodular) Lie group of dimension 2 with Lie algebra $\mathfrak{g}$.

One important fact is that the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism of $\mathfrak{g}$ onto the connected component of the identity. The exponential and logarithm maps are given by

$$\begin{aligned} \exp(\beta, \alpha) &= \left(\frac{\beta}{\alpha}(e^{\alpha} - 1), e^{\alpha}\right) \\ \log(b, a) &= \left(\frac{b}{a-1}\log(a), \log(a)\right), \end{aligned}$$

with the limit case being used if $a = 1$.

3.1.1 Representation theory of the Affine group

One of the special properties of G is that its unitary dual consists of a point with positive Plancherel measure equal to 1 and a ν-null set of one-dimensional representations (cf. [12, §6.7] where the Mackey machine is used to compute the unitary dual).

Up to a set of zero measure, $\widehat{G}$ consist of a single representation π called the **quasi-regular representation.** It acts on $\mathcal{H}_\xi = L^2(\mathbb{R})$ via

$$\pi(b,a)f(x) = |a|^{1/2}e^{-2\pi ixb}f(ax).$$

We denote the equivalence class of π in $\widehat{G}$ by the Greek letter ξ. This is a square-integrable irreducible representation since it has positive Plancherel measure.

The Duflo-Moore operator corresponding to the representation ξ is given, on its natural domain, by

$$D_\xi f(x) = |x|f(x).$$

An explicit calculation of the matrix coefficient functions for $f, g \in L^2(\mathbb{R})$ shows that

$$C_{f,g}(b,a) = |a|^{1/2}\int_{\mathbb{R}} f(x)\overline{g(ax)}e^{2\pi ibx}\,dx.$$

Let $u_a(x) = f(x)\overline{g(ax)}$. Then

$$\begin{aligned} \|C_{f,g}\|_2^2 &= \int_G |\widehat{u_a}(-b)|^2 \frac{db\,da}{|a|} \\ &= \int_G |f(b)\overline{g(ab)}|^2 \frac{db\,da}{|a|} \\ &= \|f\|_2^2 \|D_\pi^{-\frac{1}{2}} g\|_2^2, \end{aligned}$$

This is another way of seeing that π is a square-integrable representation. The

Plancherel transform is given by

$$\begin{aligned}[\widehat{f}(\xi)g](x) &= |x|^{1/2}\int_G f(b,a)g(ax)e^{2\pi ibx}\frac{db\,da}{|a|}\\ &= |x|^{1/2}\int_G \frac{1}{|a|}\mathcal{F}_1 f(x,a)g(ax)\,da\\ &= \int_G \frac{|x|^{\frac{1}{2}}}{|a|}\mathcal{F}_1 f\left(x,\frac{a}{x}\right)g(a)\,da.\end{aligned}$$

This is a Hilbert-Schmidt operator with integral kernel

$$K_f(x,a) = \frac{|x|^{\frac{1}{2}}}{|a|}\mathcal{F}_1 f\left(x,\frac{a}{x}\right), \tag{3.1}$$

where $\mathcal{F}_1$ denotes the usual Fourier transform on the real line acting in the first variable. A calculation in [12, §6.7] shows that the Plancherel formula implies

$$\begin{aligned}\|K_f\|^2_{L^2(\mathbb{R}^2)} &= \iint_{\mathbb{R}^2}\frac{|x|}{|a|^2}\left|\mathcal{F}_1 f\left(x,\frac{a}{x}\right)\right|^2\,dx\,da\\ &= \iint_{\mathbb{R}^2}|\mathcal{F}_1 f(x,a)|^2\frac{dx\,da}{|a|^2}\\ &= \|f\|^2_{L^2(G)}.\end{aligned}$$

So, as required by the general theory, the Plancherel transform implements a unitary isomorphism

$$\mathcal{P} : L^2(G) \to \mathcal{B}_2(L^2(\mathbb{R})).$$

Note that the center of the right-hand side von Neumann algebra is $\mathbb{C}\cdot \mathrm{Id}_{L^2(G)}$, so the Plancherel transform implements a central decomposition of $L^2(G)$. In particular, this calculation shows that the support of the Plancherel measure is $\{\xi\}$. There is not enough room in $L^2(G)$ for another representation, hence we identify the unitary dual of G with the singleton $\{\xi\}$.

3.1.2 The quantization for the affine group

Let $\tau(b,a) = (0,1)$ for all $(b,a) \in G$. For a symbol α formula (0.1) reads

$$[\mathrm{Op}(\alpha)u](a,b) = \iint_{\mathbb{R}^2} \frac{1}{|a'|^{3/2}} \mathrm{Tr}\left(\alpha(b,a) D_\xi^{\frac{1}{2}} \pi\left(b - \frac{a}{a'}b', \frac{a}{a'}\right)^*\right) u(b',a')\, db'\, da'.$$

If we identify symbols on G with functions $f \in L^2(G \times G)$ via the unitary map defined by equation (3.1) we may think of symbols as integral operators of the form

$$[\alpha(x,\xi)u](y) = \int_{\mathbb{R}} K_{f(x,\cdot)}(y,s)u(s)\, ds = [\mathcal{P}_2 f(x,\xi)u](y).$$

Using formula (2.8) for $\ker_a$, formula (0.1) boils down to

$$[\mathrm{Op}(\alpha)u](b,a) = \iint_{\mathbb{R}^2} \frac{1}{|a'|^{3/2}} f\left((b,a), \left(b - \frac{a}{a'}b', \frac{a}{a'}\right)\right) u(b',a')\, db'\, da'.$$

In particular the map

$$L^2(G \times G) \ni f \mapsto \ker_{\mathcal{P}_2 f} \in \mathcal{B}_2(L^2(G)),$$

is a unitary equivalence.

3.1.3 Operators that arise from the representation

Let $A = (0,1)$ and $B = (1,0)$ be the generators of $\mathfrak{g}$; they satisfy the commutation relation $[A,B] = B$. Then if $d\pi(X)u = \frac{d}{dt}\pi(e^{tX})u \mid_{t=0}$, denotes the (densely defined) induced representation of $\mathfrak{g}$ on the Hilbert space $L^2(\mathbb{R})$ we have that

$$[d\pi(A)f](x) = \frac{1}{2}f(x) + xf'(x),$$

$$[d\pi(B)f](x) = 2\pi i x f(x).$$

Clearly $[d\pi(A), d\pi(B)] = d\pi(B)$. Note also that

$$d\pi(A) = \frac{1}{2}\left(x \cdot \frac{d}{dx} + \frac{d}{dx} \cdot x\right),$$

is the infinitesimal generator of dilations of $\mathbb{R}$, a well-studied operator on $\mathbb{R}$.

3.2 Grélaud's group

Grélaud's group is one of the non-unimodular Lie groups that arises from Bianchi's classification of 3-dimensional Lie algebras [3]. Let $\theta \in \mathbb{R} \setminus \{0\}$ and let

$$A = A_\theta = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}.$$

We endow $G_\theta = \mathbb{R} \times \mathbb{R}^2$ with the multiplication law

$$(s,u)\cdot(s',u') = (s+s', e^{-s'A}u + u').$$

Here

$$e^{tA} = e^t \begin{pmatrix} \cos(t\theta) & \sin(t\theta) \\ -\sin(t\theta) & \cos(t\theta) \end{pmatrix}.$$

G_θ is a semi-direct product $\mathbb{R} \rtimes \mathbb{R}^2$ with unit $e = (0,0)$ and inverse given by

$$(s,u)^{-1} = (-s, -e^{sA}u).$$

The left Haar measure coincides with the Lebesgue measure on $\mathbb{R}^3$, but the group is not unimodular. Indeed, the modular function is given by

$$\Delta(s,u) = e^{-2s}.$$

The Lie algebra $\mathfrak{g}_\theta$ of G_θ is, as a vector space, $\mathbb{R} \times \mathbb{R}^2$ with the Lie bracket

$$[(\sigma,\mu),(\sigma',\mu')] = (0, \sigma A\mu' - \sigma' A\mu).$$

Since the commutator $[\mathfrak{g}_\theta, \mathfrak{g}_\theta]$ is contained in $\{0\} \times \mathbb{R}^2$, a commutative subalgebra, then $\mathfrak{g}_\theta$ is a two-step solvable Lie algebra. Its exponential map $\exp : \mathfrak{g}_\theta \to G_\theta$ is given by

$$\exp(\sigma, \mu) = \left(\sigma, -\frac{1}{\sigma} A^{-1} \left(e^{-\sigma A} - \mathrm{Id}\right) \mu\right).$$

The exponential map is clearly a diffeomorphim. In particular G_θ is an exponentially solvable Lie group, connected and simply connected as a topological space. Recall that exponentially solvable groups are type I. Most of the representation theory of Grélaud's group is worked out in detail in [15, §4.4], but we calculate it in Section 3.2.2 to show how Theorem 1.1 works.

3.2.1 Representation theory of G_θ

There are two families of representations of G_θ. Let $\lambda \in \mathbb{R}$, then

$$\chi_\lambda(s, u) = e^{i\lambda s}$$

is a one-dimensional unitary representation. Let $p \in S^1$ be a unit vector in the plane. Then we have the unitary representation on $\mathcal{H}_p = L^2(\mathbb{R})$

$$\pi_p(s, u) f(t) = e^{-i\langle p, e^{(s-t)A} u\rangle} f(t - s).$$

The following proposition is proven in [15, §4.4]. We also give a proof of this proposition in Section 3.2.2 using Proposition 1.1.

Proposition 3.1. *Every irreducible infinite-dimensional unitary representation of G_θ is unitarily equivalent to π_p for some $p \in S^1$. Moreover the set of classes of these representations is a ν-conull set on $\widehat{G}_\theta$, and the Plancherel measure coincides with a multiple of the Haar measure on the circle.*

From now on we identify the Plancherel measure of $\widehat{G_\theta}$ with the Haar measure on the unit circle having total measure $\nu(S^1) = 1$. We also identify π_p with $p \in S^1$.

The map $\mathcal{F}_2$ denotes the usual Fourier transform on $\mathbb{R}^2$ on the second variable.

The Duflo-Moore operators, applied to a function f lying on a dense subset of $L^2(\mathbb{R})$, are given by

$$D_p f(t) = \frac{1}{2\pi} e^{-2t} f(t).$$

With this in mind, we find that the Plancherel transformation on G_θ is given by

$$\begin{aligned}
[\widehat{f}(p)g](t) &= \frac{1}{\sqrt{2\pi}} \iint_{\mathbb{R}\times\mathbb{R}^2} f(t-s,u) e^{-i\langle e^{-sA^t}p,u\rangle} e^{-s} g(s)\, ds\, du \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}_2 f\left(t-s, \frac{1}{2\pi} e^{-sA^t} p\right) e^{-s} g(s)\, ds,
\end{aligned}$$

where A^t denotes the transpose of the matrix A. Let $p(\varphi) = (\cos(\varphi), \sin(\varphi))$. Then Plancherel's formula reads

$$\begin{aligned}
\int_{S^1} \|\widehat{f}(p)\|_{\mathcal{B}_2}^2\, dp &= \frac{1}{2\pi} \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \mathcal{F}_2 f\left(t-s, \frac{1}{2\pi} e^{-sA^t} p\right)\right|^2 e^{-2s}\, dt\, ds\, dp \\
&= \frac{1}{4\pi^2} \int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \mathcal{F}_2 f\left(t, \frac{1}{2\pi} e^{-sA^t} p(\varphi)\right)\right|^2 e^{-2s}\, dt\, ds\, d\varphi \\
&= \iint_{\mathbb{R}\times\mathbb{R}^2} |\mathcal{F}_2 f(t,u)|^2\, dt\, du \\
&= \|f\|_2^2,
\end{aligned}$$

where the second equality comes from the fact that the Jacobian of the transformation

$$u(s,\varphi) = \frac{1}{2\pi} e^{-sA^t} \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix},$$

is $e^{-2s}/4\pi^2$, i.e. $e^{-2s} ds\, d\varphi = 4\pi^2 du$.

3.2.2 Computing the unitary dual

Here we compute the unitary dual of $G = G_\theta$ using the Mackey machine (Theorem 1.1). Let $N = \{0\} \times \mathbb{R}^2$ and let $H = G/N \cong \mathbb{R}$. Note that N is a type I normal closed subgroup of G_θ since

$$(s,u)\cdot(0,v)\cdot(s,u)^{-1}=(0,e^{sA}v).$$

We identify an element $\mu\in\mathbb{R}^2$ with a representation $\sigma_\mu\in\widehat{N}$ via

$$\sigma_\mu(0,u)=e^{-i\langle\mu,u\rangle}.$$

After the above identification, the dual action of G on $\mathbb{R}^2$ is given by

$$(s,u).\mu=e^{sA^t}\mu.$$

Since $\|e^{sA^t}\mu\|=e^s\|\mu\|$, one can see that the orbits are spirals towards the origin, and the origin itself. Hence $\widehat{N}/H$ may be identified with the unit circle S^1 and a point $\{0\}$, which is a countably separated Borel space. Let $\mu\in S^1$. Note that the stabilizer in G of μ is N. Let $\pi_\mu=\mathrm{Ind}_N^G(\sigma_\mu)$ be the induced representation acting on the space $L^2(G,N,\sigma_\mu,\mu_{\mathbb{R}})$, where $\mu_{\mathbb{R}}$ is the Lebesgue measure on $\mathbb{R}$. Thus, $f\in L^2(G,N,\sigma_\mu,\mu_{\mathbb{R}})$ satisfies

$$f((s,u)\cdot(0,v))=\sigma_\mu(0,v)^*f(x)\quad\text{a.e and}\quad\int_{\mathbb{R}}|f(s,0)|^2\,ds<\infty.$$

Note that the Hilbert space on which π_μ acts is unitarily isomorphic to $L^2(\mathbb{R})$ with the following unitary isomorphism: identify a function $f\in L^2(\mathbb{R})$ with the function $f_0\in L^2(G,N,\sigma_\mu,\mu_{\mathbb{R}})$ given by

$$f_0(s,u)=e^{i\langle\mu,u\rangle}f(s).$$

Abusing notation, π_μ is given on $L^2(\mathbb{R})$ by

$$\pi_\mu(s,u)f(t)=e^{-i\left\langle\mu,e^{(s-t)A}u\right\rangle}f(t-s).$$

Having in mind that the modular function coincides with the multiplication by Δ, one also has that the Duflo-Moore operators are, up to normalization, given by

$$D_\mu f(t) = e^{-2t} f(t).$$

If we start with the orbit of $\mu = 0$, then the stabilizer of this point is all of the group, and the trivial representation extends to the whole group. Hence the representations we are looking at are of the form $\chi_\lambda = \mathrm{Ind}_G^G(1 \times \sigma_\lambda)$, where $\sigma_\lambda((s,u)N) = e^{-i\lambda s}$ is a unitary representation of $H_\mu \cong \mathbb{R}$ for some $\lambda \in \mathbb{R}$. It is easy to see that $\mathrm{Ind}_G^G(\sigma_\lambda)$ is unitarily equivalent to the one-dimensional representation

$$\chi_\lambda(s,u) = e^{-i\lambda s}.$$

Now we see that

$$\widehat{G_\theta} = \{\pi_\mu \mid \mu \in S^1\} \cup \{\chi_\lambda \mid \lambda \in \mathbb{R}\} \cong S^1 \cup \mathbb{R}.$$

To use Proposition 1.4 we chose $\Omega = \mathbb{R}^2 \setminus \{0\}$ as our G-invariant ν_N-conull subset of $\widehat{N}$, then we have the following ν-conull subset of $\widehat{G}$

$$\bigcup_{\mathcal{O}_\sigma \in \Omega/H} \{\mathrm{Ind}_N^G(\sigma \times \rho) \mid \rho \in \widehat{\{0\}}\} = \{\pi_\mu \mid \mu \in S^1\}.$$

Since for each $\sigma \in \Omega$, the dual of the stabilizer $\widehat{H_\sigma}$ consist of only a point, the calculation of the Plancherel measure amounts to find a pesudo-image of the Plancherel measure of $\widehat{N} \cong \mathbb{R}^2$ on $\Omega/H \cong S^1$ (i.e. the Lebesgue measure on $\mathbb{R}^2$). The Haar measure of the circle is a fine choice of pseudo-image. Hence the Plancherel measure of $\widehat{G_\theta}$ is just the Haar measure on the circle.

3.2.3 The quantization

Let $a \in \mathcal{S}(G_\theta)$ be a symbol and let $\tau : G \to G$ be the map defined by $\tau(s, u) = (0, 0)$ for all the elements of the group. Then, the kernel of the pseudo-differential operator with symbol a is given by

$$\ker_a(u, s, u', s') = \int_{S^1} \mathrm{Tr}\left(a(s, u, p) D_p^{\frac{1}{2}} \pi_p \left(s - s', e^{s'A} u - e^{-s'A} u'\right)\right) e^{2s'} \, dp.$$

An the symbol space can be identified with

$$\begin{aligned} \mathcal{S}(G) &= L^2(G) \otimes \int_{\widehat{G}}^{\oplus} \mathcal{B}_2(L^2(\mathbb{R})) \, d\xi \\ &\cong L^2(G) \otimes L^2(\widehat{G}) \otimes \mathcal{B}_2(L^2(\mathbb{R})) \\ &\cong L^2(G \times S^1) \otimes \mathcal{B}_2(L^2(\mathbb{R})) \\ &\cong L^2(\mathbb{R}^5 \times [0, 1]), \end{aligned}$$

the last space endowed with the Lebesgue measure.

Chapter 4

Crossed products of C^*-algebras

We introduce some tools from the theory of crossed products of C^*-algebras, this in turn shall help us to cover the bigger class of compact operators on $L^2(G)$, using the Schrödinger representation of a natural crossed product associated to G, namely $C_0(G) \rtimes G$.

4.1 C^*-dynamical systems

Definition 4.1. A **C^*-dynamical system** is a triplet $(\mathcal{A}, G, \alpha)$, where G is a locally compact group, $\mathcal{A}$ is a C^*-algebra and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is a strongly continuous representation of G.

Let $(\mathcal{A}, G, \alpha)$ be a C^*-dynamical system, to it which we associate the space $L^1(G; \mathcal{A})$ of all Bochner-integrable functions $f : G \to \mathcal{A}$. This space has the structure of a Banach *-algebra with convolution and involution laws given by formulas

$$(f \star g)(x) = \int_G f(y)\alpha_y\Big(g(y^{-1}x)\Big)\, dy,$$
$$f^\star(x) = \Delta(x)^{-1}\alpha_x\Big(f(x^{-1})^*\Big).$$

The Banach *-algebra $L^1(G;\mathcal{A})$ is naturally isomorphic to the projective tensor product $\mathcal{A} \otimes L^1(G)$. Consider the **universal norm** on $L^1(G;\mathcal{A})$ given by

$$\|f\|_{\mathcal{A}\rtimes G} = \sup_\rho \|\rho(f)\|,$$

where the supremum is taken over the set of all non-degenerate *-representations of $\mathcal{A}$. The **crossed product** $\mathcal{A} \rtimes G$ is the enveloping C^*-algebra of $L^1(G;\mathcal{A})$, that is, its completion under the norm $\|\cdot\|_{\mathcal{A}\rtimes G}$.

Example 4.1. Let $\mathcal{A}$ be a C^*-algebra, take G to be the trivial group and α to be the trivial representation, then $\mathcal{A} \rtimes G$ is naturally isomorphic to $\mathcal{A}$. Another more interesting example is when we have a continuous action of G on a topological space X. This induces a map $\alpha : G \to \mathrm{C_0(X)}$ given by $\alpha_x(f)(p) = f(x^{-1}p)$. Then $(C_0(X), G, \alpha)$ is a C^*-dynamical system and it encapsulates all the information of the group action.

Definition 4.2. A **covariant representation** of a C^*-dynamical system $(\mathcal{A}, G, \alpha)$ is composed of: a unitary representation π of G and a non-degenerate *-representation ρ of $\mathcal{A}$, both acting on a Hilbert space $\mathcal{H}$ in such a way that they satisfy the relation

$$\pi(x)\rho(f)\pi(x)^* = \rho(\alpha_x f) \qquad f \in \mathcal{A}, x \in G.$$

We denote this data as the triplet $(\rho, \pi, \mathcal{H})$.

Example 4.2. Let $(C_0(X), G, \alpha)$ be the C^*-dynamical system induced by an action of G on X. Then a covariant representations of $C_0(X) \rtimes G$ is exactly the same as a system of imprimitivity (cf. [31] §3.7). In fact there is a one-to-one correspondence between continuous actions of a group G and C^*-dynamical systems $(\mathcal{A}, G, \alpha)$ where the C^*-algebra $\mathcal{A}$ is an abelian one. This can be easily seen from the fact that every abelian C^*-algebra is of the form $C_0(X)$ for some locally compact space and there is a correspondence between strongly continuous representations $\alpha : G \to \mathrm{Aut}(C_0(X))$ and continuous actions of G on X [45, Proposition 2.7]. In particular, a system of imprimitivity is the same as a covariant representation of a C^*-dynamical system where the C^*-algebra is abelian.

Every covariant representation $(\rho, \pi, \mathcal{H})$ of a C^*-dynamical system naturally induces a non-degenerate *-representation $\rho \rtimes \pi$ of the crossed product $\mathcal{A} \rtimes G$ on $\mathcal{H}$, which is the unique extension of the representation of $L^1(G; \mathcal{A})$ given by the integral

$$\rho \rtimes \pi(f) = \int_G \rho\left(f(y)\right) \pi(y)\, dy. \tag{4.1}$$

This process sets up a bijection between the covariant representations of a C^*-dynamical system and the non-degenerate *-representations of the crossed product associated to it [45, Proposition 2.40].

4.2 The Schrödinger representation

There is a natural covariant representation associated to any left-invariant C^*-algebra of functions defined on G. We show some of its properties and how it relates to our quantization.

Let $\mathcal{A}$ be a left-invariant C^*-subalgebra of the space of bounded left uniformly continuous functions on G. For an $\mathcal{A}$-valued function F on G and elements $x, z \in G$

we make the convenient identification

$$F(x)(z) = F(z, x).$$

The triplet $(\mathcal{A}, G, \alpha)$ is a C^*-dynamical system when endowed with the action $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ given by $\alpha_x F(z, y) = F(z, x^{-1}y)$. Then our convolution and involution laws are given by

$$\begin{aligned}(F \star G)(z, x) &= \int_G F(z, y)\, G(y^{-1}z, y^{-1}x)\, dy,\\ F^\star(z, x) &= \Delta(x)^{-1}\overline{F(x^{-1}z, x^{-1})}.\end{aligned}$$

Let $\mathcal{H}$ denote the space of square integrable functions on G. Then we have a natural covariant representation of the triplet $(\mathcal{A}, G, \alpha)$ given by

$$\lambda_x u(y) = u(x^{-1}y), \qquad \mathrm{Mult}_f\, u(y) = f(y)u(y).$$

The **Schrödinger representation** is the integrated representation $\mathrm{Sch} = \mathrm{Mult} \rtimes \lambda$ of $\mathcal{A} \rtimes G$. More explicitly for a function $F \in L^1(G; \mathcal{A})$,

$$\begin{aligned}[\mathrm{Sch}(F)u](x) &= \int_G F(x, y)u(y^{-1}x)\, dy\\ &= \int_G F(x, xy^{-1})\Delta(y)^{-1}u(y)\, dy.\end{aligned}$$

The good thing about the Schrödinger representation is that, formally, one gets the following relation

$$\mathrm{Op}(f \otimes \widehat{g}) = \mathrm{Sch}(f \otimes g) \circ \mathrm{Mult}_{\Delta^{1/2}}.$$

We can estimate the norm

$$\begin{aligned}\langle \mathrm{Sch}(f)(u), v\rangle &\leq \int_G \left| \int_G f(x,y) u(y^{-1}x) v(x)\, dy \right| dx \\ &\leq \int_G \left| \int_G \|f(y)\|_\infty u(y^{-1}x) v(x)\, dy \right| dx \\ &\leq \int_G |(\|f\|_\infty * u)(x) v(x)|\, dx \\ &\leq \|f\|_{L^1(G;\mathcal{A})} \|u\|_2 \|v\|_2.\end{aligned}$$

But much more is true, from the integrated form formula (4.1) one sees that in fact one has a better estimate of this norm:

$$\|\mathrm{Sch}(f)\| \leq \|f\|_{\mathcal{A}\rtimes G}.$$

4.2.1 Interlude on Amenable groups

Let π_1, π_2 be unitary representations of G. Each one lifts to a non-degenerate *-representation $\tilde{\pi}_1$ and $\tilde{\pi}_2$ of the C^*-algebra $C^*(G)$ of the group. We say that π_1 is **weakly contained** in π_2 if and only if $\ker_{\tilde{\pi}_2} \subseteq \ker_{\tilde{\pi}_1}$.

A locally compact group G is called **amenable** if the trivial representation is weakly contained in the left regular representation.

We recall some of the equivalent definitions for amenability [7].

Proposition 4.1. *The following conditions are all equivalent.*

(i) The group G is amenable.

(ii) There is a bounded linear functional $L^\infty(G) \to \mathbb{R}$ that is positive and left-invariant.

(iii) All of the irreducible representations of G are weakly contained in the left regular

representation.

(iv) The support of any Plancherel measure is all of $\widehat{G}$.

In general a connected Lie group is amenable if and only if it has a closed normal solvable subgroup such that the quotient is compact.

By Proposition 4.1 non-compact semisimple Lie groups are not amenable.

Example 4.3. Some examples of amenable groups are: finite groups, abelian groups and connected solvable Lie groups. An extension of an amenable group by another amenable group is also amenable. Every quotient by a closed normal subgroup and every closed subgroup of an amenable group is amenable.

Example 4.4. Since Grélaud's group is a connected solvable Lie group, it is amenable. Similarly the affine group is an extension of $\mathbb{R}$ by $\mathbb{R}^{\times}$, hence it is also amenable.

Remark 4.1. In [45, §4.4] it is shown that there is epimorphism of C^*-algebras between the space of compact operators in $L^2(G)$ into $C_0(G) \rtimes G$. Part of the result is that this morphism is an isomorphism if and only if the group is amenable. In particular, for each $f \in C_0(G) \rtimes G$, the operator $\mathrm{Sch}(f)$ is a compact operator with operator norm equal to the universal norm of f. This gives us a way to extend our quantization so that we cover the more general case of compact operators. In the general case that G is not amenable, we still have that $\mathrm{Sch} : C_0(G) \rtimes G \to \mathcal{B}_0(L^2(G))$ is an onto contraction.

Chapter 5

Conclusions

There is still much room for more general quantization. For example, one could drop the type I hypothesis since the non-unimodular Plancherel theorem still works partially in this setting [8]. Another, more important aspect to improve is the generality of the symbols involved, and to get an analogue of the Hörmander symbol classes, for at least connected simply connected Lie groups. This has already been done for compact and nilpotent connected Lie groups [10, 37]. This is particularly important since almost all the operators that appear in mathematics and physics are unbounded, and our class only covers the much smaller class of Hilbert-Schmidt operators. And the usual Kohn-Nirenberg covers a big class of unbounded operators which are needed for applications.

Alphabetical Index

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