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Resumen

El principal objetivo de esta tesis es reproducir resultados de compacidad presentados en [DFG02,GI04] en el contexto de [Man]. En particular, damos condiciones para que un subconjunto de un espacio de Hilbert sea compacto en terminos de las propiedades de una familia acotada de operadores acotados; el principal ejemplo es el de la compacidad mágnetica. Finalmente, tomando en consideración [DFG02], presentamos también algunos resultados parciales para espacios de coorbitas y, motivado por [GI04], consideramos la compacidad en espacios de operadores compactos.

Abstract

This aim of this thesis is to reproduce the compactness results presented in [DFG02, GI04] in the setting of [Man12]. In particular, we give conditions for a subset of a Hilbert space to be compact in terms of the properties of bounded families of bounded operators; the main example is the case of magnetic compactness. Finally, taking acount [DFG02] we also present some parcial results considering coorbit spaces and, motivated by [GI04] we also consider compactness in spaces of compact oprators.

Introduction

The main goal of this thesis is twofold: to expose some compactness criteria, first presented in [MP13], that generalize the Kolmogorov Riesz Theorem to abstract Hilbert spaces and Banach spaces defined in terms of generalized continuous frames; secondly, to give a panorama of several topics that I came to learn working on this thesis under the supervision of professor Marius Măntoiu. The main contribution consist in make available results already present in the litterature for the classical Weyl calculus for the new magnetic Weyl calculus. In this introduction we start by presenting the structure of the document and then rapidly fix some notations.

Several classical compactness criteria are presented in Chapter 1; it is also shown how these results, presented first in a pure analytical framework, can be related to group representations. When such a structure is available, our aim is to relate it to both *representations coefficient* and the *integrated form* of such representation. In chapter 2 we will extend this construction to the case where a family of bounded operators indexed by a topological space Σ endowed with a Radon measure is given; we stress that even when the topological space is $\mathbb{R}^n$, there is no need for a relationship between $\pi(x)\pi(y)$ and $\pi(x+y)$. The framework is the one from [Man12], so it needs to be understood as an abstract quantization procedure. In chapter 3 we focus on the important example of the magnetic Weyl calculus. In particular, we generalize the results of [GI04] which where concerned only with the *integrated form* approach. In chapter 4 we study compactness in coorbit spaces, obtained from adding some Banach space of functions, (see [BS88]) to the data already available; this approach extend the one presented in [DFG02]. Finally, in chapter 5, we present the available results concerning compactness in operators spaces and then relate them with the previous chapters.

Conventions and Notations

We begin recalling the basic notions that we shall use throughout this thesis. Because all the spaces that we will study are metric, our interest will focus on finding compactness criteria linked to the *totally boundedness* property. A subset Ω of a metric space is totally bounded if for every ϵ there exists a finite cover of Ω by sets of a diameter of at most ϵ. Such a cover will be called a *ϵ-cover.* It is a known fact that in metric spaces compactness coincides, on closed sets, with total boundedness.

For a topological space Σ, assumed to be Hausdorff, we note by $\mathcal{V}(x)$ the collection of neighborhoods of x. We say that Σ is locally compact when for every $x \in \Sigma$ we can find a compact set in $\mathcal{V}(x)$. When Σ has such a topology, we denote as $C(\Sigma)$ (resp. $C_c(\Sigma)$, resp. $C_0(\Sigma)$, resp. $C_b(\Sigma)$, resp. $C_b^u(\Sigma)$) the set of complex valued continuous functions (resp. continuous functions with compact support, resp. continuous functions that vanish at infinity, resp. bounded continuous functions, resp. bounded and uniformly continuous functions). Unless otherwise stated, we will always consider the sup norm defined by $\| f \|_\infty = \sup_{x\in\Sigma} |f(x)|$. Note that $C_c(\Sigma) \subset C_0(\Sigma) \subset C_b^u(\Sigma) \subset C_b(\Sigma)$ with density in the first inclusion. When $U \subset \Sigma$ we denote the characteristic function over U by χ_U. By $\mathcal{K}(\Sigma)$ we denote the space of a characteristic function over compact sets of Σ. The Lebesgue space $L^2(\Sigma;\mu) \equiv L^2(\Sigma)$ will also be used, with scalar product $\langle u,v\rangle_{L^2(\Sigma)} =: \langle u,v\rangle_{(\Sigma)}$. For a measurable set U we denote $|U| = \int_\Sigma d\mu\chi_U$. We say that a sequence of measurable functions f_n converge in measure to f if for every $\epsilon > 0$ we have $\lim_{n\to\infty} \mu\left(\{x \in D : |f_n(x) - f(x)| \geq \epsilon\}\right) = 0$ for all $D \in \Sigma$ with $\mu(D) < \infty$.

For Banach spaces $\mathcal{X}, \mathcal{Y}$ we set $\mathbb{B}(\mathcal{X},\mathcal{Y})$ for the space of linear continuous operators from $\mathcal{X}$ to $\mathcal{Y}$ and use the abbreviation $\mathbb{B}(\mathcal{X}) := \mathbb{B}(\mathcal{X},\mathcal{X})$. The particular case $\mathcal{X}' := \mathbb{B}(\mathcal{X},\mathbb{C})$ refers to the topological dual of $\mathcal{X}$. By $\mathbb{K}(\mathcal{X},\mathcal{Y})$ we denote the compact operators from $\mathcal{X}$ to $\mathcal{Y}$. We also are going to need the Bochner space $L^1(\Sigma,\mathcal{X})$ composed of (equivalent class of) Bochner integrable functions from a measure space Σ to a Banach space $\mathcal{X}$.

If $\mathcal{H}$ is a complex separable Hilbert space, and $E \subset \mathcal{H}$ we note by $\overline{\mathrm{Sp}}(E)$ for the closure of the linear span of E. We denote by $\overline{\mathcal{H}}$ the conjugate of $\mathcal{H}$; it coincides with $\mathcal{H}$ as an additive group but it is endowed with the scalar multiplication $\alpha \cdot u := \overline{\alpha}u$ and the scalar product $\langle u,v\rangle' := \overline{\langle u,v\rangle}$. If $u,v \in \mathcal{H}$, the rank one operator $\lambda_{u,v} \equiv |v\rangle\langle u|$ is given by $\lambda_{u,v}(w) := \langle w, v\rangle u$. $\mathcal{H} \otimes \overline{\mathcal{H}}$ is the algebraic tensor product of $\mathcal{H}$ and $\overline{\mathcal{H}}$; in that case we denote by $\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}$ the completion in the Hilbert norm. The two-sided *-ideal of all Hilbert-Schmidt operators in $\mathbb{B}(\mathcal{H})$ is denoted by $\mathbb{B}_2(\mathcal{H})$; it is a Hilbert space with the scalar product $\langle S, T\rangle_{\mathbb{B}_2(\mathcal{H})} := \mathrm{Tr}(ST^*)$.

For real functions we write $f_n \uparrow f$ if $f_n \to f$ and $f_n \leq f_{n+1}$. We define analogously $\downarrow$. Given a sequence E_n of subsets of a set U, we say that the limit is $\bigcap\limits_{m=1}^{\infty} \bigcup\limits_{n=m}^{\infty} E_n =: \lim_{n\to\infty} E_n$. We also write

$$E_n \to \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n \, . \qquad (0.1)$$

Then $\uparrow$ (resp. $\downarrow$) is reserved for increasing (resp. decreasing) sequences.

Chapter 1

Classical compactness criteria

In this chapter we introduce some widely known compactness criteria, namely the Arzelà-Ascoli in section 1.1 and the Riesz-Kolmogorov in section 1.2. We introduce also some basic notions of harmonic analysis and representation theory in section 1.3 and in section 1.5, in order to show the basic approaches that motivated this thesis. In section 1.4 we reproduce the result from [GI04] and in section 1.5 the result from [DFG02] in a way that, at least we expect, should motivate our latter approach.

1.1 Compactness in spaces of continous functions

The points of departure of most compactness results in spaces of functions are linked to the classical Arzelà-Ascoli theorem that we shall describe now. This theorem characterizes the compactness of sets of continuous functions over compact sets. The theorem is built upon the notion of *equicontinuity* and *pointwise boundedness*.

Definition 1.1. *Let Σ be a topological space. A subset Ω of $C(\Sigma)$ is said to be:*

i. pointwise bounded *if for every $x \in \Sigma$ there exists $M > 0$, such as $|f(x)| \leq M$ for every $f \in \Omega$;*

ii. equicontinuous *if for every $x \in \Sigma$ and every $\epsilon > 0$ there exists V, a neighborhood of x such that $|f(x) - f(y)| < \epsilon$ holds for every $f \in \Omega$ and every $y \in V$.*

With this definition the theorem reads as follows.

Theorem 1.2 (Arzelà-Ascoli). *Let Σ be a compact set. Then, a subset Ω of $C(\Sigma)$ is totally bounded if and only if the following conditions hold true:*

1. *Ω is pointwise bounded,*

2. *Ω is equicontinuous.*

Proof. Let Ω be a pointwise bounded and equicontinuous subset of $C(\Sigma)$ and let ϵ be greater than 0. We need to find an ϵ-cover of Ω. By compactness of Σ and equicontinuity of Ω, there exists a finite cover $\{V_i\}_{i=1}^N$ of Σ, such as $|f(x) - f(y)| < \epsilon$, whenever x and y belong to the same V_i. We choose a collection $\{x_i\}_{i=1}^N$ such that $x_i \in V_i$ for every $i = 1, ..., N$. We consider the following mapping:

$$\begin{aligned} \Phi :&\Omega \longrightarrow \mathbb{C}^N, \\ &f \longrightarrow (f(x_1), ..., f(x_N)). \end{aligned} \tag{1.1}$$

The pointwise boundedness and the property of each V_i implies that $\Phi(\Omega)$ is bounded, and hence totally bounded. Taking $\{R_i\}_{i=1}^{\tilde{N}}$ a $\frac{\epsilon}{3}$-cover of $\Phi(\Omega)$, we need only to show that $\{\Phi^{-1}(R_i)\}_{i=1}^{\tilde{N}}$ is an ϵ-cover for Ω. That it is a cover follows from the fact that $\{R_i\}$ is a cover. Furthermore, for f and g in $\Phi^{-1}(R_i)$ and $x \in \Sigma$ we have

$$\begin{aligned} |f(x) - g(x)| &\leq |f(x) - f(x_j)| + |f(x_j) - g(x_j)| + |g(x_j) - g(x)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

since $x \in V_j$ for some j and $|f(x_j) - g(x_j)| \leq \| \Phi(f) - \Phi(g) \|_{\mathbb{C}^N}$.

To prove the converse we need first to remark that the total boundedness implies boundedness (i.e. uniform boundedness) which in turn implies pointwise boundedness.

To prove equicontinuity, let us fix $x \in \Omega$ and $\epsilon > 0$. From a $\frac{\epsilon}{3}$-cover $\{U_j\}$ we choose $\{g_j\} \subset \Omega$, with $g_j \in U_j$. Because every g_j is continuous, there is V_j, a neighborhood of x, such that $|g_j(x) - g_j(y)| \leq \frac{\epsilon}{3}$ for every $y \in V_j$. Setting $V = \cap V_j$, we obtain a neighborhood of x such that for every y in it and every $f \in \Omega$ we have:

$$|f(x) - f(y)| \leq |f(x) - g_j(x)| + |g_j(x) - g_j(y)| + |g_j(y) - f(y)| \leq \epsilon \tag{1.2}$$

because f needs to belong to some U_j. □

Definition 1.3. *A family of functions Ω is* tight *if for every $\epsilon > 0$ there exists a compact subset K of Σ such that* $\sup_{f\in\Omega} \| f\chi_{K^c} \|_\infty < \epsilon$.

Using this definition we have the following corollary.

Corollary 1.4. *Let Σ be a topological space. Then, a subset Ω of $C_0(\Sigma)$ is totally bounded if and only if the following conditions hold true:*

1. *Ω is pointwise bounded;*
2. *Ω is equicontinuous;*
3. *Ω is tight.*

Proof. This follows trivially applying Theorem 1.2 to $C(K)$, where K is the compact set coming from the tightness property. □

We stress that the key part of the proof of Theorem 1.2 is the definition of the mapping Φ. This motivates the following lemma (see [HOH10]).

Lemma 1.5. *Let X be a metric space. Assume that, for every $\epsilon > 0$ there exists some $\delta > 0$, a metric space W and a mapping $\Phi : X \to W$, so that $\Phi[X]$ is totally bounded and whenever $x, y \in X$ is such that $d(\Phi(x), \Phi(y)) < \delta$, then $d(x, y) < \epsilon$. Then X is totally bounded.*

Proof. Let us fix $\epsilon > 0$. We choose an $\delta(\epsilon)$-cover of $\Phi[X]$, namely $V_1, ..., V_n$. It is straightforward to check that the inverse images $\{\phi^{-1}[V_i]\}$ will form an ϵ-cover for X. □

Remark 1.6. *It is easy to see that the Φ mapping defined in* (1.1) *fulfills the assumptions of the lemma with $\delta = \frac{\epsilon}{3}$. Note also that $W = \mathbb{C}^N$ really depends on ϵ.*

1.2 The Kolmogorov Riesz Weil theorem

We turn now to study the model for the compactness criteria that we shall present in this thesis: the Kolmogorov Riesz Weil Theorem. This result was first obtained in 1931 independently by Kolmogorov and Riesz; it is the version of the latter that has become the standard one. In this section we prove a generalized version to locally compact group given by A. Weil ([Wei65]). The reason behind this choice is to present a result in a greater generality and also because this allows us to introduce some key concepts coming from "harmonic analysis".

We first recall the definition and some properties of the Haar measure (see [Fol95, Sect. 2.2]). G will always refer to a locally compact group (*i.e.* a group equipped with a locally compact topology that is compatible with the group operations).

Definition 1.7. *A Radon measure on a locally compact group is called a* Haar Measure, *if it is non zero and left invariant.*

A classical result tells us that every locally compact group has one and (up to multiplication by a factor) only one Haar measure. This enables us to study of the $L^p(G)$, for $p \geq 1$ spaces as well as the convolution of functions defined by

$$f * g(x) := \int_G dy f(y) g(y^{-1}x) \quad \text{for all } g \in L^1 \text{ and } f \in L^p . \tag{1.3}$$

Note that $f * g \in C_0(G)$. For $x \in G$ we define the left and right translations by x as $[L_x\phi](y) := \phi(x^{-1}y)$ and $[R_x\phi](y) := \phi(yx)$. We will need the following lemma.

Lemma 1.8. *Let ϕ be in $L^p(G)$. Let V be a neighborhood of the unity such that $\| L_x\phi - \phi \|_p \leq \epsilon$, whenever $x \in V$ and let $\| g \|_1 = 1$ be a positive function on G vanishing outside V. Then*

$$\| (g * \phi) - \phi \|_p \leq \epsilon . \tag{1.4}$$

Proof. The result follows from a simple computation:

$$\begin{aligned} \int_G dx (g * \phi)(x) - \phi(x) &= \int_G dx \int_G dy\, g(y)\phi(y^{-1}x) - \left(\int_G dy\, g(y) \right) \phi(x) \\ &= \int_G dy\, g(y) \left(\int_G dx \phi(y^{-1}x) - \phi(x) \right) . \end{aligned}$$

Since $|\phi(y^{-1}x) - \phi(x)| \leq \epsilon$ whenever y lies in the support of g, the proof is finished. □

If we define $\tilde{g}(x) = g(x^{-1})$ the convolution can be written as

$$f * g = \int f R_x \tilde{g} . \tag{1.5}$$

So, whenever $f \in L^p$ and $\tilde{g} \in L^q$, with $p^{-1} + q^{-1} = 1$, we have:

$$|(f * g)(x)| \leq \| f \|_p \| \tilde{g} \|_q \tag{1.6}$$

Moreover, for $f, g \in C_c(G)$, the continuity of $f * g$ can be easily checked from the strong continuity of R_x. A density argument ensures the general result. (1.6) is usually presented in $\mathbb{R}^n$ as Young's inequality, with $r = \infty$. In that context no mention of $\tilde{g}$ is made because the unimodularity of $\mathbb{R}^n$ ensures that whenever g is in L^q, we have $\tilde{g}$ in L^q (see (1.8)). See also [Fol95, Prop. 2.40].

We introduce now the *modular function* of G. We denote by $\mathbb{R}_\times$ the multiplicative group of positive real numbers. Let λ be the Haar measure of G. We remark that for a measurable set $E \in G$, $\lambda_x(E) := \lambda(Ex)$ defines a measure that is clearly left invariant if λ is such. Then it follows from the unicity of the Haar measure that there exists a positive number, denoted by $\Delta(x)$, such that $\lambda_x(E) = \Delta(x)\lambda(E)$. The mapping $\Delta : G \to \mathbb{R}_\times$ is called *the modular function of* G. We stress that the measure λ and the functional $C_c(G) \ni f \to \int d\lambda f$ need to be understood as different realizations of the same object and we shall exchange the point of view indifferently.

Proposition 1.9. *Δ is a continuous homomorphism from G to $\mathbb{R}_\times$. Moreover we have:*

$$\int dx f(xy) = \Delta(y^{-1}) \int dx f(x)\,, \tag{1.7}$$

$$\int dx f(x^{-1}) = \int dx\, \Delta(x^{-1}) f(x)\,. \tag{1.8}$$

Proof. For any $x, y \in G$ and for any a Borel set E, we have

$$\Delta(xy)\lambda(E) = \lambda(Exy) = \Delta(y)\lambda(Ex) = \Delta(x)\Delta(y)\lambda(E)\,. \tag{1.9}$$

To prove (1.7) we notice that $\chi_E(xy) = \chi_{Ey^{-1}}(x)$ so one can compute

$$\int_G d\lambda(x)\chi(Ey^{-1}) = \lambda(Ey^{-1}) = \Delta(y^{-1})\lambda(E) = \Delta(y^{-1}) \int_G d\lambda(x)\chi(E) \tag{1.10}$$

The continuity of Δ can be now studied as the continuity in y of the l.h.s. of (1.10). This continuity follows from the uniform continuity of $f \in C_c(G)$ and a density argument.

Finally, we notice that for every left invariant Haar measure λ, the relation $\rho(E) := \lambda(E^{-1})$ defines a right-invariant Radon measure. Furthermore, the computation

$$\begin{aligned}\int d\lambda(x) f(xy)\Delta(x^{-1}) &= \Delta(y)\int d\lambda(x) f(xy)\Delta((xy)^{-1}) \\ &= \int d\lambda(x) f(x)\Delta(x^{-1})\end{aligned} \tag{1.11}$$

shows that $f \to \int f(x)\Delta(x^{-1})$ defines a right-invariant functional so, by uniqueness, we have $cd\rho(x) = \Delta(x^{-1})d\lambda(x)$ for some positive number c. We need to show that $c = 1$. Let us suppose $c \neq 1$. The continuity of Δ allows us to choose a symmetric neighborhood U of the unit element of G, denoted by e_G, such that $|\Delta(x) - 1| < \frac{1}{2}|c-1|$ for every $x \in U$. Moreover, because $U = U^{-1}$, we have $\lambda(U) = \rho(U)$. This allows us to compute

$$\begin{aligned}|c-1|\lambda(U) = |c\rho(U) - \lambda(U)| &= \left|\int_U \left[\Delta(x^{-1}) - 1\right] d\lambda(X)\right| \\ &< \frac{1}{2}|c-1|\lambda(U)\end{aligned} \tag{1.12}$$

This yields the contradiction and (1.8) is proved. □

We now have all the ingredients to state the Kolmogorov Riesz Weil theorem.

Theorem 1.10. *Let Ω be a subset of $L^p(G)$, with $1 \leq p < \infty$. For Ω to be totally bounded it is necessary and sufficient that the following conditions be fulfilled.*

i. *Ω is bounded.*

ii. *For every $\epsilon > 0$ there exists a compact set $K \subset G$ such that*

$$\sup_{\phi\in\Omega} \| \chi_{K^c}\phi \|_p < \epsilon . \tag{1.13}$$

iii. *For every $\epsilon > 0$ there exists a neighborhood V of e_G such that for every $y \in V$ we have*

$$\sup_{\phi\in\Omega} \| L_y\phi - \phi \|_p < \epsilon . \tag{1.14}$$

Proof. The necessity follows from the density of $C_c(G)$ in $L^p(G)$ that allows us to choose the finite family that approximates Ω in $C_c(G)$. To prove the converse, for $\epsilon > 0$, let us fix $K(\frac{\epsilon}{6})$ and $V(\frac{\epsilon}{2})$ as in the hypothesis of the theorem and let M be the bound of Ω. We fix a positive function g, with support in V and such that $\| g \|_1 = 1$. Let $\phi \in \Omega$ be arbitrary. From (1.13) it follows that $\| \chi_K \phi - \phi \|_p < \frac{\epsilon}{6}$ which in turn implies $\| L_y(\chi_K \phi) - L_y \phi \|_p < \frac{\epsilon}{6}$. This, together with (1.14) shows that $\| L_y(\chi_K \phi) - \chi_K \phi \|_p < \frac{5\epsilon}{6}$ whenever $y \in V$. We can now apply Lemma 1.8 to notice that $\| g * (\chi_K \phi) - (\chi_K \phi) \|_p < \frac{5\epsilon}{6}$ which finally shows that $\| g * (\chi_K \phi) - \phi \|_p < \epsilon$. In order for $g(y)(\chi_K \phi)(y^{-1}x)$ to be different from zero, we need $y \in V$ and $y^{-1}x \in K$. This implies that the support of $g * (\chi_K \phi)$ is contained in VK. Note that VK can be assumed to be compact because G is locally compact. We can then set $N := \max_{x \in VK} \Delta(x^{-1})$ and compute:

$$\| \widetilde{\chi_K \phi} \|_p \leq N^{\frac{1}{p}} \, \| \chi_K \phi \|_p \leq N^{\frac{1}{p}} M \,. \tag{1.15}$$

This enables us to use (1.6) to state that $\mathfrak{L} = \{ g * (\chi_K \phi) : \phi \in \Omega \}$ is a bounded subset of $C(VK)$ with the sup norm. Furthermore we can see that:

$$|L_y(g*(\chi_K \phi)) - g*(\chi_K \phi)| = |(L_y g - g)*(\chi_K \phi)| \leq \| L_y g - g \|_q \| \widetilde{\chi_K \phi} \|_p \tag{1.16}$$

which also follows from Lemma 1.8. This inequality ensures the equicontinuity of $\mathfrak{L}$, enabling us to extract a finite family that will approximate $\mathfrak{L}$ that already aproximates Ω. To construct then a 2ϵ-cover is trivial and the proof is finished. □

Remark 1.11. *In [Wei65] the result is stated as valid for $p = \infty$. This follow from their definition of L^∞ constructed, by analogy, to other L^p spaces, as the closure of C_c in the $\| \cdot \|_\infty$-norm, namely C_0. We stick to the standard definition of L^∞ as the spaces of (classes of) essentially bounded functions.*

Remark 1.12. *It is easy to check that $\Phi_\epsilon : L^p(G) \to C_0(V(\frac{\epsilon}{2})K(\frac{\epsilon}{3}))$ defined by $\Phi(\phi) = g * (\chi_K \phi)$ fulfills the assumptions of the Lemma 1.5.*

1.3 Representations and the integrated form

We dedicated this section to gather several results in representation theory and harmonic analysis that will allow us to present a different point of view on Theorem 1.10. This point of view, linking the compacity in L^2 with the properties of some particular representations, is at the basis of the new criteria presented below. Moreover, we will stress the link of this approach with several concepts of quantum mechanics.

Throughout this section, X will denote a locally compact Abelian group; $X^\sharp$ will denote it's dual group composed of continuous homomorphism from X to $\mathbb{T}$. In $X^\sharp$ we consider the pointwise product and the topology of convergence on compact sets of X (*i.e.* $\xi_n \to \xi$ if and only if for every $K \subset X$ and $\epsilon > 0$ there exists N such that $n > N \Rightarrow \sup_{x\in K} |\xi_n(x) - \xi(x)| < \epsilon$) defining a locally compact Abelian structure. For a $\xi \in X^\sharp$, also called unitary character, we set $\xi(x) = \langle x, \xi\rangle = x(\xi)$, the latter when we want to emphasize the duality $(X^\sharp)^\sharp \cong X$, ensured by the Pontrjagin Duality Theorem (see [Fol95, Sect. 4.3]).

We introduce now the basics of the Fourier analysis over X. Given $f \in L^1(X)$ we denote by $\hat{f}$ the Fourier transform defined by

$$(\mathcal{F}f)(\xi) \equiv \hat{f}(\xi) := \int_X dx \overline{\langle x, \xi\rangle} f(x)\,. \tag{1.17}$$

With a suitable normalization on the Haar measure on $X^\sharp$ one can prove the following *Fourier inversion theorem*, see [Fol95, Theo. 4.32].

Proposition 1.13. *For $f \in L^1(X)$ such that $\hat{f} \in L^1(X^\sharp)$ we have*

$$f(x) = \int_{X^\sharp} \langle x, \xi\rangle \hat{f}(\xi) \quad \textit{for a.e. } x\,. \tag{1.18}$$

For further use we notice that $\mathcal{F}^* = \mathcal{F}^{-1}$ is defined by (1.18). The following technical lemma will be needed further below. It is presented in [GI04, Lemma 2.1] but it is a rather standard technique; see also the proof of [Fol95, Prop 4.19].

Lemma 1.14. *Let $\Lambda \in \mathcal{V}(0_X)$ and $\epsilon > 0$. There is $\psi \equiv \psi_{\Lambda,\epsilon} \in C_c(X^\sharp)$ such that $0 \le \hat{\psi} \in L^1(X)$, $\psi(0) = \int_X dx \hat{\psi}(x) = 1$ and $\int_{\Lambda^c} dx \hat{\psi}(x) \le \epsilon$. Furthermore, $\| \psi \|_\infty = 1$.*

Proof. Given a symmetric neighborhood Γ of the identity in a locally compact Abelian group we define

$$\psi_\Gamma := |\Gamma| \chi_\Gamma * \chi_\Gamma\,. \tag{1.19}$$

An easy calculation shows that $\psi_\Gamma(x) = \frac{|\Gamma \cap (x+\Gamma))|}{|\Gamma|}$. Then we have $\psi_\Gamma(0) = 1$, $0 \le \psi_\Gamma \le 1$ and $\widehat{\psi_\Gamma} \ge 0$. The last property follows from the symmetry of Γ. To show that $\widehat{\psi_\Gamma} \in L^1(X^\sharp)$ we first notice that $|\Gamma|\widehat{\psi_\Gamma} = \widehat{\chi_\Gamma}^2$. Again, the symmetry of Γ yields the positivity of $\widehat{\chi_\Gamma}^2$. We have

$$\int_{X^\sharp} d\xi \widehat{\psi_\Gamma}(\xi) = |\Gamma|^{-1} \| \widehat{\chi_\Gamma} \|_2 = |\Gamma|^{-1} \| \chi_\Gamma \|_2 = 1\,. \tag{1.20}$$

We have then a general procedure to produce functions that have almost all the properties needed. Let us fix $\Lambda \in \mathcal{V}(0_X)$ and $\epsilon > 0$. For convenience we will denote ψ_Γ when $\Gamma \in \mathcal{V}(O_{X^\sharp})$ and φ_Γ when $\Gamma \in \mathcal{V}(O_X)$. We can always construct another symmetric neighbourhood Υ of O_X such that $\Upsilon + \Upsilon \subset \Lambda$. Then φ_Υ has support inside Λ. Furthermore, there exists a compact $K \subset X^\sharp$ such that

$$\int_{K^c} \widehat{\varphi_\Upsilon} \leq \frac{\epsilon}{2}\,. \tag{1.21}$$

Then we can choose a symmetric neighborhood Γ_0 of $0_{X^\sharp}$ such that

$$1 - |\Gamma_0 \cap (\xi + \Gamma_0)| \cdot |\Gamma_0|^{-1} \leq \frac{\epsilon}{2} \quad \text{for every } \xi \in K. \tag{1.22}$$

We claim that ψ_{Γ_0} is the function that we are looking for. In fact

$$\int_{\Lambda^c} dx\, \widehat{\psi_{\Gamma_0}}(x) = \int_{\Lambda^c} dx (1 - \varphi_\Upsilon(x)) \widehat{\psi_{\Gamma_0}}(x) \tag{1.23}$$

$$\leq \int_X dx (1 - \varphi_\Upsilon(x)) \widehat{\psi_{\Gamma_0}}(x) \tag{1.24}$$

$$= \int_{X^\sharp} d\xi\, \widehat{\varphi_\Upsilon}(\xi)(1 - \psi_{\Gamma_0}(\xi)) \tag{1.25}$$

$$\leq \int_K d\xi\, \widehat{\varphi_\Upsilon}(\xi) \frac{\epsilon}{2} + \int_{K^c} d\xi\, \widehat{\varphi_\Upsilon}(\xi) \leq \epsilon \tag{1.26}$$

To go from (1.24) to (1.25) we use that $\int_X \widehat{\psi_{\Gamma_0}} = 1 = \int_{X^\sharp} \widehat{\varphi_\Upsilon}$ and Plancherel. □

Remark 1.15. *The last assertion of Lemma 1.14 was not stated in [GI04], although it was needed for a later argument. See below the proof of Theorem 1.18.*

We now describe a relation between the group theory and the bounded operators on Hilbert space. Recall that the strong topology in $\mathbb{B}(\mathcal{H})$ is defined as the topology of pointwise convergence; *i.e.* $T_n \to T$ if and only if for every $u \in \mathcal{H}$ on has $\| T_n u - Tu \|_{\mathcal{H}} \to 0$.

Definition 1.16. *A unitary representation of G in a Hilbert space $\mathcal{H}$ is a strongly continuous homomorphism π from G to $\mathcal{U}(\mathcal{H}) := \mathcal{U}(\mathbb{B}(\mathcal{H}))$ the groups of unitary operators on $\mathcal{H}$.*

Our aim is to provide a relationship between the Fourier analysis and the representation theory. In order to do that, we first show that unitary representations can be lifted to "representations" of $L^1(G)$. More precisely, if $L^1(G)$ is given the structure of Banach *-algebra (the product being the convolution defined in (1.3) and involution given by $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$), for every π unitary representation of G on $\mathcal{H}$ we define a *-representation of $L^1(G)$ formally by

$$\Pi(f) = \int_X dx\, f(x)\pi(x)^* . \tag{1.27}$$

This means that for a vector $u \in \mathcal{H}$, $\Pi(f)u$ is uniquely defined by

$$\langle \Pi(f)u, v\rangle = \int_X dx\, f(x)\langle \Pi(x)^* u, v\rangle \quad \text{for every } v \in \mathcal{H}. \tag{1.28}$$

This *-representation is called the *integrated form of* π. In a (only apparent) different point of view, Π can be seen as a functional calculus on G, $\Pi(f)$ being the operator assigned to the *symbol* f. This terminology can be made more explicit in the following example.

Let us fix X. In certain situations it can be understood as the *configuration space* of a physical system. In this setting, the *phase space* is defined as $\Sigma := X \times X^\sharp$. What we shall now do is to show that even in this abstract setting, the position and momentum operator can arise as the underlying operators of the integrated form of some particular, and physically significant, unitary representations.

Let $\mathcal{H} = L^2(X)$. We define the followings unitary representations of X and $X^\sharp$:

$$[U_x f](y) = f(y - x) , \qquad [V_\xi f](y) = \overline{\xi(y)} f(y) = \xi(-y) f(y) . \tag{1.29}$$

Let $\varphi \in C_c(X)$ and $\Psi \in \mathcal{C}_c(X^\sharp)$. We denote by M_ψ the multiplication operator by ψ in $L^2(X^\sharp)$. The following computations allow us to exhibit the behavior of the integrated representation. For u, v in $C_c(X)$ one has

$$\begin{aligned}\langle V(\hat{\varphi})u, v\rangle &= \int_{X^\sharp} \mathrm{d}\xi \hat{\varphi}(\xi)\langle V_\xi^* u, v\rangle \\ &= \int_{X^\sharp} \mathrm{d}\xi \hat{\varphi}(\xi) \int_X dy \xi(y) u(y)\overline{v(y)} \\ &= \int_X dy \varphi(y) u(y)\overline{v(y)} = \langle \varphi u, v\rangle .\end{aligned}$$

Analogously, we see that

$$\begin{aligned}\langle \mathcal{F}^* M_\psi \mathcal{F} u, v\rangle &= \int_{X^\sharp} \mathrm{d}\xi \psi(\xi)\hat{u}(\xi)\overline{\hat{v}(\xi)} \\ &= \int_{X^\sharp} \mathrm{d}\xi \psi(\xi) \left(\int_X dx \overline{\langle x, \xi\rangle} u(x)\right)\left(\int_X dy \langle y, \xi\rangle \overline{v(y)}\right) \\ &= \int_X dx \int_X dy \hat{\varphi}(x-y) u(x)\overline{v(y)} \\ &= \int_X dz \hat{\varphi}(z) \int_X dy\, u(z+y)\overline{v(y)} = \int_X dz \hat{\varphi}(z)\langle U_z^* u, v\rangle .\end{aligned}$$

What this computations show is that, if we denote by $M_\varphi =: \phi(Q)$ the operator multiplication by φ and $\psi(P)$ the operator $\mathcal{F}^* M_\psi \mathcal{F}$, we have the following (formal) relations:

$$\varphi(Q) = \int_{X^\sharp} d\xi\, \hat{\varphi}(\xi) V_\xi , \qquad \psi(P) = \int_X dx \hat{\psi}(x) U_x . \tag{1.30}$$

Obviously in this context neither P nor Q have a meaning as an operator but it is the functional calculus associated to the unitary representations that yield all the significant information. We will come back to this in Chapter 3 when the translations are conveniently transformed to encode the action of a magnetic field. In fact this approach can be generalized in the sense that for every unitary representation π the map $\Pi : \mathcal{C}_0(X^\sharp) \to \mathbb{B}(\mathcal{H})$ defined for $\hat{\psi} \in L^1(X)$ by

$$\Pi(\psi) = \int_X dx \hat{\psi}(x)\pi_x \tag{1.31}$$

is a well-defined morphism of C^*-algebras, and hence contractive. The following lemma relates the continuity of the representation at the origin with an approximation property of the functional calculus on bounded sets.

Lemma 1.17. *Let π be a unitary representation of X in $\mathcal{H}$. Let $\Omega \subset \mathcal{H}$ be a bounded set. The following assertions are equivalent:*

1. *for every $\epsilon > 0$ there exists $\Lambda \in \mathcal{V}(0_X)$ such that*

$$\sup_{u\in\Omega, x\in\Lambda} \| (\pi_x - 1)u \| < \epsilon\,; \tag{1.32}$$

2. *for every $\epsilon > 0$ there exists $\psi \in C_c(X^\sharp)$ with $\| \psi \|_\infty = 1$ such that*

$$\sup_{u\in\Omega} \| \Pi(\psi)u - u \| < \epsilon\,. \tag{1.33}$$

Proof. For simplicity we assume Ω to be bounded by 1, *i.e.* it is contained in the closed unit ball of $\mathcal{H}$.

$1 \Rightarrow 2$. Let $\epsilon > 0$. From 1 we can choose a $\Lambda \in \mathcal{V}(0_X)$ such that $\| (\pi_x u - u \| \leq \frac{\epsilon}{2}$ for every $x \in \Lambda$ and $u \in \Omega$. Thanks to Lemma 1.14 we can choose $\psi \in C_c(X^\sharp)$ such that $\psi(0) = 1$, $\int_X dx|\hat{\psi}(x)| = 1$ and $\int_{\Lambda^c} dx|\hat{\psi}(x)| \leq \frac{\epsilon}{4}$. Then for $u \in \Omega$ we have:

$$\| \Pi(\psi)u - u \| = \left| \int_X dx\hat{\psi}(x)(\pi_x - 1)u \right| \tag{1.34}$$

$$\leq \int_X dx|\hat{\psi}(x)|\; \| (\pi_x u - u) \| \tag{1.35}$$

$$\leq \int_\Lambda dx|\hat{\psi}(x)|\frac{\epsilon}{2} + \int_{\Lambda^c} dx|\hat{\psi}(x)|2 \leq \epsilon\,. \tag{1.36}$$

$2 \Rightarrow 1$. Choose a ψ corresponding to $\frac{\epsilon}{4}$. Since the support of ψ is compact, and hence equicontinuous, we can choose $\Lambda \in \mathcal{V}(0_X)$ such that $\xi(x) < \frac{\epsilon}{2}$ for every $x \in \Lambda$ and ξ in the support of ψ. Then, for every $u \in \Omega$ and $x \in \Lambda$ we have

$$\| (\pi_x - 1)u \| \leq \| (\pi_x - 1)(\Pi(\psi) - 1)u \| + \| (\pi_x - 1)\Pi(\psi)u \| \tag{1.37}$$

$$\leq 2 \cdot \frac{\epsilon}{4} + \| (\pi_x - 1)\Pi(\psi) \| \tag{1.38}$$

$$\leq \frac{\epsilon}{2} + \sup_{k\in X^\sharp} |(\xi(x) - 1)\psi(\xi)| \leq \epsilon\,, \tag{1.39}$$

finishing the proof. ☐

1.4 Integrated form and compactness in L^2.

With these results at hand we now prove a particular case of Theorem 1.10. Let $p = 2$ and X, $X^\sharp$, U and V as in the previous section. Let us set $T^\perp := 1 - T$ for an operator T. For a bounded subset Ω of $L^2(X)$, we need to exhibit a compact operator that approximates Ω in the sense of Lemma 1.5. For that matter, consider $\varphi \in C_c(X)$ and $\psi \in C_c(X^\sharp)$ and utilize the results from (1.30) to compute for $u \in L^2(X)$:

$$[\varphi(Q)\psi(P)u](y) = \varphi(y)\left(\int_X dx \hat{\psi}(x)[U_x u](y)\right) \tag{1.40}$$

$$= \int_X dx \varphi(y)\hat{\psi}(x)u(y-x) \tag{1.41}$$

$$= \int_X dx \varphi(y)\hat{\psi}(y-x)u(x)\,. \tag{1.42}$$

This show that $\varphi(Q)\psi(P)$ is a Hilbert-Schmidt Operator, and hence compact, with kernel $k(x,y) = \varphi(y)\hat{\psi}(y-x)$. With this result in mind, Theorem 1.10 can be restated as follows.

Theorem 1.18. *Let $\Omega \subset L^2(X)$ be bounded. Then we have the following equivalent assertions:*

1. *Ω is relatively compact in $L^2(X)$;*
2. $\lim_{x\to 0} \sup_{u\in\Omega} \| U_x u - u \| = 0$ *and* $\lim_{\xi\to 0} \sup_{u\in\Omega} \| V_\xi u - u \| = 0$;
3. *For every $\epsilon > 0$ there exists $\varphi \in C_c(X)$ with $\| \varphi \|_\infty = 1$ and $\psi \in C_c(X^\sharp)$ with $\| \psi \|_\infty = 1$, such that for every $u \in L^2(X)$*

$$\| \varphi(Q)^\perp u \| + \| \psi(P)^\perp u \| < \epsilon\,. \tag{1.43}$$

Proof. Let $\epsilon > 0$. From the relative compactness of Ω we can choose a finite family $K \subset \Omega$ such that for every $u \in L^2(X)$

$$\min_{v\in K} \| u - v \| < \frac{\epsilon}{4}\,. \tag{1.44}$$

From the strong continuity of U and V we can choose $\Lambda \in \mathcal{V}(0_X)$ and $\tilde{\Lambda} \in \mathcal{V}(0_{X^\sharp})$ such that

$$\sup_{v\in K, x\in\Lambda} \| U_x v - v \| < \frac{\epsilon}{2} \quad \text{and} \quad \sup_{v\in K, \xi\in\tilde{\Lambda}} \| V_\xi v - v \| < \frac{\epsilon}{2}\,. \tag{1.45}$$

Putting together (1.44) and (1.45) we have

$$\| U_x u - u \| \leq \| U_x u - U_x v \| + \| U_x v - v \| + \| v - u \| \tag{1.46}$$

if v is chosen suitably. We have then $1 \Rightarrow 2$. $2 \Rightarrow 3$ is direct consequence of Lemma 1.17 when applied to both U and V. Finally we can compute

$$\| u - \varphi(Q)\psi(P)u \| = \| u - \varphi(Q)u + \varphi(Q)u - \varphi(Q)\psi(P)u \| \tag{1.47}$$

$$= \| \varphi(Q)^{\perp}u + \varphi(Q)\psi(P)^{\perp}u \| \tag{1.48}$$

$$\leq \| \varphi(Q)^{\perp}u \| + \| \varphi \|_{\infty} \| \psi(P)^{\perp}u \| \ . \tag{1.49}$$

Then, the r.h.s. of (1.49) can be made arbitrarily small if one assumes 3. We have 1 because $\varphi(Q)\psi(P)$ is a compact operator. □

Remark 1.19. *We want to stress that the properties used in the proof do not lie in the homomorphism property of U or V. What is used is the fact that the functional calculus can properly identify some compact operators that have approximation properties and the boundness of the representation. This observation will allow us to reproduce this approach in a more abstract setting.*

In a certain sense, in this example we have more structure than is actually required. To each point in the phase space $\Sigma := X \times X^{\sharp}$ we assign an operator in $L^2(X)$ by setting $W(x,\xi) = U_{-x}V_{\xi}$. Then W, seen as map from Σ to $\mathcal{U}(\mathbb{B}(\mathcal{H}))$, is no longer a group homomorphism, because

$$\begin{aligned} W\left((x,\xi)+(y,\eta)\right) :=& U_{-x}U_{-y}V_{\xi}V_{\eta} \\ =& \xi(y)U_{-x}V_{\xi}U_{-y}V_{\eta} = \overline{\xi(y)}W(x,\xi)W(y,\eta) \end{aligned}$$

using the fundamental commutation relation $U_x V_{\xi} = \xi(x)V_(\xi)U(x)$. Such a W is called a projective representation; nonetheless, this could still be arranged to be a representation if one considers $\mathbb{H}_X := X \times X^{\sharp} \times \mathbb{T}$ with a composition suitably defined to encode the non commutativity of U and V instead. Such a group could be named a *abstract Heisenberg group*; but, to the best of our knowledge, this group is mostly studied when $X = \mathbb{R}^n$.

1.5 Representation coefficients and compactness in L^2

We present now the second approach that we shall develop on this thesis: *the representation coefficient* approach.

Definition 1.20. *Let π be a unitary representation of G in $\mathcal{H}$. The* representation coefficient $\phi^\pi : \mathcal{H} \otimes \overline{\mathcal{H}} \to C_b(G)$ *is defined by*

$$[\phi^\pi(u,v)](x) :=< \pi(x)u, v > \tag{1.50}$$

Furthermore, we ask a certain isometry condition, namely

$$\| \phi^\pi(u,v) \|_{L^2(G)} = \| u \| \| v \| \ . \tag{1.51}$$

This is called *square integrability* and is a widely studied concept in representation theory; in fact, such a representation can be seen as the irreducible subrepresentations of the left regular representation (for $f \in L^2(G)$, $(L_x f)(y) := f(x^{-1}y)$). See for instance [Car76].

For every normalized vector $v \in \mathcal{H}$, then

$$[\phi_v^\pi(u)](\cdot) := [\phi^\pi(u,v)](\cdot) \tag{1.52}$$

can be seen as an isometry from $\mathcal{H}$ to $L^2(G)$. As done before, the idea is to show compacity in $\mathcal{H}$ via the compactness criteria already known. Recall that every bounded set of a reflexive Banach space, and hence of a Hilbert space, is compact when considered with the *-weakly topology; in Chapter 4 we shall present a more general result known as the Bourbaki-Alaoglu Theorem.

Theorem 1.21. *Let G, π, $\mathcal{H}$, v be as above and $\Omega \subset \mathcal{H}$ bounded. Equivalent assertions:*

1. *Ω is relatively compact;*
2. *$\phi_v^\pi(\Omega)$ is tight in $L^2(G)$ for every normalized vector (see Definition 1.3);*
3. *There exists a normalized vector v such that $\phi_v^\pi(\Omega)$ is tight in $L^2(G)$, this can be restated: for every $\epsilon > 0$, $\exists L \subset G$ compact such that*

$$\sup_{u \in \mathcal{H}} \| \chi_{L^c} \phi_v^\pi(u) \|_2 < \epsilon \, . \tag{1.53}$$

Proof. Again, for simplicity, we shall assume that Ω is contained in the closed unit ball of $\mathcal{H}$.

$1 \Rightarrow 2$. Let $\epsilon > 0$. From the relative compactness of Ω we can choose a finite family $K \subset \Omega$ such that

$$\sup_{u \in \Omega} \min_{z \in K} \| u - z \| < \frac{\epsilon}{2} \, . \tag{1.54}$$

Then we can fix $L \subset G$ compact such that

$$\sup_{z\in K} \| \chi_{L^c}\phi_v^\pi(z) \|_2 \leq \frac{\epsilon}{2} . \tag{1.55}$$

Using (1.54) and (1.55), together with the isometry property of ϕ_v^π seen as a map from $\mathcal{H}$ to $L^2(G)$, the result follows.

$2 \Rightarrow 3$. Trivial.

$3 \Rightarrow 1$. By Riesz Theorem and Banach-Alaoglu we know that Ω is weakly compact. So, for every sequence we can extract a weakly convergent subsequence; the result will follow if we can show that every weakly convergent sequence in Ω is, in fact, norm convergent.
Let $\{u_n\} \subset \Omega$ be converging weakly to u. Then

$$\lim_{n\to\infty} [\phi_v^\pi(u_n)](x) = \lim_{n\to\infty} \langle u_n, \pi_x^* v\rangle = \langle u, \pi_x^* v\rangle = [\phi_v^\pi(u)](x) \tag{1.56}$$

i.e. $\Phi_v^\pi(u_n)$ converge pointwise to $\Phi_v^\pi(u)$. If we can improve this convergence to an L^2 convergence, the result will follow from the isometry property of ϕ_v^π. Let L be as in the hypothesis. We assume (1.53) also valid for u. Using Cauchy-Schwartz inequality we have

$$|[\phi_v^\pi(u - u_n)](x) \leq \| u - u_n \| \leq 1 + \| u \| . \tag{1.57}$$

Let us set $C := (1 + \| u \|)^2$. Then $|\phi_v^\pi(u_n)|^2$ is bounded by $C\chi_L \in L^1(G)$ in L. We can finally estimate

$$\lim_{n\to\infty} \| \phi_v^\pi(u - u_n) \| \leq \lim_{n\to\infty} \| \chi_L \phi_v^\pi(u - u_n) \| + \underset{n\to\infty}{\|} \chi_{L^c}\phi_v^\pi(u - u_n) \| . \tag{1.58}$$

The first term of the r.h.s. goes to zero by the Dominated Convergence Theorem, and the second is less than 2ϵ finishing the proof. □

This proof is a mere translation of [DFG02, Theo. 2] where G is assumed to be the reduced Heinseberg group $\mathbb{H}_r^n$; the following explanation should also bring light into the comments we made at the end of the previous section.

Definition 1.22. *The* reduced Heisenberg group *is the locally compact space* $\mathbb{R}^{2n} \times \mathbb{T}$ *equipped with the operation*

$$(x, \xi, e^{2\pi i\tau}) \cdot (y, \eta, \tau') = (x + y, \xi + \eta, e^{2\pi i(\tau+\tau')} e^{2\pi i(\xi\cdot y - \eta\cdot x)} . \tag{1.59}$$

The Schrödinger representation *is defined by*

$$\pi(x, \xi, e^{2\pi i\tau}) = e^{2\pi i\tau} U_{-x} V_\xi . \tag{1.60}$$

We recall here that the dual of $\mathbb{R}^n$ is identified with $\mathbb{R}^n$ by $\langle x, \xi \rangle = e^{-2\pi i(x\cdot\xi)}$. The Heinseberg group plays an important role in different domains of mathematics, including Signal Analysis and Pseudodifferential Operators. For a thorough study one should refer to [Fol89, Chp. 1]; see also [Grö01, Chp.9]. One useful property of the group is that the Haar measure is just the Lebesgue one. In this context, we can compute the concrete form of the representation coefficient

$$[(\phi_g^\pi)(u)](x, \xi, e^{2\pi i\tau}) = \int_{\mathbb{R}^n} dt f(t)\overline{g(t-x)}e^{-2\pi i(\xi\cdot t+\tau)}, \tag{1.61}$$

called *Short Time Fourier Transform* or *Fourier-Wigner Transform.* We can see that g plays here the role of localizing the Fourier transform of f on its support; that is why one should think of the vector v from (1.52) as a "window" to look through it, even when dealing with an abstract setting. Taking $e^{2\pi i\tau} = 1$, (1.53) now reads

2. *For every $\epsilon > 0$ a compact set $K \subset \mathbb{R}^{2n}$ exists such that*

$$\sup_{u\in\Omega}\left(\int_{K^c} dx d\xi |[(\phi_g^\pi)(u)](x,\xi)|^2\right)^{\frac{1}{2}} < \epsilon. \tag{1.62}$$

In fact this condition is known to express at the same time condition *ii.* and *iii.* of Theorem 1.10 because the "equicontinuity" of a family can be related to the tightness of their Fourier transform; see [Peg85].

Chapter 2

Bounded measurable families of bounded operators and compactness criteria in Hilbert spaces

The idea of this chapter, the most important of this thesis, is to develop a formalism that englobes the criteria exposed in Chapter 1. The setting, briefly presented in section 2.1, is the one from [Man12], even if further assumptions will be added when needed. In section 2.2 we present the compactness criteria and use it in section 2.3 to describe the pure point and continuous subspaces corresponding to a self-adjoint operator.

2.1 The framework

The framework can be condensed in the following objects $(\Sigma, \mu, \pi, \mathcal{H}, \mathcal{G})$. Σ is a Hausdorff locally compact space with a Radon measure μ. Indexed by Σ, $\{\pi(s)\}$ is a family of bounded operators acting in a Hilbert space $\mathcal{H}$. $\mathcal{G}$ plays no role until chapter 4 and until then is tacitly assumed to be $\mathcal{H}$. In order to construct a richer theory we ask $\pi : \Sigma \to \mathbb{B}(\mathcal{H})$ to be bounded and weakly continuous.

Remark 2.1. *A unitary representation is clearly bounded by* 1. *More interesting is the fact that at first glance the continuity asked to* π *seems to be more general; in fact both the weak and strong topologies coincide in* $\mathcal{U}(\mathcal{H})$ *implying that every* *weakly continuous unitary representation is in fact strong continuous. Note that even if they coincide in $\mathcal{U}(\mathcal{H})$, the respectively closures of $\mathcal{U}(\mathcal{H})$ in $\mathbb{B}(\mathcal{H})$ doesn't; see [Bla06, Sect. I.3].*

We want to formulate concepts that generalize the *integrated form* and the *representations coefficients*. For that, we first recall the isomorphism Λ between $\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}$, the Hilbert completion of the algebraic tensor product $\mathcal{H}\otimes\overline{\mathcal{H}}$, and $\mathbb{B}_2(\mathcal{H})$, the space of Hilbert-Schimdt operators with its natural Hilbert norm given by the trace. The isomorphism is completely defined by

$$\Lambda(u\otimes v) := \langle\cdot, v\rangle u\,. \tag{2.1}$$

The fact that Λ is linear and injective is clear; the surjectivity will follow from the fact that the image contains all the finite ranks operators if we show that it is an isometry. We recall that $\| \mathbb{T} \|^2_{\mathbb{B}_2(\mathcal{H})} = \sum_n \| Te_n \|^2_{\mathcal{H}}$ for every orthonormal base $\{e_n\}$. Let us fix an orthonormal basis $\{e_n\}$ of $\mathcal{H}$; then $\{e_n\otimes e_j\}$ is an orthonormal basis of $\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}$. The isometry of Λ then follows from the following computation:

$$\begin{aligned}\left\|\Lambda\left(\sum_{n,j} a_{n,j}e_n\otimes e_j\right)\right\|^2_{\mathbb{B}_2(\mathcal{H})} &= \left\|\sum_i\sum_{n,j} a_{n,j}\langle e_i, e_j\rangle e_n\right\|^2_{\mathcal{H}}\\ &= \sum_{n,j}|a_{n,j}|^2 = \left\|\sum_{n,j} a_{n,j}\, e_n\otimes e_j\right\|_{\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}}.\end{aligned}$$

Now the fact that Λ is an isomorphism easily follows from the Bounded Inverse Theorem.

Let us come back to our aim of replicating the *integrated form* and the *representations coefficient* in this general framework. The definition of the representation coefficient needs no change:

$$\phi^\pi : \mathcal{H}\otimes\overline{\mathcal{H}} \to C_b(\Sigma) \text{ defined by } [\phi^\pi(u,v)](s) := \langle\pi(s)u, v\rangle\,. \tag{2.2}$$

The central assumption of this thesis is the following.

Assumption 2.2. *The mapping ϕ^π extends to an isometric isomorphism ϕ^π : $\mathcal{H}\otimes\overline{\mathcal{H}}\to L^2(\Sigma)$.*

This encodes a square integrability of the form

$$\| \phi^\pi(u,v) \|_2 = \| u \| \| v \| \,, \tag{2.3}$$

but also a surjectivity condition that was not needed in [Man12].

As one could see in section 1.4, the key part of the proof of Theorem 1.18 consisted in show that for suitable symbols (*i.e.* functions on the phase space), the corresponding operator was Hilbert-Schimdt. With that in mind we define

$$\Pi : L^2(\Sigma) \to \mathbb{B}_2(\mathcal{H}) \text{ by } \Pi := \Lambda \circ \phi^{\pi -1} \tag{2.4}$$

What we show now is that Π, defined in a rather abstract way, can really be understood as the integrated form of π.

Proposition 2.3 ([Man12, Prop. 2.3]). *For any $f \in L^2(\Sigma)$ one has in weak sense*

$$\Pi(f) = \int_\Sigma d\mu(s) f(s) \pi(s)^* \text{ and} \tag{2.5}$$

$$\Pi(f)^* = \int_\Sigma d\mu(s) \overline{f(s)} \pi(s) \,. \tag{2.6}$$

Proof. First we notice that $\mathrm{Tr}(\Lambda(u \otimes v)) = \langle u, v \rangle$. Also notice that for $T \in \mathbb{B}(\mathcal{H})$ we have

$$T\Lambda(u \otimes v) = T(\langle \cdot, v \rangle u) = \langle \cdot, v \rangle Tu = \Lambda(Tu \otimes v) \,.$$

Then we can compute

$$\begin{aligned}
\langle \Pi(f)u, v \rangle &= \mathrm{Tr}[\Lambda(\Pi(f)u \otimes v)] \\
&= \mathrm{Tr}[\Pi(f)\Lambda(u \otimes v)] \\
&= \mathrm{Tr}[\Pi(f)\Pi(\Phi^\Pi(v \otimes u)^*] \\
&= \langle \Pi(f), \Pi(\Phi^\Pi(v \otimes u)) \rangle_{\mathbb{B}_2(\mathcal{H})} = \langle f, \Phi^\pi(v \otimes u) \rangle_{(\Sigma)} \\
&\qquad = \int_\Sigma d\mu(s) f(s) \overline{\langle v, \pi(s)^* u \rangle} \\
&\qquad = \int_\Sigma d\mu(s) f(s) \langle \pi(s)^* u, v \rangle \,.
\end{aligned}$$

Then (2.6) follows from (2.5). □

The following definition generalized the notion of tightness presented in section 1.1. Assume that the Banach space $\mathcal{Y}$ is endowed with a structure of Banach left module over a normed algebra $\mathcal{A}$, meaning that a left module structure $\mathcal{A} \times \mathcal{Y} \ni (a, y) \mapsto a \cdot y \in \mathcal{Y}$ is given and for every $a \in \mathcal{A}$ and $y \in \mathcal{Y}$ the relation $\| a \cdot y \|_{\mathcal{Y}} \leq \| a \|_{\mathcal{A}} \| y \|_{\mathcal{Y}}$ is satisfied.

Definition 2.4. *Let $\mathcal{Y}$ be a Banach left module over $\mathcal{A}$ and let $\mathcal{A}^0 \subset \mathcal{A}$; we say that* the bounded set $\Gamma \subset \mathcal{Y}$ is $\mathcal{A}^0$-tight *if for every $\epsilon > 0$ there exists $a \in \mathcal{A}^0$ with* $\sup_{y \in \Gamma} \| a \cdot y - y \|_{\mathcal{Y}} \leq \epsilon$.

Tightness in section 1.1 should then read as $\mathcal{K}(\Sigma)$-tightness with respect as the pointwise action of $C_b(\Sigma)$ on itself.

2.2 Compactness criterion

Recall that $\phi^\pi_w : \mathcal{H} \to L^2(\Sigma)$ given by $[\phi^\pi_w(u)]\,(s) := \langle \pi(s)u, w \rangle$ is well-defined and isometric for every normalized vector w of the Hilbert space $\mathcal{H}$.

Theorem 2.5. *Let Ω be a bounded subset of $\mathcal{H}$. Consider the following assertions:*

1. *Ω is relatively compact.*
2. *For every normalized $w \in \mathcal{H}$ the family $\phi^\pi_w(\Omega)$ is $\mathcal{K}(\Sigma)$-tight in $L^2(\Sigma)$.*
3. *There exists $w_0 \in \mathcal{H}$ such that the family $\phi^\pi_{w_0}(\Omega)$ is $\mathcal{K}(\Sigma)$-tight in $L^2(\Sigma)$.*
4. *For each $\epsilon > 0$ there exists $f \in C_c(\Sigma)$ with $\sup_{u \in \Omega} \| \Pi(f)u - u \| \leq \epsilon$ (i.e. Ω is $\Pi\,[C_c(\Sigma)]$-tight).*
5. *One has $\lim_{s \to s_0} \sup_{u \in \Omega} \| \pi(s)^* u - \pi(s_0)^* u \| = 0$ for every $s_0 \in \Sigma$.*
6. *For every $\epsilon > 0$ and for every $s_0 \in \Sigma$ there exists $g \in C_c(\Sigma)$ such that $\sup_{u \in \Omega} \| \Pi(g)u - \pi(s_0)^* u \| \leq \epsilon$.*

Then 1, 2, 3 and 4 are equivalent, they imply 5, which in its turn implies 6. Thus, if we assume that $\pi(s_1)^* = 1$ for some $s_1 \in \Sigma$, then all the six assertions are equivalent.

Proof. The proof of $1 \Leftrightarrow 2 \Leftrightarrow 3$ can be obtained along the same lines as the proof of Theorem 1.21. The only delicate point is the fact that there, the bound of a unitary representation we used in (1.57) and in the second term of (1.58) was 1, and here it needs to be made explicit.

$1 \Rightarrow 4$. Let $\Omega \subset \mathcal{H}$ be relatively compact and, for some $\epsilon > 0$, let F be a finite subset such that for each $u \in \Omega$ there exists $v_u \in F$ with $\| u - v_u \| \leq \epsilon/4$. The subspace $\mathcal{F}$ generated by F will be finite-dimensional and thus the corresponding projection P will be a finite-rank operator satisfying $Pv = v$ for every $v \in F$. Then for every $u \in \Omega$

$$\begin{aligned} \| Pu - u \| \leq & \; \| Pu - Pv_u \| + \| Pv_u - v_u \| + \| v_u - u \| \\ \leq & \; 2 \| u - v_u \| \leq \epsilon/2 . \end{aligned} \tag{2.7}$$

Notice that $\{ \Pi(f) \mid f \in C_c(\Sigma) \}$ is a dense set of compact operators. To see this, use the fact that $\Pi : L^2(\Sigma) \to \mathbb{B}_2(\mathcal{H})$ is an isometric isomorphism and that $C_c(\Sigma)$ is dense in $L^2(\Sigma)$; the topology of $\mathbb{B}_2(\mathcal{H})$ is stronger than that of $\mathbb{B}(\mathcal{H})$, while $\mathbb{K}(\mathcal{H})$ is the closure of $\mathbb{B}_2(\mathcal{H})$ in the operator norm. Let now $M := \sup_{u \in \Omega} \| u \|$; by density there is some $f \in C_c(\Sigma)$ with $\| P - \Pi(f) \|_{\mathbb{B}(\mathcal{H})} \leq \epsilon/2M$. From this and from (2.7) the conclusion follows immediately.

$4 \Rightarrow 1$. To prove the converse, for $\epsilon > 0$ choose $f \in C_c(\Sigma)$ such that

$$\sup_{u \in \Omega} \| \Pi(f)u - u \| \leq \epsilon/2 .$$

Since $\Pi(f)$ is a compact operator and Ω is bounded, the range $\Pi(f)\Omega$ is relatively compact, so there is a finite set G such that for each $u \in \Omega$ there is an element $v^u \in G$ with $\| \Pi(f)u - v^u \| \leq \epsilon/2$. Then for $u \in \Omega$ one has

$$\| u - v^u \| \leq \| u - \Pi(f)u \| + \| \Pi(f)u - v^u \| \leq \epsilon/2 + \epsilon/2 = \epsilon , \tag{2.8}$$

so the set Ω is totally bounded.

$4 \Rightarrow 5$. Setting $S^\perp := 1 - S$, we compute for $s_0 \in \Sigma$, $u \in \Omega$, $f \in C_c(\Sigma)$ and s belonging to a neighborhood V of s_0:

$$\begin{aligned} \| \pi(s)^* u - \pi(s_0)^* u \| \leq & \; \| [\pi(s)^* - \pi(s_0)^*] \, \Pi(f) u \| + \| [\pi(s)^* - \pi(s_0)^*] \, \Pi(f)^\perp u \| \\ \leq & \; \sup_{u \in \Omega} \| u \| \, \| [\pi(s)^* - \pi(s_0)^*] \, \Pi(f) \|_{\mathbb{B}(\mathcal{H})} + \\ & \qquad 2 \sup_{t \in V} \| \pi(t)^* \|_{\mathbb{B}(\mathcal{H})} \sup_{u \in \Omega} \| \Pi(f)^\perp u \| \end{aligned}$$

The first term is small for s belonging to a suitable neighborhood V, because Ω is bounded, π^* is strongly continuous and this is improved to norm continuity by multiplication with the compact operator $\Pi(f)$. The second term is also small for some suitable f, because of the assumption 4 and since $\| \pi^*(\cdot) \|_{\mathbb{B}(\mathcal{H})}$ is bounded on the compact set $\overline{V}$ (use the Uniform Boundedness Principle and the strong continuity of π^*).

$5 \Rightarrow 6$. Compute for any positive $g \in C_c(\Sigma)$ with $\int_\Sigma g d\mu = 1$

$$\| \Pi(g)u - \pi(s_0)^* u \| = \left\| \int_\Sigma d\mu(s) g(s) [\pi(s)^* u - \pi(s_0)^* u] \right\|$$
$$\leq \int_\Sigma d\mu(s) g(s) \, \| [\pi(s)^* - \pi(s_0)^*] \, u \|$$

and then use 5 and require g to have support inside the convenient neighbourhood of the point s_0. □

For simplicity, we are always going to assume that $\pi(s_1)^* = 1$ *for some* $s_1 \in \Sigma$. Below $\mathcal{X}_{[1]}$ denotes the closed unit ball of the Banach space $\mathcal{X}$.

Corollary 2.6. *Let $\mathcal{X}$ be a Banach space and $S \in \mathbb{B}(\mathcal{X}, \mathcal{H})$. The next assertions are equivalent.*

1. *S is a compact operator.*
2. *The set $\phi_w^\pi(S\mathcal{X}_{[1]})$ is $\mathcal{K}(\Sigma)$-tight in $L^2(\Sigma)$ for some (every) $w \in \mathcal{H}$.*

   Writing $M_{\chi_L}^\perp$ for the operator of multiplication by the function $1 - \chi_L$ in $L^2(\Sigma)$, this can be restated: for every $\epsilon > 0$ there is a compact subset L of Σ such that $\| M_{\chi_L}^\perp \circ \phi_w^\pi \circ S \|_{\mathbb{B}(\mathcal{X}, L^2)} \leq \epsilon$.
3. *For every $\epsilon > 0$ there is some $f \in C_c(\Sigma)$ such that $\| [\Pi(f) - 1] S \|_{\mathbb{B}(\mathcal{X}, \mathcal{H})} \leq \epsilon$.*
4. *The map $\Sigma \ni s \mapsto \pi(s)^* S \in \mathbb{B}(\mathcal{X}, \mathcal{H})$ is norm-continuous.*

Proof. This is a simple consequence of Theorem 2.5, since S is a compact operator if and only if $\Omega := S\mathcal{X}_{[1]}$ is relatively compact in $\mathcal{H}$; also use $\| T \|_{\mathbb{B}(\mathcal{X}, \mathcal{Y})} = \sup_{v \in \mathcal{X}_{[1]}} \| Tx \|_{\mathcal{Y}}$. □

2.3 Characterization of bound and scattering states

We consider now a self-adjoint operator H, not necessarily bounded, acting in $\mathcal{H}$. We denote by e^{itH} the strongly continuous evolution group generated by H and by E_H the spectral measure associated to it. For a vector u we denote by μ_h^u the measure on $\mathbb{R}$ defined in a Borel set J by $\mu_h^u(J) = \langle u, E_H(J)u \rangle$. For such an operator there are several ways to decompose $\mathcal{H}$ in term of his spectral properties; for our purpose, it suffices to define $\mathcal{H}_p$ as the closed linear subspace spanned by the eigenvalues of H. Then $\mathcal{H}_c := \mathcal{H} \ominus \mathcal{H}_p$. An important known fact is that if $u \in \mathcal{H}_c$, then for every $\lambda \in \mathbb{R}$ we have $\mu_h^u(\{\lambda\}) = 0$.

We start characterizing $\mathcal{H}_c$ for which we need the next lemma.

Lemma 2.7 ([GI04, Prop. 4.7]). *For an $u \in \mathcal{H}$ the following statements are equivalent:*

1. $u \in \mathcal{H}_c$;
2. $\mathfrak{C}\text{-}\lim_{t\to\infty} \langle u, e^{itH}u \rangle = 0$;
3. $w\text{-}\underline{\lim}_{t\to\infty}\, e^{itH}u = 0$;
4. $w\text{-}\mathfrak{Lim}_{t\to\infty} e^{itH}u = 0$;
5. $\mathfrak{Lim}_{t\to\infty} \| K e^{itH} u \| = 0$ *for every* $K \in \mathcal{K}(\mathcal{H})$.

Even if we don't get involved in the full proof of this Lemma, the mere statement of it deserves some comments in order to make the notion understandable and also because it allows us to present a rather beautiful connection with general topology. As can be seen from the lemma, the idea of scattering states is that, if one waits for a sufficiently long time, it get out of every compact set; that is why a week limit at infinity gets involved. However, the usual notion of limit at infinity, the one from the topology of $\mathbb{R}$, seems not to be enough to characterize $\mathcal{H}_c$ correctly; that is why in Ruelle's original paper [Rue69] a notion of limit in the mean is introduced. For reasons that we dare not to discuss here, but that are directly connected with a certain stability under translation invariant measures, this notion seem not to be optimal; we refer to the discussion in [GI04, Sec. 4.3] and the reference cited therein. We will limit ourselves here to exhibit the concrete definition of both $\mathfrak{C}$-lim, the Cesàro convergence, and $\mathfrak{Lim}$, the Lorentz convergence. In order to compare these convergences, we do not follow the usual way of defining the former (*i.e.* by taking limits of the means of the form $T^{-1}\int_0^T \varphi$) but instead we describe them in terms of filters.

Definition 2.8. *A filter on a set F is a nonempty collection $\mathfrak{F}$ of non-empty subsets of F that satisfy:*

1. *If $F_1, F_2 \in \mathfrak{F}$ then $F_1 \cap F_2 \in \mathfrak{F}$;*
2. *If $G \supset F \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

This definition enables us to define the notion of limit of a function along a filter.

Definition 2.9. *Let F be a set equipped with a filter $\mathfrak{F}$. If φ is a map of F into a topological space Y, one says that φ has limit y along $\mathfrak{F}$ if $\varphi^{-1}(V)$ for every $V \in \mathcal{V}(y)$.*

One important example of a filter is $\mathcal{V}(x)$. In that case the notion of limit along $\mathcal{V}(x)$ coincides with the usual notion of limit at x. For the case we are interested in, $F = \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, $Y = \mathbb{C}$ and $\mathfrak{F} = \mathcal{V}(\infty)$, formed of sets with bounded complement, is called the Fréchet filter. The idea is to find filters, finer than the Fréchet one, that are translation invariant.

Let us describe first the Cesàro convergence.

Definition 2.10. *The Cesàro filter, noted by $\mathfrak{C}$, is formed by the sets that contain a Borel set A such that:*

$$\lim_{T\to\infty} \frac{|A \cap [-T, T]\,|}{2T} = 1$$

We denote by $\mathfrak{C}$-$\lim$ the convergence along $\mathfrak{C}$.

Then 2.8.2 is trivial and 2.8.1 follows easily if one prove it first for intervals. This was the convergence used by Ruelle; a way to improve his result would be to identify another filter, coarser than $\mathfrak{C}$, but still translation invariant and enabling us to characterize $\mathcal{H}_c$.

Definition 2.11. *The Lorentz filter, noted by $\mathfrak{L}$, is formed by the sets that contain a Borel set A such that*

$$\lim_{T\to\infty} \inf_{t\in\mathbb{R}} \frac{|A \cap [t-T, t+T]\,|}{2T} = 1\,.$$

We denote by $\mathfrak{Lim}$ *the convergence along* $\mathfrak{L}$.

It is clear that $\mathfrak{L} \subset \mathfrak{C}$, but let us show that the inclusion is strict. Let $A \subset \mathbb{R}$ be defined by $A^c = \cup_n [t_n - T_n, t_n + T_n]$. In order to have $A \in \mathfrak{L}$ we should have $\sup_n T_n < \infty$. If not, for every $T > 0$ we could find $T_n > T$ and we would have $|A \cap [tn - T, tn + T]| = 0$. However, we can see that A defined by $A^c = \cup_n [n^2 - \sqrt{n}, n^2 + \sqrt{n}]$ is in $\mathfrak{C}$.

We can now proove some of the implications of Lemma 2.7.

Partial proof of 2.7. We begin to notice that $4 \Leftrightarrow 5$ is a known property of the compact operators, and $4 \Rightarrow 2$ follows from the previous discussion.

$3 \Rightarrow 1$. Let $u = u_c + u_p$ be defined by the decomposition $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_p$ and consider an eigenvector v corresponding to the eigenvalue λ. Then we can compute

$$e^{it\lambda}\langle u_p, v\rangle = e^{it\lambda}\langle u, v\rangle = \langle u, e^{-itH}v\rangle = \langle e^{itH}u, v\rangle \,.$$

Since the last term can be made arbitrarily small, we have $u_p \perp v$ and hence the arbitrariness of v yields $f \in \mathcal{H}_c$.

$4 \Rightarrow 3$. Suppose 3 is false. For every $g \in \mathcal{H}$ there is $c > 0$ such that for some $T > 0$ we have $|\langle e^{itH} f, g\rangle| > c$. Then its clear that the preimage of $\{z \in C : |z| < c\}$, being bounded, is not in $\mathfrak{L}$. □

Putting together 2.5 and 2.7 we can easily state the following dynamic characterization of $\mathcal{H}_c$.

Corollary 2.12. $u \in \mathcal{H}_c$ *if and only if the following condition is fulfilled:*

$$\text{For every } f \in [C_c(\Sigma)] \text{ we have } \mathfrak{Lim}_{t\to\infty} \| \Pi(f)e^{itH}u \| = 0\,.$$

Let us now characterize the pure point subspace; for this we need the following classical lemma (see [GI04, Lemma 4.8]). For $u \in \mathcal{H}$ we denote by $[u]_{\pm}^{H}$ the closure in $\mathcal{H}$ of $\{e^{itH}u | t \in \mathbb{R}_{\pm}\}$; also $[u]^{H}$ the closure of $\{e^{itH}u | t \in \mathbb{R}\}$.

Lemma 2.13. $u \in H_p^{\mathcal{H}}$ *if and only if the following equivalent assertions hold:*

1. $[u]^H$ *is compact;*
2. $[u]_{+}^{H}$ *is compact;*

3. *$[u]_-^H$ is compact.*

Proof. Let us first assume that $u \in \mathcal{H}_p$. Then there exists a finite collection of normalized eigenvectors $U := \{u_j\}_1^N$ corresponding to the eigenvalues λ_j such that $\| u - \sum_j a_j u_j \| < \frac{\epsilon}{3}$ for some scalars a_j. Then, for every $\epsilon > 0$ we can define $\Phi : [u]^H \to \overline{\mathrm{Sp}}(U)$ by $\Phi(e^{itH}u) = \sum_j a_j e^{it\lambda} u_j$. It is easy to see that it fulfills the conditions of Lemma 1.5 with $\delta = \frac{\epsilon}{3}$, and hence yields the compacity of $[u]^H$.

$1 \Rightarrow 2,3$. Trivial.

$2,3 \Rightarrow u \in \mathcal{H}_p$ Let us assume $[u]_+^H$ is compact and write $u = u_p + u_c$ in the decomposition $\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_c$. Let us fix $g \in \mathcal{H}_p$ associated to the eigenvalue λ; then, because we assume $u_c \in \mathcal{H}_c$, by condition 3 in 2.7 we have w-$\underline{\lim}_{t\to\infty} e^{itH} u_c = 0$. Furthermore, because $[u]_+^H$ and $[u_p]_+^H$ are compact, so is $[u_c]_+^H$. This means that $\lim_{t\to\infty} \| e^{itH} u_c \| = 0$ which finally implies $u_c = 0$ by unitarity of the evolution group. □

Combining 2.5 with 2.13 we trivially get:

Corollary 2.14. *$u \in \mathcal{H}_p$ if and only if:*

1. *$\phi_\omega([u]^H)$ is tight in $L^2(\Sigma)$;*
2. *$[f]^H$ is $\Pi\,[C_c(\Sigma)]$-tight.*

Chapter 3

A particular case: compactness for the magnetic Weyl calculus

In this chapter we present an important example of the framework presented in the previous chapter: the magnetic Weyl calculus. As an introduction, we present in section 3.1 the well-know formula of the standard Weyl calculus to make it easier to the reader see the connection with the magnetic one presented in section 3.2. In section 3.3 we show the magnetic compactness criteria as a corollary of 2.2, but with more characterizations due to the extra structure presented; the results are basically a generalization of [GI04] where no magnetic field was considered. The importance of this chapter lies in the fact that even if the space considered is just $\mathbb{R}^{2n}$, the family of operators indexed by it is no longer a representation, nor even a projective one. For further examples, for instance for a generalization of the magnetic formalism for nilpotent Lie groups, we refer to [BB09, BB11, Ped94].

3.1 The standard Weyl calculus

The aim of this section is to present the standard Weyl calculus, as a motivation for the subsequent presentation of the magnetic one. The setting is the one from section 1.3, but since for the magnetic case we shall need a differentiable structure, we assume $X = \mathbb{R}^n$ from the beginning. This is intend to model a non-relativistic particle moving in the n-dimensional space. The Weyl calculus is the solution to the problem of associating to a classical observable (*i.e.* a "suitable" function on the phase space) a quantum observable (*i.e.* a self-adjoint operator on a Hilbert space). For such a quantization to have a meaning, the starting point is to associate to the coordinate function $\{q_1, \dots q_n, p_1, \dots, p_n\}$ the operators $\{Q_1, \dots Q_n, P_1, \dots, P_n\}$. This is motivated by the fact that some relations for the poisson bracket are still valid when considering the commutator of operators. Explicitly we have

$$\{q_i, q_j\} = 0 = \{p_i, p_j\}, \qquad [Q_i, Q_j] = 0 = [P_i, P_j],$$
$$\{p_i, q_j\} = \delta_{ij}, \qquad [P_i, Q_j] = \delta_{ij},$$

for all $i, j = 1, \dots, n$. Then, for a general symbol, this problem can be seen as the problem of defining a functional calculus for this family of non-commuting operators, $f(Q, P)$. For a nice exposition of the importance of these commutation relations, also related with projective relations, we refer to [Ani10].

To exhibit such a functional calculus we turn now to the theory of C^*-algebras. In fact, despite the title of this section, our presentation is not the standard one, for which we refer the interested reader to [Fol89]. Instead, taking the approach from [MPR05] allows us to develop the basics of the C^*-algebraic theory that has been behind several results. We begin with some definitions.

Definition 3.1. *A C^*-algebra is a complex Banach algebra with an involution satisfying*

$$\| \alpha^* \alpha \| = \| \alpha \|^2 \tag{3.1}$$

So far, we have meet at least three C^*-algebras. First, $\mathbb{B}(\mathcal{H})$, wich in fact, by the GNS-construction, can be seen as the general C^*-algebra. Secondly for Σ compact $C(\Sigma)$ is also a C^*-algebra with the involution defined by complex conjugation. In fact this example can also be seen, this time thanks to the Gelfand theory, as the general Abelian C^*-algebra. The third one is the most suggestive for the aim of this section, but needs some preparations.

Let's recall that $L^1(X)$, with the structure already presented in the discussion following 1.16 is a Banach *-algebra. However, the L^1-norm fails to satisfy (3.1). As was already pointed out, for each unitary representation of X in $\mathcal{H}$ we can construct a non-degenerate *-representation of $L^1(X)$. Taking the supremum over all such representations we can define a norm that fulfill (3.1); taking the completion in this norm we get the C^*-algebra known as *the group C^*-algebra of X* and denoted by $C^*(X)$. In fact, using (1.31) we can see that $C^*(X) \cong C_0(X^\sharp)$.

Because a quantization procedure needs to involve the whole *phase space*, a wider framework is needed.

Definition 3.2. *A* C^*-dynamical system *is a triple* $(\mathcal{A}, \theta, X)$ *formed by an Abelian* C^**-algebra and a strongly continuous group morphism* $\theta : X \to Aut(\mathcal{A})$ *of* X *into the group of automorphism of* $\mathcal{A}$.

Definition 3.3. *A* covariant representation *of the* C^**-dynamical system* $(\mathcal{A}, \theta, X)$ *is a triple* $(\mathcal{H}, r, \pi)$ *where* $r : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ *is a non-degenerate* **-representation of* $\mathcal{A}$ *in* $\mathcal{H}$ *and* π *a strongly continuous unitary representation of* X *in* $\mathcal{H}$, *such that for all* $x \in X$ *and* $\alpha \in \mathcal{A}$ *one has* $U_x r(\alpha) U_{-x} = r[\theta_x(\alpha)]$.

The idea is to produce a C^*-algebra that contains both $C^*(X)$ and $\mathcal{A}$ in a way compatible with the action θ. The first would encode the momentum variable and the second, if one takes $\mathcal{A}$ composed by functions on X, the position variable. The Weyl calculus could the be retrieved as a representation of that C^*-algebra.

Let us consider the Banach space $L^1(X, \mathcal{A})$ with the L^1-norm

$$\| \phi \|_{L^1(X,\mathcal{A})} = \int_X dx \, \| \phi(x) \|_{\mathcal{A}} \; .$$

We define the involution by $\phi^\circ(x) = [\phi(-x)]^*$ where * stands for the involution in $\mathcal{A}$. To get an algebra structure we put

$$(\phi \diamond \varphi)(x) := \int_X dy \theta_{\frac{y-x}{2}}[\phi(y)] \theta_{\frac{y}{2}}[\varphi(x-y)] \, . \tag{3.2}$$

which generalizes the convolution in $L^1(X)$. We have to exhibit a non-degenerate *-representation of the Banach *-algebra $(L^1(X, \mathcal{A}), +, \diamond, \| \cdot \|_{L^1(X,\mathcal{A})}, ^\circ)$ in $\mathcal{H}$ for every covariant representation $(\mathcal{H}, r, \pi)$ of the C^*-dynamical system. This is given by

$$(r \rtimes \pi)(\phi) := \int_X dy r[\theta_{\frac{x}{2}}(\phi(x))] \pi(x) \, . \tag{3.3}$$

Again, we take the supremum over all such *-representations, and complete in this norm, to endow $L^1(X, \mathcal{A})$ with a C^*-algebra structure. This C^*-algebra is denoted by $\mathcal{A} \rtimes X$ and called the *crossed product of* $\mathcal{A}$ *by the action* θ *of the group* X. Note that $r \rtimes \pi$ extends to a representation of $\mathcal{A} \rtimes X$. Note also that $C^*(X) = \mathbb{C} \rtimes X$.

Remark 3.4. *It may seems that the lower indices of the action in* (3.2) *and* (3.3) *are somehow arbitrary. In fact, this is the case because they rely on the choice of a particular endomorphism of the group. Here we take* $\frac{\cdot}{2}$ *to land in the Weyl calculus, but different choices of this endomorphism will give different "quantizations" rules.*

We now particularize to the quantum mechanical framework. The C^*-dynamical system is given by $X = \mathbb{R}^N$, and θ is the action by translations on $\mathcal{A}$, composed of functions on X. To be suitable for this framework, $\mathcal{A}$ should be translation invariant and composed bounded and uniformly continuous functions, because this ensures the strong continuity of θ. The covariant representation can be recover setting $\pi(x) = U_{-x}$ and $r(a) = a(Q)$ (see section 1.4). Note also that a $\phi \in L^1(X, \mathcal{A})$ can be seen as a complex valued function on $X \times X$, so a partial Fourier transform is available. Namely

$$1 \otimes \mathcal{F} : L^1(X, \mathcal{A}) \to C_0(X^\sharp, \mathcal{A})$$

$$\phi \to [(1 \otimes \mathcal{F})\phi](\xi) = \int_X dx e^{i(x\cdot\xi)}\phi(x)$$

Computing $(r \rtimes \pi) \circ (1 \otimes \mathcal{F}^{-1}) =: \mathfrak{Op}$ for a $f \in C_0(X^\sharp, \mathcal{A})$ and $u \in L^2(X)$ we get the standard Weyl calculus:

$$[\mathfrak{Op}(f)u](x) = \int_{\mathbb{R}^{2n}} dy d\xi \, e^{i(x-y)\cdot\xi} \left[f\left(\frac{x+y}{2}\right)\right](\xi)u(y) . \tag{3.4}$$

Several remarks could be made about this construction. First note that the composition of (3.2) with the partial inverse Fourier transform yields the usual, and widely studied, *Moyal product* so part of the theory of pseudodifferential operators can be seen through the setting of the crossed-products of C^*-algebras. Also we get a good characterization of the regularity, encoded by $\mathcal{A}$, needed to achieve the calculus.

3.2 The magnetic formalism

The magnetic pseudodifferential calculus [MP04, IMP07] has as a background the problem of quantization of a physical system consisting in a spin-less particle moving in the euclidean space $X := \mathbb{R}^n$ under the influence of a magnetic field, *i.e.* a closed 2-form B on X ($dB = 0$), given by matrix-component functions

$B_{jk} = -B_{kj} : X \to \mathbb{R}$, $j, k = 1, \ldots, n$. For convenience, we are going to assume that the components B_{jk} belong to $C^\infty_{\text{pol}}(X)$, the class of smooth functions on X with polynomial bounds on all the derivatives. The magnetic field can be written in many ways as the differential $B = dA$ of some 1-form A on X called *vector potential*. One has $B = dA = dA'$ iff $A' = A + d\varphi$ for some 0-form φ (then they are called equivalent). It is easy to see that the vector potential can also be chosen of class $C^\infty_{\text{pol}}(X)$; this will be tacitly assumed.

One would like to develop a symbolic calculus $a \mapsto \mathfrak{Op}^A(a)$ taking the magnetic field into account. Basic requirements are: (i) it should reduce to the standard Weyl calculus for $A = 0$ (compare with (3.4)) and (ii) the operators $\mathfrak{Op}^A(a)$ and $\mathfrak{Op}^{A'}(a)$ should be unitarily equivalent (independently on the symbol a) if A and A' are equivalent; this is called *gauge covariance* and has a fundamental physical meaning. In this section we leave aside the C^*-algebraic approach, partially because the theory of *twisted cross-product* is far beyond the scope of this thesis, and chose to think of the emerging symbolic calculus as a functional calculus for the family of non-commuting sel-adjoint operators $(Q_1, \ldots, Q_n; P^A_1, \ldots, P^A_n)$ in $\mathcal{H} := L^2(X)$. Here Q_j is again one of the components of the position operator, but the momentum $P_j := -i\partial_j$ is replaced by *the magnetic momentum* $P^A_j := P_j - A_j(Q)$ where $A_j(Q)$ indicates the operator of multiplication with the function $A_j \in C^\infty_{\text{pol}}(X)$. Notice the commutation relations

$$i[Q_j, Q_k] = 0\,, \quad i[P^A_j, Q_k] = \delta_{jk}\,, \quad i[P^A_j, P^A_k] = B_{jk}(Q)\,. \tag{3.5}$$

One defines *the magnetic Weyl system*

$$\pi^A : \Sigma \to \mathbb{B}(\mathcal{H}), \qquad \pi^A(x, \xi) := \exp\left[i\left(x \cdot P^A - Q \cdot \xi\right)\right] \tag{3.6}$$

and gets in terms of the circulation of the 1-form A through the segment $[y, y + x] := \{y + tx \mid t \in [0, 1]\}$ the explicit formula

$$\left[\pi^A(x, \xi)u\right](y) = e^{-i\left(y + \frac{x}{2}\right)\cdot\xi} \exp\left[(-i) \int_{[y,y+x]} A\right] u(y + x)\,. \tag{3.7}$$

These operators depend strongly continuously on (x, ξ) and satisfy $\pi^A(0, 0) = 1$ and $\pi^A(x, \xi)^* = \pi^A(x, \xi)^{-1} = \pi^A(-x, -\xi)$ (thus being unitary). *However they do not form a projective representation of* $\Sigma = X \times X^\sharp$. Actually they satisfy

$$\pi^A(x, \xi)\, \pi^A(y, \eta) = \omega^B[(x, \xi), (y, \eta); Q]\, \pi^A(x + y, \xi + \eta)\,, \tag{3.8}$$

where $\omega^B[(x,\xi),(y,\eta);Q]$ only depends on the 2-form B and denotes the operator of multiplication in $L^2(X)$ by the function

$$\omega^B[(x,\xi),(y,\eta);\cdot] := \exp\left[\frac{i}{2}(y\cdot\xi - x\cdot\eta)\right]\exp\left[(-i)\int_{<\cdot,\cdot+x,\cdot+x+y>} B\right] \tag{3.9}$$

Here the distinguished factor is constructed with the flux (invariant integration) of the magnetic field through the triangle defined by the corners z, $z+x$ and $z+x+y$. A straightforward computation leads to *the magnetic Fourier-Wigner function*:

$$\begin{aligned}\left[\Phi^A(u\otimes v)\right](x,\xi) &\equiv \left[\phi_v^A(u)\right](x,\xi) := \langle\pi^A(x,\xi)u,v\rangle \\ &= \int_X dy\, e^{-iy\cdot\xi}\exp\left[(-i)\int_{[y-x/2,y+x/2]} A\right] u(y+x/2)\,\overline{v(y-x/2)}\,.\end{aligned}$$

It can be decomposed into the product of the multiplication by a function with values in the unit circle, a change of variables with unit jacobian and a partial Fourier transform. All these are isomorphisms, so $\Phi^A : L^2(X)\widehat{\otimes}L^2(X) \to L^2(\Sigma)$ defines a unitary transformation. *Thus we get a formalism which is a particular case of the previous chapter.* Therefore one can apply all the prescriptions and get the correspondence

$$f \mapsto \Pi^A(f) := \int_\Sigma f(x,\xi)\,\pi^A(-x,-\xi)\,dx d\xi\,. \tag{3.10}$$

In fact people are interested in the (symplectic) Fourier transformed version $a(Q,P^A) \equiv \mathfrak{Op}^A(a) := \Pi^A[\mathfrak{F}^{-1}(a)]$. The resulting *magnetic Weyl calculus* is given by

$$\left[\mathfrak{Op}^A(a)u\right](x) = (2\pi)^{-n}\int_X dy\int_{X^\sharp} d\xi e^{[i(x-y)\cdot\xi]} e^{\left[-i\int_{[x,y]}A\right]} a\left(\frac{x+y}{2},\xi\right)u(y) \tag{3.11}$$

An important property of (3.11) is *gauge covariance*, as hinted above: if $A' = A + d\rho$ defines the same magnetic field as A, then

$$\mathfrak{Op}^{A'}(a) = e^{i\rho}\,\mathfrak{Op}^A(a)\,e^{-i\rho}\,.$$

By killing the magnetic phase factors in all the formulae above one gets the defining relations of the usual Weyl calculus.

Due to the particular structure, one can introduce $\{U^A(x) := \pi^A(x,0) \,|\, x \in X\}$ (generalizing the group of translations for $A \neq 0$) and $\{V(\xi) := \pi^A(0,\xi) \,|\, \xi \in X^\sharp\}$ (the group generated by the position operator Q). One can also introduce $\varphi(Q) := \mathfrak{Op}^A(\varphi \otimes 1)$ and $\psi(P^A) := \mathfrak{Op}^A(1 \otimes \psi)$ for $\varphi \in L^2(X)$ and $\psi \in L^2(X^\sharp)$. One checks easily that $\varphi(Q)$ is the operator of multiplication by φ while for zero magnetic field $\psi(P^{A=0}) \equiv \psi(P)$ is the operator of convolution by the Fourier transform of ψ. Since $\varphi \otimes 1$ and $1 \otimes \psi$ are not L^2-functions in both variables, one needs the results of [MP04, IMP07] for an easy justification of these objects. Equivalently, one can use formulas as $\psi(P^A) := \int_X dx\, \widehat{\psi}(x) U^A(x)$.

The next result is inspired by [GI04, Prop. 2.2] and basically reduces to [GI04, Prop. 2.2] for $A = 0$. By $\mathcal{S}(Y)$ we denote the Schwartz space on the real finite-dimensional vector space Y.

Proposition 3.5. *The C^*-algebra $\mathbb{K}[L^2(X)]$ of compact operators in $L^2(X)$ coincides with the closed vector space $\mathfrak{C}$ generated in $\mathbb{B}[L^2(X)]$ by products $\varphi(Q)\psi(P^A)$ with $\varphi \in \mathcal{S}(X)$ and $\psi \in \mathcal{S}(X^\sharp)$.*

Proof. It is easy to check that $\varphi(Q)\psi(P^A)$ is an integral operator with kernel given for $x, y \in X$ by

$$k^A_{\varphi,\psi}(x,y) = e^{-i\int_{[x,y]} A}\, \varphi(x)\widehat{\psi}(y-x). \tag{3.12}$$

We assumed the components of A to be C^∞_{pol}-functions and this immediatly implies that the magnetic phase factor in (3.12) belongs to $C^\infty_{pol}(X \times X)$. Therefore, if $\varphi \in \mathcal{S}(X)$ and $\psi \in \mathcal{S}(X)$, then $k^A_{\varphi,\psi} \in \mathcal{S}(X \times X) \subset L^2(X \times X)$ and thus $\varphi(Q)\psi(P^A)$ is a Hilbert-Schmidt operator. From this follows $\mathbb{K}[L^2(X)] \supset \mathfrak{C}$.

Reciprocally, it is enough to show that $\mathfrak{C}$ contains all the integral operators with kernel $k \in L^2(X \times X)$ (they are the Hilbert-Schmidt operators and form a dense set in $\mathbb{K}[L^2(X)]$). Pick inside the Schwartz space $\mathcal{S}(X)$ an orthonormal base $\{e_i \,|\, i \in \mathbb{N}\}$ for $L^2(X)$. Setting

$$F^A_{ij}(x,y) := e^{-i\int_{[x,y]} A}\, e_i(x) e_j(y-x), \quad \forall\, x, y \in X,\ i,j \in \mathbb{N},$$

we get an orthonormal base $\{F^A_{ij} \mid i,j \in \mathbb{N}\}$ of $L^2(X \times X)$. So $k = \sum_{i,j} c_{ij} F^A_{ij}$, where $\sum_{i,j} |c_{ij}|^2 < \infty$ and the sum is convergent in $L^2(X \times X)$. Then the integral operator with kernel k coincides with $\sum_{i,j} c_{ij} e_i(Q) \widehat{e}_j(P^A)$. The sum converges in $\mathbb{B}_2[L^2(X)]$, thus in $\mathbb{B}[L^2(X)]$, therefore the operator belongs to $\mathfrak{C}$. □

3.3 Magnetic compactness criterion

We present now the main result of the Chapter.

Theorem 3.6. *Let Ω a bounded subset of $\mathcal{H} := L^2(X)$. The following statements are equivalent:*

1. *The set Ω is relatively compact.*
2. *For some (any) window $w \in \mathcal{H}$, the family $\phi^A_w(\Omega)$ is $\mathcal{K}(\Sigma)$-tight in $L^2(\Sigma)$.*
3. *For every $\epsilon > 0$ there exist $f \in C_c(\Sigma)$ with $\sup_{u\in\Omega} \| \left[\mathfrak{Op}^A(\widehat{f}) - 1\right] u \| \leq \epsilon$.*
4. *One has*
$$\lim_{(x,\xi)\to 0} \sup_{u\in\Omega} \| \left[\pi^A(x,\xi) - 1\right] u \| = 0 . \tag{3.13}$$
5. *One has*
$$\lim_{x\to 0} \sup_{u\in\Omega} \| \left[U^A(x) - 1\right] u \| = 0 \quad \textit{and} \quad \lim_{\xi\to 0} \sup_{u\in\Omega} \| [V(\xi) - 1] u \| = 0 . \tag{3.14}$$
6. *For every $\epsilon > 0$ there exist $\varphi \in \mathcal{S}(X)$ and $\psi \in \mathcal{S}(X^\sharp)$ with*
$$\sup_{u\in\Omega} \left(\| [\varphi(Q) - 1] u \| + \| \left[\psi(P^A) - 1\right] u \| \right) \leq \epsilon . \tag{3.15}$$

Proof. $1 \Leftrightarrow 2 \Leftrightarrow 3$ follow from Theorem 2.5 by particularization, while $4 \Leftrightarrow 5$ is trivial, taking into account the relathionships between U^A, V and π^A. The implication $3 \Rightarrow 4$ also holds, taking $s_0 = 0$ in Theorem 2.5 (and replacing s by $-s$). A careful examination of (3.8) and (3.9) would even lead to $3 \Leftrightarrow 4$, restauring the relevant convergence for arbitrary $s_0 := (x_0, \xi_0)$, but this will not be needed. $1 \Rightarrow 5$ follows trivially, because Ω can be approximated by finite sets and U^A, V are strongly continuous at the origin.

$5 \Rightarrow 6$ can be obtained along the same lines as the proof of the implication $4 \Rightarrow 5$ in Theorem 2.5, taking also into account the relations $\psi(P^A) = \int_X dx\, \widehat{\psi}(x) U^A(x)$ and $\varphi(Q) = \int_{X^\sharp} d\xi\, \widehat{\varphi}(\xi) V(\xi)$.

We finally show $6 \Rightarrow 3$. Let us set $T^\perp := 1 - T$ and compute
$$\begin{aligned} \| u - \varphi(Q)\psi(P^A)u \| &= \| \varphi(Q)\psi(P^A)^\perp u + \varphi(Q)^\perp u \| \\ &\leq \| \varphi \|_\infty \| \psi(P^A)^\perp u \| + \| \varphi(Q)^\perp u \| . \end{aligned}$$

By using the assumption 6, this can be made arbitrary small uniformly in $u \in \Omega$ if φ, ψ are chosen suitably. As in the proof of Proposition 3.5 one sees that $\varphi(Q)\psi(P^A)$ is a Hilbert-Schmidt operator. It can be approximated arbitrarily in norm by some operator $\mathfrak{Op}^A(\widehat{f})$ with $f \in C_c(\Sigma)$ and then 3 follows easily because Ω is bounded. □

Remark 3.7. *Many small variations are allowed in the results above. The Schwartz spaces $\mathcal{S}(X)$ and $\mathcal{S}(X^\sharp)$ in Proposition 3.5 or at point 6 of Theorem 3.6 can be replaced by other convenient "small" spaces. In Theorem 3.6, at point 3 one could use $\mathfrak{Op}^A(a)$ with $a \in \mathcal{S}(\Sigma)$ or with $a \in C_c^\infty(\Sigma)$.*

Chapter 4

Compactness in coorbit spaces associated to continuous frames

In this chapter we present compactness criteria for Banach spaces constructed in a way related to a Hilbert space. In section 4.3 we present a way of obtaining such spaces by means of a *tight continuous frame*, a more general framework than the one presented in section 2.1, but having a less rich mathematical structure. That is why only the *representation coefficient* approach will be available in section 4.4. As seen above, in that approach both a weakly compactness and a dominated convergence theorem were needed. In sections 4.1 and 4.2 we produce this results in a context that is useful for coorbit spaces.

4.1 The Alaoğlu-Bourbaki Theorem

In this section we want to present a well-know result that will allow us to extract a convergent subsequence in section 4.4. In order to do that we need to show that the dual of a topological vector space has a Heine-Borel type of property. Obviously this will depend on both what we will consider to be bounded sets and the topology chosen. This section is mostly based on [Rud91, Theo. 3.15].

Definition 4.1. *A* topological vector space *is a (real or complex) vector space $\mathcal{X}$ endowed with a topology such that:*

1. *every point of $\mathcal{X}$ is closed and*
2. *the vector spaces operation are continuous.*

Note that even if we don't ask $\mathcal{X}$ to be Hausdorff, it will follows from the definition. For a topological space $\mathcal{X}$ we will denote by $\mathcal{X}'_\sigma$ is dual where σ indicates that we consider on $\mathcal{X}'$ the *-weak topology. We also note by $\mathbb{K}$ the field ($\mathbb{R}$ or $\mathbb{C}$). By $\mathbb{K}^{\mathcal{X}}$ we denote the cartesian product of $\mathbb{K}$ indexed by $\mathcal{X}$ with the product topology. It can be realize as the set of $\mathbb{K}$-valued functions defined on $\mathcal{X}$ endowed with the sup-norm. The following lemma is part of the basic theory of topological vector spaces.

Proposition 4.2. *Let $x \in \mathcal{X}$ and $V \in \mathcal{V}(0_{\mathcal{X}})$. Then, there exists $r \in \mathbb{R}$ such that $x \in rV$.*

Proof. Let x be arbirary and asume that V is open. By continuity of $\mathbb{C} \ni z \to zx \in \mathcal{X}$ we know that the set of all scalars z with $zx \in V$ is open. Furthermore, it contains 0 because $0x = 0_{\mathcal{X}} \in V$. So, we can find r small enough such that $r^{-1}x \in V \Rightarrow x \in zV$. □

A set with this property is usually refer as an absorbing set; we show then that, in a topological vector space, every neighborhood of $O_{\mathcal{X}}$ is absorbing. The next definition is also standard.

Definition 4.3. *For $K \subset \mathcal{X}$ we define the polar of K by*

$$K^\circ := \{y \in \mathcal{X}' : |\langle x, y\rangle| < 1, \textit{for all } x \in K\}$$

We are ready to state and prove the next theorem.

Theorem 4.4 (Alaoğlu Bourbaki). *For $V \in \mathcal{V}(O_{\mathcal{X}})$ in a topological vector space $\mathcal{X}$, V° is *-weak compact.*

Proof. For every $x \in \mathcal{X}$, let us denote by $\gamma(x)$ a number such that $x \in \gamma(x)V$. We consider

$$P := \{f \in \mathbb{K}^{\mathcal{X}} : |f(x)| \leq \gamma(x)\}. \tag{4.1}$$

Clearly P is compact by Tychonoff's Theorem. We can now see that $V^\circ \subset \mathcal{X}'_\sigma \cap P$, so it inherits two subspace topologies, from $\mathcal{X}'_\sigma$ and from P. Let us see that the topologies coincides in V°. For this, we fix some $y_0 \in V^\circ$ and look for a base of $\mathcal{V}_{\mathcal{X}'_\sigma}(y_0)$ and of $\mathcal{V}_P(y_0)$. For every finite family $\mathfrak{F}$ of element of X we put

$$\left\{U_{\mathcal{X}'_\sigma}(\mathfrak{F}\,;\delta) := \{y \in \mathcal{X}' : |\langle x, y - y_0\rangle| < \delta\} : x \in \mathfrak{F}, \delta > 0\right\}$$

and

$$\left\{U_P(\mathfrak{F}\,;\delta) := \{f \in \mathbb{K}^{\mathcal{X}} : |f(x) - \langle x, y_0\rangle| < \delta\} : x \in \mathfrak{F}, \delta > 0\right\}.$$

Clearly they are bases $\mathcal{V}_{\mathcal{X}'_\sigma}(y_0)$ and of $\mathcal{V}_P(y_0)$ respectively and they coincides in V° yielding the desired coincidence between both subspace topology. If we can show that V° is closed in P, the result will follow from the compacity of P.

Let f_0 be in the closure of V° in P. Let us choose $x, y \in \mathcal{X}$, $a \in \mathbb{K}$ and set $\mathfrak{F} = \{x_1, x_2, ax_1 + x_2\}$. For $\epsilon > 0$ we consider the neighborhood of f_0 defined by

$$\{f \in \mathbb{K}^{\mathcal{X}} : |f(x) - f_0(x)| < \epsilon \text{ for all } x \in \mathfrak{F}\}. \tag{4.2}$$

We pick an element of this neighborhood lying in V° and we denoted by y. We compute:

$$\begin{aligned} |f_0(ax_1 + x_2) - af_0(x_1) - f_0(x_2)| =& |(f_0 - y)(ax_1 + x_2) - a(f_0 - y)(x_1) \\ & - (f_0 - y)(x_2)| \\ =& (2 + |a|)\epsilon \,. \end{aligned}$$

The arbitrariness of ϵ yields the linearity of f_0. To see that $f_0 \in V$ for an arbitrary $x \in \mathcal{X}$ we take $\mathfrak{F} = \{x\}$ and y in the intersection of this neighborhood (defined again by (4.2)) and V. Then

$$|f_0(x)| \leq |f_0(x) - \langle x, y\rangle| + |\langle x, y\rangle| < 1 + \epsilon\,. \tag{4.3}$$

Again, taking ϵ arbitrarily small we have f_0 in V° and this finish the proof. □

Note that we didn't have to prove the continuity of f_0 because the boundedness in V implies it; one can argue that 4.3 is redundant in the sense that it wasn't necessary to assume the continuity of y. We have the following corollary.

Corollary 4.5. *Let $\Omega \subset \mathcal{X}'$ be equicontinuous. Then Ω is relatively compact in the weak-* topology.*

Proof. By equicontinuity we can find some $V \in \mathcal{V}(0_{\mathcal{X}})$ such that for every $x \in V$ we have:

$$\sup_{y\in\mathcal{X}'} |\langle x, y\rangle| = \sup_{y\in\mathcal{X}'} |\langle x, y\rangle - \langle x, 0\rangle| \leq 1 \tag{4.4}$$

Then $\Omega \subset V^\circ$ and the result follows from 4.4. □

Recall that a Fréchet spaces is a metrizable locally convex complete topological vector space. Being locally convex, the topology can be described in terms of a family of seminorms (actually the are defined as the *Minkowski functionals* of a base of neighborhood of 0 formed by convex sets).

Lemma 4.6. *Let $\mathfrak{S}(\mathcal{G})$ a family of seminorm defining the topology of $\mathcal{G}$. Assume that $\mathcal{Y}$ is a normed space continuously embedded in $\mathcal{G}'_\sigma$ and let $\Omega \subset \mathcal{Y}$ be bounded.*

1. *For every $p \in \mathfrak{S}(\mathcal{G})$ there exists a positive constant D_p such that*

$$|\langle u, v\rangle| \leq D_p \| u \|_{\mathcal{Y}}\, p(v) , \qquad \forall\, v \in \mathcal{G},\, u \in \mathcal{Y} .$$

2. *Seen as a subset of $\mathcal{G}'$, the set Ω is equicontinuous and (consequently) relatively compact in the weak-* topology.*

Proof. If one assumes that $\mathcal{Y}$ is continuously embedded $\mathcal{G}'_\sigma$, then $|\langle u, \cdot\rangle|$ is continuous for every $u \in \mathcal{Y}$ given the property 1. To prove 2 we can see that a base of neighborhoods of the origin in $\mathcal{G}$ is

$$\big\{U(p\,; \delta) := \{v \in \mathcal{G} \,|\, p(v) < \delta\} \mid p \in \mathfrak{S}(\mathcal{G}), \delta > 0\big\} .$$

Assume that $\| u \|_{\mathcal{Y}} \leq M$ for every $u \in \Omega$. Let $\epsilon > 0$ and $p \in \mathfrak{S}(\mathcal{G})$. Using 1, for every $u \in U\Big(p\,; \frac{\epsilon}{MC_p}\Big)$ and every $u \in \Omega$ one gets

$$|\langle u, v\rangle| \leq D_p \| u \|_{\mathcal{X}}\, p(v) \leq D_p M p(v) \leq \epsilon ,$$

and this is equicontinuity. The last statement follows from Corollary 4.5. □

4.2 Banach function spaces

In this section we present some facts about the abstract theory of *Banach function spaces*; this theory allows us to characterize the spaces of functions over a measure space in which a version of the Dominated Convergence Theorem is available. We follow the presentation made in [BS88, Chap. 1] but the older [Lux55] is in certain aspects better. Throughout this section (Σ, μ) will denote a measure space assumed to be σ-bounded, $\overline{\mathbb{R}}_+$ the extended positive line $[0, \infty]$, and $\mathfrak{M}^+$ (resp. $\mathfrak{M}$) the set of $\overline{\mathbb{R}}_+$-valued (resp. $\overline{\mathbb{R}}$ or $\overline{\mathbb{C}}$-valued) μ-measurable functions. We start with some definitions.

Definition 4.7. *A mapping $\rho : \mathfrak{M}^+ \to \overline{\mathbb{R}}_+$, is called a* Banach function norm *if for all f, g, f_n in $\mathfrak{M}^+$, for all constants $a \geq 0$ and for all μ-measurable subsets $E \subset \Sigma$, the following properties hold:*

P1 $\rho(f) = 0 \Leftrightarrow f = 0\, \mu$*-a.e.;* $\rho(af) = a\rho(f)$; $\rho(f+g) \leq \rho(f) + \rho(g)$;

P2 $0 \leq f_n \uparrow f\, \mu$*-a.e.;* $\Rightarrow \rho(f_n) \uparrow \rho(f)$;

P3 $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$;

P4 $\mu(E) < \infty \Rightarrow \int_E d\mu f \leq C_E \rho(f)$ *for some constant* $0 < C_E < \infty$, *independent of* f.

Definition 4.8. *The collection* $\mathcal{M} = \mathcal{M}(\rho)$ *of all functions* $f \in \mathfrak{M}$ *such that* $\rho(|f|) < \infty$ *is called a* Banach function space *with norm*

$$\| f \|_{\mathcal{M}} = \rho(|f|)\,. \tag{4.5}$$

The few results that we will need are gathered in the next proposition.

Proposition 4.9. *Let be* $\mathcal{M}$ *a Banach function space. Then*

1. $\mathcal{M}$ *is a Banach space;*
2. *if* $g \leq f\; \mu$*-a.e. and* $f \in \mathcal{M}$ *then* $g \in \mathcal{M}$*;*
3. *If* μ *is a Radon measure we have* $\mathcal{K}(\Sigma) \subset \mathcal{M}$*;*
4. *If* $f_n \to f$ *in* $\mathcal{M}$ *then* $f_n \to f$ *in measure.*

Proof. Property 2 follows from **P2** and 3 from **P3**.

4. Let $f_n \to f$ in $\mathcal{M}$ and let E of finite measure. For every $\epsilon > 0$ we fix N such that $\rho(|f - f_n|) \leq \frac{\epsilon^2}{C_E}$ for every $n \geq N$ using **P3** and **P4**. Then, for $n \geq N$

$$\begin{aligned} \mu\{x \in E : |f(x)| > \epsilon\} \leq & \int_E d\mu\, \epsilon^{-1} |f - f_n| \\ \leq & \epsilon^{-1} C_E \rho(|f - f_n|) \leq \epsilon \end{aligned}$$

1. We prove only the completeness. Consider $\{f_n\} \subset \mathcal{M}$ with $\sum_{n=1}^{\infty} \| f_n \|_{\mathcal{M}}$ finite. We set $t := \sum_{n=1}^{\infty} |f_n|$, $t_N := \sum_{n=1}^{N} |f_n|$. Clearly $0 \leq t_N \uparrow t$ as $N \to \infty$, so by **P2** $\rho(t_N) \uparrow \rho(t)$. But

$$\rho(t_N) \leq \sum_{n=1}^{N} \| f_n \|_{\mathcal{M}} \leq \sum_{n=1}^{\infty} \| f_n \|_{\mathcal{M}} \tag{4.6}$$

so $\rho(t) < \infty$ and we have $t_n, t \in \mathcal{M}$ together with $t_N \to t$ in $\mathcal{M}$. By 4 we can show that $t_N \to t$ in measure and hence has a subsequence converging pointwise to t. So $\sum f_n$ is well define μ-a.e. because $\sum |f_n|$ is it (and $\mathbb{R}$ or $\mathbb{C}$ are complete). We can set

$$f := \sum_{n=1}^{\infty} f_n \quad \text{and} \quad s_N := \sum_{n=1}^{N} f_n$$

For every $m \in \mathbb{N}$ we have that $s_n - s_M \to f - s_M$ μ-a.e. as $n \to \infty$. So

$$\liminf_{n\to\infty m\geq n} \| s_m - s_M \|_{\mathcal{M}} \leq \liminf_{n\to\infty m\geq n} \sum_{j=M+1}^{m} \| f_n \|_{\mathcal{M}} = \sum_{j=M+1}^{\infty} \| f_n \|_{\mathcal{M}} .$$

Because $\inf_{m\geq n} |s_m - s_M| \uparrow |f - s_M|$, we can use **P2** to get

$$\rho(|f - s_M|) = \lim_{n\to\infty} \inf_{m\geq n} |S_m - s_M| \leq \sum_{j=M+1}^{\infty} \| f_n \|_{\mathcal{M}} .$$

So $f - s_M \in \mathcal{M}$, and hence $\| f - s_M \|_{\mathcal{M}} \to 0$. □

We said that a sequence E_n of measurable sets converges to $\emptyset$ if the set theoretic limit (see (0.1)) has measure zero. The next definition will allow us to reobtain the Dominated Convergence Theorem in this ampler setting.

Definition 4.10. *A function f in a Banach function space $\mathcal{M}$ is said to have* absolutely continuous norm *in $\mathcal{M}$ if $\| f\chi_E \|_{\mathcal{M}} \to 0$ for every $E_n \to$. The set of all functions in $\mathcal{M}$ with absolutely continuous norm is denoted by $\mathcal{M}_a$.*

Example 4.11. *We want to show that $\mathcal{M}_a$ can really be a proper subspace. It is not he case for L^p for $1 \leq p < \infty$ but the $p = \infty$ case is interesting. If μ is 0 for every singleton, as the Lebesgue measure, then $\mathcal{M}_a = \{0\}$ because for every $x \in \Sigma$ we could construct sequence of sets E_n of positive measure that converges to $\{x\}$.*

We can now state the main result of the section.

Theorem 4.12. *$f \in \mathcal{M}_a$ if and only if the followings condition holds: whenever f_n, g are μ-measurable functions satisfying $|f_n| \leq |f|$ and $f \to g$ μ-a.e., then $\| f_n - g \| \to 0$.*

For the proof we need some propositions.

Proposition 4.13. *$f \in \mathcal{M}_a$ if and only if $\| f\chi_{E_n} \|_{\mathcal{M}} \downarrow 0$ for every $E_n \downarrow \emptyset$.*

Proof. The "only if" part is trivial and the "if" follows easily if for some arbitrary sequence F_n we set $E_n = \underset{m \geq n}{\cup} F_m$. Then $E_n \downarrow 0$ and $F_n \subset E_n$ for every n. Then $0 \leq \| f\chi_{F_n} \|_{\mathcal{M}} \leq \| f\chi_{E_n} \|_{\mathcal{M}}$ and the middle term, being sandwiched by a null sequence, goes to zero. □

Proposition 4.14. *Let ϵ be greater than 0. If $f \in \mathcal{M}_a$ we can find $\delta > 0$ such that $\mu(E) < \delta \Rightarrow \| f\chi_E \|_{\mathcal{M}} < \epsilon$.*

Proof. Let ϵ be greater than zero such that for every $\delta > 0$ there exist E measurable with $\mu(E) < \delta$ and $\| f\chi_E \|_{\mathcal{M}} \geq \epsilon$. Then we can choose E_n such that $\mu(E_n) < 2^{-n}$ and $\| f\chi_E \|_{\mathcal{M}} \geq \epsilon$. Since $\mu(\bigcup E_n) \leq 1$ we have $E_n \to \emptyset \mu$-a.e. and hence $f \notin \mathcal{M}_a$. □

Proposition 4.15. *$f \in \mathcal{M}_a$ if and only if for every $f_n \downarrow 0\, \mu$-a.e. such that $f_n \leq |f|$, μ-a.e. for every n one has $\| f_n \|_{\mathcal{M}} \downarrow 0$.*

Proof. The "only if" follows by taking $f_n = f\chi_{E_n}$ for an arbitrary null sequence E_n. Suppose now that $f \in \mathcal{M}_a$. Fix some $R_n \uparrow \Sigma$ composed of sets the finite measure. Then $\Sigma \backslash R_n =: Q_n \downarrow \emptyset$ so there exists some N_1 such that $\| f\chi_{Q_n} \|_{\mathcal{M}} \leq \frac{\epsilon}{2}$ for every $n \geq N_1$. We set then $a := \frac{\epsilon}{4} \| \chi_{R_{N_1}} \|_{\mathcal{M}}$ and let $E_n := \{x \in \Sigma : f_n(x) > a\}$. Since $f \downarrow 0\, \mu$-a.e., then $f \to 0$ in measure, so we have $\mu(E_n) \downarrow 0$. By Proposition 4.14 we choose N_2 such that $\| f\chi_{E_n} \|_{\mathcal{M}} \leq \frac{\epsilon}{4}$ for every $n > N_2$. Taking $N := \max\{N_1, N_2\}$ and $n > N$ we have

$$\begin{aligned} \| f_n \|_{\mathcal{M}} &\leq \| f_n\chi_{Q_n} \|_{\mathcal{M}} + \| f_n\chi_{E_n} \|_{\mathcal{M}} + \| f_n\chi_{R_n \backslash E_n} \|_{\mathcal{M}} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + a \, \| \chi_{R_n} \|_{\mathcal{M}} \leq \epsilon \,. \end{aligned}$$

□

Proof of 4.12. The "only if" part goes as in the previous Proposition. Suppose $f \in \mathcal{M}_a$ and f_n, g are as in the hypothesis. We can set $h_n(x) = \sup_{m \geq n} |f_m(x) - g(x)|$. The h_n is as in 4.15 so we have $\| h_n \|_{\mathcal{M}} \downarrow 0$. The results then follows from

$$0 \leq \| f_n - g \| \; \mathcal{M} \leq \| h_n \|_{\mathcal{M}} \downarrow 0 \,. \tag{4.7}$$

□

When $\mathcal{M} = \mathcal{M}_a$ we say that $\mathcal{M}$ is a *Banach function space with absolutely continuous norm.*

4.3 The coorbit formalism

We recall now the concept of tight continuous frame and the construction of coorbit spaces, slightly modifying the approach of [FR05, RU11]. Let us fix a family $W := \{w(s) \,|\, s \in \Sigma\} \subset \mathcal{H}$ that is a tight continuous frame; the constant of the frame is assumed to be 1 by normalizing the measure μ. This means that the map $s \mapsto w(s)$ is assumed weakly continuous and for every $u, v \in \mathcal{H}$ one has

$$\langle u, v \rangle = \int_\Sigma d\mu(s) \langle u, w(s) \rangle \langle w(s), v \rangle . \tag{4.8}$$

Clearly W is total in $\mathcal{H}$ and defines an isometric operator

$$\phi_W : \mathcal{H} \to L^2(\Sigma) , \quad [\phi_W(u)]\,(s) := \langle u, w(s) \rangle \tag{4.9}$$

with adjoint $\phi_W^\dagger : L^2(\Sigma) \to \mathcal{H}$ given (in weak sense) by

$$\phi_W^\dagger(f) = \int_\Sigma d\mu(s) f(s) w(s) . \tag{4.10}$$

The (Gramian) kernel associated to the frame is the function $p_W : \Sigma \times \Sigma \to \mathbb{C}$ given by

$$p_W(s,t) := \langle w(t), w(s) \rangle = [\phi_W(w(t))]\,(s) = \overline{[\phi_W(w(s))]\,(t)} , \tag{4.11}$$

defining a self-adjoint integral operator $P_W = \mathfrak{Int}(p_W)$ in $L^2(\Sigma)$. One checks easily that $P_W = \phi_W \phi_W^\dagger$ is the final projection of the isometry ϕ_W, so $P_W[L^2(\Sigma)]$ is a closed subspace of $L^2(\Sigma)$. Since $\phi_W^\dagger \phi_W = 1$, one has the inversion formula

$$u = \int_\Sigma d\mu(t)\, [\phi_W(u)]\,(t)\, w(t) , \tag{4.12}$$

leading to the reproducing formula $\phi_W(u) = P_W\,[\phi_W(u)]$, i.e.

$$[\phi_W(u)]\,(s) = \int_\Sigma d\mu(t) \langle w(t), w(s) \rangle \,[\phi_W(u)]\,(t) . \tag{4.13}$$

Thus $\mathscr{P}_W(\Sigma) := P_W\,[L^2(\Sigma)]$ is a reproducing space with reproducing kernel p_W; it is composed of continuous functions on Σ.

To extend the setting above beyond the L^2-theory, one can supply an extra space of "test vectors", denoted by $\mathcal{G}$, assumed to be a Fréchet space continuously and densely embedded in $\mathcal{H}$. Applying Riesz isomorphism we are led to a

Gelfand triple $(\mathcal{G}, \mathcal{H}, \mathcal{G}'_\sigma)$. The index σ refers to the fact that on the topological dual $\mathcal{G}'$ we consider usually the weak-* topology. In certain circumstances one takes $\mathcal{G}$ to be a Banach space and sometimes it can even be fabricated from the frame W and from some extra ingredients, as below. But very often (think of the Schwartz space) the auxiliar space $\mathcal{G}$ is only Fréchet.

We shall suppose that *the family W is contained and total in $\mathcal{G}$ and that $\Sigma \ni s \mapsto w(s) \in \mathcal{G}$ is a weakly continuous function.* Then we extend ϕ_W to $\mathcal{G}'$ by $[\phi_W(u)](s) := \langle u, w(s)\rangle$, where the r.h.s. denotes now the number obtained by applying $u \in \mathcal{G}'$ to $w(s) \in \mathcal{G}$ and depends continuously on s. By the totality of the family W in $\mathcal{G}$, this extension is injective. In addition, $\Phi_W : \mathcal{G}' \to C(\Sigma)$ is continuous if one consider on $\mathcal{G}'$ the weak-* topology and on $C(\Sigma)$ the topology of pointwise convergence.

As in [FG89, FR05, RU11] and many other references treating coorbit spaces, one uses $\phi_W(\cdot)$ to pull back subspaces of functions on Σ. So let $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ be a normed space of functions on Σ (more assumptions on $\mathcal{M}$ will be imposed when necessary) and set

$$\mathsf{co}_W(\mathcal{M}) \equiv \mathsf{co}(\mathcal{M}) := \{u \in \mathcal{G}' \mid \phi_W(u) \in \mathcal{M}\}, \qquad \| u \|_{\mathsf{co}(\mathcal{M})} := \| \phi_W(u) \|_{\mathcal{M}} . \tag{4.14}$$

Recalling the totality of W in $\mathcal{G}$, one gets a normed space $\left(\mathsf{co}(\mathcal{M}), \|\cdot\|_{\mathsf{co}(\mathcal{M})}\right)$ and $\phi_W : \mathsf{co}(\mathcal{M}) \to \mathcal{M}$ is an isometry. Without extra assumptions, even when $\mathcal{M}$ is a Banach space, $\mathsf{co}(\mathcal{M})$ might not be complete, so we define $\widetilde{\mathsf{co}}(\mathcal{M})$ to be the completion. The canonical (isometric) extension of ϕ_W to a mapping : $\widetilde{\mathsf{co}}(\mathcal{M}) \to \mathcal{M}$ will also be denoted by ϕ_W. If the norm topology of $\mathsf{co}(\mathcal{M})$ happens to be stronger than the weak-* topology on $\mathcal{G}'$, then canonically $\widetilde{\mathsf{co}}(\mathcal{M}) \hookrightarrow \mathcal{G}'_\sigma$.

In this framework, coorbit spaces were defined and thoroughly investigated in [FR05, RU11]; if $\mathcal{M}$ is a Banach space then $\mathsf{co}(\mathcal{M})$ is automatically complete. The dependence of these coorbit spaces on the frame W is also studied in [FR05, RU11]; we are going to assume that the frame W is fixed.

We show now how the formalism present in section 2.1 can be considered as a particular case of the one described above. This particular case has extra structure allowing to develop a symbolic calculus and to define and study corresponding coorbit spaces of functions or "distributions" on Σ; we shall only indicate the facts that are useful for the present paper.

Let $\pi : \Sigma \to \mathbb{B}(\mathcal{H})$ be a map such that for every $u, v \in \mathcal{H}$ one has

$$\int_\Sigma d\mu(s)\, |\langle \pi(s)u, v\rangle|^2 = \|\, u \,\|^2 \,\|\, v \,\|^2 \ . \tag{4.15}$$

We set $\pi(s)u =: \pi_u(s)$ and $\pi(s)^*u \equiv \pi^*(s)u =: \pi_u^*(s)$ for every $s \in \Sigma$ and $u \in \mathcal{H}$, getting families of functions $\{\pi_u : \Sigma \to \mathcal{H} \mid u \in \mathcal{H}\}$ and $\{\pi_u^* : \Sigma \to \mathcal{H} \mid u \in \mathcal{H}\}$. One also requires π_u^* to be continuous for every u.

For every normalized vector $w \in \mathcal{H}$ the map $\phi_w^\pi : \mathcal{H} \to L^2(\Sigma)$ given by $\phi_w^\pi(u) := \phi^\pi(u, w)$ is isometric. *Fixing* w, *it is clear that we are in the above framework with the tight continuous frame defined by:*

$$W \equiv W(\pi, w) = \{w(s) := \pi(s)^* w \,|\, s \in \Sigma\} \tag{4.16}$$

Using existing notations one can write $\phi_W = \phi_w^\pi$ and $w(\cdot) = \pi_w^*(\cdot)$. After introducing a Fréchet space $\mathcal{G}$ continuously embedded in $\mathcal{H}$, one can define coorbit spaces $\mathsf{co}_w^\pi(\mathcal{M}) := \{u \in \mathcal{G}' \,|\, \phi_w^\pi(u) \in \mathcal{M}\}$ as it was done above.

4.4 Compactness criterion

Let us fix a tight continuous frame $W := \{w(s) \mid s \in \Sigma\}$ contained, total and bounded in a Fréchet space $\mathcal{G}$ that is continuously embedded in the Hilbert space $\mathcal{H}$. It is assumed that $s \mapsto \langle u, w(s)\rangle$ is continuous for every $u \in \mathcal{G}'$. For any normed space $\mathcal{M}$ of functions on Σ we have defined the coorbit space $\mathsf{co}_W(\mathcal{M}) \equiv \mathsf{co}(\mathcal{M})$ in (4.14) which will be supposed continuously embedded in $\mathcal{G}'_\sigma$.

One considers a bounded subset Ω of $\mathsf{co}(\mathcal{M})$ and investigate when this subset is relatively compact in terms of the canonical mapping $\phi_W \equiv \phi$. We are guided by [DFG02, Th. 4], but some preparations were needed due to our general setting. For instance, we need to apply 4.6 to $\mathcal{Y} = \mathsf{co}(\mathcal{M}) \hookrightarrow \mathcal{G}'_\sigma$. We also assume, as was done tacitly in [DFG02], that $\mathcal{M}$ is a *Banach space of functions with absolutely continuous norm.*

Theorem 4.16. *Let us assume that $\mathcal{M}$ is a Banach space of functions on Σ with absolutely continuous norm. Then the bounded subset Ω of $\mathsf{co}(\mathcal{M})$ is relatively compact if and only if $\phi(\Omega)$ is $\mathcal{K}(\Sigma)$-tight in $\mathcal{M}$.*

Proof. We start with the *only if part*. By relative compactness of Ω, for any $\epsilon > 0$ there is a finite subset F such that

$$\min_{v\in F} \| u - v \|_{\mathsf{co}(\mathcal{M})} \leq \frac{\epsilon}{2}, \quad \forall u \in \Omega .$$

Recalling that Σ has been assumed σ-compact, there is an increasing family $\{L_m \mid m \in \mathbb{N}\}$ of compact subsets of Σ with $\cup_m L_m = \Sigma$. Since pointwisely $|\chi_{L_m}\phi(v)| \leq |\phi(v)|$ and $\chi_{L_m}\phi(v) \xrightarrow{m\to\infty} \phi(v)$, there is a compact set $L \subset \Sigma$ with complement L^c such that

$$\max_{v\in F} \| \chi_{L^c}\phi(v) \|_{\mathcal{M}} \leq \frac{\epsilon}{2} .$$

Then, for every $u \in \Omega$, using the information above and the fact that $\phi : \mathsf{co}(\mathcal{M}) \to \mathcal{M}$ is isometric, one gets

$$\begin{aligned} \| \chi_{L^c}\phi(u) \|_{\mathcal{M}} &\leq \min_{v\in F} (\| \chi_{L^c}\phi_w(u-v) \|_{\mathcal{M}} + \| \chi_{L^c}\phi(v) \|_{\mathcal{M}}) \\ &\leq \min_{v\in F} \| \phi(u-v) \|_{\mathcal{M}} + \frac{\epsilon}{2} \\ &= \min_{v\in F} \| u-v \|_{\mathsf{co}(\mathcal{M})} + \frac{\epsilon}{2} \leq \epsilon . \end{aligned}$$

We now prove *the converse*. Knowing that $\phi(\Omega)$ is $\mathcal{K}(\Sigma)$-tight in $\mathcal{M}$, one needs to show that every sequence $(u_n)_{n\in\mathbb{N}} \subset \Omega$ has a convergent subsequence. By Lemma 4.6 the bounded set $\Omega \subset \mathsf{co}(\mathcal{M})$ is relatively compact in $\mathcal{G}'_\sigma$, so $(u_n)_{n\in\mathbb{N}}$ has a *-weakly convergent subsequence $u_j \to u_\infty \in \mathcal{G}'$:

$$\langle u_j, v\rangle \to \langle u_\infty, v\rangle \text{ for any } v \in \mathcal{G} . \tag{4.17}$$

Putting $v := w(s)$ in (4.17), we get for every $s \in \Sigma$

$$\langle u_j, w(s)\rangle = [\phi(u_j)](s) \to [\phi(u_\infty)](s) = \langle u_\infty, w(s)\rangle .$$

Therefore the sequence $(\phi(u_j))_{j\in\mathbb{N}}$ is pointwise Cauchy. We shall convert this in the norm convergence

$$\| \phi(u_j) - \phi(u_k) \|_{\mathcal{M}} \to 0 \text{ when } j, k \to \infty . \tag{4.18}$$

Then the proof would be finished since $\phi : \mathrm{co}(\mathcal{M}) \to \mathcal{M}$ is isometric: $(u_j)_{j\in\mathbb{N}}$ will be Cauchy in $\mathrm{co}(\mathcal{M})$, thus convergent (to u_∞ of course). By tightness, pick a compact subset $L \subset \Sigma$ such that $\| \chi_{L^c}\phi(u) \|_{\mathcal{M}} \leq \epsilon$ for every $u \in \Omega$; then we get

$$\| \chi_{L^c}\phi(u_j - u_k) \|_{\mathcal{M}} \leq 2\epsilon , \quad \forall j, k \in \mathbb{N} . \tag{4.19}$$

Since $\mathrm{co}(\mathcal{M})$ is continuously embedded in $\mathcal{G}'_\sigma$, for any seminorm $p \in \mathfrak{S}(\mathcal{G})$ there exist positive constants D_p, D'_p such that for every $s \in \Sigma$

$$\sup_{j,k} |\langle u_j - u_k, w(s)\rangle| \leq D_p \sup_{j,k} \| u_j - u_k \|_{\mathrm{co}(\mathcal{M})} \, p[w(s)] \leq D'_p \, p[w(s)] .$$

By our assumption on W and by the Uniform Boundedness Principle the family $\{w(s) \,|\, s \in L\}$ is bounded in $\mathcal{G}$, so we get

$$| \left[\phi(u_j - u_k)\right](s)| \leq D'_p C_{p,L} , \quad \forall j, k \in \mathbb{N}, \, s \in L .$$

Anyhow we obtain by Theorem 4.12

$$\| \chi_L \phi(u_j - u_k) \|_{\mathcal{M}} \to 0 \text{ when } j, k \to \infty . \tag{4.20}$$

Putting (4.20) and (4.19) together one gets (4.18) and thus the result. □

Remark 4.17. *Let S be an bounded operator from the Banach space $\mathcal{X}$ to $\mathrm{co}(\mathcal{M})$. Then S is a compact operator if and only if for every $\epsilon > 0$ there exists a compact set $L \subset \Sigma$ such that*

$$\| \chi_{L^c} \circ \phi_W \circ S \|_{\mathbb{B}(\mathcal{X},\mathcal{M})} \leq \epsilon . \tag{4.21}$$

This follows easily applying Theorem 4.16 to the set $\Omega := S(\mathcal{X}_{[1]})$ and using the explicit form of the operator norm. Here $\mathcal{X}_{[1]}$ denotes the closed unit ball in the Banach space $\mathcal{X}$.

Chapter 5

Compactness in spaces of operators

In this final chapter we relate the criteria presented in Chapters 2 and 4 to some general results in the space of compact operators in order to get new criteria concerning compactness in $\mathcal{K}(\mathcal{H})$ (section 5.1) and $\mathcal{K}(\mathcal{X}, \mathsf{co}(\mathcal{M}))$ (section 5.2).

5.1 Compactnes criterion for $\mathcal{K}(\mathcal{H})$

We start by presenting part of the literature concerning compactness in the space of compact operators. Interestingly, to the best of our knowledge, this literature is divided in two: a first part developed in the late 60' by Palmer and Anselone mostly concerned with relating totally boundedness of family of pre-compact operators with *collective compactness* of the family and their transpose ; a most recently second part, use a different characterization of a compact operator to get characterizations not depend on the transpose family. In this section we present the classical result from [Pal69] and postpone to the next section the second part of the discussion. We begin with some definitions. Here $\mathcal{X}$ and $\mathcal{Y}$ stand for Banach spaces (even if the completeness is not really needed) and we recall that $\mathcal{X}_{[1]}$ denotes the closed unit ball.

Definition 5.1. *Let $\mathscr{K}$ be a subset of $\mathbb{B}(\mathcal{X}, \mathcal{Y})$. It is called:*

pointwise collectively compact *if $\mathscr{K} x := \cup_{S \in \mathscr{K}} Sx$ is totally bounded in $\mathcal{Y}$ for every $x \in \mathcal{X}_{[1]}$,*

collectively compact *if $\mathscr{K} \mathcal{X}_{[1]} := \cup_{S \in \mathscr{K}} S\mathcal{X}_{[1]}$ is totally bounded in $\mathcal{Y}$.*

It is easy to see that if $\mathscr{K}$ is totally bounded then it is also collectively compact because if $S_1, \ldots, S_n$ are such that

$$\min_{j=1,\ldots,n} \sup_{S \in \mathscr{K}} \| S - S_j \|_{\mathbb{B}(\mathcal{X},\mathcal{Y})} \leq \frac{\epsilon}{2} \tag{5.1}$$

and for each $1 \leq j \leq n$ we choose x_{jl} such that

$$\min_{l=1,\ldots,N(j)} \sup_{x \in \mathcal{X}_{[1]}} \| S_j x - S_j x_{jl} \|_{\mathcal{X}} \leq \frac{\epsilon}{2 \| S_j \|} \tag{5.2}$$

we have

$$\min_{j,l} \sup_{x,S} \| Sx - S_j x_{jl} \|_{\mathcal{X}} \leq \epsilon . \tag{5.3}$$

For $\mathscr{K}$ we set $\mathscr{K}^* := \{S^* \in \mathbb{B}(\mathcal{Y}', \mathcal{X}')\}$, where S^* is the transpose operator defined for a $\eta \in \mathcal{Y}'$ by $S^*(\eta) = \eta \circ S$. Note that $\mathscr{K}$ is totally bounded if $\mathscr{K}^*$ is.

Theorem 5.2 ([Pal69, Theo. 2.1]). *A subset $\mathscr{K}$ of $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is a totally bounded set of compact operators if and only if:*

1. *$\mathscr{K}$ is collectively compact and*
2. *$\mathscr{K}^*$ is pointwise collectively compact.*

Proof. The necessity follows from the previous discussion. Let be ϵ greater than 0. By 1. $\mathscr{K}\mathcal{X}_{[1]}$ is totally bounded; then $\mathcal{Y}'_{[1]}$, being weakly compact, when restricted to $\mathscr{K}\mathcal{X}_{[1]}$ is totally bounded. We choose then $\eta_1, \ldots, \eta_n$ from an $\frac{\epsilon}{3}$-cover of $\mathcal{Y}'_{[1]}$ restricted to $\mathscr{K}\mathcal{X}_{[1]}$. By 2., for every $1 \leq j \leq n$ we choose an $\frac{\epsilon}{3}$-cover $\{\Omega_{j1}, \ldots, \Omega_{jl(j)}\}$ of $\mathscr{K}^*\eta_j$. Because each Ω_{jl} is composed of elements of the form $S^*\eta_j$, and furthermore $S^*\eta_j$ needs to be in som Ω_{jl} for every $S \in \mathscr{K}$, we can define a cover $\{\widetilde{\Omega}_{j1}, \ldots, \widetilde{\Omega}_{jl(j)}\}$ of $\mathscr{K}$ for every $1 \leq j \leq n$.Set $l_0 = \max_j l(j)$ and define $\mathscr{K}_l := \bigcap_{j=1}^{n} \widetilde{\Omega}_{jl}$ for $1 \leq l \leq l_0$. Note that for some l the intersection only runs over some j's. It is clear that $\{\mathscr{K}_l\}_{l=1}^{l_0}$ forms a cover. Take now $S_1, S_2 \in \mathscr{K}_l$, $x \in \mathcal{X}_{[1]}$, $\eta \in \mathcal{Y}_{[1]}$.Then, we have

$$\begin{aligned} |\eta(S_1 - S_2)x| &= |\eta[S_1 x] - \eta[S_2 x]| \\ &= |\eta[S_1 x] - \eta_k[S_1 x]| + |\eta_k[S_1 x] - \eta_k[S_2 x]| + |\eta_k[S_2 x] - \eta[S_2 x]| \\ &= |(\eta - \eta_k)[S_1 x]| + |[S_1^*(\eta_k) - S_2^*(\eta_k)](x)| + |(\eta - \eta_k)[S_2 x]| \\ &= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

if one choose η_k such that $|(\eta - \eta_k)[Sx]| < \frac{\epsilon}{3}$ whenever $S \in \mathscr{K}$. □

A similar proof yield the following result:

Theorem 5.3 ([Pal69, Theo. 2.2]). *A subset $\mathscr{K}$ of $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is a totally bounded set of compact operators if and only if:*

1. *$\mathscr{K}$ is pointwise collectively compact and*
2. *$\mathscr{K}^*$ is collectively compact.*

From this two theorems one could expect that a characterization of compactness in terms of pointwise compactness could be available; but this is not the case. Take for simplicity $\mathcal{X} = \mathcal{Y} = \mathcal{H}$ a Hilbert space. Let $\{e_j \mid j \in \mathbb{N}\}$ be an orthonormal base in $\mathcal{H}$ and set P_n the projection on the linear span of the first n vectors of the basis, $\{P_n\}_{n\in\mathbb{N}} = \{P_n^*\}_{n\in\mathbb{N}}$ is pointwise bounded but cannot be compact being an infinite discrete set. Combining both theorems we have:

Theorem 5.4. *A subset $\mathscr{K}$ of $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is a totally bounded set of compact operators if and only if:*

1. *$\mathscr{K}$ is collectively compact and*
2. *$\mathscr{K}^*$ is collectively compact.*

We refer now to the situation explored in Chapter 2, recalling the objects (π, ϕ_w^π, Π); for simplicity we only consider the case $\mathcal{X} = \mathcal{H}$. We want to characterize relative compactness of subsets of $\mathbb{K}(\mathcal{H})$, relying on Theorem 2.5 and Theorem 5.4. The setting is that of Chapter 2; it is also assumed that $\pi(s_1)^* = 1$ for some $s_1 \in \Sigma$.

Corollary 5.5. *Let $\mathscr{K}$ be a family of bounded operators in $\mathcal{H}$. The following assertions are equivalent:*

1. *$\mathscr{K}$ is a relatively compact family of compact operators.*
2. *For some (any) $w \in \mathcal{H}$ the family $\{\phi_w(S\mathcal{H}_{[1]}) \mid S \in \mathscr{K} \cup \mathscr{K}^*\}$ is uniformly tight in $L^2(\Sigma)$. This condition means that for every strictly positive ϵ there exists a compact subset L of Σ such that*

$$\sup_{S \in \mathscr{K} \cup \mathscr{K}^*} \| M_{\chi_L}^{\perp} \circ \phi_w \circ S \|_{\mathbb{B}(\mathcal{H}, L^2)} \leq \epsilon . \tag{5.4}$$

3. *For every* $\epsilon > 0$ *there exists* $f \in C_c(\Sigma)$ *such that*

$$\sup_{S \in \mathscr{K} \cup \mathscr{K}^*} \| [\Pi(f) - 1] S \|_{\mathbb{B}(\mathcal{H})} \leq \epsilon . \tag{5.5}$$

4. $\{\Sigma \ni s \mapsto \pi(s)^* S \in \mathbb{B}(\mathcal{H}) \mid S \in \mathscr{K} \cup \mathscr{K}^*\}$ *is an equicontinuous family.*

5.2 Compactness criterion for $\mathcal{K}(\mathcal{X}, \mathsf{co}(\mathcal{M}))$

To study compactness in $\mathcal{K}(\mathcal{X}, \mathsf{co}(\mathcal{M}))$ the previous result is not at hand because in general we do not know anything about compactness of the subsets of $\mathcal{X}'$. In [SPD06] is presented a result that only refers to $\mathscr{K}$ but a definition is needed. Note that the notion of *collectively compact* is founded on the idea that the whole family of operators should behaves as a single compact operator. But a a compact operator S can also be characterized by the fact there exist a weakly null sequence $\xi_n \subset \mathcal{X}'$ such that $\| Sx \| \leq \sup_n |\langle x, \xi_n \rangle|$ for every $x \in \mathcal{X}$. This motivate the following definition.

Definition 5.6. *Let* $\mathscr{K}$ *be a subset of* $\mathbb{B}(\mathcal{X}, \mathcal{Y})$. *It is called* equicompact *if exist a weakly null sequence* $\xi_n \subset \mathcal{X}'$ *such that*

$$\sup_{S \in \mathscr{K}} \| Sx \| \leq \sup_n |\langle x, \xi_n \rangle| \quad \textit{for every } x \in \mathcal{X} . \tag{5.6}$$

What is shown in [SPD06] is that $\mathscr{K}^*$ is collectively compact if and only if $\mathscr{K}$ is equicompact. Then 5.4 now reads.

Theorem 5.7. *A subset* $\mathscr{K}$ *of* $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ *is a totally bounded set of compact operators if and only if* $\mathscr{K}$ *is collectively compact and equicompact.*

Corollary 5.8. *Let us assume that* $\mathcal{M}$ *is a Banach space of functions on* Σ *with absolutely continuous norm, let* $\mathcal{X}$ *be a Banach space and* $\mathscr{K}$ *a subset of* $\mathbb{B}[\mathcal{X}, \mathsf{co}(\mathcal{M})]$. *Then* $\mathscr{K}$ *is a compact family of compact operators if and only if*

1. *For every* $\epsilon > 0$ *there exist a compact set* $L \subset \Sigma$ *such that*

$$\sup_{S \in \mathscr{K}} \| \chi_{L^c} \circ \phi_W \circ S \|_{\mathbb{B}(\mathcal{X}, \mathcal{M})} \leq \epsilon \textit{ and}$$

2. *There exists a sequence* $\mathcal{X}' \ni x'_n \to 0$ *such that*

$$\sup_{S \in \mathscr{K}} \| \phi_W(Sx) \|_{\mathcal{M}} \leq |\langle x'_n, x \rangle| \textit{ for every } x \in \mathcal{X} .$$

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