LOWER BOUND FOR THE DISCRIMINANT OF OCTIC NUMBER FIELDS HAVIND SIX REAL PLACES.

Tesis entregada a la

Universidad de Chile

en cumplimiento parcial de los requisitos

para optar al Grado de

Magister en Ciencias Matemáticas

Facultad de Ciencias

por

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JUNIO 1996

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UNIVERSIDAD DE CHILE

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INFORME DE APROBACION

TESIS DE MAGISTER

Se informa a la Comisión de Postgrado de la Facultad de Ciencias que la tesis de Magister presentada por el candidato:

MATIAS ATRIA LEMAITRE

ha sido aprobada por la Comisión de Evaluación de la tesis como requisito para optar al Grado de magister en Ciencias Matemáticas, en el Examen de Defensa de Tesis, rendido el día 28 de Junio de 1996.

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Lower bound for the discriminant of octic number fields having six real places

Matias Atria *

Abstract

We improve on the known bounds for the discriminant D of an octic field with 6 real places, from $|D| > 8.97^8$ to $|D| > 9.05^8$. The methods shown here should apply to other signatures.

1 Introduction

In this thesis we prove the following

Theorem 1 *If K is a number field of degree 8 having exactly six real places, then its discriminant D_K satisfies $|D_K| > 9.05^8$.*

The best previously known bound was $|D_K| > 8.9749^8$ [Ma, DyD]. Assuming the Generalized Riemann Hypothesis[1] (GRH), Odlyzko obtained $|D_K| > 9.26^8$ [Ma, Od4]. The point of this thesis is to introduce analytic techniques allowing us to improve the known bounds without assuming the GRH.

The main reason for studying these octic fields, is that for number fields of degree seven or less, or for totally real or totally complex octic fields, the minimal discriminants have been found through exhaustive computer

We are grateful for the financial support provided by Fondecyt grants 194-0353 and 196-0867 and European Community-CONICYT grant CI1-CT93-0353. We are also grateful to H. Cohen and collaborators, at Université Bordeaux I., for the development of PARI, without which this work would have been impossible.

[1]The GRH states that all the zeros of the Dedekind zeta function ζ_K of K within the critical strip $0 < \mathrm{Re}(s) < 1$ actually lie on the critical line defined by $\mathrm{Re}(s) = \frac{1}{2}$.

searches [Ma, Od4]. The methods presented here make no use of any computer searches of number fields and are expected to provide improved bounds for other degrees as well.[2]

The interest in finding lower bounds for the discriminant began when Kronecker conjectured that $|D_K| > 1$ for all number fields $K \neq \mathbb{Q}$. This was first proved by Minkowski [Mi], who also proved a lower bound for $|D_K|$ growing exponentially with $n = [K : \mathbb{Q}]$. Until the mid 1970's, most of the work on this subject still used Minkowski's geometry of numbers. The best of these results is the lower bound, due to Rogers [Ro] and Mulholland [Mu],

$$|D_K|^{\frac{1}{n}} \geq (32.561...)^{\frac{r_1}{n}}(15.775...)^{\frac{2r_2}{n}} + o(1),$$

as $n \to \infty$, where r_1 (resp., r_2) denotes the number of real (resp., complex) places of K.

Following a suggestion of H. Stark [St], Odlyzko introduced in 1976 a new analytic method for obtaining lower bounds for discriminants [Od1-3]. He was able to improve noticeably on the previously obtained bounds of Rogers and Mulholland. Further improvements came from the introduction by Serre [Se] of the explicit formulas of Guinand [Gui] and Weil [We1, We2] to discriminant bounds under the GRH. Odlyzko extended the ideas of Serre to obtain unconditional bounds (i.e., bounds that are valid without the GRH), the best known one being

$$|D_K|^{\frac{1}{n}} \geq (60.8395...)^{\frac{r_1}{n}}(22.3816...)^{\frac{2r_2}{n}} + o(1),$$

as $n \to \infty$. On the other hand, assuming the GRH, Odlyzko and Serre obtained the far better bound

$$|D_K|^{\frac{1}{n}} \geq (215.3325...)^{\frac{r_1}{n}}(44.7632...)^{\frac{2r_2}{n}} + o(1),$$

as $n \to \infty$. To understand these results, we must look at Weil's explicit formulas more closely. This formula states that if $F : \mathbb{R} \to \mathbb{R}$ is a real even function, normalized so that $F(0) = 1$ and satisfying some simple conditions which insure convergence of certain series and integrals, then [Po, We1]

$$\frac{1}{n}\log|D_K| = C_F + \frac{1}{n}\sum_{\rho}\mathrm{Re}(\Phi_F(\rho)) + \frac{2}{n}\sum_{\mathfrak{p},m}\frac{\log N\mathfrak{p}}{(N\mathfrak{p})^{m/2}}F(m\log N\mathfrak{p}), \qquad (1)$$

[2]Theorem 1 is also used in a recent paper [Chi] where the arithmetic hyperbolic 3-manifold of smallest volume is found.

where the "archimedean" term C_F depends only on F, n and r_1, the first sum is taken over the non-trivial zeros ρ of ζ_K, and in the second sum $\mathfrak{p}$ and m run over the prime ideals of K and the positive integers, respectively. The transform Φ_F is defined by

$$\Phi_F(s) = \int_{-\infty}^{\infty} e^{(s-\frac{1}{2})x} F(x) dx.$$

As long as we want a lower bound depending only on the signature of K, we cannot use any information on the distribution of zeros of ζ_K or the prime ideals of K. Thus, we would like to get from (1) an expression involving only C_F. This can be done if we take F so that the sums in (1) are non-negative, obtaining the lower bound

$$\frac{1}{n} \log |D_K| \geq C_F. \tag{2}$$

For the sum over the prime ideals, it is enough to take F non-negative. For the sum over the zeros, the condition on F varies according to whether we assume the GRH or not. In the first case, all the zeros lie on the line $\mathrm{Re}(s) = \frac{1}{2}$, so we must insure that $\mathrm{Re}(\Phi_F)$ is non-negative on this line, which is equivalent to requiring that the Fourier transform $\widehat{F}(t)$ be non-negative for all real t. If we do not assume the GRH, $\mathrm{Re}(\Phi_F)$ must be non-negative on the full critical strip, but as $\mathrm{Re}(\Phi_F)$ is harmonic, and symmetric respect to the line $\mathrm{Re}(s) = \frac{1}{2}$, it is enough to have $\mathrm{Re}(\Phi_F(1 + it)) \geq 0$ for all real t. This is the usual approach to get unconditional bounds. Note that if we do not impose this last condition, the bound (2) is still valid as long as the only violations of the GRH are in the region where $\mathrm{Re}(\Phi_F(s)) \geq 0$. In this thesis we exploit this idea, using the hypothetical presence of zeros outside the critical line to get an additional contribution to (2) that suffices to prove theorem 1.

We start with a function H such that C_H gives the best bound for $|D_K|$ under the GRH. Namely, take $y = 0.25495$ and let H to be the even function on $[-1/y, 1/y]$ defined on $[0, 1/y]$ by

$$H(x) = (1 - yx)\cos(\pi yx) + \frac{1}{\pi}\sin(\pi yx),$$

and vanishing for $|x| > 1/y$. Then we form $\widetilde{H} = \delta H_1 + (1 - \delta)H$, where H_1 is a function carefully chosen so that $\mathrm{Re}(\Phi_{\widetilde{H}})$ is negative in a region as small as possible, and $\delta > 0$ is chosen small enough as to give $C_{\widetilde{H}} \geq \log(9.207)$. Thus, if there is no zero in the region where $\mathrm{Re}(\Phi_{\widetilde{H}})$ is negative, we get an even better bound than the one given in theorem 1. Therefore, we can and do assume that there is at least one zero in this region. Then, using other auxiliary functions, we exploit the presence of this zero to obtain the desired bound.

A natural question arising at this point is how close can the unconditional bound get to that obtained under the GRH. It is certainly far harder to obtain unconditional bounds, as one has to account for the many possible positions of a zero that could violate the GRH. Nevertheless, theorem 1 suggests that there is no reason to think that the unconditional bounds are intrinsically weaker than those assuming the GRH.

Another natural question is how close the bounds obtained under the GRH are to being optimal. Martinet [Ma] gives tables comparing the smallest known values of $|D_K|^{\frac{1}{n}}$ for $2 \leq n \leq 8$ with both the unconditional and GRH bounds. Similar tables can be found in [Od4] and [DyD]. It is worth noting how small in general the discrepancies between the GRH bounds and the minimal values of $|D_K|$ are, and how far from optimal the current unconditional bounds remain, especially as n grows.

We now describe the organization of this thesis. In §2 we state Weil's formula in detail, and give the proof of theorem 1. We explain in §3 the method used to construct the auxiliary functions used in §2. In §4 we give the tedious details that are needed to insure that the numerical results given at each step are indeed correct to the precision claimed.

2 Proof of theorem 1

We keep the notation of section 1. K is a number field with discriminant $D = D_K$, having r_1 and r_2 real and complex places, respectively, $n = r_1 + 2r_2$. If $F : \mathbb{R} \to \mathbb{R}$ is an even function, we set

$$\Phi_F(s) = \int_{-\infty}^{\infty} \exp((s - 1/2)x) F(x) dx,$$

whenever this integral converges. Then, $\Phi_F(s) = \Phi_F(1-s)$ and $\Phi_F(\frac{1}{2}+it) = \widehat{F}(t)$ is the Fourier transform of F, defined by

$$\widehat{F}(t) = \int_{-\infty}^{\infty} e^{ixt} F(x)dx.$$

To simplify the formulas, we will introduce the functions

$$k(x) = \frac{1}{2\sinh(x/2)} + \frac{r_1}{n}\frac{1}{2\cosh(x/2)}$$

and

$$h(x) = \frac{1}{\sinh(x)} + \frac{r_1}{n}\frac{1}{2\cosh^2(x/2)}.$$

Now we are in position to state Weil's formula [We1, Po]

Theorem 2 *(Weil) Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a real even function, with $F(0) = 1$ and satisfying the following conditions:*

1. *There exists $\delta > \frac{1}{2}$ such that $F(x)\exp(\delta x)$ is Lebesgue integrable.*
2. *There exists $\delta > \frac{1}{2}$ such that $F(x)\exp(\delta x)$ is of bounded variation and its value at each point is the mean of its lateral limits.*
3. *The function $(1 - F(x))/x$ is of bounded variation.*

Then, the limit

$$\sum_{\rho} \Phi_F(\rho) = \lim_{T\to\infty} \sum_{|\operatorname{Im}(\rho)|<T} \Phi_F(\rho),$$

where ρ runs over the non-trivial zeros of ζ_K, exists and we have the equality

$$\begin{aligned}\frac{1}{n}\log|D_K| &= \gamma + \log(8\pi) + \frac{r_1\pi}{2n} - \frac{2}{n}\Phi_F(1) + \frac{1}{n}\sum_{\rho}\Phi_F(\rho) - \\ &\int_0^{\infty}(1-F(x))k(x)dx + \frac{2}{n}\sum_{\mathfrak{p},m}\frac{\log N\mathfrak{p}}{(N\mathfrak{p})^{(m/2)}}F(m\log N\mathfrak{p}) \quad (3)\end{aligned}$$

where $\gamma = 0.57721566...$ is Euler's constant, and $\mathfrak{p}$ and m run over the prime ideals of K and the positive integers respectively.

Remarks

1. Multiple zeros ρ in the sum are repeated according to their multiplicity.

2. If ρ is a non-trivial zero of ζ_K, then so are $\bar{\rho}$, $1-\rho$ and $1-\bar{\rho}$, so the contribution to (3) is $\frac{2}{n}\operatorname{Re}(\Phi_F(\rho))$ if $\operatorname{Re}(\rho) = 1/2$ or $\operatorname{Im}(\rho) = 0$, and $\frac{4}{n}\operatorname{Re}(\Phi_F(\rho))$ if not.

3. If the function F is positive, the sum over the prime ideals is also positive, so we can drop it from (3) and get the lower bound

$$\begin{aligned}\frac{1}{n}\log|D_K| \quad \geq \quad & \gamma + \log(8\pi) + \frac{r_1\pi}{2n} + \frac{1}{n}\sum_{\rho}\operatorname{Re}\left(\Phi_F(\rho)\right) - \\ & \frac{2}{n}\Phi_F(1) - \int_0^\infty (1-F(x))k(x)dx \end{aligned} \tag{4}$$

This is the bound we will use in the proof of theorem 1.

4. If we put $F(x) = f(x)/\cosh(x/2)$, with f non-negative, then

$$\operatorname{Re}(\Phi_F(1+it)) = \hat{f}(t),$$

so the requirement of $\operatorname{Re}(\Phi_F)$ being non-negative on this line is equivalent to $\hat{f}(t) \geq 0$ for all real t. If we make this assumption, and keep only the zeros ρ in the sum satisfying $|\operatorname{Im}(\rho)| < L$, for some $L > 0$, the formula (4) takes the form

$$\begin{aligned}\frac{1}{n}\log|D_K| \quad \geq \quad & \gamma + \log 4\pi + \frac{r_1}{n} + \frac{1}{n}\sum_{|\operatorname{Im}(\rho)|<L}\Phi_F(\rho) - \\ & \frac{4}{n}\int_0^\infty f(x)dx - \int_0^\infty (1-f(x))h(x)dx \end{aligned} \tag{5}$$

which is obtained by direct substitution. The only term that deserves some attention is the integral involving $k(x)$ in (3)

$$\int_0^\infty (1-F(x))k(x)dx = \int_0^\infty (1-f(x))h(x)dx + \int_0^\infty (\cosh(\frac{x}{2})-1)h(x)dx.$$

This last integral can be easily evaluated to be $\log 2 + \frac{r_1\pi}{2n} - \frac{r_1}{n}$. This modification to (4) will be specially useful for numerical calculations.

5. The constant C_F appearing in (1) is given by

$$C_F = \gamma + \log 8\pi + \frac{r_1}{2n} - \frac{2}{n}\Phi_F(1) - I_F,$$

where $I_F = \int_{-\infty}^{\infty}(1 - F(x))k(x)dx$.

We now begin the proof of theorem 1. Throughout the proof we use the numerical value $n = 8$ and $r_1 = 6$. Suppose that theorem 1 is false, i.e., $|D_K|^{\frac{1}{8}} \leq 9.05$. For $y > 0$, and real a and x, let

$$T_{y,a}(x) = \frac{1 + \cos(ax)}{2\cosh(x/2)}\, t(yx),$$

where

$$t(x) = \frac{9\,(\sin(x) - x\cos(x))^2}{x^6}.$$

The function $t(x)$, introduced by L. Tartar [Po], just fails to satisfy the first hypothesis of theorem 2, since $\delta = \frac{1}{2}$, but inequality (5) still holds [Po, Prop. 5]. We let $\Phi_{y,a} = \Phi_{T_{y,a}}$. In the next section we prove that $\mathrm{Re}(\Phi_{y,a}(s)) \geq 0$ for all s in the critical strip. Take $F = T_{2.52982,0}$ in inequality (5), and assume for the moment that there are $N > 1$ zeros ρ with $0 < \mathrm{Im}(\rho) < 2.77$ and $\frac{1}{2} < \mathrm{Re}(\rho) < 1$. By evaluating the minimum of the harmonic function $\mathrm{Re}(\Phi_T(\rho))$ in this rectangle, we obtain

$$C_{T_{2.52982,0}} + \frac{4N}{n} \quad \bullet \quad \mathrm{Re}(\Phi_{2.52982,0}(\rho)) \geq \log 9.05,$$

so we can assume that there is at most one zero ρ of ζ_K in the region $0 < \mathrm{Im}(\rho) < 2.77$ and $\frac{1}{2} < \mathrm{Re}(\rho) < 1$.

Now let H, as in §1, be the even function on $[-1/y, 1/y]$ defined on $[0, 1/y]$ by

$$H(x) = (1 - yx)\cos(\pi yx) + \frac{1}{\pi}\sin(\pi yx),$$

with $y = 0.25495$ and vanishing outside $[-1/y, 1/y]$. This function clearly satisfies the hypothesis of theorem 2, since it is continuous and has compact support. This was the function used by Odlyzko to get his GRH bounds. One can obtain, after an elementary calculation, the formula

$$\Phi_H(s) = \frac{4\pi^2}{y}\frac{\cosh(w) + 1}{(\pi^2 + w^2)^2},$$

where $w = (s - 1/2)/y$. Let

$$\widetilde{H}(x) = 0.995190385\, H(x) + 0.00103\, P_{12}(x) + \sum_{i=1}^{5} b_i\, T_{0.28284, a_i}(x),$$

where

$$P_c(x) = \frac{e^{-c|x|}}{\cosh(x/2)},$$

and the parameters a_i, b_i and for $1 \le i \le 5$, are given in the following table.

i	b_i	a_i
1	0.00280436622	3.8305
2	0.00062013984	5.4870
3	0.00021474837	7.1170
4	0.00009329940	8.7361
5	0.00004706117	10.3497

As explained in the introduction, the function $\widetilde{H}$ is chosen close to the GRH optimal H, but so that $\mathrm{Re}(\Phi_{\widetilde{H}}) \ge 0$ for $|\,\mathrm{Im}(s)| > 2.6$ and $0 < \mathrm{Re}(s) < 1$. Also, $C_{\widetilde{H}} > \log 9.207$ and $\frac{4}{n}\,\mathrm{Re}(\Phi_{\widetilde{H}}(\rho)) > \log 9.05 - \log 9.207$ for $\frac{1}{2} \le \mathrm{Re}(\rho) \le 1$ and $\mathrm{Im}(\rho) \ge 0$, as long as ρ lies outside the region shaded in figure 1 (we give in the next two sections the proof of this claim, and of analogous numerical statements below). As we saw above, there is at most one zero ρ in the region $\frac{1}{2} < \mathrm{Re}(s) < 1$ and $0 < \mathrm{Im}(s) < 2.6$. In view of inequality (4) and Remark 2 preceding it, we conclude that if $|D_K| \le 9.05^8$, then $\zeta_K(s)$ has exactly one zero in the shaded region of figure 1.

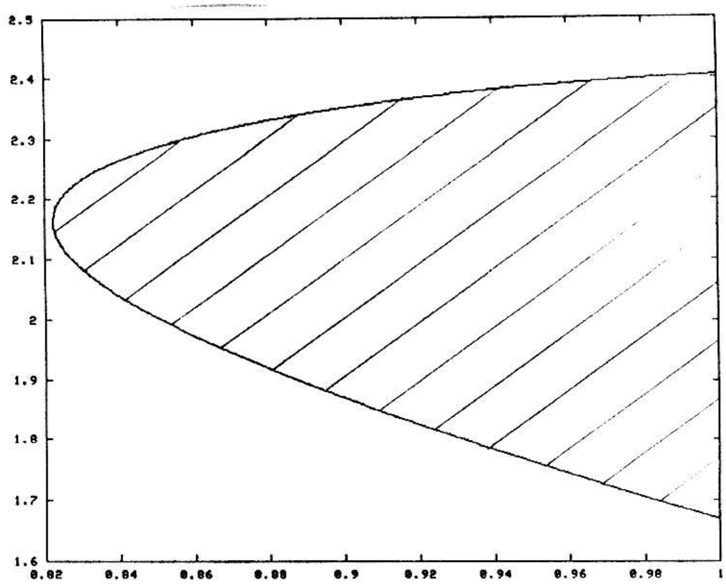

Figure 1: Region containing a hypothetical zero ρ of $\zeta_K(s)$

Now, and henceforth, we assume that ρ lies inside the region shaded in figure 1. If we take $F = T_{1.8166,0}$, we find that $C_{T_{1.8166,0}} + \frac{4}{n}\operatorname{Re}(\Phi_{1.8166,0}(\rho)) > \log 9.05$, except for ρ in the region shaded in figure 2. Next we repeat this process with the function $F = T_{1.06301,0.24}$, assuming that the zero lies in the region shaded in figure 2, obtaining the region of figure 3. Again, taking $F = T_{0.57879,2.27}$ we obtain that the zero must be in the shaded region of figure 4. The final region is obtained with $F = T_{0.55678,2.3}$, and is shown in figure 5.

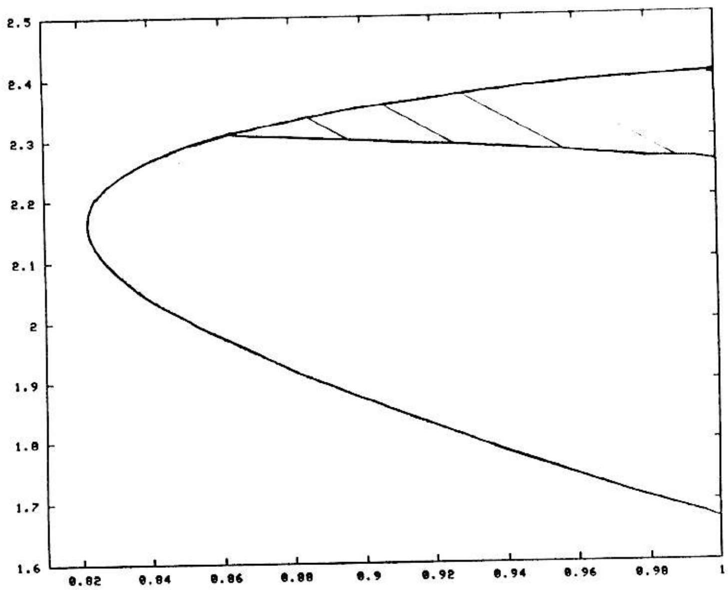

Figure 2

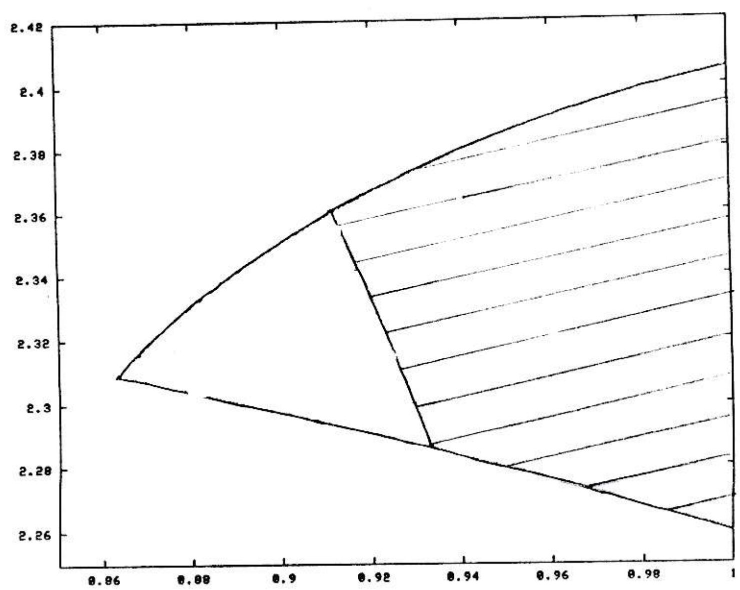

Figure 3

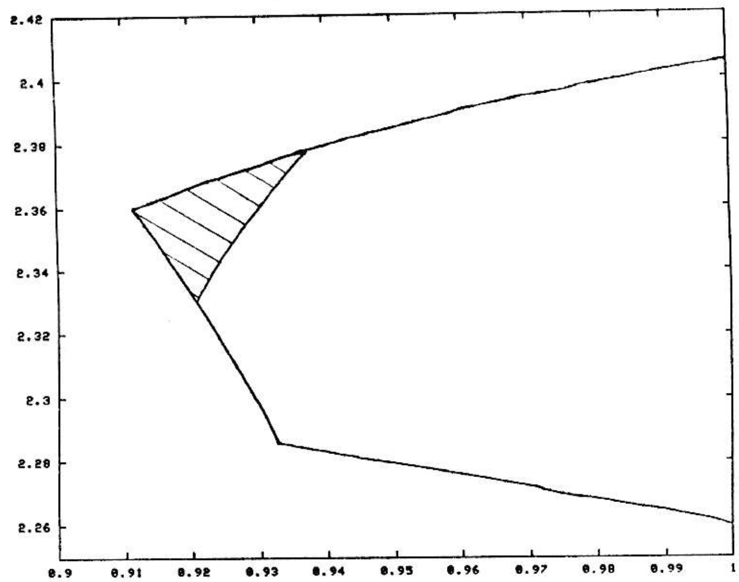

Figure 4

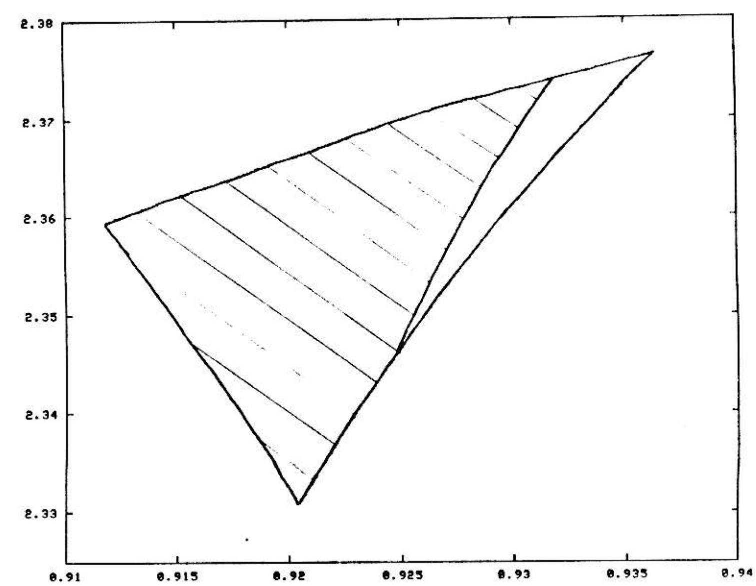

Figure 5

To exploit a zero in the region shaded in figure 5, we use the following function. Let $h = 0.0575 = 2.3/40$, and g_c be the even simple function on $[-2.3, 2.3]$, defined on $[0, 2.3]$ as

$$g_c(x) = \sum_{i=1}^{40} c_i \chi_i(x)$$

where χ_i is the characteristic function of the interval $[(i-1)h, ih]$, $1 \leq i \leq 40$, and $\sum_{i=1}^{40} c_i^2 = 1$. We take the convolution

$$G_c(x) = \frac{1}{2h}(g_c * g_c)(x),$$

and $F_c(x) = G_c(x)/\cosh(x/2)$ in (5), where $c = (c_1, \ldots, c_N)$ is given by

i	c_i	i	c_i
1	0.205343795727	21	0.169342818146
2	0.205354559903	22	0.163005723531
3	0.205368887355	23	0.156094679777
4	0.205372541340	24	0.148630446879
5	0.205344576053	25	0.140643038736
6	0.205257818499	26	0.132171622316
7	0.205079495678	27	0.123264273056
8	0.204771994407	28	0.113977599470
9	0.204293738353	29	0.104376256111
10	0.203600164563	30	0.094532371825
11	0.202644779715	31	0.084524930960
12	0.201380274779	32	0.074439161438
13	0.199759675578	33	0.064366010597
14	0.197737506095	34	0.054401839319
15	0.195270941161	35	0.044648565072
16	0.192320925429	36	0.035214710178
17	0.188853236340	37	0.026218395361
18	0.184839470019	38	0.017795157477
19	0.180257930755	39	0.010121443481
20	0.175094406846	40	0.003525971673

Note that $F_c(0) = \sum_i c_i^2 = 1$. This function was found by searching for the $c_i \geq 0$, subject to $\sum_i c_i^2 = 1$, that maximize

$$C_{F_c} + \frac{4}{n}\Phi_{F_c}(\rho_0),$$

where $\rho_0 = 0.92 + 2.33 \cdot i$ is near the lower vertex of the region shaded in figure 5. It turns out that

$$C_{F_c} + \frac{4}{n}\Phi_{F_c}(\rho) > \log 9.051$$

for ρ in the region shaded in figure 5. As $\mathrm{Re}(\Phi_{F_c}(s)) \geq 0$ for all s in the critical strip, this implies that $|D_K| > 9.05^8$, which proves theorem 1.

3 Auxiliary functions

Here we discuss the auxiliary functions used in the proof of theorem 1. We keep the notation of the previous sections.

3.1 Construction of $\widetilde{H}$

The transform $\Phi_{y,0}$ of the function $T_{y,0}$ can be calculated explicitly on the line $\mathrm{Re}(s) = 1$, and is given by

$$\mathrm{Re}(\Phi_{y,0}(1+ix) = \frac{1}{y}\hat{t}(\frac{x}{y}) = \frac{9\pi}{8y}w\left(\frac{x}{y}\right), \qquad (x \in \mathbb{R})$$

where w is the even function on $[-2, 2]$ defined on $[0, 2]$ by

$$w(x) = -\frac{1}{30}(x^5 - 20x^3 + 40x^2 - 32) = -\frac{1}{30}(x-2)^3(x^2+6x+4)$$

and $w(x) = 0$ for $x > 2$. This function has its maximum at $x = 0$ and decreases monotonically for $x \in [0, 2]$. Note that $\mathrm{Re}(\Phi_{y,0}(s)) \geq 0$, as it is harmonic and symmetric with respect to the line $\mathrm{Re}(s) = \frac{1}{2}$.

For the function P_c appearing in $\widetilde{H}$, the transform Φ_{P_c} is given by [P-B-M]

$$\Phi_{P_c}(s) = 2\{\beta(s+c) + \beta(1-s+c)\}$$

where $\beta(s) = \psi((s+1)/2) - \psi(s/2)$, and $\psi(s) = \Gamma'(s)/\Gamma(s)$. For $\mathrm{Re}(s) = 1$ this expression simplifies to

$$\Phi_{P_c}(1+it) = \frac{2c}{c^2+t^2}.$$

On the other hand, for $T_{y,a}$ we have the following identity

$$\Phi_{y,a}(s) = \frac{1}{2}\Phi_{y,0}(s) + \frac{1}{4}\Phi_{y,0}(s+ia) + \frac{1}{4}\Phi_{y,0}(s-ia) \tag{6}$$

which, for $s = 1+it$, has its maximum near $t = a$. Thus, on the line $\mathrm{Re}(s) = 1$, the effect of using $T_{y,a}$ instead of Tartar's $T_{y,0}$, is to move the maximum of $\mathrm{Re}(\Phi)$ to a point near $1+ia$. In defining $\widetilde{H}$, we chose the a_i near the first few local minima of $\mathrm{Re}(\Phi_H(1+it))$ for $t > 2.4$. The b_i were chosen to just cancel the negative contribution from $\mathrm{Re}(\Phi_H(1+it))$ at these minima, and the P_{12} was chosen so as to cancel the remaining minima for $t > 11$.

The result is a function $\mathrm{Re}(\Phi_{\widetilde{H}})$ close to $\mathrm{Re}(\Phi_H)$, but which is negative only in a small region of the critical strip. This region is contained in the region $|\,\mathrm{Im}(s)| < 2.6$.

3.2 Construction of F_c

To produce the function F_c, we fix an interval $[-A, A]$ and a number N of subdivisions of $[0, A]$. Define the even and piecewise constant function g_c as in the previous section, where we took $A = 2.3$, $N = 40$. Let $h = A/N$. Note that

$$\mathrm{Re}(\Phi_{F_c}(1+ix)) = \widehat{G_c}(x) = \frac{1}{2h}\hat{g}_c(x)^2,$$

so $\mathrm{Re}(\Phi_{F_c}(s)) \geq 0$ for all s in the critical strip. Also, the support of F_c is $[-2A, 2A]$ and we have

$$C_{F_c} = \gamma + \log 8\pi + \frac{r_1\pi}{2n} - \frac{4h}{n} + \int_h^\infty k(x)dx + \sum_{i,j=1}^{N} c_i c_j b_{ij}$$

where $b_{ij} = b_{ji}$ and

$$b_{ii} = \int_0^h \left(1 - \frac{h-x}{h\cosh(x/2)}\right) k(x)dx + I_{2i-1}$$

and, for $i < j$,

$$b_{ij} = I_{i+j-1} + I_{j-i}$$

with

$$I_j = \frac{1}{2h} \int_{h(j-1)}^{h(j+1)} \frac{(h - |x - hj|)k(x)}{\cosh(x/2)}\, dx. \qquad (j \geq 1)$$

Also, $\Phi_{G_c}(s)$ can be written as a quadratic form $c^t \varphi c$, where $\varphi = (\varphi_{ij}(s))$. As a result, we can write the bound (4) given by this function, with a given zero ρ in the interior of the critical strip as

$$K_h + c^t M(\rho) c$$

where

$$K_h = \gamma + \log 8\pi + \frac{r_1 \pi}{2n} - \frac{4h}{n} + \int_h^\infty k(x)dx,$$

and $M(\rho) = (M_{ij}(\rho))$, with

$$M_{ij}(\rho) = b_{ij} - \frac{4}{n}\varphi_{ij}(\rho).$$

Here we used $\sum_i c_i^2 = 1$, so $\Phi_{G_c}(1) = h$. Thus, we sought to maximize the quadratic form $c^t M(\rho) c$, under the constraints $\sum_i c_i^2 = 1$, and $c_i \geq 0$, to insure the positivity of G_c. The maximum of this form subject to $\sum_i c_i^2 = 1$ is given by the maximum eigenvalue of the matrix $M(\rho)$, and the vector c is its associated eigenvector. Experimentally, we found that taking $A = 2.3$ and $N = 40$, the restriction $c_i \geq 0$ is satisfied automatically.

A major disadvantage of this method is that when we increase A (as we would like), we find positive and negative entries in the eigenvector c.

4 Numerical calculations

This section is devoted to the justification of the numerical results which involve numerical integration and showing that, for various functions F, $\mathrm{Re}(\Phi_F(s))$ stays above a certain value in some regions inside the critical strip. All the calculations we are about to describe were done in a Sun SPARCstation 20, using PARI version 1.39.03.

Throughout this section, we fix the precision of our final results to 6 decimals.

Consider first $F = T_{y,a}$. By Remark 4, we need to compute the integral

$$L(y,a) = \int_0^\infty (1 - \cos(ax)t_y(x))h(x)dx$$

since

$$\int_0^\infty (1 - \frac{1+\cos(ax)}{2}t_y(x))h(x)dx = \frac{1}{2}L(y,a) + \frac{1}{2}L(y,0)$$

We calculate $L(y,a)$ using Simpson's rule. We split this integral in the intervals $[0,1]$, $[1,N]$ and $[N,\infty]$, where N is chosen so that, in this last interval, each integral is negligible to 6 places. In the interval $[1,N]$, we use Simpson's rule, estimating the fourth derivative of $(1-\cos(ax)t_y(x))h(x)$. This is used to choose the grid for the sample points. Let $\widetilde{L}(y,a)$ be the integral defining $L(y,a)$ taken in the interval $[0,1]$. Fix a and y, and let $r(x) = 1 - \cos(ax)t_y(x)$. We compute $\widetilde{L}$ by expanding $r(x)$ in its Taylor series

$$r(x) = \sum_{k=1}^{M-1} a_{2k}x^{2k} + \frac{r^{(2M)}(\theta)}{(2M)!}x^{2M},$$

where $0 \le \theta \le x \le 1$. Using the Fourier transform of t_y, we get a bound R_M for the M-th derivative of r. Then we have

$$L(y,a) = \sum_{i=1}^{M-1} a_{2k}(\alpha_{2k} + \frac{r_1}{2n}\beta_{2k}) + R'_{2M}$$

where

$$\alpha_{2k} = \int_0^1 \frac{x^{2k}}{\sinh(x)}dx, \qquad \beta_{2k} = \int_0^1 \frac{x^{2k}}{\cosh(x/2)^2}dx.$$

and $R'_{2M} < (2M)!^{-1}R_M(\alpha_{2M} + \frac{r_1}{2n}\beta_{2M})$. The integrals α_{2k} and β_{2k} can be calculated in terms of Bernoulli and Euler polynomials, which have known bounds [Ab]. Thus, we choose M such that

$$\frac{R_{2M}}{(2M)!}(\alpha_{2M} + \frac{r_1}{2n}\beta_{2M}) < 10^{-6}.$$

To calculate $\Phi_{y,a}(s)$, we use (6). Thus, we only need $\Phi_{y,a}$ for $a = 0$. In this case, using Parseval's formula

$$\int_{-\infty}^{\infty} T_{y,0}(x)\exp((s-\frac{1}{2})x)dx = \frac{9\pi}{8y}\int_{-2/y}^{2/y} \frac{w(x/y)}{\cosh(\pi(z+x))}dx,$$

where $z = -i(s-\frac{1}{2})$. Here we used the Fourier transform of $1/\cosh(x/2)$. This integral converges for $\mathrm{Re}(s) < 1$. In practice, it was used for $\frac{1}{2} < \mathrm{Re}(s) < 0.99$ and is easily calculated, for the values of y used here, with the Trapezoid rule. For $0.99 \leq \mathrm{Re}(s) \leq 1$, we compute the integral directly using Simpson's method. In this case we estimate the fourth derivative of $T_{y,0}$ using its Fourier transform, and obtain a crude upper bound for the fourth derivative of $T_{y,0}(x)\exp((s-\frac{1}{2})x)$ using Leibniz's rule.

For the function H, the only numerical integral is $(1-H(x))k(x)$, since Φ_H is given explicitly. We write this integral as

$$\int_0^\infty (1-H(x))k(x)dx = \int_{1/y}^\infty k(x)dx - \frac{r_1}{2n}\int_0^{1/y}\frac{1-H(x)}{\cosh(x/2)}dx + \frac{1}{2}I$$

where $y = 0.25495$, and

$$\begin{aligned} I &= \int_0^{1/y}\frac{1-H(x)}{\sinh(x/2)}dx \\ &= \int_0^{1/y}\frac{1-\cos(\pi yx)}{\sinh(x/2)}dx + \int_0^{1/y}\frac{xy\cos(\pi yx)}{\sinh(x/2)}dx + \frac{1}{\pi}\int_0^{1/y}\frac{\sin(\pi yx)}{\sinh(x/2)}dx. \end{aligned}$$

and calculate these integrals in the intervals $[0,1]$ and $[1,1/y]$, using the Taylor's series and the Trapezoid rule respectively, with estimations similar to those for $L(y,a)$.

Finally, for P_c, we only need to compute

$$\int_0^\infty (1-e^{-cx})h(x)dx = \int_0^\infty \frac{1-e^{-cx}}{\sinh(x)}dx + \frac{r_1}{n} + \frac{r_1}{2n}\int_0^\infty \frac{e^{-cx}}{\cosh(x/2)^2}dx.$$

The last integral is calculated explicitly as the Laplace transform of $\cosh(x/2)^{-2}$, and for the first we proceed as for $L(y,a)$.

To obtain the regions shown in §2, we proceed as follows. As $\mathrm{Re}(\Phi_F)$ is harmonic, to insure that it stays above a certain value m in regions inside the critical strip it is enough to check that $\mathrm{Re}(\Phi_F(s)) \geq m$ for s on the border of these regions. By using polygons to approximate these borders, we are reduced to showing that $\mathrm{Re}(\Phi_F) \geq m$ is satisfied in the segment joining 2 points z_1 and z_2. To do this, let $p(t) = tz_2 + (1-t)z_1$, with $0 \leq t \leq 1$, be a point in this segment, and $\varphi(t) = \mathrm{Re}(\Phi_H(p(t)))$. We put

$$\varphi(t) = \varphi(0) + t\varphi'(0) + \frac{t^2}{2}\varphi''(\tilde{t}),$$

where $0 \leq \tilde{t} \leq t$. Then, with an estimate of $\varphi''(\tilde{t})$, it is enough to take z_1 and z_2 close enough to get $\varphi(t) \geq m$.

The actual points used to approximate the regions, as well as the programs used to do all the calculations described here, are available upon request.

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