Tree containment and degree conditions

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Abstract

We survey results and open problems relating degree conditions with tree containment in graphs, random graphs, digraphs and hypergraphs, and their applications in Ramsey theory.

1 Introduction

A fundamental question in extremal graph theory is how to guarantee certain subgraphs by imposing a global condition on the host graph. Often, this is a condition on the degree sequence. Classical examples include Turán's theorem on containment of a complete subgraph, or Dirac's theorem on containment of a Hamilton cycle. One of the most intriguing open questions in the area is to determine degree conditions a graph G has to satisfy in order to ensure it contains a fixed tree T, or more generally, all trees of a fixed size.

Let us start with an easy observation. A greedy embedding argument yields that for $k \in \mathbb{N}$, a minimum degree $\delta(G)$ of at least k is enough to ensure that each tree T with k edges is a subgraph of G. Note that T is not necessarily an induced subgraph. Also note that although a copy of each k-edge tree is present, these copies need not be disjoint. For instance, if $|V(G)| = k+1$, we are considering a complete graph and its spanning trees.

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Although the minimum degree condition $\delta(G) \geq k$ is tight (it cannot be lowered to $k-1$), the condition seems quite strong. It might not be necessary that *all* vertices of the host graph have large degree. For path containment, there is a famous result relying on the *average degree*: Erdős and Gallai [42] showed in 1959 that if G has average degree $d(G) > k$ then G contains a k-edge path. Erdős and Sós conjectured in 1963 (see [40]) that this bound on the average degree should in fact guarantee *all* trees with k edges to appear as subgraphs. This conjecture, its variants and generalisations, will be one of the guiding themes of this survey.

We discuss the Erdős-Sós conjecture in Section 2 and then turn to related questions. Namely, various other conditions have been suggested that might ensure the appearance of all trees of some fixed size. One well known conjecture in this direction is the Loebl–Komós–Sós conjecture from 1995 (see [41]). This conjecture replaces the assumption on the average degree with an assumption on the median degree. We will discuss the Loebl–Komós–Sós conjecture and related results in Section 3.

More recent conjectures with the same conclusion employ a condition on a combination of the maximum and the minimum degree. The first conjecture in this direction is due to Havet, Reed, Wood and the author [70]. The idea is that a minimum degree below k may still be sufficient to find all fixed-size trees, as long as we require one vertex of large degree in the host graph. This vertex both caters for a possible large degree vertex in the tree T, and ensures we have enough space for the embedding of all of T. See Section 5 for details.

If we only wish to condition on the minimum degree of the host graph, with no assumptions on the maximum degree, and if our minimum degree condition is strictly below k, it is clearly necessary to exclude some trees, for instance stars, from our considerations. More precisely, it will make sense to add a restriction on the maximum degree of the trees we wish to find. There is a well-known result of Komlós, Sárközy and Szemerédi [91] from 1995, which had been conjectured by Bollobás [18] in 1978. It states that in large graphs G, a minimum degree slightly above $\frac{|V(G)|}{2}$ is sufficient to guarantee all bounded degree spanning trees. This result will be another recurring theme of this survey. We discuss variations of Komlós, Sárközy and Szemerédi's result in Section 4.

It has also been considered to require, apart from a minimum degree condition, additional properties in the host graph, for instance expansion (in terms of large girth, excluded subgraphs, or neighbourhood conditions). With expansion, and for bounded degree trees, the degree bounds on the host graph can be lowered. This naturally leads to considering random graphs as well. A conjecture of Kahn [86] regarding the threshold for containment of bounded degree spanning trees was recently solved by Montgomery [101]. For several of the above mentioned extremal results for tree containment (such as Komlós, Sárközy and Szemerédi's result), there are resilience versions for random graphs. Also, randomly perturbed graphs have been considered as host graphs. Expansion and random graphs will be discussed in Section 6.

Some of the above conjectures have direct applications in Ramsey theory, giving upper bounds on Ramsey numbers of trees. For most trees, however, these bounds do not seem to be sharp, and it might be that the correct numbers need to take into account the relative size of the partition classes of the tree. A conjecture of Burr [23] from 1974 for Ramsey number of trees, although asymptotically confirmed for bounded degree trees in [75], has turned out to be far from correct, leaving plenty of open questions in this area. To date, not even the two-colour Ramsey number of double stars is understood. We will give an overview of the state of the art of Ramsey theory for trees in Section 7.

Tree containment is also being studied for oriented trees in oriented graphs and digraphs. It is not sufficient to simply consider the degree of the underlying graph, so even the case of the tournament as a host graph is interesting. In Section 8, we will first look at two famous conjectures on tree containment in tournaments from the 1980's, due to Sumner [115] and Burr [24], respectively. Then, we will highlight a recent conjecture from [1] which attempts to generalise the Erdős–Sós conjecture and Burr's conjecture at the same time. Finally, we turn to results generalising the theorem of Komlós, Sárközy and Szemerédi to digraphs, and some more open questions.

Finally, tree containment problems have been translated to the hypergraph setting. We will describe this thriving area in Section 9. We cover three types of trees: Tight trees, expansions of trees and Berge trees. Each of these notions corresponds to the respective notion for hyperpaths (and these are the most commonly studied hyperpath notions). For tight trees, Kalai's conjecture (see [48]) is widely regarded as an analogue of the Erdős–Sós conjecture for hypergraphs. There is also a generalisation of Komlós, Sárközy and Szemerédi's theorem to tight hypertrees. For expansions of trees and Berge trees, we will present some Erdős–Sós type results, phrased in terms of their Turán numbers.

For the reader's convenience, we summarise here how the survey is organised: Section 2: Average degree; Section 3: Median degree; Section 4: Minimum degree; Section 5: Maximum and minimum degree; Section 6: Expanders and random graphs; Section 7: Ramsey theory; Section 8: Directed graphs; Section 9: Hypergraphs.

2 Average degree

The most prominent conjecture on tree containment is a classical conjecture of Erdős and Sós from 1963 which focuses on the average degree. It appeared for the first time in [40].

Conjecture 2.1 (Erdős–Sós conjecture, see [40])**.** *Every graph with average degree* $d(G) > k-1$ *contains every tree with* k *edges as a subgraph.*

A different way to state this conjecture would be in terms of the extremal number or *Turán number* of trees. Namely, we define as usual the Turán number $ex(n, H)$ of a graph H to be the largest number of edges an n-vertex graph may have without containing H as a subgraph. Then, Conjecture 2.1 states that

$$ex(n, T) \leq \frac{k-1}{2}n$$

for any k-edge tree T.

The Erdős–Sós conjecture is tight for every $k \in \mathbb{N}$: If k divides n, consider the n-vertex graph consisting of the union of $\frac{n}{k}$ disjoint copies of cliques on k vertices. This graph has average degree $k-1$ but it does not contain any tree with k edges since its connected components are too small. One can also consider any other $(k-1)$-regular graph, for instance the complete bipartite graph $K_{k-1,k-1}$, which does not contain the star with k edges.

A structurally different example is given by a complete graph on n vertices, in which all edges inside a set of $\lfloor \frac{k}{2} \rfloor - 1$ vertices have been deleted. This graph does not contain any balanced tree on k edges. The graph is not extremal, however, as its average degree is slightly lower than the average degree of the examples from the previous paragraph.

Before giving an overview of the known results concerning the conjecture, let us insert here a quick observation on the minimum degree we may assume the host graph from the Erdős–Sós conjecture to have. Since every graph of average degree greater than $k-1$ has a subgraph of minimum degree at least $\frac{k}{2}$ and average degree greater than $k-1$ (this subgraph can be found by successively deleting vertices of too low degree), one can assume that the host graph from the Erdős–Sós conjecture has minimum degree at least $\frac{k}{2}$.

Similarly, one can argue that if we replaced the condition $d(G) > k-1$ with the condition $d(G) > 2k-1$, then a greedy embedding of any k-edge tree into an appropriate subgraph of G will succeed, and therefore, such a version of Conjecture 2.1 trivially holds. The bound $d(G) > 2k-1$ can be lowered to $d(G) > 2\frac{k}{k+1}(k-1)$ [122].

In the early 1990's Ajtai, Komlós, Simonovits and Szemerédi announced a proof of the Erdős–Sós conjecture for large graphs. Nevertheless, many particular cases have been settled since then, or, in some cases, earlier.

The results mainly group into four types. First, the conjecture has been verified for special types of trees. Most prominently, and as we mentioned before, a classical result of Erdős and Gallai [42] from 1959 implies that the Erdős–Sós conjecture holds for paths. The Erdős–Sós conjecture is also true for stars and double stars. Indeed, for stars this is trivial, while for double stars it suffices to establish the existence of an edge between a vertex of degree $\geq k$ and a vertex of degree $\geq \frac{k}{2}$ in the host graph. Since we can assume that the minimum degree of the host graph is at least $\frac{k}{2}$, such an edge clearly exists. Moreover, it is easy to see that the Erdős–Sós conjecture holds for all trees having a vertex adjacent to at least $\frac{k}{2}$ leaves. McLennan [100] showed the conjecture holds for all trees of diameter at most 4. Fan, Hong and Liu [45] recently proved the conjecture for all *spiders*, i.e. for all trees having at most one vertex of degree exceeding 2.

Second, the Erdős–Sós conjecture has been verified for special types of host graphs. Brandt and Dobson [21] proved in 1996 that the Erdős–Sós conjecture is true for graphs with girth at least 5. Saclé and Woźniak [120] improved on this result showing in 1997 that the Erdős–Sós conjecture holds for all graphs that do not contain C_4, the cycle on 4 vertices. The conjecture also holds if we exclude certain complete bipartite subgraphs in the host graph or its complement [7, 37, **?**], and if the host graph is bipartite [122].

Third, there are results building on the relation between k and n. In particular, Conjecture 2.1 has been established for several cases when k is very close to n, the order of the host graph (note that the largest possible value of k is $k = n - 1$ and then we need to find a spanning tree in an almost complete host graph). More precisely, the conjecture holds if $k + 1 \leq n \leq k + 4$ (for all k), and even for the case $n \leq k + c$, where c is any given constant and k is sufficiently large depending on c (see [59] and references therein). Furthermore, it is shown in [16] that if we additionally assume that $k \geq 10^6$, then the Erdős–Sós conjecture holds for all graphs G with $|V(G)| \leq (1 + 10^{-11})k$ and trees T with $\Delta(T) \leq \frac{\sqrt{k}}{1000}$.

Finally, there are some recent results building on the regularity method, thus only applying to the case when k is linear in n, and n is large. In 2019, Rozhoň [119] and independently, the authors of [16] (and [14]) gave an approximate version of the Erdős–Sós conjecture for trees with linear maximum degree and large dense host graphs.

Theorem 2.2. **[16, 119]** *For each $\delta > 0$ there are n_0, γ such that for each k and for each n-vertex graph G with $n \geq k \geq \delta n \geq \delta n_0$ the following holds.*
If G satisfies $d(G) \geq (1+\delta)k$, then G contains every k-edge tree T with $\Delta(T) \leq \gamma k$.

In [16], this is used to obtain the following sharp version of Conjecture 2.1 for large dense host graphs, which unfortunately relies on the tree having constant maximum degree.

Theorem 2.3. **[16]** *For each $\delta > 0$ and Δ there is n_0 such that for each k and for each n-vertex graph G with $n \geq k \geq \delta n \geq \delta n_0$ the following holds.*
If G satisfies $d(G) > k-1$, then G contains every k-edge tree T with $\Delta(T) \leq \Delta$.

3 Median degree

A well-known variant of the Erdős–Sós conjecture, which replaces the assumption on the average degree with an assumption on the median degree, is the Loebl-Komlós-Sós conjecture from 1995. Two variants of this conjecture first appeared in [41].

Conjecture 3.1 (($\frac{n}{2}$–$\frac{n}{2}$–$\frac{n}{2}$)–Conjecture [41])**.** *Every n-vertex graph having at least $\frac{n}{2}$ vertices of degree at least $\frac{n}{2}$ contains each tree on at most $\frac{n}{2}$ vertices as a subgraph.*

The ($\frac{n}{2}$–$\frac{n}{2}$–$\frac{n}{2}$)-Conjecture has been attributed to Loebl, while according to [41], Komlós and Sós are the originators of the following variation.

Conjecture 3.2 (Komlós–Sós Conjecture [41])**.** *Every n-vertex graph having more than $\frac{n}{2}$ vertices of degree at least k contains each tree with k edges as a subgraph.*

The following amalgamation came to be called the Loebl–Komlós–Sós Conjecture.

Conjecture 3.3 (Loebl–Komlós–Sós Conjecture)**.** *Every n-vertex graph having at least $\frac{n}{2}$ vertices of degree at least k contains each tree with k edges as a subgraph.*

Note that the Loebl-Komlós-Sós conjecture neither implies nor is implied by Conjecture 2.1.

Also note that the bound on the degrees in the conjecture cannot be lowered, because because we might need to embed a star. Another example is the disjoint union of cliques of order k, which contains no tree with k edges.

As for the number of vertices of large degree, we do not know of any example making the bound $\frac{n}{2}$ sharp. The best example we know of is the following. If k is odd, consider the complete graph on $k+1$ vertices and delete all edges inside a set of $\frac{k+3}{2}$ vertices. It is easy to check that this graph has $\frac{k+1}{2}$ vertices of degree k, and it does not contain the k-edge path. Taking the disjoint union of several such graphs we obtain examples for other values of n (and we can add a disjoint small graph to reach any value of n). The total number of vertices of large degree is somewhat lower than $\frac{n}{2}$, and in [76], it was conjectured that the number given by this example might be the correct number.

Conjecture 3.4. **[76]** *Let G be a graph on n vertices having more than $\frac{n}{2} - \lfloor \frac{n}{k+1} \rfloor - (n \bmod k+1)$ vertices of degree at least k. Then G contains each k-edge tree.*

The Loebl-Komlós-Sós conjecture (Conjecture 3.3) clearly holds for stars, and it has been proved for several other special classes of trees. One of the first results of this type is due to Bazgan, Li, and Woźniak [13], who proved the conjecture for paths in 2000. Piguet and the author [109] proved that Conjecture 3.3 is true for trees of diameter at most 5, which improved earlier results of Barr and Johansson [12] and Sun [124] for smaller diameter.

Conjecture 3.3 has also been proved for special classes of host graphs. Soffer [121] showed that the conjecture is true if the host graph has girth at least 7. Dobson [37] proved the conjecture for host graphs whose complement does not contain the complete bipartite graph $K_{2,3}$.

The use of a different approach to the Loebl-Komlós-Sós conjecture based on the regularity method has been initiated by Ajtai, Komlós, and Szemerédi [3] in 1995 who solved an approximate version of Conjecture 3.1 for large graphs. Their strategy (see also [91] which appeared around the same time) relies on the regularity method, and has been replicated in similar forms in numerous articles on tree embeddings in large dense graphs. The leading idea in [3] is to cut up the tree into many tiny trees connected by a constant number of vertices (of possibly very large degree), and additionally, to find a useful matching structure in the regularised host graph. The tiny trees are then embedded into the regulars pairs corresponding to the matching, while the connecting vertices are embedded in suitable clusters that see a large amount of matching edges.

Zhao [126] used a refinement of the approach from [3] plus stability arguments to prove the exact version of Conjecture 3.1 for large graphs. Also using regularity, an approximate version of Conjecture 3.3 for k linear in n was proved by Piguet and the author [110]. Finally, adding stability arguments, Hladký and Piguet [80] and independently, Cooley [29] succeeded in proving Conjecture 3.3 for large dense graphs.

Theorem 3.5. **[29, 80]** *For every $q > 0$ there is n_0 such that for any $n > n_0$ and $k > qn$, each n-vertex graph G with at least $\frac{n}{2}$ vertices of degree at least k contains each k-edge tree.*

The regularity method described above fails in sparse host graphs. A new approach covering also this type of host graphs was explored by Hladký, Komlós, Piguet, Simonovits, Szemerédi and the present author in [76, 77, 78, 79] (for a 10-page overview of the proof see [81]). These authors introduced a decomposition technique for graphs (stemming from previous work of some of the authors on the Erdős–Sós conjecture) whose output resembles the regularity lemma if applied to a dense graph but is also meaningful in the sparse setting. This enabled them to show the following approximate version of Conjecture 3.3 for large trees.

Theorem 3.6. [76, 77, 78, 79] *For every $\varepsilon > 0$ there is k_0 such that for every $k \geq k_0$, every n-vertex graph having at least $(1+\varepsilon)\frac{n}{2}$ vertices of degree at least $(1+\varepsilon)k$ contains each k-edge tree as a subgraph.*

In [89], Klimošová, Piguet, and Rohzoň suggest an interesting generalisation of the Loebl-Komlós-Sós conjecture, inspired by a question of Simonovits. Let us say that a k-edge tree is r-skew if one of its colour classes has size at most $r(k+1)$.

Conjecture 3.7 (Skew LKS conjecture)**.** *Let G be a graph on n vertices having more than rn vertices of degree at least k. Then G contains each r-skew k-edge tree.*

The authors of [89] show an approximate version of this conjecture for large dense graphs. Conjecture 3.7 has also been verified for paths and trees of diameter at most five [118]. Examples similar to the ones given earlier in this section show the conjecture would be close to tight.

4 Minimum degree

As mentioned in the introduction, any n-vertex graph G with minimum degree at least k contains every tree with k edges. Clearly, the bound on the minimum degree is tight, as we might have to embed a star. Even if we disregard for a moment stars and other trees having vertices of very large degree, it is not possible to lower the bound on the minimum degree of the host graph. In order to see this, it suffices to consider the union of several disjoint copies of K_k which does not contain any tree with k edges. However, the latter example only works if k divides n. In particular, it fails if $k > \frac{n}{2}$. So one might suspect that for $k > \frac{n}{2}$, a lower minimum degree condition could be sufficient to ensure that G contains all k-edge trees that have bounded maximum degree.

In this direction, Bollobás [18] conjectured in 1978 that any graph on n vertices and minimum degree at least $(1+o(1))\frac{n}{2}$ would contain every spanning tree whose maximum degree is bounded by a constant. This conjecture was proved by Komlós, Sárközy and Szemerédi [91] in 1995, giving one of the earliest applications of the Blow-up lemma.

Theorem 4.1 (Komlós, Sárközy and Szemerédi [91]). *For all $\delta > 0$ and $\Delta \in \mathbb{N}$, there is n_0 such that such that every graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1+\delta)\frac{n}{2}$ contains each n-vertex tree T with $\Delta(T) \leq \Delta$.*

Subsequently, each of the two bounds in Theorem 4.1 has been improved.

Theorem 4.2 (Csaba, Levitt, Nagy-György and Szemerédi [30]). *For all $\Delta \in \mathbb{N}$, there are n_0 and c such that every graph G on $n \geq n_0$ vertices with $\delta(G) \geq \frac{n}{2} + c\log n$ contains each n-vertex tree T with $\Delta(T) \leq \Delta$.*

Theorem 4.3 (Komlós, Sárközy and Szemerédi [92]). *For all $\delta > 0$, there are n_0 and c such that such that every graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1+\delta)\frac{n}{2}$ contains each n-vertex tree T with $\Delta(T) \leq c\frac{n}{\log n}$.*

The bound on the minimum degree in Theorem 4.2 is essentially tight (see [30]). Also the bound on the maximum degree in Theorem 4.3 is essentially best possible. This can be seen by considering the random graph with edge probability $p = 0.9$ which a.a.s. does not contain a forest of stars of order $\frac{n}{\log n}$ (and thus also does not contain any tree containing such a forest).

In contrast to the results from earlier sections, the results from [30, 91, 92] are all for the case when the tree and the host graph have the same order. In view of the examples from the beginning of the section, we know that these results cannot be generalised to non-spanning trees in host graphs of smaller minimum degree. However, if in addition we require the host graph to have a connected component of size at least $k+1$, then it does at least contain the k-edge path P_k. This is the core observation behind the Erdős–Gallai Theorem. Let us state the observation here for later reference.

Observation 4.4 (Erdős-Gallai [42], Dirac (see [42])). *If $\delta(G) \geq \frac{k}{2}$, G is connected and $|V(G)| \geq k+1$ then $P_k \subseteq G$.*

In order to see that this observation is true, note that a variant of of Dirac's theorem [35] states that every 2-connected n-vertex graph G has a cycle of length at least $\min\{n, 2\delta(G)\}$. So, if G has a 2-connected component of size at least $k+1$, then this component contains a cycle of length at least k, and thus also a k-edge path (possibly using one edge that leaves the cycle). Otherwise, we can embed either the middle vertex of the path, or a vertex adjacent to the middle edge, into any cutvertex x of G, and then greedily embed the remainder of the path into two components of $G - x$, using the minimum degree of G.

This argument, however, only seems to work for the case when the tree we are looking for is the path. Already the following tree, which has only one vertex of degree > 2, cannot be embedded into all large enough connected graphs obeying the minimum degree condition from above. Assume 3 divides k and consider the tree obtained from identifying the starting vertices of three distinct $\frac{k}{3}$-edge paths. This tree is not a subgraph of the graph obtained from adding an edge between two cliques of size $\lceil \frac{k}{2} + 1 \rceil$.

Still, there is hope: It has been suggested that requiring one large degree vertex in the host graph might remedy the situation. This vertex will at the same time provide the necessary space in the host graph, and cater for a possibly existing large degree vertex of the tree. See the next section for details.

5 Maximum and minimum degree

As noted in Section 2, we may assume that the host graph from the Erdős–Sós conjecture has minimum degree at least $\frac{k}{2}$, and as we have seen in the previous section, this alone is not enough to force all k-edge trees as subgraphs. However, a graph H of average degree exceeding $k - 1$ does not only have a subgraph H' of minimum degree $\geq \frac{k}{2}$, but this subgraph H' also maintains the average degree of H (that is, $d(H') \geq d(H)$). Therefore, H' has a vertex of degree at least k. So, in Conjecture 2.1, we may assume the host graph G to obey the following three conditions: $\delta(G) \geq \frac{k}{2}$, $d(G) > k - 1$, and $\Delta(G) \geq k$.

Now, the conditions $\delta(G) \geq \frac{k}{2}$ and $\Delta(G) \geq k$ alone are not sufficient for guaranteeing all k-edge trees as subgraphs. This is because of a variation of the example given in the penultimate paragraph of Section 4: Adding a universal vertex to the disjoint union of two cliques of size $\lceil \frac{k}{2} + 1 \rceil$, we obtain a graph G satisfying the maximum and minimum degree conditions from above. But the tree obtained from joining any three trees on roughly $\frac{k}{3}$ vertices each to a new vertex (of degree 3) is not contained in G.

However, if we elevate either the bound on $\delta(G)$ or the bound on $\Delta(G)$ sufficiently, this example ceases to work. So one may suspect there is a suitable combination of conditions on the minimum and the maximum degree of the host graph that might replace the condition on the average degree in the Erdős–Sós conjecture. In this spirit, Havet, Reed, Wood and the present author [70] put forward the following conjecture.

Conjecture 5.1 ($\frac{2}{3}$-conjecture [70]). *Every graph of minimum degree at least* $\lfloor\frac{2k}{3}\rfloor$ *and maximum degree at least* k *contains each* k*-edge tree.*

In [17], Besomi, Pavez-Signé and the present author suggested another combination of bounds on the maximum and the minimum degree of the host graph.

Conjecture 5.2 ($2k$-$\frac{k}{2}$ conjecture [17]). *Every graph of minimum degree at least* $\frac{k}{2}$ *and maximum degree at least* $2k$ *contains each* k*-edge tree.*

Conjectures 5.1 and 5.2 are asymptotically best possible, as we will discuss later in this section.

Each of the conjectures is clearly true for stars and double stars. They also hold for paths, because of Observation 4.4. In [17], both Conjectures 5.1 and 5.2 were proved in an approximate form for large dense host graphs and trees whose maximum degree is bounded by $k^{\frac{1}{49}}$ and $k^{\frac{1}{67}}$, respectively. (For trees of constant maximum degree, the maximum degree of the host graph can even be lowered slightly.)

For Conjecture 5.1, more is known. Recently, Reed and the present author [113, 114] showed that the conjecture holds if we are looking for a spanning tree in a large graph.

Theorem 5.3. **[113, 114]** *There is* n_0 *such that for every* $n \geq n_0$*, every* n*-vertex graph of minimum degree at least* $\lfloor\frac{2(n-1)}{3}\rfloor$ *and maximum degree at least* $n-1$ *contains each* n*-vertex tree.*

This theorem can also be seen as an extension of Theorem 4.3: By elevating the bound on the minimum degree of the host graph, we can dispose of the bound on the maximum degree of the tree.

Moreover, in [70], Havet, Reed, Wood and the present author prove the following two variants of their conjecture.

Theorem 5.4. **[70]** *There are a function* $f : \mathbb{N} \to \mathbb{N}$ *and a constant* $\gamma > 0$ *such that if for a graph* G*, either of the following holds*

(i) $\Delta(G) > f(k)$ *and* $\delta(G) \geq \lfloor\frac{2m}{3}\rfloor$*; or*

(ii) $\Delta(G) \geq k$ *and* $\delta(G) \geq (1-\gamma)m$*,*

then G *contains each* k*-edge tree.*

Theorem 5.4 confirms that, even if the bounds suggested by Conjectures 5.1 and 5.2 should be incorrect, the idea behind the conjectures is not: It is possible to simultaneously bound the maximum and the minimum degree of a graph (with the bound on the minimum degree strictly below k) and as a result, guarantee the appearance of each tree of size k as a subgraph.

The proof of part (i) of Theorem 5.4 is relatively easy, and relies on strategically placing into a maximum degree vertex of the host graph a vertex of the tree that cuts the tree into conveniently sized components. It would be very interesting to find an extension of Theorem 5.4 (i), with the bound on the minimum degree lowered to $\frac{k}{2}$ (ideally), or to some other number strictly smaller than $\lfloor \frac{2m}{3} \rfloor$.

The following conjecture of Besomi, Pavez-Signé and the author [15] tries to correlate the bounds on maximum and the minimum degree of the host graph given in Conjectures 5.1 and 5.2.

Conjecture 5.5 (Intermediate range conjecture [15]). *For each $\alpha \in [0, \frac{1}{3})$ every graph of minimum degree at least $(1+\alpha)\frac{k}{2}$ and maximum degree at least $2(1-\alpha)k$ contains each k-edge tree.*

As for the earlier conjectures from this section, it can be seen that Conjecture 5.5 holds for stars, double stars and paths. Moreover, an approximate version for large dense host graphs and trees of bounded maximum degree is shown in [15].

Conjecture 5.5 is best possible for certain values of α. Namely, it is shown in [15] that for all odd $\ell \in \mathbb{N}$ with $\ell \geq 3$, and for all $\gamma > 0$ there are $k \in \mathbb{N}$, a k-edge tree T, and a graph G not containing T such that $\delta(G) \geq (1 + \frac{1}{\ell} - \gamma)\frac{k}{2}$ and $\Delta(G) \geq 2(1 - \frac{1}{\ell} - \gamma)k$. Let us give a quick description of the example from [15].

Example 5.6. *Consider two complete bipartite graphs $H_i = (A_i, B_i)$ with $|A_i|$ slightly below $\frac{\ell-1}{\ell}k$ and $|B_i|$ slightly below $\frac{\ell+1}{\ell} \cdot \frac{k}{2}$. Let G^* be obtained by adding a new vertex x to $H_1 \cup H_2$, such that x is connected to all of $A_1 \cup A_2$.*

Then the tree T^* formed by ℓ stars of order $\frac{k}{\ell}$ and an additional vertex v connected to the centres of the stars does not embed in G^*.

Taking $\ell = 3$, or letting ℓ converge to ∞, we obtain examples that prove the asymptotic sharpness of Conjectures 5.1 and 5.2. For Conjecture 5.1, one or both of the bipartite graphs H_i may be replaced with a conveniently sized complete graph. A structurally different example for the sharpness of Conjecture 5.1 is obtained by joining a universal vertex to a bipartite graph whose sides are both slightly below $\frac{2}{3}k$, which does not contain the tree T^* from above.

Example 5.6 only shows the asymptotic tightness of Conjecture 5.5 for values of α that are equal to $\frac{1}{\ell}$, for some odd ℓ. We are not aware of similarly good examples for other values of α, and perhaps, the degree conditions from Conjecture 5.5 could be lowered for these values, and there might be jumps. One such jump can be observed when the minimum degree bound is close to $\frac{2}{3}k$. We believe that with $\delta(G) \geq \frac{2}{3}k$, we would only need to bound $\Delta(G)$ by k (according to Conjecture 5.1). However, with the bound $\delta(G) \geq (1-\varepsilon)\frac{2}{3}k$, for any $\varepsilon > 0$, it becomes necessary to bound $\Delta(G)$ by almost $\frac{4}{3}k$ (because of Example 5.6).

Let us remark that if we exclude host graphs that are very close to the graph G^*, a different set of maximum/minimum degree conditions might be sufficient. In [15], it is shown that any large enough graph G with $\delta(G) \geq (1+\delta)\frac{k}{2}$ and $\Delta(G) \geq (1+\delta)\frac{4}{3}k$ either looks very much like the graph G^* from Example 5.6, or contains all k-edge trees T with $\Delta(T) \leq k^{\frac{1}{67}}$.

We close the section with a new conjecture due to Klimošová, Piguet, and Rohzoň which appeared in [119]. Their conjecture combines the essence of Conjecture 3.7 with the spirit of the minimum/maximum degree conjectures from the present section.

Conjecture 5.7. **[119]** *Every n-vertex graph with $\delta(G) \geq \frac{k}{2}$ and at least $\frac{n}{2\sqrt{k}}$ vertices of degree at least k contains all k-edge trees.*

Conjecture 5.7 would be tight [119].

6 Expanders and random graphs

In the results and conjectures we have seen so far, the degrees in the host graph are of the same order as the tree we wish to embed. Examples showed that this is necessary, even if we bound the maximum degree of the tree. However, adding the assumption that the host graph has some expansion properties changes the situation. Different types of expansion have been considered for this problem, among these are large girth, guarantees of large neighbourhoods of small sets of vertices, and exclusion of dense bipartite subgraphs. Also random graphs fall into this category.

An early result for tree containment in expanding graphs is due to Friedman and Pippenger [49] who extended Pósa's rotation-extension technique [111] from paths to trees and showed that if each set $X \subseteq V(G)$ with $|X| \leq 2k-2$ has at least $(\Delta+1)|X|$ neighbours then G contains all k-edge trees of maximum degree Δ. This result has been generalised in [8, **?**].

A use of expansion in the form of large girth is the result by Brandt and Dobson [21] we cited in Section 2. They showed more generally that every graph of girth at least 5 satisfying $\delta(G) \geq \frac{k}{2}$ contains every k-edge tree with $\Delta(T) \leq \Delta(G)$. A generalisation of this was conjectured by Dobson [36], and, after preliminary results by Haxell and Łuczak [73], confirmed by Jiang [83]: For any $t \in \mathbb{N}_+$, every graph G of girth at least $2t+1$ satisfying $\delta(G) \geq \frac{k}{t}$ contains every k-edge tree with

$\Delta(T) \leq \delta(G)$. This result was greatly improved by Sudakov and Vondrák [123] who used tree-indexed random walks to show that every graph G of girth at least $2t+1$ and $\delta(G) \geq d$ contains every tree with $\Delta(T) > (1-\varepsilon)d$ and $|V(T)| = cd^k$ (where c is a constant depending on ε). The same authors also show the requirement of large girth may be replaced with forbidding the host graph to contain a complete bipartite graph $K_{s,t}$ for certain s, t.

Some of these results directly apply to random graphs. One natural possibility in this setting is to replace the degree conditions with probability thresholds. Then, the main problem amounts to determining the probability threshold $p = p(n)$ for the binomial random graph[1] $G(n,p)$ to contain asymptotically almost surely (a.a.s.) each tree/all trees from a given class $\mathcal{T}_n$ of trees. Clearly, as the error probabilities for missing individual trees might add up, there is a difference between containing "each tree" and "all trees" (the latter is often referred to as *universality*).

Most of the relevant literature for tree containment in random graphs is focused on spanning trees, or almost spanning trees, and the first case to be tackled was the path. Komlós and Szemerédi [93] and Bollobás [19] showed the threshold for spanning paths is $p = (1+o(1))\frac{\log n}{n}$. The lower bound follows immediately from the fact that for smaller values of p, there are a.a.s. isolated vertices in $G(n,p)$. Kahn (see [86]) conjectured that the same threshold also applies to bounded degree trees. Namely, he conjectured that, for each Δ, there is C such that, given any sequence of n-vertex trees T_n, each with maximum degree at most Δ, the random graph $G(n, C\frac{\log n}{n})$ a.a.s. contains a copy of T. After preliminary results due to a number of authors (see e.g. [2, 5, 47, 84, 96]), Montgomery [101] recently solved Kahn's conjecture. He showed more generally the following statement.

Theorem 6.1. **[101]** *For each $\Delta > 0$, there is a $C > 0$ such that the random graph $G(n, C\frac{\log n}{n})$ almost surely contains a copy of every n-vertex tree T with maximum degree at most Δ.*

Predating [101], some results for *almost* spanning trees appeared. Most importantly, Alon, Krivelevich, and Sudakov [5] proved that for all ε and Δ there is C such that $G(n, \frac{C}{n})$ a.a.s. contains all trees of order $(1-\varepsilon)n$ of maximum degree at most Δ. The value of the constant C was improved by Balogh, Csaba, Pei and Samotij [8] by using the embedding result from [**?**]. Their result, as well as many of the results we

[1] The graph $G(n,p)$ is defined as a probability space on the set of all graphs on n (fixed) vertices where every edge appears with probability p, independently, but we also refer to an element of this space as *the random graph* $G(n,p)$. The random graph $G(n,p)$ is said to have a property $\mathcal{P}$ asymptotically almost surely (a.a.s.) if the probability of $G(n,p)$ having $\mathcal{P}$ tends to 1 as n tends to infinity.

cited in the last two paragraphs, also apply to other types of expanding graphs (not only random graphs).

Balogh, Csaba and Samotij [9] showed a result in the spirit of Theorem 4.1 for *subgraphs* of random graphs. In order to appreciate their result, let us observe that Theorem 4.1 can be stated in terms of *local resilience*: If we delete some edges from the complete graph K_n, in a way that at each vertex, at least a $(\frac{1}{2}+\delta)$-fraction of its incident edges is preserved, then the resulting graph still contains all spanning trees of maximum degree Δ (where δ, n and Δ are as in the theorem). Now, in [9] this is translated to random graphs (and *almost* spanning trees).

Theorem 6.2. **[9]** *For all Δ, ε and δ there is C such that after deleting any set of edges from $G(n, \frac{C}{n})$ in a way that at each vertex, at least a $(\frac{1}{2}+\delta)$-fraction of the original incident edges are preserved, then the resulting graph a.a.s. contains all trees of order $(1-\varepsilon)n$ and of maximum degree at most Δ.*

There are also some *global resilience* results for random graphs (in this type of result, a fraction of the edges is deleted without any restrictions on the number of edges deleted at each vertex). Balogh, Dudek and Li [10] proved a version of the Erdős–Gallai theorem for random graphs. Namely, they determine asymptotically the number of edges a subgraph of $G(n,p)$ needs to have in order to guarantee a k-edge path, for different ranges of p and k. Araújo, Moreira and Pavez-Signé [6] show a version of the Erdős–Sós conjecture for random graphs, and linear sized trees. More precisely, they show that for all Δ, ε and $t \in (0,1)$ there is C such that after deletion of at most a $(1-t-\varepsilon)$-fraction of the edges of $G(n, \frac{C}{n})$, the resulting graph still contains w.h.p. all trees of order tn and of maximum degree at most Δ. It seems not to be known whether analogues of the results from [9] and [6] for spanning trees exist.

Finally, there are some recent results for *randomly perturbed* graphs. This model relies on a graph of linear but very small minimum degree, which is 'randomly perturbed' by adding a few random edges to it. More precisely, we consider the union of a graph G_α of minimum degree at least αn and the random graph $G(n,p)$, on the same set of vertices. The study of bounded degree spanning trees in this model was initiated by Krivelevich, Kwan and Sudakov [97]. They determined the threshold $p = \frac{C}{n}$ (with C depending on α and Δ) for containment of a single tree of maximum degree Δ, and conjectured the same threshold for the corresponding universality result. This was confirmed by Böttcher, Han, Montgomery, Kohayakawa, Parczyk and Person [20]. Joos and Kim [85] show a variant of Theorem 4.1 for randomly perturbed graphs.

7 Ramsey numbers

Both Conjecture 2.1 (the Erdős–Sós conjecture) and Conjecture 3.3 (the Loebl–Komlós–Sós conjecture) have a direct application in Ramsey theory. Let us start with 2-colour Ramsey numbers. The *(2-colour) Ramsey number* $R(H_1, H_2)$ of a pair of graphs H_1, H_2 is the smallest integer n such that every 2-colouring of the edges of K_n contains a copy of H_1 in the first colour, or a copy of H_2 in the second colour. Generalising the notion to classes $\mathcal{H}_1, \mathcal{H}_2$ of graphs, we write $R(\mathcal{H}_1, \mathcal{H}_2)$ for smallest integer n such that every 2-colouring of the edges of K_n contains a copy of *each* $H_1 \in \mathcal{H}_1$ in the first colour, or a copy of *each* $H_2 \in \mathcal{H}_2$ in the second colour. We write short $R(H)$ ($R(\mathcal{H})$) for $R(H, H)$ ($R(\mathcal{H}, \mathcal{H})$).

Some of the earliest results on Ramsey numbers for trees were the following. In 1967, Gerencsér and Gyárfás [58] determined the Ramsey number of two paths. They showed that

$$R(P_k, P_\ell) = k + \lfloor \frac{\ell + 1}{2} \rfloor$$

for k-edge and ℓ-edge paths P_k and P_ℓ with $k \geq \ell \geq 2$. For stars, the Ramsey number is known to be larger. Harary [67] observed in 1972 that

$$R(K_{1,k}, K_{1,\ell}) = k + \ell$$

if at least one of k, ℓ is odd, and $R(K_{1,k}, K_{1,\ell}) = k + \ell - 1$ in the case that k and ℓ are both even.

Conjectures 2.1 and 3.3 can be applied as follows in the Ramsey setting. Given a 2-edge-coloured $K_{k+\ell}$, say with colours red and blue, it is easy to see that either the red graph has average degree greater than $k-1$, or the blue graph has average degree greater than $\ell - 1$. Also, either the red graph has median degree at least k, or the blue graph has median degree at least ℓ. Therefore, each of the two conjectures would imply that every 2-edge-colouring of $K_{k+\ell}$ with colours red and blue contains either *all* k-edge trees in red, or *all* ℓ-edge trees in blue, and therefore, $R(\mathcal{T}_k, \mathcal{T}_\ell) \leq k + \ell$, where $\mathcal{T}_j$ is the class of all trees with j edges. If k and ℓ are both even, the bound we can infer from Conjecture 2.1 is even lower: In that case $R(\mathcal{T}_k, \mathcal{T}_\ell) \leq k + \ell - 1$. Accordingly, and focusing on the case $k = \ell$, Burr and Erdős [25] conjectured in 1976 that $R(\mathcal{T}_k) \leq 2k$, and $R(\mathcal{T}_k) \leq 2k - 1$ if k is even. The bound $R(\mathcal{T}_k) \leq 2k$ has been confirmed for large k, by Zhao's solution of Conjecture 3.1 for large host graphs [126].

However, the bound $R(T_k, T_\ell) \leq k + \ell$ for a k-edge tree T_k and an ℓ-edge tree T_ℓ seems to be far from best possible for non-star trees. As noted above, the Ramsey number for paths differs significantly from the Ramsey number for stars with the same number of edges. Note that paths are (almost) completely balanced trees, while stars are the most unbalanced trees. Believing this difference to be the reason for the variation in their Ramsey numbers, Burr [23] put forward the following conjecture in 1974. He suggested that if T is a tree whose bipartition classes have sizes t_1, t_2, with $t_2 \geq t_1 \geq 2$, then the Ramsey number of T is

$$R_B(T) := \max\{2t_1 + t_2, 2t_2\} - 1.$$

Standard examples show this number would be best possible, and $R_B(T)$ matches the Ramsey numbers for paths from [58].

Haxell, Łuczak, and Tingley [75] confirmed Burr's conjecture asymptotically for trees with (linearly) bounded maximum degree in 2002. However, already shortly after the conjecture was posed, Grossman, Harary and Klawe [60] found that it was not true for certain double stars. A *double star* D_{t_1,t_2} is a union of two stars K_{1,t_1-1} and K_{1,t_2-1} whose centres are joined by an edge. The examples from [60] still allowed for the possibility that Burr's conjecture was off only by one, that is, that the Ramsey number of any tree T would be bounded by $R_B(T) + 1$. The authors of [60] conjectured this to be the truth for double stars. This has been confirmed for a range[2] of values of t_1, t_2. But recently, Norin, Sun and Zhao [106] disproved the conjecture from [60] in general by showing that the numbers $R(D_{t_1,t_2})$ and $R_B(D_{t_1,t_2})$ differ considerably if t_2 lies between $\frac{7}{4}t_1 + o(t_1)$ and $\frac{105}{41}t_1 + o(t_1)$. In particular, for the case $t_2 = 2t_1$ they find that

$$R(D_{t_1,t_2}) \geq 4.2t_1 - o(t_1)$$

while

$$R_B(D_{t_1,t_2}) = 4t_1 - 1.$$

The authors of [106] pose the following question.

Question 7.1 (Norin, Sun and Zhao [106]). *Is it true that $R(D_{t_1,t_2}) = 4.2t_1 + o(t_1)$ if $t_2 = 2t_1$?*

A question of Erdős, Faudree, Rousseau and Schelp [43], who, in 1982, asked whether $R(T) = R_B(T)$ for all trees T with colors classes of sizes t_1 and $t_2 = 2t_1$, has also been answered in the negative by the above mentioned results from [106]. The authors of [106] offer the following alternative.

[2]The current best results are $R(D_{t_1,t_2}) \leq R_B(D_{t_1,t_2}) + 1$ if $t_2 \geq 3t_1 - 2$ (obtained using ad hoc arguments [60]) and $R(D_{t_1,t_2}) \leq R_B(D_{t_1,t_2})$ if $t_2 \leq 1.699t_1 + 1$ (obtained using flag algebras [106]).

Question 7.2 (Norin, Sun and Zhao [106])**.** *Is it true that $R(T) \leq 4.2t_1 + o(t_1)$ for all trees T with colors classes of sizes t_1 and $2t_1$?*

Another natural question in this context seems to be whether there is an exact version of the asymptotic results of Haxell, Łuczak, and Tingley [75] (which would interesting even if we had to restrict the maximum degree of the tree by, say, a constant). A second question is whether their result can be extended to graphs of slightly larger maximum degree. In the main result from [75], the bound on the maximum degree of the tree is $\delta|V(T)|$, with δ depending on the approximation. On the other hand, the known counterexamples to Burr's conjecture are all double stars D of maximum degree exceeding $\frac{7}{11}|V(D)|$. So, there might be a chance that for some reasonable constant $c \leq \frac{7}{11}$, Burr's conjecture still holds for all trees T of maximum degree at most $c|V(T)|$.

Question 7.3. *Is there a constant c such that $R(T) = R_B(T)$ for all trees T with $\Delta(T) \leq c|V(T)|$?*

Let us now briefly look at results and questions for multi-colour Ramsey numbers of trees. The r-colour Ramsey number $R_r(H)$ of a graph H is defined as the smallest integer n such that every r-colouring of the edges of K_n contains a monochromatic copy of H.

Multicolour Ramsey numbers for trees have not been studied much. The most studied case is the path P_k. The 3-colour Ramsey number of the k-edge path P_k has been conjectured to be $2k$ for even k and $2k+1$ for odd k by Faudree and Schelp [46], and this is best possible. This conjecture has been confirmed for large k by Gyárfás, Ruszinkó, Sárközy and Szemerédi [63]. For more colours, less is known. Constructions based on affine planes show that $R_r(P_k) \geq (r-1)k$ if $r-1$ is a prime power. An upper bound on $R_r(P_k)$ can be obtained by applying the Erdős–Gallai theorem to the most popular colour in a given r-colouring. This yields $R_r(P_k) \leq r(k+1)$. Recently, the latter bound has been improved to $(r - \frac{1}{4})(k+1) + o(k)$ by Davies, Jenssen and Roberts [31].

Multicolour Ramsey numbers for k-edge stars were determined by Burr and Roberts [26] in 1973. They showed that $(k-1)r+1 \leq R_r(K_{1,k}) \leq (k-1)r+2$. The lower bound is tight if and only if both k and r are even.

General bounds for all trees have also been considered. Erdős and Graham [44] observed that an affirmative answer to the following question would follow from the Erdős-Sós conjecture. (A version for skew trees would also follow from Conjecture 3.7.)

Question 7.4 (Erdős and Graham [44])**.** *Is the r-colour Ramsey number for a k-edge tree T_k equal to $rk + O(1)$?*

The authors of [44] observe that $R_r(T_k)$ is bounded from above by $2rk$. This bound can be obtained using a similar argument as for the 2-colour Ramsey number, and the fact that Conjecture 2.1 holds if the average degree bound is replaced by $2k$, as we observed in Section 2.

8 Directed graphs

In this section, we will shift our focus from trees and graphs to their oriented versions, that is, oriented trees and digraphs/oriented graphs. An *oriented tree (graph)* is a tree (graph) all whose edges have been given a direction. A *digraph* may have (at most) two edges between a pair of vertices, as long as these go in opposite directions. A *tournament* is an oriented complete graph, and a *complete digraph* is a digraph having all possible edges.

Let us start with oriented graphs and trees. Before we turn to possible generalisations of the results in the earlier sections, let us illustrate how the orientations of the edges bring new difficulties. Just considering the degree in the underlying graph is clearly not enough. Indeed, it is fairly easy to construct an orientation of a complete graph K_{k+1} that does not contain the k-edge star having all its edges directed inwards, thus preventing even the easiest observation for graphs to carry over to digraphs.

So, let us consider the class of tournaments as possible host graphs. One of the first results on oriented trees in tournaments was established by Rédei [112] in 1934. It states that every tournament on $k+1$ vertices contains the *directed k-edge path* (i.e. the k-edge path having all its edges directed in the same direction). More results on oriented paths appeared in, e.g. [61, 117, 125], until in 2000, Havet and Thomassé [72] showed that with three exceptions, all oriented k-edge paths appear in any $(k+1)$-vertex tournament. Similar results have been shown for some classes of oriented trees with bounded maximum degree (see [103] and references therein).

A generalisation of these results for containment of all oriented trees of some fixed size was conjectured by Sumner in the 1980's.

Conjecture 8.1 (Sumner, see [115]). *Every tournament on $2k$ vertices contains every oriented k-edge tree.*

This conjecture is best possible, which can be seen by considering a $(k-1)$*-regular tournament* (that is, a tournament whose vertices each have in- and out-degree $k-1$) on $2k-1$ vertices, which does not contain the k-edge star with all edges directed inwards (or outwards).

Variants of Conjecture 8.1 replacing $2k$ with a larger number are known [39, 66, 68, 71]. The current best bound is $\frac{21}{8}k$, and was found by Dross and Havet [38]. Havet and Thomassé [71] showed that Conjecture 8.1 holds for *arborescences*, that is, oriented trees having all their edges directed away from (or towards) a specific vertex. After proving an approximate version [98], Kühn, Mycroft and Osthus [99] confirmed Sumner's conjecture for large n, using the regularity method. For oriented trees of bounded degree, the size of the host tournament can be lowered to $n + o(n)$ (see [99, 103, 105]).

In 1996, Havet and Thomassé proposed that the size of the host tournament can be smaller if we add a restriction on the number of leaves of the tree. This gives the following generalisation of Conjecture 8.1.

Conjecture 8.2 (Havet and Thomassé, see [69])**.** *Let T be an oriented k-edge tree with ℓ leaves. Then every tournament on $k + \ell$ vertices contains a copy of T.*

Note that for oriented stars, Conjecture 8.2 gives the same bound as Conjecture 8.1, but for other trees, the bound is lower. As we saw above, if T is a path, the tournament may be by one smaller than required by Conjecture 8.2. For progress on Conjecture 8.2 see [27, 66, 69].

Turning now to oriented graphs as possible host graphs for oriented trees, there is a natural generalisation of the results and conjectures from above, which involves the chromatic number of a digraph. An oriented graph is *n-chromatic* if the underlying graph has chromatic number n. The well-known Gallai-Hasse-Roy-Vitaver (GHRV) theorem (see e.g. [11]) states that every $(k + 1)$-chromatic oriented graph contains the directed path with k edges. As any n-vertex tournament is n-chromatic, this is a generalisation of Rédei's theorem mentioned above.

An extension of the GHRV theorem to oriented trees was suggested by Burr [24] in 1980. His conjecture would imply Sumner's conjecture.

Conjecture 8.3 (Burr [24])**.** *Every $2k$-chromatic oriented graph contains each oriented k-edge tree.*

A version of Conjecture 8.3 for large k, replacing oriented trees with oriented paths, and '$2k$-chromatic' with 'k-chromatic', is attributed in [11] to Bondy. There is also a generalisation of Conjecture 8.2 in the spirit of Conjecture 8.3: Every $(k + \ell)$-chromatic digraph contains each oriented k-edge tree having ℓ leaves (this was conjectured in [71]). Furthermore, Naia [105] conjectures that for every oriented k-edge tree T, the minimum n such that every tournament of order n contains T coincides with the minimum n such that every n-chromatic oriented graph contains T. This would imply that Conjecture 8.3 and Conjecture 8.1 are equivalent.

Conjecture 8.3 is only known for some specific classes of oriented paths (see [1] for references) and for all oriented stars [105]. Burr [24] showed that Conjecture 8.3 is true if we replace $2k$ with k^2, and Addario-Berry, Havet, Linhares Sales, Thomassé and Reed [1] improved this (roughly by a factor of 2). A better bound only for oriented graphs with large chromatic number is given in [105].

The authors of [1] also propose an interesting conjecture of their own. In order to be able to state their conjecture, we need a definition. An *antidirected* tree is an oriented tree each of whose vertices either has no incoming edges or no outgoing edges.

Conjecture 8.4 (Addario-Berry, Havet, Linhares Sales, Thomassé and Reed [1]). *Every digraph D with more than $(k-1)|V(D)|$ edges contains each antidirected k-edge tree.*

This conjecture is best possible because of the $(k-1)$-regular tournament which does not contain the k-edge out-star, or alternatively, because of the complete digraph on k vertices which does not contain *any* oriented k-edge tree. Also note that a version of Conjecture 8.4 for oriented trees that are not antidirected fails (even if we made the condition on the number of edges stronger): Consider a large complete bipartite graph $G = (A, B)$, and orient all its edges from A to B. The resulting oriented graph only has antidirected subgraphs.

The authors of [1] verify Conjecture 8.4 for antidirected trees of diameter at most 3, and they note it is not difficult to see that Conjecture 8.4 implies Conjecture 8.3 (and therefore also Conjecture 8.1) for antidirected trees. (This is because every $2k$-chromatic graph has a subgraph H of minimum degree at least $2k-1$, and thus H has more than $(k-1)|V(H)|$ edges.) They also note that if we restrict Conjecture 8.4 to symmetric digraphs (a digraph is *symmetric* if all its edges are bidirected), then the conjecture becomes equivalent to the Erdős–Sós conjecture (Conjecture 2.1). So one can interpret Conjecture 8.4 as a common generalisation of Conjecture 2.1 and Conjecture 8.3.

It seems, however, slightly dissatisfying that Conjecture 8.4 only applies to antidirected trees. We have seen above that this is necessary, as there are oriented graphs on $2n$ vertices with n^2 edges that do not even contain a two-edge directed path. In fact, any antidirected host graph with enough edges would serve as an example. Now, in order to avoid these examples, one might try requiring that the vertices of the host digraph, on average, had *both* large enough in-degree $d^-(v)$ *and* large enough out-degree $d^+(v)$.

That is, defining the *semidegree* of a vertex v as

$$d^0(v) := \min\{d^-(v), d^+(v)\},$$

we would require the average of the semidegrees, taken over all vertices v of the digraph D, to be larger than $k-1$ (or more generally, to be larger than some function of k). Although this would clearly exclude all antidirected host graphs, it is not sufficient to guarantee all oriented trees as subdigraphs. In order to see this, just consider an appropriate blow-up[3] of a $(k-1)$-edge directed path. (Observe that this example also shows that a naïve extension of the Loebl-Komlós-Sós conjecture that replaces $d(v)$ with $d^0(v)$ fails.)

Another possibility is to consider the *minimum semidegree*

$$\delta^0(D) := \min\{d^0(v) : v \in V(D)\}$$

of a digraph D. Using a greedy embedding argument, it is clear that any digraph with $\delta^0(D) \geq k$ must contain each oriented k-edge tree.

This trivial bound can be lowered if $k = n$, and the tree is a path. Indeed, results from [33, 34] imply that if D is an n-vertex digraph with $\delta^0(D) \geq \frac{n}{2}$, then D contains every orientation of the path on n vertices, and this is sharp. This might extend to oriented paths of smaller size. A first question in this direction would be whether Observation 4.4 extends to oriented graphs.

Conjecture 8.5. *Does every oriented graph D with $\delta^0(D) > \frac{k}{2}$ contain each oriented k-edge path?*

If this conjecture is true, it would be sharp. This can be seen by considering, for even k, a blow-up of the directed triangle, replacing each vertex with an independent set of size $\frac{k}{2}$. The antidirected path with k edges is not contained in this graph. Moreover, the conjecture is true for directed paths, by a result of Jackson [82]. If we replace the bound on the minimum semidegree with $\frac{3}{4}k$, it holds for antidirected paths [90]. If the host graph is a tournament, Conjecture 8.5 follows from Conjecture 8.1.

An analogous question can be asked for digraphs. Observe that now, we need to require, in addition to the minimum semidegree condition, a lower bound on the size of the largest component (in order to prevent the digraph being the union of complete digraphs of order $\frac{k}{2}+2$).

Question 8.6. *Does every digraph D with $\delta^0(D) > \frac{k}{2}$ having a component of size at least $k+1$ contain each oriented k-edge path?*
If not, can we lift the bound on the minimum semidegree (to some bound strictly below k) so that the question can be answered in the affirmative?

[3] A *blow-up* of a digraph D is obtained by replacing each vertex with an independent set of vertices, and adding all edges from such a set X to a set Y, if X and Y originated from vertices x and y belonging to an edge $\overrightarrow{xy}$ of D.

If necessary, one might additionally require a larger component (or a large strong component).

Let us shift our attention from oriented paths to oriented bounded degree trees. Mycroft and Naia [104] used the minimum semidegree notion to give an extension of Theorem 4.1 to digraphs.

Theorem 8.7 (Mycroft and Naia [104])**.** *For all positive real α, Δ there exists n_0 such that for all $n \geq n_0$ every n-vertex digraph D with $\delta^0(D) \geq (\frac{1}{2} + \alpha)n$ contains every oriented n-vertex tree of maximum degree at most Δ.*

In view of their result we feel encouraged to ask whether generalisations to digraphs, using the minimum semidegree notion, of the results and conjectures from Section 5 exist. In particular, if Conjecture 8.5 (Question 8.6) is true, one might try for results in the spirit of Theorem 5.4 and Conjectures 5.1, 5.2 and 5.5.

Question 8.8. *Are there constants $c < 1$ and C such that every oriented graph (digraph) D with $\delta^0(D) \geq ck$ that has a vertex v with $d^0(v) \geq Ck$ contains each oriented k-edge tree?*

Another possibility is to substitute the semidegree with another degree notion. One natural candidate is the *total minimum degree* $\delta_{tot}(D)$, which is defined as the minimum of the sums of the in- and out-degrees of the vertices of the digraph D. Mycroft and Naia asked the following question [104, Problem 4.1].

Question 8.9. **[104]** *Does Theorem 8.7 remain true if $\delta^0(D)$ is replaced by $\frac{\delta_{tot}(D)}{2}$?*

If this is true, one could ask for similar variants of the other open questions from this section.

We close the section with a short remark on Ramsey numbers for oriented trees. There are two natural notions. The *oriented Ramsey number* $R_k^{\rightarrow}(T)$ of an oriented tree T is the smallest integer n such that every k-coloured tournament on n vertices contains a monochromatic copy of T. The *directed Ramsey number* $R_k^{\leftrightarrow}(T)$ is defined in the same way, replacing the k-coloured tournament with a k-coloured complete digraph. Early results using these notion focused on directed paths and two colours [28, 62]. An interesting insight gives a recent work of Bucič, Letzter and Sudakov [22] who establish a difference in the order of magnitude of the two numbers, by showing that $R_k^{\rightarrow}(T) = c_k|V(T)|^k$ and $R_k^{\leftrightarrow}(T) = c_k|V(T)|^{k-1}$. As observed in [22], the former of these two equalities would also follow from Conjecture 8.3 (Burr's conjecture).

9 Hypergraphs

We will only discuss r-uniform hypergraphs, and call such hypergraphs r-graphs for short. As one might expect, there is more than one natural generalisation of trees to hypergraphs. In what follows, we will discuss *tight r-trees*, *linear r-trees*, *r-expansions* and *Berge r-trees*. We refer to [57, 88] for an overview of more Túran type results for hypergraphs.

9.1 Tight hypertrees

We start our overview with tight hypergraphs. Call an r-graph a *tight r-tree* if its edges can be ordered such that except for the first edge, every edge consists of an $(r-1)$-set contained in some previous edge, and an entirely new vertex. Note that for instance, the widely studied tight r-paths are examples of tight r-trees. (A *tight r-path* has vertices $v_1, \ldots, v_n$ and edges $\{v_i, \ldots, v_{i+k-1}\}$ for $1 \leq i \leq n-k+1$.)

For r-graphs and tight r-trees, Kalai proposed in 1984 the following natural generalisation of the Erdős-Sós conjecture (see [48]).

Conjecture 9.1 (Kalai's conjecture, see [48])**.** *Let $r \geq 2$ and let H be an r-graph on n vertices with more than $\frac{k-1}{r}\binom{n}{r-1}$ edges. Then H contains every tight r-tree T having k edges.*

As already noted in [48], it follows from constructions using a result of Rödl [116] (or alternatively, one can use designs whose existence is guaranteed by Keevash's work [87]) that Conjecture 9.1 is tight as long as certain divisibility conditions are satisfied.

It is not difficult to observe (see e.g. [51, Proposition 5.4]) that any n-vertex r-graph on n vertices with more than $(k-1)\binom{n}{r-1}$ edges contains every tight r-tree with k edges, which is a factor of r away from the conjectured bound. This bound can be proved as follows: successively delete all edges at $(r-1)$-sets of vertices that lie in few edges until arriving at a subhypergraph of large minimum 'codegree'. Then, greedily embed the tree.

Not much is known on Kalai's conjecture in general. Restricting the class of host r-graphs, it is known that the conjecture holds if the host r-graph H is r-partite [122].

Restrictions on the type of tight r-trees have led to the following results. In 1987, Frankl and Füredi [48] showed that Conjecture 9.1 holds for all 'star-shaped' tight r-trees, that is, for all tight r-trees whose first edge intersects each other edge in $r-1$ vertices. Füredi, Jiang, Kostochka, Mubayi and Verstraëte show in [53] an asymptotic version of Conjecture 9.1 for a broadened concept of 'star-shaped' (the first c edges have to intersect all other edges in $r-1$ vertices, for a constant c), and in [54] an exact result for a class of tight 3-trees. Füredi and Jiang [51] show Conjecture 9.1 for special types of tight r-trees with many leaves.

On the opposite extreme of the spectrum of tight r-trees, there are the tight r-paths. Improving on results of Patkós [107], Füredi, Jiang, Kostochka, Mubayi and Verstraëte [52] show that for tight r-paths the bound in Conjecture 9.1 can be replaced by $\frac{k-1}{2}\binom{n}{r-1}$ if r is even, and by a similar bound if r is odd. Moreover, an asymptotic version of Kalai's conjecture for tight r-paths whose order is linear in the order n of the host r-graph has been established by Allen, Böttcher, Cooley and Mycroft [4] for large n. The authors of [4] remark that they do not believe their result to be best possible, arguing that the constructions and designs from [87, 116] only exist when the order of the host graph is much larger than the order of the tight path.

It seems natural to seek extensions of other results for graphs to r-graphs and tight r-trees. This has been done for Theorem 4.1. As usual, for any r-graph H, let $\delta_i(H)$ ($\Delta_i(H)$) denote the minimum (maximum) number of edges any i-subset of $V(H)$ belongs to. With this notation, and using hypergraph regularity, Pavez-Signé, Quiroz-Camarasa, Sanhueza-Matamala and the author [108] show a version of Theorem 4.1 for hypergraphs. Namely, they show that for any $\gamma, \Delta > 0$, every large enough r-graph H with $\delta_{k-1}(H) \geq (\frac{1}{2}+\gamma)|V(H)|$ contains each r-tree T of the same order obeying $\Delta_1(T) \leq \Delta$.

One might also ask for generalisations of the results/conjectures from Section 5 to tight hypergraph trees.

Question 9.2. *Is there a function f such that every r-graph H with $\delta_{k-1}(H) \geq \frac{k}{2}$ and $\Delta_{k-1} \geq f(k)$ contains each k-edge tight r-tree?*

More cautiously, one could replace $\frac{k}{2}$ in Question 9.2 with $(1-\gamma)k$, for some fixed $\gamma > 0$. Perhaps it is also possible to extend Conjecture 3.3 to tight r-trees.

Question 9.3. *Let H be an r-graph such that at least $\frac{\binom{n}{r-1}}{2}$ of its $(r-1)$-tuples each belong to at least k edges. Does H contain each k-edge tight r-tree?*

9.2 Expansions of trees and linear paths

The *r-expansion* of a tree T is the r-uniform hypergraph obtained from T by adding to each edge $r-2$ new vertices. A *linear r-tree* is obtained from an edge by subsequently adding any number of new edges that each contain precisely one of the previous vertices. An r-expansion of a path is also called a *linear path*, as it satisfies the definition of a linear tree.

The *Turán number* $ex_r(n,H)$ of an r-graph H is defined (in complete analogy to the Turán number of a graph) as the maximum number of edges a hypergraph can have if it does not contain H. There is a considerable amount of literature on Turán numbers of expansions. For an overview we refer to the survey of Mubayi and Verstraëte [102]. One of the important results relevant for this survey is the determination of the Turán number of the linear r-path with k edges for fixed $r \geq 4$ and k and large n by Füredi, Jiang and Seiver [55] using the delta-system-method. The case $r = 3$ was solved by Kostochka, Mubayi and Verstraëte [94] using an approach based on random sampling.

Füredi [50] asymptotically determined the Turán number for r-expansions of trees for $r \geq 4$, and conjectured the corresponding asymptotics for $r = 3$; this was confirmed by Kostochka, Mubayi and Verstraëte in [95]. These results relate the Turán number of an r-expansion T with the minimum size $\sigma(T)$ of a crosscut of T (where a *crosscut* is a set of vertices met by every edge of T in exactly one vertex). More precisely, for a fixed r-expansion T, and for $r \geq 3$, the Turán number $ex_r(n,T)$ is asymptotically determined as follows [50, 95]:

$$ex_r(n,T) = \big(\sigma(T) - 1 + o(n)\big)\binom{n}{r-1}.$$

That this bound is asymptotically best possible can be seen by considering the r-graph consisting of all edges containing exactly one vertex from a fixed set of size $\sigma(T) - 1$: This r-graph does not contain T. See also [51] for some related results.

9.3 Berge hypertrees

Other recent activity has focused on Berge r-trees. A *Berge r-tree* is an r-graph H such that there is a tree T (i.e. an acyclic connected 2-graph), an injection from $V(T)$ to $V(H)$, and a bijection from $E(T)$ to $E(H)$ such that the images of the endpoints of any edge $e \in E(T)$ are contained in the image of e. This definition gives the usual definition of a Berge path if T is a path.

The Turán number for Berge r-paths $BP_k^{(r)}$ was almost completely determined by Győri, Katona and Lemons [64], with the last remaining case solved in [32]. The bound is $ex(n, BP_k^{(r)}) \leq \frac{n(k-1)}{r+1}$ if $r \geq k$ and $ex(n, BP_k^{(r)}) \leq \frac{n}{k}\binom{k}{r}$ if $r < k$, and extremal r-graphs are known.

Results for k-edge Berge r-trees $BT_k^{(r)}$ have been obtained by Gerbner, Methuku and Palmer [56] and by Győri, Salia, Tompkins and Zamora [65]. If $r \geq k(k-2)$ and the tree we are looking for is not a star, then the bound for Berge r-paths from the previous paragraph applies [65]. In the case $k > r$ the best known bound is

$ex(n, BT_k^{(r)}) \leq \frac{2(r-1)n}{k}\binom{k}{r}$, although this can be lowered by a factor of $2(r-1)$, thus reaching the bound for Berge paths from the previous paragraph, if we assume the Erdős–Sós conjecture holds [56]. These bounds are sharp under certain divisibility conditions.

It would be interesting to see extensions, both to linear trees and to Berge trees, of the results we have seen in Sections 4 and 5 for graphs, that is, results that use conditions on the minimum (and maximum) degree of the host graph, instead of the average degree (i.e. number of edges).

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