Spectral Properties of Magnetic Quantum Hamiltonians

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Ecce homo...

A Raúl del Tránsito Rozas Vergara

Nací en Santiago el año 83. Rápidamente lo abandoné para ir al pueblo de mi familia, Llay-Llay. Hice mi enseñanza básica en el colegio Filipense, aunque mucho más importante fueron las historias de mi abuelo y las pichangas con mis primos. Mi enseñanza media fue en San Felipe, en la época en que tuve que decidir entre el deporte, la música y la matemática, y por alguna razón extraña me quedé con esta última. Entré a la Universidad de Chile al mismo tiempo que era papá y ahora después de muchos años y sacrificios, ¡al fin!, egreso.

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Resumen

El objetivo principal de esta tesis es estudiar las propiedades espectrales de ciertos hamiltonianos cuánticos magnéticos, que aparecen en la teoría matemática del efecto cuántico de Hall, bajo perturbaciones que son relativamente compactas. El operador no perturbado tiene la forma $H_0 = H_{\mathrm{Landau}} + W$, donde H_{Landau} es el hamiltoniano de Landau, es decir, el operador de Schrödinger magnético en dos dimensiones con campo magnético constante y W es el llamado potencial borde, que modela las propiedades del medio, donde se supone que estas varían en sólo una dirección. Consideramos dos casos esencialmente diferentes: W monótono y W periódico. Debido a su invariancia con respecto al grupo de traslaciones unidimensionales, el operador H_0 es unitariamente equivalente a una integral directa, cuyas fibras forman una familia analítica de Kato de operadores de Schrödinger unidimensionales con espectro puramente discreto. Por lo tanto el espectro de H_0 tiene estructura de bandas.

El operador perturbado tiene la forma $H = H_0 + V$, donde el potencial decae al infinito y modela una impureza localizada.

Principalmente nos interesa la distribución asintótica del espectro discreto de H que está en una laguna abierta de su espectro esencial. En el caso monótono damos una condición suficiente, de carácter geométrico, que garantiza que el número de valores propios discretos de H en cada laguna abierta de su espectro esencial, es finito. Mientras que el caso periódico vemos que este número es siempre infinito. Si una laguna contiene infinitos valores propios de H, consideramos la convergencia de estos valores propios al borde de la laguna, la cual es descrita en términos de hamiltonianos efectivos adecuados. En el caso de V con soporte compacto obtenemos cotas asintóticas de los valores propios cerca del borde, que son precisas en cuanto al orden. El estudio de la distribución asintótica del espectro discreto requiere un estudio detallado de las funciones de banda del operador no perturbado H_0. Aquí desarrollamos este análisis y obtenemos una serie de resultados que podrían ser de interés en sí mismas, ya que ellos tienen aplicaciones potenciales en modelos y problemas relacionados.

Abstract

The main task of this thesis is to study the spectral properties of certain 2D magnetic quantum Hamiltonians, which arise in the mathematical theory of the quantum Hall effect, under relatively compact perturbations. The unperturbed operator is of the form $H_0 = H_{\text{Landau}} + W$ where H_{Landau} is the Landau Hamiltonian, i.e. the 2D magnetic Schrödinger operator with constant magnetic field, and W is the so-called edge potential modeling the properties of the media which are supposed to vary only in one direction. We consider two essentially different cases: monotone W and periodic W. Due to its invariance with respect to a one-dimensional group of translations, the operator H_0 is unitarily equivalent to a direct integral whose fibres form a Kato analytic family of 1D Schrödinger operators with purely discrete spectrum. Therefore, the spectrum of H_0 has a band structure.

The perturbed operator has the form $H = H_0 + V$ where the potential V decays at infinity, and models a localized impurity.

We are mainly interested in the asymptotic distribution of the discrete spectrum of H lying in an open gap of its essential one. In the case of a monotone W we establish a sufficient condition of geometric nature which guarantees that the number of the discrete eigenvalues of H in any open gap in its essential spectrum is finite, while in the case of periodic W we show that this number is always infinite. If a given gap contains infinitely many eigenvalues of H, we consider the convergence of these eigenvalues to the edges of the gap, which is described in the terms of appropriate effective Hamiltonians. In the case of compactly supported V we obtain asymptotic bounds of the eigenvalues near the edges of the gap, which are sharp in order. The study of the asymptotic distribution of the discrete spectrum requires a detailed study of the band functions of the unperturbed operator H_0. We perform this analysis and obtain a series of results which may be of independent interest since they have potential applications in related problems and models.

Contents

Introduction

The analysis of two-dimensional (2D) magnetic Schrödinger operators has received a lot of attention during the last 30 years. Generally, the Schrödinger operators serve as Hamiltonians of non relativistic spinless quantum particles subject to a classical electromagnetic field. The interest towards the 2D magnetic systems is due, in particular, to the fact that they play a crucial role in the explanation and the mathematical interpretation of the quantum Hall effect, i.e. the quantization of the Hall conductance (see [84]), whose discovery brought the 1985 Nobel Prize in Physics to the German physicist Klaus von Klitzing. The importance of the applications of 2D magnetic quantum Hamiltonians in physics and technology motivated the mathematical community not only to study in detail the mathematical aspects of the quantum Hall effect (see e.g. [3, 44, 23, 16]), but also to undertake a thorough and systematic study of the general spectral properties of these Hamiltonians.

This thesis is devoted to the study of operator

$$H := H_0 + V \tag{0.1}$$

with

$$H_0 := -\frac{\partial^2}{\partial x^2} + \left(-i\frac{\partial}{\partial y} - bx\right)^2 + W(x), \tag{0.2}$$

acting in $L^2(\mathbb{R}^2)$. Here $b > 0$ is the intensity of the constant magnetic field, $W \in L^\infty(\mathbb{R};\mathbb{R})$ depends only on the first variable x, while $V \in L^\infty(\mathbb{R}^2, \mathbb{R})$ decays at infinity. The function W is called *the edge potential* describing the properties of the medium which are supposed to vary only in the x-direction, while V models a localized perturbation of H_0.

The unperturbed operator H_0 is defined initially on $C_0^\infty(\mathbb{R}^2)$, and then is closed in $L^2(\mathbb{R}^2)$. Thus the operator H_0 is self-adjoint in $L^2(\mathbb{R}^2)$ (see [19]). Moreover, the perturbed operator H is obviously also self-adjoint on $\mathrm{Dom}(H_0)$.

Operators of a quite similar form have been widely used in the mathematical theory of the quantum Hall effect (see e.g. [17, 16, 33, 34]).

We are mainly interested in the asymptotic distribution of the discrete spectrum of H near the edges of its essential spectrum. In order to introduce the specific problem investigated in the thesis, we need to describe some features of the unperturbed operator H_0, fixing at the same time notations which will be used throughout the thesis.

Let $\mathcal{F}$ be the partial Fourier transform with respect to y, i.e.

$$(\mathcal{F}u)(x,k) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-iky} u(x,y)dy, \quad u \in L^2(\mathbb{R}^2). \tag{0.3}$$

Then we have

$$\mathcal{F}H_0\mathcal{F}^* = \int_{\mathbb{R}}^{\oplus} h(k)dk \tag{0.4}$$

where the operator

$$h(k) := -\frac{d^2}{dx^2} + (bx-k)^2 + W(x), \quad k \in \mathbb{R},$$

is self-adjoint in $L^2(\mathbb{R})$. The spectrum of $h(k)$ is discrete and simple for every $k \in \mathbb{R}$. Note also that $h(k)$, $k \in \mathbb{R}$, is a Kato analytic family (see [43], [73]). Note that the domain $\mathrm{Dom}(h)$ of the operator $h(k)$ is independent of $k \in \mathbb{R}$.

Fix $k \in \mathbb{R}$ and denote by $\{E_j(k)\}_{j=1}^{\infty}$ the increasing sequence of the eigenvalues of $h(k)$. The Kato analytic perturbation theory implies that for any $j \in \mathbb{N}$, $E_j(k)$ is a real analytic function of $k \in \mathbb{R}$. When we need to indicate the dependence of $E_j(k)$ on b and/or W, we will write $E_j(k; b, W)$ or $E_j(k; W)$ instead of $E_j(k)$.

It is useful to introduce another fibre operator

$$\tilde{h}(k) = -\frac{d^2}{dx^2} + b^2x^2 + W(x + k/b), \quad k \in \mathbb{R}, \tag{0.5}$$

self-adjoint in $L^2(\mathbb{R})$. Define the operator $\mathcal{U}_k : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $k \in \mathbb{R}$, unitary in $L^2(\mathbb{R})$, by

$$(\mathcal{U}_k f)(x) := f(x - k/b), \quad f \in L^2(\mathbb{R}).$$

Obviously,

$$\mathcal{U}_k^* h(k)\mathcal{U}_k = \tilde{h}(k), \quad k \in \mathbb{R}.$$

Then the operator $\tilde{h}(k)$ is unitarily equivalent to $h(k)$, and the eigenvalues of $\tilde{h}(k)$ coincide with those of $h(k)$, namely $\{E_j(k)\}_{j=1}^{\infty}$, $k \in \mathbb{R}$. We introduce the operator $\tilde{h}(k)$ since sometimes it reveals in a more transparent way the dependence of $E_j(k, b, W)$ on the variables k, b, and W.

If $W = 0$, then the eigenvalues E_j are independent of k, and their explicit form is well-known:

$$E_j(k; b, 0) = E_j(b, 0) = b(2j-1), \quad k \in \mathbb{R}, \quad j \in \mathbb{N}.$$

These are the so called Landau levels.

Further,

$$\sigma(H_0) = \bigcup_{j=1}^{\infty} \overline{E_j(\mathbb{R})}, \tag{0.6}$$

where $\sigma(H_0)$ denotes the spectrum of the operator H_0.

If we put

$$\mathcal{E}_j^- = \inf_{k\in\mathbb{R}} E_j(k), \quad \mathcal{E}_j^+ = \sup_{k\in\mathbb{R}} E_j(k), \tag{0.7}$$

evidently we have $\sigma(H_0) = \bigcup_{j=1}^{\infty} [\mathcal{E}_j^-, \mathcal{E}_j^+]$. The intervals $[\mathcal{E}_j^-, \mathcal{E}_j^+]$, $j \in \mathbb{N}$, are the bands of the spectrum $\sigma(H_0)$.

Let

$$W_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}} W(x), \quad W_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}} W(x). \tag{0.8}$$

Then the mini-max principle implies that

$$b(2j-1)+W_- \le E_j(k) \le b(2j-1)+W_+, \quad k \in \mathbb{R}, \quad j \in \mathbb{N}, \tag{0.9}$$

and $[\mathcal{E}_j^-, \mathcal{E}_j^+] \subset [b(2j-1)+W_-, b(2j-1)+W_+]$.

Throughout the thesis we assume that

$$W_- < W_+, \tag{0.10}$$

i.e. W is not identically constant. Moreover, we will assume that

$$W_+ - W_- < 2b. \tag{0.11}$$

Therefore

$$\mathcal{E}_j^+ < \mathcal{E}_{j+1}^-, \quad j \in \mathbb{N}, \tag{0.12}$$

and the intervals $(\mathcal{E}_j^+, \mathcal{E}_{j+1}^-)$, $j \in \mathbb{N}$, are open gaps in the spectrum of H_0. Thus, all the bands in the spectrum of H_0 are separated by gaps where discrete spectrum may appear under appropriate relatively compact perturbations. Note that due to the boundedness of W, the operator H_0 is lower-bounded which implies that $(-\infty, \mathcal{E}_1^-)$ is always an open gap. Also, we should note that (0.11) is not always a necessary condition for (0.12) (see the remark after Proposition 3.1.4 below).

Now, let us look closer at the perturbative electric potential V. As mentioned, we assume that

$$V \in L_0^\infty(\mathbb{R}^2) := \{u \in L^\infty(\mathbb{R}^2) \,|\, u(x,y) \to 0 \text{ as } x^2+y^2 \to \infty\}, \tag{0.13}$$

which is a simple sufficient condition for the compactness of the operator $V(-\Delta - i)^{-1}$. By the diamagnetic inequality (see [19, Section 1.3]), the operator $V(H_0 - i)^{-1}$ is also compact, and hence Weyl's theorem implies

$$\sigma_{\mathrm{ess}}(H_0 + V) = \sigma_{\mathrm{ess}}(H_0) = \bigcup_{j=1}^{\infty} [\mathcal{E}_j^-, \mathcal{E}_j^+].$$

Consequently, under condition (0.13), each open gap of $\sigma(H_0)$ may contain only discrete spectrum of $H_0 + V$.

For simplicity, we will consider perturbations of a definite sign. More precisely we will suppose that $V \ge 0$, and will consider the operators $H_\pm := H_0 \pm V$. Note that in the case of positive (resp., negative) perturbations, the discrete eigenvalues of the perturbed operator which may appear in a given open gap of the spectrum of the unperturbed operator, can accumulate only to the lower (resp., upper) edge of the gap (see Proposition 1.3.1 below).

Let T be a self-adjoint linear operator in a Hilbert space. Denote by $\mathbb{P}_{\mathcal{O}}(T)$ the spectral projection of T corresponding to the Borel set $\mathcal{O} \subseteq \mathbb{R}$. For $\lambda > 0$ set

$$\mathcal{N}_0^-(\lambda) := \operatorname{rank} \mathbb{P}_{(-\infty, \mathcal{E}_1^- - \lambda)}(H_-).$$

Next, fix $j \in \mathbb{N}$ and assume that (0.11) holds. Pick $\lambda \in (0, \mathcal{E}_{j+1}^- - \mathcal{E}_j^+)$, and set

$$\mathcal{N}_j^-(\lambda) := \operatorname{rank} \mathbb{P}_{(\mathcal{E}_j^+, \mathcal{E}_{j+1}^- - \lambda)}(H_-), \quad \mathcal{N}_j^+(\lambda) := \operatorname{rank} \mathbb{P}_{(\mathcal{E}_j^+ + \lambda, \mathcal{E}_{j+1}^-)}(H_+).$$

These functions count the number of the eigenvalues (with their multiplicities) of the operator H_- (resp., H_+) which lie in the intervals $(-\infty, \mathcal{E}_1^-)$ and $(\mathcal{E}_j^+, \mathcal{E}_{j+1}^- - \lambda)$, $j \in \mathbb{N}$, (resp., $(\mathcal{E}_j^+ + \lambda, \mathcal{E}_{j+1}^-)$, $j \in \mathbb{N}$). Then what we are going to do, is to reduce the investigation of the accumulation of the discrete eigenvalues of $H_\pm$ to the edges of the gaps of its essential spectrum, to the study of the asymptotic behavior as $\lambda \downarrow 0$ of the counting functions $\mathcal{N}_j^\pm(\lambda)$. These counting functions are closely related to the Krein spectral shift function (SSF) described in more detail in Section 1.3, and the study of them could be considered as a first step in the understanding of the asymptotic behavior of the SSF near the spectral thresholds.

We analyze the functions $\mathcal{N}_j^\pm(\lambda)$ under the assumption that W is either monotone or periodic. For definiteness, we consider only $\mathcal{N}_j^+(\lambda)$ since the function $\mathcal{N}_j^-(\lambda)$ can be investigated in a completely analogous manner.

In the monotone and in the periodic case we obtain effective Hamiltonians which govern the main asymptotic term of $\mathcal{N}_j^+(\lambda)$. If W is monotone, the effective Hamiltonian, self-adjoint in $L^2(\mathbb{R})$, has the form

$$E_j + \mathcal{V}_j^{\rm mon}. \tag{0.14}$$

Here $\mathcal{V}_j^{\rm mon}$ is an explicit pseudo-differential operator (see (2.2.3) below).

For the periodic model, under appropriate assumptions about the set where the function E_j attains its maximum, and the asymptotics of E_j in a vicinity of this set, we obtain the effective Hamiltonian

$$I \otimes \left(-\mu_j \frac{d^2}{dy^2}\right) \dot{-} \mathcal{V}_j^{\rm per} \tag{0.15}$$

self-adjoint in $l^2(\mathbb{Z}) \otimes L^2(\mathbb{R})$. Here, I is the identity operator in $l^2(\mathbb{Z})$, μ_j is a positive number, and $\mathcal{V}_j^{\rm per}$ is an infinite matrix-valued potential.

Note that the two types of effective Hamiltonians obtained have specific features that make them different from each other, and from the ones used in the study of other models related to ours.

These effective Hamiltonians can be used to analyze $\mathcal{N}_j^\pm(\lambda)$ for a wide class of perturbative potentials V. However, we concentrate our efforts on V compactly supported. One of the main reasons for this choice is that during the last decade various authors considered the distribution of the discrete spectrum of the Landau Hamiltonian (i.e. the operator H_0 with $W = 0$) with relatively compact perturbations of electric, magnetic, or geometric nature. This setting has provided interesting results like new types of asymptotics for $\mathcal{N}_j^\pm(\lambda)$ which are rather slow and not semi-classical.

The first result we obtain with the help of our effective Hamiltonian, appears in the monotone case, and concerns the conditions which guarantee that the operator H has only a finite number of eigenvalues in each gap of its essential spectrum, that is

$$\mathcal{N}_j^+(\lambda) = O(1), \quad \lambda \downarrow 0, \quad \forall\, j \in \mathbb{N}. \tag{0.16}$$

The interesting thing is that this sufficient condition depends on the geometric properties of W and V. Namely, let $x^+ := \inf\{x \in \mathbb{R} \mid W(x) = W_+\}$; then (0.16) holds true if, roughly speaking, the vertical axis through x^+ is on the right of $\operatorname{supp} V$ (see Theorem 2.2.3 below for a precise formulation). On the other hand, in the periodic model, every open gap in $\sigma(H_0)$ contains infinitely many eigenvalues of H_+, i.e. (0.16) never holds true.

When the geometric sufficient condition for the validity of (0.16) in the monotone case is not fulfilled, then generically every open gap of $\sigma(H_0)$ contains infinitely many eigenvalues. Moreover, we obtain asymptotic bounds

$$C_- \leq \liminf_{\lambda \downarrow 0} |\ln \lambda|^{-1/2} \mathcal{N}_j^+(\lambda) \leq \limsup_{\lambda \downarrow 0} |\ln \lambda|^{-1/2} \mathcal{N}_j^+(\lambda) \leq C_+ \tag{0.17}$$

with constants $0 < C_- < C_+ < \infty$ which depend on b and $\operatorname{supp} V$. Note that (0.17) implies that the asymptotic convergence of the eigenvalues of H_+ to the edge of the spectral gap is Gaussian, which is the fastest known convergence in a similar setting; for comparison, in the case of compactly supported perturbations of the Landau Hamiltonian, we have

$$\mathcal{N}_j^+(\lambda) = \frac{|\ln \lambda|}{\ln |\ln \lambda|}(1 + o(1)), \quad \lambda \downarrow 0, \tag{0.18}$$

(see [71] or [55]).

In the periodic model we again obtain asymptotic bounds of type (0.17). However now, under additional assumptions which are fulfilled, for instance, if b is large enough, we have $C_- = C_+$, so that in this case (0.17) implies the main asymptotic term as $\lambda \downarrow 0$ of $\mathcal{N}_j^+(\lambda)$.

Let us give now a brief overview of the plan of the text.

The purpose of Chapter 1 is twofold. First, here we put together some facts concerning, for instance:

- asymptotic distribution of the discrete spectrum for partial differential operators;
- compact linear operators in Hilbert spaces;
- Kato analytic perturbation theory;
- analytically fibred Hamiltonians;
- the Birman-Schwinger principle and its generalizations.

Most of these results are well known and are available in the literature, but we include them for the sake of reader's convenience since these facts are systematically used in the thesis.

At the same time, we utilize the abstract framework of the analytically fibred Hamiltonians and their relatively compact perturbations, as well as important concrete examples such as periodic Schrödinger operators and 2D magnetic Hamiltonians, in order to look at our results from a more general point of view, and to compare them with some of the existing cornerstone results in the field.

In Chapter 2 we investigate the monotone model. We start with the analysis of the properties of the band functions. We see that they are monotone just as W, and we describe the asymptotic behavior of $E_j(k)$ when k goes to infinity. Further, we construct two effective Hamiltonians for H, the first one being valid in a more general situation, and the reduced second one being useful in the case of compactly supported V. Using the first one, we prove the sufficient condition for $\mathcal{N}_j^{\pm}(\lambda)$ to be bounded, which, at heuristic level, coincides with the necessary one. In the case of compactly supported V, and infinitely many eigenvalues in each open gap, we obtain asymptotic bounds of type (0.17) for $\mathcal{N}_j^+(\lambda)$, $j \in \mathbb{N}$.

Finally, in Chapter 3 we consider the periodic case. As in the monotone model, the properties of the band functions is the first topic approached. It is quite easy to see that if W is periodic of period T, then the band functions E_j are periodic of period bT. We perform our further analysis of E_j in a more general situation than merely periodic W, and see that E_j behaves quite similarly to W when b is large. Then, in the case of periodic W we introduce the effective Hamiltonians, and for compactly supported V we obtain the asymptotic bounds for $\mathcal{N}_j(\lambda)$ of type (0.17).

Chapter 1

Preliminary Facts and Results

1.1 Eigenvalue Asymptotics of Partial Differential Operators

As stated in the Introduction, the thesis is devoted to the investigation of the asymptotic distribution of the discrete eigenvalues of 2D magnetic quantum Hamiltonians, lying in the gaps of their essential spectra.

The study of the eigenvalue asymptotics for partial differential operators has a long history. Almost one hundred years ago H. Weyl published his seminal works [86], [87] where he analyzed the asymptotic distribution of the eigenvalues of the Dirichlet (resp., Neumann) Laplacian Δ_Ω^+ (resp., Δ_Ω^-) on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with a regular boundary $\partial\Omega$. From physics point of view, these eigenvalues coincide with the squares of the eigenfrequencies of Ω under the assumption of a fixed (resp., free) boundary $\partial\Omega$; on a heuristic level the asymptotic distribution of these eigenfrequencies had been studied by the Dutch physicist P. Debye. In [86] H. Weyl proved the celebrated law

$$\mathrm{N}_\Omega^\pm(\lambda) = \frac{\omega_n}{(2\pi)^n}|\Omega|\lambda^{n/2}(1 + o(1)), \quad \lambda \to \infty, \tag{1.1.1}$$

where $\mathrm{N}_\Omega^\pm(\lambda)$ is the number of the eigenvalues of $-\Delta_\Omega^\pm$ not exceeding $\lambda \in \mathbb{R}$ and counted with their multiplicities, ω_n is the volume of the unit ball in $\mathbb{R}^n$, and $|\Omega|$ is the Lebesgue measure of Ω. Moreover, he conjectured that the two-term formula should be of the form

$$\mathrm{N}_\Omega^\pm(\lambda) = \frac{\omega_n}{(2\pi)^n}|\Omega|\lambda^{n/2} \mp \frac{1}{4}\frac{\omega_{n-1}}{(2\pi)^{n-1}}|\partial\Omega|\lambda^{(n-1)/2} + o(\lambda^{(n-1)/2}), \quad \lambda \to \infty. \tag{1.1.2}$$

H. Weyl obtained (1.1.1) using variational methods developed mainly by himself, and later by R. Courant and the St. Petersburg school represented by M. Birman, M. Solomyak, G. Rozenblum, and their students. It turned out that the variational methods are very powerful when one aims at main-term formulas of type (1.1.1) but in order to obtain sharp remainder estimates and two-term formulas of type (1.1.2), one needs appropriate Tauberian methods which do not involve the direct asymptotic analysis of the counting functions $\mathrm{N}_\Omega^\pm(\lambda)$ but the investigation of their suitable integral transforms. Applying Fourier Tauberian methods first suggested by B. Levitan, V. Ivrii proved in [36] that (1.1.2) holds true provided that the periodic billiards on Ω have measure zero.

Note that (1.1.1) and (1.1.2) are among the classical results in the field of spectral geometry relating the geometric properties of Ω and the spectral data for $-\Delta_\Omega^\pm$ and other more general elliptic partial differential operators on bounded domains Ω. One of the central problems of the spectral geometry, related to the reconstruction of a domain Ω by its spectrum is revealed in a lively manner by the title "*Can we hear the shape of a drum?*", of the seminal article of Mark Kac (see [40]).

Another large field of applications of the eigenvalue asymptotics for partial differential operators, is the quantum physics. According to M. Reed and B. Simon (see [72, Section VIII.11]), the spectral analysis of quantum Hamiltonians is one of the three main mathematical problems of quantum mechanics, and this problem includes the important task to "*estimate the position and the multiplicity of the point spectrum*". However, in contrast to the elliptic operators on bounded domains which have purely discrete spectrum, the quantum-mechanics Hamiltonians typically have non-empty essential spectrum. The best known example is the Schrödinger operator $-\hbar^2\Delta + gV$, self-adjoint in $\mathbb{R}^n$, $n \geq 1$. Here $\hbar > 0$ is the Planck constant, $V : \mathbb{R}^n \to \mathbb{R}$ is the electric potential, and $g \in \mathbb{R}$ is its coupling constant. If V decays in a suitable sense at infinity then the essential spectrum of $-\hbar^2\Delta + gV$ coincides with $[0, \infty)$. Let $\mathrm{N}(-\lambda; \hbar, g)$ be the number of the eigenvalues of $-\hbar^2\Delta + gV$ smaller than $-\lambda \leq 0$ and counted with their multiplicities. Then the asymptotics as $\lambda \downarrow 0$ of $\mathrm{N}(-\lambda; \hbar, g)$ with $\hbar$ and g fixed describes the distribution of the discrete spectrum of $-\hbar^2\Delta + gV$ below the origin which coincides with the bottom of the essential spectrum of $-\hbar^2\Delta + gV$. The asymptotics as $\hbar \downarrow 0$ and $g \to \infty$ have also important applications in quantum mechanics. The investigation of these asymptotic regimes led to a significant development of the variational methods, including the so-called Birman-Schwinger principle described in detail in the next section, as well as to cornerstone results of the spectral theory of partial differential operators such as the celebrated Cwikel-Lieb-Rozenblum estimate (see [74, 75, 53, 18]) according to which we have

$$\mathrm{N}(0; 1, 1) \leq C_n \int_{\mathbb{R}^n} (-V(x))^{n/2} dx,$$

where the constant C_n depends only on the dimension n provided that $n \geq 3$, and $V \in L^{n/2}(\mathbb{R}^n)$, $V \leq 0$.

The essential spectra of various quantum Hamiltonians with important applications in physics, such as the Schrödinger operators with periodic electromagnetic potentials, or the Landau Hamiltonian, could contain more than one connected component, and hence their boundary could contain more than one point. One of the central problems in the spectral analysis of relatively compact perturbations of such operators, is the study of the asymptotic distribution of the discrete spectrum near the edges of all the open gaps of the essential one.

A significant class of such operators which contains, in particular, the two main models considered in the thesis, are the analytically fibred Hamiltonians. Their properties are discussed in more detail in the Sections 1.3 and 1.4.

1.2 Classes of Compact Operators and Their Properties

The analysis of a sequence of discrete eigenvalues of a self-adjoint operator, converging to a given edge of its essential spectrum, usually reduces via the Birman-Schwinger principle to the study of the spectrum of a compact operator (see Proposition 1.3.1 below). For this reason we introduce in this section certain classes of compact operators and their eigenvalue counting functions, and discuss briefly their properties; the notations and the facts of this section will be used throughout the thesis, and, especially, in the proofs of our main results.

Let X_l, $l = 1,2$, be two separable Hilbert spaces. By $\mathcal{L}(X_1, X_2)$ we denote the class of bounded linear operators $T : X_1 \to X_2$. By $S_\infty(X_1, X_2)$ denote the compact operators, and by $S_p(X_1, X_2)$, $p \in [1, \infty)$, the Schatten-von Neumann class of operators $T \in S_\infty(X_1, X_2)$ for which $\|T\|_p := \left(\mathrm{Tr}\,(T^*T)^{p/2}\right)^{1/p} < \infty$.

If $X_1 = X_2 = X$, we will write $\mathcal{L}(X)$, $S_\infty(X)$, and $S_p(X)$ instead of $\mathcal{L}(X,X)$, $S_\infty(X,X)$, and $S_p(X,X)$, $p \in [1,\infty]$, respectively. Let $T = T^* \in S_\infty(X)$. For $s > 0$ set

$$n_\pm(s;T) = \mathrm{rank}\,\mathbb{P}_{(s,\infty)}(\pm T); \tag{1.2.1}$$

thus $n_\pm(\cdot;T)$ are the counting functions respectively of the positive and the negative eigenvalues of T. Let $T \in S_\infty(X_1, X_2)$. Put

$$n_*(s;T) = n_+(s^2; T^*T), \quad s > 0;$$

thus $n_*(\cdot T)$ is the counting function of the singular numbers of the operator T. We have

$$n_*(s;T) = n_*(s;T^*), \quad s > 0.$$

Moreover, if $X_1 = X_2 = X$, and $T = T^*$, we have

$$n_\pm(s;T) \leq n_*(s;T), \quad s > 0.$$

Note that the functions $n_\pm$ satisfy *the Weyl inequalities*

$$n_+(s(1+\varepsilon);T_1) - n_-(s\varepsilon;T_2) \leq n_+(s;T_1+T_2) \leq n_+(s(1-\varepsilon);T_1) + n_+(s\varepsilon;T_2), \tag{1.2.2}$$

with $s > 0$ and $\varepsilon \in (0,1)$ (see [12, Theorem 9.2.9]), while the function n_* satisfies *the Ky Fan inequalities*

$$n_*(s(1+\varepsilon);T_1) - n_*(s\varepsilon;T_2) \leq n_*(s;T_1+T_2) \leq n_*(s(1-\varepsilon);T_1) + n_*(s\varepsilon;T_2), \tag{1.2.3}$$

with $s > 0$ and $\varepsilon \in (0,1)$ (see [12, Subsection 11.1.3]). Finally, for each $s > 0$ and $p \in [1,\infty)$ we have the elementary Chebyshev type inequality

$$n_*(s;T) \leq s^{-p}\|T\|_p^p. \tag{1.2.4}$$

1.3 Analytically Fibred Hamiltonians: General Setting

By an analytically fibred operator $\mathbf{H_0}$ defined in the Hilbert space $\mathcal{H}$, we mean, following [28] or [25], an operator which satisfies

$$\mathbf{U}\mathbf{H_0}\mathbf{U}^* = \int_\Omega^\oplus \mathbf{h_0}(k)\,d\mu(k), \tag{1.3.1}$$

where:

- Ω is a real analytic manifold with a measure μ given by a positive C^∞ density;
- $\mathbf{U} : \mathcal{H} \to \int_\Omega^\oplus \mathcal{H}\, d\mu(k)$ is unitary, with $\mathcal{H}$ a separable Hilbert space;
- The operator $\mathbf{h}_0(k)$ is self-adjoint in $\mathcal{H}$ and has purely discrete spectrum for all $k \in \Omega$;
- The resolvent $(\mathbf{h}_0(k) - i)^{-1}$ is a real analytic function of $k \in \Omega$.

In what follows we will suppose also that the operator $\mathbf{H}_0$ is lower-bounded in $\mathcal{H}$.

When we have the direct integral decomposition with these properties we can analyze the spectrum of the operator $\mathbf{H}_0$ in the following way. Denote by $\{\mathbf{E}_j(k)\}_{j=1}^\infty$ the non-decreasing sequence of the eigenvalues of $\mathbf{h}_0(k)$. Then the functions $\mathbf{E}_j : \Omega \to \mathbb{C}$, $j \in \mathbb{N}$, are piecewise analytic. Define

$$\mathcal{E}_j^+ := \inf_{k\in\Omega} \mathbf{E}_j(k); \quad \mathcal{E}_j^+ := \sup_{k\in\Omega} \mathbf{E}_j(k).$$

Then

$$\sigma(\mathbf{H}) = \bigcup_{j=1}^\infty \overline{(\mathcal{E}_j^-, \mathcal{E}_j^+)}.$$

When $\mathcal{E}_j^+ < \mathcal{E}_{j+1}^-$, there is a bounded open gap $(\mathcal{E}_j^+, \mathcal{E}_{j+1}^-)$ in the spectrum of $\mathbf{H}_0$. Since $\mathbf{H}_0$ is lower-bounded, we have $\mathcal{E}_1^- > -\infty$, and hence $(-\infty, \mathcal{E}_1^-)$ is an unbounded gap in the spectrum of $\mathbf{H}_0$.

Let us consider now the discrete spectrum of $\mathbf{H}_0 \pm g\mathbf{V}$, $g \in \mathbb{R}_+$, when $V \in \mathcal{L}(\mathcal{H})$ is a non-negative relatively compact perturbation of $\mathbf{H}_0$, that is, $\mathbf{V} \geq 0$ and $\mathbf{V}(\mathbf{H}_0 - i)^{-1}$ is compact. This implies that $\mathbf{H}_0 \pm g\mathbf{V}$ can have only discrete spectrum in each gap of the spectrum of $\mathbf{H}_0$. Define the function

$$N_\pm(\lambda, g) := \sum_{0<g'<g} \dim \operatorname{Ker}(\mathbf{H}_0 \mp g'\mathbf{V} - \lambda), \quad \lambda \in \rho(\mathbf{H}_0) \cap \mathbb{R}. \tag{1.3.2}$$

The value $N_\pm(\lambda, g)$ is just the number of eigenvalues of $\mathbf{H}_0 \mp g'\mathbf{V}$ counted with their multiplicities, which cross the level λ as g' goes from 0 to g. An important property of these functions which has been extensively studied in the literature, is their asymptotic behavior as g grows to infinity.

A valuable tool in this analysis is the Birman-Schwinger principle (see e.g. [6, 32]) which implies

$$N_\pm(\lambda, g) = n_\pm(g^{-1}; \mathbf{V}^{1/2}(\mathbf{H}_0 - \lambda)^{-1}\mathbf{V}^{1/2}), \quad \lambda \in \rho(\mathbf{H}_0) \cap \mathbb{R}, \tag{1.3.3}$$

the functions $n_\pm$ being defined in (1.2.1).

Another important related asymptotics is described as follows. For $\lambda > 0$ set

$$\mathcal{N}_0^-(\lambda) := \operatorname{rank} \mathbb{P}_{(-\infty, \mathcal{E}_1^- - \lambda)}(\mathbf{H}_0 - \mathbf{V}),$$

Then $\mathcal{N}_0^-(\lambda)$ is the number of the eigenvalues of $\mathbf{H}_0 - \mathbf{V}$ counted with their multiplicities, which lie below $\mathcal{E}_1^- - \lambda$. Suppose that $(\mathcal{E}_j^+, \mathcal{E}_{j+1}^-)$, $j \in \mathbb{N}$, is a bounded gap in the spectrum of $\mathbf{H}_0$, and for $\lambda \in (0, \mathcal{E}_{j+1}^- - \mathcal{E}_j^+)$ set

$$\mathcal{N}_j^-(\lambda) := \operatorname{rank} \mathbb{P}_{(\mathcal{E}_j^+, \mathcal{E}_{j+1}^- - \lambda)}(\mathbf{H}_0 - \mathbf{V}), \quad \mathcal{N}_j^+(\lambda) := \operatorname{rank} \mathbb{P}_{(\mathcal{E}_j^+ + \lambda, \mathcal{E}_{j+1}^-)}(\mathbf{H}_0 + \mathbf{V}).$$

Obviously, these functions count the number of eigenvalues with their multiplicities, of the operators $\mathbf{H}_0 - \mathbf{V}$ or $\mathbf{H}_0 + \mathbf{V}$ inside the intervals $(\mathcal{E}_j^+, \mathcal{E}_{j+1}^- - \lambda)$ or $(\mathcal{E}_j^+ + \lambda, \mathcal{E}_{j+1}^-)$ respectively. The problem now is to calculate the asymptotic behavior of $\mathcal{N}_j^{\pm}(\lambda)$, when λ goes to zero.

Proposition 1.3.1. *For $\lambda > 0$ we have*

$$\mathcal{N}_0^-(\lambda) = n_+(1; \mathbf{V}^{1/2}(\mathbf{H}_0 - \mathcal{E}_1^- + \lambda)^{-1}\mathbf{V}^{1/2}) \tag{1.3.4}$$

Moreover, for each $j \in \mathbb{N}$ such that $(\mathcal{E}_j^+ < \mathcal{E}_{j+1}^-)$ we have

$$\mathcal{N}_j^{\pm}(\lambda) = n_{\mp}(1; \mathbf{V}^{1/2}(\mathbf{H}_0 - \mathcal{E}_j^{\pm} \mp \lambda)^{-1}\mathbf{V}^{1/2}) + O(1), \quad \lambda \downarrow 0. \tag{1.3.5}$$

Proof. Relation (1.3.4) follows from the evident identity $\mathcal{N}_0^-(\lambda) = N_+(\mathcal{E}_1^- - \lambda, 1)$ and (1.3.3). Let us prove (1.3.5) for $\mathcal{N}_j^+(\lambda)$; the argument for $\mathcal{N}_j^-(\lambda)$ is completely analogous. Fix $\lambda_0 \in (0, \mathcal{E}_{j+1}^- - \mathcal{E}_j^+)$ and pick $\delta > 0$ sufficiently small. Then we have

$$0 \le \operatorname{rank} \mathbb{P}_{(\mathcal{E}_j^+ + \lambda_0, \mathcal{E}_{j+1}^- - \delta)}(\mathbf{H}_0 + \mathbf{V}) =$$

$$N_-(\mathcal{E}_j^+ + \lambda_0, 1) - N_-(\mathcal{E}_{j+1}^- - \delta, 1) - \dim \operatorname{Ker}(\mathbf{H}_0 + \mathbf{V} - \mathcal{E}_{j+1}^- + \delta) \le$$
$$N_-(\mathcal{E}_j^+ + \lambda_0, 1).$$

Therefore, $\operatorname{rank} \mathbb{P}_{(\mathcal{E}_j^+ + \delta, \mathcal{E}_{j+1}^- - \lambda)}(\mathbf{H}_0 + \mathbf{V})$ which is a non-increasing function of δ, is uniformly bounded for $\delta > 0$ small enough., and every fixed $\lambda > 0$. Hence, the limit

$$\mathcal{N}_j^+(\lambda) = \lim_{\delta \downarrow 0} \operatorname{rank} \mathbb{P}_{(\mathcal{E}_j^+ + \lambda, \mathcal{E}_{j+1}^- - \delta)}(\mathbf{H}_0 + \mathbf{V})$$

is well-defined and finite. Now pick $\lambda \in (0, \lambda_0)$. We have

$$\mathcal{N}_j^+(\lambda) = N_-(\mathcal{E}_j^+ + \lambda, 1) - N_-(\mathcal{E}_j^+ + \lambda_0, 1) + \mathcal{N}_j^+(\lambda_0). \tag{1.3.6}$$

Now (1.3.5) for $\mathcal{N}_j^+(\lambda)$ follows immediately from (1.3.6) and (1.3.3). □

It follows that the asymptotics as $g \to \infty$ of $N_\pm(\lambda, g)$ and the asymptotics as $\lambda \downarrow$ of $\mathcal{N}_j^{\pm}(\lambda)$ can be both studied using the Birman-Schwinger principle. However, these two asymptotic regimes have some essential differences. For instance, typically the asymptotics as $g \to \infty$ of $N_+(\lambda, g)$ and $N_-(\lambda, g)$ are of different nature, while the asymptotics as $\lambda \downarrow$ of $\mathcal{N}_j^+(\lambda)$ and $\mathcal{N}_j^-(\lambda)$ are practically identical. Also, the asymptotics as $\lambda \downarrow 0$ of $\mathcal{N}_j^-(\lambda)$ or $\mathcal{N}_j^+(\lambda)$ have a local character and depend only on the band(s) adjoining the edge $\mathcal{E}_{j+1}^-$ or $\mathcal{E}_j^+$, while the asymptotics as $g \to \infty$ of $N_\pm(\lambda, g)$ are non-local and typically depend on several (in the case of $N_+(\lambda, g)$, infinitely many) bands of $\sigma(\mathbf{H}_0)$.

At this moment, it may be useful to recall the relation between $\mathcal{N}_j^{\pm}(\lambda)$ and the Krein spectral shift function (SSF). Fix the sign of the perturbation [1] of $\mathbf{H}_0$, and assume $(\mathbf{H}_0 \pm \mathbf{V} -$

[1] Of course, the SSF could be defined also without the assumption that the perturbation $\mathbf{V}$ has a definite sign. In this case, however, the SSF ξ is not obliged to have a constant sign.

$i)^{-1} - (\mathbf{H_0} - i)^{-1} \in S_1(\mathcal{H})$. Then there exists a unique $\xi_\pm \in L^1_{\rm loc}$, $\pm\xi \geq 0$, which satisfies the Lifshits-Krein formula

$$\mathrm{Tr}\,(f(H_0 \pm V) - f(H_0)) = \int_{\mathbb{R}} \xi_\pm(E) f'(E)dE, \quad \forall f \in C_0^\infty(\mathbb{R}),$$

and the normalization condition $\xi_\pm(\lambda) = 0$ for each $\lambda < \inf \sigma(H_0) \cup \sigma(H_0 \pm V)$ (see the original works [54, 51] or the monograph [88]). The function $\xi_\pm$ is called the SSF for the operator pair $(\mathbf{H_0} \pm \mathbf{V}, \mathbf{H_0})$. For almost every λ on the absolutely continuous spectrum of the operator $\mathbf{H_0}$, the SSF $\xi_\pm$ is related to the scattering matrix $S_\pm(\lambda)$ for $(\mathbf{H_0} \pm \mathbf{V}, \mathbf{H_0})$ by the celebrated Birman-Krein formula

$$\det S_\pm(\lambda) = e^{-2\pi i \xi_\pm(\lambda)}$$

(see the original work [8] or [88]), so that, up to a constant factor the SSF coincides with the scattering phase. At the same time, for almost every $\lambda \in \rho(\mathbf{H_0}) \cap \mathbb{R}$ we have

$$\xi_\pm(\lambda) = \pm n_\mp(1, \mathbf{V}^{1/2}(\mathbf{H_0} - \lambda)^{-1}\mathbf{V}^{1/2})$$

(see [80] or [60]). Thus, the asymptotics as $\lambda \downarrow 0$ of $\mathcal{N}_j^\mp(\lambda)$ is closely related to the more general and challenging problem of studying the asymptotic behavior of the SSF $\xi_\pm$ near the spectral thresholds in $\sigma(\mathbf{H_0})$.

The study of the asymptotics as $\lambda \downarrow 0$ of $\mathcal{N}_j^\mp(\lambda)$ usually requires a preliminary analysis of the following basic properties of the band functions:

- The description of the set of open gaps in $\sigma(\mathbf{H_0})$;
- The structure of the sets $\{k \in \Omega \mid \mathbf{E}_l(k) = \mathcal{E}_j^\pm\}$, $l \in \mathbb{N}$;
- The asymptotic behavior of $\mathbf{E}_l(k)$ in a vicinity of the set $\{k \in \Omega \mid \mathbf{E}_l(k) = \mathcal{E}_j^\pm\}$ for those $l \in \mathbb{N}$ for which this set is non-empty..

After we dispose of this information concerning $\mathbf{E}_j$, we are in position to attack the problem of finding the asymptotics as $\lambda \downarrow 0$ of the counting functions $\mathcal{N}_j^\mp(\lambda)$. Usually the asymptotic behavior of these functions is described in the terms of effective Hamiltonians which are simpler self-adjoint operators whose properties are known, or are easier to be studied. Note that the investigation of the band functions, and the construction of effective Hamiltonians, could be of independent interest since such analysis could lead to useful results applicable in related areas of research.

1.4 Analytically Fibred Hamiltonians: Examples

The general theory discussed in the previous section is now applied to two groups of Hamiltonians. The examples selected below are closely related to the two main models considered in the thesis. That is why we believe that they shed more light on these main models, and allow the reader to look at our results from the point of view of a more general and visible research area. At the same time, of course, we do not try by any means to make an exhaustive survey of all existing results in this relatively large and well established research area.

The general form of the Hamiltonians under consideration is

$$H(\mathbf{A}, W, V) = (-i\nabla - \mathbf{A})^2 + W + V, \tag{1.4.1}$$

self-adjoint in $L^2(\mathbb{R}^n)$, $n \geq 1$. Here $\mathbf{A} = (A_1, \ldots, A_n) : \mathbb{R}^n \to \mathbb{R}^n$ is the magnetic potential which generates a magnetic field $\mathbf{B}$. If we identify $\mathbf{A}$ with the 1-differential form $\sum_{j=1}^n A_j dx_j$, this means that $\mathbf{B} = d\mathbf{A}$. If $n = 2$, we will identify $\mathbf{B}$ with the matrix-valued function

$$\mathbf{B}(x,y) := \begin{pmatrix} 0 & b(x,y) \\ -b(x,y) & 0 \end{pmatrix}, \quad (x,y) \in \mathbb{R}^2, \tag{1.4.2}$$

where $b : \mathbb{R}^2 \to \mathbb{R}$ is a scalar function defined by $b = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$. Since $\mathbb{R}^n$ is simply-connected, the operator $H(\mathbf{A}, W, V)$ is gauge covariant (see e.g. [19, Section 6.1]), and its spectral properties depend only on the magnetic field $\mathbf{B}$, and not on the magnetic potential $\mathbf{A}$.

Moreover, W and V are electric potentials of different nature: while W is supposed to posses some ergodic properties, V is assumed to decay at infinity.

1.4.1 Periodic Systems

The first analytically fibred Hamiltonians considered were the periodic Schrödinger operators. Although direct integrals were introduced formally by J. von Neumann only in [85], the fundamental idea of how to make the decomposition in the periodic case goes back to the works of F. Bloch in the theory of solids, and even to the earlier ones of G. Floquet in relation with periodic differential equations in the real line (see the notes in [73, Chapter XIII] for references).

Let $\mathbf{A}$ and W be periodic functions in $\mathbb{R}^n$, with respect to the non-degenerate lattice Γ. Let Γ^* be the dual lattice of Γ and Q, Q^* be the respective fundamental domains of the tori $\mathbb{T} = \mathbb{R}^n/\Gamma$, $\mathbb{T}^* = \mathbb{R}^n/\Gamma^*$. Then $\mathbf{H}_0 = H(\mathbf{A}, W, 0)$ is an analytically fibred operator, and the direct integral decomposition is implemented by the so called Floquet-Bloch-Gelfand transform. This is a unitary operator defined by $\mathbf{U} : L^2(\mathbb{R}^n) \to \int_{\mathbb{T}^*}^{\oplus} L^2(\mathbb{T}) \frac{dk}{|Q^*|}$

$$(\mathbf{U}f)(x,k) = \sum_{\gamma \in \Gamma} \exp(-ik \cdot (x + \gamma)) f(x + \gamma), \quad (x,k) \in \mathbb{T} \times \mathbb{T}^*,$$

which perform identity (1.3.1) with the operator $\mathbf{h}_0(k) = (i\nabla + \mathbf{A} - \mathrm{k})^2 + W$, $k \in \mathbb{T}^*$, self-adjoint in $L^2(\mathbb{T})$.

Generically, the spectrum of $H(\mathbf{A}, W, 0)$ with periodic $(\mathbf{A}, W)$ is purely absolutely continuous, and the proof of this fact relies on the analysis of the band functions $\mathbf{E}_j(k)$. In the case $\mathbf{A} = 0$ and $n = 3$, this analysis was first done by L. Thomas in [83]. His approach could be immediately extended to the general n-dimensional (see e.g [73, 4]). The case of periodic $\mathbf{A} \neq 0$, is also typically purely absolutely continuous; we can cite here the work of M. Birman and T. Suslina [13], for two dimensions, and the article [81] of A. Sobolev for higher dimensions.

Let us pass to the discussion of the distribution of the discrete spectrum for relatively compact perturbations of periodic Schrödinger operators.

The asymptotics as $g \to \infty$ of the functions $N_{\pm}(\lambda, g)$ defined in (1.3.2) for $\mathbf{A} = 0$ and W periodic was investigated in [21] in the one-dimensional case, and in [1, 32] in the multidimensional case. Various far going generalizations of these results could be found in [9, 82]. Note also that the case $(\mathbf{A}, W)$ periodic is included as a special case in the general scheme of [7, 10] where however predominantly only negative rapidly decaying perturbations V were considered. In the one-dimensional case the asymptotics as $\lambda \downarrow 0$ of $\mathcal{N}_j^{\pm}(\lambda)$ for the operator $-d^2/dx^2 + W + V$, W periodic, V decaying i.e. the perturbed Hill operator was considered in [89, 45]. The multidimensional Schrödinger operator with perturbed periodic electric potential was studied in [67] (see also the short notes [66, 46]). In [67] it was supposed that if $\mathcal{E}_j^+ < \mathcal{E}_{j+1}^-$ then the value $\mathcal{E}_j^+$ (resp., $\mathcal{E}_{j+1}^-$) is taken only by the function $\mathbf{E}_j$ (resp., $\mathbf{E}_{j+1}$); in [50] it was proved that this hypothesis generically holds true. Further, in [67] it was assumed that there are finitely many maximal (resp., minimal) points of $\mathbf{E}_j$ (resp., $\mathbf{E}_{j+1}$), in $\mathbb{T}^*$, and all the extrema are non-degenerate. In the physics literature this condition related to the existence of the so-called effective masses, is widely believed to be fulfilled in the generic case; nevertheless, its validity is proved rigorously only in the case of the infimum of the spectrum of $-\Delta + W$ (see [47]). Denote by $k_{j,l}^+$, $l = 1, \ldots, K_j^+ < \infty$, (resp., $k_{j,l}^-$, $l = 1, \ldots, K_j^- < \infty$), the points where the band function $\mathbf{E}_j$ (resp., $\mathbf{E}_{j+1}$) reaches its maximum (resp., minimum).

The main idea of the analysis in [67] consists in restricting our attention to small vicinities of the extremal points $k_{l,j}^{\pm}$. As a result, a Schrödinger-type effective Hamiltonians of the form

$$\bigoplus_{l=1}^{K_j^{\pm}} \left(-\frac{1}{2}\langle M_{j,l}^{\pm}\nabla; \nabla\rangle + V \right), \tag{1.4.3}$$

self-adjoint in $\bigoplus_{l=1}^{K_j^{\pm}} L^2(\mathbb{R}^n)$, were introduced in [67]. Here $-M_{j,l}^+$ (resp., $M_{j,l}^-$) is the Hesse matrix of the band function $\mathbf{E}_j$ (resp., $\mathbf{E}_{j+1}$), evaluated at the maximal point $k_{j,l}^+$ (resp., at the minimal point $k_{j,l}^-$); note that in [67] for simplicity only the effective Hamiltonian corresponding to $K_j^{\pm} = 1$ was described explicitly.

Under certain hypotheses on the decay of V, which should not be too fast, the application of these effective Hamiltonians yields the following semiclassical asymptotics

$$\mathcal{N}_j^{\pm}(\lambda) = \frac{1}{(2\pi)^n} \sum_{l=1}^{K_j^{\pm}} \left| \{(y,k) \in \mathbb{R}^{2n} \,|\, \frac{1}{2}\langle M_j^{\pm}k, k\rangle \pm V(y) < -\lambda\} \right| (1 + o(1)), \quad \lambda \downarrow 0.$$

Since the spectral properties of $\mathbf{H}_0$ depend on $\mathbf{B}$, it is natural to consider also the case where $\mathbf{B}$ and W are periodic with respect to the same lattice Γ. Note the evident fact that the periodicity of $\mathbf{A}$ implies the periodicity of $\mathbf{B}$ but not vice versa. For simplicity, assume $n = 2$ and that $\Gamma = \{(m, l)|m, l \in \mathbb{Z}\}$. Then $Q = (0,1)^2$ and $Q^* = (0, 2\pi)^2$.

Here we distinguish between three cases. Let

$$\Phi = \int_Q b(x, y)dxdy$$

be the magnetic flux through the unit cell Q. First, if $\Phi = 0$ we can find a function φ, periodic in Γ such that $\Delta\varphi = b$ (see (1.4.2)) and then we are again in the situation of a periodic

$\mathbf{A} := \left(-\frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial x}\right)$, described in the previous paragraph. Suppose now that $\Phi \neq 0$,. Then we can choose by magnetic potential, $\mathbf{A}(x,y) = \tilde{\mathbf{A}}(x,y) + (0, \Phi x)$, with $\tilde{\mathbf{A}}$ periodic and $(0, \Phi x)$ linear. If $\frac{\Phi}{2\pi} \in \mathbb{Q}$, there still exists a commutative magnetic Floquet-Bloch theory. To see this, assume without loss of generality that $\frac{\Phi}{2\pi} \in \mathbb{Z}$, define the unitary operators $t_j : L(\mathbb{R}^2) \to L(\mathbb{R}^2)$, $j = 1, 2$, by

$$(t_1 f)(x,y) := e^{-i\Phi y} f(x+1, y); \quad (t_2 f)(x,y) := f(x, y+1),$$

and the unitary operator $\mathbf{U} : L^2(\mathbb{R}^2) \to \int_{\mathbb{T}^*}^{\oplus} L^2(\mathbb{T}) \frac{dk}{(2\pi)^2}$ by

$$(\mathbf{U}f)(x,y;k_1,k_2) = \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} e^{-i[k_1(x+p)+k_2(y+q)]} (t_1^p t_2^q f)(x,y).$$

Thus, in the situation of (b, W) periodic with $\frac{\Phi}{2\pi} \in \mathbb{Z}$, the analytically fibred decomposition (1.3.1) for $\mathbf{H_0} = H(\mathbf{A}, V, W)$ is related to this unitary operator $\mathbf{U}$, while $\mathbf{h_0}(k)$ is defined as the unique self-adjoint operator in $L^2(Q)$ generated by the closed quadratic form

$$\int_Q |(i\nabla - \mathbf{A} - k)u|^2 \, dxdy$$

with domain

$$\left\{u \in W_1^2(Q) : u(0,y) = e^{i\Phi y} u(1,y), y \in (0,1), u(x,0) = u(x,1), x \in (0,1)\right\}$$

(see [31, 57]). Finally in the case $\Phi \notin 2\pi\mathbb{Q}$ there exists no commutative Floquet-Bloch theory.

If Φ is rational, it is believed by many authors that the spectrum of $H(\mathbf{A}, W, 0)$ with (b, W) periodic, is typically absolutely continuous. Note, however, that the Landau Hamiltonian which corresponds to the case $n = 2$, b constant and non-vanishing, and $W = 0$ provides an evident counterexample. Another example is provided by one of the spin-down component of the 2D Pauli operator with periodic magnetic field of positive mean value (see [22, 5, 70]).

On the other hand, in the case $n = 2$, $\mathbf{B}$ constant and $\Phi \in 2\pi\mathbb{Q}$, F. Klopp showed recently in [49] that set of $W \in L^\infty(\mathbb{R}^2)$ which are periodic with respect to the same Γ, for which $\sigma(H(\mathbf{A}, W, 0))$ is absolutely continuous, is a G_δ-dense set. Even earlier. C. Beeken treated in his Ph.D. thesis [2] the same problem and obtained an explicit sufficient condition on the Fourier coefficients of W which guarantees that $\sigma(H(\mathbf{A}, W, 0))$ is absolutely continuous. As explained in the Introduction, in Chapter 3 of the thesis we treat the case $n = 2$, $\mathbf{B} \neq 0$ constant, and periodic W which depends only on one variable. As a by-product in Proposition 3.1.4 below we show that for general (not necessarily periodic) non-constant potentials $W \in C^2(\mathbb{R})$ such that $W, W', W'' \in L^\infty(\mathbb{R})$, and for any $\alpha \in \mathbb{R}$, the spectrum of $H(\mathbf{A}, W, 0)$ on $(-\infty, \alpha)$ is absolutely continuous, provided that the intensity b of the magnetic field is larger than some $b_0 = b_0(\alpha, W) > 0$.

Finally, let us comment briefly the scarce existing results on the discrete spectrum of $H(\mathbf{A}, W, V)$ for periodic $\mathbf{B}$ and W but non periodic $\mathbf{A}$. In the case of rapidly decaying negative perturbations V, the general results of [65, 10] on the asymptotics as $g \to \infty$ of $N_-(\lambda, g)$ cover such $\mathbf{A}$ and W as a very special case. In the case $W = 0$, $\mathbf{B} \neq 0$ constant, and slowly decaying, as well as positive rapidly decaying perturbations, the asymptotics as $g \to \infty$ of $N_\pm(\lambda, g)$ was considered in [65, 68]. To author's best knowledge the only special case of $\mathbf{B}$ and W periodic but $\mathbf{A}$ non periodic for which the asymptotics as $\lambda \downarrow 0$ of the functions $\mathcal{N}_j^\mp(\lambda)$ was studied, is the model examined in Chapter 3 of the thesis (see also [56]).

1.4.2 2D Magnetic Hamiltonians

Let us describe now some models which are close to the ones treated in this thesis, namely, magnetic Schrödinger operators in $L^2(\mathbb{R}^2)$. Let us start with the description of the analytic fibred decomposition which we need for all the models considered in the section. We suppose that b in (1.4.2) depends only on the x-variable, and $b(x) > 0$, $x \in \mathbb{R}$. Similarly, we assume that the electric potential W is bounded and depends only on the x-variable. Choose $\mathbf{A}(x) = (0, \beta(x))$, with $\beta(x) = \int_0^x b(t)dt$. Introduce the operator

$$\mathbf{H}_0 = (-i\nabla - \mathbf{A})^2 + W, \tag{1.4.4}$$

self-adjoint in $L^2(\mathbb{R} \times \mathcal{I})$. Here $\mathcal{I} \subset \mathbb{R}$ is a non-empty open interval; if $\mathcal{I} \neq \mathbb{R}$ then we impose appropriate boundary conditions. Then decomposition (1.3.1) is well defined with $\mathbf{U} = \mathcal{F}$, the partial Fourier transform with respect to y (see (0.3)), $\Omega = \mathbb{R}$, and the fibre

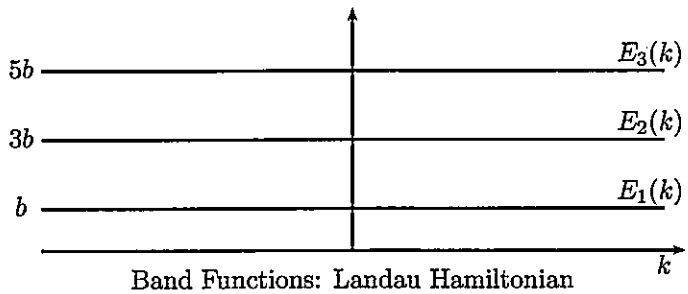
$$\mathrm{h}_0(k) = -\frac{d^2}{dx^2} + (\beta(x) - k)^2 + W(x),$$

equipped with appropriate boundary conditions if $\mathcal{I} \neq \mathbb{R}$, which is self-adjoint in $L^2(\mathcal{I})$.

The fact that this decomposition is analytic is a well known result which follows easily from the Kato perturbation theory (see [43] or [73]).

Let us consider several concrete models which have important applications in quantum physics, and represent classes of examples with different properties of the band functions.

(i) Landau Hamiltonian. As a first example let us discuss the celebrated Landau Hamiltonian i.e. $b > 0$ constant, $W = 0$, and $\mathcal{I} = \mathbb{R}$. The spectral properties of this operator were first investigated in the pioneering works of V. Fock [26] and L. Landau [52]. The decomposition into a direct integral yields band functions which are identically equal to the Landau levels $b(2j - 1)$, $j \in \mathbb{N}$.



Band Functions: Landau Hamiltonian

For the asymptotics as $g \to \infty$ of the counting functions $N^{\pm}(\lambda, g)$ in this case, we have already cited [10, 65, 68]. The asymptotics as $\lambda \downarrow 0$ of $\mathcal{N}_j^{\pm}(\lambda)$ for perturbation of power-like decay was first considered in [64] (see also the short notes [63, 37] and the monograph [38]). The effective Hamiltonian used to analyze $\mathcal{N}_j^{\pm}(\lambda)$ in [64] is a ΨDO in $L^2(\mathbb{R})$ with Weyl symbol

$$\frac{1}{2\pi}\int_{\mathbb{R}^3} \varphi_j(x)\varphi_j(x')e^{i\xi(x-x')}V(b^{-1/2}[(x+x')/2 - k], b^{-1/2}(y-\xi))dx dx' d\xi, \quad (y,k) \in T^*\mathbb{R},$$

where φ_j is the normalized jth Hermite function. Then the asymptotic behavior of $\mathcal{N}_j^{\pm}(\lambda)$ is given by

$$\mathcal{N}_j^{\pm}(\lambda) = \frac{b}{2\pi} |\{(x,y) \in \mathbb{R}^2 \mid \pm V(x,y) > \lambda\}|(1+o(1)) \asymp \lambda^{-2/\alpha}, \quad \lambda \downarrow 0,$$

where $\alpha > 0$ is the decay rate of V. Further, it was in [71] and [55] where very fast decaying potentials were studied for the first time. In [71] a complete scale of the type of the decay of V was considered. In particular, if $\lim_{|x|\to\infty} \ln V(x)/|x|^{2\beta} = -\mu > 0$, then we have

$$\mathcal{N}_j^{+}(\lambda) \sim \begin{cases} \frac{b}{2\mu} |\ln \lambda|^{1/\beta} & \text{if} \quad 0 < \beta < 1, \\ \frac{\ln \lambda}{\ln(1+2\mu/b)} & \text{if} \quad \beta = 1, \\ \frac{\beta}{\beta - 1} \frac{|\ln \lambda|}{|(\ln|\ln \lambda|)|} & \text{if} \quad \beta > 1, \end{cases} \qquad \lambda \downarrow 0. \tag{1.4.5}$$

Note that the behavior for $\beta \geq 1$ is not semiclassical. The case of V with compact support was considered in [71] as well as in [55]. As already mentioned in the Introduction, in this case asymptotic relation (0.18) holds true.

In both articles the authors based their results in the study of certain Toeplitz operators. Let P_j be the orthogonal projection onto the space $\mathrm{Ker}(\mathrm{H}_0 - b(2j-1))$, $j \in \mathbb{N}$. Then the referred Toeplitz operators are just $P_j V P_j$. These operators are the very key of the subject, and are used by many other authors in similar situations .

N.Filonov and A.Pushnitski developed in [24] the results of [71, 55] for compactly supported V, and obtained an asymptotic formula describing the convergence of the discrete eigenvalues of the perturbed operator to the Landau levels, which is more precise than (0.18). In particular, this formula recovers the logarithmic capacity of the support of V which could be considered as an original and deep result in the field of the spectral geometry.

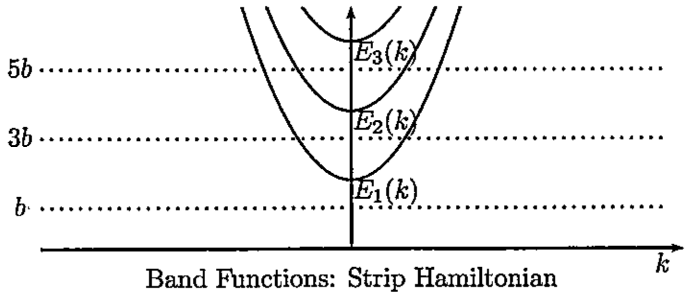
At this point we would like to mention that most of the compactly supported electric perturbations considered so far (including the ones treated in this thesis) have a definite sign. This implies that the corresponding Toeplitz operators are of a definite sign as well. Recently, the article of A. Pushnitski and G. Rozenblum [62] deals with the case of Toeplitz operators in Bargmann spaces whose symbol has a variable sign. Bounds for the asymptotic behavior of the eigenvalues of the operator are obtained and an application to the discrete spectrum of perturbed Landau Hamiltonian is given.

Magnetic perturbations of the Landau Hamiltonian have been also examined. In [76] compactly supported perturbations of the constant magnetic field combined with compactly supported electric potentials have been considered. Using appropriate creation and annihilation operators, the authors construct approximate spectral eigenspaces corresponding to a small vicinity of a given Landau level, imitating what happens in the unperturbed case of a constant magnetic field. After the analysis of the arising Toeplitz operators, it is found out that the discrete eigenvalues converge to the Landau levels in a similar way as if only compactly supported electric potentials were switched on, and that an asymptotic relation analogous to (0.18) holds true.

Finally, let us mention some geometric perturbations of the Landau Hamiltonian which leave invariant its essential spectrum, e.g. self-adjoint operators generated by the same differential operation but considered on the complement of compact set in $\mathbb{R}^2$ and equipped with appropriate boundary conditions. In [61] A. Pushnitski and G. Rozenblum studied the asymptotics as

$\lambda \downarrow 0$ of $\mathcal{N}_j^+(\lambda)$ in the case of Dirichlet boundary conditions, while in [59] M. Persson attacked the analogous problem for $\mathcal{N}_j^-(\lambda)$ and Neumann boundary conditions.

(ii) Strip Hamiltonian. Let us consider now the Hamiltonian in (1.4.4) with $b > 0$ constant, and $W = 0$, but this time with $\mathcal{I} = (-a, a)$, $a > 0$, and Dirichlet boundary conditions. The analysis of the band functions was carried out by V. Geiler and M. Senatorov in [27]. In this case $\mathbf{E}_j(k)$ behaves like k^2 as $k \to \infty$, and has a unique minimum, at $k = 0$. Since the unperturbed operator has a purely absolute continuous spectrum coinciding with $[\mathbf{E}_1(0), \infty)$, only the counting function $\mathcal{N}_0^-(\lambda)$ could be considered here. Its asymptotics as $\lambda \downarrow 0$ was investigated in [14], where actually the Krein spectral shift function was studied near each spectral threshold $\mathbf{E}_j(0)$, $j \in \mathbb{N}$.



Band Functions: Strip Hamiltonian

Let us describe the effective Hamiltonian introduced in [14] in order to investigate the behavior of the SSF near the threshold $E_j(0)$. Let $\Psi_j \in \mathrm{Ker}\,(\mathbf{h}_0(0) - \mathbf{E}_j(0))$ be a real eigenfunction normalized to one in $L^2(-a, a)$. Set also $\mu_j := 1/2\mathbf{E}_j''(0)$ and put $w_j(y) := \int_{-a}^{a} V(x, y)\Psi_j(x)^2 dx$. Then the operator

$$-\mu_j \frac{d^2}{dy^2} \pm w_j, \tag{1.4.6}$$

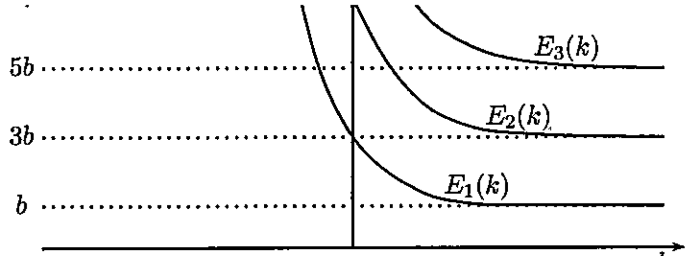
self-adjoint in $L^2(\mathbb{R})$, plays the role of an effective Hamiltonian corresponding to $\mathbf{H}_0 \pm V$ with $V \in L_0^\infty(\mathbb{R}^2)$, $V \geq 0$, and the responsible for the threshold $\mathbf{E}_j(0)$. With the help of the effective Hamiltonian $-\mu_1 \frac{d^2}{dy^2} - w_1$ which is responsible for the first spectral threshold $E_1(0)$, it was found in [14] that $\mathcal{N}_0^-(\lambda)$ behaves semiclassically, i.e.

$$\mathcal{N}_0^-(\lambda) = (2\pi)^{-1} \left|\{(y, k) \in T^*\mathbb{R} \,|\, \mu_1 k^2 - w_1(y) < -\lambda\}\right| (1 + o(1), \quad \lambda \downarrow 0,$$

under the assumption that w_1 does not decay too fast at infinity, i.e. roughly speaking, that $\lim_{|y| \to \infty} |y|^\alpha w_1(y) = \ell > 0$ with $\alpha \in (0, 2)$ and $\ell > 0$. If, on the contrary, we have $w_1(y) = O(\langle y \rangle^{-\alpha})$ with $\alpha > 2$, then $\mathcal{N}_0^-(\lambda) = O(1)$ as $\lambda \downarrow 0$, i.e. in this case the operator $\mathbf{H}_0 - V$ could have only finitely many discrete eigenvalues.

(iii) Half-Plane Hamiltonian. Consider now the magnetic Hamiltonian in (1.4.4) with $b > 0$ const, $W = 0$, and $\mathcal{I} = (0, \infty)$ with Dirichlet boundary conditions. The band functions $\mathbf{E}_j(k)$ were investigated by S. De Bièvre and J. Pulé in [20]. It was found in that article that the band functions are strictly decreasing, $\mathbf{E}_j(k)$ tends to infinity as $k \to -\infty$, and $\mathbf{E}_j(k)$ converges to

the Landau level $b(2j-1)$ as $k \to \infty$; hence, in particular, the spectrum of $\mathbf{H}_0$ is absolutely continuous. Moreover, the Landau levels are thresholds of the spectrum of $\mathbf{H}_0$.



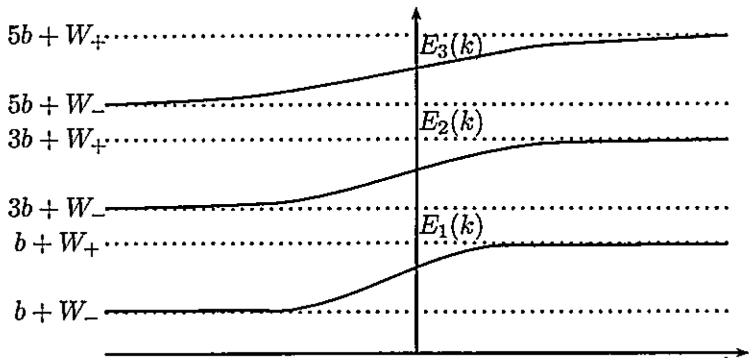
Band Functions: Half-Plane Hamiltonian

To perform an analysis of the asymptotic behavior of the SSF near the spectral thresholds similar to the one for the strip Hamiltonian is an interesting and challenging open problem.

(iv) Monotone Edge Potential. Let us discuss now the first of the main models considered in this thesis (see also [15]). This is the magnetic Hamiltonian in (1.4.4) with $b > 0$ constant $\mathcal{I} = \mathbb{R}$, and W monotone. For definiteness, we assume that W is not decreasing. Then we have $W_\pm = \lim_{x \to \pm\infty} W(x)$ (see (0.8) for the definition of $W_\pm$). The band functions are bounded, non decreasing, and we have

$$\mathcal{E}_j^\pm = \lim_{k \to \pm} E_j(k) = b(2j-1) + W_\pm.$$

Thus, the band functions do not reach their suprema and infima at finite points.

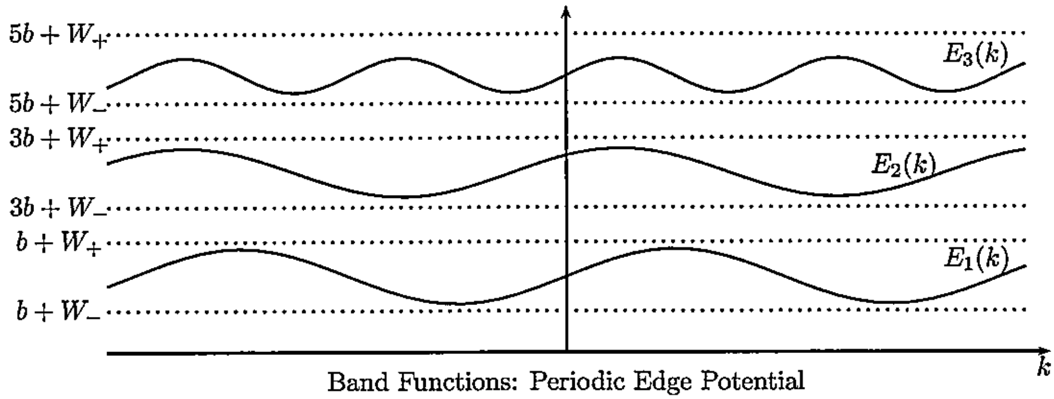


Band Functions: Monotone Edge Potential

In the k-representation the general effective Hamiltonian obtained for this model has been introduced in (0.14) (see also (2.2.3)). Its potential-energy part $\mathcal{V}_j^{\text{mon}}$ is a pseudo-differential operator with a contravariant operator written explicitly in the terms of the perturbation V. Note that in contrast to the potential-energy operators occurring in the effective Hamiltonians (1.4.3) and (1.4.6), the Weyl symbol of the operator $\mathcal{V}_j^{\text{mon}}$ depends non-trivially on both variables of $(y,k) \in T^*\mathbb{R}$. The deep reason for this important difference is that the suprema and the infima of the band functions $\mathbf{E}_j$ which coincide with the edges of the spectral gaps, are only asymptotic values never reached at finite points. Note that the operator $\mathcal{V}_j^{\rm mon}$ is unitarily equivalent to the Toeplitz operator P_jVP_j which arises as an effective Hamiltonian when we consider perturbations of the Landau Hamiltonian. In the case of non-constant monotone W, the effective Hamiltonian (0.14) however contains also a non-trivial kinetic part $\mathbf{E}_j$.

This model has another remarkable characteristic, namely, that depending on a geometric condition between W and V, we can have a finite or infinite number of eigenvalues of $\mathbf{H}$ in each gap of $\sigma(\mathbf{H}_0)$.

(v) Periodic edge potential. The second of the models studied in the thesis (see also [56]) is the Hamiltonian in (1.4.4) with $b > 0$ constant, W periodic with period T, and $\mathcal{I} = \mathbb{R}$. The band functions are bounded and periodic with period bT. An important feature of the band functions which is shown, is that they approximate well W, provided that b is large enough.



Band Functions: Periodic Edge Potential

The main novelty here comes from the fact that the band functions being periodic and defined over the whole real axis, have infinitely many points in which the extremal values are taken. Due to this the effective Hamiltonian used in Chapter 3 for the analysis of the asymptotics as $\lambda \downarrow 0$ of $\mathcal{N}_j^+(\lambda)$, is introduced in (0.15) under the assumption that 0 is the only point on $[0, bT)$ where $\mathbf{E}_j$ achieves its maximum, and this maximum is non-degenerate. Then we have $\mu_j = \frac{1}{2}\mathbf{E}_j''(0)$ in (0.15), and $\mathcal{V}_j^{\rm per}$ is an infinite matrix-valued potential written explicitly in the terms of the perturbation V.

In contrast to the case of the monotone edge potentials, in this model every open gap of the spectrum of $\mathbf{H}_0$ always contains infinitely many eigenvalues even for compactly supported perturbations.

(vi) Iwatsuka Hamiltonian. Finally, we discuss briefly the Hamiltonian in (1.4.4) with $b = b(x)$ monotone function admitting different limits of the same sign as $x \to -\infty$ and $x \to \infty$, $W = 0$, $\mathcal{I} = \mathbb{R}$. This model known as the Iwatsuka Hamiltonian, since it was introduced by A. Iwatsuka in [39].

Suppose for definiteness that b is non-decreasing, and $\lim_{x\to\pm\infty} b(x) = b_\pm$ with $0 < b_- < b_+$.

Then it is shown in [39] that

$$\mathcal{E}_j^{\pm} = b_{\pm}(2j-1), \quad j \in \mathbb{N}.$$

The main results on the asymptotic distribution of the discrete spectrum for perturbations $\mathbf{H}_0 \pm V$ of the Iwatsuka Hamiltonian were obtained by S. Shirai. In [78] he studied the asymptotics as $g \to \infty$ of the functions $N_{\pm}(\lambda, g)$, and in [77] he investigated the asymptotics as $\lambda \to \infty$ of the functions $\mathcal{N}_j^{\pm}(\lambda)$. In both cases only perturbations V of power-like decay were considered.

Chapter 2

Monotone Edge Potential

In this chapter we study the operator H defined in (0.1) - (0.2), under the assumption that the edge potential W is monotone. For definiteness we shall consider only non-decreasing W. It will be evident that a completely analogous treatment could be done in the non-increasing case.

In Section 2.1, spectral properties of the unperturbed operator H_0 are studied, in particular, the behavior of the band functions E_j, $j \in \mathbb{N}$. In Section 2.2 we state the main results of the chapter, that is, we describe effective Hamiltonias, and the asymptotic bounds of $\mathcal{N}_j^+$. The proofs of our main results could be found in Sections 2.3 – 2.6.

2.1 Basic spectral properties of H_0

We start the study of the operator H by considering first the unperturbed operator H_0, specifically, general properties of the band functions $E_j(k)$.

By analogy with the operator $\tilde{h}$ (see (0.5)), it is also useful to consider the shifted harmonic oscillator

$$\tilde{h}_\infty := -\frac{d^2}{dx^2} + b^2x^2 + W_+, \tag{2.1.1}$$

self-adjoint in $L^2(\mathbb{R})$.

Proposition 2.1.1. *Assume that W is non-decreasing and bounded. Then for each $j \in \mathbb{N}$ the eigenvalue $E_j(k)$ is a non-decreasing function of $k \in \mathbb{R}$, and*

$$\lim_{k\to-\infty} E_j(k) = b(2j-1) + W_-, \quad \lim_{k\to\infty} E_j(k) = b(2j-1) + W_+ \tag{2.1.2}$$

holds true.

Proof. The fact that E_j are non-decreasing bounded functions of k follows directly from the mini-max principle. Let us prove (2.1.2). Pick $E > -b - W_-$. Then for each $k \in \mathbb{R}$ we have $-E < b + W_- \le \inf \sigma(\tilde{h}(k))$ (see (0.9)). Moreover, $-E < b + W_+ = \inf \sigma(\tilde{h}_\infty)$. Then,

$$|(E_j(k) + E)^{-1} - (b(2j-1) + W_+ + E)^{-1}| \le$$

$$\|(\tilde{h}(k) + E)^{-1}(W_+ - W(\cdot + k/b))(\tilde{h}_\infty + E)^{-1}\| \le$$

$$\|(\tilde{h}(k)+E)^{-1}\|\|(W_+ - W(\cdot + k/b))(\tilde{h}_\infty + E)^{-1}\|. \tag{2.1.3}$$

It is also clear that

$$\|(\tilde{h}(k)+E)^{-1}\| \leq (E+b+W_-)^{-1}, \tag{2.1.4}$$

and the r.h.s. is k-independent. Further, the multiplier by $(W_+ - W(\cdot + k/b))$, $x \in \mathbb{R}$, tends strongly to zero as $k \to \infty$, while the operator $(\tilde{h}_\infty + E)^{-1}$ is compact and k-independent. Hence, the operator $(W_+ - W(\cdot + k/b))(\tilde{h}_\infty + E)^{-1}$ tends uniformly to zero as $k \to \infty$. Now, (2.1.3) – (2.1.4) imply

$$\lim_{k\to\infty}(E_j(k)+E)^{-1} = (b(2j-1)+W_+ + E)^{-1}, \quad j \in \mathbb{N},$$

which yields the second limit in (2.1.2). The first one is proved in the same manner. □

Proposition 2.1.1 implies that if $W_- < W_+$ holds, then there are no identically constant functions E_j, $j \in \mathbb{N}$. Applying the general theory of analytically fibred Hamiltonians (see e.g. [73, Section XIII.16]), we immediately obtain the following

Corollary 2.1.2. *Assume that W is non-decreasing and bounded, and $W_- < W_+$ holds true. Then the spectrum of the operator H_0 is absolutely continuous and*

$$\sigma(H_0) = \sigma_{ac}(H_0) = \bigcup_{j=1}^{\infty}[b(2j-1)+W_-, b(2j-1)+W_+].$$

Our next theorem will play a crucial role in the construction of the effective Hamiltonian introduced in the next section. For its formulation we need the following notations.

Fix $k \in \mathbb{R}$ and $j \in \mathbb{N}$, denote by $\pi_j(k)$ the orthogonal projection onto $\mathrm{Ker}\,(h(k) - E_j(k))$. Then we have

$$\pi_j(k) = \langle \cdot, \psi_j(\cdot;k)\rangle \psi_j(\cdot;k), \tag{2.1.5}$$

where $\langle \cdot,\cdot\rangle$ is the scalar product in $L^2(\mathbb{R})$, and $\psi_j(x;k)$, $x \in \mathbb{R}$, is an eigenfunction of $h(k)$ which satisfies

$$h(k)\psi_j(\cdot;k) = E_j(k)\psi_j(\cdot;k), \quad \|\psi_j(\cdot;k)\|_{L^2(\mathbb{R})} = 1. \tag{2.1.6}$$

Moreover, $\psi_j(\cdot;k)$ could be chosen to be real-valued.

Set

$$h_\infty(k) := \mathcal{U}_k \tilde{h}_\infty \mathcal{U}_k^* = -\frac{d^2}{dx^2} + (bx-k)^2 + W_+, \quad k \in \mathbb{R},$$

the operator $\tilde{h}_\infty$ being defined in (2.1.1). Denote by $\pi_{j,\infty}(k)$, $k \in \mathbb{R}$, $j \in \mathbb{N}$, the orthogonal projection onto $\mathrm{Ker}\,(h_\infty(k) - \mathcal{E}_j^+)$. Then, similarly to (2.1.5), we have

$$\pi_{j,\infty}(k) = \langle \cdot, \psi_{j,\infty}(\cdot;k)\rangle \psi_{j,\infty}(\cdot;k), \tag{2.1.7}$$

where the eigenfunction $\psi_{j,\infty}(x;k)$ satisfies

$$-\frac{\partial^2 \psi_{j,\infty}}{\partial x^2}(x;k) + (bx-k)^2\psi_{j,\infty}(x;k) = b(2j-1)\psi_{j,\infty}(x;k), \quad \|\psi_{j,\infty}(\cdot;k)\|_{L^2(\mathbb{R})} = 1, \tag{2.1.8}$$

and could be chosen to be real-valued as well. The functions $\psi_{j,\infty}$, $j \in \mathbb{N}$, admit an explicit description. Namely, if we put

$$\varphi_j(x) := \frac{H_{j-1}(x)e^{-x^2/2}}{(\sqrt{\pi}2^{j-1}(j-1)!)^{1/2}}, \quad x \in \mathbb{R}, \quad j \in \mathbb{N}, \tag{2.1.9}$$

where

$$\mathrm{H}_q(x) := (-1)^q e^{x^2} \frac{d^q}{dx^q} e^{-x^2}, \quad x \in \mathbb{R}, \quad q \in \mathbb{Z}_+,$$

are the Hermite polynomials (see e.g. [4, Chapter I, Eqs. (8.5), (8.7)]), then the real-valued function φ_j satisfies

$$-\varphi_j''(x) + x^2\varphi_j(x) = (2j-1)\varphi_j(x), \quad \|\varphi_j\|_{L^2(\mathbb{R})} = 1,$$

and we have

$$\psi_{j,\infty}(x;k) = b^{1/4}\varphi_j(b^{1/2}x - b^{-1/2}k), \quad j \in \mathbb{N}, \quad x \in \mathbb{R}, \quad k \in \mathbb{R}. \tag{2.1.10}$$

Put

$$p_j = p_j(b) := \frac{b^{-j+3/2}}{\sqrt{\pi}(j-1)!2^{j-1}}, \quad j \in \mathbb{N}. \tag{2.1.11}$$

Note that

$$\psi_{j,\infty}(x;k) = p_j^{1/2}(-k)^{j-1}e^{-(b^{-1/2}k-b^{1/2}x)^2/2}(1+o(1)) \tag{2.1.12}$$

as $k \to \infty$, uniformly with respect to x belonging to compact subsets of $\mathbb{R}$.

Theorem 2.1.3. *Fix $j \in \mathbb{N}$. Then we have*

$$\lim_{k\to\infty} \left(\mathcal{E}_j^+ - E_j(k)\right)^{-1/2} \|\pi_{j,\infty} - \pi_j(k)\|_1 = 0, \tag{2.1.13}$$

the trace-class norm being defined in Section 1.2.

We will divide the proof of the theorem into several propositions, in which we investigate separately the asymptotics of $\mathcal{E}_j^+ - E_j(k)$ and $\|\pi_{j,\infty} - \pi_j(k)\|_1 = 0$.

As in Proposition 2.1.1, it is more convenient to consider the operator $\tilde{h}(k)$ instead of $h(k)$. Let us introduce then the corresponding analogous objects.

Set

$$\tilde{\psi}_j(x;k) := \mathcal{U}_k^*\psi_j(x;k) = \psi_j(x + k/b; k), \quad x \in \mathbb{R}, \quad k \in \mathbb{R}, \quad j \in \mathbb{N},$$

the function ψ_j being defined in (2.1.6). Evidently,

$$\tilde{h}(k)\tilde{\psi}_j(x;k) = E_j(k)\tilde{\psi}_j(x;k), \quad \|\tilde{\psi}_j(\cdot;k)\|_{L^2(\mathbb{R})} = 1.$$

By analogy with (2.1.5) put

$$\tilde{\pi}_j(k) := \langle\cdot, \tilde{\psi}_j(\cdot;k)\rangle\tilde{\psi}_j(\cdot;k), \quad k \in \mathbb{R}, \quad j \in \mathbb{N}.$$

Similarly, set

$$\tilde{\psi}_{j,\infty}(x) := \mathcal{U}_k^* \psi_{j,\infty}(x;k) = b^{1/4}\varphi_j(b^{1/2}x), \quad x \in \mathbb{R}, \quad j \in \mathbb{N},$$

(see (2.1.9) for the definition of the function φ_j). Then

$$\tilde{h}_\infty \tilde{\psi}_{j,\infty}(x) = \mathcal{E}_j^+ \tilde{\psi}_{j,\infty}(x), \quad \|\tilde{\psi}_{j,\infty}\|_{L^2(\mathbb{R})} = 1.$$

Put

$$\tilde{\pi}_{j,\infty} := \langle \cdot, \tilde{\psi}_{j,\infty}\rangle \tilde{\psi}_{j,\infty}, \quad j \in \mathbb{N}.$$

Since we have

$$\mathcal{U}_k \tilde{\pi}_j(k) \mathcal{U}_k^* = \pi_j(k), \quad \mathcal{U}_k \tilde{\pi}_{j,\infty} \mathcal{U}_k^* = \pi_{j,\infty}(k), \quad k \in \mathbb{R}, \quad j \in \mathbb{N},$$

relation (2.1.13) is equivalent to

$$\lim_{k\to\infty} \left(\mathcal{E}_j^+ - E_j(k)\right)^{-1/2} \|\tilde{\pi}_{j,\infty} - \tilde{\pi}_j(k)\|_1 = 0. \tag{2.1.14}$$

Let us first examine the asymptotic behavior of $\|\tilde{\pi}_{j,\infty} - \tilde{\pi}_j(k)\|_1$ as $k \to \infty$. In order to do that we need a preliminary result.

For $z \in \mathbb{C} \setminus (\sigma(\tilde{h}_\infty) \setminus \{\mathcal{E}_j^+\})$ set

$$R_{\infty,j}^\perp(z) := (\tilde{h}_\infty - z)^{-1}(I - \tilde{\pi}_{j,\infty}).$$

Similarly, for $z \in \mathbb{C} \setminus (\sigma(\tilde{h}(k)) \setminus \{E_j(k)\})$ put

$$R_j^\perp(z) := (\tilde{h}(k) - z)^{-1}(I - \tilde{\pi}_j(k)).$$

Set

$$U_k(x) := W_+ - W(x + k/b) = \tilde{h}_\infty - \tilde{h}(k), \quad x \in \mathbb{R}, \quad k \in \mathbb{R}.$$

Lemma 2.1.4. *We have*

$$\tilde{\pi}_{j,\infty} = \tilde{\pi}_{j,\infty}\tilde{\pi}_j(k) - \tilde{\pi}_{j,\infty} U_k R_j^\perp(\mathcal{E}_j^+) = \tilde{\pi}_j(k)\tilde{\pi}_{j,\infty} - R_j^\perp(\mathcal{E}_j^+) U_k \tilde{\pi}_{j,\infty}, \tag{2.1.15}$$

$$\tilde{\pi}_j(k) = \tilde{\pi}_j(k)\tilde{\pi}_{j,\infty} + \tilde{\pi}_j(k) U_k R_{\infty,j}^\perp(E_j) = \tilde{\pi}_{j,\infty}\tilde{\pi}_j(k) + R_{\infty,j}^\perp(E_j) U_k \tilde{\pi}_j(k). \tag{2.1.16}$$

Proof. We have

$$\begin{aligned}\tilde{\pi}_{j,\infty} &= \tilde{\pi}_{j,\infty}\tilde{\pi}_j(k) + \tilde{\pi}_{j,\infty}(I - \tilde{\pi}_j(k)) \\ &= \tilde{\pi}_{j,\infty}\tilde{\pi}_j(k) + \tilde{\pi}_{j,\infty}(\tilde{h}_\infty - \mathcal{E}_j^+ - U_k)(\tilde{h}(k) - \mathcal{E}_j^+)^{-1}(I - \tilde{\pi}_j(k)).\end{aligned}$$

Since $\tilde{\pi}_{j,\infty}(\tilde{h}_\infty - \mathcal{E}_j^+) = 0$, we obtain the first equality in (2.1.15). The second equality is obtained by taking the adjoint. In relations (2.1.16) we have only exchanged the role of $\tilde{h}(k)$ and $\tilde{h}_\infty$. □

Set

$$\Phi_j(k) = \Phi_j(k; W) := \left(\int_{\mathbb{R}} U_k(x)\tilde{\psi}_{j,\infty}(x)^2 dx\right)^{1/2}, \quad k \in \mathbb{R}. \tag{2.1.17}$$

By the dominated convergence theorem we have $\lim_{k\to+\infty} \Phi_j(k) = 0$. Note that

$$\Phi_j(k) = (\mathrm{Tr}\,\tilde{\pi}_{j,\infty} U_k \tilde{\pi}_{j,\infty})^{1/2} = \|\tilde{\pi}_{j,\infty} U_k^{1/2}\|_1 = \|U_k^{1/2}\tilde{\pi}_{j,\infty}\|_1 = \|\tilde{\pi}_{j,\infty} U_k^{1/2}\| = \|U_k^{1/2}\tilde{\pi}_{j,\infty}\|. \tag{2.1.18}$$

Proposition 2.1.5. *Fix $j \in \mathbb{N}$. Then we have*

$$\|\tilde{\pi}_{j,\infty} - \tilde{\pi}_j(k)\|_1 = o(\Phi_j(k)), \quad k \to \infty. \tag{2.1.19}$$

Proof. By (2.1.15) and (2.1.16) we have

$$\tilde{\pi}_{j,\infty} - \tilde{\pi}_j(k) = -\tilde{\pi}_{j,\infty} U_k R_j^{\perp}(\mathcal{E}_j^+) - R_{\infty,j}^{\perp}(E_j) U_k \tilde{\pi}_{j,\infty} + R_{\infty,j}^{\perp}(E_j) U_k (\tilde{\pi}_{j,\infty} - \tilde{\pi}_j(k)),$$

i.e.

$$(I - R_{\infty,j}^{\perp}(E_j) U_k)(\tilde{\pi}_{j,\infty} - \tilde{\pi}_j(k)) = -\tilde{\pi}_{j,\infty} U_k R_j^{\perp}(\mathcal{E}_j^+) - R_{\infty,j}^{\perp}(E_j) U_k \tilde{\pi}_{j,\infty}.$$

Since $s - \lim_{k\to\infty} U_k = 0$ and the operator $R_{\infty,j}^{\perp}(E_j)$ is compact and uniformly bounded, we have $\lim_{k\to\infty} \|R_{\infty,j}^{\perp}(E_j) U_k\| = 0$. Therefore, the operator $I - R_{\infty,j}^{\perp}(E_j) U_k$ is invertible for sufficiently great k, and for such k we have

$$\tilde{\pi}_{j,\infty} - \tilde{\pi}_j(k) = -(I - R_{\infty,j}^{\perp}(E_j) U_k)^{-1}(\tilde{\pi}_{j,\infty} U_k R_j^{\perp}(\mathcal{E}_j^+) + R_{\infty,j}^{\perp}(E_j) U_k \tilde{\pi}_{j,\infty}).$$

Therefore,

$$\|\tilde{\pi}_{j,\infty} - \tilde{\pi}_j(k)\|_1 \le \|(I - R_{\infty,j}^{\perp}(E_j) U_k)^{-1}\|(\|U_k^{1/2} R_j^{\perp}(\mathcal{E}_j^+)\| + \|R_{\infty,j}^{\perp}(E_j) U_k^{1/2}\|)\|\tilde{\pi}_{j,\infty} U_k^{1/2}\|_1. \tag{2.1.20}$$

Arguing as above, we easily find that

$$\lim_{k\to\infty} \|U_k^{1/2} R_j^{\perp}(\mathcal{E}_j^+)\| = \lim_{k\to\infty} \|R_{\infty,j}^{\perp}(E_j) U_k^{1/2}\| = 0. \tag{2.1.21}$$

The combination of (2.1.20), (2.1.21), and (2.1.18) implies (2.1.19). □

Proposition 2.1.6. *For any $j \in \mathbb{N}$ we have*

$$\mathcal{E}_j^+ - E_j(k) = \Phi_j(k)^2(1 + o(1)), \quad k \to \infty. \tag{2.1.22}$$

Proof. Assume k large enough. Evidently,

$$\mathcal{E}_j^+ = \mathrm{Tr}\,\tilde{h}_\infty \tilde{\pi}_{j,\infty} = -\frac{1}{2\pi i}\mathrm{Tr}\int_{\Gamma_j} \tilde{h}_\infty(\tilde{h}_\infty - \omega)^{-1} d\omega = -\frac{1}{2\pi i}\mathrm{Tr}\int_{\Gamma_j} \omega(\tilde{h}_\infty - \omega)^{-1} d\omega$$

where Γ_j is a sufficiently small circle run over in the anticlockwise direction which contains in its interior $E_j(k)$ and $\mathcal{E}_j^+$ but no other points from the spectra of $\tilde{h}(k)$ and $\tilde{h}_\infty$. Similarly,

$$E_j(k) = -\frac{1}{2\pi i}\mathrm{Tr}\int_{\Gamma_j} \omega(\tilde{h}(k) - \omega)^{-1} d\omega.$$

Therefore,

$$\mathcal{E}_j^+ - E_j(k) = -\frac{1}{2\pi i}\mathrm{Tr}\int_{\Gamma_j}\omega\left((\tilde{h}_\infty-\omega)^{-1}-(\tilde{h}(k)-\omega)^{-1}\right)d\omega =$$

$$\frac{1}{2\pi i}\mathrm{Tr}\int_{\Gamma_j}\omega(\tilde{h}_\infty-\omega)^{-1}U_k(\tilde{h}(k)-\omega)^{-1}d\omega. \qquad (2.1.23)$$

Applying the Cauchy theorem, we easily get

$$\frac{1}{2\pi i}\int_{\Gamma_j}\omega(\tilde{h}_\infty-\omega)^{-1}U_k(\tilde{h}(k)-\omega)^{-1}d\omega =$$

$$\tilde{\pi}_{j,\infty}U_k\tilde{\pi}_j(k) - \mathcal{E}_j^+\tilde{\pi}_{j,\infty}U_kR_j^\perp(\mathcal{E}_j^+) - E_jR_{\infty,j}^\perp(E_j)U_k\tilde{\pi}_j(k). \qquad (2.1.24)$$

Comparing (2.1.23) and (2.1.24), and bearing in mind (2.1.18), we obtain

$$\mathcal{E}_j^+ - E_j(k) - \Phi_j(k)^2 =$$

$$\mathrm{Tr}\,\tilde{\pi}_{j,\infty}U_k(\tilde{\pi}_j(k)-\tilde{\pi}_{j,\infty}) - \mathcal{E}_j^+\mathrm{Tr}\,\tilde{\pi}_{j,\infty}U_kR_j^\perp(\mathcal{E}_j^+) - E_j\mathrm{Tr}\,R_{\infty,j}^\perp(E_j)U_k\tilde{\pi}_j(k). \qquad (2.1.25)$$

In order to complete the proof of (2.1.22), it remains to show that the three terms on the r.h.s. of (2.1.25) are of order $o(\Phi_j(k)^2)$ as $k \to \infty$.
First, we have

$$|\mathrm{Tr}\,\tilde{\pi}_{j,\infty}U_k(\tilde{\pi}_j(k)-\tilde{\pi}_{j,\infty})| \leq \|\tilde{\pi}_{j,\infty}U_k^{1/2}\|\|U_k^{1/2}\|\|\tilde{\pi}_j(k)-\tilde{\pi}_{j,\infty}\|_1 = o(\Phi_j(k)^2), \quad k\to\infty, \qquad (2.1.26)$$

by (2.1.18), (2.1.19), and the fact that $\|U_k^{1/2}\|$ is uniformly bounded with respect to $k \in \mathbb{R}$.
Next, using the trivial identities $\tilde{\pi}_{j,\infty} = \tilde{\pi}_{j,\infty}^2$ and $R_j^\perp(\mathcal{E}_j^+)\tilde{\pi}_j(k) = 0$, as well as the cyclicity of the trace, we obtain

$$\mathrm{Tr}\,\tilde{\pi}_{j,\infty}U_kR_j^\perp(\mathcal{E}_j^+) = -\mathrm{Tr}\,(\tilde{\pi}_j(k)-\tilde{\pi}_{j,\infty})\tilde{\pi}_{j,\infty}U_kR_j^\perp(\mathcal{E}_j^+). \qquad (2.1.27)$$

Therefore, similarly to (2.1.26), we have

$$|\mathcal{E}_j^+\mathrm{Tr}\,\tilde{\pi}_{j,\infty}U_kR_j^\perp(\mathcal{E}_j^+)| \leq |\mathcal{E}_j^+|\|\tilde{\pi}_j(k)-\tilde{\pi}_{j,\infty}\|_1\|\tilde{\pi}_{j,\infty}U_k^{1/2}\|\|U_k^{1/2}R_j^\perp(\mathcal{E}_j^+)\| = o(\Phi_j(k)^2) \qquad (2.1.28)$$

as $k \to \infty$. Finally, by analogy with (2.1.27) we have

$$\mathrm{Tr}\,R_{\infty,j}^\perp(E_j)U_k\tilde{\pi}_j(k) = \mathrm{Tr}\,R_{\infty,j}^\perp(E_j)U_k\tilde{\pi}_j(k)(\tilde{\pi}_j(k)-\tilde{\pi}_{j,\infty}) =$$

$$\mathrm{Tr}\,R_{\infty,j}^\perp(E_j)U_k\tilde{\pi}_{j,\infty}(\tilde{\pi}_j(k)-\tilde{\pi}_{j,\infty}) + \mathrm{Tr}\,R_{\infty,j}^\perp(E_j)U_k(\tilde{\pi}_j(k)-\tilde{\pi}_{j,\infty})^2.$$

Hence,

$$|E_j\mathrm{Tr}\,R_{\infty,j}^\perp(E_j)U_k\tilde{\pi}_j(k)| \leq$$

$$|E_j(k)|\|R_{\infty,j}^\perp(E_j)U_k^{1/2}\|\|U_k^{1/2}\tilde{\pi}_{j,\infty}\|\|\tilde{\pi}_j(k)-\tilde{\pi}_{j,\infty}\|_1 +$$

$$|E_j(k)|\|R_{\infty,j}^\perp(E_j)U_k\|\|\tilde{\pi}_j(k)-\tilde{\pi}_{j,\infty}\|_1^2 = o(\Phi_j(k)^2), \quad k\to\infty, \qquad (2.1.29)$$

by (2.1.18), (2.1.19), and the fact that $|E_j(k)|$, $\|R_{\infty,j}^\perp(E_j(k))U_k^{1/2}\|$, and $\|R_{\infty,j}^\perp(E_j(k))U_k\|$ are uniformly bounded with respect to $k \in \mathbb{R}$.
Putting together (2.1.25), (2.1.26), (2.1.28), and (2.1.29), we obtain (2.1.22). □

Now relation (2.1.14), and hence (2.1.13), follows immediately from (2.1.19) and (2.1.22), so that the proof of Theorem 2.1.3 is complete.

To end this section, we examine the explicit asymptotic behavior of $\mathcal{E}_j^+ - E_j(k)$ as $k \to \infty$ when W is a step function. This simple special case will play an important role in the proofs of our main results in the general case of bounded non-decreasing W.

Let $w_-, w_+ \in \mathbb{R}$, $w_- < w_+$, $x_0 \in \mathbb{R}$. Put

$$\mathrm{w}(x) := \begin{cases} w_+ & \text{if} \quad x \geq x_0, \\ w_- & \text{if} \quad x < x_0. \end{cases} \tag{2.1.30}$$

Proposition 2.1.7. *Assume that $w_- < w_+$. Then we have*

$$\Phi_j(k;\mathrm{w})^2 = \frac{(w_+ - w_-)}{2} p_j k^{2j-3} e^{-(b^{-1/2}k - b^{1/2}x_0)^2}(1 + o(1)), \quad k \to \infty, \tag{2.1.31}$$

the number p_j being defined in (2.1.11).

Proof. By (2.1.17),

$$\Phi_j(k;\mathrm{w})^2 = \int_{-\infty}^{x_0 - k/b} (w_+ - w_-)\tilde{\psi}_{j,\infty}(x)^2 dx.$$

Bearing in mind (2.1.10), (2.1.9), and making a change of variables, we obtain

$$\Phi_j(k;\mathrm{w})^2 = (w_+ - w_-)\frac{b^{1/2}}{\sqrt{\pi}2^{j-1}(j-1)!}\int_{-\infty}^{x_0} H_j(b^{1/2}y - b^{-1/2}k)^2\, e^{-(b^{1/2}y - b^{-1/2}k)^2} dy. \tag{2.1.32}$$

Now, (2.1.32) and (2.1.11) easily imply

$$\Phi_j(k;\mathrm{w})^2 = (w_+ - w_-)\frac{b^{3/2-j}2^{j-1}k^{2(j-1)}}{\sqrt{\pi}(j-1)!} e^{-b^{-1}k^2} \int_{-\infty}^{x_0} e^{-by^2} e^{2ky} dy (1 + o(1)), \quad k \to \infty. \tag{2.1.33}$$

The integral in (2.1.33) can be estimated by the Laplace method (see e.g. [58]) which yields

$$\int_{-\infty}^{x_0} e^{-by^2} e^{k2y} dy = \frac{e^{-bx_0^2} e^{2kx_0}}{2k}(1 + o(1)), \quad k \to \infty. \tag{2.1.34}$$

Combining (2.1.33) and (2.1.34), we obtain (2.1.31). □

With this proposition in our hands, we can put together (2.1.22) and (2.1.31) and find that

$$E_j(k,\mathrm{w}) = \mathcal{E}_j^+(\mathrm{w}) - \frac{(w_+ - w_-)}{2} p_j k^{2j-3} e^{-(b^{-1/2}k - b^{1/2}x_0)^2}(1 + o(1)), \quad k \to \infty. \tag{2.1.35}$$

Thus, in spite of the fact that the band functions $E_j(k; b, W)$, $k \in \mathbb{R}$, $j \in \mathbb{N}$, imitate in many aspects the behavior of the edge potential W (see e.g. Proposition 2.1.1 or Propositions 3.1.3 and 3.1.5 below), asymptotic relation (2.1.35) reveals an important difference: the function w is equal to its maximal value w_+ on the interval $[x_0, \infty)$, while the band functions $E_j(k; b, \mathrm{w})$, $j \in \mathbb{N}$, being analytic increasing functions, never reach their suprema $\mathcal{E}_j^+$. This purely quantum effect related to the uncertainty principle, explains many of the asymptotic results obtained in the sequel.

2.2 Main Results

This section contains the statements of the main results we shall prove in this chapter. As explained in the introduction, for definiteness, we will consider the case of positive perturbations, and respectively the asymptotic behavior as $\lambda \downarrow 0$ of $\mathcal{N}_j^+(\lambda)$, $j \in \mathbb{N}$, $\lambda \in (0, 2b + W_- - W_+)$.

2.2.1 General Effective Hamiltonian

For $A \in [-\infty, \infty)$ and $\lambda > 0$, define $S_j(\lambda; A) : L^2(A, \infty) \longrightarrow L^2(\mathbb{R}^2)$ as the operator with integral kernel

$$(2\pi)^{-1/2} V(x,y)^{1/2} e^{iky} \psi_{j,\infty}(x;k)(\mathcal{E}_j^+ - E_j(k) + \lambda)^{-1/2}, \quad k \in \mathbb{R}, \quad (x,y) \in \mathbb{R}^2, \tag{2.2.1}$$

the function $\psi_{j,\infty}$ being defined in (2.1.10).

Theorem 2.2.1. *Assume that $V \in L_0^\infty(\mathbb{R}^2; \mathbb{R})$. Fix $j \in \mathbb{N}$ and $A \in [-\infty, \infty)$. Then for any $\varepsilon \in (0,1)$ we have*

$$n_*(1+\varepsilon; S_j(\lambda; A)) + O(1) \leq \mathcal{N}_j^+(\lambda) \leq n_*(1-\varepsilon; S_j(\lambda; A)) + O(1), \quad \lambda \downarrow 0. \tag{2.2.2}$$

The proof in contained in Section 2.3.

Let us write explicitly the effective Hamiltonian encoded in (2.2.2), which is responsible for the asymptotics term as $\lambda \downarrow 0$ of $\mathcal{N}_j^+(\lambda)$.

For $(x, \xi) \in T^*\mathbb{R} = \mathbb{R}^2$ and $j \in \mathbb{N}$ set

$$\Psi_{x,\xi;j}(k) = b^{-1/2} e^{-ik\xi} \psi_{j,\infty}(x;k), \quad k \in \mathbb{R}.$$

Note that for each $j \in \mathbb{N}$ the system $\{\Psi_{x,\xi;j}\}_{(x,\xi) \in T^*\mathbb{R}}$ is overcomplete with respect to the measure $\frac{b}{2\pi} dx d\xi$, i.e. for each $f \in L^2(\mathbb{R})$ we have

$$\frac{b}{2\pi} \int_{T^*\mathbb{R}} |\langle f, \Psi_{x,\xi;j} \rangle|^2 dx d\xi = \int_{\mathbb{R}} |f(k)|^2 dk$$

(see [4, Chapter 5, Section 2] or [79, Section 24]). For $(x, \xi) \in T^*\mathbb{R}$ and $j \in \mathbb{N}$ set $P_{x,\xi;j} := \langle \cdot, \Psi_{x,\xi;j} \rangle \Psi_{x,\xi;j}$, and introduce the operator

$$\mathcal{V}_j^{\mathrm{mon}} := \frac{b}{2\pi} \int_{T^*\mathbb{R}} V(x,\xi) P_{x,\xi;j} dx d\xi \tag{2.2.3}$$

where the integral is understood in the weak sense. Then $\mathcal{V}_j^{\mathrm{mon}}$ can be interpreted as a ΨDO with contravariant (generalized anti-Wick symbol) V (see [4]). Moreover, we have

$$S_j(\lambda; -\infty)^* S_j(\lambda; -\infty) = (\mathcal{E}_j^+ - E_j + \lambda)^{-1/2} \mathcal{V}_j^{\mathrm{mon}} (\mathcal{E}_j^+ - E_j + \lambda)^{-1/2}. \tag{2.2.4}$$

Bearing this in mind, and applying the Birman-Schwinger principle, we find that (2.2.2) with $A = -\infty$ and $\varepsilon \in (0,1)$ can be re-written as

$$\operatorname{rank} \mathbb{P}_{(\mathcal{E}_j^+ + \lambda, \infty)}(E_j + (1+\varepsilon)^{-1} \mathcal{V}_j^{\mathrm{mon}}) + O(1) \leq$$

$$\mathcal{N}_j^+(\lambda) \leq$$
$$\operatorname{rank} \mathbb{P}_{(\mathcal{E}_j^+ + \lambda, \infty)}(E_j + (1-\varepsilon)^{-1}\mathcal{V}_j^{\mathrm{mon}}) + O(1), \quad \lambda \downarrow 0.$$

Due to this, we have that the operator $E_j + \mathcal{V}_j^{\mathrm{mon}}$ bounded and self-adjoint in $L^2(\mathbb{R})$, which was already introduced in (0.14), could be interpreted as the effective Hamiltonian which governs main the asymptotic term of $\mathcal{N}_j^+(\lambda)$ as $\lambda \downarrow 0$, the multiplier by E_j being its "kinetic" part, and the ΨDO $\mathcal{V}_j^{\mathrm{mon}}$ being its "potential" part.

2.2.2 Effective Hamiltonian for the Case of Compactly Supported V

In this subsection a new effective Hamiltonian is described under the assumption that V is compactly supported. More precisely, we suppose that there exist bounded domains $\Omega_\pm \subset \mathbb{R}^2$ with Lipschitz boundaries, and constants $c_0^\pm > 0$ such that

$$c_0^- \chi_{\Omega_-}(x,y) \leq V(x,y) \leq c_0^+ \chi_{\Omega_+}(x,y), \quad (x,y) \in \mathbb{R}^2, \tag{2.2.5}$$

where $\chi_{\Omega_\pm}$ denotes the characteristic function of the domain $\Omega_\pm$.

For $\delta \in (0, 1/2)$ introduce the intervals

$$I_- = I_-(\delta) := (\delta, 1-\delta), \quad I_+ = I_+(\delta) := (0, 1+\delta).$$

In addition define the operator $\Gamma_\delta^-(m) : L^2(I_-) \to L^2(\Omega_-)$ as the operator with integral kernel

$$\pi^{-1/2} m e^{-bx^2/2} e^{m(x+iy)k} k^{1/2}, \quad k \in I_-, \quad (x,y) \in \Omega_-,$$

and the operator $\Gamma_\delta^+(m) : L^2(I_+) \to L^2(\Omega_+)$ as the operator with integral kernel

$$\pi^{-1/2} m e^{-bx^2/2} e^{m(x+iy+\delta)k} k^{1/2}, \quad k \in I_+, \quad (x,y) \in \Omega_+.$$

Remark. Introduce the set

$$\mathcal{B}(\Omega_\pm) := \{u \in L^2(\Omega_\pm) \,|\, u \,\text{is analytic in}\, \Omega_\pm\} \tag{2.2.6}$$

considered as a subspace of the Hilbert space $L^2(\Omega_\pm; e^{-bx^2} dxdy)$. Note that as a functional set $\mathcal{B}(\Omega_\pm)$ coincides with the Bergman space over $\Omega_\pm$ (see e.g. [30, Subsection 3.1]). Then we can say that, up to unitary equivalence, the operators $\Gamma_\delta^\pm(m)$ maps $L^2(I_\pm)$ into the holomorphic space $\mathcal{B}(\Omega_\pm)$.

Set

$$x^+ := \inf\{x \in \mathbb{R} \,|\, W(x) = W_+\}. \tag{2.2.7}$$

Note that by the assumption $W_- < W_+$, we have $x^+ > -\infty$.

Theorem 2.2.2. *Suppose that W is a bounded non-decreasing function with $W_- < W_+$, and that $x_+ < \infty$. Assume that $V \in L_0^\infty(\mathbb{R}^2; \mathbb{R})$ satisfies* (2.2.5)*. Then we have*

$$\mathcal{N}_j^+(\lambda) \geq n_*(r(1+\varepsilon)\sqrt{(W_+ - W_-)/c_0^-}; \Gamma_\delta^-(\sqrt{b|\ln \lambda|})) + O(1), \tag{2.2.8}$$

$$\mathcal{N}_j^+(\lambda) \leq n_*(r(1-\varepsilon)\sqrt{(W_+ - W(x^+ - \delta))/c_0^+} e^{-b\delta^2/2}; \Gamma_\delta^+(\sqrt{b|\ln \lambda|})) + O(1), \tag{2.2.9}$$

as $\lambda \downarrow 0$, for all $j \in \mathbb{N}$, $A > 0$, $\varepsilon \in (0,1)$, $\delta \in (0, 1/2)$ and $r > 0$.

The proof of Theorem 2.2.2 could be found in Section 2.4.

2.2.3 Asymptotic Bounds for $\mathcal{N}_j^+$

Put

$$\mathcal{X} := \{x \in \mathbb{R} \,|\, \text{there exists } y \in \mathbb{R} \text{ such that } (x,y) \in \operatorname{ess\,supp} V\},$$

$$X^- := \inf \mathcal{X}, \quad X^+ := \sup \mathcal{X}.$$

The quantities we are interested in, namely x^+ defined in (2.2.7), and X^+ are related in two ways, as is shown on the two figures below for the case of compactly supported V.

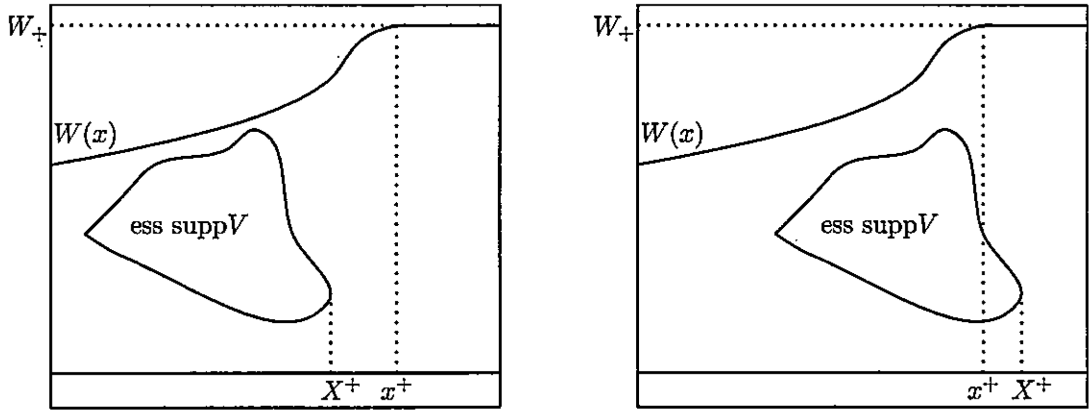


Figure 1. a   Figure 1. b

Our first theorem in this subsection contains a sufficient condition for the boundedness of $\mathcal{N}_j^+$. Roughly speaking, it concerns the situation described on Figure 1.a.

Theorem 2.2.3. *Let W be a bounded and non-decreasing function with $W_- < W_+$ and $x^+ \leq \infty$. Assume that $V \in L_0^\infty(\mathbb{R}^2)$, $V \geq 0$, $-\infty < X^- < X^+ < \infty$. Suppose in addition that* $\operatorname{ess\,sup}_{x\in\mathbb{R}} \int_{\mathbb{R}} V(x,y)dy < \infty$, *and*

$$X^+ < x^+. \tag{2.2.10}$$

Then we have

$$\mathcal{N}_j^+(\lambda) = O(1), \quad \lambda \downarrow 0, \quad j \in \mathbb{N}. \tag{2.2.11}$$

The proof of Theorem 2.2.3 is contained in Section 2.5.

Now we turn to the situation presented on Figure 1.b, namely $x^+ < X^+$. Assume that V satisfies (2.2.5). Then if we identify when appropriate $\mathbb{R}^2$ with $\mathbb{C}$ writing $z = x + iy \in \mathbb{C}$ for $(x,y) \in \mathbb{R}^2$, the quoted geometric condition can be expressed in the following manner

$$\Omega_- \cap \{z \in \mathbb{C} \,|\, \operatorname{Re} z > x^+\} \neq \emptyset. \tag{2.2.12}$$

In this situation if $x^+ = \infty$, we have $X^+ < x^+$ due to 2.2.5, and then Theorem 2.2.3 implies (2.2.11). It is then natural to assume in the next theorem $x^+ < \infty$. Since the operator H_0 is invariant under magnetic translations, we may choose $x^+ = 0$ without any loss of generality.

The estimates obtained for $\mathcal{N}_j^+(\lambda)$ contain some explicit constants related with the support of V. In order to define these constants we need the following notations. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Set

$$K_-(\Omega) := \left\{(p,q) \in \mathbb{R}^2 \,|\, p < q, \exists x \in \mathbb{R} \quad \text{such that} \quad (x, p + t(q-p)) \in \Omega, \forall t \in [0,1]\right\},$$

$$\mathbf{c}_-(\Omega) := \sup_{(p,q)\in K_-(\Omega)} (q-p).$$

In other words, $\mathbf{c}_-(\Omega)$ is just the maximal length of the vertical segments contained in $\overline{\Omega}$. Next, for $s \in [0, \infty)$ put

$$\varkappa(s) := |\{t > 0 \,|\, t \ln t < s\}|.$$

Let $B_R(\zeta) \subset \mathbb{R}^2$ be the open disk of radius $R > 0$ centered at $\zeta \in \mathbb{C}$. Set

$$K_+(\Omega) := \{(\xi, R) \in \mathbb{R} \times (0, \infty) \,|\, \exists\, \eta \in \mathbb{R} \quad \text{such that} \quad \Omega \subset B_R(\xi + i\eta)\},$$

$$\mathbf{c}_+(\Omega) := \inf_{(\xi,R)\in K_+(\Omega)} R\varkappa\left(\frac{\xi_+}{eR}\right)$$

where $\xi_+ := \max\{\xi, 0\}$. Then it is evident that

$$\mathbf{c}_+(\Omega) \geq \frac{1}{2}\mathrm{diam}\,(\Omega) \geq \frac{1}{2}\mathbf{c}_-(\Omega). \tag{2.2.13}$$

Put

$$\tilde{\Omega}_\pm := \{z \in \Omega_\pm \,|\, \mathrm{Re}\, z > 0\}.$$

Note that (2.2.12) implies $\tilde{\Omega}_\pm \neq \emptyset$.

Theorem 2.2.4. *Suppose that W is a bounded non-decreasing function with $W_- < W_+$, and $x^+ = 0$. Assume that V satisfies* (2.2.5), *and* (2.2.12) *holds true. Then the asymptotic relations*

$$\liminf_{\lambda\downarrow 0} |\ln \lambda|^{-1/2} \mathcal{N}_j^+(\lambda) \geq b^{1/2}\mathcal{C}_- \tag{2.2.14}$$

and

$$\limsup_{\lambda\downarrow 0} |\ln \lambda|^{-1/2} \mathcal{N}_j^+(\lambda) \leq b^{1/2}\mathcal{C}_+ \tag{2.2.15}$$

hold for all $j \in \mathbb{N}$, where $\mathcal{C}_- := (2\pi)^{-1}\mathbf{c}_-(\tilde{\Omega}_-)$ and $\mathcal{C}_+ := e\mathbf{c}_+(\tilde{\Omega}_+)$. In particular,

$$\lim_{\lambda\downarrow 0} \frac{\ln \mathcal{N}_j^+(\lambda)}{\ln |\ln \lambda|} = \frac{1}{2}, \quad j \in \mathbb{N}.$$

Remarks. (i) Under the hypotheses of Theorem 2.2.4 we have $0 < \mathcal{C}_- < \mathcal{C}_+$ due to (2.2.13), $\tilde{\Omega}_- \subset \tilde{\Omega}_+$, and $1/\pi < e$. To prove that the limit $\lim_{\lambda\downarrow 0} |\ln \lambda|^{-1/2} \mathcal{N}_j^+(\lambda)$ exists, and calculate this limit is an interesting open problem from the field of the spectral geometry, related to the ones considered in [24, 62].
(ii) Due to inequality (2.2.14), we have that H has infinitely many eigenvalues in each gap of $\sigma(H)$, when hypotheses of Theorem 2.2.4 are met.

The proof of Theorem 2.2.4 is contained in Section 2.6.

2.3 Proof of Theorem 2.2.1

We prove first a preliminary result. To this end we define the following operators. Pick $j \in \mathbb{N}$, $A \in [-\infty, \infty)$ and $\lambda > 0$, and set

$$\mathcal{P}_j(A) := \mathcal{F}^* \int_{(A,\infty)}^{\oplus} \pi_j(k) dk\, \mathcal{F}, \quad \mathcal{P}_{j,\infty}(A) := \mathcal{F}^* \int_{(A,\infty)}^{\oplus} \pi_{j,\infty}(k) dk\, \mathcal{F},$$

$$T_j(\lambda; A) := \mathcal{F}^* \int_{(A,\infty)}^{\oplus} (\mathcal{E}_j^+ - E_j(k) + \lambda)^{-1/2} \pi_j(k) dk\, \mathcal{F},$$

$$T_{j,\infty}(\lambda; A) := \mathcal{F}^* \int_{(A,\infty)}^{\oplus} (\mathcal{E}_j^+ - E_j(k) + \lambda)^{-1/2} \pi_{j,\infty}(k) dk\, \mathcal{F}.$$

Lemma 2.3.1. *Assume $M \in L_0^\infty(\mathbb{R}^2)$. Then the operator $MT_{j,\infty}(\lambda; A)$ is compact for any $\lambda > 0$ and $A \in [-\infty, \infty)$. Moreover, for any $A_1, A_2 \in [-\infty, \infty)$ the operator*

$$M(T_{j,\infty}(\lambda; A_1) - T_{j,\infty}(\lambda; A_2)) \tag{2.3.1}$$

extends to a uniformly bounded and continuous operator for $\lambda \geq 0$.

Proof. Denote by χ_R the characteristic function of a disk of radius R centered at the origin. For $\lambda > 0$ and $A \in [-\infty, \infty)$ write

$$MT_{j,\infty}(\lambda; A) = \chi_R M T_{j,\infty}(\lambda; A) + (1 - \chi_R) M T_{j,\infty}(\lambda; A). \tag{2.3.2}$$

The first operator at the r.h.s of (2.3.2) is Hilbert-Schmidt for any $R \in (0, \infty)$, and the norm of the second one tends to zero as $R \to \infty$. Hence, the operator $MT_{j,\infty}(\lambda; A)$ is compact. Further, the case $A_1 = A_2$ in (2.3.1) is trivial so that we suppose $A_1 \neq A_2$. Define the value for $\lambda = 0$ of the operator in (2.3.1) as

$$M\mathcal{F}^* \int_{(A_-,A_+)}^{\oplus} (\mathcal{E}_j^+ - E_j(k))^{-1/2} \pi_{j,\infty}(k) dk\, \mathcal{F}$$

with $A_- := \min\{A_1, A_2\}$ and $A_+ := \max\{A_1, A_2\}$. Now the uniform boundedness for $\lambda \geq 0$ of the operator in (2.3.1) follows from the estimates

$$\|M(T_{j,\infty}(\lambda; A_1) - T_{j,\infty}(\lambda; A_2))\| \leq \|M\|_{L^\infty(\mathbb{R}^2)} \sup_{k \in (A_-, A_+]} (\mathcal{E}_j^+ - E_j(k))^{-1/2}, \quad \lambda \geq 0,$$

while the uniform continuity of this operator for $\lambda \geq 0$ follows from the estimates

$$\|M((T_{j,\infty}(\lambda_1; A_1) - T_{j,\infty}(\lambda_1; A_2)) - (T_{j,\infty}(\lambda_2; A_1) - T_{j,\infty}(\lambda_2; A_2)))\| \leq$$

$$|\lambda_1 - \lambda_2| \|M\|_{L^\infty(\mathbb{R}^2)} \sup_{k \in (A_-, A_+]} (\mathcal{E}_j^+ - E_j(k))^{-2}, \quad \lambda_1, \lambda_2 \geq 0.$$

□

Now we are in position to prove Theorem 2.2.1.

The Birman-Schwinger principle (see Proposition 1.3.1) implies

$$\mathcal{N}_j^+(\lambda) = n_-(1; V^{1/2}(H_0 - \mathcal{E}_j^+ - \lambda)^{-1}V^{1/2}) + O(1), \quad \lambda \downarrow 0. \tag{2.3.3}$$

Pick $\tilde{A} \in \mathbb{R}$. Applying the Weyl inequalities (1.2.2), we get

$$n_+(1+s; V^{1/2}(\mathcal{E}_j^+ - H_0 + \lambda)^{-1}\mathcal{P}_{j,\tilde{A}}V^{1/2}) - n_-(s; V^{1/2}(\mathcal{E}_j^+ - H_0 + \lambda)^{-1}(I - \mathcal{P}_{j,\tilde{A}})V^{1/2}) \le$$
$$n_-(1; V^{1/2}(H_0 - \mathcal{E}_j^+ - \lambda)^{-1}V^{1/2}) \le$$
$$n_+(1-s; V^{1/2}(\mathcal{E}_j^+ - H_0 + \lambda)^{-1}\mathcal{P}_{j,\tilde{A}}V^{1/2}) + n_+(s; V^{1/2}(\mathcal{E}_j^+ - H_0 + \lambda)^{-1}(I - \mathcal{P}_{j,\tilde{A}})V^{1/2}), \tag{2.3.4}$$

for any $s \in (0,1)$. By $V \in L_0^\infty(\mathbb{R}^2; \mathbb{R})$ and the diamagnetic inequality, we easily find that

$$n_\pm(s; V^{1/2}(\mathcal{E}_j^+ - H_0 + \lambda)^{-1}(I - \mathcal{P}_{j,\tilde{A}})V^{1/2}) = O(1), \quad \lambda \downarrow 0. \tag{2.3.5}$$

Further, for any $r > 0$ we have

$$n_+(r^2; V^{1/2}(\mathcal{E}_j^+ - H_0 + \lambda)^{-1}\mathcal{P}_{j,\tilde{A}}V^{1/2}) =$$
$$n_+(r^2; V^{1/2}\mathcal{F}^* \int_{(\tilde{A},\infty)}^{\oplus} (\mathcal{E}_j^+ - E_j(k) + \lambda)^{-1}\pi_j(k)dk\mathcal{F}V^{1/2}) =$$
$$n_*(r; V^{1/2}T_j(\lambda; \tilde{A})). \tag{2.3.6}$$

Applying the Ky-Fan inequalities (1.2.3), we obtain

$$n_*(r(1+s); V^{1/2}T_{j,\infty}(\lambda; \tilde{A})) - n_*(rs; V^{1/2}(T_{j,\infty}(\lambda; \tilde{A}) - T_j(\lambda; \tilde{A}))) \le$$
$$n_*(r; V^{1/2}T_j(\lambda; \tilde{A})) \le$$
$$n_*(r(1-s); V^{1/2}T_{j,\infty}(\lambda; \tilde{A})) + n_*(rs; V^{1/2}(T_{j,\infty}(\lambda; \tilde{A}) - T_j(\lambda; \tilde{A}))). \tag{2.3.7}$$

Now note that

$$\|V^{1/2}(T_{j,\infty}(\lambda; \tilde{A}) - T_j(\lambda; \tilde{A}))\| \le$$
$$\|V\|_{L^\infty(\mathbb{R}^2)}^{1/2} \sup_{k > \tilde{A}} (\mathcal{E}_j^+ - E_j(k))^{-1/2} \|\pi_j(k) - \pi_{j,\infty}(k)\|, \tag{2.3.8}$$

uniformly with respect to $\lambda > 0$. By (2.3.8) and Theorem 2.1.3 we find that for each $q > 0$ there exists $A_0 = A_0(q)$ such that $\tilde{A} \ge A_0(q)$ implies

$$\|V^{1/2}(T_{j,\infty}(\lambda; \tilde{A}) - T_j(\lambda; \tilde{A}))\| \le q$$

for each $\lambda > 0$. Choosing $\tilde{A} \ge A_0(rs)$ in (2.3.7) we find then that

$$n_*(rs; V^{1/2}(T_{j,\infty}(\lambda; \tilde{A}) - T_j(\lambda; \tilde{A}))) = 0 \tag{2.3.9}$$

for each $\lambda > 0$. Next, the Ky-Fan inequalities imply that for any $\lambda > 0$, $r > 0$, $s \in (0,1)$ and A, $\tilde{A}$, we have

$$n_*(r(1+s); V^{1/2}T_{j,\infty}(\lambda; A)) - n_*(rs; V^{1/2}(T_{j,\infty}(\lambda; A) - T_{j,\infty}(\lambda; \tilde{A}))) \le$$

$$n_*(r; V^{1/2}T_{j,\infty}(\lambda; \tilde{A})) \leq$$
$$n_*(r(1-s); V^{1/2}T_{j,\infty}(\lambda; A)) + n_*(rs; V^{1/2}(T_{j,\infty}(\lambda; A) - T_{j,\infty}(\lambda; \tilde{A}))). \tag{2.3.10}$$

By Lemma 2.3.1 we have

$$n_*(r; V^{1/2}(T_{j,\infty}(\lambda; A) - T_{j,\infty}(\lambda; \tilde{A}))) = O(1), \quad \lambda \downarrow 0, \tag{2.3.11}$$

for any fixed $r > 0$.

Finally note that

$$T_{j,\infty}(\lambda; A)\mathcal{P}_{j;\infty}(A) = T_{j,\infty}(\lambda; A)$$

and define the operator $\mathcal{W}_0 : L^2(A, \infty) \to \mathcal{P}_{j;\infty}(A)L^2(\mathbb{R}^2)$, by $(\mathcal{W}_0 v)(x, y) := (\mathcal{F}^* \tilde{v}\psi_{j,\infty})(x, y)$, where $\tilde{v}(k)$ is the extension by zero of $v(k)$ to the whole real axis. It is easy to see that $\mathcal{W}_0$ is unitary and that

$$V^{1/2}T_{j;\infty}(A)\mathcal{W}_0 = S_j(\lambda, A),$$

then

$$n_*(r; V^{1/2}T_{j,\infty}(\lambda; A)) = n_*(r; S_j(\lambda; A)) \tag{2.3.12}$$

for each $r > 0$, $\lambda > 0$. Putting together (2.3.3) – (2.3.7), and (2.3.9) – (2.3.12), we obtain (2.2.2).

The proof of Theorem 2.2.1 is now complete.

2.4 Proof of Theorem 2.2.2

This proof needs a technical result, Lemma 2.4.1 below, and an intermediate step, Proposition 2.4.2 below.

Define the operators F_l, $l = 0, 1$. by

$$(F_l v)(z) := \int_{\mathbb{R}} e^{zk} k^l v(k) dk, \quad v \in C_0^\infty(\mathbb{R}), \quad z \in \mathbb{C}.$$

Note that $F_l v$ are entire functions in $\mathbb{C}$, and $(F_1 v)(z) = \frac{\partial (F_0 v)}{\partial z}(z)$. Moreover, the operators F_l can be extended as continuous operators from $\mathcal{D}'_{\text{comp}}(\mathbb{R})$, the space of compactly supported distributions, dual to $C^\infty(\mathbb{R})$, into the space of functions entire in $\mathbb{C}$. Set

$$f_l^\pm[v] := \int_{\Omega_\pm} e^{-bx^2} |(F_l v)(x + iy)|^2 dx dy, \quad v \in C_0^\infty(A, \infty), \quad l = 0, 1.$$

Denote by $D[f_1^\pm]$ the closure of $C_0^\infty(A, \infty)$ in the norm generated by the quadratic form $f_1^\pm$.

Lemma 2.4.1. *The quadratic form $f_0^\pm$ is closable in $D[f_1^\pm]$, and the operator $\mathbb{F}^\pm$ generated by its closure, is compact in $D[f_1^\pm]$.*

Proof. Consider $D[f_1^\pm + f_0^\pm]$, the closure of $C_0^\infty(A, \infty)$ in the norm generated by the quadratic form $f_1^\pm + f_0^\pm$. The quadratic form $f_0^\pm$ is bounded, and hence closable in $D[f_1^\pm + f_0^\pm]$. Denote by $\tilde{\mathbb{F}}^\pm$ the operator generated by its closure in $D[f_1^\pm + f_0^\pm]$. For $v \in C_0^\infty(A, \infty)$ set

$$w(x, y) := (F_0 v)(x + iy), \quad x + iy \in \mathbb{C}.$$

Then we have

$$f_0^{\pm}[v] = \int_{\Omega_\pm} e^{-bx^2}|w|^2 dxdy, \quad f_1^{\pm}[v] = 2\int_{\Omega_\pm} e^{-bx^2}|\nabla w|^2 dxdy. \tag{2.4.1}$$

Since the $\Omega_\pm$ is a bounded domain with a Lipschitz boundary, the Sobolev space $\mathrm{H}^1(\Omega_\pm)$ is compactly embedded in $L^2(\Omega_\pm)$. Hence, (2.4.1) implies that $\tilde{\mathbb{F}}^\pm$ is compact.
Let us now check that $\|\tilde{\mathbb{F}}^\pm\| < 1$. Evidently, $\|\tilde{\mathbb{F}}^\pm\| \leq 1$. Assume $\|\tilde{\mathbb{F}}^\pm\| = 1$. Since $\tilde{\mathbb{F}}^\pm$ is compact, this means that there exists $0 \neq v^\pm \in D[f_1^\pm + f_0^\pm]$ such that $f_1^\pm[v] = 0$. Let $\{v_n^\pm\}_{n\in\mathbb{N}}$ be a sequence of functions $v_n^\pm \in C_0^\infty(A,\infty) \subset C_0^\infty(\mathbb{R})$ converging to $v^\pm$ in $D[f_1^\pm + f_0^\pm]$. Set $w_n^\pm(z) = (F_0 v_n^\pm)(z)$. Evidently, for any $n \in \mathbb{N}$ we have $w_n^\pm \in \mathcal{B}(\Omega_\pm)$ (see (2.2.6)). Since $\mathcal{B}(\Omega_\pm)$ is complete, there exists $w^\pm \in \mathcal{B}(\Omega_\pm)$ such that $\lim_{n\to\infty}\|w_n^\pm - w^\pm\|_{\mathcal{B}(\Omega_\pm)} = 0$. Since $(F_1 v_n^\pm)(z) = \frac{\partial w_n^\pm}{\partial z}$, it is not difficult to check that $f_1^\pm[v^\pm] = 0$ implies that $w^\pm$ is constant in $\Omega_\pm$ (see e.g. [30, Theorem 2, Exercise 1]), and hence $w^\pm$ admits a unique analytic extension as a constant to $\mathbb{C}$. Then the distributional Paley-Wiener theorem (see e.g. [35, Theorem 1.7.7]) combined with [72, Theorem V.11] implies that $v^\pm$ is proportional to the Dirac δ-function supported at $k = 0$. Since $\operatorname{supp} v^\pm \subset [A,\infty)$ and $A > 0$ we conclude that $v^\pm = 0$ as an element of $\mathcal{D}'(\mathbb{R})$, and hence $f_1^\pm[v^\pm] + f_0^\pm[v^\pm] = 0$, which contradicts with the hypothesis that $v^\pm \neq 0$ as an element of $D[f_1^\pm + f_0^\pm]$. Therefore, $\|\tilde{\mathbb{F}}^\pm\| < 1$, and the quadratic form $f_0^\pm$ is bounded, and hence closable in $D[f_1^\pm]$. Finally, the operator $\mathbb{F}^\pm$ generated by its closure is unitarily equivalent to $(I - \tilde{\mathbb{F}}^\pm)^{-1}\tilde{\mathbb{F}}^\pm$ and therefore is compact in $D[f_1^\pm]$. □

Without lost of generality, assume that $x^+ = 0$. Define the non-decreasing functions

$$W_0^-(x) := \begin{cases} W_+ & \text{if} \quad x > 0, \\ W_- & \text{if} \quad x \leq 0, \end{cases}$$

$$W_0^+(x) = W_0^+(x;\delta) := \begin{cases} W_+ & \text{if} \quad x \geq -\delta, \\ W(-\delta) & \text{if} \quad x < -\delta, \end{cases} \qquad \delta > 0.$$

Since $x^+ = 0$ and $\delta > 0$ we have

$$W_0^-(x) \leq W(x) \leq W_0^+(x;\delta), \quad x \in \mathbb{R} \tag{2.4.2}$$

which together with the mini-max principle immediately implies

$$E_j(k;W_0^-) \leq E_j(k;W) \leq E_j(k;W_0^+). \tag{2.4.3}$$

Set

$$\omega_\pm := \{x \in \mathbb{R} \,|\, \text{there exists } y \in \mathbb{R} \text{ such that } (x,y) \in \Omega_\pm\}. \tag{2.4.4}$$

Let $\lambda > 0$, $A \in [-\infty,\infty)$. Fix $j \in \mathbb{N}$. Define $Q_j^\pm(\lambda;A) : L^2(A,\infty) \to L^2(\Omega_\pm)$ as the operator with integral kernel

$$\left(\frac{p_j}{2\pi}\right)^{1/2} e^{iky} e^{-(b^{1/2}x - b^{-1/2}k)^2/2} \left(\mathcal{E}_j^+ - E_j(k;W_0^\pm) + \lambda\right)^{-1/2} (-k)^{j-1}, \tag{2.4.5}$$

with $k \in (A,\infty)$, $(x,y) \in \Omega_\pm$, the number p_j being defined in (2.1.11).

Proposition 2.4.2. *Suppose that W is a bounded non-decreasing function with $W_- < W_+$. Assume that $V \in L_0^\infty(\mathbb{R}^2;\mathbb{R})$ satisfies* (2.2.5). *Then for every $A > 0$, $r > 0$, and $\varepsilon \in (0,1)$, we have*

$$n_*(r(1+\varepsilon); \sqrt{c_0^-}Q_j^-(\lambda; A)) + O(1) \leq n_*(r; S_j(\lambda; A)) \leq n_*(r(1-\varepsilon); \sqrt{c_0^+}Q_j^+(\lambda; A)) + O(1), \tag{2.4.6}$$

as $\lambda \downarrow 0$, where $S_j(\lambda; A)$ is the operator defined by (2.2.1), *and $c_0^\pm$ are the constants occurring in* (2.2.5).

Proof. Inequalities (2.2.5) and (2.4.3), combined with the min-max principle, imply the estimates

$$n_*(r; \sqrt{c_0^-}\tilde{S}_j^-(\lambda; A)) \leq n_*(r; S_j(\lambda; A)) \leq n_*(r; \sqrt{c_0^+}\tilde{S}_j^+(\lambda; A)) \tag{2.4.7}$$

where $\tilde{S}_j^\pm(\lambda; A) : L^2(A,\infty) \to L^2(\Omega_\pm)$ is the operator with integral kernel

$$(2\pi)^{-1/2}e^{iky}\psi_{j,\infty}(x;k)(\mathcal{E}_j^+ - E_j(k; W_0^\pm) + \lambda)^{-1/2}, \quad k \in \mathbb{R}, \quad (x,y) \in \Omega_\pm.$$

In the case $j = 1$ inequality (2.4.7) yields immediately (2.4.6) since in this case we have $\tilde{S}_1^\pm(\lambda; A) = Q_1^\pm(\lambda; A)$. Assume $j \geq 2$. Then we have

$$\psi_{j,\infty}(x;k) = p_j^{1/2}\sum_{l=0}^{j-1} P_{l,j}(x)(-k)^{j-1-l}e^{-(b^{-1/2}k-b^{1/2}x)^2/2}, \quad x \in \mathbb{R}, \quad k \in \mathbb{R}, \tag{2.4.8}$$

where $P_{l,j}$ is a polynomial of degree less than or equal to l, and $P_{0,j} = 1$. Therefore,

$$\tilde{S}_j^\pm(\lambda; A) = \sum_{l=0}^{j-1} P_{l,j}Q_j^\pm(\lambda; A)B^l$$

where the operator $B : L^2(A,\infty) \to L^2(A,\infty)$ with $A > 0$ is defined by

$$(Bu)(k) = k^{-1}u(k), \quad k \in (A,\infty), \quad u \in L^2(A,\infty).$$

Further, for each $u \in L^2(A,\infty)$ and $\eta \in (0,1)$, we have

$$\|\tilde{S}_j^-(\lambda; A)u\|^2_{L^2(\Omega_-)} = \int_{\Omega_-}\left|\sum_{l=0}^{j-1} P_{l,j}(x)(Q_j^-(\lambda, A)B^l u)(x,y)\right|^2 dxdy \geq$$

$$(1-\eta)\int_{\Omega_-}\left|P_{0,j}(x)^2(Q_j^-(\lambda, A)B^l u)(x,y)\right|^2 dxdy -$$

$$(\eta^{-1}-1)\int_{\Omega_-}\left|\sum_{l=1}^{j-1} P_{l,j}(x)(Q_j^-(\lambda, A)B^l u)(x,y)\right|^2 dxdy \geq$$

$$(1-\eta)\int_{\Omega_-}\left|P_{0,j}(x)^2(Q_j^-(\lambda, A)B^l u)(x,y)\right|^2 dxdy -$$

$$(\eta^{-1}-1)(j-1)c_2^-\sum_{l=1}^{j-1}\int_{\Omega_-}\left|(Q_j^-(\lambda,A)B^l u)(x,y)\right|^2 dxdy$$

with $c_2^- := \max_{l=1,\ldots,j-1}\sup_{x\in\omega_-} P_{l,j}(x)^2$, the set ω_- being defined in (2.4.4). Therefore,

$$\tilde{S}_j^-(\lambda;A)^*\tilde{S}_j^-(\lambda;A) \geq$$

$$(1-\eta)Q_j^-(\lambda;A)^*Q_j^-(\lambda;A)-(j-1)(\eta^{-1}-1)c_2^-\sum_{l=1}^{j-1}B^lQ_j^-(\lambda;A)^*Q_j^-(\lambda;A)B^l. \quad (2.4.9)$$

Similarly,

$$\tilde{S}_j^+(\lambda;A)^*\tilde{S}_j^+(\lambda;A) \leq$$

$$(1+\eta)Q_j^+(\lambda;A)^*Q_j^+(\lambda;A)+(j-1)(\eta^{-1}+1)c_2^+\sum_{l=1}^{j-1}B^lQ_j^+(\lambda;A)^*Q_j^+(\lambda;A)B^l \quad (2.4.10)$$

with $\eta>0$, and $c_2^+ := \max_{l=1,\ldots,j-1}\sup_{x\in\omega_+} P_{l,j}(x)^2$. Let us consider now the quadratic forms

$$a_l^\pm[u] = a_l^\pm[u;\lambda,j] := \frac{2\pi}{p_j}\|Q_j^\pm(\lambda;A)B^lu\|^2_{L^2(\Omega_\pm)} =$$

$$\int_{\Omega_\pm} e^{-bx^2}\left|\int_A^\infty e^{k(x+iy)}e^{-b^{-1}k^2/2}\left(\mathcal{E}_j^+ - E_j(k;W_0^\pm)+\lambda\right)^{-1/2}(-k)^{j-l-1}u(k)dk\right|^2 dxdy \quad (2.4.11)$$

with $u\in C_0^\infty(A,\infty)$, $\lambda>0$, $j\geq 2$, $l=0,\ldots,j-2$. Evidently, $a_l^\pm[u]\geq 0$, and $a_l^\pm[u]=0$ implies $u=0$. Denote by $D[a_l^\pm]$, $l=0,\ldots,j-2$, the completion of $C_0^\infty(A,\infty)$ in the norm generated by $a_l^\pm$.
Further, for $j\geq 2$, $l=0,\ldots,j-2$, and $\lambda>0$, define the operator $\mathcal{U}_{j,l,\lambda}$ by

$$(\mathcal{U}_{j,l,\lambda}^\pm u)(k) := e^{-b^{-1}k^2/2}\left(\mathcal{E}_j^+ - E_j(k;W_0^\pm)+\lambda\right)^{-1/2}k^{j-l-2}u(k), \quad k\in(A,\infty).$$

Note that the mapping $\mathcal{U}_{j,l,\lambda}^\pm : C_0^\infty(A,\infty)\to C_0^\infty(A,\infty)$ is bijective, and we have

$$a_l^\pm[u] = f_1^\pm[\mathcal{U}_{j,l,\lambda}^\pm u], \quad a_{l+1}^\pm[u] = f_0^\pm[\mathcal{U}_{j,l,\lambda}^\pm u], \quad u\in C_0^\infty(A,\infty), \quad l=0,\ldots,j-2. \quad (2.4.12)$$

It follows from Lemma 2.4.1 that the quadratic form $a_{l+1}^\pm$ is closable in $D[a_l^\pm]$, $l=0,\ldots,j-2$. Denote by $\mathbb{A}_l^\pm$ the operator generated in $D[a_l^\pm]$ by the closure of the quadratic form $a_{l+1}^\pm$. Since $\mathbb{A}_l^\pm = (\mathcal{U}_{j,l,\lambda}^\pm)^{-1}\mathbb{F}^\pm\mathcal{U}_{j,l,\lambda}^\pm$, i.e. the operator $\mathbb{A}_l^\pm$ is unitarily equivalent to $\mathbb{F}^\pm$, and $\mathbb{F}^\pm$ does not depend on λ, we find that $\sigma(\mathbb{A}_l^\pm)$ is independent of λ. Moreover, since $\mathbb{F}^\pm$ is compact by Lemma 2.4.1, we find that the operator $\mathbb{A}_l^\pm$ is compact as well.
Now it follows easily from (2.4.9) - (2.4.10) that for each $\varepsilon\in(0,1)$ there exist subspaces $\mathcal{H}_\pm$ of $C_0^\infty(A,\infty)$ such that the codimensions $\operatorname{codim}\mathcal{H}_\pm$ are finite and independent of λ, and

$$\|\tilde{S}_j^-(\lambda;A)u\|^2 \geq (1+\varepsilon)^{-2}\|Q_j^-(\lambda;A)u\|^2, \quad u\in\mathcal{H}_-, \quad (2.4.13)$$

$$\|\tilde{S}_j^+(\lambda;A)u\|^2 \leq (1-\varepsilon)^{-2}\|Q_j^+(\lambda;A)u\|^2, \quad u\in\mathcal{H}_+. \quad (2.4.14)$$

Combining (2.4.13) - (2.4.14) with standard variational arguments (see. e.g. [11, Lemma 1.13] and the proof of [11, Lemma 1.16]), we get

$$n_*(r; \tilde{S}_j^-(\lambda; A)) \geq n_*(r(1+\varepsilon); Q_j^-(\lambda; A)) - \operatorname{codim} \mathcal{H}_-, \tag{2.4.15}$$

$$n_*(r; \tilde{S}_j^+(\lambda; A)) \leq n_*(r(1-\varepsilon); Q_j^+(\lambda; A)) + \operatorname{codim} \mathcal{H}_+. \tag{2.4.16}$$

Putting together (2.4.15) - (2.4.16) and (2.4.7), we arrive at (2.4.6). □

Now we are in position to prove Theorem 2.2.2.

By Theorem 2.2.1 and Proposition 2.4.2 we only need to prove that for every $r > 0$, $A > 0$, $\delta \in (0, 1/2)$, and $\varepsilon \in (0,1)$

$$n_*(r; Q_j^-(\lambda; A)) \geq n_*(r(1+\varepsilon)\sqrt{W_+ - W_-}; \Gamma_\delta^-(\sqrt{b|\ln\lambda|})) + O(1), \tag{2.4.17}$$

$$n_*(r; Q_j^+(\lambda; A)) \leq n_*(r(1-\varepsilon)\sqrt{W_+ - W(-\delta)}e^{-b\delta^2/2}; \Gamma_\delta^+(\sqrt{b|\ln\lambda|}) + O(1), \tag{2.4.18}$$

as $\lambda \downarrow 0$.

Let $\lambda > 0$, $A \in [-\infty, \infty)$. Define the operators $M_{j,1}^{\pm}(\lambda; A) : L^2(\Omega_\pm) \to L^2(\Omega_\pm)$ as the operators with integral kernels

$$\frac{p_j}{2\pi} e^{-b(x^2+x'^2)/2} \int_A^\infty (\mathcal{E}_j^+ - E_j(k; W_0^\pm) + \lambda)^{-1} k^{2(j-1)} e^{-b^{-1}k^2} e^{k(x+x'+i(y-y'))} dk \tag{2.4.19}$$

with (x, y), $(x', y') \in \Omega_\pm$. Evidently, $Q_j^\pm(\lambda; A) Q_j^\pm(\lambda; A)^* = M_{j,1}^\pm(\lambda; A)$. Therefore,

$$n_+(r; Q_j^\pm(\lambda; A)^* Q_j^\pm(\lambda; A)) = n_+(r; M_{j,1}^\pm(\lambda; A)), \quad r > 0. \tag{2.4.20}$$

In the rest of the proof of the proposition we just show by successive simplifications that we can replace the operators $M_{j,1}^\pm(\lambda; A)$ by their "asymptotic values" as $\lambda \downarrow 0$, namely the operators

$$\text{const.}\, \Gamma_\delta^\pm(\sqrt{b|\ln\lambda|})\, \Gamma_\delta^\pm(\sqrt{b|\ln\lambda|})^*.$$

The main ideas of these steps are inspired by the elementary asymptotic analysis as $\lambda \downarrow 0$ of the integral in (2.4.19); here we apply essentially the results of Propositions 2.1.6 and 2.1.7 on the asymptotics of $E_j(k)$ as $k \to \infty$. The technical details of the proof become somewhat tedious since we need to ensure an adequate control on the differences of the eigenvalue counting functions for the successive approximations.

First, we concentrate at the proof of (2.4.17). Fix $\varepsilon > 0$. Then by (2.1.35) there exists $A_0^- = A_0^-(\varepsilon)$ such that $k \geq A_0^-$ implies

$$\mathcal{E}_j^+ - E_j(k; W_0^-) \leq (1+\varepsilon)\frac{W_+ - W_-}{2} p_j k^{2j-3} e^{-b^{-1}k^2}. \tag{2.4.21}$$

For $p > 0$ and $A > 0$ define $M_{j,2}^-(\lambda, A, p) : L^2(\Omega_-) \to L^2(\Omega_-)$ as the operator with integral kernel

$$\frac{p_j}{2\pi} e^{-b(x^2+x'^2)/2} \int_A^\infty (p + \lambda k^{3-2j} e^{b^{-1}k^2})^{-1} k e^{k(x+x'+i(y-y'))} dk \tag{2.4.22}$$

with (x,y), $(x',y') \in \Omega_-$. Then (2.4.21) implies that for $A_1 = \max\{A, A_0^-\}$ we have

$$n_+(r; M_{j,1}^-(\lambda; A)) \geq n_+\left(r; M_{j,2}^-(\lambda, A_1, p_j(1+\varepsilon)(W_+ - W_-)/2)\right). \tag{2.4.23}$$

Fix $\delta \in (0,1/2)$. Set $\Lambda := |\ln\lambda|^{1/2}$, and assume that $\lambda > 0$ is small that $A_1 < \delta\sqrt{b}\Lambda$. Then, by the mini-max principle,

$$n_+(r; M_{j,2}^-(\lambda, A_1, p)) \geq n_+(r; M_{j,2}^-(\lambda, \delta\sqrt{b}\Lambda, p)), \quad p > 0, \quad r > 0. \tag{2.4.24}$$

In the integral defining the kernel of the operator $M_{j,2}^-(\lambda, \delta\sqrt{b}\Lambda, p)$ (see (2.4.22)), change the variable $k = \sqrt{b}\Lambda(1+u)^{1/2}$ with $u \in (-1+\delta^2, \infty)$. Then we see that the integral kernel of $M_{j,2}^-(\lambda, \delta\sqrt{b}\Lambda, p)$ is equal to

$$\frac{p_j b\Lambda^2}{4\pi} e^{-b(x^2+x'^2)/2} \int_{-1+\delta^2}^{\infty} (p + (\sqrt{b}\Lambda(1+u)^{1/2})^{3-2j} e^{\Lambda^2 u})^{-1} e^{(x+x'+i(y-y'))\sqrt{b}\Lambda(1+u)^{1/2}} du.$$

Define $M_{j,3}^-(\lambda, \delta, p) : L^2(\Omega_-) \to L^2(\Omega_-)$ as the operator with integral kernel

$$\frac{p_j b\Lambda^2}{4\pi} e^{-b(x^2+x'^2)/2} \int_{-1+\delta^2}^{-1+(1-\delta)^2} (p + (\sqrt{b}\Lambda(1+u)^{1/2})^{3-2j} e^{\Lambda^2 u})^{-1} e^{(x+x'+i(y-y'))\sqrt{b}\Lambda(1+u)^{1/2}} du$$

with (x,y), $(x',y') \in \Omega_-$. Evidently, the mini-max principle implies

$$n_+(r; M_{j,2}^-(\lambda, \delta\sqrt{b}\Lambda, p)) \geq n_+(r; M_{j,3}^-(\lambda, \delta, p)), \quad p > 0, \quad r > 0, \quad \delta \in (0, 1/2). \tag{2.4.25}$$

Further, define $M_{j,4}^-(\lambda, \delta, p) : L^2(\Omega_-) \to L^2(\Omega_-)$ as the operator with integral kernel

$$\frac{p_j b\Lambda^2}{4\pi p} e^{-b(x^2+x'^2)/2} \int_{-1+\delta^2}^{-1+(1-\delta)^2} e^{(x+x'+i(y-y'))\sqrt{b}\Lambda(1+u)^{1/2}} du \tag{2.4.26}$$

with (x,y), $(x',y') \in \Omega_-$. By the dominated convergence theorem,

$$\lim_{\lambda\downarrow 0} \|M_{j,3}^-(\lambda, \delta, p) - M_{j,4}^-(\lambda, \delta, p)\|_2^2 = 0$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm. Fix $\varepsilon > 0$. Applying the Weyl inequalities and the Chebyshev-type estimate (1.2.4), with $p = 2$

$$n_*(s; M_{j,3}^-(\lambda, \delta, p) - M_{j,4}^-(\lambda, \delta, p)) \leq s^{-2}\|M_{j,3}^-(\lambda, \delta, p) - M_{j,4}^-(\lambda, \delta, p)\|_2^2, \quad s > 0,$$

we get

$$n_+(r; M_{j,3}^-(\lambda, \delta, p)) \geq n_+(r(1+\varepsilon); M_{j,4}^-(\lambda, \delta, p)) + O(1), \quad \lambda \downarrow 0. \tag{2.4.27}$$

In the integral defining the kernel of the operator $M_{j,4}^-(\lambda, \delta, 1)$ (see (2.4.26)), change the variable $(1+u)^{1/2} = k$ with $k \in (\delta, 1-\delta)$. Then we see that the integral kernel of $M_{j,4}^-(\lambda, \delta, 1)$ equals

$$\frac{p_j b\Lambda^2}{2\pi p} e^{-b(x^2+x'^2)/2} \int_{\delta}^{1-\delta} e^{(x+x'+i(y-y'))\sqrt{b}\Lambda k} k dk, \quad (x,y), \ (x',y') \in \Omega_-.$$

Therefore

$$M_{j,4}^-(\lambda,\delta,p)=\frac{p_j}{2p}\Gamma_\delta^-(\sqrt{b|\ln\lambda|})\Gamma_\delta^-(\sqrt{b|\ln\lambda|})^*. \tag{2.4.28}$$

Combining now (2.4.20), (2.4.23), (2.4.24), (2.4.25), (2.4.27), and (2.4.28), we obtain (2.4.17).
Let us now prove (2.4.18). The proof is quite similar to that of (2.4.17), so that we omit certain details. Set $\nu_1=0$ and $\nu_j=1$ if $j\in\mathbb{N}$, $j\geq 2$. Pick $\varepsilon\in(0,1)$. Then there exists $A_0^+=A_0^+(\varepsilon)$ such that $k\geq A_0^+$ implies

$$\mathcal{E}_j^+-E_j(k;W_0^+)\geq(1-\varepsilon)\frac{W_+-W(-\delta)}{2}p_j(k+\nu_j)^{2j-3}e^{-(b^{-1/2}k+b^{1/2}\delta)^2}. \tag{2.4.29}$$

For $p>0$ and $A>0$ define $M_{j,2}^+(\lambda,A,p):L^2(\Omega_+)\to L^2(\Omega_+)$ as the operator with integral kernel

$$\frac{p_j}{2\pi}e^{-b(x^2+x'^2)/2}\int_A^\infty(p+\lambda(k+\nu_j)^{3-2j}e^{b^{-1}k^2+2\delta k})^{-1}ke^{k(x+x'+i(y-y')+2\delta)}dk, \tag{2.4.30}$$

with (x,y), $(x',y')\in\Omega_+$. Therefore, similarly to (2.4.23), we have

$$n_+(r;M_{j,1}^+(\lambda;A))\leq n_+\left(r;M_{j,2}^+(\lambda,A_1,(1-\varepsilon)p_je^{-b\delta^2}(W_+-W(-\delta))/2)\right) \tag{2.4.31}$$

for $A_1=\max\{A,A_0^+\}$. Moreover, it is easy to check that

$$n_+(r;M_{j,2}^+(\lambda,A,p))=n_+(r;M_{j,2}^+(\lambda,0,p))+O(1),\quad \lambda\downarrow 0, \tag{2.4.32}$$

for any $A\geq 0$, $p>0$. In the integral defining the kernel of the operator $M_{j,2}^+(\lambda,0,p)$ (see (2.4.30)), change the variable $k=\sqrt{b}\Lambda(1+u)^{1/2}$ with $u\in(-1,\infty)$. Then we see that the integral kernel of $M_{j,2}^+(\lambda,0,p)$ is equal to

$$\frac{p_jb\Lambda^2}{4\pi}e^{-b(x^2+x'^2)/2}$$

$$\int_{-1}^\infty(p+(\sqrt{b}\Lambda(1+u)^{1/2}+\nu_j)^{3-2j}e^{\Lambda^2u+2\delta\sqrt{b}\Lambda(1+u)^{1/2}})^{-1}e^{(x+x'+i(y-y')+2\delta)\sqrt{b}\Lambda(1+u)^{1/2}}du.$$

Define now $M_{j,3}^+(\lambda,\delta,p):L^2(\Omega_+)\to L^2(\Omega_+)$, as the operator with integral kernel

$$\frac{p_jb\Lambda^2}{4\pi}e^{-b(x^2+x'^2)/2}$$

$$\int_{-1}^{-1+(1+\delta)^2}(p+(\sqrt{b}\Lambda(1+u)^{1/2}+\nu_j)^{3-2j}e^{\Lambda^2u+2\delta\sqrt{b}\Lambda(1+u)^{1/2}})^{-1}e^{(x+x'+i(y-y')+2\delta)\sqrt{b}\Lambda(1+u)^{1/2}}du$$

with (x,y), $(x',y')\in\Omega_+$. By the dominated convergence theorem,

$$\lim_{\lambda\downarrow 0}\|M_{j,2}^+(\lambda,\delta,p)-M_{j,3}^+(\lambda,\delta,p)\|_2^2=0.$$

Therefore, similarly to (2.4.27), we obtain

$$n_+(r;M_{j,2}^+(\lambda,\delta,p))\leq n_+(r(1-\varepsilon);M_{j,3}^+(\lambda,\delta,p))+O(1),\quad \lambda\downarrow 0, \tag{2.4.33}$$

for any $r > 0$, $\varepsilon \in (0, 1)$, $\delta > 0$, $p > 0$. Next, define $M_{j,4}^{+}(\lambda, \delta, p) : L^2(\Omega_+) \to L^2(\Omega_+)$, $\delta > 0$, as the operator with integral kernel

$$\frac{bp_j\Lambda^2}{4\pi p} e^{-b(x^2+x'^2)/2} \int_{-1}^{-1+(1+\delta)^2} e^{(x+x'+i(y-y')+2\delta)\sqrt{b}\Lambda(1+u)^{1/2}} du, \quad (x, y), (x', y') \in \Omega_+.$$

Evidently, the mini-max principle implies

$$n_+(r; M_{j,3}^{+}(\lambda, \delta, p)) \le n_+(r; M_{j,4}^{+}(\lambda, \delta, p)), \quad r > 0. \tag{2.4.34}$$

Finally, by analogy with (2.4.28), we get

$$M_{j,4}^{+}(\lambda, \delta, p) = \frac{p_j}{2p} \Gamma_\delta^{+}(\sqrt{b|\ln\lambda|}) \Gamma_\delta^{+}(\sqrt{b|\ln\lambda|})^*. \tag{2.4.35}$$

Putting together (2.4.20) and (2.4.31) – (2.4.35), we arrive at (2.4.18).

The proof of Theorem 2.2.2 is now complete.

2.5 Proof of Theorem 2.2.3

By Theorem 2.2.1, it suffices to show that

$$n_*(r; S_j(\lambda; A)) = O(1), \quad \lambda \downarrow 0, \tag{2.5.1}$$

for any fixed $r > 0$ and $A \in [-\infty, \infty)$. We have

$$n_*(r; S_j(\lambda; A)) \le r^{-2} \operatorname{Tr} S_j(\lambda; A)^* S_j(\lambda; A) = \frac{1}{2\pi r^2} \mathcal{I}_0(\lambda) \tag{2.5.2}$$

where

$$\mathcal{I}_0(\lambda) := \int_A^\infty \int_{\mathbb{R}^2} (\mathcal{E}_j^{+} - E_j(k; b, W) + \lambda)^{-1} \psi_{j,\infty}(x; k)^2 V(x, y) dx dy dk.$$

Now pick $\tilde{x} \in (X^+, x^+)$ which is possible due to (2.2.10), and set

$$\tilde{W}(x) = \begin{cases} W_+ & \text{if} \quad x \ge \tilde{x}, \\ W(\tilde{x}) & \text{if} \quad x < \tilde{x}. \end{cases} \tag{2.5.3}$$

Since $W(x) \le \tilde{W}(x)$, $x \in \mathbb{R}$, the mini-max principle implies

$$(\mathcal{E}_j^{+} - E_j(k; b, W) + \lambda)^{-1} \le (\mathcal{E}_j^{+} - E_j(k; b, \tilde{W}) + \lambda)^{-1}, \quad k \in \mathbb{R}, \quad j \in \mathbb{N}, \quad \lambda > 0.$$

Therefore,

$$\mathcal{I}_0(\lambda) \le \left(\operatorname*{ess\,sup}_{x \in \mathbb{R}} \int_{\mathbb{R}} V(x, y) dy \right) \mathcal{I}_1(\lambda) \tag{2.5.4}$$

where

$$\mathcal{I}_1(\lambda) := \int_A^\infty \int_{X^-}^{X^+} (\mathcal{E}_j^{+} - E_j(k; b, \tilde{W}) + \lambda)^{-1} \psi_{j,\infty}(x; k)^2 dx dk.$$

Taking into account (2.1.12), (2.1.22), and (2.1.31), and bearing in mind that the interval $[X^-, X^+]$ is compact, we find that for sufficiently large $A > 0$ and any $\lambda \geq 0$ we have

$$\mathcal{I}_1(\lambda) \leq 4(W_+ - W(\tilde{x}))^{-1} \max_{x\in[X^-,X^+]} e^{-b(x^2-\tilde{x}^2)} \int_A^\infty \int_{X^-}^{X^+} ke^{-2k(\tilde{x}-x)}dxdk$$

$$\leq 2(W_+ - W(\tilde{x}))^{-1} \max_{x\in[X^-,X^+]} e^{-b(x^2-\tilde{x}^2)} \int_A^\infty e^{-2k(\tilde{x}-X^+)}dk < \infty, \tag{2.5.5}$$

due to $\tilde{x} > X^+$. Putting together (2.5.2) - (2.5.5), we obtain (2.5.1), and hence, (2.2.11).

2.6 Proof of Theorem 2.2.4

2.6.1 Lower Bound

In this subsection we prove (2.2.14). Taking into account Theorem 2.2.2, we find that it suffices to show that for any $r > 0$ independent of $\lambda > 0$, we have

$$\lim_{\delta\downarrow 0} \liminf_{\lambda\downarrow 0} |\ln\lambda|^{-1/2} n_+(r; \Gamma_\delta^-(\sqrt{b|\ln\lambda|})^*\Gamma_\delta^-(\sqrt{b|\ln\lambda|})) \geq \mathcal{C}_-. \tag{2.6.1}$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and $\mathcal{I} \subset (0,\infty)$ be a bounded open non-empty interval. For $m > 0$ and $\delta \geq 0$ define the operator $\mathcal{G}_{m,\delta}(\Omega,\mathcal{I}) : L^2(\mathcal{I}) \to L^2(\mathcal{I})$ as the operator with integral kernel

$$\pi^{-1}m^2\sqrt{kk'}\int_\Omega e^{m((z+\delta)k+(\bar{z}+\delta)k')}d\mu(z), \quad k,k' \in \mathcal{I}, \tag{2.6.2}$$

where $d\mu(z) = dxdy$ is the Lebesgue measure on $\mathbb{R}^2$.

Set

$$\epsilon_- := \inf_{x\in\omega_-} e^{-bx^2}, \quad \epsilon_+ := \sup_{x\in\omega_+} e^{-bx^2}, \tag{2.6.3}$$

the sets $\omega_\pm$ being defined in (2.4.4). Then we have

$$\Gamma_\delta^-(m)^*\Gamma_\delta^-(m) \geq \epsilon_-\mathcal{G}_{m,0}(\Omega_-, I_-(\delta)), \quad m > 0. \tag{2.6.4}$$

Further, let $\mathcal{R} \subset \tilde{\Omega}_- \subset \Omega_-$ be an open non-empty rectangle whose sides are parallel to the coordinate axes. Since a translation $z \mapsto z + i\eta$, $\eta \in \mathbb{R}$, in the integral in (2.6.2) generates a unitary transformation of the operator $\mathcal{G}_{m,0}(\Omega_-, I_-(\delta))$ into an operator unitarily equivalent to it, we assume without any loss of generality that $\mathcal{R} = (\alpha,\beta)\times(-L,L)$ with $0 < \alpha < \beta < \infty$ and $L \in (0,\infty)$. Evidently,

$$\mathcal{G}_{m,0}(\Omega_-, I_-(\delta)) \geq \mathcal{G}_{m,0}(\mathcal{R}, I_-(\delta)), \quad m > 0. \tag{2.6.5}$$

For $\eta \in \mathbb{R}$ and $\delta \in (0, 1/2)$ define the operator $G_{\eta,\delta}^-(m) : L^2(I_-(\delta)) \to L^2(I_-(\delta))$ as the integral operator with kernel

$$e^{\eta m(k+k')}\frac{\sin(m(k-k'))}{\pi(k-k')}\frac{2\sqrt{kk'}}{k+k'}, \quad k,k' \in I_-(\delta).$$

Then

$$\mathcal{G}_{m,0}(\mathcal{R}, I_-(\delta)) = G^-_{\beta,\delta}(mL) - G^-_{\alpha,\delta}(mL). \tag{2.6.6}$$

Define the operator $g_{\mathcal{I}}(m) : L^2(\mathcal{I}) \to L^2(\mathcal{I})$, $m > 0$, as the operator with integral kernel

$$\frac{\sin(m(k-k'))}{\pi(k-k')}\frac{2\sqrt{kk'}}{k+k'}, \quad k, k' \in \mathcal{I}.$$

Note that we have $g_{\mathcal{I}}(m) := \gamma_{\mathcal{I}}(m)^*\gamma_{\mathcal{I}}(m)$ where $\gamma_{\mathcal{I}}(m) : L^2(\mathcal{I}) \to L^2((0,\infty) \times (-m,m))$ is the operator with integral kernel

$$\pi^{-1/2}e^{-(x+iy)k}k^{1/2}, \quad k \in \mathcal{I}, \quad x \in (0,\infty), \quad y \in (-m,m).$$

Evidently, for any finite $m > 0$, the operator $\gamma_{\mathcal{I}}(m)$ is Hilbert-Schmidt, and $\|\gamma_{\mathcal{I}}(m)\| < 1$. Therefore,

$$g_{\mathcal{I}}(m) = g_{\mathcal{I}}(m)^* \geq 0 \tag{2.6.7}$$

is a trace-class operator, and $\|g_{\mathcal{I}}(m)\| < 1$. Simple variational arguments yield

$$n_+(r; G^-_{\beta,\delta}(m) - G^-_{\alpha,\delta}(m)) \geq n_+(r(1-e^{2(\alpha-\beta)\delta m})^{-1}; G^-_{\beta,\delta}(m)) \geq$$

$$n_+(re^{-2\beta\delta m}(1-e^{2(\alpha-\beta)\delta m})^{-1}; g_{I_-(\delta)}(m)), \quad r > 0, \quad \delta \in (0,1/2). \tag{2.6.8}$$

Combining (2.6.4) – (2.6.8), we find that under the hypotheses of Theorem 2.2.4 for each $\delta \in (0,1/2)$ we have

$$n_+(r; \Gamma^-_\delta(m)^*\Gamma^-_\delta(m)) \geq n_+(re^{-2\beta\delta m}(\epsilon_-(1-e^{2(\alpha-\beta)\delta m}))^{-1}; g_{I_-(\delta)}(mL)). \tag{2.6.9}$$

In order to complete the proof of (2.6.1), we need the following

Proposition 2.6.1. *For all $l \in \mathbb{N}$ we have*

$$\lim_{m\to\infty} m^{-1}\mathrm{Tr}\, g_{\mathcal{I}}(m)^l = \frac{|\mathcal{I}|}{\pi}. \tag{2.6.10}$$

Proof. Let $l = 1$. Then, $\mathrm{Tr}\, g_{\mathcal{I}}(m) = \frac{m|\mathcal{I}|}{\pi}$. Let now $l \geq 2$. Set

$$\phi_m(k) := \frac{\sin mk}{\pi k} \quad k \in \mathcal{I}.$$

Denote by $\chi_{\mathcal{I}}$ the characteristic function of the interval $\mathcal{I}$. Then we have

$$\mathrm{Tr}\, g_{\mathcal{I}}(m)^l =$$

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \phi_m(k_1-k_2)\phi_m(k_2-k_3)\ldots\phi_m(k_{l-1}-k_l)\phi_m(k_l-k_1)\times$$

$$\frac{2^l\, k_1\ldots k_l}{(k_1+k_2)(k_2+k_3)\ldots(k_{l-1}+k_l)(k_l+k_1)}\chi_{\mathcal{I}}(k_1)\ldots\chi_{\mathcal{I}}(k_l)dk_1\ldots dk_l.$$

Changing the variables $k_1 = t_1$, $k_j = t_1 + m^{-1}t_j$, $j = 2,\ldots,l$, we get

$$\mathrm{Tr}\, g_{\mathcal{I}}(m)^l =$$

$$m \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \phi_1(-t_2)\phi_1(t_2 - t_3)\ldots\phi_1(t_{l-1} - t_l)\phi_1(t_l) \times$$

$$\frac{2^l t_1(t_1 + m^{-1}t_2)\ldots(t_1 + m^{-1}t_l)}{(2t_1 + m^{-1}t_2)(2t_1 + m^{-1}(t_2 + t_3))\ldots(2t_1 + m^{-1}(t_{l-1} + t_l))(2t_1 + m^{-1}t_l)} \times$$

$$\chi_{\mathcal{I}}(t_1)\chi_{\mathcal{I}}(t_1 + m^{-1}t_2)\ldots\chi_{\mathcal{I}}(t_1 + m^{-1}t_l)dt_1\ldots dt_l.$$

Applying the dominated convergence theorem, we get

$$\lim_{m\to\infty} m^{-1}\mathrm{Tr}\, g_{\mathcal{I}}(m)^l = |\mathcal{I}| \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \phi_1(-t_2)\phi_1(t_2 - t_3)\ldots\phi_1(t_{l-1} - t_l)\phi_1(t_l)dt_2\ldots dt_l. \quad (2.6.11)$$

Further, we have

$$\phi_1(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\xi}\chi_{(-1,1)}(\xi)d\xi, \quad t \in \mathbb{R}.$$

Therefore,

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \phi_1(-t_2)\phi_1(t_2-t_3)\ldots\phi_1(t_{l-1}-t_l)\phi_1(t_l)dt_2\ldots dt_l = \frac{1}{2\pi}\int_{\mathbb{R}} \chi_{(-1,1)}(\xi)^l d\xi = \frac{1}{\pi}. \quad (2.6.12)$$

Putting together (2.6.11) and (2.6.12), we obtain (2.6.10). □

Now Proposition 2.6.1 and estimate (2.6.7) combined with the Kac-Murdock-Szegő theorem (see the original work [41], [29, Section 11.8], or [69, Lemmas 3.1, 3.2]), imply the following

Corollary 2.6.2. *We have*

$$\lim_{m\to\infty} m^{-1} n_+(s; g_{\mathcal{I}}(m)) = \begin{cases} \frac{|\mathcal{I}|}{\pi} & \text{if} \quad s \in (0,1), \\ 0 & \text{if} \quad s > 1. \end{cases} \quad (2.6.13)$$

Now we are in position to prove (2.6.1). Fix arbitrary $s \in (0,1)$. Assume that m is so large that $re^{-2\beta\delta m}(\epsilon_-(1 - e^{2(\alpha-\beta)\delta m}))^{-1} < s$. Then (2.6.9) implies

$$n_+(r; \Gamma_\delta^-(m)^*\Gamma_\delta^-(m)) \geq n_+(s; g_{I_-(\delta)}(mL)). \quad (2.6.14)$$

Putting together (2.6.13) and (2.6.14), we find that the asymptotic estimate

$$\liminf_{\lambda\downarrow 0} |\ln\lambda|^{-1/2} n_+(r; \Gamma_\delta^-(\sqrt{b|\ln\lambda|})^*\Gamma_\delta^-(\sqrt{b|\ln\lambda|})) \geq \frac{\sqrt{b}L}{\pi}(1 - 2\delta)$$

holds for every $\delta \in (0, 1/2)$. Letting $\delta \downarrow 0$, and optimizing with respect to L we obtain (2.6.1).

2.6.2 Upper bound

In this subsection we prove (2.2.15). By analogy with (2.6.1), it suffices to show that for any $r > 0$ independent of $\lambda > 0$, we have

$$\lim_{\delta\downarrow 0}\limsup_{\lambda\downarrow 0} |\ln\lambda|^{-1/2} n_+(r; \Gamma_\delta^+(\sqrt{b|\ln\lambda|})^*\Gamma_\delta^+(\sqrt{b|\ln\lambda|})) \leq \mathcal{C}_+. \quad (2.6.15)$$

Evidently,

$$\Gamma_\delta^+(m)^*\Gamma_\delta^+(m) \le \epsilon_+ \mathcal{G}_{m,\delta}(\Omega_+; I_+(\delta)), \quad m > 0, \tag{2.6.16}$$

the integral kernel of the operator $\mathcal{G}_{m,\delta}(\Omega_+;\mathcal{I})$ being defined in (2.6.2), and the number ϵ_+ being defined in (2.6.3).

Define

$$\tilde{\Omega}_+(\delta) := \{z \in \Omega_+ \,|\, \mathrm{Re}\, z > -2\delta\}$$

for $\delta \ge 0$ so that $\tilde{\Omega}_+(0) = \tilde{\Omega}_+$. Since we have $\mathcal{G}_{m,\delta}(\Omega_+ \setminus \tilde{\Omega}_+(\delta); I_+(\delta)) \ge 0$ and

$$\lim_{m\to\infty} \mathrm{Tr}\, \mathcal{G}_{m,\delta}(\Omega_+ \setminus \tilde{\Omega}_+(\delta); I_+(\delta)) = \pi^{-1} \lim_{m\to\infty} m^2 \int_0^{1+\delta} \int_{\Omega_+\setminus\tilde{\Omega}_+(\delta)} e^{2m(\mathrm{Re}\, z+\delta)k} d\mu(z) k dk = 0,$$

we easily find that the Weyl inequalities entail

$$n_+(r; \mathcal{G}_{m,\delta}(\Omega_+; I_+(\delta))) \le n_+(r(1-\varepsilon); \mathcal{G}_{m,\delta}(\tilde{\Omega}_+(\delta); I_+(\delta))) + O(1), \quad m \to \infty, \tag{2.6.17}$$

for each $r > 0$ and $\varepsilon \in (0,1)$. Further, pick an open disk $B_R(\zeta) \subset \mathbb{R}^2$ such that $\tilde{\Omega}_+(\delta) \subset B_R(\zeta)$. Evidently,

$$n_+(r; \mathcal{G}_{m,\delta}(\tilde{\Omega}_+(\delta); I_+(\delta))) \le n_+(r; \mathcal{G}_{m,\delta}(B_R(\zeta); I_+(\delta))), \quad r > 0. \tag{2.6.18}$$

Next, put $I_* = I_*(\delta) := (0, (1+\delta)^{-1})$, and define $G_\delta^+(m) : L^2(I_*) \to L^2(I_*)$ as the operator with integral kernel

$$\pi^{-1} m^2 e^{2m(\xi+\delta)_+} \int_{B_R(0)} e^{m(zk+\bar{z}k')} d\mu(z), \quad k, k' \in I_*(\delta).$$

Changing the variable $z \mapsto z + \zeta$ in the integral defining the kernel of $\mathcal{G}_{m,\delta}(B_R(\zeta); I_+(\delta))$ (see (2.6.2)), and after that changing the variable $k \mapsto (1+\delta)^2 k$ in $I_*(\delta)$, we find that the mini-max principle implies

$$n_+(r; \mathcal{G}_{m,\delta}(B_R(\zeta); I_+(\delta))) \le n_+(r; G_\delta^+((1+\delta)^2 m)), \quad r > 0, \tag{2.6.19}$$

with $\xi = \mathrm{Re}\, \zeta$. Further, expanding the exponential functions into power series, and passing to polar coordinates, we get

$$\int_{B_R(0)} e^{m(zk+\bar{z}k')} d\mu(z) = \pi R^2 \sum_{q=0}^{\infty} \frac{(m^2 R^2 k k')^q}{(q!)^2 (q+1)}.$$

Therefore, the quadratic form of the operator $G_\delta^+(m)$ can be written as

$$\langle G_\delta^+(m) u, u\rangle_{L^2(I_*)} = e^{2m(\xi+\delta)_+} \sum_{q=0}^{\infty} \frac{(mR)^{2q+2}}{(q!)^2 (q+1)} |\tilde{u}_q|^2 \tag{2.6.20}$$

where

$$\tilde{u}_q = \int_{I_*(\delta)} k^q u(k) dk, \quad u \in L^2(I_*(\delta)), \quad q \in \mathbb{Z}_+.$$

Let $\{p_q(k)\}_{q\in\mathbb{Z}_+}$ be the system of polynomials orthonormal in $L^2(I_*(\delta))$, obtained by the Gram-Schmidt procedure from $\{k^q\}_{q\in\mathbb{Z}_+}$, $k \in I_*(\delta)$. Then,

$$k^q = \sum_{l=0}^{q} \theta_{q,l} p_l(k), \quad k \in I_*(\delta), \quad q \in \mathbb{Z}_+,$$

with appropriate $\theta_{q,l}$; in what follows we set $\theta_{q,l} = 0$ for $l > q$. Put

$$u_q = \int_{I_*(\delta)} p_q(k)u(k)dk, \quad u \in L^2(I_*(\delta)), \quad q \in \mathbb{Z}_+.$$

Then we have

$$\tilde{u}_q = \sum_{l=0}^{\infty} \theta_{q,l} u_l, \quad q \in \mathbb{Z}_+, \tag{2.6.21}$$

and

$$\|u\|^2_{L^2(I_*(\delta))} = \sum_{q=0}^{\infty} |u_q|^2. \tag{2.6.22}$$

Further, it is easy to check that

$$\sum_{q=0}^{\infty}\sum_{l=0}^{\infty} \theta_{q,l}^2 = \sum_{l=0}^{\infty} \int_{I_*(\delta)} k^{2l} dk = \sum_{l=0}^{\infty} \frac{(1+\delta)^{-2l-1}}{2l+1} < \infty.$$

Therefore, the operator $\Theta : l^2(\mathbb{Z}_+) \to l^2(\mathbb{Z}_+)$ defined by

$$(\Theta \mathbf{u})_q = \sum_{l=0}^{\infty} \theta_{q,l} u_l, \quad q \in \mathbb{Z}_+, \quad \mathbf{u} = \{u_l\}_{l\in\mathbb{Z}_+},$$

is a Hilbert-Schmidt, and hence bounded operator. Let $\rho(m) : l^2(\mathbb{Z}_+) \to l^2(\mathbb{Z}_+)$ be the diagonal operator with diagonal entries

$$e^{2m(\xi+\delta)_+} \frac{(mR)^{2q+2}}{(q!)^2(q+1)}, \quad q \in \mathbb{Z}_+. \tag{2.6.23}$$

Now (2.6.20) – (2.6.23) imply

$$n_+(s; G_\delta^+(m)) = n_+(s; \Theta^*\rho(m)\Theta), \quad s > 0. \tag{2.6.24}$$

Evidently,

$$n_+(s; \Theta^*\rho(m)\Theta) \le n_+(s; \|\Theta\|^2\rho(m)), \quad s > 0. \tag{2.6.25}$$

On the other hand, for any $s > 0$ we have

$$n_+(s; \rho(m)) = \#\left\{ q \in \mathbb{Z}_+ \,\middle|\, \frac{e^{m(\xi+\delta)_+}(mR)^{q+1}}{q!\sqrt{q+1}} > \sqrt{s} \right\}, \quad s > 0. \tag{2.6.26}$$

Applying the Stirling formula

$$q! = (2\pi)^{1/2}(q+1)^{q+1}(q+1)^{-1/2}e^{-q-1}(1+o(1)), \quad q \to \infty,$$

we find that for each $\varepsilon \in (0,1)$ there exists $q_0 \in \mathbb{Z}_+$ such that

$$\#\left\{q \in \mathbb{Z}_+ \,\middle|\, \frac{e^{m(\xi+\delta)_+}(mR)^{q+1}}{q!\sqrt{q+1}} > \sqrt{s}\right\} \leq$$

$$\#\left\{q \in \mathbb{Z}_+ \,\middle|\, \frac{(\xi+\delta)_+}{eR} > \frac{q+1}{eRm}\ln\left(\frac{q+1}{eRm}\right) + \frac{\ln\left(\sqrt{2\pi s}(1-\varepsilon)\right)}{eRm}\right\} + q_0. \tag{2.6.27}$$

Passing from Darboux sums to Riemann integrals, we find that for each constant $c \in \mathbb{R}$ we have

$$\lim_{m\to\infty} m^{-1}\#\left\{q \in \mathbb{Z}_+ \,\middle|\, \frac{(\xi+\delta)_+}{eR} > \frac{q+1}{eRm}\ln\left(\frac{q+1}{eRm}\right) + \frac{c}{m}\right\} =$$

$$eR\varkappa\left(\frac{(\xi+\delta)_+}{eR}\right). \tag{2.6.28}$$

Putting together (2.6.16) – (2.6.19) and (2.6.24) – (2.6.28), we get

$$\limsup_{\lambda\downarrow 0} |\ln\lambda|^{-1/2} n_+(r; \Gamma_\delta^+(\sqrt{b|\ln\lambda|})^* \Gamma_\delta^+(\sqrt{b|\ln\lambda|})) \leq (1+\delta)^2\sqrt{b}eR\varkappa\left(\frac{(\xi+\delta)_+}{eR}\right)$$

for any $\delta > 0$. Letting $\delta \downarrow 0$ and optimizing with respect to ξ and R, we obtain (2.6.15).

Chapter 3

Periodic Edge Potential

In this chapter we consider the operator H introduced in (0.1) – (0.2), but this time the function W is supposed to be periodic with period $T > 0$. As in Chapter 2, first we investigate in Section 3.1 the spectral properties of the unperturbed operator H_0, in particular, the behavior of the associated band functions near their extrema. In Section 3.2, we state the main results of the chapter, starting with the description of the effective Hamiltonians, and then passing to the asymptotic bounds as $\lambda \downarrow 0$ for $\mathcal{N}_j(\lambda)$. The proofs of these main results are in Sections 3.3 – 3.5

3.1 Basic Spectral Properties of H_0

We start with the simple observation that if W is periodic with period T, then the explicit expression for the operator $\tilde{h}(k)$ (see (0.5)) implies that all the band functions E_j, $j \in \mathbb{N}$, are periodic functions with period $\tau := bT$.

The asymptotics as $\lambda \downarrow 0$ of $\mathcal{N}_j^{\pm}(\lambda)$, $j \in \mathbb{N}$, is intimately related to the structure of the set

$$\mathcal{M}_j^{\pm} := \left\{ k \in [0, \tau) \,|\, E_j(k) = \mathcal{E}_j^{\pm} \right\},$$

and the behaviour of E_j in a vicinity of this set. Even though we investigate for definiteness only the asymptotics of $\mathcal{N}_j^{+}$, here it is convenient to consider both sets $\mathcal{M}_j^{\pm}$.
First of all, we assume that the band function E_j is not identically constant. Proposition 3.1.3 below contains an explicit sufficient condition for this.
Further, since the functions E_j are periodic, non-constant, and real-analytic, every set $\mathcal{M}_j^{\pm}$, $j \in \mathbb{N}$, is non empty and finite, i.e. $\mathcal{M}_j^{\pm} = \left\{ k_{\alpha,j}^{\pm} \right\}_{\alpha=1}^{A_j^{\pm}}$, $A_j^{\pm} \in \mathbb{N}$. Moreover, for each $k_{\alpha,j}^{\pm} \in \mathcal{M}_j^{\pm}$ there exists $l = l(k_{\alpha,j}^{\pm}) \in \mathbb{N}$ such that

$$\frac{d^s E_j}{dk^s}(k_{\alpha,j}^{\pm}) = 0, \quad s = 1, \ldots, 2l-1, \quad \text{and} \quad \mp \frac{d^{2l} E_j}{dk^{2l}}(k_{\alpha,j}^{\pm}) > 0.$$

If $l(k_{\alpha,j}^{\pm}) = 1$ for some $k_{\alpha,j}^{\pm} \in \mathcal{M}_j^{\pm}$, we will say that $k_{\alpha,j}^{\pm}$ is *a non-degenerate point*, and we will set

$$\mu_{\alpha,j}^{\pm} := \mp \frac{1}{2} E_j''(k_{\alpha,j}^{\pm}). \tag{3.1.1}$$

In Theorem 3.2.3 we assume that the maxima of the band function E_j are non-degenerate. Corollary 3.1.6, based on Propositions 3.1.5 and 3.1.5, offers an explicit sufficient condition which guarantees this non-degeneracy. In order to prove Propositions 3.1.3 and 3.1.5, and Corollary 3.1.6 we need some preliminary facts on the behavior of the band functions E_j for large $b > 0$. Since these auxiliary results could be of independent interest, we perform this analysis for a more general class of bounded W.

Fix $j \in \mathbb{N}$. As in Chapter 2 we denote by $\pi_j(k)$ is the orthogonal projection onto Ker $(h(k) - E_j(k))$.

Lemma 3.1.1. *Let $W \in L^\infty(\mathbb{R}, \mathbb{R})$. Fix $j \in \mathbb{N}$. Then there exists a real eigenfunction $\psi_j(\cdot; k) \in \operatorname{Ran} \pi_j(k) = \operatorname{Ker}(h(k) - E_j(k))$ such that $\|\psi_j(\cdot; k)\|_{L^2(\mathbb{R})} = 1$, and the function*

$$\mathbb{R} \ni k \mapsto \psi_j(\cdot; k) \in L^2(\mathbb{R}) \tag{3.1.2}$$

is analytic.

Proof. Our argument will follow the main lines of the proof of [39, Lemma 2.3 (v)], which on its turn is based on [73, Theorem XII.12] (see also the original work [42]). Since the coefficients of the differential operator $h(k)$ are real, there exists a real eigenfunction $\psi_j(\cdot; 0) \in \operatorname{Ran} \pi_j(0)$ such that $\|\psi_j(\cdot; 0)\|_{L^2(\mathbb{R})} = 1$. On the other hand, [73, Theorem XII.12] implies that for k in a complex vicinity of the real axis, there exists an analytic family of invertible bounded operators $\omega(k)$ such that

$$\omega_j(k)\pi_j(0) = \pi_j(k)\omega_j(k). \tag{3.1.3}$$

Moreover, for real k, the operators $\omega_j(k)$ can be chosen to be unitary. Following the argument in the proof of [39, Lemma 2.3 (v)], we find that in our case of a differential operator with real coefficients, the operator $\omega_j(k)$ can be chosen to be real and unitary for real k. Set

$$\psi_j(\cdot; k) := \omega_j(k)\psi_j(\cdot; 0).$$

Evidently, for $k \in \mathbb{R}$, the function $\psi_j(\cdot; k)$ is real, and $\|\psi_j(\cdot; k)\|_{L^2(\mathbb{R})} = 1$, while (3.1.3) implies that the function defined in (3.1.2) is analytic. □

In the sequel we will use the canonical representation

$$\pi_j(k) = \langle \cdot, \psi_j(\cdot; k)\rangle \psi_j(\cdot; k)$$

with an eigenfunction $\psi_j(\cdot; k)$ satisfying the properties described in Lemma 3.1.1. Note that for any $j \in \mathbb{N}$

$$\psi_j(x; l\tau + k) = \psi_j(x - lT; k), \quad x \in \mathbb{R}, \quad l \in \mathbb{Z}, \quad k \in \mathbb{R}. \tag{3.1.4}$$

As in Chapter 2 put

$$\tilde{\psi}_j(x; k) = \psi_j(x + k/b; k), \quad x \in \mathbb{R}, \quad k \in \mathbb{R}, \quad j \in \mathbb{N}.$$

Evidently, $\|\tilde{\psi}_j(\cdot; k)\|_{L^2(\mathbb{R})} = 1$, and

$$-\frac{\partial^2 \tilde{\psi}_j}{\partial x^2}(x; k) + b^2 x^2 \tilde{\psi}_j(x; k) + W(x + k/b)\tilde{\psi}_j(x; k) = E_j(k)\tilde{\psi}_j(x; k). \tag{3.1.5}$$

Put

$$\tilde{\pi}_j(k) := \langle \cdot, \tilde{\psi}_j(\cdot; k)\rangle \tilde{\psi}_j(\cdot; k), \quad k \in \mathbb{R}, \quad j \in \mathbb{N}.$$

Next, we deduce a suitable formula for the derivative $E_j'(k)$, $k \in \mathbb{R}$, $j \in \mathbb{N}$.

Lemma 3.1.2. *Let $W \in L^\infty(\mathbb{R};\mathbb{R}))$, and $W', W'' \in L^\infty(\mathbb{R})$. Then*

$$E_j'(k; b, W) = \frac{1}{b}\int_{\mathbb{R}} W'(x + k/b)\tilde{\psi}_j(x;k)^2 dx. \tag{3.1.6}$$

Proof. The standard Feynman-Hellmann formula (see e.g. [73, Theorem XIII 1/2]) implies

$$E_j'(k) = -2\int_{\mathbb{R}} (bx - k)\psi_j(x;k)^2 dx. \tag{3.1.7}$$

Our further manipulations of the integral in (3.1.7) could be easily justified by the well-known properties of the eigenfunction ψ_j (see e.g. [4, Theorem 4.6]).

Integrating by parts, we obtain

$$E_j'(k) = \frac{2}{b}\int_{\mathbb{R}} (bx-k)^2\psi_j(x;k)\frac{\partial \psi_j}{\partial x}(x;k)dx.$$

Bearing in mind that $\psi_j(k)$ satisfies the equation $h(k)\psi_j(k) = E_j(k)\psi_j(k)$, we find that

$$E_j'(x;k) = \frac{2}{b}\int_{\mathbb{R}}\left(\frac{\partial^2\psi_j}{\partial x^2}(x;k)(x;k) - W(x)\psi_j(x;k) + E_j(k)\psi_j(x;k)\right)\frac{\partial\psi_j}{\partial x}(x;k)dx =$$

$$\frac{1}{b}\int_{\mathbb{R}}\left(\frac{\partial}{\partial x}\left(\frac{\partial\psi_j}{\partial x}(x;k)^2\right) - W(x)\frac{\partial}{\partial x}\left(\psi_j(x;k)^2\right) + E_j(k)\frac{\partial}{\partial x}\left(\psi_j(x;k)^2\right)\right)dx =$$

$$\frac{1}{b}\int_{\mathbb{R}} W'(x)\psi_j(x;k)^2 dx.$$

Changing the variable $x \to x + k/b$ in the last integral, we obtain (3.1.6). □

Our next two propositions concern the asymptotic behavior for large b of the derivatives $E_j'(k)$ and $E_j''(k)$ respectively.

Proposition 3.1.3. *Let $W \in C^2(\mathbb{R})\cap L^\infty(\mathbb{R})$ with $W', W'' \in L^\infty(\mathbb{R})$. Suppose that $W'(x_0) > 0$ (resp., $W'(x_0) < 0$) for some x_0. Pick $j \in \mathbb{N}$. Then there exists $b_0 = b_0(W, j)$ such that $b > b_0$ implies $E_j'(bx_0; b, W) > 0$ (resp., $E_j'(bx_0; b, W) < 0$).*

Proof. Pick $b > 2\|W\|_{L^\infty}$ and denote by Γ_j the circle of radius b, centered at $b(2j-1)$. Denote by $\tilde{h}(b,0)$ the harmonic oscillator $-\frac{d^2}{dx^2} + b^2x^2$. Then the interior of Γ_j contains the eigenvalue $E_j(k;b,W)$ (resp., $b(2j-1)$) of the operator $\tilde{h}(k;b,W)$ (resp., of $\tilde{h}(b,0)$), while the rest of the spectra of these operators lie in the exterior of Γ_j. Since $\tilde{\pi}_j(k) := \langle\cdot, \tilde{\psi}_j(\cdot;k)\rangle\tilde{\psi}_j(\cdot;k)$, (3.1.6) implies

$$bE_j'(k;b,W) = \mathrm{Tr}\left(W'(\cdot + k/b)\tilde{\pi}_j(k)\right) =$$

$$\frac{1}{2\pi i}\mathrm{Tr}\left(\int_{\Gamma_j} W'(\cdot + k/b)(\tilde{h}(k;b,W) - \omega)^{-1}d\omega\right) =$$

$$\frac{1}{2\pi i}\mathrm{Tr}\left(\int_{\Gamma_j} W'(\cdot + k/b)(\tilde{h}(b,0) - \omega)^{-1}d\omega\right) -$$

$$\frac{1}{2\pi i}\mathrm{Tr}\left(\int_{\Gamma_j} W'(\cdot+k/b)(\tilde{h}(b,0)-\omega)^{-1}W(\cdot+k/b)(\tilde{h}(k;b,W)-\omega)^{-1}d\omega\right), \tag{3.1.8}$$

the contour Γ_j being run over in clockwise direction. Further, we have

$$\frac{1}{2\pi i}\mathrm{Tr}\left(\int_{\Gamma_j} W'(\cdot+k/b)(\tilde{h}(b,0)-\omega)^{-1}d\omega\right)=$$

$$b^{1/2}\int_{\mathbb{R}} W'(x+k/b)\varphi_j(b^{1/2}x)^2dx=\int_{\mathbb{R}} W'(b^{-1/2}y+b^{-1}k)\varphi_j(y)^2dy=$$

$$W'(b^{-1}k)+\int_{\mathbb{R}}(W'(b^{-1/2}y+b^{-1}k)-W'(b^{-1}k))\varphi_j(y)^2dy, \tag{3.1.9}$$

where φ_j are the normalized Hermite functions (see 2.1.9).

Combining (3.1.8) and (3.1.9), we get

$$E_j'(k;b,W)-\frac{1}{b}W'(b^{-1}k)=\frac{1}{b}(K_1+K_2) \tag{3.1.10}$$

with

$$K_1:=-\frac{1}{2\pi i}\mathrm{Tr}\left(\int_{\Gamma_j} W'(\cdot+k/b)(\tilde{h}(b,0)-\omega)^{-1}W(\cdot+k/b)(\tilde{h}(k;b,W)-\omega)^{-1}d\omega\right),$$

$$K_2:=\int_{\mathbb{R}}(W'(b^{-1/2}y+b^{-1}k)-W'(b^{-1}k))\varphi_j(y)^2dy.$$

It is easy to check that we have

$$|K_1|\le c_1b^{-1},\quad |K_2|\le c_2b^{-1/2}, \tag{3.1.11}$$

with

$$c_1:=\|W\|_{L^\infty(\mathbb{R})}\|W'\|_{L^\infty(\mathbb{R})}\left(\sum_{l=1}^{\infty}(2|l-j|-1)^{-2}\right)^{1/2}\left(\sum_{l\in\mathbb{N}:l\neq j}(2|l-j|-3/2)^{-2}+4\right)^{1/2}, \tag{3.1.12}$$

$$c_2:=\|W''\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|y|\varphi_j(y)^2dy. \tag{3.1.13}$$

Putting together (3.1.10) and (3.1.11), we get

$$\left|E_j'(k;b,W)-\frac{1}{b}W'(b^{-1}k)\right|\le c_1b^{-2}+c_2b^{-3/2}. \tag{3.1.14}$$

Now, bearing in mind that by hypothesis $W'(x_0)>0$ (resp., $W'(x_0)<0$), we find that if

$$b>b_0:=\max\left\{2\|W\|_{L^\infty(\mathbb{R})},\left(\frac{c_2+\sqrt{c_2^2+4c_1|W'(x_0)|}}{2|W'(x_0)|}\right)^2\right\}, \tag{3.1.15}$$

then $E_j'(bx_0)>0$ (resp., $E_j'(bx_0)<0$). □

proposition 3.1.3 immediately implies the following

Corollary 3.1.4. *Assume that W satisfies the hypothesis of Proposition 3.1.3. Then for each $j \in \mathbb{N}$ there exists $b_0 = b_0(j, W) > 0$ such that $b > b_0$ implies that*

$$\inf_{k\in\mathbb{R}} E_j(k; b, W) < \sup_{k\in\mathbb{R}} E_j(k; b, W). \tag{3.1.16}$$

Remark: The absolute continuity of the spectrum of the operator H_0 is equivalent to the validity of (3.1.16) for any $j \in \mathbb{N}$. Unfortunately, the constant c_2 in (3.1.13), and hence b_0 in (3.1.15), grow unboundedly as $j \to \infty$ so that Proposition 3.1.3 only implies that for any $\alpha \in \mathbb{R}$ there exists $\tilde{b}_0 = \tilde{b}_0(\alpha, W)$ such that $b > \tilde{b}_0$ implies that the spectrum of the operator $H_0(b)$ on the interval $(-\infty, \alpha)$ is absolutely continuous.
In the special case when W is periodic one, of the difficulties in the proof of the absolute continuity of $\sigma(H_0(b, W))$ for general non-constant periodic $W : \mathbb{R} \to \mathbb{R}$, is related to fact that we have

$$\lim_{j\to\infty} \left(\mathcal{E}_j^{\pm} - b(2j-1) - \langle W\rangle\right) = 0 \tag{3.1.17}$$

where $\langle W\rangle$ is the mean value of W (see [48]); in particular, $\lim_{j\to\infty}\left(\mathcal{E}_j^{+} - \mathcal{E}_j^{-}\right) = 0$. On the other hand, (3.1.17) implies as a by-product that for $j \in \mathbb{N}$ large enough, inequality (0.12) is valid even if (0.11) does not hold true.

Proposition 3.1.5. *Let $W = \overline{W} \in C^3(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $W', W'', W''' \in L^\infty(\mathbb{R})$. Suppose that $W''(x_0) > 0$ (resp., $W''(x_0) < 0$) for some x_0. Pick $j \in \mathbb{N}$. Then there exists $b_1 = b_1(W, j)$ such that $b > b_1$ implies $E_j''(bx_0; b, W) > 0$ (resp., $E_j''(bx_0; b, W) < 0$).*

Proof. First of all, note that

$$\frac{\partial\tilde{\psi}_j}{\partial k}(x;k) = \frac{\partial\psi_j}{\partial k}(x + k/b; k) + \frac{1}{b}\frac{\partial\psi_j}{\partial x}(x + k/b; k).$$

Applying Lemma 3.1.1, we conclude that $\frac{\partial\tilde{\psi}_j}{\partial k}(\cdot;k) \in L^2(\mathbb{R})$. Calculating the derivative with respect to k in (3.1.6) we have

$$E_j'(k;b,W) = \frac{1}{b^2}\int_{\mathbb{R}} W''(x+k/b)\tilde{\psi}_j(x;k)^2dx + \frac{2}{b}\int_{\mathbb{R}} W'(x+k/b)\frac{\partial\tilde{\psi}_j}{\partial k}(x;k)\tilde{\psi}_j(x;k)dx. \tag{3.1.18}$$

As in the proof of (3.1.14), we suppose that $b > 2\|W\|_{L^\infty(\mathbb{R})}$, and find that

$$\left|\int_{\mathbb{R}} W''(x + k/b)\tilde{\psi}_j(x;k)^2dx - W''(k/b)\right| \le c_3b^{-1} + c_4b^{-1/2} \tag{3.1.19}$$

where the constants c_3 and c_4 are defined by analogy with c_1 and c_2, replacing W' by W'' in (3.1.12), and W'' by W''' in (3.1.13). Further, obviously,

$$\left|\int_{\mathbb{R}} W'(x + k/b)\frac{\partial\tilde{\psi}_j}{\partial k}(x;k)\tilde{\psi}_j(x;k)dx\right| \le \|W'\|_{L^\infty(\mathbb{R})}\left\|\frac{\partial\tilde{\psi}_j}{\partial k}(\cdot;k)\right\|_{L^2(\mathbb{R})}. \tag{3.1.20}$$

Since the functions $\frac{\partial\tilde{\psi}_j}{\partial k}(\cdot;k)$ and $\tilde{\psi}_j(\cdot;k)$ are orthogonal in $L^2(\mathbb{R})$, we find that

$$\frac{\partial\tilde{\psi}_j}{\partial k}(\cdot;k) = (I - \tilde{\pi}_j(k))\frac{\partial\tilde{\psi}_j}{\partial k}(\cdot;k),$$

Deriving equation (3.1.5) with respect to k, we easily get

$$\frac{\partial\tilde{\psi}_j}{\partial k}(\cdot;k) = -\frac{1}{b}(\tilde{h}(k) - E_j(k))^{-1}(I - \tilde{\pi}_j(k))W'(\cdot + k/b)\tilde{\psi}_j(\cdot;k), \tag{3.1.21}$$

and, hence,

$$\left\|\frac{\partial\tilde{\psi}_j}{\partial k}(\cdot;k)\right\|_{L^2(\mathbb{R})} \leq \frac{1}{b^2}\|W'\|_{L^\infty(\mathbb{R})}. \tag{3.1.22}$$

Now the combination of (3.1.18) – (3.1.22) yields

$$\left|E_j''(k;b,W) - \frac{1}{b^2}W''(b^{-1}k)\right| \leq c_5 b^{-3} + c_4 b^{-5/2} \tag{3.1.23}$$

with $c_5 := c_3 + 2\|W'\|_{L^\infty(\mathbb{R})}\|W''\|_{L^\infty(\mathbb{R})}$. Therefore $W''(x_0) > 0$ (resp., $W''(x_0) < 0$), implies $E_j''(bx_0) > 0$ (resp., $E_j''(bx_0) < 0$), provided that

$$b > b_1 := \max\left\{2\|W\|_{L^\infty(\mathbb{R})}, \left(\frac{c_4 + \sqrt{c_4^2 + 4c_5|W''(x_0)|}}{2|W''(x_0)|}\right)^2\right\}. \tag{3.1.24}$$

□

Remark: Propositions 3.1.3 and 3.1.5 show that for large magnetic fields b the band functions E_j, $j \in \mathbb{N}$, behave quite similarly to the edge potential W. This behavior could be considered as *semiclassical*.

The combination of Propositions 3.1.3 and 3.1.5 easily yields the following

Corollary 3.1.6. *Let $W = \overline{W} \in C^3(\mathbb{R})$ be a T-periodic function such that $W'(x) = 0$, $x \in \mathbb{R}$, implies $W''(x) \neq 0$. Assume that the sets $\mathcal{M}_W^\pm := \{x \in [0,T) \,|\, W(x) = W_\pm\}$ consist of $A_W^\pm \in \mathbb{N}$ points. Then for each $j \in \mathbb{N}$ there exists $b_2(j,W) > 0$ such that $b > b_2$ implies that the set $\mathcal{M}_j^\pm$ contains exactly $A_W^\pm$ points, and all of them are non-degenerate.*

3.2 Main Results

In this section, we introduce the effective Hamiltonians which under suitable assumptions on W and V govern the main asymptotic term as $\lambda \downarrow 0$ of $\mathcal{N}_j^+(\lambda)$, and establish the corresponding asymptotic bounds. For the rest of the chapter, we always assume that W is periodic, $V : \mathbb{R}^2 \to \mathbb{R}$ is Lebesgue measurable, and satisfies the estimates

$$0 \leq V(x,y) \leq C_0(1+|x|)^{-m_1}(1+|y|)^{-m_2}, \quad (x,y) \in \mathbb{R}^2, \tag{3.2.1}$$

with some $C_0 \in [0,\infty)$, and $m_l \in (0,\infty)$, $l = 1,2$.

3.2.1 Effective Hamiltonians

Recall that $A_j^+ = \#\mathcal{M}_j^+$, and put $\mathcal{S}_j := \{1, \ldots, A_j^+\}$. Assume that the set $\mathcal{M}_j^+ = \left\{k_{\alpha,j}^+\right\}_{\alpha\in\mathcal{S}_j}$ contains only non-degenerate points; Corollary 3.1.6 gave sufficient conditions for this. For $\lambda > 0$ define

$$\mathcal{Q}_1(\lambda) : l^2(\mathbb{Z} \times \mathcal{S}_j) \otimes L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$$

as the operator with integral kernel

$$(2\pi)^{-1/2} V(x,y)^{1/2} \psi_j(x - lT; k_{\alpha,j}^+) e^{i(k+l\tau+k_{\alpha,j}^+)y} \left(\mu_{\alpha,j}^+ k^2 + \lambda\right)^{-1/2},$$

with $(l,\alpha) \in \mathbb{Z} \times \mathcal{S}_j$, $k \in \mathbb{R}$, and $(x,y) \in \mathbb{R}^2$, the quantities $\mu_{\alpha,j}^+ > 0$ being defined in (3.1.1). Under condition (3.2.1) with $m_1 > 1$, $m_2 > 1$, we have that $\mathcal{Q}_1(\lambda) \in S_2(l^2(\mathbb{Z} \times \mathcal{S}_j) \otimes L^2(\mathbb{R}); L^2(\mathbb{R}^2))$ for any $\lambda > 0$.

Theorem 3.2.1. *Let $W \in L^\infty(\mathbb{R};\mathbb{R})$ be a T-periodic function. Let V satisfy* (3.2.1) *with $m_1 > 1$, $m_2 > 1$. Fix $j \in \mathbb{N}$. Assume that* (0.12) *holds true, and the set $\mathcal{M}_j^+$ contains only non-degenerate points. Then for each $\varepsilon \in (0,1)$ we have*

$$n_*(1+\varepsilon; \mathcal{Q}_1(\lambda)) + O(1) \le \mathcal{N}_j^+(\lambda) \le n_*(1-\varepsilon; \mathcal{Q}_1(\lambda)) + O(1), \qquad (3.2.2)$$

as $\lambda \downarrow 0$.

The proof of Theorem 3.2.1 can be found in Section 3.3.

Now we shall give an equivalent formulation of Theorem 3.2.1 in the terms of an explicit effective Hamiltonian. Define the "diagonal" operator $\mu \in \mathcal{L}(l^2(\mathbb{Z} \times \mathcal{S}_j))$ by

$$(\mu\mathbf{u})_{l,\alpha} := \mu_{\alpha,j}^+ u_{l,\alpha}, \quad l \in \mathbb{Z}, \quad \alpha \in \mathcal{S}_j,$$

where $\mathbf{u} := \{u_{l,\alpha}\}_{(l,\alpha)\in\mathbb{Z}\times\mathcal{S}_j} \in l^2(\mathbb{Z} \times \mathcal{S}_j)$. On $l^2(\mathbb{Z} \times \mathcal{S}_j) \otimes H^2(\mathbb{R})$ define the operator

$$\mathcal{H}_0 := \mu \otimes \left(-\frac{d^2}{dy^2}\right)$$

self-adjoint in $l^2(\mathbb{Z} \times \mathcal{S}_j) \otimes L^2(\mathbb{R})$. Further, define the operator

$$\mathcal{V}_j^{\text{per}} = \mathcal{V} \in \mathcal{L}(l^2(\mathbb{Z} \times \mathcal{S}_j) \otimes L^2(\mathbb{R}))$$

by

$$(\mathcal{V}\mathbf{w})_{l,\alpha}(y) := \sum_{m\in\mathbb{Z},\, \beta\in\mathcal{S}_j} \mathcal{V}_{l,\alpha;m,\beta}(y) w_{m,\beta}(y), \quad y \in \mathbb{R}, \qquad (3.2.3)$$

where

$$\mathcal{V}_{l,\alpha;m,\beta}(y) := \frac{1}{2\pi} \int_{\mathbb{R}} V(x,y) \psi_j(x - lT; k_{\alpha,j}^+) \psi_j(x - mT; k_{\beta,j}^+) dx \; e^{-i((l-m)\tau + k_{\alpha,j}^+ - k_{\beta,j}^+)y},$$

and $\mathbf{w} \in l^2(\mathbb{Z} \times \mathcal{S}_j) \otimes L^2(\mathbb{R})$. Thus the operator $\mathcal{H}_0 - g\mathcal{V}$ with $g \ge 0$, self-adjoint on $\text{Dom}(\mathcal{H}_0)$, can be interpreted as a Schrödinger operator on the real line with infinite-matrix-valued attractive potential $-g\mathcal{V}$, and a coupling constant $g \ge 0$.

Applying the Birman-Schwinger principle and the inverse Fourier transform with respect to $k \in \mathbb{R}$, we easily find that Theorem 3.2.1 yields

$$\operatorname{rank} \mathbb{P}_{(-\infty,-\lambda)}(\mathcal{H}_0 - (1-\varepsilon)\mathcal{V}) + O(1) \le \mathcal{N}_j^+(\lambda) \le \operatorname{rank} \mathbb{P}_{(-\infty,-\lambda)}(\mathcal{H}_0 - (1+\varepsilon)\mathcal{V}) + O(1),$$

as $\lambda \downarrow 0$, for any $\varepsilon \in (0,1)$, when the hypothesis of the Theorem 3.2.1 are fulfilled.

Next, assuming a somewhat faster decay of V as $y \to \infty$, we can obtain an asymptotic estimate similar to (3.2.2) involving an operator which is simpler than $\mathcal{Q}_1(\lambda)$. Define

$$\mathcal{Q}_2 : l^2(\mathbb{Z} \times \mathcal{S}_j) \to L^2(\mathbb{R}^2)$$

as the operator with integral kernel

$$\left(\mu_{\alpha,j}^+\right)^{-1/4} V(x,y)^{1/2} \psi_j(x - lT; k_{\alpha,j}^+) e^{iy(l\tau + k_{\alpha,j}^+)}, \quad (l,\alpha) \in \mathbb{Z} \times \mathcal{S}_j, \quad (x,y) \in \mathbb{R}^2.$$

Again, if V satisfies (3.2.1) with $m_1 > 1$, $m_2 > 1$, then $\mathcal{Q}_2 \in S_2(l^2(\mathbb{Z} \times \mathcal{S}_j); L^2(\mathbb{R}^2))$.

Theorem 3.2.2. *Let $W \in L^\infty(\mathbb{R};\mathbb{R})$ be a T-periodic function. Let V satisfy* (3.2.1) *with $m_1 > 1$ and $m_2 > 3$. Fix $j \in \mathbb{N}$ and assume* (0.12). *Then for each $\varepsilon \in (0,1)$ we have*

$$n_*((1+\varepsilon)\sqrt{2\sqrt{\lambda}}; \mathcal{Q}_2) + O(1) \le \mathcal{N}_1^+(\lambda) \le n_*((1-\varepsilon)\sqrt{2\sqrt{\lambda}}; \mathcal{Q}_2) + O(1), \quad \lambda \downarrow 0. \tag{3.2.4}$$

The proof of Theorem 3.2.2 is contained in Section 3.4.

3.2.2 Asymptotics Bounds for $\mathcal{N}_j^+$

Theorems 3.2.1 – 3.2.2 can be used for the investigation of the asymptotic behaviour as $\lambda \downarrow 0$ of $\mathcal{N}_j^+(\lambda)$ for a large class of rapidly decaying perturbations V. In this subsection we concentrate on perturbations of compact support.

In order to formulate the theorem we need the following notations. For $t > 0$ set $\operatorname{Ent}(t) := \min\{l \in \mathbb{N} \,|\, l \ge t\}$. Further, let $\Omega \subset \mathbb{R}^2$ be an open, bounded, non-empty set. Let $\mathbf{V}(\Omega)$ be the set of the closed vertical intervals $\mathcal{J} \subset \Omega$ of positive length $|\mathcal{J}|$. Evidently, $\mathbf{V}(\Omega) \ne \emptyset$. Put

$$\mathcal{C}(\Omega) := \sup_{\mathcal{J} \in \mathbf{V}(\Omega)} \frac{1}{\operatorname{Ent}\left(\frac{2\pi}{bT(|\mathcal{J}|)}\right)}.$$

If $\mathcal{J} \in \mathbf{V}(\Omega)$, then elementary topological arguments imply that there exists a horizontal interval $\mathcal{I}$ of positive length, such that the rectangle $\mathcal{I} \times \mathcal{J}$ is contained in Ω.

Theorem 3.2.3. *Let $W \in L^\infty(\mathbb{R};\mathbb{R})$ be a T-periodic function. Suppose that $V : \mathbb{R}^2 \to [0,\infty)$ is a Lebesgue measurable function such that*

$$c_-\chi_{\Omega_-}(x,y) \le V(x,y) \le c_+\chi_{\Omega_+}(x,y), \quad (x,y) \in \mathbb{R}^2, \tag{3.2.5}$$

where $\chi_{\Omega_\pm}$ are the characteristic functions of the open, bounded and non-empty sets $\Omega_\pm \subset \mathbb{R}^2$, and $c_\pm \in (0,\infty)$ are constants. Fix $j \in \mathbb{N}$ and assume (0.12). *Suppose that the set $\mathcal{M}_j^+$ contains only non-degenerate points. Then we have*

$$\frac{\sqrt{2}}{\sqrt{bT}}\mathcal{C}(\Omega_-) \le \liminf_{\lambda \downarrow 0} |\ln\lambda|^{-1/2}\mathcal{N}_j^+(\lambda) \le \limsup_{\lambda \downarrow 0} |\ln\lambda|^{-1/2}\mathcal{N}_j^+(\lambda) \le \frac{\sqrt{2}}{\sqrt{bT}}A_j^+. \tag{3.2.6}$$

as ealier, $A_j^+ = \#\mathcal{M}_j^+$. *In particular, if* $A_j^+ = 1$, *and there exists a closed vertical segment* $\mathcal{J} \subset \Omega_-$ *of length* $|\mathcal{J}| \geq \frac{2\pi}{bT}$ *so that* $\mathcal{C}(\Omega_-) = 1$, *we have*

$$\lim_{\lambda \downarrow 0} |\ln \lambda|^{-1/2} \mathcal{N}_j^+(\lambda) = \frac{\sqrt{2}}{\sqrt{bT}}.$$

We prove Theorem 3.2.3 in Section 3.5.

Remarks: (i) Corollary 3.1.6 guarantees the existence of edge potentials W and magnetic fields b for which the set $\mathcal{M}_j^+$ contains only non-degenerate points, and $A_j^+ = 1$. Thus there exist explicit examples where the assumptions of Theorem 3.2.3 are met.

(ii) Theorem 3.2.3 implies that every open gap $(\mathcal{E}_j^+, \mathcal{E}_{j+1}^-)$ contains infinitely many discrete eigenvalues of the operator H for generic not identically vanishing decaying perturbations $V \geq 0$. By (3.2.6) the asymptotic rate of the convergence of these eigenvalues is not faster than Gaussian.

In principle, the analysis of the asymptotic behavior as $\lambda \downarrow 0$ of $\mathcal{N}_j^+(\lambda)$ without the non-degeneracy assumption concerning the set $\mathcal{M}_j^+$ is also feasible but much more complicated from technical point of view, so that we omit the details. However, we would just like to note that

$$(k - k_{\alpha,j}^+)^{2l} = o((k - k_{\alpha,j}^+)^2), \quad k \to k_{\alpha,j}^+, \quad \text{if } l \in \mathbb{N},\, l > 1$$

hence, the replacement of non-degenerate points $k_{\alpha,j}^+ \in \mathcal{M}_j^+$ by degenerate ones does not decrease the quantity $\lim_{\lambda \downarrow 0} |\ln \lambda|^{-1/2} \mathcal{N}_j^+(\lambda)$ (see (3.3.2), (3.3.3), and (3.3.4) bellow). Thus we find that from the proof of Theorem 3.2.1.

Corollary 3.2.4. *Let* $W \in L^\infty(\mathbb{R};\mathbb{R})$ *be a T-periodic function. Assume that* $V : \mathbb{R}^2 \to [0,\infty)$ *is a Lebesgue measurable function which satisfies* (0.13) *and the lower bound in* (3.2.5). *Fix* $j \in \mathbb{N}$. *Assume that* $\mathcal{E}_j^- < \mathcal{E}_j^+$ *and* (0.12) *holds true. Then we have*

$$0 < \liminf_{\lambda \downarrow 0} |\ln \lambda|^{-1/2} \mathcal{N}_j^+(\lambda).$$

In particular, the open gap $(\mathcal{E}_j^+, \mathcal{E}_{j+1}^-)$ *contains infinitely many discrete eigenvalues of the operator* H_+, *and the asymptotic convergence of these eigenvalues to the edge* $\mathcal{E}_j^+$ *is not faster than Gaussian.*

3.3 Proof of Theorem 3.2.1

The Birman-Schwinger principle entails

$$\mathcal{N}_j^+(\lambda) = n_-(1; V^{1/2}(H_0 - \mathcal{E}_j^+ - \lambda)^{-1} V^{1/2}) + O(1), \quad \lambda \downarrow 0, \tag{3.3.1}$$

(see (1.3.5)). Choose $\delta > 0$ so small that the intervals $\mathcal{O}_{l,\alpha}(\delta) := (l\tau + k_{\alpha,j}^+ - \delta, l\tau + k_{\alpha,j}^+ + \delta)$, $l \in \mathbb{Z}$, $\alpha \in \mathcal{S}_j$, are pairwise disjoint. Set $\mathcal{O}_\delta := \cup_{(l,\alpha) \in \mathbb{Z} \times \mathcal{S}_j} \mathcal{O}_{l,\alpha}(\delta)$. Introduce the orthogonal projection

$$P_{j,\delta} := \mathcal{F}^* \int_{\mathcal{O}_\delta}^{\oplus} \pi_j(k) dk\, \mathcal{F}$$

acting in $L^2(\mathbb{R}^2)$. Since $\mathcal{E}_j^+$ is not in the spectrum of the operator H_0 restricted to $(I-P_{j,\delta})\mathrm{Dom}(H_0)$, we find that the operator $V^{1/2}(H_0-\mathcal{E}_j^+-\lambda)^{-1}(I-P_{j,\delta})V^{1/2}$ converges in norm as $\lambda \downarrow 0$ to a compact operator. Therefore, the Weyl inequalities (1.2.2) easily imply

$$n_+(1+\varepsilon; V^{1/2}(\mathcal{E}_j^+ - H_0+\lambda)^{-1}P_{j,\delta}V^{1/2}) + O(1) \leq$$
$$n_-(1; V^{1/2}(H_0-\mathcal{E}_j^+ -\lambda)^{-1}V^{1/2}) \leq$$
$$n_+(1-\varepsilon; V^{1/2}(\mathcal{E}_j^+ - H_0+\lambda)^{-1}P_{j,\delta}V^{1/2}) + O(1), \quad \lambda \downarrow 0, \tag{3.3.2}$$

with $\varepsilon \in (0,1)$.

For $\lambda > 0$ define $\mathcal{T}_1(\lambda) : L^2(\mathbb{R}^2) \to L^2(\mathcal{O}_\delta)$ as the operator with integral kernel

$$(2\pi)^{-1/2}(\mathcal{E}_j^+ - E_j(k)+\lambda)^{-1/2}\psi_j(x;k)e^{-iky}V(x,y)^{1/2}, \quad (x,y)\in\mathbb{R}^2, \quad k\in\mathcal{O}_\delta. \tag{3.3.3}$$

Then we have

$$V^{1/2}(\mathcal{E}_j^+ - H_0+\lambda)^{-1}P_{j,\delta}V^{1/2} = \mathcal{T}_1(\lambda)^*\mathcal{T}_1(\lambda), \tag{3.3.4}$$

and hence

$$n_+(s^2; V^{1/2}(\mathcal{E}_j^+ - E_j(k)+\lambda)^{-1}P_{j,\delta}V^{1/2}) = n_*(s;\mathcal{T}_1(\lambda)) = n_*(s;\mathcal{T}_1(\lambda)^*), \quad s>0. \tag{3.3.5}$$

Let $\mathcal{W} : L^2(\mathcal{O}_\delta) \to l^2(\mathbb{Z}\times\mathcal{S}_j)\otimes L^2(-\delta,\delta)$ be the unitary operator defined by

$$(\mathcal{W}u)_{l,\alpha}(k) := u(k+l\tau+k_{\alpha,j}^+), \quad (l,\alpha)\in\mathbb{Z}\times\mathcal{S}_j, \quad k\in(-\delta,\delta),$$

with $u\in L^2(\mathcal{O}_\delta)$. Define $\mathcal{T}_2(\lambda) : l^2(\mathbb{Z}\times\mathcal{S}_j)\otimes L^2(-\delta,\delta) \to L^2(\mathbb{R}^2)$, $\lambda>0$, as the operator with integral kernel

$$(2\pi)^{-1/2}V(x,y)^{1/2}\psi_j(x-lT;k+k_{\alpha,j}^+)e^{i(k+l\tau+k_{\alpha,j}^+)y}(\mathcal{E}_j^+ - E_j(k+k_{\alpha,j}^+)+\lambda)^{-1/2},$$

where $(l,\alpha)\in\mathbb{Z}\times\mathcal{S}_j$, $k\in(-\delta,\delta)$, $(x,y)\in\mathbb{R}^2$. By (3.1.4), we have $\mathcal{T}_2(\lambda)\mathcal{W} = \mathcal{T}_1(\lambda)^*$. Therefore,

$$n_*(s;\mathcal{T}_1(\lambda)^*) = n_*(s;\mathcal{T}_2(\lambda)), \quad s>0, \quad \lambda>0. \tag{3.3.6}$$

Define $\mathcal{T}_3(\lambda) : l^2(\mathbb{Z}\times\mathcal{S}_j)\otimes L^2(-\delta,\delta) \to L^2(\mathbb{R}^2)$, $\lambda>0$, as the operator with integral kernel

$$(2\pi)^{-1/2}V(x,y)^{1/2}\psi_j(x-lT;k_{\alpha,j}^+)e^{i(k+l\tau+k_{\alpha,j}^+)y}(\mu_{\alpha,j}^+k^2+\lambda)^{-1/2},$$

with $(l,\alpha)\in\mathbb{Z}\times\mathcal{S}_j$, $k\in(-\delta,\delta)$, $(x;y)\in\mathbb{R}^2$. Then

$$\|\mathcal{T}_2(\lambda)-\mathcal{T}_3(\lambda)\|_2^2 =$$

$$(2\pi)^{-1}\sum_{(l,\alpha)\in\mathbb{Z}\times\mathcal{S}_j}\int_{\mathbb{R}^2}V(x,y)\int_{-\delta}^{\delta}\Big|\psi_j(x-lT;k+k_{\alpha,j}^+)(\mathcal{E}_j^+ - E_j(k+k_{\alpha,j}^+)+\lambda)^{-1/2}$$
$$-\psi_j(x-lT;k_{\alpha,j}^+)(\mu_{\alpha,j}^+k^2+\lambda)^{-1/2}\Big|^2\,dk\,dx\,dy \leq$$

$$\frac{C_1}{\pi}\sum_{\alpha\in\mathcal{S}_j}\left\{\int_{-\delta}^{\delta}\left|(\mathcal{E}_j^+ - E_j(k+k_{\alpha,j}^+)+\lambda)^{-1/2} - (\mu_{\alpha,j}^+ k^2+\lambda)^{-1/2}\right|^2 dk + \right.$$

$$\left.\int_{-\delta}^{\delta}\left(k^{-2}\int_{\mathbb{R}}\left|\psi_j(x;k+k_{\alpha,j}^+) - \psi_j(x;k_{\alpha,j}^+)\right|^2 dx\right) k^2(\mu_{\alpha,j}^+ k^2+\lambda)^{-1} dk\right\} \tag{3.3.7}$$

where the quantity

$$C_1 := C_0 \max_{x\in\mathbb{R}}\sum_{l\in\mathbb{Z}}(1+|x+lT|)^{-m_1}\int_{\mathbb{R}}(1+|y|)^{-m_2}dy \tag{3.3.8}$$

with C_0 being introduced in (3.2.1), is finite due to $m_1 > 1$ and $m_2 > 1$. Since we have

$$(\mathcal{E}_j^+ - E_j(k+k_{\alpha,j}^+)+\lambda)^{-1/2} - (\mu_{\alpha,j}^+ k^2+\lambda)^{-1/2} =$$

$$\frac{E_j(k+k_{\alpha,j}^+) - \mathcal{E}_j^+ + \mu_{\alpha,j}^+ k^2}{\sqrt{(\mathcal{E}_j^+ - E_j(k+k_{\alpha,j}^+)+\lambda)(\mu_{\alpha,j}^+ k^2+\lambda)}\left(\sqrt{\mathcal{E}_j^+ - E_j(k+k_{\alpha,j}^+)+\lambda}+\sqrt{\mu_{\alpha,j}^+ k^2+\lambda}\right)},$$

and

$$E_j(k+k_{\alpha,j}^+) - \mathcal{E}_j^+ + \mu_{\alpha,j}^+ k^2 = O(k^3), \quad k\to 0,$$

we find that the first term in the braces at the r.h.s of (3.3.7) is uniformly bounded with respect to $\lambda > 0$. Similarly,

$$k^2(\mu_{\alpha,j}^+ k^2+\lambda)^{-1} \le 1/\mu_{\alpha,j}^+, \quad \lambda > 0, \quad k\in\mathbb{R}. \tag{3.3.9}$$

Further, elementary calculations yield

$$k^{-2}\int_{\mathbb{R}}\left|\psi_j(x;k+k_{\alpha,j}^+) - \psi_j(x;k_{\alpha,j}^+)\right|^2 dx \le \int_0^1 \|\pi_j'(ks+k_{\alpha,j}^+)\|^2 ds. \tag{3.3.10}$$

Since the orthogonal projection $\pi_j(k)$ depends analytically on k, we find that the combination of (3.3.9) and (3.3.10) implies the uniform boundedness with respect to $\lambda > 0$ of the second term in the braces at the r.h.s. of (3.3.7). Therefore (3.3.7) yields

$$\|\mathcal{T}_2(\lambda) - \mathcal{T}_3(\lambda)\|_2 = O(1), \quad \lambda\downarrow 0. \tag{3.3.11}$$

Combining (1.2.3), (1.2.4) with $p = 2$, and (3.3.11), we get

$$n_*(s(1+\varepsilon);\mathcal{T}_3(\lambda)) + O(1) \le n_*(s;\mathcal{T}_2(\lambda)) \le n_*(s(1-\varepsilon);\mathcal{T}_3(\lambda)) + O(1), \quad \lambda\downarrow 0, \tag{3.3.12}$$

with $s > 0$ and $\varepsilon\in(0,1)$.

Finally, define $\mathcal{T}_4(\lambda) : l^2(\mathbb{Z}\times\mathcal{S}_j)\otimes L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$, $\lambda > 0$, as the operator with integral kernel

$$(2\pi)^{-1/2}V(x,y)^{1/2}\psi_j(x-lT;k_{\alpha,j}^+)e^{i(k+l\tau+k_{\alpha,j}^+)y}(\mu_{\alpha,j}^+ k^2+\lambda)^{-1/2}\chi_{(-\delta,\delta)}(k),$$

where $(l,\alpha)\in\mathbb{Z}\times\mathcal{S}_j$, $k\in\mathbb{R}$, $(x,y)\in\mathbb{R}^2$, and $\chi_{(-\delta,\delta)}$ is the characteristic function of the interval $(-\delta,\delta)$. Evidently,

$$n_*(s;\mathcal{T}_3(\lambda)) = n_*(s;\mathcal{T}_4(\lambda)), \quad s > 0, \quad \lambda > 0. \tag{3.3.13}$$

At the same time we have

$$\|\mathcal{T}_4(\lambda) - \mathcal{Q}_1(\lambda)\|_2^2 =$$

$$\frac{1}{\pi} \sum_{(l,\alpha)\in\mathbb{Z}\times\mathcal{S}_j} \int_{\mathbb{R}^2} V(x,y)\psi_j(x-lT;k_{\alpha,j}^+)^2 dx\,dy \int_\delta^\infty (\mu_{\alpha,j}^+ k^2+\lambda)^{-1}dk \leq \frac{C_1}{\pi\delta}\sum_{\alpha\in\mathcal{S}_j}(\mu_{\alpha,j}^+)^{-1},$$

the constant C_1 being introduced in (3.3.8). Arguing as in the derivation of (3.3.12), we get

$$n_*(s(1+\varepsilon);\mathcal{Q}_1(\lambda)) + O(1) \leq n_*(s;\mathcal{T}_4(\lambda)) \leq n_*(s(1-\varepsilon);\mathcal{Q}_1(\lambda)) + O(1), \quad \lambda \downarrow 0, \quad (3.3.14)$$

with $s>0$ and $\varepsilon\in(0,1)$. Putting together (3.3.1), (3.3.2), (3.3.5), (3.3.12), (3.3.13), and (3.3.14), we obtain (3.2.2).

The proof of Theorem 3.2.1 is now complete.

3.4 Proof of Theorem 3.2.2

We have

$$n_*(s;\mathcal{Q}_1(\lambda)) = n_+(s^2;\mathcal{Q}_1(\lambda)\mathcal{Q}_1(\lambda)^*), \quad s>0, \quad \lambda>0. \quad (3.4.1)$$

The operator $M_1(\lambda) := \mathcal{Q}_1(\lambda)\mathcal{Q}_1(\lambda)^* : L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ admits the integral kernel

$$\sqrt{V(x,y)V(x',y')} \sum_{(l,\alpha)\in\mathbb{Z}\times\mathcal{S}_j} \frac{e^{-\sqrt{\lambda/\mu_{\alpha,j}^+}|y-y'|}}{2\sqrt{\mu_{\alpha,j}^+\lambda}} e^{i(l\tau+k_{\alpha,j}^+)(y-y')}\psi_j(x-lT;k_{\alpha,j}^+)\psi_j(x'-lT;k_{\alpha,j}^+),$$

with $(x,y),(x',y')\in\mathbb{R}^2$. Define $M_2(\lambda): L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ as the operator with integral kernel

$$\sqrt{V(x,y)V(x',y')} \sum_{(l,\alpha)\in\mathbb{Z}\times\mathcal{S}_j} \frac{1}{2\sqrt{\mu_{\alpha,j}^+\lambda}} e^{i(l\tau+k_{\alpha,j}^+)(y-y')}\psi_j(x-lT;k_{\alpha,j}^+)\psi_j(x'-lT;k_{\alpha,j}^+),$$

with $(x,y),(x',y')\in\mathbb{R}^2$. Taking into account (3.2.1) and the elementary inequalities $0\leq 1-e^{-t}\leq t$, $t\geq 0$, we get

$$\|M_1(\lambda)-M_2(\lambda)\|_2^2 \leq$$

$$C_0^2 \max_{\alpha\in\mathcal{S}_j}(2\mu_{\alpha,j}^+)^{-2}\times$$

$$\int_{\mathbb{R}^2}(1+|x|)^{-m_1}(1+|x'|)^{-m_1}\left(\sum_{(l,\alpha)\in\mathbb{Z}\times\mathcal{S}_j}|\psi_j(x-lT;k_{\alpha,j}^+)\psi_j(x'-lT;k_{\alpha,j}^+)|\right)^2 dxdx'\times$$

$$\int_{\mathbb{R}^2}(1+|y|)^{-m_2}(1+|y'|)^{-m_2}|y-y'|^2dydy'. \quad (3.4.2)$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\int_{\mathbb{R}^2}(1+|x|)^{-m_1}(1+|x'|)^{-m_1}\left(\sum_{(l,\alpha)\in\mathbb{Z}\times\mathcal{S}_j}|\psi_j(x-lT;k_{\alpha,j}^+)\psi_j(x'-lT;k_{\alpha,j}^+)|\right)^2 dxdx' \leq$$

$$A_j^+ \left(\max_{x\in\mathbb{R}} \sum_{l\in\mathbb{Z}} (1+|x+lT|)^{-m_1} \right)^2 < \infty \tag{3.4.3}$$

since $m_1 > 1$. Similarly,

$$\int_{\mathbb{R}^2} (1+|y|)^{-m_2}(1+|y'|)^{-m_2}|y-y'|^2 dydy' < \infty \tag{3.4.4}$$

since $m_2 > 3$. Now, (3.4.2) – (3.4.4) imply

$$\|M_1(\lambda) - M_2(\lambda)\|_2 = O(1), \quad \lambda \downarrow 0.$$

Arguing again as in the derivation of (3.3.12), we get

$$n_+(s(1+\varepsilon); M_2(\lambda)) + O(1) \le n_+(s; M_1(\lambda)) \le n_+(s(1-\varepsilon); M_2(\lambda)) + O(1), \lambda \downarrow 0, \tag{3.4.5}$$

with $\varepsilon \in (0,1)$, $s > 0$. Finally,

$$M_2(\lambda) = \frac{1}{2\sqrt{\lambda}} \mathcal{Q}_2\mathcal{Q}_2^*, \quad \lambda > 0,$$

and, hence,

$$n_+(s^2; M_2(\lambda)) = n_*(s\sqrt{2\sqrt{\lambda}}; \mathcal{Q}_2), \quad s > 0, \quad \lambda > 0. \tag{3.4.6}$$

Now the combination of (3.2.2), (3.4.1), (3.4.5), and (3.4.6), yields (3.2.4).

The proof of Theorem 3.2.2 is now complete.

3.5 Proof of Theorem 3.2.3

In order to prove Theorem 3.2.3 we need the following

Lemma 3.5.1. *Let $W \in C^1(\mathbb{R})$ be real-valued periodic function. Then for any bounded interval $\mathcal{I} \subset \mathbb{R}$ of positive length, and for any $k_0 \in \mathbb{R}$ we have*

$$\lim_{\xi\to\pm\infty} \xi^{-2} \ln \int_{\mathcal{I}} \psi_j(x-\xi; k_0)^2 dx = -b. \tag{3.5.1}$$

Relation (3.5.1) follows easily from [48, Theorem 1.1], so that we omit the details.

Now we are in position to prove of Theorem 3.2.3. First, let us make another reduction step in which we use specifically the compactness condition on the support of V.

Let $\Omega \subset \mathbb{R}^2$ be an open bounded non-empty set.. Define $\mathcal{Q}_3(\Omega) : l^2(\mathbb{Z} \times \mathcal{S}_j) \to L^2(\Omega)$ as the operator with integral kernel

$$(\mu_{\alpha,j}^+)^{-1/4}\psi_j(x-lT; k_{\alpha,j}^+)e^{i(l\tau+k_{\alpha,j}^+)y}, \quad (l,\alpha) \in \mathbb{Z}\times\mathcal{S}_j), \quad (x,y) \in \Omega.$$

Then (3.2.5) combined with the mini-max principle implies

$$n_*(s; c_-\mathcal{Q}_3(\Omega_-)) \le n_+(s; \mathcal{Q}_2) \le n_*(s; c_+\mathcal{Q}_3(\Omega_+)), \quad s > 0. \tag{3.5.2}$$

3.5.1 Upper Bound

We start by proving the upper bound in (3.2.6). Since the set Ω_+ is bounded it is contained in some rectangle $\mathcal{R}_+ := \mathcal{I}_+ \times \mathcal{J}_+$ where $\mathcal{I}_+$ and $\mathcal{J}_+$ are bounded intervals of positive lengths. Evidently,

$$n_*(s; \mathcal{Q}_3(\Omega_+)) \leq n_*(s; \mathcal{Q}_3(\mathcal{R}_+)), \quad s > 0. \tag{3.5.3}$$

Let $M_3^+ \in S_\infty(l^2(\mathbb{Z} \times \mathcal{S}_j))$ be the "diagonal" operator defined by

$$(M_3^+ \mathbf{u})_{l,\alpha} = \nu_{l,\alpha}^+ u_{l,\alpha}, \quad (l, \alpha) \in \mathbb{Z} \times \mathcal{S}_j,$$

where $\mathbf{u} := \{u_{l,\alpha}\}_{(l,\alpha)\in\mathbb{Z}\times\mathcal{S}_j} \in l^2(\mathbb{Z} \times \mathcal{S}_j)$, and

$$\nu_{l,\alpha}^+ := |\mathcal{J}_+| \sum_{\beta\in\mathcal{S}_j} \left(\mu_{\beta,j}^+\right)^{-1/2} \sum_{m\in\mathbb{Z}} (m^2+1)^{-1}(l^2+1) \int_{\mathcal{I}_+} \psi_j(x - lT; k_{\alpha,j}^+)^2 dx, \quad (l, \alpha) \in \mathbb{Z} \times \mathcal{S}_j.$$

Applying the Cauchy-Schwarz inequality, we find that

$$\mathcal{Q}_3(\mathcal{R}_+)^* \mathcal{Q}_3(\mathcal{R}_+) \leq M_3^+,$$

which combined with the mini-max principle yields

$$n_*(s\sqrt{2\sqrt{\lambda}}; \mathcal{Q}_3(\mathcal{R}_+)) \leq n_+(s^2 2\sqrt{\lambda}; M_3^+) =$$

$$\#\left\{(l, \alpha) \in \mathbb{Z} \times \mathcal{S}_j \,|\, \nu_{l,\alpha}^+ > s^2 2\sqrt{\lambda}\right\}, \quad s > 0, \quad \lambda > 0. \tag{3.5.4}$$

Applying Lemma 3.5.1, we easily find that

$$\lim_{\lambda\downarrow 0} \frac{\#\left\{(l, \alpha) \in \mathbb{Z} \times \mathcal{S}_j \,|\, \nu_{l,\alpha}^+ > s\sqrt{\lambda}\right\}}{|\ln \lambda|^{1/2}} = \frac{\sqrt{2}}{\sqrt{b}T} A_j^+, \quad s > 0. \tag{3.5.5}$$

Combining now (3.2.4) with the upper bound in (3.5.2), (3.5.3), (3.5.4), and (3.5.5), we obtain the upper bound in (3.2.6).

3.5.2 Lower Bound

Let $\mathcal{J}_-$ be a closed vertical interval lo length $q \in (0, \infty)$, contained in Ω_-. Due to the invariance of H_0 with respect to y-translations, we may assume without any loss of generality that the exists a bounded interval $\mathcal{I}_-$ of length $q \in (0, \infty)$ such that $\mathcal{I}_- \times (0, q) \subset \Omega_-$. Set

$$L = L(q) := \text{Ent}\left(\frac{2\pi}{bTq}\right) = \text{Ent}\left(\frac{2\pi}{\tau q}\right).$$

Then we have $\mathcal{R}_- := \mathcal{I}_- \times (0, \frac{2\pi}{\tau L}) \subset \Omega_-$, and therefore

$$n_*(s; \mathcal{Q}_3(\Omega_-)) \geq n_*(s; \mathcal{Q}_3(\mathcal{R}_-)), \quad s > 0. \tag{3.5.6}$$

Let $M_3^- \in S_\infty(l^2(\mathbb{Z}))$ be the "diagonal" operator defined by

$$(M_3^- \mathbf{u})_m = \nu_m^- u_m, \quad m \in \mathbb{Z},$$

where $\mathbf{u} := \{u_m\}_{m\in\mathbb{Z}}$, and

$$\nu_m^- := \frac{2\pi}{\sqrt{\mu_{1,j}^+}\tau L} \int_{\mathcal{I}_-} \psi_j(x - mLT; k_{1,j}^+)^2 dx, \quad m \in \mathbb{Z}.$$

Restricting the operator $\mathcal{Q}_3(\mathcal{R}_-)$ onto the subspace

$$\left\{\mathbf{u} := \{u_{l,\alpha}\}_{(l,\alpha)\in\mathbb{Z}\times\mathcal{S}_j} \in l^2(\mathbb{Z}\times\mathcal{S}_j) \,|\, u_{l,\alpha} = 0 \quad \text{if} \quad l \notin L\mathbb{Z} \quad \text{or} \quad \alpha \neq 1\right\},$$

applying the mini-max principle, and taking into account that

$$\int_0^{\frac{2\pi}{\tau L}} e^{iL(m-m')\tau y} dy = \frac{2\pi}{\tau L}\delta_{m,m'}, \quad m, m' \in \mathbb{Z},$$

we easily find that

$$n_*(s\sqrt{2\sqrt{\lambda}}; \mathcal{Q}_3(\mathcal{R}_-)) \geq n_+(s^2 2\sqrt{\lambda}; M_3^-) = \#\left\{m \in \mathbb{Z} \,|\, \nu_m^- > s^2 2\sqrt{\lambda}\right\}, \quad s > 0, \quad \lambda > 0. \tag{3.5.7}$$

Utilizing again Lemma 3.5.1, we get

$$\lim_{\lambda\downarrow 0} \frac{\#\left\{m \in \mathbb{Z} \,|\, \nu_m^- > s\sqrt{\lambda}\right\}}{|\ln\lambda|^{1/2}} = \frac{\sqrt{2}}{\sqrt{b}TL(q)}, \quad s > 0. \tag{3.5.8}$$

Putting together (3.2.4), the lower bound in (3.5.2), (3.5.6), (3.5.7), and (3.5.8), and optimizing with respect to q, we obtain the lower bound in (3.2.6).

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