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DEDICATORIA

Dedico este trabajo

a mi hijo

Vicente

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INDICE

RESUMEN

Este trabajo trata el problema de encontrar soluciones positivas de la siguiente clase de ecuaciones semi-lineales elípticas en $\mathbb{R}^N$:

$$\Delta u + u^p + u^q = 0, \tag{1}$$

$$0 < u(x) \to 0 \text{ as } \|x\| \to +\infty, \tag{2}$$

con $1 < p < q$. A estas soluciones se les llama usualmente 'ground states'.

En particular nos interesa la existencia de soluciones radialmente simétricas de (1)-(2). Esto significa soluciones que dependen de $r = \|x - a\|$ para algún a en $\mathbb{R}^N$.

En caso que $1 < p < \frac{N+2}{N-2}$ se sabe que todas las soluciones de (1)-(2) son radialmente simétricas en torno a algún punto.

Las soluciones radialmente simétricas de este problema corresponden a aquellas de la ecuación

$$u'' + \frac{N-1}{r}u' + u_+^p + u_+^q = 0, \quad r > 0 \tag{3}$$

$$u'(0) = 0, \quad u(r) > 0 \; \forall r > 0, \quad \lim_{r \to +\infty} u(r) = 0. \tag{4}$$

donde $u_+ = \max\{u, 0\}$. Un hecho conocido, y que revisaremos, es que cuando las potencias satisfacen $\frac{N}{N-2} < p < q \leq \frac{N+2}{N-2}$ entonces no existe soluciones, mientras que si $\frac{N+2}{N-2} \leq p < q$ éstas sí existen y constituyen un continuo: cuanquier solución de (3) con valores iniciales $u'(0) = 0$, $u(0) > 0$ permanece positiva, y satisface (4).

Mucho más delicado es el caso de potencias sub y super-críticas combinadas, esto es

$$1 < p < \frac{N+2}{N-2} < q. \tag{5}$$

Un ejemplo interesante fue descubierto por Lin y Ni en este rango: si además tenemos que $q = 2p - 1$ entonces hay una solución explícita de la forma $u(r) = \left(\frac{A}{B+r^2}\right)^{\frac{1}{p-1}}$, donde A y B son constantes positivas explícitas que dependen de p y N.

Sorprendentemente, quizás, nada más se sabía hasta ahora en lo relativo a existencia o no-existencia de 'ground states' en el caso general (5). En este trabajo se

establece los siguientes resultados en este caso: Si se fija $q > \frac{N+2}{N-2}$, entonces dado cualquier entero $k \geq 1$, existe un número $p_k < \frac{N+2}{N-2}$ tal que para $p_k < p < \frac{N+2}{N-2}$, (3)-(4) posee al menos k soluciones radiales con *decaimiento rápido*, en el siguiente sentido, $u(r) = O(r^{2-N})$ cuando $r \to \infty$.

Por otra parte, si se fija $\frac{N}{N-2} < p < \frac{N+2}{N-2}$, existe un número $q_k > \frac{N+2}{N-2}$ tal que si $\frac{N+2}{N-2} < q < q_k$, entonces (3)-(4) tiene al menos k soluciones con decaimiento rápido.

También se prueba que en la situación de Lin y Ni $q = 2p - 1$, no sólo la solución explícita existe, sino también una infinidad de soluciones de decaimiento rápido, siempre y cuando cierta restricción adicional en p se cumpla. Más aun, si se fija p, entonces para todo q suficientemente cercano a $2p - 1$ existe un número arbitrariamente grande de soluciones.

Una contraparte de no-existencia de soluciones, que tambien se prueba aquí, es la siguiente: si se fija q y tomamos p suficientemente cercano (o menor) a $\frac{N}{N-2}$ entonces no existe soluciones.

El problema de existencia de soluciones con decaimiento lento o singulares, cuya aparición no se espera que sea genérica, también se analiza.

Las demostraciones de estos resultados están ampliamente basadas en un delicado análisis de espacio de fase, via métodos de sistemas dinámicos, de un sistema tri-dimensional equivalente a la ecuación original. El análisis geométrico realizado aparece como una herramienta útil que podría aplicarse a otras preguntas en este campo de estudio.

SUMMARY

This work deals with the problem of finding positive solutions of the following semi linear elliptic equations in $\mathbb{R}^N$.

$$\Delta u + u^p + u^q = 0, \tag{1}$$

$$0 < u(x) \to 0 \text{ as } \|x\| \to +\infty, \tag{2}$$

with $1 < p < q$. Such solutions are usually called *ground states.*

We are in fact interested in the existence of radially symmetric solutions to (1)-(2). This means solutions that depends on $r = \|x - a\|$ for some a in $\mathbb{R}^N$.

In case that $1 < p < \frac{N+2}{N-2}$ it is known that *all* solutions of (1)-(2) are radially symmetric around some point.

Radially symmetric solutions around the origin correspond to solutions of the equation

$$u'' + \frac{N-1}{r}u' + u_+^p + u_+^q = 0, \quad r > 0 \tag{3}$$

$$u'(0) = 0, \quad u(r) > 0 \; \forall r > 0, \quad \lim_{r \to +\infty} u(r) = 0. \tag{4}$$

where $u_+ = \max\{u, 0\}$.

A rather well known fact, which we review, is that when the powers satisfy $\frac{N}{N-2} < p < q \leq \frac{N+2}{N-2}$ then no radial ground states exist, while if $\frac{N+2}{N-2} \leq p < q$ then ground states do exist and they constitute a continuum: any solution of equation (3) with initial values $u'(0) = 0$, $u(0) > 0$ remains positive, and satisfies (4).

Much more delicate is the case of combined super-subcritical powers, namely

$$1 < p < \frac{N+2}{N-2} < q. \tag{5}$$

An interesting example was discovered by Lin and Ni in this range, if we further have that $q = 2p - 1$. In this case there is an explicit solution of the form $u(r) = \left(\frac{A}{B+r^2}\right)^{\frac{1}{p-1}}$, where A and B are explicit positive constants depending on p and N.

Perhaps surprisingly, besides this example nothing has been known so far concerning existence or nonexistence of ground states in the general range (5). Concerning this case, in this work the following facts are established:

If $q > \frac{N+2}{N-2}$ is fixed, then given any integer $k \geq 1$, there exists a number $p_k < \frac{N+2}{N-2}$ such that for $p_k < p < \frac{N+2}{N-2}$, then (3)-(4) has at least k solutions with *fast decay*, in the sense that $u(r) = O(r^{2-N})$ as $r \to \infty$.

On the other hand, if $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ is fixed, there is a number $q_k > \frac{N+2}{N-2}$ such that if $\frac{N+2}{N-2} < q < q_k$, then (3)-(4) has at least k radial solutions with fast decay.

It is also established that in Lin and Ni's situation $q = 2p - 1$, not only the explicit solution exists but also an infinite number of ground states with fast decay exists provided certain additional restriction in p holds. Moreover, if we fix p, then for all q sufficiently close to $2p - 1$ one has an arbitrarily large number of solutions.

A non-existence counterpart of these results also proven here is the following: if we fix q and then let p be close enough (or below) to $\frac{N}{N-2}$ then no ground states exist.

The question of existence of slow decay and singular ground states, whose appearance is not expected to be generic is also analyzed.

The proofs of these results are largely based on a delicate phase-space analysis, via methods of dynamical systems, of a three dimensional autonomous first order system equivalent to the original equation. The geometric analysis carried out seems to be a useful tool that may be applied to the resolution of other subtle questions arising in this field.

INTRODUCTION

Since the 1960's a lot of attention has been devoted to the study of partial differential equations of the form

$$\Delta u + f(u) = 0, \text{ in } \mathbb{R}^N \tag{0.1}$$

$$0 < u(x) \to 0 \text{ as } |x| \to +\infty, \tag{0.2}$$

which arise in a variety of fields, like stationary states of reaction-diffusion equations in Chemistry, population dynamics, or standing waves of nonlinear Schrödinger equations in nonlinear optics, among other examples. Here Δ denotes the standard Laplacian operator in $\mathbb{R}^N$, $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$, and $f(u)$ a real function with $f(0) = 0$. Solutions of (0.1) satisfying the decay condition (0.2) are usually referred to as *ground states.* The problem of classifying solutions of (0.1)-(0.2) has attracted large amounts of research, and the answers depend strongly on the particular nonlinearity $f(u)$ that is being considered. If one looks for solutions depending only on the distance to the origin, $u = u(r)$, $r = |x|$, then problem (0.1)-(0.2) becomes reduced to the ordinary differential equation

$$u'' + \frac{N-1}{r}u' + f(u) = 0, \quad r > 0 \tag{0.3}$$

$$u'(0) = 0, \quad 0 < u(r) \to 0 \text{ as } r \to +\infty. \tag{0.4}$$

Solutions of this form are called *radially symmetric.* An important question for instance is whether all solutions of (0.1)-(0.2) are radially symmetric around some point. The answer is affirmative under a variety of conditions. While a broad literature is available on the subject, which is continued to be developed until today, one can mention, among the most influential, the works [18], [6], [7], [4].

Perhaps the most celebrated equation of this form is the so-called Lane-Emden-Fowler equation

$$\Delta u + u^p = 0, \text{ in } \mathbb{R}^N \tag{0.5}$$

with $p > 1$. This equation was introduced by Lane in 1869, as a model for internal constitution of stars in Astrophysics, and then considered in the same setting by Emden in 1907 and by Eddington in 1926. It was in 1931 that Fowler [5] solved completely the problem of finding radially symmetric solutions $u = u(|x|)$. The introduction of the ingenious transformation

$$x(t) = r^{\frac{2}{p-1}} u(r)|_{r=e^t} \tag{0.6}$$

reduces the equation, for radial solutions, to the autonomous second order ordinary differential equation

$$x'' + \alpha x' + x^p - \beta x = 0, \quad -\infty < t < +\infty. \tag{0.7}$$

where

$$\alpha = N - 2 - \frac{4}{p-1}, \quad \beta = \frac{2}{p-1}(N - 2 - \frac{2}{p-1}), \tag{0.8}$$

whose trajectories can be fully understood via standard phase-plane analysis. We will carry out completely this analysis in Chapter 3 of this work. We also refer to the appendix in [11].

In particular, the following facts hold true: when $N \geq 3$ the 'critical' exponent $p = \frac{N+2}{N-2}$ sets a dramatic shift in the structure of the radial solutions of this equation. For $1 < p \leq \frac{N+2}{N-2}$ solutions need to be radially symetric around some point, see [7] and [4]. Moreover, if $1 < p < \frac{N+2}{N-2}$ no radial solution of (0.5), defined in the entire space, exists. See Chapter 1. On the other hand, if $p = \frac{N+2}{N-2}$ such solutions exist and are all of the form

$$u_\lambda(r) = a_N \left(\frac{\lambda}{\lambda^2 + r^2}\right)^{\frac{N-2}{2}}, \quad r = |x|, \ \lambda > 0.$$

When $p > \frac{N+2}{N-2}$ ground states also exist and they are constituted by a continuum of the form $u_\lambda(r) = \lambda^{\frac{2}{p-1}} u_1(\lambda r)$. There is also a difference in asymptotic behavior of the ground states between the critical and supercritical cases. In fact, observe that for $p = \frac{N+2}{N-2}$

$$u_\lambda(r) \sim a_N \lambda^{\frac{N-2}{2}} r^{-(N-2)}$$

as $r \to +\infty$, while for $p > \frac{N+2}{N-2}$

$$u_\lambda(r) \sim C_{p,N} r^{-\frac{2}{p-1}}.$$

where the constant

$$C_{p,N} = (\frac{2}{p-1}\{\frac{2}{p-1} - (N-2)\})^{\frac{1}{p-1}}$$

is precisely that making the right hand side of the above relation a (singular) solution of the equation. Observe that this singular solution still exists when $p = \frac{N+2}{N-2}$ but its decay rate is slower than that of the ground states: like $r^{-\frac{N-2}{2}}$.

In view of the above discussion, it seems natural to ask whether there exist solutions if the nonlinearity is replaced by the sum of two powers, namely the problem in $I\!R^N$

$$\Delta u + u^p + u^q = 0, \tag{0.9}$$

$$0 < u(x) \to 0 \text{ as } |x| \to +\infty, \tag{0.10}$$

with $1 < p < q$.

The study of existence of radially symmetric solutions to (0.9)-(0.10) is the purpose of the present work.

We should remark that in case that when $1 < p < \frac{N+2}{N-2}$ in fact *all* solutions of (0.9)-(0.10) are radially symmetric around some point, as established by Zou in [23].

Radially symmetric solutions of this problem correspond to those of the equation

$$u'' + \frac{N-1}{r}u' + u_+^p + u_+^q = 0, \quad r > 0 \tag{0.11}$$

$$u'(0) = 0, \quad u(r) > 0 \;\forall r > 0, \quad \lim_{r\to+\infty} u(r) = 0. \tag{0.12}$$

Here $u_+ = \max\{u, 0\}$. As we will see in Chapter 1, when the powers are both subcritical or both supercritical, the situation is fairly similar to that of the single power. In fact, if $\frac{N}{N-2} < p < q \le \frac{N+2}{N-2}$ then no radial ground states exist, while if $\frac{N+2}{N-2} \le p < q$ then ground states do exist and they constitute a continuum: any solution of equation (0.11) with initial values $u'(0) = 0$, $u(0) > 0$ remains positive, and satisfies (0.12).

Much more delicate is the case of combined super-subcritical powers, namely

$$1 < p < \frac{N+2}{N-2} < q. \tag{0.13}$$

An interesting example was discovered by Lin and Ni in [12]: in this range, if we further have $q = 2p - 1$ then there is an explicit solution of the form

$$u(r) = \left(\frac{A}{B + r^2}\right)^{\frac{1}{p-1}},$$

where A and B are explicit positive constants depending on p and N.

As for the solutions of (0.11)-(0.12), one can show that their behavior as $r \to +\infty$ may only be of one the following two types: Either $O(r^{-(N-2)})$ in whose case we say that the solution is of *fast decay*, or $\sim C_{p,N} r^{-\frac{2}{p-1}}$ as $r \to +\infty$, which we call *slow decay.* We will give the proof of this rather well-known fact in Chapter 1. This is also the case for *singular ground states.* A (radial) *singular ground state* of (0.9) is a solution $u(r) > 0$ of (0.11) which satisfies that $u(r) \to +\infty$ as $r \to 0^+$.

It is perhaps worthwhile mentioning two recent quotes concerning the question of existence of ground states in the general range of exponents (0.13).

H. Zou, Indiana Univ. Math. J., 1996: *When* $\frac{N}{N-2} < p < \frac{N+2}{N-2} < q$, *the simple looking equation becomes quite complicated and it has drawn much attention recently. It should not be surprising when one examines the nonlinearity more carefully. Indeed, this is exactly the mixed-growth case, i.e. subcritical near the origin* $u = 0$ *and supercritical near infinity.*

M. Tang, J. Differential Equations, 2000: ... *very little is known for this semilinear elliptic equation. Zou proved that any solution is radial; Serrin and Zou proved that it can admit at most one slow decay solution. Lin and Ni constructed explicitely some slow decay solutions when* $q = 2p - 1$. *On the other hand, it is unknown if it has any positive solution at all for other* (p, q) *values. Finally, it is not even known whether there are any fast decay solutions. The analysis is surprisingly difficult, and it seems that our approach may not work in this case*

Next we state our main results concerning this question. Our first result states that if q is fixed and we let p approach $\frac{N+2}{N-2}$ from below, then this problem has a large number of radial solutions. A similar fact takes place if we fix $p > \frac{N}{N-2}$ and then let q approach $\frac{N+2}{N-2}$ from above.

Theorem 0.1 (a) *Let* $q > \frac{N+2}{N-2}$ *be fixed. Then, given any integer* $k \geq 1$,*there exists a number* $p_k < \frac{N+2}{N-2}$ *such that if* $p_k < p < \frac{N+2}{N-2}$, *then* $(0.11) - (0.12)$ *has at least* k *radial ground states with fast decay.*
(b) *Let* $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ *be fixed. Then, given any integer* $k \geq 1$,*there exists a number* $q_k > \frac{N+2}{N-2}$ *such that if* $\frac{N+2}{N-2} < q < q_k$, *then* $(0.11) - (0.12)$ *has at least* k *radial ground states with fast decay.*

The existence result of Theorem 0.1 concerns fast-decay ground states. In fact, slow-decay solutions, as that found by Lin and Ni, or singular solutions are harder to be obtained. It can actually be proven that if a solution of one of those types exists, then it must be unique, see [19]. For existence of singular ground states or slow-decay ground states, we have the following result.

Theorem 0.2 (a) *Given $q > \frac{N+2}{N-2}$, there exists an increasing sequence of numbers $p_1 < p_2 < \ldots$ with $p_k \uparrow \frac{N+2}{N-2}$ such that if $p = p_k$ then there is a radial singular ground state of* (0.11), *with either slow or fast decay.*
(b) *Given $\frac{N}{N-2} < p < \frac{N+2}{N-2}$, there exists a decreasing sequence of numbers $q_1 > q_2 > \ldots$ with $q_k \downarrow \frac{N+2}{N-2}$ such that if $q = q_k$ then there is either a slow decay ground state or a slow decay singular solution.*

We observe that the slow-decay solution by Lin and Ni is not covered by the above asymptotic result. which leaves out in principle the case $q = 2p - 1$. Our next result reveals a rather striking resonance phenomenon that arises in such case, if the range of p is a bit further restricted: Not only Lin and Ni's solution exists, but also *infinitely many solutions with fast decay.* Moreover, if we fix p, then for all q sufficiently close to $2p - 1$ one has an arbitrarily large number of solutions.

Theorem 0.3 *Assume that p and q satisfy (0.13) and additionally that*

$$\frac{N + 2\sqrt{N-1}}{N + 2\sqrt{N-1} - 4} < p. \tag{0.14}$$

If $q = 2p-1$ then there exist infinitely many positive ground states with fast decay of (0.11)-(0.12). In particular, given any integer $k \geq 1$, there exists a number $\varepsilon_k > 0$ such that if $|q - (2p-1)| < \varepsilon_k$ then there are at least k radial positive ground states with fast decay of (0.11)-(0.12).

This result is actually a consequence of a more general fact which we state next.

Theorem 0.4 (a) *Assume that p and q satisfy (0.13) and that (0.14) holds. Then if there is a radial ground state of (0.11)-(0.12) with slow decay, there are infinitely many radial ground states with fast decay.*
(b) *Assume that*

$$N \leq 10 \quad \text{or} \quad q < \frac{N - 2\sqrt{N-1}}{N - 2\sqrt{N-1} - 4}. \tag{0.15}$$

Then if there is a singular radial ground state of (0.11), there are infinitely many radial ground states with fast decay.

(c) *If $\bar{p}$, $\bar{q}$ are numbers like in cases (a) or (b), then given any integer $k \geq 1$, there exists a number $\varepsilon_k > 0$ such that if*

$$|p - \bar{p}| + |q - \bar{q}| < \varepsilon_k,$$

then (0.11)-(0.12) has at least k radial solutions with fast decay.

In particular, we observe that along the sequence $p = p_k \uparrow \frac{N+2}{N-2}$ predicted by Theorem 0.2, for fixed $q > \frac{N+2}{N-2}$, infinitely many fast-decay ground states exist.

Finally, the next issue establishes that in a sense the result of part (a) of Theorem 0.1 is optimal: If we fix q and then let p be close enough (or below) to $\frac{N}{N-2}$ then no ground states exist. Let us also observe, incidentally, that for $q = 2p - 1$ there are ground states, and $\frac{N+2}{N-2} = 2\frac{N}{N-2} - 1$.

Theorem 0.5 *Let $q > \frac{N+2}{N-2}$ be fixed. Then there is a number $\bar{p} > \frac{N}{N-2}$ such that if $1 < p < \bar{p}$ then there are neither ground states nor radial singular ground states of* (0.11).

The proof of these results is largely based on a rather delicate phase-space analysis of a three dimensional autonomous first order system equivalent to the original equation, which we shall develop in the following chapters. The basic issue is that the Emden-Fowler transformation (0.6), makes the problem of finding radial ground states equivalent to that of finding positive solutions, which decay to zero as $t \to \pm\infty$, for the equation

$$x'' + \alpha x' + x_+^q + e^{\gamma t} x_+^p - \beta x = 0, \tag{0.16}$$

where

$$\alpha = N - 2 - \frac{4}{q-1}, \quad \beta = \frac{2}{q-1}(N - 2 - \frac{2}{q-1}), \quad \gamma = 2\frac{q-p}{q-1}.$$

We establish this in Lemma 1.1 in Chapter 1. Also in this chapter we deal with the comparatively simpler cases (1.19) and (1.20). In fact, in Propositions 1.2 and 1.3 we establish that the situation is quite comparable to the single power case.

In Chapter 2 we carry out a preliminary analysis of a three-dimensional autonomous system equivalent to equation (0.16), and we establish basic facts about it, as the equivalence of the problem of finding positive solutions to (0.16) with that of finding heteroclinic trajectories of the system lying simultaneously on the two-dimensional (surfaces) unstable and stable invariant manifolds of two equilibria, O_0 and O_∞, which represent respectively the asymtotic-behavior at $-\infty$ and at $+\infty$.

Chapter 3 develops the main tool in the proof of the existence result Theorem 0.1: A "topological shooting" result, Proposition 3.1, which establishes that if two trajectories, one emerging from O_0 and another ending at O_∞, wind around each other several times, then several distinct intersections of the invariant manifolds appear, thus yielding existence of several (fast-decay) ground states. In terms of equation (0.3) simply reads as follows: if u_1 and u_2 are two solutions, with u_1 coming positive from $r = 0$, and u_2 ending eventually positive as $r \to +\infty$, which cross each other $2k+1$ times, then k solutions exist. In Chapter 4 we carry out the proof of Theorem 0.1 by means of this tool.

The situation predicted in Theorem 0.2 takes place precisely when some degeneracy occurs when a certain winding number changes. We prove this in Chapter 5, while we study the case of infinitely many solutions with fast decay under the conditions of Theorem 0.4. in Chapter 6. The non-existence result of Theorem 0.5, is established in Chapter 7.

Finally, we would like to mention that methods of dynamical systems applied to this type of questions have not been extensively used, and they may indeed provide satisfactory answers in somewhat subtle questions as those here treated. We refer the reader to the works [10], [11], [3] for related questions treated with geometric methods. We believe that the method developed in this work may as well be useful in study of ground states for equations of this type in which the nonlinearity is not autonomous, such as the prescribed scalar curvature problem in $\mathbb{R}^N$, see [13], [14], [22], [11] and their references, or to quasilinear equations, involving nonlinear elliptic, rotation-invariant differential operators like the p-Laplacian, see for instance [20], [17], [21] and references therein.

PRELIMINARY RESULTS

1 Preliminary results

We are interested in solving the following problem

$$u'' + \frac{N-1}{r}u' + u_+^p + u_+^q = 0, \quad r > 0 \tag{1.1}$$

$$u'(0) = 0, \quad u(r) > 0 \, \forall \, r > 0, \quad \lim_{r \to +\infty} u(r) = 0 \tag{1.2}$$

where p and q satisfies

$$\frac{N}{N-2} < p < \frac{N+2}{N-2} < q. \tag{1.3}$$

1.1 Properties of positive solutions of (1.1)

Let us first establish some general facts valid for all positive solutions of (1.1).

Lemma 1.1 *Let u be a solution of* (1.1) *with $u(r) > 0$ for all $r > 0$. Then u is decreasing in* $(0, \infty)$. *Moreover, there is a constant $K > 0$ depending only on p and N such that for all sufficiently large $r > 0$,*

$$u(r) \leq K r^{-\frac{2}{p-1}}. \tag{1.4}$$

Proof. To see this, let us write equation (1.1) in the form

$$(r^{N-1}u')' = -r^{N-1}(u^p + u^q). \tag{1.5}$$

Hence $r^{N-1}u'$ is decreasing. Assume by contradiction that there is a number $r^* > 0$ with $u'(r^*) > 0$. Then for all $r < r^*$, $r^{N-1}u'(r) \geq r_*^{N-1}u'(r_*) > 0$. Integrating this relation we then obtain

$$u(r) \leq -C_1 r^{2-N} + C_2$$

for positive numbers C_1, C_2. This is a contradiction since u is positive for $r > 0$. Thus $u'(r) \leq 0$ for all $r > 0$.

Now we prove assertion (1.4). Since $u'(r) \leq 0$, it follows that for any $\delta > 0$ and $r > \delta$,

$$r^{N-1}u'(r) \leq -\int_\delta^r s^{N-1}(u^p(s) + u^q(s))ds. \tag{1.6}$$

Since u is decreasing we can estimate $r^{N-1}u'(r) \leq -u^p(r)\int_\delta^r s^{N-1}ds$. Since δ is arbitrary we get

$$r^{N-1}u'(r) \leq -u^p(r)\int_0^r s^{N-1}ds,$$

for all large r, hence $\frac{r}{N} \leq -u(r)^{-p}u'(r)$. Thus, integrating again, we obtain that for all sufficiently large r,

$$\frac{r^2}{2N} \leq \frac{u(r)^{1-p}}{p-1},$$

from where estimate (1.4) follows. □

We observe that the above result implies in particular that all positive solutions of (1.1) go to zero as $r \to +\infty$.

Let us now set some terminology. Since a positive solution u of (1.1) is decreasing, two possibilities arise: either u has a finite limit as $r \to 0^+$ or $u(r) \to +\infty$ as $r \to 0^+$. In the latter situation we say that u is a *singular ground state.* If u is non-singular conditions (1.2) are automatically satisfied.

Next we will define certain behaviors at infinity for a positive solution u of (1.1), which will later be established to be the only possible ones .

Definition. A positive solution u of (1.1) is said to be *of fast decay* if $u(r) = O(r^{2-N})$ as $r \to \infty$. If $\lim_{r\to\infty} r^{\frac{2}{p-1}}u(r) = \tilde{\beta}^{\frac{1}{p-1}}$, u is called *of slow decay.*

In order to prove the next result let us consider the exponent p in the Emden-Fowler transformation, namely

$$\tilde{x}(t) = r^{\frac{2}{p-1}}u(r)|_{r=e^t}. \tag{1.7}$$

Then $\tilde{x}$ satisfies

$$\tilde{x}'' - \tilde{\alpha}\tilde{x}' - \tilde{\beta}\tilde{x} + \tilde{x}_+^p + e^{-\tilde{\gamma}t}\tilde{x}_+^q = 0, \quad -\infty < t < +\infty, \tag{1.8}$$

where

$$\tilde{\alpha} = \frac{4}{p-1} - (N-2), \quad \tilde{\beta} = \frac{2}{p-1}(N-2-\frac{2}{p-1}), \quad \tilde{\gamma} = 2\frac{q-p}{p-1}$$

and these coefficients are positive, which follows from (1.3).

Proposition 1.1 *Assume (1.3) holds. Let $u(r)$ be a solution of (1.1)-(1.2). Then, u is either of fast or slow decay as $r \to +\infty$.*

Proof. We will carry out the proof by using the equation (1.8) . Multiplying equation (1.8) by $\tilde{x}'$ and integrating, we obtain the relation

$$\tilde{\alpha}\int_{-\infty}^{T}\tilde{x}'(s)^2ds =$$

$$\frac{\tilde{\gamma}}{q+1}\int_{-\infty}^{T}e^{-\tilde{\gamma}s}\tilde{x}(s)_+^{q+1}ds+\frac{e^{-\tilde{\gamma}T}}{q+1}\tilde{x}(T)_+^{q+1}+\frac{\tilde{x}'(T)^2}{2}+\frac{\tilde{x}(T)_+^{p+1}}{p+1}-\tilde{\beta}\frac{\tilde{x}(T)^2}{2}. \quad (1.9)$$

We call

$$F(\xi)=\frac{\xi_+^{p+1}}{p+1}-\tilde{\beta}\frac{\xi^2}{2}.$$

We will prove that $F(\tilde{x}(t))$ has a limit as $t\to+\infty$, after which our assertion easily follows. Let us notice that $\int_{-\infty}^{+\infty}\tilde{x}'(s)^2ds$ is finite. In fact, by (1.7), $\tilde{x}$ satisfies

$$\tilde{x}'(t)=\frac{2}{p-1}\tilde{x}(t)+e^{\frac{p+1}{p-1}t}u'(e^t).$$

Both terms in the right hand side of the equality are bounded thanks to the relation

$$r^{N-1}u'(r)=-\int_0^r s^{N-1}(u^p(s)+u^q(s))ds.$$

It follows that all terms in the right hand side of equality (1.9) are uniformly bounded, hence for certain constant C,

$$\tilde{x}'(t)^2=C-F(\tilde{x}(t))+o(1). \quad (1.10)$$

with $o(1)\to 0$ as $t\to+\infty$. We claim that $F(x(t))\to C$ as $t\to+\infty$. Let us assume the opposite: then for some $\delta>0$ there is a sequence of numbers s_n with $s_n\to+\infty$ as $n\to+\infty$ such that $C-F(\tilde{x}(s_n))>2\delta$. Since $\tilde{x}(s_n)$ is a bounded sequence, we may pass to a subsequence which we label the same way, such that $\tilde{x}(s_n)\to L$, for some number L, and which is furthermore such that

$$C-F(L)>\delta.$$

Let h be any positive number. We have that

$$\sqrt{h}\left(\int_{s_n}^{s_n+h}\tilde{x}'(t)^2dt\right)^{1/2}>|\tilde{x}(s_n+h)-\tilde{x}(s_n)|.$$

Thus

$$|\tilde{x}(s_n+h)-\tilde{x}(s_n)|\le\sqrt{h}\left(\int_{s_n}^{\infty}\tilde{x}'(t)^2dt\right)^{1/2}.$$

Since $\int_{-\infty}^{\infty} \tilde{x}'(t)^2 dt$ is finite, we get from the above expression that for any $h > 0$, and any $s \in [0, h]$, $\tilde{x}(s_n + s) \to L$. Let us now set

$$G_n(s) = C - F(\tilde{x}(s_n + s)).$$

Then G_n is uniformly bounded in $[0, h]$ and $G_n(s) \to C - F(L)$. It follows that

$$\lim_{n\to\infty} \int_0^h G_n(s)ds = \int_0^h (C - F(L))ds = h(C - F(L)).$$

Thus from relation (1.10) and our contradiction assumption,

$$\lim_{n\to\infty} \int_{s_n}^{s_n+h} \tilde{x}'(t)^2 dt = h(C - F(L)) > \delta h.$$

But this is impossible, since $\int_{-\infty}^{\infty} \tilde{x}'(t)^2 dt$ is finite. Hence $F(\tilde{x}(t)) \to C$ as $t \to +\infty$, as claimed. In particular we have that $\tilde{x}'(t) \to 0$ as $t \to +\infty$. Now because of the definition of F, we also see that only one limit point for $\tilde{x}(t)$ as $t \to +\infty$ is possible, for otherwise $F(\tilde{x}(t))$ would be along a sequence going to infinity away from the value C. Finally, coming back to the equation satisfied by $\tilde{x}$,

$$\tilde{x}'' - \tilde{\alpha}\tilde{x}' - \tilde{\beta}\tilde{x} + \tilde{x}^p + e^{-\tilde{\gamma}t}\tilde{x}^q = 0,$$

setting $L = \lim_{t\to+\infty} \tilde{x}(t)$, using that $\lim_{t\to+\infty} \tilde{x}'(t) = 0$ and choosing a sequence $s_n \to +\infty$ along which $\tilde{x}''(s_n) \to 0$, we end up with the relation

$$-\tilde{\beta}L + L^p = 0.$$

Thus either $L = 0$ or $L = \tilde{\beta}^{\frac{1}{p-1}}$ as claimed. This concludes the proof. □

Now we introduce a second Emden-Fowler transformation of (1.1)-(1.2), which is more suitable for the analysis of solutions near $r = 0$. Let us consider the change of variables

$$x(t) = r^{\frac{2}{q-1}} u(r)|_{r=e^t}. \tag{1.11}$$

Then x defined in this way satisfies

$$x'' + \alpha x' - \beta x + x_+^q + e^{\gamma t} x_+^p = 0, \quad -\infty < t < +\infty, \tag{1.12}$$

and, as we will see later,

$$x(t) > 0 \;\forall t, \quad \lim_{t\to\pm\infty} x(t) = 0, \tag{1.13}$$

where

$$\alpha = N-2-\frac{4}{q-1}, \quad \beta = \frac{2}{q-1}(N-2-\frac{2}{q-1}), \quad \gamma = 2\frac{q-p}{q-1}$$

and these coefficients are positive, which follows from (1.3).

Our next result asserts that problems (1.1)-(1.2) and (1.12)-(1.13) are equivalent under the transformation (1.11). Similarly, transformation (1.7) makes (1.1)-(1.2) equivalent to (1.8) with appropriate conditions at infinity.

Lemma 1.2 *Let $u(r)$ be a positive solution of (1.1) in $(0,\infty)$. Let $x(t)$ be given by (1.11) and $\tilde{x}$ given by (1.7). Then*

(a) $\lim_{t\to-\infty}\tilde{x}(t)=0, \quad \lim_{t\to+\infty}x(t)=0.$

(b) *$u(r)$ satisfies (1.2) if and only if* $\lim_{t\to\pm\infty}x(t)=0.$

(c) *$u(r)$ has fast decay if and only if* $\lim_{t\to+\infty}\tilde{x}(t)=0.$

(d) *$u(r)$ has slow decay if and only if* $\lim_{t\to+\infty}\tilde{x}(t)=\tilde{\beta}^{\frac{1}{p-1}}.$

(e) *$u(r)$ is a singular ground state if and only* $\lim_{t\to-\infty}x(t)=\beta^{\frac{1}{q-1}}.$

Proof. Let u be a positive solution of (1.1). Let us prove part (a). We recall that from relation (1.4), we have that $u(r)\le Kr^{-\frac{2}{p-1}}$ for large r. By definition of $x(t)$ we then have

$$x(t)\le Ke^{-\frac{2(q-p)}{(q-1)(p-1)}t}\to 0.$$

The other assertion of part (a) follows symmetrically.

Let us prove part (b) Let $x(t)$ be given by (1.11). It is immediately verified that x satisfies equation (1.12). Let us check that $x(t)\to 0$ as $t\to-\infty$. In fact, let us set $u(0)=c$. Then $u(e^t)\to c$ as $t\to-\infty$, and hence $x(t)=e^{\frac{2t}{q-1}}u(e^t)\to 0$.

Now let us assume that we have a solution $x(t)$ of (1.12)-(1.13), and set

$$u(r)=x(\log r)r^{-\frac{2}{q-1}}. \tag{1.14}$$

Since $x(t)\to 0$ as $t\to+\infty$, we get $u(r)\to 0$ as $r\to\infty$. We have to show that u has a limit as $r\to 0^+$ and that $u'(0^+)=0$. To establish this we make the following claim:

Given $\delta>0$ there exist numbers $T<0$ and $C>0$ such that

$$x(t)\le Ce^{(\frac{2}{q-1}-\delta)t} \quad \forall\, t<T. \tag{1.15}$$

Since $x(t)$ satisfies equation (1.12), and $x(t) \to 0$ as $t \to -\infty$, we find that given $\delta' > 0$, there is a $t_1 < 0$ such that

$$x'' + \alpha x' - (\beta - \delta')x \geq 0, \quad \forall\, t < t_1.$$

Let us choose δ' so that $\lambda = \frac{2}{p-1} - \delta$ be exactly the positive root of

$$\lambda^2 + \alpha\lambda - (\beta - \delta') = 0.$$

Then $z(t) = e^{(\frac{2}{q-1} - \delta)t}$ satisfies the equation

$$z'' + \alpha z' - (\beta - \delta)z = 0.$$

Let C be a positive number so that $x(t_1) - Cz(t_1) < 0$. Then $h(t) \equiv x(t) - Cz(t)$ satisfies that $h(t_1) \leq 0$ and

$$h'' + \alpha h' - (\beta - \delta)h \geq 0, \quad \forall\, t < t_1. \tag{1.16}$$

We will show next that actually $h(t) \leq 0$ for all $t < t_1$, so that estimate (1.15) holds true. To see this, we consider the functions $h_\varepsilon(t) \equiv h(t) - \varepsilon e^{-\rho t}$, where ρ is an arbitrary number with $0 < \rho < \alpha$. Since $h(t) \to 0$ as $t \to -\infty$, we can find a number $t_\varepsilon < t_1$ such that $h_\varepsilon(t) < 0$ for all $t \leq t_1$. Now, since h satisfies relation (1.16), we get then

$$h_\varepsilon'' + \alpha h_\varepsilon' - (\beta - \delta)h_\varepsilon \geq \varepsilon(-\rho^2 + \alpha\rho + (\beta - \delta)) > 0, \quad \forall\, t < t_1. \tag{1.17}$$

Now, if it happened that

$$h_\varepsilon(t_2) = \max_{t \in (t_1, t_\varepsilon)} h_\varepsilon(t) > 0,$$

with $t_2 \in (t_1, t_\varepsilon)$, then $h_\varepsilon'(t_2) = 0$, $h_\varepsilon''(t_2) \leq 0$, then relation (1.17) would be impossible. Hence $h_\varepsilon(t) \leq 0$ for all $t \leq t_1$. Since ε was arbitrary, it follows then that $h(t) \leq 0$ for $t \leq t_1$, as desired. Thus estimate (1.15) holds with $T = t_1$. Now, in terms of u given by (1.14) this means exactly that there exists a number $r_0 = e^T > 0$ such that

$$u(r) \leq Cr^{-\delta} \quad \forall\, 0 < r < r_0.$$

Let us now use the equation satisfied by u. We obtain the following relation

$$r^{N-1}u'(r) = r_0^{N-1}u'(r_0) + \int_r^{r_0} (u^p(s) + u^q(s))s^{N-1}ds.$$

Since u grows at most as a small negative power as $r \to 0^+$, it follows that the right hand side of the above expression has a limit, let us call it L, so that

$$\lim_{r\to 0} r^{N-1}u'(r) = L.$$

We claim that $L = 0$. Let us assume the opposite. Then, given $\varepsilon > 0$ we get that for some $\bar{r} > 0$

$$\frac{L-\varepsilon}{r^{N-1}} < u'(r) < \frac{L+\varepsilon}{r^{N-1}}, \quad \forall\, 0 < r < \bar{r}.$$

Hence, by integration

$$u(r) = \frac{L}{N-2}\frac{1}{r^{N-2}} + \theta(r), \quad \forall\, 0 < r < \bar{r},$$

where $|\theta(r)| < \frac{\varepsilon}{N-2} r^{2-N} + C$ for some $C > 0$. Choosing $\varepsilon < |L|$ is then attained, since $u(r) = O(r^{-\delta})$ as $r \to 0$ where δ is arbitrarily small. Hence $L = 0$. Now, coming back to the equation satisfied by u we find then that

$$-r^{N-1}u'(r) = -\delta^{N-1}u'(\delta) + \int_\delta^r (u^p(s) + u^q(s))s^{N-1}ds,$$

so that letting $\delta \to 0$,

$$u'(r) = r^{1-N}\int_0^r (u^p(s) + u^q(s))s^{N-1}ds. \tag{1.18}$$

Again using $u(s) = O(s^{-\delta})$ as $s \to 0$, for arbitrarily small δ, we find that $u'(r) \to 0$ as $r \to 0$ as required. It also follows that the limit as $r \to 0$ of $u(r)$ itself exists. The corresponding assertion (c) for $\tilde{x}$ is proved similarly. Finally, assertion (d) is already contained in the proof of Proposition 1.1, and the proof of (e) is symmetric. □

1.2 Study of solution for sub and super critical exponents

Here we consider the purely sub-critical and purely super-critical cases, which turn out to be simpler to analyze. These are the cases

$$\frac{N}{N-2} < p < q \le \frac{N+2}{N-2}, \tag{1.19}$$

and

$$\frac{N+2}{N-2} \le p < q. \tag{1.20}$$

For the sub-critical case we have the following result.

Proposition 1.2 *Assume (1.19) holds. Then no solution to problem (1.12)-(1.13) exists.*

Proof. Let us assume that there is such a solution x. Let us multiply equation (1.12) by x' and integrate between s_- and s_+. Let us also integrate by parts the last term. Then we find

$$\left[\frac{x'(s)^2}{2} + \frac{x(s)^{q+1}}{q+1} - \frac{\beta}{2}x(s)^2\right]_{s=s_-}^{s=s_+} +$$

$$\alpha \int_{s_-}^{s_+} x'(s)^2 ds + \frac{e^{\gamma s}x(s)^{p+1}}{p+1} \Big|_{s=s_-}^{s=s_+} - \frac{\gamma}{p+1}\int_{s_-}^{s_+} e^{\gamma s}x(s)^{p+1}ds = 0. \tag{1.21}$$

Now, we claim that

$$\lim_{t\to+\infty} e^{\gamma t}x(t)^{p+1} = 0. \tag{1.22}$$

In fact, by definition of x we have that

$$e^{\gamma t}x(t)^{p+1} = e^{2\frac{q+1}{q-1}t}u(e^t)^{p+1}.$$

Now, from the proof of Lemma 1.1, we have that u satisfies estimate (1.4). This relation implies

$$u(e^t)^{p+1} \leq K^{p+1}e^{-2\frac{p+1}{p-1}t},$$

and hence

$$e^{\gamma t}x(t)^{p+1} \leq K^{p+1}e^{-2\rho t},$$

where $\rho = \frac{p+1}{p-1} - \frac{q+1}{q-1} > 0$. Thus (1.22) holds. Now, let us choose sequences $s_- = s_-^n \to -\infty$ and $s_+ = s_+^n \to +\infty$, such that $x'(s_\pm^n) \to 0$. Such sequences exist since $x(s) \to 0$ as $s \to \pm\infty$. Then from (1.22) and (1.21), we obtain after letting $n \to \infty$,

$$\alpha \int_{-\infty}^{+\infty} x'(s)^2 ds - \frac{\gamma}{p+1}\int_{-\infty}^{+\infty} e^{\gamma s}x(s)^{p+1}ds = 0.$$

Since $\gamma > 0$ and $\alpha \leq 0$ if (1.19) is assumed, we get that $x(t)$ vanishes identically, and we have reached a contradiction which finishes the proof. □

Let us consider now the super-critical case. In this case we will prove that there are indeed solutions to (1.1)-(1.2). These solutions form a continuum and have slow decay. More precisely, let us consider the initial value problem

$$u'' + \frac{N-1}{r}u' + u_+^p + u_+^q = 0, \quad r > 0 \tag{1.23}$$

$$u'(0) = 0, \quad u(0) = c > 0. \tag{1.24}$$

This problem is indeed solvable. In fact, let us consider the integral equation

$$u(r) = c - \int_0^r \rho^{1-N} d\rho \int_0^\rho s^{N-1}(u_+^p(s) + u_+^q(s))ds \equiv \Phi(u)(r).$$

It is easily checked that if $\delta > 0$ is chosen sufficiently small then Φ applies the set

$$B_\delta = \{u \in C[0,\delta] \,/\, \|u - c\|_\infty \leq 1\}$$

into itself. Here

$$\|v\|_\infty = \sup_{0 \leq r \leq \delta} |v(r)|.$$

Besides, Φ is a contraction mapping of the complete metric space B_δ with the metric induced by $\| \; \|_\infty$. Thus (1.23)-(1.24) has a (unique) local solution. This local solution becomes decreasing. Since the nonlinearity is uniformly bounded along the whole range of u, it follows that the solution can actually be extended to the whole real line.

Proposition 1.3 *Assume that the exponents p, q satisfy (1.20). Then for any $c > 0$ the solution of (1.23)-(1.24) satisfies that $u(r) > 0$ for all $r > 0$. Moreover $u(r)$ has slow decay in the sense that there is a $k > 0$ such that for all sufficiently large r,*

$$u(r) \leq k r^{-\frac{2}{p-1}}. \tag{1.25}$$

Proof. Let $c > 0$ be fixed, and u the solution of the initial value problem (1.23)-(1.24). We consider the transformation $\tilde{x}(t)$ given by (1.7).

We recall that the equation now satisfied is

$$\tilde{x}'' - \tilde{\alpha}\tilde{x}' - \tilde{\beta}\tilde{x} + \tilde{x}^p + e^{-\tilde{\gamma}t}\tilde{x}^q = 0.$$

Multiplying the above equation by $\tilde{x}'$, integrating from $-\infty$ to T, and also integrating by parts the last term we find

$$|\tilde{\alpha}| \int_{-\infty}^T \tilde{x}'(s)^2 ds + \frac{\tilde{\gamma}}{q+1} \int_{-\infty}^T e^{-\tilde{\gamma}s}\tilde{x}(s)_+^{q+1} ds + \frac{\tilde{\gamma}}{q+1} e^{-\tilde{\gamma}T}\tilde{x}(T)_+^{q+1} =$$

$$-\frac{\tilde{x}'(T)^2}{2} + \tilde{\beta}\frac{\tilde{x}(T)^2}{2} - \frac{\tilde{x}(T)_+^{p+1}}{p+1}.$$

We observe that it is impossible that $\tilde{x}(T) \leq 0$ unless $\tilde{x} \equiv 0$. On the other hand, for a similar reason, we cannot have the existence of a sequence $T_n \to +\infty$ with $\tilde{x}(T_n) \to 0$. We conclude that $\tilde{x}(t) \geq c > 0$ for all large t. This implies the validity of relation (1.25), thus concluding the proof. □

As a consequence of this chapter, we mention that our problem in what follows is to prove existence of solutions of problem (1.12)-(1.13), which correspond to non-singular ground states with fast or slow decay. We will be also interested in singular solutions. We will consider that p, q satisfies (1.3).

2 The phase space analysis

Recall that we are looking for solution of

$$x'' + \alpha x' + x_+^q + e^{\gamma t} x_+^p - \beta x = 0, \quad -\infty < t < +\infty. \tag{2.1}$$

$$x(t) > 0 \; \forall t, \quad \lim_{t \to \pm\infty} x(t) = 0. \tag{2.2}$$

where

$$\alpha = N - 2 - \frac{4}{q-1}, \quad \beta = \frac{2}{q-1}(N - 2 - \frac{2}{q-1}), \quad \gamma = 2\frac{q-p}{q-1}.$$

Also recall that we are considering that p and q satisfy the relations

$$\frac{N}{N-2} < p < \frac{N+2}{N-2} < q,$$

from where it follows that the coefficients α, β and γ are positive.

For our purpose we introduce the new variables $y = x'$ and $z = e^{\gamma t}$. Equation (2.1) becomes then equivalent to the autonomous first order system

$$\begin{cases} x' = y, \\ y' = -\alpha y + \beta x - x_+^q - z x_+^p, \\ z' = \gamma z. \\ z \geq 0 \end{cases} \tag{2.3}$$

The results of last chapter imply that in terms of these variables our task is equivalent to finding solutions $\mathbf{x}(t) = (x(t), y(t), z(t))$ of this system, with $x(t) > 0$, such that $\mathbf{x}(t) \to (0,0,0)$ as $t \to -\infty$, while $(x(t), y(t), z(t)) \to (0, 0, +\infty)$ as $t \to +\infty$. We will establish this fact in the following lemma.

Lemma 2.1 *$x(t)$ satisfies (2.1)-(2.2) if and only if $\mathbf{x}(t) = (x(t), y(t), z(t))$ satisfies (2.3) with the conditions $x(t) > 0 \; \forall t$, $\mathbf{x}(t) \to (0,0,0)$ as $t \to -\infty$ and $(x(t), y(t), z(t)) \to (0, 0, +\infty)$ as $t \to +\infty$.*

We will be concerned with the phase space analysis of (2.3). This means the study of the orbit structure of this system. We also introduce some notations that will be helpful for us in what follows.

General remarks for the phase space analysis of (2.3):

1. The positive z-axis is an orbit for the flow.

2. The plane $z = 0$ is invariant under the flow. We carry out this phase plane analysis below.

3. Orbits cross transversally the plane $x = 0$. For $y > 0$, they go into the region $x > 0$ while the opposite happens for $y < 0$.

4. The two-dimensional foliation $\mathcal{F}$ given by

$$\mathcal{F} = \{\{(x, y, z) \ / \ z = c\} \ , \ c \in I\!R_+\},$$

is invariant under the flow. The planes $z = c$ move upwards in time. In fact $z(t) = z_0 e^{\gamma t}$, γ positive.

5. The system has two singularities $O_0 = (0,0,0)$ and $P_0 = (\beta^{\frac{1}{q-1}}, 0, 0)$, which are hyperbolic saddle points.

6. The unstable manifold of O_0, $W^u(O_0)$, is two-dimensional and transversal to the plane $z = 0$ and its stable manifold, $W^s(O_0)$, is one-dimensional and lies in $z = 0$. The unstable manifold of P_0, $W^u(P_0)$, is one-dimensional and is transversal to the plane $z = 0$ and its stable manifold, $W^s(P_0)$, is two-dimensional and lies in $z = 0$.

7. The unstable manifold of O_0, $W^u(O_0)$, contains the z-axis for $z \geq 0$. This semi-axis separates $W^u(O_0)$ into two components invariant under the flow, one of them is a half-plane contained in $x < 0$ (observe that for $x \leq 0$ the resulting system is linear). The other component, $W^u_+(O_0)$, is a surface not necessarily fully contained in $x \geq 0$.

Since we are looking for solutions $\mathbf{x}$ of (2.3) with $\mathbf{x}(t) \to (0, 0, +\infty)$ as $t \to +\infty$ we will study the "point" $(0, 0, +\infty)$ at infinity. For this purpose, we introduce the following coordinate system at infinity:

$$\begin{cases} \tilde{x} = x z^{\frac{1}{p-1}}, \\ \tilde{y} = (y + \frac{\gamma x}{p-1}) z^{\frac{1}{p-1}}, \\ \tilde{z} = \frac{1}{z^{\frac{q-1}{p-1}}}. \end{cases} \tag{2.4}$$

In coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ the system (2.3) is written as

$$\begin{cases} \tilde{x}' = \tilde{y}, \\ \tilde{y}' = \tilde{\alpha}\tilde{y} + \tilde{\beta}\tilde{x} - \tilde{x}_+^p - \tilde{z}\tilde{x}_+^q, \\ \tilde{z}' = -\tilde{\gamma}\tilde{z} \\ \tilde{z} \geq 0 \end{cases} \tag{2.5}$$

where now

$$\tilde{\alpha} = \frac{4}{p-1} - (N-2), \quad \tilde{\beta} = \frac{2}{p-1}(N-2-\frac{2}{p-1}), \quad \tilde{\gamma} = 2\frac{q-p}{p-1}$$

are all positive coefficients.

This change of coordinates corresponds exactly to use the Emden-Fowler transformation with the exponent p, rather than q, in the original equation.

Similarly as with system (2.3), we have now the following.

General remarks for the phase space analysis of (2.5):

1. The positive $\tilde{z}$-axis is an orbit for the flow.

2. The plane $\tilde{z} = 0$ is invariant under the flow. We carry out this phase plane analysis below.

3. Orbits cross transversally the plane $\tilde{x} = 0$. For $\tilde{y} > 0$, they go into the region $\tilde{x} < 0$ while the opposite happens for $\tilde{y} < 0$.

4. The two-dimensional foliation $\tilde{\mathcal{F}}$ given by

$$\tilde{\mathcal{F}} = \{\{(\tilde{x}, \tilde{y}, \tilde{z}) \ / \ \tilde{z} = c\} \ , \ c \in \mathbb{R}^+\},$$

   is invariant under the flow. The planes $\tilde{z} = c$ move downwards in time. In fact $\tilde{z}(t) = \tilde{z}_0 \tilde{e}^{-\tilde{\gamma}t}$, $\tilde{\gamma}$ positive.

5. The system has two singularities $O_\infty = (0,0,0)$ and $P_\infty = (\tilde{\beta}^{\frac{1}{p-1}}, 0, 0)$, which are hyperbolic saddle points.

6. The stable manifold of O_∞, $\tilde{W}^s(O_\infty)$, is two-dimensional and transversal to the plane $\tilde{z} = 0$ and its unstable manifold, $W^u(O_\infty)$, is one-dimensional and lies in $\tilde{z} = 0$. The stable manifold of P_∞, $\tilde{W}^s(P_0)$, is one-dimensional and is transversal to the plane $\tilde{z} = 0$, and its unstable manifold, $\tilde{W}^u(P_\infty)$, is two-dimensional and lies in $\tilde{z} = 0$.

7. The stable manifold of O_∞, $\tilde{W}^s(O_\infty)$, contains the $\tilde{z}$-axis for $\tilde{z} \geq 0$. This semi-axis separates $\tilde{W}^u(O_\infty)$ into two components invariant under the flow, one of them is a half-plane contained in $\tilde{x} < 0$ (observe that for $\tilde{x} \leq 0$ the resulting system is linear). The other component, $\tilde{W}^u_+(O_\infty)$, is a surface not necessarily fully contained in $\tilde{x} \geq 0$.

Next we describe the phase plane diagram associated to systems (2.3) and (2.5) when restricted respectively to the planes $z = 0$ and $\tilde{z} = 0$.

We analyze the situation corresponding to different ranges of the exponents p and q.

2.1 The phase plane analysis

Now we will be concerned with the following two-dimensional system.

$$\begin{cases} x' = y, \\ y' = -\alpha y + \beta x - x_+^a \end{cases} \tag{2.6}$$

where $a > \frac{N}{N-2}$ and

$$\alpha = (N-2) - \frac{4}{a-1}, \quad \beta = \frac{2}{a-1}(N-2-\frac{2}{a-1}),$$

with $N \geq 3$.

Since $a > \frac{N}{N-2}$, we have $\beta > 0$ and there are two singularities, $O = (0,0)$ which is a hyperbolic saddle point and $P = (\beta^{\frac{1}{a-1}}, 0)$, which has different nature depending of the different values of a. Linearizing we see that O has an associated one unstable eigenvalue $\frac{2}{a-1} - (N-2) < 0$ with associated eigenvector $(1, \frac{2}{a-1} - (N-2))$, and one stable eigenvalue, $\frac{2}{a-1}$ with eigenvector $(1, \frac{2}{a-1})$.

(i) The case $a = \frac{N+2}{N-2}$. Here $\alpha = 0$. The singularity P is a center. In fact the system in this special case is Hamiltonian, with energy given by

$$H(x,y) = \frac{y^2}{2} + \frac{x_+^{a+1}}{a+1} - \beta\frac{x^2}{2}. \tag{2.7}$$

H is constant along all orbits, and those locally surrounding P are periodic orbits enclosed by a homoclinic orbit of O. See Figure 1.

(ii) The case $a > \frac{N+2}{N-2}$. Now $\alpha > 0$. P is a hyperbolic attractor, with two stable eigenvalues,

$$\frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta(a-1)}}{2}.$$

We see that these eigenvalues are complex if and only if

$$N \leq 10 \quad \text{or} \quad a < \frac{N - 2\sqrt{N-1}}{N - 2\sqrt{N-1} - 4}, \tag{2.8}$$

In this case P is an attractor spiral focus. If instead

$$a > \frac{N - 2\sqrt{N-1}}{N - 2\sqrt{N-1} - 4},$$

the two eigenvalues are real and negative and P becomes a stable node. Also, these singularities are connected by a heteroclinic orbit from O to P, constituted exactly by the unstable manifold of O. This is checked directly using the fact that the functional H given by (2.7) is now strictly decreasing along non constant orbits, see Figure 2 for details.

(iii) The case $a < \frac{N+2}{N-2}$. In this situation $\alpha < 0$. P is a hyperbolic repulsor, with two unstable eigenvalues,

$$\frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta(a-1)}}{2}.$$

These eigenvalues are complex if and only if

$$a > \frac{N + 2\sqrt{N-1}}{N + 2\sqrt{N-1} - 4}. \tag{2.9}$$

In this case P is an repulsive spiral focus. If instead

$$a < \frac{N + 2\sqrt{N-1}}{N + 2\sqrt{N-1} - 4},$$

the two eigenvalues are real and positive and P becomes a unstable node. These singularities are connected by a heteroclinic orbit from P to 0, constituted exactly by the stable manifold of O. To check this one uses the fact that the functional H given by (2.7) is now strictly increasing along non constant orbits, see Figure 3.

2.2 Completing the phase space analysis

Now we consider the full systems (2.3) and (2.5). Concerning (2.3), we restrict ourselves to analyze trajectories that lie in the half-space $z \geq 0$. The plane $z = 0$ contains the two singularities of the flow, $O_0 = (0,0,0)$ and $P_0 = (\beta^{\frac{1}{q-1}}, 0, 0)$, which are hyperbolic. Additionally to the eigenvalues and eigenvectors considered in the planar case (ii) in the previous subsection, the singularity O_0 possesses the unstable eigenvalue $\gamma = 2\frac{(q-p)}{q-1}$ with corresponding eigenvector $(0,0,1)$. Thus, from standard invariant manifold theory, see for instance [9], O_0 has a two dimensional unstable manifold $W^u(O_0)$, constituted by all trajectories approaching O_0 as $t \to -\infty$, whose tangent plane at O_0 is spanned by the two unstable eigenvectors. Moreover, it coincides with this plane for $x < 0$, since $x_+ \equiv 0$ and the flow is linear on this region. $W^u(O_0)$ contains the entire z-axis as well as the heteroclinic orbit on $z = 0$ connecting O_0 and P_0. It is also transversal to the planes $z = 0$ and $x = 0$.

Concerning the singularity P_0 we obtain the unstable eigenvalue γ with associated eigenvector

$$(1, -\gamma, \frac{\beta(p-1) + \alpha\gamma + \gamma^2}{\beta^{\frac{q}{p-1}}})$$

and, as we have seen from the planar analysis (ii), that P_0 has two stable eigenvalues

$$\frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta(p-1)}}{2}.$$

It thus follows that the unstable manifold of P_0 for $z \geq 0$, which we call $W^u(P_0)$, is one-dimensional, and constituted by a single orbit. Its stable manifold is two-dimensional and lies in $z = 0$. We recall that P_0 is a spiral focus attractor for the flow restricted to this plane if (2.8) holds, and a stable node otherwise.

We consider now the flow for system (2.5). Similarly as before, the plane $\tilde{z} = 0$ is invariant under this new flow whose singularities are the points $O_\infty = (0,0,0)$ and $P_\infty = (\tilde{\beta}^{\frac{1}{p-1}}, 0, 0)$, which are again hyperbolic. As we have seen in the planar case (iii), for the flow restricted to $\tilde{z} = 0$, O_∞ is a hyperbolic saddle point and P_∞ is a hyperbolic repulsor which are connected by a heteroclinic orbit. Linearizing around O_∞ one obtains an extra stable eigenvalue $-\tilde{\gamma} = 2\frac{(q-p)}{p-1}$ with associated eigenvectors $(0,0,1)$. Thus, O_∞ has a two dimensional stable manifold $\tilde{W}^s(O_\infty)$, constituted by all trajectories approaching O_∞ as $t \to +\infty$, whose tangent plane at O_∞ is spanned by the two stable eigenvectors and it coincides with this plane for $\tilde{x} < 0$. $\tilde{W}^s(O_\infty)$ contains the entire $\tilde{z}$-axis as well as the heteroclinic orbit on $\tilde{z} = 0$ connecting P_∞ and O_∞. It is also transversal to the planes $\tilde{z} = 0$ and $\tilde{x} = 0$. Now, linearizing around the singularity P_∞ we obtain one stable eigenvalue $-\tilde{\gamma}$ with associated eigenvector

$$(-1, -\tilde{\gamma}, \frac{\tilde{\beta}(q-1) + \tilde{\alpha}\tilde{\gamma} + \tilde{\gamma}^2}{\tilde{\beta}^{\frac{p}{q-1}}})$$

and two unstable eigenvalues

$$\frac{\tilde{\alpha} \pm \sqrt{\tilde{\alpha}^2 - 4\tilde{\beta}(q-1)}}{2}.$$

We recall that these eigenvalues are complex if and only if relation (2.8) holds with a replaced by p and the unstable manifold is two-dimensional and lies in $\tilde{z} = 0$.

Next we specify an important notational point. Let us recall that the coordinates $\tilde{\mathbf{x}}$ and $\mathbf{x}$ are related by the transformation

$$\begin{cases} \tilde{x} = xz^{\frac{1}{q-1}}, \\ \tilde{y} = (y + \frac{\gamma x}{q-1})z^{\frac{1}{q-1}}, \\ \tilde{z} = \frac{1}{z^{\frac{p-1}{q-1}}}. \end{cases} \tag{2.10}$$

Let us write by brevity $\tilde{\mathbf{x}} = T(\mathbf{x})$. In the sequel we will denote by $W^s(O_\infty)$ the object $\tilde{W}^s(O_\infty)$ expressed in the $\mathbf{x}$ coordinates, and similarly for other objects of these kind. Thus we set

$$W^s(O_\infty) \equiv T^{-1}(\tilde{W}^s(O_\infty)), \quad W^s(P_\infty) \equiv T^{-1}(\tilde{W}^s(P_\infty)).$$

Let now state some general properties of the orbit structure that we will use later and let also translate our original problem into the language of invariant manifolds of the above discussed systems.

Lemma 2.2 *Let* $\mathbf{x}(t) = (x(t), y(t), e^{\gamma t})$ *be a solution of system (2.3).* $\tilde{\mathbf{x}}(t) = (\tilde{x}(t), \tilde{y}(t), e^{-\tilde{\gamma} t})$ *a solution of (2.5).*
(a) *Assume that* $x(t) > 0$ *for all* $-\infty < t < t_0$. *Then the orbit of* $\mathbf{x}$ *is either contained in* $W^u(O_0)$ *or it coincides with* $W^u(P_0)$.
(b) *Assume that* $\tilde{x}(t) > 0$ *for all* $t_0 < t < \infty$. *Then the orbit of* $\tilde{\mathbf{x}}$ *is either contained in* $\tilde{W}^s(O_\infty)$ *or it coincides with* $\tilde{W}^s(P_\infty)$.

Lemma 2.3 *(a) Solution with fast decay of (1.1)-(1.2) corresponds to an orbit which lies in* $W^u(O_0) \cap W^s(O_\infty)$.

(b) Solution with slow decay of (1.1)-(1.2) corresponds to an orbit which lies in $W^u(O_0) \cap W^s(P_\infty)$.

(c) Positive singular solution with fast decay of (1.1) corresponds to an orbit which lies in $W^u(P_0) \cap W^s(O_\infty)$.

(d) Positive singular solution with slow decay of (1.1) corresponds to an orbit which lies in $W^u(P_0) \cap W^s(P_\infty)$.

Lemma 2.4 *The unstable manifold of* P_0, $W^u(P_0)$, *is contained in the closure of the unstable manifold of* O_0, $W^u(O_0)$. *Similarly, the stable manifold of* P_∞, $\tilde{W}^s(P_\infty)$, *is contained in the closure of the stable manifold of* O_∞, $\tilde{W}^s(O_\infty)$.

The Proofs of Lemmas 2.2 and 2.3 are easily carried out from the results of Chapter 1, after a change of notation. For the proof of Lemma 2.4 we will make use of the well-known Palis' λ-lemma, see for instance [15] or [8]. Its statement in the context we are dealing with is as follows.

Lemma 2.5 *Consider a system of the form* $x' = f(x)$, f *of class* C^1 . *Denote by* $\varphi(x,t)$ *its associated flow. Let* P *be a hyperbolic saddle point of* $f(x)$ *with stable and unstable manifolds* $W^s(P)$, $W^u(P)$, *with respective dimensions* n_s *and* n_u. *Let* D *be a small* n_u*-dimensional disk transversal to the flow which intersects* $W^s(P)$ *at the point* Q. *Let* B^u *be any compact disk inside* $W^u(P)$ *which contains* P. *Let us denote* $D_t = \varphi(t, D)$. *Then, given* $\varepsilon > 0$, *there exists a* $t_0 > 0$ *such that for each* $t \geq t_0$ *there exists a disk* $\bar{D}_t$ *contained in* D_t *which contains* $\varphi(t, Q)$ *and is* C^1 ε*-close to* B^u.

Proof of Lemma 2.4 As we have seen, P_0 is a hyperbolic singularity whose unstable manifold $W^u(P_0)$ is a one dimensional curve. Take a short segment transversal to the $z = 0$ plane which lies entirely in the two dimensional manifold $W^u(0_0)$ (taken for instance close and almost parallel to the z-axis). By virtue of the above lemma, the flow will take this segment into a one dimensional segment, still contained in $W^u(O_0)$, which gets arbitrarily uniformly close to any given finite piece of the curve $W^u(P_0)$. This proves that $W^u(P_0)$ lies on the boundary of $W^u(O_0)$. The symmetric assertion that $\tilde{W}^s(P_\infty)$ is contained in the boundary of $\tilde{W}^s(O_\infty)$ follows similarly.See Figure 4. □

3 The topological argument for the proof of Theorem 0.1

The following result is the key step in the proof of Theorem 0.1, which we will carry out in Chapter 4.

Proposition 3.1 *Let* $\mathbf{x}_0(t) = (x_0(t), y_0(t), z_0(t))$ *be an orbit in* $W^u(O_0)$ *and* $\mathbf{x}_\infty(t) = (x_\infty(t), y_\infty(t), z_\infty(t))$ *be an orbit in* $W^s(O_\infty)$*. Assume that* $x_0(t) > 0$ *in* $(-\infty, T_0)$*,* $x_\infty(t) > 0$ *in* $(T_\infty, +\infty)$ *and that* $x_0 - x_\infty$ *has at least* $2k+1$ *zeroes in* (T_∞, T_0) *for some* $k \geq 1$*. Then there exist at least* $k-1$ *orbits in* $W^u(O_0) \cap W^s(O_\infty)$*.*

Remark. We can translate this Proposition in terms of equation (1.1) as follows:

Assume that equation (1.1) *has a solution* $u_0(r)$ *defined and positive on an interval* $(0, R_0)$ *and a solution* $u_\infty(r)$ *defined and positive on an interval* (R_∞, ∞)*. Assume also that* $R_\infty < R_0$*,* $u_0 \not\equiv u_\infty$ *and that* $u_0 - u_\infty$ *has at least* $2k+1$ *zeroes in* (R_∞, R_0) *for some* $k \geq 1$*. Then there exist at least* $k-1$ *radial ground states with fast decay of* (0.9)*.*

For instance the proof of part (a) of Theorem 0.1 will be reduced to showing that for each number k the assumptions of this result indeed hold if we fix q supercritical and then let p be close enough from below to the critical exponent. Similarly for part (b).

We have divided the proof of Proposition 3.1 into three lemmas. First we need some preliminary observations, as well as some notation and definitions.

We may assume that only a finite number of orbits lie simultaneously in $W^u(O_0)$ and in $W^s(O_\infty)$ (otherwise an infinite number of ground states with fast decay automatically exist), then slightly perturbing $\mathbf{x}_i(t)$ $(i = 0, \infty)$ to neighboring trajectories in $W^u(O_0)$, respectively in $W^s(O_\infty)$, we may also assume without loss of generality that these trajectories do not lie simultaneously in the two manifolds.

We recall that the z-axis separates the manifold $W^u(O_0)$ into two components invariant under the flow, one of them a half-plane contained in $x < 0$, and the other a surface $W_+^u(O_0)$, which we define so that it contains the z-axis. Observe that $W_+^u(O_0)$ is not necessarily contained in $x \geq 0$. Let us observe that the trajectory $\mathbf{x}_0$ splits $W_+^u(O_0)$ into two components. Let us call H_0 the closure of the component which contains the z-axis.

Let $U(z_0) = H_0 \cap \{z = z_0\}$, for any $z_0 > 0$, be the section of H_0 in the $\{z = z_0\}$ plane. Then $U(z_0)$ is a C^1 curve without self-intersections, whose endpoints are $(0, 0, z_0)$ and the point of the trajectory $\mathbf{x}_0$ in the plane $\{z = z_0\}$.

Similarly, the z-axis separates the manifold $W^s(O_\infty)$ into two components invariant under the flow, one of them a half-plane contained in $x < 0$, the other a surface $W_+^s(O_\infty)$, which we define so that it contains the z-axis. Now, the trajectory $\mathbf{x}_\infty$ splits $W_+^s(O_\infty)$ into two components. Let us call H_∞ the closure of the component which contains the z-axis. We denote $S(z_0) = H_\infty \cap \{z = z_0\}$, for any $z_0 > 0$, the section of H_∞ in the $\{z = z_0\}$ plane.

Remark. Our goal is to prove that for certain z_0 the curves $U(z_0)$ and $S(z_0)$ intersect at least at $k-1$ points. Due to the form of system (2.3), we conclude that the same is true for all $z > 0$. Observe that these intersections correspond to $k-1$ distinct trajectories lying simultaneously in $W^s(O_\infty)$ and $W^u(O_0)$. In order to do this we need the next definitions.

We can lift a planar curve $\sigma(s)$, $s \in [0, 1]$, in $\mathbb{R}^2 \setminus \{(x_0, y_0)\}$, to a curve $\bar{\sigma}(s) = (\theta(s), \rho(s))$ in the polar coordinates plane via the relation

$$\sigma(s) = (x_0 + \rho(s) \sin \theta(s), y_0 + \rho(s) \cos \theta(s))$$

Definition. The *winding number* of the curve σ around (x_0, y_0) is the number

$$W(\sigma, (x_0, y_0)) = [\frac{1}{2\pi}(\theta(1) - \theta(0))],$$

where $[\cdot]$ denotes integral part.

Next we consider two disjoint curves ϕ_1 and ϕ_2 in the 3-dimensional space, parametrized by the z-coordinate in the form

$$\phi_i(z) = (x_i(z), y_i(z), z), \quad i = 1, 2, \quad z \in [z_1, z_2].$$

Definition. The *linking number* of ϕ_1, ϕ_2 in $[z_1, z_2]$ is the integer $W(\sigma, (0, 0))$, where $\sigma(z) = (x_1(z) - x_2(z), y_1(z) - y_2(z))$, $z \in [z_1, z_2]$.

The linking number is obviously invariant under homotopies which preserve endpoints of the curves, keep the curves disjoint and preserve their z-coordinates.

Let $\phi_i(z)$ be a parametrization of the trajectories $\mathbf{x}_i$, $i = 0, \infty$ via the z-coordinate, namely $\phi_i(z) = \mathbf{x}_i(\gamma^{-1}\log z)$. Fix a number $z > 0$. Let $\sigma_z(s)$, $s \in [0,1]$, be a one-to-one parametrization of $U(z)$ such that $\sigma_z(0) = (0,0,z)$ and $\sigma_z(1) = \phi_0(z)$. See Figure 5 for a description of the linking situation we are concerned with.

Proposition 3.1 will be a direct consequence of the three lemmas we state and prove next.

Let us denote by z_-, z_+ the numbers given by $z_- = \gamma^{-1}e^{T_\infty}$, $z_+ = \gamma^{-1}e^{T_0}$, where T_∞ and T_0 are the numbers mentioned in Proposition 3.1. Notice that $z_- < z_+$.

Lemma 3.1 *Let k be the number in the assumptions of Proposition 3.1. Let $0 < z_1 < z_2$ be numbers such that $z_1 < z_-$ and $z_+ < z_2$. Then the linking number of the curves ϕ_0 and ϕ_∞ in $[z_1, z_2]$, is at least k.*

Observe that there is a unique value of z for which $\phi_\infty(z)$ crosses the plane $x = 0$. We define $\bar{z}_1$ to be this value.

Lemma 3.2 *Let $z_2 > \bar{z}_1$.Then the winding number of the curve σ_{z_2}, contained in the plane $z = z_2$, around the point $\phi_\infty(z_2)$, $W(\sigma_{z_2}, \phi_\infty(z_2))$, is equals $m-1$ or m, where m is the linking number of the curves ϕ_0 and ϕ_∞ in $[\bar{z}_1, z_2]$.*

Lemma 3.3 *If z_2 is chosen sufficiently large, then the curves $U(z_2)$ and $S(z_2)$ intersect at least $W(\sigma_{z_2}, \phi_\infty(z_2))$ times.*

We will devote the rest of this chapter to the proof of these results.

Proof of Lemma 3.1. We will show that the linking number of ϕ_0 and ϕ_∞ in $[z_1, z_2]$ where $2k+1$ zeros of $x_0 - x_\infty$ exist in the interval $[t_{z_1}, t_{z_2}]$, is at least k. Here $t_z = \gamma^{-1}\log z$.

By definition of the linking number, it equals the winding number around the origin of the curve $\sigma(z) = \phi_0(z) - \phi_\infty(z)$ in $[z_1, z_2]$. Let $h = x_0 - x_\infty$, then $\sigma(z) = (h, h', 0)(\gamma^{-1}\log z)$. Since this number is invariant under a reparametrization of the curve σ, it equals that of $\hat{\sigma}(t) = (h, h')(t)$, $t \in [t_{z_1}, t_{z_2}]$. Note that this curve does not touch the point $(0,0)$ since $\mathbf{x}_i$, $i = 0, \infty$ can not have intersection. Hence the winding number around the point $W(\hat{\sigma}, (0,0))$ is indeed well defined. Let us also observe that whenever h vanishes, $\hat{\sigma}$ crosses transversally the line $h = 0$ in the clockwise direction. Let $t_{z_1} < t_1 < t_2 < \ldots < t_{2k+1} < t_{z_2}$ be the zeroes of h, and consider a lifting $(\rho(t), \theta(t))$ of $\hat{\sigma}$, so that $\hat{\sigma}(t) = (\rho(t)\sin\theta(t), \rho(t)\cos\theta(t))$. Without loss of generality we may assume $\theta(t_{z_1}) \in (0, \pi)$. Then $\theta(t_j) = j\pi$ for $j = 1, \ldots, 2k+1$.It follows that $(2k+1)\pi < \theta(t_{z_2})$ and hence

$$W(\hat{\sigma}, (0,0)) \geq [\frac{1}{2\pi}(\theta(t_{z_2}) - \theta(t_{z_1})] = k.$$

This concludes the proof. □

Remark. We observe from the above proof that the linking number of ϕ_0 and ϕ_∞ is nondecreasing as a function of the interval where it is measured, namely the linking number in $[z_1', z_2']$ is larger than or equal to that in $[z_1, z_2]$ whenever $[z_1, z_2] \subset [z_1', z_2']$.

Proof of Lemma 3.2. The proof consists on the contruccion of a suitable homotopy of ϕ_0 and ϕ_∞, which will make it easier to handle the relation between linking and winding numbers of the objects in the statement of the lemma. By definition of $\bar{z}_1$, we have that $\phi_\infty(z) \in \{x > 0\}$ for all $z > \bar{z}_1$.

Let us consider the surface H_0 defined earlier in this chapter. Then the set

$$U(\bar{z}_1) \cup U(z_2) \cup \{(0,0,z) \mid \bar{z}_1 \leq z \leq z_2\} \cup \phi_0([\bar{z}_1, z_2]),$$

is the boundary of $\hat{H}_0 = H_0 \cap \{\bar{z}_1 \leq z \leq z_2\}$, in manifold sense, see Figure 5. Let $\varphi(t, \mathbf{x})$ denote the solution of (2.3) with $\varphi(0, \mathbf{x}) = \mathbf{x}$.

Let us define the curve ϕ_0^ε in $\hat{H}_0$ as:

$$\phi_0^\varepsilon(z) = \begin{cases} \varphi(\frac{z-\bar{z}_1}{\varepsilon}\gamma^{-1}\log\frac{z}{\bar{z}_1}, \sigma_{\bar{z}_1}(1 - \frac{z-\bar{z}_1}{\varepsilon})) \text{ if } \bar{z}_1 \leq z < \bar{z}_1 + \varepsilon \\ (0,0,z) \quad \text{if } \bar{z}_1 + \varepsilon \leq z \leq z_2 - \varepsilon, \\ \varphi((1 - \frac{z-z_2+\varepsilon}{\varepsilon})\gamma^{-1}\log\frac{z}{z_2}, \sigma_{z_2}(\frac{z-z_2+\varepsilon}{\varepsilon})) \quad \text{if } z_2 - \varepsilon < z \leq z_2. \end{cases}$$

We check next that for ε small, ϕ_0 and ϕ_0^ε are homotopic inside $\hat{H}_0$, with invariant z-coordinate. It is straightforward·to check that there is a homeomorphism $F : \hat{H}_0 \to [0,1] \times [\bar{z}_1, z_2]$ which leaves the z-coordinate invariant and satisfies the following properties

$$F(U(\bar{z}_1)) = [0,1] \times \{\bar{z}_1\}, \quad F(U(z_2)) = [0,1] \times \{z_2\},$$

$$F(\{(0,0,z) \mid \bar{z}_1 \leq z \leq z_2\}) = \{0\} \times [\bar{z}_1, z_2], \quad F(\phi_0([\bar{z}_1, z_2]) = \{1\} \times [\bar{z}_1, z_2].$$

On the other hand, it can also be checked that the curves $F(\phi_0(z))$ and $F(\phi_0^\varepsilon(z))$, $z \in [\bar{z}_1, z_2]$ are homotopic inside the rectangle $[0,1] \times [\bar{z}_1, z_2]$, with a homotopy G

which leaves the endpoints of these curves as well as their z-coordinates invariant. $F^{-1}oG$ is a homotopy in $\hat{H}_0$ with the desired properties. See Figure 6.

It follows that the linking number of ϕ_0 and ϕ_∞ equals that of ϕ_0^ε and ϕ_∞.

Let us write

$$\phi_\infty(\bar{z}_1) = (0, \bar{y}_1, \bar{z}_1), \quad \phi_\infty(z_2) = (x_2, y_2, z_2).$$

We define ϕ_∞^ε as the polygonal curve

$$\phi_\infty^\varepsilon(z) = \begin{cases} (0, \bar{y}_1, z) \text{ if } \bar{z}_1 \leq z < \bar{z}_1 + \varepsilon \\ (0, \bar{y}_1, \bar{z}_1 + \varepsilon) + \frac{z - \bar{z}_1 - \varepsilon}{z_2 - \bar{z}_1 - 2\varepsilon}\mathbf{d} \text{ if } \bar{z}_1 + \varepsilon \leq z \leq z_2 - \varepsilon \\ (x_2, y_2, z) \text{ if } z_2 - \varepsilon < z \leq z_2, \end{cases}$$

where $\mathbf{d} = (x_2, y_2 - \bar{y}_1, z_2 - \bar{z}_1 - 2\varepsilon)$. Then ϕ_∞ and ϕ_∞^ε are homotopic inside

$$\{\bar{z}_1 \leq z \leq z_2\} \setminus \{\phi_0^\varepsilon([\bar{z}_1, \bar{z}_2]\},$$

leaving endpoints fixed and z-coordinate invariant. Indeed, if we choose δ and ε sufficiently small, we obtain that $\phi_\infty([\bar{z}_1, z_2])$ and $\phi_\infty^\varepsilon([\bar{z}_1, z_2])$ are contained in the set

$$\mathcal{R} = \{|(x, y) - (0, \bar{y}_1)| < \delta,\ \bar{z}_1 \leq z < \bar{z}_1 + \varepsilon\} \cup$$

$$\{x > 0,\ \bar{z}_1 + \varepsilon \leq z < z_2 - \varepsilon\} \cup \{|(x, y) - (x_2, y_2)| < \delta,\ z_2 - \varepsilon < z \leq z_2\},$$

and $\phi_0^\varepsilon([\bar{z}_1, z_2]) \cap \mathcal{R} = \emptyset$. It is checked that a homotopy in $\mathcal{R}$ with the desired properties can be built up.

Hence the linking number of the curves ϕ_0, ϕ_∞ in $[\bar{z}_1, z_2]$ equals that of ϕ_0^ε and ϕ_∞^ε

We claim that the winding number of σ_{z_2} around the point $\phi_\infty(z_2)$, measured in the plane $z = z_2$ is equal to $m - 1$ or m, where m is the linking number of ϕ_0^ε and ϕ_∞^ε. We have that

$$\phi_0^\varepsilon(z) - \phi_\infty^\varepsilon(z) = \begin{cases} \sigma_{\bar{z}_1}(1 - \frac{z - \bar{z}_1}{\varepsilon}) - (0, \bar{y}_1, \bar{z}_1) \text{ if } \bar{z}_1 \leq z < \bar{z}_1 + \varepsilon \\ -(0, \bar{y}_1, 0) - \frac{z - \bar{z}_1 - \varepsilon}{z_2 - \bar{z}_1 - 2\varepsilon}(x_2, y_2 - \bar{y}_1, 0)\ \bar{z}_1 + \varepsilon \leq z \leq z_2 - \varepsilon, \\ \sigma_{z_2}(\frac{z - (z_2 - \varepsilon)}{\varepsilon}) - (x_2, y_2, z_2) \text{ if } z_2 - \varepsilon < z \leq z_2. \end{cases}$$

Let us call $\tilde{\phi}(z)$ the x-y component of $\phi_0^\varepsilon(z) - \phi_\infty^\varepsilon(z)$. Let us write

$$\tilde{\phi}(z) = (\rho(z) \sin\theta(z), \rho(z) \cos\theta(z)).$$

We observe that $W(\tilde{\phi}, (0, 0))$ corresponds precisely to the linking number of ϕ_0^ε and ϕ_∞^ε,

$$W(\tilde{\phi}, (0, 0)) = [\frac{1}{2\pi}(\theta(z_2) - \theta(\bar{z}_1))] =$$

$$[\frac{1}{2\pi}(\theta(\bar{z}_1+\varepsilon)-\theta(\bar{z}_1)+\theta(z_2-\varepsilon)-\theta(\bar{z}_1+\varepsilon)+\theta(z_2)-\theta(z_2-\varepsilon))]. \tag{3.1}$$

We claim that

$$|\theta(\bar{z}_1+\varepsilon)-\theta(\bar{z}_1)|<\pi \tag{3.2}$$

and

$$|\theta(z_2-\varepsilon)-\theta(\bar{z}_1+\varepsilon)|<\pi. \tag{3.3}$$

We check first (3.2). We recall that $\sigma_{\bar{z}_1}$ is a parametrization of $U(\bar{z}_1)=H_0\cap\{z=\bar{z}_1\}$. Because of the form of vector field defining system (2.3), $W^u_+(O_0)$ cannot intersect the set $\{x=0,y>0\}$. On the other hand, $W^u(O_0)$ splits into a half-plane H contained in $\{x<0,y<0\}$ and $W^u_+(O_0)$. From these facts it follows that $U(\bar{z}_1)$ does not intersect $\{x<0,y>0\}$ and hence (3.2) holds true. Now, since between $\bar{z}_1+\varepsilon$ and $z_2-\varepsilon$, the curve $\phi(z)$ is a line segment, inequality (3.3) readily follows. From (3.2), (3.3) and (3.1) it follows that

$$n\le W(\tilde{\phi},(0,0))\le n+1$$

where

$$n=[\frac{1}{2\pi}(\theta(z_2)-\theta(z_2-\varepsilon))].$$

But n is precisely the winding number we want to estimate and $W(\tilde{\phi},(0,0))=m$. Thus the claim follows, and hence the lemma. □

Proof of Lemma 3.3. We recall that for $z_2>0$, the section $S(z_2)$ is given by $H_\infty\cap\{z=z_2\}$, which is a curve with endpoints $(0,0,z_2)$ and $\mathbf{x}_\infty(\gamma^{-1}\log z_2)$. We have that the orbit of $\mathbf{x}_\infty$ lies in $W^s(O_\infty)$, so that in coordinates (2.4), $\tilde{\mathbf{x}}(t)\to 0$ as $t\to+\infty$. Hence if $\tilde{x}_2$ is sufficiently small, $\tilde{S}(\tilde{z}_2)$ is well approximated by the segment joining its endpoints. Now, the image of this segment via transformation (2.4) is a line segment joining $(0,0,z_2)$ and $\mathbf{x}_2(\gamma^{-1}\log z_2)$, where $z_2=\tilde{z}_2^{-\frac{p-1}{q-1}}$. Thus, if $\tilde{z}_2$ is small enough, we also have that $S(z_2)$ is well approximated by the segment joining its endpoints.

Let $\eta_{z_2}(s)$, $s\in[0,1]$ be a parametrization of $S(z_2)$ such that $\eta_{z_2}(0)=(0,0,z_2)$, $\eta_{z_2}(1)=\phi_\infty(z_2)\equiv(x_2,y_2,z_2)$. Let us call η the vector in $\mathbb{R}^2$ whose components are the x,y coordinates of η_{z_2}. Since $S(z_2)$ does not have self-intersections, we may choose η to be one-to-one. $\eta(s)$, $s\in[0,1)$, can be lifted to a curve $\bar{\eta}(s)=(\theta_\eta(s),\rho_\eta(s))$ in the polar coordinates plane, so that

$$\eta(s)=(\rho_\eta(s)\sin\theta_\eta(s)+x_2,\rho_\eta(s)\cos\theta_\eta(s)+y_2).$$

Now, as we have mentioned, the z-axis separates $W^u(O_0)$ into two components, one of them a half-plane contained in $x < 0$, $y < 0$ and the other the surface we called $W^u_+(O_0)$. Thus $U(z_2)$ does not intersect this half-plane. We denote by l the parametrization of (x, y)-coordinates of the line constituting the intersection of the half plane and $z = z_2$, let us say $l(s) = -(s, as)$, $s > 0$, for some $a > 0$. We call $\bar{l} = (\theta_l, \rho_l)$ its lifting to polar coordinates around (x_2, y_2) as above. We call σ the planar curve whose components are the (x, y)-components of σ_{z_2}, and $\bar{\sigma} = (\theta_\sigma, \rho_\sigma)$ its lifting to polar coordinates around (x_2, y_2). Since $\eta(0) = \sigma(0) = l(0^+) = (0, 0)$, then $\bar{\eta}(0) = \bar{\sigma}(0) = \bar{l}(0) = (\theta_0, \rho_0)$.

Let us call m the integer $m = W(\sigma, (x_2, y_2))$. Then there exist numbers $0 = s_0 < s_1 < \ldots < s_m \leq 1$ such that $\theta_\sigma(s_j) = 2j\pi + \theta_0$. Now, if z_2 is sufficiently large, we have that $\sigma(1) \in \{x < 0\}$ and we can conclude $\theta_\sigma(1) \geq 2(m+1)\pi + \theta_0$. Thus, if we set $s_{m+1} = 1$, then $s_m < s_{m+1}$. We will show that for each $1 \leq j \leq m$ $\bar{\sigma}((s_{j-1}, s_{j+1}))$ intersects the curve $\bar{\eta}(s) + (2j\pi, 0)$.

Let us observe that since η is well approximated by the segment joining $(0, 0)$ and (x_2, y_2), then

$$\theta_0 - \pi < \theta_\eta(s) < \theta_0 + \pi \quad \text{for all } s \in [0, 1).$$

We also have that

$$\theta_0 - \pi < \theta_l(s) < \theta_0 + \pi \quad \text{for all } s \in (0, \infty).$$

Besides, $\rho_l(s) \to \infty$ if $s \to +\infty$. Thus, the curve L_j obtained by joining the curves $\bar{\eta} + (2j\pi, 0)$ and $\bar{l} + (2j\pi, 0)$ is contained in the set

$$((2j-1)\pi + \theta_0, (2j+1)\pi + \theta_0) \times (0, \infty).$$

L_j does not have self-intersections, so that it separates the half-plane $\rho > 0$ into two components, one of them containing the set $\{\theta \leq (2j-1)\pi + \theta_0\}$ and the other $\{\theta \geq (2j+1)\pi + \theta_0\}$. Therefore, for all $1 \leq j \leq m$, $\bar{\sigma}(s_j, s_{j+1})$ intersects L_j. Since σ does not intersect l, $\bar{\sigma}$ does not intersect $\bar{l} + (2j\pi, 0)$. Hence $\bar{\sigma}(s_j, s_{j+1})$ intersects $\bar{\eta} + (2j\pi, 0)$, and the claim is thus proven, see Figure 7. Next we see that these correspond to distinct intersections the original coordinates. Now, let $a_j \in (s_{j-1}, s_{j+1})$ be such that $\bar{\sigma}(a_j)$ lies on the curve $\bar{\eta} + (2j\pi, 0)$. We have that if $j_1 \neq j_2$ then $\bar{\sigma}(a_{j_1}) \neq \bar{\sigma}(a_{j_2})$. In fact, if otherwise, the curves $\bar{\eta} + (2j_1\pi, 0)$ and $\bar{\eta} + (2j_2\pi, 0)$ would intersect, and then the curve η would self intersect, and this does not happen. Thus, for all j, there is a b_j such tat $\bar{\sigma}(a_j) = \bar{\eta}(b_j) + (2j\pi, 0)$, so that $\sigma(a_j) = \eta(b_j)$. Since σ is one-to-one, all points $\sigma(a_j)$ are distinct, and hence $U(z_2)$ and $S(z_2)$ intersect at least at $m - 1$ points. This concludes the proof. □

4 Existence of ground states: Proof of Theorem 0.1

In this chapter we will carry out the proof of Theorem 0.1, which follows as a corollary of the results of the previous chapter. Let us consider first the situation in part (a) of the theorem. Part (b) is actually symmetric. Then we fix a number q with $q > \frac{N+2}{N-2}$. Proposition 3.1 tells us that part (a) holds true if we show that given $k \geq 1$, there is a number $p_k < \frac{N+2}{N-2}$ such that for $p_k < p < \frac{N+2}{N-2}$ there exist orbits $\mathbf{x}_0(t) = (x_0(t), y_0(t), z_0(t))$ in $W^u(O_0)$ and $\mathbf{x}_\infty(t) = (x_\infty(t), y_\infty(t), z_\infty(t))$ in $W^s(O_\infty)$ with the properties that $x_0(t) > 0$ in $(-\infty, T_0)$, $x_\infty(t) > 0$ in $(T_\infty, +\infty)$ and $x_0 - x_\infty$ is not identically zero and has at least $2k+1$ zeroes in (T_∞, T_0) . To do this, we need first the following fact.

Lemma 4.1 *Assume $p = \frac{N+2}{N-2}$ and $q > p$. Let $(x(t), y(t), z(t))$ be any trajectory in $W^u(O_0)$ with $z(t) > 0$ and $x(t) > 0$ as $t \to -\infty$. Then*
(i) $x(t) > 0$ *for all $t \in \mathbb{R}$.*
(ii) $\tilde{x}(t)$ *defined by transformation* (2.4) *is uniformly bounded and remains away from zero as $t \to \infty$.*

Proof. This result is nothing but a restatement of Proposition 1.3 for the case $p = \frac{N+2}{N-2} < q$, in terms of the solution after changing variables. □.

Proof of Theorem 0.1 part (a). We will check the validity of the assumptions of Proposition 3.1 with and arbitrary large k provided that p is sufficiently close to the critical exponent. Let us fix first $p = \frac{N+2}{N-2}$. Let $\mathbf{x}_\infty(t)$ be the only trajectory of (2.3) with z-component $e^{\gamma t}$ whose orbit coincides with $W^s(P_\infty)$. Consider also any (fixed) trajectory $\mathbf{x}(t)$ in $W^u(O_0)$ which does not coincide with $\mathbf{x}_\infty(t)$. Let $\tilde{x}_\infty(t)$ and $\tilde{x}(t)$ be their respective first coordinates in the transformation (2.4). We claim that $\tilde{x} - \tilde{x}_\infty$ has an infinite number of zeros. We present two different proofs of this fact:

Proof 1. Let t_n be any sequence with $t_n \to +\infty$. Let us set $\tilde{x}_n(t) = \tilde{x}(t_n + t)$. Then from the previous lemma, $\tilde{x}_n(t)$ is uniformly bounded above, and below away from zero. $\tilde{x}_n$ satisfies the equation

$$\tilde{x}_n'' + \tilde{x}_n^{\frac{N+2}{N-2}} + e^{-\tilde{\gamma}t}\delta_n \tilde{x}_n^q - \tilde{\beta}\tilde{x}_n = 0, \quad -\infty < t < +\infty, \tag{4.1}$$

with $\delta_n = e^{-\tilde{\gamma}t_n} \to 0$. By a standard compactness argument, it follows that, passing to a subsequence $\tilde{x}_n \to \bar{x}$, uniformly on compact intervals, where $\bar{x}$ solves

$$\bar{x}'' + \bar{x}^{\frac{N+2}{N-2}} - \tilde{\beta}\bar{x} = 0, \quad -\infty < t < +\infty, \tag{4.2}$$

$\bar{x}$ is bounded above and below away from zero. Besides, since $\tilde{x}$ and $\tilde{x}_\infty$ do not coincide, and x_∞ is the only trajectory in $W^u(P_\infty)$ and $\tilde{x}_\infty(t+t_n) \to \beta^{\frac{1}{p-1}}$ uniformly on compacts, then $\bar{x}$ is non constant. But the only solutions positive and bounded away from zero of the above equation are periodic, and cross the constant value $\beta^{\frac{1}{p-1}}$ an infinite number of times. This proves the claim. □

Proof 2. When $p = \frac{N+2}{N-2}$ we have that in the ˜-coordinates there exists a foliation constituted by cylinders, which is invariant for the flow associated to system (2.5), which have as their cross-section in the plane $\tilde{z} = 0$, each of the periodic orbits of the two-dimensional flow when restricted to that plane. This happens because this system is *normally contractive*, see [16]. On the other hand, the orbits remain bounded, and also away from the plane $x = 0$, as follows from Proposition 1.3 . Therefore each orbit in $W^u(O_0)$ must accumulate into one of those periodic orbits. This shows that each orbit which does not coincide with $W^s(P_\infty)$ winds around that one-dimensional manifold infinitely many times, as desired. □

Now we continue with the proof of the theorem, part (a). Let us consider an interval $[t_1, t_2]$ where one sees $2k+1$ zeros of $\tilde{x} - \tilde{x}_\infty$. For fixed q, we take a number p slightly smaller than $\frac{N+2}{N-2}$. Then in the ˜ coordinates, $W^s(P_\infty)$ remains as close as we wish on each given compact interval of the $\tilde{z}$-coordinate to the trajectory $\tilde{\mathrm{x}}_\infty$ if one chooses p close enough to critical. Similarly, one can find a trajectory in $W^u(O_0)$ very close to $\tilde{\mathrm{x}}$ for all p near critical. Since the $2k+1$ zeros of $\tilde{x} - \tilde{x}_\infty$ are simple, the same will be true for those close-by trajectories, in the interval (t_1, t_2) for p sufficiently close to critical. In this way, the assumption of Proposition 3.1 do hold in the situation described in (a) of Theorem 0.1 and the result hence follows. □.

The proof of assertion (b) of Theorem 0.1 is actually symmetric. It can be understood as basically a reflection of the situation just described. We need the following analogue of Lemma 4.1.

Lemma 4.2 *Assume* $q = \frac{N+2}{N-2}$ *and* $\frac{N}{N-2} < p < q$. *Let* $(x(t), y(t), z(t))$ *be any trajectory in* $W^s(O_\infty)$ *with* $z(t) > 0$ *and* $x(t) > 0$ *as* $t \to +\infty$. *Then*
(i) $x(t) > 0$ *for all* $t \in \mathbb{R}$.
(ii) $x(t)$ *is uniformly bounded and remains away from zero as* $t \to -\infty$.

Proof. This assertion follows in a symmetric manner to that of Lemma 4.1, after a slight modification of Proposition 1.3. □

After this result, the proof of part (b) of the theorem follows by a similar perturbation analysis as that carried out in part (a), except that now we consider $t \to -\infty$. This concludes the proof of Theorem 0.1. □

SINGULAR AND SLOW-DECAY GROUND STATES

5 Singular and slow-decay ground-states: Proof of Theorem 0.2

In this chapter we will carry out the proof of Theorem 0.2, for which the main ingredient are again the arguments developed in Chapter 3. We will prove part (a) of the theorem. Part (b) follows from a symmetric argument.

Let us fix $q > \frac{N+2}{N-2}$. We recall that a singular ground state with slow decay exists if and only if the one dimensional manifolds $W^u(P_0)$ and $W^s(P_\infty)$ coincide, while a singular ground state with fast decay is present whenever $W^u(P_0)$ is contained in $W^s(O_\infty)$. Our task is then to show that there is a sequence $p_k \uparrow \frac{N+2}{N-2}$ so that one of these two possibilities takes place at $p = p_k$.

Let us consider the solutions $\mathrm{x}_0(t)$ and $\mathrm{x}_\infty(t)$ with z-component $e^{\gamma t}$ whose trajectories coincide respectively with $W^u(P_0)$ and $W^s(P_\infty)$.

Referring to the notation introduced in Chapter 3, especially in the proof of Lemma 3.3, we consider for a $z > 0$ the unstable and stable z-sections $U^p(z)$ and $S^p(z)$ of $W^u(O_0)$ and $W^s(O_\infty)$ respectively. We consider one-to-one parametrizations, σ^p and η^p, of $U^p(z)$ and $S^p(z)$ with $\sigma^p(0) = \eta^p(0) = (0,0,z)$ and $\sigma^p(1) = \phi_0(z) \equiv P^p$ and $\eta^p(1) = \phi_\infty(z) \equiv Q^p$. Let also $l(s)$ be the half line constituted by the z-section of the plane branch of $W^u(O_0)$, contained in $x < 0$.

Let us consider liftings to polar coordinates around the point Q^p,

$$\bar{\sigma}^p(s) = (\theta^p_\sigma(s), \rho^p_\sigma(s)), \quad \bar{\eta}^p(s) = (\theta^p_\eta(s), \rho^p_\eta(s)), \quad \bar{l}^p(s) = (\theta^p_l(s), \rho^p_l(s))$$

of these curves, selected so that

$$(\theta^p_0, \rho^p_0) \equiv \bar{\sigma}^p(0) = \bar{\eta}^p(0) = \bar{l}^p(0)$$

defines a continuous function of p.

Let us consider a number $p_0 > \frac{N}{N-2}$ such that

$$W^u(P_0) \cap (W^s(P_\infty) \cup W^s(0_\infty)) = \emptyset. \tag{5.1}$$

Let $N(p_0)$ be the total linking number in $(0,\infty)$ of the curves ϕ_0 and ϕ_∞. Then $N(p_0) < +\infty$. From the proof of Theorem 0.1 we know that $N(p)$ grows to infinity as $p \uparrow \frac{N+2}{N-2}$. Let us choose a number $p_0 < p_1 < \frac{N+2}{N-2}$ with $N(p_1) \geq N(p_0) + 4$ and such that (5.1) also holds at p_1. The claim, from which the result of part (a) of the theorem readily follows, is that there must exist a number $p \in (p_0, p_1)$ such that either $P^p = Q^p$ or $P^p \in S^p(z_0)$. We will show this, making a suitable choice of z_0.

Let us observe first that there is a number $M > 0$ such that for all $p \in [p_0, p_1]$, $z_0 \geq 1$, $s \in [0,1]$, $|\rho_\sigma^p(s)| \leq M$. On the other hand, since $\mathbf{x}_0(t)$ does not correspond to a singular ground state for any $p \in [p_0, p_1]$, it must cross the $x = 0$ plane. It follows that if we fix z_0 large enough we may also assume that $|Q^p - P^p| = \rho_\sigma^p(1) > M$ for all $p \in [p_0, p_1]$. Let us fix such a z_0.

Let n_0 be the winding number $n_0 \equiv W(\sigma^{p_0}, Q^{p_0})$. Then, enlarging z_0 if necessary, we may also assume from Lemma 3.1 that $N(p_0) \leq n_0 \leq N(p_0) + 1$ Now, from our choice of p_1 we then have that

$$W(\sigma^{p_1}, Q^{p_1}) \geq n_0 + 3. \tag{5.2}$$

Let us consider, the translates of the curve $\bar{l}^p$, $\bar{l}_n^p(s) = \bar{l}^p(s) + (2n\pi, 0)$. Then if $M > 0$ is chosen large enough, the curves l_n^p separate the region $\rho > M$ into two connected components for all $p \in [p_0, p_1]$. Now, $\theta_l(s) \in (\theta_0^p - \pi, \theta_0^p + \pi)$. Let us assume that the point $\bar{\sigma}^{p_0}(1)$ is between $l_n^{p_0}$ and $l_{n+1}^{p_0}$. (Actually $n = n_0$ or $n = n_0 - 1$). Then, by continuity, $\bar{\sigma}^p(1)$ is between $\bar{l}_n^p$ and $\bar{l}_{n+1}^p$ for all $p \in [p_0, p_1]$ since this point always is in $\rho > M$, see Figure 8. We conclude that

$$\theta_\sigma^{p_1}(1) \leq \theta_0(p_1) + \pi + 2\pi(n+1) \leq 2\pi(n_0 + 2)$$

and hence the winding number

$$W(\sigma^{p_1}, Q^{p_1}) \leq n_0 + 2.$$

We have reached a contradiction with (5.2), and hence the assertion of Theorem 0.2 in its part (a) holds. The proof of part (b) of the theorem is analogous. □

6 Infinitely many ground states: Proof of Theorem 0.4

We want to find infinitely many fast decay solutions to (0.9)-(0.10) under the assumptions of Theorem 0.4. As we have already seen, this means that we look for intersections between the invariant manifolds $\tilde{W}^s_+(O_\infty)$ and $\tilde{W}^u_+(O_0)$.

Our task is to show that under the conditions of Theorem 0.4, parts (a) and (b), there are infinitely many distinct trajectories lying in $\tilde{W}^s_+(O_\infty) \cap \tilde{W}^u_+(O_0)$. Recall that such trajectories must remain positive in their x-coordinate.

The strategy to establish this is as follows: we stand on a neighborhood of P_∞ and prove that the curves corresponding to the sections of these manifolds, $S(\tilde{z})$ and $U(\tilde{z})$, for small $\tilde{z}_0$, actually intersect infinitely many times, thus giving rise to infinitely many of the seeked trajectories. The proof is based on analysis of the linearization of system (2.5) in a neighborhood of P_∞. In the linear system $S(\tilde{z})$ is seen as a spiral, while $U(\tilde{z})$ is almost seen as a segment crossing at the vortex of $S(\tilde{z})$. Thus, in this small neighborhood of P_∞ one finds infinitely many intersections between $S(\tilde{z})$ and $U(\tilde{z})$, which represent infinitely many ground states.

We say that two systems $x' = f(x)$ and $y' = g(y)$ with respective singularities P_0 and Q_0 are C^1-equivalent around these points, if there is a C^1-diffeomorphism between respective neighborhoods of these points which transforms trajectories of the first system into trajectories of the other, preserving orientations. The following fact will be important for our purposes.

Lemma 6.1 *System (2.5) is C^1-equivalent to its linearized system around P_∞. So is system (2.3) around P_0, provided that relation (2.9) holds. Moreover, the associated diffeomorphisms preserve orientation.*

Proof. To this end we employ the following result, due to Belitskij, [1], [2].

Lemma 6.2 *Suppose we have a system of the form $x' = f(x)$ with $f(x_0) = 0$ and f smooth in a neighborhood of x_0. Assume also that x_0 is a hyperbolic saddle* *point of f with eigenvalues $\lambda_1, \ldots, \lambda_n$. Assume also that none of the relations $Re\,\lambda_i = Re\,\lambda_j + Re\,\lambda_k$ is fulfilled. Then the system is C^1-equivalent in a neighborhood of x_0 to its linear part.*

This result applies immediately to system (2.5) around P_∞ if the unstable eigenvalues are not real, since we have two eigenvalues with the same, negative real parts, and a third eigenvalue which is positive. Then no of the relations $Re\,\lambda_i = Re\,\lambda_j + Re\,\lambda_k$ is possible. The same happens in system (2.3) around P_0 if its stable eigenvalues are not real. If the unstable eigenvalues of P_∞ are real, then we see that the only possibility to have one of these combinations is that

$$-\tilde{\gamma} + \frac{\tilde{\alpha} + \sqrt{\tilde{\alpha}^2 - 4\tilde{\beta}(p-1)}}{2} = \frac{\tilde{\alpha} - \sqrt{\tilde{\alpha}^2 - 4\tilde{\beta}(p-1)}}{2},$$

namely

$$\tilde{\gamma} = \sqrt{\tilde{\alpha}^2 - 4\tilde{\beta}(p-1)}.$$

But this is impossible since $\tilde{\gamma} > \tilde{\alpha}$, as checked from the definitions of these numbers. □

We prove first the assertion of part (a) of Theorem 0.4. From Lemma 6.1, we know that in the situation here considered (2.5) is C^1-equivalent in a neighborhood of P_∞ to the following linear flow:

$$\begin{cases} \bar{x}' = \bar{y}, \\ \bar{y}' = a\bar{x} + b\bar{y} + c\bar{z} \\ \bar{z}' = d\bar{z} \end{cases} \tag{6.1}$$

where

$$a = \tilde{\beta}(1-p), \quad b = \tilde{\alpha}, \quad c = -\tilde{\beta}^{\frac{q}{p-1}}, \quad d = -\tilde{\gamma}.$$

After a suitable linear transformation, we check that this system is also linearly equivalent to a linear system of the form

$$\begin{cases} \bar{x}' = \bar{y}, \\ \bar{y}' = \tilde{a}\bar{x} + \tilde{b}\bar{y} \\ \bar{z}' = \tilde{d}\bar{z}. \end{cases} \tag{6.2}$$

Let $\Phi : \mathcal{V} \to \mathcal{U}$ be the diffeomorphism setting the equivalence between (2.5) and (6.2), where $\mathcal{V}$ is a neighborhood of P_∞ and $\mathcal{U}$ one of O. As in the original system (2.3), the origin in (6.2) is a repelling focus when the flow is restricted to the plane $\bar{z} = 0$ and the $\bar{z}$-axis is its corresponding stable manifold.

Let us recall that the stable manifold of O_∞ contains the stable manifold of P_∞ in its closure, hence in the neighborhood $\mathcal{V}$ of P_∞, $\tilde{W}^s(O_\infty) \cap \mathcal{V} \neq \emptyset$. To this intersection, it corresponds, through the equivalence Φ, an invariant manifold $\mathcal{M}$ of system (6.2) inside $\mathcal{U}$. Let $\psi(t)$ be the image through Φ restricted to $\{\tilde{z} = 0\} \cap \mathcal{V}$, of the orbit corresponding to the heteroclinic trajectory which connects P_∞ to O_∞. It is checked, after making $\mathcal{U}$ and $\mathcal{V}$ smaller if necessary, that the manifold $\mathcal{M}$ must be constituted exactly by the set of points of the form $(\psi(t), \bar{z})$ which lie inside $\mathcal{U}$, see Figure 9.

Let $\tilde{\mathrm{x}}(t)$ be the solution of (2.5) which corresponds to a slowly decaying solution of (1.1)-(1.2), which exists by hypothesis. Recall that this trajectory corresponds precisely to the (one-dimensional) stable manifold of P_∞, hence its image through Φ for all sufficiently large t is precisely the part of the $\bar{z}$-axis inside $\mathcal{U}$.

Let us consider a small $\tilde{z}_0$ so that the plane $\tilde{z} = \tilde{z}_0$ intersects $\mathcal{V}$. Let us consider also the image of $\tilde{W}^u(O_0)$ through the equivalence Φ, near P_∞. This two-dimensional manifold lies on a transversal section to the linear flow of (6.2), given by $\mathcal{S} = \Phi(\mathcal{V} \cap \{\tilde{z} = \tilde{z}_0\})$. Now let us consider the transition map which goes from $\mathcal{S}$ to the plane $\bar{z} = \delta$ with a sufficiently small δ. Summarizing, we have a function H defined on $\mathcal{V} \cap \{\tilde{z} = \tilde{z}_0\}$ with values in $\mathcal{U} \cap \{\bar{z} = \delta\}$ where H is the composition of Φ and the transition map. We observe that H is a C^1-diffeomorphism. Since trajectories cross transversally the plane $\tilde{z} = \tilde{z}_0$, so does $\tilde{W}^u(O_0)$. Hence $\tilde{W}^u(O_0) \cap \{\tilde{z} = \tilde{z}_0\}$ defines a C^1 curve, except possibly at its endpoint, which corresponds to

$$\{P_1\} = \tilde{W}^u(P_0) \cap \{\tilde{z} = \tilde{z}_0\}.$$

Since H is a diffeomorphism, it follows that the set $H(\tilde{W}^u(O_0) \cap \mathcal{V} \cap \{\tilde{z} = \tilde{z}_0\})$ is a C^1 curve, which we call γ inside the planar section $\mathcal{U} \cap \{\bar{z} = \delta\}$ which contains the point $P_2 = (0, 0, \delta)$. Note that $P_2 \neq H(P_1)$ since P_1 is not in $\tilde{W}^u(O_0)$.

Let us recall that the image through the equivalence of $\tilde{W}^s(O_\infty)$ intersected with $\mathcal{U} \cap \{\bar{z} = \delta\}$ is precisely the curve $\sigma(t) = (\psi(t), \delta)$, a spiral around the point $P = (0, 0, \delta)$ which can be explicitely computed. Summarizing, we need to show that the spiral curve σ around P and the C^1 curve γ which contains the point P in its interior do intersect. Each intersection will correspond to a trajectory in $\tilde{W}^u(O_0) \cap \tilde{W}^s(O_\infty)$. It follows from Lemma 6.3, stated and proved below (the case $\theta(0^+)$ finite) that these curves must indeed intersect an infinitely many times, each of which corresponds to a solution of (0.9)-(0.10), see Figure 10.

Now, the remark after Lemma 6.3 and the continuity under parameters of the solutions of the system, implies that if p and q are slightly perturbed, a large number of these intersections will persist. This proves the assertion of part (c) of the theorem in case that the conditions of part (a) hold.

Let us now consider the case of part (b). In case that (0.15) holds, and there exists a singular solution with fast decay, the proof is symmetric to the one above.

Thus we assume that there is a singular solution with slow decay (a very degenerate case, which we cannot a priori discard) and that (0.15) holds. This means that the unstable manifold of P_0 coincides with the stable manifold of P_∞. We consider a function H defined similarly as in the proof of part (a), except that now the endpoint of the curve γ is assumed to coincide with $(0, 0, \delta)$. The stable eigenvalues of P_0 are complex, so the curve gamma is a spiral. In fact, this follows again from the C^1-equivalence orientation preserving with the linearization around P_0, in a neighborhood of these point, which implies this character in a section $\tilde{z} = \delta$, a small δ. then the flow lifts this section diffeomorphically toward $\tilde{z} = +\infty$, namely to $z = 0^-$. If the unstable eigenvalues of P_∞ are complex, then as before the corresponding curve σ is an spiral. Both spirals have same endpoint, they however wind *in opposite directions*, as it is easily argued, see Figure 11. In such a case, Lemma 6.3 for $\theta(0^+) = -\infty$ now applies. If the eigenvalues of P_∞ turned out to be real, σ would not be a spiral but a C^1 curve up to its endpoint. This is again a consequence of C^1-equivalence with the linearization, as stated in Lemma 6.1. This situation makes again Lemma 6.3 applicable.

Finally the remaining part of assertion (c) follows again from the remark after Lemma 6.3. This concludes the proof of the theorem. □

Next we prove the topological facts used in the proof of the theorem, which are included in Lemma 6.3 below. Let P_0 be a point in the plane. We consider a spiral curve σ around P_0 of the following form.

$$\sigma(t) = P_0 + \rho(t)(\cos \mu(t), \sin \mu(t)), \quad t \in [0, \infty).$$

We assume ρ and μ are continuous, that $0 < \rho(t) < \rho(0)$ for $t > 0$, that $\rho(t) \to 0$ as $t \to \infty$, and also that $\mu(0) = 0 \leq \mu(t)$ with $\mu(t) \to +\infty$ as $t \to \infty$. Let now $\gamma(s)$, $s \in (0, 1]$ be a continuous curve of the form

$$\gamma(s) = P_0 + r(s)(\cos \theta(s), \sin \theta(s)), \quad s \in (0, 1].$$

with $r(s) > 0$, $\theta(s)$ continuous functions in $(0, 1]$ such that $r(0^+) = 0$. $\theta(s)$ satisfies that either $\theta(0^+) = \theta_0$, a finite number, or $\theta(0^+) = -\infty$. We also assume that both σ and γ do not have self-intersections.

We have the validity of the following fact.

Lemma 6.3 *Let σ and γ be curves as above. Assume additionally that $r(s) < \rho(0)$ for all $t \in [0,1]$ and that $\gamma(1) \neq P_0$ and does not lie on the curve σ. Then the curves γ and σ intersect an infinite number of times.*

Proof. We lift the curves γ and σ to the universal covering of the plane without P_0, the polar coordinates plane around P_0,(r,θ), $r > 0$, $\theta \in I\!R$. Then the a lifting of σ is of course given by

$$\tilde{\sigma}(t) = (\rho(t), \mu(t))$$

and that of γ by $\tilde{\gamma}(s) = (r(s), \theta(s))$. From the assumptions made, Jordan's theorem implies that the curve $\tilde{\gamma}$ separates the strip $(0, \rho(0)] \times I\!R$ of the r-θ plane into two components A_- and A_+ to the "left" and to the "right" of the curve γ respectively. Also, the curve $\tilde{\gamma}$ lies entirely inside this strip. Consider the family of translates $\tilde{\gamma}_k = \tilde{\gamma} + (0, 2k\pi)$, which are also liftings of γ. Given a number n, consider a t_n such that $\mu(t) > \mu(t_n)$ $\rho(t) < 1/n$ for all $t > t_n$. The curve σ splits again the part of the A_n strip with $\theta > \mu(t_n)$ into two components A_n^- and A_n^+. If $\theta(0^+) = \theta_0$ is finite, we extend $\tilde{\gamma}$ with the half of the θ-axis to the left of $\theta(0)$, Then, if k is chosen sufficiently large, the following happens: there are points of $\tilde{\gamma}_k$ which lie on A_n^-, while necessarily $\tilde{\gamma}_k(1)$ lies on A_n^+. it follows by connectedness that the curve $\tilde{\gamma}_k$ intersects $\tilde{\sigma}$ somewhere in A_n. Since the r-coordinate of this point is less than $\frac{1}{n}$, and n is arbitrary, this inherits infinitely many intersections of the original curves γ and σ, as desired. □.

Remark. The topological nature of the above argument yields its "stability" in the following sense: There exist numbers ε_k, s_k, t_k such that for any continuous curves $\sigma_1(t)$, $t \in (0, \infty]$ and $\gamma_1(s)$, $s \in (0,1]$ such that

$$|\sigma_1(t) - \sigma(t)| + |\gamma_1(s) - \gamma(s)| < \varepsilon_k$$

for all $t \in (0, t_k]$, $s \in [s_k, 1]$, there are at least k distinct intersection points of the curves σ_1 and γ_1.

7 Nonexistence of ground states: Proof of Theorem 0.5

In this chapter we will perform the proof of the nonexistence result Theorem 0.5. Thus, we fix $q > \frac{N+2}{N-2}$. We will show that if p is taken sufficiently close to $\frac{N}{N-2}$, then no radial ground states of (0.9) (singular or non-singular) exist. The argument presented below is one of perturbation: No such a solution exists if $p = \frac{N}{N-2}$. This situation in fact remains true for small perturbations in p.

We consider the initial value problem

$$u'' + \frac{N-1}{r}u' + u_+^p + u_+^q = 0, \quad r > 0 \tag{7.1}$$

$$u'(0) = 0, \quad u(0) = \alpha > 0. \tag{7.2}$$

Let $u_\alpha(r)$ be the unique solution of this initial value problem, whose existence was proved in Chapter 1. Our purpose below is to analyze the behavior of these solutions separately in the cases $\alpha \to +\infty$ and $\alpha \to 0$.

Let us denote by $x_\alpha(t)$ and $\tilde{x}_\alpha(t)$ their Emden-Fowler transformations, namely

$$x_\alpha(t) = e^{\frac{2t}{q-1}} u_\alpha(e^t), \quad \tilde{x}_\alpha(t) = e^{\frac{2t}{p-1}} u_\alpha(e^t).$$

As have seen, $x_\alpha(t)$ has associated a trajectory of system (2.3) in $W^u(O_0)$, $\mathbf{x}_\alpha(t) = (x_\alpha(t), y_\alpha(t), e^{\gamma t})$. We discuss an important fact about the behavior of $\mathbf{x}_\alpha(t)$. The following holds: as $\alpha \to +\infty$ a part of the orbit, $\mathbf{x}_\alpha(t)$, gets very close to the heteroclinic orbit connecting O_0 and P_0. To see this, let us notice that $\hat{u}_\alpha(r) = \alpha^{-1} u_\alpha(\alpha^{-\frac{q-1}{2}} r)$ satisfies

$$u'' + \frac{N-1}{r}u' + u_+^q + \alpha^{-(q-p)} u_+^p = 0, \quad r > 0 \tag{7.3}$$

$$u'(0) = 0, \quad u(0) = 1. \tag{7.4}$$

Letting $\alpha \to +\infty$ we get by continuity under parameters of this initial value problem, that $\hat{u}_\alpha(r)$ approaches over compacts of $[0, \infty)$ as $\alpha \to +\infty$ the positive solution of

$$u'' + \frac{N-1}{r}u' + u_+^q = 0, \quad r > 0 \tag{7.5}$$

$$u'(0) = 0, \quad u(0) = 1. \tag{7.6}$$

which after the Emden-Fowler transformation corresponds precisely to the heteroclinic orbit connecting O_0 and P_0 in the plane $z = 0$. However the relation between the corresponding transformed vector $\hat{\mathbf{x}}_\alpha(t)$ of $\hat{u}_\alpha$ and $\mathbf{x}_\alpha(t)$ is simply a translation:

$$\hat{\mathbf{x}}_\alpha(t) = \mathbf{x}_\alpha(t - t_\alpha)$$

where $t_\alpha = \frac{q-1}{2} \log \alpha$. This means that a piece of the orbit associated to $\mathbf{x}_\alpha$ approaches the heteroclinic. As a consequence, we have that there are points of this orbit which become arbitrarily close to P_0, as $\alpha \to \infty$. Now, let us consider the unique trajectory $\mathbf{x}_*(t)$ with z-component $e^{\gamma t}$, corresponding to the one-dimensional unstable manifold of P_0, $W^u(P_0)$. Associated to this is then the (unique) singular solution of (7.1) given by $u_\infty(r) = r^{-\frac{2}{q-1}} x_*(\log r)$, where x_* is the x-component of $\mathbf{x}_*$.

For large α, the trajectory $\mathbf{x}_\alpha$ enters a neighborhood of P_0 where the dynamics of the system is well described by its linear part, in the sense of C_0-equivalence, see Hartman and Grobman Theorem, e.g. Theorem 1.1.3 in [8].

Let us recall that P_0 is a hyperbolic attractor on the $z = 0$ plane, either a focus or a node, while it has one expanding direction transversal to this plane, precisely the tangent line to the one-dimensional unstable manifold of P_0. Examination of the linear system yields that an orbit not contained in the $z = 0$ plane which gets close to P_0, turns upwards, staying close to $W^u(P_0)$ in an entire neighborhood of P_0. Since this neighborhood is independent of α, the conclusion is that in a neighborhood of P_0, the trajectory of $\mathbf{x}_\alpha$ gets uniformly close to $W^u(P_0)$ as $\alpha \to +\infty$. Continuity in the initial conditions of the initial value problem associated to the system implies then that given any compact subset of the half-space $z > 0$, large alpha implies $\mathbf{x}_\alpha$ stays uniformly close to $\mathbf{x}_*$ on compact subsets. Summarizing we have proven,

Lemma 7.1 *Given $\varepsilon > 0$, $0 < \delta < 1$ and the region $A_\delta = \{\delta \leq z \leq \delta^{-1}\}$, there is a number $\bar{\alpha}$ such that for any $\alpha \geq \bar{\alpha}$ the part of the curve, $\mathbf{x}_\alpha$, contained in A_δ lies within an ε-neighborhood of $\mathbf{x}_*$.*

Now, let us consider the situation in which $\alpha \to 0$. The following fact holds.

Lemma 7.2 *Given numbers $\frac{N}{N-2} \leq \bar{p} < \frac{N+2}{N-2} < q$ there is a positive number a such that for any $\frac{N}{N-2} \leq p \leq \bar{p}$ and all $\alpha \leq a$ there is a unique point t_α with $\tilde{x}(t_\alpha) = 0$.*

Proof. We consider now directly problem (7.1)-(7.2). Let us set

$$\tilde{u}_\alpha(r) = \alpha^{-1} u_\alpha(\alpha^{-\frac{p-1}{2}} r).$$

Then $\tilde{u}_\alpha$ satisfies

$$\tilde{u}_\alpha'' + \frac{N-1}{r}\tilde{u}_\alpha' + \tilde{u}_{\alpha+}^p + \alpha^{q-p}\tilde{u}_{\alpha+}^q = 0, \quad r > 0$$

$$\tilde{u}_\alpha'(0) = 0, \quad \tilde{u}_\alpha(0) = 1.$$

It follows by continuity of the solution of this problem in α, that $\tilde{u}_\alpha \to u_0$ uniformly over compacts, where u_0 is the unique solution of the initial value problem

$$u'' + \frac{N-1}{r}u' + u_+^p = 0, \quad r > 0$$

$$u'(0) = 0, \quad u(0) = 1.$$

This solution vanishes exactly once at certain number $r^* > 0$, with $u_0'(r^*) < 0$ since p is subcritical. r^* is bounded by some number depending only on $\bar{p}$. Hence for all α sufficiently small, the same will happen at certain point r_α. This concludes the proff. □

Proof of Theorem 0.5. Let us fix q supercritical, and consider first the case $p = \frac{N}{N-2}$. We claim that no solution of (7.1) positive in the interval $(0, \infty)$ exists in this situation. In this case $\tilde{\beta} = 0$, hence the equation satisfied in the ˜ coordinates is

$$\tilde{x}'' - \tilde{\alpha}\tilde{x}' + \tilde{x}^p + e^{-\tilde{\gamma}t}\tilde{x}^q = 0$$

Let us observe that this solution satisfies that $\tilde{x}(t) \to 0$ and $\tilde{x}'(t) \to 0$ as $t \to -\infty$, hence integrating the equation from $-\infty$ to t we obtain the relation

$$\tilde{x}'(t) - \tilde{\alpha}\tilde{x}(t) + \int_{-\infty}^{t} \tilde{x}^p(\tau)d\tau \leq 0. \tag{7.7}$$

We have that $\tilde{x}(t)$ and $\tilde{x}'(t)$ are uniformly bounded. The proof is similar to that carried out at the beginning of Chapter 1. For instance boundedness of $\tilde{x}$ is equivalent to that of the function $r^{\frac{2}{p-1}}u(r)$. Integrating (7.1) we obtain that

$$-u'(r) \geq \frac{1}{r^{N-1}} \int_0^r u^p(s)s^{N-1}ds.$$

In particular u is decreasing, so that,

$$-u'(r) \geq \frac{r}{N}u^p(r).$$

From here it easily follows that $u(r) \leq Cr^{-\frac{2}{p-1}}$ and $u'(r) \leq Cr^{-\frac{p+1}{p-1}}$ which imply that $\tilde{x}$ and $\tilde{x}'$ are bounded.

Coming back to relation (7.7), we obtain from the boundedness of $\tilde{x}$ and $\tilde{x}'$ that $\int_{-\infty}^{\infty} \tilde{x}^p(\tau)d\tau < +\infty$. Hence there is a sequence t_n such that $\tilde{x}(t_n) \to 0$ and $\tilde{x}'(t_n) \to 0$. But, invoking again relation (7.7) at $t = t_n$ and letting $n \to \infty$ we obtain $\int_{-\infty}^{\infty} \tilde{x}^p(\tau)d\tau = 0$, hence $\tilde{x} \equiv 0$, a contradiction which proves the claim.

Let us now proceed to the proof of the theorem. We see that the singular solution $\mathbf{x}_*$ crosses transversally the plane $x = 0$ at some height $z = \bar{z}$. From Lemma 7.1, it follows that for each p close to $\frac{N}{N-2}$ and all $\mathbf{x}_\alpha$'s with sufficiently large α, let us say $\alpha \geq b > 0$, also cross $x = 0$ before reaching height $2\bar{z}$. On the other hand, from Lemma 7.2, we see that all $\tilde{\mathbf{x}}'_\alpha s$ with sufficiently small α, say $0 < \alpha < a$, also cross the plane $\tilde{x} = 0$ and the distance from the crossing point to the $\tilde{z}$-axis is bounded below, away from zero.

Let us now consider x_α with $\alpha \in [a, b]$. All x_α's vanish before infinity if $p = \frac{N}{N-2}$. Continuity of the solution of the initial value problem in p then implies that for all p sufficiently close to $N/(N-2)$, and all $\alpha \in [a, b]$, x_α also vanishes. Summarizing, we have shown that no solution of problem (7.1) – (7.2) can remain positive for all $r > 0$ if p is sufficiently close to $\frac{N}{N-2}$. This concludes the proof of the theorem. □

FIGURES

Figure 1

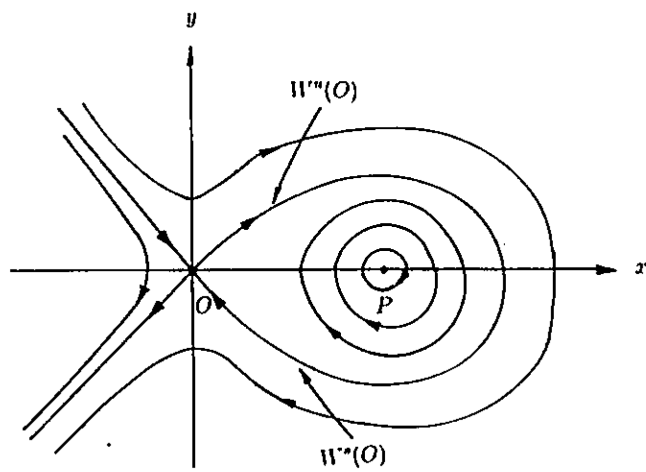


Figure 2

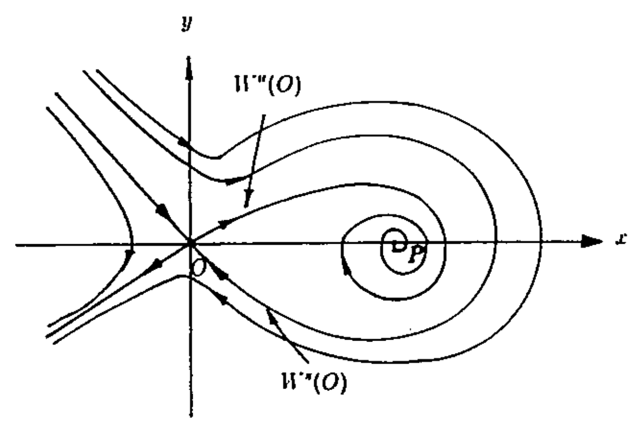
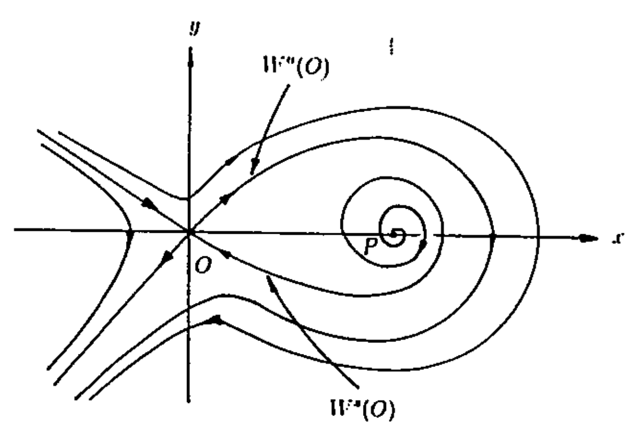


Figure 3



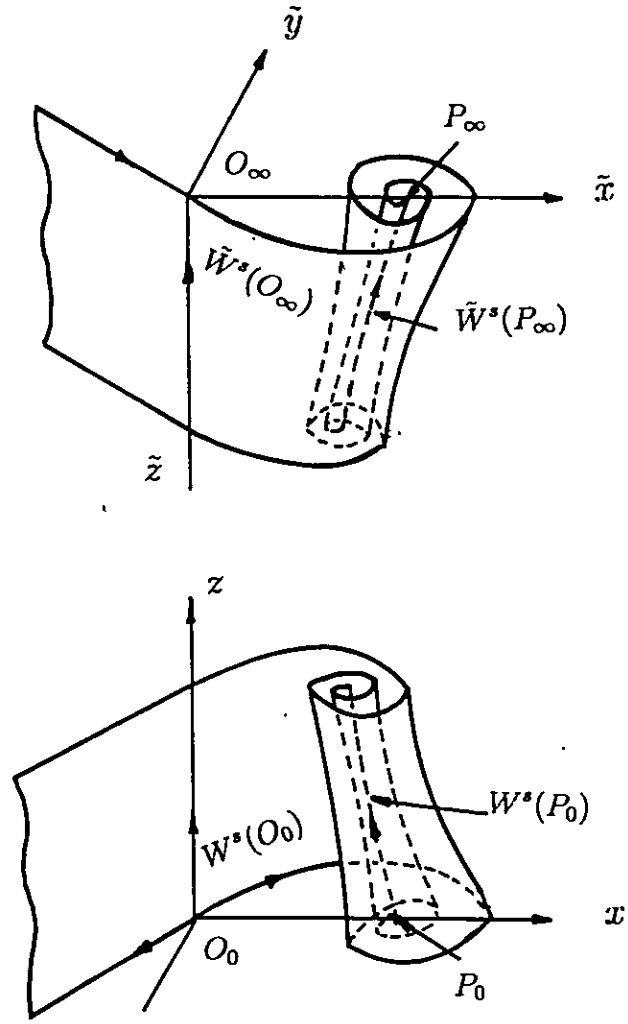

$\tilde{y}$
P_∞
O_∞
$\tilde{x}$
$\tilde{W}^s(O_\infty)$
$\tilde{W}^s(P_\infty)$
$\tilde{z}$
z
$W^s(P_0)$
$W^s(O_0)$
x
O_0
P_0

Figure 4

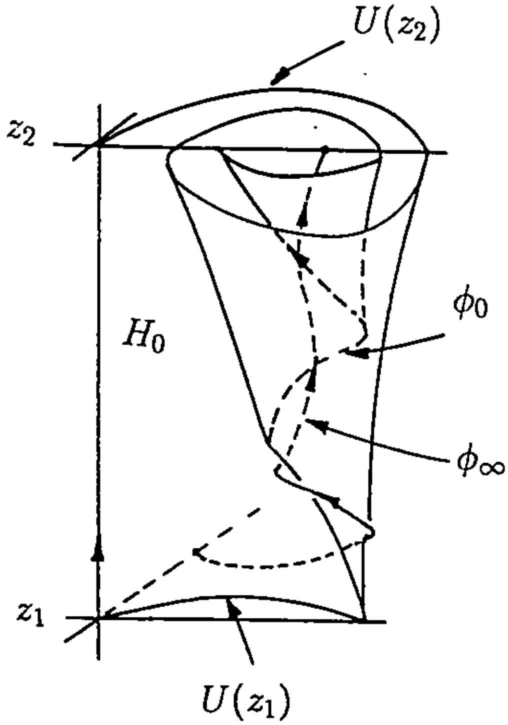


Figure 5

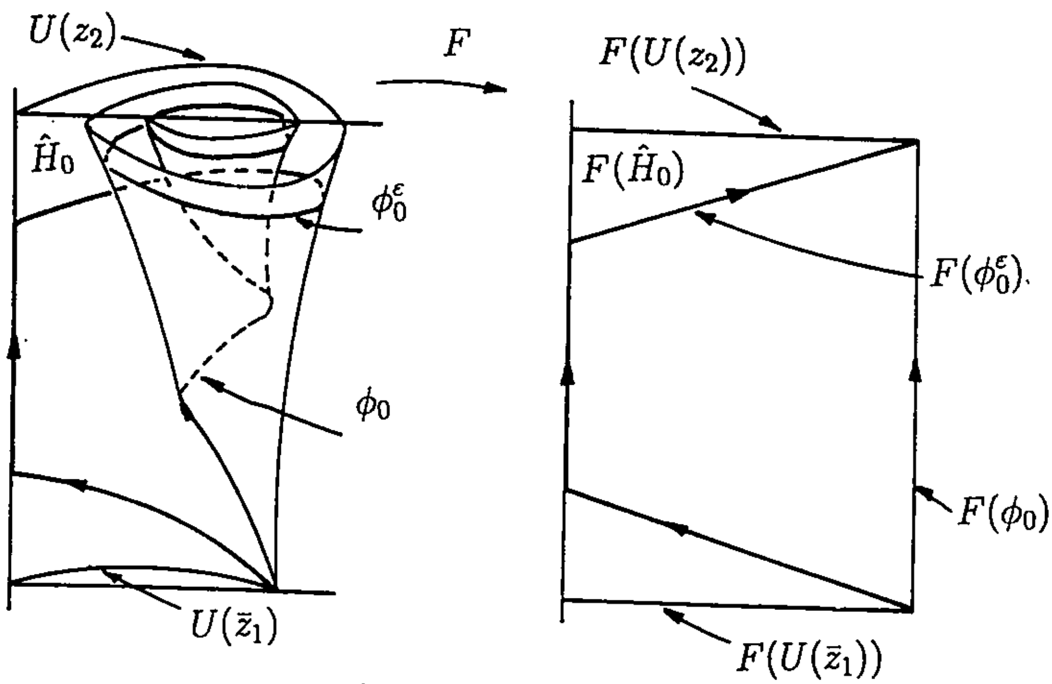
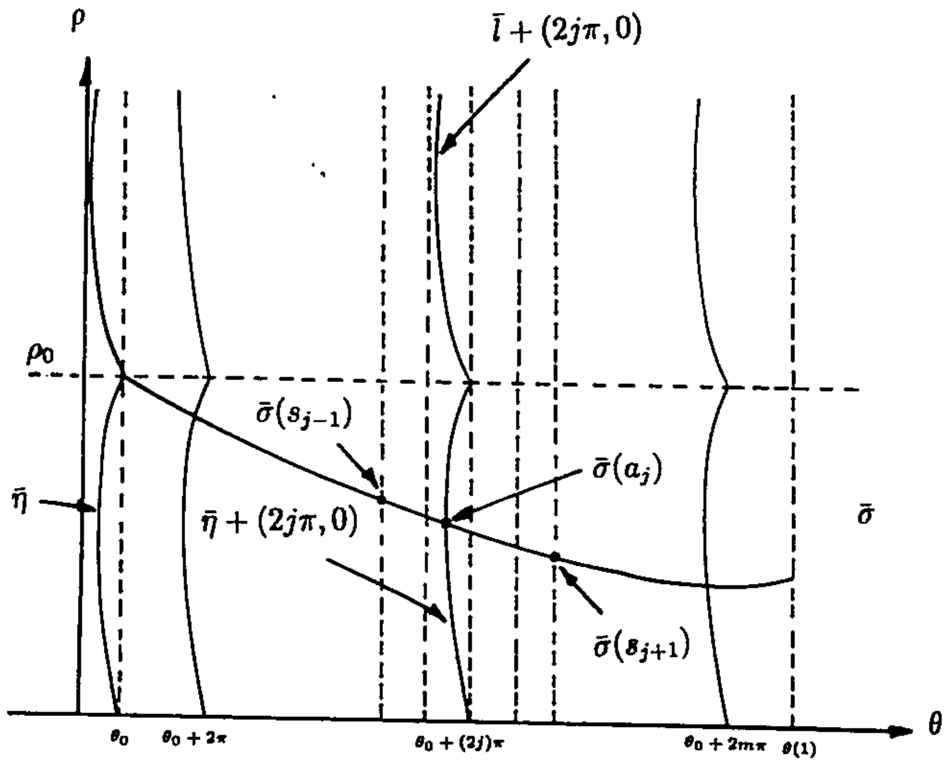


Figure 6



ρ
$\bar{l} + (2j\pi, 0)$
ρ_0
$\bar{\sigma}(s_{j-1})$
$\bar{\sigma}(a_j)$
$\bar{\eta}$
$\bar{\eta} + (2j\pi, 0)$
$\bar{\sigma}$
$\bar{\sigma}(s_{j+1})$
θ
θ_0
$\theta_0 + 2\pi$
$\theta_0 + (2j)\pi$
$\theta_0 + 2m\pi$
$\theta(1)$

Figure 7

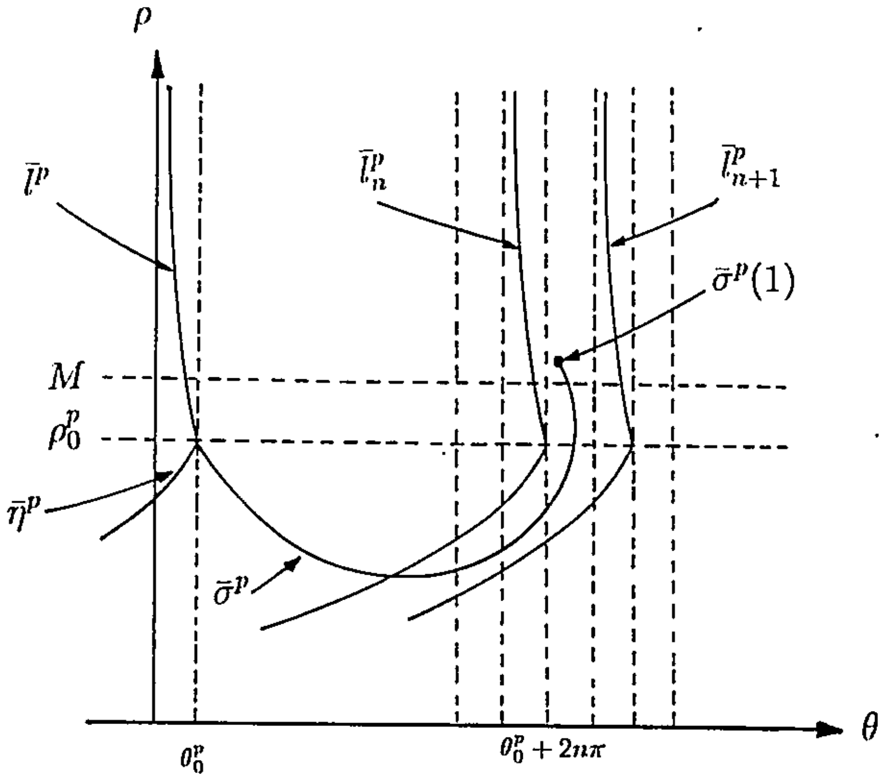

ρ
$\bar{l}^p$
$\bar{l}_n^p$
$\bar{l}_{n+1}^p$
$\bar{\sigma}^p(1)$
M
ρ_0^p
$\bar{\eta}^p$
$\bar{\sigma}^p$
θ_0^p
$\theta_0^p + 2n\pi$
θ

Figure 8

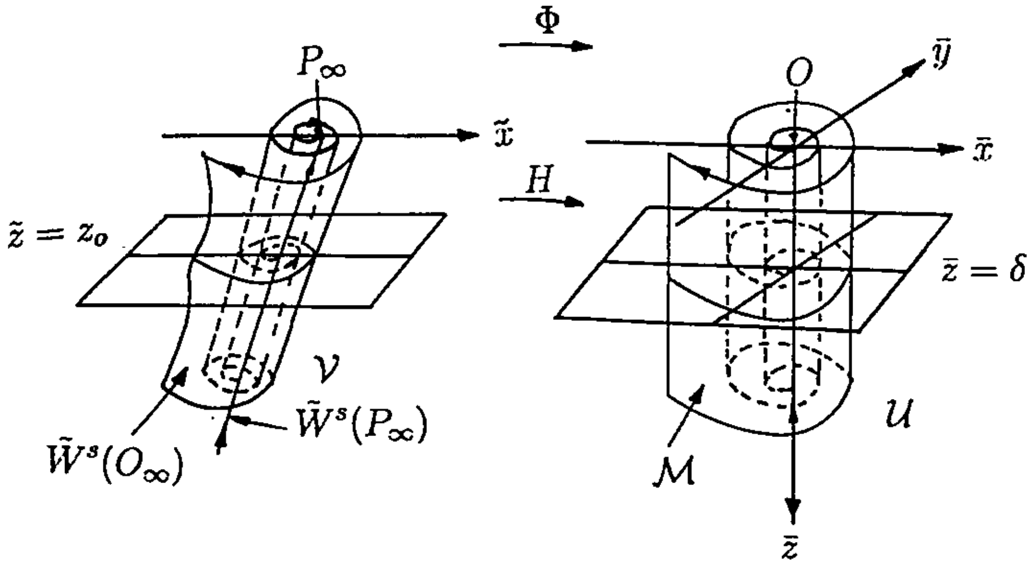


Figure 9

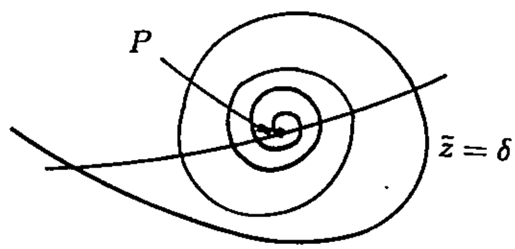


Figure 10

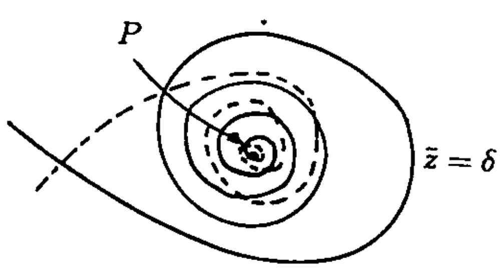


Figure 11

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