Alturas relativas y cota de Minkowski para cuerpos de números

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A mi madre

Ana Luisa

y mis hijas

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y

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Resumen

Cotas para la menor norma en una clase de ideales.

Damos un método analítico para obtener cotas superiores para la menor norma entre todas las normas de ideales enteros en una clase de ideales de un cuerpo de números. Aplicando esto a cuerpos de grados pequeño, pudimos mejorar sustancialmente las mejores cotas conocidas hasta el momento.

Contando puntos de altura relativa acotada.

Sean L/K una extensión de cuerpos de números, $\mathcal{O}^*_{L/K}$ el subgrupo del grupo de las unidades $\mathcal{O}^*_L$, consistente en los elementos que son raíces de unidades de $\mathcal{O}_K$. Consideremos la acción de $\mathcal{O}^*_{L/K}$ sobre el espacio proyectivo 1-dimensional $\mathbf{P}^1(L)$, dado por $u \cdot [x,y] = [ux,y]$, para $u \in \mathcal{O}^*_{L/K}$ y $[x,y] \in \mathbf{P}^1(L)$. Sea $H_L(K,P)$ la altura relativa a K para puntos P en el espacio proyectivo 1-dimensional $\mathbf{P}^1(L)$. Sea $N(L/K,B)$ el número de puntos P en $P^1(L^*)/\mathcal{O}^*_{L/K}$ con altura relativa $H_L(K,P) \leq B$.

Demostramos la fórmula $N(L/K,B) = CB^2 + O(B^{2-\frac{1}{[L:\mathbb{Q}]}})$, donde C es una constante que depende de invariantes aritméticos de L/K tales como el regulador, número de clases de ideales y discriminante.

Abstract

Bounds for the smallest norm in an ideal class.

We develop an analytic method for obtaining upper bounds for the smallest norm among all norms of integral ideals in an ideal class of a number field. Applying this to number fields of small degree, we are able to substantially improve on the best previously known bounds.

Counting points of bounded relative height.

Let L/K be an extension of number fields, let $\mathcal{O}^*_{L/K}$ be the subgroup of the unit group $\mathcal{O}^*_L$ consisting of elements that are roots of units of $\mathcal{O}^*_K$. Let $\mathcal{O}^*_{L/K}$ act on the 1-dimensional projective space $P^1(L)$ over L by $u \cdot [x,y] = [ux,y]$, for $u \in \mathcal{O}^*_{L/K}$ and $[x,y] \in P^1(L)$. Let $H_L(K,P)$ be the height relative to K for a point P in $P^1(L)$. Let $N(L/K,B)$ be the number of points P in $P^1(L^*)/\mathcal{O}^*_{L/K}$ with relative height $H_L(K,P) \leq B$.

We proved the formula $N(L/K,B) = CB^2 + O(B^{2-\frac{1}{[L:\mathbb{Q}]}})$, where C is a constant depending on invariants of L/K such as the regulator, class number and, discriminant.

Introducción

En esta tesis se resuelven dos problemas, uno de teoría analítica de números y el otro de teoría algebraica de números. Debido a que estos dos problemas son independientes, en cada sección se dará una introducción más detallada, con sus principales resultados junto con su propia bibliografía.

Consideremos un cuerpo de números L con $[L : \mathbb{Q}] = n = r_1 + 2r_2$, donde L tiene r_1 incrustaciones reales y $2r_2$ incrustaciones complejas. Minkowski prueba que existe una constante $C(r_1, r_2)$, que depende solo de r_1 y r_2, tal que para cada clase de ideales $\mathcal{C}$, existe un ideal entero $\mathfrak{a}_{\mathcal{C}} \in \mathcal{C}$ que satisface $N(\mathfrak{a}_{\mathcal{C}}) \leq C(r_1, r_2)\sqrt{|d_L|}$, donde N es la norma y d_L es el discriminante del cuerpo L.

Obtener una buena cota superior para la constante $C(r_1, r_2)$ tiene gran importancia para el cálculo del número de clases de ideales de un cuerpo. Además, ya que $N(\mathfrak{a}_{\mathcal{C}}) \geq 1$, se obtiene una buena cota inferior para el discriminante del cuerpo. En [N, pp. 82, 129] se encuentra una extensa bibliografía de trabajos relacionados con esta cota.

Minkoswki obtiene la cota [N, p. 96]

$$C(r_1, r_2) \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^{r_2}.$$

Roger y Mulholland [R], [M], obtienen que para n suficientemente grande

$$C(r_1, r_2) \leq (32.5)^{-r_1 2}(15.7)^{-r2}.$$

Luego Zimmert [Zi] usando métodos analiticos, obtiene la mejor cota dada hasta el momento para n suficientemente grande, a saber

$$C(r_1, r_2) \leq (50.7)^{-\frac{r_1}{2}} 19.9^{-r_2}.$$

Además Zimmert obtiene las mejores cotas conocidas hasta ahora, cuando el grado de L es pequeño.

Modificando el método de Zimmert, se obtiene en esta tesis un método, que al aplicarse a cuerpos de grados pequeños, mejora sustancialmente las cotas dadas por Zimmert. En la introducción de la primera sección se expone este método, junto con una tabla con las cotas obtenidas.

El segundo problema tratado en esta tesis se refiere a contar el número de puntos en el espacio proyectivo $\mathbf{P}^1(L)$ sobre un cuerpo de número L, con altura relativa a un subcuerpo K acotada. Bergé y Martinet [B-M] definen en el espacio proyectivo n-dimensional $\mathbf{P}^n(L)$ una altura relativa a una extensión L/K de cuerpos de números. Esta altura tiene la propiedad de ser invariante bajo cierta acción de un subgrupo $\mathcal{O}^*_{L/K}$ del grupo de unidades de L. Bergé y Martinet demuestran que el número de puntos de $\mathbf{P}^n(L)$ módulo esta acción, con altura relativa acotada, es finito. Estos resultados plantean el problema de contar los puntos, módulo la acción de $\mathcal{O}^*_{L/K}$, con altura relativa acotada.

Debido a la complejidad de esta altura, en esta tesis se estudia el caso $n = 1$. Para obtener este número, nos basamos en el trabajo de Schanuel [S] para la altura clásica. Schanuel demostró que el número de puntos $P \in \mathbf{P}^1(L)$ con altura clásica acotada por B es

$$D_L B^2 + O(B^{2-\frac{1}{[L:\mathbf{Q}]}}),$$

donde D_L es un término que involucra las constantes clásicas de L: el discriminante, regulador y el número de clases. Interesa generalizar este cálculo al caso relativo L/K para conocer las constantes análogas a las clásicas en el caso relativo. Resultan ser, en este caso, el discriminante, regulador y el número de clases clásicos de L y el regulador relativo de L/K introducido por Bergé y Martinet.

Señalamos que en el caso relativo, los cálculos y problemas son bastante más engorrosos que en el caso absoluto ($K = \mathbf{Q}$). En la introducción de la segunda sección se expone una síntesis con los resultados obtenidos relativos a este problema.

Indice

1 Bounds for the smallest norm in an ideal class

1.1 Introduction

Let K be a number field with $[K : \mathbf{Q}] = r_1 + 2r_2$, where K has r_1 real embeddings and $2r_2$ complex embeddings. Minkowski proved that there exists a constant $C(r_1, r_2)$, which depends only on r_1 and r_2, such that for any ideal class $\mathcal{C}$ of K, there exists an integral ideal $\mathfrak{a}_\mathcal{C} \in \mathcal{C}$ satisfying $N(\mathfrak{a}_\mathcal{C}) \leq C(r_1, r_2)\sqrt{|d_K|}$. Here N is the absolute norm and d_K is the discriminant of the field K.

By results of C. A. Rogers [R] and H. P. Mulholland [M], for $[K : \mathbf{Q}]$ large,

$$N(\mathfrak{a}_\mathcal{C}) \leq \left((32.5)^{\frac{r_1}{2}}(15.7)^{r_2}\right)^{-1}\sqrt{|d_K|}.$$

The best bounds so far for the constant $C(r_1, r_2)$ were given by Zimmert [Zi] in 1981,

$$N(\mathfrak{a}_\mathcal{C}) \leq ((50.7)^{\frac{r_1}{2}}(19.9)^{r_2})^{-1}\sqrt{|d_K|},$$

again for $[K : \mathbf{Q}]$ large. He also obtained the best known bounds when the degree of K is small.

Before Zimmert, the bound was always obtained using methods from the geometry of numbers [N, p. 129]. Zimmert introduced in [Zi] an interesting analytic method to obtain his bound. In this section of the thesis, we will modify Zimmert's method to obtain, for fields of small degree, a bound which improves on Zimmert's. In Table 1 below we give both Zimmert's bound and the new bound found for each case.

The main technique for obtaining the new bounds is given by Theorem 1 and its Corollary.

Theorem 1 . *Let K be a number field. Given an ideal class $\mathcal{C}$, let $\mathfrak{a}_\mathcal{C}$ be an integral ideal in $\mathcal{C}$ with minimal norm, and let $\zeta_\mathcal{C}(s) = \sum_{m=N(\mathfrak{a}_\mathcal{C})}^{\infty} a_m m^{-s}$ be the partial zeta function corresponding to $\mathcal{C}$. Then for any $y \in \mathbf{R}$,*

$$t_0 e^y - \frac{\sqrt{d_K}}{N(\mathfrak{a}_\mathcal{C})} \leq B \sum_{m=N(\mathfrak{a}_\mathcal{C})}^{\infty} a_m F\left(y - \log\left(\frac{m}{N(\mathfrak{a}_\mathcal{C})}\right)\right),$$

where

$$F(y) = \frac{1}{2\pi i}\int_{-\delta_1-i\infty}^{-\delta_1+i\infty}(e^y)^{1-s}T(s)R(s)ds, \tag{1}$$

$$T(s) = \left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}+\gamma\right)}\right)^{r_1+r_2}\left(\frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{s}{2}+1+\gamma\right)}\right)^{r_2}, \qquad (\gamma > 0)$$

$R(s)$ is a rational function such that there exist e_i, a_{ij}, $l \geq 0$ and $n_i > 0$, with

$$R(s)\left(1+\frac{1+2\gamma}{s}\right)^a\left(1+\frac{1+2\gamma}{s+1}\right)^b = \sum_{i=0}^{l} e_i \prod_{j=0}^{n_i}(s+a_{ij})^{-1},$$

$\delta_1 > 0$ such that $R(s)$ has no pole in the strip $-\delta_1 \leq \mathrm{Re}\ s \leq 0$. Furthermore B and t_0 are positive constants that depend on r_1, r_2, $R(s)$ and the choice of γ in $T(s)$.

The next result is an immediate consequence.

Corollary. *Suppose that there exists a $y_2 \in \mathbf{R}$ such that*

$$F(y) \leq 0, \text{ for } -\infty < y \leq y_2. \tag{*}$$

Then

$$N(\mathfrak{a}_C) \leq (t_0 e^{y_2})^{-1}\sqrt{|d_k|}.$$

In Table 1 below, we give Zimmert's lower bound $\sqrt{|d_k|}/N(\mathfrak{a}_C) \geq Z(r_1, r_2)$ and our new lower bound $Z_1(r_1, r_2)$. In the last column we give the smallest $\sqrt{|d_K|}$ known for K of the given signature (r_1, r_2) [O 2, p.133]. Taking C to be the trivial class, for which $N(\mathfrak{a}_C) = 1$, we see that no general lower bound for $\sqrt{|d_K|}/N(\mathfrak{a}_C)$ could exceed the last column.

Table 1.

n	r_1	r_2	$Z(r_1,r_2)$	$Z_1(r_1,r_2)$	known $\sqrt{\lvert d\rvert}$
2	2	0	1.760	2.137	2.236
2	0	1	1.400	1.651	1.732
3	3	0	4.636	6.235	7.0
3	1	1	3.355	4.340	4.795
4	4	0	14.45	21.21	26.92
4	2	1	9.749	13.76	16.58
4	0	2	6.792	9.250	10.81
5	5	0	50.21	79.19	121.0
5	3	1	32.12	49.57	67.16
5	1	2	21.11	31.02	40.11
6	6	0	188.1	315.0	547.8
6	0	3	46.74	70.98	98.72
8	8	0	3088	5644	16801
8	0	4	385.5	635.5	1121
10	10	0	58540	121120	?
10	0	5	3560	6443	14464

To obtain these bounds by the above Corollary, we need to find a suitable y_2. Unfortunately, very little is known in general about the function F in Theorem 1. Analyzing Zimmert's technique, we are able to show that some y_2 exists. However, to obtain new bounds we need a far larger value of y_2 than the one given by Zimmert's proof. To do this we must numerically calculate $F(y)$ (see Lemma 4 below) and also develop an algorithm to insure that for all $y \leq y_2$, we have $F(y) \leq 0$. To approximate $F(y)$ we use an idea of Friedman [F 2], given in the following Lemma.

Lemma 4. *For any integer $m \geq 1$, $F(y)$ has the form*

$$F(y) = \sum_{j=1}^{m} (e^y)^{1-j} P_j(y) + \epsilon(m,y),$$

where $\epsilon(m,y) = \dfrac{1}{2\pi i} \displaystyle\int_{m+\frac{1}{2}-i\infty}^{m+\frac{1}{2}+i\infty} (e^y)^{1-s} R(s)T(s)ds$ *tends to zero as* $m \to \infty$, *and $P_j(y)$ is a polynomial in y of degree at most $r_1 + r_2$.*

The above Lemma allows us to quickly calculate $F(y)$ numerically for any given y, since $|\epsilon(m, y)|$ can be bounded explicitly and the polynomials P_j can be determined recursively.

It should be noted that given a real number y, we actually prove that $F(y) \leq -\epsilon$, for some $\epsilon > 0$. The reason for this is that we use a numerical approximation of the function $F(s)$, which is quite small in the target zone. In fact, to prove that a given point y_2 satisfies the condition $(*)$, i.e. $F(y) < 0$, for all $-\infty < y \leq y_2$, seems to be quite difficult. We need to work carefully with the computer and numerical estimates. We use *PARI* [C] to calculate $F(y)$ and then to obtain the bound y_2.

We now describe the organization of this part of the thesis:

- In §1.2, we present the basic idea of Zimmert's method (Lemma 1) in detail, the proof of Theorem 1.
- In §1.3, we give an approximation of the function $F(y)$ (Lemma 4).
- In §1.4, we give an algorithm to prove inequality $(*)$ above.

1.2 Zimmert's method

Zimmert's method uses the functional equation of the zeta function of an ideal class. We present his method, slightly reformulated.

Given an ideal class $\mathcal{C}$ of K, denote by $\mathcal{C}' = \overline{\partial_K}\mathcal{C}^{-1}$ the conjugate class of $\mathcal{C}$, where ∂_K is the different of K. The zeta function of the ideal class $\zeta_{\mathcal{C}} = \sum_{\mathfrak{a}\in\mathcal{C}}(N(\mathfrak{a}))^{-s}$ satisfies the functional equation $\Delta(s, \mathcal{C}') = \Delta(1-s, \mathcal{C})$, where

$$\begin{aligned} \Delta(s,\mathcal{C}) &= \left(\sqrt{\tfrac{|d_K|}{\pi^n}}\right)^s \Gamma_{a,b}(s)\, \zeta_{\mathcal{C}}(s), \\ \Gamma_{a,b}(s) &= \Gamma\left(\tfrac{s}{2}\right)^a \Gamma\left(\tfrac{s+1}{2}\right)^b, \qquad (2) \\ n &= [K:\mathbb{Q}]\,, \quad a = r_1 + r_2, \quad b = r_2. \end{aligned}$$

For $\gamma > 0$, consider the auxiliary functions:

$$\begin{aligned} P(s) &= \frac{\Gamma_{a,b}(s)}{\Gamma_{a,b}(s+2\gamma+1)} = \left(\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{s+1}{2}+\gamma)}\right)^a \left(\frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s}{2}+1+\gamma)}\right)^b, \\ T(s) &= \frac{\Gamma_{a,b}(1-s)}{\Gamma_{a,b}(s+2\gamma+1)} = \left(\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s+1}{2}+\gamma)}\right)^a \left(\frac{\Gamma(\frac{2-s}{2})}{\Gamma(\frac{s}{2}+1+\gamma)}\right)^b, \end{aligned} \tag{3}$$

and a rational function $R(s)$ such that:

$$R(s)\left(1+\frac{1+2\gamma}{s}\right)^a\left(1+\frac{1+2\gamma}{s+1}\right)^b = \sum_{i=0}^{l} e_i \prod_{j=0}^{n_i}(s+a_{ij})^{-1}, \tag{4}$$

with $e_i,\ a_{ij},\ l \geq 0$ and $n_i > 0$.

Lemma 1 (Zimmert). *Let $f(s)$ be a Dirichlet series with non-negative coefficients, convergent in the halfplane $Re(s) > 1$. Then for any $x > 0$ and $\tau > 1$,*

$$0 \leq \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} x^s R(s)P(s)f(s)ds. \tag{5}$$

Proof. (Zimmert [Zi]) To prove the Lemma, it suffices to prove that

$$0 \leq \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} x^s R(s)P(s)ds. \tag{6}$$

The convergence of (6) is given by the condition (4) for $R(s)$, because we have $|R(\tau+it)| \leq \frac{M}{1+t^2}$, for some constant M. This is because in (4) above, $n_i > 0$ by hypothesis. Consider the infinite product representation for Γ [G-R, p. 944]

$$\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^s\left(1+\frac{s}{n}\right)^{-1}\right].$$

Using this and denoting $(a+b)(\frac{1}{2}+\gamma)$ by c, we have that $P(s)$ is equal to

$$\left(1+\frac{1+2\gamma}{s}\right)^a\left(1+\frac{1+2\gamma}{s+1}\right)^b\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-c}\left(1+\frac{1+2\gamma}{s+2n}\right)^a\left(1+\frac{1+2\gamma}{s+2n+1}\right)^b.$$

On the vertical line Re $(s) = \tau$, the partial products $P_m(s)$ converge uniformly to $P(s)$, where $P_m(s)$ is equal to

$$\left(1+\frac{1+2\gamma}{s}\right)^a \left(1+\frac{1+2\gamma}{s+1}\right)^b \prod_{n=1}^{m} \left(1+\frac{1}{n}\right)^{-c} \left(1+\frac{1+2\gamma}{s+2n}\right)^a \left(1+\frac{1+2\gamma}{s+2n+1}\right)^b.$$

It is clear that for all m, the function $t \to |P_m(\tau+it)|$ is monotonically decreasing for $t \geq 0$. The same is valid for the function $t \to |P(\tau+it)|$. Hence $|P(\tau+it)-P_m(\tau+it)| \leq P(\tau)+P_m(\tau)$. On the line Re $s = \tau$, $|P-P_m|$ is uniformly bounded, because $P_m(\tau)$ converges to $P(\tau)$.

Using this, we obtain

$$\lim_{m\to\infty} \int_{\tau-i\infty}^{\tau+i\infty} x^s R(s) P_m(s) ds = \int_{\tau-i\infty}^{\tau+i\infty} x^s R(s) P(s) ds.$$

Hence, in order to prove the Lemma, it is enough to prove that

$$0 \leq \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} x^s R(s) P_m(s) ds.$$

We need the following:

Claim: *Let $a_i \geq 0$ and $n \geq 1$. Then for $x > 0$,*

$$\frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} x^s \prod_{j=0}^{n} (s+a_j)^{-1} ds \geq 0.$$

In fact, we can assume that $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$. If we write

$$\prod_{j=0}^{n} (s+a_j)^{-1} = (s+a_n)^{-n-1} \prod_{j=0}^{n} \left(1+\frac{a_n-a_j}{s+a_n}\right)^{-1},$$

and multiply out, then we see that $\prod_{j=0}^{n} (s+a_j)^{-1}$ has the form $\sum_{j=2}^{M} b_j (s+a_n)^{-j}$, with $b_j \geq 0$. Using the formula, valid for $\tau > 0$,

$$\frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} x^s (s+a_n)^{-j} ds = \begin{cases} x^{-a_n} \frac{(\log x)^{j-1}}{(j-1)!} & \text{for } x \geq 1 \\ 0 & \text{for } x \leq 1 \end{cases},$$

the claim follows, and so (6) is proved. □

Zimmert uses the classical [L, p. 266]

Lemma 2. *Let $f(s)$, $f_2(s)$ be as in the above Lemma and suppose that both $f(s)$ and $f_2(s)$ can be extended analyticaly to the whole plane, except for a simple pole at $s = 1$. Furthermore suppose $s(s-1)A^s\Gamma_{a,b}(s)f(s)$ is entire of order 1. Here $\Gamma_{a,b}$ is as in (2), with a, b non-negative integers and $a > 0$. If for some constant $A > 0$,*

$$A^s\,\Gamma_{a,b}(s)f(s) = A^{1-s}\,\Gamma_{a,b}(1-s)f_2(1-s),$$

then $\lim_{t\to\infty} t^{-c}f(\delta+it) = 0$, *whenever*

$$\begin{cases} c > 0 & \text{if} \quad \delta \geq 1, \\ c > \frac{a+b}{2}(1-\delta) & \text{if} \quad 0 \leq \delta \leq 1, \\ c > \frac{a+b}{2}(1-2\delta) & \text{if} \quad \delta \leq 0. \end{cases}$$

Proof. The (positive) Dirichlet series $f(s)$ and $f_2(s)$ converge absolutely for $\mathrm{Re}\, s > 1$. By the functional equation above and the convexity theorem below, we have the Lemma.

Theorem([L, p. 265-266]): *Let $f(s)$ be holomorphic in the strip $a_1 \leq \mathrm{Re}\, s \leq a_2$. For each $\delta \in (a_1, a_2)$ assume that $f(\delta+it)$ grows at most like a power of $|t|$, and let $\psi(\delta)$ be the least number ≥ 0 for which $f(\delta+it) << |t|^{\psi(\delta)+\epsilon}$ for every $\epsilon > 0$. Assume also that $f(\delta+it) << e^{|t|^\alpha}$ in the strip, with some $\alpha \geq 1$. Then $\psi(\delta)$ is convex as a function of δ.*

Indeed, $\psi(\delta) = 0$ for $\delta \geq 1$, since the Dirichlet series converges. As ψ can be computed for the $\Gamma-$factors, Lemma 2 follows from the functional equation and convexity.

Theorem 1. *Let K be a number field. Given an ideal class C, let $\mathfrak{a}_C$ be an integral ideal in C with minimal norm and let $\zeta_C(s) = \sum_{m=N(\mathfrak{a}_C)}^{\infty} a_m m^{-s}$ be the partial zeta function corresponding to C. Then, for any $y \in \mathbf{R}$,*

$$t_0 e^y - \frac{\sqrt{d_K}}{N(\mathfrak{a}_C)} \leq B \sum_{m=N(\mathfrak{a}_C)}^{\infty} a_m F\left(y - \log\left(\frac{m}{N(\mathfrak{a}_C)}\right)\right),$$

where notation is as in §1.1.

Proof. In the Lemma above take $f(s) = \zeta_{C'}(s)$ and any $\tau > 1$. By Lemma 2 and the asymptotic formula $|\Gamma(x+iy)| \sim e^{-\frac{\pi}{2}|y|}|y|^{x-\frac{1}{2}}$, uniformly for x in an interval and $|y| >> 0$ [G-S, p. 945], we can shift the line of integration in

(5), from Re $s = \tau$ to Re $s = -\delta_1$. Thus we pick up the residue at $s = 0$ and $s = 1$ corresponding to the (simple) poles of $\Delta(s, \mathcal{C})$. If we use the functional equation for this function, we have:

$$0 \leq \kappa\Big(xR(1)P(1) - AR(0)T(0)\Big) + \frac{x}{A2\pi i}\int_{-\delta_1 - i\infty}^{-\delta_1 + i\infty}\left(\frac{A^2}{x}\right)^{1-s} R(s)T(s)\zeta_{\mathcal{C}}(1-s)ds, \tag{7}$$

where $\kappa = \frac{2^{r_1+r_2}\pi^{r_2}R_K}{w_K\sqrt{d_K}}$, $A = \sqrt{\frac{|d_K|}{\pi^n}}$ as above, R_K is the regulator of K and w_K is the number of roots of unity in K.

Hence

$$\frac{AR(0)T(0)}{xR(1)P(1)} - 1 \leq \frac{1}{\kappa A R(1)P(1)}\sum_{m=N(\mathfrak{a}_{\mathcal{C}})}^{\infty} a_m \frac{1}{2\pi i}\int_{-\delta_1 - i\infty}^{-\delta_1 + i\infty}\left(\frac{A^2}{xm}\right)^{1-s} R(s)T(s)ds. \tag{8}$$

Let

$$t_0 = \frac{R(0)T(0)}{R(1)P(1)}\sqrt{\pi^n} = \frac{R(0)}{R(1)}\left(\frac{\Gamma(1+\gamma)}{\Gamma(\frac{1}{2}+\gamma)}\right)^{r_1}\left(\frac{1}{2}+\gamma\right)^{r_2}\sqrt{\pi^n}, \tag{9}$$

and put $y = \log(\frac{A^2}{xN(\mathfrak{a}_{\mathcal{C}})})$. We rewrite (8) as follow :

$$t_0 e^y - \frac{\sqrt{|d_K|}}{N(\mathfrak{a}_{\mathcal{C}})} \leq \frac{\sqrt{\pi^n}}{\kappa\ R(1)P(1)N(\mathfrak{a}_{\mathcal{C}})}\sum_{m=N(\mathfrak{a}_{\mathcal{C}})}^{\infty} a_m F\left(y - \log\left(\frac{m}{N(\mathfrak{a}_{\mathcal{C}})}\right)\right).$$

We note that the hypothesis (4) on $R(s)$, implies that $R(t) > 0$ for $t > 0$. Hence $B = \frac{\sqrt{\pi^n}}{\kappa R(1)P(1)N(\mathfrak{a}_{\mathcal{C}})} > 0$. □

Zimmert takes the function $R(s) = \frac{(s+\alpha)}{(s+\beta)\ (s+2\gamma-\beta)\ (s+2\gamma-\alpha)}$, where $0 \leq \alpha < \beta < \gamma$. By estimating of the integral in (7) and taking the limit $\beta \to \gamma$, he obtained a bound $Z(r_1, r_2) \leq \frac{\sqrt{|d_k|}}{N(\mathfrak{a}_{\mathcal{C}})}$, where $Z(r_1, r_2) = t_0 e^{Y(r_1, r_2, \gamma)}$,

$$Y(r_1, r_2, \gamma) = -r_1\frac{\Gamma'}{\Gamma}\left(\frac{1+\gamma}{2}\right) - 2r_2\left(\frac{\Gamma'}{\Gamma}(1+\gamma) - \log(2)\right) - \frac{2}{\gamma - \alpha},$$

and t_0 is as in (9). For each signature (r_1, r_2), he chooses γ and α in order to obtain his bound.

Proposition 1. *For all $s \in \mathbb{C}$ with* Re $s \geq -\gamma$ *and* Re $s \neq 1, 2, 3, \cdots$, *we have*

$$|T(s)| \leq |T(\text{Re } s)|. \tag{10}$$

Proof. We define

$$G(s,\gamma) \;=\; \frac{\Gamma(s)}{\Gamma(1+\gamma-s)}, \tag{11}$$

and we obtain

$$T(s) \;=\; G\left(\frac{1-s}{2},\gamma\right)^{a}\; G\left(\frac{1}{2}+\frac{1-s}{2},\; 1+\gamma\right)^{b}. \tag{12}$$

To prove the proposition we need the following:

Claim *For all $\gamma > 0$ and $s \in \mathbf{C}$ with Re $s \leq \frac{1+\gamma}{2}$ and Re $s \neq 0, -1, -2\cdots$, we have*

$$|G(s,\gamma)| \leq |G(\mathrm{Re}\; s,\gamma)|.$$

For $s = x+iy$, with x and y real and $x \neq 0, -1, -2, \cdots$, we have by [G-R, 8.326]

$$\left|\frac{\Gamma(x+iy)}{\Gamma(x)}\right|^2 = \prod_{n=0}^{\infty}\left(1+\left(\frac{y}{x+n}\right)^2\right)^{-1}.$$

Hence

$$\left|\frac{G(x+iy,\gamma)}{G(x,\gamma)}\right|^2 = \frac{\prod_{n=0}^{\infty}\left(1+\left(\frac{y}{1+\gamma-x+n}\right)^2\right)}{\prod_{n=0}^{\infty}\left(1+\left(\frac{y}{x+n}\right)^2\right)} \;\leq\; 1 \quad \text{for} \quad x \;\leq\; \frac{1+\gamma}{2}.$$

This proves the claim. Using the claim and (12) we obtain the proposition. □

In the next Lemma we obtain, by a method analogous to Zimmert's, a point $y_1 = y_1(\delta_2)$ that satisfies (*). In general, y_1 is bad bound, but this point is important in the algorithm to obtain new bounds.

Lemma 3. *Let $\delta_2 \leq \gamma$ be chosen so that the rational funcion $R(s)$, has the unique simple pole $-\beta$ in the strip $-\delta_2 \leq \mathrm{Re}\; s \leq \; -\delta_1$, where $R(s)$ satisfies (4), and $\delta_1 > 0$ as in Theorem 1. Suppose furthermore that* $\mathrm{Res}_{s=-\beta}(R(s))$ is negative. *Let $y_1 = y_1(\delta_2)$ be given by*

$$y_1 := \frac{1}{\delta_2-\beta}\left(\log\Big(-\mathrm{Res}_{\;s=-\beta}\Big(R(s)\Big)T(-\beta)\Big) - \log\left(\frac{|T(-\delta_2)|}{2\pi}\int_{-\delta_2-i\infty}^{-\delta_2+i\infty}|R(s)ds|\right)\right). \tag{13}$$

Then $F(y) < 0$ *for all* $-\infty < y \le y_1$.

Proof. We shift the line of integration in (1) from $\mathrm{Re}\, s = -\delta_1$ to $\mathrm{Re}\, s = -\delta_2$. Then

$$F(y) = (e^y)^{1+\beta}\mathrm{Res}_{\,s=-\beta}\Big(R(s)\Big)T(-\beta) + \frac{1}{2\pi i}\int_{-\delta_2-i\infty}^{-\delta_2+i\infty} (e^y)^{1-s} R(s)T(s)ds.$$

Using $\delta_2 \le \gamma$ in proposition 1, we have

$$\left|\frac{1}{2\pi i}\int_{-\delta_2-i\infty}^{-\delta_2+i\infty} (e^y)^{1-s} R(s)T(s)ds\right| \le (e^y)^{1+\delta_2}\frac{|T(-\delta_2)|}{2\pi}\int_{-\delta_2-i\infty}^{-\delta_2+i\infty} |R(s)ds|.$$

As $F(y) \le 0$ if

$$y \le \frac{1}{\delta_2-\beta}\left(\log\left(-\mathrm{Res}_{\,s=-\beta}\Big(R(s)\Big)T(-\beta)\right) - \log\left(\frac{|T(-\delta_2)|}{2\pi}\int_{-\delta_2-i\infty}^{-\delta_2+i\infty} |R(s)ds|\right)\right).$$

□

Remark 1 *Let* $R(s) = \frac{(s+\alpha)}{(s+\beta)(s+\alpha_1)(s+\alpha_2)}$ *with*

$$\alpha < \beta < \gamma < \alpha_1 \le \alpha_2. \tag{14}$$

Let δ_2 *be such that* $\alpha_1 \ge \delta_2 \ge \beta$. *The point* $y_1(\delta_2)$ *given in Lemma* 3, *satisfies*

$$y_1 \ge \frac{1}{\delta_2-\beta}\log\left(\frac{(\beta-\alpha)T(-\beta)}{(\alpha_1-\beta)(\alpha_2-\beta)}\right) - \log\left(\frac{(\delta_2-\alpha)|T(-\delta_2)|}{2(\delta_2-\beta)(\alpha_1-\delta_2)}\right). \tag{15}$$

Proof. First, note that by (14), $R(s)$ satisfies (4). By contition of δ_2, $R(s)$ has the unique pole $-\beta$ in the strip $\delta_2 \le \mathrm{Re}\, s \le 0$.

Furthermore by the condition $0 \le \alpha < \beta \le \delta_2$ we see that at $t = 0$, $\frac{(\delta_2-\alpha)^2+t^2}{(\delta_2-\beta)^2+t^2}$ has a maximum. Furthermore, by the condition $\delta_2 \le \alpha_1 \le \alpha_2$, one has $|R(s)| \le \frac{(\delta_2-\alpha)}{(\delta_2-\beta)((\alpha_1-\delta_2)^2+t^2)}$ for $\mathrm{Re}\, s = -\delta_2$. Hence

$$\int_{-\delta_2-i\infty}^{-\delta_2+i\infty} |R(s)ds| \le \frac{(\delta_2-\alpha)\pi}{(\delta_2-\beta)(\alpha_1-\delta_2)}.$$

□

1.3 An approximation of F

We will give an approximation of the function

$$F(y) = \frac{1}{2\pi i}\int_{-\delta_1-i\infty}^{-\delta_1+i\infty}(e^y)^{1-s}R(s)T(s)ds,$$

given in (1). Recall that

$$T(s) = \left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}+\gamma\right)}\right)^{r_1+r_2}\left(\frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{s}{2}+1+\gamma\right)}\right)^{r_2},$$

with $\gamma > 0$ as in (2).

Lemma 4. *For any integer $m \geq 1$, $F(y)$ has the form*

$$F(y) = \sum_{j=1}^{m}(e^y)^{1-j}P_j(y) + \epsilon(m,y), \tag{16}$$

where

$$\epsilon(m,y) = \frac{1}{2\pi i}\int_{m+\frac{1}{2}-i\infty}^{m+\frac{1}{2}+i\infty}(e^y)^{1-s}R(s)T(s)ds,$$

$$P_j(y) = \sum_{k=0}^{t_j-1} c_{k,j}\, d_{-(k+1),j}\, y^k,$$

$t_j = a$ for j odd and $t_j = b$ for j even, and where $c_{k,j}$ and $d_{k,j}$ are given by the expansions near $s = j$:

$$(e^y)^{j-s}R(s) = \sum_{k=0}^{\infty} c_{k,j}\, y^k(s-j)^k \quad \textit{and} \quad T(s) = \sum_{k=-t_j}^{\infty} d_{k,j}\,(s-j)^k.$$

Proof. The function $(e^y)^{1-s}R(s)$ is analytic in the halfplane $\mathrm{Re}\, s \geq -\delta_1$ and $T(s)$ has only poles of order a at $s = 1,3,5,\cdots$ and of order b at $s = 2,4,6,\cdots$. Hence if we shift the line of integration from $\mathrm{Re}\,(s) = -\delta_1$ to $\mathrm{Re}\,(s) = m+\frac{1}{2}$, we pick up the residues of these poles.

Given a pole at $s = j$ of order t_j, if we write $(e^y)^{1-s}R(s) = (e^y)^{1-j+(j-s)}R(s)$ then

$$\mathrm{Res}_{s=j}\left((e^y)^{1-s}R(s)T(s)\right) = (e^y)^{1-j}\sum_{k=0}^{t_j-1} c_{k,j}\, d_{-k-1,j}\, y^k. \qquad \square$$

We shall show that $|\epsilon(m,y)|$ tends to 0, and can be considered an error term. We will prove first that the polynomial P_j can be found recursively. For this we recall that

$$G(s,\gamma)=\frac{\Gamma(s)}{\Gamma(1+\gamma-s)},\quad T(s)=G\left(\frac{1-s}{2},\gamma\right)^a\ G\left(\frac{1}{2}+\frac{1-s}{2},\ 1+\gamma\right)^b,$$

as in (11) and (12).

Proposition 2.

a) $G(s+1,\gamma)=(\gamma-s)sG(s,\gamma)$.

b) $G(s,\gamma)\ G\left(\frac{1}{2}+s,\gamma\right)=2^{1+2\gamma-4s}G(2s,2\gamma)=\frac{2^{1+2\gamma-4s}}{4(\gamma-s)s}G\left(2\left(s+\frac{1}{2}\right),2\gamma\right).$

Proof. The formula $\Gamma(s+1)=s\Gamma(s)$ and the duplication formula $2^{2s-1}\Gamma(s)\Gamma(\frac{1}{2}+s)=\sqrt{\pi}\Gamma(2s)$ [G-R, p. 946], yield a) and b). □

Proposition 3. *If we denote* $\frac{1-s}{2}$ *by* w, *then*

$$T(s)\ =\ 2^{2(1+\gamma-2w)b}\ \frac{G(w,\gamma)^{a-b}\ G(2w,2\gamma)^b}{(1+2\gamma-2w)^b}, \tag{17}$$

and

$$T(s)\ =\ \frac{2^{(1+2\gamma-4(\frac{1}{2}+w))a}\ G(\frac{1}{2}+w),\gamma)^{b-a}G(2(\frac{1}{2}+w),2\gamma)^a}{[(\frac{1}{2}+\gamma-(\frac{1}{2}+w))((\frac{1}{2}+w)-\frac{1}{2})]^a\ (1+\gamma-(\frac{1}{2}+w))^b}. \tag{18}$$

Proof. By (12) and using the property $\Gamma(1+s)=s\Gamma(s)$ in (11) (with $\frac{1}{2}+\gamma-w$), we have

$$T(s)\ =\ 2^b\ \frac{G(w,\gamma)^a\ G(\frac{1}{2}+w,\gamma)^b}{(1+2\gamma-2w)^b}.$$

Using b) of the Proposition above, we have the Proposition. □

Proposition 4.

a) If $G(s,\gamma)=\sum_{j=-1}^{\infty}a_j(s+k)^j$ *for* s *near* $-k$ $(k=0,1,2\cdots)$, *then near* $s=-(k+1)$

$$G(s,\gamma)=-\frac{\sum_{j=0}^{\infty}\left(\frac{s+k+1}{k+1}\right)^j\sum_{j=0}^{\infty}\left(\frac{s+k+1}{\gamma+k+1}\right)^j}{(k+1)(\gamma+k+1)}\sum_j a_j(s+k+1)^j.$$

b) For all k, if $G(2s,2\gamma) = \sum_{j=-1}^{\infty} b_j(s+k)^j$ near $s=-k$, then near $s=-(k+1)$

$$G(2s,2\gamma) = \frac{\sum_{j=0}^{\infty}\left(\frac{s+k+1}{k+1}\right)^j \sum_{j=0}^{\infty}\left(\frac{2(s+k+1)}{2k+1}\right)^j}{2(k+1)(2k+1)} \frac{\sum_{j=0}^{\infty}\left(\frac{s+k+1}{\gamma+k+1}\right)^j}{2(\gamma+k+1)}$$

$$\cdot \frac{\sum_{j=0}^{\infty}\left(\frac{2(s+k+1)}{2\gamma+2k+1}\right)^j}{(2\gamma+2k+1)} \sum_j b_j(s+k+1)^j.$$

Proof. a) For each $k \geq 0$, we only need to rewrite a) in Proposition 2 as

$$G(s,\gamma) = \frac{-G(s+1,\gamma)}{(\gamma+k+1-(s+k+1))(1+k-(s+k+1))}.$$

We note that when s is near $-(k+1)$, $s+1$ is near $-k$. Hence we obtain a).

b) As in a), we write

$$G(2s,2\gamma) = \frac{G(2(s+1),2\gamma)}{(2\gamma-2s)\,2s\,(2\gamma-(2s+1))\,(2s-1)},$$

and change $-2s$ by $2(k+1)-2(s+k+1)$. □

Lemma 5. *The polynomials P_j given in* (16)*, can be found recursively.*

Proof. To obtain the coefficients $c_{k,j}$ of the series expansion of $(e^y)^{j-s}R(s)$, we only need the series

i) $s+a = (a-j) + (s+j)$,

ii) $\displaystyle \frac{1}{s+a} = \frac{1}{a-j}\sum_{k=0}^{\infty}\left(\frac{-1}{a-j}\right)^k (s+j)^k$,

iii) $\displaystyle (e^y)^{j-s} = \sum_{k=0}^{\infty}\frac{(-y)^k}{k!}(s-j)^k$.

By Lemma 4, we only need to prove that the coefficient $d_{k,j}$ of the series expansion of $T(s)$ near $s=j$, can be found recursively.

By (15) in Proposition 4, to obtain the series expansion of $T(s)$ near $s=1$, we only need the series expansions of $2^{2(s+\gamma)}$, and $\frac{1}{s+2\gamma}$ near $s=1$, and the series expansions of $G(s,\gamma)$ and $(2s,2\gamma)$ near $s=0$. But, by Proposition 4, if we have the above series, we obtain the series expansions for $T(s)$ near $s=3,5,7\cdots$. Note that the expansion series of $\frac{1}{s+2\gamma}$ near $s=j$ is given by *ii*) above, and the series expansion $2^{2(s+\gamma)}$ is

$$iv)\quad 2^{2(s+\gamma)} = \sum_{k=0}^{\infty} \frac{1}{2^{2j}} \left(\frac{2^{2\gamma}(2\log 2)^k}{k!} \right) (s+j)^k.$$

Analogously, given the series expansions of $G(s,\gamma)$ and $(2s,2\gamma)$ near 0, we obtain by (16), the series expansion of $T(s)$ near 2 (the other series are given by *ii*) or *iv*) above). By Proposition 4, we obtain the series expansions for $T(s)$ near $s=4,6,8\cdots$. □

Lemma 6.

a) For each $m \in \mathbf{N}$ *and any* $y \in \mathbf{R}$, *we have*

$$|\epsilon(m,y)| \;\leq\; (e^y)^{\frac{1}{2}-m}\,\frac{|T(m+\frac{1}{2})|}{2\pi}\int_{m+\frac{1}{2}-i\infty}^{m+\frac{1}{2}+i\infty} |R(s)ds|.$$

b) If we take $R(s) = \frac{(s+\alpha)}{(s+\beta)(s+\alpha_1)(s+\alpha_2)}$ *with* $0 \leq \alpha < \beta < \gamma < \alpha_1 \leq \alpha_2$, *we have*

$$|\epsilon(m,y)| \leq (e^y)^{\frac{1}{2}-m}\frac{|T(m+\frac{1}{2})|}{2(\alpha_1+m+\frac{1}{2})}.$$

c) For each $m \in \mathbf{N}$, *we have*

$$\left|T\left(m+\frac{1}{2}\right)\right| = \frac{(\sqrt{2}\pi)^{a+b}}{\left(\Gamma(\frac{m}{2}+\frac{3}{4})\Gamma(\frac{m}{2}+\frac{3}{4}+\gamma)\right)^a\left(\Gamma(\frac{m}{2}+1+\frac{1}{4})\Gamma(\frac{m}{2}+\frac{1}{4}+\gamma)\right)^b}.$$

Proof.

a) We have by (10) and (16),

$$|\epsilon(m,y)| \;\leq\; (e^y)^{\frac{1}{2}-m}\,\frac{|T(m+\frac{1}{2})|}{2\pi}\int_{m+\frac{1}{2}-i\infty}^{m+\frac{1}{2}+i\infty} |R(s)ds|.$$

b) When $R(s) = \frac{(s+\alpha)}{(s+\beta)(s+\alpha_1)(s+\alpha_2)}$, we bound $\int_{m+\frac{1}{2}-i\infty}^{m+\frac{1}{2}+i\infty} |R(s)ds|$ as in to Remark 1, but bounding $\frac{(\alpha+m+\frac{1}{2})^2+t^2}{(\beta+m+\frac{1}{2})^2+t^2}$ by 1. Hence,

$$\int_{m+\frac{1}{2}-i\infty}^{m+\frac{1}{2}+i\infty} |R(s)ds| \leq \frac{\pi}{\alpha_1 + m + \frac{1}{2}}.$$

c) Using $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$, we have

$$\left|T\left(m+\frac{1}{2}\right)\right| = \left|\frac{\Gamma(\frac{1-(m+\frac{1}{2})}{2})}{\Gamma(\frac{m}{2}+\frac{3}{4}+\gamma)}\right|^a \left|\frac{\Gamma(1-(\frac{m}{2}+\frac{1}{4}))}{\Gamma(1+\frac{m}{2}+\frac{1}{4}+\gamma)}\right|^b$$

$$= \left(\frac{\sqrt{2}\pi}{\Gamma(\frac{m}{2}+\frac{3}{4})\Gamma(\frac{m}{2}+\frac{3}{4}+\gamma)}\right)^a \left(\frac{\sqrt{2}\pi}{\Gamma(\frac{m}{2}+\frac{1}{4})\Gamma(\frac{m}{2}+1+\frac{1}{4}+\gamma)}\right)^b.$$

Note that, $|T(m+\frac{1}{2})| \to 0$ quickly as $m \to \infty$. □

1.4 The algorithm

In this section, we will describe the algorithm used to find a largest possible point y^* that satisfies (∗). We implemented the algorithm using *PARI* [C].

Proposition 12. *Suppose* $|\epsilon(m,y)| \leq \frac{\varepsilon}{2}$ *for all* y *in an interval* $[x_1, x_2]$, *with* $\epsilon(m,y)$ *as in* (16). *Let* $a_1 \in [x_1, x_2]$ *be such that there exists* $\delta = \delta(a_1) > 0$, *satisfying*

$$[a_1, a_1+\delta] \subseteq [x_1, x_2],$$

and

$$\sum_{j\in A} g_j(a_1+\delta) + \sum_{j\in B\cup C} g_j(a_1) + \delta c M \leq \frac{-\varepsilon}{2}, \tag{19}$$

where

$g_j(y) := e^{y(1-j)}P_j(y)$ *for* $1 \leq j \leq m$, P_j *as in Lemma 4*,

$$\begin{aligned} A &= \{1 \leq j \leq m |\ g_j \text{ is increasing on } [x_1,x_2]\}, \\ B &= \{1 \leq j \leq m |\ g_j \text{ is decreasing on } [x_1,x_2]\}, \\ C &= \{1 \leq j \leq m |\ g_j \text{ is not monotone on } [x_1,x_2]\}, \end{aligned}$$

$c = \#C$ and for each $j \in C$, $|g_j'(y)| < M$ for $x_1 \le y \le x_2$.

Then all $y \in [a_1, a_1 + \delta]$ satisfy $(*)$, provided a_1 satisfies $(*)$.

Proof. Note that by (16), $F(y) = \sum_{j=1}^{m} g_j(y) + \epsilon(m, y)$. If $y \in [a_1, a_1 + \delta]$, we have by the mean value theorem and the definition of A, B and C and (19),

$$F(y) \le \sum_{j\in A} g_j(a_1 + \delta) + \sum_{j\in B} g_j(a_1) + \sum_{j\in C} g_j(y) + \epsilon(m, y) \le$$

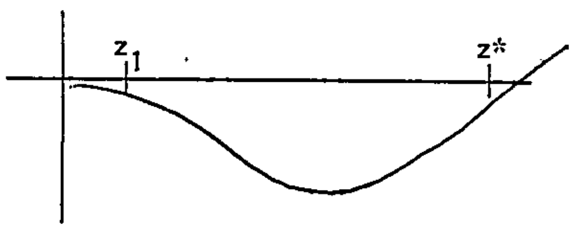
$$\le \sum_{j\in A} g_j(a_1+\delta) + \sum_{j\in B\cup C} g_j(a_1) + \delta c\, M + \epsilon(m, y) \le 0. \qquad \square$$

We work with the rational function

$$R(s) \;=\; \frac{(s+\alpha)}{(s+\beta)(s+2(2\gamma-\beta))(s+2(2\gamma-\alpha))},$$

where $\alpha = \gamma - \frac{\gamma(\gamma+1)}{\sqrt{1+3\gamma(\gamma+1)}}$ as in [Zi, p. 373], and $0 \le \alpha < \beta < \gamma$. Note that $R(s)$ satisfies (14).

Changing the variable y to $z = t_0 e^y$, we have empirically, in general that $F(\log(\frac{z}{t_0}))$ seems to have the following shape



where $z_1 = t_0 e^{y_1}$ with y_1 given by (13) or (15), and $z^* = t_0 e^{y^*}$ is the bound sought.

To obtain y^* rigorously, we find a point y_1 given by (13) or by (15). We choose $0 \le \varepsilon < -\frac{F(y_1)}{2}$. Using Lemma 6, we choose m_1 such that $|\epsilon(m_1, y)| < \frac{\varepsilon}{2}$ for each $y \ge y_1$. Then using Proposition 12 successively to move from $a_1 = y_1$ in the interval $[y_1, y_2]$ to some largest y_2, we obtain y^*.

In Table 2 below, we give γ, $z_1 = t_0 e^{y_1}$, an upper bound for $F(y_1)$, m, and the bound $z^* = t_0 e^{y^*}$.

Table 2.

n	r_1	r_2	γ	z_1	upper bound for $F(y_1)$	m	z^*
2	2	0	2.48	0.20360	$-5.9027 \cdot 10^{-7}$	30	2.1379
2	0	1	3.09	0.15003	$-4.7945 \cdot 10^{-9}$	40	1.6518
3	3	0	1.63	0.60690	$-7.0926 \cdot 10^{-5}$	12	6.2350
3	1	1	1.92	0.41994	$-4.327 \cdot 10^{-6}$	16	4.3407
4	4	0	1.25	2.0029	$-1.781 \cdot 10^{-3}$	8	21.219
4	2	1	1.41	1.3195	$-1.651 \cdot 10^{-4}$	9	13.768
4	0	2	1.61	0.89110	$-1.146 \cdot 10^{-5}$	11	9.2504
5	5	0	1.04	7.1184	$-2.062 \cdot 10^{-2}$	6	79.190
5	3	1	1.15	4.5012	$-2.821 \cdot 10^{-3}$	7	49.572
5	1	2	1.27	2.9145	$-3.284 \cdot 10^{-4}$	8	31.025
6	6	0	0.91	26.716	$-1.543 \cdot 10^{-1}$	5	315.00
6	0	3	1.16	6.5421	$-6.095 \cdot 10^{-4}$	7	70.987
8	8	0	0.74	424.17	-6.005	4	5644.0
8	0	4	0.94	54.767	$-1.032 \cdot 10^{-2}$	5	635.51
10	10	0	0.64	7452.2	-182.064	4	112120
10	0	5	0.82	98.560	$-6.604 \cdot 10^{-3}$	5	6443.8

1.5 References

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2 Counting points of bounded relative height.

2.1 Introduction

Given a number field L, let $\{\sigma_1, \ldots, \sigma_{s_L}\}$ be the real embeddings of L and let $\{\sigma_{s_L+1}, \cdots, \sigma_{s_L+2t_L}\}$ be the complex embeddings, with $\sigma_{s_L+i} = \overline{\sigma}_{s_L+t_L+i}$ for $1 \le i \le t_L$, where $^-$ denotes complex conjugation. We write $\mathcal{O}_L$ for the ring of algebraic integers of L, $\mathcal{O}_L^*$ for the group of units of $\mathcal{O}_L$.

For each $P = [x_0, x_1, \cdots, x_n]$ in the n-dimensional projective space $\mathbf{P}^n(L)$ over L, with $x_i \in L$ not all equal to 0, define the height

$$H_L(P) = N(\mathfrak{a})^{-1} \prod_{j=1}^{[L:\mathbf{Q}]} \max_{1 \le k \le n+1} \{|\sigma_j(x_k)|\}, \qquad (*)$$

where $\mathfrak{a} = (x_0, x_1 \cdots, x_n)$ is the fractional ideal generated by the x_j's, and N is the absolute norm of ideals. Given a positive real number B, let $\nu(L, B, n)$ be the number of points $P \in \mathbf{P}^n(L)$ with height $H_L(P) \le B$.

We note that, when $L = \mathbf{Q}$ and $n = 1$, $H_{\mathbf{Q}}([1, a]) = \max(|p|, |q|)$, where $a = p/q$, with p and q relatively prime integers and $q > 0$. Hence $\nu(\mathbf{Q}, B, 1)$ is the numbers of pairs (p, q) with p and q as above and with absolute value less than or equal to B. In this case, $\nu(\mathbf{Q}, B, 1) = \frac{12}{\pi^2}B^2 + O(B \log B)$. This is equivalent to the classical fact [H-W, Theorem 331] that $\frac{6}{\pi^2} = \frac{1}{\zeta(2)}$ is the probability that two large random positive integers be relatively prime.

Schanuel [S] obtained that $\nu(L, B, n-1)$ is

$$\frac{n^{s_L+t_L-1} h_L R_L}{\zeta_L(n) W_L} \left(\frac{2^{s_L}(2\pi)^{t_L}}{d_L^{1/2}} \right)^n B^n + \begin{cases} O(B \log B) & \text{if } n = 2 \text{ and } L = \mathbf{Q}, \\ O(B^{n - \frac{1}{[L:\mathbf{Q}]}}) & \text{otherwise}, \end{cases} \qquad (**)$$

where R_L is the regulator of L, d_L is the absolute value of the discriminant, h_L is the class number, ζ_L is the Dedekind zeta-function and W_L is the number of roots of unity in L.

For $P \in \mathbf{P}^n(\overline{\mathbf{Q}})$, the absolute height $H(P)$ is given by $H_M(P)^{\frac{1}{[M:\mathbf{Q}]}}$, where M is any number field such that $P \in \mathbf{P}^n(M)$. Bergé and Martinet [B-M, p. 159] defined for $P \in \mathbf{P}^n(L)$, the height $H_L(K, P)$ relative to a subfield K of L by

$$H_L(K, P) := \inf_{j, \varepsilon_k} \{H([\varepsilon_0^{1/j} x_0, \ldots, \varepsilon_n^{1/j} x_n])\}^{[L:\mathbf{Q}]},$$

where H is the absolute height, j runs over $\mathbf{Z}$ and ε_k runs over $\mathcal{O}_K^*$. For $\theta \in L^*$, we define $H_L(K,\theta)$ by $H_L(K,[\theta,1])$. We note that $H_L(\mathbf{Q},\theta) = H_L(\theta)$.

Let $\mathcal{O}_{L/K}^*$ be the subgroup of $\mathcal{O}_L^*$, consisting of those $u \in \mathcal{O}_L^*$ such that there exists $j \in \mathbf{Z}$, $j \neq 0$ such that $u^j \in \mathcal{O}_K^*$. We define the action of $(\mathcal{O}_{L/K}^*)^n$ on $P^n(L)$ by $(u_0,\cdots,u_{n-1}) \cdot [x_0,\cdots,x_n] = [u_0x_0,\cdots,u_{n-1}x_{n-1},x_n]$, for each $(u_0,\cdots,u_{n-1}) \in (\mathcal{O}_{L/K}^*)^n$ and $[x_0,\cdots,x_n] \in \mathbf{P}^n(L)$.

Bergé and Martinet [B-M 1, p. 159] point out that, for all $\varepsilon \in (\mathcal{O}_{L/K}^*)^n$ and $P \in \mathbf{P}^n(L)$, $H_L(K,\varepsilon \cdot P) = H_L(K,P)$, and furthermore proved that the number of points in $\mathbf{P}^n(L)/(\mathcal{O}_{L/K}^*)^n$ with bounded height is finite [B-M 1, p. 174].

For $n=1$ and a positive real number B, we denote by $N(L/K,B,1)$ the number of points P in $\mathbf{P}^1(L)/(\mathcal{O}_{L/K}^*)$, with $H_L(K,P) \leq B$. Thus $\nu(L,B,1) = W_L\ N(L/\mathbf{Q},B,1)$, as Schanuel does not divide by the action of $\mathcal{O}_{L/\mathbf{Q}}^* = \mu_L$, the group of roots of unity in L. We are able to generalize Schanuel's result (**) as follows.

Theorem 1. *For an extension L/K of number fields, and a large positive real number B, we have*

$$N(L/K,B,1) = C_{L/K}\,\frac{R_L R_K h_L}{I_{L/K}\zeta_L(2)}\Big(\frac{2^{s_L}(2\pi)^{t_L}}{W_L d_L^{1/2}}\Big)^2 B^2 + \begin{cases} O(B\log B) & \text{if } L=\mathbf{Q}, \\ O(B^{n-\frac{1}{[L:\mathbf{Q}]}}) & \text{otherwise,}\end{cases}$$

where $I_{L/K} = [\mathcal{O}_{L/K}^* : \mathcal{O}_K^*\mu_L]$, and the constant $C_{L/K}$ depends only on the ramification pattern of the archimedean places of L and K.

Bergé and Martinet [B-M 1] defined the relative regulator $R_{L/K} = \frac{R_L\ I_{L/K}}{R_K}$. In terms of $R_{L/K}$ one can write

$$N(L/K,B,1) = \frac{h_L\kappa_L^2}{\zeta_L(2)}\frac{C_{L/K}}{R_{L/K}}B^2 \;+\; O\left(B^{2-\frac{1}{[L:\mathbf{Q}]}}\right),$$

where $\kappa_L = \frac{2^{s_L}(2\pi)^{t_L}R_L}{W_L d_L^{1/2}}$ is the residue at $s=1$ of any ideal class zeta function of L. In $\mathbf{P}^n(L)$ we expect a formula of the type

$$N(L/K,B,n) \sim \frac{h_L\kappa_L^{n+1}}{\zeta_L(n+1)}\frac{C_{L/K}(n)}{R_{L/K}^n}B^{n+1}.$$

In the next two theorems, we compute $C_{L/K}$ when L is either a totally real or a totally complex number field. Put $\epsilon_{L/K} = \frac{\epsilon_L}{\epsilon_K}$, where for a field M we define $\epsilon_M = 1$ if M is totally real field and $\epsilon_M = 2$ otherwise.

Theorem 2. *Given an extension L/K of number fields, so that each is either totally real or totally complex, we have*

$$C_{L/K} = \frac{2^{t_L - t_K}}{2\epsilon_{L/K}\,\epsilon_L^{r_L - r_K}} \left(\frac{[L:K]}{\epsilon_{L/K}}\right)^{r_K} u\left(r_K, \frac{[L:K]}{\epsilon_{L/K}}\right),$$

where r_K is the free rank of the unit group of K and

$$u(m,n) := 2 + \sum_{k=1}^{n-1} \binom{n}{k} \binom{n-1}{k}^m \sum_{i=0}^{m} \left(\frac{k}{n-k}\right)^i.$$

Theorem 3. *Given an extension L/K of number fields, so that L is a totally complex field and K is neither totally real nor totally complex, we have*

$$C_{L/K} = \frac{[L:K]^{r_K}}{2^{s_L+1}}\, v\left(s_K, t_K, \frac{1}{2}[L:K]\right),$$

where

$$v(l,m,n) := 2 + \sum_{k=1}^{2n-1} \binom{2n}{k} \binom{2n-1}{k}^{m-1} \binom{n}{\left[\frac{k}{2}\right]}^{l} \sum_{j=0}^{l+m-1} \left(\frac{k}{2n-k}\right)^j,$$

where $[x]$ denotes the integer part of x.

Note that $[L:K]$ is even under the asumptions in Theorem 3.

These theorems are based on Theorem 5 below. We need first a variation of a formula, due to Bergé and Martinet, which gives an expression for $H_L(K,\theta)$ analogous to $(*)$ above.

Theorem 4. *Let $\theta \in L^*$, put $\theta = x_1 x_2^{-1}$, with $x_1, x_2 \in \mathcal{O}_L$, and denote the $\mathcal{O}_L$-ideal $(x_1) + (x_2)$ by $\mathfrak{a}$. For any embedding τ of K into $\mathbf{C}$, consider the $[L:K]$ embeddings $\sigma_i^{(\tau)}$ of L into $\mathbf{C}$ that extend τ, ordered so that:*

$$|\sigma_1^{(\tau)}(\theta)| \geq |\sigma_2^{(\tau)}(\theta)| \geq \ldots \geq |\sigma_{[L:K]}^{(\tau)}(\theta)|.$$

Then

$$H_L(K,\theta) = N(\mathfrak{a})^{-1} \prod_{i=1}^{[L:K]} \max\left(\prod_{\tau} |\sigma_i^{(\tau)}(x_1)|, \prod_{\tau} |\sigma_i^{(\tau)}(x_2)|\right).$$

Corollary. *If $L = K$, or if K is a totally real number field and L is a totally complex quadratic extension of K, then for $\theta \in L^*$,*

$$H_L(K,\theta) = \max(N(\mathfrak{c}), N(\mathfrak{b})),$$

where the ideal $(\theta) = \mathfrak{c}\mathfrak{b}^{-1}$, with $\mathfrak{b}$ and $\mathfrak{c}$ being relatively prime integral ideals.

We note that Bergé and Martinet [B-M 1, p. 167] obtained this corollary when $L = K$.

Define $\mathbf{T} : L^* \times L^* \to \mathbf{R}$ for $(x,y) \in L^* \times L^*$ by

$$\mathbf{T}(x,y) = \prod_{i=1}^{[L:K]} \max(\prod_{\tau} |\sigma_i^{(\tau)}(x_1)|, \prod_{\tau} |\sigma_i^{(\tau)}(x_2)|),$$

where for each embedding τ of K, we order the embeddings $\sigma_i^{(\tau)}$ that extend τ as in Theorem 4, for $\theta = xy^{-1}$. Hence, for non zero $x, y \in \mathcal{O}_L$, $H_L(K, xy^{-1}) = N((x) + (y))^{-1}\mathbf{T}(x,y)$.

We define $\mathbf{t} : (\mathbf{R}^{*s} \times \mathbf{C}^{*2t})^2 \to \mathbf{R}$ to be the continuous extension of $\mathbf{T}$ to $(\mathbf{R}^{*s} \times \mathbf{C}^{*2t})^2$, where $L \times L \hookrightarrow (\mathbf{R}^s \times \mathbf{C}^{2t})^2$ is induced by the geometric embedding $\varphi_L : L \to \mathbf{R}^s \times \mathbf{C}^{2t}$ given by $(\varphi_L(\theta))_i = \sigma_i(\theta)$. We consider the action of $\mathcal{O}^*_{L/K} \times \mathcal{O}^*_L$ on $(\mathbf{R}^{*s} \times \mathbf{C}^{*2t})^2$ which for $(u,\epsilon) \in \mathcal{O}^*_{L/K} \times \mathcal{O}^*_L$, takes $(x,y) \in (\mathbf{R}^{*s} \times \mathbf{C}^{*2t})^2$ to $(\varphi_L(\varepsilon)\varphi_L(u)x, \varphi_L(\varepsilon)y)$, where we are considering $\mathbf{R}^{*s} \times \mathbf{C}^{*2t}$ with component-wise multiplication.

Let $\mathcal{I}^*$ be the closure of $\varphi_L(L^*)$ in $\mathbf{R}^{*s} \times \mathbf{C}^{*2t}$. Given a subset $\mathcal{A}$ of $\mathcal{I}^* \times \mathcal{I}^*$ and a positive number B, define

$$T_{\mathcal{A}}(B) := \{(x,y) \in \mathcal{A} \mid \mathbf{t}(x,y) \leq B\}.$$

Recall that a subset $T \subset \mathbf{R}^n$ is said to be k-Lipschitz parametrizable if there exists a finite number of Lipschitz maps $\phi_j : I^k \to T$, where I^k denotes the unit cube in $\mathbf{R}^k$ [L 1, p. 128].

With the above definitions, we can formulate

Theorem 5. *Let $\mathcal{A}$ be a cone which is also a fundamental domain for $\mathcal{I}^* \times \mathcal{I}^*$ with respect to the above action of $\mathcal{O}^*_{L/K} \times \mathcal{O}^*_L$. Suppose furthermore that the boundary of $T_{\mathcal{A}}(B)$ is $(2[L:K]-1)$-Lipschitz parametrizable. Then, the* *number of points* $x \in \mathbf{P}^1(L)/\mathcal{O}^*_{L/K}$, *with* $H_L(K,x) \le B$ *is*

$$\frac{2^{2t_L} h_L}{d_L \zeta_L(2)} Vol(T_{\mathcal{A}}(1))B^2 + \begin{cases} O(B\log B) & \text{if } K = L = \mathbf{Q}, \\ O(B^{2-\frac{1}{[L:\mathbf{Q}]}}) & \text{otherwise.} \end{cases}$$

Next, we describe the organization of this part of the thesis:

- In §2.2, we give the proof of Theorem 4, of its corollary, and an important special case of Theorem 1.
- In §2.3, we give the proof of Theorem 5.
- In §2.4, we give a fundamental domain $\mathcal{A}$ as in Theorem 5.
- In §2.5, we give the proof of Theorem 1.
- In §2.6, we give the proof of Theorems 2 and 3 .

2.2 A height formula and an example.

Let L/K be an extension of number fields.

Theorem 4. *Let* $\theta \in L^*$, *put* $\theta = x_1 x_2^{-1}$, *with* $x_1, x_2 \in \mathcal{O}_L$, *and denote the* $\mathcal{O}_L$*-ideal* $(x_1) + (x_2)$ *by* $\mathfrak{a}$. *For any embedding* τ *of* K *into* $\mathbf{C}$, *consider the* $[L:K]$ *embeddings* $\sigma_i^{(\tau)}$ *of* L *into* $\mathbf{C}$ *that extend* τ, *ordered so that:*

$$|\sigma_1^{(\tau)}(\theta)| \ge |\sigma_2^{(\tau)}(\theta)| \ge \ldots \ge |\sigma_{[L:K]}^{(\tau)}(\theta)|. \tag{1}$$

Then

$$H_L(K,\theta) = N(\mathfrak{a})^{-1} \prod_{i=1}^{[L:K]} \max\left(\prod_{\tau} |\sigma_i^{(\tau)}(x_1)|, \prod_{\tau} |\sigma_i^{(\tau)}(x_2)|\right), \tag{2}$$

where N *is the absolute norm of ideals.*

Proof. For $\theta \in L^*$, consider the principal ideal $(\theta) = \mathfrak{c}\mathfrak{b}^{-1}$, with $\mathfrak{c}$ and $\mathfrak{b}$ being relatively prime integral ideals. We use an equivalent formulation of $H_L(K,\theta)$ given in [B-M 2, p. 4], namely

$$H_L(K,\ \theta) = N(\mathfrak{b}) \prod_{i=1}^{[L:K]} \max\left(1,\ \prod_{\tau} |\sigma_i^{(\tau)}(x_1 x_2^{-1})|\right), \tag*{$(2)_B$}$$

(there is an obvious typographic mistake in [B-M 1, p. 165, line 16], where the above formula is cited from [B-M 2]). Hence, by the product rule [Frö-T, p. 113], we have

$$H_L(K,\ \theta) = N(\mathfrak{b})N((x_2))^{-1}\prod_{i=1}^{[L:K]} \max\left(\prod_\tau |\sigma_i^{(\tau)}(x_1)|,\ \prod_\tau |\sigma_i^{(\tau)}(x_2)|\right).$$

But if $(x_1)+(x_2)=\mathfrak{a}$ and $(x_1x_2^{-1})=\mathfrak{c}\mathfrak{b}^{-1}$ with $\mathfrak{b},\mathfrak{c}$ relatively integral ideals, then $(x_1)=\mathfrak{a}\mathfrak{c}$ and $(x_2)=\mathfrak{a}\mathfrak{b}$. □

Corollary. *If $L=K$, or if K is a totally real number field and L is a totally complex quadratic extension of K, then*

$$H_L(K,\theta) \ = \ \max\Big(\ N(\mathfrak{c}),\ N(\mathfrak{b})\ \Big),$$

where $(\theta)=\mathfrak{c}\mathfrak{b}^{-1}$, with $\mathfrak{c}$ and $\mathfrak{b}$ being relatively prime integral ideals.

Proof. In the second case, let $\theta = x_1x_2^{-1},\ (x_1)=\mathfrak{a}\mathfrak{c},\ (x_2)=\mathfrak{a}\mathfrak{b}$. Then

$$H_L(K,\theta) \ = \ N(\mathfrak{a})^{-1}\ \prod_{i=1}^{2}\max\ \left(\prod_\tau |\sigma_i^{(\tau)}(x_1)|,\ \prod_\tau |\sigma_i^{(\tau)}(x_2)|\ \right) \ =$$

$$= N(\mathfrak{a})^{-1}\ \max\left(\prod_\tau |\sigma_1^{(\tau)}(x_1)|^2, \prod_\tau |\sigma_1^{(\tau)}(x_2)|^2\right) = N(\mathfrak{a})^{-1}\ \max\Big(N(\mathfrak{a}\mathfrak{c}), N(\mathfrak{a}\mathfrak{b})\Big) =$$

$$= \max\Big(N(\mathfrak{c}), N(\mathfrak{b})\Big).$$

The case $L=K$ is similar [B-M 1, p. 167]. □

In the introduction we defined:

- $\mathcal{O}^*_{L/K} = \left\{u\in\mathcal{O}^*_L \mid \exists j\in\mathbf{Z} \text{ such that } j\neq 0 \text{ and } u^j\in\mathcal{O}^*_K\right\},$ (3)
- The action of $(\mathcal{O}^*_{L/K})^n$ on $P^n(L)$ given by

$$(u_0,\cdots,u_{n-1})\cdot[x_0,\cdots,x_n] = [u_0x_0,\cdots,u_{n-1}x_{n-1},x_n], \tag{4}$$

We denote:

the ideal class group of L by C_L, the set of integral ideals by I_L and the ideal class of $\mathfrak{a}$ by $[\mathfrak{a}]$. (5)

Remark 1. *It follows that, when $L = K$, or when K is a totally real number field and L is a totally complex quadratic extension of K, the number of $P \in \mathbf{P}^1(L)/\mathcal{O}^*_{L/K}$ such that $H_L(K,P) \leq B$ is the same as the number of pairs of ideals $(\mathfrak{c},\mathfrak{b}) \in I_L \times I_L$ so that $[\mathfrak{c}] = [\mathfrak{b}]$, $N(\mathfrak{c}), N(\mathfrak{b}) \leq B$ and $\mathfrak{c}, \mathfrak{b}$ are relatively prime.*

We note that in the both cases $\mathcal{O}^*_{L/K} = \mathcal{O}^*_L$. We define

$$\begin{array}{rcl} P_B & = & \{P \in \mathbf{P}^1(L)/\mathcal{O}^*_{L/K} \mid H_L(K,P) \leq B \text{ and } p \neq [1,0], P \neq [0,1]\}, \\ A_B & = & \{(\mathfrak{c},\mathfrak{b}) \in I_L \times I_L \mid \ [\mathfrak{c}] = [\mathfrak{b}], (\mathfrak{c},\mathfrak{b}) = 1 \text{ and } N(\mathfrak{c}),\ N(\mathfrak{b}) \leq B\}. \end{array} \tag{6}$$

Note that if $[x_1,x_2] = P \in \mathbf{P}^1(L)/\mathcal{O}^*_{L/K}$, then $x_1 \neq 0$ and $x_2 \neq 0$. Given $P \in P_B$, let $[x_1,x_2] = P$, with x_1, $x_2 \in \mathcal{O}_L$ non zero. There exists a unique pair $(\mathfrak{c},\mathfrak{b}) \in A_B$ such that $(x_1) = \mathfrak{a}\mathfrak{c}$ and $(x_2) = \mathfrak{a}\mathfrak{b}$ (It is clear that $[\mathfrak{b}] = [\mathfrak{c}]$, and by the Corollary above, $N(\mathfrak{c}), N(\mathfrak{b}) \leq B$). The pair $(\mathfrak{c},\mathfrak{b}) \in A_B$ is independent of the choice of x_1, $x_2 \in \mathcal{O}_L$, because, for each non zero $\alpha \in L^*$, for each $u \in \mathcal{O}^*_{L/K}$, we have $(u\alpha x_1) = ((\alpha)\mathfrak{a})\mathfrak{c}$ and $(\alpha x_2) = ((\alpha)\mathfrak{a})\mathfrak{b}$. If $(\alpha)\mathfrak{a} = \mathfrak{m}\mathfrak{f}^{-1}$, with $\mathfrak{m}$ and $\mathfrak{f}$ being relatively prime integer ideals, then $\mathfrak{f}|\mathfrak{c}$ and $\mathfrak{f}|\mathfrak{b}$, but $(\mathfrak{c},\mathfrak{b}) = 1$, hence $(\alpha)\mathfrak{a}$ is an integer ideal.

Let $\chi : P_B \to A_B$ be so that $\chi(P) = (\mathfrak{c},\mathfrak{b})$ as above. This is a bijective function. In fact:

- If $\chi(P_1) = \chi(P_2) = (\mathfrak{c},\mathfrak{b})$, let $P_1 = [x_1,x_2]$ and $P_2 = [x_3,x_4]$, with $x_i \in \mathcal{O}_L$ and $x_i \neq 0$. Then $\mathfrak{c}\mathfrak{b}^{-1} = (x_1x_2^{-1}) = (x_3x_4^{-1})$. Hence there exists $u \in \mathcal{O}^*_L$ such that $x_1x_2^{-1} = ux_3x_4^{-1}$. Then $[x_1,x_2] = [x_3,x_4]$ in $\mathbf{P}^1(L)/\mathcal{O}^*_{L/K}$ (remember that $\mathcal{O}^*_L = \mathcal{O}^*_{L/K}$).
- To show surjectivity, let $(\mathfrak{c},\mathfrak{b}) \in A_B$. Then there exists $x \in L^*$ such that $\mathfrak{c}\mathfrak{b}^{-1} = (x)$. Hence by the corollary above, $H_K(L,[x,1]) \leq B$ and then $\chi[x,1] = (\mathfrak{c},\mathfrak{b})$.

Example. *When $L = K$, or when K is a totally real number field and L is a totally complex quadratic extension of K, the number of points x in $\mathbf{P}^1(L)/\mathcal{O}^*_{L/K}$, with $H_L(K,x) \leq B$ is*

$$\left(\frac{2^s(2\pi)^t R_L}{d_L^{1/2} W_L}\right)^2 \frac{h_L}{\zeta_L(2)} B^2 + \begin{cases} O(B\log B) & \text{if } K = L = \mathbf{Q}, \\ O(B^{2-\frac{1}{[L:\mathbf{Q}]}}) & \text{otherwise.} \end{cases}$$

To prove the example, we will use the above Remark.

For each ideal class $\mathcal{C}$ of L and $B \in \mathbf{R}_{>0}$, we define:

$$\begin{aligned}\widetilde{A}(\mathcal{C},B) &= \{(\mathfrak{a}_1,\mathfrak{a}_2) \in I_L \times I_L \mid \mathfrak{a}_1,\mathfrak{a}_2 \in \mathcal{C}, N(\mathfrak{a}_1), N(\mathfrak{a}_2) \leq B\},\\ A(\mathcal{C},B) &= \{(\mathfrak{a}_1,\mathfrak{a}_2) \in A_B \mid \mathfrak{a}_1,\mathfrak{a}_2 \in \mathcal{C}\},\end{aligned}$$

with A_B as in (6) and I_L as in (5). We have

$$\widetilde{A}(\mathcal{C},B) = \bigcup_{\mathfrak{a}\in I_L} \bigcup_{(\mathfrak{b}_1,\mathfrak{b}_2)\in A([\mathfrak{a}]^{-1}\mathcal{C},\frac{B}{N(\mathfrak{a})})} \{(\mathfrak{a}\mathfrak{b}_1,\mathfrak{a}\mathfrak{b}_2)\}.$$

We denote by $M(\mathcal{C},B)$ (respectively $\widetilde{M}(\mathcal{C},B)$) the cardinality of $A(\mathcal{C},B)$ (respectively $\widetilde{A}(\mathcal{C},B)$). Hence, noting that $M\left([\mathfrak{a}]^{-1}\mathcal{C},\frac{B}{N(\mathfrak{a})}\right) = 0$ if $N(\mathfrak{a}) > B$, we obtain

$$\widetilde{M}(\mathcal{C},B) = \sum_{N(\mathfrak{a})\leq B} M\left([\mathfrak{a}]^{-1}\mathcal{C},\frac{B}{N(\mathfrak{a})}\right).$$

By Möbius inversion (see Proposition 1 below),

$$M(\mathcal{C},B) = \sum_{N(\mathfrak{a})\leq B} \mu(\mathfrak{a})\widetilde{M}\left([\mathfrak{a}]^{-1}\mathcal{C},\frac{B}{N(\mathfrak{a})}\right), \tag{7}$$

where μ is the Möbius function on ideals.

Now, Abel summation [A, p. 72] yields

$$\sum_{N(\mathfrak{b})\leq B} \frac{\mu(\mathfrak{b})}{N(\mathfrak{b})} = O(\log B). \tag{8}$$

Indeed, let $a(n) = \#\{\mathfrak{a} \mid N(\mathfrak{a}) = n\}$ and $f(n) = \frac{1}{n}$. Then

$$\sum_{N(\mathfrak{b})\leq B} \frac{1}{N(\mathfrak{b})} = \sum_{n\leq B} a(n)f(n) = A(B)f(B) - A(1)f(1) - \int_1^B A(t)f'(t)dt,$$

where $A(x) = \sum_{n\leq x} a(n)$. Since the number $A(x)$ of integral ideals of L, with norm at most x is $\rho h x + O(x^{1-1/[L\mathbf{Q}]})$ [L 1, p. 132], where ρ is a constant that depends on L, we obtain that this expression is $O(\log B)$.

For fixed $s > 1$, we have that

$$\sum_{N(\mathfrak{b})\le B} \frac{\mu(\mathfrak{b})}{N(\mathfrak{b})^s} = \frac{1}{\zeta_L(s)} + O(B^{1-s}), \tag{9}$$

because

$$\sum_{N(\mathfrak{b})\le B} \frac{\mu(\mathfrak{b})}{N(\mathfrak{b})^s} = \frac{1}{\zeta_L(s)} - \sum_{N(\mathfrak{b})>B} \frac{\mu(\mathfrak{b})}{N(\mathfrak{b})^s},$$

and the latter term is bounded by $\sum_{N(\mathfrak{b})>B} \frac{1}{N(\mathfrak{b})^s}$. Indeed, we can prove as above that, $\sum_{N(\mathfrak{b})>B} \frac{1}{N(\mathfrak{b})^s} = O(B^{1-s})$. Returning to (7), w distinguish two cases:

- When $K = L = \mathbf{Q}$, $\widetilde{M}(\mathcal{C}, x) = [x]^2 = (x + O(1))^2$. Hence by (7) and (8),

$$M(\mathcal{C}, B) = \sum_{n\le B} \left(\frac{B}{n} + O(1)\right)^2 \mu(n) \;=\; \frac{B^2}{\zeta_L(2)} + O(B \log B).$$

- Otherwise, we note that $\widetilde{M}(\mathcal{C}, x) = (M_{L,\mathcal{C}}(x))^2$, where $M_{L,\mathcal{C}}(x)$ is the number of integral ideals $\mathfrak{b}$ of L in $\mathcal{C}$, with $N(\mathfrak{b}) \le x$. By [L 1, p. 132], $M_{L,\mathcal{C}} = \rho x + O(x^{1-1/N})$, where the O constant depends only on L and $\rho = \frac{2^s (2\pi)^t R_L}{d_L^{1/2} W_L}$. Hence by (7) $M(C, B)$ is

$$\sum_{N(\mathfrak{a})\le B} \mu(\mathfrak{a}) \left[\frac{\rho^2 B^2}{N(\mathfrak{a})^2} + 2\rho\, O\left(\left(\frac{B}{N(\mathfrak{a})}\right)^{2-1/N}\right) + O\left(\left(\frac{B}{N(\mathfrak{a})}\right)^{2-2/N}\right)\right]. \tag{10}$$

Now, for fixed $t > 1$ we have that

$$\sum_{N(\mathfrak{a})\le B} \mu(\mathfrak{a}) O\left(\left(\frac{B}{N(\mathfrak{a})}\right)^t\right) = O\left(B^t \sum_{N(\mathfrak{a})\le B} \frac{\mu(\mathfrak{a})}{N(\mathfrak{a})^t}\right) = O(B^t),$$

because the sum is bound by $\zeta_L(t)$. Hence using (9) and (10) we have

$$M(C, B) = \frac{\rho^2 B^2}{\zeta_L(2)} + O(B^{2-\frac{1}{N}}),$$

for each class C.

As A_B and P_B in (6) are bijection, we conclude that the numbers of elements $x \in \mathbf{P}^1(L)/\mathcal{O}^*_{L/K}$ with $H_L(K,x) \le B$ is

$$\left(\frac{2^s(2\pi)^t R_L}{d_L^{1/2} W_L}\right)^2 \frac{h_L}{\zeta_L(2)} B^2 + \begin{cases} O(B\log B) & \text{if } K = L = \mathbf{Q}, \\ O(B^{2-\frac{1}{[L:\mathbf{Q}]}}) & \text{otherwise.} \end{cases}$$

Proposition 1. *Let* $f,\ g : C_L \times \mathbf{R}_{>0} \to \mathbf{R}$ *be functions satisfying*

$$f(\mathcal{C},B) \;=\; \sum_{N(\mathfrak{a})\le B} g\left([\mathfrak{a}]^{-1}\mathcal{C}, \frac{B}{N(\mathfrak{a})}\right),$$

for $B \in \mathbf{R}_{>0}$ *and* $\mathcal{C} \in C_L$. *Then*

$$g(\mathcal{C},B) \;=\; \sum_{N(\mathfrak{a})\le B} \mu(\mathfrak{a}) f\left([\mathfrak{a}]^{-1}\mathcal{C}, \frac{B}{N(\mathfrak{a})}\right),$$

where μ *is the Möbius function defined by:*

$$\begin{array}{lcll} \mu(\wp) & = & -1, & \textit{for all prime ideals } \wp, \\ \mu(\wp^j) & = & 0, & \textit{for all } j > 1, \\ \mu(\mathcal{O}_L) & = & 1, & \\ \mu(\mathfrak{a}\mathfrak{b}) & = & \mu(\mathfrak{a})\mu(\mathfrak{b}) & \textit{for all relatively prime ideals } \mathfrak{a} \textit{ and } \mathfrak{b}. \end{array}$$

Proof.

$$\sum_{N(\mathfrak{a}_1)\le B} \mu(\mathfrak{a}_1) f\left([\mathfrak{a}_1]^{-1}\mathcal{C}, \frac{B}{N(\mathfrak{a}_1)}\right) = \sum_{N(\mathfrak{a}_1)\le B} \mu(\mathfrak{a}_1) \sum_{N(\mathfrak{a}_2)\le \frac{B}{N(\mathfrak{a}_1)}} g\left([\mathfrak{a}_1\mathfrak{a}_2]^{-1}\mathcal{C}, \frac{B}{N(\mathfrak{a}_1\mathfrak{a}_2)}\right) =$$

$$= \sum_{N(\mathfrak{a})\le B} g\left([\mathfrak{a}]^{-1}\mathcal{C}, \frac{B}{N(\mathfrak{a})}\right) \sum_{\mathfrak{a}_1\mathfrak{a}_2 = \mathfrak{a}} \mu(\mathfrak{a}_1) = g\left(\mathcal{C},B\right),$$

because

$$\sum_{\mathfrak{b}|\mathfrak{a}} \mu(\mathfrak{b}) = \begin{cases} 1 & \text{if } \mathfrak{a} = \mathcal{O}_L, \\ 0 & \text{otherwise.} \end{cases} \qquad (11)$$

□

Similary, we have

Proposition 2. *Let* $f,\ g : I_L \times \mathbf{R}_{>0} \to \mathbf{R}$ *be functions satisfying*

$$f(\mathfrak{a}, B) = \sum_{N(\mathfrak{b}) \le \frac{B}{N(\mathfrak{a})}} g(\mathfrak{a}\mathfrak{b}, B),$$

for $B \in \mathbf{R}_{>0}$ *and* $\mathfrak{a} \in I_L$. *Then*

$$g(\mathfrak{a}, B) = \sum_{N(\mathfrak{b}) \le \frac{B}{N(\mathfrak{a})}} \mu(\mathfrak{b}) f(\mathfrak{a}\mathfrak{b}, B).$$

Proof.

$$\sum_{N(\mathfrak{a}_1) \le \frac{B}{N(\mathfrak{a})}} \mu(\mathfrak{a}_1) f(\mathfrak{a}\mathfrak{a}_1, B) = \sum_{N(\mathfrak{a}_1) \le \frac{B}{N(\mathfrak{a})}} \mu(\mathfrak{a}_1) \sum_{N(\mathfrak{a}_2) \le \frac{B}{N(\mathfrak{a}\mathfrak{a}_1)}} g(\mathfrak{a}\mathfrak{a}_1\mathfrak{a}_2, B) =$$

$$= \sum_{N(\mathfrak{b}) \le \frac{B}{N(\mathfrak{a})}} g(\mathfrak{a}\mathfrak{b}, B) \sum_{\mathfrak{a}_1\mathfrak{a}_2 = \mathfrak{b}} \mu(\mathfrak{a}_1) = g(\mathfrak{a}, B),$$

by (11). □

2.3 Proof of Theorem 5.

Given $(x, y) \in L^* \times L^*$, for each embedding τ of K into $\mathbf{C}$, order the embeddings of L into $\mathbf{C}$ that extend τ, as in (1) for $\theta = xy^{-1}$. Define

$$\mathbf{T}(x, y) := \prod_{i=1}^{[L:K]} \max\left(\prod_{\tau} |\sigma_i^{(\tau)}(x)|\ ,\ \prod_{\tau} |\sigma_i^{(\tau)}(y)| \right). \tag{12}$$

Note that for non zero $x,\ y \in \mathcal{O}_L$, if $(x) + (y) = \mathfrak{a}$, then by (2),

$$H_L(K, [x, y]) = N(\mathfrak{a})^{-1} \mathbf{T}(x, y). \tag{13}$$

Furthermore, for non zero $\alpha \in O_L$, one has

$$N((x) + (y))^{-1} \mathbf{T}(x, y) = N((\alpha x) + (\alpha y))^{-1} \mathbf{T}(\alpha x, \alpha y).$$

For $\mathcal{O}^*_{L/K}$ as in (3), define an action of $\mathcal{O}^*_{L/K} \times \mathcal{O}^*_L$ on $L^* \times L^*$ by

$$(u, \varepsilon)(x, y) = (\varepsilon u x, \varepsilon y), \tag{14}$$

for $(u,\epsilon) \in \mathcal{O}^*_{L/K} \times \mathcal{O}^*_L$, and $(x,y) \in L^* \times L^*$.

We denote s_L and t_L by s and t, keeping the subscript only when necessary. Let

$$C_{s,t} = \mathbf{R}^s \times \mathbf{C}^{2t}, \quad C^*_{s,t} = \mathbf{R}^{*s} \times \mathbf{C}^{*2t}, \tag{15}$$

and let $\varphi_L : L \to C_{s,t}$ be the geometric embedding of L in $C_{s,t}$, given by

$$\varphi_L(x) \;=\; (\sigma_1(x), \cdots, \sigma_{s+2t}(x)). \tag{16}$$

Note that φ_L is not quite the usual embedding: Here we consider all the embeddings of L into $\mathbf{C}$.

For $u \in \mathcal{O}^*_{L/K}$ and $\varepsilon \in \mathcal{O}^*_L$, we define linear transformations ϑ_u and θ_ε on $C_{s,t} \times C_{s,t}$ given by

$$\vartheta_u(x,y) \;=\; (\varphi_L(u)x, y) \quad \text{and} \quad \theta_\varepsilon(x,y) \;=\; (\varphi_L(\varepsilon)x, \varphi_L(\varepsilon)y),$$

where multiplication in $C_{s,t}$ is taken component-wise.

We define the action of $\mathcal{O}^*_{L/K} \times \mathcal{O}^*_L$ on $C_{s,t} \times C_{s,t}$ as follows. For $(u,\epsilon) \in \mathcal{O}^*_{L/K} \times \mathcal{O}^*_L$, and $(x,y) \in C_{s,t} \times C_{s,t}$, the action is given by $(\theta_\varepsilon \circ \vartheta_u)(x,y)$. Note that

$$\theta_\varepsilon \circ \vartheta_u(x,y) = (\varphi_L(\varepsilon)\varphi_L(u)x \;,\; \varphi_L(\varepsilon)y). \tag{17}$$

This action is compatible with the action of $\mathcal{O}^*_{L/K} \times \mathcal{O}^*_L$ on $L^* \times L^*$ given in (14).

We define $\mathbf{t} : C^*_{s,t} \times C^*_{s,t} \to \mathbf{R}$ as the unique continuous function with domain $C^*_{s,t} \times C^*_{s,t}$ and such that, for all $x,\ y \in L^*$, $\mathbf{t}(\varphi(x), \varphi(y)) = \mathbf{T}(x,y)$, with $\mathbf{T}$ as in (12).

Noting that $C_{s,t}$ is isomorphic to $\prod_\tau \prod_{\sigma|\tau} L_\sigma$ and is isomorphic to each reordering of the L_σ, where L_σ denotes the closure of $\sigma(L)$ in $\mathbf{C}$, we can write $x \in C_{s,t}$ as $(x_{\sigma_i^{(\tau)}})$ or simply (x_i^τ). The function $\mathbf{t}$ is such that given $(x,y) \in C^*_{s,t} \times C^*_{s,t}$

$$\mathbf{t}(x,y) := \prod_{i=1}^{[L:K]} \max\left(\prod_\tau |x_i^\tau| \;,\; \prod_\tau |y_i^\tau|\right), \tag{18}$$

where for each embedding τ of K into $\mathbf{C}$, we order the embeddings of L into $\mathbf{C}$ that extend τ so that,

$$|x_i^\tau (y_i^\tau)^{-1}| \;\geq |x_{i+1}^\tau (y_{i+1}^\tau)^{-1}|. \tag{19}$$

Note that for each $(\varepsilon, u) \in \mathcal{O}_{L/K}^* \times \mathcal{O}_L^*$ and $(x,y) \in C_{s,t}^* \times C_{s,t}^*$,

$$\mathrm{t}(\theta_\varepsilon \circ \vartheta_u(x,y)) = \mathrm{t}(x,y), \tag{20}$$

because, if we take $l = \#\{i \mid \prod_\tau |x_i^\tau| \geq \prod_\tau |y_i^\tau|\}$ and j such that $u^j \in \mathcal{O}_K^*$, we have

$$\mathrm{t}(\theta_\varepsilon \circ \vartheta_u(x,y)) = \left(\prod_{i=1}^{[L:K]} \prod_\tau |\sigma_i^\tau(\varepsilon)|\right) \left(\prod_\tau |\tau(u^j)|\right)^{\frac{l}{j}} \mathrm{t}(x,y) = \mathrm{t}(x,y).$$

Note that the closure $\mathcal{I}$ of $\varphi_L(L)$ is isomorphic to $\mathbf{R}^s \times \mathbf{C}^t$ and thus to $\mathbf{R}^N$, where $N = [L:\mathbf{Q}]$. Put

$$\mathcal{I}^* = \mathcal{I} \bigcap C_{s,t}^*.. \tag{21}$$

Thus, $\mathcal{I}^*$ is the closure of $\varphi_L(L^*)$ in $\mathbf{R}^{*s} \times \mathbf{C}^{*2t}$. Given a subset $\mathcal{A}$ of $\mathcal{I}^* \times \mathcal{I}^*$, for $B \in \mathbf{R}_{>0}$ we define the sets $T(B)$ and $T_\mathcal{A}(B)$ by

$$T(B) := \{(x,y) \in \mathcal{I}^* \times \mathcal{I}^* \mid \mathrm{t}(x,y) \leq B\}, \quad T_\mathcal{A}(B) = T(B) \cap \mathcal{A}. \tag{22}$$

Lemma 1. *Let $\mathcal{A}$ be a fundamental domain for $(\mathcal{I}^* \times \mathcal{I}^*)/(\mathcal{O}_{L/K}^* \times \mathcal{O}_L^*)$. For each ideal $\mathfrak{a}$ of L and B a real positive number, denote by $M(\mathfrak{a}, B)$ the number of pairs $(x,y) \in (\mathfrak{a} \times \mathfrak{a}) \bigcap T_\mathcal{A}(B)$. We have for $\alpha \in L^*$,*

$$M(\alpha\mathfrak{a}, B) = M\left(\mathfrak{a}, \frac{B}{|N(\alpha)|}\right).$$

Proof. Let $E(\mathfrak{a}, B) = (\mathfrak{a} \times \mathfrak{a}) \bigcap \{(x,y) \in \mathcal{A} \mid \mathrm{t}(x,y) = B\}$ and $b(\mathfrak{a}, B) = \#E(\mathfrak{a}, B)$. We have that for each $\alpha \in L^*$, $b(\alpha\mathfrak{a}, B) = b(\mathfrak{a}, \frac{B}{|N(\alpha)|})$. In fact, given $(x,y) \in E(\alpha\mathfrak{a}, B)$ (here $\alpha\mathfrak{a}$ is taken as lattice in $C_{s,t}$), there exist $a,\ b \in \mathfrak{a}$, such that $(x,y) = (\varphi_L(\alpha)\varphi_L(a), \varphi_L(\alpha)\varphi_L(b))$. Furthermore by (18), $\mathrm{t}(x,y) = |N(\alpha)|\mathrm{t}(\varphi_L(a), \varphi_L(b))$, hence $\mathrm{t}(\varphi_L(a), \varphi_L(b)) = \frac{B}{|N(\alpha)|}$. But $\mathcal{A}$ is a fundamental domain for $(\mathcal{I}^* \times \mathcal{I}^*)/(\mathcal{O}_{L/K}^* \times \mathcal{O}_L^*)$, hence there exist $(u, \varepsilon) \in \mathcal{O}_{L/K}^* \times \mathcal{O}_L^*$ such that $\theta_\varepsilon \circ \vartheta_u(\varphi_L(a), \varphi_L(b)) = (z,w) \in \mathcal{A}$. It is clear that $\mathrm{t}(z,w) = \mathrm{t}(\varphi_L(a), \varphi_L(b))$ and $(z,w) = (\varphi_L(\varepsilon u a), \varphi_L(\varepsilon b)) \in (\mathfrak{a} \times \mathfrak{a})$. Hence $b(\alpha\mathfrak{a}, B) \leq b(\mathfrak{a}, \frac{B}{|N(\alpha)|})$. Taking α^{-1}, $\alpha\mathfrak{a}$ and $\frac{B}{|N(\alpha)|}$ instead of α, $\mathfrak{a}$ and B, we obtain $b(\mathfrak{a}, \frac{B}{|N(\alpha)|}) \leq b(\alpha\mathfrak{a}, B)$. Hence the Lemma. □

A subset T of some Euclidean space is said to be k-Lipschitz parametrizable if there exists a finite number of Lipschitz maps $\phi_j : I^k \to T$, where I^k denote the unit cube in k-space, such that $T = \cup_j \phi_j(I^k)$.

Theorem 5. *Let $\mathcal{A}$ be a cone which is also a fundamental domain for $\mathcal{I}^* \times \mathcal{I}^*$ with respect to the action described in* (17) *above. Suppose furthermore that the boundary of $T_{\mathcal{A}}(1)$ is $(2[L:K]-1)$-Lipschitz parametrizable. Then the number $N(L/K, B)$ of points $x \in \mathbf{P}^1(L)/\mathcal{O}^*_{L/K}$, with $H_L(K, x) \le B$ is*

$$\frac{2^{2t} h_L}{d_L \zeta_L(2)} Vol(T_{\mathcal{A}}(1)) B^2 + \begin{cases} O(B \log B) & \text{if } K = L = \mathbf{Q}, \\ O(B^{2 - \frac{1}{[L:\mathbf{Q}]}}) & \text{otherwise,} \end{cases}$$

where $T_{\mathcal{A}}(1)$ is as in (22), *d_L is the absolute value of the discriminant of L, h_L is the class number, and ζ_L is the Dedekind zeta-function.*

Proof. By the example, when $K = L = \mathbf{Q}$ we have that $N(L/K, B)$ is $\frac{B^2}{\zeta_L(2)} + O(B \log B)$. Furthermore in this case we can take

$$\mathcal{A} = \{((x, y) \in \mathbf{R}^* \times \mathbf{R}^* \mid 0 < x \text{ and } 0 < y\}.$$

In this case $T_{\mathcal{A}}(1) = \{(x, y) \in \mathbf{R}^* \times \mathbf{R}^* \mid 0 < x,\ y \le 1\}$ and Vol $(T_{\mathcal{A}}(1)) = 1$. We will exclude this case.

For an integral ideal $\mathfrak{a}$ and a positive real number B, we denote by:

- $M(\mathfrak{a}, B)$ the number of pairs $(x, y) \in (\mathfrak{a} \times \mathfrak{a})/(\mathcal{O}^*_{L/K} \times \mathcal{O}^*_L)$ such that $\mathbf{T}(x, y)) \le B$, where the action is given by (14).
- $M^*(\mathfrak{a}, B)$ the number of pairs $(x, y) \in (\mathfrak{a} \times \mathfrak{a})/(\mathcal{O}^*_{L/K} \times \mathcal{O}^*_L)$ such that $(x) + (y) = \mathfrak{a}$ and $\mathbf{T}(x, y)) \le B$.
- $\widetilde{M}([\mathfrak{a}], B)$ the number of points $[x, y] \in \mathbf{P}^1(L)/\mathcal{O}^*_{L/K}$ such that $[(x) + (y)] = [\mathfrak{a}]$ and $H_L(K, [x, y]) \le B$, where the action is given by (4).

We note that $\widetilde{M}([\mathfrak{a}], B)$ is finite by [B-M 1, p. 174]. We shall prove that $M(\mathfrak{a}, B)$ and $M^*(\mathfrak{a}, B)$ are also finite. If we regard $\mathfrak{a} \times \mathfrak{a} \subset C_{s,t} \times C_{s,t}$, on which the action was already defined by (17), we have that $M(\mathfrak{a}, B) = \#\Big((\mathfrak{a} \times \mathfrak{a}) \bigcap T_{\mathcal{A}}(B)\Big)$, as in Lemma 1. Furthermore by [L 1, p. 115], $\mathfrak{a} \times \mathfrak{a}$ is a lattice in $\mathbf{R}^{2N}$ with a fundamental domain F with $\mathrm{Vol}(F) = \left(\frac{N(\mathfrak{a}) d_L^{1/2}}{2^t}\right)^2$.

For $\alpha \in \mathbf{Q}^*$ and $(x,y) \in C^*_{s,t} \times C^*_{s,t}$, we have by (18), $\mathfrak{t}(\alpha x, \alpha y) = |\alpha|^N \mathfrak{t}(x,y)$. By continuity, we have this for each $\alpha \in \mathbf{R}^*$. Hence $T_{\mathcal{A}}(B) = B^{\frac{1}{N}} T_{\mathcal{A}}(1)$, as $\mathcal{A}$ is a cone and Vol $(T_{\mathcal{A}}(B)) = B^2$ Vol $(T_{\mathcal{A}}(1))$.

Let $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_h\}$ be ideals representing the class group of L. By standard results in the geometry of numbers [L1, p. 128],

$$M\left(\mathfrak{a}_i, \frac{B}{|N(\alpha)|}\right) = \mathrm{Vol}\Big(T_{\mathcal{A}}(1)\Big)\left(\frac{2^t}{N(\mathfrak{a}_i)d_L^{1/2}}\right)^2\left(\frac{B}{|N(\alpha)|}\right)^2 + O\left(\left(\frac{B}{|N(\alpha)|}\right)^{2-1/N}\right),$$

where the O-constant depends on $\mathfrak{a}_i$, L and $\mathcal{A}$. Denote this constant by $c_{\mathfrak{a}_i}$. If we put

$$c_L = \max_{1\le j\le h} c_{\mathfrak{a}_j}\Big(N(\mathfrak{a}_j)\Big)^{2-\frac{1}{N}},$$

we have

$$c_{\mathfrak{a}_i}\left(\frac{B}{|N(\alpha)|}\right)^{2-\frac{1}{N}} \le c_L\left(\frac{B}{N(\mathfrak{a}_i)|N(\alpha)|}\right)^{2-\frac{1}{N}}.$$

Given $\mathfrak{a}$ an ideal, put $\mathfrak{a} = \alpha\mathfrak{a}_i$ for some $\alpha \in L^*$ and $\mathfrak{a}_i \in \{\mathfrak{a}_1, \cdots, \mathfrak{a}_h\}$. Using Lemma 1, we have

$$M(\mathfrak{a}, B) = \mathrm{Vol}\,(T_{\mathcal{A}}(1))\left(\frac{2^t}{N(\mathfrak{a})d_L^{1/2}}\right)^2 B^2 + O\left(\left(\frac{B}{|N(\mathfrak{a})|}\right)^{2-1/N}\right), \tag{23}$$

where the O constant only depend on L and $\mathcal{A}$.

If we rewrite $M^*(\mathfrak{a}, B)$ as the number of pairs $(x,y) \in (\mathfrak{a}\times\mathfrak{a})/(\mathcal{O}^*_{L/K}\times\mathcal{O}^*_L)$ $(x \neq 0,\ y \neq 0)$ such that $(x)+(y) = \mathfrak{a}$ and $H_L(K,[x,y]) \le BN(\mathfrak{a})^{-1}$ (by (13)), then we obtain

$$\widetilde{M}([\mathfrak{a}], B) = M^*\Big(\mathfrak{a}, BN(\mathfrak{a})\Big). \tag{24}$$

Furthermore, by definition of M and M^*, we obtain

$$M(\mathfrak{a}, B) = \sum_{N(\mathfrak{b})\le\frac{B}{N(\mathfrak{a})}} M^*(\mathfrak{a}\mathfrak{b}, B),$$

where we used the fact that, if $N(\mathfrak{a}\mathfrak{b}) > B$ then $M^*(\mathfrak{a}\mathfrak{b}, B) = 0$. Indeed $M^*(\mathfrak{a}\mathfrak{b}, B) = \widetilde{M}([\mathfrak{a}\mathfrak{b}], \frac{B}{N(\mathfrak{a}\mathfrak{b})})$ by (24), and $H_L(K,[x,y]) \ge 1$. Using Proposition 2,

$$M^*(\mathfrak{a}, B) = \sum_{N(\mathfrak{b})\le\frac{B}{N(\mathfrak{a})}} M(\mathfrak{a}\mathfrak{b}, B)\mu(\mathfrak{b}).$$

Using (23), we obtain

$$M^*(\mathfrak{a},B) = \sum_{N(\mathfrak{b})\le \frac{B}{N(\mathfrak{a})}} \left(\mathrm{Vol}(T_{\mathcal{A}}(1)) \left(\frac{2^t}{N(\mathfrak{ab})d_L^{1/2}}\right)^2 B^2 + O\left(\left(\frac{B}{N(\mathfrak{ab})}\right)^{2-\frac{1}{N}}\right)\right)\mu(\mathfrak{b})$$

$$= \mathrm{Vol}(T_{\mathcal{A}}(1)) \left(\frac{2^t B}{N(\mathfrak{a})d_L^{1/2}}\right)^2 \sum_{N(\mathfrak{b})\le \frac{B}{N(\mathfrak{a})}} \frac{\mu(\mathfrak{b})}{N(\mathfrak{b})^2} + O\left(\left(\frac{B}{N(\mathfrak{a})}\right)^{2-1/N}\right) \sum_{N(\mathfrak{b})\le \frac{B}{N(\mathfrak{a})}} \frac{\mu(\mathfrak{b})}{N(\mathfrak{b})^{2-\frac{1}{N}}},$$

because the O-constant in (23) does not depend on the lattice.

Hence, using (9), we have

$$M^*(\mathfrak{a},B) = \mathrm{Vol}\Big(T_{\mathcal{A}}(1)\Big) \frac{2^{2t}}{N(\mathfrak{a})^2 d_L \zeta_L(2)} B^2 + O\left(\left(\frac{B}{N(\mathfrak{a})}\right)^{2-1/N}\right). \tag{25}$$

By (24) and (25), we have

$$\widetilde{M}([\mathfrak{a}],B) = M^*(\mathfrak{a},BN(\mathfrak{a})) = \frac{2^{2t}\mathrm{Vol}\Big(T_{\mathcal{A}}(1)\Big)B^2}{d_L\zeta_L(2)} + O(B^{2-1/N}).$$

This number is independent of the ideal class $[\mathfrak{a}]$. Hence, the number of points $x \in \mathbf{P}^1(L)/\mathcal{O}^*_{L/K}$ with $H_L(K,x) \le B$ is

$$N(L/K,B) = \frac{2^{2t_L}\mathrm{Vol}\Big(T_{\mathcal{A}}(1)\Big)h_L}{d_L\zeta_L(2)} B^2 + O(B^{2-1/N}). \qquad \square$$

In Lemma 5 of the following section we obtain a fundamental domain as required for Theorem 5.

2.4 Fundamental domain for Theorem 5.

We will first construct in Proposition 6 a fundamental domain $\mathcal{D} = \bigcup_k \mathcal{D}^{(k)}$, which is a cone but has in general, an unbounded intersection with $T(1) = \{(x,y) \in \mathcal{I}^* \times \mathcal{I}^* \mid \mathrm{t}(x,y) \le 1\}$. We then modify $\mathcal{D}$ to obtain in Proposition 8 a fundamental domain $\mathcal{A}$ as required in Theorem 5. Define $\psi : C^*_{s,t} \to \mathbf{R}^{s+2t}$ by

$$\Big(\psi(x)\Big)_i = \log|x_i|, \tag{26}$$

where $C^*_{s,t} = \mathbf{R}^{*s} \times \mathbf{C}^{*2t}$ as in (15). Note that $\psi \circ \varphi_L$ is not quite the usual logarithmic embedding when $t > 0$, as we repeat complex conjugate embeddings. Define

$$\phi_2 := \psi \times \psi \quad \text{and} \quad V = \phi_2(\mathcal{I}^* \times \mathcal{I}^*), \tag{27}$$

with $\mathcal{I}^*$ as in (21). Note that $V \subset \mathbf{R}^{s+2t} \times \mathbf{R}^{s+2t}$ has dimension $2(s+t)$. Put

$$\mathcal{L} : L^* \to \mathbf{R}^{s+2t}, \quad \mathcal{L} := \psi \circ \varphi_L,$$

with φ_L and ψ as in (16) and (26) respectively. Define the action of $\mathcal{O}^*_{L/K} \times \mathcal{O}^*_L$ on $\mathbf{R}^{s+2t} \times \mathbf{R}^{s+2t}$ by

$$(u, \varepsilon) \cdot (x, y) = (\mathcal{L}(\varepsilon), \mathcal{L}(\varepsilon)) + (\mathcal{L}(u), 0) + (x, y), \tag{28}$$

for $(u, \varepsilon) \in \mathcal{O}^*_{L/K} \times \mathcal{O}^*_L$ and $(x, y) \in \mathbf{R}^{s+2t} \times \mathbf{R}^{s+2t}$. Note that

$$(u, \varepsilon) \cdot \phi_2(x, y) = \phi_2((\theta_\varepsilon \circ \vartheta_u)(x, y)). \tag{29}$$

Let $W = W_1 \oplus W_2 \subset V$, where W_1, W_2 be spanned by

$$W_1 = \langle (\mathcal{L}(\varepsilon), \mathcal{L}(\varepsilon)) \mid \varepsilon \in \mathcal{O}^*_L \rangle \quad \text{and} \quad W_2 = \langle (\mathcal{L}(u), 0) \mid u \in \mathcal{O}^*_{L/K} \rangle. \tag{30}$$

Let $\{\varepsilon_i\}_{i=1}^{r_L}$ be a system of fundamental units of L and let $\{u_i\}_{i=1}^{r_K}$ be a set of free generators of $\mathcal{O}^*_{L/K}$ modulo torsion, where $r_L = s + t - 1$ and $r_K = s_K + t_K - 1$. The union of $\{E_i \mid 1 \leq i \leq r_L\}$ and $\{U_j \mid 1 \leq j \leq r_K\}$ is a basis for W, where

$$E_i = ((\mathcal{L}(\varepsilon_i), \mathcal{L}(\varepsilon_i)) \quad \text{and} \quad U_j = (\mathcal{L}(u_j), 0). \tag{31}$$

Lemma 2. *Let W' be a subspace of V such that $V = W' \oplus W$, where V is as in (27) and $W = W_1 \oplus W_2$ as in (30). Let $\Delta = \Delta(W')$ be the subset of V given by*

$$(x, y) \in \Delta \textit{ iff } (x, y) = w' + \sum_{i=1}^{r_L} \alpha_i E_i + \sum_{i=1}^{r_K} \beta_i U_i \textit{ with } w' \in W' \textit{ and } 0 \leq \alpha_i, \beta_i < 1.$$

If $S \subset C^*_{s,t} \times C^*_{s,t}$ is closed under the action of $\mathcal{O}^*_{L/K} \times \mathcal{O}^*_L$ given in (17), then

$$\phi_2^{-1}(\Delta) \bigcap \left\{ (x, y) \in S \mid 0 \leq \arg(x_1),\ \arg(y_1) < \frac{2\pi}{W_L} \right\},$$

is a fundamental domain for S with respect to this action, if ϕ_2 is as in (27), *E_i and U_i as in* (31).

Proof. Given $(x,y) \in S$, we write

$$(\psi_L(x), \psi_L(y)) = w' + \sum_{i=1}^{r_L}(\alpha_i + c_i)E_i + \sum_{i=1}^{r_L}(\beta_i + d_i)U_i,$$

where $w' \in W'$, $0 \le \alpha_i, \ \beta_i < 1$ and $c_i, \ d_i \in \mathbf{Z}$.

Put $\eta = \varepsilon_1^{c_1} \cdots \varepsilon_{r_L}^{c_{r_L}}$, $\mu = u_1^{d_1} \cdots u_{r_K}^{d_{r_K}}$ and $(x', y') = \theta_{\eta^{-1}} \circ \vartheta_{\mu^{-1}}(x,y)$. It is clear that $(x',y') \in \phi_2^{-1}(\Delta)$. Let ρ_L be a primitive root of unity in L of order W_L. We find $k_1, \ k_2 \in \mathbf{Z}$ such that $0 \le \arg(y_1' \ \rho_L^{-k_1}) < \frac{2\pi}{W_L}$, and $0 \le \arg(x_1' \ \rho_L^{-k_1-k_2}) < \frac{2\pi}{W_L}$. Let $(z,w) = (\varphi_L(\rho_L^{-(k_1+k_2)})x', \varphi_L(\rho_L^{-k_1})y')$. Using (29) we have

$$\phi_2(z,w) = \phi_2(\theta_{\rho_L^{-k_1}} \circ \vartheta_{\rho_L^{-k_2}}(x',y')) = (\rho_L^{-k_2}, \rho_L^{-k_1}) \cdot \phi_2(x',y').$$

But $\mathcal{L}(\rho_L^{k_2}) = \mathcal{L}(\rho_L^{k_1}) = 0$. Hence by (28), $\phi_2(z,w) = \phi_2(x',y')$ and so $(z,w) \in \phi_2^{-1}(\Delta)$. Is clear that $(z,w) \in S$, because S closed under the action. Furthermore $0 \le \arg(z_1), \ \arg(w_1) < \frac{2\pi}{W_L}$. Hence we have found (z,w) in $\phi_2^{-1}(\Delta) \cap \left\{(x,y) \in S \ \mid \ 0 \le \arg(x_1), \ \arg(y_1) < \frac{2\pi}{W_L}\right\}$ such that $(x,y) = \theta_{\rho_L^{k_1}\eta} \circ \vartheta_{\rho_L^{k_2}\mu}(z,w)$. The proof of uniqueness of (z,w) is straightforward (cf. [B-C, p. 349]). □

Lemma 2 gives a tool for obtaining fundamental domains for $\mathcal{I}^* \times \mathcal{I}^*$, with respect to the action given in (17). But given a fundamental domain $\mathcal{A}$ for $\mathcal{I}^* \times \mathcal{I}^*$, $\mathcal{A}$ may not satisfy the hypothesis of Theorem 5, or it may be difficult to calculate $\mathrm{Vol}(T_{\mathcal{A}}(1))$. An example of this occurs, when we take W' in Lemma 2 as $W^{\perp}$ with respect to the natural Euclidean structure. Instead, we partition $\mathcal{I}^* \times \mathcal{I}^*$ into convenient subsets which are invariant under the action of $\mathcal{O}_{L/K}^* \times \mathcal{O}_L^*$ as follows.

For each embedding τ of K, let g_τ be an ordering of the embeddings σ that extend τ. We assume that if σ is a complex embedding of L which extends a real embedding τ of K, then $\{\sigma, \ \overline{\sigma}\} = \{\sigma_i^\tau, \ \sigma_{i+1}^\tau\}$ for some i. Define the set $C = C_g$, with $g = (g_\tau)_\tau$ by

$$C = \{(x,y) \in \mathcal{I}^* \times \mathcal{I}^* \ \mid |x_i^\tau (y_i^\tau)^{-1}| \ge |x_{i+1}^\tau (y_{i+1}^\tau)^{-1}| \text{ for all } i \text{ and } \tau, \}, \quad (32)$$

where $\mathcal{I}^*$ is given in (21) and the ordering of the coordinates is the one corresponding to g. Thus $\mathcal{I}^* \times \mathcal{I}^* = \cup_g C_g$, where the C_g overlap only on the boundary (given by equality in (32) for some i and τ).

We note that for each $(x,y) \in C_g$, there exists k, with $0 \leq k \leq [L:K]$, such that

$$\prod_\tau |x_k^\tau (y_k^\tau)^{-1}| \ \geq \ 1 > \prod_\tau |x_{k+1}^\tau (y_{k+1}^\tau)^{-1}|.$$

For $0 \leq k \leq [L:K]$, we define the set $B^{(k)} = B_g^{(k)}$ by

$$B^{(k)} = \left\{ (x,y) \in C \,\middle|\, \prod_\tau |(x_k^\tau (y_k^\tau)^{-1})| \geq 1 > \prod_\tau |(x_{k+1}^\tau (y_{k+1}^\tau)^{-1})| \right\}. \tag{33}$$

Proposition 3. *For each $0 \leq k \leq [L:K]$, the set $B^{(k)}$ is closed under the action of $\mathcal{O}_{L/K}^* \times \mathcal{O}_L^*$ given in (17).*

Proof. If $0 \leq k \leq [L:K]$, $(x,y) \in B^{(k)}$ and $(u,\varepsilon) \in \mathcal{O}_{L/K}^* \times \mathcal{O}_L^*$, we put $(z,w) = \theta_\varepsilon \circ \vartheta_u(x,y) = (\varphi_L(\varepsilon u)x, \varphi_L(\varepsilon)y)$. Thus, for each τ,

$$|z_i^\tau \ (w_i^\tau)^{-1}| = |\sigma_i^\tau(\varepsilon u) \ x_i^\tau \ (\sigma_i^\tau(\varepsilon) \ y_i^\tau)^{-1}| = |\sigma_i^\tau(u) \ x_i^\tau \ (y_i^\tau)^{-1}|.$$

But, $\sigma_j^\tau(u) = \sigma_i^\tau(u)$, because $u \in \mathcal{O}_{L/K}^*$. Hence $(x,y) \in C$ implies $(z,w) \in C$. Also, $\prod_\tau |z_i^\tau (w_i^\tau)^{-1}| = \prod_\tau |x_i^\tau (y_i^\tau)^{-1}|$ for each i. Hence, if $(x,y) \in B^{(k)}$, then $(z,w) \in B^{(k)}$. □

Note that for each $(x,y) \in B^{(k)}$,

$$\mathrm{t}(x,y) = \prod_\tau \left(\prod_{i=1}^{k} |x_i^\tau| \prod_{i=k+1}^{[L:K]} |y_i^\tau| \right), \tag{34}$$

where t is as in (18).

Since $\phi_2(B^{(k)})$ is invariant under translation by elements in W (by (28), (29) and the proof of proposition 3), we will define such translations on the space V to obtain a suitable subspace $V_k = W'$ to apply lemma 2 with $S = B^{(k)}$. We need first some definitions. Let

$$\epsilon(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is a real embedding ,} \\ 2 & \text{otherwise.} \end{cases} \tag{35}$$

Given k with $1 \leq k \leq [L:K]$, define $P_k = P_k^g$ by

$$P_k = \{\tau \mid \epsilon(\tau) = \epsilon(\sigma_k^\tau)\}\,.$$

We fix $\tau_1 = \tau_{1,k}^g$ any embedding of K such that: (36)

- If $P_k \neq \phi$ then choose any $\tau_1 \in P_k$.
- If $P_k = \phi$ and $\{\tau \mid \sigma_{k+1}^\tau \neq \overline{\sigma_k^\tau}\} \neq \phi$ then choose τ_1 such that $\sigma_{k+1}^{\tau_1} \neq \overline{\sigma_k^{\tau_1}}$.
- If $P_k = \phi$ and $\{\tau \mid \sigma_{k+1}^\tau \neq \overline{\sigma_k^\tau}\} = \phi$ then take any τ_1, because in this case $B^{(k)}$ is empty.

With this choice of τ_1 we define linear function $w_k,\ v_k : V \to W$, where $W = W_1 + W_2$ as in (30), such that for each $(x,y) \in V$:

$$w_k(x,y) = (z,z) \text{ and } w_k(x,y) \in W_1. \tag{37}$$

Here z is given by:

$$z_k^\tau \;=\; x_k^\tau, \quad \text{for } \tau \notin \{\tau_1, \overline{\tau_1}\},$$

$$z_k^\tau \;=\; -\frac{1}{\epsilon(\sigma_k^{\tau_1})}\left(\sum_{\tau' \notin \{\tau_1,\overline{\tau_1}\}} \sum_{i=1}^{[L:K]} z_i^{\tau'} + \sum_{\tau' \in \{\tau_1,\overline{\tau_1}\}} \sum_{\{i \neq k \mid \sigma_i^{\tau'} \neq \overline{\sigma_k^{\tau'}}\}} z_i^{\tau'}\right), \quad \text{for } \tau \in \{\tau_1, \overline{\tau_1}\},$$

$$\begin{aligned}
z_i^\tau &= x_i^\tau && \text{for} \quad 1 \leq i \leq k-2,\\
z_i^\tau &= y_i^\tau && \text{for} \quad k+2 \leq i \leq [L:K],\\
z_{k-1}^\tau &= x_{k-1}^\tau && \text{if} \quad \sigma_{k-1}^\tau \neq \overline{\sigma_k^\tau},\\
z_{k+1}^\tau &= y_{k+1}^\tau && \text{if} \quad \sigma_{k+1}^\tau \neq \overline{\sigma_k^\tau}\\
z_i^\tau &= z_k^\tau && \text{if} \quad i \in \{k-1,\ k+1\} \text{ and } \sigma_i^\tau = \overline{\sigma_k^\tau}.
\end{aligned}$$

Similary,

$$v_k(x,y) = (z,0) \text{ and } v_k(x,y) \in W_2, \tag{38}$$

where for each $1 \leq i \leq [L:K]$, z is given by:

$$z_i^\tau = y_k^\tau, \qquad \text{for all } \tau \notin \{\tau_1, \overline{\tau_1}\},$$

$$z_i^\tau = -\tfrac{1}{\epsilon(\tau_1)} \textstyle\sum_{\tau' \notin \{\tau_1, \overline{\tau_1}\}} y_k^{\tau'}, \quad \text{for } \tau \in \{\tau_1, \overline{\tau_1}\}.$$

For $1 \leq k \leq [L:K]$, using $W \subset V$ we define the transformation

$$f_k : V \to V, \quad f_k := Id - w_k. \tag{39}$$

Let

$$\mathcal{N} = (\mathrm{n}, \mathrm{n}) \in V, \quad \mathbf{w}_k = (\mathrm{w}, \mathrm{w}) \in W \tag{40}$$

be given by

$$\mathrm{n}_i^\tau = \frac{1}{[L:\mathbf{Q}]}, \quad \mathrm{w}_i^\tau = \begin{cases} \frac{1}{[L:\mathbf{Q}]} - \frac{1}{\epsilon(\sigma_k^{\tau_1})} & \text{if } \tau \in \{\tau_1, \overline{\tau_1}\} \text{ and } (i = k \text{ or } \sigma_i^\tau = \overline{\sigma_k^\tau}), \\ \frac{1}{[L:\mathbf{Q}]} & \text{otherwise .} \end{cases}$$

Note that

$$\mathcal{N} - \mathbf{w}_k = \frac{1}{\epsilon(\sigma_k^{\tau_1})} (\mathbf{v}_k, \mathbf{v}_k), \tag{41}$$

where

$$(\mathbf{v}_k)_i^\tau = \begin{cases} 1 & \text{if } \tau \in \{\tau_1, \overline{\tau_1}\} \text{ and } (i = k \text{ or } \sigma_i^\tau = \overline{\sigma_k^\tau}), \\ 0 & \text{otherwise .} \end{cases} \tag{42}$$

Define

$$k_0 = \begin{cases} 2 & \text{if } K \text{ is totally real and } L \text{ is totally complex,} \\ 2 & \text{if } L/K \text{ is mixed (see below),} \\ 1 & \text{otherwise,} \end{cases} \tag{43}$$

where we say that L/K is mixed if K is totally real, L is neither totally real nor totally complex, and for all embeddings τ, σ_k^τ is complex.

Lemma 3. *For each $1 \leq k \leq [L:K]$ and $(x,y) \in V$ there exist a_i^τ, b_i^τ $(1 \leq i \leq [L:K])$ and $A_k(x,y)$ in $\mathbf{R}$ such that $f_k(x,y) + f_k\Big(v_k(f_k(x,y))\Big)$ can be written as*

$$\sum_\tau \left(\sum_{i=1}^{k-1} a_i^\tau (0, e_i^\tau) + \sum_{i=k+1}^{[L:K]} a_i^\tau (e_i^\tau, 0) \right) + a_k^{\tau_1}(0, \mathrm{v}_k) + a_{\mathcal{N}}(\mathcal{N} - \mathbf{w}_k), \tag{44}$$

where $\{e_i^\tau\}_{i,\tau}$ *is the canonical basis for* $\mathbf{R}^N = \mathbf{R}^{s+2t}$, *and* f_k, v_k *are given by* (39) *and* (38).

If $(x,y) = \phi_2(z,w)$ *with* $(z,w) \in B^{(k)}$, *where* $B^{(k)}$ *and* ϕ_2 *are as in* (33) *and* (27) *respectively, then* $a_{\mathcal{N}} = \log\Big(\mathrm{t}(z,w)\Big)$, *where* t *is the function given in* (18).

If $(x,y) = \phi_2(z,w)$ *with* $(z,w) \in B^{(0)}$, *taking* $k = k_0$, *we have that* $f_k(x,y) + f_k(v_k(f_k(x,y)))$ *can be written as*

$$\sum_\tau \sum_{i=k+1}^{[L:K]} b_k^\tau(e_i^\tau, 0) + a_k^{\tau_1}(\mathbf{v}_k, 0) + a_{\mathcal{N}}(\mathcal{N} - \mathbf{w}_k),$$

where $a_{\mathcal{N}} = \log\Big(\mathrm{t}(z,w)\Big)$.

Proof. Let $1 \le k \le [L:K]$ and $(x,y) \in V$. We need only write down the functions f_k, v_k and w_k given in (39), (38) and (37) respectively, to obtain that $f_k(x,y) + f_k(v_k(f_k(x,y)))$ has the form

$$\sum_\tau \left(\sum_{i=1}^{k-1} a_i^\tau(0, e_i^\tau) + \sum_{i=k+1}^{[L:K]} b_i^\tau(e_i^\tau, 0) \right) + c_1(0, \mathbf{v}_k) + c_2(\mathbf{v}_k, 0),$$

where $\mathbf{v}_k$ is as in (42). By (41) we have

$$c_1(0, \mathbf{v}_k) + c_2(\mathbf{v}_k, 0) = a_k^{\tau_1}(0, \mathbf{v}_k) + a_{\mathcal{N}}(\mathcal{N} - \mathbf{w}_k),$$

for some a_k, and $a_{\mathcal{N}}$. Hence $f_k(x,y) + f_k(v_k(f_k(x,y)))$ has the form (44) claimed.

Since $(x,y) = f_k(x,y) + f_k(v_k(f_k(x,y))) + (w_1, w_1) + (w_2, 0)$, where $(w_1, w_1) \in W_1$ and $(w_2, 0) \in W_2$, we have that if $(x,y) = \phi_2(z,w)$, with $(z,w) \in B^{(k)}$, then by (34), $\log(\mathrm{t}(z,w))$ is

$$\sum_\tau \left(\sum_{i=1}^{k} x_i^\tau + \sum_{i=k+1}^{[L:K]} y_i^\tau \right) = a_{\mathcal{N}} \left(\sum_\tau \sum_{i=1}^{[L:K]} \frac{1}{N} \right) + \sum_\tau \left(\sum_{i=1}^{[L:K]} (w_1)_i^\tau + \sum_{i=1}^{k} (w_2)_i^\tau \right),$$

where $N = [L:\mathbf{Q}]$. Hence we obtain $N\frac{a_{\mathcal{N}}}{N} = a_{\mathcal{N}} = \log\Big(t(z,w)\Big)$, because

$$\sum_\tau \sum_{i=1}^{[L:K]} (w_1)_i^\tau = 0, \text{ and } \sum_\tau (w_2)_i^\tau = 0 \text{ for each } i.$$

On the other hand, if $(x,y) = \phi_2(z,w)$ with $(z,w) \in B^{(0)}$, then taking $k = k_0$,

$$c_1(0, \mathrm{v}_k) + c_2(\mathrm{v}_k, 0) = a_k^{\tau_1}(\mathrm{v}_k, 0) + a_{\mathcal{N}}(\mathcal{N} - \mathrm{w}_k),$$

for some $a_k^{\tau_1}$ and $a_{\mathcal{N}}$. Hence $f_k(x,y) + f_k(v_k(f_k(x,y)))$ has the form claimed. As before, $N\frac{a_{\mathcal{N}}}{N} = \log\Big(t(z,w)\Big)$. □

For $1 \leq k \leq [L:K]$, let V_k be the subspace of V whose elements can be written as

$$\sum_{\tau}\left(\sum_{i=1}^{k-1} a_i^{\tau}(0, e_i^{\tau}) + \sum_{i=k+1}^{[L:K]} a_i^{\tau}(e_i^{\tau}, 0)\right) + a_k^{\tau_1}(0, \mathrm{v}_k) + a_{\mathcal{N}}\mathcal{N}, \qquad (45)$$

with all coefficient a real and such that, if $\sigma_i^{\tau} = \overline{\sigma_j^{\tau'}}$ then $a_i^{\tau} = a_j^{\tau'}$, as we are requiring $V_k \subset V$ (see (27)). Denote by V_0 the subspace of V whose elements can be written as

$$\sum_{i=k+1}^{[L:K]} a_i^{\tau}(e_i^{\tau}, 0) + a_k^{\tau_1}(\mathrm{v}_k, 0) + a_{\mathcal{N}}\mathcal{N}, \qquad (46)$$

where $k = k_0$, with k_0 as in (43).

Proposition 4. *For each $0 \leq k \leq [L:K]$, we have $V = V_k \oplus W$, where $W = W_1 \oplus W_2$ as in (30).*

Proof. For each $1 \leq k \leq [L:K]$ and $(x,y) \in V$, we have using $f_k = Id - w_k$,

$$(x,y) = f_k(x,y) + f_k(v_k(f_k(x,y))) + \Big[w_k(x,y) - v_k(f_k(x,y)) + w_k\Big(v_k(f_k(x,y))\Big)\Big].$$

Using Lemma 3, it is clear that $V = V_k + W$, as $Im(w_k)$, $Im(v_k) \subset W$.

For $k = 0$, we have $V = V_0 + W$ as above with $f_{k_0} = Id - w_{k_0}$.

For each k, the dimension of V_k is

$$\sum_{\tau \text{ real}}\left(\#\{\sigma \text{ real}, \sigma|\tau\} + \frac{\#\{\sigma \text{ complex}, \sigma|\tau\}}{2} - 1\right) + \sum_{\tau \text{ complex}} \frac{[L:K]-1}{2} + 2$$

$= s_L + t_L - s_K - t_K + 2 = r_L - r_K + 2$. As $\dim(W) = r_L + r_K$, we have

$$\dim(V_k) + \dim(W) = (r_L - r_K + 2) + (r_L + r_K) = 2(r_L + 1) = 2(s_L + t_L) = \dim V.$$

Hence $V = V_k \oplus W$. □

For each V_k, put $\Delta^{(k)} = \Delta(V_k)$ as in Lemma 2. Let

$$\mathcal{D}^{(k)} = \left\{(x,y) \in \phi_2^{-1}(\Delta^{(k)}) \cap B^{(k)} \mid 0 \leq \arg(x_1),\ \arg(y_1) < \frac{2\pi}{W_L}\right\}, \quad (47)$$

with $B^{(k)}$ as in (33).

Note that when $(x,y) \in \mathcal{D}^{(k)}$, $\phi_2(x,y)$ can be written for $k \neq 0$ as

$$\sum_\tau \left(\sum_{i=1}^{k-1} a_i^\tau(0,e_i^\tau) + \sum_{i=k+1}^{[L:K]} a_i^\tau(e_i^\tau,0)\right) + a_k^{\tau_1}(0,\mathrm{v}_k) + a_{\mathcal{N}}\mathcal{N} + \sum_{i=1}^{r_L} \alpha_i E_i + \sum_{i=1}^{r_K} \beta_i U_i, \quad (48)$$

and for $k = 0$, as

$$\sum_\tau \left(\sum_{i=k_0+1}^{[L:K]} a_i^\tau(e_i^\tau,0)\right) + a_k^{\tau_1}(\mathrm{v}_k,0) + a_{\mathcal{N}}\mathcal{N} + \sum_{i=1}^{r_L} \alpha_i E_i + \sum_{i=1}^{r_K} \beta_i U_i, \quad (49)$$

with $0 \leq \alpha_i,\ \beta_i < 1$, and where v_k and $\mathcal{N}$ are as in (42) and (40), E_i, U_j are as in (31), and k_0 is as in (43). Furthermore $a_{\mathcal{N}} = \log\Big(\mathrm{t}(x,y)\Big)$.

Proposition 5. *For each $0 \leq k \leq [L:K]$, $\mathcal{D}^{(k)}$ is a cone and a fundamental domain for the action of $\mathcal{O}_{L/K}^* \times \mathcal{O}_L^*$ on $B^{(k)}$.*

Proof. Lemma 2 shows that $\mathcal{D}^{(k)}$ is a fundamental domain for $B^{(k)}$.

For $(x,y) \in C_{s,t}^* \times C_{s,t}^*$ and $\lambda > 0$, $\phi_2(\lambda x, \lambda y) = (N \log|\lambda|)\mathcal{N} + \phi_2(x,y)$. Hence $\phi_2(\lambda x, \lambda y) \in \Delta^{(k)}$ if and only if $\phi_2(x,y) \in \Delta^{(k)}$ (recall $\mathcal{N} \in V_k$). Since $B^{(k)}$ is clearly a cone and $\arg(x) = \arg(\lambda x)$ for $\lambda < 0$, we see that $\mathcal{D}^{(k)}$ is a cone. □

Lemma 4. *For $0 \leq k \leq [L:K]$. $(x,y) \in \mathcal{D}^{(k)}$ if and only if*

(i) $0 \leq \arg x_1,\ \arg y_1 < \frac{2\pi}{W_L}$,

(ii) $\phi_2(x,y)$ *be written as in* (48) *or* (49) *(depending on k).*

(iii) The coefficients a_i^τ of (48) *or* (49), *satisfy the inequalities:*

- *For $k \neq 0$:*

$$\begin{array}{ll} 1) & a_1^\tau \leq a_2^\tau \leq \cdots \leq a_{k-1}^\tau. \\ 2) & a_{k-1}^{\tau_1} \leq a_k^{\tau_1}, \text{ and for } \tau \notin \{\tau_1, \overline{\tau_1}\},\ a_{k-1}^\tau \leq 0. \\ 3) & a_k^{\tau_1} \leq 0. \\ 4) & a_{[L:K]}^\tau \leq a_{[L:K]-1}^\tau \leq \cdots \leq a_{k+1}^\tau. \\ 5) & a_{k+1}^{\tau_1} \leq -a_k^{\tau_1}, \text{ and for } \tau \notin \{\tau_1, \overline{\tau_1}\},\ a_{k+1}^\tau \leq 0. \\ 6) & \sum_\tau a_{k+1}^\tau < 0. \end{array} \tag{50}$$

- *For $k = 0$:*

$$\begin{array}{ll} 1') & a_{[L:K]}^\tau \leq a_{[L:K]-1}^\tau \leq \cdots \leq a_{k_0+1}^\tau. \\ 2') & a_{k_0+1}^{\tau_1} \leq a_{k_0}^{\tau_1} \text{ and for } \tau \notin \{\tau_1, \overline{\tau_1}\},\ a_{k_0+1}^\tau \leq 0. \\ 3') & a_{k_0}^{\tau_1} \leq 0, \end{array}$$

where k_0 is as in (43).

Furthermore, for $B > 0$, $\phi_2\Big(T_{\mathcal{D}^{(k)}}(B)\Big)$ is the intersection of $\phi_2(\mathcal{D}^{(k)})$ with the region given by $a_{\mathcal{N}} \leq \log B$, where $a_{\mathcal{N}}$ is the coefficient of $\mathcal{N}$ in formulas (48) and (49).

Proof. Given $(x, y) \in \mathcal{D}^{(k)}$, using the definition of $\mathcal{D}^{(k)}$ given in (47), we have $0 \leq \arg x_1,\ \arg y_1 < \frac{2\pi}{W_L}$. Also, $\phi_2(x, y) \in \Delta^{(k)}$ can be written as in (48) for $k \neq 0$, or as in (49) for $k = 0$. Furthermore, since $\phi_2(B^{(k)})$ is invariant under translation by elements in W (cf. Proposition 3, (28) and (29)) and $\phi_2 = \psi \times \psi$, with $(\psi(x))_i = \log |x_i|$, we have:

- The inequalities given for C in (32), be become under ϕ_2 the inequalities 1), 2), 4) and 5) (respectively in the inequalities 1') and 2')) for the coefficients a_i^τ when $k \neq 0$ (respectively, when $k = 0$).
- The inequalities given for $B^{(k)}$ in (33), be become the inequalities 3) and 6) (respectively, the inequality 3') when $k \neq 0$ (respectively, when $k = 0$).

Note that we have used all the conditions that define $\mathcal{D}^{(k)}$.

Conversely, if (x,y) satisfies the conditios (i), (ii) and (iii), then, by (ii), $(x,y) \in \phi_2^{-1}\left(\Delta^{(k)}\right)$. Hence together with the inequalities given in (50), we have that $(x,y) \in B^{(k)}$. Hence, by (i), and (47), we have $(x,y) \in \mathcal{D}^{(k)}$.

On the other hand, if $(x,y) \in \mathcal{D}^{(k)}$ with $\mathbf{t}(x,y) \leq B$, then by lemma 3, $a_{\mathcal{N}} = \log \mathbf{t}(x,y)$. Hence $a_{\mathcal{N}} \leq \log B$. □

Proposition 6. *For $B > 0$, let $T_{\mathcal{D}^{(k)}}(B)$ be as in (22). Then $T_{\mathcal{D}^{(k)}}(B)$ is bounded for $k = 0$ or $k = [L:K]$.*

For $k \neq 0$, $k \neq [L:K]$, and $(x,y) \in T_{\mathcal{D}^{(k)}}$, we have:

- *For $\tau \notin \{\tau_1, \overline{\tau_1}\}$, $|x_i^\tau|$ and $|y_i^\tau|$ are bounded for all i.*
- *For $\tau \in \{\tau_1, \overline{\tau_1}\}$, $|x_i^\tau|$ is bounded for $1 \leq i \leq k$, and $|y_i^\tau|$ is bounded for $k \leq i \leq [L:K]$.*

Proof: It suffices to show that $\log|x_i^\tau|$ and $\log|y_i^\tau|$ are bounded above for τ and i as in the proposition. Using (48) , we have for $k \neq 0$

$$\log|x_i^\tau| = a_{\mathcal{N}} + \sum_{j=1}^{r_L} \alpha_j \log|\sigma_i^\tau(\varepsilon_j)| + \sum_{j=1}^{r_K} \beta_j \log|\sigma_i^\tau(u_j)| + \begin{cases} 0 & \text{if } i \geq k \text{ and } k \neq 0, \\ a_i^\tau & \text{otherwise,} \end{cases}$$

and

$$\log|y_i^\tau| = a_{\mathcal{N}} + \sum_{j=1}^{r_L} \alpha_j \log|\sigma_i^\tau(\varepsilon_j)| + \begin{cases} a_i^\tau & \text{if } i \geq k+1 \text{ and } k \neq 0, \\ a_k^{\tau_1} & \text{if } (i,\tau) = (k,\tau_1) \text{ or } \sigma_i^\tau = \overline{\sigma_k^{\tau_1}}, \\ 0 & \text{otherwise.} \end{cases}$$

For $k = 0$, we obtain analogous formulas from (49).

Hence, by the above and lemma 4, there exists a constant c' that depends only on L and K such that for $c = c' + \log B$, we have:

- For $k \neq 0$ and for $\tau \notin \{\tau_1, \overline{\tau_1}\}$:
  1. $\log|x_i^\tau| \leq c$, for $1 \leq i \leq k+1$,
  2. $-c' + \log|x_{i+1}^\tau| \leq \log|x_i^\tau| \leq c' + \log|x_{i-1}^\tau|$, for $k+2 \leq i \leq [L:K]-1$,
  3. $-c' + \log|y_{i-1}^\tau| \leq \log|y_i^\tau| \leq c' + \log|y_{i+1}^\tau|$, for $2 \leq i \leq k-2$,
  4. $\log|y_i^\tau| \leq c$, for $k-1 \leq i \leq [L:K]$.

  Hence $\log|x_i^\tau|$ and $\log|y_i^\tau|$ are bounded for all i and $\tau \neq \{\tau_1, \overline{\tau_1}\}$.

- For $k \neq 0$ and for $\tau \in \{\tau_1, \overline{\tau_1}\}$:

  5. $\log|x_i^\tau| \leq c$, for $1 \leq i \leq k$,
  6. $\log|y_i^\tau| \leq c$, for $k \leq i \leq [L:K]$.

  Note that for $k = [L:K]$, we have furthermore that for $\tau \in \{\tau_1, \overline{\tau_1}\}$,

  $$-c' + \log|y_{i-1}^\tau| \leq \log|y_i^\tau| \leq c' + \log|y_{i+1}^\tau|, \text{ for } 2 \leq i \leq [L:K]-1.$$

  Hence $\log|y_i^\tau|$ is bounded above. Note in this case that $\log|x_i^\tau|$ is bounded above because $k = [L:K]$ in inequality 5.

- For $k = 0$, taking k_0 as in (43), we have that for all τ:

  1. $\log|x_i^\tau| \leq c$, for $1 \leq i \leq k_0 + 1$,
  2. $-c' + \log|x_{i+1}^\tau| \leq \log|x_i^\tau| \leq c' + \log|x_{i-1}^\tau|$, $k_0 + 2 \leq i \leq [L:K]-1$,
  3. $\log|y_i^\tau| \leq c$, for $1 \leq i \leq [L:K]$.

Hence $\log|x_i^\tau|$ and $\log|y_i^\tau|$ are bounded above for all τ and i. □

Using $\sum_\tau a_{k+1}^\tau < 0$ $\Big((6)$ in Lemma 4$\Big)$, we have:

Remark 3. *For each $1 \leq k \leq [L:K]-1$, the set $\mathcal{D}^{(k)}$ can be written as:*

$$\mathcal{D}^{(k)} = D_{k,0} \,\dot\cup\, D_{k,1} \,\dot\cup \cdots \dot\cup\, D_{k,[K:\mathbf{Q}]-1},$$

where $\dot\cup$ denotes disjoint union of sets,

$$\begin{aligned} D_{k,0} &= \{(x,y) \in \mathcal{D}^{(k)} \mid a_{k+1}^{\tau_1} \leq 0\}, \\ D_{k,i} &= \{(x,y) \in \mathcal{D}^{(k)} \mid \textstyle\sum_{j=1}^{i} a_{k+1}^{\tau_j} > 0 \text{ and } \sum_{j=1}^{i+1} a_{k+1}^{\tau_j} \leq 0\}, \end{aligned} \tag{51}$$

and the a_i^τ are the coefficients of $\phi_2(x,y)$ as in (48).

Note that in the case $K = \mathbf{Q}$ that Schanuel [S] studied the above decomposition has only one term. It is a special feature of the relative case that we must decompose $\mathcal{D}^{(k)}$ further to take into account the units in $\mathcal{O}_{L/K}^*$.

For each $1 \leq i \leq [K:\mathbf{Q}]-1$, define $\Phi^i : D_{k,i} \to V$ by

$$\Phi^i(x,y) := \phi_2(x,y) + \sum_{j=1}^{i} a_{k+1}^{\tau_j}\left(0, \sum_{t=1}^{[L:K]} e_t^{\tau_j}\right) - \left(\sum_{j=1}^{i} a_{k+1}^{\tau_j}\right)\left(0, \sum_{t=1}^{[L:K]} e_t^{\tau_{i+1}}\right)$$

$$+\left(\sum_{j=1}^{i} a_{k+1}^{\tau_j}\right)\left(\sum_{t=1}^{[L:K]-k} e_{k+t}^{\tau_{i+1}}, \sum_{t=1}^{[L:K]-k} e_{k+t}^{\tau_{i+1}}\right) - \sum_{j=1}^{i} a_{k+1}^{\tau_j}\left(\sum_{t=1}^{[L:K]-k} e_{k+t}^{\tau_j}, \sum_{t=1}^{[L:K]-k} e_{k+t}^{\tau_j}\right),$$

where the a_t^τ are the coefficients of $\phi_2(x,y)$ as in (48). Note that $\Phi^i(x,y) = \phi_2(x,y) + w$, with $w \in W$.

Proposition 7. *For $(x,y) \in D_{k,i}$ and $1 \le i \le [K : \mathbf{Q}] - 1$, $\Phi^i(x,y)$ can be written as:*

$$\Phi^i(x,y) = \sum_{j=1}^{[K:\mathbf{Q}]} \sum_{t=1}^{k-1} b_t^{\tau_j}(0, e_t^{\tau_j}) + \sum_{j=1}^{[K:\mathbf{Q}]} \sum_{t=k+2}^{[L:K]} b_t^{\tau_j}(e_t^{\tau_j}, 0) + a_{\mathcal{N}}\mathcal{N} +$$

$$+ \sum_{j=1}^{i+1} b_k^{\tau_j}(0, e_k^{\tau_j}) + \sum_{j=i+1}^{[K:\mathbf{Q}]} b_{k+1}^{\tau_j}(e_{k+1}^{\tau_j}, 0) + \sum_{j=1}^{r_L} \alpha_j E_j + \sum_{j=1}^{r_K} \beta_j U_j,$$

where $a_{\mathcal{N}}$, α_j and β_j are as in (48).

Furthermore, for all t and τ, b_t^τ is bounded above.

Proof: A calculation, using definition of Φ^i and (48), yields

$$b_t^{\tau_j} = \begin{cases} a_t^{\tau_j} + c_t a_{k+1}^{\tau_j} & \text{if } \begin{cases} 1 \le j \le i \text{ and } t \notin \{k, k+1\}, \\ \text{or} \\ j = 1 \text{ and } t = k, \end{cases} \\ a_t^{\tau_{i+1}} - c_t \sum_{l=1}^{i} a_{k+1}^{\tau_l} & \text{if } j = i+1 \text{ and } t \ne k, \\ -\sum_{l=1}^{i} a_{k+1}^{\tau_l} & \text{if } j = i+1 \text{ and } t = k, \\ a_{k+1}^{\tau_j} & \text{if } 2 \le j \le i \text{ and } t = k, \\ a_t^{\tau_j} & \text{if } i+2 \le j \le [K:\mathbf{Q}] \text{ and any } t, \end{cases} \tag{52}$$

with

$$c_t = \begin{cases} 1 & \text{for } 1 \le t \le k, \\ -1 & \text{for } k+1 \le t \le [L:K]. \end{cases}$$

By the inequalities given in (50), we obtain that $b_t^{\tau_j} \le 0$ for $j \ne i+1$. Using furthermore the definition of $D_{k,i}$ in (51), we obtain that $b_k^{\tau_{i+1}} < 0$ and together (50) we obtain $b_t^{\tau_{i+1}} \le 0$ for each t. □

For $1 \leq k \leq [L:K]-1$ and $1 \leq i \leq [K:\mathbf{Q}]-1$, define the subspace $\mathbf{V}_{k,i}$ of V given by vectors that can be written as

$$\sum_{j=1}^{[K:\mathbf{Q}]}\sum_{t=1}^{k-1} b_t^{\tau_j}(0,e_t^{\tau_j})+\sum_{j=1}^{[K:\mathbf{Q}]}\sum_{t=k+2}^{[L:K]} b_t^{\tau_j}(e_t^{\tau_j},0)+a_{\mathcal{N}}\mathcal{N}+\sum_{j=1}^{i+1} b_k^{\tau_j}(0,e_k^{\tau_j})+\sum_{j=i+1}^{[K:\mathbf{Q}]} b_{k+1}^{\tau_j}(e_{k+1}^{\tau_j},0). \tag{53}$$

For each $\mathbf{V}_{k,i}$, put $\Delta_{k,i}=\Delta(\mathbf{V}_{k,i})$, as in Lemma 2. Denote by $A_{k,i}$ the subset of $\phi_2^{-1}(\Delta_{k,i})\cap B^{(k)}$ given by

$$\left\{(x,y)\;\middle|\; 0\leq \arg(x_1),\ \arg(y_1)<\frac{2\pi}{W_L},\quad b_k^{\tau_{i+1}}<0 \text{ and } b_{k+1}^{\tau_{i+1}}\leq 0\right\}. \tag{54}$$

Put $A_{k,0}:=D_{k,0}$.

Proposition 8. *For each $1\leq k\leq [L:K]-1$ the set*

$$\mathcal{A}^{(k)}=A_{k,0}\,\dot\cup\cdots\dot\cup\,A_{k,[K:\mathbf{Q}]-1}, \tag{55}$$

is a fundamental domain for $B^{(k)}$ and each $A_{k,j}$ is a cone.

Furthermore for each real positive number B, the set $T_{\mathcal{A}^{(k)}}(B)$ is bounded, with $T_{\mathcal{A}^{(k)}}(B)$ as in (22).

Proof: For each $1\leq i\leq [K:\mathbf{Q}]-1$, using the definition of Φ^i, Proposition 7 and definition of $V_{k,i}$, we obtain $V=V_{k,i}\oplus W$. Taking in Lemma 2 the set S given by

$$S=S_i=\bigcup_{u\in\mathcal{O}^*_{L/K}}\ \bigcup_{\epsilon\in\mathcal{O}^*_L}\theta_\epsilon\circ\vartheta_u(D_{k,i}),$$

we obtain that $A_{k,i}$ is a fundamental domain for S_i. Hence using that $A_{k,0}=D_{k,0}$ and Remark 3, we have $\mathcal{A}^{(k)}$ is a fundamental domain for $B^{(k)}$.

$A_{k,j}$ is a cone basically because $\mathcal{N}\in\mathbf{V}_{k,i}$. Furthermore, by Proposition 7, the set $\{(x,y)\in A_{k,i}\mid t(x,y)\leq B\}$ is bounded. The condition $b_k^{\tau_{i+1}}<0$ and the expression for the elements of $A_{k,i}$ given in (53) insure that $A_{k,i}\cap A_{k,j}=\phi$ when $j\neq i$. □

We will prove for $\mathcal{A}=\cup_g\cup_{k=0}^{[L:K]}(\mathcal{A}_g)^{(k)}$, that $T_{\mathcal{A}}(1)$ has a boundary which is $(2N-1)$-Lipschitz parametrizable, where $T_{\mathcal{A}}(1)$ is as in (22), $(\mathcal{A}_g)^{(k)}$ as in (55) for the ordering g, and we let $\mathcal{A}^{(0)}=\mathcal{D}^{(0)}$ and $\mathcal{A}^{([L:K])}=\mathcal{D}^{([L:K])}$.

Since $T_{\mathcal{A}}(1)=\mathcal{A}\cap T(1)$, we have

$$\partial T_{\mathcal{A}}(1)\subset\left(\partial\mathcal{A}\cap\overline{T(1)}\right)\cup\left(\overline{\mathcal{A}}\cap\partial T(1)\right),$$

where ∂A denotes the boundary of A and $\overline{A}$ denotes the closure of A in $\mathcal{I} \times \mathcal{I}$, where $\mathcal{I}$ is as in (21).

By Proposition 8, $T_{\mathcal{A}}(1)$ is bounded. Hence it suffices to Lipschitz-parametrize $\partial\mathcal{A} \cap \overline{T(1)}$ and a compact subset of $\partial T(1)$.

Proposition 9. *Any compact subset of $\partial T(1)$ is $(2N-1)$-Lipschitz parametrizable.*

Proof. For each ordering g and each $0 \le k \le [L:K]$, in view of (34), we consider the function $t_{g,k} : \mathcal{I} \times \mathcal{I} \to \mathbf{R}$ given by

$$t_{g,k}(x,y) = \prod_{\tau} \left(\prod_{i=1}^{k} |x_i^{\tau}| \prod_{i=k+1}^{[L:K]} |y_i^{\tau}| \right), \tag{56}$$

where for each τ the embeddings σ_i^{τ} are ordered by g.

Since t is continuous in $\mathcal{I}^* \times \mathcal{I}^*$ by (18), and the $t_{g,k}$ are continuous in $\mathcal{I} \times \mathcal{I}$, we have

$$\partial T(1) \subset \overline{\mathrm{t}^{-1}(1)} \subset \bigcup_{g,k} t_{g,k}^{-1}(1).$$

It is clear by (56) that $t_{g,k}$ is continuously differentiable with non vanishing derivates in $t_{g,k}^{-1}(1)$. By the implicit function theorem, for each $\mathcal{X} \in t_{g,k}^{-1}(1)$ there is an open neighborhood $W(\mathcal{X})$ of $\mathcal{X}$ and a differentiable map $\psi_{\mathcal{X}} : [0,1]^{2N-1} \to \mathcal{I} \times \mathcal{I}$ satisfying $W(\mathcal{X}) \cap t_{g,k}^{-1}(1) \subset \psi_{\mathcal{X}}([0,1]^{2N-1})$.

Now, if A is compact subset of $\partial T(1)$, for each g, k we can find a finite number of neighborhoods $W(\mathcal{X})$ that cover $A \cap t_{g,k}^{-1}(1)$. Hence there are a finite number of continuously differentiable maps $\phi_{\mathcal{X}}$ with the property that $\cup Im\ \phi_{\mathcal{X}} = t_{g,k}^{-1}(1)$. The $\phi_{\mathcal{X}}$ are Lipschitz by the mean value theorem. □

Proposition 10. *The closure of $\mathcal{A} \cap T(1) = \{(x,y) \in \mathcal{A} \mid \mathrm{t}(x,y) \le 1\}$ is contained in $[0,1](\overline{\mathcal{A} \cap S(1)}) = \{sa \mid 0 \le s \le 1,\ a \in \overline{\mathcal{A} \cap S(1)}\}$, where*

$$S(1) = \{(x,y) \in \mathcal{I}^* \times \mathcal{I}^* \mid \mathrm{t}(x,y) = 1\}. \tag{57}$$

Proof. For $(x,y) \in \overline{\mathcal{A} \cap T(1)}$, there exists a sequence $(x_n, y_n) \in \mathcal{A} \cap T(1)$ with $(x_n, y_n) \to (x,y)$. Since $\mathcal{A}$ is a finite union of $(A_g)_{k,i}$, where $(A_g)_{k,i}$ is as in (54) for the ordering g, there exists a subsequence $(x_{n'}, y_{n'})$ with $\{(x_{n'}, y_{n'})\} \subset (A_g)_{k,i}$ for some g, k, i. But $(x_{n'}, y_{n'}) \in t_{g,k}^{-1}([0,1])$, with $t_{g,k}$ as in (56). Hence $(x,y) \in t_{g,k}^{-1}([0,1])$ and $t_{g,k}(x_{n'}, y_{n'}) \to t_{g,k}(x,y)$.

- If $t_{g,k}(x,y) \neq 0$, put $\lambda = t_{g,k}(x,y) \leq 1$ and $\lambda_{n'} = t_{g,k}(x_{n'},y_{n'})$. Then $\lambda_{n'}^{-\frac{1}{N}}(x_{n'},y_{n'}) \to \lambda^{-\frac{1}{N}}(x,y)$, where $N = [L:\mathbf{Q}]$ and $\lambda_{n'}^{-\frac{1}{N}}(x_{n'},y_{n'}) \in S(1)$. Hence $(x,y) = \lambda^{\frac{1}{N}}\left(\lambda^{\frac{-1}{N}}(x,y)\right) \in (0,1](\overline{\mathcal{A} \cap S(1)})$.

- If $t_{g,k}(x,y) = 0$, $a_{\mathcal{N}} = \log\left(t_{g,k}(x_{n'},y_{n'})\right)$, where $a_{\mathcal{N}}$ is a coefficient corresponding to $\mathcal{N}$ for $\phi_2(x_{n'},y_{n'})$, with $\mathcal{N}$ as in (40). Hence $a_{\mathcal{N}}$ diverges to $-\infty$. Thus all coordinates of $\phi_2(x_{n'},y_{n'})$ diverge to $-\infty$. Hence $(x_{n'},y_{n'}) \to 0$ and $(x,y) = 0 \in [0,1](\overline{A \cap S(1)})$. □

Proposition 11. *$\partial\mathcal{A} \cap \overline{T(1)}$ is $(2N-1)$-Lipschitz parametrizable.*

Proof. Since $\mathcal{A}$ is a cone, $\lambda\partial\mathcal{A} = \partial\mathcal{A}$ for $\lambda > 0$. By proposition 10,

$$\partial\mathcal{A} \cap \overline{T(1)} = \partial\mathcal{A} \cap \overline{\mathcal{A} \cap T(1)} \subset \partial\mathcal{A} \cap [0,1](\overline{\mathcal{A}} \cap \overline{S(1)}) = [0,1](\partial\mathcal{A} \cap \overline{S(1)}),$$

where $S(1)$ as in (57). Is suffices to find a Lipschitz parametrization of $\partial\mathcal{A} \cap \overline{S(1)}$. Indeed if $\psi : [0,1]^{2N-2} \to \mathcal{I} \times \mathcal{I}$ Lipschitz parametrizes $\partial\mathcal{A} \cap \overline{S(1)}$, then $\Phi : [0,1]^{2N-1} \to \mathcal{I} \times \mathcal{I}$ Lipschitz parametrizes $\partial\mathcal{A} \cap \overline{T(1)}$, where $\Phi(x,t) = t\psi(x)$ for $x \in [0,1]^{2N-2}$ and $t \in [0,1]$.

For (g,k,i) fixed, where g is an ordering of the embeddings of L into $\mathbf{C}$, $1 \leq k \leq [L:K]$ and $1 \leq i \leq [K:\mathbf{Q}]$, consider the subset F of $\mathcal{I} \times \mathcal{I}$, given by

$$F := \overline{(A_g)_{k,i}}. \tag{58}$$

Since $(x,y) \in (A_g)_{k,i}$, the coefficients $\alpha_l(x,y)$ and $\beta_t(x,y)$ corresponding to E_l and U_t for $\phi_2(x,y)$, belong to the interval $[0,1)$, define for (l,t,δ) fixed, with $1 \leq l \leq r_L$, $1 \leq t \leq r_K$ and $\delta \in \{0,1\}$, the functions $\mu_1,\ \mu_2 : F \to \mathbf{R}$ as follows:

- First define the functions on $\mathcal{I}^* \times \mathcal{I}^*$ by

$$\mu_1(x,y) = \alpha_l(x,y) - \delta \quad \text{and} \quad \mu_2(x,y) = \beta_t(x,y) - \delta.$$

  Clearly μ_1 and μ_2 are continuous on $\mathcal{I}^* \times \mathcal{I}^*$.

- Extend μ_1 and μ_2 to F continuity, which is possible by (58).

Hence $H_1 = \mu_1^{-1}(0)$ and $H_2 = \mu_2^{-1}(0)$ are closed in F. (59)

Now, define $C = C_{\mathrm{j}}$ by

$$C = \left(H_1 \bigcup H_2\right) \bigcap \left(\partial(A_g)_{k,i} \bigcap \overline{S(1)}\right),$$

where $\mathrm{j} = (g, k, i, l, t, \delta)$. C is compact in $\mathcal{I} \times \mathcal{I}$, because $(H_1 \cup H_2) \cap \partial(A_g)_{k,i}$ is closed in $\mathcal{I} \times \mathcal{I}$ (by (57) and (59)) and $\overline{S(1)} \cap \partial(A_g)_{k,i}$ is compact (it is a closed an bounded subset of $\overline{T(1) \cap (A_g)_{k,i}}$).

Furthermore, by definition of $\overline{S(1)}$, H_1 and H_2 we have by the implicit function theorem, that for each $\mathcal{X} \in C$, there is an open neighborhood $W(\mathcal{X})$ and a differentiable map $\psi_{\mathcal{X}} : I^{2N-2} \to \mathcal{I} \times \mathcal{I}$ satisfying $W(\mathcal{X}) \cap t_{g,k}^{-1}(1) \subset \psi_{\mathcal{X}}(I^{2N-2})$. But

$$\partial\mathcal{A} \bigcap \overline{S(1)} \subset \bigcup_{\mathrm{j}} C_{\mathrm{j}},$$

where j runs over all (g, k, i, l, t, δ), is a finite union of compact sets. Hence there are a finite number of differentiable functions $\psi_{\mathcal{X}}$ such that the union of $\phi_X(I^{2N-2})$ cover $\partial\mathcal{D} \cap \overline{T(1)}$. Again, $\psi_X(I^{2N-2})$ is Lipschitz map by the mean value theorem. □

Proposition 12. *For each ordering g of the embeddings of L and for each $0 \leq k \leq [L : K]$,*

$$Vol\left(T_{(\mathcal{A}_g)^{(k)}}(B)\right) = Vol\left(T_{(\mathcal{D}_g)^{(k)}}(B)\right),$$

with $(\mathcal{A}_g)^{(k)}$ and $(\mathcal{D}_g)^{(k)}$ as in (55) *and* (47) *respectively (for the ordering g).*

Proof. Let g and k be fixed. We have

$$\mathcal{D}^{(k)} = D_{k,0} \dot{\cup} D_{k,1} \dot{\cup} \cdots \dot{\cup} D_{k,[K:\mathbf{Q}]-1},$$

with $D_{k,i} \cap D_{k,j} = \phi$ for $i \neq j$, $D_{k,i}$ as in (51), $\phi_2(\mathcal{D}^{(k)}) \subset \Delta(V_k)$, with Δ as in Lemma 2 and V_k as in (45).

On the other hand,

$$\mathcal{A}^{(k)} = A_{k,0} \dot{\cup} \cdots \dot{\cup} A_{k,[K:\mathbf{Q}]} - 1,$$

where for each $1 \leq i \leq [K : \mathbf{Q}] - 1$, $\phi_2(A_{k,i}) \subset \Delta(\mathbf{V}_{k,i})$, with $\mathbf{V}_{k,i}$ as in (53) and $A_{k,0} = D_{k,0}$.

It is easy to prove that $\text{Vol}\Big(T_{(A_g)_{k,i}}(B)\Big) = \text{Vol}\Big(T_{(D_g)_{k,i}}(B)\Big)$, using the linear transformation $\chi : V_k \to \mathbf{V}_{k,i}$ given by $\chi(a\ \mathcal{N}) = a\ \mathcal{N}$ and

$$\chi\left(\sum_{\tau}\left(\sum_{i=1}^{k-1} a_i^{\tau}(0, e_i^{\tau}) + \sum_{i=k+1}^{[L:K]} a_i^{\tau}(e_i^{\tau}, 0)\right) + a_k^{\tau_1}(0, \mathrm{v}_k)\right) =$$

$$= \sum_{j=1}^{[K:\mathbf{Q}]}\sum_{t=1}^{k-1} b_t^{\tau_j}(0, e_t^{\tau_j}) + \sum_{j=1}^{[K:\mathbf{Q}]}\sum_{t=k+2}^{[L:K]} b_t^{\tau_j}(e_t^{\tau_j}, 0) + \sum_{j=1}^{i+1} b_k^{\tau_j}(0, e_k^{\tau_j}) + \sum_{j=i+1}^{[K:\mathbf{Q}]} b_{k+1}^{\tau_j}(e_{k+1}^{\tau_j}, 0),$$

where $b_t^{\tau_j}$ as given in (52). □

Lemma 5. *A fundamental domain for $\mathcal{I}^* \times \mathcal{I}^*$ with respect to the action of $\mathcal{O}_{L/K}^* \times \mathcal{O}_L^*$ described in* (17) *is*

$$\mathcal{A} = \bigcup_g \dot{\bigcup_k} (\mathcal{A}_g)^{(k)},$$

where $(\mathcal{A}_g)^{(k)}$ is as in (55) *and g is an ordering of the embeddings of L. Furthermore, $\mathcal{A}$ is a cone such that $T_{\mathcal{A}}(1)$ is $(2N-1)$-Lipschitz parametrizable, where $N = [L : \mathbf{Q}]$ and*

$$Vol(T_{\mathcal{A}}(1)) = \sum_g \sum_{k=0}^{[L:K]} Vol\left(T_{(\mathcal{D}_g)^{(k)}}(1)\right), \tag{60}$$

with $(\mathcal{D}_g)^{(k)}$ as in (47).

Proof. By proposition 8, we have that $\mathcal{A}$ is a fundamental domain for $\mathcal{I}^* \times \mathcal{I}^*$ and it is a cone. Using Propositions 9, 10 and 11, we have that $\partial T_{\mathcal{A}}(1)$is $(2N-1)$-Lipschitz parametrizable. By proposition 12, we obtain that

$$\sum_{k=0}^{[L:K]} \text{Vol}(T_{(\mathcal{A}_g)^{(k)}}(1)) = \sum_{k=0}^{[L:K]} \text{Vol}(T_{(\mathcal{D}_g)^{(k)}}(1)).$$

Furthermore, by the definition of C_g in (32), we have that if $g \neq g'$ then $\text{Vol}(C_g \cap C_{g'}) = 0$ because $C_g \cap C_{g'}$ is at most $(2N-1)-$dimensional. Hence we was obtained the Lemma. □

2.5 Proof of Theorem 1.

In this section we will prove Theorem 1. First, we need some notation.

Given an ordering g of the embeddings of L into $\mathbf{C}$, and k with $1 \leq k \leq [L:K]$, we have as in (45),

$$V_k = \left\{ \sum_{\tau} \left(\sum_{i=1}^{k-1} a_i^{\tau}(0, e_i^{\tau}) + \sum_{i=k+1}^{[L:K]} a_i^{\tau}(e_i^{\tau}, 0) \right) + a_k^{\tau_1}(0, \mathbf{v}_k) + a_{\mathcal{N}}\mathcal{N} \,\middle|\, a_i^{\tau},\ a_{\mathcal{N}} \in \mathbf{R} \right\},$$

where $\mathbf{v}_k$ and $\mathcal{N}$ are as in (40) and (42), respectively. Hence a natural basis for V_k is:

$$\begin{aligned}
&\{(e_i^{\tau}, 0) \mid k+1 \leq i \leq [L:K],\ \ \sigma_i^{\tau} \text{ real}\} \\
&\cup \{(0, e_i^{\tau}) \mid 1 \leq i \leq k-1,\ \ \sigma_i^{\tau} \text{ real}\} \\
&\cup \{(e_i^{\tau} + e_{i+1}^{\tau}, 0) \mid k+1 \leq i \leq [L:K]-1,\ \ \sigma_{i+1}^{\tau} = \overline{\sigma_i^{\tau}}\} \\
&\cup \{(0, e_i^{\tau} + e_{i+1}^{\tau}) \mid 1 \leq i \leq k-2,\ \ \sigma_{i+1}^{\tau} = \overline{\sigma_i^{\tau}}\} \ \cup\ \{(0, \mathbf{v}_k),\ \mathcal{N}\}.
\end{aligned}$$

Let

$$\{v_i^{\tau} \mid i \neq k,\ \sigma_i^{\tau} \in P(L)\} \ \cup\ \{v_k^{\tau_1},\ \mathcal{N}\}, \tag{61}$$

be this basis, ordered as above, where $P(L) = \{\sigma_1, \cdots, \sigma_{s_L+t_L}\}$ is a set of embeddings representing the arquimedean places of L.

For V_0, we let our basis be

$$\{(e_i^{\tau} + e_{i+1}^{\tau}, 0) \mid k_0+1 \leq i \leq [L:K]-1,\ \ \sigma_{i+1}^{\tau} = \overline{\sigma_i^{\tau}}\}$$

$$\cup\{(e_i^{\tau}, 0) \mid k_0+1 \leq i \leq [L:K],\ \ \sigma_i^{\tau} \text{ real}\} \cup \{(\mathbf{v}_k, 0),\ \mathcal{N}\},$$

where k_0 is as in (43).

Now, we can prove Theorem 1.

Theorem 1. *For an extension L/K of number fields and a large positive real number B, the number $N(L/K,B)$ of points $P \in \mathbf{P}^1(L)/\mathcal{O}_{L/K}^*$, with height $H_L(K,P) \leq B$ is*

$$N(L/K,B) = C_{L/K}\ \frac{R_L R_K h_L}{I_{L/K} W_L^2 \zeta_L(2)} \left(\frac{2^{s_L}(2\pi)^{t_L}}{d_L^{1/2}} \right)^2 B^2\ +\ O(B^{2-\frac{1}{[L:\mathbf{Q}]}}),$$

where notation is as in section 1.

Proof. In the previous section we obtained a fundamental domain $\mathcal{A}$ for $\mathcal{I}^* \times \mathcal{I}^*$, with respect to the action described in (17) that satisfies the conditions of Theorem 5. Hence we only need to calculate $\mathrm{Vol}\Big(T_{\mathcal{A}}(1)\Big)$ to obtain Theorem 1. By lemma 5, we must calculate $\sum_g \sum_k \mathrm{Vol}\Big(T_{\mathcal{D}_g^{(k)}}(1)\Big)$, where $\mathcal{D}_g^{(k)}$ is as in (47).

For a fixed g and k, put $S = T_{\mathcal{D}_g^{(k)}}(1)$ and let:

$$\widehat{S} = \bigcup_{\zeta,\zeta' \in \mu_L} (\theta_\zeta \circ \vartheta_{\zeta'})S,$$

$$\overline{S} = \{(x,y) \in \widehat{S} \mid \text{ for } i \leq s_L, \ \ x_i, \ y_i \geq 0\},$$

where $(\theta_\varepsilon \circ \vartheta_u)(x,y)$ is the action of $(\varepsilon, u) \in \mathcal{O}_{L/K}^* \times \mathcal{O}_L^*$ on $(x,y) \in \mathcal{I}^* \times \mathcal{I}^*$ given in (17). It is clear that $\mathrm{Vol}(\widehat{S}) = 2^{2s_L}\mathrm{Vol}(\overline{S}) = W_L^2 \mathrm{Vol}(S)$. Hence

$$\mathrm{Vol}(T_{\mathcal{D}_g^{(k)}}(1)) = \mathrm{Vol}(S) = \frac{2^{2s_L}}{W_L^2}\mathrm{Vol}(\overline{S}). \tag{62}$$

We will calculate $\mathrm{Vol}(\overline{S})$. Since $\overline{S} \subset (\mathbf{R}_{\geq 0})^{2s_L} \times \mathbf{C}^{2t_L}$, we let $R \subset (\mathbf{R}_{\geq 0})^{2(s_L+t_L)} \times [0, 2\pi)^{2t_L}$, correspond to S under the change of coordinates from $x_\sigma := x_i^\tau$ and $y_\sigma := y_i^\tau$ to polar coordinates $(r_\sigma, \ r'_\sigma, \ \phi_\sigma, \ \phi'_\sigma)$ (respectively, (r_σ, r'_σ)) when σ is a complex embedding (respectively, when σ is a real embedding). Then

$$\mathrm{Vol}(\overline{S}) = (2\pi)^{2t_L} \int_R \prod_{\sigma \in P(L)_{\mathrm{complex}}} (r_\sigma r'_\sigma) \ dV_R, \tag{63}$$

where $dV_R = dr_{\sigma_1} \cdots dr_{\sigma_{s_L+t_L}} dr'_{\sigma_1} \cdots dr'_{\sigma_{s_L+t_L}}$.

If we change the variables r_σ and r'_σ to $w_\sigma = \log r_\sigma$ and $w'_\sigma = \log r'_\sigma$, note that

$$\prod_{\{\sigma \in P(L)_{\mathrm{complex}}\}} (r_\sigma r'_\sigma) \prod_{\sigma \in P(L)} (r_\sigma r'_\sigma) = \prod_{\sigma \in P(L)} (r_\sigma r'_\sigma)^{\epsilon(\sigma)} = \exp\Big(\sum_{\sigma \in P(L)} \epsilon(\sigma)(w_\sigma + w'_\sigma)\Big), \tag{64}$$

with $\epsilon(\sigma)$ as in (35). Thus we obtain

$$\mathrm{Vol}(\overline{S}) = (2\pi)^{2t_L} \int_{\phi_2(\overline{S})} \exp\Big(\sum_{\sigma \in P(L)} \epsilon(\sigma)(w_\sigma + w'_\sigma)\Big) dV_{R'},$$

where $dV_{\mathcal{R}'} = dw_{\sigma_1} \cdots dw_{\sigma_{s_L+t_L}} dw'_{\sigma_1} \cdots dw'_{\sigma_{s_L+t_L}}$.

Now, for $(x,y) \in \overline{S}$, $\phi_2(x,y) = \Big((w_\sigma),(w'_\sigma)\Big)$ can be written as in (48) (or (49), depending on k). Thus,

$$\phi_2(x,y) = a_k^{\tau_1} v_k^{\tau_1} + a_{\mathcal{N}}\mathcal{N} + \sum_{i\neq k}\sum_{\sigma_i^\tau \in P(L)} a_i^\tau v_i^\tau + \sum_{i=1}^{r_L} \alpha_i E_i + \sum_{i=1}^{r_K} \beta_i U_i,$$

where the v_i^τ are as in (61), $0 \le \alpha_i,\ \beta_i \le 1$, $E_i = (\mathcal{L}(\varepsilon_i), \mathcal{L}(\varepsilon_i))$ and $U_i = (\mathcal{L}(u_i), 0)$ as in (31).

Changing the variables w_σ, w'_σ to the variables $a_{\mathcal{N}}$, a_i^τ, α_i, β_i, we transform the region $\phi_2(\overline{S})$ to a region $\mathcal{R} \times [0,1]^{r_L+r_K}$, where $\mathcal{R}$ is given by the inequalities given in (50), with $a_{\mathcal{N}} \le \log(B)$.

If we put

$$(z_1, z_2) = a_k^{\tau_1} v_k^{\tau_1} + a_{\mathcal{N}}\mathcal{N} + \sum_{i\neq k}\sum_{\sigma_i^\tau \in P(L)} a_i^\tau\ v_i^\tau,$$

we have

$$\Big((z_1)_{\sigma_i^\tau}, (z_2)_{\sigma_i^\tau}\Big) = \begin{cases} (\frac{a_{\mathcal{N}}}{N},\ a_i^\tau\ +\ \frac{a_{\mathcal{N}}}{N}) & \text{if} \quad i \le k-1, \\ (\frac{a_{\mathcal{N}}}{N},\ a_i^\tau\ +\ \frac{a_{\mathcal{N}}}{N}) & \text{if} \quad (i,\tau) = (k,\tau_1), \\ (a_i^\tau\ +\ \frac{a_{\mathcal{N}}}{N},\ \frac{a_{\mathcal{N}}}{N}) & \text{if} \quad i \ge k+1, \\ (\frac{a_{\mathcal{N}}}{N}, \frac{a_{\mathcal{N}}}{N}) & \text{if} \quad i = k \text{ and } \tau \neq \tau_1, \end{cases} \tag{65}$$

and

$$\frac{\partial(w_\sigma)}{\partial(a_i^\tau)} = \begin{cases} 1 & \text{if } \sigma = \sigma_i^\tau \text{ and } i > k \\ 0 & \text{otherwise} \end{cases}, \qquad \frac{\partial(w'_\sigma)}{\partial(a_i^\tau)} = \begin{cases} 1 & \text{if } \sigma = \sigma_i^\tau \text{ and } i \le k \\ 0 & \text{otherwise} \end{cases},$$

$$\frac{\partial(w_\sigma)}{\partial(a_{\mathcal{N}})} = \frac{1}{N}, \qquad \frac{\partial(w'_\sigma)}{\partial(a_{\mathcal{N}})} = \frac{1}{N},$$

$$\frac{\partial(w_\sigma)}{\partial(\alpha_i)} = \log|\sigma(\varepsilon_i)|, \qquad \frac{\partial(w'_\sigma)}{\partial(\alpha_i)} = \log|\sigma(\varepsilon_i)|$$

$$\frac{\partial(w_\sigma)}{\partial(\beta_i)} = \log|\sigma(u_i)|, \qquad \frac{\partial(w'_\sigma)}{\partial(\beta_i)} = 0.$$

Hence, the corresponding Jacobian determinant J is

$$J = \frac{1}{N\prod_{i=1}^{r_K} m_i} |\det \mathcal{A} \ \det \mathcal{B}|,$$

where for each $1 \leq i \leq r_K$, m_i is such that, $\{u_i^{m_i}\}_{i=1}^{r_K}$ is a set of generators of $\mathcal{O}_K^*$ modulo torsion, the matrix $\mathcal{A} = (a_{ij})$ is $(r_L+1)\times(r_L+1)$ and $\mathcal{B} = (b_{ij})$ is $(r_K+1)\times(r_K+1)$, where

$$a_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } j=1 \\ \log|\sigma_i(\varepsilon_{j-1})| & \text{if } j>1 \end{array} \right\}, \quad b_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } (i,j)=(1,1) \\ 0 & \text{if } i\neq 1,\ j=1 \\ \log|\tau_i(u_{j-1}^{m_j})| & \text{if } j>1 \end{array} \right\}.$$

We have $|\det \mathcal{A} \ \det \mathcal{B}| = N\frac{\epsilon(\tau_1)R_L R_K}{2^{t_L+t_K}}$. Hence

$$J = \frac{\epsilon(\tau_1)R_L R_K}{I_{L/K}2^{t_L+t_K}}. \tag{66}$$

On other hand, remembering that $(w_\sigma, w'_\sigma) = (z_1,z_2) + \sum_{i=1}^{r_L}\alpha_i E_i + \sum_{i=1}^{r_K}\beta_i U_i$, we have

$$\sum_{\sigma\in P(L)} \epsilon(\sigma)(w_\sigma + w'_\sigma) = \sum_{\sigma\in P(L)} \epsilon(\sigma)((z_1)_\sigma + (z_2)_\sigma) =$$

$$= \epsilon(\sigma_k^{\tau_1})a_k^{\tau_1} + 2a_{\mathcal{N}} + \sum_{i\neq k}\sum_{\sigma_i^\tau\in P(L)} \epsilon(\sigma_i^\tau)a_i^\tau.$$

Hence, using (63), (64) and lemma 4, we obtain

$$\mathrm{Vol}(\overline{S}) = (2\pi)^{2t}\int_{\mathcal{R}}\int_{-\infty}^{\log B} J\exp\left(\epsilon(\sigma_k^{\tau_1})a_k^{\tau_1} + 2a_{\mathcal{N}} + \sum_{i\neq k}\sum_{\sigma_i^\tau\in P(L)} \epsilon(\sigma_i^\tau)a_i^\tau\right)da_{\mathcal{N}}dV_{\mathcal{R}}, \tag{67}$$

where $\mathcal{R}$ is the region determined by the inequalities given in (50), and $dV_{\mathcal{R}} = da_k^{\tau_1}\prod_{i\neq k,\sigma_i^\tau\in P(L)} da_i^\tau$.

Note in (66) that $\frac{J}{\epsilon(\tau_1)}$ is independent of g and k. Hence by (62) and (67),

$$\mathrm{Vol}(T_{\mathcal{D}}(1)) = \frac{2^{2s_L}}{W_L^2}\left[\frac{(2\pi)^{2t_L}R_L R_K}{I_{L/K}}\right]\frac{C_{L/K}}{2^{2t_L}},$$

where

$$C_{L/K}=2^{t_L-t_K-1}\sum_g\sum_k\epsilon(\tau_1)\int_{\mathcal{R}_g^{(k)}}\exp\left(\epsilon(\sigma_k^{\tau_1})a_k^{\tau_1}+\sum_{i\neq k}\sum_{\sigma_i^{\tau}\in P(L)}\epsilon(\sigma_i^{\tau})a_i^{\tau}\right)dV_{\mathcal{R}_g^{(k)}}. \tag{68}$$

Here, for each ordering g and $0\leq k\leq[L:K]$, $dV_{\mathcal{R}_g^{(k)}}=da_k^{\tau_1}\prod_{i\neq k,\sigma_i^{\tau}\in P(L)}da_i^{\tau}$, $\mathcal{R}_g^{(k)}$ is the subset of $\mathbf{R}^{r_L-r_K+1}$ given by the inequalities (50).

Note that the constant $C_{L/K}$ depends only on the ramification pattern of the arquimedean places of L and K.

Hence, by Theorem 5, we have proved Theorem 1. □

2.6 Theorems 2 and 3.

In this section, we will calculate $C_{L/K}$ when L is a totally real or totally complex field.

We note first that when L is totally real or totally complex, for each pair g, g' of orderings, $\mathrm{Vol}\left(T_{\mathcal{D}_g^{(k)}}(1)\right)=\mathrm{Vol}\left(T_{\mathcal{D}_{g'}^{(k)}}(1)\right)$, because for different g, g', $\mathcal{D}_g^{(k)}$ and $\mathcal{D}_{g'}^{(k)}$ are isometric, where $\mathcal{D}_g^{(k)}$ is as in (47) (for g). Furthermore $\epsilon(\tau_1)=\epsilon(K)$ (see (36)), where

$$\epsilon(K)=\begin{cases}1 & \text{if } K \text{ is totally real,}\\ 2 & \text{otherwise.}\end{cases} \tag{69}$$

Hence we consider any fixed ordering g and omit it in subscripts.

We consider separately the case when K is totally real or totally complex, and the case when it is neither.

2.6.1 K totally real or totally complex.

In this subsection we also asumme that K is totally real or totally complex. Let

$$n_1:=s_K+t_K=\frac{[K:\mathbf{Q}]}{\epsilon(K)},\quad n_2:=[L:K],\quad n_3:=\frac{n_2}{\epsilon_{L/K}}, \tag{70}$$

where $\epsilon_{L/K}=\frac{\epsilon(L)}{\epsilon(K)}$ and ϵ is as in (69).

We note that, when L is totally complex and K is totally real, $B^{(k')}$ is empty for k' odd, so $P(L)=\{\sigma_{2j}^{\tau}\mid 1\leq j\leq n_3,\ \tau\in P(K)\}$. Hence in this case we replace the variables a_{2j}^{τ} by a_j^{τ}.

With the above change of index, for K as in this subsection, we have

$$\{a_i^\tau \mid \sigma_i^\tau \in P(L)\} = \{a_i^\tau \mid 1 \le i \le n_3,\ \tau \in P(K)\}.$$

Calculation of the integral

Noting that $\epsilon(\sigma_i^\tau) = \epsilon(L)$ for each i, τ, the integral that appears in (68) is equal to

$$\frac{1}{\epsilon(L)^{r_L - r_K + 1}} \int_{\mathcal{R}} \exp\left(a_k^{\tau_1} + \sum_{\tau \in P(K)} \sum_{i \neq k} a_i^\tau\right) dV_{\mathcal{R}}, \tag{71}$$

where $dV = da_k^{\tau_1} \prod_{\tau \in P(K)} \prod_{i \neq k} da_i^\tau$, and $\mathcal{R}$ is given by (50).

Put

$$I^{(k)} = \int_{\mathcal{R}} \exp\left(a_k^{\tau_1} + \sum_{\tau \in P(K)} \sum_{i \neq k} a_i^\tau\right) dV. \tag{72}$$

- When $k = 0$ or when $k = n_3$, using $n_1 = r_K + 1$ and the fact that for each j,

$$\int_{-\infty}^{x_{j+1}} \int_{-\infty}^{x_j} \cdots \int_{-\infty}^{x_2} \exp\left(\sum_{l=1}^{j} x_l\right) dx_1 dx_2 \cdots dx_j = \frac{1}{(j-1)!} \int_{-\infty}^{x_{j+1}} \exp(j\, x_j) dx_j, \tag{73}$$

  we obtain

$$I^{(k)} = \frac{1}{n_3((n_3-1)!)^{n_1}} = \frac{n_3^{n_1 - 1}}{(n_3!)^{n_1}}. \tag{74}$$

- When $k \neq 0,\ k \neq n_3$, we put

$$\mathcal{R} = R_- \cup R_+ \tag{75}$$

  where R_- is the intersection of $\mathcal{R}$ with the region given by $a_{k+1}^{\tau_1} \le 0$ and R_+ is the intersection of $\mathcal{R}$ with the region $a_{k+1}^{\tau_1} > 0$.

It is clear by (73) that

$$\int_{R_-} = \frac{1}{k((n_3-k)!(k-1)!)^{n_1}}. \tag{76}$$

Note that R_+ is the subregion of $\mathcal{R}$ such that $-\infty \le a_k^{\tau_1} \le -a_{k+1}^{\tau_1}$ and $0 \le a_{k+1}^{\tau_1} \le -\sum_{\tau \neq \tau_1} a_{k+1}^\tau$. Hence the integral $\int_{R_+}$ is

$$C \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \int_{0}^{-\sum_{\tau \neq \tau_1} a_{k+1}^\tau} \int_{-\infty}^{-a_{k+1}^{\tau_1}} \exp\left((k a_k^{\tau_1} + \sum_\tau (n_3 - k) a_{k+1}^\tau\right) da_k^{\tau_1} dV,$$

where $C = \frac{1}{((n_3-k-1)!(k-1)!)^{n_1}}$ and $dV = da_{k+1}^{\tau_1} \cdots da_{k+1}^{\tau_{n_1}}$. Hence

$$\int_{R_+} = \frac{C}{k} \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \int_{0}^{-\sum_{\tau\neq\tau_1} a_{k+1}^{\tau}} \exp\left((n_3-2k)a_{k+1}^{\tau_1} + \sum_{\tau\neq\tau_1} (n_3-k)a_{k+1}^{\tau}\right) dV. \tag{77}$$

To calculate the integral in (77), we consider two cases, when $k = \frac{n_3}{2}$ and when $k \neq \frac{n_3}{2}$.

When $k \neq \frac{n_3}{2}$:

$$\int_{R_+} = \frac{C}{(n_3-2k)k} \left(\frac{1}{k^{n_1-1}} - \frac{1}{(n_3-k)^{n_1-1}} \right). \tag{78}$$

Hence, by (75), we add (76) to (78) to obtain

$$I^{(k)} = \left(\frac{1}{k((n_3-k)!(k-1)!)^{n1}} + \frac{(n_3-k)((n_3-k)^{n_1-1} - k^{n_1-1})}{(n_3-2k)((n_3-k)!(k)!)^{n_1}} \right) =$$

$$= \left(\frac{(n_3-k)^{n_1} - k^{n_1}}{((n_3-k)!k!)^{n_1}(n_3-2k)} \right) = \frac{\begin{pmatrix} n_3 \\ k \end{pmatrix} (n_3-k)^{n_1-1}}{(n_3!)((n_3-k)!k!)^{n_1-1}} \sum_{j=0}^{n_1-1} \left(\frac{k}{n_3-k} \right)^j,$$

because we have the geometric sum

$$(a-b)^{n-1} \sum_{j=0}^{n-1} \left(\frac{b}{a-b} \right)^j = \frac{(a-b)^n - b^n}{a-2b}. \tag{79}$$

Hence, when $k \neq \frac{n_3}{2}$,

$$I^{(k)} = \frac{(n_3)^{n_1-1}}{(n_3!)^{n_1}} \begin{pmatrix} n_3 \\ k \end{pmatrix} \begin{pmatrix} n_3-1 \\ k \end{pmatrix}^{n_1-1} \sum_{j=0}^{n_1-1} \left(\frac{k}{n_3-k} \right)^j. \tag{80}$$

When $k = \frac{n_3}{2}$:

We note first that for $a \geq 1$,

$$\int_{-\infty}^{0} \cdots \int_{-\infty}^{0} (\sum_{j=1}^{i} x_j) \exp(a \sum_{j=1}^{i} x_j) dx_1 \cdots dx_i = \frac{-i}{a^{i+1}}. \tag{81}$$

Hence by (77) and the above, we have

$$\int_{R_+} = \frac{C}{k}\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}(-\sum_{\tau\neq\tau_1} a_{k+1}^{\tau})\exp(k\sum_{\tau\neq\tau_1} a_{k+1}^{\tau})dV = C\frac{(n_1-1)}{k^{n_1+1}}.$$

Hence, since $n_3 - k = k$, we have that

$$\int_{R_+} = \frac{(n_1-1)}{k((n_3-k)!(k-1)!)^{n_1}}. \tag{82}$$

Hence by (75), we add (76) to (82) to obtain

$$I^{(k)} = \left(\frac{n_1}{k((n_3-k)!(k-1)!)^{n_1}}\right).$$

If we note that when $k = \frac{n_3}{2}$,

$$\sum_{j=0}^{n_1-1}\left(\frac{k}{n_3-k}\right)^j = n_1 \quad\text{and}\quad \binom{n_3-1}{k} = \binom{n_3-1}{k-1},$$

we find that (80) is also true for $k = \frac{n_3}{2}$.

Now, for each ordering g, we have by the choice of τ_1 in (36), $\epsilon(\tau_1) = \epsilon(K)$. Furthermore the number of orderings g is $(n_3!)^{n_1}$. Hence, by (68)and (71), we have

$$C_{L/K} = \frac{(n_3!)^{n_1}2^{t_L-t_K}\epsilon(K)}{2\epsilon(L)^{r_L-r_K+1}}\sum_{k=0}^{n_3} I^{(k)}.$$

By formulas (73) and (80), we have

$$C_{L/K} = \frac{n_3^{n_1-1}2^{t_L-t_K}}{2\epsilon_{L/K}\epsilon(L)^{r_L-r_K}}\left(2+\sum_{k=1}^{n_3-1}\binom{n_3}{k}\binom{n_3-1}{k}^{n_1-1}\sum_{j=0}^{n_1-1}\left(\frac{k}{n-k}\right)^j\right).$$

Noting that $n_3 = \frac{[L:K]}{\epsilon_{L/K}}$ and $n_1 - 1 = r_K$, we have proved

Theorem 2. *Given an extension L/K of number fields, so that each is either totally real or totally complex, we have*

$$C_{L/K} = \frac{2^{t_L-t_K}}{2\epsilon_{L/K}\ \epsilon(L)^{r_L-r_K}}\left(\frac{[L:K]}{\epsilon_{L/K}}\right)^{r_K} u\left(r_K, \frac{[L:K]}{\epsilon_{L/K}}\right),$$

where $C_{L/K}$ is given in Theorem 1, $\epsilon_{L/K}$ as in (70), r_K is the free rank of the unit group of K and

$$u(m,n) := 2 + \sum_{k=1}^{n-1} \binom{n}{k} \binom{n-1}{k}^m \sum_{i=0}^{m} \left(\frac{k}{n-k}\right)^i.$$

Remark 4. $C_{L/K}$ has a simple form in the next cases:

- If $L = K$, we have that $C_{L/K} = \frac{1}{2}u(r_K, 1)$ and $u(r_K, 1) = 2$. Hence $C_{L/K} = 1$ in this case, in agreement wih the example in section 1.1.
- If K is totally real and L is a totally complex quadratic extension of K, we have that $C_{L/K} = 2^{t_L-2}u(r_K, 1) = 2^{t_L-1}$, also in accordance with example in section 2.2., using $R_L = \frac{2^{t_L-1}R_K}{I_{L/K}}$.
- If $r_K = 0$, we have

$$u(r_K, n) = 2 + \sum_{k=1}^{n} \binom{n}{k} = 2^n.$$

  1. When $K = \mathbf{Q}$ and L is totally real or totally complex, we have $\frac{[L:K]}{\epsilon_{L/K}} = s_L + t_L$. Hence

$$C_{L/K} = \frac{2^{t_L}}{2\epsilon(L)^{r_L+1}} 2^{s_L+t_L}.$$

     Noting that when L is totally real or totally complex $\frac{2^{t_L}}{\epsilon(L)^{r_L+1}} = 1$, we have that $C_{L/\mathbf{Q}} = 2^{s_L+t_L-1}$ in according with Schanuel's result (**) in the introduction.

  2. When K is totally complex and $[K : \mathbf{Q}] = 2$, we have $r_L = t_L - 1$ and $C_{L/K} = \frac{1}{2}u(0, [L : K])$. Hence $C_{L/K} = 2^{[L:K]-1}$

- If $r_K = 1$, we have

$$u(1, n) = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

This can proved by comparing coefficient $x^n y^n$ on expanding $(x+y)^{2n}$ by the binomial theorem and as $(x+y)^n(x+y)^n$.

In each case, taking $n_2 = [L:K]$, we have

$$C_{L/K} = \frac{n_2}{2}\binom{2n_2}{n_2}.$$

2.6.2 K mixed.

In this subsection, we consider the cases where L is totally complex and K is such that $s_K \neq 0$ and $t_K \neq 0$. In this case put

$$n_1 := s_K + t_K, \quad n_2 := [L:K], \quad n_2 = 2n_3. \tag{83}$$

Note that in this case, n_2 is even, because for each real embedding τ of K, there are only complex embeddings σ of L that extend τ.

Given k, let

$$i_k = \left[\frac{k+1}{2}\right], \tag{84}$$

where $[x]$ is the integer part of x i.e. i_k is such that $2i_k = k$ or $2i_k = k+1$.

We have

$$P(L) = \{\sigma_i^\tau \mid 1 \leq i \leq n_2,\ \tau \in P(K) \text{ complex}\} \bigcup \{\sigma_{2i}^\tau \mid 1 \leq i \leq n_3, \ \text{ real}\}.$$

Hence, when τ is real, we replace the variables a_{2j}^τ by a_j^τ. With the above change of index (for τ real), what was written $\sigma_j^\tau, j \neq k$, can now be written (for τ real) $\sigma_j^\tau, j \neq i_k$. Furthermore we have that

$$\{a_i^\tau \mid \sigma_i^\tau \in P(L)\} = \{a_i^\tau \mid 1 \leq i \leq \epsilon(\tau) n_3,\ \tau \in P(K)\}.$$

Calculation of the integral

Note that τ_1 is complex by the choice of τ_1 in (36).

Also, $\epsilon(\sigma_i^\tau) = 2$ for all i, τ. Hence, the integral in (68) is equal to

$$\frac{1}{2^{r_L - r_K + 1}} \int_{\mathcal{R}} \exp\left(a_k^{\tau_1} + \sum_{\tau \text{ real}} \sum_{j \neq i_k} a_j^\tau + \sum_{\tau \text{complex}} \sum_{j \neq k} a_j^\tau\right) dV_{\mathcal{R}}, \tag{85}$$

where $dV_{\mathcal{R}} = da_k^{\tau_1} \prod_{\tau \mathrm{real}} \prod_{j \neq i_k} a_j^{\tau} \prod_{\tau \in P(K)_{\mathrm{complex}}} \prod_{j \neq k} a_j^{\tau}$ and the region $\mathcal{R}$ is given by (50) (but replacing k by i_k when τ is real). Put

$$I^{(k)} = \int_{\mathcal{R}} \exp\left(a_k^{\tau_1} + \sum_{\tau \mathrm{real}} \sum_{j \neq i_k} a_j^{\tau} + \sum_{\tau \in P(K)_{\mathrm{complex}}} \sum_{j \neq k} a_j^{\tau}\right) dV_{\mathcal{R}}, \qquad (86)$$

- When $k = 0$, or when $k = n_2$, using (73) we obtain

$$I^{(k)} = \frac{1}{n_2((n_3-1)!)^{s_K}(n_2-1)!)^{t_K}} = \frac{n_2^{t_K-1} n_3^{s_K}}{(n_2!)^{t_K}(n_3!)^{s_K}} \qquad (87)$$

- When $k \neq 0$, $k \neq n_2$ we subdivide $\mathcal{R}$ as in (75).

It is clear by (73) that

$$\int_{R_-} = \frac{1}{k((n_2-k)!(k-1)!)^{t_K}((n_3-i_k)!(i_k-1)!)^{s_K}}. \qquad (88)$$

Note that, in this case when τ is real, we have $\overline{\sigma_{k-1}^{\tau}} = \sigma_k^{\tau}$. Hence R_+ is the subregion of $\mathcal{R}$ such that:

$$-\infty \leq a_k^{\tau_1} \leq -a_{k+1}^{\tau_1}.$$

$$0 \leq a_{k+1}^{\tau_1} \leq -\Big(\sum_{\tau \neq \tau_1 \ \mathrm{complex}} a_{k+1}^{\tau} + \frac{1}{2}\sum_{\tau \mathrm{real}} a_{i_k+1}^{\tau}\Big). \qquad (k \text{ even})$$

$$0 \leq a_{k+1}^{\tau_1} \leq -\sum_{\tau \neq \tau_1 \ \mathrm{complex}} a_{k+1}^{\tau}. \qquad (k \text{ odd })$$

Since the integral $\int_{R_+}$ depends on the parity of k, we will consider the two case separately. **When k is even.**

By the choice of i_k we have, $k = 2i_k$. Using (73), $\int_{R_+}$ is

$$C \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \int_{0}^{-\Sigma} \int_{-\infty}^{-a_{k+1}^{\tau_1}} \exp\left(k a_k^{\tau_1} + \sum_{\tau \mathrm{complex}} (n_2-k) a_{k+1}^{\tau} + \sum_{\tau \mathrm{real}} (n_3-i_k) a_{i_k+1}^{\tau}\right) da_k^{\tau_1} dV,$$

where

$$C = \frac{1}{((n_2-k-1)!(k-1)!)^{t_K}((n_3-i_k-1)!(i_k-1)!)^{s_K}}, \ dV = \prod_{\tau \in P(K)} da_{k+1}^{\tau},$$

and $\Sigma = \sum_{\tau\neq\tau_1,\ \text{complex}} a^{\tau}_{k+1} + \frac{1}{2}\sum_{\tau\text{real}} a^{\tau}_{i_k+1}$.

Integrating with respect to $a^{\tau_1}_k$ and letting $a^{\tau}_{k+1} = \frac{1}{2}a^{\tau}_{i_k+1}$, we have

$$\int_{R_+} = C\frac{2^{s_K}}{k}\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}\int_{0}^{-\Sigma'} \exp\left(\sum_{\tau\neq\tau_1}(n_2-k)a^{\tau}_{k+1} + (n_2-2k)a^{\tau_1}_{k+1}\right) dV, \tag{89}$$

where $\Sigma' = \sum_{\substack{\tau\in P(K)\\ \tau\neq\tau_1}} a^{\tau}_{k+1}$.

To calculate the integral in (89), we consider two cases: when $k = \frac{n_2}{2}$ and $k \neq \frac{n_2}{2}$.

When $k \neq \frac{n_2}{2}$, we have

$$\int_{R_+} = \frac{C\ 2^{s_K}}{k(n_2-2k)}\left(\frac{1}{k^{n_1-1}} - \frac{1}{(n_2-k)^{n_1-1}}\right).$$

Hence, using $k = 2i_k$ and $n_2 - k = 2(n_3 - i_k)$, we have

$$\int_{R_+} = \left(\frac{2^{s_K}(n_2-k)((n_2-k)^{n_1-1} - k^{n_1-1})}{2^{2s_K}((n_2-k)!k!)^{t_K}((n_3-i_k)!i_k!)^{s_K}(n_2-2k)}\right). \tag{90}$$

Adding (88) to (90) we obtain

$$I^{(k)} = \left(\frac{(n_2-k)^{n_1} - k^{n_1}}{2^{s_K}((n_2-k)!k!)^{t_K}((n_3-i_k)!i_k!)^{s_K}(n_2-2k)}\right).$$

Using (79) and noting that $\frac{n_2-k}{2} = n_3 - i_k$ and $\frac{k}{2} = i_k$, we have

$$I^{(k)} = \frac{n_2^{t_K-1}n_3^{s_K}}{n_2!(n_2)!^{t_K-1}(n_3)!^{s_K}}\begin{pmatrix} n_2 \\ k\end{pmatrix}\begin{pmatrix} n_2-1 \\ k\end{pmatrix}^{t_K-1}\begin{pmatrix} n_3-1 \\ \left[\frac{k}{2}\right]\end{pmatrix}^{s_K}\sum_{j=0}^{n_1-1}\left(\frac{k}{n_2-k}\right)^{j}. \tag{91}$$

When $k = \frac{n_2}{2}$, using (81), and (89), we have

$$\int_{R_+} = \frac{n_1-1}{k((k-1)!(n_2-k)!)^{t_K}((i_k-1)!(n_3-i_k)!)^{s_K}}. \tag{92}$$

Adding (88) to (92) we have

$$I^{(k)} = \frac{n_1}{k((i_k-1)!(n_3-i_k)!)^{s_K}((k-1)!(n_2-k)!)^{t_K}}.$$

If we note that, when $k = \frac{n_3}{2}$,

$$\sum_{j=0}^{n_1-1}\left(\frac{k}{n_3-k}\right)^j = n_1, \quad \binom{n_2-1}{k} = \binom{n_2-1}{k-1}, \quad \binom{n_3-1}{i_k} = \binom{n_3-1}{i_k-1},$$

we find that formula (91) is holds.

When k is odd

By the choice of i_k, we have $k = 2i_k - 1$. From (73), we have

$$\int_{R_+} = C\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}\int_{0}^{-\Sigma}\int_{-\infty}^{-a_{k+1}^{\tau_1}} \exp\left(ka_k^{\tau_1} + \sum_{\tau \text{complex}}(n_2-k)a_{k+1}^{\tau}\right) da_k^{\tau_1}\, dV,$$

where

$$C = \frac{1}{((n_2-k-1)!(k-1)!)^{t_K}((n_3-i_k)!(i_k-1)!)^{s_K}}, \quad \Sigma = \sum_{\substack{\tau\in P(K),\ \text{complex}\\ \tau\neq\tau_1}} a_{k+1}^{\tau},$$

and $dV = \prod_{\tau\in P(K),\ \text{complex}} da_{k+1}^{\tau}$.

Integrating with respect to $a_k^{\tau_1}$, we have

$$\int_{R_+} = \frac{(n_2-k)((n_2-k)^{t_K-1} - k^{t_K-1})}{((n_2-k)!k!)^{t_K}((n_3-i_k)!(i_k-1)!)^{s_K}(n_2-2k)}. \tag{93}$$

Adding (88) to (93),

$$I^{(k)} = \frac{(n_2-k)^{t_K} - k^{t_K}}{((n_2-k)!k!)^{t_K}((n_3-i_k)!(i_k-1)!)^{s_K}(n_2-2k)}.$$

Using (79) we have

$$I^{(k)} = \frac{1}{n_2!((n_2-1)!)^{t_K-1}n_3!^{s_K}}\binom{n_2}{k}\binom{n_2-1}{k}^{t_K-1}\binom{n_3}{\left[\frac{k}{2}\right]}^{s_K}\sum_{j=0}^{n_1-1}\left(\frac{k}{n_2-k}\right)^j.$$

Hence formula (91) is also true when k is odd.

Now, for each ordering g of the embeddings, we have by the choice of τ_1 in (33), $\epsilon(\tau_1) = 2$. Furthermore the number of orderings g is $(n_2!)^{t_K}(n_3!)^{t_K}$. Hence, by (68),

$$C_{L/K} = \frac{2^{t_L - t_K + 1}(n_3!)^{s_K}(n_2!)^{t_K}}{2^{r_L - r_K + 1}} \sum_{k=0}^{n_2} I^{(k)}.$$

Hence, by formulas (87) and (91), we have

$$C_{L/K} \left(\frac{2^{t_L - t_K} n_3^{s_K} n_2^{t_K - 1}}{2^{r_L - r_K + 1}} \right)^{-1} =$$

$$\left(2 + \sum_{k=0}^{2n_3 - 1} \binom{n_2}{k} \binom{n_2 - 1}{k}^{t_k - 1} \binom{n_3}{\left[\frac{k}{2}\right]}^{s_K} \sum_{j=0}^{n_1 - 1} \left(\frac{k}{n_2 - k} \right)^j \right).$$

Noting that $2n_3 = n_2 = [L : K]$ we have proved

Theorem 3. *Given an extension L/K of number fields, so that L is a totally complex field and K is neither totally real nor totally complex, we have*

$$C_{L/K} = \frac{[L : K]^{r_K}}{2^{s_L + 1}} v\left(s_K, t_K, \frac{1}{2}[L : K]\right),$$

where

$$v(l, m, n) := 2 + \sum_{k=1}^{2n-1} \binom{2n}{k} \binom{2n-1}{k}^{m-1} \binom{n}{\left[\frac{k}{2}\right]}^{l} \sum_{j=0}^{l+m-1} \left(\frac{k}{2n - k} \right)^j,$$

and $[x]$ denotes the integer part of x.

Remark 5. Note that

$$v(0, m, n) = u(m - 1, 2n),$$

where u (respectively v) is as in theorem 2 (respectively theorem 3).

Since when K and L are totally complex, we have $\epsilon_{L/K} = 1$ and $\frac{1}{2^{s_L+1}} = \frac{2^{t_L - t_K}}{2\epsilon_{L/K}\epsilon(L)^{r_L - r_K}}$, the formula for $C_{L/K}$ given in Theorem 2, satisfies the formula given in Theorem 3.

2.7 References

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