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... a mi familia y a mi amor...

"la matemática es el trabajo del espíritu humano
que está destinado tanto a estudiar como a conocer,
tanto a buscar la verdad como a encontrarla"

EVARISTE GALOIS

Biografía

Nací el 19 de Agosto de 1988 en Santiago. Mi familia está conformada por mis padres, Luis y Liliana, y mi hermano Sebastián, que me han acompañado siempre. Desde niña, en el colegio, tenía una inclinación hacia las matemáticas, me parecían entretenidas y siempre destaqué con notas en ese ramo, pero nunca pensé en ser matemático hasta el último año de enseñanza media donde un profesor de matemáticas del preuniversitario que asistía me comentó: "si te gustan las matemáticas no estudies ingeniería sino que licenciatura, allí se ve lo que realmente es hacer y aprender matemáticas"; y fue así como el 2007 conocí y entré a la Facultad de Ciencias de la Universidad de Chile. Durante los años de licenciatura, mi gusto por las matemáticas fue creciendo a tal punto de elegir el camino de investigación matemática el 2011 a través del programa de Doctorado en la misma facultad. A pesar de las dificultades en un comienzo, cada una de las etapas las disfruté y aproveché al máximo. Ahora me encuentro terminando esta etapa y a puertas de comenzar una nueva para lograr establecerme profesionalmente en investigación y educación matemática.

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Resumen

En esta tesis estudiamos la formación de singularidades durante la evolución por curvatura de curvas en el espacio $\mathbb{R}^3$. Nociones de curvatura total de la curva y superficie mínima asociada a ésta son utilizadas para definir radios isoperimétricos que dependen de cantidades geométricas tales como largos y áreas. Probamos que bajo ciertas hipótesis de la curva y superficie mínima en cuestión se descarta la formación de singularidades tipo II. Además, mostramos que cerca del punto donde se produce la singularidad es posible aproximar una curva cualquiera en $\mathbb{R}^3$, que evolucione por curvatura, por una curva plana con mejores propiedades que la anterior.

Abstract

In this thesis, we study the formation of singularities through the curve shortening flow in $\mathbb{R}^3$. Notions of total curvature of the curve and its associated minimal surface are used to define isoperimetric ratios that depends of geometric quantities such as lengths and areas. We prove that, under certain assumptions of the curve and minimal surface, the formation of type II singularities is discarded. Moreover, we show that, close to the singularity, it is possible to approximate any curve in $\mathbb{R}^3$ that evolves by its curvature by a planar curve with better properties than previous curve.

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Introduction

The Curve Shortening Flow is a geometric heat equation that can be used to deform curves. More precisely, given an initial curve γ_0, the flow produces a family of curves that satisfies a certain law of motion. The aim is to analyze the shape of the solutions of this sort of flow.

One way to motivate the Curve Shortening Flow is to try to write down as naively as possible a heat equation for γ. We want to obtain a flow that is independent of the choice of parametrization of the curve and this can be achieved by picking a canonical choice of parametrization, the arc-length parameter s. Then we can simply write down the heat equation for this parametrization as follows

$$\frac{\partial \gamma}{\partial t} = \frac{\partial^2 \gamma}{\partial s^2}. \tag{HE}$$

Note that the arc-length parameter is defined using the map γ, and it changes as the curve changes. Moreover, as the coefficients depend on the derivatives of γ the equation has the structure of a quasilinear heat equation.

Using Frenet's equations (see (1.1.1)) we can write the equation in a more geometric way and the Curve Shortening Flow equation (HE) becomes

$$\frac{\partial \gamma}{\partial t} = k \cdot N. \tag{CSF}$$

That is, the curve moves in the normal direction with speed equal to the curvature at each point.

This flow was proposed in 1956 by Mullins to model the motion of idealized grain boundaries. In the context of geometric measure theory Brakke studied weak solutions of the Mean Curvature Flow in 1978, of which the Curve Shortening Flow is the 1-dimensional case. In 1986, Grayson, Gage and Hamilton renewed interest in the Curve Shortening Flow arising from work on planar curves.

In addition, the Curve Shortening Flow acts to decrease the length of the closed curve at the fastest rate possible (relative to the total speed of motion as measured in the square-integral sense). Let us make this precise: Given an immersion $\gamma_0 : S^1 \to \mathbb{R}^3$, we can consider smooth variations $\gamma(x,t)$ which satisfy

$$\frac{\partial \gamma}{\partial t}(x,t) = V(x,t) \quad \text{and} \quad \gamma(x,0) = \gamma_0(x),$$

where V is an arbitrary (smooth) map from S^1 to $\mathbb{R}^3$.

If L denotes the length of γ, by a straightforward estimation we obtain

$$\frac{\partial}{\partial t}L \geq -\left(\int_{S^1} |V|^2\right)^{\frac{1}{2}} \left(\int_{S^1} k^2\right)^{\frac{1}{2}},$$

with equality if and only if V is a multiple of $k \cdot N$. Hence, we say that the Curve Shortening Flow is the gradient flow of the length functional and it arises naturally in problems where a curve energy is relevant.

For closed compact initial curve the smooth solution to (CSF) exists on a maximal time interval $[0, \omega)$, with $\omega > 0$ finite. At time ω the curve becomes singular and intuitively, this singularity occurs when the maximum curvature becomes unbounded.

The natural extension of the Curve Shortening Flow to higher dimensions is the Mean Curvature Flow. That is a one-parameter family $F : \mathcal{M} \times [0, \omega) \to \mathbb{R}^{n+1}$ of smooth immersions of the n-dimensional hypersurface $\mathcal{M}$, where $F(\cdot, 0) = F_0$ is a smooth immersion and F satisfies

$$\frac{\partial F}{\partial t}(p,t) = H(p,t) \cdot \nu(p,t), \quad \text{for every } (p,t) \in \mathcal{M} \times [0, \omega), \qquad \text{(MCF)}$$

where $H(p,t)$ and $\nu(p,t)$ are the mean curvature and the outer normal, respectively, at the point $F(p,t)$ of the surface $\mathcal{M}_t = F(\cdot, t)(\mathcal{M})$.

A motivation to study the Mean Curvature Flow is that, in analogy with the Ricci Flow of metrics on abstract Riemannian manifols [24], it can be used to obtain classification results for hypersurfaces satisfying certain curvature conditions [15]. It also has been used to derive isoperimetric inequalities [23] and to produce minimal surfaces [6].

In the case of compact surfaces, it is known that there exists a smooth solution to (MCF) on a maximal time interval $[0, \omega)$, with $\omega > 0$ finite, and the mean curvature of the surfaces becomes unbounded as $t \to \omega$.

In the last decades several different notions of weak solutions have been introduced to define a flow after the singular time ω. In [9] a new approach based on a "surgery procedure" was considered to extend the flow after singularities. Compared with other notions of weak solutions existing in the literature, the flow with surgeries has the advantage that it keeps track of the changes of topology of the evolving surface and this can be applied to classify possible geometries of the initial manifold. The surgery construction was inspired by a procedure originally introduced by Hamilton in [24] for the Ricci flow, which deforms metrics on a Riemannian manifold. In three dimensions the Ricci flow of Hamilton was employed by Perelman, in conjunction with a different surgery procedure, to prove Thurston's geometrization conjecture [10].

It is important to note that the results above on the Mean Curvature Flow were obtained for manifolds of codimension 1. In that setting a fundamental tool is the comparison principle which, among others things, ensures that embeddedness is preserved. In contrast, the comparison principle cannot be applied for higher codimension manifolds evolving by the Mean Curvature Flow and this makes it more difficult to study such solutions. Consequently, fewer results are known in that context and usually it is necessary to have other preserved quantities. For example, in the study of the Lagrangean Mean Curvature Flow [1, 2] the existence of a preserved Lagrangean is exploited, while in [3] it is fundamental that a curvature pinching condition is preserved.

The main goal in this thesis is to gain understanding of the Curve Shortening Flow in $\mathbb{R}^3$. We are interested in proposing in the future a type of surgery in the evolution of space curves, following the work of Huisken and Sinestrari in [9]. To attain this objective we need to understand in detail the formation of singularities.

The formation of singularities is fully studied for planar curves that are smooth and embedded. In [19] it was proved that if the initial curve is closed, convex and embedded in $\mathbb{R}^2$, then the solution to (CSF) stays convex in the evolution, and when $t \to \omega$ it shrinks to a point. A generalization of that result can be seen in [16] where it was proved that if the initial curve is closed, embedded and possibly not convex in $\mathbb{R}^2$, then the solution to (CSF) becomes a convex curve before the time ω. Thus it converges to a point as well. These theorems were simplified by Huisken in [8] using extrinsic and intrinsic distances and defining certain isoperimetric ratios. Note that in this case there is no formation of singularities until the time when the curve shrinks to a point.

The first result in this thesis is related to an isoperimetric ratio defined by Grayson in [16].

Theorem A *Suppose that $\gamma(\cdot,t)$ satisfies the equation (CSF) and the total curvature of $\gamma(\cdot,0)$ is less than 4π. If $\gamma(\cdot,t)$ remains embedded for all $t \in [0,\omega)$ with finite singular time ω, then the isopermetric ratio defined by* (2.0.2) *is uniformly bounded for all $t \in [0,\omega)$. In particular, if $\gamma(\cdot,t)$ develops a Type I singularity, then the isoperimetric ratio converges to 4π.*

A standard way of classifying singularities is into two classes called Type I and Type II (see Definition 1.3.1). In the case of planar curves if the initial curve is an embedding, then the formation of Type II singularities is discarded.

On the other hand, for spaces curves, embeddedness is not preserved by the flow and other types of singularities can be formed. Intuitively, since the torsion of a space curve is not zero, the curve could move in more directions; even if the initial curve lies in a plane its evolution could be outside of this plane. However, in this thesis we will prove that the formation of Type II singularities can be discarded for space curves under suitable conditions. We will prove

Theorem B *Suppose that $\gamma(\cdot,t)$ satisfies the equation (CSF) and the total curvature of $\gamma(\cdot,0)$ is less than 4π. Let X_t be the minimal surface enclosed by $\gamma(\cdot,t)$. If its Gaussian curvature $K(\cdot,t)$ is uniformly bounded and $\gamma(\cdot,t)$ remains embedded for all $t \in [0,\omega)$ with finite singular time ω and does not shrink to a point, then there is no formation of Type II singularities.*

This theorem complements the result proved in [11], which establishes that if the initial curve satisfies that its total curvature is less than 4π, and the singularity formed in the spatial curve shortening flow is of Type I, then the curve shrinks to a round point when approaching the maximum time ω.

The main idea of the proof is to consider a curve γ with total curvature less than 4π and inspired by [8], we will use isoperimetric ratios to analyze its behavior through the evolution by the Curve Shortening Flow. To define isoperimetric quantities we were motivated by the classic isoperimetric problem, which relates the length with the area enclosed by a curve. However, in $\mathbb{R}^3$ it is not clear how to define the enclosed area and to circumvent this issue we will consider the solution to Plateau's Problem, which associates a minimal surface to a curve γ. In 1931 Jesse Douglas, simultaneously with Tibor Rado, showed that every Jordan curve in $\mathbb{R}^n$ bounds at least one minimal surface of disc-type. Additionally, in [21] it was shown that the minimal surface is unique if its boundary is an unknotted curve. Following these results, the area enclosed by an unknotted curve γ is well defined and we will use it in the definition of isoperimetric quantities.

Others existing results that we will use in the proof of Theorem B are given in [12, 13] and will be described in Chapter 1.

In Chapter 3 we will show that if γ evolves by its curvature, then it can be approximated by a "nice" curve $\hat{\gamma}$ close to a singularity. Since this approximation could extend the time of evolution of γ, the computation of the evolution of these curves and estimation of their curvatures may give us the first step to perform a simple surgery. We hope to address this in the future. The main tool of Chapter 3 will be a result in [12], which states that singularity formation is a planar phenomenon. Consequently, we will prove that there exists a plane where γ will be approximated by a graph of a curve over this plane close to the singularity and we will show that their curvatures are directly related.

This work is organized as follows:

In the first chapter we collect basic facts and notation about the Curve Shortening Flow, total curvature, singularities and minimal surfaces. Also, we state the results that will be used freely in the thesis.

The second chapter contains the definitions of isoperimetric ratios using minimal surfaces for unknotted curves. Also we will show monotonicity properties of these ratios and theirs rescalings. Next, we will study the properties of the Grim Reaper (see Definition 1.3.4) and we will define certain isoperimetric ratios related to that curve. We conclude the chapter with the proof of the main theorem of this thesis.

In the last chapter we define a neighborhood of the singularity to show that the curve γ restricted to this neighborhood has some properties, in the sense that the curve is close to some plane. Secondly, we will show that this curve is approximated by a space curve that is a graph of a planar curve. Finally, we estimate the relation among their respective curvatures.

Chapter 1

Preliminaries

We denote by B the open disk

$$B = \{w = (u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\};$$

and its boundary by

$$S^1 = \{p = (u, v) : u^2 + v^2 = 1\}.$$

1.1 Curve Shortening Flow

In order to fix notation, the Frenet matrix for a space curve γ with arc-length parameter s will be written as:

$$\frac{\partial}{\partial s}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}; \tag{1.1.1}$$

where T, N and B are the tangent, normal and binormal vectors, respectively. The quantities k and τ are the curvature and torsion of γ.

Definition 1.1.1. *The space curve γ evolves by the **Curve Shortening Flow** or by its curvature if it satisfies the equation (CSF), where $\gamma : S^1 \times [0, \omega) \to \mathbb{R}^3$ is a one-parameter smooth family of curves and $\gamma(\cdot, 0) = \gamma_0$ is a smooth space curve.*

Moreover, we can rewrite the equation (CSF) as:

$$\frac{\partial \gamma}{\partial t} = \frac{\partial T}{\partial s} = \frac{\partial^2 \gamma}{\partial s^2}. \tag{1.1.2}$$

Thus, the Curve Shortening Flow is a system of quasilinear parabolic equations.

Some examples of curves which evolve by the Curve Shortening Flow are the following:

- The ***straight line***: Since its curvature k is identically 0 it does not change in time.

- The *circle*: Considering the parametrization

$$\gamma(x,t) = (r(t)\cos(x), r(t)\sin(x), 0),$$

the evolution of the circle can be compute by the evolution of its radio

$$\frac{\partial r}{\partial t} = \frac{-1}{r(t)}.$$

We obtain the solution $r(t) = \sqrt{r_0^2 - 2t}$, where r_0 is the initial radio. Thus, we get a maximal time of evolution given by $\omega = \frac{r_0^2}{2}$. Therefore, the circle evolves until it becomes a single point.

- The *helix*: Consider a curve parametrized as

$$\gamma(x,t) = (A(t)\cos(x), A(t)\sin(x), B(t)x),$$

where A and B are functions depending on time. The evolution of the helix is given by

$$\left(\frac{\partial A}{\partial t}\cos(x), \frac{\partial A}{\partial t}\sin(x), \frac{\partial B}{\partial t}x\right) = \frac{-A}{A^2+B^2}(\cos(x), \sin(x), 0);$$

where the functions A and B can be determined by

$$\begin{aligned} \frac{A(t)^2}{2} + B^2\ln(A(t)) &= -t + \frac{A(0)^2}{2} + B^2\ln(A(0)), \\ B(t) &= c \in \mathbb{R}. \end{aligned}$$

Note that the curvature and torsion of the helix are given by:

$$k = \frac{A}{A^2+B^2} \longrightarrow 0 \text{ when } t \to \infty;$$
$$\tau = \frac{B}{A^2+B^2} \longrightarrow B^{-1} \text{ when } t \to \infty.$$

Therefore, the limit curve of the evolution is a straight line, although the torsion is not approaching zero.

There are two important theorems about existence of the Curve Shortening Flow. From [19] one has short time existence of solutions on a small open interval in time:

Theorem 1.1.2. *Let γ_0 be a smooth, immersed and closed curve in $\mathbb{R}^3$. There exists $\varepsilon > 0$ such that solutions $\gamma : S^1 \times [0,\varepsilon) \to \mathbb{R}^3$ to the Curve Shortening Flow exists. Furthermore, these solutions are smooth.*

In [13] it has been shown that solutions to the space curve flow exist until the curvature becomes unbounded. More precisely

Theorem 1.1.3. *If the curvature of γ is uniformly bounded on the time interval $[0, \alpha)$, there exists an $\varepsilon > 0$ such that $\gamma(\cdot, t)$ exists and is smooth on the extended time interval $[0, \alpha + \varepsilon)$.*

Proposition 1.1.4 (Geometric Invariance under Tangential Perturbations). *If a smooth family of curves $\gamma : S^1 \times [0, \omega) \to \mathbb{R}^3$ satisfies*

$$\begin{aligned} \frac{\partial \gamma}{\partial t} &= kN + \mathfrak{a}T \\ \gamma(s, 0) &= \gamma_0(s) \end{aligned} \tag{1.1.3}$$

where T, N and k are the tangent vector, the normal vector and the curvature of the point $\gamma(s, t)$, respectively, and also if $\mathfrak{a} : S^1 \times [0, \omega) \to \mathbb{R}$ is a smooth function then there exists a family of reparametrizations of $\gamma(\cdot, t)$ which satisfies (CSF).

Conversely, if a smooth family of curves $\gamma : S^1 \times [0, \omega) \to \mathbb{R}^3$ satisfies (CSF), *then any reparametrization satisfies the system* (1.1.3) *for some $\mathfrak{a}(s, t)$.*

Proof. Let $\varphi : S^1 \times [0, \omega) \to S^1$ be a smooth family of diffeomorphisms of S^1 with $\varphi(s, 0) = s$ for every $s \in S^1$ and

$$\frac{\partial}{\partial t}\varphi(s, t) = -\mathfrak{a}(\varphi(s, t), t).$$

By the existence and uniqueness theorem for ODE's on S^1 this family exists, it is unique and smooth. Considering the reparametrizations $\tilde{\gamma}(x, t) = \gamma(\varphi(x, t), t)$, one has

$$\begin{aligned} \frac{\partial \tilde{\gamma}}{\partial t}(s, t) &= \frac{\partial \gamma}{\partial t}(\varphi(s, t), t) + T(\varphi(s, t), t) \cdot \frac{\partial \varphi}{\partial t}(s, t) \\ &= k(\varphi(s, t), t)N(\varphi(s, t), t) + \mathfrak{a}(\varphi(s, t), t)T(\varphi(s, t), t) - \mathfrak{a}(\varphi(s, t), t)T(\varphi(s, t), t) \\ &= k(\varphi(s, t), t)N(\varphi(s, t), t) \\ &= \tilde{k}(s, t)\tilde{N}(s, t). \end{aligned}$$

Hence, $\tilde{\gamma}$ satisfies (CSF) and $\tilde{\gamma}_0 = \gamma_0$.

Conversely, if γ satisfies (CSF), then $\tilde{\gamma}(x, t) = \gamma(\varphi(x, t), t)$ satisfies

$$\begin{aligned} \frac{\partial \tilde{\gamma}}{\partial t}(s, t) &= \frac{\partial \gamma}{\partial t}(\varphi(s, t), t) + T(\varphi(s, t), t) \cdot \frac{\partial \varphi}{\partial t}(s, t) \\ &= k(\varphi(s, t), t)N(\varphi(s, t), t) - \mathfrak{a}(\varphi(s, t), t)T(\varphi(s, t), t). \end{aligned}$$

Therefore, $\tilde{\gamma}$ satisfies (1.1.3). □

Since a reparametrization does not geometrically change the curve, an invariant way of writing (CSF), which factors out the issue od dependence on parametrizations, is

Definition 1.1.5. *The space curve γ evolves by **Curve Shortening Flow** if it satisfies*

$$\frac{\partial \gamma}{\partial t}(p, t) \cdot N(p, t) = k(p, t). \tag{1.1.4}$$

Here $\gamma : S^1 \times [0, \omega) \to \mathbb{R}^3$ is a one-parameter smooth family of curves and $\gamma(\cdot, 0) = \gamma_0$ is a smooth space curve.

The definition above will be mainly used in Chapter 3, while in Chapter 2 we will often assume that the curve γ satisfies (CSF).

This definition has the following properties

- ***Invariance under isometries of*** $\mathbb{R}^3$

  If $\gamma : S^1 \times [0, \omega) \to \mathbb{R}^3$ is a solution to (1.1.4) and $A : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry then $A \circ \gamma$ is a solution to (1.1.4).

- ***Invariance under rescaling space-time***

  If $\gamma : S^1 \times [0, \omega) \to \mathbb{R}^3$ is a solution to (1.1.4) and for fixed $\lambda > 0$ we define $\gamma_\lambda : S^1 \times [0, \lambda^2\omega) \to \mathbb{R}^3$ by

  $$\gamma_\lambda(p, t) = \lambda\gamma(p, \lambda^{-2}t),\ p \in S^1$$

  then γ_λ is solution to the Curve Shortening Flow in the sense (1.1.4).

1.2 Total Curvature

Given a space curve γ, we define its curvature (always non-negative) as

$$k = \left\| \frac{\partial^2 \gamma}{\partial s^2} \right\|. \tag{1.2.1}$$

Note that, when γ is a planar curve, for every $s \in S^1$ it is possible to give a sign to its curvature $k(s)$ as follow: Let $\{e_1, e_2\}$ be the natural basis of $\mathbb{R}^2$. We choose the normal vector $N(s)$ such that the basis $\{T(s), N(s)\}$ possesses the same orientation as the basis $\{e_1, e_2\}$. Thus the curvature $k(s)$ is defined by

$$\frac{\partial T}{\partial s}(s) = k(s) \cdot N(s),$$

that might be either positive or negative (depending on the chosen orientation for γ or $\mathbb{R}^2$).

Thus, we define

Definition 1.2.1. *The total curvature of γ is given by:*

$$\int_\gamma |k| ds.$$

Remark 1.2.2. *It is clear that $|k|$ coincides with equation (1.2.1). Thus, for space curves we can consider the total curvature as*

$$\int_\gamma k ds.$$

In [12, Th.5.1] it was shown that if a curve evolves by its curvature, then its total curvature is a decreasing function in time:

Theorem 1.2.3. *Let γ be a solution to the Curve Shortening Flow. Then we have*

$$\frac{\partial}{\partial t}\int_{\gamma}|k|ds \leq -\int_{\gamma}\tau^{2}|k|ds < 0. \tag{1.2.2}$$

Additionally, following [18, Sec.5-7,Th.3] we have:

Theorem 1.2.4 (Fenchel's Theorem). *If γ is an embedded curve then the total curvature $\int_{\gamma}|k| \geq 2\pi$ and equality holds if and only if the curve is a planar convex curve.*

Proof. First, we construct a tubular neighborhood of radius r about γ, forming a toroidal surface S that is parametrized by:

$$X(s,v) = \gamma(s) + r\cos(v)N(s) + r\sin(v)B(s).$$

Here N and B denote the normal and binormal vector, respectively.

Using Frenet's equations, we have

$$\begin{aligned}
X_s &= (1 - rk(s)\cos(v))T(s) - r\tau(s)\sin(v)N(s) + r\tau(s)\cos(v)B(s);\\
X_v &= -r\sin(v)N(s) + r\cos(v)B(s);\\
X_s \times X_v &= -r\cos(v)(1 - rk(s)\cos(v))N(s) - r\sin(v)(1 - rk(s)\cos(v))B(s);\\
|X_s \times X_v| &= r|(1 - rk(s)\cos(v))|.
\end{aligned}$$

Since the surface S is embedded we may choose a radius $r < \frac{1}{\max(k(s))}$. This guarantees that $|X_s \times X_v| > 0$.

Now we compute the coefficients of the first and second fundamental form and normal vector of the surface S :

$$\begin{aligned}
E &= X_s \cdot X_s = (1 - rk(s)\cos(v))^2 + r^2\tau(s)^2;\\
F &= X_s \cdot X_v = r^2\tau(s);\\
G &= X_v \cdot X_v = r^2;\\
\nu &= \frac{X_s \times X_v}{|X_s \times X_v|} = -\cos(v)N(s) - \sin(v)B(s);\\
e &= \nu \cdot X_{ss} = -k(s)\cos(v)(1 - rk(s)\cos(v)) + r\tau(s)^2;\\
f &= \nu \cdot X_{sv} = r\tau(s);\\
g &= \nu \cdot X_{vv} = r.
\end{aligned}$$

Note that in the factors $eg - f^2$ and $EG - F^2$ the torsion τ does not appear. Thus, using the formula $K = \frac{eg-f^2}{EG-F^2}$ we obtain that the Gaussian curvature is well defined, and:

$$K = \frac{-k(s)\cos(v)}{r(1 - rk(s)\cos(v))}.$$

Moreover, the Gaussian curvature K is positive if $\cos(v) \leq 0$, i.e. when $\frac{\pi}{2} \leq v \leq \frac{3\pi}{2}$. We shall denote by S_+ and S_- the subsets of S where the Gaussian curvature K is positive and negative, respectively. Since S is a torus with Euler characteristic 0, the Gauss-Bonnet Theorem implies that

$$\begin{aligned}\int_{S_+} KdA &= \int_{S_+} K|X_s \times X_v| ds dv = \int_{S_+} -k(s)\cos(v) ds dv \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-\cos(v)) dv \int_\gamma k ds \\ &= 2\int_\gamma k ds.\end{aligned}$$

Therefore, the total Gaussian curvature of S_+ is equal to twice the total curvature of the curve γ.

On the other hand, the Gauss map $M : S \to S^2$ restricted to S_+ is surjective. This follows from observing that for any given $q \in S^2$ there is a plane P parallel to the tangent plane at q that satisfies $P \cap S = \emptyset$. By sliding P to the first point of tangency $p \in S$, the tube S must lie entirely on one side of P (since otherwise it would not hit the point p first). This implies that $K(p) \geq 0$, i.e. $p \in S_+$. This shows that $M(S_+) = S^2$.

Moreover, the Gaussian curvature K is the Jacobian of the Gauss map for an orientable surface. Therefore, using a change of variables we get that $\int_{S_+} KdA \geq 4\pi$.

Finally, we can conclude that the total curvature is positive and greater or equal than 2π.

A proof of $\int_\gamma |k| = 2\pi$ if and only if γ is a planar convex curve can be found in [18]. □

We consider two types of embedded curves: unknotted and knotted curves.

Definition 1.2.5. *[7] An embedded curve $\gamma : S^1 \to \mathbb{R}^3$ is* ***unknotted*** *if there is an orientation-preserving homeomorphism of $\mathbb{R}^3$ onto itself which maps γ onto a planar circle in $\mathbb{R}^3$, i.e. onto S^1; otherwise, γ is* ***knotted*** *or is a* ***knot****.*

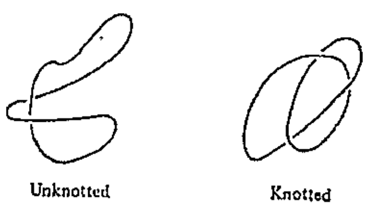

Figure 1.1

From [20] we have the following:

Lemma 1.2.6. *If the curve γ is knotted its total curvature is greater or equal than 4π. However, if the total curvature of γ is less than 4π then γ is an unknotted curve.*

Thus, we can prove the following proposition:

Proposition 1.2.7. *If the initial curve γ_0 has total curvature less than 4π and its evolution by curvature γ_t stays embedded then γ_t will be unknotted for every $t \in [0,\omega)$.*

Proof. We have that

$$\int_{\gamma_0} |k| ds < 4\pi.$$

Hence, as g_t is embedded for every $t \in [0,\omega)$, using the Theorem 1.2.3 we obtain

$$\int_{\gamma_t} |k| ds \leq \int_{\gamma_0} |k| ds < 4\pi.$$

Thus, for every $t \in [0,\omega)$ the total curvature of γ_t is less than 4π. Therefore, $\gamma(\cdot,t)$ is an unknotted curve for every $t \in [0,\omega)$. □

Another important result that we will use in this thesis is the well known *Gauss-Bonnet Theorem in* $\mathbb{R}^3$:

Theorem 1.2.8. *Suppose M is a compact two-dimensional surface in $\mathbb{R}^3$ with boundary ∂M. Let K be the Gaussian curvature of M, and let k_g be the geodesic curvature of ∂M. Then*

$$\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M), \tag{1.2.3}$$

where dA is the element of area of the surface, and ds is the line element along the boundary of M. Here $\chi(M)$ denote the Euler characteristic of M.

A proof of this theorem can be found in [18].

1.3 Singularities

Theorem 1.1.3 states that if the curvature is bounded then it is possible to extend the time of evolution. Thus, there exists a maximal time when the curvature becomes unbounded. Therefore, we say that the evolution of γ by curvature forms a singularity at time ω when the curvature $k(\cdot,\omega)$ "blows-up", i.e. when the curvature $k(\cdot,t)$ tends to infinity as $t \to \omega$.

Singularity formation can be classified according to the following definitions.

Definition 1.3.1.

- *The singularity formation is of Type I if* $\lim_{t\to\omega}(\sup k^2(\cdot,t))(\omega - t)$ *is bounded.*
- *The singularity formation is of Type II if* $\lim_{t\to\omega}(\sup k^2(\cdot,t))(\omega - t)$ *is unbounded.*

To study the formation of these singularities we define two types of sequences:

Definition 1.3.2.

- $\{(p_n, t_n)\} \in S^1 \times [0, \omega)$ *is a blow-up sequence if*

$$\lim_{n\to\infty} t_n = \omega \quad \textit{and} \quad \lim_{n\to\infty} k^2(p_n, t_n) = \infty;$$

- $\{(p_n, t_n)\} \in S^1 \times [0, \omega)$ *is an essential blow-up sequence if it is a blow-up sequence and there exists* $\rho \in \mathbb{R}^+$, *independent of* n, *such that*

$$\rho\,(\sup\, k^2(\cdot, t)) \leq k^2(p_n, t_n) \textit{ when } t \leq t_n.$$

In [12, Th.6.1] it was proved that if $\{(p_n, t_n)\}$ is an essential blow-up sequence, then the formation of singularities is a planar phenomenon in the following sense.

Theorem 1.3.3. *If* $\{(p_n, t_n)\}$ *is an essential blow-up sequence then*

$$\lim_{n\to\infty} \frac{\tau}{k}(p_n, t_n) = 0.$$

Analogously to the planar case, the curve can be rescaled in space and in time to obtain a limit curve with bounded curvature. According to [12] the limit of rescaled solutions can be classified in the following way:

- If γ forms a Type I singularity, then γ is asymptotic to a planar solution which is homothetically shrinking, i.e. it is a contracting self-similar solution. These planar solutions were studied and classified by Abresch-Langer [25] (see Fig.1.2).
- If γ forms a Type II singularity, then there exists an essential blow-up sequence $\{(p_n, t_n)\}$ such that a rescaling of γ converges along a subsequence of $\{(p_n, t_n)\}$ to a convex eternal solution γ_∞. Next, in [12] it was shown that a solution of this type is a graph that moves by translation and satisfies a parabolic differential equation. Such solution is known as the *Grim Reaper* (see Fig.1.3).

Definition 1.3.4. *([16]) The Grim Reaper is the planar curve defined by*

$$\gamma_\infty(x, t) = (x, -\ln(\cos(x))) + (0, t),$$

where $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *and it moves upwards with constant speed in time.*

Remark 1.3.5. *The Grim Reaper is a convex curve with bounded curvature and total curvature equal to* π. *Moreover,* γ_∞ *is a solution to the Curve Shortening Flow following* (1.1.4).

1.4 Minimal Surfaces

Minimal surfaces can be defined by several equivalent ways in $\mathbb{R}^3$. In this thesis we use the following:

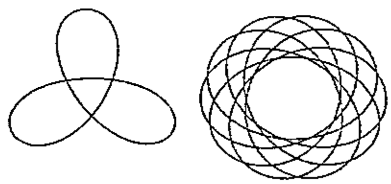

Figure 1.2: Abresch-Langer Curves

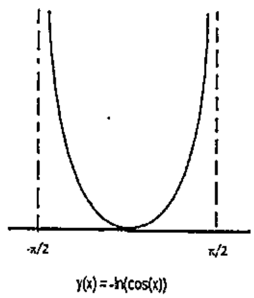

Figure 1.3: Grim Reaper

Definition 1.4.1. *A surface X is minimal if it satisfies $H = 0$ with H its mean curvature.*

Remark 1.4.2. *Given a minimal surface it is possible to find (locally) a conformal parametrization $X : B \to \mathbb{R}^3$ that satisfies*

$$\Delta X = \frac{\partial^2 X}{\partial u^2} + \frac{\partial^2 X}{\partial v^2} = 2H \cdot \nu = 0, \tag{1.4.1}$$

where $\nu := \frac{X_u \times X_v}{|X_u \times X_v|}$ denotes the normal vector to the tangent space and Δ is the Laplace-Beltrami operator of the surface. Thus, it is possible to choose a harmonic parametrization of a minimal surface.

Given an unknotted curve γ the problem of finding a minimal surface with boundary γ is known as the ***Plateau Problem***. In the second chapter, we will use a parametrization of a disc-type minimal surface with this boundary, given by

Theorem 1.4.3. *Consider the following system of non-linear partial differential equations:*

$$\begin{aligned} &\Delta X = 0 \ \text{ i.e. } X \textbf{ is harmonic;} \\ &|X_u|^2 - |X_v|^2 = X_u \cdot X_v = 0 \ \text{ i.e. } X \textbf{ is conformal;} \\ &X|_{\partial B} : \partial B \to \gamma \ \textbf{ is a parametrization of } \gamma. \end{aligned} \tag{1.4.2}$$

Then there exists $X \in H^{1,2}(B, \mathbb{R}^3) \cap C^0(\bar{B}, \mathbb{R}^3)$ such that it is a solution of (1.4.2) and defines a minimal surface with norm given by

$$\|X\|_1^2 = \int_B \left(|X|^2 + |\nabla X|^2\right) dw.$$

Here $H^{1,2}(B,\mathbb{R}^3)$ denotes to the Sobolev space of L^2-functions and $C^0(\bar{B},\mathbb{R}^3)$ denotes the space of continuous functions $X:\bar{B}\to\mathbb{R}^3$. A proof of this Theorem can be found in [14].

Remark 1.4.4. *We may choose X_u and X_v non identically zero. If one of them were identically zero, for example X_u, using the relations* (1.4.3) *we get that X_v is identically zero as well; this implies that X is a constant function but $X|_{S^1}=\gamma$ and γ is not constant.*

Now, we want to define the area associated to an unknotted and embedded curve γ. Using Theorem 1.4.3 we have that given a curve γ there exists a minimal surface with boundary γ. Moreover, by [21] we have that if the boundary is unknotted then there exists a ***unique*** minimal surface that satisfies (1.4.2).

Thus, the area associated to the curve will be defined as the area of that minimal surface. More specifically,

Definition 1.4.5. *Let γ be an unknotted space curve. If $X:B\to\mathbb{R}^3:(u,v)\to X(u,v)$ is a parametrization of the minimal surface with boundary γ, then its enclosed area is given by:*

$$A=\int_B\left|\frac{\partial X}{\partial u}(u,v)\times\frac{\partial X}{\partial v}(u,v)\right|dudv. \tag{1.4.3}$$

Remark 1.4.6. *Note that if we assume that γ_0 is unknotted, Proposition 1.2.7 implies that its evolution γ_t will be unknotted for every $t\in[0,\omega)$. Thus, from [21] we will have that there exists a unique associated minimal surface X_t. Therefore, for every t the area A_t will be well-defined.*

Furthermore, in [26] it was shown that every minimal surface with boundary satisfies the following ***isoperimetric inequality***.

Theorem 1.4.7. *Let $X\in C^2(B,\mathbb{R}^3)$ be a minimal surface, assume that X is of class $H^{1,2}(B,\mathbb{R}^3)$. If $L(X)$ denotes the length of the boundary of X and $A(X)$ denotes the area of X we have that if $L(X)<\infty$, then its area $A(X)<\infty$ and*

$$4\pi A(X)\leq L^2(X). \tag{1.4.4}$$

Equality is attained if and only if $X:B\to\mathbb{R}^3$ represents a (simply covered) disk.

Chapter 2

Type II Singularity

The aim of this chapter is to prove the main theorems of this thesis:

Theorem A *Suppose that $\gamma(\cdot,t)$ satisfies the equation (CSF) and the total curvature of $\gamma(\cdot,0)$ is less than 4π. If $\gamma(\cdot,t)$ remains embedded for all $t \in [0,\omega)$ with finite singular time ω then the isopermetric ratio defined by* (2.0.2) *is uniformly bounded for all $t \in [0,\omega)$. In particular, if $\gamma(\cdot,t)$ develops a Type I singularity then the isoperimetric ratio converges to 4π.*

Theorem B *Suppose that $\gamma(\cdot,t)$ satisfies the equation (CSF) and the total curvature of $\gamma(\cdot,0)$ is less than 4π. Let X_t be the minimal surface enclosed by $\gamma(\cdot,t)$. If its Gaussian curvature $K(\cdot,t)$ is uniformly bounded and $\gamma(\cdot,t)$ remains embedded for all $t \in [0,,\omega)$ with finite singular time ω and does not shrink to a point, then there is no formation of Type II singularities.*

The proof of these results relies on certain isoperimetric ratios.

We first note that the area of the surface $X : B \to \mathbb{R}^3$ is given by $A(X) = \int_X d\mu$, where μ is the canonical measure associated to the metric induced by the immersion X. From Definition 1.4.5 and Proposition 1.2.7 we get that for every t there exists an unique minimal surface $X(\cdot,t)$ with boundary $\gamma(\cdot,t)$. Thus, we can define for every $t \in [0,\omega)$ the area of such surface:

Definition 2.0.8. *Let $\gamma(\cdot,t)$ be an unknotted space curve that evolves by its curvature. If $X(\cdot,t) : B \to \mathbb{R}^3 : (u,v) \to X(u,v,t)$ is a parametrization of the minimal surface enclosed by $\gamma(\cdot,t)$, we can define its enclosed area by:*

$$A_t = \int_B \left| \frac{\partial X}{\partial u}(u,v,t) \times \frac{\partial X}{\partial v}(u,v,t) \right| dudv. \tag{2.0.1}$$

Remark 2.0.9. *Since the limit curve of the rescaled solution may not be closed, it is convenient to assume that each γ_n is defined on the real line as a periodic map. Thus, we assume that $\gamma(\cdot,t) : \left(\frac{-\pi}{2},\frac{\pi}{2}\right] \to \mathbb{R}^3$.*

Thus, we define

(i) Let $\gamma(\cdot,t)$ be a solution of (CSF) and $X(\cdot,t)$ its enclosed minimal surface. If L_t is the length of $\gamma(\cdot,t)$ and A_t is the area of $X(\cdot,t)$, we define the isoperimetric ratio

$$\mathcal{I}(t) := \frac{L_t^2}{A_t}. \tag{2.0.2}$$

Remark 2.0.10. *Note that this isoperimetric ratio generalizes the isoperimetric ratios employed by Hamilton [23] and Grayson [16].*

(ii) Let $\gamma(\cdot,t)$ be a solution of (CSF) and $X(\cdot,t)$ its enclosed minimal surface. Given two different points $\gamma(a,t)$ and $\gamma(b,t)$ such that

$$s(b) := \int_0^b ds(t) < \frac{L_t}{4} \quad \text{and} \quad -s(a) := \int_a^0 ds(t) < \frac{L_t}{4}, \tag{2.0.3}$$

we consider the piecewise geodesic $\Gamma_{ab}(\cdot,t)$ on X_t that minimizes the distance between $\gamma(a,t)$ and $\gamma(b,t)$. In particular,

$$\Gamma_{ab}(0,t) = \gamma(b,t), \quad \Gamma_{ab}(1,t) = \gamma(a,t).$$

We will denote the length of $\Gamma_{ab}(\cdot,t)$ as d_{ab}^t and the length of $\gamma(\cdot,t)$ between a and b as l_{ab}^t. We define the isoperimetric ratio

$$\mathcal{G}(a,b,t) := \frac{l_{ab}^t}{d_{ab}^t}. \tag{2.0.4}$$

Remark 2.0.11. *Note that $\mathcal{G}(a,b,t) \geq 1$ and it generalizes the isoperimetric ratio employed by Huisken [8]. Moreover, if the curve $\gamma(\cdot,t)$ is not embedded then this ratio would be unbounded.*

The rest of this chapter is organized as follows: In section 2.1 we will study the isoperimetric ratio defined by (2.0.2), we will compute its evolution in time and we will show that it is invariant under rescalings. These computations imply Theorem A which relates to a result of Grayson for planar curves [16]. In the second section we will study the isoperimetric ratio defined by (2.0.4), we will estimate its spatial variation, its evolution in time and we will show that it is invariant under rescalings as well. In section 2.3 we will use known results to ensure the convergence of a family of surfaces. The last section is devoted to prove Theorem B, using the properties of the isoperimetric ratios to discard the formation of Type II singularities.

2.1 Isoperimetric Inequality I

The aim of this section is to consider the isoperimetric ratio of $\gamma(\cdot,t)$ defined by equation (2.0.2) and to find bounds on this ratio through its evolution. Also we will compute the evolution by curvature of the length and area associated to the curve $\gamma(\cdot,t)$.

We start by computing the evolution of the length L_t. Following [13, Lem. 1.4] we obtain

Lemma 2.1.1. *If $\gamma(\cdot,t)$ evolves by curvature then the evolution of its length is given by:*

$$\frac{\partial}{\partial t}L_t = -\int_\gamma k^2 ds. \tag{2.1.1}$$

Proof. If w is a time independent parametrization of the circle, then the arc-length parameter is given by: $ds = \left|\frac{\partial\gamma}{\partial w}\right| dw$. Thus,

$$L_t = \int_\gamma ds(t) = \int_{S^1}\left|\frac{\partial\gamma_t}{\partial w}\right| dw \tag{2.1.2}$$

Therefore,

$$\begin{aligned}
\frac{\partial L_t}{\partial t} &= \int_{S^1}\frac{\partial}{\partial t}\left|\frac{\partial\gamma_t}{\partial w}\right| dw = \int_{S^1}\frac{\frac{\partial}{\partial t}\frac{\partial\gamma}{\partial w}\cdot\frac{\partial\gamma}{\partial w}}{\left|\frac{\partial\gamma}{\partial w}\right|}dw = \int_{S^1}\frac{\frac{\partial}{\partial w}\frac{\partial\gamma}{\partial t}\cdot\frac{\partial\gamma}{\partial w}}{\left|\frac{\partial\gamma}{\partial w}\right|}dw \\
&= \int_{S^1}\left(\frac{\partial k}{\partial s}N - k^2T + k\tau B\right)\cdot\frac{\partial\gamma}{\partial w}dw \\
&= \int_{S^1}\left(\frac{\partial k}{\partial s}N - k^2T + k\tau B\right)\cdot T\left|\frac{\partial\gamma}{\partial w}\right| dw \\
&= -\int_\gamma k^2 ds.
\end{aligned}$$

□

Let $X : B \to \mathbb{R}^3$ be a minimal surface and $\varepsilon > 0$. If $X(\cdot,t)$ is the minimal surface associated to the evolution $\gamma(\cdot,t)$ of the boundary γ, then $X(\cdot,t)$ is a variation of X, i.e. is a function $X : B\times(-\varepsilon,\varepsilon)\to\mathbb{R}^3$ such that $X(\cdot,0) = X$ and $X(\cdot,t)\in C^{1,\alpha}$.

If $Y = \frac{\partial X}{\partial t}$ is a vector field along the surface X, from [17, Chap. 1.2] we will compute the evolution of the area A_t.

Lemma 2.1.2. *If $\gamma(\cdot,t)$ evolves by curvature, then the evolution of its enclosed area is given by:*

$$\frac{\partial}{\partial t}A_t = -\int_\gamma k_g ds, \qquad \text{where } k_g \text{ is the geodesic curvature.} \tag{2.1.3}$$

Proof. We know that the area as function at time t is:

$$A_t = A(X_t) = \int_B\sqrt{\det(g_{ij})}dudv = \int_B\sqrt{\det\left(\left\langle\frac{\partial X_t}{\partial u},\frac{\partial X_t}{\partial v}\right\rangle\right)}dudv.$$

From [17] we know that if ν is the inward unit normal vector to the surface and Y^T is the tangent component of Y, then for any surface in motion we have

$$\frac{\partial}{\partial t}\sqrt{\det(g_{ij})} = \sqrt{\det(g_{ij})}\left(\operatorname{div}(Y^T) - H\langle Y,\nu\rangle\right). \tag{2.1.4}$$

Thus,

$$
\begin{aligned}
\frac{\partial}{\partial t}A_t &= \frac{\partial}{\partial t}\int_B d\mu_t \\
&= \int_B \left(\operatorname{div}(Y^T) - H\langle Y,\nu\rangle\right)\sqrt{\det(g_{ij})}dudv \\
&= \int_X \left(\operatorname{div}(Y^T) - H\langle Y,\nu\rangle\right)d\mu \\
&= \int_X \operatorname{div}(Y^T)d\mu, \text{ (since the surface is minimal, } H=0) \\
&= \int_\gamma Y^T\cdot\nu d\mu, \text{ (by the Divergence Theorem)} \\
&= \int_\gamma (kN)^T\cdot\nu ds = -\int_\gamma k_g ds.
\end{aligned}
$$

□

Finally, we can prove

Theorem 2.1.3. $\frac{L_t^2}{A_t}$ *is decreasing in time.*

Proof. We start by showing that $L_t^2 - 4\pi A_t$ is decreasing. Using equations (2.1.1) and (2.1.3):

$$
\begin{aligned}
\frac{\partial}{\partial t}(L_t^2 - 4\pi A_t) &= 2L_t\frac{\partial}{\partial t}L_t - 4\pi\frac{\partial}{\partial t}A_t \\
&= -2L_t\int_\gamma k^2 ds + 4\pi\int_\gamma k_g ds.
\end{aligned}
$$

Moreover, Hölder's inequality implies that:

$$
\left(\int_\gamma |k|ds\right)^2 \le L\int_\gamma k^2 ds, \tag{2.1.5}
$$

and since $k^2 = k_g^2 + k_N^2$ with k_N the normal curvature, we have that:

$$
|k_g| \le |k|. \tag{2.1.6}
$$

Thus,

$$
\begin{aligned}
\frac{\partial}{\partial t}(L_t^2 - 4\pi A_t) &\le -2\left(\int_\gamma |k|ds\right)^2 + 4\pi\int_\gamma |k|ds \\
&= -2\int_\gamma |k|ds\left(\int_\gamma |k|ds - 2\pi\right) \\
&= -2\int_\gamma kds\left(\int_\gamma kds - 2\pi\right) \\
&\le 0.
\end{aligned}
$$

The last step followed from Theorem 1.2.4.

Therefore,

$$2L_t\frac{\partial}{\partial t}L_t - 4\pi\frac{\partial}{\partial t}A_t \leq 0. \tag{2.1.7}$$

On the other hand, using inequality (2.1.7) and the isoperimetric inequality given by Theorem 1.4.7, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{L_t^2}{A_t}\right) &= \frac{2L_t}{A_t}\frac{\partial}{\partial t}L_t - \frac{L_t^2}{A_t^2}\frac{\partial}{\partial t}A_t \\ &\leq \frac{1}{A_t}\frac{\partial}{\partial t}(L_t^2 - 4\pi A_t) \\ &\leq 0.\end{aligned}$$

□

2.1.1 Rescaled Solution

The goal here is to define a rescaled solution and its associated minimal surface for γ along a blow-up sequence. We also observe that the isoperimetric ratio defined in this section is invariant by rescaling.

Definition 2.1.4. *A rescaled solution γ_n of γ along a blow-up sequence $\{(p_n, t_n)\} \in S^1 \times [0, \omega)$ is a curve $\gamma_n : S^1 \times [-\lambda_n^2 t_n, \lambda_n^2(\omega - t_n)) \to \mathbb{R}^3$ defined by:*

$$\gamma_n(\cdot, \bar{t}) = \lambda_n(O_n\gamma(\cdot, t) + B_n); \quad \bar{t} = \lambda_n^2(t - t_n),$$

where $\lambda_n \in \mathbb{R}^+, O_n \in SO(3), B_n \in \mathbb{R}^3$ are chosen so that γ_n is a solution of the Curve Shortening Flow and:

- $\gamma_n(p_n, 0) = 0 \in \mathbb{R}^3$;
- *the unit tangent vector* $T_n(p_n, 0) = (1, 0, 0)$;
- $k_n \cdot N_n(p_n, 0) = (0, 1, 0)$.

Remark 2.1.5. *Since we will use geometric quantities such as length and area, and we know which they are invariant under rotations or translations, we assume that $\gamma_n(\cdot, \bar{t}) = \lambda_n\gamma(\cdot, t)$, with $\bar{t} = \lambda_n^2(t - t_n)$, then we can compute*

$$\begin{aligned}\frac{\partial\gamma_n}{\partial\bar{t}} &= \lambda_n^{-2}\frac{\partial\gamma_n}{\partial t} = \lambda_n^{-1}\frac{\partial\gamma}{\partial t} = \frac{k}{\lambda_n}\cdot N; \\ \frac{\partial\gamma_n}{\partial s} &= \lambda_n\frac{\partial\gamma}{\partial s} = \lambda_n T; \\ ds_n(\bar{t}) &= \left\|\frac{\partial\gamma_n}{\partial s}\right\| ds = \lambda_n ds(t); \\ T_n(\cdot, \bar{t}) &= T(\cdot, \bar{t}); \\ \frac{\partial^2\gamma_n}{\partial s_n^2} &= \frac{k}{\lambda_n}\cdot N; \\ N_n(\cdot, \bar{t}) &= N(\cdot, \bar{t}).\end{aligned}$$

Therefore, the rescaled curve $\gamma_n(\cdot,t)$ evolves by its curvature and

$$k_n(\cdot,\bar{t}) = \frac{k(\cdot,t)}{\lambda_n}. \tag{2.1.8}$$

Remark 2.1.6. *In addition, by definition of rescaled solution we have that $k_n^2(p_n,0)=1$, then*

$$\lambda_n^2 = k^2(p_n,t_n). \tag{2.1.9}$$

Therefore, if $\{(p_n,t_n)\}_n$ is a blow-up sequence then λ_n^2 converges to infinity as n tends to infinity.

We define a minimal surface associated to the rescaled curve $\gamma_n(\cdot,\bar{t})$ by the parametrization:

$$X_{n,\bar{t}} : B \to \mathbb{R}^3 : (u,v) \to X_{n,\bar{t}}(u,v) = \lambda_n X_t(u,v). \tag{2.1.10}$$

Lemma 2.1.7. *The rescaled surface parameterized by $X_{n,\bar{t}}$, with boundary $\gamma_n(\cdot,\bar{t})$, is minimal.*

Proof. This is obtained directly from the equivalence with existence of harmonic coordinates:

(i) Since X_t is harmonic

$$\begin{aligned}
\Delta X_{n,\bar{t}} &= \frac{\partial^2 X_{n,\bar{t}}}{\partial u^2} + \frac{\partial^2 X_{n,\bar{t}}}{\partial v^2} = \frac{\partial}{\partial u}\left(\lambda_n \frac{\partial}{\partial u} X_t\right) + \frac{\partial}{\partial v}\left(\lambda_n \frac{\partial}{\partial v} X_t\right) \\
&= \lambda_n \frac{\partial^2}{\partial u^2} X_t + \lambda_n \frac{\partial^2}{\partial v^2} X_t = \lambda_n \left(\frac{\partial^2}{\partial u^2} X_t + \frac{\partial^2}{\partial v^2} X_t\right) \\
&= \lambda_n(\Delta X_t) \\
&= 0.
\end{aligned}$$

Therefore, $X_{n,\bar{t}}$ is harmonic.

(ii) Since X_t is conformal

$$\begin{aligned}
\left|\frac{\partial X_{n,\bar{t}}}{\partial u}\right|^2 - \left|\frac{\partial X_{n,\bar{t}}}{\partial v}\right|^2 &= \lambda_n^2 \left|\frac{\partial}{\partial u} X_t\right|^2 - \lambda_n^2 \left|\frac{\partial}{\partial v} X_t\right|^2 = 0, \\
\frac{\partial X_{n,\bar{t}}}{\partial u} \cdot \frac{\partial X_{n,\bar{t}}}{\partial v} &= \lambda_n^2 \left(\frac{\partial}{\partial u} X_t \cdot \frac{\partial}{\partial v} X_t\right) = 0.
\end{aligned}$$

Therefore, $X_{n,\bar{t}}$ is conformal.

(iii) $X_{n,\bar{t}}|_{\partial B} : \partial B \to \gamma_n(\cdot,\bar{t})$ parameterize $\gamma_n(\cdot,\bar{t})$.

$$X_{n,\bar{t}}|_{\partial B} = (\lambda_n X_t)|_{\partial B} = \lambda_n X_t|_{\partial B} = \lambda_n \gamma(\cdot,t) = \gamma_n(\cdot,\bar{t}).$$

□

In what follows, we denote by $A_{n,\bar{t}}$ as the area of $X_{n,\bar{t}}$.

Lemma 2.1.8. *Let $\gamma_n(\cdot,\bar{t})$ be a rescaled solution of $\gamma(\cdot,t)$ and $L_{n,\bar{t}}$, $A_{n,\bar{t}}$ its length and minimal enclosed area, respectively, then*

$$\frac{(L_{n,\bar{t}})^2}{A_{n,\bar{t}}} = \frac{(L_t)^2}{A_t} \quad \text{for all } n \in \mathbb{N}.$$

Proof. Since the length and enclosed area are invariant by rotation and translation, we assume that:

$$\gamma_n(\cdot,\bar{t}) = \lambda_n \gamma(\cdot,t), \qquad \bar{t} = \lambda_n^2(t - t_n).$$

Clearly,

$$L_{n,\bar{t}} = \int_{\gamma_n} ds_n(\bar{t}) = \int_{\gamma} \lambda_n ds(t) = \lambda_n L_t.$$

We know that the area of a minimal surface is $\text{Area}(X) = \int_B \left|\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}\right| dudv$. Then, if $A_{n,\bar{t}}$ is the area of minimal surface $X_{n,\bar{t}}$ (with boundary $\gamma_n(\cdot,\bar{t})$), we have:

$$A_{n,\bar{t}} = \int_B \lambda_n^2 \left|\frac{\partial X_t}{\partial u} \times \frac{\partial X_t}{\partial v}\right| dudv = \lambda_n^2 A_t.$$

Finally,

$$\frac{(L_{n,\bar{t}})^2}{A_{n,\bar{t}}} = \frac{\lambda_n^2 (L_t)^2}{\lambda_n^2 A_t} = \frac{(L_t)^2}{A_t} \quad \text{for all } n \in \mathbb{N}.$$

□

Following Theorem 2.1.3 and Lemma 2.1.8 we conclude

Theorem A. *Suppose that $\gamma(\cdot,t)$ satisfies the equation (CSF) and the total curvature of $\gamma(\cdot,0)$ is less than 4π. If $\gamma(\cdot,t)$ remains embedded for all $t \in [0,\omega)$ with finite singular time ω then the isopermetric ratio defined by (2.0.2) is uniformly bounded for all $t \in [0,\omega)$. In particular, if $\gamma(\cdot,t)$ develops a Type I singularity then the isoperimetric ratio converges to 4π.*

2.2 Isoperimetric Inequality II

In this section we will study the isoperimetric ratio defined by equation (2.0.4). The aim is to find bounds for this ratio through its evolution.

If ω is the singular time, suppose that for fixed $t_0 < \omega$ the maximum of $\mathcal{G}$ is attainted at (a, b, t_0).

Consider a_ε and b_ε, variations of a and b, respectively, such that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} s(a_\varepsilon) = \mathcal{A}, \quad \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} s(b_\varepsilon) = \mathcal{B} \quad \text{and} \quad \left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} s(a_\varepsilon) = \left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} s(b_\varepsilon) = 0.$$

Here s is the arc-length parameter of $\gamma(\cdot, t_0)$, i.e.

$$s(a_\varepsilon) := \int_0^{a_\varepsilon} ds(t_0) \quad \text{and} \quad s(a_\varepsilon) := \int_0^{b_\varepsilon} ds(t_0).$$

Let $\Gamma_\varepsilon(\cdot, t_0) : [0,1] \to X_{t_0}$ be a variation of $\Gamma_{ab}(\cdot, t_0)$ such that

$$\Gamma_\varepsilon(0, t_0) = \gamma(b_\varepsilon, t_0), \quad \Gamma_\varepsilon(1, t_0) = \gamma(a_\varepsilon, t_0) \quad \text{and} \quad \Gamma_0(x, t_0) = \Gamma_{ab}(x, t_0).$$

If $d_\varepsilon^{t_0}$ denotes the length of $\Gamma_\varepsilon(\cdot, t_0)$ and $l_\varepsilon^{t_0}$ denotes the length of $\gamma(\cdot, t_0)$ between a_ε and b_ε, then we define

$$\mathcal{G}_\varepsilon^{t_0} := \mathcal{G}(a_\varepsilon, b_\varepsilon, t_0) = \frac{l_\varepsilon^{t_0}}{d_\varepsilon^{t_0}}.$$

Remark 2.2.1. *Since $X(\cdot, t_0)$ is a minimal surface, we have that*

$$X_{uu}(\cdot, t_0) - X_{vv}(\cdot, t_0) = 0 \quad \text{and} \quad X_u(\cdot, t_0) \cdot X_v(\cdot, t_0) = 0 = |X_u(\cdot, t_0)|^2 - |X_v(\cdot, t_0)|^2,$$

where u, v are given by the isothermal parametrization of $X(\cdot, t_0)$.

This relations implies

$$\begin{aligned} X_u(\cdot, t_0) \cdot X_{uv}(\cdot, t_0) = & -X_{uu}(\cdot, t_0) \cdot X_v(\cdot, t_0) &= X_v(\cdot, t_0) \cdot X_{vv}(\cdot, t_0) \\ X_{uv}(\cdot, t_0) \cdot X_v(\cdot, t_0) = & -X_u(\cdot, t_0) \cdot X_{vv}(\cdot, t_0) &= X_u(\cdot, t_0) \cdot X_{uu}(\cdot, t_0). \end{aligned}$$

Moreover, if $T_\varepsilon(\cdot, t_0)$ is the tangent vector of $\Gamma_\varepsilon(\cdot, t_0)$ we have that

$$\begin{aligned} T_\varepsilon(\cdot, t_0) &= \frac{u_x}{\sqrt{u_x^2 + v_x^2}} \frac{X_u(\cdot, t_0)}{|X_u(\cdot, t_0)|} + \frac{v_x}{\sqrt{u_x^2 + v_x^2}} \frac{X_v(\cdot, t_0)}{|X_v(\cdot, t_0)|} \\ \frac{\partial}{\partial x} X_u(\cdot, t_0) &= u_x X_{uu}(\cdot, t_0) + v_x X_{uv}(\cdot, t_0) \\ \frac{\partial}{\partial x} X_v(\cdot, t_0) &= u_x X_{uv}(\cdot, t_0) + v_x X_{vv}(\cdot, t_0). \end{aligned}$$

We start by computing the first and second spatial variations of these lengths.

Proposition 2.2.2. *The first spatial variation of $d_{ab}^{t_0}$ is given by*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} d_\varepsilon^{t_0} = \mathcal{A}T_0(1, t_0) \cdot T_\gamma(a, t_0) - \mathcal{B}T_0(0, t_0) \cdot T_\gamma(b, t_0).$$

Proof. We have that $\Gamma_\varepsilon(x, t_0) = X(u(x, t_0, \varepsilon), v(x, t_0, \varepsilon), t_0)$ and $d_\varepsilon^{t_0} = \int_0^1 ds_\varepsilon(t_0)$.

Thus, following Remark 2.2.1 we compute

$$\begin{aligned}
\frac{d}{d\varepsilon}d_\varepsilon^{t_0} &= \frac{d}{d\varepsilon}\int_0^1 \left\|\frac{\partial}{\partial x}\Gamma_\varepsilon(\cdot,t_0)\right\| dx \\
&= \frac{d}{d\varepsilon}\int_0^1 |X_u|\sqrt{u_x^2+v_x^2}\,dx \\
&= \int_0^1 \frac{d}{d\varepsilon}|X_u|\sqrt{u_x^2+v_x^2} + |X_u|\frac{d}{d\varepsilon}\sqrt{u_x^2+v_x^2}\,dx \\
&= \int_0^1 \frac{\sqrt{u_x^2+v_x^2}}{|X_u|}\frac{dX_u}{d\varepsilon}\cdot X_u + |X_u|\frac{(u_x u_{x\varepsilon}+v_x v_{x\varepsilon})}{\sqrt{u_x^2+v_x^2}}dx \\
&= \int_0^1 \frac{u_x^2+v_x^2}{\sqrt{u_x^2+v_x^2}}(u_\varepsilon X_{uu}+v_\varepsilon X_{uv})\cdot\frac{X_u}{|X_u|} + T_\varepsilon(\cdot,t_0)\cdot(u_{x\varepsilon}X_u+v_{x\varepsilon}X_v)\,dx \\
&= \int_0^1 T_\varepsilon(\cdot,t_0)\cdot\left(u_\varepsilon\frac{\partial X_u}{\partial x}+v_\varepsilon\frac{\partial X_v}{\partial x}\right) + T_\varepsilon(\cdot,t_0)\cdot(u_{x\varepsilon}X_u+v_{x\varepsilon}X_v)\,dx \\
&= \int_0^1 T_\varepsilon(\cdot,t_0)\cdot\frac{\partial}{\partial x}(u_\varepsilon X_u+v_\varepsilon X_v)\,dx \\
&= \int_0^1 T_\varepsilon(\cdot,t_0)\cdot\frac{\partial}{\partial x}\frac{d}{d\varepsilon}\Gamma_\varepsilon(\cdot,t_0)dx \\
&= T_\varepsilon(\cdot,t_0)\cdot\frac{d}{d\varepsilon}\Gamma_\varepsilon(\cdot,t_0)\Big|_0^1 - \int_0^1 \frac{\partial}{\partial x}T_\varepsilon(\cdot,t_0)\cdot\frac{d}{d\varepsilon}\Gamma_\varepsilon(\cdot,t_0)dx \\
&= T_\varepsilon(\cdot,t_0)\cdot\frac{d}{d\varepsilon}\Gamma_\varepsilon(\cdot,t_0)\Big|_0^1 \quad (\text{ since } \Gamma_\varepsilon \text{ is a piecewise geodesic}).
\end{aligned}$$

Since $\Gamma_\varepsilon(\cdot,t_0)$ is a piecewise geodesic, we obtain

$$\frac{d}{d\varepsilon}d_\varepsilon^{t_0} = \left(\frac{d}{d\varepsilon}s(a_\varepsilon)\right)T_\varepsilon(1,t_0)\cdot T_\gamma(a_\varepsilon,t_0) - \left(\frac{d}{d\varepsilon}s(b_\varepsilon)\right)T_\varepsilon(0,t_0)\cdot T_\gamma(b_\varepsilon,t_0). \quad (2.2.1)$$

Thus,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} d_\varepsilon^{t_0} = \mathcal{A}T_0(1,t_0)\cdot T_\gamma(a,t_0) - \mathcal{B}T_0(0,t_0)\cdot T_\gamma(b,t_0).$$

□

Note that $T_0(1,t_0)$ is the unit tangent vector to $\Gamma(\cdot,t_0)$ at $\gamma(b,t_0)$, $T_0(0,t_0)$ is the unit tangent vector to $\Gamma(\cdot,t_0)$ at $\gamma(a,t_0)$, $T_\gamma(b,t_0)$ is the unit tangent vector to $\gamma(\cdot,t_0)$ at b and $T_\gamma(a,t_0)$ is the unit tangent vector to $\gamma(\cdot,t_0)$ at a.

Proposition 2.2.3. *The first spatial variation of $l_{ab}^{t_0}$ is given by*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} l_\varepsilon^{t_0} = \mathcal{B}-\mathcal{A}.$$

Proof. We compute

$$\frac{d}{d\varepsilon}l_\varepsilon^{t_0} = \frac{d}{d\varepsilon}\int_{a_\varepsilon}^{b_\varepsilon} ds(t_0) = \frac{d}{d\varepsilon}\left(s(b_\varepsilon) - s(a_\varepsilon)\right) = \frac{d}{d\varepsilon}s(b_\varepsilon) - \frac{d}{d\varepsilon}s(a_\varepsilon). \qquad (2.2.2)$$

Thus,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} l_\varepsilon^{t_0} = \mathcal{B} - \mathcal{A}.$$

□

Lemma 2.2.4. *Let a, b be a sin (2.0.3). Then, the tangent vectors $T_\gamma(\cdot, t_0)$ and $T_0(\cdot, t_0)$ satisfy*

$$T_0(1, t_0) \cdot T_\gamma(a, t_0) = T_0(0, t_0) \cdot T_\gamma(b, t_0) = -\frac{d_0^{t_0}}{l_0^{t_0}}.$$

Proof. The first variation of this isoperimetric ratio is given by

$$\frac{d}{d\varepsilon}\left(\frac{l_\varepsilon^{t_0}}{d_\varepsilon^{t_0}}\right) = \frac{1}{d_\varepsilon^{t_0}}\frac{d}{d\varepsilon}l_\varepsilon^{t_0} - \frac{l_\varepsilon^{t_0}}{\left(d_\varepsilon^{t_0}\right)^2}\frac{d}{d\varepsilon}d_\varepsilon^{t_0}. \qquad (2.2.3)$$

Since the maximum of $\mathcal{G}_\varepsilon^{t_0}$ is attained at $\varepsilon = 0$, then

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{G}_\varepsilon^{t_0} = \frac{1}{d_0^{t_0}}\left(\mathcal{B} - \mathcal{A}\right) - \frac{l_0^{t_0}}{\left(d_0^{t_0}\right)^2}\left(\mathcal{A}T_0(1, t_0) \cdot T_\gamma(a, t_0) - \mathcal{B}T_0(0, t_0) \cdot T_\gamma(b, t_0)\right).$$

Thus, if we make $\mathcal{B} = \mathcal{A} = 1$, we obtain

$$T_0(1, t_0) \cdot T_\gamma(a, t_0) = T_0(0, t_0) \cdot T_\gamma(b, t_0),$$

and if we set $\mathcal{B} = 1,\ \mathcal{A} = 0$, we obtain

$$T_0(0, t_0) \cdot T_\gamma(b, t_0) = -\frac{d_0^{t_0}}{l_0^{t_0}}.$$

□

From (2.2.1) we obtain that the second spatial variation of $d_\varepsilon^{t_0}$ is given by

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}d_\varepsilon^{t_0} &= \left(\frac{d^2}{d\varepsilon^2}s(a_\varepsilon)\right)T_\varepsilon(1,t_0)\cdot T_\gamma(a_\varepsilon,t_0)+\left(\frac{d}{d\varepsilon}s(a_\varepsilon)\right)\left(\frac{d}{d\varepsilon}T_\varepsilon(1,t_0)\right)\cdot T_\gamma(a_\varepsilon,t_0)\\ &+\left(\frac{d}{d\varepsilon}s(a_\varepsilon)\right)T_\varepsilon(1,t_0)\cdot\left(\frac{d}{d\varepsilon}T_\gamma(a_\varepsilon,t_0)\right)-\left(\frac{d^2}{d\varepsilon^2}s(b_\varepsilon)\right)T_\varepsilon(0,t_0)\cdot T_\gamma(b_\varepsilon,t_0)\\ &-\left(\frac{d}{d\varepsilon}s(b_\varepsilon)\right)\left(\frac{d}{d\varepsilon}T_\varepsilon(0,t_0)\right)\cdot T_\gamma(b_\varepsilon,t_0)-\left(\frac{d}{d\varepsilon}s(b_\varepsilon)\right)T_\varepsilon(0,t_0)\cdot\left(\frac{d}{d\varepsilon}T_\gamma(b_\varepsilon,t_0)\right)\\ &= \left(\frac{d^2}{d\varepsilon^2}s(a_\varepsilon)\right)T_\varepsilon(1,t_0)\cdot T_\gamma(a_\varepsilon,t_0)+\left(\frac{d}{d\varepsilon}s(a_\varepsilon)\right)\left(\frac{d}{d\varepsilon}T_\varepsilon(1,t_0)\right)\cdot T_\gamma(a_\varepsilon,t_0)\\ &-\left(\frac{d^2}{d\varepsilon^2}s(b_\varepsilon)\right)T_\varepsilon(0,t_0)\cdot T_\gamma(b_\varepsilon,t_0)-\left(\frac{d}{d\varepsilon}s(b_\varepsilon)\right)\left(\frac{d}{d\varepsilon}T_\varepsilon(0,t_0)\right)\cdot T_\gamma(b_\varepsilon,t_0)\\ &+\left(\frac{d}{d\varepsilon}s(a_\varepsilon)\right)^2k_\gamma(a_\varepsilon,t_0)T_\varepsilon(1,t_0)\cdot N_\gamma(a_\varepsilon,t_0)\\ &-\left(\frac{d}{d\varepsilon}s(b_\varepsilon)\right)^2k_\gamma(b_\varepsilon,t_0)T_\varepsilon(0,t_0)\cdot N_\gamma(b_\varepsilon,t_0).\end{aligned}$$

Therefore,

$$\begin{aligned}\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0}d_\varepsilon^{t_0} =&\mathcal{A}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}T_\varepsilon(1,t_0)\right)\cdot T_\gamma(a,t_0)-\mathcal{B}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}T_\varepsilon(0,t_0)\right)\cdot T_\gamma(b,t_0)\\ &+\mathcal{A}^2k_\gamma(a,t_0)T_0(1,t_0)\cdot N_\gamma(a,t_0)-\mathcal{B}^2k_\gamma(b,t_0)T_0(0,t_0)\cdot N_\gamma(b,t_0).\end{aligned}$$

On the other hand, following (2.2.2) we obtain

$$\frac{d^2}{d\varepsilon^2}l_\varepsilon^{t_0}=\frac{d^2}{d\varepsilon^2}s(b_\varepsilon)-\frac{d^2}{d\varepsilon^2}s(a_\varepsilon).$$

Thus,

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0}l_\varepsilon^{t_0}=0.$$

Lemma 2.2.5. *The curvature of $\gamma(\cdot,t_0)$, the tangent vector of $\Gamma_{ab}(\cdot,t_0)$ and the normal vector of $\gamma(\cdot,t_0)$ satisfy*

$$k_\gamma(a,t_0)T_0(1,t_0)\cdot N_\gamma(a,t_0)-k_\gamma(b,t_0)T_0(0,t_0)\cdot N_\gamma(b,t_0)\geq 0.$$

Proof. The second variation of the isoperimetric ratio is given by

$$\frac{d^2}{d\varepsilon^2}\left(\frac{l_\varepsilon^{t_0}}{d_\varepsilon^{t_0}}\right) = \frac{1}{d_\varepsilon^{t_0}}\frac{d^2}{d\varepsilon^2}l_\varepsilon^{t_0}-\frac{2}{\left(d_\varepsilon^{t_0}\right)^2}\frac{d}{d\varepsilon}d_\varepsilon^{t_0}\frac{d}{d\varepsilon}l_\varepsilon^{t_0}-\frac{l_\varepsilon^{t_0}}{\left(d_\varepsilon^{t_0}\right)^2}\frac{d^2}{d\varepsilon^2}d_\varepsilon^{t_0}+\frac{2l_\varepsilon^{t_0}}{\left(d_\varepsilon^{t_0}\right)^3}\left(\frac{d}{d\varepsilon}d_\varepsilon^{t_0}\right)^2.$$

Since the maximum of $\mathcal{G}_\varepsilon^{t_0}$ is attained at $\varepsilon = 0$, then

$$\begin{aligned}
0 &\geq \left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \mathcal{G}_\varepsilon^{t_0} \\
&= -\frac{2}{\left(d_0^{t_0}\right)^2}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} d_\varepsilon^{t_0}\right)\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} l_\varepsilon^{t_0}\right) - \frac{l_0^{t_0}}{\left(d_0^{t_0}\right)^2}\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} d_\varepsilon^{t_0} + \frac{2l_0^{t_0}}{\left(d_0^{t_0}\right)^3}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} d_\varepsilon^{t_0}\right)^2 \\
&= -\frac{1}{\left(d_0^{t_0}\right)^2}\Bigg[2\left(\mathcal{B}-\mathcal{A}\right)\left(\mathcal{A}T_0(1,t_0)\cdot T_\gamma(a,t_0) - \mathcal{B}T_0(0,t_0)\cdot T_\gamma(b,t_0)\right) \\
&\qquad + l_0^{t_0}\mathcal{A}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} T_\varepsilon(1,t_0)\right)\cdot T_\gamma(a,t_0) - l_0^{t_0}\mathcal{B}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} T_\varepsilon(0,t_0)\right)\cdot T_\gamma(b,t_0) \\
&\qquad + l_0^{t_0}\mathcal{A}^2 k_\gamma(a,t_0)T_0(1,t_0)\cdot N_\gamma(a,t_0) - l_0^{t_0}\mathcal{B}^2 k_\gamma(b,t_0)T_0(0,t_0)\cdot N_\gamma(b,t_0) \\
&\qquad - \frac{2l_0^{t_0}}{d_0^{t_0}}\left(\mathcal{A}T_0(1,t_0)\cdot T_\gamma(a,t_0) - \mathcal{B}T_0(0,t_0)\cdot T_\gamma(b,t_0)\right)^2\Bigg].
\end{aligned}$$

Thus, if we make $\mathcal{A} = 1$, $\mathcal{B} = -1$, we obtain

$$\begin{aligned}
0 &\geq -\frac{1}{\left(d_0^{t_0}\right)^2}\Bigg[-4\left(T_0(1,t_0)\cdot T_\gamma(a,t_0) + T_0(0,t_0)\cdot T_\gamma(b,t_0)\right) \\
&\qquad + l_0^{t_0}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} T_\varepsilon(1,t_0)\right)\cdot T_\gamma(a,t_0) + l_0^{t_0}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} T_\varepsilon(0,t_0)\right)\cdot T_\gamma(b,t_0) \\
&\qquad + l_0^{t_0} k_\gamma(a,t_0)T_0(1,t_0)\cdot N_\gamma(a,t_0) - l_0^{t_0} k_\gamma(b,t_0)T_0(0,t_0)\cdot N_\gamma(b,t_0) \\
&\qquad - \frac{2l_0^{t_0}}{d_0^{t_0}}\left(T_0(1,t_0)\cdot T_\gamma(a,t_0) + T_0(0,t_0)\cdot T_\gamma(b,t_0)\right)^2\Bigg].
\end{aligned}$$

Using Lemma 2.2.4, we get

$$\begin{aligned}
k_\gamma(a,t_0)T_0(1,t_0)\cdot N_\gamma(a,t_0) - k_\gamma(b,t_0)T_0(0,t_0)\cdot N_\gamma(b,t_0) \geq &-\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} T_\varepsilon(1,t_0)\cdot T_\gamma(a,t_0) \\
&-\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} T_\varepsilon(0,t_0)\cdot T_\gamma(b,t_0).
\end{aligned} \tag{2.2.4}$$

On the other hand, if we set $\mathcal{A} = -1$, $\mathcal{B} = 1$, we obtain

$$\begin{aligned}
0 &\geq -\frac{1}{\left(d_0^{t_0}\right)^2}\Bigg[4\left(-T_0(1,t_0)\cdot T_\gamma(a,t_0) - T_0(0,t_0)\cdot T_\gamma(b,t_0)\right) \\
&\qquad - l_0^{t_0}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} T_\varepsilon(1,t_0)\right)\cdot T_\gamma(a,t_0) - l_0^{t_0}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} T_\varepsilon(0,t_0)\right)\cdot T_\gamma(b,t_0) \\
&\qquad + l_0^{t_0} k_\gamma(a,t_0)T_0(1,t_0)\cdot N_\gamma(a,t_0) - l_0^{t_0} k_\gamma(b,t_0)T_0(0,t_0)\cdot N_\gamma(b,t_0) \\
&\qquad - \frac{2l_0^{t_0}}{d_0^{t_0}}\left(-T_0(1,t_0)\cdot T_\gamma(a,t_0) - T_0(0,t_0)\cdot T_\gamma(b,t_0)\right)^2\Bigg].
\end{aligned}$$

Using Lemma 2.2.4, we get

$$k_\gamma(a,t_0)T_0(1,t_0)\cdot N_\gamma(a,t_0) - k_\gamma(b,t_0)T_0(0,t_0)\cdot N_\gamma(b,t_0) \geq \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon(1,t_0)\cdot T_\gamma(a,t_0) + \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon(0,t_0)\cdot T_\gamma(b,t_0). \quad (2.2.5)$$

Therefore, if we add (2.2.4) and (2.2.5) we conclude

$$k_\gamma(a,t_0)T_0(1,t_0)\cdot N_\gamma(a,t_0) - k_\gamma(b,t_0)T_0(0,t_0)\cdot N_\gamma(b,t_0) \geq 0. \quad (2.2.6)$$

□

The evolution in time of the isoperimetric ratio $\mathcal{G}(a;b,t)$ is computed below.

Following the first variation of $d_\varepsilon^{t_0}$ and that γ evolves by its curvature, we compute the evolution of d_0^t

Lemma 2.2.6. *If $X(\cdot,t)$ is the parametrization of the minimal surface with boundary $\gamma(\cdot,t)$, not necessarily isothermal, then the evolution in time of d_0^t is given by*

$$\frac{d}{dt}\Big|_{t=t_0} d_0^t = k_\gamma(a,t_0)T_0(1,t_0)\cdot N_\gamma(a,t_0) - k_\gamma(b,t_0)T_0(0,t_0)\cdot N_\gamma(b,t_0) - G_1.$$

Here G_1 is a function that depends of the evolution of X and it is given by

$$G_1 = \int_0^1 T_0\cdot\frac{\partial}{\partial x}\frac{\partial}{\partial t}X(\cdot,t)dx.$$

Proof. We know that

$$d_0^t = \int_0^1 \left\|\frac{\partial}{\partial x}\Gamma_0\right\| dx = \int_0^1 \sqrt{|X_u|^2u_x^2 + 2u_xv_xX_u\cdot X_v + |X_v|^2v_x^2}\, dx.$$

Thus,

$$\begin{aligned}
\frac{\partial}{\partial t}d_0^t &= \int_0^1 \frac{\partial}{\partial t}\sqrt{|X_u|^2u_x^2 + 2u_xv_xX_u\cdot X_v + |X_v|^2v_x^2}\, dx \\
&= \frac{1}{\|\frac{\partial}{\partial x}\Gamma_0\|}\Big[|X_u|u_x^2\frac{\partial}{\partial t}|X_u| + |X_u|^2u_xu_{xt} + u_{xt}v_xX_u\cdot X_v + u_xv_{xt}X_u\cdot X_v + u_xv_xX_{ut}\cdot X_v \\
&\quad + u_xv_xX_u\cdot X_{vt} + |X_v|^2v_xv_{xt} + |X_v|v_x^2\frac{\partial}{\partial t}|X_v|\Big]\, dx.
\end{aligned}$$

On the other hand, note that

$$\begin{aligned}
\frac{\partial}{\partial x}\Gamma_0 &= X_uu_x + X_vv_x; \\
T_0 &= \frac{1}{\sqrt{|X_u|^2u_x^2 + 2u_xv_xX_u\cdot X_v + |X_v|^2v_x^2}}\left(X_uu_x + X_vv_x\right); \\
\frac{\partial}{\partial t}\frac{\partial}{\partial x}\Gamma_0 &= u_xX_{ut} + u_{xt}X_u + v_xX_{vt} + v_{xt}X_v.
\end{aligned}$$

It is easy to see that

$$\frac{\partial}{\partial t} d_0^t = \int_0^1 T_0 \cdot \frac{\partial}{\partial t}\frac{\partial \Gamma_0}{\partial x} dx.$$

In addition, note that

$$\frac{\partial}{\partial t}\frac{\partial \Gamma_0}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \Gamma_0}{\partial t} - \frac{\partial}{\partial x}\frac{\partial X}{\partial t}.$$

Thus, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} d_0^t &= \int_0^1 T_0 \cdot \frac{\partial}{\partial x}\frac{\partial \Gamma_0}{\partial t} dx - \int_0^1 T_0 \cdot \frac{\partial}{\partial x}\frac{\partial X}{\partial t} dx \\
&= T_0 \cdot \frac{\partial \Gamma_0}{\partial t}\Big|_0^1 - \int_0^1 \frac{\partial T_0}{\partial x} \cdot \frac{\partial \Gamma_0}{\partial t} dx - \int_0^1 T_0 \cdot \frac{\partial}{\partial x}\frac{\partial X}{\partial t} dx \\
&= k_\gamma(a,t) T_0(1,t) \cdot N_\gamma(a,t) - k_\gamma(b,t) T_0(0,t) \cdot N_\gamma(b,t) - \int_0^1 T_0 \cdot \frac{\partial}{\partial x}\frac{\partial X}{\partial t} dx.
\end{aligned}$$

□

Remark 2.2.7. *Note that if we assume that $|K(\cdot,t)|$ is uniformly bounded on time implies that*

$$G_1 = \int_0^1 T_0 \cdot \frac{\partial}{\partial x}\frac{\partial}{\partial t} X(\cdot,t) dx \leq d_0^t \cdot \left(\sup_{t \in [0,\omega)} |K(\cdot,t)| \right). \qquad (2.2.7)$$

For further details, we refer the reader to Appendix A.

On the other hand, we know that

$$\frac{d}{dt}\Big|_{t=t_0} l_0^t = -\int_a^b k_\gamma^2(\cdot,t_0) ds(t_0).$$

Therefore, the evolution of the isoperimetric ratio $\mathcal{G}(a,b,t)$ is given by

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=t_0} \left(\frac{l_0^t}{d_0^t} \right) &= -\frac{1}{d_0^t} \int_a^b k_\gamma^2(\cdot,t_0) ds(t_0) \\
&\quad - \frac{l_0^t}{(d_0^t)^2} \left(k_\gamma(a,t_0) T_0(1,t_0) \cdot N_\gamma(a,t_0) - k_\gamma(b,t_0) T_0(0,t_0) \cdot N_\gamma(b,t_0) - G_1 \right).
\end{aligned}$$

Following Lemma 2.2.5 we obtain

$$\frac{d}{dt}\Big|_{t=t_0} \left(\frac{l_0^t}{d_0^t} \right) \leq \frac{l_0^t}{(d_0^t)^2} G_1. \qquad (2.2.8)$$

Using Remark 2.2.7, we conclude

Lemma 2.2.8. *The isoperimetric ratio defined by* (2.0.4) *satisfies*

$$\frac{l_{ab}^t}{d_{ab}^t} \leq \left(\frac{l_{ab}^0}{d_{ab}^0}\right) \exp\left(Ct\right), \quad C = \sup_{t\in[0,\omega)} |K(\cdot,t)|.$$

Remark 2.2.9. *Note that the assumption on* $|K(\cdot,t)|$ *is related with embeddedness of the minimal surface.*

Using this result, we can prove the following proposition

Proposition 2.2.10. *The isoperimetric ratio* $\mathcal{G}(a,b,t)$ *is bounded for every* a, b *as in* (2.0.3) *and* $t \in [0,\omega)$.

Proof. Using Lemma 2.2.8, since t is finite, only we need to prove that $\mathcal{G}(a,b,0)$ is finite.

Suppose that there exist two points $\gamma_0(a)$ and $\gamma_0(b)$ such that $\mathcal{G}(a,b,0)$ is infinity.

Since l_{ab}^0 is finite for every points, then $d_{ab}^0 = 0$. However, the embeddedness of X_0 implies that $d_{ab}^0 = 0$ if and only if $l_{ab}^0 = 0$, that means which $\gamma_0(a)$ and $\gamma_0(b)$ tend to the same point $\gamma_0(P)$.

Note that this point P satisfies (2.0.3), w.l.o.g we will assume that $P = 0$.

If $P = 0$, using Mean Value Theorem, we have that there exist $\xi, \xi' \in (a,b)$ such that

$$\frac{l_{ab}^0}{d_{ab}^0} \leq \frac{l_{ab}^0}{\|\gamma_0(b)-\gamma_0(a)\|} = \frac{\|\gamma_0'(\xi)\||b-a|}{\|\gamma_0'(\xi')\||b-a|} = \frac{\|\gamma_0'(\xi)\|}{\|\gamma_0'(\xi')\|}.$$

Since γ_0 is embedded, a, b tend to 0 implies that ξ, ξ' tend to the same point.

Therefore, $\mathcal{G}(a,b,0)$ tends to 1 if $\gamma_0(a)$ and $\gamma_0(b)$ tend to the same point $\gamma_0(0)$.

Note that if the maximum of $\mathcal{G}(a,b,0)$ is attained at the endpoints in the condition (2.0.3) then the isoperimetric ratio $\mathcal{G}(a,b,0)$ cannot approach infinity because the curve $\gamma(\cdot,0)$ is embedded up to the singular time. Thus, that bound of the this ratio cannot be explicitly computed in terms of the initial data. □

Now, we prove properties of the isoperimetric ratio defined by (2.0.4) if the curves $\gamma(\cdot,t)$ and $\Gamma_{ab}(\cdot,t)$ are rescaled along a blow-up sequence (Definition 2.1.4).

Lemma 2.2.11. *The isoperimetric ratio defined by* (2.0.4) *is invariant under this rescaling.*

Proof. Since $\Gamma_{ab}(\cdot,t)$ lies on X_t, following Section 2.1.1, if $X_{n,\bar{t}}$ is the minimal surface associated to the rescaled curve $\gamma_n(\cdot,\bar{t})$ then we can consider $(\Gamma_{ab})_n(\cdot,\bar{t}) = \lambda_n\Gamma_{ab}(\cdot,t)$ as the rescaling of $\Gamma_{ab}(\cdot,t)$.

Therefore, it is easy to see that

$$(\mathcal{G}(a,b,\bar{t}))_n = \frac{\left(l_{ab}^{\bar{t}}\right)_n}{\left(d_{ab}^{\bar{t}}\right)_n} = \frac{l_{ab}^t}{d_{ab}^t} = \mathcal{G}(a,b,\bar{t}),$$

where $\left(l_{ab}^{\bar{t}}\right)_n$ is the length of the rescaled curve $\gamma_n(\cdot,\bar{t})$ and $\left(d_{ab}^{\bar{t}}\right)_n$ is the length of the rescaled curve $(\Gamma_{ab})n(\cdot,\bar{t})$. □

Proposition 2.2.12. *If the Gaussian curvature $K(\cdot, t)$ of $X(\cdot, t)$ is uniformly bounded in time then the curvature of $(\Gamma_{ab}(\cdot, \bar{t}))_n$ tends to zero when n tends to infinity.*

Proof. If $(k_{ab})_n(\cdot, \bar{t})$ denotes the curvature of $(\Gamma_{ab}(\cdot, \bar{t}))_n$, then we have

$$(k_{ab})_n(\cdot, \bar{t}) = \frac{k_{ab}(\cdot, t)}{\lambda_n},$$

where $k_{ab}(\cdot, t)$ is the curvature of $\Gamma_{ab}(\cdot, t)$ and $\lambda_n = k_\gamma^2(p_n, t_n)$.

On the other hand, as $\Gamma_{ab}(\cdot, t)$ is a piecewise geodesic then $(k_{ab}(\cdot, t))^2$ is equal to its normal curvature $((k_{ab})_N(\cdot, t))^2$.

Following Euler's Theorem we know that there exists $\theta \in [0, 2\pi)$ such that

$$k_N = k_1 \cos^2(\theta) + k_2 \sin^2(\theta),$$

where k_1 and k_2 are the principal curvatures.

Moreover, as $X(\cdot, t)$ is a minimal surface for every t then we have that $k_1 = -k_2$ and its Gaussian Curvature is given by $K(\cdot, t) = -k_1^2(\cdot, t)$.

Thus, we obtain

$$k_N^2(\cdot, t) \leq k_1^2(\cdot, t) = |K(\cdot, t)|.$$

Therefore, as $|K(\cdot, t)|$ is uniformly bounded, we conclude

$$((k_{ab})_n(\cdot, \bar{t}))^2 = \frac{((k_{ab})_N(\cdot, t))^2}{\lambda_n^2} \leq \frac{|K(\cdot, t)|}{\lambda_n^2} \underset{n \to \infty}{\longrightarrow} 0.$$

□

2.3 Convergence of Minimal Surfaces

This section will be essential to prove that a sequence of minimal surfaces with total boundary curvature at most 4π and evolving according (CSF), converges to a planar surface. Recalling a result of Courant about the convergence of the areas of these surfaces.

From [4, Th.25] we obtain:

Theorem 2.3.1. *Suppose that $\{M_n\} \subset \mathbb{R}^3$ are orientable, embedded, simply connected, minimal surfaces and that $\int_{M_n} |K_n| < \lambda < \infty$. Then (after passing to a subsequence) $\{M_n\}$ converges smoothly in $\mathbb{R}^3$ to a limit minimal surface M.*

Remark 2.3.2. *Note that this convergence is considered in the Sobolev space defined in Theorem 1.4.3.*

Proposition 2.3.3. *Let X_t be a minimal surface where its boundary $\gamma(\cdot, t)$ has total curvature less than 4π, then $\int_{X_t} |K(\cdot, t)| dA < 2\pi$ for all $t \in [0, \omega)$, where $K(\cdot, t)$ is the Gaussian curvature of the surface X_t.*

Proof. Since X_t is a minimal surface, its Gaussian Curvature $K(\cdot,t)$ will be non-positive. Using the Gauss-Bonnet Theorem we get

$$\int_{X_t} |K(\cdot,t)|dA = -\int_{X_t} K(\cdot,t)dA = \int_{\gamma(\cdot,t)} k_g(\cdot,t)ds - 2\pi \leq \int_{\gamma(\cdot,t)} |k_g(\cdot,t)|ds - 2\pi,$$

where $k_g(\cdot,t)$ is the geodesic curvature of $\gamma(\cdot,t)$.

On the other hand, if $k(\cdot,t)$ is the curvature, $k^2(\cdot,t) = k_N^2(\cdot,t) + k_g^2(\cdot,t)$, where $k_N(\cdot,t)$ is the normal curvature and $k_g(\cdot,t)$ is the geodesic curvature. Thus, $|k_g(\cdot,t)| \leq |k(\cdot,t)|$.

In addition, we have $\int_{\gamma(\cdot,t)} |k(\cdot,t)|ds < 4\pi$ for all $t \in [0,\omega)$. □

Theorem 2.3.4. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of the associated minimal surfaces of the rescaled solution $\gamma_n(\cdot,0)$ defined previously. Then, that sequence converges smoothly to a minimal surface. Moreover, this limit surface is a planar surface.*

Proof. Firstly, it is easy to compute that $|K_n(\cdot,0)| = \lambda_n^{-2}|K(\cdot,t_n)|$. Then, using Proposition 2.3.3 we get

$$\int_{X_n} |K_n(\cdot,0)|dA_n = \int_{X_{t_n}} |K(\cdot,t_n)|\, dA_{t_n} < 2\pi. \tag{2.3.1}$$

Moreover, $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of minimal surfaces with total Gaussian curvature less than 2π, as the boundaries have total curvatures less than 4π. Using Theorem 1.4.3 we obtain that the minimal surfaces $\{X_n\}$ are of disk-type, i.e. embedded and simply connected. Therefore, using Theorem 2.3.1 we get that the sequence $\{X_n\}_{n\in\mathbb{N}}$ converges to a minimal surface.

On the other hand, from equation (2.3.1) we can conclude that $K_n(\cdot,0) \to 0$ a.e. when $n \to \infty$. Thus, by the relation between the square norm of second fundamental form and the Gaussian curvature (in the case of minimal surfaces in $\mathbb{R}^3$), $|A_n(\cdot,0)|^2 = -2K_n(\cdot,0)$, we obtain that the limit is a planar minimal surface. □

In the next section we want to estimate the isoperimetric ratios of the rescaled curve $\gamma_n(\cdot,0)$ and of regions of the limit curve γ_∞. For this we will use an important result proved by Courant:

Theorem 2.3.5. *([22, Th.3.6]) Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of minimal surfaces with boundaries γ_n of bounded lengths L_n. If the surface X_n tends to a minimal surface X whose boundary γ has length $L = \lim_{n\to\infty} L_n$, then the areas A_n of the minimal surfaces X_n tend to the area A of X.*

2.4 Main Theorem

From [12, Th. 7.3] we know that if $\{(p_n,t_n)\}$ is an essential blow-up sequence and

$$\omega_n = \lambda_n^2(\omega - t_n), \text{ with } \omega_\infty = \lim_{n\to\infty} \omega_n,$$

then there exists a subsequence such that the rescaled solutions γ_n converge uniformly on compact sets of $\mathbb{R}\times[-\infty,\omega_\infty]$ to a smooth, nontrivial limit γ_∞. Moreover, the convergence is C^1 (uniformly on compact sets) at least and the solution γ_∞ exists at least on the interval $[-\infty, 0]$.

It is important to recall that as the limit curve may not be closed, we are assuming that each γ_n is defined on the real line as a periodic map.

In [12] was proved that the limit γ_∞ is a family of convex planar curves and if we assume that $\gamma(\cdot,t)$ forms a Type II singularity at time ω (here $\omega_\infty = \infty$), then there exists an essential blow-up sequence $\{(p_n,t_n)\}$ such that a limit of rescalings along this sequence converges uniformly on compact subsets of $\mathbb{R}\times[-\infty,\infty]$ to the Grim Reaper (1.3.4).

Since we know that $\mathbb{R}$ is homeomorphic to any open interval, we will consider that γ_n converges uniformly on compact sets of $(-\frac{\pi}{2},\frac{\pi}{2})\times[-\infty,\infty]$ to γ_∞. Moreover, as we assume that γ_n is defined on the real line as a periodic curve, we can suppose that the parametrization of γ_n is given by:

$$\gamma_n : \left(-\frac{\pi}{2},\frac{\pi}{2}\right] \times [-\lambda_n^2 t_n, \lambda_n^2(\omega - t_n)) \longrightarrow \mathbb{R}^3. \tag{2.4.1}$$

The purpose of this section is to prove the main result of this thesis

Theorem B. *Suppose that $\gamma(\cdot,t)$ satisfies the equation (CSF) and the total curvature of $\gamma(\cdot,0)$ is less than 4π. Let X_t be the minimal surface enclosed by $\gamma(\cdot,t)$. If its Gaussian curvature $K(\cdot,t)$ is uniformly bounded and $\gamma(\cdot,t)$ remains embedded for all $t\in[0,\omega)$ with finite singular time ω and does not shrink to a point, then there is no formation of Type II singularities.*

To show this theorem, we will use the results in sections 2.1, 2.2, 2.3 and the results proved in [12]. Also, following the definitions given by (2.0.2) and (2.0.4) we will define isoperimetric ratios for the Grim Reaper.

Consider the following compact subset of $(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}$,

$$\mathcal{K}_m = \left[-\frac{\pi}{2}+\frac{1}{m},\frac{\pi}{2}-\frac{1}{m}\right]\times\{0\}.$$

For the endpoints of $\mathcal{K}_m$, $-\alpha(m) := (-\frac{\pi}{2}+\frac{1}{m})$ and $\alpha(m) := (\frac{\pi}{2}-\frac{1}{m})$, there are two possibilities:

$$\int_0^{\alpha(m)} ds(t) \geq \frac{L_{t_n}}{4} \quad \text{or} \quad \int_{-\alpha(m)}^0 ds(t) \geq \frac{L_{t_n}}{4}, \quad \text{for some } t \leq t_n \tag{2.4.2}$$

or

$$\int_0^{\alpha(m)} ds(t) < \frac{L_{t_n}}{4} \quad \text{and} \quad \int_{-\alpha(m)}^0 ds(t) < \frac{L_{t_n}}{4}, \quad \text{for every } t \leq t_n. \tag{2.4.3}$$

If we suppose in addition that the evolution of $\gamma(\cdot,t)$ forms a Type II singularity, from [12] we know that given $\varepsilon > 0$ there exists $N(m) \in \mathbb{N}$ such that $\forall n \geq N(m)$ we have that $\gamma_n|_{\mathcal{K}_m}$ converges uniformly in C^1 to $\gamma_\infty|_{\mathcal{K}_m}$, i.e. for fixed m

$$\lim_{n\to\infty} \gamma_n|_{\mathcal{K}_m} = \gamma_\infty|_{\mathcal{K}_m} \quad \text{and} \quad \lim_{n\to\infty} \gamma_n'|_{\mathcal{K}_m} = \gamma_\infty'|_{\mathcal{K}_m}, \tag{2.4.4}$$

where the Grim Reaper $\gamma_\infty : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\times\mathbb{R} \to \mathbb{R}^2$ is given by $\gamma_\infty(x,0) = (x, -\ln(\cos(x)))$.

The proof of this theorem will be divided in two cases given by the conditions (2.4.2) and (2.4.3).

<u>**The first case**</u> Throughout this case we will denote by L_n the length of the curve $\gamma_n(\cdot,0)$, by $ds_n(0)$ the arc-length parameter of $\gamma_n(\cdot,0)$ and by A_n the area of the minimal surface $X_{n,0}$.

We have that the endpoints of the compact set $\mathcal{K}_m$ satisfy

$$\int_{-\alpha(m)}^{\alpha(m)} ds(t_n) \leq L_{t_n}.$$

Thus, we obtain

$$\int_{-\alpha(m)}^{\alpha(m)} ds_n \leq L_n. \tag{2.4.5}$$

If the evolution of the curve γ forms a Type II singularity, then $\gamma_n|_{\mathcal{K}_m}$ converges uniformly in C^1 to $\gamma_\infty|_{\mathcal{K}_m}$ when n tends to infinity.

Denote by $L_{n,m}$ the length of $\gamma_n|_{\mathcal{K}_m}$, i.e.

$$L_{n,m} = \int_{-\alpha(m)}^{\alpha(m)} ds_n.$$

Throughout this section we will denote by $\Gamma_m(\cdot,t_n)$ the piecewise geodesic on X_{t_n} that joins $\gamma(-\alpha(m),t_n)$ with $\gamma(\alpha(m),t_n)$ and its rescaled curve as $(\Gamma_m)_n(\cdot,0)$.

Since $(\Gamma_m)_n(\cdot,0)$ stays on the minimal surface X_n, the curve $\gamma_n|_{\mathcal{K}_m} \cup (\Gamma_m)_n(\cdot,0)$ enclosed a region $X_{n,m}$ of this surface. We will denote by $A_{n,m}$ as the area of this region.

In this case we will consider the isoperimetric ratio defined by (2.0.2)

$$\mathcal{I}(n,m) := \frac{L_{n,m}^2}{A_{n,m}}.$$

Lemma 2.4.1. *The isoperimetric ratio $\mathcal{I}(n,m)$ is uniformly bounded from above for every $n \in \mathbb{N}$ and for every $m \in \mathbb{N}$ that satisfies* (2.4.2).

Proof. Following Theorem 2.1.8 we have that the isoperimetric ratio is invariant under rescalings, thus

$$\frac{L_n^2}{A_n} = \frac{L_{t_n}^2}{A_{t_n}}.$$

Moreover, Theorem A shows that this isoperimetric ratio is uniformly bounded for every $t \in [0, \omega)$. Thus,

$$4\pi < \frac{L_{t_n}^2}{A_{t_n}} < \mathcal{B},$$

where $\mathcal{B}$ is a finite constant independent of n and m. Therefore, using the relation (2.4.5) we obtain

$$\frac{L_{n,m}^2}{A_{n,m}} < \frac{L_n^2}{A_{n,m}} < \mathcal{B}\frac{A_n}{A_{n,m}}.$$

Since $A_{n,m} \leq A_n$, the region $X_{n,m} \subset X_n$ and the points $-\alpha(m), \alpha(m)$ satisfy (2.4.2), then there exists $0 < \tilde{\mathcal{B}} \leq 1$ such that

$$A_{n,m} = \tilde{\mathcal{B}} A_n.$$

Thus, we obtain

$$\frac{L_{n,m}^2}{A_{n,m}} < \frac{\mathcal{B}}{\tilde{\mathcal{B}}} < \infty.$$

□

On the other hand, if we consider $(\Gamma_m)_\infty$ as the straight line that joins the points $\gamma_\infty(-\frac{\pi}{2} + \frac{1}{m}, 0)$ with $\gamma_\infty(\frac{\pi}{2} - \frac{1}{m}, 0)$, we obtain a bounded region associated to the Grim Reaper, denoted by $X_{\infty,m}$, such that its boundary is $\gamma_\infty|_{\mathcal{K}_m} \cup (\Gamma_m)_\infty$. We will denote by $L_{\infty,m}$ the length of $\gamma_\infty|_{\mathcal{K}_m}$ and by $A_{\infty,m}$ the area of this bounded region.

Lemma 2.4.2. *The isoperimetric ratio $\frac{L_{\infty,m}^2}{A_{\infty,m}}$ converges to infinity when m tends to infinity.*

Proof. We know that

$$\begin{aligned} \gamma_\infty(x) &= (x, -\ln(\cos(x))); \\ \gamma_\infty'(x) &= (1, \tan(x)); \\ \|\gamma_\infty'(x)\| &= \sec(x). \end{aligned}$$

Moreover, it is easy to see that the straight line is given by

$$(\Gamma_m)_\infty(x) = (x, -\ln(\cos(\alpha(m)))).$$

Therefore, we compute

$$L_{\infty,m} = \int_{\mathcal{K}_m} \|\gamma_\infty'\| dx = \int_{-\alpha(m)}^{\alpha(m)} \sec(x) dx;$$

and

$$\begin{aligned} A_{\infty,m} &= \int_{-\alpha(m)}^{\alpha(m)} -\ln(\cos(\alpha(m))) + \ln(\cos(x)) dx \\ &= -2\alpha(m)\ln(\cos(\alpha(m))) + \int_{-\alpha(m)}^{\alpha(m)} \ln(\cos(x)) dx. \end{aligned}$$

Clearly, $L_{\infty,m}$ and $A_{\infty,m}$ converge to infinity when m tends to infinity. Thus, using L'hôpital's rule we obtain

$$\lim_{m\to\infty}\frac{A_{\infty,m}}{L^2_{\infty,m}} = \lim_{m\to\infty}\frac{A'_{\infty,m}}{2L_{\infty,m}L'_{\infty,m}}.$$

From the fundamental theorem of calculus we can compute:

$$\begin{aligned} L'_{\infty,m} &= \sec(\alpha(m))\alpha'(m) + \sec(-\alpha(m))\alpha'(m) \\ &= 2\sec(\alpha(m))\alpha'(m), \end{aligned}$$

and

$$\begin{aligned} A'_{\infty,m} &= -2\alpha'(m)\ln(\cos(\alpha(m))) - 2\alpha(m)\tan(\alpha(m))\alpha'(m) \\ &\quad + \ln(\cos(\alpha(m)))\alpha'(m) + \ln(\cos(-\alpha(m)))\alpha'(m) \\ &= -2\alpha(m)\tan(\alpha(m))\alpha'(m). \end{aligned}$$

Hence,

$$\frac{A'_{\infty,m}}{L'_{\infty,m}} = -\sin(\alpha(m))\alpha(m) \underset{m\to\infty}{\longrightarrow} -\frac{\pi}{2}.$$

Therefore,

$$\lim_{m\to\infty}\frac{A_{\infty,m}}{L^2_{\infty,m}} = \lim_{m\to\infty}\frac{A'_{\infty,m}}{2L_{\infty,m}L'_{\infty,m}} = 0.$$

□

Remark 2.4.3. *Theorem 2.3.4 states that the surface X_n converges to a planar surface away from its boundary implying that the curve on the surface $(\Gamma_m)_n(\cdot,0)$ converges (C^1 at least) to a planar curve $(\Gamma_m)_\infty$.*

However, as in this case we assume that the Gaussian curvature of $X(\cdot,t_n)$ is uniformly bounded, we obtain that $(\Gamma_m)_n(\cdot,0)$ converges (C^1 at least) to a planar curve $(\Gamma_m)_\infty$ up to the boundary.

Moreover, from Proposition 2.2.12 we have that the limit curve $(\Gamma_m)_\infty$ is a straight line that joins the points $\gamma_\infty(-\alpha(m),0)$ with $\gamma_\infty(\alpha(m),0)$.

For further details, we refer the reader to Appendix A.

Lemma 2.4.4. *The isoperimetric ratio $\frac{L^2_{n,m}}{A_{n,m}}$ converges to $\frac{L^2_{\infty,m}}{A_{\infty,m}}$ as n converges to infinity.*

Proof. From the relation (2.4.4) we compute

$$\begin{aligned} \lim_{n\to\infty} L_{n,m} &= \lim_{n\to\infty}\int_{\mathcal{K}_m}\|\gamma'_n\|\,dx \\ &= \int_{\mathcal{K}_m}\lim_{n\to\infty}\|\gamma'_n\|\,dx = \int_{\mathcal{K}_m}\|\gamma'_\infty\|\,dx \\ &= L_{\infty,m}. \end{aligned}$$

Moreover, following Remark 2.4.3 we have that $(\Gamma_m)'_n$ uniformly converges to $(\Gamma_m)'_\infty$. Thus, for m fixed, the length of the boundary of $X_{n,m}$ converges to the length of the boundary of $X_{\infty,m}$ as n converges to infinity.

Therefore, using Theorem 2.3.5 we obtain that for m fixed $A_{n,m}$ converges to $A_{\infty,m}$ as n converges to infinity. □

Proposition 2.4.5. *Suppose that $\gamma(\cdot,t)$ satisfies the equation (CSF), the total curvature of $\gamma(\cdot,0)$ is less than 4π and the endpoints of $\mathcal{K}_m$ satisfy (2.4.2). If $\gamma(\cdot,t)$ remains embedded for all $t\in[0,\omega)$ with finite singular time ω and does not shrink to a point, then there is no formation of Type II singularities.*

Proof. We will prove the result by contradiction, that is, given a curve $\gamma(\cdot,t)$ that evolves by curvature we will assume that $\gamma(\cdot,0)$ has total curvature less than 4π and that $\gamma(\cdot,t)$ remains embedded for all $t\in[0,\omega)$, but there is formation of Type II singularities.

From [12] we know that there exists a subsequence along an essential blow-up sequence $\{(p_n,t_n)\}$ such that the rescaled solutions γ_n converge uniformly on compact subsets of $(\frac{-\pi}{2},\frac{\pi}{2})\times[-\infty,\infty]$ to the Grim Reaper γ_∞. Therefore, for K_m defined above, we have

$$\lim_{n\to\infty}\gamma_n|_{\mathcal{K}_m}=\gamma_\infty|_{\mathcal{K}_m}\quad\text{and}\quad\lim_{n\to\infty}\gamma_n'|_{\mathcal{K}_m}=\gamma_\infty'|_{\mathcal{K}_m}.$$

For m fixed, we estimate

$$\left|\frac{L^2_{\infty,m}}{A_{\infty,m}}\right|\le\left|\frac{L^2_{\infty,m}}{A_{\infty,m}}-\frac{L^2_{n,m}}{A_{n,m}}\right|+\left|\frac{L^2_{n,m}}{A_{n,m}}\right|. \tag{2.4.6}$$

Using Lemma 2.4.1 we have that there exist constants $\mathcal{B},\tilde{\mathcal{B}}$ independent of n and m such that

$$\left|\frac{L^2_{n,m}}{A_{n,m}}\right|=\frac{L^2_{n,m}}{A_{n,m}}<\frac{\mathcal{B}}{\tilde{\mathcal{B}}}<\infty.$$

From Lemma 2.4.4 we get that for every $\varepsilon>0$ and for m fixed we have

$$\left|\frac{L^2_{\infty,m}}{A_{\infty,m}}-\frac{L^2_{n,m}}{A_{n,m}}\right|<\varepsilon.$$

Thus, using (2.4.6) we obtain

$$\left|\frac{L^2_{\infty,m}}{A_{\infty,m}}\right|\le\varepsilon+\frac{\mathcal{B}}{\tilde{\mathcal{B}}}<\infty.$$

However, from Lemma 2.4.2, if we let m tend to infinity we obtain a contradiction, since $\mathcal{B}$ is finite and $\tilde{\mathcal{B}}\in(0,1]$. □

<u>The second case</u>

Throughout this case we will denote the length of the piecewise geodesic $(\Gamma_m)_n(\cdot,0)$ by $(d_m)_n$ and the length of $\gamma_n(\cdot,0)$ between $-\alpha(m)$ and $\alpha(m)$ by $(l_m)_n$.

Using Lemma 2.1.1 and condition (2.4.3), it is easy to see that $\alpha(m)$ and $-\alpha(m)$ satisfy

$$\begin{aligned}\int_0^{\alpha(m)} ds(t) &< \frac{L_{t_n}}{4} < \frac{L_t}{4}, \quad \forall t \leq t_n,\\ \int_{-\alpha(m)}^{0} ds(t) &< \frac{L_{t_n}}{4} < \frac{L_t}{4}, \quad \forall t \leq t_n.\end{aligned}$$

Therefore, for every $t \leq t_n$ the points $-\alpha(m)$ and $\alpha(m)$ satisfy the condition (2.0.3).

If we denote the length of $(\Gamma_m)_\infty$ by $(d_m)_\infty$ and the length of $\gamma_\infty(\cdot,0)$ between $-\alpha(m)$ and $\alpha(m)$ by $(l_m)_\infty$, then we prove

Lemma 2.4.6. *The isoperimetric ratio $\frac{(l_m)_\infty}{(d_m)_\infty}$ converges to infinity when m tends to infinity.*

Proof. Since $(\Gamma_m)_\infty$ is the straight line that joins $-\alpha(m)$ with $\alpha(m)$ then

$$(d_m)_\infty = \pi - \frac{2}{m} \underset{m\to\infty}{\longrightarrow} \pi.$$

On the other hand, we have

$$(l_m)_\infty = \int_{-\alpha(m)}^{\alpha(m)} \sec(x)dx \underset{m\to\infty}{\longrightarrow} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec(x)dx = l_\infty,$$

where l_∞ is the length of the Grim Reaper γ_∞ and it is infinity. □

Lemma 2.4.7. *The isoperimetric ratio $\frac{(l_m)_n}{(d_m)_n}$ converges to $\frac{(l_m)_\infty}{(d_m)_\infty}$ when n tends to infinity.*

Proof. Using Remark 2.4.3, for m fixed we have that $(\Gamma_m)'_n$ uniformly converges to $(\Gamma_m)'_\infty$, then

$$\lim_{n\to\infty}(d_m)_n = \lim_{n\to\infty}\int_0^1 \|(\Gamma_m)'_n\|dx = \int_0^1 \lim_{n\to\infty}\|(\Gamma_m)'_n\|dx = (d_m)_\infty.$$

On the other hand, following (2.4.4) we have

$$\lim_{n\to\infty}(l_m)_n = \lim_{n\to\infty}\int_{-\alpha(m)}^{\alpha(m)} \|\gamma'_n\|dx = \int_{-\alpha(m)}^{\alpha(m)} \lim_{n\to\infty}\|\gamma'_n\|dx = (l_m)_\infty.$$

□

Proposition 2.4.8. *Suppose that $\gamma(\cdot,t)$ satisfies the equation (CSF), the total curvature of $\gamma(\cdot,0)$ is less than 4π and the endpoints of $\mathcal{K}_m$ satisfy* (2.4.3). *If the Gaussian curvature $K(\cdot,t)$ of X_t is uniformly bounded and $\gamma(\cdot,t)$ remains embedded for all $t\in[0,\omega)$ with finite singular time ω and does not shrink to a point, then there is no formation of Type II singularities.*

Proof. We will prove the result by contradiction, that is, given a curve $\gamma(\cdot,t)$ that evolves by curvature we will assume that $\gamma(\cdot,0)$ has total curvature less than 4π, X_t has Gaussian curvature uniformly bounded in time and that $\gamma(\cdot,t)$ remains embedded for all $t\in[0,\omega)$, but there is formation of Type II singularities.

Thus, from [12] we know that there exists a subsequence along an essential blow-up sequence $\{(p_n,t_n)\}$ such that the rescaled solutions γ_n converge uniformly on compact subsets of $(\frac{-\pi}{2},\frac{\pi}{2})\times[-\infty,\infty]$ to the Grim Reaper γ_∞. Therefore, for $\mathcal{K}_m$ defined above, we have

$$\lim_{n\to\infty}\gamma_n|_{\mathcal{K}_m}=\gamma_\infty|_{\mathcal{K}_m}\quad\text{and}\quad\lim_{n\to\infty}\gamma_n'|_{\mathcal{K}_m}=\gamma_\infty'|_{\mathcal{K}_m}.$$

If the endpoints of $\mathcal{K}_m$ satisfy (2.4.3), then we consider the isoperimetric ratio defined by (2.0.4). Thus, for every n and m we estimate

$$\left|\frac{(l_m)_\infty}{(d_m)_\infty}\right|\le\left|\frac{(l_m)_\infty}{(d_m)_\infty}-\frac{(l_m)_n}{(d_m)_n}\right|+\left|\frac{(l_m)_n}{(d_m)_n}\right|. \tag{2.4.7}$$

Using Lemma 2.2.11 and Proposition 2.2.10 we have that there exists a constant C independent of n and m such that

$$\left|\frac{(l_m)_n}{(d_m)_n}\right|=\frac{(l_m)_n}{(d_m)_n}<C.$$

From Lemma 2.4.7 we have that for every $\varepsilon>0$ there exists $N(m)>0$ such that for every $n\ge N(m)$ we have

$$\left|\frac{(l_m)_\infty}{(d_m)_\infty}-\frac{(l_m)_n}{(d_m)_n}\right|<\varepsilon.$$

Thus, using (2.4.7) we obtain that for n big enough

$$\left|\frac{(l_m)_\infty}{(d_m)_\infty}\right|\le\varepsilon+C.$$

However, from Lemma 2.4.6, if we let m tend to infinity we obtain a contradiction, since C is finite. □

Proof of the Main Theorem B. Combining Proposition 2.4.5 and 2.4.8 we obtain the result. □

Remark 2.4.9. *It is important to recall that the hypothesis on total curvature, is used in the existence and uniqueness of the minimal surface $X(\cdot,t)$ along of Chapter 2.*

Chapter 3

Planarity

3.1 Approximation by a graph

The purpose of this section is to show that if γ evolves by curvature and it develops a singularity then the curve γ restricted to a neighborhood of the singularity can be approximated by a graph over an open planar curve. To prove this, we will first show that the curve in this neighborhood is "quasi-planar". Next, we will define the planar curve and the associated graph. Finally, we will show that this graph approximates the space curve close to the singularity.

These results are the first step to realize a simple surgery close to the singularity, which we hope to do in the future.

We start by giving some definitions,

Definition 3.1.1. *A space curve is said to be planar at a point $p \in S^1$ if $\frac{\tau}{k}(p) = 0$, where k is its curvature and τ is its torsion.*

Remark 3.1.2. *This definition was first given in [12].*

We consider $d > 0$, $(p_n, t_n) \in S^1 \times [0, \omega)$ and the neighborhood of $\gamma(p_n, t_n)$ given by

$$N(p_n, t_n, d) = \left\{ (p,t) \in S^1 \times [t_n, \omega) : \left| \int_{p_n}^{p} ds(t_n) \right| \leq \sqrt{\frac{d}{M_{t_n}}} \text{ and } |t_n - t| \leq \frac{d}{M_{t_n}} \right\}, \quad (3.1.1)$$

where $M_{t_n} = \sup k^2(\cdot, t_n)$.

In [12, Th. 6.1] Altschuler proved that $\lim_{n\to\infty} \frac{\tau}{k}(p_n, t_n) = 0$ along an essential blow-up sequence $\{(p_n, t_n)\}$. Thus, for every $\varepsilon > 0$ there exists a neighborhood $\mathcal{V}_\varepsilon(p_n, t_n)$ such that

$$\left| \frac{\tau}{k}(p,t) \right| < \varepsilon, \quad \forall (p,t) \in \mathcal{V}_\varepsilon(p_n, t_n).$$

Remark 3.1.3. *We can choose $d_\varepsilon > 0$ such that $N(p_n, t_n, d_\varepsilon) \subset \mathcal{V}_\varepsilon(p_n, t_n)$. Thus, we obtain that*

$$\left| \frac{\tau}{k}(p,t) \right| < \varepsilon, \quad \forall (p,t) \in N(p_n, t_n, d_\varepsilon).$$

Note that when $\varepsilon \to 0$ the neighborhood $\mathcal{V}_\varepsilon(p_n, t_n)$ is shrinking to the point (p_n, t_n). Thus, we have that $d_\varepsilon \to 0$ as $\varepsilon \to 0$.

Throughout this chapter, for every $\varepsilon > 0$ we will consider $d_\varepsilon \in \mathbb{R}^+$ defined above.

The idea of this chapter is to approximate γ close to the singularity by a "nice" curve.

Firstly, we will prove that for every $\varepsilon > 0$ and for $(p_1, t), (p_2, t) \in N(p_n, t_n, d_\varepsilon)$, the angle between their binormal vectors is close to zero. Since the variation of this angle is the torsion, we will estimate

$$\Theta(p_1, p_2, t) = \int_{p_1}^{p_2} |\tau| ds(t) = \int_{p_1}^{p_2} |\tau(p,t)| \left\| \frac{\partial \gamma_t}{\partial p} \right\| dp. \tag{3.1.2}$$

Theorem 3.1.4. *Let $\varepsilon > 0$ and $(p_1, t), (p_2, t) \in N(p_n, t_n, d_\varepsilon)$. If $\Theta(p_1, p_2, t)$ denotes the angle between the binormal vectors of $\gamma(p_1, t)$ and $\gamma(p_2, t)$, then*

$$\Theta(p_1, p_2, t) < \varepsilon.$$

Proof. From (3.1.2) we have

$$\begin{aligned} \Theta(p_1, p_2, t) &= \int_{p_1}^{p_2} |\tau(p,t)| \left\| \frac{\partial \gamma_t}{\partial p} \right\| dp \\ &= \int_{p_1}^{p_2} \left| \frac{\tau(p,t)}{k(p,t)} \right| |k(p,t)| ds(t) dt \\ &< \varepsilon \int_{p_1}^{p_2} |k(p,t)| ds(t) dt. \end{aligned}$$

Thus, by Theorem 1.2.3, we have that there exists $C_0 > 0$ such that

$$\Theta(p_1, p_2, t) < \varepsilon C_0.$$

□

Since $\Theta(p_1, p_2, t) < \varepsilon$ we have that $\cos(\Theta(p_1, p_2, t)) \sim 1$. Moreover, using that

$$\cos(\Theta(p_1, p_2, t)) = B(p_1, t) \cdot B(p_2, t) \quad \text{for every } (p_1, t), (p_2, t) \in N(p_n, t_n, d_\varepsilon),$$

we obtain

$$B(p_1, t) \cdot B(p_2, t) \sim 1 \text{ and } \|B(p_1, t) - B(p_2, t)\| \sim 0. \tag{3.1.3}$$

On the other hand, for every $\varepsilon > 0$ and for every $(p_1, t), (p_2, t) \in N(p_n, t_n, d_\varepsilon)$ we get

$$\begin{aligned}\|T(p_1,t)-T(p_2,t)\| \le & k(\cdot,t)\|\gamma(p_1,t)-\gamma(p_2,t)\| \\ \le & k(\cdot,t)\Big[\|\gamma(p_1,t)-\gamma(p_1,t_n)\| + \|\gamma(p_1,t_n)-\gamma(p_2,t_n)\| \\ & + \|\gamma(p_2,t_n)-\gamma(p_2,t)\|\Big] \\ \le & k(\cdot,t)\left[k(\cdot,t)|t-t_n| + \int_{p_1}^{p_2} ds(t_n) + k(\cdot,t)|t-t_n|\right] \\ \le & k(\cdot,t)\left[2k(\cdot,t)\frac{d_\varepsilon}{M_{t_n}} + \sqrt{\frac{d_\varepsilon}{M_{t_n}}}\right] \\ = & 2k(\cdot,t)^2\frac{d_\varepsilon}{M_{t_n}} + k(\cdot,t)\sqrt{\frac{d_\varepsilon}{M_{t_n}}}.\end{aligned}$$

From [12, Cor.3.14] we obtain that if $\rho > 0$ is the constant in Definition 1.3.2 then there exists a constant $\alpha_1 < \infty$, depending only on ρ, such that for $t \in \left[t_n, t_n + \frac{3\rho}{64M_{t_n}}\right]$ we have $k^2 \le \alpha_1 M_{t_n}$. Thus; for every $\varepsilon > 0$ we have

$$\|T(p_1,t)-T(p_2,t)\| \le 2\alpha_1 d_\varepsilon + \sqrt{\alpha_1}\sqrt{d_\varepsilon}. \tag{3.1.4}$$

Therefore, using equations (3.1.3) and (3.1.4), if $\varepsilon \to 0$ we conclude that for every $(p_1,t),(p_2,t) \in N(p_n,t_n,d_\varepsilon)$ we have

$$\|T(p_1,t)-T(p_2,t)\| \sim 0 \quad \text{and} \quad \|N(p_1,t)-N(p_2,t)\| \sim 0. \tag{3.1.5}$$

Moreover, from [13] we have the following formulae for the evolution of a space curve:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} + k^2\frac{\partial}{\partial s}; \tag{3.1.6}$$

$$\frac{\partial T}{\partial t} = \frac{\partial k}{\partial s}N + k\tau B; \tag{3.1.7}$$

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3 - k\tau^2; \tag{3.1.8}$$

$$\frac{\partial \tau}{\partial t} = \frac{\partial^2 \tau}{\partial s^2} + \frac{2}{k}\frac{\partial k}{\partial s}\frac{\partial \tau}{\partial s} + \frac{2\tau}{k}\frac{\partial^2 k}{\partial s^2} - \frac{2\tau}{k^2}\left(\frac{\partial k}{\partial s}\right)^2 + 2\tau k^2. \tag{3.1.9}$$

We can compute from the previous equations the evolution of the normal vector in the following way. First, we know from Frenet's equations that $\frac{\partial T}{\partial s} = kN$, so

$$\frac{\partial}{\partial t}\frac{\partial T}{\partial s} = \frac{\partial k}{\partial t}N + k\frac{\partial N}{\partial t}.$$

On the other hand, using equation (3.1.6) we have

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial T}{\partial s} &= \frac{\partial}{\partial s}\frac{\partial T}{\partial t} + k^2\frac{\partial T}{\partial s} \\
&= \frac{\partial}{\partial s}\left[\frac{\partial k}{\partial s}N + k\tau B\right] + k^3 N \\
&= \frac{\partial^2 k}{\partial s^2}N + \frac{\partial k}{\partial s}\frac{\partial N}{\partial s} + \frac{\partial k}{\partial s}\tau B + k\frac{\partial \tau}{\partial s}B + k\tau\frac{\partial B}{\partial s} + k^3 N \\
&= \frac{\partial^2 k}{\partial s^2}N - k\frac{\partial k}{\partial s}T + 2\tau\frac{\partial k}{\partial s}B + k\frac{\partial \tau}{\partial s}B - k\tau^2 N + k^3 N.
\end{aligned}$$

Thus,

$$\frac{\partial N}{\partial t} = \frac{1}{k}\left[\left(\frac{\partial^2 k}{\partial s^2} - k\tau^2 + k^3 - \frac{\partial k}{\partial t}\right)N + 2\tau\frac{\partial k}{\partial s}B - k\frac{\partial k}{\partial s}T + k\frac{\partial \tau}{\partial s}B\right].$$

Therefore, the evolution of the normal vector is given by

$$\frac{\partial N}{\partial t} = \left(\frac{2\tau}{k}\frac{\partial k}{\partial s} + \frac{\partial \tau}{\partial s}\right)B - \frac{\partial k}{\partial s}T. \qquad (3.1.10)$$

To compute the evolution of the binormal vector we use the following relations:

$$B\cdot B = 1, \quad B\cdot N = 0, \quad B\cdot T = 0.$$

Thus,

$$\begin{aligned}
\frac{\partial B}{\partial t}\cdot B &= 0; \\
\frac{\partial B}{\partial t}\cdot N &= -B\cdot\frac{\partial N}{\partial t} = -\left(\frac{2\tau}{k}\frac{\partial k}{\partial s} + \frac{\partial \tau}{\partial s}\right); \\
\frac{\partial B}{\partial t}\cdot B &= -B\cdot\frac{\partial T}{\partial t} = -k\tau.
\end{aligned}$$

Hence,

$$\frac{\partial B}{\partial t} = -k\tau T - \left(\frac{2\tau}{k}\frac{\partial k}{\partial s} + \frac{\partial \tau}{\partial s}\right)N. \qquad (3.1.11)$$

Using the equations of evolution (3.1.7), (3.1.10) and (3.1.11), we can prove the following proposition:

Proposition 3.1.5. *The norm of the evolution of the tangent, normal and binormal vector are bounded by multiples of* M_{t_n}.

Proof. Using the equations of evolution of T, N and B we obtain:

$$\begin{aligned}
\left\|\frac{\partial T}{\partial t}\right\|^2 &= \left(\frac{\partial k}{\partial s}\right)^2 + (k\tau)^2 \\
\left\|\frac{\partial N}{\partial t}\right\|^2 &= \left(\frac{\partial k}{\partial s}\right)^2 + \frac{1}{k^2}\left(2\frac{\partial k}{\partial s}\tau + k\frac{\partial \tau}{\partial s}\right)^2 \\
\left\|\frac{\partial B}{\partial t}\right\|^2 &= (k\tau)^2 + \frac{1}{k^2}\left(2\frac{\partial k}{\partial s}\tau + k\frac{\partial \tau}{\partial s}\right)^2.
\end{aligned}$$

From [12, Cor.3.14] we find constants $\alpha_2, \alpha_3 < \infty$, depending only on ρ in Definition 1.3.2 such that for $t \in [t_n, t_n + \frac{3\rho}{64M_{t_n}}]$ we have

$$\begin{aligned} (k\tau)^2 &\leq \alpha_2 M_{t_n}^2 \\ \left(\frac{\partial k}{\partial s}\right)^2 &\leq \alpha_2 M_{t_n}^2 \\ \left(2\frac{\partial k}{\partial s}\tau + k\frac{\partial \tau}{\partial s}\right)^2 &\leq \alpha_3 M_{t_n}^3. \end{aligned}$$

Hence,

$$\begin{aligned} \left\|\frac{\partial T}{\partial t}\right\|^2 &\leq 2\alpha_2 M_{t_n}^2 \\ \left\|\frac{\partial N}{\partial t}\right\|^2 &\leq \alpha_2 M_{t_n}^2 + \frac{1}{k^2}\alpha_3 M_{t_n}^3 \leq (\alpha_2+\alpha_3) M_{t_n}^2 \\ \left\|\frac{\partial B}{\partial t}\right\|^2 &\leq \alpha_2 M_{t_n}^2 + \frac{1}{k^2}\alpha_3 M_{t_n}^3 \leq (\alpha_2+\alpha_3) M_{t_n}^2. \end{aligned}$$

□

In addition, if $(p, t_1), (p, t_2) \in N(p_n, t_n, d_\varepsilon)$ we obtain

$$\begin{aligned} \|T(p,t_1) - T(p,t_2)\| &\leq \left\|\frac{\partial T}{\partial t}\right\| |t_1 - t_2| < 2\sqrt{2\alpha_2} d_\varepsilon, \\ \|N(p,t_1) - N(p,t_2)\| &\leq \left\|\frac{\partial N}{\partial t}\right\| |t_1 - t_2| < 2\sqrt{\alpha_2+\alpha_3} d_\varepsilon, \\ \|B(p,t_1) - B(p,t_2)\| &\leq \left\|\frac{\partial B}{\partial t}\right\| |t_1 - t_2| < 2\sqrt{\alpha_2+\alpha_3} d_\varepsilon. \end{aligned}$$

Therefore, if $\varepsilon \to 0$ we get

$$\|T(p,t_1) - T(p,t_2)\| \sim 0, \quad \|N(p,t_1) - N(p,t_2)\| \sim 0 \quad \text{and} \quad \|B(p,t_1) - B(p,t_2)\| \sim 0.$$

Thus, we conclude

Theorem 3.1.6. *For every $\varepsilon > 0$ and $(s_1, t_1), (s_2, t_2) \in N(p_n, t_n, d_\varepsilon)$ we have*

$$\|T(s_1,t_1) - T(s_2,t_2)\| \sim 0, \tag{3.1.12}$$
$$\|N(s_1,t_1) - N(s_2,t_2)\| \sim 0, \tag{3.1.13}$$
$$\|B(s_1,t_1) - B(s_2,t_2)\| \sim 0. \tag{3.1.14}$$

Now, we want to approximate the curve $\gamma(\cdot, t)$, where t satisfies $|t - t_n| < \frac{d_\varepsilon}{M_{t_n}}$. First, we will prove that if $\gamma(\cdot, t)$ evolves by its curvature then its norm is uniformly bounded.

Proposition 3.1.7. *If γ evolves by its curvature and $\|\gamma(\cdot,0)\|$ is uniformly bounded, then $\|\gamma(\cdot,t)\|$ is uniformly bounded for every $t \in [0,\omega)$.*

Proof. We compute

$$\begin{aligned}
\frac{\partial}{\partial t}\|\gamma_t\| &= \frac{\frac{\partial \gamma_t}{\partial t}\cdot\gamma_t}{\|\gamma_t\|}\\
\frac{\partial}{\partial s}\|\gamma_t\| &= \frac{\frac{\partial \gamma_t}{\partial s}\cdot\gamma_t}{\|\gamma_t\|}\\
\frac{\partial^2}{\partial s^2}\|\gamma_t\| &= \frac{\frac{\partial^2 \gamma_t}{\partial s^2}\cdot\gamma_t}{\|\gamma_t\|} + \frac{1}{\|\gamma_t\|} - \frac{\left(\frac{\partial \gamma_t}{\partial s}\cdot\gamma_t\right)^2}{\|\gamma_t\|^3}.
\end{aligned}$$

Since γ_t evolves by its curvature we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\|\gamma_t\| &= \frac{\partial^2}{\partial s^2}\|\gamma_t\| - \frac{1}{\|\gamma_t\|} + \frac{\left(\frac{\partial \gamma_t}{\partial s}\cdot\gamma_t\right)^2}{\|\gamma_t\|^3}\\
&\leq \frac{\partial^2}{\partial s^2}\|\gamma_t\|.
\end{aligned}$$

The maximum principle implies that the maximum of $\|\gamma_t\|$ is decreasing in time. Thus,

$$\|\gamma(\cdot,t)\| \leq \max\|\gamma(\cdot,0)\|.$$

□

On the other hand, it is known that Frenet's frame $\{T(s,t), N(s,t), B(s,t)\}$ is an orthonormal basis of $\mathbb{R}^3$, so, for every $(s,t)\in N(p_n,t_n,d_\varepsilon)$ the space curve $\gamma(s,t)$ can be written as linear combination of $T(s,t)$, $N(s,t)$ and $B(s,t)$:

$$\gamma(s,t) = a(s,t)T(s,t) + b(s,t)N(s,t) + c(s,t)B(s,t), \tag{3.1.15}$$

where a, b, c are continuous real functions.

Let $T_0 = T(s_0,t_0)$ be a fixed tangent unit vector, let $N_0 = N(s_0,t_0)$ be a fixed normal unit vector and let $B_0 = B(s_0,t_0)$ be a fixed binormal unit vector at some point $(s_0,t_0)\in N(p_n,t_n,d_\varepsilon)$. Consider the plane generated by T_0 and N_0.

We define ***the planar curve*** $\bar{\gamma}$ as the projection of the curve γ on the plane $\langle T_0, N_0\rangle$; i.e.

$$\bar{\gamma}(s,t) = (\gamma(s,t)\cdot T_0)T_0 + (\gamma(s,t)\cdot N_0)N_0,$$

and, $\hat{\gamma}$ as the following curve

$$\hat{\gamma}(s,t) = \bar{\gamma}(s,t) + c(s,t)B_0,$$

where $c(s,t)$ is given by (3.1.15).

Remark 3.1.8. *Note that $\hat{\gamma}$ can be seen as a graph over the plane $\langle T_0, N_0\rangle$.*

Theorem 3.1.9. *The curve $\gamma(\cdot,t)$ in $N(p_n,t_n,d_\varepsilon)$ is approximated by $\hat{\gamma}(\cdot,t)$ in the following sense,*

$$\|\gamma(s,t)-\hat{\gamma}(s,t)\|\sim 0, \quad \textit{for every } (s,t)\in N(p_n,t_n,d_\varepsilon).$$

Proof. Let $\varepsilon>0$ and $(s,t)\in N(p_n,t_n,d_\varepsilon)$. First, note that

$$\begin{aligned} c(s,t) =&\gamma(s,t)\cdot B(s,t)\\ =&\gamma(s,t)\cdot(B(s,t)-B_0)+\gamma(s,t)\cdot B_0, \end{aligned} \tag{3.1.16}$$

and, consider

$$\gamma(s,t)=(\gamma(s,t)\cdot T_0)T_0+(\gamma(s,t)\cdot N_0)N_0+(\gamma(s,t)\cdot B_0)B_0. \tag{3.1.17}$$

Since c is a continuous function, we can show that $\hat{\gamma}$ approximates γ as follows

$$\begin{aligned} \|\hat{\gamma}(s,t)-\gamma(s,t)\| &= |c(s,t)-(\gamma(s,t)\cdot B_0)|\|B_0\| \\ &= |\gamma(s,t)\cdot(B(s,t)-B_0)| \text{ (use (3.1.16))} \\ &\le \|\gamma(s,t)\|\|(B(s,t)-B_0)\|. \end{aligned}$$

Moreover, using Proposition 3.1.7 and equation (3.1.14) we obtain

$$\|\hat{\gamma}(s,t)-\gamma(s,t)\|\sim 0.$$

□

In conclusion, for every $\varepsilon>0$ and given a space curve that forms a singularity at the point (p,ω) there exists an essential blow-up sequence $\{(p_n,t_n)\}$ such that $\gamma(\cdot,t)$ is approximated by $\hat{\gamma}(\cdot,t)$ in the neighborhood $N(p_n,t_n,d_\varepsilon)$.

Remark 3.1.10. *Using the relations (3.1.12) and (3.1.13) we compute*

$$\begin{aligned} |T\cdot N_0| &= |T\cdot(N_0-N+N)|\le\|N_0-N\|\sim 0; \\ |N\cdot T_0| &= |N\cdot(T_0-T+T)|\le\|T_0-T\|\sim 0; \\ N\cdot N_0 &= 1-\frac{\|N-N_0\|^2}{2}\sim 1; \\ T\cdot T_0 &= 1-\frac{\|T-T_0\|^2}{2}\sim 1. \end{aligned}$$

Geometrically that means, which any Frenet frame $\{T,N,B\}$ at a point in the neighborhood $N(p_n,t_n,d_\varepsilon)$ is close to the fixed Frenet frame $\{T_0,N_0,B_0\}$.

3.2 Evolution of the planar curve

The aim in this section is to study the evolution of the planar curve $\bar{\gamma}$, keeping in mind that γ evolves by its curvature, and to find a relation between the curvature of γ, the curvature of $\bar{\gamma}$ and the curvature of $\hat{\gamma}$. We have

$$\bar{\gamma}(s,t) = (\gamma(s,t) \cdot T_0)T_0 + (\gamma(s,t) \cdot N_0)N_0. \tag{3.2.1}$$

Since our interest is in the evolution by curvature of the curve then following Proposition 1.1.4, we want to estimate $\frac{\partial \bar{\gamma}}{\partial t} \cdot \bar{N}$. Note that

$$\frac{\partial \bar{\gamma}}{\partial t} = \left(\frac{\partial \gamma}{\partial t} \cdot T_0\right) T_0 + \left(\frac{\partial \gamma}{\partial t} \cdot N_0\right) N_0 = (kN \cdot T_0)T_0 + (kN \cdot N_0)N_0. \tag{3.2.2}$$

Moreover, if s is the arc-length parameter of γ, we compute

$$\begin{aligned} \frac{\partial \bar{\gamma}}{\partial s} &= \left(\frac{\partial \gamma}{\partial s} \cdot T_0\right) T_0 + \left(\frac{\partial \gamma}{\partial s} \cdot N_0\right) N_0 = (T \cdot T_0)T_0 + (T \cdot N_0)N_0, \\ \left|\frac{\partial \bar{\gamma}}{\partial s}\right| &= \sqrt{(T \cdot T_0)^2 + (T \cdot N_0)^2} = \bar{K}. \end{aligned}$$

Remark 3.2.1. *Note that by Remark 3.1.10 we have that $\bar{K} \sim 1$.*

Using the computations above, we obtain the tangent vector $\bar{T}$ and the arc-length parameter of $\bar{\gamma}$,

$$\begin{aligned} \bar{T} &= \bar{K}^{-1} \left((T \cdot T_0)T_0 + (T \cdot N_0)N_0\right), \\ d\bar{s} &= \bar{K} ds. \end{aligned}$$

Moreover,

$$\frac{\partial \bar{K}}{\partial s} := \bar{K}_s = \bar{K}^{-1} \left((T \cdot T_0)(kN \cdot T_0) + (T \cdot N_0)(kN \cdot N_0)\right).$$

Thus,

$$\begin{aligned} \frac{\partial^2 \bar{\gamma}}{\partial \bar{s}^2} &= \bar{K}^{-1} \left[\frac{\partial \bar{T}}{\partial s}\right] \\ &= \bar{K}^{-1} \left[\left(-\bar{K}^{-2}\bar{K}_s\right) \left((T \cdot T_0)T_0 + (T \cdot N_0)N_0\right) + \bar{K}^{-1} \left(\frac{\partial}{\partial s} \left((T \cdot T_0)T_0 + (T \cdot N_0)N_0\right)\right)\right] \\ &= \bar{K}^{-1} \Big[- \bar{K}^{-3} \left((T \cdot T_0)(kN \cdot T_0) + (T \cdot N_0)(kN \cdot N_0)\right) \left((T \cdot T_0)T_0 + (T \cdot N_0)N_0\right) \\ &\quad + \bar{K}^{-1} \left((kN \cdot T_0)T_0 + (kN \cdot N_0)N_0\right) \Big] \\ &= -\bar{K}^{-4} \left((T \cdot T_0)(kN \cdot T_0) + (T \cdot N_0)(kN \cdot N_0)\right) \left((T \cdot T_0)T_0 + (T \cdot N_0)N_0\right) \\ &\quad + \bar{K}^{-2} \left((kN \cdot T_0)T_0 + (kN \cdot N_0)N_0\right). \end{aligned}$$

Note that

$$
\begin{aligned}
\left(\frac{\partial^2\bar{\gamma}}{\partial\bar{s}^2}\right)\cdot T_0 =& -\bar{K}^{-4}\left[(T\cdot T_0)^2(kN\cdot T_0)+(T\cdot T_0)(T\cdot N_0)(kN\cdot N_0)\right]+\bar{K}^{-2}(kN\cdot T_0)\\
=& -\bar{K}^{-2}\left[\frac{(T\cdot T_0)^2(kN\cdot T_0)+(T\cdot T_0)(T\cdot N_0)(kN\cdot N_0)}{\bar{K}^2}-(kN\cdot T_0)\right]\\
=& -\bar{K}^{-2}\left[\frac{(T\cdot T_0)^2(kN\cdot T_0)+(T\cdot T_0)(T\cdot N_0)(kN\cdot N_0)}{(T\cdot T_0)^2+(T\cdot N_0)^2}-(kN\cdot T_0)\right]\\
=& -\bar{K}^{-2}\left[\frac{(T\cdot T_0)(T\cdot N_0)(kN\cdot N_0)-(kN\cdot T_0)(T\cdot N_0)^2}{(T\cdot T_0)^2+(T\cdot N_0)^2}\right]\\
=& -\bar{K}^{-2}(T\cdot N_0)\left[\frac{(T\cdot T_0)(kN\cdot N_0)-(kN\cdot T_0)(T\cdot N_0)}{(T\cdot T_0)^2+(T\cdot N_0)^2}\right]\\
=& -\bar{K}^{-4}(T\cdot N_0)\left[(T\cdot T_0)(kN\cdot N_0)-(kN\cdot T_0)(T\cdot N_0)\right].
\end{aligned}
$$

And also

$$
\begin{aligned}
\left(\frac{\partial^2\bar{\gamma}}{\partial\bar{s}^2}\right)\cdot N_0 =& -\bar{K}^{-4}\left[(T\cdot T_0)(T\cdot N_0)(kN\cdot T_0)+(T\cdot N_0)^2(kN\cdot N_0)\right]+\bar{K}^{-2}(kN\cdot N_0)\\
=& -\bar{K}^{-2}\left[\frac{(T\cdot T_0)(T\cdot N_0)(kN\cdot T_0)+(T\cdot N_0)^2(kN\cdot N_0)}{\bar{K}^2}-(kN\cdot N_0)\right]\\
=& -\bar{K}^{-2}\left[\frac{(T\cdot T_0)(T\cdot N_0)(kN\cdot T_0)+(T\cdot N_0)^2(kN\cdot N_0)}{(T\cdot T_0)^2+(T\cdot N_0)^2}-(kN\cdot N_0)\right]\\
=& -\bar{K}^{-2}\left[\frac{(T\cdot T_0)(T\cdot N_0)(kN\cdot T_0)-(T\cdot T_0)^2(kN\cdot N_0)}{(T\cdot T_0)^2+(T\cdot N_0)^2}\right]\\
=& -\bar{K}^{-2}(T\cdot T_0)\left[\frac{(T\cdot N_0)(kN\cdot T_0)-(T\cdot T_0)(kN\cdot N_0)}{(T\cdot T_0)^2+(T\cdot N_0)^2}\right]\\
=& -\bar{K}^{-4}(T\cdot T_0)\left[(T\cdot N_0)(kN\cdot T_0)-(T\cdot T_0)(kN\cdot N_0)\right].
\end{aligned}
$$

If we let $\psi=(kN\cdot T_0)(T\cdot N_0)-(T\cdot T_0)(kN\cdot N_0)$, we obtain

$$
\begin{aligned}
\left(\frac{\partial^2\bar{\gamma}}{\partial\bar{s}^2}\right)\cdot T_0 &= \bar{K}^{-4}(T\cdot N_0)\psi;\\
\left(\frac{\partial^2\bar{\gamma}}{\partial\bar{s}^2}\right)\cdot N_0 &= -\bar{K}^{-4}(T\cdot T_0)\psi.
\end{aligned}
$$

Hence,

$$
\frac{\partial^2\bar{\gamma}}{\partial\bar{s}^2} = \frac{\psi}{\bar{K}^4}\left((T\cdot N_0)T_0-(T\cdot T_0)N_0\right), \tag{3.2.3}
$$

and

$$
\left\|\frac{\partial^2\bar{\gamma}}{\partial\bar{s}^2}\right\|^2 = \left(\frac{\psi}{\bar{K}^4}\right)^2\left((T\cdot N_0)^2+(T\cdot T_0)^2\right)=\frac{\psi^2}{\bar{K}^6}.
$$

Moreover, as $\bar{\gamma}$ is a planar curve, we have that

$$\begin{aligned}\bar{k} &= \left\|\frac{\partial^2\bar{\gamma}}{\partial\bar{s}^2}\right\|,\\ \bar{N} &= \frac{1}{\bar{k}}\cdot\frac{\partial^2\bar{\gamma}}{\partial\bar{s}^2}.\end{aligned}$$

Thus, using equations (3.2.2) and (3.2.3), we compute

$$\frac{\partial\bar{\gamma}}{\partial t}\cdot\frac{\partial^2\bar{\gamma}}{\partial\bar{s}^2} = \frac{\psi^2}{\bar{K}^4}.$$

We conclude that

$$\frac{\partial\bar{\gamma}}{\partial t}\cdot\frac{\partial^2\bar{\gamma}}{\partial\bar{s}^2} - \bar{K}^2\left\|\frac{\partial^2\bar{\gamma}}{\partial\bar{s}^2}\right\|^2 = 0.$$

Therefore, the evolution of $\bar{\gamma}$ in the normal direction has velocity:

$$\frac{\partial\bar{\gamma}}{\partial t}\cdot\bar{N} = \bar{K}^2\bar{k}. \tag{3.2.4}$$

In conclusion, if a space curve γ evolves by its curvature then its projection on the plane $\langle T_0, N_0\rangle$ evolves by equation (3.2.4). Moreover, using Remark 3.2.1 we obtain that the evolution of the planar curve $\bar{\gamma}$ is "*quasi by Curve Shortening Flow*", i.e. $\bar{\gamma}$ satisfies

$$\frac{\partial\bar{\gamma}}{\partial t}\cdot\bar{N} \sim \bar{k}.$$

In addition, using Remark 3.1.10, we have that the curvature of the space curve γ is related to the curvature of the planar curve $\bar{\gamma}$ by

$$\bar{k} = \frac{\psi}{\bar{K}^3} = \frac{k\Big[(N\cdot T_0)(T\cdot N_0) - (T\cdot T_0)(N\cdot N_0)\Big]}{\Big[(T\cdot T_0)^2 + (T\cdot N_0)^2\Big]^{\frac{3}{2}}} \sim -k, \tag{3.2.5}$$

where we used (3.1.10).

Therefore, if γ forms a singularity then $\bar{\gamma}$ forms a singularity as well.

On the other hand, we can compute the curvature of $\hat{\gamma} = \bar{\gamma} + cB_0$ as follows If $\bar{s}$ denote the arc-length parameter of $\bar{\gamma}$ and $\hat{s}$ denote the arc-length parameter of $\hat{\gamma}$ then

$$\begin{aligned}\frac{\partial\hat{\gamma}}{\partial\bar{s}} &= \bar{T} + \frac{\partial c}{\partial\bar{s}}B_0;\\ \left\|\frac{\partial\hat{\gamma}}{\partial\bar{s}}\right\| &= \sqrt{1 + \left(\frac{\partial c}{\partial\bar{s}}\right)^2} = \hat{K};\\ \hat{T} &= \hat{K}^{-1}\left(\bar{T} + \frac{\partial c}{\partial\bar{s}}B_0\right);\\ d\hat{s} &= \hat{K}d\bar{s}.\end{aligned}$$

Moreover,

$$\frac{\partial \hat{K}}{\partial \bar{s}} := \hat{K}_{\bar{s}} = \hat{K}^{-1}\left(\frac{\partial c}{\partial \bar{s}}\right)\left(\frac{\partial^2 c}{\partial \bar{s}^2}\right).$$

Thus,

$$\begin{aligned}\frac{\partial^2 \hat{\gamma}}{\partial \hat{s}^2} &= \hat{K}^{-1}\left[-\hat{K}^{-2}\hat{K}_{\bar{s}}\left(\bar{T}+\frac{\partial c}{\partial \bar{s}}B_0\right)+\hat{K}^{-1}\bar{k}\bar{N}+\hat{K}^{-1}\frac{\partial^2 c}{\partial \bar{s}^2}B_0\right]\\ &= \hat{K}^{-1}\left[-\hat{K}^{-3}\left(\frac{\partial c}{\partial \bar{s}}\right)\left(\frac{\partial^2 c}{\partial \bar{s}^2}\right)\left(\bar{T}+\frac{\partial c}{\partial \bar{s}}B_0\right)+\hat{K}^{-1}\bar{k}\bar{N}+\hat{K}^{-1}\frac{\partial^2 c}{\partial \bar{s}^2}B_0\right].\end{aligned}$$

Note that $\{\bar{T}, \bar{N}, B_0\}$ is an orthogonal basis, i.e. $\bar{T}\cdot B_0 = 0$, $\bar{N}\cdot B_0 = 0$ and $\bar{T}\cdot\bar{N} = 0$. Therefore,

$$\frac{\partial^2 \hat{\gamma}}{\partial \hat{s}^2} = -\hat{K}^{-4}\left(\frac{\partial c}{\partial \bar{s}}\right)\left(\frac{\partial^2 c}{\partial \bar{s}^2}\right)\bar{T}+\hat{K}^{-2}\bar{k}\bar{N}+\hat{K}^{-4}\left(\frac{\partial^2 c}{\partial \bar{s}^2}\right)B_0.$$

The curvature of $\hat{\gamma}$ is given by

$$\begin{aligned}\hat{k}^2 &= \left\|\frac{\partial^2 \hat{\gamma}}{\partial \hat{s}^2}\right\|^2\\ &= \hat{K}^{-8}\left(\frac{\partial c}{\partial \bar{s}}\right)^2\left(\frac{\partial^2 c}{\partial \bar{s}^2}\right)^2+\hat{K}^{-4}\bar{k}^2+\hat{K}^{-8}\left(\frac{\partial^2 c}{\partial \bar{s}^2}\right)^2\\ &= \hat{K}^{-4}\bar{k}^2+\hat{K}^{-8}\left(\frac{\partial^2 c}{\partial \bar{s}^2}\right)^2\left[\left(\frac{\partial c}{\partial \bar{s}}\right)^2+1\right]\\ &= \hat{K}^{-4}\bar{k}^2+\hat{K}^{-6}\left(\frac{\partial^2 c}{\partial \bar{s}^2}\right)^2.\end{aligned} \tag{3.2.6}$$

Furthermore, we have

$$\frac{\partial c}{\partial \bar{s}} = \bar{K}^{-1}\frac{\partial c}{\partial s};$$

and

$$\begin{aligned}\frac{\partial^2 c}{\partial \bar{s}^2} &= \bar{K}^{-1}\left[\frac{\partial}{\partial s}\left(\bar{K}^{-1}\frac{\partial c}{\partial s}\right)\right]\\ &= \bar{K}^{-1}\left[-\bar{K}^{-2}\bar{K}_s\frac{\partial c}{\partial s}+\bar{K}^{-1}\frac{\partial^2 c}{\partial s^2}\right]\\ &= \bar{K}^{-1}\left[-\bar{K}^{-3}\frac{\partial c}{\partial s}\left((T\cdot T_0)(kN\cdot T_0)+(T\cdot N_0)(kN\cdot N_0)\right)+\bar{K}^{-1}\frac{\partial^2 c}{\partial s^2}\right]\\ &\sim \bar{K}^{-2}\frac{\partial^2 c}{\partial s^2} \text{ (use (3.1.10)).}\end{aligned}$$

Therefore,

$$\hat{K} = \sqrt{1+\bar{K}^{-2}\left(\frac{\partial c}{\partial s}\right)^2} \sim \sqrt{1+\left(\frac{\partial c}{\partial s}\right)^2}.$$

Hence, using (3.2.6) and (3.2.5) we obtain

$$\hat{k}^2 \sim \frac{k^2}{\left(1+\left(\frac{\partial c}{\partial s}\right)^2\right)^2} + \frac{\frac{\partial^2 c}{\partial s^2}}{\left(1+\left(\frac{\partial c}{\partial s}\right)^2\right)^3}. \tag{3.2.7}$$

Therefore, we obtain a direct relation between the curvature of γ and the curvature of the planar curve $\bar{\gamma}$ in (3.2.5). Also, we obtain a relation depending on $c(s,t)$ between the curvature of γ and the curvature of $\hat{\gamma}$ in (3.2.7).

Our aim in the future would be to consider a not necessarily embedded space curve γ and to extend its evolution through singularities. The strategy that we intend to use is to consider $\hat{\gamma}$ as an approximation of γ choosing appropriately the function c such that $c \sim 0$ close to the singularity. We expect that the relations obtained in this chapter combined with Chapter 2 will give us control on singularities.

Appendix A

Remarks on the Gaussian curvature of $X(\cdot, t)$

In the Chapter 2 we assumed two conditions for the minimal surfaces $X(\cdot, t)$

- Its Gaussian curvature $K(\cdot, t)$ is uniformly bounded in time;
- $X(\cdot, t)$ converges in C^1 up to the boundary.

In this appendix we will start showing that the first condition is enough to ensure Remark 2.2.7. Next, we will discuss the reasons to hope that these conditions are satisfied.

A.1 Evolution of $X(\cdot, t)$

Similarly to Proposition 1.1.4 we can reparametrize of $X(\cdot, t)$ such that its evolution in time is only in the normal direction, in the following way:

For a minimal surface $X : B \times [0, \omega) \to \mathbb{R}^3$ such that

$$\frac{\partial}{\partial t}X(u,v,t) = \mathcal{V}(u,v,t)\vec{n}(u,v,t) + \mathcal{W}(u,v,t)X_u(u,v,t) + \mathcal{Z}(u,v,t)X_v(u,v,t),$$

consider $\varphi(u,v,t) = (\varphi_1(u,v,t), \varphi_2(u,v,t)) \in B$ a smooth family of diffeomorphisms of B with $\varphi(u,v,0) = (u,v)$ for every $(u,v) \in \partial B$ and

$$\frac{\partial}{\partial t}\varphi(u,v,t) = -(\mathcal{W}(u,v,t), \mathcal{Z}(u,v,t)).$$

If $\tilde{X}(u,v,t) = X(\varphi(u,v,t),t)$, then we obtain that

$$\frac{\partial}{\partial t}\tilde{X}(u,v,t) = \tilde{\mathcal{V}}(u,v,t)\tilde{\vec{n}}(u,v,t).$$

Therefore, we can assume that the evolution of the minimal surface X_t in time is given by

$$\frac{\partial}{\partial t}X(\cdot,t) = \mathcal{V}(\cdot,t)\vec{n}(\cdot,t),$$

where $\vec{n}(\cdot, t)$ is the unit normal vector to the surface $X(\cdot, t)$.

A.2 Remark 2.2.7

The aim in this remark is to obtain a lower bound of G_1 defined by

$$G_1 = \int_0^1 T_0 \cdot \frac{\partial}{\partial x}\frac{\partial}{\partial t}X(\cdot,t)dx.$$

Proposition A.2.1. *If k_0^N is the normal curvature of Γ_0, then*

$$G_1 = \int_0^1 \mathcal{V} \cdot k_0^N(\cdot,t)ds.$$

Proof. Following Section A.1, we have that

$$\begin{aligned}
\int_0^1 T_0 \cdot \frac{\partial}{\partial x}\frac{\partial X}{\partial t}dx &= \int_0^1 T_0 \cdot \frac{\partial}{\partial x}\mathcal{V}\vec{n}dx \\
&= \int_0^1 T_0 \cdot \left(\frac{\partial \mathcal{V}}{\partial x}\vec{n} + \mathcal{V}\frac{\partial \vec{n}}{\partial x}\right) dx \\
&= \int_0^1 T_0 \cdot \left(\frac{\partial \mathcal{V}}{\partial x}\vec{n} + u_x\mathcal{V}\vec{n}_u + v_x\mathcal{V}\vec{n}_v\right) dx \\
&= \int_0^1 \frac{\mathcal{V}\cdot\left(e\cdot u_x^2 + 2f\cdot u_xv_x + g\cdot v_x^2\right)}{\sqrt{|X_u|^2u_x^2 + 2u_xv_xX_u\cdot X_v + |X_v|^2v_x^2}}dx \\
&= \int_0^1 \mathcal{V}\cdot\frac{II(u_x,v_x)}{I(u_x,v_x)}ds \\
&= \int_0^1 \mathcal{V}\cdot k_0^N(\cdot,t)ds.
\end{aligned}$$

□

Proposition A.2.2. *The normal component of the evolution of $X(\cdot,t)$ satisfies*

$$\left(\frac{\partial}{\partial t}X(\cdot,t)\cdot\vec{n}(\cdot,t)\right)^2 = \mathcal{V}^2(\cdot,t) \le |K(\cdot,t)|.$$

Proof. We will prove that $\mathcal{V}$ satisfies an elliptic equation.

Note that

$$\begin{aligned}
\Delta\left(\frac{\partial}{\partial t}X\cdot\vec{n}\right) &= \frac{1}{E}\left(\frac{\partial^2}{\partial u^2}\left(\frac{\partial}{\partial t}X\cdot\vec{n}\right) + \frac{\partial^2}{\partial v^2}\left(\frac{\partial}{\partial t}X\cdot\vec{n}\right)\right) \\
&= \frac{1}{E}\left(\vec{n}\cdot\frac{\partial}{\partial t}X_{uu} + \mathcal{V}\left\|\frac{\partial}{\partial u}\vec{n}\right\|^2 + \vec{n}\cdot\frac{\partial}{\partial t}X_{vv}\right) + \mathcal{V}\left\|\frac{\partial}{\partial v}\vec{n}\right\|^2 \\
&= \frac{\mathcal{V}}{E}\left(\left\|\frac{\partial}{\partial u}\vec{n}\right\|^2 + \left\|\frac{\partial}{\partial v}\vec{n}\right\|^2\right) \\
&= 2\mathcal{V}|K|,
\end{aligned}$$

where Δ is the Laplace Beltrami operator.

Therefore, if we consider the elliptic operator $\mathcal{L}(u) = \Delta u - 2|K|u$, then we have

$$\begin{cases} \mathcal{L}(\mathcal{V}) = 0 & \text{in } B \\ \mathcal{V} = k_0^N & \text{on } \partial B. \end{cases}$$

Thus, using the maximum principle we get

$$\mathcal{V}^2 = \left(\frac{\partial}{\partial t} X \cdot \vec{n}\right)^2 \leq \left(\frac{\partial}{\partial t}\gamma \cdot \vec{n}\right)^2 = (k\, N \cdot \vec{n})^2 = (k_0^N)^2 \leq |K|.$$

□

Therefore, we conclude that

$$G_1 = \int_0^1 \mathcal{V} \cdot k_0^N(\cdot, t) ds \leq \int_0^1 |K(\cdot, t)| \leq d_0^t \cdot \left(\sup_{t \in [0,\omega)} |K(\cdot, t)|\right). \qquad \text{(A.2.1)}$$

A.3 Properties away from the boundary

- Following Section 2.3, we have that the sequence of rescaled minimal surface $\{X(\cdot, 0)\}_n$ along a blow-up sequence converges to a planar surface. Therefore, $|K_n(\cdot, 0)|$ along this sequence tends to zero.
- The condition $|K(\cdot, t)|$ be uniformly bounded in t holds away from the boundary without further assumptions. This follows from a result in [5], where the authors prove that the density is uniformly bounded by the total curvature of the boundary.

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