Approximation and Existence of Solutions of Delay Differential Equations

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"Prolongue una línea recta al infinito: ¿Qué encontrará al final?."

– J. L. Martínez, *La Nueva Novela, 1977.*

Resumen Biográfico

Daniel Enrique Sepúlveda Oehninger (Santiago, 1982). En el año 2000 entra a estudiar pedagogía en matemática en la Universidad Metropolitana de Ciencias de la Educación, obteniendo el grado de Licenciado en Educación Matemática el año 2003 y luego, en 2005, el título de Profesor de Matemática y Esdiística Educacional. Luego continúa estudios de postgrado en la Facultad de Ciencias de la Universidad de Chile, obteniendo el grado de Magíster en Ciencias mención Matemáticas el año 2009, fruto de la investigación realizada en ese tiempo junto al profesor Manuel Pinto, se dio forma al artículo h-asymptotic stability by fixed point in neutral nonlinear differential equations with delay el año 2011. El año 2016 publicó el artículo Weighted pseudo almost periodic functions, convolutions and abstract integral equations, en conjunto con Dr. Anibal Coronel y Dr. Manuel Pinto.

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Resumen

Esta investigación es acerca de ecuaciones diferenciales con retardo y las aproximaciones numéricas de las soluciones de estas, y en particular, el problema de aproximar soluciones sobre un intervalo no acotado y la transferencia de propiedades cualitativas entre las soluciones de una ecuación diferencial con retardo y su correspondiente ecuación en diferencias.

Obtenemos resultados de aproximación y transferencia de propiedades de estabilidad exponencial de las soluciones de ecuaciones diferenciales con retardo variable no autonoma y de ecuaciones diferenciales funcionales retardadas con retroalimentación, a sus respectivas ecuaciones en diferencias, apoyandonos en el uso de argumento constante a trozos, teoría de ecuaciones diferenciales funcionales y desigualdades integrales entre otros.

También probamos un resultado tipo Cauchy-Peano de existencia global, por medio de la convergencia de una sucesión de soluciones aproximadas, para una ecuación diferencial semi lineal con retardo dependiente del tiempo y del estado. De esta manera estudiamos la transferencia de propiedades de existencia desde la ecuación en diferencias a la correspondiente ecuación diferencial funcional.

Abstract

This research is about differential equations with delay and numerical approximations of the solutions of these, particularly, the problem of approximate solutions over an unbounded interval and transference of qualitative properties between solutions of a differential equation with delay and the corresponding difference equation.

We obtain results for approximation and transference of exponential stability properties of solutions of non-autonomous differential equations with variable delay and retarded functional differential equations with feedback, respectively, to the corresponding difference equations by using piecewise constant argument, theory of functional differential equations and integral inequality among other.

Also we prove a global existence of Cauchy-Peano type theorem, by convergence of an approximate sequence, for a semi-linear differential equation with time- and state-dependent delay. This way we study the transference of existence from difference equation to the corresponding continuous functional differential equations.

1

Introduction

1.1 Introduction

"Numerical Analysis is the study of algorithms for the problems of continuous mathematics."

– L. N. Trefethen, *The definition of Numerical Analysis, 1992.*

The study about chaotic behavior of some dynamical systems of continuous time was motivated by unexpected results of numerical approximations of the solutions of system differential equations, for example the non-linear system related with weather prediction by Lorenz (1963). In Sparrow (1982) several numerical approximations to the Lorenz equation are computed, trying to collect as amount of information as possible. Sparrow was fully aware of the limitations of the study of chaos by using numerical simulations. Usually it is expected that qualitative properties of both continuous and discrete time dynamical systems are not significantly different. However, well-known bounds for the error of numerical methods is $e^{cT}h^p$, where h is the step size, p is the order of the method and T is the length of the time interval. From the above it follows that: if we compute approximations on a long interval of time we cannot ensure that the approximation is close to the solution neither that have similar qualitative features. Several authors have addressed the problem of finding necessary conditions to ensure that the distance between the approximate and actual solutions of differential equations is small, regardless of the length of the time interval. The study of the relation between the continuous and discrete dynamical systems is an interesting problem, see Cooke and Györi (1994); Cryer (1972); Grüne (2002, 2003); Györi (1991); Kloeden and Lorenz (1986); Kloeden and Schmalfuß (1996). This type of result are equivalent to find conditions to ensure that discretization of differential equations does not significantly alter the basic qualitative features of the solutions. For ordinary differential equations it is well known that if has an asymptotically stable steady or periodic solution, then comparisons between solutions, for a long time, are possible.

Recently, several researchers have discretized systems of differential equations using piecewise constant argument, they obtained an Euler's approximation of the solution of original system, see Mohamad and Gopalsamy (2003); Huang et al. (2006); Abbas and Xia (2013). These paper follow the ideas developed by Györi (1991), who was interested in the convergence of several approximation, by piecewise constant argument, to the actual solution of a linear differential equation with delay. Actually, that work belongs to a series of papers about approximations of solutions of delayed differential equations (see Györi, 1988, 1991; Cooke and Györi, 1994; Györi et al., 1995; Györi and Hartung, 2002, 2008). In the theory of functional differential equations numerical approximation is a key topic since numerical computations is the main technique to visualize the solutions that, in general, can not be expressed explicitly, even for non autonomous scalar linear equations. Numerical methods for differential equations with delay are well-known, see Bellen and Zennaro (2013). Numerical analysis for the study of stability of delayed differential equations have been developed recently, see Breda et al. (2015). However, the study of numerical approximation of a solution of functional differential equations over an unbounded interval is in correspondence with the problem of transference of qualitative properties between a continuous dynamical system and the corresponding discrete dynamical system. For delay differential equations, in the scalar case, there are results about presence of spurious dynamics in numerical approximation, i.e., dynamics that are not present in the actual solutions (see Györi et al., 1996; Hartung and Turi, 1995; Cooke and Ivanov, 2000). Therefore, numerical approximation is closely related with the problem of existence and stability of solutions. Since our bibliographic review we conclude that:

Theorems of approximation and transference of stability properties of differential equations with variable delay could be generalization of the results in Cooke and Györi (1994); Györi and Hartung (2002) to larger cases. These kind of results will contribute to the development of existing theory at present and can be applied in identification of parameters in differential equations (see Hartung and Turi, 1997; Hartung et al., 1998, 2000).

Recently Benchohra et al. (2013) have studied the global existence of solutions of functional differential equations with state-dependent delay. They obtained an existence of Cauchy-Peano type for equations differential with time- and state- dependent delay by Schauder fixed point. Theorems of global existence of Cauchy-Peano type, proved by convergence of a sequence of functions, for differential equations with state-dependent delay would be a extension of the local existence results of Tavernini (1978) and Györi et al. (1995) that could be applied for neutral differential equation with state-dependent delay Hartung et al. (1997).

Therefore the overall goals of this research are:

1. Establish theorems of approximation and transference of stability properties of solutions of differential equations with variable delay using piecewise constant argument.

2. Establish theorems of approximation and transference of stability properties of solutions of differential equations with distributed delay using piecewise constant argument.

3. Relate results of existence, Cauchy-Peano type of solutions of differential equations with delay transfer Results existence between solutions of a difference equation and differential equation with delay.

1.2 Basic theory of differential equations with delay

Suppose $q \geq 0$ is a given real number, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the field of real numbers with norm $\|\cdot\|$; $C([a,b], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a,b] = [-q,0]$ we let $\mathcal{C} = C([-q,0], \mathbb{R}^n)$ and designate the norm of an element φ in $\mathcal{C}$ by $\|\varphi\|_q = \sup_{-q\leq\theta\leq 0} \|\varphi(\theta)\|$. If

$$t_0 \in \mathbb{R}, A \geq 0, \text{ and } x \in C([t_0 - q, t_0 + A], \mathbb{R}^n),$$

then for any $t \in [t_0, t_0 + A]$, we let $x_t(\theta) = x(t+\theta), -q \leq \theta \leq 0$. If D is a subset of $\mathbb{R} \times \mathcal{C}, f : D \to \mathbb{R}$ is a given function and " $'$ " represents the right-hand derivative, we say that the relation

$$x'(t) = f(t, x_t) \tag{1.1}$$

is a *differential equation with delay* or a *retarded functional differential equation* on D. A function x is said to be a *solution* of equation (1.1) on $[t_0 - q, t_0 + A)$ if there are $t_0 \in \mathbb{R}$ and $A > 0$ such that $x \in C([t_0 - q, t_0 + A), \mathbb{R}^n), (t, x_t) \in D$ and $x(t)$ satisfies equation (1.1) for $t \in [t_0, t_0 + A)$. For given $t_0 \in \mathbb{R}$, $\phi \in \mathcal{C}$, we say $x(t_0, \phi, f)$ is a solution of equation (1.1) *with initial value* ϕ at t_0 if there is an $A > 0$ such that $x(t_0, \phi, f)$ is a solution of equation (1.1) on $[t_0 - q, t_0 + A]$ and $x_{t_0}(t_0, \phi, f) = \phi$. The theory of differential equations with delay can be found in Driver (1977) and Hale and Lunel (1993).

1.2.1 Existence

To study existence and uniqueness for differential equation with delay (1.1) with initial value ϕ at t_0 by converting into an integral equation we need a *continuity condition* over $f(t, x_t)$, i.e. $f(t, x_t)$ is continuous with respect to t in $[t_0, t_0 + A)$ for each given continuous function $x \in \mathcal{C}$.

If $f : D \to \mathbb{R}$ satisfies continuity condition then a continuous function $x \in \mathcal{C}$ is a solution of equation (1.1) with initial value ϕ at t_0, over $[t_0 - q, t_0 + A_1)$ for some $0 < A_1 \leq A$, if and only if

$$x(t) = \begin{cases} \phi(t - t_0) & \text{for } t_0 - q \leq t \leq t_0 \\ \phi(0) + \int_{t_0}^{t} f(s, x_s)ds & \text{for } t_0 \leq t \leq t_0 + A_1. \end{cases}$$

Definition 1.1. *The functional $f : D \to \mathbb{R}^n$ is said to be locally Lipschitz if for each given $(\tau, \phi) \in D$ there exist numbers $a > 0$ and $b > 0$ such that:*

$$M = [\tau - a, \tau + a] \times \{\psi \in \mathcal{C} : \|\psi - \phi\|_q \leq b\}$$

and f is Lispchitz on $M \cap D$.

Next we recall some fundamental theorems

Theorem 1.1 (Uniqueness). *Let $f : D \to \mathbb{R}^n$ satisfy the continuity condition and let it be locally Lipschitz. Then, given any $\phi \in \mathcal{C}$ (1.1) have at most one solution with initial condition ϕ at t_0 on $[t_0 - q, A_1)$ for some $0 < A_1 \leq A$.*

Theorem 1.2 (Dependence on initial conditions). *Let $f : D \to \mathbb{R}^n$ satisfy the continuity condition and let it be (globally) Lipschitz. Let ϕ and $\psi \in \mathcal{C}$ be given and let x and y be unique solutions of Eq. (1.1) with $x_{t_0} = \phi$ and $y_{t_0} = \psi$. If x and y are both defined on $[t_0 - q, A_1)$ then*

$$\|x(t) - y(t)\| \leq \|\phi - \psi\|_q e^{K(t-t_0)} \text{ for } t_0 \leq t < A_1.$$

Theorem 1.3 (Local Existence). *Let $f : D \to \mathbb{R}^n$ satisfy the continuity condition and let it be locally Lipschitz. Then, for each $\phi \in \mathcal{C}$, equation (1.1) with $x_= \phi$ have a unique solution on $[t_0 - r, t_0 + \Delta)$ for some $\Delta > 0$.*

The above results can be found in Driver (1977).

1.2.2 Stability

We recall definition of stability of the solution $x = 0$ for differential equation with delay (1.1) from Hale and Lunel (1993).

Definition 1.2. *Suppose $f(t, 0) = 0$ for all $t \in \mathbb{R}$. The solution $x = 0$ of equation (1.1) is said to be stable if for any $t_0 \in \mathbb{R}, \epsilon > 0$, there is a $\delta = \delta(\epsilon, t_0)$ such that phi satisfies $\|\phi\|_q < \delta$ implies $\|x_t(t_0, \phi)\|_q < \epsilon$ for $t \geq t_0$. The solution $x = 0$ of equation (1.1) is said to be asymptotically stable if it is stable and there is $b_0 = b_0(t_0) > 0$ such that if $\|\phi\|_q < b_0$ implies $x(t; t_0, \phi) \to 0$ as $t \to \infty$. The solution $x = 0$ of equation (1.1) is said to be uniformly stable if the number δ in the definition is independent of t_0. The solution $x = 0$ of equation (1.1) is uniformly asymptotically stable if it is uniformly stable and there is $b_0 > 0$ such that for every $\eta > 0$, there is a $T(\eta)$ such that if $\|\phi\|_q < b_0$ implies $x_t(t_0, \phi) < \eta$ for $t \geq t_0 + T$ for every $t_0 \in \mathbb{R}$.*

For homogeneous linear system of functional differential equations the uniformly asymptotically stability of the solution $x = 0$ and the exponentially asymptotically stability of the solution $x = 0$ are equivalent (see Hale and Lunel, 1993, pp. 185).

1.2.3 Variation of parameters

In the theory of linear evolution equations an important tool is formula of variation of parameter. We follow (Driver, 1977) and consider the linear delay differential system

$$x'(t) = L(t, x_t) + h(t), \tag{1.2}$$

where $L : [0, A) \times \mathcal{C} \to \mathbb{R}^n$ is linear for all $t \in [0, A)$, h is a continuous n-vector-valued function on $[0, A)$. The variation of parameter give us an expression of $x(t; t_0, \phi)$ in terms of solutions of the linear homogeneous equation

$$y'(t) = L(t, y_t). \tag{1.3}$$

Next, we define the *unit step function* $u : [-q, 0] \to \mathbb{R}$ by

$$u(\sigma) = \begin{cases} 0 & \text{for } -q \leq \sigma < 0, \\ 1 & \text{for } \sigma = 0. \end{cases} \tag{1.4}$$

Theorem 1.4. *Let $L : [0, A) \times C \to \mathbb{R}^n$ is linear for all $t \in [0, A)$, h is a continuous n-vector-valued function on $[0, A)$. Then for each $\phi \in C$ the unique solution of equation* (1.2) *is given by:*

$$x(t; t_0, \phi) = y(t; t_0, \phi) + \int_{t_0}^{t} y(t; s, h(s)u_s)ds, \tag{1.5}$$

where $y(t; t_0, \phi)$ is the solution of (1.3) *with initial function ϕ.*

1.2.4 Halanay Inequality

Halanay (1966) proved an asymptotic formula for the solutions of a differential inequality with delay, and applied it in the stability theory of delayed linear systems. Since beginning of the twenty-first century several authors have called Halanay inequality to such inequality (see Baker and Tang, 2000; Mohamad and Gopalsamy, 2000; Liz and Trofimchuk, 2000; Liz and Ferreiro, 2002; Liz et al., 2005).

Theorem 1.5. *Let t_0 be a real number and q be a non-negative number. If $v : [t_0 - q, \infty) \to \mathbb{R}^+$ satisfies*

$$\frac{d}{dt}v(t) \leq -\alpha v(t) + \beta \left[\sup_{s \in [t-q, t]} v(s) \right]; \quad t \geq t_0$$

where α and β are constants with $\alpha > \beta > 0$, then

$$v(t) \leq \|v_{t_0}\|_q \, e^{-\eta(t-t_0)} \text{ for } t \geq t_0, \tag{1.6}$$

where η is the unique positive solution of

$$\eta = \alpha - \beta e^{\eta q}. \tag{1.7}$$

Proof. We follow the proof of Driver (1977). We start showing that equation (1.7) has a unique positive solution, we consider the function defined by:

$$\triangle(t) = t - \alpha + \beta e^{tq}.$$

Since $\triangle(0) < 0, \triangle(\alpha) > 0$, and $\triangle'(t) = 1 + \beta q e^{tq} > 0$, it follows that there is a unique $\eta \in \mathbb{R}$ for which $\triangle(\eta) = 0$. Moreover $0 < \eta < \alpha$.

Define $w(t) = \|v_{t_0}\|_q \, e^{-\eta(t-t_0)}$ for $t \in [t_0, \infty)$ and let $k > 1$ be arbitrary. Then $v(t) < kw(t)$ for $t_0 - q \leq t \leq t_0$. Now suppose that $v(t) = kw(t)$ for some $t \in (t_0, \infty)$. Then since v and w are continuous functions, there must exist some $t_1 \in (t_0, \infty)$ such that:

$$v(t) < kw(t) \text{ for } t_0 - q \leq t < t_1 \text{ and } v(t_1) = kw(t_1).$$

This could not occur if $v'(t_1)$ were lest to $kw'(t_1)$. But, on the other hand, we find

$$\begin{aligned} v'(t_1) &\leq -\alpha v(t_1) + \beta \left\| v_{t_1} \right\|_q \\ &< -\alpha k w(t_1) + \beta k w(t_1 - q) \\ &= k \left\| v_{t_0} \right\|_q e^{-\eta(t_1 - t_0)} (-\alpha + \beta e^{\eta q}) \\ &= k \left\| v_{t_0} \right\|_q e^{-\eta(t_1 - t_0)} (-\eta) \\ &= k w'(t_1), \end{aligned}$$

a contradiction. Thus we conclude that

$$v(t) < kw(t) \text{ for } t_0 \leq t.$$

Finally let $k \to 1$ to find $v(t) \leq w(t)$, which is (1.6). □

Baker and Tang (2000) were, to our knowledge, the first to get a Halanay type inequalities for non-autonomous functional differential equations. Mohamad and Gopalsamy (2000) they also obtained a Halanay type inequalities for non-autonomous functional differential equations. To conclude this section we show an integral version of Halanay inequality.

Lemma 1.1. *Consider*

$$v(t) \leq \left\| v_{t_0} \right\|_q e^{-\sigma(t-t_0)} f(t) + \int_{t_0}^{t} e^{-\sigma(t-s)} K \left[\sup_{u \in [s-q,s]} v(u) \right] ds, \quad t \geq t_0,$$

where σ, q and K are positive real numbers, and $f \in C(\mathbb{R})$ is a positive non decreasing function. If $\sigma > K > 0$, then there exist $\eta > 0$ and $M > 0$ such that

$$v(t) \leq \left\| v_{t_0} \right\|_q f(t) e^{-\eta(t-t_0)}, \quad t \geq t_0,$$

where η is the real solution of

$$\eta = \sigma - K e^{\eta q}.$$

Proof. We define $w(t) := \frac{v(t)}{f(t)}$, so we have

$$w(t) \leq \left\| v_{t_0} \right\|_q e^{-\sigma(t-t_0)} + \int_{t_0}^{t} e^{-\sigma(t-s)} K \frac{\left[\sup_{u \in [s-q,s]} f(u) \right]}{f(t)} \left[\sup_{u \in [s-q,s]} w(u) \right] ds.$$

Since the function f is non decreasing we have

$$w(t) \leq \left\| v_{t_0} \right\|_q e^{-\sigma(t-t_0)} + \int_{t_0}^{t} e^{-\sigma(t-s)} K \left[\sup_{u \in [s-q,s]} w(u) \right] ds.$$

Now define

$$\mu(t) := \begin{cases} \|v_{t_0}\|_q & \text{for } t_0 - q \leq t \leq t_0 \\ \|v_{t_0}\|_q e^{-\sigma(t-t_0)} + \int_{t_0}^{t} e^{-\sigma(t-s)} K \left[\sup_{u\in[s-q,s]} w(u)\right] ds & \text{for } t_0 \leq t. \end{cases}$$

Then μ is continuous and nonegative, and $w(t) \leq \mu(t)$ for $t_0 - q \leq t$. Moreover, for $t_0 \leq t$

$$\mu'(t) \leq -\sigma\mu(t) + K\left[\sup_{u\in[t-q,t]} \mu(u)\right].$$

Since $\sigma > K > 0$, by Halanay's inequality, there exists $\eta > 0$ such that

$$\mu(t) \leq e^{-\eta(t-t_0)}, \quad t \geq t_0.$$

Therefore

$$w(t) \leq \mu(t) \leq \|v_{t_0}\|_q e^{-\eta(t-t_0)}, \quad t \geq t_0.$$

and

$$v(t) \leq \|v_{t_0}\|_q f(t) e^{-\eta(t-t_0)}, \quad t \geq t_0.$$

□

1.3 Basic theory of differential equations with Piecewise Constant Argument

Differential equations with piecewise continuous arguments (DEPCA) arise when looking for an extension of the theory of functional differential equations (FDE) with continuous arguments to differential equations with discontinuous argument.

The theory of scalar DEPCA of type

$$x'(t) = f(t, x(t), x(h(t))),$$

where the argument $h(t)$ is constant on certain intervals, for example $h(t)$ is the gratest-integer function. The study of DEPCA was initiated by Wiener (1983, 1984); Cooke and Wiener (1984). DEPCA has received attention of several researcher because include impulsive and loaded equations of control theory, and some biomedical models have used DEPCA as successful mathematical model, see Busenberg and Cooke (1988); Byrne and Gourley (2001); Gourley (2003).

DEPCA are hybrid equations, because they combine the characteristics of continuous and discrete equations. The continuity of a solution in points that unite two consecutive intervals, implies the existence of recursive relations for the solution in these points.

Akhmet (2007) introduced differential equations with piecewise constant argument of generalized type (DEPCAG), these kind of functional differential equation have been widely developed (see for instance Akhmet, 2008, 2010; Pinto, 2009, 2011, and references therein).

1.3.1 Approximation using piecewise constant argument

Given a real normed vectorial space $\mathbb{V}$, and the vectorial delay differential equations,

$$y'(t) = f(t, y(t), y(t-\tau)), \quad \tau > 0, \tag{1.8}$$

with the initial condition

$$y(t) = \varphi(t), \quad -\tau \le t \le 0, \quad \varphi \in C \equiv C([-\tau, 0], \mathbb{V}), \tag{1.9}$$

where $f : \mathbb{R}^+ \times \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is a continuous function and $\tau > 0$ is a real number. In Györi (1991) was introduced an approximating delay differential equation with piecewise constant argument related to equation (1.8) in the following form:

$$z_h'(t) = f\left(t, z_h([t]_h), z_h([t]_h - [\tau]_h)\right), \tag{1.10}$$

with the initial condition

$$z_h(nh) = \varphi(nh), \quad n = -k, \cdots, 0, \tag{1.11}$$

where $h = \frac{\tau}{k}$ and $k \ge 1$ is an integer, and $[\cdot]_h = \left[\frac{\cdot}{h}\right] h$ is the usual greatest integer function. By a solution of initial value problem (1.10)-(1.11) we mean a function z_h defined on $\{-kh :$ $k = 0, 1, \cdots,$ such that $-\lambda \le -kh \le 0\}$ by (1.11), which satisfy the following properties on $\mathbb{R}^+$:

1. The function z_h is continuous on $\mathbb{R}^+$,
2. the derivative $z_h'(t)$ exists at each point $t \in \mathbb{R}^+$ with the possible exception of the points $kh(k = 0, 1, 2, \cdots)$ where finite one-sided derivatives exist, and
3. the function z_h satisfies (1.10) on each interval $I_{(k,h)} := [kh, (k+1)h)$ for $k = 0, 1, 2, \cdots$.

The equation (1.10) it is equivalent to

$$z_h'(t) = f\left(t, z_h(hn), z_h(h(n-k))\right), \quad t \in I_{(n,h)}, \ n \ge 0,$$

with the initial condition (1.11). We have that the solution of (1.10)-(1.11) must be a solution of

$$z_h(t) = z_h(h_n) + \int_{h_n}^{t} f(s, a(n), a(n-k))ds, t \in I_{(n,h)},$$

where $h_n = hn$; and $a(n) := a_\varphi(n)$ is a family of initial value that must solve the delayed system of difference equations

$$a(n+1) = z_h(h_n) + \int_{h_n}^{h_{n+1}} f(s, a(n), a(n-k))ds, n \in \mathbb{N}, \tag{1.12}$$

with the initial condition

$$a(n) = \varphi(h_n), \quad n = -k, \cdots, 0. \tag{1.13}$$

It follows from our discussion that is necessary for the approximation method the existence of unique solutions of delayed nonlinear equations (1.8)-(1.9), (1.10)-(1.11) and (1.12)-(1.13).

2

Non-autonomous differential equations with variable delay

The study of transference of uniform asymptotic stability between solutions of linear differential equation with delay and the corresponding discrete difference equation started with Cooke and Györi (1994).

Györi et al. (1996) approximated several differential equations with time- and state-dependent delay. However, when they computed the approximation for the functional differential equation

$$y'(t) = y(t - |y(t)|) + \sin(2t) - \sin^2(t - \sin^2(t)), t \geq 0,$$

with initial function $\phi(t) = \sin^2(t)$, they noted differences between asymptotic behaviours of numerical and actual solutions (see Hartung and Turi, 1995).

Cooke and Ivanov (2000) studied the dynamic of the solutions of a singular difference equation with delay which can be interpreted as an Euler discretizations of a singularly perturbed differential equations with delay. They stated, in the conclusion of their paper, that numerical approximation of solutions of singularly perturbed delay differential equations maybe showing dynamics which are irrelevant to the actual dynamics in these equations. These type of difficulties, namely spurious fixed point, are well-known for Runge-Kutta methods of numerical approximation for ordinary differential equations.

Mohamad and Gopalsamy (2000) proved Halanay-type inequalities for nonautonomous scalar systems with discrete and distributed delays. They use piecewise constant argument to obtain a discrete non-autonomous difference systems and then show discrete time inequalities, which are analogues of continuous time inequalities. Therefore, in some sense, they also proved the transference of stability of differential equations with delay and the corresponding difference equation.

Liz and Ferreiro (2002) proved a discrete version of Halanay's inequality and used it to obtain the transference of asymptotic stability between the solutions of

$$x'(t) = -ax(t) + b(t)f(t, x_t), \quad x_0 = \phi,$$

and

$$\frac{x_{n+1} - x_n}{h} = -ax_n + b(t_n)f(t_n, \phi_n).$$

Győri and Hartung (2002) studied the transference of uniform asymptotic stability between solutions of linear neutral differential equation with constant delay and the corresponding discrete difference equation, this way they extended and complement the result of Cooke and Győri (1994).

Mohamad and Gopalsamy (2003) studied the asymptotic behaviour of continuous-time cellular neural networks with discrete delays. Using piecewise constant argument they obtained discrete-time analogues of the continuous-time cellular neural networks and proved that the asymptotic behaviour of the continuous-time systems are preserved by the discrete-time analogues.

In this chapter we study discretization, approximation and transference of uniform asymptotic stability of a non-autonomous linear differential equations with variable delay.

2.1 Discretization by piecewise constant argument

We consider the homogeneous non-autonomous linear differential equations with variable delay

$$x'(t) = -a(t)x(t - r(t)), \tag{2.1}$$

where $a : [0, \infty) \to [0, \infty)$ and $r : [0, \infty) \to [0, q]$, with initial condition

$$x(t) = \varphi(t), \quad -q \le t \le 0, \quad \varphi \in \mathcal{C} \equiv C([-q, 0], \mathbb{R}), \tag{2.2}$$

where $q := \sup_{t\in\mathbb{R}_0^+}\{r(t)\}$ is a positive real number. The set of DEPCA corresponding to equation (2.1) is:

$$z_h'(t) = -a(t)z_h\left(\left[t - \left[r\left([t]_h\right)\right]_h\right]_h\right), \tag{2.3}$$

where $[\cdot]_h = [\frac{\cdot}{h}]h$ with $[\cdot]$ the usual greatest integer function. The initial condition for differential equation (2.3) is

$$z_h(nh) = \varphi(nh), \quad n = -k, \cdots, 0, \tag{2.4}$$

where h is a positive number in the interval $(0, q]$. In fact we can consider $h = \frac{q}{k}$ where $k \ge 1$ is an integer. By a solution of (2.3)-(2.4) we mean a function z_h defined on $\{ih : i = -k, \cdots, 0\}$ by (2.4), which satisfy the following properties on $\mathbb{R}^+$:

i) The function z_h is continuous on $\mathbb{R}^+$,

ii) the derivative $z_h'(t)$ exists at each point $t \in \mathbb{R}^+$ with the possible exception of the points $ih (i = 0, 1, 2, \cdots)$ where finite one-sided derivatives exist, and

iii) the function z_h satisfies (2.3) on each interval $I_{(j,h)} := [jh, (j+1)h)$ for $j = 0, 1, 2, \cdots$.

Note that for every positive h close to zero, it is expected that solutions of (2.3)-(2.4) has similar qualitative features to the solutions of (2.1)-(2.2), since $[t]_h \to t$ uniformly on $\mathbb{R}$, as $h \to 0$. But this can be false, even for Euler's method, in the setting of singular functional differential equations, see Cooke and Ivanov (2000).

If we denote

$$\gamma_h(t,r) := \left[\frac{t}{h} - \left[\frac{r([\frac{t}{h}]h)}{h}\right]\right]h,$$

then equation (2.3) can be write like

$$z_h'(t) = -a(t)z_h(\gamma_h(t,r)).$$

For $t \in I_{(i,h)}$ the function $r([\frac{t}{h}]h)$ take just one value, therefore the function $\left[\frac{r([\frac{t}{h}]h)}{h}\right] := k_i$ is a fixed integer for $t \in I_{(i,h)}$. It follows that

$$\begin{aligned}
\left[\frac{t}{h} - \left[\frac{r([\frac{t}{h}]h)}{h}\right]\right] &= \left[\frac{t}{h} - k_i\right] \\
&= \left[\frac{t - ih + ih}{h} - k_i\right] \\
&= \left[\frac{t - ih}{h} + i - k_i\right],
\end{aligned}$$

since $t \in I_{(i,h)}$ we have $0 \leq \frac{t-ih}{h} \leq 1$, so

$$\left[\frac{t}{h} - \left[\frac{r([\frac{t}{h}]h)}{h}\right]\right] = i - k_i.$$

It is follows that

$$\gamma_h(t,r) = h(i - k_i), \text{ for } t \in I_{(i,h)}.$$

Therefore (2.3) is equivalent to

$$z_h'(t) = -a(t)z_h(h(i - k_i)), \quad t \in I_{(i,h)},\ i \geq 0. \tag{2.5}$$

Note that, for $ih \leq t \leq (i+1)h$, we integrate (2.5) and obtain

$$z_h(t) = z_h(ih) - \int_{ih}^{t} a(s)ds z_h(h(i - k_i)).$$

Making $t \to (i+1)h^-$, from the continuity of z_h, we obtain

$$z_h((i+1)h)ds = z_h(ih) - \int_{ih}^{(i+1)h} a(s)ds z_h(h(i - k_i)).$$

Therefore the sequence $\mathfrak{z}_h(i) := z_h(ih)$ satisfy the linear difference equation with variable delay

$$\mathfrak{z}_h(n+1) = \mathfrak{z}_h(n) - \int_{nh}^{(n+1)h} a(s)ds \mathfrak{z}_h(n - k_n), \tag{2.6}$$

with initial conditions

$$\mathfrak{z}_h(n) = \varphi(nh), \quad n = 0, 1, \cdots, \quad -q \le -nh \le 0. \tag{2.7}$$

Note that (2.6)-(2.7) is a discretization of the differential equations with variable delay (2.1)-(2.2) that coincides with Euler's approximation method for autonomous differential equations with variable delay. From the recurrence relation (2.6) and initial conditions, we have

$$\begin{aligned}
\mathfrak{z}_h(0) &= \varphi(0) \\
\mathfrak{z}_h(1) &= \mathfrak{z}_h(0) - \int_0^h a(s)ds\mathfrak{z}_h(0 - k_0), \\
\mathfrak{z}_h(2) &= \mathfrak{z}_h(1) - \int_h^{2h} a(s)ds\mathfrak{z}_h(1 - k_1) \\
&= \mathfrak{z}_h(0) - \int_0^h a(s)ds\mathfrak{z}_h(0 - k_0) - \int_h^{2h} a(s)ds\mathfrak{z}_h(1 - k_1).
\end{aligned}$$

Therefore the sequence $\mathfrak{z}_h$ solution of (2.6)-(2.7) is well-defined, and satisfy

$$\mathfrak{z}_h(n) = \varphi(0) - \sum_{i=0}^{n-1} \int_{ih}^{(i+1)h} a(s)ds\mathfrak{z}_h(i - k_i), \quad k \ge 0. \tag{2.8}$$

From (2.8), it follows that the solution of (2.3)-(2.4) for $t \ge 0$ can be written

$$z_h(t) = \mathfrak{z}_h(n) + \int_{nh}^t a(s)ds\mathfrak{z}_h\,(n - k_n)\,,$$

or

$$z_h(t) = \varphi(0) - \sum_{i=0}^{n-1} \int_{ih}^{(i+1)h} a(s)ds\mathfrak{z}_h(i - k_i) - \int_{nh}^t a(s)ds\mathfrak{z}_h\,(n - k_n)\,,$$

where $n = n(t)$ is such that $nh \le t < (n+1)h$. Thus, we have proved

Theorem 2.1. *The initial value problem* (2.3)-(2.4) *has a unique solution in the form*

$$z_h(t) = \varphi(0) - \sum_{i=0}^{n-1} \int_{ih}^{(i+1)h} a(s)ds\mathfrak{z}_h(i - k_i) - \int_{nh}^t a(s)ds\mathfrak{z}_h\,(n - k_n)$$

for $t \ge 0$ and the sequence $\mathfrak{z}_h(\cdot)$ satisfies the nonlinear difference equations (2.6) *with initial conditions* (2.7).

2.2 Aproximation over compact interval

We begin this section by addressing the problem of approximation of solutions of initial value problem (2.1)-(2.2) over compact interval. We follows some ideas of Györi (1991) to obtain.

Theorem 2.2. *If $r : [0, \infty) \to [0, q]$ is a continuous function then, for any $\varphi \in C$, the solutions $x(\varphi)(t)$ and $z_h(\varphi)(t)$ of the initial value problems (2.1)-(2.2) and (2.3)-(2.4), respectively, satisfy the following relations for all $T > 0$*

$$\lim_{h\to 0} \max_{0\le t\le T} \|x(\varphi)(t) - z_h(\varphi)(t)\| = 0,$$

namely

$$\max_{0\le t\le T} \|x(\varphi)(t) - z_h(\varphi)(t)\| \le \left[e^{\int_0^T a(s)ds} \int_0^T a(s)ds\right] w_x\left(w_r(h;T) + 2h; T\right). \tag{2.9}$$

where $w_r(h;T)$ and $w_x(h;T)$ are defined by

$$\begin{aligned} w_r(h;t) &= \max\left\{|r(t_2) - r(t_1)| : \ 0 \le t_1, t_2 \le t, \ |t_2 - t_1| \le h\right\}, \\ w_x\left(w_r(h;t) + 2h; t\right) &= \max\left\{|x(t_2) - x(t_1)| : \ -q \le t_1, t_2 \le t, \ |t_2 - t_1| \le 2h + w_r(h,t)\right\}. \end{aligned}$$

Proof. Consider the solutions $x(t) = x(\varphi)(t)$ and $z_h(t) = z_h(\varphi)(t)$ of initial value problems (2.1)-(2.2) and (2.3)-(2.4), respectively. Then from (2.1) and (2.3) we find

$$x'(t) - z_h'(t) = -a(t)\left[x(t - r(t)) - z_h(\gamma_h(t,r))\right],$$

for all $t \ge 0$. Thus the function $\varepsilon_h(t) = x(t) - z_h(t)$ satisfies

$$\varepsilon_h'(t) = -a(t)\varepsilon_h(\gamma_h(t,r)) - a(t)\left[x(t - r(t)) - x(\gamma_h(t,r))\right],$$

for all $t \ge 0$ with $\varepsilon_h(0) = 0$ and $\varepsilon_h(\gamma_h(t,r)) = x(\gamma_h(t,r)) - z_h(\gamma_h(t,r))$. We integrate over $[0,t]$ and obtain

$$\begin{aligned} |\varepsilon_h(t)| &\le \int_0^t a(s)|\varepsilon_h(\gamma_h(s,r))|ds + \int_0^t a(s)\left|x(s - r(s)) - x(\gamma_h(s,r))\right| ds \\ &\le \int_0^t a(s)|\varepsilon_h(s, r(s))|ds + f_h(t), \end{aligned}$$

where

$$f_h(t) := \int_0^t a(s)\left|x(s - r(s)) - x(\gamma_h(s,r))\right| ds, \quad t \ge 0.$$

On the other hand,

$$\gamma_h(s,r) \le s \text{ for all } s \ge 0,$$

and from initial conditions we have that

$$|\varepsilon_h(\gamma_h(s,r))| = |\varphi(\gamma_h(s,r)) - x_h(\gamma_h(s,r))| = 0,$$

for all $s \ge 0$ such that $\gamma_h(s - r) \le 0$. Therefore function $\xi(t) = \max_{0\le s\le t} |\varepsilon_h(s)|$ satisfies the inequality

$$\xi(t) \le \int_0^t a(s)\xi(\gamma_h(s,r))ds + f_h(t) \le \int_0^t a(s)\xi(s))ds + f_h(t), \quad t \ge 0,$$

since the integral term and $f_h(t)$ are monotone increasing functions. Now, from Gronwall-Bellman inequality we find

$$\xi(t) \leq f_h(t) e^{\int_0^t a(s)ds}, \quad t \in [0,T]. \tag{2.10}$$

Now, we note that $|t - r(t) - \gamma_h(t,r)| = |t - r(t) - (i - k_i)h|$ where $i = [\frac{t}{h}]$ and $k_i = \left[\frac{r([\frac{t}{h}]h)}{h}\right]$, so

$$\begin{aligned} |t - r(t) - \gamma_h(t,r)| &\leq |t - ih| + |r(t) - k_i h| \\ &= \left| t - \left[\frac{t}{h}\right] h \right| + \left| r(t) - \left[\frac{r([\frac{t}{h}]h)}{h}\right] h \right| \\ &\leq h + \left| r(t) - r([\frac{t}{h}]h) \right| + \left| r([\frac{t}{h}]h) - \left[\frac{r([\frac{t}{h}]h)}{h}\right] h \right| \\ &\leq h + \left| r(t) - r([\frac{t}{h}]h) \right| + h \\ &\leq 2h + w_r(h;t), \end{aligned} \tag{2.11}$$

where $w_r(h;t) = \max\{|r(t_2) - r(t_1)| : \ 0 \leq t_1, t_2 \leq t, \ |t_2 - t_1| \leq h\}$. Note that for uniformly continuous function r, $w_r(h,t)$ tends to zero as $h \to 0$. Set

$$w_x(w_r(h;t) + 2h; t) = \max\{|x(t_2) - x(t_1)| : \ -q \leq t_1, t_2 \leq t, \ |t_2 - t_1| \leq 2h + w_r(h,t)\}.$$

Then from (2.11) it is follows that

$$|x(s - r(s)) - x(\gamma_h(s,r))| \leq w_x(w_r(h;t) + 2h; t),$$

for all $0 \leq s \leq t$ and for all r. Also,

$$f_h(t) \leq \int_0^t a(s)ds \ w_x(w_r(h;t) + 2h; t), \quad t \geq 0,$$

and clearly (2.10) yields

$$\xi(t) \leq e^{\int_0^t a(s)ds} \int_0^t a(s)ds \ w_x(w_r(h;T) + 2h; T), \quad t \in [0,T]. \tag{2.12}$$

Since (2.12) and $|\varepsilon_h(t)| = |x(t) - z_h(t)| = |x(\varphi)(t) - z_h(\varphi)(t)| \leq \xi(t)$ we obtain (2.9) for all $h = \frac{q}{k} > 0$ and $t \in [0,T]$. So, for all $T > 0$

$$\max_{0 \leq t \leq T} |x(\varphi)(t) - z_h(\varphi)(t)| \leq \left[e^{\int_0^T a(s)ds} \int_0^T a(s)ds \right] \ w_x(w_r(h;T) + 2h; T) \to 0,$$

as $h \to 0$, from the uniform continuity of the functions x and r on $[0,T]$.

□

2.3 Transference of stability properties

In this section we obtain a necessary conditions for transference of uniform asymptotic stability of the zero solution of (2.1) to the zero solution (2.6). We assume that:

(A1) The zero solution of (2.1) is uniformly asymptotically stable, (see definition 1.2),

(A2) the function $a(t)$ is bounded, namely

$$a_0 = \sup_{t \geq 0} |a(t)| < \infty. \tag{2.13}$$

(A3) The function $r(t)$ is uniformly continuous on $[0, \infty)$.

Next we will obtain an estimate for the distance between the solutions of initial value problems (2.1)-(2.2) and (2.3)-(2.4) on $[0, \infty)$.

Theorem 2.3. *If the assumptions (A1), (A2) and (A3) holds. Then for h small enough, and for every $\varphi \in C$ the solutions $x(\varphi)(t)$ and $z_h(\varphi)(t)$ of the linear differential equations with delay* (2.1) *and* (2.3)*, respectively, satisfy*

$$|x(\varphi)(t) - z_h(\varphi)(t)| \leq \left[K \sup_{t_0 - q \leq s \leq t_0} |x(\varphi)(s) - z_h(\varphi)(s)| + K_1(h) M_1 \|\varphi\|_q t \right] e^{-\eta(t-t_0)}, \quad t \geq t_0,$$

where $\eta > 0$, $t_0 = 3q + w_r(q)$ and $K_1(h) = a_0^2 \left[2h + w_r(h)\right] K$, with a_0 defined in (2.13) *and*

$$w_r(\ell) = \max\left\{ |r(t_2) - r(t_1)| : \ 0 \leq t_1, t_2; \ |t_2 - t_1| \leq \ell \right\},$$

and $K_1(h) \to 0$ as $h \to 0$.

Proof. Consider the solutions $x(t) = x(\varphi)(t)$ and $z_h(t) = z_h(\varphi)(t)$ of differential equations with delay (2.1) and (2.3), respectively. We define the error function $E_h(\cdot) := x(\cdot) - z_h(\cdot)$, it follows that

$$E_h'(t) = -a(t) \left[x(t - r(t)) - z_h(\gamma_h(t, r)) \right],$$

for all $t \geq 0$. Adding and substracting $a(t)z(t - r(t))$ we obtain

$$E_h'(t) = -a(t) E_h(t - r(t)) - a(t) \left[z_h(t - r(t)) - z_h(\gamma_h(t, r)) \right],$$

and, by fundamental theorem of calculus, we have

$$E_h'(t) = -a(t) E_h(t - r(t)) - a(t) \int_{\gamma_h(t,r)}^{t - r(t)} z_h'(\xi) d\xi.$$

Now, from (2.3), we obtain

$$E_h'(t) = -a(t) E_h(t - r(t)) - a(t) \int_{\gamma_h(t,r)}^{t - r(t)} a(\xi) z_h(\gamma_h(\xi, r)) d\xi,$$

it is follows that

$$E_h'(t) = -a(t)E_h(t - r(t)) + a(t)\int_{\gamma_h(t,r)}^{t-r(t)} a(\xi)E_h(\gamma_h(\xi,r))d\xi + g_h(t),$$

where

$$g_h(t) := -a(t)\int_{\gamma_h(t,r)}^{t-r(t)} a(\xi)x(\gamma_h(\xi,r))d\xi. \tag{2.14}$$

Since variation-of-constants formula, (see Driver, 1977, pp. 334), we have

$$E_h(t) = U(t;t_0,E_{h_{t_0}}) + \int_{t_0}^{t} U\left(t;s,\left[a(s)\left(\int_{\gamma_h(s,r)}^{s-r(s)} a(\xi)E_h(\gamma_h(\xi,r))d\xi\right) + g_h(s)\right]u\right)ds,$$

where $U(t;t_0,E_{h_{t_0}})$ is the unique solution of equation (2.1) with initial value $E_{h_{t_0}}$ at t_0, and u is the unit step function $u:[-q,0]\to\mathbb{R}$ defined by (1.4). Thus $|E_h(t)|$ for all $t\geq t_0$ satisfies

$$|E_h(t)| \leq |U(t;t_0,E_{h_{t_0}})| + \int_{t_0}^{t}\left|U\left(t;s,\left[a(s)\left(\int_{\gamma_h(s,r)}^{s-r(s)} a(\xi)E_h(\gamma_h(\xi,r))d\xi\right) + g_h(s)\right]u\right)\right|ds. \tag{2.15}$$

Since we assume that zero solution of (2.1) is uniformly asymptotically stable, there are constants $\sigma > 0$ and $K > 0$, (see Hale and Lunel, 1993, pp. 185), such that for each $\phi\in\mathcal{C}$ we have

$$|U(t;s,\phi)| \leq K\left\|\phi\right\|_q e^{-\sigma(t-s)}, \quad t\geq s. \tag{2.16}$$

Moreover, there exists a constant M_0 such that

$$|x(t)| \leq M_0\left\|\varphi\right\|_q e^{-\sigma t}, \quad t\geq 0. \tag{2.17}$$

In order to use (2.17) to estimate $g_h(t)$ we need find a positive real number t_0 such that for $t\geq t_0$ then

$$0\leq \gamma_h(\xi,t), \text{ whenever } \xi\geq\gamma_h(t,r).$$

We recall (2.11), i.e., $|t-r(t)-\gamma_h(t,r)| = |\gamma_h(t,r)-t+r(t)| \leq 2h + w_r(h)$, it is follows

$$t - r(t) - w_r(h) - 2h \leq \gamma_h(t,r) \leq t - r(t) + w_r(h) + 2h.$$

Since $h\in(0,q]$ and r is uniformly continuous on $[0,\infty)$ it follows

$$t - 3q - w_r(q) \leq t - r(t) - w_r(q) - 2q \leq t - r(t) - w_r(h) - 2h \leq \gamma_h(t,r). \tag{2.18}$$

Now we use (2.17) and (2.13) in (2.14) to estimate $g_h(t)$ for $t\geq t_0 := 3q + w_r(q)$ and obtain:

$$\begin{aligned} |g_h(t)| &\leq a_0^2\int_{\gamma_h(t,r)}^{t-r(t)} |x(\gamma_h(\xi,r))|d\xi \\ &\leq a_0^2\int_{\gamma_h(t,r)}^{t-r(t)} M_0\left\|\varphi\right\|_q e^{-\sigma\gamma_h(\xi,r))}d\xi \\ &= a_0^2\int_{\gamma_h(t,r)}^{t-r(t)} M_0\left\|\varphi\right\|_q e^{-\sigma\xi}e^{\sigma[\xi-\gamma_h(\xi,r))]}d\xi, \end{aligned}$$

since (2.11) and uniform continuity of r we have

$$
\begin{aligned}
|g_h(t)| &\leq a_0^2 \int_{\gamma_h(t,r)}^{t-r(t)} M_0 \|\varphi\|_q e^{-\sigma\xi} e^{\sigma[w_r(h)+2h]} d\xi \\
&\leq e^{-\sigma\gamma_h(t,r))} a_0^2 \int_{\gamma_h(t,r)}^{t-r(t)} M_0 \|\varphi\|_q e^{\sigma[w_r(h)+2h]} d\xi. \qquad (2.19)
\end{aligned}
$$

Next, using (2.11), we estimate $s - \gamma(s, r)$

$$
\begin{aligned}
s - \gamma(s,r) &= s - r(s) - \gamma(s,r) + r(s) \\
&\leq q + 2h + w_r(h) \\
&\leq 3q + w_r(q). \qquad (2.20)
\end{aligned}
$$

Since $t \geq t_0 := 3q + w_r(q)$, (2.11), (2.18) and (2.20) inequality (2.19) become into

$$
|g_h(t)| \leq e^{-\sigma t} a_0^2 M_1 \|\varphi\|_q [2h + w_r(h)], \qquad (2.21)
$$

where

$$
M_1 := M_0 e^{\sigma[5q+2w_r(q)]}.
$$

Using estimations (2.16) and (2.21) for $t_0 := 3q + w_r(q)$, in (2.15) we obtain for $t \geq t_0$

$$\begin{aligned}
|E_h(t)| &\leq |U(t;t_0,E_{h_{t_0}})| + \int_{t_0}^{t} \left| U\left(t;s,\left[a(s)\left(\int_{\gamma_h(s,r)}^{s-r(s)} a(\xi)E_h(\gamma_h(\xi,r))d\xi\right) + g_h(s)\right]u\right)\right| ds \\
&\leq K\left\|E_{h_{t_0}}\right\|_q e^{-\sigma(t-t_0)} \\
&\quad + \int_{t_0}^{t} K \left\| \left[a(s)\left(\int_{\gamma_h(s,r)}^{s-r(s)} a(\xi)E_h(\gamma_h(\xi,r))d\xi\right) + g_h(s)\right]u\right\|_q e^{-\sigma(t-s)}ds \\
&\leq K\left\|E_{h_{t_0}}\right\|_q e^{-\sigma(t-t_0)} \\
&\quad + \int_{t_0}^{t} K \left(\left|a(s)\left(\int_{\gamma_h(s,r)}^{s-r(s)} a(\xi)E_h(\gamma_h(\xi,r))d\xi\right)\right| + |g_h(s)|\right) e^{-\sigma(t-s)}ds \\
&\leq K\left\|E_{h_{t_0}}\right\|_q e^{-\sigma(t-t_0)} + a_0^2\int_{t_0}^{t} K\left|\int_{\gamma_h(s,r)}^{s-r(s)} E_h(\gamma_h(\xi,r))d\xi\right| e^{-\sigma(t-s)}ds \\
&\quad + \int_{t_0}^{t} Ke^{-\sigma s}a_0^2 M_1\|\varphi\|_q\left[2h + w_r(h)\right]e^{-\sigma(t-s)}ds \\
&\leq K\left\|E_{h_{t_0}}\right\|_q e^{-\sigma(t-t_0)} + a_0^2\int_{t_0}^{t} K\left(\sup_{\gamma_h(s,r)\leq\varsigma\leq s-r(s)}|E_h(\varsigma)|\right)\int_{\gamma_h(s,r)}^{s-r(s)} d\xi e^{-\sigma(t-s)}ds \\
&\quad + a_0^2 K M_1\|\varphi\|_q\left[2h + w_r(h)\right]e^{-\sigma t}t \\
&= K\left\|E_{h_{t_0}}\right\|_q e^{-\sigma(t-t_0)} + a_0^2\int_{t_0}^{t} K\left(\sup_{\gamma_h(s,r)\leq\varsigma\leq s-r(s)}|E_h(\varsigma)|\right)\left(s - r(s) - \gamma_h(s,r)\right)e^{-\sigma(t-s)}ds \\
&\quad + a_0^2 K M_1\|\varphi\|_q\left[2h + w_r(h)\right]e^{-\sigma t}t \\
&\leq K\left\|E_{h_{t_0}}\right\|_q e^{-\sigma(t-t_0)} + a_0^2\left[2h + w_r(h)\right]\int_{t_0}^{t} K\left(\sup_{s-3q-w_r(h)\leq\varsigma\leq s}|E_h(\varsigma)|\right)e^{-\sigma(t-s)}ds \\
&\quad + a_0^2 K M_1\|\varphi\|_q\left[2h + w_r(h)\right]e^{-\sigma t}t \\
&= e^{-\sigma(t-t_0)}\left[K\left\|E_{h_{t_0}}\right\|_q + K_1(h)M_1\|\varphi\|_q e^{\sigma t_0}t\right] + K_1(h)\int_{t_0}^{t} e^{-\sigma(t-s)}\sup_{s-t_0\leq\varsigma\leq s}|E_h(\varsigma)|ds,
\end{aligned}$$

where $K_1(h) := a_0^2\left[2h + w(r;h)\right]K$. If h is small enough such that:

$$\sigma > K_1(h),$$

then by Halanay type inequality, Lemma1.1, there exists $\eta > 0$ such that

$$|E_h(t)| \leq \left[K\left\|E_{h_{t_0}}\right\|_q + K_1(h)M_1\|\varphi\|_q e^{\sigma t_0}t\right]e^{-\eta(t-t_0)}, \quad t \geq t_0, \tag{2.22}$$

where η is the positive solution of

$$\eta = \sigma - K_1(h)e^{\eta q}.$$

□

Thus we have proved the fundamental theorem of this chapter. There are similar results to our results, however the novelty of our theorem lies in the considered delayed differential equation and the technique used in the proof. Both Cooke and Györi (1994) and Györi and Hartung (2002) they relied on the Gronwall-Bellman inequality to obtain an estimating exponential decay. Using Gronwall-Bellman inequality (2.22) become into

$$|E_h(t)| \leq \left[K \left\|E_{h_{t_0}}\right\|_q + K_1(h)M_1 \|\varphi\|_q e^{\sigma t_0} t\right] e^{-\sigma_0(t-t_0)}, \quad t \geq t_0,$$

where $\sigma_0 = \sigma - K_1(h)e^{\sigma q}$. Therefore we need h small enough such that

$$\sigma > K_1(h)e^{\sigma q} = a_0^2 \left[2h + w(r;h)\right] K e^{\sigma q}.$$

On the other hand, the necessary condition to use Halanay type inequality is: h small enough such that

$$\sigma > K_1(h) = a_0^2 \left[2h + w(r;h)\right] K,$$

then

$$|E_h(t)| \leq \left[K \left\|E_{h_{t_0}}\right\|_q + K_1(h)M_1 \|\varphi\|_q e^{\sigma t_0} t\right] e^{-\eta(t-t_0)}, \quad t \geq t_0,$$

where η is the positive real solution of

$$\eta = \sigma - K_1(h)e^{\eta q}.$$

We note that the size of h is independent of the delay size q, and the number $-\eta$ is the unique real solution of the characteristic equation

$$\lambda = -\sigma + K_1(h)e^{-\lambda q}, \tag{2.23}$$

corresponding to the differential equations with delay

$$y'(t) = -\sigma y(t) + K_1(h)y(t-q).$$

In fact $-\eta$ is the eigenvalue of the characteristic equation (2.23) with the greatest real part.

Now we can prove that the solutions of (2.6)-(2.7) approximate uniformly the solutions of (2.1)-(2.2), and also that zero solution of (2.6)-(2.7) is uniformly asymptotically stable.

Corollary 2.1. *Under the conditions of the theorem 2.3, we have that:*

1. $|x(t) - z_h([t]_h)| \to 0$ *as* $h \to 0$, *for* $t > 0$;

2. *the zero solution of the difference equations with delay* (2.6)-(2.7) *is uniform asymptotically stable.*

Proof. In section 2.1 we have shown that (2.6)-(2.7) correspond to a discrete version of differential equation (2.1)-(2.2). We recall that $z_h(hn) = \mathfrak{z}_h(n)$ for n any positive integer. If $t < t_0$ we use theorem 2.2. If $t \geq t_0$ then

$$|x(t) - z_h([t]_h)| \leq |x(t) - x([t]_h)| + |x([t]_h) - z_h([t]_h)|.$$

Then, from inequality (2.22), we have

$$|x(t) - z_h([t]_h)| \leq |x(t) - x([t]_h)| + \left[K \left\| E_{h_{t_0}} \right\|_q + K_1(h) M_1 \|\varphi\|_q [t]_h \right] e^{-\eta([t]_h - t_0)}.$$

Next, for $\epsilon > 0$ there are positive constants h_1, h_2 and h_3 such that:
If $h < h_1$ then $|x(t) - x([t]_h)| < \frac{\epsilon}{3}$, since continuity of x. If $h < h_2$ then $\left\| E_{h_{t_0}} \right\|_q < \frac{\epsilon}{3K}$, from theorem 2.2. If $h < h_3$ then $K_1(h) < \frac{\epsilon}{3M_1\|\varphi\|_q} \frac{\eta e}{e^{\eta t_0}}$. Therefore, for $h < \min\{h_1, h_2, h_3\}$ it is follows

$$|x(t) - z_h([t]_h)| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

We have shown part 1. Since inequality (2.22) we have also

$$|x(nh) - \mathfrak{z}_h(n)| \leq \left[K \left\| E_{h_{t_0}} \right\|_q + K_1(h) M_1 \|\varphi\|_q nh \right] e^{-\eta(nh - t_0)} \to 0 \text{ as } n \to \infty.$$

Since the zero solution of (2.1)-(2.2) is uniformly asymptotically stable, and $\|x(nh) - \mathfrak{z}_h(n)\|$ decay rate exponentially to zero, it is follows that $\mathfrak{z}_h(n)$ tends to zero exponentially, for every initial conditions φ. Since the difference equation (2.6) is lineal, and the zero solution is exponentially stable, it follows that the zero solution of (2.6)-(2.7) is uniformly asymptotically stable. □

Thus we have shown that under the hypotheses of theorem 2.3 the numerical approximations of equation (2.6) are good for all $T > 0$, independent of the size of T. Moreover the corresponding discrete difference equation is uniformly asymptotically stable also. Furthermore, our result is independent of delay size, this was possible thanks to our use of inequality Halanay. Thus we extend and improve the results of Cooke and Györi (1994). We use piecewise constant argument, theory of functional differential equations and Halanay-type inequality in the proof. We emphasize our use of Halanay-type inequality at the proof of our result because given us the estimate (2.22) an optimal inequality for transference of uniform stability, which improve the results of Cooke and Györi (1994) obtained using Gronwall-Bellman inequality.

2.4 Applications and examples

In our result we assume that the zero solution of (2.1) is uniformly asymptotically stable, the problem of find necessary conditions for uniform stability of non-autonomous differential equations with variable delay called the attention of several authors because its difficulty. Next we recall some stability criteria for equation (2.1) and apply our result to obtain stability criteria for difference equation.

A classic result of stability for functional differential equations can be found in Yorke (1970). A consequence of Yorke's theorem is:

Theorem A (Yorke). *If the function* $a(\cdot)$ *satisfy*

$$0 < a(t) \leq \alpha, \quad t \geq 0;$$

for a positive constant α *such that*

$$0 < \alpha q < \frac{3}{2}.$$

Then the zero solution of (2.1) *is uniformly asymptotically stable.*

Example 2.1. *We consider the non-autonomous linear differential equations with delay*

$$x'(t) = -\left[1 + \frac{\sin(t)}{3}\right] x\left(t - |\cos(t)|\right). \tag{2.24}$$

Since $0 < 1 + \frac{\sin(t)}{3} < \frac{4}{3}$ *and* $0 \leq |\cos(t)| \leq 1$, *it follows that* $\alpha q = \frac{4}{3} < \frac{3}{2}$, *therefore from theorem A the zero solution of* (2.24) *is uniformly asymptotically stable so (A1) and (A2) holds. Since the function* $\cos(x)$ *is uniformly continuous on* $\mathbb{R}$ *(A3) holds. So theorem 2.3 and corollary 2.1 are valid. Therefore we can approximate the solution of* (2.24) *by the family of difference equations* (2.6) *corresponding to* (2.24)

$$\mathfrak{z}_h(n+1) = \mathfrak{z}_h(n) - \mathfrak{a}_h(n)\mathfrak{z}_h(n-k_n),$$

where

$$\mathfrak{a}_h(n) = \int_{nh}^{(n+1)h} \left[1 + \frac{\sin(s)}{3}\right] ds = h - \frac{\cos((n+1)h) - \cos(nh)}{3},$$

and

$$k_n = \left[\frac{|\cos(nh)|}{h}\right].$$

It follows that the zero solution of

$$\mathfrak{z}_h(n+1) = \mathfrak{z}_h(n) - \left\{h - \frac{\cos((n+1)h) - \cos(nh)}{3}\right\} \mathfrak{z}_h\left(n - \left[\frac{|\cos(nh)|}{h}\right]\right), \tag{2.25}$$

is uniformly asymptotically stable. We note that, since mean value theorem, (2.25) *is equivalent to*

$$\begin{aligned}
\mathfrak{z}_h(n+1) - \mathfrak{z}_h(n) &= -\left\{h - \frac{\cos((n+1)h) - \cos(nh)}{3}\right\} \mathfrak{z}_h\left(n - \left[\frac{|\cos(nh)|}{h}\right]\right) \\
&= -h\left\{1 - \frac{1}{3}\frac{\cos((n+1)h) - \cos(nh)}{h}\right\} \mathfrak{z}_h\left(n - \left[\frac{|\cos(nh)|}{h}\right]\right) \\
&= -h\left\{1 + \frac{\sin(c_{n+1})}{3}\right\} \mathfrak{z}_h\left(n - \left[\frac{|\cos(nh)|}{h}\right]\right),
\end{aligned}$$

for some $c_{n+1} \in (nh, nh+h)$.

$$\frac{\mathfrak{z}_h(n+1) - \mathfrak{z}_h(n)}{h} = -\left[1 + \frac{\sin(c_{n+1})}{3}\right] \mathfrak{z}_h\left(n - \left[\frac{|\cos(nh)|}{h}\right]\right). \tag{2.26}$$

Figure 2.1: Approximate solution of (2.24) with initial function $\varphi \equiv 5$ and $h = 0.5$

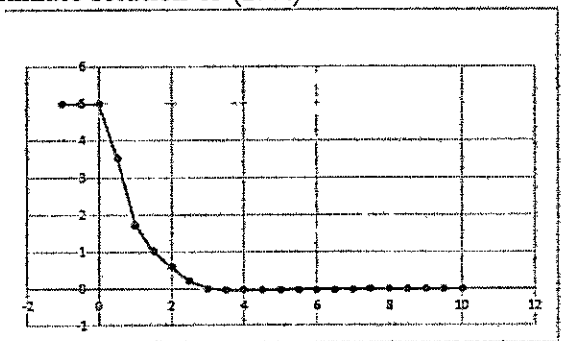


Figure 2.2: Approximate solution of (2.24) with initial function $\varphi \equiv 5$ and $h = 0.\overline{3}$

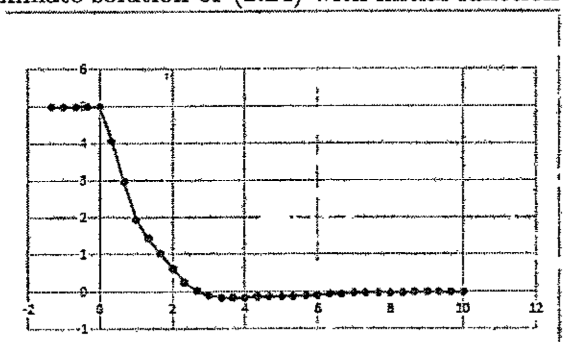
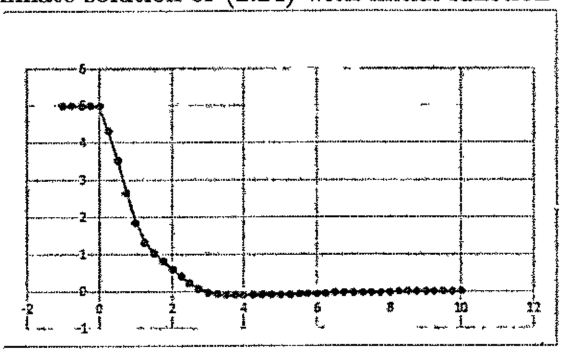


Figure 2.3: Approximate solution of (2.24) with initial function $\varphi \equiv 5$ and $h = 0.25$



3

Retarded functional differential equation with feedback

Differential equations with delay arise like useful mathematical predictive model, among other sciences, in the theory of growth of a single species. For instance, the works of Nicholson (1954, 1957) involving the Australian sheep blowfly *Lucila cuprina* has become a classic example of model of population with delay. The experiments exhibit clear patterns of sustained large oscillations of nearly constant frequency. May in 1974, used a delayed logistic differential equation to obtain a predictive model of Nicholson's Blowflies. However, the May's successful model considered a delay equals to 8.52 days while the directly measured values vary between 11 to 14 days. Gurney et al. (1980), devised other model with the aim to solve the difference between computed and observed values of the time delay. They consider the delay differential equation with feedback

$$N'(t) = -\delta N(t) + PN(t-q)\exp(-\alpha N(t-q)), \tag{3.1}$$

where $q > 0, P \geq 0$ and $\delta > 0$, to describe the dynamics of Nicholson's blowflies. A good presentation and a interesting review of this topic can be founds at Banks (1994) and Berezansky et al. (2010), respectively.

Mackey and Glass considered a family of delay differential equations with feedback

$$x'(t) = -\delta x(t) + \beta\frac{x(t-q)}{1+x^n(t-q)}, \tag{3.2}$$

where δ, β, q and n are positive real numbers to illustrate the appearance of complex dynamics in physiological control systems. They suggested that some physiological disorders can be characterized by change in qualitative features of dynamics (Mackey and Glass, 1977; Glass and Mackey, 1979). Moore about this equation can be found at Glass and Mackey (2010).

The above models consider delay differential equation with feedback, this kind of differential equations have been studied by several researcher (Liz et al., 2002, 2005; Röst and Wu, 2007; Liz and Röst, 2009). We note that there is some Lotka-Volterra systems which presents similar dynamics to delay functional differential equation with feedback, (Kuang and Smith, 1993).

In this chapter we consider the retarded functional differential equation with unimodal feedback

$$x'(t) = -\delta x(t) + f\left(\int_{-q}^{0} \theta(\sigma)x(t+\sigma)d\sigma\right), \quad x \geq 0, \tag{3.3}$$

where δ, q are positive constants, $\theta : [-q, 0] \to \mathbb{R}^+$ such that $\int_{-q}^{0} \theta(\sigma)d\sigma = 1$, and f is the delay feedback function $f : \mathbb{R}^+ \to \mathbb{R}^+$.

Our aims are study the existence of equilibria of (3.3) and its stability, obtain a discrete equation corresponding to (3.3) by piecewise constant argument, and finally study the transference of stability properties of the steady state.

3.1 Equilibria and stability

In this section we shall find the equilibria of Equation (3.3) and condition for the local exponential stability of them. We start with the definition of unimodal function

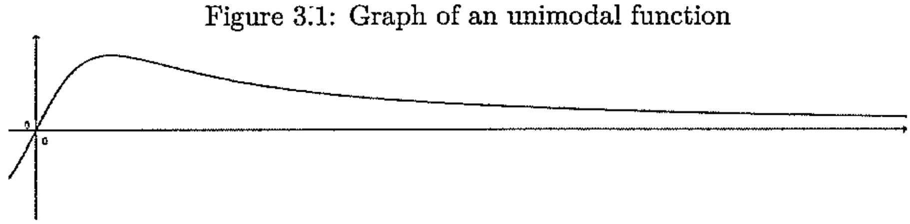
Definition 3.1. *The function $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called unimodal if:*

1. $f(x) \geq 0$ *for all* $x \geq 0$, $f(0) = 0$;
2. *there is a unique* $x_0 > 0$ *such that*

$$f'(x) > 0 \quad \text{if } 0 \leq x \leq x_0, f'(x_0) = 0 \text{ and } f'(x) < 0 \text{ if } x > x_0,$$

3. $f''(x) < 0$ *if* $0 \leq x \leq x_0$ *and* $\lim_{x\to\infty} f(x) = 0$.

Figure 3.1: Graph of an unimodal function



Since the assumption

$$\int_{-q}^{0} \theta(\sigma)d\sigma = 1, \tag{3.4}$$

the equilibria of equation (3.3) correspond to solutions of the equation

$$-\delta\xi + f(\xi) = 0. \tag{3.5}$$

Since the definition of unimodal function it follows that zero is a solution of (3.5). If $f'(0) \le \delta$, then zero is the unique solution. In the other hand, if $f'(0) > \delta$ there is a positive equilibrium K. We denote $x(t)$ a solution of Equation (3.3), and define a function $y(t) = x(t) - \xi$, whith $\xi \in \{0, K\}$. It is follows that y satisfy the functional differential equation

$$y'(t) = x'(t) = -\delta y(t) + f\left(\int_{-q}^{0} \theta(\sigma) y(t+\sigma) d\sigma + \xi\right) - f(\xi), \tag{3.6}$$

since ξ is solution of (3.5). The corresponding linear variational equation, corresponding to equation (3.6) about ξ, for $v(t)$ is

$$v'(t) = -\delta v(t) + f'(\xi) \int_{-q}^{0} \theta(\sigma) v(t+\sigma) d\sigma. \tag{3.7}$$

Proposition 3.1. *If*

$$\delta > f'(\xi) > 0, \tag{3.8}$$

then the zero solution of Equation (3.7) *is exponentially stable.*

Proof. To state the stability of the zero solution of Equation (3.7), we use a Halanay Inequality and the upper right Dini derivative $D^+|v(t)|$. Since Lemma 3 of Chen and Cao (2003) we have

$$\begin{aligned}
D^+|v(t)| &= \frac{v(t)}{|v(t)|} v'(t) \\
&= \frac{v(t)}{|v(t)|}\left[-\delta v(t) + f'(\xi)\int_{-q}^{0} \theta(\sigma) v(t+\sigma) d\sigma\right] \\
&= -\delta |v(t)| + f'(\xi) \frac{v(t)}{|v(t)|} \int_{-q}^{0} \theta(\sigma) v(t+\sigma) d\sigma \\
&\le -\delta |v(t)| + f'(\xi) \int_{-q}^{0} \theta(\sigma) |v(t+\sigma)| d\sigma \\
&\le -\delta |v(t)| + f'(\xi) \int_{-q}^{0} \theta(\sigma) d\sigma \sup_{t-q \le \sigma \le t} |v(\sigma)| \\
&= -\delta |v(t)| + f'(\xi) \sup_{t-q \le \sigma \le t} |v(\sigma)|.
\end{aligned}$$

Since the assumption (3.8) and Halanay inequality there exists $\mu > 0$ such that:

$$|v(t)| \le \|v_0\| \, e^{-\mu t} \tag{3.9}$$

where

$$\mu = \delta - f'(\xi) e^{\mu q}.$$

Then the zero solution of (3.7) is exponentially stable. □

If $f'(0) \leq \delta$ the condition (3.8) holds, then zero is a locally exponentially stable equilibrium of (3.3). If $f'(0) > \delta$ then the zero solution is unstable, and to apply Proposition 3.1 is necessary that $f'(K) > 0$. In consequence, if $f'(0) > \delta$ and K belongs to the interval $(0, x_0)$ the condition (3.8) holds, so K is a locally exponentially stable equilibrium of (3.3).

There is a third possibility, $f'(0) > \delta$ and $K > x_0$. In this case $f'(x) < 0$ for $x > x_0$, so we can not apply the Proposition 3.1. We focus in the first two cases, where the equilibrium is locally exponentially stable.

Figure 3.2: Case 1: $y = \delta x$ and $y = f(x)$, with $f'(0) \leq \delta$

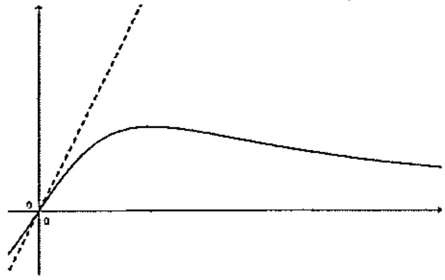
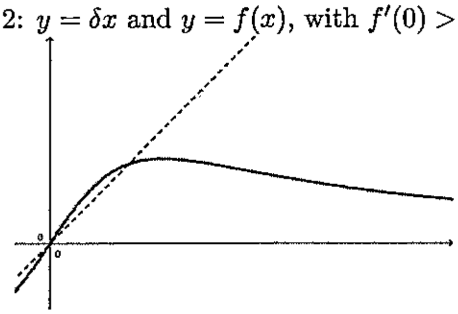


Figure 3.3: Case 2: $y = \delta x$ and $y = f(x)$, with $f'(0) > \delta$ and $K \in (0, x_0)$



3.2 Discretization by piecewise constant argument for distributed delay

In this section, we discretize the retarded functional differential equation with unimodal feedback (3.6) using piecewise constant, i.e. we consider the equation:

$$y'(t) \quad = \quad -\delta y(t) + f\left(\int_{-q}^{0} \theta(\sigma) y([t]_h + [\sigma]_h) d\sigma + \xi\right) - f(\xi).$$

The corresponding differential equation with piecewise constant arguments, since ξ is an equilibrium of (3.3), is given by

$$v_h'(t) = -\delta y_h(t) + f\left(\xi + \sum_{j=-k}^{-1} \Theta_h(j)\, y_h(([t/h]+j)h)\right) - f(\xi), \qquad (3.10)$$

where $\Theta_h(j) := \int_{jh}^{(j+1)h} \theta(\sigma) d\sigma$, for $h \in (0, q]$. We can consider $h = \frac{q}{k}$ for $k \geq 1$ an integer. By the solution of above equation (3.10), we mean a function $y_h(t)$, defined on $\{jh : j = -k, \cdots, -1, 0\}$ such that

1. The function y_h is continuous on $\mathbb{R}^+$.
2. The function y_h satisfies (3.10) on each interval $I_{(i,h)} = [ih, (i+1)h)$ for $i = 0, 1, \cdots$.
3. The corresponding derivatives of $y_h(t)$ should exist at each point $t \in \mathbb{R}^+$, with the possible exception of the points ih $(i = 0, 1, 2 \cdots)$, where finite one-sided derivatives exists.

We can rewrite equation (3.10) for $t \in [rh, (r+1)h)$ as

$$\begin{aligned} y_h'(t) &= -\delta y_h(t) + f\left(\xi + \sum_{j=-k}^{-1} \Theta_h(j)\, y_h((r+j)h)\right) - f(\xi). \\ y_h(nh) &= \phi(nh), \quad n = -k, \cdots, 0. \end{aligned}$$

Using variation of constants formula, for $rh \leq t < (r+1)h$, the solution of (3.10) can be written as

$$\begin{aligned} y_h(t) &= e^{-\delta(t-rh)} y_h(rh) + \int_{rh}^{t} e^{-\delta(t-s)} \left[f\left(\xi + \sum_{j=-k}^{-1} \Theta_h(j)\, y_h((r+j)h)\right) - f(\xi)\right] ds \\ &= e^{-\delta(t-rh)} y_h(rh) + \frac{1 - e^{-\delta(t-rh)}}{\delta} \left[f\left(\xi + \sum_{j=-k}^{-1} \Theta_h(j)\, y_h((r+j)h)\right) - f(\xi)\right], \end{aligned}$$

for $r \geq 0$. Replacing $t = (r+1)h$, integrating and using continuity of y_h, we obtain

$$y_h((r+1)h) = e^{-\delta h} y_h(rh) + \frac{1 - e^{-\delta h}}{\delta} \left[f\left(\xi + \sum_{j=-k}^{-1} \Theta_h(j)\, y_h((r+j)h)\right) - f(\xi)\right],$$

for $r \geq 0$. Define sequence $\mathfrak{y}_h(r) := y_h(rh)$. This sequence satisfies the following non-linear delay difference equation

$$\mathfrak{y}_h(r+1) = e^{-\delta h} \mathfrak{y}_h(r) + \frac{1 - e^{-\delta h}}{\delta} \left[f\left(\xi + \sum_{j=-k}^{-1} \Theta_h(j)\, \mathfrak{y}_h(r+j)\right) - f(\xi)\right], \quad (3.11)$$

with initial condition

$$\mathfrak{y}_h(r) = \phi(rh) \quad r = -k, -k+1, \cdots . \tag{3.12}$$

The initial value problem (3.11)-(3.12) is a discretization of the original problem (3.3). From the above recurrence relation and initial conditions, we obtain

$$\begin{aligned}
\mathfrak{y}_h(0) &= \phi(0), \\
\mathfrak{y}_h(1) &= e^{-\delta h}\mathfrak{y}_h(0) + \frac{1-e^{-\delta h}}{\delta}\left[f\left(\xi + \sum_{j=-k}^{-1}\Theta_h(j)\,\mathfrak{y}_h(1+j)\right) - f(\xi)\right], \\
\mathfrak{y}_h(2) &= e^{-\delta h}\mathfrak{y}_h(1) + \frac{1-e^{-\delta h}}{\delta}\left[f\left(\xi + \sum_{j=-k}^{-1}\Theta_h(j)\,\mathfrak{y}_h(2+j)\right) - f(\xi)\right] \\
&= e^{-\delta 2h}\mathfrak{y}_h(0) + \frac{e^{-\delta h} - e^{-2\delta h}}{\delta}\left[f\left(\xi + \sum_{j=-k}^{-1}\Theta_h(j)\,\mathfrak{y}_h(1+j)\right) - f(\xi)\right] \\
&\quad + \frac{1-e^{-\delta h}}{\delta}\left[f\left(\xi + \sum_{j=-k}^{-1}\Theta_h(j)\,\mathfrak{y}_h(2+j)\right) - f(\xi)\right].
\end{aligned}$$

From the initial conditions, the above sequence $\mathfrak{y}_i^h(r)$ is well defined.

Theorem 3.1. *The initial value problem* (3.10) *has a unique solution in the form*

$$y_h(t) = e^{-\delta(t-rh)}\mathfrak{y}_h(n) + \frac{1-e^{-\delta(t-nh)}}{\delta}\left[f\left(\xi + \sum_{j=-k}^{-1}\Theta_h(j)\,\mathfrak{y}_h(n+j)\right) - f(\xi)\right], \tag{3.13}$$

for $t \geq 0$ *where* $n := [t/h]$ *and the sequence* $\mathfrak{y}_h(\cdot)$ *satisfies the non linear difference equations* (3.11) *with initial conditions* (3.12).

3.3 Transference of stability properties of equilibria

Next we shall find necessary conditions for transference of local exponential stability of equilibria of (3.3) in the cases 1 and 2.

Theorem 3.2. *If* $\delta > f'(\xi) > 0$ *then, the zero solution of equations* (3.6) *and* (3.11) *are both locally exponentially stable.*

Proof. From proposition 3.1 it follows that the condition $\delta > f'(\xi) > 0$ implies that the zero solution of (3.6) is locally exponentially stable. In order to prove that the local stability of the zero solution of (3.11) we consider the variational equation

$$\mathfrak{v}_h(r+1) = e^{-\delta h}\mathfrak{v}_h(r) + \frac{1-e^{-\delta h}}{\delta} f'(\xi) \sum_{j=-k}^{-1}\Theta_h(j)\,\mathfrak{v}_h(r+j). \tag{3.14}$$

The equation (3.14) can be written in the form

$$\begin{aligned}\Delta\mathfrak{v}_h(r) &= \mathfrak{v}_h(r+1)-\mathfrak{v}_h(r)\\ &= -(1-e^{-\delta h})\mathfrak{v}_h(r)+\frac{1-e^{-\delta h}}{\delta}f'(\xi)\sum_{j=-k}^{-1}\Theta_h(j)\,\mathfrak{v}_h(r+j).\end{aligned}$$

From variation of parameter formula for difference equation, for $n \geq 0$, we have

$$\mathfrak{v}_h(n) = (e^{-\delta h})^n\,\mathfrak{v}_h(0)+\sum_{i=0}^{n-1}(e^{-\delta h})^{n-i-1}\frac{1-e^{-\delta h}}{\delta}f'(\xi)\sum_{j=-k}^{-1}\Theta_h(j)\,\mathfrak{v}_h(i+j). \quad (3.15)$$

Now we can estimate $|\mathfrak{v}_h(n)|$ to obtain

$$\begin{aligned}|\mathfrak{v}_h(n)| &\leq \left(e^{-\delta h}\right)^n|\mathfrak{v}_h(0)|+\sum_{i=0}^{n-1}\left(e^{-\delta h}\right)^{n-i-1}\frac{1-e^{-\delta h}}{\delta}f'(\xi)\sum_{j=-k}^{-1}\Theta_h(j)\,|\mathfrak{v}_h(i+j)|\\ &\leq \left(e^{-\delta h}\right)^n|\mathfrak{v}_h(0)|+\sum_{i=0}^{n-1}\left(e^{-\delta h}\right)^{n-i-1}\frac{1-e^{-\delta h}}{\delta}f'(\xi)\max_{i-k\leq j\leq i}|\mathfrak{v}_h(j)|,\end{aligned}$$

since variation of parameter formula we obtain

$$\Delta|\mathfrak{v}_h(n)| = -(1-e^{-\delta h})|\mathfrak{v}_h(n)|+\frac{1-e^{-\delta h}}{\delta}f'(\xi)\max_{i-k\leq j\leq i}|\mathfrak{v}_h(j)|. \quad (3.16)$$

From the assumptions $\delta > f'(\xi) > 0$ follows that

$$1 \leq 1-e^{-\delta h} > \left(1-e^{-\delta h}\right)\frac{f'(\xi)}{\delta} > 0.$$

Therefore (3.16) satisfy the assumption of discrete version of Halanay inequality Liz and Ferreiro (2002), then there exists a positive constant λ_0 such that

$$|\mathfrak{v}_h(n)| \leq \left(\max_{-k\leq j\leq 0}|\mathfrak{v}_h(j)|\right)\lambda_0^n.$$

Then the zero solution of (3.14) is exponentially stable. Therefore the zero solution (3.11) is locally exponentially stable. □

The novelties of our theorem 3.2 are:

1. We have adapted the method developed bi Mohamad and Gopalsamy (2000) and Liz and Ferreiro (2002) to retarded functional differential equations with unimodal feedback, up to we know, there is not similar result in the literature.

2. We conclude that the local exponential stability of equilibrium holds for both equations (3.3) and (3.11), i.e. we have proved transference of stability properties of equilibria from continuous dynamical system to discrete.

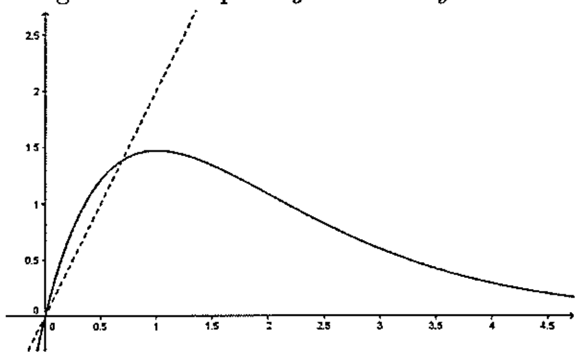
3.3.1 Applications and examples

Example 3.1. *To illustrate our theorem 3.2 we consider a the following Nicholson differential equation with continuously distributed delay*

$$N'(t) = -2N(t) + 4\int_{-1}^{0} N(t+\sigma)d\sigma e^{-\int_{-1}^{0} N(t+\sigma)d\sigma}. \tag{3.17}$$

We note that (3.17) *corresponds to case 2, see figure 3.4. The positive equilibrium of* (3.17)

Figure 3.4: Graph of $y = 2x$ and $y = 4xe^{-x}$



is $K = \ln(2)$.

Therefore we can approximate the solution of (3.17) *by the family of difference equations (3.11)-(3.12), to obtain*

$$\mathfrak{v}_h(r+1) = e^{-2h}\mathfrak{v}_h(r) + \frac{1-e^{-2h}}{2}\left[2\left(\ln(2) + h\sum_{j=-k}^{-1}\mathfrak{v}_h(r+j)\right)e^{-\sum_{j=-k}^{-1} h\mathfrak{v}_h(r+j)} - 2\ln(2)\right],$$

Figure 3.5: Approximate solution of (3.17) with initial function $\phi \equiv 5 + \ln(2)$ and $h = 0.5$

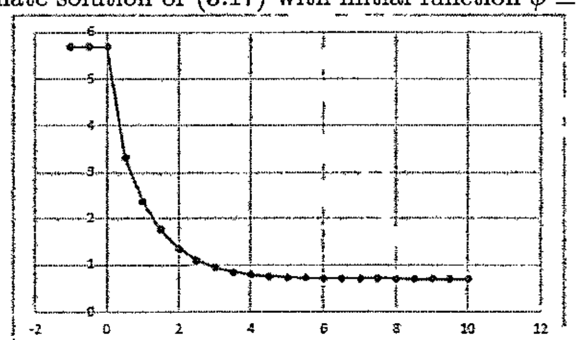


Figure 3.6: Approximate solution of (3.17) with initial function $\phi \equiv 5 + \ln(2)$ and $h = 0.25$

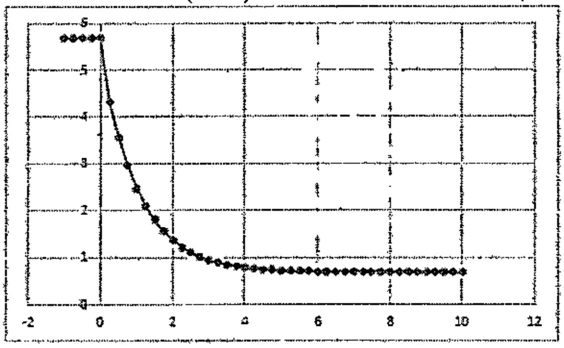


Figure 3.7: Approximate solution of (3.17) with initial function $\phi \ln(2) - 0.4$ and $h = 0.5$

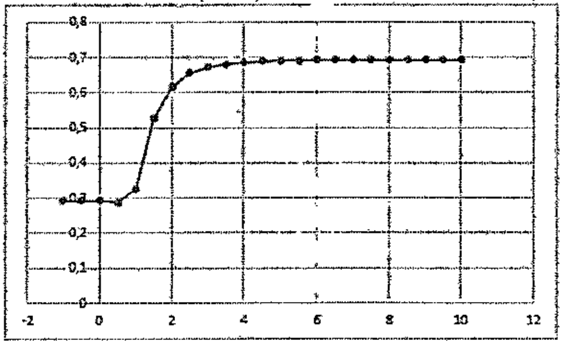
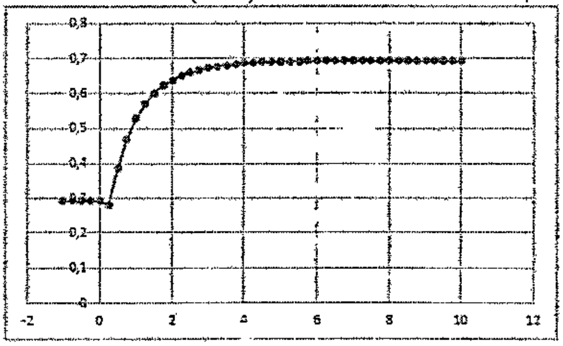


Figure 3.8: Approximate solution of (3.17) with initial function $\phi \ln(2) - 0.4$ and $h = 0.25$



3.3.2 Exponential decay rate of the error

In this subsection we obtain an analogous result to theorem 2.3. In order to do that we rewrite equation (3.6) as

$$y'(t) = L(y_t) + f\left(\int_{-q}^{0}\theta(\sigma)y(t+\sigma)d\sigma + \xi\right) - f(\xi) - f'(\xi)\int_{-q}^{0}\theta(\sigma)y(t+\sigma)d\sigma, \tag{3.18}$$

where

$$L(y_t) = -\delta y(t) + f'(\xi)\int_{-q}^{0}\theta(\sigma)y(t+\sigma)d\sigma. \tag{3.19}$$

Assumption (3.8) and Proposition 3.1 implies that solution $x = 0$ of the linear homogeneous functional differential equation

$$x'(t) = L(x_t) \tag{3.20}$$

is $\mu > 0$ exponentially stable. Now we consider the function

$$g(u) := f(u+\xi) - f(\xi) - f'(\xi)u. \tag{3.21}$$

From Definition 3.1 the function f is twice continuously differentiable over the interval $(0, x_0)$. So, Taylor's theorem holds, and from Lagrange form for the remainders we have that:

$$\begin{aligned} g(u_1) - g(u_2) &= f(u_1 - u_2 + u_2 + \xi) - f'(\xi)u_1 - f(u_2+\xi) + f'(\xi)u_2 \\ &= f(u_2+\xi) + f'(u_2+\xi)(u_1-u_2) \\ &\quad + \frac{f''(\vartheta)}{2}(u_1-u_2)^2 - f'(\xi)(u_1-u_2) - f(u_2+\xi) \\ &= \left[f'(u_2+\xi) - f'(\xi) + \frac{f''(\vartheta)}{2}(u_1-u_2)\right](u_1-u_2) \\ &= \left[f''(\vartheta_1)u_2 + \frac{f''(\vartheta)}{2}(u_1-u_2)\right](u_1-u_2). \end{aligned}$$

where ϑ is between $u_2 + \xi$ and $u_1 - u_2$, and ϑ_1 between $u_2 + \xi$ and ξ. Therefore the function $g(\cdot)$ is locally Lipschitz.

The above situation motivates us to study a semi linear functional differential equation

$$y'(t) = L(y_t) + g\left(\int_{-q}^{0}\theta(\sigma)y(t+\sigma)d\sigma\right). \tag{3.22}$$

where the linear functional differential equation (3.20) is μ-exponentially stable, and $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $g(0) = 0$. The DEPCA corresponding (3.22) is

$$z_h'(t) = L(z_{h_t}) + g\left(\int_{-q}^{0}\theta(\sigma)z_h([t]_h + [\sigma]_h)d\sigma\right). \tag{3.23}$$

In order to estimate the error $|y(t) - z_h(t)|$, we need some previous result.

Proposition 3.2. *We assume the zero solution of* (3.20) *is μ-exponentially stable and the function $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $g(0) = 0$, i.e. there exists a positive constant* K *such that*

$$|g(u_1) - g(u_2)| \leq \mathrm{K}|u_1 - u_2|.$$

If $\mu > \mathrm{K} > 0$, then the function $y(t)$ solution of (3.22) *satisfy*

$$|y(t)| \leq \|y_{t_0}\|_q \, e^{-\lambda(t-t_0)} \tag{3.24}$$

where λ is the positive solution

$$\lambda = \mu - e^{\mathrm{K}q}.$$

Proof. We use formula of variation of parameter in equation (3.22) and obtain

$$y(t) = U(t; t_0, y_{t_0}) + \int_{t_0}^{t} U\left(t; s, g\left(\int_{-q}^{0} \theta(\sigma) y(s+\sigma) d\sigma u\right)\right) ds.$$

So, from the μ-exponential stability of the zero solution, we have

$$\begin{aligned}
|y(t)| &\leq e^{-\mu(t-t_0)} \|y_{t_0}\|_q + \int_{t_0}^{t} e^{-\mu(t-s)} \left| g\left(\int_{-q}^{0} \theta(\sigma) y(s+\sigma) d\sigma\right)\right| ds \\
&\leq e^{-\mu(t-t_0)} \|y_{t_0}\|_q + \int_{t_0}^{t} e^{-\mu(t-s)} \mathrm{K} \left|\int_{-q}^{0} \theta(\sigma) y(s+\sigma) d\sigma\right| ds \\
&\leq e^{-\mu(t-t_0)} \|y_{t_0}\|_q + \int_{t_0}^{t} e^{-\mu(t-s)} \mathrm{K} \sup_{s-q \leq \sigma \leq s} |y(\sigma)| \, ds.
\end{aligned}$$

Since the assumption $\mu > \mathrm{K} > 0$, from Halany inequality, there exists a positive λ such that

$$|y(t)| \leq \|y_{t_0}\|_q \, e^{-\lambda(t-t_0)},$$

where λ is the positive solution

$$\lambda = \mu - e^{\mathrm{K}q}.$$

□

As consequence of the above proposition we state the following Lemma.

Lemma 3.1. *If the solution of* (3.22) *satisfy* (3.24), *for any initial function $\varphi \in C$, then*

$$|y(s+\sigma) - y([s]_h + [\sigma]_h))| \leq M_2(\varphi, \sigma, h) e^{-\lambda s}, \quad s \geq 0,$$

where

$$M_2(\varphi, \sigma, h) := \|\varphi\|_q \, e^{\lambda(\sigma+2h+t_0)} (\delta + [f'(\xi) + \mathrm{K}] e^{\lambda q}) 2h, \tag{3.25}$$

and $M_2(\varphi, \sigma, h) \to 0$ as $h \to 0$.

Proof. From variation of parameter, triangular inequality and equation (3.19) we obtain

$$\begin{aligned}
|y(s+\sigma)-y([s]_h+[\sigma]_h))| &= \left|\int_{[s]_h+[\sigma]_h}^{s+\sigma} y'(u)du\right| \\
&= \left|\int_{[s]_h+[\sigma]_h}^{s+\sigma} \left(L(y_u)+g\left(\int_{-q}^{0}\theta(\sigma)y(u+\sigma)d\sigma\right)\right)du\right| \\
&\le \int_{[s]_h+[\sigma]_h}^{s+\sigma} \left|L(y_u)+g\left(\int_{-q}^{0}\theta(\sigma)y(u+\sigma)d\sigma\right)\right|du \\
&\le \int_{[s]_h+[\sigma]_h}^{s+\sigma} \left(\delta|y(u)|+f'(\xi)\sup_{u-q\le\sigma\le u}|y(\sigma)|+\mathrm{K}\sup_{u-q\le\sigma\le u}|y(\sigma)|\right)du \\
&= \int_{[s]_h+[\sigma]_h}^{s+\sigma} \left(\delta|y(u)|+[f'(\xi)+\mathrm{K}]\sup_{u-q\le\sigma\le u}|y(\sigma)|\right)du.
\end{aligned}$$

Since $y(t)$ satisfy (3.24), we have:

$$\begin{aligned}
|y(s+\sigma)-y([s]_h+[\sigma]_h))| &\le \delta\|\varphi\|_q\int_{[s]_h+[\sigma]_h}^{s+\sigma} e^{-\lambda(u-t_0)}du \\
&\quad +[f'(\xi)+\mathrm{K}]\|\varphi\|_q\int_{[s]_h+[\sigma]_h}^{s+\sigma} e^{-\lambda(u-q-t_0)}du \\
&\le \|\varphi\|_q e^{-\lambda([s]_h+[\sigma]_h-t_0)}(\delta+[f'(\xi)+\mathrm{K}]e^{\lambda q})\int_{[s]_h+[\sigma]_h}^{s+\sigma} du \\
&\le \|\varphi\|_q e^{-\lambda(s-\sigma-2h-t_0)}(\delta+[f'(\xi)+\mathrm{K}]e^{\lambda q})2h.
\end{aligned}$$

Since (3.25), the lemma is proved. □

Now we can prove that the error in our approximation has an exponential decay rate to zero, as t tends to infinity.

Theorem 3.3. *Under the assumptions Proposition 3.2. If the parameters involve satisfy*

$$\lambda > \mathrm{K} > 0,$$

then for h small enough, and for every $\varphi \in C$ the solutions $y(\varphi)(t)$ and $z_h(\varphi)(t)$ of the functional differential equations (3.22) *and* (3.23)*, respectively, satisfy*

$$|E_h(t)| \le \left(\left\|E_{h_{t_0}}\right\|_q + \mathrm{K}M_2(\varphi,q,h)e^{-\lambda t_0}t\right)e^{-\mu_0(t-t_0)}, \quad t \ge t_0.$$

With $M_2(\varphi,q,h)$ defined by (3.25) *and*

$$0 < \mu_0 = \lambda - \mathrm{K}e^{\mu_0 q}.$$

Proof. For every $h \in (0, h_0)$ from equations (3.22) and (3.23), it follows that $E_h(t) := y(t) - z_h(t)$ satisfy the differential equation

$$E_h'(t) \;=\; L(E_{h_t}) + g\left(\int_{-q}^{0} \theta(\sigma) y(t+\sigma) d\sigma\right) - g\left(\int_{-q}^{0} \theta(\sigma) z_h([t]_h + [\sigma]_h) d\sigma\right).$$

Using variation of parameter formula and elementary estimations we obtain

$$\begin{aligned}
|E_h(t)| \;&\leq\; e^{-\mu(t-t_0)} \left\| E_{h_{t_0}} \right\|_q \\
&\quad + \int_{t_0}^{t} e^{-\mu(t-s)} \left| g\left(\int_{-q}^{0} \theta(\sigma) y(s+\sigma) d\sigma\right) - g\left(\int_{-q}^{0} \theta(\sigma) z_h([s]_h + [\sigma]_h) d\sigma\right) \right| ds \\
&\leq\; e^{-\mu(t-t_0)} \left\| E_{h_{t_0}} \right\|_q \\
&\quad + \int_{t_0}^{t} e^{-\mu(t-s)} \mathbf{K} \left| \int_{-q}^{0} \theta(\sigma) y(s+\sigma) d\sigma - \int_{-q}^{0} \theta(\sigma) z_h([s]_h + [\sigma]_h) d\sigma \right| ds \\
&\leq\; e^{-\mu(t-t_0)} \left\| E_{h_{t_0}} \right\|_q + \int_{t_0}^{t} e^{-\mu(t-s)} \mathbf{K} \left| \int_{-q}^{0} \theta(\sigma) E_h([s]_h + [\sigma]_h) d\sigma \right| ds \\
&\quad + \int_{t_0}^{t} e^{-\mu(t-s)} \mathbf{K} \left| \int_{-q}^{0} \theta(\sigma) [y(s+\sigma) - y([s]_h + [\sigma]_h)] d\sigma \right| ds. \qquad (3.26)
\end{aligned}$$

Since $\mu > \lambda > 0$, we have

$$\begin{aligned}
|E_h(t)| \;&\leq\; e^{-\lambda(t-t_0)} \left\| E_{h_{t_0}} \right\|_q + \int_{t_0}^{t} e^{-\lambda(t-s)} \mathbf{K} \sup_{s-q \leq \sigma \leq s} |E_h(\sigma)| \, ds \\
&\quad + \int_{t_0}^{t} e^{-\lambda(t-s)} \mathbf{K} \sup_{-q \leq \sigma \leq 0} |y(s+\sigma) - y([s]_h + [\sigma]_h)| \, ds. \qquad (3.27)
\end{aligned}$$

Now, from Lemma 3.1 we obtain

$$\begin{aligned}
|E_h(t)| \;&\leq\; e^{-\lambda(t-t_0)} \left\| E_{h_{t_0}} \right\|_q + \int_{t_0}^{t} e^{-\lambda(t-s)} \mathbf{K} \sup_{s-q \leq \sigma \leq s} |E_h(\sigma)| \, ds \\
&\quad + \int_{t_0}^{t} e^{-\lambda(t-s)} \mathbf{K} M_2(\varphi, q, h) e^{-\lambda s} ds \\
&\leq\; e^{-\lambda(t-t_0)} \left(\left\| E_{h_{t_0}} \right\|_q + \mathbf{K} M_2(\varphi, q, h) e^{-\lambda t_0} t \right) + \int_{t_0}^{t} e^{-\lambda(t-s)} \mathbf{K} \sup_{s-q \leq \sigma \leq s} |E_h(\sigma)| \, ds.
\end{aligned}$$

Therefore, from assumption $\lambda > \mathbf{K} > 0$ and Halanay inequality, there exists a positive constant μ_0 such that

$$|E_h(t)| \leq \left(\left\| E_{h_{t_0}} \right\|_q + \mathbf{K} M_2(\varphi, q, h) e^{-\lambda t_0} t \right) e^{-\mu_0(t-t_0)}, \quad t \geq t_0. \qquad (3.28)$$

where $\mu_0 = \lambda - \mathbf{K} e^{-\mu_0 q}$. □

Thus we have proved our theorem 3.3, the novelty of this result lies in consider a semilinear retarded functional differential equation and extended the technique used by Cooke and Győri (1994) and Győri and Hartung (2002) to the case where the delay is continuously distributed. We note the use of Halanay inequality as key technique to prove our result.

4

Differential equation with state dependent delay

Many situations of basic sciences depends on external or internal influence, a possible mathematical model to consider in that case is a differential equation with variable delay. When there is internal influence the model corresponding is a functional differential equation with state-dependent delay. This kind of differential equations with delays are an interesting field for the mathematicians (Walther, 2014). For example the simplest functional differential equations

$$x'(t) = -\alpha x(t - d(x_t)) \tag{4.1}$$

with a non-constant delay functional $d : C \to [0, q]$, and $\alpha \in \mathbb{R} - \{0\}$ it is not linear. The existence and uniqueness theory for differential equation with state-dependent delay is quite different to the developed by Hale and Lunel (1993). To illustrate the differences between both theories, we consider the initial value problem

$$x'(t) = -x(t - |x(t)|), \quad t \geq 0, \tag{4.2}$$

$$\psi(t) = \begin{cases} -1, & \text{for } t \leq -1, \\ -1 + \frac{3}{2}(t+1)^{1/3}, & \text{for } -1 < t \leq -\frac{7}{8}, \\ 1 + \frac{10}{7}t, & \text{for } -\frac{7}{8} < t \leq 0. \end{cases} \tag{4.3}$$

The functions $x(t) = t + 1$ and $x(t) = t + 1 - t^{3/2}$ are both solutions of the initial value problem (4.2)-(4.3), for t in a neighbourhood of zero. Therefore, continuous initial data do not guaranteed uniqueness of solutions of differential equation with state-dependent delay. Winston (1974) gave the above example and Driver (1963) developed a fundamental theory for state-dependent delay differential equation.

Tavernini (1978) studied the numerical approximation of solutions of functional differential equation like

$$u'(t) = f(t, u_t), \quad t \geq 0, \tag{4.4}$$

where the delay may be state-dependent. Tavernini proved the convergence of Euler polygonal and obtained an uniqueness-existence theorem. Győri et al. (1995) developed a numerical approximation for a class of differential equations with time- and state- dependent delay, and gave a version of Cauchy-Peano existence theorem for this class of differential equations based in the previous work of Győri et al. (1993). Hartung et al. (1997) extended the previous work for a class of neutral functional differential equations with state dependent delay described by

$$\frac{d}{dt}\Big(x(t)+a(t)x(t-d_1(t,x(t)))\Big)=f\Big(t,x(t),x(t-d_1(t,x(t)))\Big),\quad t\geq 0,\tag{4.5}$$

Benchohra et al. (2013) have studied the global existence of solutions of functional differential equations with state-dependent delay

$$y'(t)=Ay(t)+f\left(t,y_{\rho(t,y_t)}\right),\quad \text{a.e. } t\geq 0,\tag{4.6}$$

where $f:[0,\infty)\times\mathcal{B}\to E$ is a given function, $A:D(A)\subset E\to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t),t\in[0,\infty)$, $\mathcal{B}$ is an abstract phase space, $\rho:[0,\infty)\times\mathcal{B}\to\mathbb{R}$, and E is a real Banach space. The authors used Schauder fixed point theorem as key technique, therefore the uniqueness is not ensured, so they obtained a Cauchy-Peano existence global theorem.

In this chapter we study transference of qualitative properties between differential equations with state-dependent delay and the corresponding difference equation, we shall prove a global Cauchy-Peano existence theorem for a differential equation with state-dependent delay. Our key techniques are approximations by piecewise constant argument and Arzela-Ascoli theorem. In this way we state transference of existence of solutions between a difference equation and the corresponding differential equations with delay.

4.1 Discretization by piecewise constant argument

In this section we consider the scalar differential equations with state-dependent delay

$$y'(t)=-by(t)+f\left(t,y(t-r(t,y(t)))\right),\quad t\geq 0,\tag{4.7}$$

where $b\in\mathbb{R}^+$, $f:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ is a continuous function and $r:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+$ is a bounded continuous function. The initial conditions are

$$y(t)=\varphi(t),\quad t\in[-\lambda,0];\tag{4.8}$$

where

$$\lambda:=\sup\{r(t,u):t\geq 0,u\in\mathbb{R}\}.$$

A function y is said to be a *solution* of equation (4.7) on $[-q,A)$ if there is $A>0$ such that $y\in C([-q,A),\mathbb{R})$, and $y(t)$ satisfies equation (4.7) for $t\in[0,A)$. For given $\phi\in\mathcal{C}$, we say

$y(t;0,\phi)$ is a solution of equation (4.7) *with initial value* ϕ at 0 if there is an $A > 0$ such that $y(t;0,\phi)$ is a solution of equation (4.7) on $[-q, A]$ and $y_0(0,\phi) = \phi$.

We shall prove a Cauchy-Peano existence theorem for the initial value problem (4.7)-(4.8) by the convergence of solutions of the corresponding difference equation. We consider the family of piecewise constant functions that $[\cdot]_h = \left[\frac{\cdot}{h}\right] h$, for $h > 0$, where $[\cdot]$ is the great integer function. Note that $[t]_h \to t$ as $h \to 0^+$, uniformly on $\mathbb{R}$. This family allows us to consider the family of differential equations with piecewise constant argument

$$z_h'(t) = -bz_h(t) + f\left(t, z_h\left(\gamma_h(t)\right)\right), \quad t \geq 0, \tag{4.9}$$

and the initial conditions

$$z_h(k) = \varphi(k), \quad k = 0, 1, \cdots, \quad -\lambda \leq -kh \leq 0, \tag{4.10}$$

where

$$\gamma_h(t) := [t]_h - \left[r\left([t]_h, z_h([t]_h)\right)\right]_h.$$

By a solution of initial value problem (IVP) (4.9)-(4.10) we mean a function z_h defined on $\{-kh \ : \ k = 0, 1, \cdots, \ -\lambda \leq -kh \leq 0\}$ by (4.10), which satisfies the following properties on $\mathbb{R}^+$:

(i) the function z_h is continuous on $\mathbb{R}^+$,

(ii) the derivative $z_h'(t)$ exists at each point $t \in \mathbb{R}^+$ with the possible exception of the points kh $(k = 0, 1, 2, \cdots)$ where finite one-sided derivatives exist, and

(iii) the function z_h satisfies (4.9) on each interval $I_{(k,h)} := [kh, (k+1)h)$ for $k = 0, 1, 2, \cdots$.

Next we obtain theorem about existence of solutions of the initial value problem (4.9)-(4.10), also we obtain the discrete equations associated with the differential equation with time- and state-dependent delay (4.7)-(4.8). Next we consider the notation $\mathfrak{a}_h(k) \equiv z_h(kh)$.

Theorem 4.1. *The IVP (4.9)-(4.10) has a unique solution in the form*

$$z_h(t) = e^{-bt}\varphi(0) + \sum_{i=0}^{k-1} \int_{I_{(i+1,h)}} e^{-b(t-s)} f\left(s, \mathfrak{a}_h\left(i - d_i\right)\right) ds + \int_{kh}^{t} e^{-b(t-s)} f\left(s, \mathfrak{a}_h\left(k - d_k\right)\right) ds, \tag{4.11}$$

for $t \geq 0$. The difference equation corresponding to the discrete version of IVP (4.7)-(4.8) is

$$\mathfrak{a}_h(k+1) = e^{-bh}\mathfrak{a}_h(k) + \int_{I_{(k,h)}} e^{-b((k+1)h-s)} f\left(s, \mathfrak{a}_h(k - d_k)\right) ds, \quad k \geq 0, \tag{4.12}$$

where $d_k \equiv [r(kh, \mathfrak{a}_h(k))/h]$, and $\mathfrak{a}_h(k) \equiv z_h(kh)$.

Proof. For $t \in I_{(k,h)}$ we have that

$$z_h'(t) = -bz_h(t) + f(t, z_h(\gamma_h(t))),$$

from the variation of parameters formula we have that the solution of (4.9) on $I_{(k,h)}$ is

$$z_h(t) = e^{-b(t-kh)} z_h(kh) + \int_{kh}^{t} e^{-b(t-s)} f\left(s, z_h\left(\gamma_h(s)\right)\right) ds.$$

Making $t \to (k+1)h^-$, from the continuity of z_h, we obtain

$$z_h((k+1)h) = e^{-bh} z_h(kh) + \int_{I_{(k,h)}} e^{-b((k+1)h)-s)} f\left(s, z_h((k-d_k)h)\right) ds, \quad k \geq 0.$$

Therefore the sequence $\mathfrak{a}_h(k) := z_h(kh)$ satisfy the nonlinear delay difference equation

$$\mathfrak{a}_h(k+1) = e^{-bh} \mathfrak{a}_h(k) + \int_{I_{(k,h)}} e^{-b((k+1)h)-s)} f\left(s, \mathfrak{a}_h(k-d_k)\right) ds, \quad k \geq 0. \tag{4.13}$$

With initial conditions

$$\mathfrak{a}_h(k) = \varphi(kh), \quad k = 0, 1, \cdots, \quad -\lambda \leq -kh \leq 0. \tag{4.14}$$

From the recurrence relation and initial conditions, we have

$$\begin{aligned}
\mathfrak{a}_h(0) &= \varphi(0), \\
\mathfrak{a}_h(1) &= e^{-bh} \mathfrak{a}_h(0) + \int_{I_{(1,h)}} e^{-b(h-s)} f\left(s, \mathfrak{a}_h(-[r(0, \mathfrak{a}_h(0))/h])\right) ds \\
&= e^{-bh} \varphi(0) + \int_{I_{(1,h)}} e^{-b(h-s)} f\left(s, \mathfrak{a}_h(-[r(0, \mathfrak{a}_h(0))/h])\right) ds, \\
\mathfrak{a}_h(2) &= e^{-bh} \mathfrak{a}_h(1) + \int_{I_{(2,h)}} e^{-b(2h-s)} f\left(s, \mathfrak{a}_h(1-[r(1, \mathfrak{a}_h(1))/h])\right) ds \\
&= e^{-bh} \left[e^{-bh} \varphi(0) + \int_{I_{(1,h)}} e^{-b(h-s)} f\left(s, \mathfrak{a}_h(-[r(0, \mathfrak{a}_h(0))/h])\right) ds \right] \\
&\quad + \int_{I_{(2,h)}} e^{-b(2h-s)} f\left(s, \mathfrak{a}_h(1-[r(1, \mathfrak{a}_h(1))/h])\right) ds \\
&= e^{-b(2h)} \varphi(0) + \int_{I_{(1,h)}} e^{-b(2h-s)} f\left(s, \mathfrak{a}_h(-[r(0, \mathfrak{a}_h(0))/h])\right) ds \\
&= + \int_{I_{(2,h)}} e^{-b(2h-s)} f\left(s, \mathfrak{a}_h(1-[r(1, \mathfrak{a}_h(1))/h])\right) ds.
\end{aligned}$$

Therefore

$$\mathfrak{a}_h(k) = e^{-b(kh)} \varphi(0) + \sum_{i=0}^{k-1} \int_{I_{(i+1,h)}} e^{-b(kh-s)} f\left(s, \mathfrak{a}_h(i-d_i)\right) ds, \quad k \geq 0. \tag{4.15}$$

The sequence $\mathfrak{a}_h(k)$ is well-defined, because $-\lambda \leq (k - d(\mathfrak{a}_h))h \leq kh$ for every $k = 0, 1, 2, \cdots$. It follows that the solution of (4.9)-(4.10) for $t \in I_{k,h}$ can be written

$$z_h(t) = e^{-b(t-kh)}\mathfrak{a}_h(k) + \int_{kh}^{t} e^{-b(t-s)} f\left(s, \mathfrak{a}_h\left(k - d_k\right)\right) ds,$$

or

$$z_h(t) = e^{-bt}\varphi(0) + \sum_{i=0}^{k-1} \int_{I_{(i+1,h)}} e^{-b(t-s)} f\left(s, \mathfrak{a}_h\left(i - d_i\right)\right) ds + \int_{kh}^{t} e^{-b(t-s)} f\left(s, \mathfrak{a}_h\left(k - d_k\right)\right) ds.$$

□

4.2 Cauchy-Peano type theorem

In order to conclude the existence of solution of (4.7)-(4.8) by the convergence of solution of (4.12)-(4.11), as h tends to zero, we state our main assumptions

(H1) The delay function r is bounded and locally Lipschitz-continuous on $\mathbb{R}^+ \times \mathbb{R}$, i.e.,

(a) For any $t \geq 0$ and $u \in \mathbb{R}$ there exists a positive constant Q such that:

$$|r(t, u)| \leq Q.$$

(b) For every $T > 0$ and $M > 0$ there exists a constant $L_2 = L_2(T, M)$ such that:

$$|r(t_1, u_1) - r(t_2, u_2)| \leq L_2(|t_1 - t_2| + |u_1 - u_2|), \quad t_i \in [0, T], u_i \in [0, M], i = 1, 2.$$

(H2) The function $f(t, u)$ is Lipschitz-continuous on the second variable, i.e., there exists a constant $L_1 \in \mathbb{R}^+$ such that:

$$|f(t, u_1) - f(t, u_2)| \leq L_1|u_1 - u_2|, \quad t \in \mathbb{R}^+, u_i \in \mathbb{R}, i = 1, 2;$$

and $f(\cdot, 0) = 0$ for every $t \in \mathbb{R}^+$.

(H3) The coefficients of delay differential equation (4.7) satisfy:

$$b > L_1 > 0. \tag{4.16}$$

We state some properties of the family of functions

$$\Omega_h := \{z_h \text{ solutions of IVP (4.9)-(4.10), for } h \in (0, q]\}. \tag{4.17}$$

Proposition 4.1. *Assume (H2), (H3) holds. Then the set Ω_h is uniformly bounded by a convergent to zero function. Actually, for any $h \in (0, q]$ the family of functions z_h satisfy:*

$$\|z_h(t)\| \leq \|\varphi\|_q \, e^{-\mu t},$$

where μ is the real solution of

$$\mu = b - L_1 e^{\mu q}. \tag{4.18}$$

Proof. For any $h \in (0, q]$, from (H2), we have:

$$\begin{aligned}
\|z_h(t)\| &\leq e^{-bt}\|z_h(0)\| + \int_0^t e^{-b(t-s)}\|f(s, z_h(\gamma_h(s)))\|ds \\
&\leq e^{-bt}\|z_h(0)\| + \int_0^t e^{-b(t-s)}\|f(s, z_h(\gamma_h(s))) - f(s, 0) + f(s, 0)\|ds \\
&\leq e^{-bt}\|\varphi\|_q + \int_0^t e^{-b(t-s)} L_1 \|z_h(\gamma_h(s))\|ds.
\end{aligned}$$

Since (H3) holds, from Halanay Inequality, there exists $\mu > 0$ such that:

$$\|z_h(t)\| \leq \|\varphi\|_q e^{-\mu t}, \quad t \geq 0,$$

where μ is the real solution $\mu = b - L_1 e^{\mu q}$. The Proposition is proved. □

Proposition 4.2. *If (H2), (H3) and φ is continuous on $[-q, 0]$; then the set of continuous functions Ω_h is an equi-continuous set of functions on $[0, \infty)$.*

Proof. We shall prove that there exists a increasing function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that for every $\epsilon > 0$, we have

$$|z_h(t_0) - z_h(t_1)| < \epsilon, \quad \forall h \in (0, q),$$

whenever t_0 and t_1 belongs to $[-\tau, \infty)$ and $|t_1 - t_0| < \delta(\epsilon)$. First we assume that $0 \leq t_1 \leq t_2$. From (4.9) it follows for $t \geq 0$ that

$$\begin{aligned}
|z_h'(t)| &\leq b|z_h(t)| + |f(t, z_h(\gamma_h(t)))| && (4.19)\\
&\leq b|z_h(t)| + L_1|z_h(\gamma_h(t))|, && (4.20)
\end{aligned}$$

therefore

$$\|z_h'\|_\infty \leq b\|z_h\|_\infty + L_1\|z_h\|_\infty.$$

It follows there exists a positive constant M_2 such that $\|z_h'\|_\infty \leq M_2$. If $0 \leq t_1 \leq t_2$, by Mean Value theorem, then

$$|z_h(t_1) - z_h(t_2)\| = \|z_h'(C)\| \cdot \|t_1 - t_2\| \leq M_2 \cdot \|t_1 - t_2\|.$$

In consequence there exists a increasing function $\delta_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\delta_1(\epsilon) = \frac{\epsilon}{M_2}$. If $t_1 \leq t_2 \leq 0$ then the assumed continuity of φ implies that φ is uniformly continuous over

$[-q,0]$, therefore there is a increasing function $\delta_\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for every $\epsilon > 0$, if $|t_1 - t_0| < \delta_\varphi(\epsilon)$ then

$$|\varphi(t_0) - \varphi(t_1)| < \epsilon.$$

Finally, if $t_1 \leq 0 \leq t_2$ then, if $|t_2 - t_1| < \min\{\delta_1(\epsilon/2), \delta_\varphi(\epsilon/2)\}$, from the above estimations we have

$$\begin{aligned} \|z_h(t_1) - z_h(t_2)\| &\leq \|z_h(t_1) - z_h(0)\| + \|z_h(0) - z_h(t_2)\| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So, we consider a increasing function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\delta(\epsilon) = \min\{\delta_1(\epsilon/2), \delta_\varphi(\epsilon/2)\}$. Then, for every $\epsilon > 0$, if $|t_1 - t_0| < \delta(\epsilon)$ then

$$|z_h(t_0) - z_h(t_1)| < \epsilon, \quad \forall h \in (0, q),$$

whenever t_0 and t_1 belongs to $[-\tau, \infty)$. □

The prove of Cauchy-Peano existence theorem require some compactness criteria for an specific space of function. The Arzela-Ascoli theorem is a key result for prove that a subset of functions, with compact domain, has property of relative compactness. But for a set of functions without compact domain, the Arzela-Ascoli theorem it is not enough. The equiconvergence, in some sense, must be considered (see Gallardo and Pinto, 1996).

Theorem 4.2. *If (H1), (H2) and (H3) holds, and* $\varphi \in C([-q,0],\mathbb{R})]$ *is continuous on* $[-q,0]$. *Then the initial value problem (4.7)-(4.8) has, at least, one solution on* $[-q,\infty)$.

Proof. Take a sequence h_k such that: $h_k \to 0$ as $k \to \infty$. For every $h = h_k$ we consider the initial value problem (4.9)-(4.10). Since theorem 4.1, the initial value problem (4.9)-(4.10) has a solution z_{h_k} on $[0,\infty)$. Since Propositions 4.1 and 4.2 it is follows that $\{z_{h_k}\}$ is a sequence of uniformly bounded and equi-continuous functions on $[0,\infty)$ so, by the Arzela-Ascoli theorem, it follows that we can take a convergent subsequence $\{z^1_{h_k}\} \subset \{z_{h_k}\}$, on the interval $[0,1]$; next we consider the convergent subsequence $\{z^1_{h_k}\}$, over $[0,2]$; and there exists a convergent subsequence $\{z^2_{h_k}\} \subset \{z^1_{h_k}\}$, on the interval $[0,2]$; recursively we can get a convergent subsequence $\{z^n_{h_k}\} \subset \{z^{n-1}_{h_k}\}$ over $[0,n]$, for any n a positive integers. We take the diagonal sequence $\{z^k_{h_k}\}$ and by construction the uniform convergence of the sequence on any compact interval it follows, and from Proposition 4.1, the sequence is equiconvergent to zero, we conclude that

$$\lim_{k\to\infty} z^k_{h_k} = y, \qquad \text{uniformly on } \mathbb{R}.$$

For simplicity in the notation, we denote again by z_{h_k} the diagonal convergent subsequence. We extend the functions $z^k_{h_k}$ and $y(t)$ to $[-q,0]$ by $z^k_{h_k} = \varphi(t)$ and $y(t) = \varphi(t)$. Clearly $z^k_{h_k}(t)$ satisfies the initial conditions (4.10). $y(t)$ is a continuous function on $[-q,\infty)$ and for all $t \in [-q,\infty), z^k_{h_k}(t) \to y(t)$ as $k \to \infty$.

Next we need to show that $y(t)$ satisfy (4.7) on $[0,\infty)$. We recall that $y(t)$ satisfy (4.7) if and only if

$$y(t) = e^{-bt}\varphi(0) + \int_0^t e^{-b(t-s)} f\left(s, y\left(s - r(s, y(s))\right)\right) ds. \tag{4.21}$$

From theorem 4.1, z_{h_k} satisfy integral equations (4.21), and from $z_{h_k}(t) \to y(t)$ as $k \to \infty$, to conclude (4.21), we need to prove that

$$\lim_{k\to\infty}\left[\sum_{i=0}^{n-1}\int_{I_{(i,h_k)}} e^{-b(t-s)} f\left(s, a_{h_k}(i-d_i)\right) ds + \int_{nh_k}^t e^{-b(t-s)} f\left(s, a_{h_k}(n-d_n)\right) ds\right] = \int_0^t e^{-b(t-s)} f\left(s, y\left(s - r(s, y(s))\right)\right) ds,$$

for $t \in \mathbb{R}_0^+$. We shall use the Lebesgue's dominated convergence theorem over $f\left(s, z_{h_k}(\gamma_{h_k}(s))\right) - f\left(s, y\left(s - r(s, y(s))\right)\right)$. In order to estimate this difference of functions, We note that for all $0 < \delta < h$, we can find numbers t_δ, τ_δ, such that $|t_\delta - \tau_\delta| \leq \delta$ but $|[t_\delta]_h - [\tau_\delta]_h| = h$, therefore we can obtain $|[t_\delta]_h - [\tau_\delta]_h| \leq |t_\delta - \tau_\delta| + h$. Now we estimate

$$|z_{h_k}([s]_{h_k}) - z_{h_k}(s)| \leq w_{z_{h_k}}(h_k; s) := \max\{|z_{h_k}(t_2) - z_{h_k}(t_2)| \ : 0 \leq t_1, t_2 \leq s, |t_1 - t_2| \leq h_k\}.$$

Note that $w_{z_{h_k}}(h_k; s)$ tends to zero as h_k tends to zero. We also estimate

$$\begin{aligned}
|\gamma_{h_k}(s) - s + r(s, y(s))| &= |[s]_{h_k} - [r([s]_{h_k}, z_{h_k}([s]_{h_k}))]_{h_k} - s + r(s, y(s))| \\
&\leq |[s]_{h_k} - s| + |[r([s]_{h_k}, z_{h_k}([s]_{h_k}))]_{h_k} - r(s, y(s))| \\
&\leq |s - s| + h_k + |r([s]_{h_k}, z_{h_k}([s]_{h_k})) - r(s, y(s))| + h_k \\
&\leq 2h_k + L_2\left(|[s]_{h_k} - s| + |z_{h_k}([s]_{h_k}) - y(s)|\right) \\
&\leq 2h_k + L_2\left(h_k + |z_{h_k}([s]_{h_k}) - z_{h_k}(s)| + |z_{h_k}(s) - y(s)|\right) \\
&\leq 2h_k + L_2\left(h_k + w_{z_{h_k}}(h_k; s) + ||z_{h_k} - y||_\infty\right)
\end{aligned}$$

So, finally, we estimate $|f\left(s, z_{h_k}(\gamma_{h_k}(s))\right) - f\left(s, y\left(s - r(s, y(s))\right)\right)|$ for simplicity we denote $\Delta_f(h_k) := |f\left(s, z_{h_k}(\gamma_{h_k}(s))\right) - f\left(s, y\left(s - r(s, y(s))\right)\right)|$ therefore

$$\begin{aligned}
\Delta_f(h_k) &\leq L_1|z_{h_k}(\gamma_{h_k}(s)) - y\left(s - r(s, y(s))\right)| \\
&\leq L_1\left\{|z_{h_k}(\gamma_{h_k}(s)) - z_{h_k}\left(s - r(s, y(s))\right)| + |z_{h_k}\left(s - r(s, y(s))\right) - y\left(s - r(s, y(s))\right)|\right\} \\
&\leq L_1\left\{w_{z_{h_k}}(2h_k + L_2\left(h_k + w_{z_{h_k}}(h_k; s) + ||z_{h_k} - y||_\infty\right); s)| + ||z_{h_k} - y||_\infty\right\}.
\end{aligned}$$

It follows that

$$|f\left(s, z_{h_k}(\gamma_{h_k}(s))\right) - f\left(s, y\left(s - r(s, y(s))\right)\right)| \to 0 \text{ as } k \to \infty, \tag{4.22}$$

and that

$$\Delta_f(h_k) \leq L_1\left\{w_{z_{h_1}}(2h_1 + L_2\left(h_1 + w_{z_{h_1}}(h_1; s) + ||z_{h_1} - y||_\infty\right); s)| + ||z_{h_1} - y||_\infty\right\}. \tag{4.23}$$

Therefore, there exists a measurable function

$$g(t,s) := e^{-b(t-s)}L_1\left\{w_{z_{h_1}}(2h_1 + L_2\left(h_1 + w_{z_{h_1}}(h_1;s) + \|z_{h_1} - y\|_\infty\right);s)| + \|z_{h_1} - y\|_\infty\right\}$$

such that $\int_0^t g(t,s)ds$ is well defined for $t \in \mathbb{R}_0^+$. Now, from Lebesgue's dominated convergence theorem it follows that (4.21) holds, so we conclude that $y(t)$ satisfy the integral equation (4.21) that is equivalent to (4.7). □

Example 4.1. *We consider the semi-linear differential equations with state-dependent delay*

$$y'(t) = -by(t) + \sin\left(y(t - |cos(y(t))|)\right). \tag{4.24}$$

We check that assumptions of theorem 4.2 holds. Since $r(t, y(t)) = r(y(t)) = |\cos(y(t))|$ *follows that* $r(u)$ *is a bounded function and for real numbers* u_1 *and* u_2 *we have*

$$|r(u_1) - r(u_2)| \le |\cos(u_1) - \cos(u_2)| \le |u_1 - u_2|, \quad u_i \in \mathbb{R}, i = 1, 2.$$

Therefore (H1) holds. Since $f(t,u) = f(u) = \sin(u)$ *(H2) holds with* $L_1 = 1$*. So, for any real number* $b > 1$ *(H3) holds. We conclude that for any* $b > 1$ *and a given initial function* $\varphi \in C$ *equation* (4.24) *has at least a solution over the interval* $[0, \infty)$*, from theorem 4.2.*

5

Conclusion

In order to get our overall goals we need identify the main assumptions, techniques and methods used in papers about approximation and transference of stability properties between solutions of differential equations with delay and the corresponding difference equations.

About the techniques and methods used, we note that the results of Cooke and Györi (1994) and Györi and Hartung (2002) rest on the functional differential equations that the error function $E_h(t)$ satisfy. Actually $E_h(t)$ satisfy a linear non-homogeneous or semi-linear functional differential equations. If the linear homogeneous differential equations with delay is uniformly asymptotically stable, then is possible conclude the exponential decay rate of the error by using an integral inequality. In the literature review on this subject, authors considered Gronwall-Bellman inequality, however we use Halanay inequality since is more appropriate for delay differential equations. We also note that Mohamad and Gopalsamy (2000) and Liz and Ferreiro (2002) used Halanay inequality as key technique to prove that the zero solution of both continuous functional differential equations and discrete equations are exponentially stable. However, this technique does not state any estimation of the error.

Our theorems 2.3 and 3.3 state the approximation and transference of exponential stability properties of solutions of non-autonomous differential equations with variable delay and retarded functional differential equation with feedback, respectively, to the corresponding difference equation by using piecewise constant argument, the techniques and methods above mentioned. Theorem 3.2 state the transference of local exponential stability properties of the equilibrium of retarded functional differential equation with feedback to the corresponding difference equation, by using directly Halanay Inequality.

In order to relate existence Cauchy-Peano type theorems for differential equations with state-dependent delay with the transference of existence between a difference equation to functional differential equation. We prove theorem 4.2 a global existence result of this kind, by convergence of a sequence of approximation. We have extended the existence result of Györi et al. (1995) and Tavernini (1978) (see Hartung et al., 1997).

Natural applications of our results can be found in the systems of differential equations used to model cellular neural networks (see Mohamad and Gopalsamy, 2003; Abbas and Xia,

2013) and identification of parameters in functional differential equations (see Hartung and Turi, 1997; Hartung et al., 1998, 2000). We have applied the techniques presented in our work to a model of cellular neural network, actually in a system of differential equations of fractional order with delay Tyagi et al. (2016). Our results in Chapter 3 can be applied in several model of a single population (see Liz et al., 2005), however, we have the conviction that the procedure developed there can be successful in many models that consider semi-linear functional differential equations.

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