SUPERFICIES DE RIEMANN Y SUS DESSINS D'ENFANTS

Tesis
entregada a la
Universidad de Chile
en cumplimiento parcial de los requisitos
para optar al grado de

Doctor en Ciencias con mención en Matemáticas

Facultad de Ciencias

Por

Angélica María Vega Moreno

Abril, 2017

Director de tesis: Dr. Rubén A. Hidalgo

FACULTAD DE CIENCIAS

UNIVERSIDAD DE CHILE

INFORME DE APROBACIÓN

TESIS DE DOCTORADO

Se informa a la escuela de Postgrado de la Facultad de Ciencias que la Tesis de Doctorado presentada por la candidata

Angélica María Vega Moreno

ha sido aprobada por la Comisión de Evaluación de la tesis como requisito parcial para optar al grado de Doctor en Ciencias con mención en Matemáticas en el examen de Defensa de Tesis rendido el día 12 ABR 2017.

Director de Tesis:
Dr. Rubén A. Hidalgo (UFRO) ....

Co-Director de Tesis
Dra. Anita Rojas (U. de Chile) ....

Comisión de Evaluación de Tesis

Dr. Antonio Behn (U. de Chile) ....

Dra. Mariela Carvacho (UTFSM) ....

Dra. Rubí Rodríguez (UFRO) ....

To my daughter, husband and parents.

Nací el 14 de octubre de 1981 en Bogotá. A la edad de 8 años viajé a la entonces Unión Soviética, junto a mi padre, donde viví por un periodo de 7 años y donde desarrolle mi interés por las matemáticas. Al regresar a Colombia, después de aquel viaje, terminé con mis estudios de educación básica e inicié con mi carrera profesional en la Universidad Nacional de Colombia. En ésta me gradué con el título de Matemática y decidí continuar con mis estudios de postgrado en la misma disciplina. Tentada por estudiar y viajar al mismo tiempo, tomé la decisión de viajar a Santiago de Chile y aceptar una oferta educativa en la Universidad Católica de Chile de donde, años después, obtuve mi título de Magíster en Matemáticas. Una vez terminé con el Magíster y habiéndome adaptado a Chile, inicié con el Doctorado en Ciencias Mención Matemáticas en la Universidad de Chile en donde estoy por culminar mis estudios.

ACKNOWLEDGEMENTS

Throw next lines I wish to express my gratitude to all people who makes me possible the culmination of this Ph. D programm. But, since this is something emotional, I prefer to do it in Spanish, my maternal language.

A todas las personas e instituciones que han hecho posible la realización de uno de mis más grandes sueños, la culminación de mis estudios de doctorado, les quiero manifestar mi total gratitud:

Al profesor Rubén A. Hidalgo por su tiempo, comprensión, paciencia, dedicación, y sobre todo por su disposición.

Al departamento de Matemáticas de la Universidad de Chile por la oportunidad que me brindo al aceptarme en calidad de alumna de su programa de Doctorado en Ciencias Mensión en Matemáticas permitiéndome usar sus recursos materiales y humanos. En particular, manifiesto mi gratitud hacia los profesores del departamento por su disposición, apoyo y calidez.

Al programa de becas nacionales MECESUP por su apoyo económico durante el periodo de tres años.

Al profesor Eduardo Friedmann por su seguimiento, presencia, apoyo, disposición y consejos. Siempre creyó en mi y me hizo ver luz cuando yo veía obscuridad.

A Sebastián Reyes por sus comentarios, sugerencias y correcciones a esta tesis.

A la comisión evaluadora conformada por los profesores Antonio Behn, Mariela Carvacho y Rubí Rodríguez por sus correcciones, comprensión, disposici´n y tiempo.

A mi esposo Julián Agredo por su enorme comprensión, amor y apoyo incondicional. Gracias por tus sacrificios y bondades. Sin duda alguna, me siento muy afortunada al tenerte a mi lado como esposo, confidente y colega!

A mi hijita Sofía por hacerme sentir plena y llenarme de tantas alegrías y motivaciones.

A mis padres y a mi suegra por tantas ayudas y tanto cariño.

A Gabriel Zoolotochin, Ana María Moncada, Mariela Carvacho y mis compañeros de trabajo de la USM por su amistad y hacer de mi estadía en Chile una estancia realmente placentera, amigable, como en casa.

Finalmente, a mi familia, a Deissy Moreno, compañeros de estudio y amigos, que de una u otra forma me han ayudado en este capítulo de mi vida.

RESUMEN

Sean $\beta_j : S_j \to S_0$, $j = 1, 2$, dos recubrimientos holomorfos no constantes dados entre superficies de Riemann compactas. Asociado a ellos esta su producto fibrado (en la categoría de conjuntos) el cual puede o no ser conexo y cuando es conexo puede o no ser irreducible. Se da una descripción Fuchsiana de las componentes irreducibles del producto fibrado y, como consecuencia, mostramos que si uno de los recubrimientos β_j es regular, entonces todas las componentes irreducibles son isomorfas. En el caso en que el producto fibrado sea conexo (por ejemplo, por los resultados de Fulton-Hansen, cuando S_0 tiene género cero), proveemos condiciones suficientes para que éste sea irreducible; se dan ejemplos para ver que estas condiciones no son necesarias en general. Definimos el cuerpo (fuerte) de moduli del producto fibrado y vemos que éste coincide con el cuerpo más pequeño que contiene a los cuerpos de moduli de los pares iniciales (S_1, β_1) y (S_2, β_2).

ABSTRACT

Given non-constant holomorphic coverings $\beta_j : S_j \to S_0$, $j = 1, 2$, between compact Riemann surfaces, there is associated its fiber product (in the set theoretical sense), which may or not be connected and when it is connected it may or not be irreducible. A Fuchsian group description of the irreducible components of the fiber product is given and, as a consequence, we show that if one of the coverings β_j is regular, then all irreducible components are isomorphic. In the case that the fiber product is connected (for instance, by results of Fulton-Hansen, if S_0 has genus zero), we provide sufficient conditions for it to be irreducible; examples are provided to see that these conditions are not necessary in general. We define the (strong) field of moduli of the fiber product and see that it coincides with the minimal field containing the fields of moduli of the starting pairs (S_1, β_1) and (S_2, β_2).

Contents

CHAPTER 1

Introduction

In the category of sets there is a construction called the fiber product which satisfies certain universality property. Such a construction cannot be realized in every subcategory. For example, in the category of Riemann surfaces the fiber product is not always again a Riemann surface. But for the category of algebraic varieties (or schemes) this is known to be possible [Ii]. Since the fiber product has been a main tool in the construction of interesting examples and counterexamples in algebraic geometry, and compact Riemann surfaces can be thought as complex algebraic varieties, we will work with the category of singular Riemann surfaces (which contains the Riemann surfaces) for which this kind of construction is possible. This will allow us to continue with the study started in [**Hid**] of the fiber product at the level of compact Riemann surfaces, which is our interest.

Let us consider three compact Riemann surfaces S_0, S_1 and S_2 together with two non-constant holomorphic maps $\beta_1 : S_1 \to S_0$ and $\beta_2 : S_2 \to S_0$. The fiber product of the two pairs (S_1, β_1) and (S_2, β_2) is given by

$$S_1 \times_{(\beta_1,\beta_2)} S_2 = \{(z_1, z_2) \in S_1 \times S_2 : \beta_1(z_1) = \beta_2(z_2)\}$$

and for it there is a natural function $\beta : S_1 \times_{(\beta_1,\beta_2)} S_2 \to S_0$ defined by $\beta(z_1, z_2) = \beta_1(z_1) = \beta_2(z_2)$. In general, this fiber product might not be irreducible, non-singular or even connected and therefore it might not be a compact Riemann surface. But it is a singular Riemann surface with a finite number of irreducible components $R_1, \dots, R_n$, each one a compact Riemann surface [1] and satisfies the following universal property:

If R is a compact Riemann surface and $p_j : R \to S_j$ for $j = 1, 2$ are non-constant holomorphic functions such that $\beta_1 \circ p_1 = \beta_2 \circ p_2$ then there exists a non-constant holomorphic map $t : R \to R_k$ for some $1 \leq k \leq n$ such that $p_j = \pi_j \circ t$ (this determines uniquely $t(r) = \big(p_1(r), p_2(r)\big)$) where $\pi_j : S_1 \times_{(\beta_1,\beta_2)} S_2 \to S_j$ is the projection $\pi_j(z_1, z_2) = z_j$ for $j = 1, 2$.

Chapter 2 contains all the tools needed to follow this thesis and in Chapter 3 we work out all the details of the above construction and discussion. We will also provide a Fuchsian group description of the irreducible components of the fiber product and, as a consequence, we will show that if one of the functions β_j is a regular (branched) covering, then all of them are isomorphic Riemann surfaces (see Corollary 3.2.4). Besides, we will provide examples of non-connected fiber products and when the contrary occurs, that is when $S_1 \times_{(\beta_1,\beta_2)} S_2$

[1] It is known from [**Hid**] that any two of the irreducible components of lowest genus of $S_1 \times_{(\beta_1,\beta_2)} S_2$ (if different from one) define isomorphic compact Riemann surfaces (when they have genus one they are isogenous elliptic curves).

is connected, we establish sufficient conditions for it to be irreducible. At the end of this third chapter we will give an explicit application to dessins d'enfants.

In Chapter 4 we define an algebraic invariant called the *strong field of moduli of the fiber product* and we prove there that it is the smallest field containing the fields of moduli of the corresponding pairs (S_1, β_1) and (S_2, β_2).

Finally, in Chapter 5 we exhibit a variety of examples including all the concepts and results of the Chapters 2, 3 and 4 with all its details.

CHAPTER 2

Preliminaries

Along this chapter we will develop basic concepts and results needed for the rest of this thesis. In Sections 2.1 and 2.2 we provide definitions and results related to Riemann surfaces and holomorphic functions between them. In Section 2.3 we recall some notions and results of a basic course in algebraic topology as are the fundamental group and coverings. Also we establish their connections with the concepts of the preceding sections. In section 2.4 we use the uniformization theorem to classify compact Riemann surfaces and give other description of how to obtain (compact) Riemann surfaces by means of actions of Fuchsian groups on the upper half plane $\mathbb{H}$. In Section 2.5 we discus the concept of monodromy because we also are interested in having a tool to decide whether two coverings are isomorphic. Section 2.6 is dedicated to the Riemann- Roch's theorem and some of its consequences. Section 2.7 deals with the concepts and results of Belyi pairs and dessins d'enfants; we exhibit the equivalence between these concepts. Finally, in Section 2.8 we present the Galois action on polynomials (and thus on algebraic curves) and rational functions and we make a list of its properties when this action is over Belyi pairs.

2.1. Riemann surfaces.

DEFINITION 2.1.1. A *Riemann surface* is a connected topological Hausdorff space X with an open cover $\{U_\alpha\}_{\alpha\in I}$ of X and $\{\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{C}, \alpha \in I\}$ a collection of *compatible* homeomorphisms (called *charts*) over open subsets V_α of $\mathbb{C}$, that is, a collection of homeomorphisms satisfying the following property:

- Whenever $U_\alpha \cap U_\beta \neq \emptyset$, the function $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ (called *transition function*)

$$\begin{array}{ccc} & U_\alpha \cap U_\beta & \\ \varphi_\alpha \swarrow & & \searrow \varphi_\beta \\ \varphi_\alpha(U_\alpha \cap U_\beta) & \xrightarrow{\varphi_\beta\circ\varphi_\alpha^{-1}} & \varphi_\beta(U_\alpha \cap U_\beta) \end{array}$$

  is analytic.

A collection of charts fulfilling this property is called an (holomorphic) *atlas* and the inverse functions φ_α^{-1} are called *parametrizations.*

REMARK. If X is a Riemann surface then:

i) Two atlases of X are compatible if their union is again an atlas for X.
ii) By Zorn's lemma, every atlas of X is contained in a unique maximal atlas, called a *complex structure* of X.
iii) Given $p \in X$ and $z_0 \in \mathbb{C}$, then it is always possible to add, to an existing atlas, a chart (U_p, ψ) such that $p = \psi^{-1}(z_0)$ (composing by a translation of $\mathbb{C}$).
iv) As a consequence of the previous item, it is always possible to assume each chart is *centered at p* (this meaning that $\varphi_\alpha(p) = 0$).

EXAMPLE 2.1.2. Let X be the complex plane $\mathbb{C}$ endowed with the usual topology. Then $\{(\mathbb{C}, id)\}$ and $\{(\mathbb{C}, id), (\mathbb{C}, z \to z - 1)\}$ are compatible atlases.

EXAMPLE 2.1.3. The simplest of the examples of compact Riemann surfaces is the *Riemann sphere* $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ endowed with the topology of open sets and complements of compact sets in $\mathbb{C}$, and with complex structure induced by the atlas $\{(\mathbb{C}, \varphi_1), (\widehat{\mathbb{C}} \setminus \{0\}, \varphi_2)\}$ where

$$\varphi_1(z) = z \quad \text{and} \quad \varphi_2(z) = \begin{cases} 1/z, & z \in \mathbb{C}^* \\ 0, & z = \infty \end{cases}$$

Defining $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, in this case, $\varphi_1^{-1} \circ \varphi_2 : \mathbb{C}^* \to \mathbb{C}^*$ with $\varphi_1^{-1} \circ \varphi_2(z) = 1/z$ is obviously analytic.

Let us observe that another atlas for the sphere is given by $\{(\mathbb{C}, J \circ \varphi_1), (\widehat{\mathbb{C}} \setminus \{0\}, J \circ \varphi_2)\}$, where $J(z) = \overline{z}$. This atlas is clearly non-compatible with the above one.

DEFINITION 2.1.4. The cartesian product $\mathbb{C}^n$, of $\mathbb{C}$ n-times, is the *complex affine n-space* or simply *affine n-space*. In particular, it is called *the affine line* when $n = 1$ and *the affine plane* when $n = 2$.

DEFINITION 2.1.5. Let us denote by $\mathbb{C}[X_1, X_2, \ldots, X_n]$ the ring of polynomials in n variables over $\mathbb{C}$. If $f \in \mathbb{C}[X_1, X_2, ..., X_n]$, a point $p = (x_1, x_2, ..., x_n)$ in $\mathbb{C}^n$ is said to be a zero of f if $f(p) = 0$. If f is not constant, the set of its zeros is called the *hypersurface* defined by f and is denoted by V_f. In the particular case $n = 2$, a hypersurface is called an *affine algebraic curve*. More generally, if $f_1, ..., f_r \in \mathbb{C}[X_1, X_2, ..., X_n]$

$$V_{f_1,\ldots,f_r} := \{p \in \mathbb{C}^n : f_1(p) = \ldots = f_r(p) = 0\} = \bigcap_{i=1,\ldots,r} V_{f_i}$$

is an *affine algebraic set* or simply an *algebraic set*.

REMARK. It is obvious that $V_{fg} = V_f \cup V_g$ for any polynomials f and g.

DEFINITION 2.1.6. For an algebraic set $V \subseteq \mathbb{C}^n$ we say that:

(1) V is *reducible* if it is the union of non empty algebraic sets V_1 and V_2 in $\mathbb{C}^n$ such that $V_i \neq V$ $(i = 1, 2)$. Otherwise V is *irreducible.*

(2) $V = V_1 \cup ... \cup V_m$ is the *decomposition of V into irreducible components* if V_i is irreducible for every $i = 1, ..., m$.

DEFINITION 2.1.7. Let $f(X,Y) \in \mathbb{C}[X,Y]$ and let C_f be the affine algebraic curve defined by f. A point $p \in C_f$ is *non-singular* if $\frac{\partial f}{\partial X}\big|_p \neq 0$ or $\frac{\partial f}{\partial Y}\big|_p \neq 0$. Otherwise it is *singular.* We say that C_f is *non-singular* or simply *smooth* if all its points are non-singular.

EXAMPLE 2.1.8. The algebraic set $X = \{(z,w) \in \mathbb{C}^2 : z^4 - w^2 = 0\}$ is reducible because it can be decomposed as the union of the non empty algebraic sets

$$X = \{(z,w) \in \mathbb{C}^2 : z^2 + w = 0\} \cup \{(z,w) \in \mathbb{C}^2 : z^2 - w = 0\}.$$

Besides, it has only one singular point in $\mathbb{C}^2$, namely $(0,0)$. Note that the above two components are non-singular and irreducible.

EXAMPLE 2.1.9. **(Affine algebraic curves).** An interesting type of Riemann surface is the one considered by a non-singular (connected by Proposition 2.1.10 below) affine algebraic curve $C_f := \{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\}$ endowed with the topology induced by the usual topology on $\mathbb{C}^2$ and complex structure defined as follows:

For a point $p = (y_0, z_0)$ in C_f:

- If $\frac{\partial f}{\partial y}\big|_p \neq 0$, by the implicit function theorem, there is a holomorphic complex function $g(x)$, defined in a non empty neighbourhood V_{x_0} of x_0 in $\mathbb{C}$ such that:
  - $g(x_0) = y_0$.
  - $f(x, g(x)) = 0$ and $g'(x) = -\frac{\partial f/\partial x}{\partial f/\partial y}\big|_x$ for all $x \in V_{x_0}$.

  Therefore, there is a non empty neighbourhood $V_p^1 = \{(x, g(x)) : x \in V_{x_0}\}$ of p in C_f such that (V_p^1, π_1), where π_1 denotes the projection on the first coordinate, is a chart for p.
- If $\frac{\partial f}{\partial x}|_p \neq 0$, arguing in the same way, there is an holomorphic complex function $h(y)$ defined in a non empty neighbourhood V_{y_0} of y_0 in $\mathbb{C}$ such that:
  - $h(y_0) = x_0$.

– $f(h(y), y) = 0$ and $h'(y) = -\frac{\partial f/\partial y}{\partial f/\partial x}\big|_z$ for all $y \in V_{y_0}$.
Therefore, there is a non empty neighbourhood $V_p^2 = \{(h(y), y) : y \in V_{y_0}\}$ of p in C_f such that (V_p^2, π_2), where π_2 denotes the projection on the second coordinate, is a chart for p.

These charts are compatible and therefore they make C_f a Riemann surface. Actually, if $p \in C_f$ is such that $\frac{\partial f}{\partial x}\big|_p \neq 0$ and $\frac{\partial f}{\partial y}\big|_p \neq 0$ then there is a non empty neighbourhood $V_p = V_p^1 \cap V_p^2$ of p in C_f for which the transition functions

$$\pi_2 \circ \pi_1^{-1}(x) = \pi_2(x, g(x)) = g(x) \text{ and } \pi_1 \circ \pi_2^{-1}(y) = \pi_1(h(y), y) = h(y)$$

are holomorphic.

PROPOSITION 2.1.10. *If $f(X, Y) \in \mathbb{C}[X, Y]$ is non-singular and irreducible then the affine curve C_f defined by f is a Riemann surface.*

PROOF. Irreducibility of f implies connectedness of C_f. See [**SHAF**], for example. □

EXAMPLE 2.1.11. Suppose $h(Y) \in \mathbb{C}[Y]$ and let $f(X, Y) = X^2 - h(Y)$. The following properties are satisfied:

- If $h(Y)$ is not a perfect square, $f(X, Y)$ is irreducible. Indeed, if this were not true, we could write
$$f(X, Y) = (X - b(Y))(X - k(Y)) = X^2 - (b(Y) + k(Y))X + b(Y)k(Y)$$
where $b + k = 0$ and $-bk = h$. Thus $b = -k$ and $h = b^2$ which is a contradiction.
- If all the roots of $h(Y)$ are distinct, $f(X, Y)$ is non-singular. This assertion follows by noting that $\frac{\partial f}{\partial Y} = -h'(Y)$ and $\frac{\partial f}{\partial X} = 2X$.

EXAMPLE 2.1.12. The curve $\{(z, w) \in \mathbb{C}^2 : z^2 + w = 0\}$ is a Riemann surface because it is non-singular and irreducible.

In order to provide more examples of Riemann surfaces, as projective plane curves and complete intersection curves, we need to pass through additional terminology.

DEFINITION 2.1.13. An *n-dimensional complex analytic manifold* is a connected Hausdorff topological space X with an open cover $\{U_\alpha\}_{\alpha \in I}$ of X and $\{\phi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{C}^n, \alpha \in I\}$ a collection of *compatible* homeomorphisms (called *n-dimensional complex charts*) onto open subsets V_α of $\mathbb{C}^n$; that is, a collection of homeomorphisms satisfying the following property:

- Whenever $U_\alpha \cap U_\beta \neq \emptyset$, the function $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ (called *transition function*) is holomorphic; that is, it is holomorphic in each of the n variables separately at every point.

A collection of n-dimensional complex charts fulfilling this property is called an (analytic) *n-dimensional complex atlas* and the inverse functions φ_α^{-1} are called *parametrizations.*

REMARK. If X is an n-dimensional complex manifold then:

i) Two n-dimensional complex atlases of X are compatible if their union is again a n-dimensional complex atlas for X.
ii) By Zorn's lemma, every n-dimensional complex atlas of X is contained in a unique maximal n-dimensional complex atlas, called a *n-dimensional complex structure* of X.
iii) Riemann surfaces are precisely the 1-dimensional complex manifolds.

DEFINITION 2.1.14. Consider the equivalence relation between two non-zero vectors in $\mathbb{C}^{n+1}$ for which $(x_0, x_1, ..., x_n) \sim (y_0, y_1, ..., y_n)$ if and only if there is $\lambda \neq 0 \in \mathbb{C}$ such that $(x_0, x_1, ..., x_n) = \lambda(y_0, y_1, ..., y_n)$ (in other words, the one-dimensional complex subvector spaces they generate coincide). The set of equivalent classes of this equivalence relation is called *the n-dimensional complex projective space* or simply *the n-dimensional projective space* and is denoted by $\mathbb{P}^n_{\mathbb{C}}$. Its elements are called *homogeneous coordinates* and are denoted by $[x_0 : x_1 : ... : x_n]$ for each $(x_0, x_1, ..., x_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. In other words, $\mathbb{P}^n_{\mathbb{C}}$ is the space that parametrizes one-dimensional subvector spaces of $\mathbb{C}^{n+1}$.

REMARK 2.1.15. *$\mathbb{P}^n_{\mathbb{C}}$ endowed with the quotient topology is Hausdorff, connected, simply connected and compact.*

i) *For each $i = 0, ...n$, the set $U_i = \{[x_0 : x_1 : ... : x_n] : x_i \neq 0\}$ is an open subset of $\mathbb{P}^n_{\mathbb{C}}$, U_i is canonically isomorphic to $\mathbb{C}^n$ by mean of the homeomorphism*

$$\begin{array}{cccc} \phi_i & : & U_i & \to & \mathbb{C}^n \\ & & [x_0 : ... : x_{i-1} : x_i : x_{i+1} : ... : x_n] & \mapsto & \left(\frac{x_0}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_n}{x_i}\right) \end{array}$$

or equivalently

$$\begin{array}{cccc} \phi_i & : & U_i & \to & \mathbb{C}^n \\ & & [x_0 : ... : x_{i-1} : 1 : x_{i+1} : ... : x_n] & \mapsto & (x_0, ..., x_{i-1}, x_{i+1}, ..., x_n) \end{array}.$$

In particular, as $U_0 \cong \mathbb{C}^n$ and $\mathbb{P}^n_{\mathbb{C}} \setminus U_0 \cong \mathbb{P}^{n-1}_{\mathbb{C}}$, it can be seen that $\mathbb{P}^n_{\mathbb{C}} \cong \mathbb{C}^n \cup \mathbb{P}^{n-1}_{\mathbb{C}}$ and, by an induction procedure, that $\mathbb{P}^n_{\mathbb{C}} \cong \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup ... \cup \mathbb{C} \cup \{\infty\}$

(in Example 2.2.3 it will be shown that $\mathbb{P}^1_{\mathbb{C}} \simeq \widehat{\mathbb{C}}$). Of course we also have $\mathbb{P}^n_{\mathbb{C}} = U_0 \cup ... \cup U_n$.

ii) *By means of the following diagram*

$$\begin{array}{ccc} & U_i \cap U_j & \\ {}^{\phi_i}\swarrow & & \searrow^{\phi_j} \\ \phi_i(U_i \cap U_j) & \overset{\phi_j \circ \phi_i^{-1}}{\dashrightarrow} & \phi_j(U_i \cap U_j) \end{array}$$

we can see that $\mathbb{P}^n_{\mathbb{C}}$ is an n-dimensional analytic manifold since $\phi_j \circ \phi_i^{-1}$ is analytic.

DEFINITION 2.1.16. $\mathbb{P}^1_{\mathbb{C}}$ and $\mathbb{P}^2_{\mathbb{C}}$ are called the *complex projective line* and the *complex projective plane*, respectively.

EXAMPLE 2.1.17. (**$\mathbb{P}^1_{\mathbb{C}}$ is a Riemann surface**). The set $\{(U_0, \varphi_0), (U_1, \varphi_1)\}$, where $\varphi_0 = \phi_0$ and $\varphi_1 = \phi_1$ of the Remark 2.1.15, is an atlas for the complex projective line. In fact, it is not difficult to see that for $U_0 \cap U_1$ the transition functions are holomorphic.

DEFINITION 2.1.18. A polynomial $f(X_1, ..., X_n)$ of degree d in $\mathbb{C}[X_1, ..., X_n]$ is *homogeneous* if $f(\lambda X_1, ..., \lambda X_n) = \lambda^d f(X_1, ..., X_n)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

EXAMPLE 2.1.19. For a positive integer n the polynomial $f(X, Y, Z) = X^n + Y^n + Z^n$ is homogeneous of degree n.

DEFINITION 2.1.20. If $f(X_1, ..., X_n)$ is a homogeneous polynomial of degree d, then for every $i \in \{1, ..., n\}$ we can obtain a new d-degree polynomial f_i by taking $X_i = 1$ in f. This process is usually called *dehomogenizing with respect to the i-th component.* Conversely, if $g(X_1, ..., X_n)$ is any polynomial of degree d then we also can obtain degree d the homogeneous polynomial $X^d_{n+1} g\left(\frac{X_1}{X_{n+1}}, ..., \frac{X_n}{X_{n+1}}\right)$. This process is often called *homogenizing with respect to the component $n+1$.*

EXAMPLE 2.1.21. Homogenizing the polynomial $f(X, Y, Z) = 4XY + 7X^2Y^3 - Z^2Y^8$ with respect to the variable U provides the polynomial

$$\widetilde{f}(X, Y, Z, U) = 4XYU^8 + 7X^2Y^3U^5 - Z^2Y^8$$

and, noting that $\widetilde{f}$ is homogeneous, dehomogenizing $\widetilde{f}$ with respect to the variable Z we obtain the polynomial

$$\widetilde{f}_3 = 4XYU^8 + 7X^2Y^3U^5 - Y^8.$$

REMARK 2.1.22. *If $f(X,Y,Z) \in \mathbb{C}[X,Y,Z]$ is homogeneous and irreducible (or a unit) in $\mathbb{C}[X,Y,Z]$ then, using the same notation as in the previous definition, it is not difficult to see that the polynomials f_1, f_2 and f_3 are irreducible or units.*

DEFINITION 2.1.23. A *projective algebraically variety* V in $\mathbb{P}^n_{\mathbb{C}}$ is a set consisting of all the common zeros of a finite collection of homogenous polynomials (not necessarily of the same degree) $f_1, ..., f_r$ in $\mathbb{C}[X_0, ..., X_n]$.

DEFINITION 2.1.24. A projective algebraically variety V in $\mathbb{P}^n_{\mathbb{C}}$ is *reducible* if there are non-empty different varieties $V_1, V_2 \subsetneq \mathbb{P}^n_{\mathbb{C}}$ such that $V = V_1 \cup V_2$. Otherwise, V is *irreducible.*

EXAMPLE 2.1.25. The variety $V = \{[z : w : t] \in \mathbb{P}^2_{\mathbb{C}} : z^3(t^3 - z^3) = w^3(t^3 - w^3)\}$ is reducible since

$$\begin{aligned} z^3(t^3 - z^3) - w^3(t^3 - w^3) &= (z^3 - w^3)t^3 - (z^6 - w^6) \\ &= (z^3 - w^3)(t^3 - z^3 - w^3) \\ &= (z - w)(z - e^{2\pi i/3}w)(z - e^{4\pi i/3}w)(t^3 - z^3 - w^3) \end{aligned}$$

and therefore the decomposition of V into irreducible components is given by

$$V = V_1 \cup V_2 \cup V_3 \cup V_4,$$

where $V_1 = \{[z : w : t] \in \mathbb{P}^2_{\mathbb{C}} : z - w = 0\}$, $V_2 = \{[z : w : t] \in \mathbb{P}^2_{\mathbb{C}} : z - e^{2\pi i/3}w = 0\}$, $V_3 = \{[z : w : t] \in \mathbb{P}^2_{\mathbb{C}} : z - e^{4\pi i/3}w = 0\}$ and $V_4 = \{[z : w : t] \in \mathbb{P}^2_{\mathbb{C}} : t^3 - z^3 - w^3 = 0\}$.

EXAMPLE 2.1.26. **(Projective plane curves).** Consider an homogenous polynomial $f(X,Y,Z)$ in $\mathbb{C}[X,Y,Z]$ and the projective algebraically variety

$$X_f := \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : f(x,y,z) = 0\}$$

defined by f. We make the following intersections

$$\begin{aligned} X_f \cap U_0 &= \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : x \neq 0, f(x,y,z) = 0\} \\ &= \{[1 : y : z] \in \mathbb{P}^2_{\mathbb{C}} : f(1,y,z) = 0\}. \end{aligned}$$

$X_f \cap U_1 = \{[x:1:z] \in \mathbb{P}^2_{\mathbb{C}} : f(x,1,z)=0\}$, $X_f \cap U_2 = \{[x:y:1] \in \mathbb{P}^2_{\mathbb{C}} : f(x,y,1)=0\}$
and define the sets

$$X_0 := \{(y,z) \in \mathbb{C}^2 : f_0(y,z)=0\}$$
$$X_1 := \{(x,z) \in \mathbb{C}^2 : f_1(x,z)=0\}$$
$$X_2 := \{(x,y) \in \mathbb{C}^2 : f_2(x,y)=0\}$$

where $f_0(y,z) := f(1,y,z)$, $f_1(x,z) := f(x,1,z)$ and $f_2(x,y) = f(x,y,1)$.

In this case, $\phi_0(X_f \cap U_0) = X_0$, $\phi_1(X_f \cap U_1) = X_1$ and $\phi_2(X_f \cap U_2) = X_2$ for ϕ_0, ϕ_1, ϕ_2 as in remark 2.1.15.

Until now we have not defined charts to make X_f a Riemann surface, but X_0 ,X_1 and X_2 are affine curves. If they are non-singular and irreducible, then by Proposition 2.1.10 and the Implicit Function Theorem, we can endow X_f with a complex structure as follows:

For any point $p = [1:y_0:z_0] \in X_f \cap U_0$ we have:

- $X_f \cap U_0$ is an open subset of X_f homeomorphic to X_0 by means of ϕ_0.
- $\phi_0(p) = (y_0, z_0)$ is a point in X_0.
- If $\frac{\partial f_0}{\partial Z}|_{\phi_0(p)} \neq 0$ then there is a holomorphic complex function $g(y)$ defined in a non empty neighbourhood V_{y_0} of y_0 in $\mathbb{C}$ such that $g(y_0) = z_0$ and $f_0\big(y, g(y)\big) = 0$ for all $y \in V_{y_0}$. Therefore there is a non empty neighbourhood V^1_p of $\phi_0(p)$ in X_0 such that $V^1_p = \big\{\big(y, g(y)\big) : y \in V_{y_0}\big\}$. So if π_1 denotes the projection on the first coordinate then $\pi_1|_{V^1_p}$ is a homeomorphism and $\varphi_1 = \pi_1 \circ \phi_0 : \phi_0^{-1}(V^1_p) \to \mathbb{C}$ would be a chart for p.
- If $\frac{\partial f_0}{\partial Y}|_{\phi_0(p)} \neq 0$ then there is a holomorphic complex function $h(z)$ defined in a non empty neighbourhood V_{z_0} of z_0 in $\mathbb{C}$ such that $h(z_0) = y_0$ and $f_0\big(h(z), z\big) = 0$ for all $z \in V_{z_0}$. Therefore there is a non empty neighbourhood V^2_p of $\phi_0(p)$ in X_0 such that $V^2_p = \big\{\big(h(z), z\big) : z \in V_{z_0}\big\}$. So if π_2 denotes the projection on the second coordinate then $\pi_2|_{V^2_p}$ is a homeomorphism and $\varphi_2 = \pi_2 \circ \phi_0 : \phi_0^{-1}(V^1_p) \to \mathbb{C}$ would be a chart for p.

It is not difficult to see that after applying the same procedure to provide charts for the remaining points of X_f; that is, for points in $X_f \cap U_1$ and in $X_f \cap U_2$, all these charts together with the previous ones are compatible and therefore we can consider X_f as a Riemann surface if it is connected.

DEFINITION 2.1.27. A homogeneous polynomial $f(X,Y,Z) \in \mathbb{C}[X,Y,Z]$ is *non-singular* if there is no common solution of

$$f = \frac{\partial f}{\partial X} = \frac{\partial f}{\partial Y} = \frac{\partial f}{\partial Z} = 0$$

in the projective plane $\mathbb{P}^2_{\mathbb{C}}$.

EXAMPLE 2.1.28. For a positive integer n, the homogeneous polynomial

$$f(X,Y,Z) = X^n + Y^n + Z^n$$

is non-singular. Actually, $\partial f/\partial X = nX^{n-1}$, $\partial f/\partial Y = nY^{n-1}$ and $\partial f/\partial Z = nZ^{n-1}$. Thus we observe that for $p = [x:y:z] \in \mathbb{P}^2_{\mathbb{C}}$, when $n=1$, $\frac{\partial f}{\partial X}|_p = \frac{\partial f}{\partial Y}|_p = \frac{\partial f}{\partial Z}|_p = 1 \neq 0$, and when $n > 1$ then $\frac{\partial f}{\partial X}|_p = \frac{\partial f}{\partial Y}|_p = \frac{\partial f}{\partial Z}|_p = 0$ is equivalent to $x = y = z = 0$ but $[0:0:0] \notin \mathbb{P}^2_{\mathbb{C}}$.

PROPOSITION 2.1.29. *With the same notations as in Example 2.1.26, a homogeneous polynomial $f(X,Y,Z) \in \mathbb{C}[X,Y,Z]$ is non-singular if and only if every affine plane curve X_i defined by its associated f_i (i=0,1,2) is non-singular ([MIR] p. 15).*

REMARK 2.1.30. *A non-singular homogeneous polynomial $f(X,Y,Z) \in \mathbb{C}[X,Y,Z]$ is irreducible (see [MIR] p. 15).*

So, we have the following

REMARK 2.1.31. *If $f(X,Y,Z) \in \mathbb{C}[X,Y,Z]$ is a non-singular homogeneous polynomial, then its associated projective plane curve $X_f := \{[x:y:z] \in \mathbb{P}^2_{\mathbb{C}} : f(x,y,z) = 0\}$ is a compact Riemann surface. Moreover, at every point of X_f one can take as a chart a ratio of two of the homogeneous coordinates ([GG] p. 16).*

EXAMPLE 2.1.32. The curve $\{[x_1:x_2:x_3] \in \mathbb{P}^2_{\mathbb{C}} : x_1^n + x_2^n + x_3^n = 0\}$ (for a positive integer n) is a compact Riemann surface (it is a direct consequence of the Example 2.1.28 and the above remark), as well as is the curve $\{[x:y:z] \in \mathbb{P}^2_{\mathbb{C}} : y^3 - x^2z + xz^2\}$. To see it, we define the homogeneous polynomial $g(X,Y,Z) = Y^3 - X^2Z + XZ^2$ and verify that it is non-singular:

$$\partial g/\partial X \overset{(a)}{=} Z(Z-2X), \quad \partial g/\partial Y \overset{(b)}{=} 3Y^2 \quad \text{and} \quad \partial g/\partial Z \overset{(c)}{=} X(2Z-X)$$

Thus, if for some $p = [x:y:z] \in \mathbb{P}^2_{\mathbb{C}}$

$$g(p) = \frac{\partial g}{\partial X}\Big|_p = \frac{\partial g}{\partial Y}\Big|_p = \frac{\partial g}{\partial Z}\Big|_p = 0$$

by (a) we would have two options:

- Case $z = 0$.

  In this case, by (b) and (c) we would have $y = 0$ and $x = 0$. But $[0:0:0] \notin \mathbb{P}^2_{\mathbb{C}}$ and therefore this option is not possible.

- Case $z = 2x$.

  In this case, again by (b) and (c) we would have $y = 0$ and $x = 0$. Therefore this option is not possible either.

In conclusion, $\frac{\partial g}{\partial X}\big|_p \neq 0$ for all $p = [x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$ and this implies that g is non-singular. It can be seen that g is irreducible.

DEFINITION 2.1.33. Assume $C_f = \{(x, y) \in \mathbb{C}^2 : f(x, y)\}$ is a plane curve defined by $f(X, Y) \in \mathbb{C}[X, Y]$ and $F(X, Y, Z)$ is the resulting polynomial after homogenizing f. Then the projective curve

$$X_F := \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : F(x, y, z) = 0\}$$

defined by F is the *projective closure* of C_f.

Now, since $\mathbb{P}^n_{\mathbb{C}}$ is locally homeomorphic to an open set in $\mathbb{C}^n$ and is an n-dimension complex manifold, to cut down the complex dimension by one, one has to impose an equation. Thus, in order to define a Riemann surface in this space, we will naively need to take $n-1$ homogeneous polynomials in $\mathbb{C}[X_0, ..., X_n]$ and analyse the set of all its common zeros, and hence we need the analogue of the non-singularity condition on it.

DEFINITION 2.1.34. A projective algebraic curve $X \in \mathbb{P}^n_{\mathbb{C}}$ is called a *complete intersection curve* if there are $n - 1$ homogeneous polynomials $f_1, ..., f_{n-1} \in \mathbb{C}[X_0, ..., X_n]$ such that X is the set of all its common zeros. And it is *smooth* if the $(n-1) \times (n+1)$ matrix of partial derivatives $\partial f_i / \partial X_j$ has maximal rank $n - 1$ at every point of X.

Similarly, as for projective plane curve, we have the next result.

PROPOSITION 2.1.35. *A smooth complete intersection curve X is a compact Riemann surface. Moreover, at every point of X one can take a ratio x_i/x_j as a chart of the homogeneous coordinates ([MIR] p. 17).*

REMARK. The condition on partial derivatives is the hypothesis of the multi-variable Implicit Function Theorem.

EXAMPLE 2.1.36. For $\lambda \in \mathbb{C} \setminus \{0, 1\}$ the curve, which appears in Example 3.3.1 to exhibit the existence of a disconnected fiber product,

$$S_0 = \{[y_1 : y_2 : y_3 : y_4 : y_5] \in \mathbb{P}^4_{\mathbb{C}} : y_3^2 = y_1 y_2,\ y_5(y_1 + y_2) + y_4^2 = 0,\ \lambda y_1 + y_2 + y_5 = 0\}$$

is smooth and therefore compact.

In fact, let $Q_1(Y_1,Y_2,Y_3,Y_4,Y_5) := Y_3^2 - Y_1Y_2$, $Q_2(Y_1,Y_2,Y_3,Y_4,Y_5) := Y_5Y_1 + Y_5Y_2 + Y_4^2$, $Q_3(Y_1,Y_2,Y_3,Y_4,Y_5) := \lambda Y_1 + Y_2 + Y_5$ and $p = [y_1 : y_2 : y_3 : y_4 : y_5] \in S_0$. The matrix of partial derivatives of Q_1, Q_2 and Q_3 evaluated at p is

$$\begin{pmatrix} -y_2 & -y_1 & 2y_3 & 0 & 0 \\ y_5 & y_5 & 0 & 2y_4 & y_1+y_2 \\ \lambda & 1 & 0 & 0 & 1 \end{pmatrix}$$

which is now equivalent to

$$\begin{pmatrix} -y_2 & -y_1 & 2y_3 & 0 & 0 \\ (1-\lambda)y_5 & 0 & 0 & 2y_4 & y_1+y_2-y_5 \\ \lambda & 1 & 0 & 0 & 1 \end{pmatrix}.$$

If this matrix were not of maximal rank, $y_3 = y_4 = 0$. Because of this,

$$\begin{cases} Q_1(p) = 0 \\ Q_2(p) = 0 \\ Q_3(p) = 0 \end{cases} \equiv \begin{cases} y_1y_2 = 0 & (1) \\ y_5(y_1+y_2) = 0 & (2) \\ \lambda y_1 + y_2 + y_5 = 0 & (3) \end{cases}$$

and by (1) we would have two possibilities:

- If $y_1 = 0$,
$$\begin{cases} (1) \\ (2) \\ (3) \end{cases} \equiv \begin{cases} 0 = 0 \\ y_5y_2 = 0 & \rightarrow -y_5^2 = 0 \rightarrow y_5 = 0 \\ y_2 + y_5 = 0 \rightarrow y_2 = -y_5 & \rightarrow y_2 = 0 \end{cases}$$
which implies that $[y_1 : y_2 : y_3 : y_4 : y_5] = [0:0:0:0:0]$. But this is not possible so this case can not occur.
- If $y_2 = 0$,
$$\begin{cases} (1) \\ (2) \\ (3) \end{cases} \equiv \begin{cases} 0 = 0 \\ y_5y_1 = 0 & \rightarrow -\lambda y_1^2 = 0 \rightarrow y_1 = 0 \\ \lambda y_1 + y_5 = 0 \rightarrow y_5 = -\lambda y_1 & \rightarrow y_5 = 0 \end{cases}$$
which also implies that $[y_1 : y_2 : y_3 : y_4 : y_5] = [0:0:0:0:0]$. But this is not possible so this case cannot occur.

In conclusion, S_0 is smooth and by the above proposition it is a compact Riemann surface.

2.2. Holomorphic maps

DEFINITION 2.2.1. Let X and Y be two Riemann surfaces and let $F : X \rightarrow Y$ be a continuous function.

(1) The map F is said to be *holomorphic* (or a *holomorphic morphism*) at the point $p \in X$ if there exist charts (U, φ) at p and (W, ψ) at $F(p)$ such that $F(U) \subset W$

and its *local version*, $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$, is holomorphic at $\varphi(p)$

$$\begin{array}{ccc} U & \xrightarrow{F} & W \\ {\scriptstyle\varphi}\updownarrow & & \updownarrow{\scriptstyle\psi} \\ \varphi(U) & \overset{\widetilde{F}}{\dashrightarrow} & \psi(W) \end{array}$$

(2) If F is holomorphic at every point of X, then F is said to be *holomorphic* on X.
(3) If F is a bijective holomorphic morphism, then we say that it is an *isomorphism* and, in that case, we say that X and Y are *isomorphic* ($X \simeq Y$).
(4) An isomorphism from a Riemann surface X to itself is called an *automorphism* of X. The set of automorphisms of X form a group which will be denoted by $Aut(X)$.

REMARK 2.2.2. *The composition of holomorphic maps is again a holomorphic map.*

EXAMPLE 2.2.3. $\mathbb{P}^1_{\mathbb{C}} \simeq \widehat{\mathbb{C}}$.

In fact, for $F : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ defined by $F([z:w]) = z/w$, we have (see Remark 2.1.15 and Example 2.1.17):

- F is well defined and is bijective.
- For a point $p = [z:w] \in U_0$:

$$\begin{array}{ccc} U_0 & \xrightarrow{F} & \widehat{\mathbb{C}} \setminus \{0\} \\ {\scriptstyle\varphi_0}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbb{C} & \overset{\widetilde{F}}{\dashrightarrow} & \mathbb{C} \end{array}$$

  where $\varphi_0([z:w]) = w/z$ and $\psi(z) = 1/z$, the local form $\widetilde{F}$ of F is of the form $w \mapsto w$ (because $w \overset{\varphi_0^{-1}}{\longmapsto} [1:w] \overset{F}{\longmapsto} 1/w \overset{\psi}{\longmapsto} w$) which is analytic. In particular, we have $F([1:0]) = \infty$.
- For a point $p = [z:w] \in U_1$:

$$\begin{array}{ccc} U_1 & \xrightarrow{F} & \mathbb{C} \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle id_{\mathbb{C}}} \\ \mathbb{C} & \overset{\widetilde{F}}{\dashrightarrow} & \mathbb{C} \end{array}$$

  where $\varphi_1([z:w]) = z/w$, the local form $\widetilde{F}$ of F is also analytic because it is of the form $z \mapsto z$ ($z \overset{\varphi_1^{-1}}{\longmapsto} [z:1] \overset{F}{\longmapsto} z \overset{id_{\mathbb{C}}}{\longmapsto} z$).

REMARK. It is clear, by the definition, that if f is holomorphic at p then it is holomorphic on a neighbourhood of p.

EXAMPLE 2.2.4. Let $C_f = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\}$ be a smooth affine plane curve. Then its not hard to see that:

- The usual projections π_1 and π_2 on the first and second variable respectively are holomorphic.
- $g \in \mathbb{C}[X, Y]$ is holomorphic as a function from C_f to $\widehat{\mathbb{C}}$.
- If $g, h \in \mathbb{C}[X, Y]$ then g/h (viewed from C_f to $\widehat{\mathbb{C}}$) is holomorphic at every point $p = (x, y) \in C_f$ such that $h(x, y) \neq 0$.

REMARK 2.2.5. *The groups of automorphisms of* $\mathbb{P}^1_{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$ *(the upper halfplane) and* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ *(the unitary disc) are as follows (*[GG] *p. 27-28):*

- $Aut(\mathbb{P}^1_{\mathbb{C}}) = \left\{z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C},\ ad - bc \neq 0\right\} = \mathbb{P}SL(2, \mathbb{C})$, *the group of Möbius transformations.*
- $Aut(\mathbb{C}) = \{z \mapsto az + b :\ a, b \in \mathbb{C}, a \neq 0\}$.
- $Aut(\mathbb{D}) = \{z \mapsto e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z} : \alpha \in \mathbb{D}, \theta \in \mathbb{R}\} = \left\{z \mapsto \frac{\overline{a}z+\overline{b}}{bz+a} : a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1\right\}$.
- $Aut(\mathbb{H}) = \{z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R},\ ad - bc = 1\} = \mathbb{P}SL(2, \mathbb{R})$.

THEOREM 2.2.6. *(Local form and multiplicity) Let* $F : X \to Y$ *be a non constant holomorphic morphism between Riemann surfaces and let* $p \in X$*. Then there is a unique integer* $m \geq 1$ *such that for every centered chart at* $F(p)$, (W, ψ), *there exists a centered chart* (U, φ) *at* p, *such that* F *is locally* $z \mapsto z^m$ *(*$\psi \circ F \circ \varphi^{-1}(z) = z^m$ *for all* $z \in \varphi(U)$*).*

PROOF. Fix any chart (W, ψ) centered at $F(p)$ and choose any chart (U_1, φ_1) centered at p. Then $g = \psi \circ F \circ \varphi_1^{-1}$ is analytic. Since $g(0) = 0$ and g is not a constant we can write $g(z) = z^m S(z)$ with $S(z)$ a complex analytic function at 0 such that $S(0) \neq 0$ and some integer $m \geq 1$. In these conditions, there is a complex analytic function $R(z)$ such that $R^m(z) = S(z)$ and therefore $g(z) = \big(zR(z)\big)^m$. Next, define $T(z) = zR(z)$ and observe that T is analytic and invertible near 0, by the Implicit Function Theorem, since $T(0) = 0$ and $T'(0) = R(0) \neq 0$. Then, there is a neighbourhood $U \subseteq U_1$ of p such that $T \circ \varphi_1|_U$ is a homeomorphism. In this way, (U, φ) is also a chart centered at p and the new local version of F is given by $\psi \circ F \circ \varphi^{-1}(z) = (\psi \circ F \circ \varphi_1^{-1}) \circ T^{-1}(z) = \big(T\big(T^{-1}(z)\big)\big)^m = z^m$.

Now, since the number of pre-images of a point does not depend of the choice of the chart, the uniqueness of m is established. ∎

DEFINITION 2.2.7. The integer m of the previous theorem is called the *multiplicity of* F *at* p, and it is denoted by $mult_pF$. The points $p \in X$ with $mult_pF \geq 2$ will be called *ramification* (or *critical*) *points*. We shall say that F is a *ramified morphism* that *ramifies* at those points. The images of the critical points will be called *critical values.*

REMARK 2.2.8. *If* $F : X \to Y$ *is a non-constant holomorphic function between Riemann surfaces and* $p \in X$ *is not one of its critical values (that is,* $mult_pF = 1$*), then there is an open neighbourhood* V_p *of* p *in* X *for which* $f|_{V_p}$ *is a homeomorphism.*

Actually, due to the above theorem, for every centered chart (W, ψ) *at* $F(p)$ *exists a centered chart* (U, φ) *at* p *for which* F *is locally the identity* $(z \mapsto z)$*. Moreover, since* φ *and* ψ *are homeomorphisms it is easily seen that* $F|_U$ *is a homeomorphism and* U *is the required neighbourhood.*

PROPOSITION 2.2.9. *Let* $F, G : X \to Y$ *be two holomorphic morphisms between Riemann surfaces. Then the following hold.*

(1) *If* F *is non-constant, then it is an open map and the fiber* $F^{-1}(y)$ *of* $y \in Y$ *is a discrete set in* X*.*
(2) *If* F *is non-constant and* X *is compact, then* F *is surjective,* Y *is compact and, for every* $y \in Y$*,* $F^{-1}(y)$ *is a non empty finite set.*
(3) *If* F *is injective, then* F *is an isomorphism onto its image.*
(4) *If* $F = G$ *in a set* $S \subseteq X$ *with an accumulation point in* X*, then* $F = G$ *in* X*.*

PROOF. (1) Let F be non-constant. Since its local version at any point p is $f = \psi \circ F \circ \varphi^{-1}$, then locally $F = \psi^{-1} \circ f \circ \varphi$ and, by mean of this, it is easy to see that $F(U)$ is open in Y for every chart (U, φ) and therefore F is open.

Now, suppose $y \in Y$ with $F^{-1}(y) \neq \emptyset$. If $x \in F^{-1}(y)$, then we may choose φ and ψ to be centered charts (respectively at x and y) so that $f(z) = z^n$, for a suitable positive integer n. Then the second parts follows.

(2) Since F is open and continuous and Y is Hausdorff, $F(X)$ is an open and closed subset of Y. Then, because of the connectedness of Y, $F(X) = Y$. The second assertion is obvious.
(3) Immediate.
(4) If $F = G$ in a set $S \subseteq X$ with an accumulation point q in X, there are centered charts $(U \subseteq X, \varphi), (W \subseteq Y, \psi)$ for which $q \in U$ and locally $\psi \circ F \circ \varphi^{-1} = \psi \circ G \circ \varphi^{-1}$. Thus $F|_U = G|_U$ because φ and ψ are homeomorphisms. Now, let

$$B = \{x \in X : \exists \text{ neighbourhood } V_x \text{ of } x \text{ such that } F|_{V_x} = G|_{V_x}\}.$$

Then $B \neq \emptyset$ since $U \subseteq B$ and B is open by definition. Similarly, one can check that the complement of B is open and therefore $B = X$ by connectivity of X.

□

PROPOSITION 2.2.10. *Let $F : X \to Y$ be a non-constant holomorphic function between Riemann surfaces, $p \in X$ and (U, φ) and (W, ψ) any charts for p and $F(p)$ respectively. If $h(z)$ is the local version of F in these charts and $z_0 = \varphi(p)$, then $mult_p F = 1 + ord_{z_0} h'(z)$ ([MIR] p. 45).*

REMARK. With the above notations, we have the following facts.

i) The set of critical points of F is discrete.
ii) If X is compact, the set of critical points as well as the set of critical values is finite.

REMARK 2.2.11. *For a smooth affine plane curve C_f given by $f(x, y) \in \mathbb{C}[x, y]$ we have that*

$$\begin{array}{rccc} \pi_1 : & C_f & \to & \mathbb{C} \\ & (x,y) & \mapsto & x \end{array} \qquad \left(\begin{array}{rccc} \pi_2 : & C_f & \to & \mathbb{C} \\ & (x,y) & \mapsto & y \end{array} \right)$$

ramifies at $p = (x_0, y_0) \in X$ if and only if $\partial f / \partial y|_p = 0$ ($\partial f / \partial x|_p = 0$).

In fact, if $\partial f / \partial y|_p \neq 0$, by mean of the Implicit Function Theorem, we have $y = y(x)$ in some neighbourhood $V_p \subseteq C_f$ of p, and in this case we can see that π_1 restricted to V_p is a homeomorphism. Therefore π_1 does not ramify at p. Inversely, if $\partial f / \partial y|_p = 0$ then $\partial f / \partial x|_p \neq 0$ and again, by the Implicit Function Theorem, $x = x(y)$ in some neighbourhood $V_p \subseteq C_f$ of p and π_2 can be taken as a chart. In this case, the local version of π_1 is of the form $y \mapsto x(y)$ and $mult_p \pi_1 = 1 + ord_{y_0} \frac{dx}{dy}(y_0)$. But $\frac{dx}{dy}(y_0) = -(\frac{\partial f}{\partial y}\big|_p)/(\frac{\partial f}{\partial x}\big|_p) = 0$ which means that $ord_{y_0} \frac{dx}{dy}(y_0) \geq 1$ and thus $mult_p \pi_1 \geq 2$ (the statement for π_2 can also be proved in a similar way).

THEOREM 2.2.12. *Let $F : X \to Y$ a non-constant holomorphic function between Riemann surfaces. Then $\sum_{p \in F^{-1}(y_1)} mult_p F = \sum_{p \in F^{-1}(y_2)} mult_p F$ for any $y_1, y_2 \in Y$ ([MIR] p. 47-48).*

DEFINITION 2.2.13. The *degree* of a non-constant holomorphic function $F : X \to Y$ between compact Riemann surfaces is the number $\sum_{p \in F^{-1}(y)} mult_p F$ for any $y \in Y$ and is denoted by $deg(F)$.

COROLLARY. A non-constant holomorphic function between compact Riemann surfaces F is an isomorphism if and only if $deg(F) = 1$.

EXAMPLE 2.2.14. Let $\beta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\beta(z) = z(z-1)/(z-2)$. We will see that β is holomorphic, has degree 2, ramifies only at the points $2 \pm \sqrt{6}$ with $mult_{2\pm\sqrt{6}}\beta = 2$ and its critical values are $\beta(2 \pm \sqrt{6}) = (6 \pm 4\sqrt{6})/3$.

- For a point $p \in \mathbb{C} \setminus \{2\}$:

$$\begin{array}{ccc} \mathbb{C}\setminus\{2\} & \xrightarrow{\beta} & \mathbb{C} \\ {\scriptstyle id_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle id_{\mathbb{C}}} \\ \mathbb{C} & \overset{\widetilde{\beta}}{\dashrightarrow} & \mathbb{C} \end{array}$$

  - The local form $\widetilde{\beta}(z) = z(z-1)/(z-2)$ is obviously analytic.
  - $\widetilde{\beta}'(z) = \frac{z^2-4z+2}{(z-2)^2} = \frac{(z-2+\sqrt{6})(z-2-\sqrt{6})}{(z-2)^2}$ and $\widetilde{\beta}'(z) = 0 \leftrightarrow z = 2 \pm \sqrt{6}$ thus $mult_{2\pm\sqrt{6}}\beta = 2$ and $mult_p\beta = 1$ for the rest points in $\mathbb{C} \setminus \{2\}$.
  - $\beta(2+\sqrt{6}) = \frac{6+4\sqrt{6}}{6}$ and $\beta(2-\sqrt{3}) = \frac{6-4\sqrt{6}}{3}$
- For $p = 2$:
  In a neighbourhood V of 2,

$$\begin{array}{ccc} V & \xrightarrow{\beta} & \widehat{\mathbb{C}}\setminus\{0\} \\ {\scriptstyle id_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathbb{C} & \overset{\widetilde{\beta}}{\dashrightarrow} & \mathbb{C} \end{array}$$

  where $\psi(z) = 1/z$:
  - The local form $\widetilde{\beta}(z) = \frac{z-2}{z(z-1)}$ is analytic.
  - $\widetilde{\beta}'(z) = -\frac{z^2-4z+2}{z^2(z-1)^2} = -\frac{(z-(2+\sqrt{6}))(z-(2-\sqrt{6}))}{z^2(z-1)^2}$ and thus $mult_2\beta = 1$.
- For $p = \infty$:
  In a neighbourhood of ∞,

$$\begin{array}{ccc} \widehat{\mathbb{C}}\setminus\{0\} & \xrightarrow{\beta} & \widehat{\mathbb{C}}\setminus\{0\} \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathbb{C} & \overset{\widetilde{\beta}}{\dashrightarrow} & \mathbb{C} \end{array}$$

  where $\varphi(z) = \psi(z) = 1/z$:
  - The local form $\widetilde{\beta}(z) = \left(\dfrac{\frac{1}{z}(\frac{1}{z}-1)}{\frac{1}{z}-2}\right)^{-1} = \dfrac{z(1-2z)}{1-z^2}$ is analytic.
  - $\widetilde{\beta}'(z) = \frac{1}{(1-z^2)^2}$ and thus $mult_\infty\beta = 1$ (since $\beta(\infty) = \infty$ and the chart φ is centered).

Moreover, since $\beta^{-1}(\infty) = \{2, \infty\}$ we deduce that $deg(\beta) = 2$.

DEFINITION 2.2.15. Let S be a compact real topological 2-manifold. A *triangulation* of S is a decomposition of S into closed subsets, each homeomorphic to a triangle, such that any two triangles are either disjoint, meet only at a simple vertex or meet only along a single edge. If S can be *triangulated*; that is, if there is a triangulation of S and v, e and f denotes the number of vertices, edges and faces (or triangles) respectively, then the *Euler number* of S is the number $\gamma(S) = v - e + f$.

PROPOSITION 2.2.16. *If S is a triangulated compact real 2-manifold then its Euler number do not depend of the choice of triangulation. It is a topological invariant and therefore an intrinsic property of S.*

PROOF. See [**MIR**] and the first chapter of [**MASS**]. □

REMARK. A compact Riemann surface is an oriented compact real 2-manifold and can be triangulated ([**FK**] p.179).

REMARK. It is well known from a course in elementary topology that every oriented and compact real surface S is triangulated and is homeomorphic to a connected sum of the sphere and exactly g torus where $g \geq 0$. (This is a part of the so called *topological classification theorem* of surfaces). In addition, we have $\gamma(S) = 2 - 2g$ (see for example [**KOS**] and the first chapter of [**MASS**]).

DEFINITION 2.2.17. The *genus* of an oriented and compact surface S is the number g of the above remark and will be denoted by $gen(S)$.

THEOREM 2.2.18. *(Riemann-Hurwitz formula) Let X and Y be two compact Riemann surfaces and $F : X \to Y$ be a non-constant holomorphic function. Then*

$$gen(X) = deg(F)\big(gen(Y) - 1\big) + 1 + \frac{1}{2}\sum_{p \in X}(mult_p F - 1).$$

PROOF. See [**MIR**], pages 52-53. The proof follows by taking a triangulation of Y so that all critical values are vertices of such a triangulation and lifting it to obtain a triangulation for X. Then compare both Euler numbers. □

EXAMPLE 2.2.19. For a positive integer n the genus of the compact Riemann surface X_f in $\mathbb{P}^2_{\mathbb{C}}$ defined by the polynomial $f(X,Y,Z) = X^n + Y^n + Z^n$ is $(n-1)(n-2)/2$.

To see this, first observe that $X_f = (X_f \cap U_0) \cup (X_f \cap U_2)$ (following the notations of the Example 2.1.26). In fact, for a point $p = [x : y : z] \in X_f$ when $z = 0$ we must have $x \neq 0$ (otherwise we would have $x = y = z = 0$ which is not possible) and

$$X_f \cap U_2^c = \{[1 : e^{\pi(2k+1)/n} : 0] : 0 \leq k \leq n-1\}.$$

Secondly, consider the function

$$\begin{array}{rccc} \Pi : & X_f & \to & \mathbb{P}^1_{\mathbb{C}} \\ & [x:y:z] & \mapsto & [x:z] \end{array}$$

and using again the notations of the remark 2.1.15 we have:

- For $p = [x : y : 1] \in X_f \cap U_2$:
  If $\pi_1 : X_2 = \{(x,y) \in \mathbb{C}^2 : f_2(x,y) = 0\} \to \mathbb{C}$, where $f_2(x,y) = f(x,y,1)$, is the projection on the first coordinate, then the following diagram is commutative

$$\begin{array}{ccc} X_f \cap U_2 & \xrightarrow{\Pi} & \mathbb{P}^1_{\mathbb{C}} \\ \phi_2 \downarrow & & \downarrow \varphi_0 \\ X_2 & \xrightarrow{\pi_1} & \mathbb{C} \end{array}$$

  - Since ϕ_2 and φ_1 are homeomorphisms and π_1 is holomorphic, we conclude that Π also is holomorphic. Actually, Π and π_1 have the same local forms.
  - Looking for critical points of Π is equivalent to looking for critical points of π_1 and, as we know, for that we must look for solutions of $\frac{\partial f_2}{\partial Y}\big|_{(x,y)} = 0$ (equivalently $ny^{n-1} = 0$), which are exactly the points $(x, 0)$ for which x^n equals -1. In other words π_1 ramifies at the points $(e^{\pi(2k+1)/n}, 0)$ with $0 \leq k \leq n-1$ and, equivalently, Π ramifies at the points $[e^{\pi(2k+1)/n} : 0 : 1]$ with $0 \leq k \leq n-1$.
  - By the above item, we conclude that, near the points $(x,y) = (e^{\pi(2k+1)/n}, 0)$ for $0 \leq k \leq n-1$, the partial derivative $\frac{\partial f_2}{\partial X}\big|_{(x,y)} \neq 0$, and thus π_2 will work as a centered chart at these points for π_1, and $\phi_2 \circ \pi_2^{-1}$ will work as a centered chart at the points $[e^{\pi(2k+1)/n} : 0 : 1]$ where $0 \leq k \leq n-1$ for Π. By the Implicit Function Theorem and considering the following diagram

$$\begin{array}{ccc} [h(y) : y : 1] & \overset{\Pi}{\longmapsto} & [h(y) : 1] \\ \uparrow \phi_2^{-1} & & \downarrow \varphi_1 \\ (h(y), y) & \overset{\pi_1}{\longmapsto} & h(y) \\ \uparrow \pi_2^{-1} & & \downarrow id_{\mathbb{C}} \\ y & \overset{\widetilde{\Pi}}{\dashrightarrow} & h(y) \end{array}$$

we can find the local form $\widetilde{\Pi}(y) = h(y)$ of Π. Here h is an analytic function with $h'(y) = -\frac{\partial f_2/\partial Y}{\partial f_2/\partial X}\big|_{(x,y)} = -\frac{y^{n-1}}{x^{n-1}}$ (see example 2.1.9). This implies $ord_0 h'(y) = n-1$ and therefore $mult_{p_k}\Pi = n$ for all $0 \leq k \leq n-1$.

- For $p = [1 : y : z] \in X_f \cap U_0$:
  If $\pi_2 : X_0 = \{(y,z) \in \mathbb{C}^2 : f_0(x,y) = 0\} \to \mathbb{C}$, where $f_0(x,y) = f(1,y,z)$, is the projection on the second coordinate the following diagram is commutative

$$\begin{array}{ccc} X_f \cap U_0 & \xrightarrow{\Pi} & \mathbb{P}^1_{\mathbb{C}} \\ \phi_0 \downarrow & & \downarrow \varphi_1 \\ X_0 & \xrightarrow{\pi_2} & \mathbb{C} \end{array}$$

  - Since ϕ_0, φ_0 are homeomorphisms and π_2 is holomorphic we conclude that Π is also holomorphic. Actually, Π and π_2 have the same local forms.
  - Looking for critical points of Π is equivalent to looking for critical points of π_2 and, as we know, for that we must look for solutions of $\frac{\partial f_0}{\partial Y}\big|_{(y,z)} = 0$ ($ny^{n-1} = 0$) which are exactly the points $(0,z)$ for which $z^n = -1$. In other words, π_2 ramifies at the points $(0, e^{\pi(2k+1)/n})$ where $0 \leq k \leq n-1$ and, equivalently, Π ramifies at the points $[1 : 0 : e^{\pi(2k+1)/n}]$ where $0 \leq k \leq n-1$, but these coincide with those found before.
  - For $0 \leq k \leq n-1$ $mult_{[1:e^{\pi(2k+1)/n}:0]} = 1$ with $\Pi([1 : e^{\pi(2k+1)/n} : 0]) = [1 : 0]$.
- $deg(\Pi) = n$ since $mult_{[1:e^{\pi(2k+1)/n}:0]} = 1$ for all $0 \leq k \leq n-1$ and

$$\Pi^{-1}([1:0]) = \left\{([1 : e^{\pi(2k+1)/n} : 0]) : 0 \leq k \leq n-1\right\}.$$

Finally, the Riemann-Hurwitz formula allows us to calculate the genus of X_f:

$$gen(X_f) = n(0-1) + 1 + \frac{1}{2}n(n-1) = \frac{(n-1)(n-2)}{2}.$$

2.3. The fundamental group and coverings.

DEFINITION 2.3.1. Let X and Y be two compact topological surfaces.

(1) A *loop with base point* p in X is a path (continuous function) $\alpha : [0,1] \to X$ satisfying $\alpha(0) = \alpha(1) = p$

(2) Two loops $\alpha, \beta : [0,1] \to X$ with the same base point p are *homotopically equivalent* if there is a continuous map

$$\begin{array}{rccc} \gamma : & [0,1] \times [0,1] & \to & X \\ & (t,s) & \mapsto & \gamma(t,s) := \gamma_s(t) \end{array}$$

such that:

(a) $\gamma_s(0) = \gamma_s(1) = p$ for all $s \in [0,1]$.

(b) $\gamma_0(t) = \alpha(t)$ and $\gamma_1(t) = \beta(t)$ for all $t \in [0,1]$.

(3) If $\alpha, \beta : [0,1] \to X$ are two loops with the same base point, then we can define in the set of homotopy classes the product $[\alpha] * [\beta] = [\alpha\beta]$ where

$$\alpha\beta(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq 1/2 \\ \beta(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

and therefore endow it with a group structure. This group is called the *fundamental group* of X relative to the base point p, and it is denoted by $\pi_1(X,p)$

REMARK. i) The identity element of $\pi_1(X,p)$ is the equivalence class of the constant loop $t \to p$. And the inverse of $[\alpha]$, where α is a loop with base point p, is the equivalence class of the loop $t \to \alpha(1-t)$.

ii) $\pi_1(X,p)$ also can be denoted just by $\pi_1(X)$ as a consequence of the arc-connectedness of X of the previous definition. Actually, it can be proved that for any two points $p, q \in X$, $\pi_1(X,p)$ and $\pi_1(X,q)$ are isomorphic.

EXAMPLE 2.3.2. If S^1 denotes the unit circumference in $\mathbb{R}^2$, its fundamental group is isomorphic to $\mathbb{Z}$ ($\pi_1(S^1) \simeq \mathbb{Z}$) and the fundamental group of the torus $S^1 \times S^1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Both are abelian but, for example, the fundamental group of the double torus is not as it is isomorphic to the group $\langle \alpha_1, \alpha_2, \beta_1, \beta_2 : \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1} = 1 \rangle$ ([MUN] p. 345-346,372,374,452).

DEFINITION 2.3.3. A continuous mapping $f : X \to Y$ between two topological surfaces X and Y is a *covering map*, or simply a *covering*, if for every $y \in Y$ there is a neighbourhood V_y of y such that V_y is well-covered by p. In other words, $p^{-1}(V_y)$ is a union of pairwise disjoint sets U_i and each restriction $p|_{U_i} : U_i \to V_y$ is an homeomorphism.

PROPOSITION 2.3.4. *Let $f : X \to Y$ be a covering between two topological surfaces X and Y.*

(1) *The fibres of f are always discrete sets and, if Y is connected, then they also have the same cardinality, called the degree of the covering.*
(2) *f satisfies the path lifting property: every path $\gamma : [0,1] \to Y$ can be lifted to X, in other words, for every path $\gamma : [0,1] \to Y$ and every $p \in X$ so that $f(p) = \gamma(0)$, there is a unique path $\widetilde{\gamma} : [0,1] \to X$ such that $p \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = p$ ([MUN] p. 342-343).*

EXAMPLE 2.3.5. Let

$$\begin{array}{rccc} f: & X=\{(\cos 2\pi t, \sin 2\pi t, t): t\in\mathbb{R}\}\subset\mathbb{R}^3 & \to & S^1 \\ & (\cos 2\pi t, \sin 2\pi t, t) & \mapsto & (\cos 2\pi t, \sin 2\pi t). \end{array}$$

Then f and $f\times f: X\times X \longrightarrow S^1\times S^1$ are coverings maps of infinite degree. Moreover, for example, the path $\alpha:[0,1]\to S^1$ beginning at $(1,0)$ given by $\alpha(s)=(\cos\pi s, \sin\pi s)$ lifts to the path $\widetilde{\alpha}:[0,1]\to X$ given by $\widetilde{\alpha}(s)=(\cos\pi s, \sin\pi s, s/2)$ beginning at $(1,0,0)$ and ending at $(-1,0,1/2)$.

REMARK. Let us consider a covering map $f: X\to Y$, where X and Y are topological oriented surfaces. If Y has a complex structure (that is, if Y is a Riemann surface), then X inherits a unique complex structure that makes f holomorphic. This is possible by mean of charts of the form $(U_i, \psi_j\circ f)$, where (V_j,ψ_j) is a chart in Y and $V_j=f(U_i)$ is a well-covered neighbourhood. In fact:

i) The transition functions are then analytic $(\psi_k\circ f)\circ(\psi_j\circ f)^{-1}=\psi_k\circ\psi_j^{-1}$.
ii) If f is also analytic with respect to another chart (U,φ) then $\psi_j\circ f\circ\varphi^{-1}$ is analytic but this is the same as saying that (U,φ) is also compatible with any of the above charts.

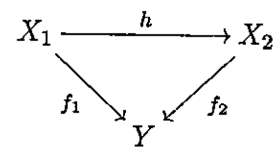
PROPOSITION 2.3.6. *Let $f: X\to Y$ be a non-constant morphism of compact Riemann surfaces and let B the set of all critical values of f. Then the restriction*

$$f: X\setminus f^{-1}(B)\to Y\setminus B$$

is a covering.

PROOF. We already know that if $x_1,...x_d$ are the preimages of a non-critical value y then there are neighbourhoods $V, U_1,...,U_d$ of $y, x_1,...,x_d$ respectively such that the restrictions $f|_{U_i}: U_i\to V$ are isomorphisms, and therefore, homeomorphisms (see remark 2.2.8). Now, if we assume the assertion to be false; that is, if V can not be taken such that $f^{-1}(V)=\bigsqcup_{i=1}^d U_i$ (or equivalently $f^{-1}(V)-\bigsqcup_{i=1}^d U_i\neq\emptyset$) then we would have a sequence of points $\{y_n\}\subseteq Y$ approaching y such that each fiber $f^{-1}(y_n)$ contains a point $x'_n\notin\bigsqcup_{i=1}^d U_i$. Let $x\in X$ be a limit point of $\{x'_n\}_n$. By continuity of f we would have $f(x)=y$ and then $x=x_j$ for some $j=1,...,d$. Thus for n large enough we would have $x'_n\in U_j$ but this is a contradiction. □

DEFINITION 2.3.7. Two coverings $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ of Y are *isomorphic* if there is a homeomorphism $h : X_1 \to X_2$ such that the following diagram commutes



If $X_1 = X_2 = X$ and $f_1 = f_2 = f$, then h is an *automorphism* of (X, f). In this case, the *covering group* or *deck group* of the cover map $f : X \to Y$ (or of the pair (X, f)) is the group of all these automorphisms, and is denoted by $Aut(X, f)$ or $Deck(f : X \to Y)$.

DEFINITION 2.3.8. Let G be a subgroup of automorphisms of a topological surface X. We say that G acts on X

(1) *freely* if the nontrivial elements of G do not fix any point of X.
(2) *properly discontinuously* if for each $x \in X$ there are at most finitely many transformations $g_1 = Id, ..., g_r \in G$ fixing x, and there exists a neighbourhood U_x of x such that $g(U_x) \cap U_x = \emptyset$ for all $g \in G - \{g_1, ...g_r\}$.

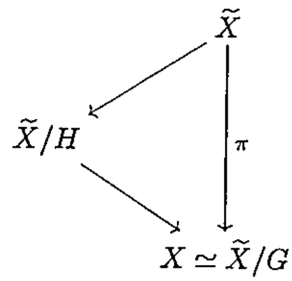
EXAMPLE 2.3.9. The group of automorphisms $G = \langle z \mapsto z + n + im : n, m \in \mathbb{Z} \rangle$ of $\mathbb{C}$ acts freely and properly discontinuously on it, and $\mathbb{C}/G$ is homeomorphic to the torus $S^1 \times S^1$.

THEOREM 2.3.10. *Let X be a topological surface.*

(1) *There is always a covering $\pi : \widetilde{X} \to X$ with $\widetilde{X}$ connected and simply connected.*
(2) *$\widetilde{X}$ is called the universal covering space of X and is unique up to isomorphisms of coverings.*
(3) *If $\pi : \widetilde{X} \to X$ is the universal covering of X, the group $G = Aut(\widetilde{X}, \pi)$ can be identified with the fundamental group $\pi_1(X)$.*
(4) *The action of G on $\widetilde{X}$ is free and properly discontinuous, preserves every fibre of π, and is transitive in each of them. In particular π induces a homeomorphism*

$$\begin{array}{ccc} \widetilde{X}/G & \to & X \\ [\tilde{x}] & \mapsto & \pi(\tilde{x}) \end{array}$$

(5) *Every covering of X is isomorphic to a covering of the form $\widetilde{X}/H \to \widetilde{X}/G \simeq X$, where H is a subgroup of G.*



(6) *If X is a Riemann surface and $\pi : \widetilde{X} \to X$ is the universal holomorphic covering, then the covering group G is a group of holomorphic transformations, the map $\pi : \widetilde{X}/G \to X$ is a biholomorphism, and any other holomorphic covering of X is isomorphic to a projection of the form $\widetilde{X}/H \to \widetilde{X}/G \simeq X$.*

PROOF. See [GG] pages 65-69. □

REMARK 2.3.11. *Due to proposition 2.3.6 the term covering is used, in the theory of Riemann surfaces, to refers to any non constant morphism (ramified or not) between compact Riemann Surfaces and from now we will use it here too.*

PROPOSITION 2.3.12. *Let G be a finite subgroup of automorphisms of a compact Riemann surface X. Then the following hold.*

(1) *The set of points in X with nontrivial stabilizer is discrete with no accumulation points.*
(2) *If $gen(X)$ and $gen(X/G)$ denote the genus of X and X/G respectively, then, for the canonical projection $\pi : X \to X/G$, the Riemann-Hurwit's formula can be reformulated as follows*

$$2gen(X) - 2 = |G|\left(2gen(X/G) - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{|I(p_i)|}\right)\right)$$

where $\{p_1, ..., p_r\}$ is a maximal set of representatives of critical points in different orbits and $I(p_i)$ denotes the stabilizer of p_i in G.

PROOF. See [MIR]. For (1) pages 76-77 and for (2) page 80. □

PROPOSITION 2.3.13. *Let Y be a compact Riemann surface, $B \subset Y$ be a finite subset, and $Y^* = Y \setminus B$. Assume that $f^* : X^* \to Y^*$ is an unramified holomorphic covering of finite degree. Then there exists a unique compact Riemann surface $X \supseteq X^*$ such that f^* extends to a unique morphism $f : X \to Y$. Moreover, $X \setminus X^*$ is a finite set ([GG] p. 75-77).*

PROPOSITION 2.3.14. *Let X_1 and X_2 be two compact Riemann surfaces, $B_1 \subseteq X_1$ and $B_2 \subseteq X_2$ be finite subsets. If $X_1^* = X_1 \setminus B_1$ and $X_2^* = X_2 \setminus B_2$ are isomorphic then X_1 and X_2 must be isomorphic too ([GG] p. 77).*

EXAMPLE 2.3.15. In this example we will show that for the curve

$$S = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : x^n + y^n + z^n = 0\}$$

the function

$$\begin{array}{cccc} f: & S & \to & \widehat{\mathbb{C}} \\ & [x : y : z] & \mapsto & -(\frac{y}{x})^n \end{array}$$

is a degree n^2 holomorphic covering, then when $n > 1$ it ramifies only at the points

$$[1 : e^{\frac{\pi(2k+1)}{n}} : 0],\ [0 : 1 : e^{\frac{\pi(2k+1)}{n}}] \text{ and } [1 : 0 : e^{\frac{\pi(2k+1)}{n}}]$$

each one with multiplicity n, and that its critical values are

$$f\big([1 : e^{\frac{\pi(2k+1)}{n}} : 0]\big) = 1,\ f\big([0 : 1 : e^{\frac{\pi(2k+1)}{n}}]\big) = \infty \text{ and } f\big([1 : 0 : e^{\frac{\pi(2k+1)}{n}}]\big) = 0$$

$(0 \leq k \leq n-1)$.

In fact, following the notations of Remark 2.1.15 and Example 2.1.26, we see that $f|_{S \cap U_0} = g \circ \pi_1 \circ h$ where

$$\begin{array}{ccccccc} S \cap U_0 & \xrightarrow{h} & \mathbb{P}^1_{\mathbb{C}} & \xrightarrow{\pi_1} & \widehat{\mathbb{C}} & \xrightarrow{g} & \widehat{\mathbb{C}} \\ [1 : y : z] & \mapsto & [y : z] & \mapsto & y & \mapsto & -y^n, \end{array}$$

and if we define $S_0 := \{(y, z) \in \mathbb{C}^2 : f_0(y, z) = 0\}$ and $\Pi_1 : S_0 \to \widehat{\mathbb{C}}$ by $\Pi_1(y, z) = y$, the next diagram is commutative

$$\begin{array}{ccccc} S \cap U_0 & \xrightarrow{h} & \mathbb{P}^1_{\mathbb{C}} & \xrightarrow{\pi_1} & \widehat{\mathbb{C}} \\ \downarrow \phi_0 & & & \nearrow_{\Pi_1} & \\ S_0 & & & & \end{array}$$

Moreover:

- h is holomorphic if and only if Π_1 is holomorphic, and we already know that Π_1 is holomorphic.

- Noting that π_1 is an isomorphism (see Example 2.2.3), h ramifies at a point $[1:y:z] \in S \cap U_0$ if and only if Π_1 ramifies at $\phi_0([1:y:z]) = (y,z) \in S_0$, and this occurs when $\frac{\partial f_0}{\partial Z}\big|_{(y,z)} = 0$. Since $\frac{\partial f_0}{\partial Z}\big|_{(y,z)} = nz^{n-1}$, z should be equal to zero and y should be any of the roots of $y^n = -1$. Thus h ramifies at the points $[1:e^{\frac{\pi(2k+1)}{n}}:0]$ where $0 \le k \le n-1$.
- If $p = [0:y:z] \in S$ then $y^n = -z^n$, or equivalently $y = e^{\frac{\pi(2k+1)}{n}} z$, which means
$$S \setminus U_0 = \big\{[0:e^{\frac{\pi(2k+1)}{n}}:1] : 0 \le k \le n-1\big\}.$$
In this way we need to add exactly n points to make $S \cap U_0$ compact and extend $\pi_1 \circ h$ to S (see Propositions 2.3.13 and 2.3.14).
- The holomorphic extension $\widehat{\pi_1 \circ h}$ of $\pi_1 \circ h$ to S will send the points $[0:e^{\frac{\pi(2k+1)}{n}}:1]$, for $0 \le k \le n-1$, to ∞.
- $f = g \circ \widehat{\pi_1 \circ h}$ and therefore it is holomorphic.
- Since $\widehat{\pi_1 \circ h}([1:e^{\frac{\pi(2k+1)}{n}}:0]) = e^{\frac{\pi(2k+1)}{n}} \neq 2$ for $0 \le k \le n-1$, 2 is not a critical value of $\widehat{\pi_1 \circ h}$. Besides, it is not hard to see that $\left|\widehat{\pi_1 \circ h}^{-1}(2)\right| = n$ and therefore $deg(\widehat{\pi_1 \circ h}) = n$.
- Due to the preceding items and noting that $\widehat{\pi_1 \circ h}^{-1}(e^{\frac{\pi(2k+1)}{n}}) = [1:e^{\frac{\pi(2k+1)}{n}}:0]$ we can conclude that $\widehat{\pi_1 \circ h}$ ramifies only at the points $[1:e^{\frac{\pi(2k+1)}{n}}:0]$ with $mult_{[1:e^{\frac{\pi(2k+1)}{n}}:0]}\widehat{\pi_1 \circ h} = n$ for all $0 \le k \le n-1$.

On the other hand, analyzing separately the function g for $n > 1$ we know that it is a degree n holomorphic covering which ramifies only at the points 0 and ∞ with $g(0) = 0$, $g(\infty) = \infty$ and $mult_0 g = mult_\infty g = n$.

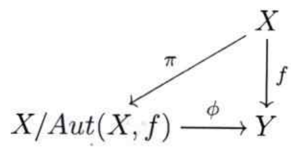
The assertions made at the beginning of this example for f hold putting together all the previous observations and noting that:

- $g^{-1}(1) = \{e^{\frac{\pi(2k+1)}{n}} : 0 \le k \le n-1\}$ and $\widehat{\pi_1 \circ h}^{-1}(e^{\frac{\pi(2k+1)}{n}}) = \big\{[1:e^{\frac{\pi(2k+1)}{n}}:0]\big\}$ for each $0 \le k \le n-1$.
- $g^{-1}(0) = 0$ and $\widehat{\pi_1 \circ h}^{-1}(0) = \big\{[1:0:e^{\frac{\pi(2k+1)}{n}}] : 0 \le k \le n-1\big\}$.
- $g^{-1}(\infty) = \infty$ and $\widehat{\pi_1 \circ h}^{-1}(\infty) = \big\{[0:1:e^{\frac{\pi(2k+1)}{n}}] : 0 \le k \le n-1\big\}$.

DEFINITION 2.3.16. A Riemann surface of *finite type* is a surface obtained from a compact Riemann surface by removing a finite number of points.

DEFINITION 2.3.17. A covering $f : X \to Y$ between two compact Riemann surfaces is *regular* (or *normal*, or *Galois*) if the covering group $Aut(X, f)$ acts transitively on each fibre.

A regular covering $f : X \to Y$ can be seen as a covering of the form $X \to X/Aut(X, f)$. More precisely, there is a commutative diagram



where ϕ is the isomorphism defined by $\phi([p]_{Aut(X,f)}) = f(p)$.

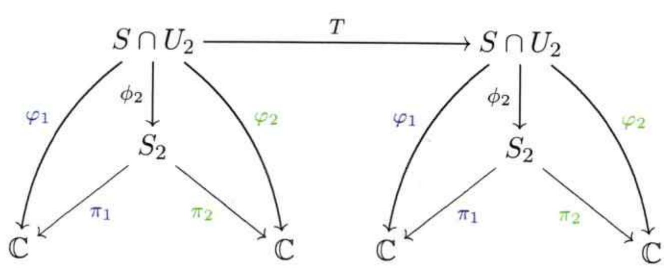
REMARK 2.3.18. *A covering $f : X \to Y$ between two compact Riemann surfaces is regular if and only if $deg(f) = |Aut(X, f)|$ or, equivalently, if and only if $deg(f) = |Deck(f)|$ (see* [**GG**] *p.171-172).*

EXAMPLE 2.3.19. Let $S = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : x^n + y^n + z^n = 0\}$. Then the covering $f : S \longrightarrow \widehat{\mathbb{C}}$ defined by $f([x : y : z]) = -(\frac{y}{x})^n$ is regular.

This assertion is clear for $n = 1$ (see the Example 2.3.15); therefore, here we will treat only the case when $n > 1$. In order to do this, let us consider the functions $T, W : S \to S$ defined by $T\big([x : y : z]\big) = [e^{2\pi i/n}x : y : z]$ and $W\big([x : y : z]\big) = [x : e^{2\pi i/n}y : z]$. It is clear that T and W are well defined and bijective. In addition:

- T is an automorphism.

  In fact, if $S_2 = \{(x, y) \in \mathbb{C}^2 : f_2(x, y) := f(x, y, 1) = 0\}$, following the notations of Remark 2.2.11 and Example 2.1.26, it is possible to draw a diagram as below

  in which $\pi_1(x, y) = x$, $\pi_2(x, y) = y$, $\varphi_1 := \pi_1 \circ \phi_2$ and $\varphi_2 := \pi_2 \circ \phi_2$. In this way, for a point $p = [x : y : 1] \in S \cap U_2$ we can "read"it as follow (see Examples 2.1.9 and 2.1.26):

  – If $\frac{\partial f_2}{\partial Y}\big|_{(x,y)} \neq 0$ there is a neighbourhood V of p for which (V, φ_1) is a chart and we can assume that $\phi_2(V) = \big\{\big(x, h(x)\big) : x \in \varphi_1(V)\big\}$ for some holomorphic function h.

– If $\frac{\partial f_2}{\partial X}\Big|_{(x,y)} \neq 0$ there is a neighbourhood V of p for which (V, φ_2) is a chart and we can assume that $\phi_2(V) = \{(k(y), y) : y \in \varphi_2(V)\}$ for some holomorphic function k.

Thus, we only have the next four options to describe the local form of T:

$$\begin{aligned}
\varphi_2 \circ T \circ \varphi_1^{-1}(w) &= \pi_2 \circ \phi_2 \circ T \circ \phi_2^{-1} \circ \pi_1^{-1}(w) \\
&= \pi_2 \circ \phi_2 \circ T \circ \phi_2^{-1}(w, h(w)) \\
&= \pi_2 \circ \phi_2 \circ T([w : h(w) : 1]) \\
&= \pi_2 \circ \phi_2([e^{2\pi i/n}w : h(w) : 1]) \\
&= \pi_2(e^{2\pi i/n}w, h(w)) \\
&= h(w) \\
\varphi_2 \circ T \circ \varphi_2^{-1}(w) &= w \\
\varphi_1 \circ T \circ \varphi_1^{-1}(w) &= e^{2\pi i/n}w \\
\varphi_1 \circ T \circ \varphi_2^{-1}(w) &= e^{2\pi i/n}k(w)
\end{aligned}$$

All of them are holomorphic.

- W is also an automorphism (we can show it in a similar way to the above item).
- T and $W \in Aut(S, f)$.

  In fact, $f \circ T = f$ and $f \circ W = f$ because

$$f \circ T([x : y : z]) = f([e^{2\pi i/n}x : y : z]) = -(\frac{y}{x})^n$$

and

$$f \circ W([x : y : z]) = f([x : e^{2\pi i/n}y : z]) = -(\frac{y}{x})^n.$$

- f is regular.

  Actually, $\langle T, W\rangle \simeq \mathbb{Z}_n^2$ since the groups $\langle T\rangle$ and $\langle W\rangle$ are each isomorphic to $\mathbb{Z}_n$ and

$$T \circ W([x : y : z]) = [e^{2\pi i/n}x : e^{2\pi i/n}y : z] = W \circ T.$$

Therefore $|Aut(S, f)| = n^2 = deg(f)$, with $Aut(S, f) = \langle T, W\rangle$ and the assertion follows from Remark 2.3.18.

2.4. Uniformization and Fuchsian groups.

THEOREM 2.4.1. *Uniformization Theorem. Every simply connected Riemann surface is isomorphic to either $\mathbb{D}, \mathbb{C}$ or $\widehat{\mathbb{C}}$ ([GG] p. 95 or [FK] chapter 4).*

The uniformization theorem provides the three possible candidates for the universal covering space of a Riemann surface. We may observe that the above are pairwise non-isomorphic; one of them is compact and the other two cannot be isomorphic by Liouville's theorem. A Riemann surface is called *hyperbolic* if its universal cover is isomorphic to $\mathbb{H}$.

According to this, for the compact ones, we have the next result ([GG] p. 96).

THEOREM 2.4.2. *Compact Riemann surfaces can be classified as follows (up to isomorphisms):*

(1) $\widehat{\mathbb{C}}$ *is the only compact Riemann surface of genus 0.*
(2) *Every compact Riemann surface of genus 1 can be described in the form* $\mathbb{C}/\Lambda$, *where* Λ *is a lattice of the form* $w_1\mathbb{Z} \oplus w_2\mathbb{Z}$, *for* w_1, w_2 *two complex numbers* $\mathbb{R}$*-linearly independent, acting on* $\mathbb{C}$ *as a group of translations.*
(3) *Every compact Riemann surface of genus greater than one is isomorphic to a quotient* $\mathbb{H}/K$, *where* $K \subseteq \mathbb{P}SL(2,\mathbb{R})$ *acts freely and properly discontinuously on* $\mathbb{H}$.

But this is not the only way to describe a compact Riemann surface. Actually, it is more frequent to find examples of quotients $\mathbb{H}/K$ where $K \subseteq \mathbb{P}SL(2,\mathbb{R})$ acts properly discontinuously but not freely on $\mathbb{H}$. If $\Gamma < \mathbb{P}SL(2,\mathbb{R})$ acts properly discontinuously on $\mathbb{H}$, by definition, given a point $z \in \mathbb{H}$ there is a neighbourhood U_z such that $\gamma(U_z) \cap U_z = \emptyset$ for all $\gamma \in \Gamma$ not in the stabilizer $I(z) = \{\gamma_1, ..., \gamma_n\}$ of z (or equivalently for all $\gamma \in \Gamma$ such that $\gamma(z) \neq z$). Now, if M is an isomorphism from $\mathbb{H}$ to $\mathbb{D}$ sending z to $0 \in \mathbb{D}$, then $MI(z)M^{-1}$ is a finite group of rotations around the origin and thus it is a cyclic group generated by a rotation $w \mapsto e^{2\pi/n}w$. It follows that $I(z)$ is also cyclic and U_z can be chosen to be $I(z)$-invariant. It also follows that the set of fixed points is discrete. Otherwise, there would be an infinite sequence of elements $\gamma_i \in \Gamma$ and of points $z_i \in \mathbb{H}$ such that $z_i \to z$ and $\gamma_i(z_i) = z_i$. But this would mean that for every $\epsilon > 0$ there would be infinitely many elements $\gamma \in \Gamma$ such that $\gamma(U_z) \cap U_z \neq \emptyset$, because each γ_i has at most two fixed points and $\{z_i\}$ is infinite.

Now, from a different point of view, $\mathbb{P}SL(2,\mathbb{R})$ is naturally endowed with the usual topology of $M_{2\times 2}(\mathbb{R}) \simeq \mathbb{R}^4$ which makes it a topological group and this allows us to talk about Fuchsian groups, which will be defined next.

DEFINITION 2.4.3. A *Fuchsian group* is a discrete subgroup Γ of $\mathbb{P}SL(2,\mathbb{R})$.

The remaining results of this section are proved in chapter 2 of [GG] using arguments of hyperbolic geometry, also treated there. In this work we will need only the results, so we will leave those arguments and details to interested readers.

PROPOSITION 2.4.4. *$\Gamma < \mathbb{P}SL(2,\mathbb{R})$ is a Fuchsian group if and only if it acts properly discontinuously on $\mathbb{H}$.*

EXAMPLE 2.4.5. $\mathbb{P}SL(2,\mathbb{Z})$ is a Fuchsian group and therefore so is any subgroup.

PROPOSITION 2.4.6. *If Γ is a Fuchsian group then we have:*

(1) *The quotient $\mathbb{H}/\Gamma$ has a natural complex structure of Riemann surface, for which the canonical projection $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ is holomorphic and $mult_p\pi = |I(z)|$ for any $p \in \mathbb{H}$.*
(2) *Γ acts freely on $\mathbb{H}$ if and only if Γ is torsion free (that is; if it has no nontrivial elements of finite order).*

PROPOSITION 2.4.7. *Let $S_1 = \mathbb{H}/\Gamma_1$ and $S_2 = \mathbb{H}/\Gamma_2$ be two Riemann surfaces (compact or not) uniformized by freely acting Fuchsian groups Γ_1 and Γ_2. Then S_1 and S_2 are isomorphic if and only if there exist $T \in \mathbb{P}SL(2,\mathbb{R})$ such that $T \circ \Gamma_1 \circ T^{-1} = \Gamma_2$.*

EXAMPLE 2.4.8. The *principal congruence subgroup of level 2*

$$\Gamma(2) := \left\langle T_1(z) = z+2, T_2(z) = \frac{-z}{2z-1}, T_3(z) = \frac{z-2}{2z-3} \right\rangle$$

is a subgroup of the *modular group* $\Gamma(1) = \mathbb{P}SL(2,\mathbb{Z})$ with index 6, and T_1, T_2 and T_3 fix only the points $\infty, 0$ and 1 respectively, the relation $T_1T_2T_3 = Id$ is satisfied, and $\Gamma(2)$ acts freely on $\mathbb{H}$.

Due to this last property, $\Gamma(2)$ must be isomorphic to the fundamental group of $\mathbb{H}/\Gamma(2)$, which is the free group of rank two as the quotient is isomorphic to the sphere with three points deleted, namely $0, 1, \infty \in \widehat{\mathbb{C}}$. Besides, T_1, T_2 and T_3 are identified, by mean of this isomorphism, as simple loops around $\infty, 0$ and 1 respectively.

PROPOSITION 2.4.9. *If $H \leq K$ is an inclusion of Fuchsian groups then for the corresponding morphism $\mathbb{H}/H \longrightarrow \mathbb{H}/K$ the following statements hold:*

(1) *Its degree is the index $[K : H]$.*
(2) *It is regular if and only if H is a normal subgroup of K.*

2.5. Monodromy.

Let $f : S_1 \to S$ be a degree d ramified covering between two compact Riemann surfaces and let $\{y_1, ..., y_n\}$ be the set of critical values of f. We already know that the restriction

$$f : S_1 \setminus f^{-1}(y_1, ..., y_n) \longrightarrow S \setminus \{y_1, ..., y_n\}$$

is a topological cover. If $y \in S$ is a noncritical value of f and γ is a loop in $S \setminus \{y_1, ..., y_n\}$ based on y, we can lift γ to a path $\widetilde{\gamma}$ with initial point at any given point $x \in f^{-1}(y)$ and end point at certain $x' \in f^{-1}(y)$. Thus we can associate to f the group homomorphism

$$\begin{array}{rccc} M_f : & \pi_1(S \setminus \{y_1, ..., y_n\}, y) & \longrightarrow & Bij(f^{-1}(y)) \\ & \gamma & \mapsto & M_f(\gamma) = \sigma_\gamma^{-1} \end{array}$$

where $\sigma_\gamma(x) := x'$.

Of course $Bij(f^{-1}(y)) \simeq \Sigma_d$ (the symmetric group on d letters), and therefore we can consider M_f as a group homomorphism from the fundamental group to Σ_d.

DEFINITION 2.5.1. We will refer to the above representation of the fundamental group $\pi_1(S \setminus \{y_1, ..., y_n\})$ in the symmetric group Σ_d as the *monodromy of f*. The image of M_f in Σ_d will be the *monodromy group* of f and will be denoted by $Mon(f)$; that is,

$$Mon(f) = \{\sigma_\gamma : \gamma \in \pi_1(S \setminus \{y_1, ..., y_n\})\}.$$

REMARK. The statements below hold ([**GG**] p. 163-164).

i) M_f depends on the ordering of the points in $f^{-1}(y)$ only up to conjugation in Σ_d.
ii) Different choices of the base point $y \in S \setminus \{y_1, ..., y_n\}$ give rise to conjugate monodromy groups.
iii) $Mon(f)$ is a transitive subgroup of Σ_d because of the connectedness of the surface.

REMARK 2.5.2. *If $S \setminus \{y_1, ..., y_n\}$ is a hyperbolic surface then we can consider the Fuchsian group representation of the unramified covering $f : S_1 \setminus \{y_1, ..., y_n\} \to S \setminus \{y_1, ..., y_n\}$, provided by covering space theory, and obtain the following commutative diagram*

$$\begin{array}{ccc} \mathbb{H} & & \\ \downarrow{\scriptstyle \pi_{\Gamma_1}} & \searrow^{\pi_\Gamma} & \\ \mathbb{H}/\Gamma_1 & \xrightarrow{\pi} & \mathbb{H}/\Gamma \\ \updownarrow\wr & & \updownarrow\wr \\ S_1 \setminus f^{-1}(\{y_1, ..., y_n\}) & \xrightarrow{f \wr} & S \setminus \{y_1, ..., y_n\}, \end{array}$$

where Γ is a Fuchsian group isomorphic to the fundamental group $\pi_1(S \setminus \{y_1, ..., y_n\})$ and Γ_1 is a subgroup of Γ isomorphic to the fundamental group of $\pi_1\Big(S_1 \setminus f^{-1}(\{y_1, ..., y_n\})\Big)$. By mean of this representation, it is possible to guarantee that $Mon(f) \cong \Gamma/\Gamma_1$ when Γ_1 is a normal subgroup of Γ (see Section 2.7.1 of [**GG**]*). In a more general situation, the natural action of Γ on the coset space Γ/Γ_1 is given by permutations, and this action is given by the monodromy group.*

THEOREM 2.5.3. *If $f_i : S_i \to S$ $(i = 1, 2)$ are two morphisms of some degree d with the same set of critical values $\{y_k\}$ in S, then f_1 and f_2 have conjugate monodromies if and only if they are isomorphic coverings (*[**GG**] *p. 167).*

EXAMPLE 2.5.4. Let n and m be two positive integers with no common factors. As a consequence of the theory developed in Section 3.3 we will see, in Example 5.1.10, that the variety

$$C = \{(x, y, z, w) \in \mathbb{C}^4 : 1 + z^m + w^m = 0, x^n = z^m, y^n = w^m\}$$

is irreducible, and, even more, that it is a Riemann surface of finite type (removing all its $3nm$ singular points). We will also see there that $\beta : C \longrightarrow \mathbb{C}$ with $\beta(x, y, z, w) = -x^n$ is a degree n^2m^2 regular covering with critical values 0, 1 and ∞.

On the other hand, we have seen that $S = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : x^{nm} + y^{nm} + z^{nm} = 0\}$ is a smooth Riemann surface, and that the covering $f : S \longrightarrow \mathbb{C}$ defined by $f\big([x : y : z]\big) = -(\frac{y}{x})^{nm}$ is also regular, has degree n^2m^2 and critical values 0, 1 and ∞.

Therefore, we can conclude that β and f are isomorphic coverings, since the monodromies $Mon(\beta)$ and $Mon(f)$ are conjugate groups in $\Sigma_{n^2m^2}$. Actually, both $Mon(\beta)$ and $Mon(f)$ are isomorphic to $\mathbb{Z}_n^2 \times \mathbb{Z}_m^2$ since β and f are normal (see [**GG**] Section 2.9).

2.6. Riemann-Roch's theorem.

As we stated in the Preliminaries, this section is dedicated to exhibit the Riemann-Roch theorem and some of its consequences. In order to do this, we will first pass through the concepts of meromorphic functions, holomorphic and meromorphic 1-forms and divisors (all of them on a Riemann surface), among others. As well, we will recall some results related to these concepts, to finally state the Riemann-Roch's theorem and some of its most important consequences.

DEFINITION 2.6.1. Let X be a Riemann surface and $f : X \to \mathbb{C}$ be a function which is holomorphic in a punctured neighbourhood of a point $p \in X$. Let us consider a chart (U, ϕ) at p. We will say that:

(1) f has a *removable singularity* (resp. *pole*) at p if $f \circ \phi^{-1}$ has a removable singularity (resp. pole) at $\phi(p)$.
(2) f is *meromorphic at* p if it is either holomorphic, has a removable singularity or has a pole at p.

In this case, $f \circ \phi^{-1}$ has the Laurent series expansion about $\phi(p)$

$$f \circ \phi^{-1}(z) = \sum_{k=n}^{+\infty} c_k \big(z - \phi(p)\big)^k,$$

where

$$c_k = \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f \circ \phi^{-1}(z)}{(z - \phi(p))^{k+1}} dz$$

with

$$\gamma_r(\theta) := \phi(p) + re^{i\theta} \quad (\theta \in (0, 2\pi)) \quad \text{in} \quad \phi(U) \quad \text{and} \quad c_n \neq 0,$$

which we will call *the Laurent series expansion of* f *about* p *with respect to* ϕ. The integer n is called the *order of* f *at* p and is denoted by $ord_p(f)$.
(3) f is *meromorphic on an open set* $W \subseteq X$ if it is meromorphic at each point of W.
(4) The field of all meromorphic functions on a Riemann surface X will be denoted by $\mathcal{M}(X)$.

REMARK. With the above notations:

i) f has a removable singularity (resp. pole) at p if and only if for every chart (U, ϕ) at p the function $f \circ \phi^{-1}$ has a removable singularity (resp. pole) at $\phi(p)$.
ii) The Laurent series expansion of f about p depends on the choice of the chart. Nevertheless, the order of f at p is independent of the choice of the chart at p ([**MIR**] p. 26).

PROPOSITION 2.6.2. $\mathcal{M}(\mathbb{P}^1) = \mathcal{M}(\widehat{\mathbb{C}}) = \mathbb{C}(z)$, *the field of rational functions in one variable ([**GG**] p.26).*

REMARK. The proof of existence of meromorphic functions on a compact Riemann surface X of genus $g \geq 1$ and the fact that they are *a lot* (in this case a *lot* means that they separate points in X) can be found in [**GG**] Section 2.2. Let us note that if X is a compact Riemann surface, a non constant meromorphic map must have poles.

PROPOSITION 2.6.3. *Let* X *be a Riemann surface and suppose* $f : X \to \mathbb{C}$ *is a non constant meromorphic function at* $p \in X$. *Then:*

(1) *f is holomorphic at p if and only if $ord_p(f) \geq 0$.*
(2) *$f(p) = 0$ if and only if $ord_p(f) > 0$.*
(3) *f has a pole at p if and only if $ord_p(f) < 0$.*

DEFINITION 2.6.4. Let X be a Riemann surface and suppose $f : X \to \widehat{\mathbb{C}}$ is a non constant meromorphic function at $p \in X$. We will say that f has a *zero of order n* at p if $ord_p(f) = n \geq 1$, and f has a *pole of order n* at p if $ord_p(f) = -n < 0$.

PROPOSITION 2.6.5. *Let X be a Riemann surface and let $f, g : X \to \mathbb{C}$ be two non constant meromorphic functions at $p \in X$. Then:*
(1) $ord_p(fg) = ord_p(f) + ord_p(g)$.
(2) $ord_p(1/f) = -ord_p(f)$.
(3) $ord_p(f \pm g) \geq min\{ord_p(f), ord_p(g)\}$.

REMARK. Let $f : X \to Y$ be a morphism between two Riemann surfaces and p a point in X. Then $mult_p f = 1 + ord_p(\psi \circ f)'$ for any chart ψ at $f(p)$.

EXAMPLE 2.6.6. Let

$$\begin{array}{cccc} f: & \widehat{\mathbb{C}} & \to & \widehat{\mathbb{C}} \\ & z & \mapsto & f(z) = \frac{z+1}{z^2+3}. \end{array}$$

Since $(z+1)/(z^2+3) = (z+1)(z+i\sqrt{3})^{-1}(z-i\sqrt{3})^{-1}$ and $f(1/z) = z(z+1)(1+i\sqrt{3}z)^{-1}(1-i\sqrt{3}z)^{-1}$, it follows that f has two zeros of order 1, namely -1 and ∞, and two poles of order 1, namely $\pm i\sqrt{3}/3$. In fact,

$$ord_p(f) = \begin{cases} 1, & p = -1, \infty \\ -1, & p = \pm i\sqrt{3}/3 \\ 0, & \text{in other case.} \end{cases}$$

The next three results are inherited immediately from the corresponding theorems of functions defined on open sets in the complex plane and can be found in [MIR] at page 29.

PROPOSITION 2.6.7. *Let f be a meromorphic function defined on a connected open set W of a Riemann surface X. If f is not identically zero then the zeros and poles of f form a discrete subset of W.*

COROLLARY 2.6.8. *The set of zeros and poles of a non-identically zero meromorphic function f on a compact Riemann surface is finite.*

PROPOSITION 2.6.9. *Let f and g two meromorphic functions defined on a connected open set W of a Riemann surface. If $f = g$ on a subset $S \subseteq W$ which has a limit point in W then $f = g$ on W.*

PROPOSITION 2.6.10. *If f is a non constant meromorphic function on a compact Riemann surface X then*

$$\sum_{p \in X} ord_p(f) = 0$$

([MIR] *p.31).*

DEFINITION 2.6.11. A *holomorphic* (resp. *meromorphic*) *1-form* on an open set $V \subseteq \mathbb{C}$ is an expression ω of the form

$$\omega = f(z)dz,$$

where f is a holomorphic (resp. meromorphic) function on V. In this case we will say that ω is a holomorphic (resp. meromorphic) 1-form *in the coordinate z.*

REMARK. Let us remember that we can write

$$z = x + iy \quad \text{and} \quad \overline{z} = x - iy$$

and

$$x = (z + \overline{z})/2 \quad \text{and} \quad y = (z - \overline{z})/2i,$$

and by means of the differentials

$$dz = dx + idy \quad \text{and} \quad d\overline{z} = dx - idy$$

and

$$dx = (dz + d\overline{z})/2 \quad \text{and} \quad d\overline{y} = (dz - d\overline{z})/2i$$

we can deduce the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

With this notation, the condition $\partial f/\partial\overline{z} = 0$ is equivalent to the Cauchy-Riemann equations and thus a C^∞ function f is holomorphic on an open set $V \subseteq \mathbb{C}$ if and only if $\partial f/\partial\overline{z} = 0$ on V.

DEFINITION 2.6.12. Let $\omega_1 = f(z)dz$ and $\omega_2 = g(w)dw$ be two holomorphic (resp. meromorphic) 1-forms defined on the open sets V_1 and V_2 of $\mathbb{C}$ respectively, and let $z = T(w)$ define a holomorphic function from V_2 to V_1. We will say that ω_1 *transforms to* ω_2 *under* T if $g(w) = f\big(T(w)\big)T'(w)$.

REMARK. If T is invertible, then ω_1 transforms to ω_2 under T if and only if ω_2 transforms to ω_1 under T^{-1}.

DEFINITION 2.6.13. A *holomorphic* (resp. *meromorphic*) *1-form on X a Riemann surface X* is a collection of holomorphic (resp. meromorphic) 1-forms $\{\omega_\phi\}$, one for each chart $\phi : U \to V$ in the coordinate of the target V, such that if two charts $\phi_j : U_j \to V_j$ $(j = 1, 2)$ have overlapping domains, then the associated holomorphic (resp. meromorphic) 1-form ω_{ϕ_1} transforms to ω_{ϕ_2} under the change of coordinates $T = \phi_1 \circ \phi_2^{-1}$.

The following result states that in order to define a holomorphic (resp. meromorphic) 1-form on a Riemann surface it is sufficient to give a holomorphic (resp. meromorphic) 1-form on the charts of a given atlas instead of giving a holomorphic (resp. meromorphic) 1-form on every chart of the surface.

PROPOSITION 2.6.14. *Let X be a Riemann surface and $\mathcal{A}$ a complex atlas on X. Suppose that holomorphic (resp. meromorphic) 1-forms are given for each chart of $\mathcal{A}$, which transform to each other on their common domains. Then there is a unique holomorphic (resp. meromorphic) 1-form on X extending these holomorphic (resp. meromorphic) 1-forms on each of the charts of $\mathcal{A}$ (*[MIR] *p.106,107).*

DEFINITION 2.6.15. The *space of all holomorphic 1-forms* and the *space of all meromorphic 1-forms* on a Riemann surface X will be denoted by $\mathscr{H}^1(X)$ and $\mathcal{M}^1(X)$ respectively; they are complex vector spaces.

THEOREM 2.6.16. *Let X be a compact Riemann surface of genus g. Then*

$$dim \mathscr{H}^1(X) = g$$

([GRIF] *p.102).*

DEFINITION 2.6.17. Let X be a Riemann surface and ω be a meromorphic 1-form defined in a punctured neighbourhood of $p \in X$. Choosing a local coordinate, that is a local chart, centered at p, it is possible to write $\omega = f(z)dz$ where f is a meromorphic function at 0. In this way, it is possible to define the *order of* ω *at* p, denoted by $ord_p(\omega)$, as the order of f at 0.

REMARK. With the notations of the above definition:

i) $ord_p(\omega)$ is well defined and is independent of the choice of the local chart.
ii) ω is holomorphic at p if and only if $ord_p(\omega) \geq 0$.

DEFINITION 2.6.18. Let X be a Riemann surface and ω be a meromorphic 1-form defined in a punctured neighbourhood of $p \in X$. We will say p is a *zero of* ω *of order* n if $ord_p(\omega) = n > 0$, and is a *pole of* ω *of order* n if $ord_p(\omega) = -n < 0$.

REMARK. The set of zeros and poles of a non zero meromorphic 1-form defined in a punctured neighbourhood of a point of a Riemann surface X is discrete. In particular, if X is compact this set is finite.

THEOREM 2.6.19. ***The Poincaré-Hopf index formula for meromorphic differentials.*** *If X is a compact Riemann surface of genus g and $\omega \in \mathcal{M}^1(X)$ then*

$$\sum_{p \in X} ord_p(w) = 2g - 2$$

([GRIF] *p.23).*

DEFINITION 2.6.20. Let X be a Riemann surface:

(1) A *divisor on* X is a function $D : X \to \mathbb{Z}$ whose support, that is, the set of all points $p \in X$ such that $D(p) \neq 0$, is a discrete subset of X, and we will denote it using the summation notation by

$$D = \sum_{p \in X} D(p) \cdot p.$$

The set of all divisors on X endowed with pointwise addition forms a group, which will be denoted by $Div(X)$.

In particular, if X is compact, its support is finite, so in this case this allows to define the *degree of a divisor* D, denoted by $deg(D)$, as the sum of the values of D:

$$deg(D) = \sum_{p \in X} D(p).$$

(2) The *divisor of a nonzero meromorphic function* f on X, denoted by $div(f)$, is the divisor

$$div(f) = \sum_{p} ord_p(f) \cdot p.$$

Any divisor of this form is called a *principal divisor* on X, and the set of all principal divisors on X will be denoted by $PDiv(X)$.

(3) The *divisor of a nonzero meromorphic 1-form* ω on X, denoted by $div(\omega)$, is the divisor

$$div(\omega) = \sum_{p} ord_p(\omega) \cdot p.$$

Any divisor of this form is called a *canonical divisor* on X, and the set of all canonical divisors on X will be denoted by $KDiv(X)$.

REMARK. Let X be a Riemann surface. If $f, g \in \mathcal{M}(X)$ are non constant and ω is a non zero meromorphic 1-form then:

i) $div(fg) = div(f) + div(g)$.
ii) $div(1/f) = -div(f)$.
iii) $div(f\omega) = div(f) + div(\omega)$.

If in addition X is compact then $deg\big(div(f)\big) = 0$ (see Proposition 2.6.10).

EXAMPLE 2.6.21. Let $X = \widehat{\mathbb{C}}$, $f(z) = c\sum_j (z - \lambda_j)^{e_j}$ be a rational function on z and $\omega = f(z)dz$. Then the divisor of the meromorphic 1-form ω is

$$div(\omega) = \sum_j e_j \cdot \lambda_j - \big(2 + \sum_j e_j\big) \cdot \infty$$

and has degree -2. For instance, if $\omega = dz$, then $div(\omega) = -2 \cdot \infty$.

DEFINITION 2.6.22. Let X be a compact Riemann surface and

$$D = \sum_{j=1}^{k} D(p_j) \cdot p_j \in Div(X).$$

It will be said that D is an *effective divisor* if $D(p_j) \geq 0$ for all $j = 1, \ldots, k$, and this will be denoted by $D \geq 0$.

DEFINITION 2.6.23. The set

$$\mathscr{L}(D) := \{f \in \mathcal{M}(X) : div(f) + D \geq 0\} \cup \{0\}$$

forms a vector space over $\mathbb{C}$, called *the space of meromorphic functions with poles bounded by D*; its dimension will be denoted by $l(D)$. The set

$$\mathscr{L}^1(D) := \{\omega \in \mathcal{M}^1(X) : div(\omega) \geq D\} \cup \{0\}$$

also forms a vector space over $\mathbb{C}$, called *the space of meromorphic 1-forms with poles bounded by D*; its dimension will be denoted by $i(D)$.

THEOREM 2.6.24. ***(The Riemann-Roch theorem)*** *Let X be a compact Riemann surface of genus g and let $D \in Div(X)$. Then*

$$l(D) = deg(D) - g + i(D) + 1$$

([**GRIF**] *p.117).*

COROLLARY 2.6.25. *If X is a compact Riemann surface of genus* 0*, then it is isomorphic to $\mathbb{P}^1$ (*[**GRIF**] *p.125).*

COROLLARY 2.6.26. *If X is a compact Riemann surface of genus* 1*, then it can be represented by a smooth algebraic curve of degree* 3 *in $\mathbb{P}^2$ (*[**GRIF**] *p.126).*

DEFINITION 2.6.27. Suppose X is a compact Riemann surface of genus $g \geq 2$ and $\{\omega_1, \ldots, \omega_g\}$ is a basis of $\mathscr{H}^1(X)$. Then the function

$$\begin{array}{rccc} \Phi_K : & X & \to & \mathbb{P}^{g-1} \\ & p & \mapsto & [\omega_1(p) : \cdots : \omega_g(p)] \end{array}$$

for $\omega_j(p) := f_j\big(z(p)\big)$ $(j = 1, \ldots, g)$, where z is a local coordinate around the point $p \in X$ and $\omega_j = f_j(z)dz$ $(j = 1, \ldots, g)$, is called the *canonical embedding* of X. The image $\Phi_K(X) \subseteq \mathbb{P}^{g-1}$ is called a *canonical model curve* of X.

REMARK. The above definition of Φ_K is independent of the choice of the local coordinate around p and it is easy to check that Φ_K is nondegenerate.

DEFINITION 2.6.28. A compact Riemann surface X of genus $g > 1$ is called *hyperelliptic* if there is a degree 2 holomorphic function from X to $\mathbb{P}^1$; otherwise it is called *nonhyperelliptic.* By the Riemann-Hurwitz formula, the number of critical values of the above two-fold holomorphic function is equal to $2g + 2$.

PROPOSITION 2.6.29. *If X is a nonhyperelliptic compact Riemann surface, then the canonical embedding Φ_K is injective and $\Phi_K(X)$ is smooth ([GRIF] p.135).*

PROPOSITION 2.6.30. *Any hyperelliptic compact Riemann surface of genus g can be represented as the normalization of a plane algebraic curve C, that is the desingularization and compactification of C, of degree $2g + 2$ ([GRIF] p.137).*

Finally, as a consequence of the above, the following result may be deduced.

COROLLARY 2.6.31. *Every compact Riemann surface is isomorphic to a connected smooth projective algebraic curve.*

2.7. Belyi pairs, dessins d'enfants and their equivalence.

As we stated before, in this section we will treat the concepts of Belyi pairs and dessins d'enfants and we will give the explanation of their equivalence.

DEFINITION 2.7.1. A compact Riemann surface S is a *Belyi curve* if there exist a non-constant meromorphic map $\beta : S \longrightarrow \widehat{\mathbb{C}}$ whose critical set (the set of all its critical values) is contained inside $\{0, 1, \infty\}$. In this case, β is called a *Belyi map* and the pair (S, β) a *Belyi pair.*

REMARK. As the group of Möbius transformations acts transitively on triples of distinct points, the existence of a non-constant meromorphic map $h : S \longrightarrow \widehat{\mathbb{C}}$ with at most three critical values is sufficient to guarantee that S is a Belyi curve. In other words, if the set of critical values of h is contained in $\{a, b, c\}$, then $\beta = \alpha \circ h : S \longrightarrow \widehat{\mathbb{C}}$, where $\alpha(z) = \frac{(z-a)(b-c)}{(a-b)(c-z)}$, is a Belyi map. Actually β and h not only have the same number of critical values but also the same critical points.

DEFINITION 2.7.2. We will say that two Belyi pairs (S_1, β_1) and (S_2, β_2) are *equivalent* if there is an isomorphism $h : S_1 \longrightarrow S_2$ such that $\beta_1 = \beta_2 \circ h$, or, equivalently, if the following diagram is commutative

$$\begin{array}{ccc} S_1 & \xrightarrow{\quad h \quad} & S_2 \\ & {\scriptstyle \beta_1}\searrow \quad \swarrow{\scriptstyle \beta_2} & \\ & \widehat{\mathbb{C}} & \end{array}$$

We already know that if S is a compact Riemann surface, then there is some irreducible non-singular projective algebraic curve C, defined over the field of complex numbers $\mathbb{C}$, such that $S \simeq C$ as a compact Riemann surface (see Section 2.6).

THEOREM 2.7.3 (Belyi's theorem [B]). *A compact Riemann surface S is a Belyi curve if and only if it can be represented by a curve C defined over the field of algebraic numbers $\overline{\mathbb{Q}}$. Even more, if (S, β) is a Belyi pair, we may also assume that there is a (rational) Belyi map $\beta_C : C \to \mathbb{C}$ defined over $\overline{\mathbb{Q}}$ such that (S, β) is equivalent to (C, β_C) (see also* [GG] *Chapter 3).*

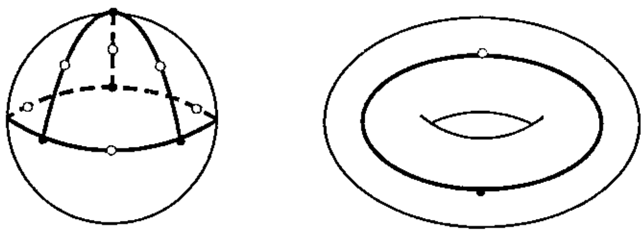
DEFINITION 2.7.4. A *dessin d'enfant*, or simply a *dessin*, is a pair $(X, \mathcal{D})$ where X is an oriented compact topological surface, and $\mathcal{D} \subset X$ is a finite graph such that:

(1) $\mathcal{D}$ is connected.
(2) $\mathcal{D}$ is bipartite, that is, its vertices have been given either white or black colour and vertices connected by an edge have different colours.
(3) $X \setminus \mathcal{D}$ is the union of finitely many topological discs, which we will call *faces* of $\mathcal{D}$.

The *genus of* $(X, \mathcal{D})$ is simply the genus of the topological surface X.

REMARK. Observe that condition (1) in the above definition is superfluous as it is consequence of condition (3). Also, the bipartite condition asserts that the graph $\mathcal{D}$ has no loops (edges with both end vertices being the same) and every closed path on it has an even number of edges. Each face of $\mathcal{D}$ must be bounded by an even number of edges if we count twice an edge which meets the same face at both sides (internal edges).

EXAMPLE 2.7.5. In the next figure (a) is a dessin d'enfant and (b) is not a dessin d'enfant.



(a) (b)

DEFINITION 2.7.6. The *degree of a vertex* of $\mathcal{D}$ is the number of edges incident with it and the *degree of a face* is half the number of edges bounding it considering the above remark (that is, each internal edge is counted twice).

DEFINITION 2.7.7. Two dessins $(X_1, \mathcal{D}_1)$ and $(X_2, \mathcal{D}_2)$ are *equivalent* if there is an orientation preserving homeomorphism from X_1 to X_2 whose restriction to $\mathcal{D}_1$ induces an isomorphism between the bipartite graphs $\mathcal{D}_1$ and $\mathcal{D}_2$ (that is, black vertices are sent to black vertices and white vertices are sent to white vertices).

There is an amazing connection between Belyi pairs and dessins d'enfants. Actually these sets of classes are equivalent as we anticipated. In order to see it, let us first construct a Belyi function and therefore a Belyi pair from a dessin d'enfant and viceversa.

Given a dessin $(X, \mathcal{D})$ we can associate to it a special *triangle decomposition* $\mathcal{T}(\mathcal{D})$ of X which will be constructed following the next indications:

(1) For each face F we choose a point v_F in its interior (center of the face) and mark it with the symbol $\times$.
(2) Per each face F and the above chosen center v_F, we draw topological segments from v_F to each of the black and white vertices in the boundary of F, such that they are not allowed to intersect any other edge of $\mathcal{D}$ or any other segment constructed in the same way except at v_F.

This triangulation consist of an even number of triangles (each one containing one vertex of each type $\circ, \bullet$ and $\times$) which we can classify into two types $+$ and $-$. Let j be an edge of a triangle T of $\mathcal{T}(\mathcal{D})$. If the circuit $\circ \to \bullet \to \times \to \circ$ follows the positive orientation on the border of T we denote it by T_j^+, otherwise we denote it by T_j^-.

REMARK. See section 4.2.1 of [GG].

i) Every edge of $\mathcal{D}$ belongs to exactly two triangles of different type.
ii) Two adjacent triangles are of different type (+ or −).
iii) Each face of $\mathcal{D}$ is decomposed as a union of an even number of triangles, half of them of each type.

Now, if we define $\overline{\mathbb{H}}^+ := \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ (the closure of the upper halfplane) we can choose a triangle T_j^+ of type + and a homeomorphism $f_j^+ : T_j^+ \to \overline{\mathbb{H}}^+$ such that

$$f_j^+ : \begin{cases} \partial T_j^+ & \to \mathbb{R} \cup \{\infty\} \\ \circ & \to 0 \\ \bullet & \to 1 \\ \times & \to \infty \end{cases}$$

REMARK. The existence of such homeomorphism is due to two facts. The first one is that every homeomorphism from an edge of a triangle T to a segment in $\partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$ extends to a homeomorphism from the whole ∂T to $\partial\overline{\mathbb{H}}$, and the second one is that every homeomorphism $\partial T \longrightarrow \partial\overline{\mathbb{H}}$ extends to a homeomorphism $T \longrightarrow \overline{\mathbb{H}}$ ([GG] p. 232).

Next, take the triangle T_j^- adjacent to T_j^+ along the edge label j, and map T_j^- to the closure of the lower halfplane $\overline{\mathbb{H}}^-$ by a homeomorphism $f_j^- : T_j^- \to \overline{\mathbb{H}}^-$ that coincides with f_j^+ in the intersection $T_j^+ \cap T_j^-$ and verifies

$$f_j^- : \begin{cases} \partial T_j^- & \longrightarrow \mathbb{R} \cup \{\infty\} \\ \circ & \longrightarrow 0 \\ \bullet & \longrightarrow 1 \\ \times & \longrightarrow \infty \end{cases}$$

Actually, we just *glue* the triangles T_j^+ and T_j^-. But we can also glue together the collection of all homeomorphisms $f_j^\pm$ to construct a continuous function $f_{\mathcal{T}(\mathcal{D})} : X \longrightarrow \widehat{\mathbb{C}}$ whose restriction $f_{\mathcal{T}(\mathcal{D})} : X^* = X \setminus f_{\mathcal{T}(\mathcal{D})}^{-1}\{0,1,\infty\} \longrightarrow \widehat{\mathbb{C}} \setminus \{0,1,\infty\}$ is a topological covering. As a consequence of this, we can endow X^* with the only Riemann surface structure that makes $f_{\mathcal{T}(\mathcal{D})}$ holomorphic and therefore convert X into a Riemann surface denoted by $S_{\mathcal{T}(\mathcal{D})}$. Clearly, $f_{\mathcal{T}(\mathcal{D})}$ also becomes a morphism from $S_{\mathcal{T}(\mathcal{D})}$ to $\widehat{\mathbb{C}}$.

We have to point out that, modulo equivalence of coverings, the pair $(S_{\mathcal{T}(\mathcal{D})}, f_{\mathcal{T}(\mathcal{D})})$ depends only on the dessin but not on the particular choice of triangle decomposition, nor on the choice of the collection of local homeomorphisms $f_j^\pm : T_j^\pm \longrightarrow \overline{\mathbb{H}}^\pm$ ([GG] p. 232-233). Therefore, from now on, we shall write $(S_\mathcal{D}, f_\mathcal{D})$ instead of writing $(S_{\mathcal{T}(\mathcal{D})}, f_{\mathcal{T}(\mathcal{D})})$ and we will referred to it as the *Belyi pair associated to the dessin* $\mathcal{D}$.

REMARK. $f_{\mathcal{D}}$ satisfies the following properties ([GG] p. 233-234):

i) $f_{\mathcal{D}}$ ramifies only at the vertices of $\mathcal{T}(\mathcal{D})$, that is, at the points $\circ, \bullet$ and $\times$.
ii) $f_{\mathcal{D}}$ is a Belyi map.
iii) $deg(f_{\mathcal{D}})$ agrees with the number of edges of $\mathcal{D}$, or, equivalently, with the cardinality of $f_{\mathcal{D}}^{-1}(1/2)$.
iv) The multiplicity of $f_{\mathcal{D}}$ at a vertex v of $\mathcal{D}$ is half the number of triangles surrounding v, since each pair of adjacent triangles covers a complete neighbourhood of $f_{\mathcal{D}}(v)$ and thus it agrees with the degree of the vertex.
v) The multiplicity of $f_{\mathcal{D}}$ at the center $\times$ of a face of $\mathcal{D}$ is half the number of triangles surrounding it and thus it is also half the number of edges of that face.
vi) By construction, $f_{\mathcal{D}}^{-1}([0,1]) = \mathcal{D}$.

Inversely, we have the following result.

PROPOSITION 2.7.8. *If (S, f) is a Belyi pair and we colour white the points $f^{-1}(0)$ and black the points $f^{-1}(1)$ and set $\mathcal{D}_f := f^{-1}([0,1])$ then ([GG] p. 239-240):*

(1) *$\mathcal{D}_f$ is a dessin d'enfant .*
(2) *Each of the sets $f^{-1}([-\infty,0]), f^{-1}([0,1])$ and $f^{-1}([1,\infty])$ is a union of topological segments. All of them together form the complete set of edges of a triangle decomposition $\mathcal{T}(\mathcal{D}_f)$.*
(3) *$f = f_{\mathcal{D}_f}$.*

Finally, we are ready to state the equivalence we anticipated before so many times.

THEOREM 2.7.9. *The two correspondences*

$$\begin{array}{ccc} \{\textit{Equiv. clases of dessins}\} & \longrightarrow & \{\textit{Equiv. clases of Belyi pairs}\} \\ (X,\mathcal{D}) & \mapsto & (S_{\mathcal{D}}, f_{\mathcal{D}}) \\ (S,\mathcal{D}_f) & \leftarrow & (S,f) \end{array}$$

which send equivalent dessins to equivalent Belyi pairs and viceversa induce well defined mutually inverse maps ([GG] p. 235,240-242).

Now, we will describe the so called regular dessins.

DEFINITION 2.7.10. A dessin d'enfant for which all faces have the same degree and the same occurs with all white vertices as well as all black ones is called *uniform*.

DEFINITION 2.7.11. If $(X, \mathcal{D})$ is a dessin d'enfant, then the set of orientation preserving homeomorphisms of X which preserve $\mathcal{D}$ as a bipartite graph will be denoted by

$Homeo^+(X, \mathcal{D})$. We can define an equivalence relation on it by saying that $H_1 \sim H_2$ if $H_1 \circ H_2^{-1}$ preserves setwise each edge of $\mathcal{D}$. The equivalence classes for this relation will be called *automorphisms of the dessin*, and the set of all such automorphisms will be denoted by $Aut(X, \mathcal{D})$ or simply by $Aut(\mathcal{D})$.

PROPOSITION 2.7.12. *If $(X, \mathcal{D})$ is a dessin d'enfant with $\mathcal{T}(\mathcal{D})$ and $f_{\mathcal{T}(\mathcal{D})}$ as before, then ([GG] p. 258-259):*

(1) *For any $H \in Homeo^+(X, \mathcal{D})$ there is a unique $H_0 \in Homeo^+(X, \mathcal{D})$ such that $H \sim H_0$ and $f_{\mathcal{T}(\mathcal{D})} \circ H_0 = f_{\mathcal{T}(\mathcal{D})}$.*
(2) *The map*

$$\begin{array}{ccc} Aut(\mathcal{D}) & \longrightarrow & Aut(S_{\mathcal{D}}, f_{\mathcal{D}}) \\ H & \mapsto & H_0 \end{array}$$

is a group isomorphism.

DEFINITION 2.7.13. A dessin $(X, \mathcal{D})$ is called *regular* if $Aut(\mathcal{D})$ acts transitively on the edges of $\mathcal{D}$.

PROPOSITION 2.7.14. *Regular dessins are uniform, but there are uniform dessins which are non-regular ([GG] p. 261).*

THEOREM 2.7.15. *A dessin is regular if and only if the corresponding Belyi function is a Galois (branched/ramified) covering of the sphere.*

PROOF. This is a consequence of proposition 2.7.12. □

2.8. Galois action.

Suppose $K \leq \mathbb{C}$ is a subfield, $\sigma : K \to \mathbb{C}$ a monomorphism and $\alpha \in \mathbb{C}$. Then recall we have two options:

(1) If α is algebraic over K and β is any root of the minimal polynomial of α over K, there is an extension $\overline{\sigma} : K(\alpha) \to \mathbb{C}$ characterized by sending α to β.
(2) If α is transcendental over K and β is also transcendental over K, there is also an extension $\overline{\sigma} : K(\alpha) \to \mathbb{C}$ characterized by sending α to β. In this case $K(\alpha)$ is isomorphic to $K(\beta)$.

The first property together with Zorn's lemma imply that any monomorphism $\sigma : K \to \mathbb{C}$ can be extended to an automorphism of the algebraic closure $\overline{K}$ of K, and then, by the second property, it also can be extended to an automorphism of $\mathbb{C}$.

In what follows, given a field extension $F \leq E$, $Gal(E/F)$ will denote the group of automorphisms of E whose restriction to F is the identity.

DEFINITION 2.8.1. Let $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ (or $\sigma \in Gal(\mathbb{C}/\mathbb{Q})$) and $a \in \overline{\mathbb{Q}}$ (or $a \in \mathbb{C}$). We shall write a^σ instead of $\sigma(a)$. Accordingly we shall employ the following notations:

(1) If $P = \sum a_{i_1,...i_n} X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}$ is a polynomial in $\overline{\mathbb{Q}}[X_1,...X_n]$ (or in $\mathbb{C}[X_1,...X_n]$), then
$$P^\sigma := \sum a^\sigma_{i_1,...i_n} X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}.$$
In addition, if $R(X_1,...,X_n) = P(X_1,...,X_n)/Q(X_1,...,X_n)$ is a rational function, then
$$R^\sigma := P^\sigma/Q^\sigma.$$
In this way, σ induces automorphisms of $\overline{\mathbb{Q}}[X_1,...,X_n]$ (or $\mathbb{C}[X_1,...,X_n]$) and of the field of rational functions $\overline{\mathbb{Q}}(X_1,...,X_n)$ (or $\mathbb{C}(X_1,...,X_n)$), respectively.

(2) If S is a Riemann surface isomorphic to a (affine or projective) curve $C_{F_1,...F_m}$ for which the polynomials $F_1, F_2, \cdots, F_m$ are in $\overline{\mathbb{Q}}[X_1,...,X_n]$ (or in $\mathbb{C}[X_1,...,X_n]$), then we set $S^\sigma \simeq C_{F_1^\sigma,...F_m^\sigma}$.

(3) If $R = P/Q : C_{F_1,...F_m} \longrightarrow \mathbb{C}$ is a rational morphism, then the morphism $R^\sigma = P^\sigma/Q^\sigma : C_{F_1^\sigma,...F_m^\sigma} \longrightarrow \mathbb{C}$ is also rational.

(4) Due to the second and third items of this definition, for a Belyi pair (S,β) (or ramified coverings of the projective line) equivalent to $(C_{F_1,...F_m}, R)$ we may set $(S,\beta)^\sigma = (C_{F_1^\sigma,...F_m^\sigma}, R^\sigma)$, which is again a Belyi pair (see Theorem 2.8.2 below).

REMARK. See [GG] Sections 3.3 and 3.4.

i) This Galois action is well defined (see also [GG] p. 197-198).

ii) If $\sigma \in Gal(\mathbb{C}/\mathbb{Q})$ and $p = (x_1,...,x_n) \in \mathbb{C}^n$ (resp. $p = [x_0 : ... : x_n] \in \mathbb{P}^n_{\mathbb{C}}$) satisfies $F(p) = 0$ for a polynomial F in $\mathbb{C}[X_1,...X_n]$ (resp. in $\mathbb{C}[X_0,...X_n]$), then the point $p^\sigma := (x_1^\sigma,...,x_n^\sigma)$ (resp. $p^\sigma := [x_0^\sigma,...,x_n^\sigma]$) satisfies $F^\sigma(p^\sigma) = 0$. Thus we can see that the correspondence $p \mapsto p^\sigma$ defines a bijection between the curves $C_{F_1,...F_m}$ and $C_{F_1^\sigma,...F_m^\sigma}$. Now, if $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ we achieve the same result in case of having a Belyi curve. For if, we can extend σ to a $\overline{\sigma} \in Gal(\mathbb{C}/\mathbb{Q})$ and obtain a bijection between $C_{F_1,...F_m}$ and $C_{F^{\overline{\sigma}}_1,...F^{\overline{\sigma}}_m}$ since a Belyi curve can be defined over $\overline{\mathbb{Q}}$ and therefore $C_{F^{\overline{\sigma}}_1,...F^{\overline{\sigma}}_m} = C_{F_1^\sigma,...F_m^\sigma}$.

THEOREM 2.8.2. *The action of* $Gal(\mathbb{C}/\mathbb{Q})$ *on Belyi pairs* (S, β) *enjoys the following properties ([GG] p. 210-211):*

1. $deg(\beta^\sigma) = deg(\beta)$.
2. $\left(\beta(p)\right)^\sigma = \beta^\sigma(p^\sigma)$ *for any* $p \in S$.
3. $mult_{p^\sigma}(\beta^\sigma) = mult_p(\beta)$.
4. $a \in \widehat{\mathbb{C}}$ *is a critical value of* β *if and only if* a^σ *is a critical value of* β^σ.
5. *If* (S, β) *is a Belyi pair, then* $(S, \beta)^\sigma = (S^\sigma, \beta^\sigma)$ *is also a Belyi pair.*
6. S *and* S^σ *have the same genus.*
7. *The rule*
$$\begin{array}{ccc} Aut(S,\beta) & \longrightarrow & Aut(S^\sigma, \beta^\sigma) \\ h & \mapsto & h^\sigma \end{array}$$
is a group isomorphism.

Due to the equivalence between Belyi pairs and dessins d'enfants, the absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ also acts on dessins themselves. The transform $\mathcal{D}^\sigma$ of a dessin $\mathcal{D}$ by an element $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is defined by the rule

$$\begin{array}{ccc} \mathcal{D} & \dashrightarrow & \mathcal{D}^\sigma \\ \downarrow & & \uparrow \\ (S_\mathcal{D}, f_\mathcal{D}) & \longrightarrow & (S^\sigma_\mathcal{D}, f^\sigma_\mathcal{D}) \end{array}$$

In other words, if $(S_\mathcal{D}, f_\mathcal{D})$ is the Belyi pair corresponding to $\mathcal{D}$, then $\mathcal{D}^\sigma$ is the dessin corresponding to the conjugate Belyi pair $(S^\sigma_\mathcal{D}, f^\sigma_\mathcal{D})$.

THEOREM 2.8.3. *The following properties of a dessin* $\mathcal{D}$ *remain invariant under the action of the absolute Galois group.*

1. *The number of edges.*
2. *The number of white vertices, black vertices and faces.*
3. *The degree of the white vertices, black vertices and faces.*
4. *The genus.*
5. *The automorphism group.*
6. *The regularity and the uniformity.*

PROOF. This is a direct consequence of the Theorem 2.8.2 and the Remark 2.5.2. □

THEOREM 2.8.4. *The action of* $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ *on dessins d'enfants of genus* g *is faithful for every* g *([GG] p. 268-273).*

Actually, in a more recent work, it was proven that the absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on a smaller set of dessins as are the regular ones (see for example [GJ] or [Gui]).

CHAPTER 3

Fiber Products of Riemann surfaces.

This chapter is devoted to describing the fiber product of Riemann surfaces and investigating some of its properties: In section 3.1 we will make such description passing through the definition of a singular Riemann surface and we will expose the universal property, already mentioned in the introduction, of the fiber product. In section 3.2 we will give a Fuchsian group description of the fiber product. Section 3.3 is assigned to the properties of the fiber product such as the connectivity and the irreducibility. As we stated before, the fiber product is not always connected or irreducible and as testimony of this we will exhibit examples that show it. In addition, we will also give an upper bound to the number of the possible irreducible components and we will give sufficient conditions for a fiber product to be irreducible in case it is connected (although these conditions are not necessary as we will see in some examples of chapter 5). Finally, in section 3.4 we make an application to the case of the fiber products of dessins d'enfants.

3.1. The fiber product of Riemann surfaces.

DEFINITION 3.1.1 (Singular Riemann surfaces). Let $R_1, \dots, R_n$ be compact Riemann surfaces and let $C_1, \dots, C_m \subset R_1 \cup \cdots \cup R_n$ be pairwise disjoint finite sets. A *singular Riemann surface* is the space obtained after identifying all points belonging to the same set C_j $(j = 1, \dots, m)$ in $R_1 \cup \cdots \cup R_n$. Its *irreducible components* are the starting Riemann surfaces R_i $(i = 1, \dots, n)$ and its *singularities* are the m equivalence classes $C_1, \dots, C_m$. We will say that the singular Riemann surface is *reducible* if $n \geq 2$. Otherwise, it is *irreducible.*

REMARK. Observe that a singular Riemann surface has a structure of a compact topological space (which might or might not be connected) and each connected component of the complement of the singular points has the structure of an analytically finite Riemann surface. Moreover, the singular points have neighbourhoods homeomorphic to a finite union of cones with common vertex.

DEFINITION 3.1.2 (Fiber product of Riemann surfaces). Let S_0, S_1 and S_2 be three compact Riemann surfaces and let $\beta_1 : S_1 \to S_0$ and $\beta_2 : S_2 \to S_0$ be two non-constant holomorphic functions.

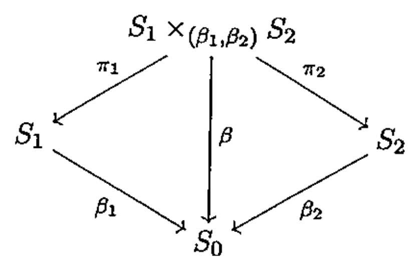
(1) We can associate to the pairs (S_1, β_1) and (S_2, β_2) the set

$$S_1 \times_{(\beta_1,\beta_2)} S_2 := \{(z_1, z_2) \in S_1 \times S_2 : \beta_1(z_1) = \beta_2(z_2)\} \subseteq S_1 \times S_2$$

called *the fiber product*. The fiber product inherits the induced topology from the complex surface $S_1 \times S_2$. Within this topology, the fiber product is a compact Hausdorff space.

(2) If $\pi_j : S_1 \times_{(\beta_1,\beta_2)} S_2 \to S_j$ denotes the projection $\pi_j(z_1, z_2) = z_j$, for $j = 1, 2$, then $\beta = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2$ is a surjective function from the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ to S_0

(3.1)



EXAMPLE 3.1.3. Consider $S_0 = S_1 = S_2 = \widehat{\mathbb{C}}$ and let $\beta_1(z) = z^n$ and $\beta_2(z) = z^m$ for any positive integers n and m. Then, in this case, the fiber product is the set

$$S_1 \times_{(\beta_1,\beta_2)} S_2 = \{(z, w) \in \mathbb{C}^2 : z^n = w^m\} \cup \{(\infty, \infty)\}$$

which, for example, may be or not be irreducible depending on the choices of n and m. Actually, it consist of $d = mcd(n, m)$ (the maximum common divisor of n and m) irreducible components of genus zero all them glued at the points $(0, 0)$ and (∞, ∞) (this would be clear after reading this section).

We have the following remark.

REMARK 3.1.4. *For the projections π_1 and π_2 the following statements hold:*

i) *π_1 and π_2 are continuous and therefore β is also continuous (immediate).*

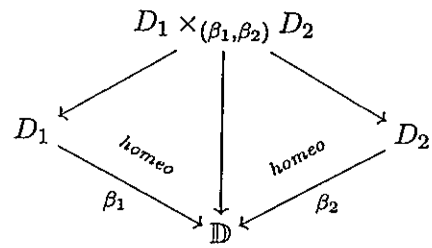
ii) *If B denotes the collection of points in S_0 which are critical values of either β_1 or of β_2, the projections $\pi_1|_{(S_1\times_{(\beta_1,\beta_2)}S_2)\setminus\beta^{-1}(B)}$ and $\pi_2|_{(S_1\times_{(\beta_1,\beta_2)}S_2)\setminus\beta^{-1}(B)}$ are topological coverings of degree $d_2 = deg(\beta_2)$ and $d_1 = deg(\beta_1)$ respectively.*

To see this we first have to prove that $\pi_j|_{(S_1\times_{(\beta_1,\beta_2)}S_2)\setminus\beta^{-1}(B)}$ (j=1,2) is locally injective.

- *Let $z \in S_0 \setminus B$. In this case, z has d_j pre-images by β_j in S_j (j=1,2) and $d_1 d_2$ pre-images by β in $S_1 \times_{(\beta_1,\beta_2)} S_2$. Let (z_1, z_2) be one of these pre-images in $S_1 \times_{(\beta_1,\beta_2)} S_2$. Then z_1 and z_2 are not ramification points of β_1 and β_2 respectively and*

  $$z_1 \text{ has } d_2 \text{ pre-images by } \pi_1 \text{ in } S_1 \times_{(\beta_1,\beta_2)} S_2$$
  $$\text{and}$$
  $$z_2 \text{ has } d_1 \text{ pre-images by } \pi_2 \text{ in } S_1 \times_{(\beta_1,\beta_2)} S_2.$$

  So, there are open neighbourhoods $D_1 \subset S_1$ and $D_2 \subset S_2$ of z_1 and z_2 such that $D_1 \simeq \beta_1(D_1)$ and $D_2 \simeq \beta_2(D_2)$ are homeomorphic to $\mathbb{D}$ and $\beta_1(D_1) \cap \beta_2(D_2) \neq \emptyset$. Even more, we can choose D_1 and D_2 such that $\beta_1(D_1) = \beta_2(D_2) \simeq \mathbb{D}$ and $\beta_j|_{D_j}$ (j=1,2) are homeomorphisms. Therefore, we can locally see the diagram 3.1 as

  If we suppose π_1 is not locally injective around (z_1, z_2) then for every open neighborhood V of (z_1, z_2) in the fiber product, there would be two points $(x_1, x_2), (y_1, y_2) \in V$ such that

  $$(x_1, x_2) \neq (y_1, y_2) \quad \text{and} \quad x_1 = \pi_1(x_1, x_2) = \pi_1(y_1, y_2) = y_1.$$

  In other words, if we suppose π_1 is not locally injective around (z_1, z_2), then for every open neighborhood V of (z_1, z_2) in the fiber product, there would be two points $(x_1, x_2), (y_1, y_2) \in V$ such that $x_1 = y_1$, $x_2 \neq y_2$ and $\beta_2(x_2) = \beta_1(x_1) = \beta_1(y_1) = \beta_2(y_2)$. In particular, for every $n \in \mathbb{N}$ the set

  $$V_n := \left[\beta_1^{-1}(D(0,1/n)) \times \beta_2^{-1}(D(0,1/n))\right] \cap D_1 \times_{(\beta_1,\beta_2)} D_2,$$

  where $D(0,1/n) := \{z \in \mathbb{C} : 0 \leq |z| < 1/n\}$, is an open neighborhood of (z_1, z_2) in the fiber product and there would be two points $(x_1^n, x_2^n), (x_1^n, y_2^n)$ such that $x_2^n \neq y_2^n$ and $\beta_2(x_2^n) = \beta_2(y_2^n)$. Therefore, for each $n \in \mathbb{N}$ there would be two points x_2^n and y_2^n in $\beta_2^{-1}(D(0,1/n)) \subseteq D_2$ such that $\beta_2(x_2^n) = \beta_2(y_2^n)$. But this contradicts the fact that $\beta_2|_{D_2}$ is a homeomorphism from D_2 onto $\mathbb{D}$. Therefore π_1 is locally injective in $(S_1 \times_{(\beta_1,\beta_2)} S_2) \setminus \beta^{-1}(B)$.
  Analogously, it can be proved that π_2 is locally injective restricting its domain to $(S_1 \times_{(\beta_1,\beta_2)} S_2) \setminus \beta^{-1}(B)$.

Now, let z_1 be any point in $S_1 \setminus \beta_1^{-1}(B)$. Then it has d_2 pre-images in $S_1 \times_{(\beta_1,\beta_2)} S_2$, say $(z_1, z_1^1), \ldots, (z_1, z_{d_2}^1)$. Since π_1 is locally injective we can choose open neighborhoods $U_1, \ldots, U_{d_2}$ pairwise disjoints of $(z_1, z_1^1), \ldots, (z_1, z_{d_2}^1)$ respectively *in the fiber product such that for every $1 \leq k \leq d_2$ each U_k does not intersect $\beta^{-1}(B)$ and $\pi_1|_{U_k} : U_k \to \pi_1(U_k)$ is an homeomorphism. Defining*

$$V = \bigcap_{k=1}^{d_2} \pi_1(U_k)$$

we obtain an open neighbourhood of z_1 in S_1 for which there are pairwise disjoint neighbourhoods $\pi_1^{-1}(V) \cap U_k$ $(1 \leq k \leq d_2)$ of (z_1, z_k^1) in $S_1 \times_{(\beta_1,\beta_2)} S_2$ such that

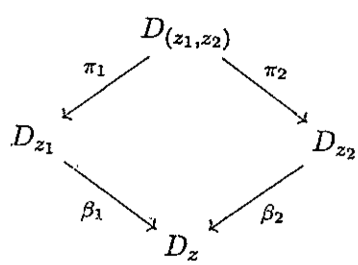
$$\pi_1 : \pi_1^{-1}(V) \cap U_k \longrightarrow \pi_1\big(\pi_1^{-1}(V) \cap U_k\big)$$

is a homeomorphism. In other words, $\pi_1|_{(S_1 \times_{(\beta_1,\beta_2)} S_2) \setminus \beta^{-1}(B)}$ is a topological covering of degree d_2. The other case is analogous.

iii) *$\beta|_{(S_1 \times_{(\beta_1,\beta_2)} S_2) \setminus \beta^{-1}(B)}$ is also a topological covering (this is an immediate consequence of the above item).*

Using the above remark it is possible to endow the space $(S_1 \times_{(\beta_1,\beta_2)} S_2) \setminus \beta^{-1}(B)$ with a complex structure as follows:

Defining $S_1^* := S_1 \setminus \beta_1^{-1}(B)$, $S_2^* := S_2 \setminus \beta_1^{-1}(B)$ and $S_0^* := S_0 \setminus B$, since π_1, π_2, β_1 and β_2 restricted to $(S_1 \times_{(\beta_1,\beta_2)} S_2) \setminus \beta^{-1}(B)$ are locally homeomorphisms, for a point $(z_1, z_2) \in (S_1 \times_{(\beta_1,\beta_2)} S_2) \setminus \beta^{-1}(B)$ with $z = \beta(z_1, z_2)$ there are open neighbourhoods D_{z_1}, D_{z_2}, D_z and $D_{(z_1,z_2)}$ in S_1^*, S_2^*, S_0^* and $S_1^* \times_{(\beta_1,\beta_2)} S_2^*$ [1] respectively, each homeomorphic to the unit disc $\mathbb{D}$ in $\mathbb{C}$, such that the following diagram of homeomorphisms is commutative



Actually, these neighbourhoods can also be chosen such that $\beta_j : D_{z_j} \longrightarrow D_z$ are isomorphic coverings. In this way, for D_{z_1,z_2} in $S_1^* \times_{(\beta_1,\beta_2)} S_2^*$ we can define the charts $(D_{(z_1,z_2)}, \psi_i^1 \circ \pi_1)$ or $(D_{(z_1,z_2)}, \psi_i^2 \circ \pi_2)$ where ψ_i^1 and ψ_i^2 are charts for D_{z_1} and D_{z_2} in S_1^* and S_2^* respectively. These charts are compatible because

$$\begin{aligned}(\psi_k^2 \circ \pi_2) \circ (\psi_i^1 \circ \pi_1)^{-1} &= \psi_k^2 \circ (\pi_2 \circ \pi_1^{-1}) \circ (\psi_i^1)^{-1} \\ &= \psi_k^2 \circ (\beta_2^{-1} \circ \beta_1) \circ (\psi_i^1)^{-1} \\ &= (\psi_k^2 \circ \beta_2^{-1}) \circ (\beta_1 \circ \psi_i^1)^{-1},\end{aligned}$$

$(\psi_i^1 \circ \pi_1) \circ (\psi_k^1 \circ \pi_1)^{-1} = \psi_i^1 \circ (\psi_k^1)^{-1}$ and $(\psi_i^2 \circ \pi_2) \circ (\psi_k^2 \circ \pi_2)^{-1} = \psi_i^2 \circ (\psi_k^2)^{-1}$ are all holomorphic.

[1] $S_1^* \times_{(\beta_1,\beta_2)} S_2^* = (S_1 \times_{(\beta_1,\beta_2)} S_2) \setminus \beta^{-1}(B)$

REMARK 3.1.5. *With the above complex structure, the function* $\pi_j|_{S_1^* \times_{(\beta_1,\beta_2)} S_2^*}$ $(j=1,2)$ *is holomorphic (actually, using these charts it can be seeing that the identity is their local form) and therefore* $\beta|_{S_1^* \times_{(\beta_1,\beta_2)} S_2^*}$ *is holomorphic too.*

On the other hand, if $(z_1, z_2) \in S_1 \times_{(\beta_1,\beta_2)} S_2$, remembering that the pairs (S_1, β_1) and (S_2, β_2) can be seen as pairs of algebraic curves with rational functions defined over the complex numbers $\mathbb{C}$, and considering local coordinates (centered charts) for z_1 and z_2, the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ can be locally seen as the affine curve

$$U = \{(z,w) \in \mathbb{D}^2 : z^{m_1} = w^{m_2}\}$$
$$= \left\{(z,w) \in \mathbb{D}^2 : \prod_{k=0}^{d-1} (z^{\widehat{m_1}} - e^{2\pi i k/d} w^{\widehat{m_2}}) = 0\right\}$$

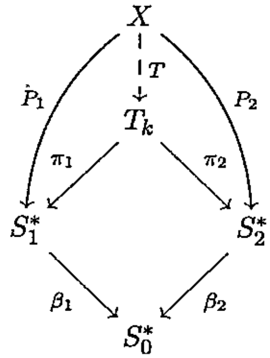
where $m_1 = mult_{z_1}\beta_1$, $m_2 = mult_{z_2}\beta_2$, $d = mcd(m_1, m_2)$ is the maximum common divisor of m_1 and m_2, $m_1 = d\widehat{m_1}$ and $m_2 = d\widehat{m_2}$. Observe that $(0,0)$ (respectively, (z_1, z_2)) is the only point of U (respectively, near (z_1, z_2)) which can be a singular point, and U (respectively, some neighbourhood of (z_1, z_2)) is homeomorphic to a collection of d cones with common vertex $(0,0)$ (respectively, (z_1, z_2)). In particular, this asserts that for $d = 1$ the fiber product looks locally as a topological real surface. Besides, the complex structure defined above is the same as the usual complex structure for curves (see Examples 2.1.9 and 2.1.26) and, as a consequence, $S_1^* \times_{(\beta_1,\beta_2)} S_2^*$ is a finite collection of connected components, each one an analytically finite Riemann surface, say $T_1, T_2 \ldots, T_n$, which after compactifying, correspond to the irreducible components of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$.

Summarizing all of the above, we have: **The fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is a compact singular Riemann surface with a finite collection of irreducible components and whose singularities are exactly the points (z_1, z_2) for which both z_1 and z_2 are ramification points of β_1 and β_2 respectively[2].**

PROPOSITION 3.1.6. ***(The universal property of the fiber product).*** *With the above notations, if X is an analytically finite Riemann surface and $P_j : X \to S_j^*$ $(j = 1,2)$ are coverings such that $\beta_1 \circ P_1 = \beta_2 \circ P_2$ then there is a covering T from X to some irreducible component T_k of $S_1 \times_{(\beta_1,\beta_2)} S_2$, such that $P_j = \pi_j \circ T$ or, equivalently, such that*

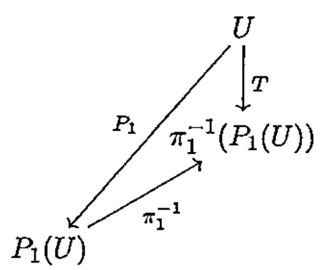
[2]Here, besides of the singularities contemplated in the definition of a singular Riemann surface, we have consider as singular points those points $(z_1, z_2) \in S_1 \times_{(\beta_1,\beta_2)} S_2$ such that both z_1 and z_2 are critical points of β_1 and β_2 respectively and $d = 1$. Around this kind of points the fiber product looks like a single cone and we know that after the des-singularization of each of these kind of points we will obtain, in strictly rigor, a singular Riemann surface. For this reason we do not include this kind of singularities in the definition of a singular Riemann surface, but we do it in a context of the fiber products with aim of study this objects as well as possible.

the following diagram is commutative.



PROOF. If $T : X \to S_1^* \times_{(\beta_1,\beta_2)} S_2^*$ is defined by $T(x) = \big(P_1(x), P_2(x)\big)$, it is clear that $P_j = \pi_j \circ T$ on X. Now, since π_1 is locally homeomorphic, for any point $T(x)$ there is an open neighbourhood V of $T(x) = \big(P_1(x), P_2(x)\big)$ in $S_1^* \times_{(\beta_1,\beta_2)} S_2^*$ for which $\pi_1(V)$ is an open neighbourhood of $P_1(x)$ in S_1^* and $\pi_1|_V : V \to \pi_1(V)$ is biholomorphic.

On the other hand, since P_1 is locally homeomorphic, there is an open neighbourhood U of x in X such that $P_1(U) \subseteq \pi_1(V)$ is an open neighbourhood of $P_1(x)$ in S_1^* and $P_1|_U : U \to P_1(U)$ is an isomorphism. Thus, we have the following commutative diagram



from which, due to our previous observations, it is deducible that T is a local homeomorphism and therefore continuous. Even more, it is a topological covering which is locally holomorphic, with $T(X)$ connected (because of the connectedness of X). Therefore, T is a covering from X to some irreducible component T_k of $S_1 \times_{(\beta_1,\beta_2)} S_2$. □

Before ending this section, it is important to observe and keep in mind that the fiber product depends not only on the starting compact Riemann surfaces but also on the starting coverings. The next example shows this situation (all the details and calculus are exposed in Example 5.1.1).

EXAMPLE 3.1.7. Consider the curves $S_1 = \{(x,y) \in \mathbb{C}^2 : y^2 = x(x-1)(x-2)\} \cup \{\infty\}$ and $S_2 = \{(x,y) \in \mathbb{C}^2 : y^2 = x(x-1)(x-4)\} \cup \{\infty\}$, and consider the coverings $\beta_1 : S_1 \to \widehat{\mathbb{C}}$ and $\beta_2, \widetilde{\beta_2} : S_2 \to \widehat{\mathbb{C}}$ defined by $\beta_1(x,y) = x$, $\beta_2(x,y) = x$ and $\widetilde{\beta_2}(x,y) = \frac{2x+1}{3x-1}$. It can be seen that the fiber products $S_1 \times_{(\beta_1,\beta_2)} S_2$ and $S_1 \times_{(\beta_1,\widetilde{\beta_2})} S_2$ are connected and irreducible singular Riemann surfaces which, after its des-singularization and compactification, become into compact Riemann surfaces; the first one has genus two whereas the second one has genus five.

3.2. A Fuchsian group description of the fiber product.

Let us assume S_0^* is a hyperbolic surface (the case of non-hyperbolic situation can be carry out in a similar way by replacing the hyperbolic plane) and let Γ_0 be a Fuchsian group acting on the hyperbolic plane $\mathbb{H}$ such that S_0^* is isomorphic to $\mathbb{H}/\Gamma_0$. As a consequence of covering theory, for $j = 1, 2$ there is a finite index subgroup Γ_j of Γ_0 for which the covering $\beta_j : S_j^* \to S_0^*$ is realized by any of the subgroups $\gamma\Gamma_j\gamma^{-1}$ where $\gamma \in \Gamma_0$ (see Sections 2.3 and 2.4).

Let us fix the subgroups $K_1 = \gamma_1\Gamma_1\gamma_1^{-1}$ and $K_2 = \gamma_2\Gamma_2\gamma_2^{-1}$, where $\gamma_1, \gamma_2 \in \Gamma_0$. Defining $K := K_1 \cap K_2$ and $R^* := \mathbb{H}/K$, we have the covering maps $Q_j : R^* \to S_j^*$ $(j = 1, 2)$ induced by the inclusion of K inside K_j. Clearly, in this case, $\beta_1 \circ Q_1 = \beta_2 \circ Q_2$ and, by the universal property of the fiber product (see Proposition 3.1.6), the surface $\mathbb{H}/K$ is one of the components T_k. Inversely, each T_k is obtained in the above way by suitable choices of γ_1 and γ_2.

Now, notice that for every $\gamma_1, \gamma_2 \in \Gamma_0$, the groups

$$\gamma_1\Gamma_1\gamma_1^{-1} \cap \gamma_2\Gamma_2\gamma_2^{-1} \quad \text{and} \quad \Gamma_1 \cap (\gamma_1^{-1} \circ \gamma_2)\Gamma_2(\gamma_1^{-1} \circ \gamma_2)^{-1}$$

provides isomorphic surfaces. In fact,

$$\alpha \in \gamma_1\Gamma_1\gamma_1^{-1} \cap \gamma_2\Gamma_2\gamma_2^{-1} \leftrightarrow \gamma_1^{-1}\alpha\gamma_1 \in \Gamma_1 \cap (\gamma_1^{-1}\gamma_2)\Gamma_2(\gamma_1^{-1}\gamma_2)^{-1}$$

which means that $\gamma_1\Gamma_1\gamma_1^{-1} \cap \gamma_2\Gamma_2\gamma_2^{-1}$ and $\Gamma_1 \cap (\gamma_1^{-1} \circ \gamma_2)\Gamma_2(\gamma_1^{-1} \circ \gamma_2)^{-1}$ are conjugate in Γ_0 (see theorem 2.4.7). This, in particular, allow us to state the following result.

THEOREM 3.2.1. *Every component T_k is isomorphic to $\mathbb{H}/(\Gamma_1 \cap \gamma\Gamma_2\gamma^{-1})$ for some $\gamma \in \Gamma_0$, and every quotient of the form $\mathbb{H}/(\Gamma_1 \cap \gamma\Gamma_2\gamma^{-1})$, where $\gamma \in \Gamma_0$, is isomorphic to some component T_k.*

REMARK 3.2.2. *Of course, from the above it is also deducible that:*

i) $\bigcup_{\gamma\in\Gamma_0} \mathbb{H}/(\gamma\Gamma_1\gamma^{-1} \cap \Gamma_2) \simeq S_1^* \times_{(\beta_1,\beta_2)} S_2^* \simeq \bigcup_{\gamma\in\Gamma_0} \mathbb{H}/(\Gamma_1 \cap \gamma\Gamma_2\gamma^{-1})$.

ii) *Every component T_k is isomorphic to one of the quotients $\mathbb{H}/(\gamma\Gamma_1\gamma^{-1} \cap \Gamma_2)$, where $\gamma \in \Gamma_0$, and every quotient of the form $\mathbb{H}/(\gamma\Gamma_1\gamma^{-1} \cap \Gamma_2)$, where $\gamma \in \Gamma_0$, is isomorphic to some component T_k.*

COROLLARY 3.2.3. *If one of the subgroups Γ_1 or Γ_2 is a normal subgroup of Γ then all the components T_k are isomorphic Riemann surfaces.*

Because each irreducible component of the fiber product, after desingularization, corresponds to one of the analytically finite Riemann surfaces T_k, the above can be stated as follows.

COROLLARY 3.2.4. *If one of the maps $\beta_j : S_j \to S_0$ $(j = 1, 2)$ is a regular branched covering, then all the irreducible components of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ are isomorphic.*

PROOF. Without loss of generality, let us assume that $\beta_2 : S_2 \to S_0$ is a regular branched covering. Then $\beta_2 : S_2^* \to S_0^*$ is a regular unbranched covering and thus Γ_2 is a normal subgroup of Γ_0. Therefore $(\Gamma_1 \cap \gamma\Gamma_2\gamma^{-1}) = \Gamma_1 \cap \Gamma_2$ for every $\gamma \in \Gamma_0$, and it follows that all components T_k are isomorphic. □

3.3. On the irreducibility of the fiber product.

Until now, we have analyzed how the fiber product locally looks like, which are its singular points and have made a Kleinian group description of it. But other questions still need to be answered. For example, we would like to know when it is connected or irreducible. The first big step to move along this way was given by W. Fulton and J. Hansen in [**FH**]. They proved there that when S_0 has genus zero the fiber product is connected. Nevertheless, when the genus of S_0 is positive, in general, this is not always true, as we will exhibit it in the next example (all the details and calculations are presented in Example 5.1.2).

EXAMPLE 3.3.1. Examples of non-connected fiber products.

(1) An example when S_0 has genus at least two:
    Let S and S_0 be two compact Riemann surfaces of genus at least two and let $\pi : S \to S_0$ be an unbranched covering of degree $d \geq 2$. Considering $S_1 = S_2 = S$ and $\beta_1 = \beta_2 = \pi$ we obtain a non-connected fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$.
(2) An example when S_0 has genus one:
    For $\lambda \in \mathbb{C} \setminus \{0, 1\}$ consider

$$S_1 = S_2 = S = \{[x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^3_{\mathbb{C}} : x_1^2 + x_2^2 + x_3^2 = 0,\ \lambda x_1^2 + x_2^2 + x_4^2 = 0\},$$
$$S_0 = \{[y_1 : y_2 : y_3 : y_4 : y_5] \in \mathbb{P}^4_{\mathbb{C}} : y_3^2 = y_1 y_2,\ y_5(y_1 + y_2) + y_4^2 = 0, \lambda y_1 + y_2 + y_5 = 0\},$$

and let

$$\beta_1 = \beta_2 = \pi : \begin{array}{ccc} S & \to & S_0 \\ [x_1 : x_2 : x_3 : x_4] & \mapsto & [x_1^2 : x_2^2 : x_1 x_2 : x_3 x_4 : x_4^2]. \end{array}$$

We claim that in this case the corresponding fiber product is the union of two disjoint Riemann surfaces A and B, each one isomorphic to S, where

$$A = \{([x_1 : x_2 : x_3 : x_4], [x_1 : x_2 : x_3 : x_4]) \in (\mathbb{P}^3_{\mathbb{C}})^2 : \\ x_1^2 + x_2^2 + x_3^2 = 0,\ \lambda x_1^2 + x_2^2 + x_4^2 = 0\}$$

and

$$B = \{([x_1 : x_2 : x_3 : x_4], [x_1 : x_2 : -x_3 : -x_4]) \in (\mathbb{P}^3_{\mathbb{C}})^2 : \\ x_1^2 + x_2^2 + x_3^2 = 0,\ \lambda x_1^2 + x_2^2 + x_4^2 = 0\}.$$

On the other hand, with respect to the irreducibility, even in the case that the fiber product is connected, it might be reducible. The next example shows this situation. For details and calculus see Example 5.1.3.

EXAMPLE 3.3.2. If $S_1 = S_2 = S_0 = \widehat{\mathbb{C}}$ and $\beta_1(z) = \beta_2(z) = z(z^3 + z^2 + 1)$, then $S_1 \times_{\beta_1,\beta_2} S_2$ consists of two irreducible components which, after its des-singularization and compactification, become into compact Riemann surfaces; one of them is the Riemann sphere and the other is a genus one Riemann surface provided by the compactification of the curve

$$1 + x^2 + x^3 + xy + x^2y + y^2 + xy^2 + y^3 = 0.$$

Consider the curves $S_1 = \{(x, y) \in \mathbb{C}^2 : y^2 = x(x-1)(x-2)\} \cup \{\infty\}$ and $S_2 = \{(x, y) \in \mathbb{C}^2 : y^2 = x(x-1)(x-4)\} \cup \{\infty\}$, and consider the coverings $\beta_1 : S_1 \to \widehat{\mathbb{C}}$ and $\beta_2, \widetilde{\beta_2} : S_2 \to \widehat{\mathbb{C}}$ defined by $\beta_1(x, y) = x$, $\beta_2(x, y) = x$ and $\widetilde{\beta_2}(x, y) = \frac{2x+1}{3x-1}$. It can be seen that the fiber products $S_1 \times_{(\beta_1,\beta_2)} S_2$ and $S_1 \times_{(\beta_1,\widetilde{\beta_2})} S_2$ are connected and irreducible singular Riemann surfaces which, after its des-singularization and compactification, become into compact Riemann surfaces; the first one has genus two whereas the second one has genus five.

Now, in order to go further, in the following theorem we provide sufficient conditions for the fiber product, when it is connected, to be irreducible.

THEOREM 3.3.3. *Let $\beta_1 : S_1 \to S_0$ and $\beta_2 : S_2 \to S_0$ be two non-constant holomorphic maps between compact Riemann surfaces and suppose its fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is connected. For each $j = 1, 2$ and each $q \in S_0$ set*

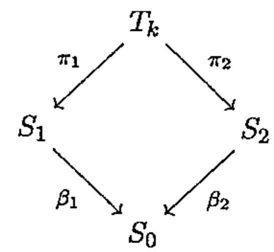
$$a_q^{(j)} := lcm(mult_z \beta_j : \beta_j(z) = q)$$

where "lcm" stands for "least common multiple". If either

1. *$gcd(deg(\beta_1), deg(\beta_2)) = 1$; or*
2. *$gcd(a_q^{(1)}, a_q^{(2)}) = 1$ for every $q \in S_0$,*

where "gcd" stands for "greatest common divisor", the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is irreducible and therefore a compact Riemann surface.

PROOF. Let us suppose (1) holds and let T_k be any of the irreducible components of $S_1 \times_{(\beta_1,\beta_2)} S_2$. If $d_{1,k} := deg(\pi_1|_{R_k})$ and $d_{2,k} := deg(\pi_2|_{R_k})$ then $d_{1,k}\ deg(\beta_1) = d_{2,k}.deg(\beta_2)$ and the following diagram commutes



In this way, $deg(\beta_2)$ divides $d_{1,k}$ and $deg(\beta_1)$ divides $d_{2,k}$. Now, as

$$\beta := \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2 : S_1 \times_{(\beta_1,\beta_2)} S_2 \to \widehat{\mathbb{C}}$$

has degree $deg(\beta_1)deg(\beta_2)$ the condition $mcd\big(deg(\beta_1), deg(\beta_2)\big) = 1$ asserts that

$$d_{1,k}deg(\beta_1) \leq deg(\beta) \to d_{1,k} \leq deg(\beta_2)$$
$$d_{2,k}deg(\beta_2) \leq deg(\beta) \to d_{2,k} \leq deg(\beta_1)$$

and thus one has $d_{1,k} = deg(\beta_2)$ and $d_{2,k} = deg(\beta_1)$. In other words, $S_1 \times_{(\beta_1,\beta_2)} S_2 = T_k$.

On the other hand, let us now assume (2) holds. Under this hypothesis, each one of the singular points of $S_1 \times_{(\beta_1,\beta_2)} S_2$ has a neighborhood homeomorphic to a disc. As we already know that $S_1 \times_{(\beta_1,\beta_2)} S_2$ is connected, the result follows. □

Trough the above theorem provides sufficient conditions for the fiber product of a two pairs to be irreducible, these conditions are not necessary (see Example 5.1.4). Nevetheless, without assuming such conditions, an upper bound for the number of irreducible components can be provided.

LEMMA 3.3.4. *The number of irreducible components of the fiber product of the two pairs* (S_1, β_1) *and* (S_2, β_2) *is at most* $gcd\big(deg(\beta_1), deg(\beta_2)\big)$.

PROOF. Let us use the same notation as in Section 3.2 and let N_j be the normalizer of Γ_j in Γ_0 $(j = 1, 2)$. By Remark 3.2.2, the number of irreducible components of the fiber product is at most the index $[N_j : \Gamma_j]$ of Γ_j in N_j, which divides $deg(\beta_j)$ $(j = 1, 2)$. Therefore, the number of irreducible components of the fiber product is at most $gcd\big(deg(\beta_1), deg(\beta_2)\big)$. □

In Section 5 we provide examples where this bound is sharp and where it is not sharp (see Examples 5.1.6 and 5.1.7).

3.4. Fiber product of dessins d'enfants.

Let us now consider two Belyi pairs (S_1, β_1) and (S_2, β_2). As we know, we can assume S_j is given as an algebraic curve over $\overline{\mathbb{Q}}$ and β_j as a rational map also defined over $\overline{\mathbb{Q}}$ (see Section 2.6). Then its fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is a connected, possibly reducible, algebraic curve defined over $\overline{\mathbb{Q}}$, and the map $\beta : S_1 \times_{(\beta_1,\beta_2)} S_2 \to \widehat{\mathbb{C}}$ defined by $\beta(z_1, z_2) = \beta_j(z_j)$ $(j = 1, 2)$ is also a rational map defined over $\overline{\mathbb{Q}}$. Each irreducible component turns out to be a Belyi curve and the restriction of β to it a Belyi map. By Theorem 3.3.3, we have simple sufficient conditions for the fiber product to be irreducible and so for $(S_1 \times_{(\beta_1,\beta_2)} S_2, \beta)$ to be a Belyi pair. In this way, due to the equivalence between Belyi pairs and dessins d'enfants, we will be able to construct new dessins d'enfants.

Let $\mathcal{D}$ be a dessin d'enfant on a compact oriented surface of genus g; we assume the vertices are colored in either black or white. Associated to $\mathcal{D}$ is its valence

$$val(\mathcal{D}) = (a_1, ..., a_\alpha; b_1, ..., b_\beta; c_1, ..., c_\gamma)$$

where

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_\alpha$$
$$1 \leq b_1 \leq b_2 \leq \cdots \leq b_\beta$$
$$1 \leq c_1 \leq c_2 \leq \cdots \leq c_\gamma$$

with $a_1,..., a_\alpha$ the degrees of the black vertices of $\mathcal{D}$ (D has exactly α black vertices), $b_1,..., b_\beta$ the degrees of the white vertices of $\mathcal{D}$ (D has exactly β black vertices) and $c_1,..., c_\gamma$ the degrees of the centers of the faces of $\mathcal{D}$ (D has exactly γ faces). Remember that the degree of a face is half the number of boundary edges of it. By Euler's characteristic formula,

$$2 - 2g = \alpha + \beta + \gamma - n$$

where $n = a_1 + \cdots + a_\alpha = b_1 + \cdots + b_\beta = c_1 + \cdots + c_\gamma$ is the number of edges of $\mathcal{D}$.

As we mentioned above, due to the equivalence of the categories of Belyi pairs and dessins d'enfants, the above allows us to talk about the fiber product of two given dessins. But, as already noted, such a fiber product might not be a dessin d'enfant. Theorem 3.3.3 may be stated, in terms of dessins d'enfants, as follows.

THEOREM 3.4.1. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two dessin d'enfants with*

$$val(\mathcal{D}_1) = \left(a_1^{(1)}, ..., a_{\alpha_1}^{(1)}; b_1^{(1)}, ..., b_{\beta_1}^{(1)}; c_1^{(1)}, ..., c_{\gamma_1}^{(1)}\right)$$
$$val(\mathcal{D}_2) = \left(a_1^{(2)}, ..., a_{\alpha_2}^{(2)}; b_1^{(2)}, ..., b_{\beta_2}^{(2)}; c_1^{(2)}, ..., c_{\gamma_2}^{(2)}\right)$$

and for $j = 1, 2$ let

$$A_j := lcm\left(a_1^{(j)}, ..., a_{\alpha_j}^{(j)}\right), \quad B_j := lcm\left(b_1^{(j)}, ..., b_{\beta_j}^{(j)}\right), \quad C_j := lcm\left(c_1^{(j)}, ..., c_{\gamma_j}^{(j)}\right)$$

$$\textit{and} \quad n_j := a_1^{(j)} + \cdots + a_{\alpha_j}^{(j)} = b_1^{(j)} + \cdots + b_{\beta_j}^{(j)} = c_1^{(j)} + \cdots + c_{\gamma_j}^{(j)}.$$

In this way, if either

1. $gcd(n_1, n_2) = 1$, *or*
2. $gcd(A_1, A_2) = gcd(B_1, B_2) = gcd(C_1, C_2) = 1$,

then the fiber product of the two dessins $\mathcal{D}_1$ and $\mathcal{D}_2$ is again a dessin d'enfant.

PROOF. For $j = 1, 2$ let $(S_{\mathcal{D}_j}, f_{\mathcal{D}_j})$ be the Belyi pair associated to the desssin $(X_j, \mathcal{D}_j)$. Therefore (see section 1.7):

- Since the black vertices of $\mathcal{D}_j$ are the preimages of 1 by $f_{\mathcal{D}_j}$, $|f_{\mathcal{D}_j}^{-1}(1)| = \alpha_j$ and the multiplicities at those points of $f_{\mathcal{D}_j}$ are $a_1^{(j)}, ..., a_{\alpha_j}^{(j)}$.
- Since the white vertices of $\mathcal{D}_j$ are the preimages of 0 by $f_{\mathcal{D}_j}$, $|f_{\mathcal{D}_j}^{-1}(0)| = \beta_j$ and the multiplicities at those points of $f_{\mathcal{D}_j}$ are $b_1^{(j)}, ..., b_{\beta_j}^{(j)}$.
- Since the centers of the faces of $\mathcal{D}_j$ are the preimages of ∞ by $f_{\mathcal{D}_j}$, $|f_{\mathcal{D}_j}^{-1}(\infty)| = \gamma_j$ and the multiplicities at those points of $f_{\mathcal{D}_j}$ are $c_1^{(j)}, ..., c_{\gamma_j}^{(j)}$.

In this way, following the notations of the Theorem 3.3.3, we have that:

- Condition (1) of the Theorem 3.3.3 is equivalent to the condition $gcd(n_1, n_2) = 1$ since $n_j = a_1^{(j)} + \cdots + a_{\alpha_j}^{(j)} = deg(f_{\mathcal{D}_1})$, for example.
- The condition (2) of the Theorem 3.3.3 is equivalent to the condition

$$gcd(A_1, A_2) = gcd(B_1, B_2) = gcd(C_1, C_2) = 1$$

since for $q \in \widehat{\mathbb{C}}$

$$a_q^{(j)} = lcm(mult_z f_{\mathcal{D}_j} : f_{\mathcal{D}_j}(z) = q) = \begin{cases} A_j = lcm(a_1^{(j)}, ..., a_{\alpha_j}^{(j)}), & q = 0 \\ B_j = lcm(b_1^{(j)}, ..., b_{\beta_j}^{(j)}), & q = 1 \\ C_j = lcm(c_1^{(j)}, ..., c_{\gamma_j}^{(j)}), & q = \infty \\ 1, & \text{in other cases} \end{cases}$$

and

$$gcd(a_0^{(1)}, a_0^{(2)}) = gcd(A_1, A_2)$$
$$gcd(a_1^{(1)}, a_1^{(2)}) = gcd(B_1, B_2)$$
$$gcd(a_\infty^{(1)}, a_\infty^{(2)}) = gcd(C_1, C_2).$$

□

REMARK 3.4.2. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the corresponding dessins d'enfants as in the above theorem. Then:*

i) *The black (respectively, white) vertices of the fiber product dessin d'enfant $\mathcal{D}$ are those pairs (x, y) where $x \in \mathcal{D}_1$ and $y \in \mathcal{D}_2$ are black (respectively, white) vertices. (Immediate)*

ii) *The centers of faces are given by the pairs (x, y) where x is a center of a face of $\mathcal{D}_1$ and y is a center of a face of $\mathcal{D}_2$. (Immediate)*

iii) *The valence of a pair (either a black vertex or a white vertex or a center of face) is given as the product of the corresponding valences. (Immediate)*

iv) *An edge between a black vertex (x_1, x_2) and a white vertex (y_1, y_2) in $\mathcal{D}$ exists if and only if there exist an edge between x_1 and y_1 in $\mathcal{D}_1$ and an edge between x_2 and y_2 in $\mathcal{D}_2$.*

In fact:

If there is an edge between a black vertex (x_1, x_2) and a white vertex (y_1, y_2) in $\mathcal{D}$, it is because there is a lifting $\widetilde{\delta}$ of the path $\delta : I = [0,1] \to \widehat{\mathbb{C}}$ where $\delta(t) = t$ such that the following diagram is commutative

$$\begin{array}{ccc} & & S_{\mathcal{D}_1} \times_{(f_{\mathcal{D}_1}, f_{\mathcal{D}_2})} S_{\mathcal{D}_2} \\ & \overset{\widetilde{\delta}}{\nearrow} & \downarrow f_{\mathcal{D}} \\ I & \xrightarrow{\ \delta\ } & \widehat{\mathbb{C}} \end{array}$$

for $f_{\mathcal{D}} = f_{\mathcal{D}_1} \circ \pi_1 = f_{\mathcal{D}_2} \circ \pi_2$ with $\widetilde{\delta}(0) = (y_1, y_2)$ and $\widetilde{\delta}(1) = (x_1, x_2)$.

Now, for each $j = 1, 2$, if $\widetilde{\delta}_j := \pi_j \circ \widetilde{\delta}$ then

$$\begin{array}{cccc} \widetilde{\delta}_j : & I & \to & S_{\mathcal{D}_j} \\ & t & \mapsto & \pi_j(\widetilde{\delta}(t)) \end{array}$$

is continuous, and for each $t \in I$

$$f_{\mathcal{D}_j} \circ \widetilde{\delta}_j(t) = f_{\mathcal{D}_j}\big(\pi_j \circ \widetilde{\delta}(t)\big) = (f_{\mathcal{D}_j} \circ \pi_j) \circ \widetilde{\delta}(t) = f_{\mathcal{D}} \circ \widetilde{\delta}(t) = \delta(t) = t.$$

In this way, $\widetilde{\delta}_1$ and $\widetilde{\delta}_2$ are liftings of δ to $S_{\mathcal{D}_1}$ and $S_{\mathcal{D}_2}$ for the coverings $f_{\mathcal{D}_1}$ and $f_{\mathcal{D}_2}$ respectively

$$\begin{array}{ccc} & & S_{\mathcal{D}_j} \\ & \overset{\widetilde{\delta}_j}{\nearrow} & \downarrow f_{\mathcal{D}_j} \\ I & \xrightarrow{\ \delta\ } & \widehat{\mathbb{C}} \end{array} \qquad \text{for } j=1,2$$

such that for $j = 1, 2$

$$\widetilde{\delta}_j(0) = \pi_j \circ \widetilde{\delta}(0) = \pi_j(y_1, y_2) = y_j$$
$$\widetilde{\delta}_j(1) = \pi_j \circ \widetilde{\delta}(1) = \pi_j(x_1, x_2) = x_j;$$

that is, there are an edge between x_1 and y_1 in $\mathcal{D}_1$ and an edge between x_2 and y_2 in $\mathcal{D}_2$.

Inversely, if there are an edge between x_1 and y_1 in $\mathcal{D}_1$ and an edge between x_2 and y_2 in $\mathcal{D}_2$, it is because there are two liftings $\widetilde{\delta}_1$ and $\widetilde{\delta}_2$ of the path $\delta : I \to \widehat{\mathbb{C}}$

defined by $\delta(t) = t$ *to* $S_{\mathcal{D}_1}$ *and* $S_{\mathcal{D}_2}$ *of the coverings* $f_{\mathcal{D}_1}$ *and* $f_{\mathcal{D}_2}$ *respectively*

$$\begin{array}{ccc} & & S_{\mathcal{D}_j} \\ & \overset{\widetilde{\delta}_j}{\nearrow} & \downarrow f_{\mathcal{D}_j} \\ I & \xrightarrow{\delta} & \widehat{\mathbb{C}} \end{array} \qquad \textit{for } j=1,2$$

such that $\widetilde{\delta}_j(0) = y_j$ *and* $\widetilde{\delta}_j(1) = x_j$ $(j = 1, 2)$. *Thus,* $f_{\mathcal{D}_j} \circ \widetilde{\delta}_j(t) = \delta(t) = t$ *and therefore* $\widetilde{\delta}_j(t) \in f_{\mathcal{D}_j}^{-1}(t)$ $(j = 1, 2)$. *In particular,*

$$y_1 = \widetilde{\delta}_1(0) \in f_{\mathcal{D}_1}^{-1}(0) \qquad x_1 = \widetilde{\delta}_1(1) \in f_{\mathcal{D}_1}^{-1}(1)$$
$$y_2 = \widetilde{\delta}_2(0) \in f_{\mathcal{D}_2}^{-1}(0) \qquad x_2 = \widetilde{\delta}_2(1) \in f_{\mathcal{D}_2}^{-1}(1)$$

Now, note that the image of the continuous map

$$\begin{array}{rcl} I = [0,1] & \to & S_{\mathcal{D}_1} \times S_{\mathcal{D}_2} \\ t & \mapsto & \big(\widetilde{\delta}_1(t), \widetilde{\delta}_2(t)\big) \end{array}$$

is actually in $S_{\mathcal{D}_1} \times_{(f_{\mathcal{D}_1}, f_{\mathcal{D}_2})} S_{\mathcal{D}_2}$. *Thus*

$$\begin{array}{rrcl} \widetilde{\delta}: & I = [0,1] & \to & S_{\mathcal{D}_1} \times_{(f_{\mathcal{D}_1}, f_{\mathcal{D}_2})} S_{\mathcal{D}_2} \\ & t & \mapsto & \big(\widetilde{\delta}_1(t), \widetilde{\delta}_2(t)\big) \end{array}$$

is also continuous.

Besides, $\widetilde{\delta}(0) = \big(\widetilde{\delta}_1(0), \widetilde{\delta}_2(0)\big) = (y_1, y_2)$, $\widetilde{\delta}(1) = \big(\widetilde{\delta}_1(1), \widetilde{\delta}_2(1)\big) = (x_1, x_2)$ *and for* $j = 1, 2$

$$f_{\mathcal{D}} \circ \widetilde{\delta}(t) = f_{\mathcal{D}}\big(\widetilde{\delta}_1(t), \widetilde{\delta}_2(t)\big) = f_{\mathcal{D}_j} \circ \pi_j\big(\widetilde{\delta}_1(t), \widetilde{\delta}_2(t)\big) = f_{\mathcal{D}_j} \circ \widetilde{\delta}_j(t) = \delta(t) = t.$$

Therefore, $\widetilde{\delta}$ *is a lifting of* δ *to* $S_{\mathcal{D}_1} \times_{(f_{\mathcal{D}_1}, f_{\mathcal{D}_2})} S_{\mathcal{D}_2}$ *of* $f_{\mathcal{D}}$ *that joints the points* (y_1, y_2) *and* (x_1, x_2). *In other words, there is an edge between the vertices* (y_1, y_2) *and* (x_1, x_2).

v) *A similar situation for faces holds. Actually, two faces are adjacent in the fiber product dessins d'enfants* $\mathcal{D}$ *if and only if its projections to each coordinate are also adjacent.*

In fact, in order to see it we will divide the arguments in two parts:

First, note that a black vertex (x_1, x_2) *(respectively, white vertex* (y_1, y_2)*) is a vertex of the face with center* (z_1, z_2) *in* $\mathcal{D}$ *if and only if* x_1 *is a black vertex (respectively* y_1 *is a white vertex) of the face with center* z_1 *in* $\mathcal{D}_1$ *and* x_2 *is a black vertex (respectively* y_2 *is a white vertex) of the face with center* z_2 *in* $\mathcal{D}_2$. *This occurs because to be a black (respectively, white) vertex of a face of a dessin d'enfant it is necessary the existence of a lifting of the path* $[1, \infty] \to \widehat{\mathbb{C}}$ *defined by* $t \mapsto t$ *(respectively* $[0, \infty] \to \widehat{\mathbb{C}}$ *defined by* $t \mapsto t$*) with respect to its associate Belyi map, which joins that vertex with the center of that face. With this in mind, the statement can be proved, as we did in the previous item, replacing* (y_1, y_2)

(respectively (x_1, x_2)) by (z_1, z_2) and replacing the interval $[0, 1]$ by the interval $[1, \infty]$ (respectively $[0, \infty]$) [3].

Second, two faces of $\mathcal{D}$ whose centers are the points (z_1, z_2) and (w_1, w_2) have in common the edge which joins the black vertex (x_1, x_2) and the white vertex (y_1, y_2) if and only if both

a) *the faces in $\mathcal{D}_1$ whose centers are z_1 and w_1 have in common the edge which joins the black vertex x_1 the white vertex y_1; and*

b) *the faces in $\mathcal{D}_2$ whose centers are z_2 and w_2 have in common the edge which joins the black vertex x_1 and the white vertex y_1*

holds (this follows from the previous argument).

3.4.1. Fiber product of regular dessins d'enfants. Let $(X_1, \mathcal{D}_1)$ and $(X_2, \mathcal{D}_2)$ be two regular dessins d'enfants and let (S_1, β_1) and (S_2, β_2) be its corresponding associated Belyi pairs respectively. If we denote by $G_j = Aut(S_j, \beta_j) < Aut(S_j)$ the deck group of β_j for $j = 1, 2$, the direct product $G_1 \times G_2$ acts naturally on the fiber product $S_1 \times_{(\beta_1, \beta_2)} S_2$ by the rule

$$(g_1, g_2)(z_1, z_2) = \big(g_1(z_1), g_2(z_2)\big).$$

If, in addition, $S_1 \times_{(\beta_1, \beta_2)} S_2$ is irreducible, then $\beta = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2 : S_1 \times_{(\beta_1, \beta_2)} S_2 \to \widehat{\mathbb{C}}$ satisfies that:

1. $\beta(g_1, g_2) = \beta$ for every $(g_1, g_2) \in G_1 \times G_2$ and therefore $G_1 \times G_2 \leq Aut(S_1 \times_{(\beta_1, \beta_2)} S_2, \beta)$.

   In fact, $\beta\big((g_1, g_2)(z_1, z_2)\big) = \beta\big(g_1(z_1), g_2(z_2)\big) = \beta_1\big(g_1(z_1)\big) = \beta_1(z_1) = \beta(z_1, z_2)$ for all $(z_1, z_2) \in S_1 \times_{(\beta_1, \beta_2)} S_2$.
2. It is regular if β_1 and β_2 (equivalently $\mathcal{D}_1$ and $\mathcal{D}_2$) are regular. In this case, the corresponding deck group is the direct product of the deck groups of β_1 and β_2. Actually, $\beta(z, w) = \beta(x, y)$ if and only if $(z, w) = (g_1, g_2)(x, y)$ for some $(g_1, g_2) \in G_1 \times G_2$ since

$$\begin{aligned}
&\beta(z,w) = \beta(x, y)\\
&\quad \leftrightarrow \beta_1 \circ \pi_1(z, w) = \beta_1 \circ \pi_1(x, y) \quad \wedge \quad \beta_2 \circ \pi_2(z, w) = \beta_2 \circ \pi_2(x, y)\\
&\quad \leftrightarrow \beta_1(z) = \beta_1(x) \quad \wedge \quad \beta_2(w) = \beta_2(y)\\
&\quad \leftrightarrow z = g_1(x) \text{ for some } g_1 \in G_1 \quad \wedge \quad w = g_2(y) \text{ for some } g_2 \in G_2\\
&\quad \leftrightarrow (z, w) = \big(g_1(x), g_2(y)\big) = (g_1, g_2)(x, y) \text{ for some } (g_1, g_2) \in G_1 \times G_2.
\end{aligned}$$

[3] Note that the liftings of the paths $[1, \infty] \to \widehat{\mathbb{C}}$, $[0, \infty] \to \widehat{\mathbb{C}}$ and $[0, 1] \to \widehat{\mathbb{C}}$, all defined by $t \mapsto t$, are the edges of the triangles of the triangle decomposition associated to a dessin.

CHAPTER 4

The strong field of moduli of the fiber product of pairs

In this chapter we will develop the concept of the strong field of moduli of the fiber product of pairs. It is an algebraic invariant for the fiber product and it is different from the concept of the field of moduli. We will also prove that the strong field of moduli of the fiber product of pairs coincides with the smallest field containing the corresponding fields of moduli of the starting pairs.

4.1. The field of moduli of pairs.

DEFINITION 4.1.1. Let us consider a pair (R, η) where R is a compact Riemann surface (seen as an algebraic curve) and $\eta : R \to \widehat{\mathbb{C}}$ is a non-constant holomorphic map (seen as a rational map).

(1) If $\sigma \in Gal(\mathbb{C}/\mathbb{Q})$, we will say that (R^σ, η^σ) is *isomorphic* to (R, η) if there is an isomorphism $f_\sigma : R \to R^\sigma$ such that $\eta^\sigma \circ f_\sigma = \eta$ or, equivalently, if the following diagram is commutative

$$\begin{array}{ccc} R & \xrightarrow{\;f_\sigma\;} & R^\sigma \\ & {\scriptstyle\eta}\searrow \quad \swarrow{\scriptstyle\eta^\sigma} & \\ & \widehat{\mathbb{C}} & \end{array} \qquad ;$$

we will denote this by $(R^\sigma, \eta^\sigma) \equiv (R, \eta)$ (see sections 2.6 and 2.8).

(2) The *field of moduli* of the pair (R, η) is the fixed field of the group

$$G = \{\sigma \in Gal(\mathbb{C}/\mathbb{Q}) : (R^\sigma, \eta^\sigma) \equiv (R, \eta)\}.$$

It is a well known fact that this field is contained in any field of definition of (R, η) but it might not be itself a field of definition of (R, η) (see, for instance, **[E]**, **[Hid1]**, **[Hid2]**, **[H]**, **[K]** and **[Shi]**). As a consequence of Weil's descent theorem, the field of moduli is a field of definition if R has no non-trivial automorphisms (see **[Weil]**). Another result, due to Wolfart **[Wlf]**, asserts that if R is quasiplatonic (that is if $R/Aut(R)$ has genus zero and exactly three cone points) then the field of moduli is also a field of definition.

4.2. The strong field of moduli of the fiber product of pairs.

DEFINITION 4.2.1. Let us consider two pairs (S_1, β_1) and (S_2, β_2), where S_1 and S_2 are compact Riemann surfaces and $\beta_1 : S_1 \to \widehat{\mathbb{C}}$ and $\beta_2 : S_2 \to \widehat{\mathbb{C}}$ are non-constant holomorphic maps, and its corresponding fiber product $(S_1 \times_{(\beta_1,\beta_2)} S_2, \beta)$ where $\beta = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2$ (π_j is the corresponding projection on the j factor). If $\sigma \in Gal(\mathbb{C}/\mathbb{Q})$, we have then the pairs $(S_1^\sigma, \beta_1^\sigma)$, $(S_2^\sigma, \beta_2^\sigma)$ and their corresponding fiber product $(S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma, \beta^\sigma)$ [1] (see Sections 2.6 and 2.8).

(1) We will say that $(S_1 \times_{(\beta_1,\beta_2)} S_2, \beta)$ is *strongly isomorphic* to $(S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma, \beta^\sigma)$, denoting this by $(S_1 \times_{(\beta_1,\beta_2)} S_2, \beta) \equiv^s (S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma, \beta^\sigma)$, if there are isomorphisms

$$F : S_1 \times_{(\beta_1,\beta_2)} S_2 \to S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma, \quad F_1 : S_1 \to S_1^\sigma, \quad \text{and} \quad F_2 : S_2 \to S_2^\sigma$$

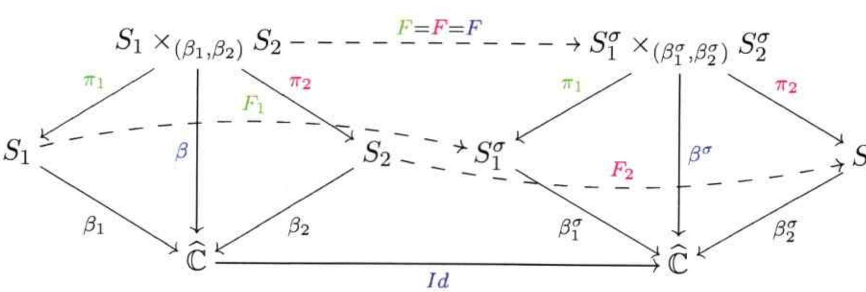
such that

$$\beta^\sigma \circ F = \beta$$

and

$$\pi_j \circ F = F_j \circ \pi_j \quad \text{for} \quad j = 1, 2 \quad \text{(abusing of language)};$$

or, equivalently, such that the following "green, blue and magenta diagrams" (all in one) are commutative



(2) The *strong field of moduli* of the fiber product $(S_1 \times_{(\beta_1,\beta_2)} S_2, \beta)$ is defined as the fixed field of the group

$$G = \left\{ \sigma \in Gal(\mathbb{C}/\mathbb{Q}) : (S_1 \times_{(\beta_1,\beta_2)} S_2, \beta) \equiv^s (S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma, \beta^\sigma) \right\}.$$

LEMMA 4.2.2. *If $\sigma \in Gal(\mathbb{C}/\mathbb{Q})$ is such that $(S_1 \times_{(\beta_1,\beta_2)} S_2, \beta)$ and $(S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma, \beta^\sigma)$ are strongly isomorphic then $(S_1, \beta_1) \equiv (S_1^\sigma, \beta_1^\sigma)$ and $(S_2, \beta_2) \equiv (S_2^\sigma, \beta_2^\sigma)$.*

[1] $(S_1 \times_{(\beta_1,\beta_2)} S_2, \beta)^\sigma = (S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma, \beta^\sigma)$

PROOF. If $z_1 \in S_1$, then there is at least one point $z_2 \in S_2$ such that $\beta_1(z_1) = \beta_2(z_2)$. Thus the pair (z_1, z_2) is in $S_1 \times_{(\beta_1,\beta_2)} S_2$ and

$$\begin{aligned}\beta_1^\sigma \circ F_1(z_1) &= (\beta_1^\sigma \circ F_1) \circ \pi_1(z_1, z_2) = \beta_1^\sigma \circ (F_1 \circ \pi_1)(z_1, z_2)\\ &= \beta_1^\sigma \circ (\pi_1 \circ F)(z_1, z_2) = (\beta_1^\sigma \circ \pi_1) \circ F(z_1, z_2)\\ &= \beta^\sigma \circ F(z_1, z_2) = \beta(z_1, z_2) = \beta_1(z_1).\end{aligned}$$

In other words, $\beta_1 = \beta_1^\sigma \circ F_1$. The other case can be proved analogously. □

REMARK 4.2.3. *If in our previous definition we exclude the existence of the isomorphisms F_1 and F_2, then we obtain the field of moduli of the fiber product. So, we see that the field of moduli of the fiber product is a subfield of its strong field of moduli. Nevertheless, the strong field of moduli contains of all the information in the construction of the fiber product.*

The following result states the relation between the strong field of moduli of the fiber product and the fields of moduli of the two starting pairs.

THEOREM 4.2.4. *The strong field of moduli of $(S_1 \times_{(\beta_1,\beta_2)} S_2, \beta)$ is equal to the smallest field containing the fields of moduli of the two starting pairs.*

PROOF. For $j = 1, 2$ let us consider the subgroup

$$G_j = \left\{\sigma \in Gal(\mathbb{C}/\mathbb{Q}) : (S_j^\sigma, \beta_j^\sigma) \equiv (S_j, \beta_j)\right\}$$

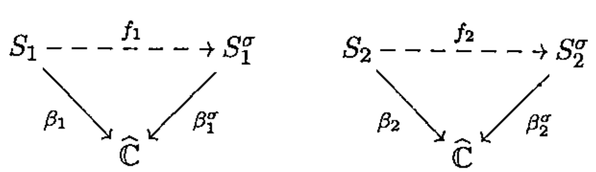
with its fixed field

$$\mathbb{K}_j = Fix(G_j)$$

(the field of moduli of the pair (S_j, β_j)) and let $\mathbb{K}$ be the smallest field containing $\mathbb{K}_1$ and $\mathbb{K}_2$. We claim first that the strong field of moduli $\mathbb{K}_s$ of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is contained in $\mathbb{K}$. To see it, if we take $\sigma \in G_1 \cap G_2$ then for each $j = 1, 2$ there is an isomorphism

$$f_j : S_j \to S_j^\sigma \text{ such that } \beta_j^\sigma \circ f_j = \beta_j$$

and thus the following diagrams



are commutative.

Now, considering the isomorphism

$$\begin{array}{rccc} f: & S_1 \times_{(\beta_1,\beta_2)} S_2 & \to & S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma \\ & (z_1, z_2) & \mapsto & \left(f_1(z_1), f_2(z_2)\right) \end{array}$$

we have that:

- f is well defined.

  In fact, for $(z_1, z_2) \in S_1 \times_{(\beta_1,\beta_2)} S_2$, $\big(f_1(z_1), f_2(z_2)\big) \in S_1^\sigma \times S_2^\sigma$ and

$$\begin{aligned}\beta_1^\sigma \circ \pi_1\big(f_1(z_1), f_2(z_2)\big) &= \beta_1^\sigma \circ f_1(z_1) = \beta_1(z_1) = \beta_2(z_2) = \beta_2^\sigma \circ f_2(z_2)\\ &= \beta_2^\sigma \circ \pi_2\big(f_1(z_1), f_2(z_2)\big),\end{aligned}$$

  which means $\big(f_1(z_1), f_2(z_2)\big) \in S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma$.

- f is bijective.

  In fact:

  - f is injective since

$$\begin{aligned}f(z_1, z_2) = f(w_1, w_2) &\leftrightarrow \big(f_1(z_1), f_2(z_2)\big) = \big(f_1(w_1), f_2(w_2)\big)\\ &\leftrightarrow f_1(z_1) = f_1(w_1) \quad \wedge \quad f_2(z_2) = f_2(w_2)\\ &\leftrightarrow z_1 = w_1 \quad \wedge \quad z_2 = w_2\\ &\leftrightarrow (z_1, z_2) = (w_1, w_2).\end{aligned}$$

  - f is epijective since both f_1 and f_2 are also epijective. In other words, for (w_1, w_2) in $S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma$ there are $z_1 \in S_1$ and $z_2 \in S_2$ such that $f_1(z_1) = w_1$ and $f_2(z_2) = w_2$ respectively and

$$\begin{aligned}\beta_1(z_1) &= \beta_1^\sigma \circ f_1(z_1) = \beta_1^\sigma(w_1) = \beta_2^\sigma(w_2)\\ &= \beta_2^\sigma \circ f_2(z_2) = \beta_2(z_2),\end{aligned}$$

    which means $(z_1, z_2) \in S_1 \times_{(\beta_1,\beta_2)} S_2$.

- f is continuous (immediate).
- $\pi_1 \circ f = f_1 \circ \pi_1$ and $\pi_2 \circ f = f_2 \circ \pi_2$ since for $(z_1, z_2) \in S_1 \times_{(\beta_1,\beta_2)} S_2$,

$$\pi_j \circ f(z_1, z_2) = \pi_j\big(f_1(z_1), f_2(z_2)\big) = f_j(z_j) = f_j \circ \pi_j(z_1, z_2) \quad (j = 1, 2).$$

- f is an isomorphism.

  Actually, note that if B denotes the collection of points in $\widehat{\mathbb{C}}$ which are either critical values of β_1 or critical values of β_2, for every point $(z_1, z_2) \in \big((S_1 \times_{(\beta_1,\beta_2)} S_2)\setminus\beta^{-1}(B)\big)$ we can choose open neighbourhoods $U \subseteq \big((S_1 \times_{(\beta_1,\beta_2)} S_2)\setminus\beta^{-1}(B)\big)$ and $V \subseteq \big((S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma) \setminus (\beta^\sigma)^{-1}(B)\big)$ of (z_1, z_2) and $f(z_1, z_2)$ respectively such that (see Section 2.8):
  - $U \subseteq f^{-1}(V)$.
  - (U, β) and (V, β^σ) are local charts for (z_1, z_2) and $f(z_1, z_2)$ in $(S_1 \times_{(\beta_1,\beta_2)} S_2)\setminus \beta^{-1}(B)$ and $(S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma) \setminus (\beta^\sigma)^{-1}(B)$ respectively.
  - $\pi_1|_U$, $\pi_1|_V$, $\beta_1|_{\pi_1(U)}$ and $\beta_1^\sigma|_{\pi_1(V)}$ are biholomorphisms.

  In this way, the local version $\beta^\sigma \circ f \circ \beta^{-1}$ of f is holomorphic since for every $z \in \beta(U)$

$$\beta^\sigma \circ f \circ \beta^{-1}(z) = (\beta_1^\sigma \circ \pi_1) \circ f \circ (\pi_1^{-1} \circ \beta_1^{-1})(z)$$

$$= \beta_1^\sigma \circ (\pi_1 \circ f \circ \pi_1^{-1}) \circ \beta_1^{-1}(z)$$
$$= \beta_1^\sigma \circ f_1 \circ \beta_1^{-1}(z)$$

and β_1^σ, f_1 and β_1^{-1} are holomorphic.

- $\beta = \beta^\sigma \circ f$.

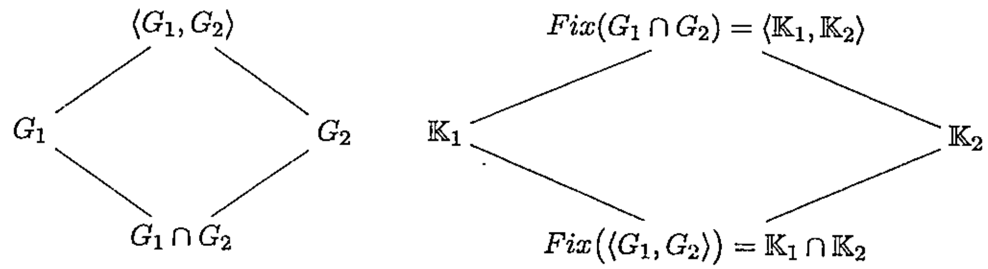
  In fact, for $(z_1, z_2) \in S_1 \times_{(\beta_1,\beta_2)} S_2$

$$\beta^\sigma \circ f(z_1,z_2) = \beta^\sigma\big(f_1(z_1), f_2(z_2)\big) = \beta_1^\sigma \circ \pi_1\big(f_1(z_1), f_2(z_2)\big) = \beta_1^\sigma\big(f_1(z_1)\big)$$
$$= \beta_1(z_1) = \beta_1 \circ \pi_1(z_1,z_2) = \beta(z_1,z_2).$$

In this way, if

$$G_s := \{\sigma \in Gal(\mathbb{C}/\mathbb{Q}) : (S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma, \beta^\sigma) \equiv^s (S_1 \times_{(\beta_1,\beta_2)} S_2, \beta)\}$$

then $G_1 \cap G_2 \leq G_s$ and therefore $Fix(G_1 \cap G_2) \geq Fix(G_s) = \mathbb{K}_s$. Even more, by Galois theory we have that $Fix(G_1 \cap G_2) = \langle \mathbb{K}_1, \mathbb{K}_2 \rangle = \mathbb{K}$



and so $\mathbb{K}_s$ is contained in $\mathbb{K} = \langle \mathbb{K}_1, \mathbb{K}_2 \rangle$ (the smallest field containing $\mathbb{K}_1$ and $\mathbb{K}_2$).

Conversely, we can see that $G_s \leq G_1$ and $G_s \leq G_2$ (see Remark 4.2.2) which implies that $\mathbb{K}_s \geq \mathbb{K}_1$ and $\mathbb{K}_s \geq \mathbb{K}_1$. Therefore, $\mathbb{K}_s \geq \mathbb{K}$. □

We next show an example where the strong field of moduli is an extension of degree two of the field of moduli for a fiber product (all details and calculations are made in Example 5.1.12).

EXAMPLE 4.2.5. Let (S_1, β_1) be a Belyi pair whose field of definition and field of moduli is $\mathbb{Q}(i)$ and suppose $Gal(\mathbb{Q}(i)/\mathbb{Q}) = \langle \tau \rangle$ where $\tau(i) = -i$. Let $\widetilde{\tau}$ be any extension to $\mathbb{C}$ of τ. By defining $S_2 = S_1^{\widetilde{\tau}}$ and $\beta_2 = \beta_1^{\widetilde{\tau}}$, one has that the field of moduli and the strong field of moduli of the corresponding fiber product are $\mathbb{Q}$ and $\mathbb{Q}(i)$ respectively.

CHAPTER 5

Examples.

In this chapter we provide a series of examples of fiber products. Example 5.1.1 shows the dependence not only on the starting compact Riemann surfaces but also on the starting coverings. Example 5.1.2 exhibit examples of fiber products which are non-connected when S_0 has positive genus. A fiber product which is connected but reducible and, even more, its irreducible components are not isomorphic, is exposed in Example 5.1.3. There the conditions of Theorem 3.3.3 are not hold. Example 5.1.4 shows that the conditions of Theorem 3.3.3 are not necessary whereas Example 5.1.5 is an example where those conditions are hold (this last example is also an example of a fiber product of two normal Belyi pairs). In Example 5.1.6 the upper bound in Lemma 3.3.4 is sharp while in Example 5.1.7 this bound is not sharp. Fiber product of cyclic gonal curves is exhibited in Example 5.1.8 (the upper bound in Lemma 3.3.4 is attained), of Fermat curves in Example 5.1.10 and of regular Belyi pairs in Example 5.1.11. Finally, the field of moduli and the strong field of moduli of a fiber product may differ as is exposed in Example 5.1.12.

5.1. Examples of fiber products

EXAMPLE 5.1.1. *Consider the curves* $S_1 = \{(x,y) \in \mathbb{C}^2 : y^2 = x(x-1)(x-2)\} \cup \{\infty\}$ *and* $S_2 = \{(x,y) \in \mathbb{C}^2 : y^2 = x(x-1)(x-4)\} \cup \{\infty\}$, *and consider the coverings* $\beta_1 : S_1 \to \widehat{\mathbb{C}}$ *and* $\beta_2, \widetilde{\beta_2} : S_2 \to \widehat{\mathbb{C}}$ *defined by* $\beta_1(x,y) = x$, $\beta_2(x,y) = x$ *and* $\widetilde{\beta_2}(x,y) = \frac{2x+1}{3x-1}$. *It can be seen that the fiber products* $S_1 \times_{(\beta_1,\beta_2)} S_2$ *and* $S_1 \times_{(\beta_1,\widetilde{\beta_2})} S_2$ *are connected and irreducible singular Riemann surfaces which, after its des-singularization and compactification, become into compact Riemann surfaces; the first one has genus two whereas the second one has genus five.*

Actually:

- For a complex number $\lambda \neq 0, 1$, the curve
$$\{(x,y) \in \mathbb{C}^2 : y^2 = x(x-1)(x-\lambda)\}$$
is a Riemann surface which becomes into a compact Riemann surface adding a single point, namely ∞ (see [GG] pg. 15-17). Therefore, S_1 and S_2 are Riemann surfaces whose compactifications are $\widehat{S_1} = S_1 \cup \{\infty\}$ and $\widehat{S_2} = S_2 \cup \{\infty\}$ respectively.

- The extension of β_1 to $\widehat{S_1}$ is a degree two normal covering which ramifies at the points $(0,0),(1,0),(2,0)$ and ∞ with

$$mult_{(0,0)}\beta_1 = mult_{(1,0)}\beta_1 = mult_{(2,0)}\beta_1 = mult_{\infty}\beta_1 = 2$$

and whose critical points are $\beta_1\big((0,0)\big) = 0$, $\beta_1\big((1,0)\big) = 1$, $\beta_1\big((2,0)\big) = 2$ and $\beta_1\big(\infty\big) = \infty$ (abusing of language).

- The extension of β_2 to $\widehat{S_2}$ is a degree two normal covering which ramifies at the points $(0,0),(1,0),(4,0)$ and ∞ with

$$mult_{(0,0)}\beta_2 = mult_{(1,0)}\beta_2 = mult_{(4,0)}\beta_2 = mult_{\infty}\beta_2 = 2$$

and whose critical points are $\beta_2\big((0,0)\big) = 0$, $\beta_2\big((1,0)\big) = 1$, $\beta_2\big((4,0)\big) = 4$ and $\beta_2\big(\infty\big) = \infty$ (abusing of language).

- Noting that $\widetilde{\beta_2} = T\circ\beta_2$, where T is the Möbius transformation defined by $T(z) = \frac{2x+1}{3x-1}$, it is deducible that the extension of $\widetilde{\beta_2}$ to $\widehat{S_1}$ is a degree two normal covering which ramifies at the points $(0,0),(1,0),(4,0)$ and ∞ with

$$mult_{(0,0)}\widetilde{\beta_2} = mult_{(1,0)}\widetilde{\beta_2} = mult_{(2,0)}\widetilde{\beta_2} = mult_{\infty}\widetilde{\beta_2} = 2$$

and whose critical points are $\widetilde{\beta_2}\big((0,0)\big) = -1$, $\widetilde{\beta_2}\big((1,0)\big) = 3/2$, $\widetilde{\beta_2}\big((4,0)\big) = 9/11$ and $\widetilde{\beta_2}\big(\infty\big) = 2/3$.

- $S_1 \times_{(\beta_1,\widetilde{\beta_2})} S_2$ has no singular points (see Section 3.1), is irreducible (see Theorem 3.3.3) and has genus 5.

  In fact, if $\widetilde{\beta} = \pi_1\circ\beta_1 = \pi_2\circ\widetilde{\beta_2}$ then $\widetilde{\beta}$ has multiplicity equals to two at 16 points in $S_1 \times_{(\beta_1,\widetilde{\beta_2})} S_2$ (corresponding to pre-images of $-1, 0, 2/3, 9/11, 1, 3/2, 4$ and ∞) and has multiplicity one at the other points. Therefore, by Riemann-Hurwitz formula $gen(S_1 \times_{(\beta_1,\widetilde{\beta_2})} S_2) = 5$.

- The only singular points of $S_1 \times_{(\beta_1,\beta_2)} S_2$ are: $\big((0,0),(0,0)\big)$, $\big((1,0),(1,0)\big)$ and (∞,∞). Around each of them the fiber product looks like 2 cones glued only at that point. In this way, in order to des-singularized and compactify $S_1 \times_{(\beta_1,\beta_2)} S_2$ it is necessary to remove all these singular points and fill the remaining punctures by adding 6 different points, namely $p_1,\ldots,p_6$, where p_1 and p_2 are pre-images of 0 by β, p_3 and p_4 are pre-images of 1 by β, p_5 and p_6 are pre-images of ∞ by β.
- $S_1 \times_{(\beta_1,\beta_2)} S_2$ is connected (see Section 3.3) and irreducible.

  In fact, by Lemma 3.3.4, this fiber product could have at most two irreducible components. Without loss of generality, if it had two irreducible components, say T_1 and T_2 (T_1 and T_2 should be isomorphic by Corollary 3.2.4), we can assume p_1, p_3 and p_5 are in T_1, and p_2, p_4 and p_6 are in T_2. In addition, if $\beta_2^{-1}(2) = \{q_1,q_2\}$ and $\beta_1^{-1}(4) = \{r_1,r_2\}$, we can also assume that $\big((2,0),q_1\big)$ and $\big(r_1,(4,0)\big)$ are in T_1, and $\big((2,0),q_2\big)$ and $\big(r_2,(4,0)\big)$ are in T_2. In this way,

$$mult_{\left((2,0),q_1\right)}\beta|_{T_1} = mult_{\left(r_1,(4,0)\right)}\beta|_{T_1} = 2$$

and

$$mult_{\big((2,0),q_2\big)}\beta|_{T_2} = mult_{\big(r_2,(4,0)\big)}\beta|_{T_2} = 2.$$

Thus, by the Riemann-Hurwitz formula $(2(gen(T_j)-1) = 2*2(0-1)+3+2)$ the genus of each irreducible component of $S_1 \times_{(\beta_1,\beta_2)} S_2$ should be $3/2$, which is a contradition.

- $gen(S_1 \times_{(\beta_1,\beta_2)} S_2) = 2$.

  In fact, if $\beta_2^{-1}(2) = \{q_1, q_2\}$ and $\beta_1^{-1}(4) = \{r_1, r_2\}$ this assertion follows noting that

$$mult_p\beta = \begin{cases} 2, & p = \big((2,0),q_1\big), \big((2,0),q_2\big), \big(r_1,(4,0)\big), \big(r_2,(4,0)\big) \\ 2, & p = p_1, \dots, p_6 \\ 1, & \text{in other cases} \end{cases}$$

  and applying the Riemann-Hurwitz formula.

EXAMPLE 5.1.2 (Examples of non-connected fiber products).

(1) **An example when S_0 has genus at least two.**

Let us consider two compact Riemann surfaces S_0 and S, both of genus at least two, and let $\pi : S \to S_0$ be an unbranched (not ramified) covering of degree $d \geq 2$. Take $S_1 = S_2 = S$ and $\beta_1 = \beta_2 = \pi$. In these conditions we claim that the fiber product $X = S_1 \times_{(\beta_1,\beta_2)} S_2$ is not connected.

Suppose X is connected. Because β does not ramify, neither π_1 nor π_2 ramify and $\deg(\pi_1) = deg(\pi_2) = d$. In this way, by the Riemann-Hurwitz formula,

$$gen(X) - 1 = d(gen(S) - 1).$$

Now, since $d \geq 2$ and $gen(S) - 1 > 0$, we have

$$gen(X) - 1 \geq 2(gen(S) - 1) > gen(S) - 1$$

and thus the genus of X is strictly bigger than of S.

On the other hand, taking $P_j : S \to S_j$ (for $j = 1, 2$) equal to the identity we see that $\beta_1 \circ P_1 = \beta_2 \circ P_2$ and, by the universal property of the fiber product, there is a holomorphic covering $T : S \to X$ such that $P_j = \pi_j \circ T$. In addition, since P_1 and P_2 are injective, T is also injective. Therefore $gen(X) = gen(S)$, providing a contradiction.

(2) **An example when S_0 has genus one.**

For $\lambda \in \mathbb{C} \setminus \{0, 1\}$ consider

$$S_1 = S_2 = S = \{[x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^3_{\mathbb{C}} : x_1^2 + x_2^2 + x_3^2 = 0,\ \lambda x_1^2 + x_2^2 + x_4^2 = 0\},$$
$$S_0 = \{[y_1 : y_2 : y_3 : y_4 : y_5] \in \mathbb{P}^4_{\mathbb{C}} : y_3^2 = y_1 y_2,\ y_5(y_1 + y_2) + y_4^2 = 0,\ \lambda y_1 + y_2 + y_5 = 0\}$$

and let

$$\begin{array}{rccc}\beta_1=\beta_2=\pi: & S & \to & S_0\\ & [x_1:x_2:x_3:x_4] & \mapsto & [x_1^2:x_2^2:x_1x_2:x_3x_4:x_4^2].\end{array}$$

We claim that, in this case, the fiber product X is the union of two disjoint Riemann surfaces A and B, each one isomorphic to S, where

$$A=\{([x_1:x_2:x_3:x_4],[x_1:x_2:x_3:x_4])\in(\mathbb{P}^3_{\mathbb{C}})^2:\\ x_1^2+x_2^2+x_3^2=0,\ \lambda x_1^2+x_2^2+x_4^2=0\}$$

and

$$B=\{([x_1:x_2:x_3:x_4],[x_1:x_2:-x_3:-x_4])\in(\mathbb{P}^3_{\mathbb{C}})^2:\\ x_1^2+x_2^2+x_3^2=0,\ \lambda x_1^2+x_2^2+x_4^2=0\}.$$

Actually:

- It may be proved, in a similar way as we did for S_0 (see Example 2.1.36), that S is a smooth complete intersection curve and thus a compact Riemnnn surface.
- π is well defined and it is a non constant holomorphic function (immediate).
- $\pi: S\to S_0$ is an unbranched two-fold covering whose deck group is the cyclic group generated by the involution $\tau([x_1:x_2:x_3:x_4])=[-x_1:-x_2:x_3:x_4]$ acting freely on S.

  In fact:
  - The map $\tau: S\to S$ with $\tau([x_1:x_2:x_3:x_4])=[-x_1:-x_2:x_3:x_4]$ is an automorphism of S. It is immediate to see that it is well defined and holomorphic. It only remains to prove that it is injective and this occurs because

    $$[-x_1:-x_2:x_3:x_4]=[-z_1:-z_2:z_3:z_4]$$

    $$\leftrightarrow\begin{cases}-x_1=-tz_1\\-x_2=-tz_2\\x_3=tz_3\\x_4=tz_4\end{cases}\text{for some } t\in\mathbb{C}\backslash\{0\}\ \leftrightarrow\begin{cases}x_1=tz_1\\x_2=tz_2\\x_3=tz_3\\x_4=tz_4\end{cases}\text{for some } t\in\mathbb{C}\backslash\{0\}$$

    $$\leftrightarrow [x_1:x_2:x_3:x_4]=[z_1:z_2:z_3:z_4].$$
  - $\langle\tau\rangle\simeq\mathbb{Z}_2$.
  - $\langle\tau\rangle$ acts freely on S.

    In order to see it, let's determine if τ fixes points or, equivalently, let's determine in what cases $[-x_1:-x_2:x_3:x_4]=[x_1:x_2:x_3:x_4]$ for $[x_1:x_2:x_3:x_4]\in S$.
    * If $x_4\neq 0$ or $x_3\neq 0$, $x_1=x_2=0$ but $[0:0:x_3:x_4]\notin S$. Therefore $x_3=x_4=0$.

* Since $[-x_1 : -x_2 : x_3 : x_4] = [x_1 : x_2 : -x_3 : -x_4]$, if $x_2 \neq 0$ or $x_1 \neq 0$ then $x_3 = x_4 = 0$. But $[x_1 : x_2 : 0 : 0] \notin S$. Therefore $x_1 = x_2 = 0$.

In this way, it is deducible that $[-x_1 : -x_2 : x_3 : x_4] = [x_1 : x_2 : x_3 : x_4]$ never occurs in S and thus $\langle\tau\rangle$ acts freely on S.

– $\pi \circ \tau = \pi$ since

$$\begin{aligned}\pi \circ \tau([x_1 : x_2 : x_3 : x_4]) &= \pi([-x_1 : -x_2 : x_3 : x_4]) \\ &= [x_1^2 : x_2^2 : x_1x_2 : x_3x_4 : x_4^2] \\ &= \pi([x_1 : x_2 : x_3 : x_4])\end{aligned}$$

for all $[x_1 : x_2 : x_3 : x_4]$ in S.

– π is an unbranched two-fold covering.

In fact, since $\tau \in Aut(S, \pi)$, $\langle\tau\rangle$ is an order two group acting freely on S and since any covering has a finite number of critical points and values, it is enough to find infinite points in S_0 with exactly two pre-images by π to prove that π is an unbranched two-fold covering.

Let $y_2 \neq 0, -1, -\lambda, i, \lambda i$ and let $\sqrt{y_2}$, $\sqrt{\lambda + y_2}$ and $\sqrt{1 + y_2}$ be fixed square roots of y_2, $\lambda + y_2$ and $1 + y_2$ respectively. It is not hard to see that the point

$$p_{y_2} = \left[1 : y_2 : \sqrt{y_2} : \sqrt{\lambda + y_2}\sqrt{1 + y_2} : -(\lambda + y_2)\right]$$

is in S_0. Now, if $q_{y_2} = [x_1 : x_2 : x_3 : x_4] \in S$ is a pre-image by π of p_{y_2} then

$$\begin{cases} x_1^2 = 1 \\ x_2^2 = y_2 \\ x_1x_2 = \sqrt{y_2} \\ x_4^2 = -(\lambda + y_2) \\ x_3x_4 = \sqrt{\lambda + y_2}\sqrt{1 + y_2} \end{cases} \rightarrow \begin{cases} x_1 = \pm 1 \\ x_2 = \pm\sqrt{y_2} \\ x_1x_2 \overset{(*)}{=} \sqrt{y_2} \\ x_4 = \pm i\sqrt{\lambda + y_2} \\ x_3x_4 \overset{(**)}{=} \sqrt{\lambda + y_2}\sqrt{1 + y_2}. \end{cases}$$

Since $q_{y_2} \in \mathbb{P}^3_{\mathbb{C}}$, we can assume $x_1 = 1$ and, by $(*)$ and $(**)$, q_{y_2} has exactly two different possibilities:

$$q_{y_2} = \left[1 : \sqrt{y_2} : \sqrt{1 + y_2}i : -\sqrt{\lambda + y_2}i\right]$$

or

$$q_{y_2} = \left[1 : \sqrt{y_2} : -\sqrt{1 + y_2}i : \sqrt{\lambda + y_2}i\right]$$

and both of them are in S.

- X is the union of two disjoint Riemann surfaces A and B, each one isomorphic to S, where $A = \{(x, x) : x \in S\}$ and $B = \{(x, \tau(x)) : x \in S\}$.

Actually:

$$
\begin{aligned}
X &= S_1 \times_{(\beta_1,\beta_2)} S_2 \\
&= \Big\{ \big([x_1 : x_2 : x_3 : x_4], [z_1 : z_2 : z_3 : z_4] \big) \in (\mathbb{P}^3_{\mathbb{C}})^2 : x_1^2 + x_2^2 + x_3^2 = 0, \\
&\qquad \lambda x_1^2 + x_2^2 + x_4^2 = 0,\ z_1^2 + z_2^2 + z_3^2 = 0,\ \lambda z_1^2 + z_2^2 + z_4^2 = 0, \\
&\qquad [x_1^2 : x_2^2 : x_1x_2 : x_3x_4 : x_4^2] \overset{(*)}{=} [z_1^2 : z_2^2 : z_1z_2 : z_3z_4 : z_4^2] \Big\}
\end{aligned}
$$

$\big($*note that by* $(*)$ $x_1 = 0 \leftrightarrow z_1 = 0\big)$

$$
\begin{aligned}
&= \Big\{ \big([0 : x_2 : x_3 : x_4], [0 : z_2 : z_3 : z_4] \big) \in (\mathbb{P}^3_{\mathbb{C}})^2 : x_2^2 + x_3^2 = 0,\ x_2^2 + x_4^2 = 0, \\
&\qquad z_2^2 + z_3^2 = 0,\ z_2^2 + z_4^2 = 0, [0 : x_2^2 : 0 : x_3x_4 : x_4^2] = [0 : z_2^2 : 0 : z_3z_4 : z_4^2] \Big\} \\
&\qquad \bigcup \\
&\quad \Big\{ \big([1 : x_2 : x_3 : x_4], [1 : z_2 : z_3 : z_4] \big) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1 + x_2^2 + x_3^2 = 0, \\
&\qquad \lambda + x_2^2 + x_4^2 = 0,\ 1 + z_2^2 + z_3^2 = 0,\ \lambda + z_2^2 + z_4^2 = 0, \\
&\qquad [1 : x_2^2 : x_2 : x_3x_4 : x_4^2] = [1 : z_2^2 : z_2 : z_3z_4 : z_4^2] \Big\} \\
&= \Big\{ \big([0 : x_2 : x_3 : x_4], [0 : z_2 : z_3 : z_4] \big) \in (\mathbb{P}^3_{\mathbb{C}})^2 : x_2^2 + x_3^2 = 0,\ x_3^2 = x_4^2, \\
&\qquad z_2^2 + z_3^2 = 0,\ z_3^2 = z_4^2, [0 : x_2^2 : 0 : x_3x_4 : x_4^2] \overset{(**)}{=} [0 : z_2^2 : 0 : z_3z_4 : z_4^2] \Big\} \\
&\qquad \bigcup \\
&\quad \Big\{ \big([1 : x_2 : x_3 : x_4], [1 : z_2 : z_3 : z_4] \big) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1 + x_2^2 + x_3^2 = 0, \\
&\qquad \lambda + x_2^2 + x_4^2 = 0,\ 1 + z_2^2 + z_3^2 = 0,\ \lambda + z_2^2 + z_4^2 = 0, \\
&\qquad x_2 = z_2,\ x_3x_4 = z_3z_4,\ x_4^2 = z_4^2 \Big\}
\end{aligned}
$$

$\big($*note that by* $(**)$ $x_2 = 0 \leftrightarrow z_2 = 0\big)$

$$
\begin{aligned}
&= \Big\{ \big([0 : 1 : x_3 : x_4], [0 : 1 : z_3 : z_4] \big) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1 + x_3^2 = 0,\ x_3 = \pm x_4, \\
&\qquad 1 + z_3^2 = 0,\ z_3 = \pm z_4,\ x_3x_4 = z_3z_4,\ x_4 = \pm z_4 \Big\} \\
&\qquad \bigcup \\
&\quad \Big\{ \big([1 : x_2 : x_3 : x_4], [1 : x_2 : z_3 : z_4] \big) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1 + x_2^2 + x_3^2 = 0, \\
&\qquad \lambda + x_2^2 + x_4^2 = 0,\ x_3^2 = z_3^2,\ x_4^2 = z_4^2,\ x_3x_4 = z_3z_4 \Big\} \\
&= \Big\{ \big([0 : 1 : x_3 : x_4], [0 : 1 : z_3 : z_4] \big) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1 + x_3^2 = 0,\ 1 + z_3^2 = 0, \\
&\qquad x_3 = \pm x_4,\ x_4 = \pm z_4,\ x_3 = \pm z_3,\ z_3 = \pm z_4,\ x_3x_4 = z_3z_4 \Big\} \\
&\qquad \bigcup \\
&\quad \Big\{ \big([1 : x_2 : x_3 : x_4], [1 : x_2 : z_3 : z_4] \big) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1 + x_2^2 + x_3^2 = 0, \\
&\qquad \lambda + x_2^2 + x_4^2 = 0,\ x_3 = \pm z_3,\ x_4 = \pm z_4,\ x_3x_4 = z_3z_4 \Big\}
\end{aligned}
$$

$$= \left\{ \left([0:1:x_3:x_4],[0:1:x_3:x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1+x_3^2=0,\ 1+x_4^2=0 \right\}$$
$$\bigcup$$
$$\left\{ \left([0:1:x_3:x_4],[0:1:-x_3:-x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1+x_3^2=0,\ 1+x_4^2=0 \right\}$$
$$\bigcup$$
$$\left\{ \left([1:x_2:x_3:x_4],[1:x_2:x_3:x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1+x_2^2+x_3^2=0, \lambda+x_2^2+x_4^2=0 \right\}$$
$$\bigcup$$
$$\left\{ \left([1:x_2:x_3:x_4],[1:x_2:-x_3:-x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1+x_2^2+x_3^2=0, \lambda+x_2^2+x_4^2=0 \right\}$$

$$= \left\{ \left([0:x_2:x_3:x_4],[0:1:x_3:x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : x_2^2+x_3^2=0,\ x_2^2+x_4^2=0 \right\}$$
$$\bigcup$$
$$\left\{ \left([0:x_2:x_3:x_4],[0:1:-x_3:-x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : x_2^2+x_3^2=0,\ x_2^2+x_4^2=0 \right\}$$
$$\bigcup$$
$$\left\{ \left([1:x_2:x_3:x_4],[1:x_2:x_3:x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1+x_2^2+x_3^2=0, \lambda+x_2^2+x_4^2=0 \right\}$$
$$\bigcup$$
$$\left\{ \left([1:x_2:x_3:x_4],[1:x_2:-x_3:-x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1+x_2^2+x_3^2=0, \lambda+x_2^2+x_4^2=0 \right\}$$

(*reordering*)

$$= \left\{ \left([0:x_2:x_3:x_4],[0:x_2:x_3:x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : x_2^2+x_3^2=0,\ x_2^2+x_4^2=0 \right\}$$
$$\bigcup$$
$$\left\{ \left([1:x_2:x_3:x_4],[1:x_2:x_3:x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1+x_2^2+x_3^2=0, \lambda+x_2^2+x_4^2=0 \right\}$$
$$\bigcup$$
$$\left\{ \left([0:x_2:x_3:x_4],[0:x_2:-x_3:-x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : x_2^2+x_3^2=0,\ x_2^2+x_4^2=0 \right\}$$
$$\bigcup$$
$$\left\{ \left([1:x_2:x_3:x_4],[1:x_2:-x_3:-x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : 1+x_2^2+x_3^2=0, \lambda+x_2^2+x_4^2=0 \right\}$$

(*rewriting*)

$$= \left\{ \left([x_1:x_2:x_3:x_4],[x_1:x_2:x_3:x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : x_1^2+x_2^2+x_3^2=0,\ \lambda x_1^2+x_2^2+x_4^2=0 \right\}$$
$$\bigcup$$
$$\left\{ \left([x_1:x_2:x_3:x_4],[x_1:x_2:-x_3:-x_4]\right) \in (\mathbb{P}^3_{\mathbb{C}})^2 : x_1^2+x_2^2+x_3^2=0,\ \lambda x_1^2+x_2^2+x_4^2=0 \right\}$$

(*in our above notation*)

$$= A \cup B.$$

EXAMPLE 5.1.3. *If* $S_1 = S_2 = S_0 = \widehat{\mathbb{C}}$ *and* $\beta_1(z) = \beta_2(z) = z(z^3+z^2+1)$, *then* $S_1 \times_{\beta_1,\beta_2} S_2$ *consists of two irreducible components which, after its des-singularization and compactification, become into compact Riemann surfaces; one of them is the Riemann* *sphere and the other is a genus one Riemann surface provided by the compactification of the curve*

$$1 + x^2 + x^3 + xy + x^2y + y^2 + xy^2 + y^3 = 0.$$

Actually:

- β_1 and β_2 are holomorphic (immediate).
- Because the genus of S_0 is zero, the fiber product is connected and

$$\begin{aligned} S_1^* \times_{(\beta_1,\beta_2)} S_2^* &\simeq \{(x,y) \in \mathbb{C}^2 : x(x^3+x^2+1) = y(y^3+y^2+1)\} \\ &\simeq \{(x,y) \in \mathbb{C}^2 : x^4 - y^4 + x^3 - y^3 + x - y = 0\} \\ &\simeq \{(x,y) \in \mathbb{C}^2 : (x-y)(1+x^2+x^3+xy+x^2y+y^2+xy^2+y^3) = 0\} \\ &\simeq \{(x,y) \in \mathbb{C}^2 : x = y\} \bigcup \\ &\quad \{(x,y) \in \mathbb{C}^2 : 1+x^2+x^3+xy+x^2y+y^2+xy^2+y^3 = 0\} \end{aligned}$$

- $C_1 := \{(x,y) \in \mathbb{C}^2 : x = y\}$ is irreducible and non-singular and therefore a Riemann surface.
- The compactification $\widehat{C_1}$ of C_1 is isomorphic to $\widehat{\mathbb{C}}$ and thus $gen(\widehat{C_1}) = 0$.

  In fact, the map $f : C_1 \to \mathbb{C}$ defined by $f(x,y) = x$ is, naturally, an injective holomorphic function and, by Proposition 2.3.13, it can be extended to $\widehat{C_1}$ adding only one point whose image by f is the point ∞ in $\widehat{\mathbb{C}}$.
- The curve $C_2 := \{(x,y) \in \mathbb{C}^2 : 1+x^2+x^3+xy+x^2y+y^2+xy^2+y^3 = 0\}$ is irreducible[1] and therefore its compactification $\widehat{C_2}$ also is irreducible. Besides, C_2 is non-singular and thus $\widehat{C_2}$ is a compact Riemann surface.

  In fact,

$$\frac{\partial}{\partial X}(1+X^2+X^3+XY+X^2Y+Y^2+XY^2+Y^3) = 2X+3X^2+Y+2XY+Y^2$$
$$\frac{\partial}{\partial Y}(1+X^2+X^3+XY+X^2Y+Y^2+XY^2+Y^3) = X+X^2+2Y+2XY+3Y^2.$$

  Thus, if $(x,y) \in \mathbb{C}^2$ is such that

$$2x+3x^2+y+2xy+y^2 \overset{(1)}{=} 0 \quad \text{and} \quad x+x^2+2y+2xy+3y^2 \overset{(2)}{=} 0,$$

  then (substracting)

$$x+2x^2-y-2y^2 = 0 \leftrightarrow (x-y)(1+2x+2y) = 0 \leftrightarrow x = y \vee y = -x - \frac{1}{2}$$

  and we obtain two possibilities: $x = y$ or $y = -x - 1/2$.

  - When $x = y$:

$$1+x^2+x^3+xy+x^2y+y^2+xy^2+y^3 = 1+3x^2+4x^3$$

    and its zeros are -1 and $\frac{1\pm\sqrt{15}i}{8}$.

[1] We used the program [**GAP**] to verify it.

Therefore the points $(-1,-1)$, $\left(\frac{1+\sqrt{15}i}{8}, \frac{1+\sqrt{15}i}{8}\right)$ and $\left(\frac{1-\sqrt{15}i}{8}, \frac{1-\sqrt{15}i}{8}\right)$ are possibly singular points of C_2. But, in this case, none of these points satisfies neither (1) nor (2) so none of them is singular.

- When $x \neq y$ and $y = -x - 1/2$:
  * From (1) we have that

$$
\begin{aligned}
2x + 3x^2 + (-x - 1/2) + 2x(-x - 1/2) + (-x - 1/2)^2 = 0 \\
\leftrightarrow x^2 - 1/2 + (4x^2 + 4x + 1)/4 = 0 \\
\leftrightarrow 8x^2 + 4x - 1 = 0 \\
\leftrightarrow x = \frac{-1 \pm \sqrt{3}}{4}
\end{aligned}
$$

  * From (2) we have that

$$
\begin{aligned}
x + x^2 + 2(-x - 1/2) + 2x(-x - 1/2) + 3(-x - 1/2)^2 = 0 \\
\leftrightarrow -x^2 - 2x - 1 + (12x^2 + 12x + 3)/4 = 0 \\
\leftrightarrow 8x^2 + 4x - 1 = 0 \\
\leftrightarrow x = \frac{-1 \pm \sqrt{3}}{4}.
\end{aligned}
$$

  Then from both (1) and (2) we obtain the points $\left(\frac{-1\pm\sqrt{3}}{4}, \frac{-1\mp\sqrt{3}}{4}\right)$. But none of these two point are in C_2.

Hence we conclude that C_2 is non-singular.

- In order to determine the singular points of the fiber product and its shape, we first have to analyze $\beta_1 = \beta_2$ from which we will know that β_1 has degree 4, ramifies only at the points -1, $\frac{1\pm\sqrt{15}i}{8}$ and ∞ with

$$mult_{-1}\beta_1 = mult_{\frac{1\pm\sqrt{15}i}{8}}\beta_1 = 2 \text{ and } mult_{\infty}\beta_1 = 4$$

and its critical values are $\beta_1(-1) = -1$, $\beta_1(\infty) = \infty$, $\beta_1\left(\frac{1+\sqrt{15}i}{8}\right) = \frac{37+50\sqrt{15}i}{8^3}$ and $\beta_1\left(\frac{1-\sqrt{15}i}{8}\right) = \frac{37-45\sqrt{15}i}{8^3}$.

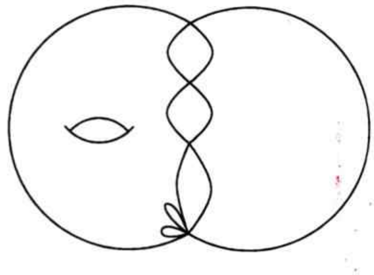
In fact:

- For a point $p \in \mathbb{C}$:
  $\beta_1'(z) = 4z^3 + 3z^2 + 1 = (z+1)(4z^2 - z + 1)$ and thus

$$\beta_1'(z) = 0 \leftrightarrow z = -1, \frac{1 \pm \sqrt{15}i}{8}.$$

  This means that $mult_{-1}\beta_1 = mult_{\frac{1\pm\sqrt{15}i}{8}}\beta_1 = 2$ and for the other points p in $\mathbb{C}$ $mult_p\beta_1 = 1$.
- For a point $p = \infty$:
  In a neighbourhood of ∞ the local form of β_1 is $\widetilde{\beta_1}(z) = \frac{z^4}{z^3+z+1}$ and thus $mult_{\infty}\beta_1 = 4$.
- Since $\beta_1^{-1}(\infty) = \{\infty\}$ we deduce that $deg(\beta_1) = 4$.

- From the previous item we conclude that the singular points of the fiber product are $(-1,-1)$, $\left(\frac{1+\sqrt{15}i}{8}, \frac{1+\sqrt{15}i}{8}\right)$, $\left(\frac{1-\sqrt{15}i}{8}, \frac{1-\sqrt{15}i}{8}\right)$ and (∞,∞). Locally, around each of the first three of them it looks like two glued cones (at that point) and around (∞,∞) it looks like four glued cones (at that point).
- $gen(\widehat{C_2}) = 1$ and the fiber product looks like



Actually, if we make

$$\beta_1^{-1}(\infty) = \{s = \infty\}$$
$$\beta_1^{-1}(-1) = \{p_1, p_2, p_3 = (-1,-1)\}$$
$$\beta_1^{-1}\left(\frac{37-45\sqrt{15}i}{8^3}\right) = \left\{q_1, q_2, q_3 = \left(\frac{1-\sqrt{15}i}{8}, \frac{1-\sqrt{15}i}{8}\right)\right\}$$
$$\beta_1^{-1}\left(\frac{37+50\sqrt{15}i}{8^3}\right) = \left\{r_1, r_2, r_3 = \left(\frac{1+\sqrt{15}i}{8}, \frac{1+\sqrt{15}i}{8}\right)\right\}$$

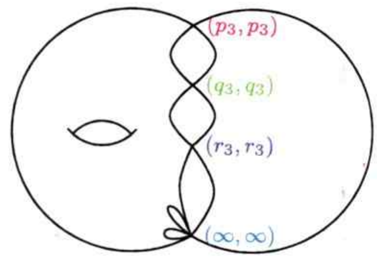
it can be seen that

$$\beta^{-1}(\infty) = \{(s,s)\}$$
$$\beta^{-1}(-1) = \{(p_k, p_l) : 1 \le k, l \le 3\}$$
$$\beta^{-1}\left(\frac{37-45\sqrt{15}i}{8^3}\right) = \{(q_k, q_l) : 1 \le k, l \le 3\}$$
$$\beta^{-1}\left(\frac{37+50\sqrt{15}i}{8^3}\right) = \{(r_k, r_l) : 1 \le k, l \le 3\}.$$

We also know that $\widehat{C_1} \cap \widehat{C_2} = \{(p_3,p_3),(q_3,q_3),(r_3,r_3),(s,s)\}$ and its elements are the only singular points of the fiber product since around the remaining twenty four points of

$$S_1^* \times_{(\beta_1,\beta_2)} S_2^* = \beta^{-1}\left(\left\{-1, \frac{37-45\sqrt{15}i}{8^3}, \frac{37+50\sqrt{15}i}{8^3}, \infty\right\}\right)$$

the fiber product looks like $\{(x,y) \in \mathbb{C}^2 : x - y = 0\}$ or $\{(x,y) \in \mathbb{C}^2 : x^2 - y = 0\}$ (around the point $(0,0)$ which is smooth). Besides, six points of $(S_1^* \times_{(\beta_1,\beta_2)} S_2^*) \setminus (\widehat{C_1} \cap \widehat{C_2})$ (those of the form (p_k,p_k), (q_k,q_k) and (r_k,r_k) where $k = 1,2$) are exclusively in $\widehat{C_1}$ and the other eighteen are exclusively in $\widehat{C_2}$. The fiber product then looks like



On the other hand, since 0 is not a critical value of β_1 then it has four pre-images by β_1, namely t_1, t_2, t_3 and t_4. Thus, 0 has sixteen pre-images by β in $S_1 \times_{(\beta_1,\beta_2)} S_2$ and four of them are exclusively in $\widehat{C_1}$ and the other twelve are exclusively in $\widehat{C_2}$. Therefore $\beta|_{\widehat{C_1}}$ has degree 4 and $\beta|_{\widehat{C_2}}$ has degree 12. Without loss of generality and abusing of language after removing the singular points and replacing them by new ones except for the point (s, s), in the process of des-singularization and compactification of both singular components of the fiber product, we can call those new ones in the same form as before in both $\widehat{C_1}$ and $\widehat{C_2}$. But, after removing the point (s, s) it is needed to add three different points $\infty_1, \infty_2, \infty_3$ to C_2 and another one ∞_4 to C_1 in order to cover the holes of the resting cones (which were meeting at (s, s) before the des-singularization of these singular components).

Now, from the above discussion of this item, it is deducible that:

$$mult_x\beta|_{\widehat{C_1}} = \begin{cases} 1, & x = (p_1, p_1), (p_2, p_2), (q_1, q_1), (q_2, q_2), (r_1, r_1), (r_2, r_2) \\ 2, & x = (p_3, p_3), (q_3, q_3), (r_3, r_3) \\ 4, & x = \infty_4 \\ 1, & \text{in other cases} \end{cases}$$

$$mult_x\beta|_{\widehat{C_2}} = \begin{cases} 1, & x = (p_1, p_2), (p_2, p_1), (q_1, q_2), (q_2, q_1), (r_1, r_2), (r_2, r_1) \\ 2, & x = (p_1, p_3), (p_2, p_3), (p_3, p_1), (p_3, p_2), (p_3, p_3), \\ & \quad (q_1, p_3), (q_2, q_3), (q_3, q_1), (q_3, q_2), (q_3, q_3), \\ & \quad (r_1, r_3), (r_2, r_3), (r_3, r_1), (r_3, r_2), (r_3, r_3) \\ 4, & x = \infty_1, \infty_2, \infty_3 \\ 1, & \text{in other cases} \end{cases}$$

and using the Riemann-Hurwitz formula we obtain, for example, the genus of $\widehat{C_2}$ as follows:

$$2(gen(\widehat{C_2}) - 1) = 2 \cdot (0 - 1) + 5 + 5 + 5 + 9$$

from where $gen(\widehat{C_2}) = 1$.

- The conditions of the Theorem 3.3.3 do not hold since $mcd(deg(\beta_1), deg(\beta_2)) = 4$ and

$$a_q^{(1)} = a_q^{(2)} = \begin{cases} mcm(1,1,2) = 1, & q = -1, \frac{37+50\sqrt{15}i}{8^3}, \frac{37-45\sqrt{15}i}{8^3} \\ 4, & q = \infty \\ 1, & \text{in other case} \end{cases}$$

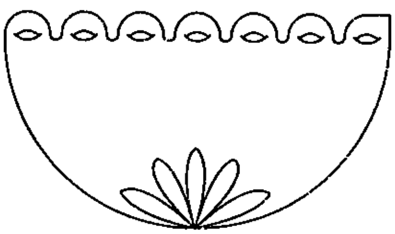
(using the same notation as in the theorem).

EXAMPLE 5.1.4. **An example where the conditions of Theorem 3.3.3 are not necessary.**

Let us consider $S_1 = S_2 = \widehat{\mathbb{C}}$, $\beta_1(z) = 4z^3(1-z^3)$ *and* $\beta_2(w) = -27w^4(w^2-1)/4$. *In this case, both conditions of Theorem 3.3.3 do not hold, but the fiber product*

$$S_1 \times_{(\beta_1,\beta_2)} S_2 = \{[z:w:t] \in \mathbb{P}^2_{\mathbb{C}} : 16z^3(t^3-z^3) - 27w^4(w^2-t^2) = 0\}$$

is an irreducible genus 7 *curve which has a singular point with a neighborhood being* 6 *cones glued at their centers and another singular point with a neighborhood being a disc.*



Actually:

- β_1 has degree 6, ramifies at the points $0, \frac{1}{\sqrt[3]{2}}, \frac{e^{2\pi i/3}}{\sqrt[3]{2}}, \frac{e^{4\pi i/3}}{\sqrt[3]{2}}$ and ∞ with

$$mult_z\beta_1 = \begin{cases} 2, & z = 1/\sqrt[3]{2},\ e^{2\pi i/3}/\sqrt[3]{2},\ e^{4\pi i/3}/\sqrt[3]{2} \\ 3, & z = 0 \\ 6, & z = \infty \end{cases}$$

and its critical values are

$$\beta_1(0) = 0,\ \beta_1(1/\sqrt[3]{2}) = \beta_1(e^{2\pi i/3}/\sqrt[3]{2}) = \beta_1(e^{4\pi i/3}/\sqrt[3]{2}) = 1 \text{ and } \beta_1(\infty) = \infty.$$

In fact:

– For a point $p \in \mathbb{C}$:

$$\beta_1'(z) = 24z^2\left(\frac{1}{\sqrt[3]{2}} - z\right)\left(\frac{e^{2\pi i/3}}{\sqrt[3]{2}} - z\right)\left(\frac{e^{4\pi i/3}}{\sqrt[3]{2}} - z\right)$$

thus $mult_0\beta_1 = 3$ and $mult_{\frac{1}{\sqrt[3]{2}}}\beta_1 = mult_{\frac{e^{2\pi i/3}}{\sqrt[3]{2}}}\beta_1 = mult_{\frac{e^{4\pi i/3}}{\sqrt[3]{2}}}\beta_1 = 2$.

- For the point $p = \infty$:
  In a neighbourhood of ∞ the local form of β_1 is $\widetilde{\beta_1}(z) = \frac{z^6}{4(z^3-1)}$ thus $mult_\infty\beta_1 = 6$.
- Since $\beta_1^{-1}(\infty) = \{\infty\}$ it is deducible that $deg(\beta_1) = 6$.
- $\beta_1^{-1}(0) = \{0, 1, e^{2\pi i/3}, e^{4\pi i/3}\}$, $\beta_1^{-1}(1) = \{1/\sqrt[3]{2},\ e^{2\pi i/3}/\sqrt[3]{2},\ e^{4\pi i/3}/\sqrt[3]{2}\}$ and $\beta_1^{-1}(\infty) = \{\infty\}$.

• β_2 also has degree 6, ramifies at the points $0, \pm\sqrt{\frac{2}{3}}$ and ∞ with

$$mult_0\beta_2 = 4, \quad mult_{\pm\sqrt{\frac{2}{3}}}\beta_2 = 2 \quad \text{and} \quad mult_\infty\beta_2 = 6$$

and its critical values are $\beta_2(0) = 0$, $\beta_2\left(\pm\sqrt{\frac{2}{3}}\right) = -1$ and $\beta_2(\infty) = \infty$.

In fact:
- For a point $p \in \mathbb{C}$:
  $\beta_2'(w) = \frac{81}{2}w^3\left(w - \sqrt{\frac{2}{3}}\right)\left(w + \sqrt{\frac{2}{3}}\right)$ thus $mult_0\beta_2 = 4$ and $mult_{\pm\sqrt{\frac{2}{3}}}\beta_2 = 2$.
- For the point $p = \infty$:
  In a neighbourhood of ∞ the local form of β_2 is $\widetilde{\beta_2}(w) = \frac{4w^6}{27(1-w^2)}$ thus $mult_\infty\beta_2 = 6$.
- Since $\beta_2^{-1}(\infty) = \{\infty\}$ it is deducible that $deg(\beta_2) = 6$.
- $\beta_2^{-1}(0) = \{0, \pm 1\}$, $\beta_2^{-1}(-1) = \left\{\pm\sqrt{\frac{2}{3}}, \pm\sqrt{\frac{1}{3}}i\right\}$ and $\beta_2^{-1}(\infty) = \{\infty\}$.

• Since $mcd\big(deg(\beta_1), deg(\beta_2)\big) = 6 \neq 1$ and $mcd\big(a_\infty^{(1)}, a_\infty^{(2)}\big) = mcd(6,6) = 6 \neq 1$ (using the same notations as in the theorem 3.3.3) both conditions of the theorem 3.3.3 do not hold.

• After des-singularized and compactify the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ we obtain a Riemann surface (its irreducibility can be analyzed using [**GAP**]) of genus 7 which is isomorphic to the des-singularization and compactification of the curve

$$\left\{(z,w) \in \mathbb{C}^2 : 16z^3(1-z^3) + 27w^4(w^2-1) = 0\right\}$$

In fact, it is necessary to note that the fiber product has only two singular points. One of them is the point (∞,∞) (corresponding to the preimage by β of ∞) with a neighborhood being 6 cones glued at this point and the other one is the point $(0,0)$ (one of the preimages of 0) with a neighborhood being a single cone. As in the Example 5.1.3, this is something that we have to keep in mind in order to calculate its genus. In other words, after removing the singular points of the fiber product (the des-singularization of the fiber product) in order to obtain a compact Riemann surface, it is necessary to add six points, namely $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6$ to cover the holes of these six cones which were meeting at (∞,∞) and to add another point which we will denote by $(0,0)$ (abusing of language) to cover the hole of the resting cone. Therefore, if we set $\rho := e^{2\pi i/3}$, $\beta_1^{-1}(-1) = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ and $\beta_2^{-1}(1) = \{q_1, q_2, q_3, q_4, q_5, q_6\}$ we obtain:

$$mult_x\beta = \begin{cases} 12, & x = (0,0) \\ 3, & x = (0,1), (0,-1) \\ 4, & x = (1,0), (\rho,0), (\rho^2,0) \\ 1, & x = (1,1), (1,-1), (\rho,1), (\rho,-1), (\rho^2,1), (\rho^2,-1) \\ & \left(p_k, \frac{i}{\sqrt{3}}\right), \left(p_k, -\frac{i}{\sqrt{3}}\right) \\ 2, & x = \left(p_k, \sqrt{\frac{2}{3}}\right), \left(p_k, -\sqrt{\frac{2}{3}}\right) \\ & \left(\frac{1}{\sqrt[3]{2}}, q_k\right), \left(\frac{\rho}{\sqrt[3]{2}}, q_k\right), \left(\frac{\rho^2}{\sqrt[3]{2}}, q_k\right) \\ 6, & x = \infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6 \\ 1, & \text{in other cases} \end{cases}$$

where $1 \leq k \leq 6$ and, by the Riemann-Hurwitz formula, if $g = gen(S_1 \times_{(\beta_1,\beta_2)} S_2)$,

$$2g - 2 = 2 \cdot 36(0-1) + 11 + 4 + 9 + 12 + 18 + 30$$

from which $g = 7$.

EXAMPLE 5.1.5. **An example where the necessary conditions of Theorem 3.3.3 hold.**

Consider the compact Riemann surfaces (see Example 2.1.32)

$$S_1 = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : y^3 - x^2 z + xz^2 = 0\}$$
$$S_2 = \{[x_1 : x_2 : x_3] \in \mathbb{P}^2_{\mathbb{C}} : x_1^2 + x_2^2 + x_3^2 = 0\}$$

and the functions $\beta_1 : S_1 \to \widehat{\mathbb{C}}$ *and* $\beta_2 : S_2 \to \widehat{\mathbb{C}}$ *defined by* $\beta_1([x : y : z]) = \frac{x}{z}$ *and* $\beta_2([x_1 : x_2 : x_3]) = -\left(\frac{x_2}{x_1}\right)^2$. *Here we have that* S_1 *is a genus one curve,* $S_2 \simeq \widehat{\mathbb{C}}$, *and* (S_1, β_1) *and* (S_2, β_2) *are regular Belyi pairs provided by the groups*

$$\langle [x : y : z] \mapsto [x : e^{2\pi i/3} y : z] \rangle \cong \mathbb{Z}_3$$

and

$$\langle [x_1 : x_2 : x_3] \mapsto [-x_1 : x_2 : x_3], [x_1 : x_2 : x_3] \mapsto [x_1 : -x_2 : x_3] \rangle \cong \mathbb{Z}_2^2$$

respectively. The fiber product

$$S_1 \times_{(\beta_1,\beta_2)} S_2$$
$$\|$$
$$\{([x : y : z], [x_1 : x_2 : x_3]) \in \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} : x_1^2 + x_2^2 + x_3^2 = 0,\ y^3 - x^2 z + xz^2 = 0,\ xx_1^2 = -zx_2^2\},$$

is irreducible (because the conditions of Theorem 3.3.3 hold) and it is isomorphic to the following irreducible curve of genus 4

$$R = \{[y : v : w : t] \in \mathbb{P}^3_{\mathbb{C}} : y^3 t - v^4 - v^2 t^2 = 0, t^2 + v^2 + w^2 = 0\} \subset \mathbb{P}^3_{\mathbb{C}}$$

which has the following automorphisms:

$$T\big([y : v : w : t]\big) = [e^{2\pi i/3} y : v : w : t]$$
$$A\big([y : v : w : t]\big) = [y : -v : w : t]$$
$$B\big([y : v : w : t]\big) = [y : v : -w : t]$$

such that $\langle T, A, B\rangle = \langle T\rangle \times \langle A, B\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_2^2$. *The map*

$$F : R \to \widehat{\mathbb{C}} : [y : v : w : t] \mapsto x = \frac{w}{iv - t}$$

provides a regular branched covering with deck group $\langle T\rangle$. *The branch values of* F *are given by the points* ∞, 0, $\pm i$, ± 1. *It follows that* R *can be also described by the cyclic* 3*-gonal curve*

$$y^3 = x(x^4 - 1),$$

and the group $\langle A, B\rangle$, *under the map* F, *corresponds in this model to the group*

$$\big\langle a(x, y) = (1/x, -y/x^2), b(x) = (-x, -y)\big\rangle.$$

Actually:

- S_1 and S_2 are compact Riemann surfaces (see Example 2.1.32).
- As we did for β_2 in Example 2.3.15, it can be proved that β_1 is a non-constant holomorphic covering of degree $deg(\beta_1) = 3$ which ramifies only at the points $[0 : 0 : 1]$, $[1 : 0 : 0]$ and $[1 : 0 : 1]$ with

  $$mult_{[0:0:1]}\beta_1 = mult_{[1:0:0]}\beta_1 = mult_{[1:0:1]}\beta_1 = 3$$

  and critical values

  $$\beta_1\big([0 : 0 : 1]\big) = 0,\ \beta_1\big([1 : 0 : 0]\big) = \infty \text{ and } \beta_1\big([1 : 0 : 1]\big) = 1.$$

  From this item it is deducible that β_1 is a Belyi map for S_1.
- It is a direct consequence of the application of the Riemann-Hurwitz formula that $gen(S_1) = 1$ and $gen(S_2) = 0$.
- Similarly as we proved that (S_2, β_2) is a regular Belyi pair (see Example 2.3.19) provided by the group $\langle T_1, T_2\rangle \cong \mathbb{Z}_2^2$ where $T_1, T_2 : S_2 \to S_2$ are defined by

  $$T_1\big([x_1 : x_2 : x_3]\big) = [-x_1 : x_2 : x_3] \quad T_2\big([x_1 : x_2 : x_3]\big) = [x_1 : -x_2 : x_3],$$

  it can be proved that (S_1, β_1) is a regular Belyi pair provided by the group $\langle T\rangle \cong \mathbb{Z}_3$ where $T : S_1 \to S_1$ is defined by $T\big([x : y : z]\big) = [x : e^{2\pi i/3} : z]$.
- $S_1 \times_{(\beta_1, \beta_2)} S_2$ is irreducible (both conditions of Theorem 3.3.3 are hold) and is non-singular.

- $S_1 \times_{(\beta_1,\beta_2)} S_2 \simeq R := \{[y:v:w:t] \in \mathbb{P}^3_{\mathbb{C}} : y^3 t - v^4 - v^2 t^2 = 0, t^2 + v^2 + w^2 = 0\}$.

  In fact:

  – Since the following diagram

$$\begin{array}{ccc} U_2 \cap S_1 = \{[x:y:1] \in \mathbb{P}^2_{\mathbb{C}} : y^3 - x^2 + x = 0\} & \overset{f}{\longleftrightarrow} & \{(x,y) \in \mathbb{C}^2 : y^3 - x^2 + x = 0\} \\ \downarrow \beta_1 & & \downarrow \\ \widehat{\mathbb{C}} & \overset{id_{\widehat{\mathbb{C}}}}{\longleftrightarrow} & \widehat{\mathbb{C}} \end{array}$$

  where

$$\begin{array}{ccc} [x:y:1] & \overset{f}{\longleftrightarrow} & (x,y) \\ \beta_1 \downarrow & & \downarrow \\ x & \overset{id_{\widehat{\mathbb{C}}}}{\longleftrightarrow} & x \end{array}$$

  is commutative, we can argue that f is an isomorphism and thus

$$\big(U_2 \cap S_1, \beta_1\big) \equiv \big(\{(x,y) \in \mathbb{C}^2 : y^3 - x^2 + x = 0\}, (x,y) \mapsto x\big)$$

  – Since the following diagram

$$\begin{array}{ccc} U_0 \cap S_2 = \{[1:x_2:x_3] \in \mathbb{P}^2_{\mathbb{C}} : 1 + x_2^2 + x_3^2 = 0\} & \overset{g}{\longleftrightarrow} & \{(x_2,x_3) \in \mathbb{C}^2 : 1 + x_2^2 + x_3^2 = 0\} \\ \downarrow \beta_2 & & \downarrow \\ \widehat{\mathbb{C}} & \overset{id_{\widehat{\mathbb{C}}}}{\longleftrightarrow} & \widehat{\mathbb{C}} \end{array}$$

  where

$$\begin{array}{ccc} [1:x_2:x_3] & \overset{g}{\longleftrightarrow} & (x_2,x_3) \\ \beta_2 \downarrow & & \downarrow \\ -(x_2)^2 & \overset{id_{\widehat{\mathbb{C}}}}{\longleftrightarrow} & -(x_2)^2 \end{array}$$

  is commutative, we can argue that g is an isomorphism and thus

$$\begin{aligned} \big(U_0 \cap S_2, \beta_2\big) &\equiv \Big(\{(x_2,x_3) \in \mathbb{C}^2 : 1 + x_2^2 + x_3^2 = 0\}, (x_2,x_3) \mapsto -(x_2)^2\Big) \\ &\equiv \Big(\{(v,w) \in \mathbb{C}^2 : 1 + v^2 + w^2 = 0\}, (v,w) \mapsto -v^2\Big) \end{aligned}$$

  – From the above we obtain that $S_1 \times_{(\beta_1,\beta_2)} S_2$ minus a finite set of points is isomorphic to

$$\begin{aligned} &\{(x,y,v,w) \in \mathbb{C}^4 : y^3 - x^2 + x = 0,\ 1 + v^2 + w^2 = 0,\ x = -v^2\} \\ &= \{(-v^2,y,v,w) \in \mathbb{C}^4 : y^3 - v^4 - v^2 = 0,\ 1 + v^2 + w^2 = 0\} \leq \mathbb{C}^4 \end{aligned}$$

  and can be seen as

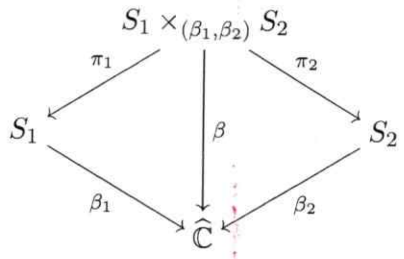
$$\{(y,v,w) \in \mathbb{C}^3 : y^3 - v^4 - v^2 = 0,\ 1 + v^2 + w^2 = 0\} \leq \mathbb{C}^3,$$

whose projective version is

$$\{[y:v:w:t] \in \mathbb{P}^3_{\mathbb{C}} : y^3t - v^4 - v^2t^2 = 0,\ t^2 + v^2 + w^2 = 0\} \leq \mathbb{P}^3_{\mathbb{C}}.$$

- $gen(S_1 \times_{(\beta_1,\beta_2)} S_2) = gen(R) = 4$.

In fact, from the above items we can conclude that the following commutative diagram



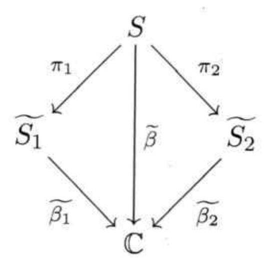
with

$$\beta_1^{-1}(0) = \{[0:0:1]\} \qquad \beta_2^{-1}(0) = \{[1:0:\pm i]\}$$
$$\beta_1^{-1}(1) = \{[1:0:1]\} \qquad \beta_2^{-1}(1) = \{[1:\pm i:0]\}$$
$$\beta_1^{-1}(\infty) = \{[1:0:0]\} \qquad \beta_2^{-1}(\infty) = \{[0:1:\pm i]\}$$

and

$$mult_{[0:0:1]}\beta_1 = mult_{[1:0:1]}\beta_1 = mult_{[1:0:0]}\beta_1 = 3$$
$$mult_{[1:0:\pm i]}\beta_2 = mult_{[1:\pm i:0]}\beta_2 = mult_{[0:1:\pm i]}\beta_2 = 2$$

is equivalent to the following commutative diagram



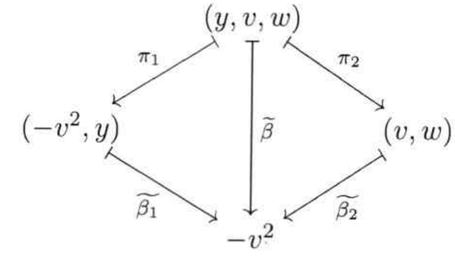
where

$$\widetilde{S_1} := \{(x,y) \in \mathbb{C}^2 : y^3 - x^2 + x = 0\}, \quad \widetilde{S_2} := \{(v,w) \in \mathbb{C}^2 : 1 + v^2 + w^2 = 0\},$$

$$\widetilde{\beta_1}(x,y) = x, \quad \widetilde{\beta_2}(v,w) = -v^2,$$

$$S = \{(x,y,v,w) \in \mathbb{C}^4 : y^3 - x^2 + x = 0, 1 + v^2 + w^2 = 0, x = -v^2\}$$
$$\simeq \{(y,v,w) \in \mathbb{C}^3 : y^3 - v^4 - v^2 = 0, 1 + v^2 + w^2 = 0\}$$

is the corresponding fiber product of $(\widetilde{S_1}, \widetilde{\beta_1})$ and $(\widetilde{S_2}, \widetilde{\beta_2})$ and



with

$$\widetilde{\beta_1}^{-1}(0) = \{(0,0)\} \qquad \widetilde{\beta_2}^{-1}(0) = \{(0, \pm i)\}$$
$$\widetilde{\beta_1}^{-1}(1) = \{(1,0)\} \qquad \widetilde{\beta_2}^{-1}(1) = \{(\pm i, 0)\}$$
$$\widetilde{\beta_1}^{-1}(\infty) = \{\infty\} \qquad \widetilde{\beta_2}^{-1}(\infty) = \{\infty_1, \infty_2\}$$

and

$$mult_{(0,0)}\widetilde{\beta_1} = mult_{(1,0)}\widetilde{\beta_1} = mult_{\infty}\widetilde{\beta_1} = 3$$
$$mult_{(0,\pm i)}\widetilde{\beta_2} = mult_{(\pm i,0)}\widetilde{\beta_2} = mult_{\infty_1,\infty_2}\widetilde{\beta_2} = 2.$$

Note that the points ∞ and ∞_1, ∞_2 were added to compactify $\widetilde{S_1}$ and $\widetilde{S_2}$ and extend $\widetilde{\beta_1}$ and $\widetilde{\beta_2}$ respectively. In this way $\widetilde{\beta}$, and therefore β, has six critical points $\big((0,0),(0,\pm i)\big)$, $\big((1,0),(\pm i,0)\big)$, (∞, ∞_1) and (∞, ∞_2), each one with multiplicity 6. By Riemann-Hurwitz formula,

$$2gen(S_1 \times_{(\beta_1,\beta_2)} S_2) - 2 = 2 \cdot 3 \cdot 4(0-1) + 5 + 5 + 5 + 5 + 5 + 5$$

from which the assertion is true.

- R has singularities only at the points $[0:1:0:\pm i]$ and $[1:0:0:0]$ (these are the only points where the 2×4 matrix of partial derivatives does not have maximal rank).
- R has the following automorphisms:

$$T\big([y:v:w:t]\big) = [e^{2\pi i/3}y : v : w : t]$$
$$A\big([y:v:w:t]\big) = [y : -v : w : t]$$
$$B\big([y:v:w:t]\big) = [y : v : -w : t]$$

such that $\langle T, A, B\rangle = \langle T\rangle \times \langle A, B\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_2^2$.

In fact:

- It is immediate that T, A and B are well defined, holomorphic and non-constant. Besides, it is not hard to prove that they are also injective.

– $T, A, B \in Aut(R, \beta)$:

Remember that $R = \{[y:v:w:t] \in \mathbb{P}^3_{\mathbb{C}} : y^3t - v^4 - v^2t^2 = 0, t^2+v^2+w^2 = 0\}$ and note that $(S_1 \times_{(\beta_1,\beta_2)} S_2, \beta)$ is equivalent to $\left(R, [y:v:w:t] \mapsto -(\frac{v}{t})^2\right)$. Thus, abusing of the notation, we will have

$$\beta \circ T([y:v:w:t]) = \beta([e^{2\pi i/3}y:v:w:t]) = -\left(\frac{v}{t}\right)^2 = \beta([y:v:w:t])$$
$$\beta \circ A([y:v:w:t]) = \beta([y:-v:w:t]) = -\left(\frac{v}{t}\right)^2 = \beta([y:v:w:t])$$
$$\beta \circ B([y:v:w:t]) = \beta([y:v:-w:t]) = -\left(\frac{v}{t}\right)^2 = \beta([y:v:w:t])$$

with $\langle T\rangle \cong \mathbb{Z}_3$, $\langle A\rangle \cong \langle B\rangle \cong \mathbb{Z}_2^2$ and

$$A \circ B([y:v:w:t]) = [y:-v:-w:t] = B \circ A$$
$$T \circ A([y:v:w:t]) = [e^{2\pi i/3}y:-v:w:t] = A \circ T$$
$$T \circ B([y:v:w:t]) = [e^{2\pi i/3}y:v:-w:t] = B \circ T.$$

- The map

$$F : R \to \widehat{\mathbb{C}} : [y:v:w:t] \mapsto \frac{w}{iv-t}$$

provides a regular branched covering with deck group $\langle T\rangle$ and branching values ∞, 0, $\pm i$, ± 1.

In fact:

– The covering F can be seeing as the composition of the coverings F_1 and F_2 where $F_1 : R \to Z := \{[v:w:t] \in \mathbb{P}^2_{\mathbb{C}} : v^2+w^2+t^2 = 0\}$ and $F_2 : Z \to \widehat{\mathbb{C}}$ are defined as $F_1([y:v:w:t]) = [v:w:t]$ and $F_2([v:w:t]) = \frac{w}{iv-t}$.

– Note that, in our notation, F_1 is our starting projection π_2 which we know is a degree three covering and, by done in the above items, it has the only three branching values $[1:0:\pm i]$, $[1:\pm i:0]$ and $[0:1:\pm i]$. Besides, it is immediate that $T \in Aut(R, F_1)$.

– The affine version $\widetilde{F_2}$ of F_2 is the function

$$\begin{array}{rccc} \widetilde{F_2}: & \{(v,w) \in \mathbb{C}^2 : v^2+w^2+1 = 0\} & \to & \widehat{\mathbb{C}} \setminus \{\pm 1\} \\ & (v,w) & \mapsto & w/(iv-1) \end{array}$$

which is obviously holomorphic.

– F_2 is actually an isomorphism.

  * Let a be any complex number different from ± 1. In order to find the possible pre-images $(v,w) \in \{(v,w) \in \mathbb{C}^2 : v^2+w^2+1 = 0\}$ of a by

$\widetilde{F_2}$, it is necessary to consider $v \neq -i$. This way, if $\widetilde{F_2}\big((v,w)\big) = a$ then $w = a(iv-1)$ and thus

$$\begin{aligned} v^2+w^2+1=0 &\leftrightarrow v^2+a^2(iv-1)^2+1=0 \\ &\leftrightarrow (1-a^2)v^2-2a^2iv+(1+a^2)=0 \end{aligned}$$

from which $v = \frac{i(1+a^2)}{1-a^2}$ is the only possible value.

  * The only possible pre-image of ∞ by $\widetilde{F_2}$ is $(-i,0)$.
- The branching values of F are ∞, 0, ± 1 and $\pm i$ since $F_2\big([1:0:i]\big) = \infty$, $F_2\big([1:0:-i]\big) = 0$, $F_2\big([1:\pm i:0]\big) = \pm 1$ and $F_2\big([0:1:\pm i]\big) = \pm i$.

- R can be also described by the cyclic 3-gonal curve

$$y^3 = x(x^4-1).$$

In fact:

- The curve $E := \{(x,y) \in \mathbb{C}^2 : y^3 = x(x^4-1)\}$ becomes into a compact Riemann surface $\widehat{E}$ adding a single point, namely (∞,∞).

  Actually, E can be seen as the fiber product of the pairs $(\widehat{\mathbb{C}}, f_1)$ and $(\widehat{\mathbb{C}}, f_2)$ where $f_1, f_2 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are defined by $f_1(y) = y^3$ and $f_2(x) = x(x^4-1)$ for which we know that:

  * f_1 is a degree 3 covering which ramifies only at the points 0 and ∞ with

$$mult_0 f_1 = mult_\infty f_1 = 3$$

  and whose critical values are $f_1(0) = 0$ and $f_1(\infty) = \infty$.

  * f_2 is a degree 5 covering which ramifies only at the points $\frac{\sqrt{2}}{2\sqrt[4]{3}}(1 \pm i)$, $\frac{\sqrt{2}}{2\sqrt[4]{3}}(-1 \pm i)$ and ∞ with

$$mult_{\frac{\sqrt{2}}{2\sqrt[4]{3}}(1\pm i)} f_2 = mult_{\frac{\sqrt{2}}{2\sqrt[4]{3}}(-1\pm i)} f_2 = 2, \quad mult_\infty = 5$$

  and whose critical values are $f_2(\infty) = \infty$, $f_2\Big(\frac{\sqrt{2}}{2\sqrt[4]{3}}(1\pm i)\Big) = -\frac{2\sqrt{2}}{3\sqrt[4]{3}}(1\pm i)$ and $f_2\Big(\frac{\sqrt{2}}{2\sqrt[4]{3}}(-1\pm i)\Big) = \frac{2\sqrt{2}}{3\sqrt[4]{3}}(1 \mp i)$.

  The conditions of Theorem 3.3.3 for the pairs $(\widehat{\mathbb{C}}, f_1)$ and $(\widehat{\mathbb{C}}, f_2)$ are satisfied and thus $\widehat{\mathbb{C}} \times_{(f_1,f_2)} \widehat{\mathbb{C}}$ is non-singular. Besides, this fiber product around the only singular point (∞,∞) looks like a single cone. Thus, in order to compactify E we need to remove this singular point of $\widehat{\mathbb{C}} \times_{(f_1,f_2)} \widehat{\mathbb{C}}$ and add a single point which, abusing of language, we will also denote by (∞,∞). Therefore,

$$\widehat{E} = E \cup \{(\infty,\infty)\}$$

  is a smooth compactification of E.

- The projection $\Pi_1 : \widehat{E} \to \widehat{\mathbb{C}}$ defined by $\Pi_1(x,y) = x$ is a regular covering of degree 3 with deck group $\langle (x,y) \mapsto (x, e^{2\pi i/3}) \rangle \cong \mathbb{Z}_3$ and whose critical values are 0, ± 1, $\pm i$ and ∞ (see Remark 2.2.11).
- R is isomorphic to E since the coverings F and Π_1 have the same degree, critical values, and its monodromies $Mon(F)$ and $Mon(\Pi_1)$ are conjugate in Σ_3. Actually $Mon(F) \cong \mathbb{Z}_3 \cong Mon(\Pi_1)$ (see the last theorem of Section 2.5).

EXAMPLE 5.1.6. **An example where the upper bound in Lemma 3.3.4 is attained.**

Let us consider the compact Riemann surfaces

$$S_1 = \{[x:y:z] \in \mathbb{P}^2_{\mathbb{C}} : y^2 - x^2 + xz = 0\} \simeq \widehat{\mathbb{C}},$$
$$S_2 = \{[x_1:x_2:x_3] \in \mathbb{P}^2_{\mathbb{C}} : x_1^2 + x_2^2 + x_3^2 = 0\} \simeq \widehat{\mathbb{C}}$$

and the coverings $\beta_1 : S_1 \to \widehat{\mathbb{C}}$ and $\beta_2 : S_2 \to \widehat{\mathbb{C}}$ defined by

$$\beta_1([x:y:z]) = \frac{x}{z} \text{ and } \beta_2([x_1:x_2:x_3]) = -\left(\frac{x_2}{x_1}\right)^2 \text{ (see Example 2.2.19).}$$

In this example both conditions of Theorem 3.3.3 do not hold, (S_1,β_1) and (S_2,β_2) are regular Belyi pairs provided by the groups $\langle [x:y:z] \mapsto [x:-y:z] \rangle \cong \mathbb{Z}_2$ and $\langle [x_1:x_2:x_3] \mapsto [-x_1:x_2:x_3], [x_1:x_2:x_3] \mapsto [x_1:-x_2:x_3] \rangle \cong \mathbb{Z}_2^2$ respectively (see Example 2.3.19). The fiber product is given by

$$S_1 \times_{(\beta_1,\beta_2)} S_2$$
$$||$$
$$\{([x:y:z],[x_1:x_2:x_3]) : x_1^2 + x_2^2 + x_3^2 = 0,\ y^2 - x^2 + xz = 0,\ xx_1^2 = -zx_2^2\} \subset \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}.$$

The above fiber product is reducible and consists of two irreducible components, respectively isomorphic to the following ones (both are isomorphic under the isomorphism $L([u:v:w:t]) = [u:v:-w:t]$):

$$R_1 = \{[u:v:w:t] \in \mathbb{P}^3_{\mathbb{C}} : uw^2 + ut^2 + iwt^2 = 0, v^2 + w^2 + t^2 = 0\} \simeq \widehat{\mathbb{C}}$$

$$R_2 = \{[u:v:w:t] \in \mathbb{P}^3_{\mathbb{C}} : uw^2 + ut^2 - iwt^2 = 0, v^2 + w^2 + t^2 = 0\} \simeq \widehat{\mathbb{C}}.$$

Actually:

- S_1 is a compact Riemann surface since the homogeneous polynomial $Y^2 - X^2 + XZ$ in $\mathbb{C}[X,Y,Z]$ is no singular. Besides, similarly as we did for the previous examples, it can be proved that β_1 is a non-constant holomorphic covering of degree 2 which ramifies only at the points $[0:0:1]$ and $[1:0:1]$ with critical values $\beta_1([0:0:1]) = 0$ and $\beta_1([1:0:1]) = 1$. Therefore (S_1,β_1) is a Belyi pair.

- S_2 is a compact Riemann surface and β_2 is a degree 4 Belyi covering which ramifies only at the points $[1:0:\pm i]$, $[0:1:\pm i]$ and $[1:\pm i:0]$ with

$$mult_{[1:0:\pm i]}\beta_2 = mult_{[0:1:\pm i]}\beta_2 = mult_{[1:\pm i:0]}\beta_2 = 2$$

and critical values $\beta_2([1:0:\pm i]) = 0$, $\beta_2([0:1:\pm i]) = \infty$ and $\beta_2([1:\pm i:0]) = 1$ (see Example 2.3.15). Therefore (S_1, β_1) is also a Belyi pair.
- It is a direct consequence of the application of the Riemann-Hurwitz formula that $gen(S_1) = 0$ and therefore $S_1 \simeq \widehat{\mathbb{C}}$.
- It is not hard to see that (S_1, β_1) is a regular Belyi pair provided by the group

$$\langle [x:y:z] \mapsto [x:-y:z] \rangle \cong \mathbb{Z}_2$$

and (S_2, β_2) is a regular Belyi pair provided by the group

$$\langle [x_1:x_2:x_3] \mapsto [-x_1:x_2:x_3],\ [x_1:x_2:x_3] \mapsto [x_1:-x_2:x_3] \rangle \cong \mathbb{Z}_2^2$$

.
- Both conditions of Theorem 3.3.3 do not hold since (using the same notations as in the Theorem 3.3.3)

$$mcd(deg(\beta_1), deg(\beta_2)) = 2 \neq 1 \text{ and } mcd(a_0^{(1)}, a_0^{(2)}) = mcd(2,2) = 2 \neq 1.$$

EXAMPLE 5.1.7. **An example where the upper bound in Lemma 3.3.4 is not attained.**

Let us consider $S_1 = S_2 = \widehat{\mathbb{C}}$, $\beta_1(z) = 4z^3(1-z^3) = \beta_2(z)$ *(this covering was analyzed in detail in Example 5.1.4). In this case*

$$S_1 \times_{(\beta_1,\beta_2)} S_2 = \{[z:w:t] : z^3(t^3 - z^3) = w^3(t^3 - w^3)\}$$

which consists of four irreducible components, these being:

$$R_1 = \{[z:w:t] \in \mathbb{P}^2_{\mathbb{C}} : z = w\} \simeq \widehat{\mathbb{C}}$$
$$R_2 = \{[z:w:t] \in \mathbb{P}^2_{\mathbb{C}} : z = e^{2\pi i/3}w\} \simeq \widehat{\mathbb{C}}$$
$$R_3 = \{[z:w:t] \in \mathbb{P}^2_{\mathbb{C}} : z = e^{4\pi i/3}w\} \simeq \widehat{\mathbb{C}}$$
$$R_4 = \{[z:w:t] \in \mathbb{P}^2_{\mathbb{C}} : z^3 + w^3 = t^3\} \quad \textit{(a genus one curve).}$$

Nevertheless $mcd(deg(\beta_1), deg(\beta_2)) = 6 \neq 4$ *since* $deg(\beta_1) = deg(\beta_2) = 6$.

Actually:

- $S_1 \times_{(\beta_1,\beta_2)} S_2 = R_1 \cup R_2 \cup R_3 \cup R_4$ since

$$\begin{aligned} z^3(t^3 - z^3) = w^3(t^3 - w^3) &\leftrightarrow z^3t^3 - w^3t^3 - z^6 + w^6 = 0 \\ &\leftrightarrow (z^3 - w^3)(t^3 - z^3 - w^3) = 0 \\ &\leftrightarrow (z-w)(z - e^{2\pi i/3}w)(z - e^{4\pi i/3}w)(t^3 - z^3 - w^3) = 0. \end{aligned}$$

- It is immediate that R_1, R_2 and R_3 are irreducible and non-singular (and thus compact Riemann surfaces). R_4 is also irreducible and non-singular (and thus a compact Riemann surface) because the homogeneous polynomial $T^3 - Z^3 - W^3$ is non-singular.
- Only remains to check that R_4 is a genus one curve.

  In fact, considering that

$$R_4 \setminus \{[-1:1:0], [-e^{2\pi i/3}:1:0], [e^{4\pi i/3}:1:0]\} \simeq \{(z,w) \in \mathbb{C}^2 : z^3 + w^3 - 1 = 0\},$$

by mean of the projection on the first coordinate π_1 we will be capable to calculate its genus:

  - π_1 ramifies only at the points $(1,0)$, $(e^{2\pi i/3}, 0)$ and $(e^{4\pi i/3}, 0)$ (see remark 2.2.11) and its critical values are $\pi_1((1,0)) = 1$, $\pi_1((e^{2\pi i/3}, 0)) = e^{2\pi i/3}$ and $\pi_1((e^{4\pi i/3}, 0)) = e^{4\pi i/3}$.
  - Since π_1 does not ramify over 0 and $\pi_1^{-1}(0)$ has only three elements, the degree of π_1 is 3.

Thus, to compactify $\{(z,w) \in \mathbb{C}^2 : z^3 + w^3 - 1 = 0\}$ and extend π_1 to its compactification it is necessary to add three points to $\{(z,w) \in \mathbb{C}^2 : z^3 + w^3 - 1 = 0\}$ (which in R_4 will be correspond to $[-1:1:0]$, $[-e^{2\pi i/3}:1:0]$ and $[e^{4\pi i/3}:1:0]$) whose image under π_1 will be ∞. Finally, using the Riemann-Hurwitz formula, it can be deducible that $gen(R_4) = 1$.

EXAMPLE 5.1.8. **Fiber product of cyclic gonal curves.** *Let $n, m \geq 2$ be integers and set*

$$S_1 = \{[x:y:z] \in \mathbb{P}^2_{\mathbb{C}} : y^n = x(x-z)z^{n-2}\}$$
$$S_2 = \{[u:w:v] \in \mathbb{P}^2_{\mathbb{C}} : w^m = u(u-v)v^{m-2}\}$$

$$\beta_1([x:y:z]) = x/z$$

$$\beta_2([u:w:v]) = u/v.$$

In this case,

$$S_1 \times_{(\beta_1,\beta_2)} S_2$$
$$\|$$
$$\{([x:y:z],[u:w:v]) \in (\mathbb{P}^2_{\mathbb{C}})^2 : y^n = x(x-z)z^{n-2},\ w^m = u(u-v)v^{m-2},\ xv = zu\}.$$

In affine coordinates ($z = v = 1$), the above fiber product can be seen as follows:

$$X_{n,m} := \{(x,y,x,w) \in \mathbb{C}^4 : y^n = x(x-1),\ w^m = y^n\}.$$

However, if $D_{n,m} = mcd(deg(\beta_1), deg(\beta_2)) = mcd(n,m)$, $a = n/D_{n,m}$ and $b = m/D_{n,m}$, we can write

$$X_{n,m} := \left\{ (x,y,x,w) \in \mathbb{C}^4 : y^n = x(x-1),\ \prod_{j=0}^{D_{n,m}-1} \left(w^b - (\rho_{n,m})^j y^a\right) = 0 \right\},$$

where $\rho_{n,m} = e^{2\pi i/D_{n,m}}$. It follows that $X_{n,m}$ (and so $S_1 \times_{(\beta_1,\beta_2)} S_2$) contains exactly $D_{n,m}$ connected components, these being given by

$$X_{n,m,j} := \left\{ (x,y,x,w) \in \mathbb{C}^4 : y^n = x(x-1),\ w^b = (\rho_{n,m})^j y^a \right\}$$

where $0 \le j \le D_{n,m} - 1$. All these irreducible components are isomorphic and

$$gen(X_{n,m,j}) = \begin{cases} \frac{bn-1}{2} & \text{if } n \text{ and } m \text{ are odd} \\ \frac{bn-2}{2} & \text{in other case.} \end{cases}$$

In particular, in the following table the last column provides the genus of the corresponding irreducible components in the fiber product.

n	m	$D_{n,m}$	*genus*
6	4	2	5
6	9	3	8
12	18	6	17

Actually:

- In order to make the details of this example, it is enough to analyze only S_1 and β_1 because the analysis for S_2 and β_2 is analogue. The affine version of S_1 is $E := \{(x,y) \in \mathbb{C}^2 : y^n = x(x-1)\}$ which becomes into a compact Riemann surface adding a single point if n is odd and two points if n is even.

  In fact:
  - E can be seen as the fiber product of the pairs $(\widehat{\mathbb{C}}, f_1)$ and $(\widehat{\mathbb{C}}, f_2)$ where $f_1, f_2 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are defined by $f_1(x) = x^2 - x$ and $f_2(y) = y^n$ for which we know that:
    * f_1 is a degree two covering which ramifies only at the points $1/2$ and ∞ with
      $$mult_{1/2} f_1 = mult_\infty f_1 = 2$$
      and whose critical values are $f_1(1/2) = -1/4$ and $f_1(\infty) = \infty$.
    * f_2 is a degree n covering which ramifies only at the points 0 and ∞ with
      $$mult_0 f_2 = mult_\infty f_2 = n$$
      and whose critical values are $f_2(0) = 0$ and $f_2(\infty) = \infty$.
  - The conditions of Theorem 3.3.3 are not satisfied for the pairs $(\widehat{\mathbb{C}}, f_1)$ and $(\widehat{\mathbb{C}}, f_2)$ but it can be easily proved, in the standard way, that E is non-singular and therefore irreducible.

– The fiber product $\widehat{\mathbb{C}} \times_{(f_1,f_2)} \widehat{\mathbb{C}}$ has only one singular point (the point (∞,∞)) and:
  * If n is odd, around it the fiber product looks like a single cone. Thus, in order to compactify E we need to remove the singular point (∞,∞) of $\widehat{\mathbb{C}} \times_{(f_1,f_2)} \widehat{\mathbb{C}}$ and add a single point which we will denote by ∞. Therefore,
$$\widehat{E} = E \cup \{\infty\}$$
is a smooth compactification of E.
  * If n is even, around it the fiber product looks like two glued (at (∞,∞)) cones. Thus, in order to compactify E we need to remove the singular point (∞,∞) of $\widehat{\mathbb{C}} \times_{(f_1,f_2)} \widehat{\mathbb{C}}$ and add two different points which we will denote by ∞_1 and ∞_2. Therefore,
$$\widehat{E} = E \cup \{\infty_1, \infty_2\}$$
is a smooth compactification of E.

- β_1 is a degree n regular Belyi covering with deck group
$$\langle [x:y:z] \mapsto [x : e^{2\pi i/n} y : z] \rangle \cong \mathbb{Z}_n.$$
In the affine version, when n is odd, β_1 has only three critical points, namely $(0,0)$, $(1,0)$ and ∞ with
$$mult_{(0,0)}\beta_1 = mult_{(1,0)}\beta_1 = mult_{\infty}\beta_1 = n$$
and three critical values $\beta_1(0,0) = 0$, $\beta_1(1,0) = 1$ and $\beta_1(\infty) = \infty$; conversely, when n is even, β_1 has exactly four critical points, namely $(0,0)$, $(1,0)$, ∞_1 and ∞_2 with
$$mult_{(0,0)}\beta_1 = mult_{(1,0)}\beta_1 = n \text{ and } mult_{\infty_1}\beta_1 = mult_{\infty_2}\beta_1 = n/2$$
and three critical values $\beta_1(0,0) = 0$, $\beta_1(1,0) = 1$ and $\beta_1(\infty_1) = \beta_1(\infty_2) = \infty$.

In fact:
  – In the affine version, β_1 is not else than the projection on the first coordinate of E. Thus, abusing of language, β_1 ramifies at the points $(0,0)$ and $(1,0)$ in E (see Remark 2.2.11) with $\beta_1(0,0) = 0$ and $\beta_1(1,0) = 1$.
  – $deg(\beta_1) = n$ (since -1 is not a critical value of β_1) and
$$\beta_1^{-1}(-1) = \{(-1,y) \in \mathbb{C}^2 : y^n = 2\}$$
has n elements.
  – Since $\beta_1^{-1}(0) = \{(0,0)\}$ and $\beta_1^{-1}(1) = \{(1,0)\}$,
$$mult_{(0,0)}\beta_1 = mult_{(1,0)}\beta_1 = n.$$
  – It is not hard to see that β_1 is a regular covering with deck group
$$\langle [x:y:z] \mapsto [x : e^{2\pi i/n} y : z] \rangle \cong \mathbb{Z}_n.$$

  - In order to extend holomorphically β_1 to $\widehat{E}$, it is necessary, in case n is odd, to make $\beta_1(\infty) = \infty$ and thus $mult_\infty \beta_1 = n$; and in case n is even, to make $\beta_1(\infty_1) = \beta_1(\infty_2) = \infty$ and thus $mult_{\infty_1}\beta_1 = mult_{\infty_2}\beta_1 = n/2$.
- As a consequence of the Riemann-Hurwitz formula and the above items

$$gen(S_1) = gen(\widehat{E}) = \begin{cases} \frac{n-1}{2}, & n \text{ is odd} \\ \frac{n-2}{2}, & n \text{ is even.} \end{cases}$$

- The components $X_{n,m,j}$ for $0 \leq j \leq D_{n,m} - 1$ are isomorphic (see Corollary 3.2.4) and are irreducible (see Lemma 3.3.4).
- For $0 \leq j, k \leq D_{n,m} - 1$ with $j \neq k$, $X_{n,m,j} \cap X_{n,m,k} = \{(0,0,0,0), (1,0,1,0)\}$.
  In fact:

$$\begin{aligned}(x,y,x,w) \in X_{n,m,j} \cap X_{n,m,k} &\to y^n = x(x-1) \wedge w^b = \rho^j_{D_{n,m}} y^a \wedge w^b = \rho^k_{D_{n,m}} y^a \\ &\to y^n = x(x-1) \wedge w^b = \rho^j_{D_{n,m}} y^a \wedge (\rho^{j-k}_{D_{n,m}} - 1) y^a = 0 \\ &\to y = w = 0 \wedge (x = 0 \vee x = 1) \\ &\to (x,y,x,w) = (0,0,0,0) \vee (x,y,x,w) = (1,0,1,0).\end{aligned}$$

- The genus of the irreducible components $X_{n,m,j}$ can be calculate by mean of the equation

$$gen(X_{n,m,j}) = \begin{cases} \frac{bn-1}{2} & \text{if } n \text{ and } m \text{ are odd} \\ \frac{bn-2}{2} & \text{in other case.} \end{cases}$$

  In fact:

  Let $\Pi : X_{n,m,0} \to E$ defined by $\Pi(x,y,x,w) = (x,y)$. It is immediate that Π is a covering of degree b and, noting that $T : X_{n,m,0} \to X_{n,m,0}$ defined by $T(x,y,x,w) = (x,y,x,e^{2\pi i/b}w)$ is in $Aut(X_{n,m,0}, \Pi)$ with $(0,0,0,0)$ and $(1,0,1,0)$ the only points with non-trivial stabilizer (in particular they are fixed points by T), it is also regular. Next, we will use this covering and the Riemann-Hurwitz formula to calculate the genus of $X_{n,m,j}$ in all cases.
  - If n and m are odd, the only singular points in the affine version $X_{n,m}$ of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ are $(0,0,0,0)$, $(1,0,1,0)$ and (∞,∞) (or equivalently $\big((0,0),(0,0)\big)$, $\big((1,0),(1,0)\big)$ and (∞,∞)). Around each of them the fiber product looks like $D_{n,m}$ cones glued only at that point. In this way, in order to des-singularized and compactify $X_{n,m}$ it is necessary to remove all these singular points and fill the remaining punctures by adding $3D_{n,m}$ points. In particular, since all the irreducible components are isomorphic and for $0 \leq j,k \leq D_{n,m} - 1$ with $j \neq k$, $X_{n,m,j} \cap X_{n,m,k} = \{(0,0,0,0),(1,0,1,0)\}$, it is follows that to des-singularized and compactify $X_{n,m,0}$ we only need to add three points, namely $(0,0,0,0)$, $(1,0,1,0)$ and $\widetilde{\infty_1}$ (again abusing of language). In other words,

$$\widehat{X_{n,m,0}} = X_{n,m,0} \cup \{\widetilde{\infty_1}\}$$

is a non-singular compactification of $X_{n,m,0}$. Now, to extend Π to $\widehat{X_{n,m,0}}$ one has to make $\Pi(\widetilde{\infty_1}) = \infty$ (remember that $\widehat{E} = E \cup \{\infty\}$ when n is odd) and therefore $(0,0,0,0)$, $(1,0,1,0)$ and $\widetilde{\infty_1}$ are the only critical points of Π with

$$mult_{(0,0,0,0)}\Pi = mult_{(1,0,1,0)}\Pi = mult_{\widetilde{\infty_1}}\Pi = b$$

and the result follows, as stated before, applying the Riemann-Hurwitz formula.

– If n is odd and m is even, the only singular points in the affine version $X_{n,m}$ of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ are $(0,0,0,0)$, $(1,0,1,0)$, (∞,∞_1) and (∞,∞_2). Around $(0,0,0,0)$ or $(1,0,1,0)$ the fiber product looks like $D_{n,m}$ cones glued only at that point; and around (∞,∞_1) or (∞,∞_2) the fiber product looks like $mcd(n,m/2) = D_{n,m}$ cones glued only at that point. In this way, in order to des-singularized and compactify $X_{n,m}$ it is necessary to remove all these singular points and fill the remaining punctures by adding $4D_{n,m}$ points. In particular, as before, to des-singularized and compactify $X_{n,m,0}$ we only need to add four points, namely $(0,0,0,0)$, $(1,0,1,0)$, $\widetilde{\infty_1}$ and $\widetilde{\infty_2}$ (again abusing of language). In other words,

$$\widehat{X_{n,m,0}} = X_{n,m,0} \cup \{\widetilde{\infty_1}, \widetilde{\infty_2}\}$$

is a non-singular compactification of $X_{n,m,0}$. Now, to extend Π to $\widehat{X_{n,m,0}}$ one has to make $\Pi(\widetilde{\infty_1}) = \Pi(\widetilde{\infty_2}) = \infty$ (remember that $\widehat{E} = E \cup \{\infty\}$ when n is odd) and therefore $(0,0,0,0)$, $(1,0,1,0)$, $\widetilde{\infty_1}$ and $\widetilde{\infty_2}$ are the only critical points of Π with

$$mult_{(0,0,0,0)}\Pi = mult_{(1,0,1,0)}\Pi = b \quad \text{and} \quad mult_{\widetilde{\infty_1}}\Pi = mult_{\widetilde{\infty_2}}\Pi = b/2$$

and the result follows applying the Riemann-Hurwitz formula.

– If n is even and m is odd, the only singular points in the affine version $X_{n,m}$ of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ are $(0,0,0,0)$, $(1,0,1,0)$, (∞_1,∞) and (∞_2,∞). Around $(0,0,0,0)$ or $(1,0,1,0)$ the fiber product looks like $D_{n,m}$ cones glued only at that point; and around (∞_1,∞) or (∞_2,∞) the fiber product looks like $mcd(n/2,m) = D_{n,m}$ cones glued only at that point. In this way, in order to des-singularized and compactify $X_{n,m}$ it is necessary to remove all these singular points and fill the remaining punctures by adding $4D_{n,m}$ points. In particular, as before, to des-singularized and compactify $X_{n,m,0}$ we only need to add four points, namely $(0,0,0,0)$, $(1,0,1,0)$, $\widetilde{\infty_1}$ and $\widetilde{\infty_2}$ (again abusing of language). In other words,

$$\widehat{X_{n,m,0}} = X_{n,m,0} \cup \{\widetilde{\infty_1}, \widetilde{\infty_2}\}$$

is a non-singular compactification of $X_{n,m,0}$. Now, to extend Π to $\widehat{X_{n,m,0}}$ one has to make $\Pi(\widetilde{\infty_1}) = \infty_1$ and $\Pi(\widetilde{\infty_2}) = \infty_2$ ($\widehat{E} = E \cup \{\infty_1,\infty_2\}$ because n is even) and therefore $(0,0,0,0)$, $(1,0,1,0)$, $\widetilde{\infty_1}$ and $\widetilde{\infty_2}$ are the only critical points of Π with

$$mult_{(0,0,0,0)}\Pi = mult_{(1,0,1,0)}\Pi = mult_{\widetilde{\infty_1}}\Pi = mult_{\widetilde{\infty_2}}\Pi = b$$

and again the result follows applying the Riemann-Hurwitz formula.

– If n and m are even, the only singular points in the affine version $X_{n,m}$ of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ are $(0,0,0,0)$, $(1,0,1,0)$, (∞_1,∞_1), (∞_1,∞_2), (∞_2,∞_1) and (∞_2,∞_2). Around $(0,0,0,0)$ or $(1,0,1,0)$ the fiber product looks like $D_{n,m}$ cones glued only at that point; and around each of the points (∞_1,∞_1), (∞_1,∞_2), (∞_2,∞_1) and (∞_2,∞_2) the fiber product looks like $mcd(n/2,m/2) = D_{n,m}/2$ cones glued only at that point. In this way, in order to des-singularized and compactify $X_{n,m}$ it is necessary to remove all these singular points and fill the remaining punctures by adding $4D_{n,m}$ points. In particular, as before, to des-singularized and compactify $X_{n,m,0}$ we only need to add four points, namely $(0,0,0,0)$, $(1,0,1,0)$, $\widetilde{\infty_1}$ and $\widetilde{\infty_2}$ (again abusing of language). In other words,

$$\widehat{X_{n,m,0}} = X_{n,m,0} \cup \{\widetilde{\infty_1}, \widetilde{\infty_2}\}$$

is a non-singular compactification of $X_{n,m,0}$. Now, to extend Π to $\widehat{X_{n,m,0}}$ one has to make $\Pi(\widetilde{\infty_1}) = \infty_1$ and $\Pi(\widetilde{\infty_2}) = \infty_2$ ($\widehat{E} = E \cup \{\infty_1, \infty_2\}$ because n is even) and therefore $(0,0,0,0)$, $(1,0,1,0)$, $\widetilde{\infty_1}$ and $\widetilde{\infty_2}$ are the only critical points of Π with

$$mult_{(0,0,0,0)}\Pi = mult_{(1,0,1,0)}\Pi = mult_{\widetilde{\infty_1}}\Pi = mult_{\widetilde{\infty_2}}\Pi = b$$

and again the result follows applying the Riemann-Hurwitz formula.

EXAMPLE 5.1.9. **Fiber product of gonal curves.**

Let $n, m \geq 2$ be integers, $a, b \in \{1, ..., n\}$, $c, d \in \{1, ..., m-1\}$, polynomials $P(y)$ of degree at most $n-1$, $Q(w)$ of degree at most $m-1$, and set

$$S_1 = \left\{[x:y:z] \in \mathbb{P}^2_{\mathbb{C}} : y^n + P(y) = x^a(x-z)^b z^{n-a-b}\right\}$$

$$S_2 = \left\{[u:w:v] \in \mathbb{P}^2_{\mathbb{C}} : w^m + Q(w) = u^c(u-v)^d v^{m-c-d}\right\}$$

$$\beta_1([x:y:z]) = x/z$$

$$\beta_2([u:w:v]) = u/v$$

This kind of examples can be worked out similarly as for the previous example. For instance, for $n=6$, $m=4$, $a=1$, $b=4$, $c=d=1$, $P(y)=0$ and $Q(w)=0$, we get that the fiber product is irreducible of genus 9.

EXAMPLE 5.1.10. **Fiber product of Fermat curves.**
In this example we consider the Fermat curves

$$S_1 = \{[x_1 : x_2 : x_3] \in \mathbb{P}^2_{\mathbb{C}} : x_1^n + x_2^n + x_3^n = 0\}$$
$$S_2 = \{[y_1 : y_2 : y_3] \in \mathbb{P}^2_{\mathbb{C}} : y_1^m + y_2^m + y_3^m = 0\}$$

and the coverings

$$\beta_1\big([x_1 : x_2 : x_3]\big) = -\Big(\frac{x_2}{x_1}\Big)^n$$
$$\beta_2\big([y_1 : y_2 : y_3]\big) = -\Big(\frac{y_2}{y_1}\Big)^m.$$

The pairs (S_1, β_1) and (S_2, β_2) are regular Belyi pairs whose deck groups are the Abelian groups $\mathbb{Z}_n^2$ and $\mathbb{Z}_m^2$ respectively (see Examples 2.1.32 and 2.3.15). The fiber product is given by

$$S_1 \times_{(\beta_1,\beta_2)} S_2$$
$$||$$
$$\Big\{\big([x_1 : x_2 : x_3], [y_1 : y_2 : y_3]\big) \in (\mathbb{P}^2_{\mathbb{C}})^2 : x_1^n + x_2^n + x_3^n = 0,\ y_1^m + y_2^m + y_3^m = 0,\ x_2^n y_1^m = x_1^n y_2^m\Big\}$$

and has $3nm$ singular points.

I. *When $mcd(n, m) = 1$, we know that $S_1 \times_{(\beta_1,\beta_2)} S_2$ is irreducible (by Theorem 3.3.3). Therefore, the surface $S_1 \times_{(\beta_1,\beta_2)} S_2$ represents the Belyi curve of genus $(2 + n^2m^2 - 3nm)/2$ given by the Fermat curve*

$$\{[z_1 : z_2 : z_3] \in \mathbb{P}^2_{\mathbb{C}} : z_1^{nm} + z_2^{nm} + z_3^{nm} = 0\}$$

and the Belyi map β is represented, in this case, by the covering

$$f\big([z_1 : z_2 : z_3]\big) = -\Big(\frac{z_2}{z_1}\Big)^{nm}.$$

II. *When $mcd(n, m) = d > 1$ one has that one of the irreducible components of $S_1 \times_{(\beta_1,\beta_2)} S_2$ is given by the Fermat curve*

$$\{[w_1 : w_2 : w_3] \in \mathbb{P}^2_{\mathbb{C}} : w_1^{nm/d} + w_2^{nm/d} + w_3^{nm/d} = 0\}.$$

All other irreducible components are isomorphic to the previous one by Corollary 3.2.4. For instance, if we set $n = 4$ and $m = 2$ then the fiber product provides 4 (isomorphic) irreducible components, each one a curve of genus 3 (see Example 2.2.19), these being

$$R_1 = \big\{[x : y : w : u : v] \in \mathbb{P}^4_{\mathbb{C}} : x^2 + yw = 0,\ u^2 + vw = 0,\ y^2 + v^2 + w^2 = 0\big\}$$
$$R_2 = \big\{[x : y : w : u : v] \in \mathbb{P}^4_{\mathbb{C}} : x^2 + yw = 0,\ u^2 - vw = 0,\ y^2 + v^2 + w^2 = 0\big\}$$
$$R_3 = \big\{[x : y : w : u : v] \in \mathbb{P}^4_{\mathbb{C}} : x^2 - yw = 0,\ u^2 + vw = 0,\ y^2 + v^2 + w^2 = 0\big\}$$
$$R_4 = \big\{[x : y : w : u : v] \in \mathbb{P}^4_{\mathbb{C}} : x^2 - yw = 0,\ u^2 - vw = 0,\ y^2 + v^2 + w^2 = 0\big\}.$$

An isomorphism between R_1 and R_2 is given by taking u to iu, an isomorphism between R_1 and R_3 is given by taking x to ix and an isomorphism between R_1 and R_4 is given by taking u to iu and x to ix. Moreover, R_1 is isomorphic to

$$\{[x : u : w] : x^4 + u^4 + w^4 = 0\}$$

by seeing that

$$y = -\frac{x^2}{w} \quad \text{and} \quad v = -\frac{u^2}{w}$$

Actually:

- Let us remember that (see Example 2.3.15):
  - β_1 is a degree n^2 regular covering with deck group $\langle T_n, W_n \rangle$ where the automorphisms $T_n, W_n : S_1 \to S_1$ are defined by

    $$T_n([x_1 : x_2 : x_3]) = [e^{2\pi i/n} x_1 : x_2 : x_3]$$
    $$W_n([x_1 : x_2 : x_3]) = [x_1 : e^{2\pi i/n} x_2 : x_3].$$

    When $n > 1$, its critical points are

    $[1 : 0 : e^{\pi(2k+1)/n}]$, $[1 : e^{\pi(2k+1)/n} : 0]$ and $[0 : 1 : e^{\pi(2k+1)/n}]$ $(0 \leq k \leq n-1)$

    each one with multiplicity n and its critical values are

    $$\beta_1([1 : 0 : e^{\pi(2k+1)/n}]) = 0$$
    $$\beta_1([1 : e^{\pi(2k+1)/n} : 0]) = 1$$
    $$\beta_1([0 : 1 : e^{\pi(2k+1)/n}]) = \infty.$$

  - β_2 is a degree m^2 regular covering with deck group $\langle T_m, W_m \rangle$ where the automorphisms $T_m, W_m : S_1 \to S_1$ are defined by

    $$T_m([y_1 : y_2 : y_3]) = [e^{2\pi i/m} y_1 : y_2 : y_3]$$
    $$W_m([y_1 : y_2 : y_3]) = [y_1 : e^{2\pi i/m} y_2 : y_3].$$

    When $m > 1$, its critical points are

    $[1 : 0 : e^{\pi(2j+1)/m}]$, $[1 : e^{\pi(2j+1)/m} : 0]$ and $[0 : 1 : e^{\pi(2j+1)/m}]$ $(0 \leq j \leq m-1)$

    each one with multiplicity m and its critical values are

    $$\beta_2([1 : 0 : e^{\pi(2j+1)/m}]) = 0$$
    $$\beta_2([1 : e^{\pi(2j+1)/m} : 0]) = 1$$
    $$\beta_2([0 : 1 : e^{\pi(2j+1)/m}]) = \infty.$$

In this way, the fiber product $S_1 \times_{(\beta_1, \beta_2)} S_2$ has the following $3nm$ singular points:

$$([1 : 0 : e^{\pi(2k+1)/n}], [1 : 0 : e^{\pi(2j+1)/m}])$$
$$([1 : e^{\pi(2k+1)/n} : 0], [1 : e^{\pi(2j+1)/m} : 0])$$
$$([0 : 1 : e^{\pi(2k+1)/n}], [0 : 1 : e^{\pi(2j+1)/m}]),$$

whose images are

$$\beta\Big(\big([1:0:e^{\pi(2k+1)/n}],[1:0:e^{\pi(2j+1)/m}]\big)\Big)=0$$
$$\beta\Big(\big([1:e^{\pi(2k+1)/n}:0],[1:e^{\pi(2j+1)/m}:0]\big)\Big)=1$$
$$\beta\Big(\big([0:1:e^{\pi(2k+1)/n}],[0:1:e^{\pi(2j+1)/m}]\big)\Big)=\infty$$

$(0\leq k\leq n-1$ and $0\leq j\leq m-1)$ and

$$\beta^{-1}(0)=\big\{\big([1:0:e^{\pi(2k+1)/n}],[1:0:e^{\pi(2j+1)/m}]\big):0\leq k\leq n-1,0\leq j\leq m-1\big\}$$
$$\beta^{-1}(1)=\big\{\big([1:e^{\pi(2k+1)/n}:0],[1:e^{\pi(2j+1)/m}:0]\big):0\leq k\leq n-1,0\leq j\leq m-1\big\}$$
$$\beta^{-1}(\infty)=\big\{\big([0:1:e^{\pi(2k+1)/n}],[0:1:e^{\pi(2j+1)/m}]\big):0\leq k\leq n-1,0\leq j\leq m-1\big\}$$

Thus, the multiplicity of β in each one of those points is nm and around each one of them the fiber product looks like d glued cones (only at that point). Therefore, in order to des-singularized and compactify $S_1\times_{(\beta_1,\beta_2)}S_2$ it is necessary to remove all these singular points and fill the remaining punctures by adding $3nmd$ points from which dnm of them have image 0, others dnm have image 1 and the rest dnm points have image ∞.

- When $d=mcd\big(deg(\beta_1),deg(\beta_2)\big)=1$, $S_1\times_{(\beta_1,\beta_2)}S_2$ is irreducible and β is normal (see Section 3.4.1).
- An affine version of $S_1\times_{(\beta_1,\beta_2)}S_2$ is

$$\{(x_2,x_3,y_2,y_3)\in\mathbb{C}^4:1+x_2^n+x_3^n=0,\ 1+y_2^m+y_3^m=0,\ x_2^n=y_2^m\}$$

which is equivalent to

$$\{(x_2,x_3,y_2,y_3)\in\mathbb{C}^4:1+y_2^m+y_3^m=0,\ x_2^n=y_2^m,\ x_3^n=y_3^m\},$$

and in this last version β is defined by

$$\beta(x_2,x_3,y_2,y_3)=-x_2^n$$

or, equivalently, by

$$\beta(x_2,x_3,y_2,y_3)=-y_2^m.$$

- When $d=1$, β and f are isomorphic coverings (see Example 2.5.4) and thus the genus of $S_1\times_{(\beta_1,\beta_2)}S_2$ is $(2+n^2m^2-3nm)/2$ (see Example 2.2.19).
- When $d>1$, the affine version of the fiber product $S_1\times_{(\beta_1,\beta_2)}S_2$ allows us to decompose it into the union of d^2 irreducible components $X_{n,m,r,s}$, namely

$$X_{n,m,r,s}$$
$$||$$
$$\{(x_2,x_3,y_2,y_3)\in\mathbb{C}^4:1+y_2^m+y_3^m=0,\ x_2^{n_1}-\rho^r y_2^{m_1}=0,\ x_3^{n_1}-\rho^s y_3^{m_1}=0\}$$

where $n=dn_1$, $m=dm_1$, $\rho=e^{2\pi i/d}$ and $0\leq r,s\leq d-1$. In this case, it is not difficult to verify that for $0\leq r,s,t,u\leq d-1$ with $r\neq t$ and $s\neq u$ the pairwise intersection $X_{n,m,r,s}\cap X_{n,m,u,t}$ is always contained in the set of singular points of the fiber product. Besides, since all irreducible components of the fiber product are isomorphic by Corollary 3.2.4, we deduce that $\beta|_{X_{n,m,r,s}}$ $(0 \le r, s \le d-1)$ is a degree $d^2n_1^2m_1^2$ covering.

On the other hand, the group $\langle T, W\rangle$, where $T, W : X_{n,m,0,0} \to X_{n,m,0,0}$ are defined by

$$T(x_2, x_3, y_2, y_3) = (e^{2\pi i/n}x_2, x_3, e^{2\pi i/m}y_2, y_3)$$
$$W(x_2, x_3, y_2, y_3) = (x_2, e^{2\pi i/n}x_3, y_2, e^{2\pi i/m}y_3),$$

is isomorphic to $\mathbb{Z}_{mcm(n,m)} \times \mathbb{Z}_{mcm(n,m)} = \mathbb{Z}_{dn_1m_1} \times \mathbb{Z}_{dn_1m_1}$ and is the deck group of $\beta|_{X_{n,m,0,0}}$. Therefore, by the last theorem of section 2.5 and Example 2.5.4, one has that the irreducible components of $S_1 \times_{(\beta_1,\beta_2)} S_2$ are given by the Fermat curve

$$\left\{[w_1 : w_2 : w_3] \in \mathbb{P}^2_{\mathbb{C}} : w_1^{nm/d} + w_2^{nm/d} + w_3^{nm/d} = 0\right\}$$

$(Mon(\beta|_{X_{n,m,0,0}}) \cong (Z_{dn_1m_1})^2)$.

- Finally, in the particular case when $n = 4$ and $m = 2$, $d = mcd(n, m) = 2$ and by the above items the affine version of $S_1 \times_{(\beta_1,\beta_2)} S_2$ is

$$\begin{aligned}
&X_{4,2,0,0} \cup X_{4,2,0,1} \cup X_{4,2,1,0} \cup X_{4,2,1,1}\\
&\quad = \{(x_2, x_3, y_2, y_3) \in \mathbb{C}^4 : 1 + y_2^2 + y_3^2 = 0,\ x_2^2 - y_2 = 0,\ x_3^2 - y_3 = 0\}\\
&\qquad \cup\\
&\quad \{(x_2, x_3, y_2, y_3) \in \mathbb{C}^4 : 1 + y_2^2 + y_3^2 = 0,\ x_2^2 - y_2 = 0,\ x_3^2 + y_3 = 0\}\\
&\qquad \cup\\
&\quad \{(x_2, x_3, y_2, y_3) \in \mathbb{C}^4 : 1 + y_2^2 + y_3^2 = 0,\ x_2^2 + y_2 = 0,\ x_3^2 - y_3 = 0\}\\
&\qquad \cup\\
&\quad \{(x_2, x_3, y_2, y_3) \in \mathbb{C}^4 : 1 + y_2^2 + y_3^2 = 0,\ x_2^2 + y_2 = 0,\ x_3^2 + y_3 = 0\}\\
&\text{(replacing } x_2 \text{ by } x,\ x_3 \text{ by } u,\ y_2 \text{ by } y \text{ and } y_3 \text{ by } v)\\
&\quad = \{(x, u, y, v) \in \mathbb{C}^4 : 1 + y^2 + v^2 = 0,\ x^2 - y = 0,\ u^2 - v = 0\}\\
&\qquad \cup\\
&\quad \{(x, u, y, v) \in \mathbb{C}^4 : 1 + y^2 + v^2 = 0,\ x^2 - y = 0,\ u^2 + v = 0\}\\
&\qquad \cup\\
&\quad \{(x, u, y, v) \in \mathbb{C}^4 : 1 + y^2 + v^2 = 0,\ x^2 + y = 0,\ u^2 - v = 0\}\\
&\qquad \cup\\
&\quad \{(x, u, y, v) \in \mathbb{C}^4 : 1 + y^2 + v^2 = 0,\ x^2 + y = 0,\ u^2 + v = 0\}.
\end{aligned}$$

EXAMPLE 5.1.11. *In this example we consider*

$$S_1 = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : y^7 = xz^4(x - z)^2\}$$

$$S_2 = \{[u : w : v] \in \mathbb{P}^2_{\mathbb{C}} : w^3 = uv(u-v)\}$$
$$\beta_1([x : y : z]) = \frac{x}{z}$$
$$\beta_2([u : w : v]) = \frac{u}{v}.$$

The pair (S_1, β_1) (respectively, (S_2, β_2)) is a regular Belyi pair whose deck group is the Abelian group $\mathbb{Z}_7$ (respectively, $\mathbb{Z}_3$). The fiber product in this case has exactly 3 singular points; one over ∞, other over 0 and the other over 1, and it is irreducible of genus 10. Therefore, this provides a regular Belyi pair with deck group $\mathbb{Z}_{21}$ (see Section 3.4.1). An affine model of this fiber product is given by

$$\{(x, y, w) : y^7 = x(x-1)^2,\ w^3 = x(x-1)\}$$

and a projective model is given by

$$\{[x : y : w : t] \in \mathbb{P}^3_{\mathbb{C}} : y^7 - x^3t^4 + 2x^2t^5 - xt^6 = 0,\ w^3 - x^2t + xt^2 = 0\}.$$

in which the singular points are $[0:0:0:1]$, $[1:0:0:1]$ and $[1:0:0:0]$.

Actually:

- The affine version of S_1 is $E_1 := \{(x, y) \in \mathbb{C}^2 : y^7 = x(x-1)^2\}$ which becomes into a compact Riemann surface after removing the only singular point $(1, 0)$ and adding two points, namely $(1, 0)$ (abusing of language) and ∞.
  In fact:
  - Similarly as we did in Example 5.1.8, E_1 can be seen as the fiber product of the Belyi pairs $(\widehat{\mathbb{C}}, f_1)$ and $(\widehat{\mathbb{C}}, f_2)$ where $f_1, f_2 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are defined by $f_1(x) = x(x-1)^2$ and $f_2(y) = y^7$ for which we know that:
    * f_1 is a degree three covering which ramifies only at the points 1, 1/3 and ∞ with
    $$mult_1 f_1 = mult_{1/3} f_1 = 2 \text{ and } mult_\infty f_1 = 3$$
    and whose critical values are
    $$f_1(1) = 0,\ f_1(1/3) = 4/27 \text{ and } f_1(\infty) = \infty.$$
    * f_2 is a degree seven covering which ramifies only at the points 0 and ∞ with
    $$mult_0 f_2 = mult_\infty f_2 = 7$$
    and critical values are $f_2(0) = 0$ and $f_2(\infty) = \infty$.
    * The first condition of Theorem 3.3.3 is satisfied and thus E is irreducible.
    * The fiber product $\widehat{\mathbb{C}} \times_{(f_1, f_2)} \widehat{\mathbb{C}}$ has only two singular points ($(1, 0)$ and (∞, ∞)) and around each of them the fiber product looks like a single cone. Thus, in order to des-singularized and compactify E_1 we need to remove these singular points and add another two which, abusing of language, we will denote by $(1, 0)$ and ∞. Therefore,
    $$\widehat{E_1} = E_1 \cup \{\infty\}$$

is a smooth compactification of E_1.

- Similarly, as in Example 5.1.8, it can be proved that β_1 is a degree 7 regular Belyi covering with deck group $\langle [x : y : z] \mapsto [x : e^{2\pi i/7}y : z] \rangle \cong \mathbb{Z}_7$, has exactly three critical points, namely $(0, 0)$, $(1, 0)$ and ∞ (in the affine version), with
$$mult_{(0,0)}\beta_1 = mult_{(1,0)}\beta_1 = mult_{\infty}\beta_1 = 7$$
and three critical values $\beta_1(0, 0) = 0$, $\beta_1(1, 0) = 1$ and $= \beta_1(\infty) = \infty$.
- The affine version of S_2 is $E_2 = \{(x, y) \in \mathbb{C}^2 : w^3 = u(u-1)\}$ which is non-singular and
$$\widehat{E_2} = E_2 \cup \{\infty\}$$
is a smooth compactification of E_2 (this was done in Example 5.1.8).
- β_2 is a degree 3 regular Belyi covering with deck group
$$\langle [u : w : v] \mapsto [u : e^{2\pi i/3}w : v] \rangle \cong \mathbb{Z}_3$$
(it was done in Example 5.1.8), has exactly three critical points, namely $(0, 0)$, $(1, 0)$ and ∞ (in the affine version), with
$$mult_{(0,0)}\beta_2 = mult_{(1,0)}\beta_2 = mult_{\infty}\beta_2 = 3$$
and three critical values $\beta_2(0, 0) = 0$, $\beta_2(1, 0) = 1$ and $\beta_2(\infty) = \infty$.
- $S_1 \times_{(\beta_1,\beta_2)} S_2$ is irreducible since any of the conditions of Theorem 3.3.3 is satisfied.
- An affine model of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is given by
$$\{(x, y, w) \in \mathbb{C}^3 : y^7 = x(x-1)^2, w^3 = x(x-1)\}$$
since the curve $\{(x, y, u, w) \in \mathbb{C}^4 : y^7 = x(x-1)^2, w^3 = u(u-1), x = u\}$ is isomorphic to the curve $\{(x, y, w) \in \mathbb{C}^3 : y^7 = x(x-1)^2, w^3 = x(x-1)\}$.
- As a consequence of the above items, $S_1 \times_{(\beta_1,\beta_2)} S_2$ has exactly three singular points, namely $(0, 0, 0, 0)$, $(1, 0, 1, 0)$ and (∞, ∞) (in the affine version). Around each of them the fiber product looks like a single cone ($mcd(7, 3) = 1$). Thus, $S_1 \times_{(\beta_1,\beta_2)} S_2$ becomes into a compact Riemann surface after removing its three singular points and replace them by three new ones which, abusing of language, we will denote by $(0, 0, 0, 0)$, $(1, 0, 1, 0)$ and (∞, ∞). In this way, β is a regular Belyi covering with deck group isomorphic to $\mathbb{Z}_{21} \cong \mathbb{Z}_7 \times \mathbb{Z}_3$ (see Section 3.4.1) which has only three critical points, namely $(0, 0, 0, 0)$, $(1, 0, 1, 0)$ and (∞, ∞), with
$$mult_{(0,0,0,0)}\beta = mult_{(1,0,1,0)}\beta = mult_{(\infty,\infty)}\beta = 21$$
and three critical values $\beta\big((0, 0, 0, 0)\big) = 0$, $\beta\big((1, 0, 1, 0)\big) = 1$ and $\beta\big((\infty, \infty)\big) = \infty$.
- $gen(S_1 \times_{(\beta_1,\beta_2)} S_2) = 10$.

In fact, by Riemann -Hurwitz formula,
$$2gen(S_1 \times_{(\beta_1,\beta_2)} S_2) - 2 = 2 \cdot 21(0 - 1) + 3 \cdot 20$$
from which the assertion follows.

- A projective model of $S_1 \times_{(\beta_1,\beta_2)} S_2$ is given by
  $$\{[x:y:w:t] \in \mathbb{P}^3_{\mathbb{C}} : y^7 - x^3t^4 + 2x^2t^5 - xt^6 = 0,\ w^3 - x^2t + xt^2 = 0\}$$
  and in this model the three singular points are given by $[0:0:0:1]$, $[1:0:0:1]$ and $[1:0:0:0]$.

  In fact,
  $$y^7 = x(x-1)^2 \leftrightarrow y^7 - x^3 + 2x^2 - x = 0$$
  $$w^3 = x(x-1) \leftrightarrow w^3 - x^2 + x = 0$$
  from which the statement follows.

EXAMPLE 5.1.12. **An example where the field of moduli differs from the strong field of moduli of a fiber product.** *Let (S_1, β_1) be a Belyi pair whose field of definition and field of moduli is $\mathbb{Q}(i)$ and suppose $Gal(\mathbb{Q}(i)/\mathbb{Q}) = \langle\tau\rangle$ where $\tau(i) = -i$. If $\widetilde{\tau}$ is any extension to $\mathbb{C}$ of τ defining $S_2 = S_1^{\widetilde{\tau}}$ and $\beta_2 = \beta_1^{\widetilde{\tau}}$ one has that the field of moduli (if it is irreducible) and the strong field of moduli of the corresponding fiber product are $\mathbb{Q}$ and $\mathbb{Q}(i)$ respectively.*

Actually:

- Note first that if $\mathbb{K}_1$ and $\mathbb{K}_2$ denotes the fields of moduli of (S_1, β_1) and (S_2, β_2) respectively then $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{Q}(i)$.
- $\widetilde{\tau} \in G := \{\sigma \in Gal(\mathbb{C}/\mathbb{Q}) : (S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma, \beta^\sigma) \equiv (S_1 \times_{(\beta_1,\beta_2)} S_2, \beta)\}$.

  In fact, the map
  $$\begin{array}{rccc} f: & S_1 \times_{(\beta_1,\beta_2)} S_2 & \to & S_1^{\widetilde{\tau}} \times_{(\beta_1^{\widetilde{\tau}},\beta_2^{\widetilde{\tau}})} S_2^{\widetilde{\tau}} = S_2 \times_{(\beta_2,\beta_1)} S_1 \\ & (z_1, z_2) & \mapsto & f(z_1,z_2) = (z_2, z_1). \end{array}$$
  is an isomorphism (it can be easily checked) which satisfies that $\beta^{\widetilde{\tau}} \circ f = \beta$ since
  $$\beta^{\widetilde{\tau}} \circ f(z_1,z_2) = \beta^{\widetilde{\tau}}(z_2,z_1) = \beta_1^{\widetilde{\tau}}(z_2) = \beta_2(z_2) = \beta_2 \circ \pi_2(z_1,z_2) = \beta(z_1,z_2)$$
  for all $(z_1, z_2) \in S_1 \times_{(\beta_1,\beta_2)} S_2$.
- Since there is no possible isomorphism between S_1 and $S_1^{\widetilde{\tau}}$, we may see that $\widetilde{\tau}$ is not in $G_s := \{\sigma \in Gal(\mathbb{C}/\mathbb{Q}) : (S_1^\sigma \times_{(\beta_1^\sigma,\beta_2^\sigma)} S_2^\sigma, \beta^\sigma) \equiv^s (S_1 \times_{(\beta_1,\beta_2)} S_2, \beta)\}$ and thus $G_s \lneqq G$. In other words, if $\mathbb{K}$ and $\mathbb{K}_s$ denotes the field of moduli and the strong field of moduli of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$, since we know that $\mathbb{K} = Fix(G)$ and $\mathbb{K}_s = Fix(G_s)$, $\mathbb{K}_s \gneqq \mathbb{K}$ or, equivalently, the field of moduli of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is a proper subfield of its strong field of moduli.
- $\mathbb{K}_s = \mathbb{Q}(i)$ since $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{Q}(i)$ (see Theorem 3.4.1). Therefore, by the above item, $\mathbb{K} = \mathbb{Q}$.

Bibliography

[B] Belyĭ, G. V. *On Galois Extensions of a Maximal Cyclotomic Field*, Mathematics of the USSR-Izvestiya 14 (2) (1980), 247. doi:10.1070/IM1980v014n02ABEH001096

[E] C. J. Earle *On the moduli of closed Riemann surfaces with symmetries* Advances in the Theory of Riemann Surfaces, (1971), 119–130. Ed. L.V. Ahlfors et al, Princeton Univ. Press, Princeton.

[FH] W. Fulton and J. Hansen *A connectness theorem for projective verieties, with applications to intersections and singularities of mappings*, Annls of Math. 110 (1979), 159-166.

[FK] H. M. Farkas, I. Kra *Riemann surfaces*, Springer-Verlag, New York Heidelberg Berlin, 1980.

[GAP] The GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.8.3; 2016. (http://www.gap-system.org).

[GG] E. Girondo, G. González-Diez *Introduction to Compact Riemann Surfaces and Dessins d'Enfants*, London Mathematical Society Student Texts **79**. Cambridge University Press, Cambridge, 2012..

[GJ] G. González-Diez, A. Jaikin-Zapirain *The absolute Galois group acts faithfully on regular dessins and on Beauville surfaces*, Lond. Math. Soc. (3) 111 (2015), no. 4, 775-796.

[GRIF] P. Griffiths *Introduction to Algebraic Curves*, American Mathematical Society, Providence, Rhod Island, 1989.

[Gui] P. Guillot *An elementary approach to dessins d'enfants and the Grothendieck-Teichmller group* Enseign. Math. vol 60, 2014.

[H] B. Huggins *Fields of moduli of hyperelliptic curves* Math. Res. Lett 14 No. 2 (2007), 249–262.

[Hid] R. A. Hidalgo *The fiber product of Riemann surfaces: A Kleinian group point of view* Analysis and Mathematical Physics 1 (2011), 37–45.

[Hid1] R. A. Hidalgo *Non-hyperelliptic Riemann surfaces with real field of moduli but not definable over the reals* Archiv der Mathematik **93** (2009), 219–222.

[Hid2] R. A. Hidalgo *Erratum to: Non-hyperelliptic Riemann surfaces with real field of moduli but not definable over the reals* Archiv der Math **98** (2012), 449–451.

[Ii] S. Iitaka *Algebraic geometry: An introduction to the birational geometry of algebraic varieties*, Springer-Verlag, 1982.

[K] A. Kontogeorgis *Field of moduli versus field of definition for cyclic covers of the projective line* J. de Theorie des Nombres de Bordeaux **21** (2009) 679–692.

[KOS] Kosniowski *A first course in algebraic topology*, Cambridge University Press, Cambridge (England), 1986.

[MASS] W. S. Massey *Algebraic Topology: An Introduction*, Springer-Verlag, New York Berlin Heidelberg, 1967.

[MIR] R. Miranda *Algebraic Curves and Riemann surfaces*, Grad. Studies in Maths., vol 5, American Math. Society, 1997.

[MUN] J. Munkres *Topology*, Massachusets Institute of Technology, second edition, Prentice Hall, 2000.

[SHAF] I.R. Shafarevich *Basic Algebraic Geometry*, Springer-Verlag, Berlin, 1977.

[Shi] G. Shimura *On the field of rationality for an abelian variety* Nagoya Math. J. **45** (1972), 167–178.

[Weil] A. Weil *The field of definition of a variety* Amer. J. Math. **78** (1956), 509–524.

[Wlf] J. Wolfart *ABC for polynomials, dessins d'enfants and uniformization—a survey. Elementare und analytische Zahlentheorie* 313–345, Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main, 20, Franz Steiner Verlag Stuttgart, Stuttgart, 2006.