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QUANTUM ELECTRODYNAMICS SYSTEM FOR A LASER: FROM CAVITY QED TO WAVEGUIDE QED

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN CIENCIAS, MENCIÓN FÍSICA

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RESUMEN DE LA TESIS PARA OPTAR
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SISTEMAS ELECTRODINÁMICOS CUÁNTICOS PARA UN LÁSER: DESDE LAS CAVIDADES HASTA LAS GUÍAS DE ONDAS

La electrodinámica cuántica de cavidades se ha convertido en la herramienta canónica para el estudio de la interacción entre la luz y la materia, en particular, en aquellos regímenes en donde emerge el comportamiento cuántico de la luz, el cual puede estar asociado a aplicaciones en metrología o computación cuántica. En los últimos años, una nueva plataforma ha llamado la atención de los investigadores, la electrodinámica cuántica en guías de ondas.

Esta nueva plataforma, al igual que el caso de las cavidades, permite el control de la cooperatividad, es decir, la razón entre el acoplamiento de la materia (emisores) y los modos del campo electromagnético, y los procesos de disipación del sistema. Además, esta plataforma presenta gran flexibilidad para ser implementada con las tecnologías de fibra óptica que han sido ampliamente desarrolladas y adoptadas para distintas aplicaciones. Más aún, dichos sistemas presentan interacciones de largo alcance, lo que promueve la aparición de fenómenos colectivos como la superradiancia o subradiancia, en los cuales, los modos guiados del campo electromagnético sirven como medio para generar correlaciones entre los emisores, potenciando o inhibiendo la emisión.

Este trabajo plantea un modelo de un láser generado por un sistema electrodinámico cuántico en guías de ondas. Dicho modelo, describe tanto los componentes del láser, átomos y campo electromagnético emitido (sistema de interés), así como también, el reservorio que consiste en el mecanismo de bombeo y los modos del campo electromagnético no considerados como parte del sistema de interés. Posteriormente, se describe la dinámica del sistema de interés a través de las ecuaciones de Heisenberg-Langevin, a partir de las cuales, se estudia el régimen estacionario de emisión del láser. Finalmente, se presentan dos mecanismos para obtener el ancho espectral de emisión. El primer mecanismo considera que el campo puede ser descrito como su amplitud estacionaria más pequeñas fluctuaciones que serán ignoradas. Los resultados de este mecanismo muestran que el ancho espectral esta determinado por los procesos de decaimiento del campo asociados a la guía de ondas, mientras que su potencia depende del parámetro de bombeo y la cantidad de átomos que presenta el sistema. El segundo mecanismo, considera que el campo tiene una forma definida por una amplitud y una fase, y el ancho espectral estará determinado por los procesos de difusión de dicha fase. Los resultados de este mecanismo que considera fluctuaciones, muestran dos regímenes de comportamiento que dependen tanto de los procesos de decaimiento del campo como de los átomos.

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The Cavity Quantum Electrodynamics system is the canonical tool for studying the interaction between light and matter, motivated by applications in quantum metrology or quantum computing. In the last years, a new platform is taking importance, the waveguide Quantum Electrodynamics systems.

This platform, as well as the cavity case, is used to control the cooperativity of the system, i.e., the rate between the coupling strength of matter (emitters) and the electromagnetic field modes and the dissipation process of the system. Furthermore, this platform presents great versatility in connectivity to other systems, such as fiber optics technologies. Moreover, waveguide QED systems present long-range interaction, promoting collective effects such as superradiance and subradiance, where the guided modes of the electromagnetic field generate correlations between the emitters, enhancing or inhibiting the emission.

In this thesis, we present a model of a waveguide QED system laser. This model describes a system of interest that considers the atoms and the modes of the electromagnetic field of the waveguide. This system of interest interacts with a reservoir that considers the pumping mechanism and electromagnetic field modes out of the waveguide. From this model, we derive the Heisenberg-Langevin equation of the system of interest operator. Finally, we present two methodologies to study the spectrum of emission. The first mechanism considers that the electric field is determined by an amplitude defined by the steady-state solution of the bosonic operator's mean-field equation and fluctuations. These fluctuations are ignored. The results of this methodology show that the spectral linewidth is determined by the decay process of the waveguide, and the power of the emission is determined by the pumping parameter and the number of atoms. The second mechanism considers that the field is determined by an amplitude and a phase, and the spectral linewidth is determined by the diffusion process of the phase. The results of this methodology, which consider the effects of fluctuations, show two regimes where the spectrum depends on the decay process of the waveguide and the atoms.

Roads? Where we're going,
we don't need roads.

Dr. Emmet Brown

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El primer agradecimiento es a mi familia, que ha sido mi pilar más fundamental. Me imagino que debe ser difícil ver como alguien que quieres toma un rumbo incierto y, pese a creer que no es una buena decisión, aun así apoyar. Gracias por todo.

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¡GRACIAS TOTALES!"

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List of Variables

Variable	Symbol	Dimension
Discrete atomic-waveguide modes coupling	g_k	Hz
Continuous atomic-waveguide modes coupling	$g(\omega)$	$\text{Hz}^{1/2}$
Discrete bosonic operator	a_k	1
Continuous bosonic operator	$a(\omega_k)$	$\text{Hz}^{-\frac{1}{2}}$
Atomic Pauli operators	$\sigma^{+/-/z}$	1
Colective Atomic Pauli operators	$S^{+/-/z}$	1

Table 1: System of Interest Variables

Variable	Symbol	Dimension
Discrete Atomic-reservoir modes coupling	$\kappa_{\mu,i}$	Hz
Discrete Atomic-pumping modes coupling	$\tilde{\kappa}_{\beta,i}$	$\text{Hz}^{\frac{1}{2}}$
Discrete Effective waveguide modes-reservoir modes coupling	$\bar{\kappa}_{\mu,i}$	Hz
Continuous Atomic-reservoir modes coupling	$\kappa(\omega)$	$\text{Hz}^{\frac{1}{2}}$
Continuous Atomic-pumping modes coupling	$\tilde{\kappa}(\omega)$	1
Continuous Effective waveguide modes-reservoir modes coupling	$\zeta(\omega)$	Hz
Reservoir bosonic operator	$\tilde{r}_\mu/\tilde{r}_\mu^\dagger$	1
Pumping operator	$\mathcal{P}_\beta^+/\mathcal{P}_\beta^-$	1
Continuous Reservoir bosonic operator	$\tilde{r}(\omega)/\tilde{r}^\dagger(\omega)$	$\text{Hz}^{-\frac{1}{2}}$
Continuous Pumping operator	$\mathcal{P}(\omega)^+/\mathcal{P}(\omega)^-$	$\text{Hz}^{-\frac{1}{2}}$

Table 2: Reservoir Variables

Introduction

One of Quantum Optics's main challenges is manipulating light's quantum states. To this aim, it is necessary to control the coupling between the system of interest, e.g., an emitter and one or several modes of the electromagnetic field, and the coupling of the system of interest and the reservoir. In particular, when the control of quantum fluctuation enables technological applications. This tool has been studied in different regions of the electromagnetic field, from the microwave by Haroche[1] to optical by Kimble[2] and Reiserer et al. [3].

The cavity QED systems are the milestone platforms used to control electromagnetic field properties[4, 5]. This platform uses mirrors to modify the vacuum modes of the field, inducing preferential interaction between an emitter and a particular mode. The control of this preferred interaction leads to the observation of phenomena such as the Purcell effect[6]- the modification of spontaneous emission rates-, photon-mediated interaction, or synchronization processed between the emitters, which exhibits collective behavior. The cavity QED systems are the milestone platforms used to control electromagnetic field properties[4, 5]. This platform uses mirrors to modify the vacuum modes of the field, inducing preferential interaction between an emitter and a particular mode. The control of this preferred interaction leads to the observation of phenomena such as the Purcell effect[6]- the modification of spontaneous emission rates-, photon-mediated interaction, or synchronization processed between the emitters, which exhibits collective behavior.

To characterize the ratio between the coupling with these preferred modes compared with the interaction with the reservoir, a parameter known as cooperativity is defined. This cooperativity parameter is determined by the coupling strength g, between the preferred cavity mode and the emitter, the decay rate κ, which characterizes the effect of the cavity walls, and the emitter's spontaneous emission rate γ.

$$C = \frac{g^2}{\gamma\kappa} \tag{0.1}$$

The regime when $C > 1$ is known as a strong coupling regime, where the preferred interaction is larger than the interaction with the reservoir.

In recent years, a new platform known as waveguide QED systems has emerged as a promising tool to modify the electromagnetic field. A waveguide QED system consists of an ensemble of emitters that could be coupled to nanophotonic waveguides or microwave waveguides, where the electric field is confined, increasing amplitude and promoting the interaction with the emitters. This tool could be designed to control the decay rate between the guided modes and the free modes. Other promising properties are the presence of long-range interactions between the emitters mediated by the guided modes[7], as well as the versatility to be coupled to conventional optical fiber technologies.

Recent research developed in waveguide QED systems has shown technological applications. For example, the generation of Fock states[8], the generation of multiphotonic states generated by the collective behavior of the emitters[9, 10], and entanglement between qubits mediated by waveguides[11, 12].

Coherent interaction between a large number of emitters has been recently studied. Theoretical studies predict the emergence of collective behavior, such as the generation of entanglement states and collective enhancement or suppression of the emission by superradiance or subradiance regimes.

This thesis aims to model waveguide QED as a platform of implementation for a laser. In this model, the waveguide defines the lasing mode, atoms around it act as a gain medium, and an external optical field enables a pump. Moreover, the main goal is to study how the presence of collective behavior mediated by the platform affects the properties of the laser, in particular, the linewidth of the emission.

This thesis is organized into 5 chapters.

Chapter 1, presents fundamental concepts necessary to characterize the behavior of a laser. We will start presenting theoretical tools of quantum optics and open quantum systems. Then we will introduce the basic elements of a laser and the canonical example of a model for a laser in a cavity QED system.

Chapter 2, presents the model developed in this project in order to represent a laser implemented in a waveguide QED system. In this chapter, we will use an open quantum system approach to describe a total system, which consists of a system of interest and a reservoir. The system of interest consists of the electromagnetic modes inside the waveguide and the atoms acting as an active medium. The reservoir considers the pumping mechanism and the free electromagnetic field modes. At the end of this chapter, we will present the Hamiltonian of the total system in two cases. A discrete model that could represent a multimodal cavity QED system. And a continuous model that represents the waveguide QED system will be used in the following chapters.

Chapter 3, presents the derivation of the equation of motion of the system of interest operators. In particular, we will use the Heisenberg-Langevin formalism to derive the equation of motion, which considers the effects of the fluctuation generated by the interaction with the reservoir.

The last two chapters introduce two different methodologies used to characterize the laser linewidth of the waveguide QED system.

In chapter 4, we present a methodology to obtain the spectrum of the laser by defining an electric field as a mean-field term plus fluctuations. The mean-field term will be derived from the steady-state solutions of the Heisenberg-Langevin equations of motion. At the end of this chapter, we will study the spectrum of the field ignoring the fluctuating terms.

Finally, in chapter 5, we will present a methodology to derive the spectrum of emission of the waveguide QED system laser considering the effects of fluctuations. To this aim, we will derive the equations of motions of the phase of each mode of the electromagnetic field inside the waveguide, and use it to obtain an expression for the coherence function as the sum of each mode contribution. At the end of this chapter, we will study the spectrum of this emission by numerically applying the Wiener-Khintchine Theorem. The numerical results will be studied in two regimes, which we will call "generalized good/bad cavity regimes".

Chapter 1

Fundamental Concepts

This chapter will present the building blocks necessary to develop a fully quantum mechanical model for a laser in a waveguide QED system.

We will start with the description of how light and matter interact with each other. In section 1.1, we will introduce the basic concept of quantum optics, starting with the quantization of the electromagnetic field, presenting some states of this quantized field, and a model of interaction between this quantized field and the quantized matter.

Then, in order to consider that almost all quantum optics systems are not entirely isolated from their environment, in section 1.2, we will present the open quantum system formalism. This formalism is used to describe the interaction between a system of interest and its reservoir, considered as all other degrees of freedom that are not included in the system of interest. Moreover, we will present the equation of motion of this system of interest, considering the effects of the reservoir.

Once we understand the general idea of how an open quantum optics system could be described, we will focus on the main topic of this thesis, the laser. In section 1.3, we will present the importance of the laser as a motivation for this work.

Finally, in section 1.4, we will present a model of a cavity QED system of a laser and a method to obtain the spectral linewidth of the laser emission based on the study of the effects of fluctuation in the diffusion of the phase of the field.

1.1. Quantum Optics

Quantum optics is the physics area dedicated to studying the interaction between light and matter, both intrinsically quantum. The quantum nature of matter determines its discretized levels of energy. When matter interacts with light through absorption or emission[13], it is called an emitter. There are several types of emitters, such as atoms or molecules, superconductor qubits, or NV-centers in diamonds[14, 15]. On the other hand, the quantum nature of light is manifested by quantum fluctuations, which satisfies the Heisenberg uncertainty principle.

In this section, we will present several quantum mechanics formalisms used to describe the quantum nature of the electromagnetic field and its interaction with matter. In section 1.1, we derive the quantization of the electromagnetic field in order to obtain an expression for the Hamiltonian and the electric and magnetic quantized fields. In section 1.1.2, we introduce two types of quantum states of the electromagnetic field, the number or Focks states and the coherent states, each of one in the unimodal and multimodal cases. Section 1.1.3 present the generalization of the discrete coherent states to the continuous case, with great importance in this thesis. Finally, section 1.1.4 present a model to represent the interaction between the quantized light described in the previous sections and the quantized matter in the dipole approximation.

1.1.1. Quantization of the Electromagnetic Field

We will consider the Columb gauge conditions $\nabla \cdot \mathbf{A}(r,t) = 0$ to describe the electromagnetic field, where $\mathbf{A}$ is the vector potential that satisfies the wave equation [16, 17].

$$\nabla^2 \mathbf{A} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{1.1}$$

Then, the electric and magnetic fields in the free space with no sources of radiation are determined by:

$$\mathbf{E}(r,t) = -\frac{\partial \mathbf{A}(r,t)}{\partial t} \tag{1.2}$$

$$\mathbf{B}(r,t) = \boldsymbol{\nabla} \times \mathbf{A}(r,t) \tag{1.3}$$

We will define a cavity of size L with a perfectly reflecting mirror, where L will be large compared with the wavelength of the light. The vector potential takes the form of a sum of plane waves:

$$\mathbf{A}(r,t) = \sum_{k,s} \hat{e}_{k,s} \left[A_{k,s}(t) e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}} + A^*_{k,s}(t) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}} \right] \tag{1.4}$$

Where $A_{k,s}$ is the complex amplitude and $\hat{e}_{k,s}$ is a real polarization vector, the index k and s represent the wavenumber and the two independent polarization, respectively.

The equation (1.1) becomes

$$\frac{d^2}{dt} A_{k,s} + \omega_k^2 A_{k,s} = 0 \tag{1.5}$$

with $\omega_k = ck$.

Moreover, the electric and magnetic fields take the form:

$$\mathbf{E}(r,t) = i \sum_{k,s} \omega_k \hat{e}_{k,s} \left[A_{k,s} e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega_k t)} - A^*_{k,s} e^{-i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega_k t)} \right] \tag{1.6}$$

$$\mathbf{B}(r,t) = \frac{i}{c} \sum_{k,s} \omega_k \left(\frac{\vec{\mathbf{k}} \times \hat{e}_{k,s}}{|\vec{\mathbf{k}}|} \right) \left[A_{k,s} e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega_k t)} - A^*_{k,s} e^{-i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega_k t)} \right] \tag{1.7}$$

The energy of the field is determined by the following:

$$H = \frac{1}{2}\int_V \left(\epsilon \mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right) dV \tag{1.8}$$

Where:

$$\int_V dV e^{(\pm i(\vec{\mathbf{k}}-\vec{\mathbf{k'}})\cdot\vec{\mathbf{r}})} = V\delta_{k,k'} \tag{1.9}$$

$$\int_V dV e^{(\pm i(\vec{\mathbf{k}}+\vec{\mathbf{k'}})\cdot\vec{\mathbf{r}})} = V\delta_{k,-k'} \tag{1.10}$$

Then, the energy takes the form [17]

$$\begin{aligned} E(r,t) = \frac{1}{2}\sum_k\sum_{s,s'} V[(A_{k,s}A^*_{k,s'} + A^*_{k,s}A_{k,s})(\epsilon_0\omega_k^2\hat{e}_{k,s}\cdot\hat{e}_{k,s'} + \mu_0^{-1}\vec{\mathbf{k}}\times\hat{e}_{k,s}\cdot\vec{\mathbf{k}}\times\hat{e}_{k,s'}) \\ -(A_{k,s}A_{-k,s'}e^{-2i\omega_k t} + A^*_{k,s}A^*_{-k,s'}e^{2i\omega_k t})(\epsilon_0\omega_k^2\hat{e}_{k,s}\cdot\hat{e}_{-k,s'} + \mu_0^{-1}\vec{\mathbf{k}}\times\hat{e}_{k,s}\cdot\vec{\mathbf{k}}\times\hat{e}_{-k,s'})] \end{aligned} \tag{1.11}$$

If we use the vector product property $\vec{\mathbf{k}}\times\hat{e}_{k,s}\cdot\vec{\mathbf{k}}\times\hat{e}_{\pm k,s'} = k^2\hat{e}_{k,s}\cdot\hat{e}_{\pm k,s'}$, then the energy could be rewritten as:

$$E = \sum_k\sum_s \epsilon_0 V\omega_k^2(A_{k,s}A^*_{k,s} + A^*_{k,s}A_{k,s}). \tag{1.12}$$

If we introduce the change of variables:

$$A_{k,s} = \frac{1}{2\omega_k\sqrt{\epsilon_0 V}}\left(\omega_k q_k + ip_{k,s}\right) \tag{1.13}$$

$$A^*_{k,s} = \frac{1}{2\omega_k\sqrt{\epsilon_0 V}}\left(\omega_k q_k - ip_{k,s}\right) \tag{1.14}$$

Where the q_k and p_k could be recognized as the canonical position and momentum variables. If we replace these variables in the Hamiltonian, it gets the form of a simple harmonic oscillator:

$$H = \frac{1}{2}\sum_{k,s}\left(p_{k,s}^2 + \omega_{k,s}^2 q_{k,s}^2\right). \tag{1.15}$$

Promoting the canonical variables to operators, $p \rightarrow \hat{p}$ and $q \rightarrow \hat{q}$, which satisfies the canonical commutation relation:

$$[\hat{q}_{k,s}, \hat{q}_{k',s'}] = 0 = [\hat{p}_{k,s}, \hat{p}_{k',s'}] \tag{1.16}$$

$$[\hat{q}_{k,s}, \hat{p}_{k,s}] = i\hbar\delta_{k,k'}\delta_{s,s'} \tag{1.17}$$

Defining the creation and annihilation operators:

$$\hat{a}_{k,s} = \frac{1}{2\pi\hbar{\omega_k}^{1/2}} [\omega_k\hat{q}_{k,s} + i\hat{p}_{ks}] \tag{1.18}$$

$$\hat{a}^{\dagger}_{k,s} = \frac{1}{2\pi\hbar{\omega_k}^{1/2}} [\omega_k\hat{q}_{k,s} - i\hat{p}_{ks}] \tag{1.19}$$

which satisfies the canonical commutation relations

$$[\hat{a}_{k,s}, \hat{a}_{k',s'}] = 0 = [a^{\dagger}_{k,s}, a^{\dagger}_{k',s'}] = \delta_{k,k'}\delta_{s,s'}, \tag{1.20}$$

$$[\hat{a}_{k,s}, \hat{a}^{\dagger}_{k',s'}] = \delta_{k,k'}\delta_{s,s'}. \tag{1.21}$$

Finally, the Hamiltonian operator of the free quantized electromagnetic field:

$$\hat{H} = \sum_{k,s} \hbar\omega_k \left(\hat{a}^{\dagger}_{k,s}\hat{a}_{k,s} + \frac{1}{2}\right) = \sum_{k,s} \hbar\omega_k \left(\hat{n}_{k,s} + \frac{1}{2}\right) \tag{1.22}$$

The last term of the RHS of equation (1.22) is known as zero energy point and is related to phenomena such as the Casimir effect. In the following sections, we will neglect this term because it does not affect the dynamics of the systems studies in this thesis.

The quantized electric and magnetic fields take the form:

$$\hat{\mathbf{E}}(r,t) = \hat{\mathbf{E}}^{+}(r,t) + \hat{\mathbf{E}}^{-}(r,t) \tag{1.23}$$

$$\hat{\mathbf{B}}(r,t) = \hat{\mathbf{B}}^{+}(r,t) + \hat{\mathbf{B}}^{-}(r,t) \tag{1.24}$$

Where the positive frequency part of the electric and magnetic field:

$$\hat{\mathbf{E}}^{+}(r,t) = i\sum_{k,s} \hat{e}_{k,s} \left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{\frac{1}{2}} \hat{a}_{k,s} e^{-i(\omega_k t - \vec{\mathbf{k}}\cdot\vec{\mathbf{r}})} \tag{1.25}$$

$$\hat{\mathbf{B}}^{+}(r,t) = i\sum_{k,s}\vec{\mathbf{k}}\times\hat{e}_{k,s}\left(\frac{\hbar}{2\epsilon_0\omega_k V}\right)^{\frac{1}{2}}\hat{a}_{k,s}e^{-i(\omega_k t-\vec{\mathbf{k}}\cdot\vec{\mathbf{r}})} \tag{1.26}$$

1.1.2. States of the Electromagntic Field

From the Hamiltonian of the free electromagnetic field (1.22)

$$\hat{H} = \sum_{k,s}\hbar\omega_k\left(\hat{a}^{\dagger}_{k,s}\hat{a}_{k,s}+\frac{1}{2}\right) = \sum_{k,s}\hbar\omega_k\left(\hat{n}_{k,s}+\frac{1}{2}\right) \tag{1.27}$$

Where $\hat{n}_{k,s}$ is the photon number operator of the mode with k wave number and polarization s, with eigenvalues of the form:

$$\hat{n}_{k,s}\left|n_{k,s}\right\rangle = \hat{a}^{\dagger}_{k,s}\hat{a}_{k,s}\left|n_{k,s}\right\rangle = n_{n,s}\left|n_{k,s}\right\rangle \tag{1.28}$$

The orthonormal eigenstates $\left|n_{k,s}\right\rangle$ are the photon number states or Fock states of the electromagnetic field [17].

The state of the total field is the product of the states of each unimodal case.

$$\left|n_{k_1,1},n_{k_1,2},n_{k_2,1},n_{k_2,2},..\right\rangle = \left|n_{k_1,1}\right\rangle\left|n_{k_1,2}\right\rangle\left|n_{k_2,1}\right\rangle\left|n_{k_2,2}\right\rangle\left|n_{k_3,1}\right\rangle .. = \left|\{n_{k,s}\}\right\rangle \tag{1.29}$$

Another example of electromagnetic field states of essential importance is the coherent state, denoted by $\left|\{\alpha\}\right\rangle$. Those states are distinguished because they are one of the quantum-mechanical states with the closest properties to classical electromagnetic waves. In the limit of strong excitation, the electric field variation of a coherent state approaches a classical wave with stable amplitude and fixed phase.

In the unimodal case, we could define coherent states in the number states basis:

$$\left|\alpha\right\rangle = e^{(-\frac{1}{2}|\alpha|^2)}\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}\left|n\right\rangle \tag{1.30}$$

where α is a complex number.

Coherent states are defined as right eigenstates of the destruction operator:

$$\hat{a}\left|\alpha\right\rangle = e^{(-\frac{1}{2}|\alpha|^2)}\sum_{n}\frac{\alpha^n}{(n!)^{1/2}}n^{1/2}\left|n-1\right\rangle = \alpha\left|\alpha\right\rangle \tag{1.31}$$

From the last expression, we could rewrite the equation (1.30).

$$\left|\alpha\right\rangle = e^{(-\frac{1}{2}|\alpha|^2)}\sum_{n}\frac{(\alpha\hat{a}^{\dagger})^n}{n!}\left|0\right\rangle = e^{(\alpha\hat{a}^{\dagger}-\frac{1}{2}|\alpha|^2)}\left|0\right\rangle \tag{1.32}$$

The generalization to the multimodal case is:

$$|\{\alpha\}\rangle = |\alpha_{k_1}\rangle \, |\alpha_{k_2}\rangle \, |\alpha_{k_3}\rangle \, |\alpha_{k_4}\rangle \, .. \tag{1.33}$$

where $\{\alpha\}$ denotes the complete set of complex amplitudes for each specific excited mode[17].

It follows that the electric field in the multimodal case could be rewritten as:

$$\hat{\mathbf{E}}^{-}(z,t)\,|\{\alpha\}\rangle = \sum_k \left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{\frac{1}{2}} \alpha_k e^{-i(\omega_k t - kz)}\,|\{\alpha\}\rangle\,. \tag{1.34}$$

The multimode coherent state is an eigenstate of the positive frequency part of the electric field operator.

Coherent states will play a central role in this thesis. In particular, this work aims to characterize the emission of a laser, which is determined by coherent state excitation in the case of single-mode lasers that operate above the threshold.

1.1.3. Continuos-mode Field Operators

This thesis aims to extend the canonical implementation of a laser in a cavity QED system to a waveguide QED system. The main characteristic of this new platform is that it presents open boundary conditions. The modes inside the waveguide could be described as continuous instead of the discrete case, as in a cavity.

To this aim, we will generalize the discrete coherent state, presented in the previous section, to the continuous case. We will consider a quantization volume V, defined as a transversal area A perpendicular to the z-direction and a length $L \to \infty$ in the z-direction. This configuration defines one-dimensional continuous modes characterized by a wavevector k and a frequency ω, with a mode spacing $\Delta\omega = 2\pi c L$, which tends to zero when the length L tends to infinity.

To obtain the fields operators in a continuous form, we will extend the sum to an integral:

$$\sum_k \to \frac{1}{\Delta\omega}\int d\omega_k. \tag{1.35}$$

The continuous-mode creation and destruction operators $\hat{a}(\omega_k)$ and $\hat{a}^\dagger(\omega_k)$,

$$\hat{a}_k \to (\Delta\omega)^{1/2}\hat{a}(\omega_k) \qquad \hat{a}_k^\dagger \to (\Delta\omega)^{1/2}\hat{a}^\dagger(\omega_k). \tag{1.36}$$

The generalized continuous-mode commutation relations:

$$[\hat{a}(\omega_k), \hat{a}^\dagger(\omega_{k'})] = \delta(\omega_k - \omega_{k'}). \tag{1.37}$$

Then, the continuous-modes electric and magnetic field operators are:

$$\hat{\mathbf{E}}^{+}(z,t) = i\int d\omega_k \left(\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}\right)^{\frac{1}{2}} \hat{a}(\omega_k)e^{-i(\omega_k t-\frac{z}{c})} \tag{1.38}$$

$$\hat{\mathbf{B}}^{+}(z,t) = i\int d\omega_k \left(\frac{\hbar\omega_k}{4\pi\epsilon_0 c^3 A}\right)^{\frac{1}{2}} \hat{a}(\omega_k)e^{-i(\omega_k t-\frac{z}{c})} \tag{1.39}$$

Finally, the continuous-modes Hamiltonian is:

$$\hat{H} = \int d\omega_k \hbar\omega_k \hat{a}^{\dagger}(\omega_k)\hat{a}(\omega_k) \tag{1.40}$$

where we have neglected the vacuum energy.

1.1.4. Light-Matter Interaction

Quantum Optics is the study area of light and matter interaction. Throughout this thesis, the interaction between the quantized field (described in the previous section) and emitters (quantized matter) will be restricted to the dipole approximation.

To understand this interaction, we will consider a toy model, which consists of an atom interacting with one resonant electromagnetic field mode of a cavity. A two-level system emitter will represent the atom. The free Hamiltonian of the emitter will be expressed in terms of the pseudo-spin operator, and the bosonic operator will model the free Hamiltonian of the electromagnetic field on the cavity.

$$H_{free} = \frac{1}{2}\hbar\omega_A\sigma^z + \hbar\omega a^{\dagger}a + \frac{1}{2}\hbar\omega \tag{1.41}$$

The difference in energy between the atomics levels is $\hbar\omega_A = E_e - E_g$, where E_g is the energy of the ground state and E_e is the energy of the excited state, and ω_A is the resonance frequency of the atomic transition. The ω term is the resonance frequency of the cavity mode. The last term of the equation (1.41) represents the energy of vacuum fluctuation of the electromagnetic field.

The dipolar approximation assumes that the wavelength of the electromagnetic field mode is much longer than the characteristic length of the emitter. The interaction Hamiltonian takes the form:

$$H_{int} = -\hat{\mathbf{d}}\cdot\hat{\mathbf{E}} \tag{1.42}$$

If the matter is described as a two-level system, the dipole operator $\hat{d}$ is given by:

$$\hat{\mathbf{d}} = \langle 0|\,\hat{\mathbf{d}}\,|1\rangle\,(|0\rangle\langle 1| + |1\rangle\langle 0|) = \langle 0|\,\hat{\mathbf{d}}\,|1\rangle\,(\sigma^- + \sigma^+) = \mathcal{D}(\sigma^- + \sigma^+) \tag{1.43}$$

where σ^- and σ^+ are the rising and lowering Pauli operators and $\mathcal{D} = \langle 0|\,\hat{\mathbf{d}}\,|1\rangle$ is the dipole operator matrix element.

While the resonant electric field mode of a unimodal cavity in the Schrödinger picture is given by equation (1.25):

$$\hat{\mathbf{E}} = i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{\frac{1}{2}}\hat{\mathbf{e}}\left(\hat{a}-\hat{a}^{\dagger}\right) \tag{1.44}$$

Then, the interaction Hamiltonian takes the form:

$$\begin{aligned} H_{int} = & \quad -\mathcal{D}\cdot\mathcal{E}_{\mathbf{0}}(\sigma^{+}+\sigma^{-})\left(\hat{a}-\hat{a}^{\dagger}\right) \\ = & \quad -\hbar g(\sigma^{+}+\sigma^{-})\left(\hat{a}-\hat{a}^{\dagger}\right) \end{aligned} \tag{1.45}$$

where $\mathcal{E}_{\mathbf{0}} = i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{\frac{1}{2}}\hat{\mathbf{e}}$ and we define the coupling streght $g = \mathcal{D}\cdot\mathcal{E}_{\mathbf{0}}$.

1.2. Open Quantum System

In quantum optics, it is common to find problems where the system of interest is subject to dissipation processes; those situations are known as open quantum systems. This section deals with the dissipation of open quantum systems from the system-plus-reservoir approach, where the system of interest consisting of only a few relevant degrees of freedom is in contact with an environment with a very large or infinite number of degrees of freedom. The evolution of the system of interest could be studied from the Schrödinger picture through a quantum master equation or the Heisenberg picture using the Heisenberg-Langevin evolutions of the system operators.

This section is divided as follows. Subsection 1.2.1 presents an apparent inconsistency between the Heisenberg uncertainty principle with dissipation in quantum systems and how the role of the environment explains this situation. Subsection 1.2.2 describe the derivation of a master equation of a damped two-level system. Finally, subsection 1.2.3 studies the same problem from the perspective of Heisenberg-Langevin formalism.

1.2.1. System-Plus-Reservoir Approach

To understand dissipation in quantum mechanics, we will consider the classical dissipation treatment in a harmonic oscillator[18].

The classical harmonic oscillator Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q \tag{1.46}$$

With equations of motion:

$$\dot{q} = \frac{p}{m} \qquad \dot{p} = -m\omega^2 q. \tag{1.47}$$

We will include dissipation through a velocity-dependent force of the form $-\gamma p$, and the dissipative equation becomes:

$$\dot{q} = \frac{p}{m} \qquad \dot{p} = -\gamma p - m\omega^2 q. \tag{1.48}$$

Or equivalent:

$$\ddot{q} + \gamma\dot{q} + \omega q^2 = 0. \tag{1.49}$$

If we quantize this system by promoting the canonical variables to the operator, $p \to \hat{p}$ and $q \to \hat{q}$, which satisfy the commutation relation $[\hat{q}, \hat{p}] = i\hbar$, then the equations (1.48)

$$\dot{\hat{q}} = \frac{\hat{p}}{m} \qquad \dot{\hat{p}} = -m\omega^2 \hat{q}. \tag{1.50}$$

However, if we study the evolution of the uncertainty relation:

$$\frac{d}{dt}[\hat{q}, \hat{p}] = \frac{d}{dt}(\hat{q}\hat{p} - \hat{p}\hat{q}) = \dot{\hat{q}}\hat{p} + \hat{q}\dot{\hat{p}} - \dot{\hat{p}}\hat{q} - \hat{p}\dot{\hat{q}} \tag{1.51}$$

With a solution,

$$[\hat{q}(t), \hat{p}(t)] = e^{-\gamma t}[\hat{q}(0), \hat{p}(0)] = i\hbar e^{-\gamma t} \tag{1.52}$$

This situation becomes an inconsistency because of the decay behavior of the uncertainty principle [1]. We will consider the fundamental relationship between fluctuation and dissipation to overcome this misleading.

The system-plus-reservoir approach states that a small system of interest interacts with a large environment in the thermal equilibrium. This environment exerts a fluctuating force in the system of interest and induces dissipation. From this perspective, damping is a consequence of the coupling of the system with its environment.

The equation of motion (1.49) takes the form:

$$\ddot{q} + \gamma\dot{q} + \omega q^2 = \frac{F(t)}{m}. \tag{1.53}$$

In the harmonic oscillator case, defining a reservoir interacting with the system of interest is necessary. For simplicity, we will consider the that this reservoir takes the form of another harmonic oscillator. Then, the Hamiltonian of the total system (system of interest plus reservoir) becomes:

$$\hat{H} = \hbar\hat{a}^\dagger\hat{a} + \hbar\hat{b}^\dagger\hat{b} + \hbar\kappa\left(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger\right) \tag{1.54}$$

[1] This inconsistency is not only valid in the quantum regime. In classical systems, which include dissipation, it is necessary to consider the effect of the environment through a fluctuation that leads to dissipation.

The Heisenberg evolution equation of $\hat{a}$ and $\hat{b}$:

$$\dot{\hat{a}} = -\frac{i}{\hbar}[\hat{a}, \hat{H}] = -i\omega\hat{a} - i\kappa\hat{b} \tag{1.55}$$

$$\dot{\hat{b}} = -\frac{i}{\hbar}[\hat{b}, \hat{H}] = -i\omega\hat{b} - i\kappa\hat{a} \tag{1.56}$$

With solutions

$$\hat{a}(t) = e^{-i\omega t}[\hat{a}(0)\cos(\kappa t) - i\hat{b}(0)\sin(\kappa t)] \tag{1.57}$$

$$\hat{b}(t) = e^{-i\omega t}[\hat{b}(0)\cos(\kappa t) - i\hat{a}(0)\sin(\kappa t)] \tag{1.58}$$

So, the commutation relation:

$$[\hat{a}, \hat{a}^\dagger] = [\hat{a}(0)\cos(\kappa t) - i\hat{b}(0)\sin(\kappa t), \hat{a}^\dagger(0)\cos(\kappa t) + i\hat{b}(0)\sin(\kappa t)] = 1 \tag{1.59}$$

The consideration of an environment acting over the damped system through a fluctuating force introduces thermal fluctuations and preserves the commutation relations.

1.2.2. Master Equation

A closed system is characterized by its states, usually represented by a density operator defined on a Hilbert space, and a Hamiltonian determines its dynamics. The density operator evolves by the Liouville-von Neumann equation:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\mathcal{H}, \rho] \tag{1.60}$$

In the case of an open system, we will be interested in a subsystem of the more extensive closed system. The total system consists of a system of interest and its environment or reservoir, with states defined in the system and the reservoir Hilbert space[19–21]. The dynamic is governed by a total system Hamiltonian of the form:

$$H_T = H_S + H_R + V, \tag{1.61}$$

where H_S is the Hamiltonian of our system of interest, H_R is the reservoir Hamiltonian, and V is the interaction between the system of interest and the reservoir, each acting in the system, the reservoir and the product Hilbert space.

The system of interest and the reservoir states are obtained by tracing the total system state

$$\rho_S = Tr_R[\rho] \text{ and } \rho_R = Tr_S[\rho]. \tag{1.62}$$

To obtain the evolution equation of the system of interest density matrix, we will study the Liouville-von Neumann equation of the total state in the interaction picture

$$\frac{d}{dt}\tilde{\rho}(t) = -i[\tilde{V}(t), \tilde{\rho}(t)] \tag{1.63}$$

where $\tilde{\rho} = U(t)\rho(t)U^{\dagger}(t)$ and $\tilde{V} = U(t)V(t)U^{\dagger}(t)$ are the density matrix of the system of interest and the interaction Hamiltonian in the interaction picture, with $U(t) = e^{[i(H_s+H_R)t]}$.

The solution of the equation (1.63) is:

$$\tilde{\rho}(t) = \tilde{\rho}(0) - i\int_0^t dt'[\tilde{V}(t'), \tilde{\rho}(t')]. \tag{1.64}$$

If we replace the solution in the Liouville-von Neumann equation:

$$\frac{d}{dt}\tilde{\rho}(t) = -i[\tilde{V}(t), \tilde{\rho}(0)] - \int_0^t dt'[\tilde{V}(t'), [\tilde{V}(t'), \tilde{\rho}(t')]]. \tag{1.65}$$

Tracing out the reservoir degrees of freedom, we obtain:

$$\frac{d}{dt}\tilde{\rho}_s = -iTr_R\{[\tilde{V}(t), \tilde{\rho}(0)]\} - Tr_R\{\int_0^t dt'[\tilde{V}(t'), \tilde{\rho}(t')]\} \tag{1.66}$$

This equation is exact, but now we will take the Born-Markov approximation. The Born approximation assumes that the reservoir is large compared to the system of interest and that the coupling is weak. The time dependence of the reservoir could be neglected, and the total system density matrix takes the form:

$$\tilde{\rho}(t) = \tilde{\rho}_S(t) \otimes \tilde{\rho}_R. \tag{1.67}$$

The Markov approximation considers that the reservoir is in a stationary state with short correlation times compared with the system of interest time scales. The system of interest only depends on the present time of the environment, and the changes imprinted in the reservoir in the past can not return. Then, the memory effects could be neglected.

Finally, the dynamic of the system of interest is described by:

$$\frac{d}{dt}\tilde{\rho}_s(t) = -\int_0^\infty ds Tr_R\{[\tilde{V}(t), [\tilde{V}(t-s), \tilde{\rho}_S(t) \otimes \tilde{\rho}_R]]\} \tag{1.68}$$

where we have replaced $\tilde{\rho}(t') \to \tilde{\rho}(t)$, extended the limit of time to infinity since the time scales of the reservoir is shorter than the system scales $t_R << t_S$, and vanished the system-reservoir correlations.

If we consider a specific interaction of the form:

$$V = \hbar \sum_i s_i \Gamma_i \tag{1.69}$$

where s_i are the system of interest operator defined in the Hilbert space $\mathcal{H}_S$ and the Γ_i are the reservoir operator defined in the Hilbert space $\mathcal{H}_R$.

Then, the master equation takes the form:

$$\begin{aligned} \frac{d}{dt}\tilde{\rho}_s(t) = -\sum_{i,j}\int_0^t dt' \quad & \{[\tilde{s}_i(t)\tilde{s}_j(t')\tilde{\rho}(t) - \tilde{s}_j(t')\tilde{\rho}(t)\tilde{s}_i(t)]\langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\rangle_R \\ & + [\tilde{\rho}(t)\tilde{s}_j(t')\tilde{s}_i(t) - \tilde{s}_i(t)\tilde{\rho}(t')\tilde{s}_j(t')]\langle\tilde{\Gamma}_i(t')\tilde{\Gamma}_j(t)\rangle_R\} \end{aligned} \tag{1.70}$$

Where the correlation function decay rapidly compare with the time scale of which $\tilde{\rho}_s(t)$ varies, $\langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\rangle_R \propto \delta(t-t')$.

1.2.2.1. Master Equation of the Damped Harmonic Oscillator

A canonical example in Quantum Optics is the case of a quantum harmonics oscillator (e.g., an electromagnetic field cavity mode) interacting with a reservoir of quantum harmonic oscillators[18].

The Hamiltonian of the system of interest is

$$H_S = \hbar\omega_0 a^\dagger a \tag{1.71}$$

where ω_0 is the resonance cavity mode frequency, and a and $a^\dagger$ are the creation and annihilation operators or system of interest operators defined in H_s.

The reservoir Hamiltonian is

$$H_R = \sum_j \hbar\omega_j r_j^\dagger r_j, \tag{1.72}$$

where ω_j is the frequency of each quantum harmonic oscillator in the reservoir, and r_j and $r_j^\dagger$ are the creation and annihilation reservoir operators defined in H_R.

The interaction Hamiltonian takes the form:

$$V = \sum_j \hbar\left(\kappa_j^* a r_j^\dagger + \kappa_j a^\dagger r_j\right) = \hbar\left(a\Gamma^\dagger + a^\dagger\Gamma\right) \tag{1.73}$$

where κ is the coupling constant in the rotating wave approximation, and we have defined $\Gamma = \sum_j \kappa_j r_j$ and $\Gamma^\dagger = \sum_j \kappa_j^* r_j^\dagger$.

Then, the master equation (1.70) in the interaction picture takes the form [18]:

$$\begin{aligned}\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^t dt' \quad & \{[aa\tilde{\rho}(t') - a\tilde{\rho}(t')a]e^{-i\omega_0(t+t')}\langle\tilde{\Gamma}^\dagger(t)\tilde{\Gamma}^\dagger(t')\rangle_R + h.c. \\ & + [a^\dagger a^\dagger\tilde{\rho}(t') - a^\dagger\tilde{\rho}(t')a^\dagger]e^{i\omega_0(t+t')}\langle\tilde{\Gamma}(t)\tilde{\Gamma}(t')\rangle_R + h.c. \\ & + [aa^\dagger\tilde{\rho}(t') - a^\dagger\tilde{\rho}(t')a]e^{-i\omega_0(t-t')}\langle\tilde{\Gamma}^\dagger(t)\tilde{\Gamma}(t')\rangle_R + h.c. \\ & + [a^\dagger a\tilde{\rho}(t') - a\tilde{\rho}(t')a^\dagger]e^{i\omega_0(t-t')}\langle\tilde{\Gamma}(t)\tilde{\Gamma}^\dagger(t')\rangle_R + h.c.\}.\end{aligned} \tag{1.74}$$

Where the reservoir correlations are defined as:

$$\langle\tilde{\Gamma}_i^\dagger(t)\tilde{\Gamma}_j^\dagger(t')\rangle_R = \langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\rangle_R = 0 \tag{1.75a}$$

$$\langle\tilde{\Gamma}_i^\dagger(t)\tilde{\Gamma}_j(t')\rangle_R = \sum_j |\kappa_j|^2 e^{i\omega_j(t-t')}\tilde{n}(\omega_j, T) \tag{1.75b}$$

$$\langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j^\dagger(t')\rangle_R = \sum_j |\kappa_j|^2 e^{i\omega_j(t-t')}\left(\tilde{n}(\omega_j, T) + 1\right) \tag{1.75c}$$

with $\tilde{n}(\omega_j, T) = \frac{e^{-\hbar\omega_j/k_bT}}{1-e^{-\hbar\omega_j/k_bT}}$ the mean photon number of an oscillator with frequency ω_j, where T is the temperature and k_b is the Boltzmann constant, and we have assumed a thermal reservoir.

If we consider a continuous reservoir, then we need to change the sum to an integral of equations (1.75) by introducing the density of states $\mathbb{D}(\omega)$. Then, the master equation:

$$\begin{aligned}\frac{d}{dt}\tilde{\rho} = -\int_0^t d\tau\{&[aa^\dagger\tilde{\rho}(t-\tau) - a^\dagger\tilde{\rho}(t-\tau)a]e^{-\omega_0\tau}\langle\tilde{\Gamma}^\dagger(t)\tilde{\Gamma}(t-\tau)\rangle_R + h.c. \\ &+ [a^\dagger a\tilde{\rho}(t-\tau) - a\tilde{\rho}(t-\tau)a^\dagger]e^{i\omega_0\tau}\langle\tilde{\Gamma}(t)\tilde{\Gamma}^\dagger(t-\tau)\rangle_R + h.c.\end{aligned} \tag{1.76}$$

Where the nonzero reservoir correlation functions are:

$$\langle\tilde{\Gamma}^\dagger(t)\tilde{\Gamma}(t-\tau)\rangle_R = \int_0^\infty e^{i\omega\tau}\mathbb{D}(\omega)|\kappa(\omega)|^2\tilde{n}(\omega, T) \tag{1.77a}$$

$$\langle\tilde{\Gamma}(t)\tilde{\Gamma}^\dagger(t-\tau)\rangle_R = \int_0^\infty e^{-i\omega\tau}\mathbb{D}(\omega)|\kappa(\omega)|^2[\tilde{n}(\omega, T) + 1] \tag{1.77b}$$

with $\tilde{n}(\omega, t) = \frac{e^{-\hbar\omega/k_bT}}{1-e^{-\hbar\omega/k_bT}}$.

Considering the Markov approximation ($\tilde{\rho}(t-\tau) \to \tilde{\rho}(t)$), the master equation could be rewritten as:

$$\frac{d}{dt}\tilde{\rho}(t) = \alpha\left(a\tilde{\rho}a^{\dagger} - a^{\dagger}a\tilde{\rho}\right) + \beta\left(a\tilde{\rho}a^{\dagger} + a^{\dagger}\tilde{\rho}a - a^{\dagger}a\tilde{\rho} - \tilde{\rho}aa^{\dagger}\right) + h.c. \tag{1.78}$$

with

$$\alpha \equiv \int_0^t \int_0^{\infty} d\omega e^{-i(\omega-\omega_0)\tau}\mathbb{D}(\omega)|\kappa(\omega)|^2 \tag{1.79}$$

$$\beta \equiv \int_0^t dt \int_0^{\infty} e^{-i(\omega-\omega_0)\tau}\mathbb{D}(\omega)|\kappa(\omega)|^2\tilde{n}(\omega, T) \tag{1.80}$$

Using the expression:

$$\lim_{t\to\infty}\int_0^{\infty} d\tau e^{-i(\omega_0-\omega)\tau} = \pi\delta(\omega-\omega_0) + i\frac{P}{\omega_0-\omega} \tag{1.81}$$

where P is the Cauchy principal value.

Then, α and β take the form:

$$\alpha = \pi\mathbb{D}(\omega)|\kappa(\omega_0)|^2 + iP\int_0^{\infty}\frac{\mathbb{D}(\omega)|\kappa(\omega)|^2}{\omega_0-\omega} = \pi\mathbb{D}(\omega)|\kappa(\omega_0)|^2 + i\Delta, \tag{1.82}$$

$$\beta = \pi\mathbb{D}(\omega_0)|\kappa(\omega_0)|^2\tilde{n}(\omega_0,t) + iP\int_0^{\infty}\frac{\mathbb{D}(\omega)|\kappa(\omega)|^2}{\omega_0-\omega}\tilde{n}(\omega,t) = \pi\mathbb{D}(\omega_0)|\kappa(\omega_0)|^2\tilde{n}(\omega_0,t) + i\Delta' \tag{1.83}$$

where we have defined $\Delta = P\int_0^{\infty}\frac{\mathbb{D}(\omega)|\kappa(\omega)|^2}{\omega_0-\omega}$ and $\Delta' = P\int_0^{\infty}\frac{\mathbb{D}(\omega)|\kappa(\omega)|^2}{\omega_0-\omega}\tilde{n}(\omega,t)$.

Finally, the master equation of the damped harmonic oscillator:

$$\frac{d}{dt}\tilde{\rho}(t) = -i\Delta[a^{\dagger}a,\tilde{\rho}] + \frac{\gamma}{2}\left(2a\tilde{\rho}a^{\dagger} - a^{\dagger}a\tilde{\rho} - \tilde{\rho}a^{\dagger}a\right) + \gamma\tilde{n}\left(a\tilde{\rho}a^{\dagger} + a^{\dagger}\tilde{\rho}a - a^{\dagger}a\tilde{\rho} - \tilde{\rho}aa^{\dagger}\right) \tag{1.84}$$

where we have defined the decay rate $\gamma \equiv 2\pi\mathbb{D}(\omega_0)|\kappa(\omega_0)|^2$ and $\tilde{n} = \tilde{n}(\omega_0, t)$.

In the Schrodinger Picture, equation (1.84) is usually performed in the Lindblad form

$$\dot{\rho} = -i\omega_0'[a^{\dagger}a,\rho] + \frac{\gamma}{2}(\tilde{n}+1)(2a\rho a^{\dagger} - a^{\dagger}a\rho - \rho a^{\dagger}a) + \frac{\gamma}{2}\tilde{n}(2a^{\dagger}\rho a - aa^{\dagger}\rho - \rho aa^{\dagger}) \tag{1.85}$$

where $\omega_0' \equiv \omega_0 + \Delta$.

1.2.3. Heisenberg-Langevin Equation of Motion

This section will be focused on the description of open quantum systems in the Heisenberg picture. To this aim, we will study the Heisenberg evolution of the system of interest and reservoir operators. In order to obtain the dynamic of the system operator considering the effects of the reservoir.

Motivated by the classical Langevin expression, we will consider that the interaction with the reservoir induces a fluctuating force on the system of interest expressed by a quantum noise operator. This fluctuation is responsible for the decay effects.

Based on the Gardiner and Zoller's work[22], we will consider an arbitrary system of interest interacting with a reservoir of harmonic oscillators, with a total system Hamiltonian of the form:

$$H_T \qquad = H_S + H_R + H_{int} \tag{1.86}$$

Where:

$$H_R = \int_{-\infty}^{\infty} d\omega \hbar\omega b^\dagger(\omega) b(\omega) \tag{1.87}$$

$$H_{int} = i\int_{-\infty}^{\infty} d\omega\kappa(\omega)[b^\dagger(\omega)c - c^\dagger b(\omega)], \tag{1.88}$$

$\kappa(\omega)$ is the coupling strength, b and $b^\dagger$ are the reservoir operators, and c and $c^\dagger$ are the system of interest operators.

Here two main approximations have been considered. The first is the rotating wave approximation, and the second is the extension of the integral limit from $[0, \infty]$ to $[-\infty, \infty]$. The first approximation considers neglecting the rapidly oscillating terms.

The Heisenberg equations of the system of interest and the reservoir equation take the form:

$$\dot{b}(\omega) = -i\omega b(\omega) + \kappa(\omega)c \tag{1.89}$$

$$\dot{a} = -\frac{i}{\hbar}[a, H_S] + \int d\omega\kappa(\omega)\{b^\dagger(\omega)[a, c] - [a, c^\dagger b(\omega)]\} \tag{1.90}$$

where a is one system of interest operator of the collection of all c possible system operators.

The solution of the equation (1.89):

$$b(\omega) = e^{-i\omega(t-t_0)} b_{t_0}(\omega) + \kappa(\omega)\int_{t_0}^{t} e^{-i(t-t')} c(t')dt' \tag{1.91}$$

where b_{t_0} is $b(\omega)$ at time t_0.

When we replace the solution (1.91) in the equation (1.89):

$$\begin{aligned} \dot{a} = \quad & -\frac{i}{\hbar}[a, H_S] + \int d\omega\kappa(\omega)\{e^{i(t-t_0)}b_{t_0}(\omega)[a, c] - [a, c^\dagger]e^{-i\omega(t-t_0)}b_{t_0}(\omega)\} \\ & \int d\omega \int_{t_0}^{t} dt\kappa^2(\omega)\{e^{i\omega(t-t')}c^\dagger(t')[a, c] - [a, c^\dagger]e^{i\omega(t-t')}c(t')\} \end{aligned} \tag{1.92}$$

Now, we will consider the first Markov approximation, where the coupling $\kappa(\omega) = \sqrt{\gamma/2\pi}$, and defining $b_{in} = \frac{1}{\sqrt{\omega}} \int d\omega e^{-i\omega(t-t_0)}b_{t_0}(\omega)$ which satisfies the commutation relation $[b_{in}(t), b_{in}^\dagger(t')] = \delta(t - t')$.

Then the quantum Langevin equation takes the form:

$$\begin{aligned} \dot{a} = \quad & -\frac{i}{\hbar}[a, H_s] - \sqrt{\frac{\gamma}{2\pi}}[a, c^\dagger] \int d\omega e^{-i\omega(t-t_0)}b_{t_0}(\omega) + \sqrt{\frac{\gamma}{2\pi}} \int d\omega e^{-i\omega(t-t_0)}b_{t_0}(\omega)[a, c] \\ & -[a, c^\dagger]\left(\frac{\gamma}{2\pi}\right)\int_{t_0}^{t} dt'\delta(t - t')c(t') + \left(\frac{\gamma}{2\pi}\right)\int_{t_0}^{t} \delta(t - t')c^\dagger(t')[a, c] \end{aligned} \tag{1.93}$$

$$\dot{a} = -\frac{i}{\hbar}[a, H_S] - [a, c^\dagger]\left(\sqrt{\gamma}b_{in}(t) + \frac{\gamma}{2}c(t)\right) + \left(\sqrt{\gamma}b_{in}^\dagger(t) + \frac{\gamma}{2}c^\dagger(t)\right)[a, c] \tag{1.94}$$

1.2.3.1. Heisenberg-Langevin Equation for an Harmonic Oscillator

In the previous section, we studied the canonical damped harmonics oscillator immersed in a reservoir of harmonic oscillators in the Schrödinger picture. This section will study the same system in the Heisenberg picture based on the work developed by Meystre[23].

As in section 1.2.2.1 , we will consider the total Hamiltonian of the form:

$$H_T = H_S + H_R + H_{int} = \hbar\omega_0 a^\dagger a + \sum_j \hbar\omega_j r_j^\dagger r_j + \sum_j \hbar\left(\kappa_j^* a r_j^\dagger + \kappa_j r_j a^\dagger\right) \tag{1.95}$$

Where a and $a^\dagger$ are the system of interest bosonic operators, r and $r^\dagger$ are the reservoir bosonic operator, and κ_j are the coupling strength between the system mode and the j-mode of the reservoir.

The Heisenberg equations of the system of interest and the reservoir operator are:

$$\dot{a}(t) = -\frac{i}{\hbar}[a, H_T] = -\omega_0 a - i\sum_j \kappa_j r_j \tag{1.96}$$

$$\dot{r}_j(t) = -\frac{i}{\hbar}[r_j(t), H_T] = -i\omega_j r_j(t) - i\kappa_j^* a(t) \tag{1.97}$$

The solution of the equation (1.97) is:

$$r_j(t) = r_j(t_0)e^{-i\omega_j(t-t_0)} - i\kappa_j^* \int_{t_0}^{t} dt' e^{-i\omega_j(t-t')} \tag{1.98}$$

We replace the solution in the Heisenberg equation of the system operator Heisenberg equation (1.96):

$$\dot{a}(t) = -i\omega_0 a - i\sum_j \kappa_j r_j(t_0)e^{-i\omega_j(t-t_0)} - \sum_j |\kappa_j|^2 \int_{t_0}^{t} dt' a(t')e^{-i\omega_j(t-t')} \tag{1.99}$$

Defining the slowly varying system operator:

$$A(t) = a(t)e^{i\omega_0 t} \tag{1.100}$$

which satisfies the commutator relation $[A(t), A^\dagger(t)] = 1$.
The evolution of this slowly varying operator is:

$$\dot{A}(t) = -i\sum_j \kappa_j r_j e^{-i(\omega_j-\omega_0)(t-t_0)} - \sum_j |\kappa_j|^2 \int_{t_0}^{t} dt' A(t')e^{-i(\omega_j-\omega_0)(t-t')} \tag{1.101}$$

From the expression, we could define the noise operator:

$$\mathcal{F}(t) = -i\sum_j \kappa_j r_j(t_0)e^{i(\omega_j-\omega_0)(t-t_0)} \tag{1.102}$$

where $\langle \mathcal{F}(t) \rangle = 0$ and $\left\langle \mathcal{F}^\dagger(t)\mathcal{F}(t') \right\rangle = \delta(t-t')$.

Now, we will consider the approximation $A(t') \to A(t)$, because the exponential term in the integral evolves rapidly compared with the slowly varying operator, and we will extend the limit of the integral to infinity. Then, the last term of the equation (1.101) becomes:

$$\begin{aligned}
\sum_j |\kappa_j|^2 \int_{t_0}^{t} dt' A(t')e^{-i(\omega_j-\omega_0)(t-t')} &= A(t)\sum_j |\kappa_j|^2 \int_{t_0}^{t} dt' e^{-i(\omega_j-\omega_0)(t-t')} \\
&= A(t)\int_{-\infty}^{\infty} d\omega_j \mathbb{D}(\omega_j)|\kappa(\omega_j)|^2 \int_0^{\infty} dt' e^{-i(\omega_0-\omega_j)t'} \\
&= A(t)\left(\int_{-\infty}^{\infty} d\omega_j \mathbb{D}(\omega_j)|\kappa(\omega_j)|^2 \pi\delta(\omega_j-\omega_0) + iP\int_{-\infty}^{\infty} d\omega_j \frac{|\kappa(\omega_j)|^2}{\omega_0-\omega_j}\right) \\
&= \frac{\kappa}{2}A(t) + iP\int_{-\infty}^{\infty} d\omega_j \frac{|\kappa(\omega_j)|^2}{\omega_0-\omega_j}A(t)
\end{aligned} \tag{1.103}$$

where $\mathbb{D}(\omega_j)$ is the density of states, we have used the relation (1.81) and defined the decay rate $\kappa = 2\pi\mathbb{D}(\omega_0)|\kappa(\omega_0)|^2$.

Then, we obtain the Heisenberg-Langevin equation:

$$\dot{A}(t) = -\frac{\kappa}{2}A(t) + \mathcal{F}(t) \tag{1.104}$$

where we have neglected the principal value.

1.3. Laser

At the beginning of the XX century, Einstein published his paper "On the Quantum Theory of Radiation,"[13] where he explained the mechanisms of light-matter interactions and established the basic ideas of revolutionary technology, the laser. Since its first implementation in 1960 by Maiman [24], the laser has become one of the most versatile tools with many technological applications, such as in industry, science, and even art. Some of the most remarkable applications are related to Nobel Prize ideas. In 2018 the Nobel Prize in Physics was awarded to Arthur Ashkin, Gerard Mourou, and Donna Strickland *"for groundbreaking inventions in the field of laser physics,"* such as *"optical tweezers"* and *"the generations of high-intensity ultra-short optical pulses"* [25]. While in 2017, Kip Thorne, Rainer Weiss, and Barry C. Barrish won the Nobel Prize in Physics *"for decisive contribution to the LIGO detector and the observations of gravitational waves."*[26] Another important application in science is the atomics clocks, where the laser's linewidth must be narrow to guarantee the best performance of the clock. This work will be focused on this property, the linewidth, and how we could manipulate the electromagnetic field to satisfy this narrow condition. In particular, we will study the waveguide QED system platform as an extension of the canonical case of a cavity QED system developed by Maiman in 1960.

1.4. Linewidth of a Cavity QED Laser

This section is based on the work developed by Minghui[27] and Meiser[28] and uses the same notation as those references. We present this model as a canonical example of a laser on a cavity and as an example of a methodology to obtain the linewidth of the laser using quantum electrodynamics tools applied to the case of a unimodal cavity. In the following chapters, we will extend this methodology to the case of a waveguide, where the main challenge is the extrapolation of the QED tools used in this section to the multimodal system.

This section is divided as follows. Section 1.4.1 presents a cavity QED system, the Hamiltonian, the master equation, and the Heisenberg-Langevin equations of motion of the system. Section 1.4.2 studies the semiclassical limits of the system equations and obtain the steady-state solutions in the mean-field. Section 1.4.3 presents a methodology to obtain the laser linewidth based on a fully quantum mechanical description and the study of the first-order coherence function.

1.4.1. Model of a Laser on a Cavity QED system

Based on the work of Minghui[27], we present a quantum electrodynamics description of a system of interest that consists of an ensemble of atoms inside a cavity. Those atoms are modeled as two-level systems of the same resonance frequency ω_a. The electromagnetic field inside the cavity is represented as one quantum harmonics oscillator of frequency ω_c. Because of the closed boundary conditions of the cavity, the electromagnetic field has discretized energy levels. We will consider that the separation between those energy levels is enough to guarantee that only one frequency is resonant with the atoms inside its spectral linewidth [29].

The system of interest evolution is determined by the Tavis-Cumming Hamiltonian.

$$\hat{H} = \hbar\omega_a \sum_{j=1}^{N} \hat{\sigma}_j^+ \hat{\sigma}_j^- + \hbar\omega_c \hat{a}^\dagger \hat{a} + \frac{\hbar g}{2} \sum_{j=1}^{N} (\hat{a}^\dagger \hat{\sigma}_j^- + \hat{\sigma}_j \hat{a}) \tag{1.105}$$

As mentioned in section 1.2, almost all quantum optical systems are not isolated, meaning an open quantum system methodology must be applied. Here we will consider the system-plus-reservoir approach, where the system described before is coupled to its environment, and this coupling generates decay and decoherence process. The system-environment interaction consists of three parts. The first part is the interaction between the ensemble of atoms and the electromagnetic field modes, except the cavity mode. The second part is the pumping mechanism; this interaction guarantees the population inversion of the system's atoms. The third part is the effective interaction between the cavity mode and all other modes of the electromagnetic field, which is mediated by the atoms of the cavity wall. Based on the reference [18], the total system interaction Hamiltonian is described by the equation (1.106).

$$\begin{aligned} H_{total}^{\mathcal{I}} = \quad & H^F + H^A + H^{\mathcal{P}} \\ = \quad & \hbar\left(a\Gamma^\dagger + a^\dagger\Gamma\right) + \sum_j \hbar\left(\sigma_j^-\Gamma_j^\dagger + \sigma_j^+\Gamma_j\right) + \sum_j \hbar\left(\sigma_j^-\Gamma_j^{\mathcal{P}\dagger} + \sigma_j^+\Gamma_j^{\mathcal{P}}\right) \end{aligned} \tag{1.106}$$

where:

$$\Gamma^\dagger = \sum_j \kappa_j^* r_j^\dagger \tag{1.107a}$$

$$\Gamma_j^\dagger = \sum_{k,\lambda} \bar{\kappa}_{k,\lambda}^{j*} r_{k,\lambda}^\dagger \tag{1.107b}$$

$$\Gamma_j^{\mathcal{P}+} = \sum_k \tilde{\kappa}_{j,k}^* \mathcal{P}_{j,k}^+ . \tag{1.107c}$$

where $r_j^\dagger$ are the reservoir operators associated with the degrees of freedom of the electromagnetic fields modes that could interact with the atoms through spontaneous emission, $r_{k,\lambda}^\dagger$ are the reservoir operators associated with the degrees of freedom of the electromagnetic field modes that interacts with the resonant mode of the cavity through the degrees of freedom of the cavity walls, and $\mathcal{P}_{j,k}^+$ are the operator associated to the degrees of freedom of the reservoir which characterize the pump mechanism.

The master equation could be deduced from the Hamiltonian (1.106) by the same method applied in section 1.2.2.

$$\frac{d}{dt}\hat{\rho} = \frac{1}{i\hbar}[\hat{H}, \hat{\rho}] + \kappa\mathcal{L}[\hat{a}]\hat{\rho} + \sum_{j=1}^{N}\gamma\mathcal{L}[\hat{\sigma}_j^-]\hat{\rho} + w\mathcal{L}[\hat{\sigma}_j^+]\hat{\rho} \tag{1.108}$$

Where κ, γ, and w represent the different decay rates. The first term, κ, is the decay rate associated with the interaction mediated by the atoms of the cavity walls between the cavity mode and the other electromagnetic field modes. The γ term is the spontaneous emission rate of the atoms to the modes of the electromagnetic field that are not considered in the system of interest. The w is the pumping rate related to the incoherent effective process of population inversion of the two-levels systems.

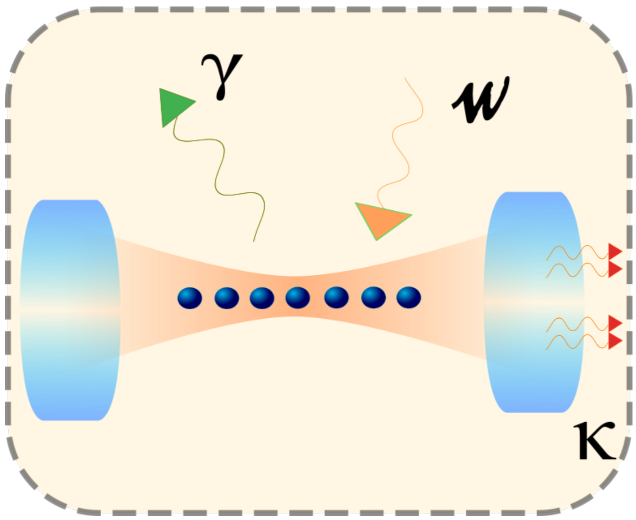

Figure 1.1: Diagram of the different decay processes of a laser in a cavity. Where γ, w,κ represent the decay rates associated with the spontaneous emission, pumping, and decay generated by the walls of the cavity.

From the same Hamiltonian 1.106, we could study the system in the Heisenberg picture. Following the sale procedure of the toy model of the section 1.2.3, applied to the system operators a. We obtain the Heisenberg-Langevin equations.

$$\frac{d}{dt}\hat{a} = -\frac{1}{2}(\kappa + 2i\omega_c - 2i\omega)\hat{a} - \frac{iNg}{2}\hat{S}^- + \hat{F}^a \tag{1.109a}$$

$$\frac{d}{dt}\hat{S}^- = -\frac{1}{2}(\Gamma + 2i\omega_a - 2i\omega)\hat{S}^- + \frac{ig}{2}\hat{a}\hat{S}^z + \hat{F}^- \tag{1.109b}$$

$$\frac{d}{dt}\hat{S}^z = -(w+\gamma)\left(\hat{S}^z - d_0\right) + ig\left(\hat{a}^\dagger\hat{S}^- - \hat{a}\hat{S}^+\right) + \hat{F}^z \tag{1.109c}$$

Where we have defined the collective atomics ladder operators as and $\hat{S}^- = \sum_{j=1}^{N}\hat{\sigma}_j^-/N$ and $\hat{S}^+ = \sum_{j=1}^{N}\hat{\sigma}_j^+/N$, and the collective inversion operator $\hat{S}^z = \sum_{j=1}^{N}\hat{\sigma}_j^z/N$. Here we also define the generalized single-atom decoherence as $\Gamma = w + \gamma$ and the atomic inversion of a single atom in absence of a cavity as $d_0 = (w-\gamma)/(w+\gamma)$.

Equations (1.109) also consider the definition of the quantum noise operators as $\hat{F}^\mu(t)$, which are characterized by a zero mean $\left\langle \hat{F}^\mu \right\rangle(t) = 0$, and the second-order correlations satisfy

$$\left\langle \hat{F}^{\mu}(t)\hat{F}^{\nu}(t')\right\rangle = 2D_{\mu\nu}(t)\delta(t-t'). \tag{1.110}$$

Here, $D_{\mu\nu}$ are the diffusion coefficients, which are obtained by the generalized Einstein relations[30]:

$$2D^{aa^{\dagger}} = \kappa \tag{1.111a}$$

$$2D^{+-} = \frac{w}{N} \tag{1.111b}$$

$$2D^{-+} = \frac{\gamma}{N} \tag{1.111c}$$

$$2D^{+z} = -\frac{2w}{N}\left\langle \hat{S}^{+}\right\rangle \tag{1.111d}$$

$$2D^{z+} = \frac{2\gamma}{N}\left\langle \hat{S}^{+}\right\rangle \tag{1.111e}$$

$$2D^{-z} = \frac{2\gamma}{N}\left\langle \hat{S}^{-}\right\rangle \tag{1.111f}$$

$$2D^{z-} = -\frac{2w}{N}\left\langle \hat{S}^{-}\right\rangle \tag{1.111g}$$

$$2D^{zz} = \frac{2\gamma}{N}(1+\left\langle \hat{S}^{z}\right\rangle) + \frac{2w}{N}(1-\left\langle \hat{S}^{z}\right\rangle) \tag{1.111h}$$

The master equation and the Heisenberg-Langevin equation consist of a fully quantized description of the laser system on a Cavity.

1.4.2. Steady state solutions of the Mean Field Equations of a Cavity QED Laser

To simplify the treatment of the cavity QED system equations of motions, we will consider a mean-field approximation of the dynamics[27]. This approximation is applied considering that the noise terms scale as $\sqrt{N}$, while the expectations values scale as N. Then, taking the limits of a large number of atoms, the noise terms could be neglected. The equations of motion take the form of expressions [27, 28].

$$\frac{d}{dt}a_0 = -\frac{1}{2}(\kappa + 2i\omega_c - 2i\omega)a_0 - \frac{iNg}{2}S_0^- \tag{1.112a}$$

$$\frac{d}{dt}S_0^- = -\frac{1}{2}(\Gamma + 2i\omega_a - 2i\omega)S_0^- + \frac{ig}{2}\hat{a}S_0^z \tag{1.112b}$$

$$\frac{d}{dt}S_0^z = -(w+\gamma)\left(S_0^z - d_0\right) + ig\left(\hat{a}^* S_0^- - a_0 S_0^+\right) \tag{1.112c}$$

Studying the steady-state solution of the mean field equations, we obtain an expression for the atomic coherence and the atomic inversion as:

$$S_0^- = \frac{igS_0^z a_0}{(\Gamma + 2i\omega_a - 2i\omega)} \tag{1.113}$$

$$S_0^z = \frac{d_0}{1 + \frac{g^2|a_0|^2}{(\gamma+w)}\frac{2\Gamma}{\Gamma^2+(\omega_a-\omega)^2}} \tag{1.114}$$

The steady-state solution for the bosonic operator equation of motion in the mean-field must satisfy the condition:

$$\left(1 - \frac{Ng^2}{(\kappa + 2i\omega_c - 2i\omega)(\Gamma + 2i\omega - 2i\omega)}\frac{d_0}{1 + \frac{g^2|a_0|^2}{\gamma+w}\frac{2\Gamma}{\Gamma^2+(\omega_a-\omega)^2}}\right)a_0 = 0 \tag{1.115}$$

This expression has two possible solutions to $|a_0|^2$. The first solution is the trivial $a_0 = 0$, where there is no lasing behavior, and the second solution takes the form:

$$|a_0|^2 = \frac{\gamma + w}{g^2}\frac{\Gamma + 4(\omega_A - \omega)^2}{2\Gamma}\left(\frac{d_0 Ng^2}{(\kappa + 2i\omega_c - 2i\omega)(\Gamma + 2i\omega_a - 2i\omega)} - 1\right). \tag{1.116}$$

This second solution must satisfy the positivity condition of $|a_0|^2$, which implies that the rotating frequency takes the form $\omega = \frac{\kappa\omega_a + \Gamma\omega_c}{\kappa+\Gamma}$. If we take the limit when $|\omega_c - \omega_a| << \kappa + \Gamma$, the equation (1.116) takes the form

$$|a_0|^2 \approx \frac{(w+\gamma)\Gamma}{2g^2}\left(\frac{d_0 Ng^2}{\kappa\Gamma} - 1\right) = n_0\left(\mathcal{G} - 1\right), \tag{1.117}$$

where $n_0 = \frac{(w+\gamma)\Gamma}{2g^2}$ is called the saturation photon number, and $\mathcal{G} = \frac{d_0 Ng^2}{\kappa\Gamma}$ is the generalized cooperativity and defines the threshold where the laser works. In this generalized cooperativity is the rate between the system of interest coupling determined by $d_0 Ng^2$ and the decays process of the cavity κ and the atom γ, and is the generalization of the relation (0.1).

1.4.3. Phase Diffusion Linewidth

This section shows a methodology to obtain the laser linewidth by studying the evolution of the phase of the bosonic operator of the system. This work is based on the references [27–29, 31].

First, we will consider an ansatz of the form

$$\hat{a} = (a_0 + \hat{r})e^{i\hat{\phi}}. \tag{1.118}$$

In this expression, the field is defined as the product of an amplitude and a phase. If we neglect the amplitude fluctuation, then the two-times correlation function takes the form(see section [optics]):

$$g^{(1)}(t) = \frac{\langle E^-(t)E^+(0)\rangle}{\langle E^-(0)E^+(0)\rangle} = \frac{\left\langle a_0^* e^{-i\hat{\phi}(t)} a_0 e^{i\hat{\phi}(0)}\right\rangle}{\left\langle a_0^* e^{-i\hat{\phi}(0)} a_0 e^{i\hat{\phi}(0)}\right\rangle} = \left\langle e^{i[\hat{\phi}(t)-\hat{\phi}(0)]}\right\rangle \tag{1.119}$$

In order to obtain the laser linewidth, we will consider the quantum mechanical equation of the bosonic operator (1.109a) and differentiate it.

$$\ddot{\hat{a}} \approx -\frac{1}{2}(\kappa + \Gamma)\dot{\hat{a}} - \frac{\Gamma\kappa}{4}\hat{a}\hat{S}^z + \frac{Ng^2}{4}\hat{a}\hat{S}^z + \hat{F} \tag{1.120}$$

Where $\hat{F} = \frac{\Gamma}{2}\hat{F}^a - \frac{iNg}{2}\hat{F}^- + \dot{\hat{F}}^a$. Studying the resonance case $\omega_c = \omega_A$ and when the rotating frame frequency satisfy $\omega = \frac{\kappa\omega_a + \Gamma\omega_c}{\kappa+\Gamma}$, we get

$$\ddot{\hat{a}} \approx -\frac{1}{2}(\kappa + \Gamma)\dot{\hat{a}} - \frac{\Gamma\kappa}{4}\hat{a}\hat{S}^z + \frac{Ng^2}{4}\hat{a}\hat{S}^z + \hat{F} \tag{1.121}$$

Replacing the ansatz in the equation (1.121), we could get an equation for the phase evolution by taking the imaginary part.

$$\ddot{\hat{\phi}} = -\frac{1}{2}(\kappa + \Gamma)\dot{\hat{\phi}} + \frac{1}{a_0}Im[\hat{F}] \tag{1.122}$$

Integrating the last expression, we get the solution of phase evolution.

$$\hat{\phi(t)} - \hat{\phi}(0) = \frac{2}{a_0(\kappa + \Gamma)}\int_0^t dt' Im[\frac{\Gamma}{2}\hat{F}^a - \frac{iNg}{2}\hat{F}^-] \tag{1.123}$$

The expression 1.123, depends on the gaussian operators $\hat{F}^a$ and $\hat{F}^-$, which satisfy the relation(see appendix correlation).

$$\left\langle e^{i[\hat{\phi}(t)-\phi(0)]}\right\rangle = e^{-\frac{1}{2}\left\langle(\hat{\phi}(t)-\phi(0))^2\right\rangle} = e^{-\frac{1}{2}\Delta\nu t} \tag{1.124}$$

From the relation (1.124), the Appendix A studies the argument of the exponential at the RHS, and obtain the expression:

$$\left\langle(\hat{\phi}(t) - \hat{\phi}(0))^2\right\rangle = \frac{(\mathcal{G}/d_0 + 1)}{2(\mathcal{G} - 1)}\frac{\Gamma}{(w + \gamma)}\frac{g^2\kappa}{(\kappa + \Gamma)^2}t. \tag{1.125}$$

Then two-times correlation function takes the form

$$
\begin{aligned}
g^{(1)}(t) = & \left\langle e^{i[\hat{\phi}(t)-\hat{\phi}(0)]}\right\rangle \\
= & e^{-\frac{1}{2}\left\langle(\hat{\phi}(t)-\phi(0))^2\right\rangle} \\
= & e^{-\frac{1}{2}\left(\frac{(\mathcal{G}/d_0+1)}{2(\mathcal{G}-1)}\frac{\Gamma}{(w+\gamma)}\frac{g^2\kappa}{(\kappa+\Gamma)^2}\right)t}
\end{aligned}
\tag{1.126}
$$

From this last equation, we could recognize the spectral linewidth $\Delta\nu$

$$
\Delta\nu = \frac{(\mathcal{G}/d_0+1)}{2(\mathcal{G}-1)}\frac{\Gamma}{(w+\gamma)}\frac{g^2\kappa}{(\kappa+\Gamma)^2} \tag{1.127}
$$

This expression could be studied in two regimes. The first regime is known as the "good cavity regime,"where the cavity decay rate is smaller than the atomic decay rate, i.e., $\kappa << \Gamma$. In this regime, the emission is determined by the coherence of the light field, where the equation (1.127) is approximated to the Shawlow-Townes linewidth [32]:

$$
\Delta\nu \approx \frac{(\mathcal{G}/d_0+1)}{2(\mathcal{G}-1)}\frac{\kappa}{(w+\gamma)\Gamma/g^2} \propto \frac{\kappa}{|a_0|^2}. \tag{1.128}
$$

The second regime is known as the "bad cavity regime,"where $\kappa >> \Gamma$. In this regime, the emission is determined by the correlation between the emitters. Then, the equation (1.127) is approximated to:

$$
\Delta\nu \approx \frac{(\mathcal{G}/d_0+1)}{2(\mathcal{G}-1)}C\gamma \propto C\gamma \tag{1.129}
$$

where C is the cooperativity parameter defined in equation (0.1).

To sum up, this section recapitulates the work developed in references [27, 30, 33] to present a mechanism to obtain the spectral linewidth of the field in the cavity QED system. In the following chapters, we will extend this procedure to the case of a waveguide QED system, where the electromagnetic is not discretized, so the method must be generalized to the continuous case.

Chapter 2

Model of a Laser on a Waveguide QED

This work aims to extend the laser behavior to the waveguide QED system platform. Waveguide QED systems have become a promising platform for the fundamental study of light and matter interaction and for technological applications such as quantum metrology, quantum networks [34, 35], and quantum computers [36].

This chapter presents a fully quantum mechanical model for a laser in a waveguide QED system. We will consider a system of interest that consists of a collection of atoms or emitters interacting with the guided-electromagnetic field modes of the waveguide. This system of interest is not isolated and interacts with a reservoir, which generates dissipation processes, so an open quantum system approach must be applied.

We will present a discretized model to represent the interaction of the system and its reservoir. Then, it will be extended to the continuous case considering that in the limit where the waveguide is long compared with the characteristic length of the system, the guided mode could be considered continuous. Finally, from this interaction model, we will obtain the equations that characterize the dynamics of our system of interest.

This chapter is divided as follows. Section 2.1, presents the total system Hamiltonian. In section 2.1.1 we will present the system of interest, the atoms and the guided modes, and their interaction. Then, in section 2.1.2, we will study the interaction of the system of interest with the reservoir. Section 2.1.2.1 describes the mechanism of pumping and its Hamiltonian. Section 2.1.2.2 describes the interaction between the waveguide modes and the electromagnetic field outside the waveguide. Then, section 2.2 presents the discrete and continuous version of the interaction Hamiltonian which describes the dynamics of our model for a waveguide QED system laser.

2.1. Open Quantum System Approach

2.1.1. System of interest

As is shown in the figure 2.1, we will consider a waveguide that could be implemented as an optical nanofiber, a coplanar waveguide in circuits QED or nanophotonic waveguides [37], which interacts with emitters such as atoms, superconducting qubits, quantum dots, or NV-centers in diamonds. The strength of this interaction depends on the platform. In the case of optical nanofibers, the electromagnetic field is confined to a reduced volume promoting an increase in the coupling to the emitters through the evanescent field. In the case of circuit QED, a coplanar waveguide is capacitively coupled with a Josephson junction, which behaves as an emitter thanks to its not linear behavior.

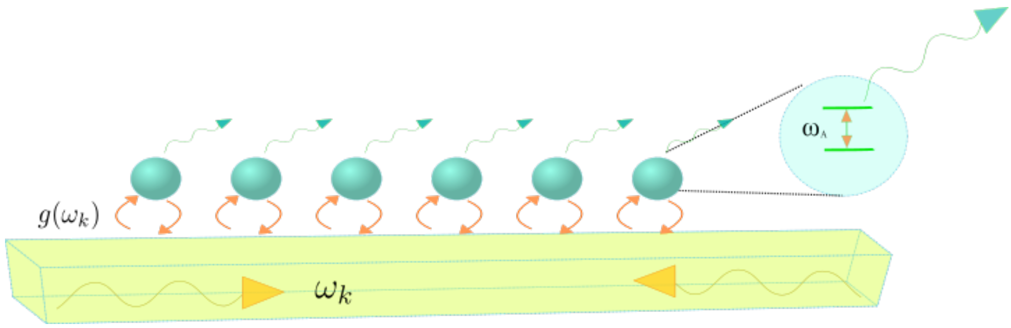

Figure 2.1: Rectangular Waveguide

Those platforms could be represented by a multimodal waveguide (continuous or discretized) that interacts with an ensemble of emitters with a coupling strength of the form:

$$g(\omega_k) = \mathcal{D}\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon c A}}, \tag{2.1}$$

where $\mathcal{D}$ is the dipole operator matrix element.

Here we have considered a dipole approximation, which means that the characteristic length of the emitter is smaller than the wavelength of the electromagnetic field modes considered. (There are some cases where the dipole approximation is not valid, e.g., giant atom [38].)

Here, we will consider a system of interest interaction Hamiltonian in the discretized case as:

$$H_{\mathcal{I}}^{S} = \sum_{j,k} \hbar \left(g_k^* a_k^\dagger \sigma_j^- e^{i(\omega_k - \omega_A)t} + g_k \sigma_j^+ a_k e^{-i(\omega_k - \omega_A)t} \right) \tag{2.2}$$

This expression could be extended to the continuous case:

$$\begin{aligned} H_I^{\mathcal{S}} &= \sum_j \int d\omega_k \hbar \left(g^*(\omega_k) a^\dagger(\omega_k) \sigma_j^- e^{i(\omega_k - \omega_A)t} + g(\omega_k) \sigma_j^+ a(\omega_k) e^{-i(\omega_k - \omega_A)t} \right) \\ &= \int d\omega_k \hbar \left(g^*(\omega_k) a^\dagger(\omega_k) N S^- e^{i(\omega_k - \omega_A)t} + g(\omega_k) N S^+ a(\omega_k) e^{-i(\omega_k - \omega_A)t} \right) \end{aligned} \tag{2.3}$$

Nevertheless, in order to describe a laser, this system of interest could not be isolated. Moreover, in practical implementations, noise sources are inherent in the systems. We must consider some incoherent processes, such as a mechanism of population inversion and spontaneous emission to modes that are not part of the system of interest.

The following section will describe the total system of the waveguide QED laser platform and how the reservoir interacts with the system of interest.

2.1.2. Reservoir

2.1.2.1. Pumping System

This section presents the pumping mechanism, the incoherent process that guarantees the inversion of the population of the gain of the laser.

A two-level system cannot achieve population inversion because any coherent interaction saturates the excited state population at $N/2$, where N is the number of atoms. Then a third state must be considered to obtain an effective population inversion between the two states of interest. Figure (2.2) shows the three-level system, its energy levels, the ground, the excited, and the upper states, and the transition processes between them.

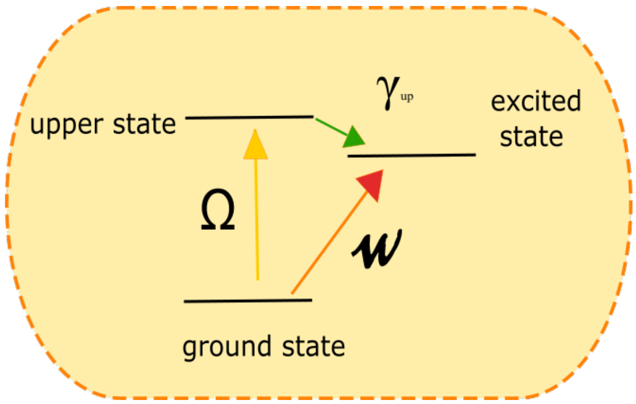

Figure 2.2: Diagram of the pumping process. This picture shows a possible mechanism to obtain population inversion through a three-level atom, generating an effective pumping rate w.

The underlying process follows the following stages. A coherent source of light, resonant with the $g-u$ transition, could populate the upper state. Then, a spontaneous emission process could generate an incoherent transition between the upper and the excited state. The effective result is an incoherent process between the excited and upper states, which can reach the inversion of the population.

Here, we will consider that the degrees of freedom associated with the pumping mechanism could be represented by a Pauli pseudo-spin operators, $\mathcal{P}^{+/-/z}(\omega)$. The advantage of considering this kind of reservoir is the characterization of a negative temperature reservoir, promoting population inversion.

The Hamiltonian of the degrees of freedom of the pumping mechanism in the continuous limits takes the form:

$$H_I^{\mathcal{P}} = \frac{1}{2}\int d\omega_\nu \hbar\omega_\nu \mathcal{P}^z(\omega_\nu) \tag{2.4}$$

This expression could be studied in the discrete case:

$$H_I^{\mathcal{P}} = \frac{1}{2}\sum_\beta \hbar\omega_\beta \mathcal{P}^z_\beta. \tag{2.5}$$

2.1.2.2. Free Electromagnetic Field

In order to consider the decay process associated with the spontaneous emission of the system of interest atoms and the decay process associated with the guided modes decay because of the interaction with the waveguide atoms, we will define a reservoir of the electromagnetics field modes outside the waveguide. This free electromagnetic field reservoir will be divided into two subsystems. First, we will consider those modes interacting with the atoms of the system of interest. The second subsystem considers those modes that interact with the degrees of freedom of the waveguide's dielectric body, which generates an effective mode-mode interaction between the guided modes and the modes outside of the waveguide.

The Hamiltonian of this free electromagnetic field reservoir in the interaction picture takes the form:

$$H_I^R = \hbar \sum_{\mu} \omega_{\mu} \tilde{r}_{\mu}^{\dagger} \tilde{r}_{\mu} + \hbar \sum_{\alpha} \omega_{\alpha} r_{\alpha}^{\dagger} r_{\alpha} \tag{2.6}$$

where $\tilde{r}_{\mu}$ and $\tilde{r}_{\mu}^{\dagger}$ are the degrees of freedom of the electromagnetic field which could interact with the atoms of the system of interest through spontaneous emission, and r_{α} and $r_{\alpha}^{\dagger}$ are the operators that represent the degrees of freedom of the electromagnetic field that could effectively interact with the guided modes through the traced degrees of freedom of the material of the waveguide.

2.2. The Interaction Hamiltonian of the Total System

This section will present the Interaction Hamiltonian of the total system in the interaction picture and the Hamiltonian of the system of interest. First, it will be presented in the discrete case, which could be a model of a multimode cavity. Then, this multimode model will be extended to the continuous case, which could be the model of the waveguide QED system Laser.

2.2.1. Discrete Model

The canonical case of a laser in a unimodal cavity is extended to a cavity QED multimodal case. In this system, we considered that the cavity length is large enough to assume that there is not only one resonant mode of the cavity interacting resonantly with the emitter. If the length of the cavity increase, the separation between the resonant modes frequencies of the cavity decrease, as was shown in the reference Pierre et al. [39]. Then, we must consider more modes in our system of interest.

The interaction Hamiltonian of the system of interest in the interaction picture takes the form:

$$\begin{aligned} H_I^{\mathcal{S}} = & \sum_{j,k} \hbar \left(g_k^* a_k^{\dagger} \sigma_j^{-} e^{i(\omega_k - \omega_A)t} + g_k \sigma_j^{+} a_k e^{-i(\omega_k - \omega_A)t} \right) \\ = & \sum_{k} \hbar \left(g_k^* a_k^{\dagger} N S^{-} e^{i(\omega_k - \omega_A)t} + g_k N S^{+} a_k e^{-i(\omega_k - \omega_A)t} \right) \end{aligned} \tag{2.7}$$

Where a_k and $a_k^\dagger$ are the creation and destruction operators of the cavity mode of frequency ω_k,σ_j^- sigma is the coherence atomic operator, and we have defined S^- as the collective coherence operator.

The interaction Hamiltonian of the system of interest and the reservoir, in the interaction picture, takes the form:

$$\begin{aligned}
H_I^{\mathcal{SR}} = & \sum_{\mu,i} \hbar \left(\kappa_{\mu,i}^* \tilde{r}_\mu^\dagger \sigma_i^- e^{i(\omega_\mu - \omega_A)t} + \kappa_{\mu,i} \sigma_i^+ \tilde{r}_\mu e^{-i(\omega_\mu - \omega_A)t} \right) \\
& + \sum_{\beta,i} \hbar \left(\tilde{\kappa}_{\beta,i}^* \mathcal{P}_\beta^+ \sigma_i^+ e^{-i(\omega_\beta - \omega_A)t} + \tilde{\kappa}_{\beta,i} \sigma_i^- \mathcal{P}_\beta^- e^{i(\omega_\beta - \omega_A)t} \right) \\
& + \sum_{\alpha,\beta} \hbar \left(\bar{\kappa}_{\alpha,\beta}^* r_\alpha^\dagger a_\beta e^{i(\omega_\alpha - \omega_\beta)t} + \bar{\kappa}_{\alpha,\beta} a_\beta^\dagger r_\alpha e^{-i(\omega_\alpha - \omega_\beta)t} \right)
\end{aligned} \tag{2.8}$$

2.2.2. Continuous Model

Motivated by the idea of implementing a laser in a waveguide QED system, we will generalize the case described in the previous section to the continuous case. We will consider that the guided modes are continuous in the waveguide system. This assumption considers the generalization of the idea presented in reference [39]. If we understand the waveguide as a large cavity, the separation between the frequencies of the modes will decrease. In the case of infinite large, we obtain a continuous spectrum of frequencies.

The system of interest interaction Hamiltonian in the interaction picture is:

$$\begin{aligned}
H_I^{\mathcal{S}} = & \sum_j \int d\omega_k \hbar \left(g^*(\omega_k) a^\dagger(\omega_k) \sigma_j^- e^{i(\omega_k - \omega_A)t} + g(\omega_k) \sigma_j^+ a(\omega_k) e^{-i(\omega_k - \omega_A)t} \right) \\
= & \int d\omega_k \hbar \left(g^*(\omega_k) a^\dagger(\omega_k) N S^- e^{i(\omega_k - \omega_A)t} + g(\omega_k) N S^+ a(\omega_k) e^{-i(\omega_k - \omega_A)t} \right)
\end{aligned} \tag{2.9}$$

Where we have defined a continuous term of coupling strength $g(\omega_k)$, which includes the density of states of the cavity/waveguide modes, and $a(\omega_k)$ and $a^\dagger(\omega_k)$ are the continuous bosonic operators of the form of the relations (1.36).

The interaction Hamiltonian in the interaction picture is:

$$\begin{aligned}
H_I^{\mathcal{SR}} = & \sum_i \int d\omega_\mu \hbar \left(\kappa^*(\omega_\mu) \tilde{r}(\omega_\mu)^\dagger \sigma_i^- e^{i(\omega_\mu - \omega_A)t} + \kappa(\omega_\mu) \sigma_i^+ \tilde{r}(\omega_\mu) e^{-i(\omega_\mu - \omega_A)t} \right) \\
& + \sum_i \int d\omega_\nu \hbar \left(\tilde{\kappa}^*(\omega_\nu) \mathcal{P}^+(\omega_\nu) \sigma_i^- e^{-i(\omega_\nu - \omega_A)t} + \tilde{\kappa}(\omega_\nu) \sigma_i^+ \mathcal{P}^-(\omega_\nu) e^{i(\omega_\nu - \omega_A)t} \right) \\
& + \int d\omega_\alpha \int d\omega_\beta \hbar \left(\zeta^*(\omega_\alpha, \omega_\beta) r(\omega_\alpha)^\dagger a(\omega_\beta) e^{i(\omega_\alpha - \omega_\beta)t} + \zeta(\omega_\alpha, \omega_\beta) a^\dagger(\omega_\beta) r(\omega_\alpha) e^{-i(\omega_\alpha - \omega_\beta)t} \right)
\end{aligned} \tag{2.10}$$

Where $\kappa(\omega_\mu)$ is the coupling between the atoms and the outside of the cavity/waveguide modes, and $\tilde{\kappa}(\omega_\nu)$ is the coupling between the atoms and the pumping mechanism reservoir modes.

The coupling term $\zeta(\omega_\alpha, \omega_\beta)$ of the last line in equation (2.10) could be understood as an adimensional decay coefficient of the waveguide, which models the effective interaction between the guided modes and the out-of-the-waveguide modes of the electromagnetic field. The underlying process of this effective interaction considers two processes. The first process is the interaction between the guided modes with the atoms of the waveguide. The second process is the subsequent interaction of these waveguide degrees of freedom with the outside of the waveguide modes. Because this model has traced those degrees of freedom associated with the waveguide dielectric body, we get an effective mode-mode interaction between the ω_α-frequency mode with the ω_β-frequency mode, expressed by the coupling term $\zeta(\omega_\alpha, \omega_\beta)$.

Chapter 3

Heisenberg-Langevin Equation of Motion of a Waveguide QED system Laser

In this chapter, we will derive the equations of motion of the laser waveguide QED system model of the last chapter. First, we will consider the continuous Hamiltonian of the total system, reservoir plus system of interest, to derive its Heisenberg equations of motion. Then, we will apply the Heisenberg-Langevin formalism shown in section (Langevin) to obtain the dynamic equations of the system of interest operators.

This chapter is divided as follows. Section 3.1 presents the Heisenberg equation of motion of all the total system operators. In section 3.2, we will use the methodology presented in section 1.2.3 to derive the Heisenberg-Langevin equation of motion of the system of interest operators. Finally, 3.3 is a summary of our model of a waveguide QED system that will be used in the following chapter.

3.1. Heisenberg Equation of the total system operators

Considering the continuous Hamiltonian of the waveguide QED system laser in the interaction picture:

$$
\begin{aligned}
H_I = \quad & \sum_j \int d\omega_k \hbar \left(g^*(\omega_k) a^\dagger(\omega_k) \sigma_j^- e^{i(\omega_k - \omega_A)t} + g(\omega_k) \sigma_j^+ a(\omega_k) e^{-i(\omega_k - \omega_A)t} \right) \\
& + \sum_i \int d\omega_\mu \hbar \left(\kappa^*(\omega_\mu) \tilde{r}(\omega_\mu)^\dagger \sigma_i^- e^{i(\omega_\mu - \omega_A)t} + \kappa(\omega_\mu) \sigma_i^+ \tilde{r}(\omega_\mu) e^{-i(\omega_\mu - \omega_A)t} \right) \\
& + \sum_i \int d\omega_\nu \hbar \left(\tilde{\kappa}^*(\omega_\nu) \mathcal{P}^+(\omega_\nu) \sigma_i^- e^{-i(\omega_\nu - \omega_A)t} + \tilde{\kappa}(\omega_\nu) \sigma_i^+ \mathcal{P}^-(\omega_\nu) e^{i(\omega_\nu - \omega_A)t} \right) \\
& + \int d\omega_\alpha \int d\omega_\beta \hbar \left(\zeta^*(\omega_\alpha, \omega_\beta) r(\omega_\alpha)^\dagger a(\omega_\beta) e^{i(\omega_\alpha - \omega_\beta)t} + \zeta(\omega_\alpha, \omega_\beta) a^\dagger(\omega_\beta) r(\omega_\alpha) e^{-i(\omega_\alpha - \omega_\beta)t} \right)
\end{aligned}
\tag{3.1}
$$

In this section, we will study the evolution of all operators which influence the dynamic of our system of interest. Using the general expression of the evolution of an operator A:

$$\dot{A}(t) = \frac{-i}{\hbar}\left[A(t), H\right] + \left(\frac{\partial A}{\partial t}\right) \tag{3.2}$$

In the case of the reservoir and the system operators, there is no explicit dependency on time, then the last term of the RHS of equation (3.2) vanishes.

Applying the expression (3.2) to the system of interest operators. In the case of the bosonic operators of the waveguide modes, we obtain:

$$\dot{a}(\omega_k) = \tfrac{-i}{\hbar}\left[a(\omega_k), H_I\right] \tag{3.3}$$

$$\dot{a}(\omega_k) = -i\int d\omega_\alpha \zeta(\omega_\alpha, \omega_k) r(\omega_\alpha) e^{-i(\omega_\alpha - \omega_k)t} - i\sum_i g^*(\omega_k)\sigma_i^- e^{i(\omega_k - \omega_A)t} \tag{3.4}$$

In the same way, we could obtain the evolution of the atomics operators for coherence and inversion:

$$\begin{aligned}\dot{\sigma}_j^- = \; & -i\int \omega_\mu \kappa(\omega_\mu)\sigma_j^z \tilde{r}(\omega_\mu) e^{-i(\omega_\mu - \omega_A)t} - i\int d\omega_\beta \tilde{\kappa}(\omega_\beta)\sigma_j^z \mathcal{P}^+(\omega_\beta) e^{-i(\omega_\beta - \omega_A)t} \\ & -i\int d\omega_k g(\omega_k)\sigma_j^z a(\omega_k) e^{-i(\omega_k - \omega_A)t}\end{aligned} \tag{3.5}$$

$$\begin{aligned}\dot{\sigma}_j^z = \; & 2i\int d\omega_\mu \left(\kappa^*(\omega_\mu)\tilde{r}^\dagger \sigma_j^- e^{i(\omega_\mu - \omega_A)t} - \kappa(\omega_\mu)\sigma_j^+ \tilde{r}(\omega_\mu) e^{-i(\omega_\mu - \omega_A)t}\right) \\ & -2i\int d\omega_\beta \left(\tilde{\kappa}^*(\omega_\beta)\sigma_j^+ \mathcal{P}^+(\omega_\beta) e^{-i(\omega_\beta - \omega_A)t} - \tilde{\kappa}(\omega_\beta)\mathcal{P}^-(\omega_\beta)\sigma_j^- e^{i(\omega_\beta - \omega_A)t}\right) \\ & +2i\int d\omega_k \left(g^*(\omega_k) a^\dagger(\omega_k)\sigma_j^- e^{i(\omega_k - \omega_A)t} - g(\omega_k)\sigma_j^+ a(\omega_k) e^{-i(\omega_k - \omega_A)t}\right)\end{aligned} \tag{3.6}$$

Applying the same procedure to the reservoir operators. In the case of the bosonic reservoir operators $\tilde{r}(\omega_l)$, which represent the modes interacting with the atoms of the system of interest, we get:

$$
\begin{aligned}
\dot{\tilde{r}}(\omega_l) = \quad & \frac{-i}{\hbar}\left[\tilde{r}(\omega_l), H_I\right] \\
= \quad & \frac{-i}{\hbar}\left[\tilde{r}(\omega_l), \sum_i \int d\omega_\mu \hbar \left(\kappa^*(\omega_\mu)\tilde{r}(\omega_\mu)^\dagger \sigma_i^- e^{i(\omega_\mu-\omega_A)t} + \kappa(\omega_\mu)\sigma_i^+ \tilde{r}(\omega_\mu) e^{-i(\omega_\mu-\omega_A)t}\right)\right] \\
= \quad & \frac{-i}{\hbar}, \sum_i \int d\omega_\mu \hbar \left(\kappa^*(\omega_\mu)\left[\tilde{r}(\omega_l), \tilde{r}(\omega_\mu)^\dagger\right]\sigma_i^- e^{i(\omega_\mu-\omega_A)t}\right) \\
= \quad & \frac{-i}{\hbar} \sum_i \int d\omega_\mu \hbar \left(\kappa^*(\omega_\mu)\delta(\omega_l-\omega_\mu)\sigma_i^- e^{i(\omega_\mu-\omega_A)t}\right) \\
= \quad & -i \sum_i \kappa^*(\omega_l)\sigma_i^- e^{i(\omega_l-\omega_A)t}
\end{aligned}
\tag{3.7}
$$

In the case of the bosonic reservoir operators $r(\omega_\nu)$, which represent the electromagnetic field modes interacting with the waveguide modes, we get:

$$
\begin{aligned}
\dot{r}(\omega_\nu) = \quad & \frac{-i}{\hbar}\left[r(\omega_\nu), H_I\right] \\
= \quad & \frac{-i}{\hbar}\left[r(\omega_\nu), \int d\omega_\alpha \int d\omega_\beta \hbar \left(\zeta^*(\omega_\alpha, \omega_\beta) r^\dagger(\omega_\alpha) a(\omega_\beta) e^{i(\omega_\alpha-\omega_\beta)t}\right)\right] \\
= \quad & \frac{-i}{\hbar} \int d\omega_\alpha \int d\omega_\beta \hbar \left(\zeta^*(\omega_\alpha, \omega_\beta)\left[r(\omega_\nu), r^\dagger(\omega_\alpha)\right] a(\omega_\beta) e^{i(\omega_\alpha-\omega_\beta)t}\right) \\
= \quad & \frac{-i}{\hbar} \int d\omega_\alpha \int d\omega_\beta \hbar \left(\zeta^*(\omega_\alpha, \omega_\beta)\delta(\omega_\nu-\omega) a(\omega_\beta) e^{i(\omega_\alpha-\omega_\beta)t}\right) \\
= \quad & -i \int d\omega_\beta \zeta^*(\omega_\nu, \omega_\beta) a(\omega_\beta) e^{i(\omega_\nu-\omega_\beta)t}
\end{aligned}
\tag{3.8}
$$

Finally, the evolution of the pumping operators $\mathcal{P}^-(\omega_\mu)$ and $\mathcal{P}^z(\omega_\mu)$ is determined by:

$$
\begin{aligned}
\dot{\mathcal{P}}^-(\omega_\mu) = \quad & \frac{-i}{\hbar}\left[\mathcal{P}^-(\omega_\mu), H_I\right] \\
= \quad & \frac{-i}{\hbar}\left[\mathcal{P}^-(\omega_\mu), \sum_i \int d\omega_\nu \hbar \left(\tilde{\kappa}^*(\omega_\nu)\mathcal{P}^+(\omega_\nu)\sigma_i^- e^{-i(\omega_\nu-\omega_A)t} + \tilde{\kappa}(\omega_\nu)\sigma_i^+ \mathcal{P}^-(\omega_\nu) e^{i(\omega_\nu-\omega_A)t}\right)\right] \\
= \quad & \frac{-i}{\hbar} \sum_i \int d\omega_\nu \hbar \left(\tilde{\kappa}^*(\omega_\nu)\left[\mathcal{P}^-(\omega_\mu), \mathcal{P}^+(\omega_\nu)\right]\sigma_i^- e^{-i(\omega_\nu-\omega_A)t}\right) \\
= \quad & \frac{-i}{\hbar} \sum_i \int d\omega_\nu \hbar \tilde{\kappa}^*(\omega_\nu)\left(-\delta(\omega_\mu-\omega_\nu)\mathcal{P}^z(\omega_\nu)\right)\sigma_i^- e^{-i(\omega_\nu-\omega_A)t} \\
= \quad & i \sum_i \int d\omega_\mu \tilde{\kappa}^*(\omega_\mu)\mathcal{P}^z(\omega_\mu)\sigma_i^- e^{-i(\omega_\mu-\omega_A)t}
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
\dot{\mathcal{P}}^z(\omega_\mu) = \quad & \frac{-i}{\hbar}\left[\mathcal{P}^z(\omega_\mu), H_I\right] \\
= \quad & \frac{-i}{\hbar}\left[\mathcal{P}^z(\omega_\mu), \sum_i \int d\omega_\nu \hbar \left(\tilde{\kappa}^*(\omega_\nu)\mathcal{P}^+(\omega_\nu)\sigma_i^- e^{-i(\omega_\nu-\omega_A)t} + \tilde{\kappa}(\omega_\nu)\sigma_i^+\mathcal{P}^-(\omega_\nu)e^{i(\omega_\nu-\omega_A)t}\right)\right] \\
= \quad & -i\sum_i \int d\omega_\nu \left(\tilde{\kappa}^*(\omega_\nu)\left[\mathcal{P}^z(\omega_\mu), \mathcal{P}^+(\omega_\nu)\right]\sigma_i^- e^{-i(\omega_\nu-\omega_A)t} + \tilde{\kappa}(\omega_\mu)\sigma_i^+\left[\mathcal{P}^z(\omega_\mu), \mathcal{P}^+(\omega_\nu)\right]e^{i(\omega_\mu-\omega_A)}\right. \\
= \quad & -i\sum_i \int d\omega_\nu \left(\tilde{\kappa}^*(\omega_\nu)(2\delta(\omega_\mu-\omega_\nu)\mathcal{P}^+(\omega_\nu))\sigma_i^- e^{-i(\omega_\nu-\omega_A)t} + \tilde{\kappa}(\omega_\mu)\sigma_i^+(-2\delta(\omega_\mu-\omega_\nu)\mathcal{P}^-(\omega_\nu))e\right. \\
= \quad & -2i\sum_i \left(\tilde{\kappa}^*(\omega_\mu)\mathcal{P}^+(\omega_\mu)\sigma_i^- e^{-i(\omega_\mu-\omega_A)t} - \tilde{\kappa}(\omega_\mu)\sigma_i^+\mathcal{P}^-(\omega_\mu)e^{i(\omega_\mu-\omega_A)t}\right)
\end{aligned}
$$

3.1.1. Solutions of Heisenberg Equations of the Reservoir operators

We need to integrate the reservoir equation of motion (3.7), (3.8), (3.9) and (3.10) to obtain the Heisenberg-Langevin evolution equation of the system of interest operators.

The solution of the equation (3.7) is

$$
\tilde{r}_t(\omega_l) = \tilde{r}_{t_0}(\omega_l) - i\sum_i \int dt' \kappa^*(\omega_l)\sigma_i^- e^{i(\omega_l-\omega_A)t'} \tag{3.11}
$$

The evolution of the $r(\omega_\nu)$reservoir operator

$$
r_t(\omega_\nu) = r_{t_0}(\omega_\nu) - i\int dt' \int d\omega_\beta \zeta^*(\omega_\nu, \omega_\beta)a_{t'}(\omega_\beta)e^{i(\omega_\nu-\omega_\beta)t'} \tag{3.12}
$$

And the solutions of the pumping operator equations (3.9) and (3.10) are

$$
\mathcal{P}_t^-(\omega_\mu) = \mathcal{P}_{t_0}^-(\omega_\mu) + i\sum_i \int dt' \tilde{\kappa}^*(\omega_\mu)\mathcal{P}_{t'}^z(\omega_\mu)\sigma_i^+(t')e^{-i(\omega_\mu-\omega_A)t'} \tag{3.13}
$$

$$
\begin{aligned}
\mathcal{P}_t^z(\omega_\mu) = \mathcal{P}_{t_0}^z(\omega_\mu) - 2i\sum_i \int dt' \tilde{\kappa}^*(\omega_\mu)\sigma_i^+(t')\mathcal{P}_{t'}^+(\omega_\mu)e^{-i(\omega_\mu-\omega_A)t'} \\
+2i\sum_i \int dt' \tilde{\kappa}(\omega_\mu)\mathcal{P}_{t'}^z(\omega_\mu)\sigma_i^-(t')e^{i(\omega_\mu-\omega_A)t}
\end{aligned} \tag{3.14}
$$

3.2. Deduction of the Heisenberg-Langevin equations of motion

3.2.1. Deduction of the atomic coherence operator Heisenberg-Langevin Equation of Motion

From the equation of motion of the coherence atomic operator 3.5, we will replace the solution of the reservoir bosonic operator which interacts with the atoms (3.11) and the pumping operator equation solution (3.13)

$$\begin{aligned}\dot{\sigma}_j^- = \quad & -i\int d\omega_k g(\omega_k)\sigma_j^z a(\omega_k)e^{-i(\omega_k-\omega_A)t} \\ & -i\int \omega_\mu \kappa(\omega_\mu)\sigma_j^z\left(\tilde{r}_{t_0}(\omega_\mu) - i\sum_i \int dt'\kappa^*(\omega_\mu)\sigma_i^- e^{i(\omega_\mu-\omega_A)t'}\right)e^{-i(\omega_\mu-\omega_A)t} \\ & -i\int d\omega_\beta \tilde{\kappa}(\omega_\beta)\sigma_j^z\left(\mathcal{P}_{t_0}^+(\omega_\mu) - i\sum_i \int dt'\tilde{\kappa}^*(\omega_\beta)\mathcal{P}_{t'}^z(\omega_\beta)\sigma_i^-(t')e^{-i(\omega_\beta-\omega_A)t'}\right)e^{-i(\omega_\beta-\omega_A)t}\end{aligned} \tag{3.15}$$

Defining the noise operator associated to the atomic coherence $\mathcal{F}_{(-)}^R = -i\int \omega_\mu\kappa(\omega_\mu)\sigma_j^z\left(\tilde{r}_{t_0}(\omega_\mu)\right)e^{-i(\omega_\mu-\omega_A)t}$ and $\mathcal{F}_{(-)}^{\mathcal{P}} = -i\int d\omega_\beta\tilde{\kappa}(\omega_\beta)\sigma_j^z\left(\mathcal{P}_{t_0}^+(\omega_\mu)\right)e^{-i(\omega_\beta-\omega_A)t}$, where $\mathcal{F}_{(-)} = \mathcal{F}_{(-)}^{\mathcal{P}} + \mathcal{F}_{(-)}^R$.

$$\begin{aligned}\dot{\sigma}_j^- = \quad & -i\int d\omega_k g(\omega_k)\sigma_j^z a(\omega_k)e^{-i(\omega_k-\omega_A)t} + \mathcal{F}_{(-)} \\ & -\sum_i \int \omega_\mu\kappa(\omega_\mu)\kappa^*(\omega_\mu)\sigma_j^z(t)\left(\int dt'\sigma_i^-(t')e^{i(\omega_\mu-\omega_A)t'}\right)e^{-i(\omega_\mu-\omega_A)t} \\ & -\sum_i \int d\omega_\beta\tilde{\kappa}(\omega_\beta)\tilde{\kappa}^*(\omega_\beta)\sigma_j^z(t)\left(\int dt'\sigma_i^-(t')\mathcal{P}_{t'}^z(\omega_\beta)e^{-i(\omega_\beta-\omega_A)t'}\right)e^{-i(\omega_\beta-\omega_A)t}\end{aligned} \tag{3.16}$$

Now, we will consider that the term $\sigma_i^-(t')$ and $\mathcal{P}_{t'}^z(\omega_\beta)\sigma_i^-(t')$ in the integrand varies slowly compared with the exponential term so that we will apply the approximation $\sigma_i^-(t') \to \sigma_i^-(t)$ and $\mathcal{P}_{t'}^z(\omega_\beta)\sigma_i^-(t') \to \mathcal{P}_t^z(\omega_\beta)\sigma_i^-(t)$, respectively.

$$\begin{aligned}\dot{\sigma}_j^- = \quad & -i\int d\omega_k g(\omega_k)\sigma_j^z a(\omega_k)e^{-i(\omega_k-\omega_A)t} + \mathcal{F}_{(-)} \\ & -\sum_i \int \omega_\mu\kappa(\omega_\mu)\kappa^*(\omega_\mu)\sigma_j^z(t)\sigma_i^-(t)\left(\int dt' e^{i(\omega_\mu-\omega_A)t'}\right)e^{-i(\omega_\mu-\omega_A)t} \\ & -\sum_i \int d\omega_\beta\tilde{\kappa}(\omega_\beta)\tilde{\kappa}^*(\omega_\beta)\sigma_j^z(t)\sigma_i^-(t)\mathcal{P}_t^z(\omega_\beta)\left(\int dt' e^{-i(\omega_\beta-\omega_A)t'}\right)e^{-i(\omega_\beta-\omega_A)t}\end{aligned} \tag{3.17}$$

We will use the relation

$$\lim_{t\to\infty}\int_0^t dt' e^{i(\omega-\omega_0)t'} = \pi\delta(\omega-\omega_0) + i\frac{P}{\omega_0-\omega} \tag{3.18}$$

The, we obtain

$$
\begin{aligned}
\dot{\sigma}_j^- = \quad & -i\int d\omega_k g(\omega_k)\sigma_j^z a(\omega_k)e^{-i(\omega_k-\omega_A)t} + \mathcal{F}_{(-)} \\
& -\sum_i \int \omega_\mu \kappa(\omega_\mu)\kappa^*(\omega_\mu)\sigma_j^z(t)\sigma_i^-(t)\left(\pi\delta(\omega_\mu-\omega_A) - i\frac{P}{\omega_\mu-\omega_A}\right)e^{-i(\omega_\mu-\omega_A)t} \\
& -\sum_i \int d\omega_\beta \tilde{\kappa}(\omega_\beta)\tilde{\kappa}^*(\omega_\beta)\sigma_j^z(t)\sigma_i^-(t)\mathcal{P}_t^z(\omega_\beta)\left(\pi\delta(\omega_\beta-\omega_A) + i\frac{P}{\omega_\beta-\omega_A}\right)e^{-i(\omega_\beta-\omega_A)t}
\end{aligned}
\tag{3.19}
$$

Defining the decay rates of the atomic coherence associated with the spontaneous emission and the pumping mechanism:

$$
\gamma = 2\pi|\kappa(\omega_A)|^2 \tag{3.20}
$$

$$
w = 2\pi|\tilde{\kappa}(\omega_A)|^2\mathcal{P}_t^z(\omega_A) \tag{3.21}
$$

And the terms associated with the Lamb shift:

$$
\Delta_{(-)} = P\sum_i \int \omega_\mu \frac{|\kappa(\omega_\mu)|^2}{\omega_\mu-\omega_A}e^{-i(\omega_\mu-\omega_A)t} - P\sum_i \int d\omega_\beta \frac{|\tilde{\kappa}(\omega_\beta)|^2}{\omega_\beta-\omega_A}\mathcal{P}_t^z(\omega_\beta)e^{-i(\omega_\beta-\omega_A)t} \tag{3.22}
$$

Finally, we get the Heisenberg-Langevin equation of the atomic coherence operator:

$$
\dot{\sigma}_j^- = -i\int d\omega_k g(\omega_k)\sigma_j^z a(\omega_k)e^{-i(\omega_k-\omega_A)t} + \mathcal{F}_{(-)} - \frac{\gamma+w}{2}\sum_i \sigma_j^z(t)\sigma_i^-(t) + i\Delta\sigma_j^z(t)\sigma_i^-(t) \tag{3.23}
$$

3.2.2. Deduction of the atomic inversion operator Heisenberg-Langevin Equation of Motion

From the equation of motion of the atomic inversion operator (3.6), we will replace the solution of the reservoir bosonic operator, which interacts with the atoms (3.7) and the pumping operator equation solution (3.13)

$$
\begin{aligned}
\dot{\sigma}_j^z = \quad & 2i\int d\omega_\mu \left(\kappa^*(\omega_\mu)\left(\tilde{r}_{t_0}^\dagger(\omega_\mu) + i\sum_i \int dt'\kappa(\omega_\mu)\sigma_i^+(t')e^{-i(\omega_\mu-\omega_A)t'}\right)\sigma_j^-(t)e^{i(\omega_\mu-\omega_A)t}\right) \\
& -2i\int d\omega_\mu \left(\kappa(\omega_\mu)\sigma_j^+(t)\left(\tilde{r}_{t_0}(\omega_\mu) - i\sum_i \int dt'\kappa^*(\omega_\mu)\sigma_i^-(t')e^{i(\omega_\mu-\omega_A)t'}\right)e^{-i(\omega_\mu-\omega_A)t}\right) \\
& -2i\int d\omega_\beta \left(\tilde{\kappa}^*(\omega_\beta)\sigma_j^+(t)\left(\mathcal{P}_{t_0}^+(\omega_\beta) - i\sum_i \int dt'\tilde{\kappa}^*(\omega_\beta)\sigma_i^-(t')\mathcal{P}_{t'}^z(\omega_\beta)e^{i(\omega_\mu-\omega_A)t'}\right)e^{-i(\omega_\beta-\omega_A)t}\right) \\
& +2i\int d\omega_\beta \left(\tilde{\kappa}(\omega_\beta)\left(\mathcal{P}_{t_0}^-(\omega_\beta) + i\sum_i \int dt'\tilde{\kappa}^*(\omega_\beta)\mathcal{P}_{t'}^z(\omega_\beta)\sigma_i^+(t')e^{-i(\omega_\mu-\omega_A)t'}\right)\sigma_j^-(t)e^{i(\omega_\beta-\omega_A)t}\right) \\
& +2i\int d\omega_k \left(g^*(\omega_k)a^\dagger(\omega_k)\sigma_j^- e^{i(\omega_k-\omega_A)t} - g(\omega_k)\sigma_j^+ a(\omega_k)e^{-i(\omega_k-\omega_A)t}\right) \qquad (3.24)
\end{aligned}
$$

Defining the noise operator associated to the atomic inversion $\mathcal{F}_{(z)} = \mathcal{F}_{(z)}^R + \mathcal{F}_{(z)}^{\mathcal{P}}$, where

$$
\mathcal{F}_{(z)}^R = 2i\int d\omega_\mu \left(\kappa^*(\omega_\mu)\tilde{r}_{t_0}^\dagger(\omega_\mu)\sigma_j^- e^{i(\omega_\mu-\omega_A)t}\right) - 2i\int d\omega_\mu \left(\kappa(\omega_\mu)\sigma_j^+\tilde{r}_{t_0}(\omega_\mu)e^{-i(\omega_\mu-\omega_A)t}\right) \qquad (3.25)
$$

$$
\mathcal{F}_z^{\mathcal{P}} = -2i\int d\omega_\beta \left(\tilde{\kappa}^*(\omega_\beta)\sigma_j^+\mathcal{P}_{t_0}^+(\omega_\beta)e^{-i(\omega_\beta-\omega_A)t}\right) + 2i\int d\omega_\beta \left(\tilde{\kappa}(\omega_\beta)\mathcal{P}_{t_0}^-(\omega_\beta)\sigma_j^- e^{i(\omega_\beta-\omega_A)}t\right) \qquad (3.26)
$$

Then, the inversion equation takes the form of (3.27).

$$
\begin{aligned}
\dot{\sigma}_j^z = \quad & \mathcal{F}_{(z)} + 2i\int d\omega_k \left(g^*(\omega_k)a^\dagger(\omega_k)\sigma_j^- e^{i(\omega_k-\omega_A)t} - g(\omega_k)\sigma_j^+ a(\omega_k)e^{-i(\omega_k-\omega_A)t}\right) \\
& -2\sum_i \int d\omega_\mu \left(|\kappa(\omega_\mu)|^2\left(\int dt'\sigma_i^+(t')e^{-i(\omega_\mu-\omega_A)t'}\right)\sigma_j^-(t)e^{i(\omega_\mu-\omega_A)t}\right) \\
& -2\sum_i \int d\omega_\mu \left(|\kappa(\omega_\mu)|^2\sigma_j^+(t)\left(\int dt'\sigma_i^-(t')e^{i(\omega_\mu-\omega_A)t'}\right)e^{-i(\omega_\mu-\omega_A)t}\right) \\
& -2\sum_i \int d\omega_\beta \left(|\tilde{\kappa}(\omega_\beta)|^2\sigma_j^+(t)\left(\int dt'\sigma_i^-(t')\mathcal{P}_{t'}^z(\omega_\beta)e^{i(\omega_\mu-\omega_A)t'}\right)e^{-i(\omega_\beta-\omega_A)t}\right) \\
& -2\sum_i \int d\omega_\beta \left(|\tilde{\kappa}(\omega_\beta)|^2\left(\int dt'\mathcal{P}_{t'}^z(\omega_\beta)\sigma_i^+(t')e^{-i(\omega_\mu-\omega_A)t'}\right)\sigma_j^-(t)e^{i(\omega_\beta-\omega_A)t}\right)
\end{aligned} \qquad (3.27)
$$

Now, we will consider that the terms $\sigma_i^-(t')$,$\sigma_i^+(t')$, $\mathcal{P}_{t'}^z(\omega_\beta)\sigma_i^-(t')$ and $\sigma_i^+(t')\mathcal{P}_{t'}^z(\omega_\beta)$ in the integrand varies slowly compared with the exponential term, so we will apply the approximation $\sigma_i^-(t') \to \sigma_i^-(t)$,$\sigma_i^+(t') \to \sigma_i^+(t)$, $\mathcal{P}_{t'}^z(\omega_\beta)\sigma_i^-(t') \to \mathcal{P}_t^z(\omega_\beta)\sigma_i^-(t)$ and $\sigma_i^+(t')\mathcal{P}_{t'}^z(\omega_\beta) \to \sigma_i^+(t)\mathcal{P}_t^z(\omega_\beta)$.

$$
\begin{aligned}
\dot{\sigma}_j^z = \quad & \mathcal{F}_{(z)} + 2i \int d\omega_k \left(g^*(\omega_k) a^\dagger(\omega_k) \sigma_j^- e^{i(\omega_k - \omega_A)t} - g(\omega_k) \sigma_j^+ a(\omega_k) e^{-i(\omega_k - \omega_A)t} \right) \\
& -2 \sum_i \int d\omega_\mu \left(|\kappa(\omega_\mu)|^2 \left(\int dt' e^{-i(\omega_\mu - \omega_A)t'} \right) \sigma_i^+(t) \sigma_j^-(t) e^{i(\omega_\mu - \omega_A)t} \right) \\
& -2 \sum_i \int d\omega_\mu \left(|\kappa(\omega_\mu)|^2 \sigma_j^+(t) \sigma_i^-(t) \left(\int dt' e^{i(\omega_\mu - \omega_A)t'} \right) e^{-i(\omega_\mu - \omega_A)t} \right) \\
& -2 \sum_i \int d\omega_\beta \left(|\tilde{\kappa}(\omega_\beta)|^2 \sigma_j^-(t) \sigma_i^+(t) \mathcal{P}_t^z(\omega_\beta) \left(\int dt' e^{i(\omega_\mu - \omega_A)t'} \right) e^{-i(\omega_\beta - \omega_A)t} \right) \\
& -2 \sum_i \int d\omega_\beta \left(|\tilde{\kappa}(\omega_\beta)|^2 \left(\int dt' e^{-i(\omega_\mu - \omega_A)t'} \right) \mathcal{P}_t^z(\omega_\beta) \sigma_i^+(t) \sigma_j^-(t) e^{i(\omega_\beta - \omega_A)t} \right)
\end{aligned}
\tag{3.28}
$$

Using the relation (3.18), we could obtain the time integral of the last equation.

$$
\begin{aligned}
\dot{\sigma}_j^z = \quad & \mathcal{F}_{(z)} + 2i \int d\omega_k \left(g^*(\omega_k) a^\dagger(\omega_k) \sigma_j^- e^{i(\omega_k - \omega_A)t} - g(\omega_k) \sigma_j^+ a(\omega_k) e^{-i(\omega_k - \omega_A)t} \right) \\
& -2 \sum_i \int d\omega_\mu \left(|\kappa(\omega_\mu)|^2 \left(\pi\delta(\omega_\mu - \omega_A) + i \frac{P}{\omega_\mu - \omega_a} \right) \sigma_i^+(t) \sigma_j^-(t) e^{i(\omega_\mu - \omega_A)t} \right) \\
& -2 \sum_i \int d\omega_\mu \left(|\kappa(\omega_\mu)|^2 \sigma_j^+(t) \sigma_i^-(t) \left(\pi\delta(\omega_\mu - \omega_A) - i \frac{P}{\omega_\mu - \omega_a} \right) e^{-i(\omega_\mu - \omega_A)t} \right) \\
& -2 \sum_i \int d\omega_\beta \left(|\tilde{\kappa}(\omega_\beta)|^2 \sigma_j^-(t) \sigma_i^+(t) \mathcal{P}_t^z(\omega_\beta) \left(\pi\delta(\omega_\beta - \omega_A) - i \frac{P}{\omega_\beta - \omega_a}) \right) e^{-i(\omega_\beta - \omega_A)t} \right) \\
& -2 \sum_i \int d\omega_\beta \left(|\tilde{\kappa}(\omega_\beta)|^2 \left(\pi\delta(\omega_\beta - \omega_A) + i \frac{P}{\omega_\beta - \omega_a} \right) \mathcal{P}_t^z(\omega_\beta) \sigma_i^+(t) \sigma_j^-(t) e^{i(\omega_\beta - \omega_A)t} \right)
\end{aligned}
\tag{3.29}
$$

Now, we will use the definition of the decay rates γ and w of the equations (3.21) and (3.20).

$$
\begin{aligned}
\dot{\sigma}_j^z = \quad & \mathcal{F}_{(z)} + 2i \int d\omega_k \left(g^*(\omega_k) a^\dagger(\omega_k) \sigma_j^- e^{i(\omega_k - \omega_A)t} - g(\omega_k) \sigma_j^+ a(\omega_k) e^{-i(\omega_k - \omega_A)t} \right) \\
& -\gamma \sum_i \left(\sigma_i^+(t) \sigma_j^-(t) + \sigma_j^+(t) \sigma_i^-(t) \right) - w \sum_i \left(\sigma_j^-(t) \sigma_i^+(t) + \sigma_i^+(t) \sigma_j^-(t) \right) \\
& -2iP \sum_i \int d\omega_\mu \left(\frac{|\kappa(\omega_\mu)|^2}{\omega_\mu - \omega_a} \right) \left(\sigma_i^+(t) \sigma_j^-(t) e^{i(\omega_\mu - \omega_A)t} - \sigma_j^+(t) \sigma_i^-(t) e^{-i(\omega_\mu - \omega_A)t} \right) \\
& +2iP \sum_i \int d\omega_\beta \left(\frac{|\tilde{\kappa}(\omega_\beta)|^2}{\omega_\beta - \omega_a} \right) \left(\sigma_j^+(t) \sigma_i^-(t) \mathcal{P}_t^z(\omega_\beta) e^{-i(\omega_\beta - \omega_A)t} - \mathcal{P}_t^z(\omega_\beta) \sigma_i^+(t) \sigma_j^-(t) e^{i(\omega_\beta - \omega_A)t} \right)
\end{aligned}
\tag{3.30}
$$

3.2.3. Deduction of the Continuous bosonic Heisenberg-Langevin Equation of Motion

From the equation of motion of the bosonic operator (3.4), we will replace the solution of the reservoir bosonic operator (3.12)

$$\dot{a}(\omega_k) = \quad -ig^*(\omega_k)NS^-e^{i(\omega_k-\omega_A)t} - i\int d\omega_\alpha\zeta(\omega_\alpha,\omega_k)\left(r_\alpha(t_0) - i\pi\zeta^*(\omega_\alpha,\omega_\alpha)a(\omega_\alpha)\right) \tag{3.31}$$

$$\begin{aligned}\dot{a}(\omega_k) = \quad & -ig^*(\omega_k)NS^-e^{i(\omega_k-\omega_A)t} - i\int d\omega_\alpha\zeta(\omega_\alpha,\omega_k)r_\alpha(t_0) \\ & -\pi\int d\omega_\alpha\zeta(\omega_\alpha,\omega_k)\zeta^*(\omega_\alpha,\omega_\alpha)a(\omega_\alpha)\end{aligned} \tag{3.32}$$

Here, we could define $\mathcal{F}_a(\omega_k) = -i\int d\omega_\alpha\zeta(\omega_\alpha,\omega_k)r_\alpha(t_0)$ as the noise operator of the bosonic system operator equation.

$$\dot{a}(\omega_k) = \quad -ig^*(\omega_k)NS^-e^{i(\omega_k-\omega_A)t} + \mathcal{F}_a(\omega_k) - \pi\int d\omega_\alpha\zeta(\omega_\alpha,\omega_k)\zeta^*(\omega_\alpha,\omega_\alpha)a(\omega_\alpha) \tag{3.33}$$

Assuming that the term of the coupling is only considerable near resonance, then we could define a rate associated with the mode with frequency ω_k as $\kappa(\omega_k) \equiv \pi|\bar{\kappa}(\omega_k)|^2$ and $\zeta(\omega_\alpha,\omega_k)\zeta^*(\omega_\alpha,\omega_\alpha) \approx |\bar{\kappa}(\omega_k)|^2\delta(\omega_k-\omega_\alpha)$

$$\begin{aligned}\pi\int d\omega_\alpha\zeta(\omega_\alpha,\omega_k)\zeta^*(\omega_\alpha,\omega_\alpha)a(\omega_\alpha) \quad & \approx \pi\int d\omega_\alpha|\bar{\kappa}(\omega_k)|^2\delta(\omega_k-\omega_\alpha)a(\omega_\alpha) \\ & \approx \pi|\bar{\kappa}(\omega_k)|^2 a(\omega_k)\end{aligned} \tag{3.34}$$

Finally, we get the Heisenberg-Langevin equation of motion of the bosonic operator

$$\dot{a}(\omega_k) = \quad -ig^*(\omega_k)NS^-e^{i(\omega_k-\omega_A)t} + \mathcal{F}_a(\omega_k) - \kappa(\omega_k)a(\omega_k). \tag{3.35}$$

3.3. Heisenberg-Langevin Equations of Motion of the Atom-Field System operators

This section presents a summary of the equation of motion of the waveguide QED system laser model derived in the previous sections. Explicit calculations of the derivation of those equations are presented in Appendix B and C.

3.3.1. Equations of Motion in the Rotating Frame

Defining the new variables in the rotating frame $\tilde{a}(\omega_k) = a(\omega_k)e^{-i(\omega_k-\omega)t}$ and $\tilde{S}^- = S^- e^{-i(\omega_a-\omega)t}$, acording to the equations (C.1,C.2,C.3), we get:

$$\dot{a}(\omega_k) = \tilde{\mathcal{F}}_{(a)} - ig(\omega_k)^* N S^- - \frac{1}{2}\left(\kappa - 2i(\omega_k - \omega)\right) a(\omega_k) \tag{3.36a}$$

$$\dot{S}^- = \mathcal{F}_{(-)} - \frac{1}{2}\left[\Gamma + 2i(\omega_A - \omega)\right] S^- + i\int d\omega_k g(\omega_k) S^z a(\omega_k) \tag{3.36b}$$

$$\dot{S}^z = \mathcal{F}_z - \Gamma\left(S^z - d_0\right) + 2i\int d\omega_k \left(g^*(\omega_k) a^\dagger(\omega_k) S^- - g(\omega_k) S^+ a(\omega_k)\right) \tag{3.36c}$$

This set of equations is the generalization of the set of equations (1.109), from the discrete case of a laser in a cavity QED system to the continuous case that models a laser in a waveguide QED system.

Figure 3.1 is a diagram of the model presented in this chapter. It presents the different decays process that affects the dynamics of the system operators expressed by equations (3.36).

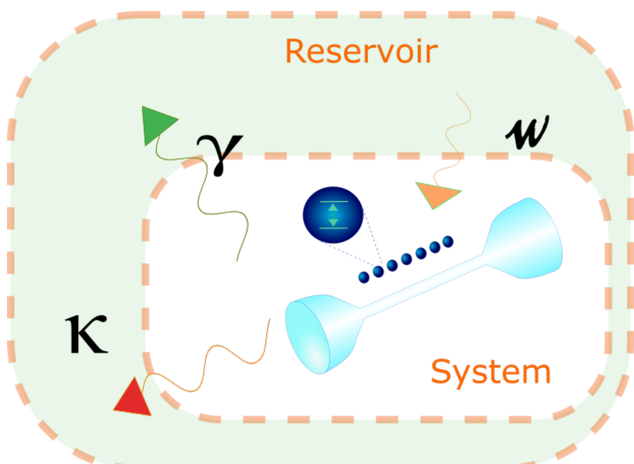

Figure 3.1: Optical nanofiber system and reservoir diagram, with κ decay process associated with the waveguide, γ represents the emission of the atoms, and w is the decay rate associated with the pumping mechanism.

Chapter 4

Linewidth of a Waveguide QED system Laser

In this chapter, we will present a method to obtain the linewidth of the field generated by a laser developed in a waveguide QED system. To obtain the laser linewidth, we will consider that the electric field is determined by the stationary solution of the bosonic operator of the waveguide in the mean-field approximation, as is shown in the equation (4.1).

$$\left\langle E^{+}(t)\right\rangle = \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} \left\langle a(\omega_k)\right\rangle_{ss} e^{-i\omega_k t} \tag{4.1}$$

We could obtain the emission spectrum from this definition and apply the Wiener-Khinchin Theorem.

This chapter is divided as follows. First, section 4.1 presents the equations of motions in the mean-field approximation and derives the steady-state solutions of those equations. Section 4.2 will obtain an expression for the Electric field operator based on the assumption (4.1). Section 4.3 derives an expression for the emission spectrum by applying the Wiener-Khinchin Theorem. Finally, subsection 4.3.2 shows the results of the power and the linewidth of the laser emission.

4.1. Mean-Field Equation of Motion

In this section, we will consider the mean-field approximation. In this approximation, the correlations of two operators could be considered independent, $\langle S^z a(\omega_k)\rangle = \langle S^z\rangle \langle a(\omega_k)\rangle$. Then, the equations of motion 3.36 derived in the previous chapter take the form:

$$\langle \dot{a}(\omega_k)\rangle = -ig(\omega_k)^* N \left\langle S^-\right\rangle - \frac{1}{2}\left[\kappa - 2i(\omega_k - \omega)\right]\langle a(\omega_k)\rangle \tag{4.2a}$$

$$\left\langle \dot{S}^-\right\rangle = -\frac{1}{2}\left[\Gamma + 2i(\omega_A - \omega)\right]\left\langle S^-\right\rangle + i\int d\omega_k g(\omega_k)\langle S^z\rangle\langle a(\omega_k)\rangle \tag{4.2b}$$

$$\left\langle \dot{S}^z \right\rangle = -\Gamma\left[\langle S^z\rangle - d_0\right] + 2i\int d\omega_k \left(g^*(\omega_k)\left\langle a^\dagger(\omega_k)\right\rangle \left\langle S^-\right\rangle - g(\omega_k)\left\langle S^+\right\rangle \langle a(\omega_k)\rangle\right) \tag{4.2c}$$

In order to obtain an expression for the electric field to derive the emission spectrum of the laser in the waveguide QED system, we will study the steady-state solutions of the set of equations 4.2.

4.1.1. Steady State Solutions in the Mean Field

Imposing the steady state condition on the coherence equation (4.2b)

$$\left\langle S^- \right\rangle_{ss} = \frac{2i}{\Gamma + 2i(\omega_A - \omega)} \int d\omega'_k g(\omega'_k) \langle S^z\rangle_{ss} \langle a(\omega'_k)\rangle \tag{4.3}$$

Replacing the last expression in the inversion mean field equation, and considering the steady state condition

$$\langle S^z\rangle_{ss} = d_0 + \frac{2i}{\Gamma}\int d\omega_k \left(g^*(\omega_k)\left\langle a^\dagger(\omega_k)\right\rangle \left\langle S^-\right\rangle_{ss} - g(\omega_k)\left\langle S^+\right\rangle_{ss} \langle a(\omega_k)\rangle\right) \tag{4.4}$$

After some algebra we obtain

$$\langle S^z\rangle_{ss}\left[1 + \left(\frac{8\langle S^z\rangle_{ss}}{\Gamma^2 - 4(\omega_A-\omega)^2}\right)\int d\omega_k g^*(\omega_k)\left\langle a^\dagger(\omega_k)\right\rangle \int d\omega'_k g(\omega'_k)\langle a(\omega'_k)\rangle\right] = d_0 \tag{4.5}$$

Considering that $g(\omega_k) = \mathcal{D}\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}$, where $\mathcal{D}$ is the dipole element of matrix of the dipole operator. Then

$$\int d\omega_k g(\omega'_k)\langle a(\omega'_k)\rangle = \int d\omega'_k \mathcal{D}\sqrt{\frac{\hbar\omega'_k}{4\pi\epsilon_0 cA}}\langle a(\omega'_k)\rangle = \mathcal{D}\left\langle E^+\right\rangle \tag{4.6}$$

Replacing in the expresion for the steady state solution in the mean field of the inversion operator

$$\langle S^z\rangle_{ss}\left(1 + \frac{8}{\Gamma^2 - 4(\omega_A-\omega)^2}\int d\omega_k g^*(\omega_k)\left\langle a^\dagger(\omega_k)\right\rangle \int d\omega'_k g(\omega'_k)\langle a(\omega'_k)\rangle\right) = d_0 \tag{4.7}$$

Where we could reconized the definition of the Electric field

$$\langle S^z\rangle_{ss} = \frac{d_0\left(\Gamma^2 - 4(\omega_A-\omega)^2\right)}{\Gamma^2 - 4(\omega_A-\omega)^2 + 8|\mathcal{D}|^2\langle E^-\rangle\langle E^+\rangle} \tag{4.8}$$

Replacing equation (4.8) on the coherence equation (4.3), the steady state solution of the mean field coherence equation

$$\left\langle S^{-}\right\rangle_{ss} = \left(\frac{2id_0\Gamma}{\Gamma^2 + 8|\mathcal{D}|^2 \langle E^-\rangle \langle E^+\rangle}\right)\left\langle E^+\right\rangle \tag{4.9}$$

Finally, the steady state solution of the mean field equation of motion of the bosonic operator

$$\langle a(\omega_k)\rangle_{ss} = \frac{-2ig(\omega_k)^* N d_0}{\kappa + 2i\Delta_k} \left[\frac{2i(\Gamma - 2i\Delta_A)}{\Gamma^2 + 4\Delta_A^2 + 8\int d\omega_k g^*(\omega_k)\langle a^\dagger(\omega_k)\rangle \int d\omega_k' g(\omega_k')\langle a(\omega_k')\rangle}\right] \int d\omega_k'' g(\omega_k'')\langle a(\omega_k'')\rangle \tag{4.10}$$

4.2. Electric Field Equation of Motion in the Heisenberg Picture

The main goal of this section is to obtain an expression for the electric field operator. First, we will consider the electric field definition in the Heisenberg picture. Then, we will present the electric field operator in the steady-state regime based on the steady-state solution of the bosonic operator Heisenberg-Langevin equation of motion in the mean-field approximation. Finally, we present an expression of the electric field operator defined as a mean field amplitude and a fluctuating term.

4.2.1. Steady-state Solution of the Mean-field equation of the Electric Field operator

In the Heisenberg picture, the electric field operator will be determined by the solution of the Heisenberg-Langevin equation of the waveguide bosonic operator.

$$E^+(t) = \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon c A}}\, a(\omega_k) \tag{4.11}$$

Where $a(\omega_k)$ is the solution of the Heisenberg equation of the bosonic operator.

The steady-state solution of the electric field operator in the mean-field approximations takes the form:

$$\left\langle E^+\right\rangle_{ss} = \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon c A}}\, \langle a(\omega_k)\rangle_{ss} \tag{4.12}$$

We could obtain an expression analogous to the condition equation of the bosonic operator (4.10) for the electric field by taking (4.10) and multiplying it by the one-photon amplitude and integrating over ω_k.

$$\left\langle E^+\right\rangle_{ss} = \left[\frac{4Nd_0(\Gamma-2i(\omega_A-\omega))}{\Gamma^2+4(\omega_A-\omega)^2+8|\mathcal{D}|^2\left\langle E^-\right\rangle_{ss}\left\langle E^+\right\rangle_{ss}}\right]\left\langle E^+\right\rangle_{ss}\int d\omega_k\left(\frac{|g(\omega_k)|^2}{\kappa+2i(\omega_k-\omega)}\right) \tag{4.13}$$

If we consider that the rotating frequency is resonant with the atomic frequency, $\omega_A = \omega$, then:

$$\left\langle E^+\right\rangle_{ss} = \left[\frac{4Nd_0(\Gamma)}{\Gamma^2+8|\mathcal{D}|^2\left\langle E^-\right\rangle_{ss}\left\langle E^+\right\rangle_{ss}}\right]\left\langle E^+\right\rangle_{ss}\int d\omega_k\left(\frac{|g(\omega_k)|^2}{\kappa+2i(\omega_k-\omega_A)}\right) \tag{4.14}$$

Studing the integral of the equation (4.14)

$$\int d\omega_k\left(\frac{|g(\omega_k)|^2}{\kappa+2i(\omega_k-\omega_A)}\right) = \frac{1}{2i}\int d\omega_k\left(\frac{|g(\omega_k)|^2}{\omega_k-\omega_A-i\kappa/2}\right) \tag{4.15}$$

In the limit of $\kappa << \omega_A$, based on the complex variable treatment of the reference [40], we obtain the relation:

$$\lim_{\kappa\to 0}\int d\omega_k\left(\frac{|g(\omega_k)|^2}{\omega_k-\omega_A-i\kappa/2}\right) = 2\pi i|g(\omega_A)|^2 \tag{4.16}$$

If we replace the result of the last expression in the equation (4.14)

$$\left\langle E^+\right\rangle_{ss} = \left[\frac{4Nd_0\Gamma}{\Gamma^2+8|\mathcal{D}|^2\left\langle E^-\right\rangle_{ss}\left\langle E^+\right\rangle_{ss}}\right]\left\langle E^+\right\rangle_{ss}\pi|g(\omega_A)|^2. \tag{4.17}$$

Here, we have defined the spontaneous emission rate in the waveguide as $\gamma^{(1D)} = 2\pi|g(\omega_A)|^2$. Then, we obtain a condition expression to obtain the steady-state solution of the electric field in the mean-field approximation:

$$\left\langle E^+\right\rangle_{ss}\left(1-\left[\frac{2Nd_0\Gamma\gamma^{(1D)}}{\Gamma^2+8|\mathcal{D}|^2\left\langle E^-\right\rangle_{ss}\left\langle E^+\right\rangle_{ss}}\right]\right) = 0 \tag{4.18}$$

The equation (4.18) has two solutions. The first is the trivial solution $\left\langle E^+\right\rangle_{ss} = 0$. The second solution is obtained by taking the parenthesis equal to zero. Studying the parenthesis condition, we get:

$$\left\langle E^{-}\right\rangle_{ss}\left\langle E^{+}\right\rangle_{ss}=\frac{\Gamma^{2}}{8|\mathcal{D}|^{2}}\left[\frac{2Nd_{0}\gamma^{(1D)}}{\Gamma}-1\right] \tag{4.19}$$

Where we could define $\mathcal{G}=\frac{2Nd_0\gamma^{(1D)}}{\Gamma}$ as the generalized cooperativity, and obtain

$$\left\langle E^{-}\right\rangle_{ss}\left\langle E^{+}\right\rangle_{ss}=\frac{\Gamma^{2}}{8|\mathcal{D}|^{2}}\left[\mathcal{G}-1\right], \tag{4.20}$$

this expression could be understood as the generalization of equation (1.117).

Finally, the steady-state solution of the electric field in the mean-field approximation takes the form:

$$\left\langle E^{+}\right\rangle_{ss}=\frac{\Gamma\mathcal{D}^{*}}{2\sqrt{2}|\mathcal{D}|^{2}}\sqrt{\mathcal{G}-1} \tag{4.21}$$

We could find the steady-state solution of the bosonic operator equation in the mean-field approximation by replacing (4.21) in (4.10):

$$\langle a(\omega_k)\rangle_{ss}=\sqrt{\frac{\hbar\omega_k}{2\pi\epsilon_0 cA}}\frac{4d_0N\Gamma}{\kappa+2i(\omega_k-\omega_A)}\frac{|\mathcal{D}|^2}{\Gamma^2+\Gamma^2[\mathcal{G}-1]}\left\langle E^{+}\right\rangle_{ss} \tag{4.22}$$

After some algebra:

$$\langle a(\omega_k)\rangle_{ss}=\frac{2d_0N\mathcal{D}^{*}}{\kappa+2i(\omega_k-\omega_A)}\sqrt{\frac{\hbar\omega_k}{2\pi\epsilon_0 cA}}\sqrt{\frac{\mathcal{G}-1}{2\mathcal{G}^2}}=\frac{2d_0Ng(\omega_k)}{\kappa+2i(\omega_k-\omega_A)}\sqrt{\frac{\mathcal{G}-1}{2\mathcal{G}^2}} \tag{4.23}$$

4.2.2. Electric Field Operator above the threshold

Finally, we now could define the mean-field approximation electrical field operator of the form:

$$\begin{aligned}E_0^{+}(t)= &\quad \left\langle E^{+}(t)\right\rangle=\int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}\langle a(\omega_k)\rangle_{ss}e^{-i\omega_k t}\\ = &\quad 2d_0N\sqrt{\frac{\mathcal{G}-1}{2\mathcal{G}^2}}\mathcal{D}^{*}\int d\omega_k\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}\left(\frac{e^{-i\omega_k t}}{\kappa+2i(\omega_k-\omega_A)}\right)\end{aligned} \tag{4.24}$$

We will assume that the electric field could be expressed by the mean-field amplitude plus a fluctuation term determined by a zero mean and delta-correlated operator $\hat{\xi}$.

$$\hat{E}^{+}(t)=E_0^{+}(t)+\hat{\xi}(t). \tag{4.25}$$

In the following section, we will use this expression to obtain the emission spectrum of the laser.

4.3. The Spectrum

In the previous section, we obtained an expression for the electric field of the laser emission above the threshold. This expression is determined by a mean-field approximation plus a fluctuating term. In this section, we will study the spectrum of laser emission considering only the mean-field term and ignoring the fluctuating terms. In section 4.3.1, we will apply the Wiener-Khintchine Theorem by taking the Fourier transform of the two times correlation function of the field. In section 4.32, we will show the power of the laser emission and the numerical results of its dependency on the pumping rate and the number of atoms in the system.

4.3.1. Wiener-Khintchine Theorem

In this section, we will make use of the Wiener-Khintchine Theorem [17, 23, 41, 42], which states that the power spectral density $S(\omega)$ could be obtained as the Fourier transform of the two times correlation function. In the case of the electric field, the spectrum could be derivated as follows:

$$S(\omega) = \int_0^\infty \left\langle E^-(\tau)E^+(0)\right\rangle e^{-i\omega\tau} + \int_0^\infty \left\langle E^-(0)E^+(\tau)\right\rangle e^{i\omega\tau} \tag{4.26}$$

If we replace the expression (4.25) in the relation (4.26), we will get the following expression.

$$S(\omega) = \int_0^\infty \left\langle \left(E_0^-(\tau) + \hat{\xi}^\dagger(\tau)\right)\left(E_0^+(t) + \hat{\xi}(0)\right)\right\rangle e^{-i\omega\tau} + \int_0^\infty \left\langle \left(E_0^-(0) + \hat{\xi}^\dagger(0)\right)\left(E_0^+(\tau) + \hat{\xi}(\tau)\right)\right\rangle e^{i\omega\tau} \tag{4.27}$$

Where we can neglect the terms $\left\langle E_0^+\hat{\xi}\right\rangle$ and $\left\langle \hat{\xi}^\dagger E_0^-\right\rangle$, because $\hat{\xi}$ has zero mean.

$$\begin{aligned} S(\omega) = \quad & \int_0^\infty \left\langle E_0^-(\tau)E_0^+(t)\right\rangle e^{-i\omega\tau} + \int_0^\infty \left\langle E_0^-(\tau)E_0^+(t)\right\rangle e^{i\omega\tau} \\ & + \int_0^\infty \left\langle \hat{\xi}^\dagger(\tau)\hat{\xi}(0)\right\rangle e^{-i\omega\tau} + \int_0^\infty \left\langle \hat{\xi}^\dagger(\tau)\hat{\xi}(0)\right\rangle e^{i\omega\tau} \end{aligned} \tag{4.28}$$

If we define the first two terms of the RHS of last equation as $S_1(\omega)$ and $S_2(\omega)$, respectively, then $S_1(\omega)$:

$$\begin{aligned} S_1(\omega) = 4d_0^2N^2\left(\frac{\mathcal{G}-1}{2\mathcal{G}^2}\right)|\mathcal{D}|^2\int d\omega_k\int d\omega_k' \quad & \left[\left(\frac{\hbar}{4\pi\epsilon_0 cA}\right)^2\omega_k\omega_k'\left(\frac{1}{\kappa-2i(\omega_k-\omega_A)}\right)\right] \\ & \left[\left(\frac{1}{\kappa+2i(\omega_k'-\omega_A)}\right)\right]\int_0^\infty d\tau e^{-i(\omega-\omega_k)\tau} \end{aligned} \tag{4.29}$$

The time integral has the form of equation (3.18), ignoring the principal value term, we get:

$$S_1(\omega) = 8\pi d_0^2 N^2 \left(\frac{\mathcal{G}-1}{2\mathcal{G}^2}\right) |\mathcal{D}|^2 \int d\omega_k' \left[\left(\frac{\hbar}{4\pi\epsilon_0 cA}\right)^2 \omega\omega_k' \left(\frac{1}{\kappa - 2i(\omega-\omega_A)}\right)\left(\frac{1}{\kappa + 2i(\omega_k'-\omega_A)}\right)\right] \tag{4.30}$$

After some algebra, and considering the relation $g(\omega_k) = \mathcal{D}\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}$.

$$S_1(\omega) = 8\pi d_0^2 N^2 \left(\frac{\mathcal{G}-1}{2\mathcal{G}^2}\right)\left(\frac{\hbar\omega}{4\pi\epsilon_0 cA}\right)\left(\frac{1}{\kappa - 2i(\omega-\omega_A)}\right) \int d\omega_k' \left[\left(\frac{|g(\omega_k')|^2}{\kappa + 2i(\omega_k'-\omega_A)}\right)\right] \tag{4.31}$$

From the last expression, we could reconize that $\gamma^{(1D)} = \int d\omega_k' \left(\frac{|g(\omega_k')|^2}{\kappa+2i(\omega_k'-\omega_A)}\right)$ (see the previous section).

$$S_1(\omega) = 4\pi d_0^2 N^2 \gamma^{(1D)} \left(\frac{\mathcal{G}-1}{2\mathcal{G}^2}\right)\left(\frac{\hbar\omega}{4\pi\epsilon_0 cA}\right)\left(\frac{1}{\kappa - 2i(\omega-\omega_A)}\right) \tag{4.32}$$

If we repite the same procedure with $S_2(\omega)$, we get:

$$S(\omega) = 4\pi d_0^2 N^2 \gamma^{(1D)} \left(\frac{\mathcal{G}-1}{2\mathcal{G}^2}\right)\left(\frac{\hbar\omega}{4\pi\epsilon_0 cA}\right)\left(\frac{1}{\kappa - 2i(\omega-\omega_A)} + \frac{1}{\kappa + 2i(\omega-\omega_A)}\right) \tag{4.33}$$

$$S(\omega) = 4\pi d_0^2 N^2 \gamma^{(1D)} \left(\frac{\mathcal{G}-1}{2\mathcal{G}^2}\right)\left(\frac{\hbar\omega}{4\pi\epsilon_0 cA}\right)\left(\frac{2\kappa}{\kappa^2 + 4(\omega-\omega_A)^2}\right) \tag{4.34}$$

$$S(\omega) = \frac{\pi\Gamma^2}{\gamma^{(1D)}} (\mathcal{G}-1)\left(\frac{\hbar\omega}{4\pi\epsilon_0 cA}\right)\left(\frac{\kappa}{\kappa^2 + 4(\omega-\omega_A)^2}\right) \tag{4.35}$$

But $\gamma^{(1D)} = 2\pi|g(\omega_A)|^2 = 2\pi|\mathcal{D}|^2 \frac{\hbar\omega}{4\pi\epsilon_0 cA}$

$$S(\omega) = \frac{\Gamma^2}{2|\mathcal{D}|^2} (\mathcal{G}-1)\left(\frac{\omega}{\omega_A}\right)\left(\frac{\kappa}{\kappa^2 + 4(\omega-\omega_A)^2}\right) \tag{4.36}$$

Finally, the spectrum could be rewritten as:

$$S(\omega) = \quad 4\left\langle E^{-}\right\rangle\left\langle E^{+}\right\rangle\left(\frac{\omega}{\omega_A}\right)\left(\frac{\kappa}{\kappa^2+4(\omega-\omega_A)^2}\right) \tag{4.37}$$

4.3.2. Laser Power and Spectrum

In this section, we will study the spectrum obtained in the previous section. First, we will compare the expression (4.37) to a Lorentzian shape. Then we will study the Power of the emission spectrum.

The equation (4.37), which shows the spectrum as a function of the intensity of the field, $\langle E^{-}\rangle\langle E^{+}\rangle$, could be written as a function of the decay rate of the system, γ, w, and $\gamma^{(1D)}$.

$$S(\omega) = \quad \frac{(\gamma+w)^2}{2|\mathcal{D}|^2}\left(\frac{2Nd_0\gamma^{(1D)}}{\gamma+w}-1\right)\left(\frac{\omega}{\omega_A}\right)\left(\frac{\kappa}{\kappa^2+4(\omega-\omega_A)^2}\right) \tag{4.38}$$

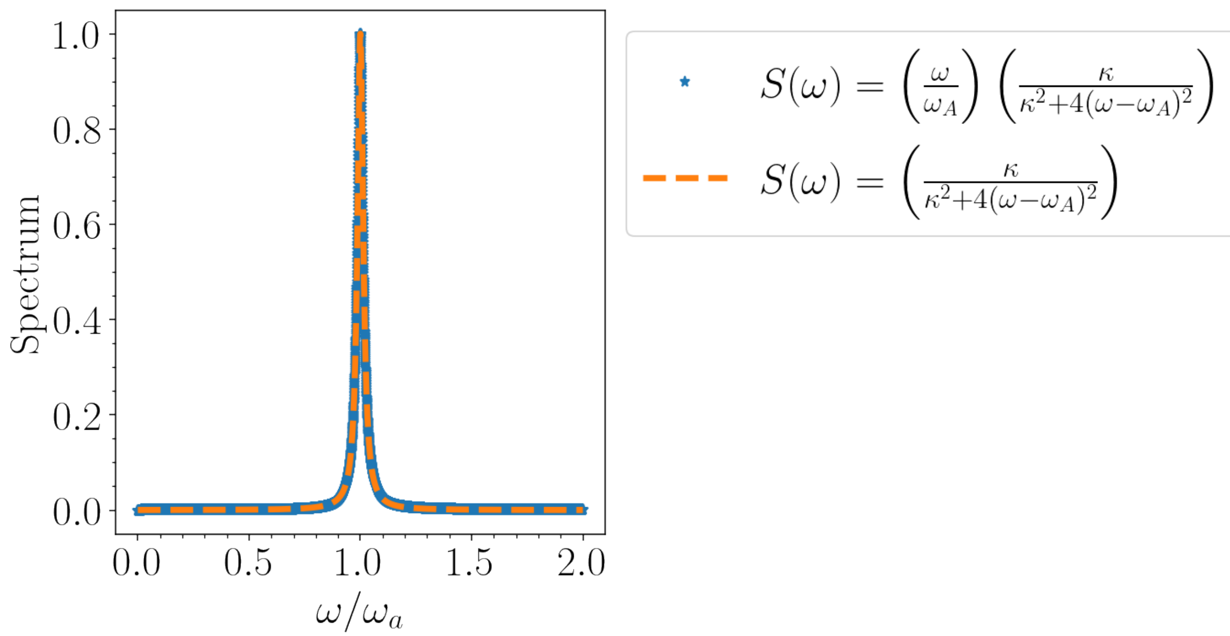

Figure 4.1: This figure shows the normalized version of the spectrum. Here, we compare two curves. The blue line considers the term ω/ω_A, and the orange curve approximates this term to one. The two curves are completely overlapped for $\kappa << \omega_a$, here we used $\kappa/\omega_a \sim 0.01$.

Figure 4.1 presents a normalized form of the emission spectrum of the equation 4.39. In this figure, we considered the case when ω_A is larger compared with the others time scales of the system. In this regime, the two spectrums shown in the figure could be considered to be approximately the same. From now, we will consider that the following expression approximately describes the spectrum:

$$S(\omega) \approx \quad \frac{(\gamma+w)^2}{2|\mathcal{D}|^2}\left(\frac{2Nd_0\gamma^{(1D)}}{\gamma+w}-1\right)\left(\frac{\kappa}{\kappa^2+4(\omega-\omega_A)^2}\right). \tag{4.39}$$

This equation states that a Lorentzian curve could approximately describe the spectrum. The linewidth of this curve will be directly defined by the field decay rate κ. Figure 4.2, presents different spectrums and shows how the linewidth varies for different values of κ.

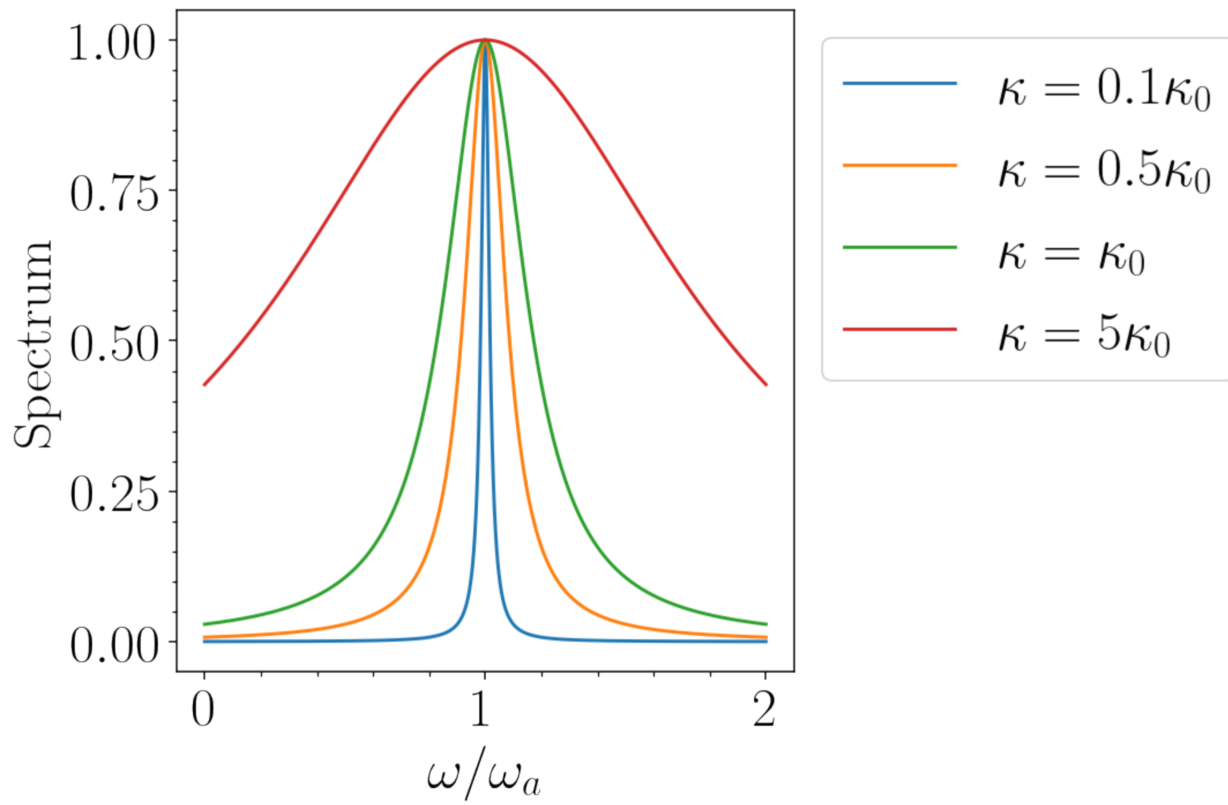

Figure 4.2: This figure shows the spectrum of the laser emission. Here we have considered different values for κ, where $\kappa_0 = 2\pi \times 11MHz$ based on the reference value in the cavity case of reference [43].

From the expression 4.39, we could study the Power of the emission spectrum of the laser, which is determined by:

$$P = \frac{(\gamma + w)^2}{2|\mathcal{D}|^2}\left(\frac{2Nd_0\gamma^{(1D)}}{\gamma + w} - 1\right) = \frac{\Gamma^2}{2|\mathcal{D}|^2}\left(\frac{2Nd_0\gamma^{(1D)}}{\Gamma} - 1\right) \tag{4.40}$$

It is important to consider that in order to have emission, the system must satisfy the threshold condition. This condition states that the parenthesis of the equation (4.40) must be positive, and $d_0 = \frac{w-\gamma}{w+\gamma} > 0$, so $w > \gamma$ to guarantee population inversion.

Figure 4.3 illustrates the Power of the emission spectrum as a function of the pumping parameter. There are two important aspects to derive from this figure. First, it shows that the Power has a maximum, which depends on the number of atoms. The second aspect is that the emission is confined between the threshold conditions. The close-up of the figure shows that there is no emission before a certain value, representing the condition $w > \gamma$. Moreover, the graphic shows that there is a maximum value of the pumping from which there is no more emission. This maximum value also depends on the number of atoms in the system.

In figure 4.4, we studied the Power of the emission as a function of the number of atoms in the system. Here we can see that the number of atoms necessary to satisfy the threshold condition (illustrated with the vertical lines) depends on the rate of emission of the atoms to the waveguide modes, $\gamma^{(1D)}$. Moreover, $\gamma^{(1D)}$ not only determines the threshold, but it also determines the slope of the curve. Then, this model states that if we increase the coupling between the atoms and the waveguide, the emission will be more intense, and the number of atoms necessary to overcome the threshold will decrease.

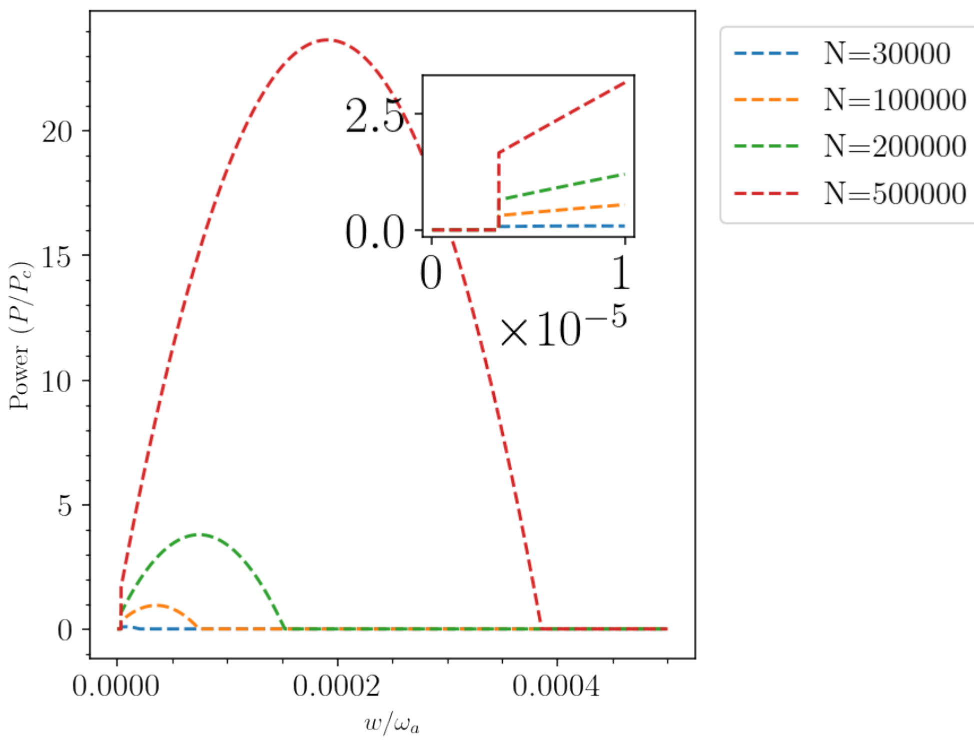

Figure 4.3: This Figure shows the dependency of the Power of the laser emission on the number of atoms and the pumping rate. The close-up of the figure shows the threshold from which the laser starts the emission. Here we considered a normalization term of the form $P_c = \hbar\omega_a\gamma^{(1D)}$.

Finally, we present Figure 4.5. This figure shows the dependency of the Power as a function of the number of atoms and the pumping rate simultaneously. At the bottom of the figure, we can observe the threshold associated with the condition $w > \gamma$. At the top left side of the figure, we can observe that when the number of atom increase, the pumping rate necessary to satisfy the threshold condition also increases. The top right side illustrates that the emission intensity will increase if we increase the pumping rate or the number of atoms.

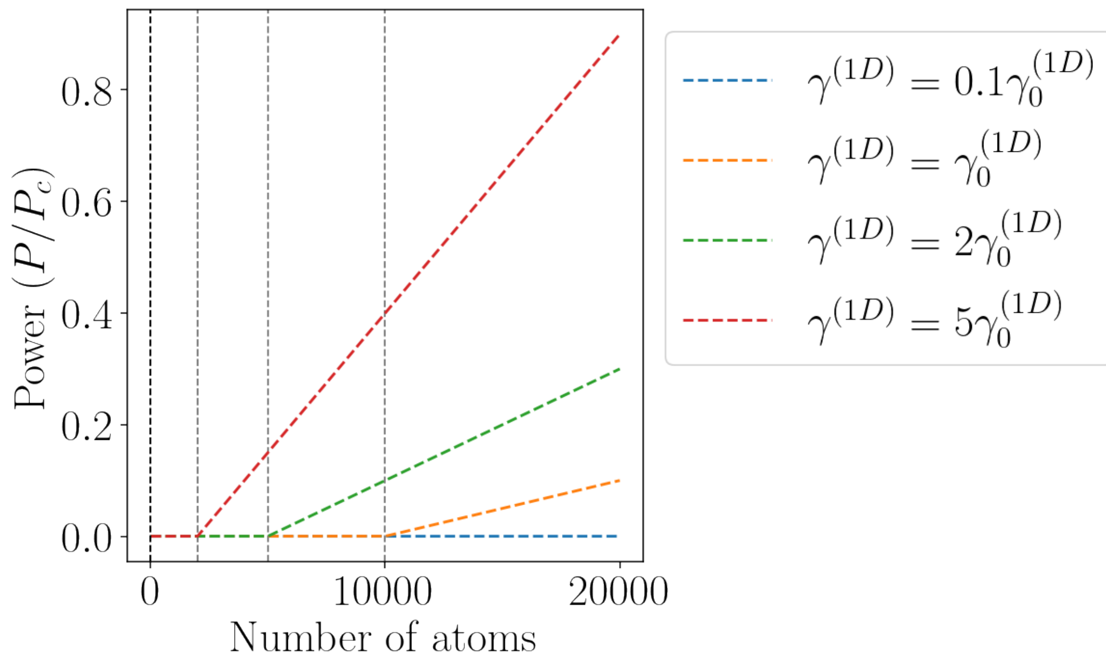

Figure 4.4: This Figure shows the dependency of the Power of the emission spectrum on the Numer of atoms of the system. The different curves represent different values of the rate of emission to the atoms in the waveguide, $\gamma^{(1D)}$. Where the normalization term $P_c = \hbar\omega_a\gamma^{(1D)}$. The vertical lines show the number of atoms necessary to satisfy the threshold condition.

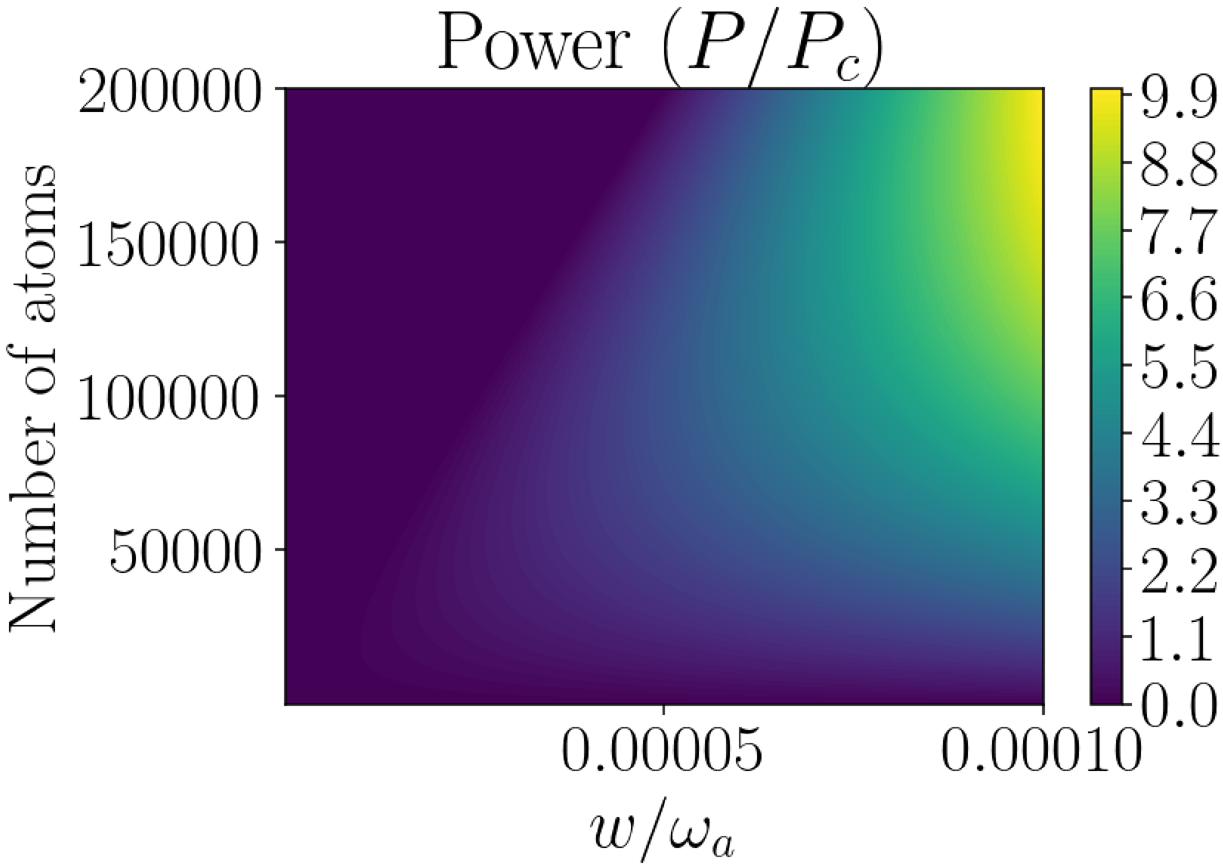

Figure 4.5: This Figure shows the dependency of the Power of the laser emission on the number of atoms and the pumping rate. The regions of the figure colored blue imply that the threshold condition is not satisfied, and there is no emission of the laser.Here we considered a normalization term of the form $P_c = \hbar\omega_a\gamma^{(1D)}$.

Chapter 5

Waveguide Phase Equation Method for the Linewidth

In the previous chapter, we presented a mechanism to obtain the linewidth based on the definition of the electric field operator as an amplitude determined by the mean steady-state solution of the Heisenberg-Langevin equation on the mean-field approximation and a fluctuating term that was ignored. In this chapter, we will present a different method to obtain the linewidth of the laser emission, taking into account the effects of fluctuations.

To this aim, we will study the first-order coherence function:

$$g^{(1)}(\tau) = \frac{\langle E^{-}(t)E^{+}(t+\tau)\rangle}{\langle E^{-}(t)E^{+}(t)\rangle} = \frac{\int d\omega_k \int d\omega_k' g(\omega_k) g^*(\omega_k') a_0^\dagger(\omega_k) a_0(\omega_k') \left\langle e^{i(\phi_k(t)-\phi_{k'}(t+\tau))}\right\rangle}{\int d\omega_k \int d\omega_k' g(\omega_k) g^*(\omega_k') a_0^\dagger(\omega_k) a_0(\omega_k') \left\langle e^{i(\phi_k(t)-\phi_{k'}(t))}\right\rangle} \tag{5.1}$$

Here the field is determined by an amplitude $a_0(\omega_k)$, which is determined by the stationary solution of the bosonic operator equation obtained in the previous chapter, and a phase which is determined by the fluctuating terms.

Because the fluctuating terms are gaussian, the exponential of the last expression satisfies the relation:

$$\left\langle e^{i[\phi_k(t)-\phi_{k'}(0)]}\right\rangle = e^{-\frac{1}{2}\left\langle(\phi_k(t)-\phi_{k'}(0))^2\right\rangle} \tag{5.2}$$

Then the coherence function takes the form:

$$g^{(1)}(\tau) = \frac{\int d\omega_k \int d\omega_k' g(\omega_k) g^*(\omega_k') a_0^\dagger(\omega_k) a_0(\omega_k') e^{-\frac{1}{2}\left\langle(\phi_k(\tau)-\phi_{k'}(0))^2\right\rangle}}{\int d\omega_k \int d\omega_k' g(\omega_k) g^*(\omega_k') a_0^\dagger(\omega_k) a_0(\omega_k') e^{-\frac{1}{2}\langle(\phi_k(0)-\phi_{k'}(0))^2\rangle}} \tag{5.3}$$

In order to obtain the first-order coherence function, in section 5.1, we will derive an expression for the evolution of the phase term. Based on this equation of motion, we will get the form of the argument of the exponential in the equation (coherence).

Solving the phase equation of motion of the phase, we will obtain the contribution of fluctuations of each mode of the electric field. In particular, we will study the effects of these contributions in different regimes of coupling.

In section 5.3, we will obtain the expression of the first-order coherence function based on the integration of the contribution of all waveguide modes. Finally, in section 5.4, we will present the numerical results of the spectrum obtained as the Fourier Transform of the coherence function.

5.1. Waveguide Phase Equation

In this section, we will assume that the solution of the bosonic operator of the waveguide modes takes the form:

$$a(\omega_k) = \langle a(\omega_k) \rangle_{ss} e^{i\phi_k(t)} = a_0(\omega_k) e^{i\phi_k} \tag{5.4}$$

Where $a_0(\omega_k)$ is the steady-state solution of the mean-field equation of the bosonic operator and ϕ_k is defined as a phase term. Then, the electric field is defined as:

$$E^{-}(t) = \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 c A}} a_0(\omega_k) e^{i\phi_k(t)} = \int d\omega_k \frac{g(\omega_k)}{\mathcal{D}} a_0(\omega_k) e^{i\phi_k(t)} \tag{5.5}$$

This section aims to derive the evolution of the phase term to obtain the argument of the exponential in the first-order coherence function.

$$g^{(1)}(\tau) = \frac{\int d\omega_k \int d\omega'_k g(\omega_k) g^*(\omega'_k) a_0^\dagger(\omega_k) a_0(\omega'_k) \left\langle e^{i(\phi_k(t) - \phi_{k'}(t+\tau))} \right\rangle}{\int d\omega_k \int d\omega'_k g(\omega_k) g^*(\omega'_k) a_0^\dagger(\omega_k) a_0(\omega'_k) \left\langle e^{i(\phi_k(t) - \phi_{k'}(t))} \right\rangle} \tag{5.6}$$

To this aim, we first take the equation of motion of the bosonic operator and replace the ansatz (5.4). Then, we will study the imaginary part. Finally, we will obtain the coherence function based on the two times correlations functions of the fluctuations operators.

5.1.1. Derivation of the Phase Equation

This section generalizes the procedure developed in reference Minghui [27] presented in the section 1.4 to the case of the continuous multimodal model developed in chapter 2. The main idea is to obtain the equation of the evolution of the phase.

To this aim, first, we will consider the equation of motion of the bosonic operator in the rotating frame derived in chapter 3.

$$\dot{a}(\omega_k) = \mathcal{F}_\kappa - i g^*(\omega) N S^- - \frac{1}{2} \left[\kappa + 2i(\omega_k - \omega) \right] a(\omega) \tag{5.7}$$

Derivating the last equation

$$\begin{aligned}\ddot{a}(\omega_k) = \quad & \dot{\mathcal{F}}_\kappa - ig^*(\omega_k)N\mathcal{F}_{(-)} + \frac{1}{2}\left[\Gamma + 2i(\omega_A - \omega)\right]\mathcal{F}_\kappa \\ & -\frac{1}{2}\left[\Gamma + 2i(\omega_A - \omega)\right]\dot{a}(\omega_k) - \frac{1}{2}[\kappa + 2i(\omega_k - \omega)]\dot{a}(\omega_k) \\ & -\frac{1}{4}\left[\Gamma + 2i(\omega_A - \omega)\right][\kappa + 2i(\omega_k - \omega)]a(\omega_k) + g^*(\omega_k)N\int d\omega_k' g(\omega_k')S^z a(\omega_k')\end{aligned} \tag{5.8}$$

Where S^z is the solution of the Heisenberg-Langevin equation of the inversion

$$S^z = \int dt' \int d\omega_k'' e^{\Gamma(t-t')}\left(\Gamma + \mathcal{F}_{(z)} - \frac{2}{N}\left[\frac{d}{dt}(a^\dagger(\omega_k'')a(\omega_k'')) + \kappa a^\dagger(\omega_k'')a(\omega_k'') - a^\dagger(\omega_k'')\mathcal{F}_\kappa - \mathcal{F}_\kappa^\dagger a(\omega_k'')\right]\right) \tag{5.9}$$

Now, we will define a noise operator of the form $\mathcal{F}(\omega_k) = \dot{\mathcal{F}}_\kappa - ig^*(\omega_k)N\mathcal{F}_{(-)} + \frac{1}{2}\left[\Gamma + 2i(\omega_A - \omega)\right]\mathcal{F}_\kappa$. If we replace this noise operator and the expression of S^z of the equation in the equation 5.8:

$$\begin{aligned}\ddot{a}(\omega_k) = \quad & \mathcal{F}(\omega_k) - \frac{1}{2}\left[\Gamma + 2i(\omega_A - \omega)\right]\dot{a}(\omega_k) - \frac{1}{2}[\kappa + 2i(\omega_k - \omega)]\dot{a}(\omega_k) \\ & -\frac{1}{4}\left[\Gamma + 2i(\omega_A - \omega)\right][\kappa + 2i(\omega_k - \omega)]a(\omega_k) \\ & + \int d\omega_k' \int dt' \int d\omega_k'' e^{\Gamma(t-t')} g^*(\omega_k)g(\omega_k')N \\ & \left(\Gamma + \mathcal{F}_{(z)} - \frac{2}{N}\left[\frac{d}{dt}(a^\dagger(\omega_k'')a(\omega_k'')) + \kappa a^\dagger(\omega_k'')a(\omega_k'') - a^\dagger(\omega_k'')\mathcal{F}_\kappa - \mathcal{F}_\kappa^\dagger a(\omega_k'')\right]\right) a(\omega_k')\end{aligned} \tag{5.10}$$

From this point, we will study the resonance case $\omega_A = \omega$, so the rotating frame moves in the atomic transition frequency.

$$\begin{aligned}\ddot{a}(\omega_k) = \quad & \mathcal{F}(\omega_k) - \frac{1}{2}[\kappa + \Gamma + 2i(\omega_k - \omega_A)]\dot{a}(\omega_k) - \frac{\Gamma}{4}[\kappa + 2i(\omega_k - \omega_A)]a(\omega_k) \\ & + g^*(\omega_k)N\int d\omega_k' g(\omega_k')a(\omega_k')\int dt' e^{\Gamma(t-t')} \\ & \left(\Gamma + \mathcal{F}_{(z)} - \frac{2}{N}\int d\omega_k''\left[\frac{d}{dt}(a^\dagger(\omega_k'')a(\omega_k'')) + \kappa a^\dagger(\omega_k'')a(\omega_k'') - a^\dagger(\omega_k'')\mathcal{F}_\kappa - \mathcal{F}_\kappa^\dagger a(\omega_k'')\right]\right)\end{aligned} \tag{5.11}$$

As was mentioned, we will assume an ansatz of the form (5.4):

$$a(\omega_k) = \langle a(\omega_k)\rangle_0 e^{i\phi_k(t)} \tag{5.12}$$

When we replace this ansatz in the equation (5.11):

$$\left[-\dot{\phi}_k^2 + i\ddot{\phi}_k\right] a_0(\omega_k) e^{i\phi_k(t)} =$$
$$\mathcal{F}(\omega_k) - \frac{1}{2}[\kappa + \Gamma + 2i(\omega_k - \omega_A)] \left(i\dot{\phi}_k(t) a_0(\omega_k) e^{i\phi_k}\right) - \frac{\Gamma}{4}[\kappa + 2i(\omega_k - \omega_A)] a_0(\omega_k) e^{i\phi_k}$$
$$+2g^*(\omega_k) N \int d\omega_k' g(\omega_k') a_0(\omega_k') e^{i\phi_{k'}} \int dt' e^{\Gamma(t-t')}$$
$$\left(\Gamma + \mathcal{F}_{(z)} - \frac{2}{N} \int d\omega_k'' \left[\frac{d}{dt}|a_0(\omega_k'')|^2 + \kappa |a(\omega_k'')|^2 - a_0^*(\omega_k'') e^{-i\phi_{k''}} \mathcal{F}_\kappa - \mathcal{F}_\kappa^\dagger a_0(\omega_k'') e^{i\phi_{k''}}\right]\right) \tag{5.13}$$

We will study the imaginary part:

$$i\ddot{\phi}_k a_0(\omega_k) e^{i\phi_k(t)} =$$
$$Im[\mathcal{F}(\omega_k)] - \frac{1}{2}[\kappa + \Gamma + 2i(\omega_k - \omega_A)] \left(\dot{\phi}_k(t) a_0(\omega_k) e^{i\phi_k}\right) - Im[\frac{\Gamma}{4}[\kappa + 2i(\omega_k - \omega_A)] a_0(\omega_k) e^{i\phi_k}]$$
$$+Im[2g^*(\omega_k) N \int d\omega_k' g(\omega_k') a_0(\omega_k') e^{i\phi_{k'}} \int dt' e^{\Gamma(t-t')} (\Gamma + \mathcal{F}_{(z)})$$
$$-Im[2g^*(\omega_k) N \int d\omega_k' g(\omega_k') a_0(\omega_k') e^{i\phi_{k'}} \int dt' e^{\Gamma(t-t')} \left(\frac{2}{N} \int d\omega_k'' \left[a_0^*(\omega_k'') e^{-i\phi_{k''}} \mathcal{F}_\kappa + \mathcal{F}_\kappa^\dagger a_0(\omega_k'') e^{i\phi_{k''}}\right]\right)$$
$$-Im[2g^*(\omega_k) N \int d\omega_k' g(\omega_k') a_0(\omega_k') e^{i\phi_{k'}} \int dt' e^{\Gamma(t-t')} \left(\frac{2}{N} \int d\omega_k'' \left[\kappa |a(\omega_k'')|^2\right]\right) \tag{5.14}$$

In the overdamped regime:

$$\frac{1}{2}[\kappa + \Gamma + 2i(\omega_k - \omega_A)] \left(\dot{\phi}_k(t) a_0(\omega_k) e^{i\phi_k}\right) =$$
$$Im[\mathcal{F}(\omega_k)] - Im[\frac{\Gamma}{4}[\kappa + 2i(\omega_k - \omega_A)] a_0(\omega_k) e^{i\phi_k}]$$
$$+Im[2g^*(\omega_k) N \int d\omega_k' g(\omega_k') a_0(\omega_k') e^{i\phi_{k'}} \int dt' e^{\Gamma(t-t')} (\Gamma + \mathcal{F}_{(z)})$$
$$-Im[2g^*(\omega_k) N \int d\omega_k' g(\omega_k') a_0(\omega_k') e^{i\phi_{k'}} \int dt' e^{\Gamma(t-t')} \left(\frac{2}{N} \int d\omega_k'' \left[a_0^*(\omega_k'') e^{-i\phi_{k''}} \mathcal{F}_\kappa + \mathcal{F}_\kappa^\dagger a_0(\omega_k'') e^{i\phi_{k''}}\right]\right)$$
$$-Im[2g^*(\omega_k) N \int d\omega_k' g(\omega_k') a_0(\omega_k') e^{i\phi_{k'}} \int dt' e^{\Gamma(t-t')} \left(\frac{2}{N} \int d\omega_k'' \left[\kappa |a(\omega_k'')|^2\right]\right) \tag{5.15}$$

We will drop the real terms and integrate the last relation:

$$\phi_k(t) - \phi_k(0) = \frac{2}{a_0(\omega_k)(\Gamma + \kappa)} \int_0^t Im[\mathcal{F}(\tilde{\omega_k})] - \int_0^t \frac{\Gamma}{\Gamma + \kappa} (\omega_k - \omega_A) \tag{5.16}$$

Then, the evolution of the phase is determined by:

$$\phi_k(t) = \frac{2}{a_0(\omega_k)(\Gamma+\kappa)} \int_0^t Im[\mathcal{F}(\tilde{\omega_k})] - \int_0^t \frac{\Gamma}{\Gamma+\kappa}(\omega_k - \omega_A) + \phi_k(0) \tag{5.17}$$

5.2. Fluctuating terms of the Phase

In the previous section, we obtained an expression for the phase of the bosonic operator, which is determined by fluctuating terms. In order to obtain the exponential term in the coherence equation (5.6), we will use the relation:

$$\left\langle e^{i[\phi_k(t)-\phi_{k'}(0)]} \right\rangle = e^{-\frac{1}{2}\left\langle (\phi_k(t)-\phi_{k'}(0))^2 \right\rangle} \tag{5.18}$$

This expression is a generalization of the unimodal relation presented in section 1.4.3 to the multimodal case. (The derivation of this relation is presented in Appendix E.1). In other words, equation (5.18) represents the contribution to the linewidth of the phase fluctuations of each mode of the electromagnetic field in the waveguide, in analogy to the unimodal case studied by Minghui [27].

In this section, we will calculate the argument of the RHS of the equation (5.18) by replacing the solution of the phase (5.17). Then we will determine the noise operators' correlations contributing to the phase correlations.

5.2.1. Phase Correlations

Studying the argument of the exponential in the RHS of equation (5.18), we get:

$$\begin{aligned}
\left\langle (\phi_k(t) - \phi_{k'}(0))^2 \right\rangle = & \left\langle (\phi_k(t) - \phi_{k'}(0))^\dagger (\phi_k(t) - \phi_{k'}(0)) \right\rangle \\
= & \left\langle (\phi_k(t)^\dagger - \phi_{k'}(0)^\dagger)(\phi_k(t) - \phi_{k'}(0)) \right\rangle \\
= & \left\langle \phi_k^\dagger(t)\phi_k(t) \right\rangle - \left\langle \phi_k^\dagger(t)\phi_{k'}(0) \right\rangle - \left\langle \phi_{k'}^\dagger(0)\phi_k(t) \right\rangle + \left\langle \phi_{k'}^\dagger(0)\phi_{k'}(0) \right\rangle
\end{aligned} \tag{5.19}$$

Replacing the solution obtained in the last section for the phase evolution, equation (5.17), in the last expression(See Appendix E.1):

$$\begin{aligned}
\left\langle (\phi_k(t) - \phi_{k'}(0))^2 \right\rangle = & \frac{1}{|a_0(\omega_k)|^2(\Gamma+\kappa)^2} \int dt' \int dt'' \left\langle \mathcal{F}^\dagger(t')\mathcal{F}(t'') \right\rangle + \int dt' \int dt'' \left\langle \mathcal{F}(t')\mathcal{F}^\dagger(t'') \right\rangle \\
& + \frac{\Gamma^2}{(\Gamma+\kappa)^2}(\omega_k - \omega_A)^2 t^2,
\end{aligned} \tag{5.20}$$

where we have assumed that $\left\langle \phi_k^\dagger(0)\phi_{k'}(0) \right\rangle = \left\langle \phi_{k'}^\dagger(0)\phi_k(0) \right\rangle = 0$ and that $\left\langle \phi_{k'}^\dagger(0)\phi_{k'}(0) \right\rangle + \left\langle \phi_k^\dagger(0)\phi_k(0) \right\rangle = 2\left\langle \phi_k^\dagger(0)\phi_k(0) \right\rangle$.

Considering the definition of the fluctuation operator in chapter 3, we could derive the correlation presented in the first term of the equation (this derivation is presented in Appendix

E):

$$\int dt' \int dt'' \left\langle \mathcal{F}^\dagger(t')\mathcal{F}(t'')\right\rangle + \int dt' \int dt'' \left\langle \mathcal{F}(t')\mathcal{F}^\dagger(t'')\right\rangle = |g(\omega_k)|^2 N (\gamma + w)\, t + \frac{\Gamma^2}{4}\pi|\zeta(\omega_k,\omega_k)|^2 t$$
$$= |g(\omega_k)|^2 N\Gamma t + \frac{\Gamma^2 \kappa(\omega_k)}{4\vartheta} t \tag{5.21}$$

Where we have considered the definition $\kappa(\omega_k) \equiv \pi|\zeta(\omega_k,\omega_k)|^2\vartheta$, with ϑ a dimensionalization term.

Finally, we obtain an expression for the argument of the exponential in the coherence function:

$$\left\langle (\phi_k(t) - \phi_{k'}(0))^2 \right\rangle = \frac{1}{|a_0(\omega_k)|^2(\Gamma+\kappa)^2}\left(|g(\omega_k)|^2 N\Gamma + \frac{\Gamma^2\kappa(\omega_k)}{4\vartheta}\right) t + \frac{\Gamma^2}{(\Gamma+\kappa)^2}(\omega_k - \omega_A)^2 t^2 \tag{5.22}$$

Defining the coefficients η_k^2 and $\Delta\nu_k$ as:

$$\eta_k = \frac{\Gamma}{(\Gamma+\kappa)}(\omega_k - \omega_A) \tag{5.23}$$

$$\Delta\nu_k = \frac{1}{|a_0(\omega_k)|^2(\Gamma+\kappa)^2}\left(|g(\omega_k)|^2 N\Gamma + \frac{\Gamma^2\kappa(\omega_k)}{4\vartheta}\right) \tag{5.24}$$

Where we had used de steady state solution of the photon number derived in the chapter 4, $|a_0(\omega_k)|^2 = \frac{4d_0^2N^2|g(\omega_k)|^2}{\kappa(\omega_k)^2+4(\omega_k-\omega_A)^2}(\frac{\mathcal{G}-1}{2\mathcal{G}^2})$ and defined the generalized cooperativity term $\mathcal{G}(\omega_k) = \frac{4|g(\omega_k)|^2N\vartheta}{\Gamma\kappa(\omega_k)}d_0$.

The expression (5.22) could be rewritten as:

$$\left\langle (\phi_k(t) - \phi_{k'}(0))^2 \right\rangle = \Delta\nu_k t + \eta_k^2 t^2 \tag{5.25}$$

Here we recognize that the first term, which has a linear dependency on time, could be understood as an analogous term of the unimodal case presented in equation (1.127). Nevertheless, the generalization to the multimodal case presents a new term with quadratic time dependence. This term is not uncommon, since it also appears when there is inhomogeneous broadening, such as Doppler broadening.

5.3. Coherence Function

In this section, we will present the coherence function of our model for a waveguide QED system laser in the case where fluctuation effects are considered. This coherence function is the sum of the fluctuation of the phase of each field mode. The effects of those phase fluctuations will be described through the correlation term calculated in the last section. At the end of this section, we will obtain a coherence function with a form of a Bessel Function.

5.3.1. Derivation of the Coherence Function

In order to get the coherence function of the waveguide laser emission, we will use the definition:

$$g^{(1)}(\tau) = \frac{\int d\omega_k \int d\omega'_k a_0^\dagger(\omega_k) a_0(\omega'_k) e^{-\frac{1}{2}\left\langle(\phi_k(\tau)-\phi_{k'}(0))^2\right\rangle}}{\int d\omega_k \int d\omega'_k a_0^\dagger(\omega_k) a_0(\omega'_k) e^{-\frac{1}{2}\langle(\phi_k(0)-\phi_{k'}(0))^2\rangle}} \tag{5.26}$$

Here $a_0(\omega_k) = \sqrt{\frac{4d_0^2 N^2 g(\omega_k)|^2}{\kappa(\omega_k)^2+4(\omega_k-\omega_A)^2}}\sqrt{\left(\frac{\mathcal{G}-1}{2\mathcal{G}^2}\right)}$ is the steady state solution of the bosonic operator equation derived in chapter 4. The exponential term of equation (5.26) considers the fluctuation effects. As was shown in the last section, the correlations of the phases are related to the noise operator correlation. Those correlations could be written as the relation (5.25). If we replace $a_0(\omega_k)$ and equation (5.25) in the equation (5.26), we get:

$$g^{(1)}(\tau) = \frac{\int d\omega_k \sqrt{\frac{4d_0^2 N^2 |g(\omega_k)|^2}{\kappa(\omega_k)^2+4(\omega_k-\omega_A)^2}}\sqrt{\left(\frac{\mathcal{G}-1}{2\mathcal{G}^2}\right)} e^{-\frac{1}{2}(\Delta\nu_k t+\eta_k^2 t^2)}}{\int d\omega_k \sqrt{\frac{4d_0^2 N^2 |g(\omega_k)|^2}{\kappa(\omega_k)^2+4(\omega_k-\omega_A)^2}}\sqrt{\left(\frac{\mathcal{G}-1}{2\mathcal{G}^2}\right)}} \tag{5.27}$$

If we ignore the normalization term and the term associated with the initial condition of the phase, the coherence function could be written as:

$$g^{(1)}(\tau) \propto \int d\omega_k \sqrt{\frac{|g(\omega_k)|^2}{\kappa(\omega_k)^2 + 4(\omega_k-\omega_A)^2}} e^{-\frac{1}{2}(\Delta\nu_k t+\eta_k^2 t^2)} \tag{5.28}$$

The integrand of equation (5.28) has the form of an amplitude and an exponential term. The amplitude is a square root of a Lorentzian distribution. This amplitude could be understood as a weight where only the values near from resonance are significant. This implies that we could approximate the values of $\kappa(\omega_k)$ and $|g(\omega_k)|^2$ by constants evaluated in the resonant frequency ω_A. Then $|g(\omega_k)|^2 \approx 2\pi|g(\omega_A)|^2 \equiv \gamma^{(1D)}$ and $\kappa(\omega_k) \approx \kappa(\omega_A) \equiv \kappa$. The amplitude term takes the form:

$$g^{(1)}(\tau) \propto \int d\omega_k \sqrt{\frac{\gamma^{(1D)}}{\kappa^2 + 4(\omega_k-\omega_A)^2}} e^{-\frac{1}{2}(\Delta\nu_k t+\eta_k^2 t^2)} \tag{5.29}$$

This approximation is not only valid for the amplitude, but the terms of the phase are also affected, then $\Delta\nu_k$ takes the form:

$$\begin{aligned}
\Delta\nu_k = &\quad \frac{1}{|a_0(\omega_k)|^2(\Gamma+\kappa)^2}\left(|g(\omega_k)|^2 N\Gamma + \frac{\Gamma^2\kappa(\omega_k)}{4\vartheta}\right) \\
= &\quad \left(\frac{2\mathcal{G}^2}{\mathcal{G}-1}\right)\left(\frac{\kappa(\omega_k)^2+4(\omega_k-\omega_A)^2}{4d_0^2N^2|g(\omega_k)|^2}\right)\frac{\Gamma}{(\Gamma+\kappa)^2}\left[|g(\omega_k)|^2 N + \frac{\Gamma\kappa(\omega_k)}{4\vartheta}\right] \\
\approx &\quad \left(\frac{4\gamma^{(1D)}N\vartheta+\Gamma\kappa}{2\vartheta(2Nd_0\gamma^{(1D)}-\Gamma)}\right)\left[\frac{\kappa^2+4(\omega_k-\omega_A)^2}{(\Gamma+\kappa)^2}\right]\gamma^{(1D)}
\end{aligned} \tag{5.30}$$

Where we used de steady state solution of the photon number derived in the chapter 4, $|a_0(\omega_k)|^2 = \frac{4d_0^2N^2|g(\omega_k)|^2}{\kappa(\omega_k)^2+4(\omega_k-\omega_A)^2}(\frac{\mathcal{G}-1}{2\mathcal{G}^2})$ and the approximation $|g(\omega_k)|^2 \approx 2\pi|g(\omega_A)|^2 \equiv \gamma^{(1D)}$ and $\kappa(\omega_k) \approx \kappa(\omega_A) \equiv \kappa$.

If we replace equation (5.30) in equation (5.29), the coherence function takes the form:

$$g^{(1)}(\tau) \propto \int d\omega_k \sqrt{\frac{\gamma^{(1D)}}{\kappa^2+4(\omega_k-\omega_A)^2}}\, e^{-\left(\frac{4\gamma^{(1D)}N\vartheta+\Gamma\kappa}{2\vartheta(2Nd_0\gamma^{(1D)}-\Gamma)}\right)\left[\frac{\kappa^2+4(\omega_k-\omega_A)^2}{(\Gamma+\kappa)^2}\right]\gamma^{(1D)}t-\frac{1}{2}\frac{\Gamma^2}{(\Gamma+\kappa)^2}(\omega_k-\omega_A)^2t^2} \tag{5.31}$$

Equation (5.31) represents the contribution of each mode which determines the coherence function. The integral over all the modes frequencies could be written in a compact form by defining $\Delta = (\omega_k - \omega_A)$ and $\lambda(t) = \frac{4\gamma^{(1D)}d_0N\vartheta+\Gamma\kappa}{\vartheta(2d_0N\gamma^{(1D)}-\Gamma)}\frac{\gamma^{(1D)}t}{(\Gamma+\kappa)^2} + \frac{\Gamma^2t^2}{2(\Gamma+\kappa)^2}$.

$$\int d\omega_k \sqrt{\frac{1}{\kappa^2+4(\omega_k-\omega_A)^2}}\, e^{-\left[\left(\frac{4\gamma^{(1D)}N\vartheta+\Gamma\kappa}{\vartheta(2Nd_0\gamma^{(1D)}-\Gamma)}\right)\frac{\gamma^{(1D)}t}{(\Gamma+\kappa)^2}-\frac{\Gamma^2}{2(\Gamma+\kappa)^2}t^2\right](\omega_k-\omega_A)^2} = \int d\Delta \frac{e^{-\lambda(t)\Delta^2}}{\sqrt{\kappa^2+4\Delta^2}} \tag{5.32}$$

The solution of the RHS of equation (5.23) is a modified Bessel Function, defined as $K_\alpha(x) = \int_0^\infty e^{-xcosh(z)}cosh(\alpha z)dz$

$$\int d\Delta \frac{e^{-\lambda(t)\Delta^2}}{\sqrt{\kappa^2+4\Delta^2}} = \frac{1}{2}e^{\frac{\lambda(t)\kappa^2}{2}}K_0(\frac{\lambda(t)\kappa^2}{2}) \tag{5.33}$$

Finally, we obtain an expression for the coherence function of the form:

$$g^{(1)}(\tau) \propto e^{\left(\frac{\kappa^2\Gamma^2t^2}{4(\Gamma+\kappa)^2}\right)}K_0\left(\frac{\kappa^2}{2}\left[\frac{4\gamma^{(1D)}d_0N\vartheta+\Gamma\kappa}{\vartheta(2d_0N\gamma^{(1D)}-\Gamma)}\frac{\gamma^{(1D)}t}{(\Gamma+\kappa)^2}+\frac{\Gamma^2t^2}{2(\Gamma+\kappa)^2}\right]\right) \tag{5.34}$$

5.4. Spectrum of the Phase Equation Method

In this section, we will present the emission spectrum of the waveguide QED system laser, which considers the effects of the fluctuations. First, we will apply the Kitchine-Wiener Theorem by taking the Fourier transform of the coherence function presented in the previous section. In section 5.3.2, numerical results will be presented, where the spectrum obtained is compared with the standard Lorentzian distribution. In section 5.3.3, we will study the linewidth of the spectrum and its dependency on the system variables.

5.4.1. Wiener-Khintchine Theorem

In section 4.3.1, we applied the Wiener-Khintchine Theorem for the two times correlation function of the field, which was considered in a mean-field approximation. In this section, we will use the same formalism by taking the Fourier transform to the coherence function derived in the previous section.

The definition of the Wiener-Khintchine Theorem states the relation:

$$S(\omega) = \int dt g^{(1)}(t) e^{i\omega t} \tag{5.35}$$

If we define $\alpha = \frac{\kappa^2}{2}\left(\frac{(4\gamma^{(1D)} d_0 N\vartheta + \Gamma\kappa)}{\vartheta(2\gamma^{(1D)} d_0 N - \Gamma)}\right)\frac{\gamma^{(1D)}}{(\Gamma+\kappa)^2}$ and $\beta = \frac{\kappa^2\Gamma^2}{4(\Gamma+\kappa)^2}$, we could obtain a simpler expression for the coherence function:

$$g^{(1)}(\tau) \propto e^{\beta t^2} K_0[\alpha t + \beta t^2] = e^{\beta t^2}\int_0^\infty e^{-(\alpha t+\beta t^2)cosh(z)} dz \tag{5.36}$$

where we used the definition of the modified Bessel function, $K_0(x) = \int_0^\infty e^{-xcosh(z)}dz$.

Now, when we replace the equation (5.36) in the definition of the Wiener-Khintchine Theorem (5.35):

$$S(\omega) = \int dt \left[e^{\beta t^2}\int_0^\infty e^{-(\alpha t+\beta t^2)cosh(z)} dz\right] e^{i\omega t} \tag{5.37}$$

5.4.2. Spectrum and Power in the Phase Equation Method

In the previous section, we derived an expression for the spectrum of emission of the waveguide QED system laser, which considers the effect of the interaction of the system with the environment through the study of the phase fluctuation of each mode of the electromagnetic field inside the waveguide. In this section, we will present some numerical results to solve the integral of the equation (5.37).

The idea of this section is to study the behavior of the emission spectrum in two regimes of parameters. The first regime, which we will call the "generalized good cavity regime", considers that the decay rate of the field inside the waveguide is narrow compared with the decoherence term $\Gamma = w + \gamma$. The second regime studies the other range of parameters, so the field decay rate, which models the effect of the waveguide atoms, is larger compared with the decoherence of the atoms of the system. This regime will be called the "generalized bad cavity regime."Those names were chosen not only because they are used in the standard cavity case but instead because our model could be easily extended to the case of a multimodal cavity if we consider the discrete model presented in chapter 2.

5.4.2.1. Generalized Good Cavity Regime

In this subsection, we will study the region of the parameter where the decay rate of the electromagnetic field of the waveguide is smaller than the decay process associated with the spontaneous emission of the atoms and the pumping mechanism, $\kappa << \gamma + w = \Gamma$. This means that we will consider the approximation where the β of equation (5.36) takes the form:

$$\beta = \frac{\kappa^2\Gamma^2}{4(\Gamma+\kappa)^2} \approx \frac{\kappa^2}{4}. \tag{5.38}$$

When we replace this expression in the spectrum equation (5.37) and use numerical integration methods to calculate it, we obtain a spectrum shape described in the figure 5.1. This figure presents three curves, each representing the spectrum for different values of the field decay term κ. It is important to note that the spectrum linewidth maintains unchanged when the value of κ decreases. The green curve shows that there is a critical value from which the shape of the curve starts to change its linewidth.

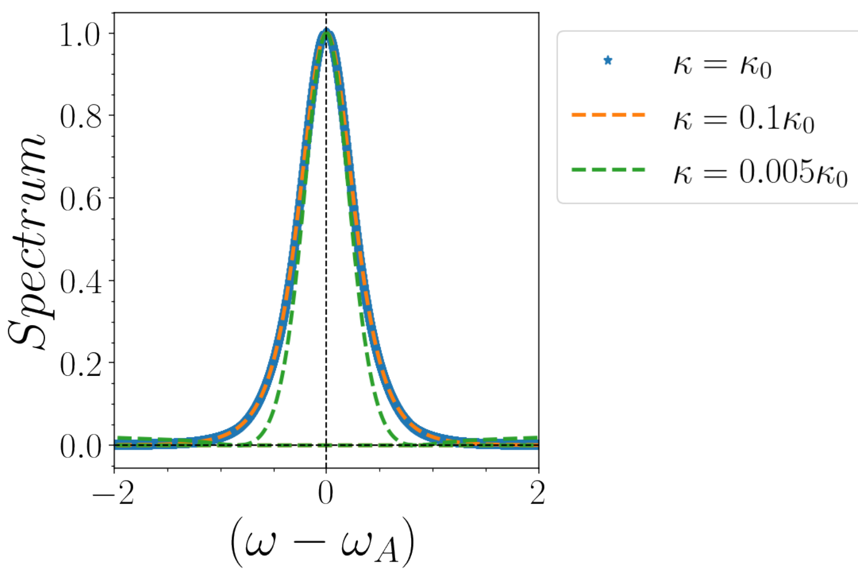

Figure 5.1: **Generalized Good Cavity Regime.** This figure shows three Spectral curves, each with a different value of κ. The blue and the orange curves have the same linewidth, while the green curve represents the critical point from which the linewidth starts to change with κ. Here we considered $\kappa_0 = \omega_A \times 10^{-4}$ and $\omega_A = 10^9$.

Figure 5.2 shows that if the value of κ is less than the critical value of the showed in figure 5.1, two new peaks emerge in the spectrum. While the central peak decreases its linewidth by decreasing the value of κ, the lateral peaks increase their amplitudes.

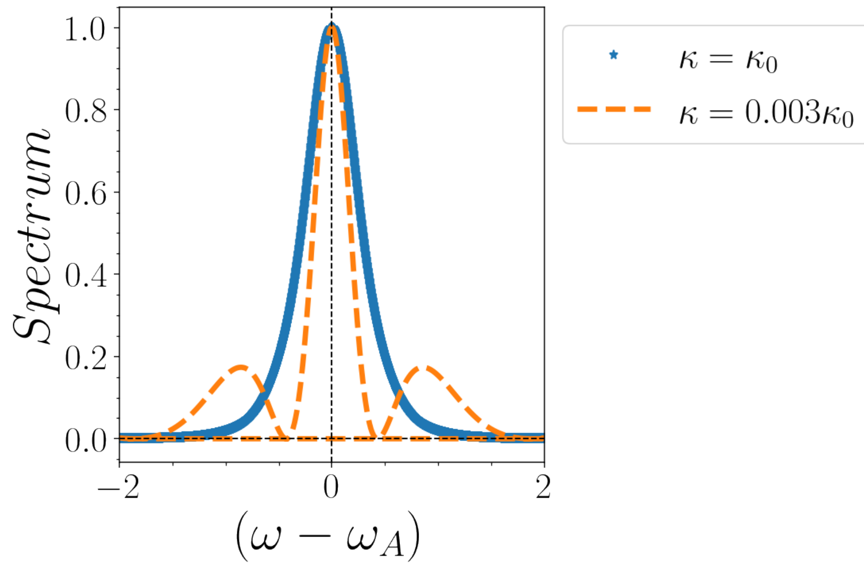

Figure 5.2: **Generalized Good Cavity Regime.** This figure shows the spectrum of emission below the critical point shown in figure 5.2. The orange curve presents three peaks, and the central peak has a narrower linewidth than the behavior above the critical point.

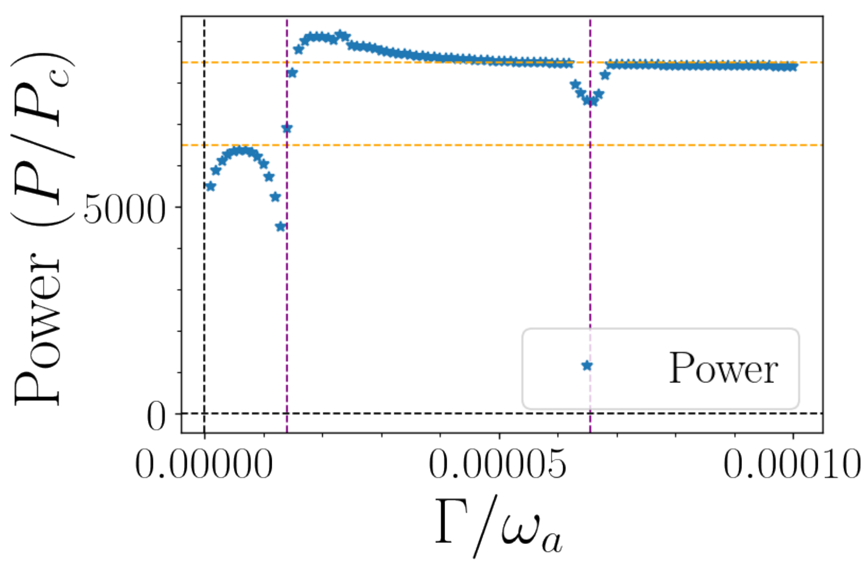

Figure 5.3: **Generalized Good Cavity Regime.** This figure shows the Power of the spectrum as a function of the atomic decay term $\Gamma = \gamma + w$. The two vertical lines are in $\Gamma/\omega_a = 0.000015$ and $\Gamma/\omega_a = 0.000063$. The bottom horizontal line shows the local maximum of the left side of the graphic, and the top horizontal line shows the asymptotic behavior of the curve on the right side. Here we used $P_c = \hbar\omega_a\gamma^{(1D)}$.

We could find different behaviors if we study the Power of the spectrum in the generalized good cavity regime as a function of the atom decay Γ. As we can observe on the left side of Figure 5.3, the Power shows a negative concavity behavior with a local maximum. Then, after some critical point, the Power increases until a maximum value and decreases asymptotically to a particular value.

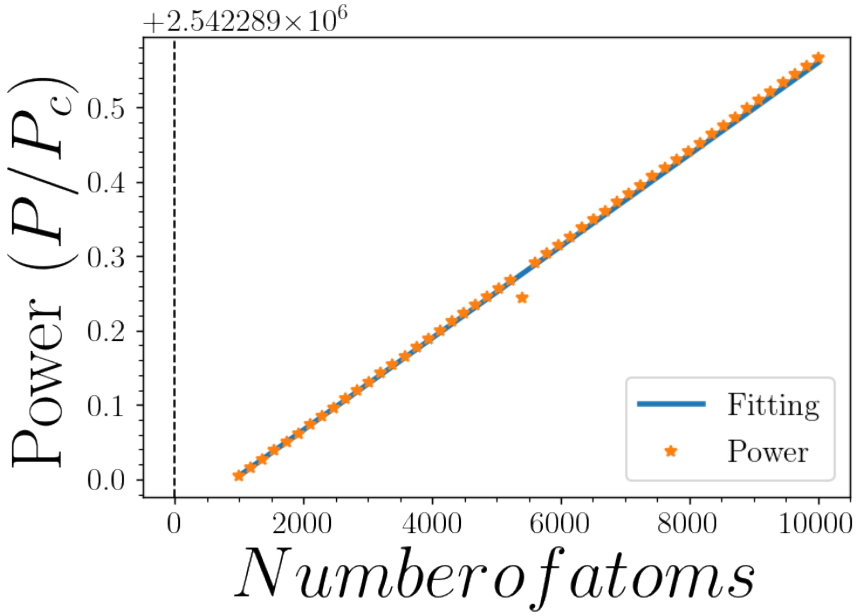

Figure 5.4: **Generalized Good Cavity Regime.** This figure shows the Power of the spectrum as a function of the number of atoms in the generalized good cavity regime. A curve of orange stars represents the numerical results, and the blue line represents a linear fitting curve. Here we used $P_c = \hbar\omega_a\gamma^{(1D)}$.

The dependency of the Power of the emission spectrum on the number of atoms is shown in figure 5.4. Here, we obtained that the Power increase linearly with the number of atoms in the generalized good cavity regime.

5.4.2.2. Generalized Bad Cavity Regime

This subsection will consider the regime when the decay processes associated with the field are bigger than the processes associated with the decay of the atom, $\kappa >> \Gamma$. This regime implies that the terms α and β of the equation (5.37) could be approximated to:

$$\alpha \approx \frac{\gamma^{(1D)}}{2}\left(\frac{(4\gamma^{(1D)}d_0N\vartheta + \Gamma\kappa)}{\vartheta(2\gamma^{(1D)}d_0N - \Gamma)}\right) \tag{5.39}$$

$$\beta \approx \frac{\Gamma^2}{4}. \tag{5.40}$$

Here we will study the dependency of the spectrum and the Power of the spectrum of the emission as a function of the system decay rates. The numerical results show that the spectrum of the emission has a Lorentzian shape.

Figure 5.5 shows the normalized emission spectrum in the generalized bad cavity regime. From this figure, we can deduce that the linewidth of the spectrum in this regime does not depend on the field decay rate κ.

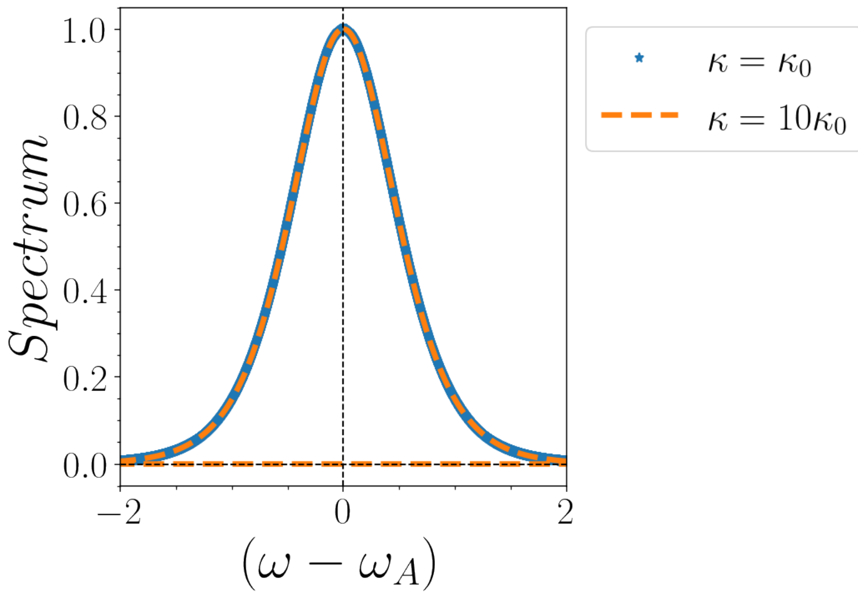

Figure 5.5: **Generalized Bad Cavity Regime.** This Figure shows the spectrum of emission of the laser in the generalized bad cavity regime. In this figure, there are two overlapping curves with different values of the κ parameter.

The Power of the spectrum as a function of the decay rate of the field is shown in figure 5.6. This figure shows that there is an inversed relation between the Power and κ.

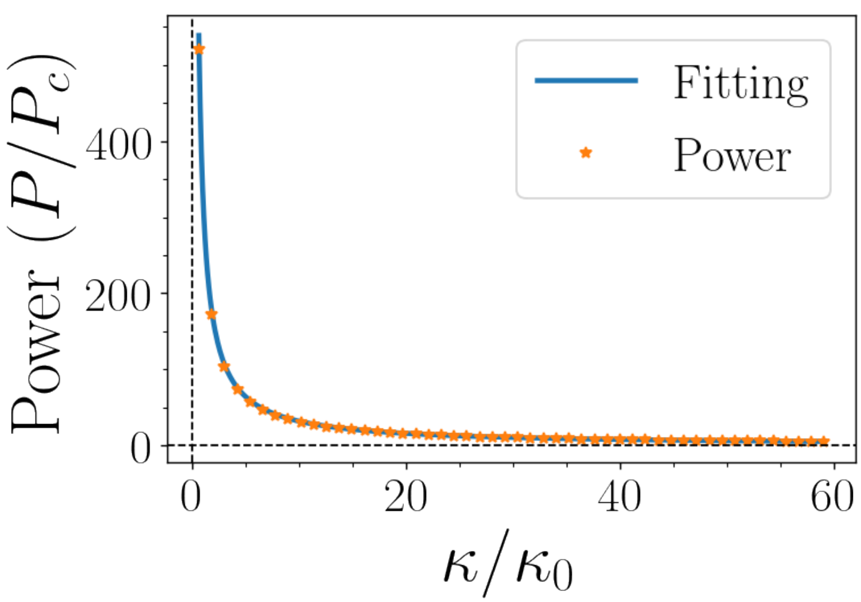

Figure 5.6: **Generalized Bad Cavity Regime.** This Figure shows the numerical results of the Power of the emission in the "generalized bad cavity regime,represented by orange stars. In contrast, the continuous blue line represents an inverse proportionality of the Power and the field decay rate, which fits the numerical results. Here we used $P_c = \hbar\omega_a\gamma^{(1D)}N$, with $N = 10^6$ the number of atoms.

Conclusions

In this thesis, we developed a fully quantum mechanical model for a laser implemented in a waveguide QED system. This quantum mechanical treatment was motivated by the potential applications of the study of the Spectral linewidth of a laser, which could not be derived from a semi-classical description. Although a laser could be considered a macroscopic system because of the number of components, this model will describe the system starting from the microscopic interactions between its elements. Here we have used an open quantum system approach, where we considered a system of interest consisting of the atoms that act as the active medium and the electromagnetic field modes inside the waveguide. The system of interest interacts with a reservoir. This reservoir consists of a pump mechanism, which guarantees the population inversion of the atoms and electromagnetic field modes outside the waveguide.

In chapter 1, we presented the background knowledge necessary to understand the behavior of the laser. First, we presented quantum optical tools which describe the interaction between matter and the quantized electromagnetic field. We introduced the two quantum states of light, in particular, the coherent state used to describe the laser emission. Here we presented the extension from the unimodal cavity case to the continuous multimodal case of the waveguide field. Then, we introduced the Heisenberg-Langevin formalism, which was the methodology used to describe the dynamics of the system of interest, which considers the effect of the interaction with the reservoir. The motivation of this work is to extend the case of a laser developed in the standard platform of cavity QED to this new platform, the waveguide QED system. To this aim, we considered some tools developed in the cavity QED system case. In particular, the study of the linewidth in the case of the unimodal cavity QED system developed by Minghui [27]. In that model, the effects of the interaction with the reservoir are considered by studying the phase fluctuations of the cavity mode. Here, we extended this idea from the unimodal cavity mode to the multimodal case, so we developed a theoretical tool to represent a multimodal cavity or, in the continuous case, a waveguide.

In chapter 2, we presented our model for the waveguide QED system. Here we used the open quantum system approach to define a system of interest as the atoms plus the field inside the waveguide. The reservoir considered the degree of freedom associated with the electromagnetic field outside the waveguide and the pumping mechanism. This mechanism is modeled by a collection of two-level systems that generate a negative temperature reservoir necessary to obtain population inversion. One important aspect to highlight of this chapter is the definition of the adimensional effective coupling term $\zeta(\omega_\alpha, \omega_\beta)$, which could be understood as a transmission coefficient of the waveguide, and models the effective interaction, mediated by the atoms of the waveguide, between the modes inside the waveguide and the free electromagnetic field modes of the reservoir.

In chapter 3, we implemented the Heisenberg-Langevin formalism in order to obtain the dynamic evolution of the system of interest. In particular, in the case of the waveguide electromagnetic field modes equations, where the $\zeta(\omega_\alpha, \omega_\beta)$ term couples the modes inside and outside the waveguide, an effective mode-mode interaction could be created between two waveguide modes, mediated by the reservoir. In this work, we assumed that those situations could be ignored because the reservoir is markovian. For future works, the study of those mode-mode effective interactions is proposed, particularly in the non-markovian description of the reservoir.

In chapter 4, we studied the steady-state solution of the mean-field approximation of the Heisenberg-Langevin equation of motion. Future works could consider the numerical treatment of this set of equations, in particular, to simulate the multimodal cavity QED system, where the discrete model presented in chapter 2 could be considered, and because the number of degrees of freedom is less compared with the continuous case, the numerical costs of implementation are reduced. Chapter 4 also studied the spectrum of emission of the laser. To this, we defined a form of the electric field as an amplitude determined by the steady-state solutions of the mean field equations of motion of the bosonic operators and a fluctuating term. The analytical results for the spectrum of emission show that the linewidth is determined by the term of decay of the field κ. The Power of the spectrum showed to have a linear dependency on the number of atoms, with a slope determined by the $\gamma^{(1D)}$, the spontaneous emission to the waveguide modes. It is important to highlight the presence of the threshold, which defines the activation of the lasing behavior, where both conditions $2Nd_0\gamma^{(1D)} > \Gamma$ and $w > \gamma$ must be satisfied to generate the laser emission.

Finally, in chapter 5, we generalized the study developed in reference Minghui [27], from the case of a unimodal cavity to the case of a waveguide. Here we have obtained the phase equation of each mode inside the waveguide and then studied the diffusion of this phase through its two times correlations. Then, we obtained a form for the coherence function of the electric field emitted by the laser as the sum of the contributions of each mode. This coherence function has the form of a generalized Bessel function. For future works, we propose to find a new way to treat this expression and its Fourier transform, analytically or numerically.

The numerical results of chapter 5 were studied in two regimens of parameters, which we called the "generalized good/bad cavity regime.İn the generalized good cavity regime, the spectral shape does not depend on the κ term until a critical point from which the spectrum presents two new peaks. The central peak, which is in resonance with the atomic frequency, has a narrower linewidth than the spectrum above the critical point. For future works, we propose to understand if this configuration of a narrower central linewidth could be helpful to technological applications.

The Power of the emission showed a linear dependency on the number of atoms. In the generalized bad cavity regime, the decay of the field does not influence the shape of the spectrum, but the Power has an inverse proportionality relation with κ.

We could find a relation between the results obtained in the generalized good cavity regime with the results shown in chapter 4, for the method without fluctuations. However, the results of the generalized bad cavity regime showed a new behavior that is only considered in the method with fluctuation.

One open question of this work is how to interpret the total field spectrum as the sum of each mode singular spectrum, each with its characteristic linewidth. What does each mode spectrum represent? What does it mean to sum over all these modes to have the total spectrum?

Bibliography

[1] Haroche, S., "Nobel lecture: Controlling photons in a box and exploring the quantum to classical boundary," Rev. Mod. Phys., vol. 85, pp. 1083–1102, 2013, doi:10.1103/RevModPhys.85.1083.

[2] Kimble, H. J., Polzik, E., Rempe, G., y Thompson, R. J., "Structure and dynamics in cavity quantum electrodynamics," en Quantum Electronics and Laser Science Conference, p. QTuA2, Optica Publishing Group, 1992, https://opg.optica.org/abstract.cfm?URI=QELS-1992-QTuA2.

[3] Reiserer, A. y Rempe, G., "Cavity-based quantum networks with single atoms and optical photons," Rev. Mod. Phys., vol. 87, pp. 1379–1418, 2015, doi:10.1103/RevModPhys.87.1379.

[4] Haroche, S. y Raimond, J.-M., Exploring the quantum atoms, cavities, and photons. Oxford University Press, 2006.

[5] Solano, P., Grover, J. A., Hoffman, J. E., Ravets, S., Fatemi, F. K., Orozco, L. A., y Rolston, S. L., "Optical nanofibers," en Advances In Atomic, Molecular, and Optical Physics, pp. 439–505, Elsevier, 2017, doi:10.1016/bs.aamop.2017.02.003.

[6] Anonymous, "Proceedings of the american physical society," Phys. Rev., vol. 69, pp. 674–674, 1946, doi:10.1103/PhysRev.69.674.

[7] P.Solano, P.Barberis-Blostein, Fatemi, F., Orozco, L., y Rolston, S., "Superradiance reveals infinite-range dipole interactions through a nanofiber," nature communications.

[8] Perarnau-Llobet, M., González-Tudela, A., y Cirac, J. I., "Multimode fock states with large photon number: effective descriptions and applications in quantum metrology," Quantum Science and Technology, vol. 5, p. 025003, 2020, doi:10.1088/2058-9565/ab6ce5.

[9] Paulisch, V., Waveguide Quantum Electrodynamics Frontiers in the Generation and Scattering of Multiphoton State. Ph.D. thesis, Ludwig-Maximilians-Universitat Munchen, 2018.

[10] Gonzalez-Tudela, A., Paulisch, V., Chang, D., Kimble, H. J., y Cirac, J., "Deterministic generation of arbitrary photonic states assisted by dissipation.," Physical review letters, vol. 115 16, p. 163603, 2015.

[11] Gonzalez-Tudela, A., Martin-Cano, D., Moreno, E., Martin-Moreno, L., Tejedor, C., y Garcia-Vidal, F. J., "Entanglement of two qubits mediated by one-dimensional plasmonic waveguides," Phys. Rev. Lett., vol. 106, p. 020501, 2011, doi:10.1103/PhysRevLett.106.020501.

[12] Gonzalez-Tudela, A., Martin-Cano, D., Moreno, E., Martin-Moreno, L., Garcia-Vidal,

F. J., y Tejedor, C., “Exploring qubit-qubit entanglement mediated by one-dimensional plasmonic nanowaveguides,” physica status solidi (c), vol. 9, no. 5, pp. 1303–1308, 2012, doi:10.1002/pssc.201100197.

[13] Einstein, A., “Zur Quantentheorie der Strahlung. (German) [On the Quantum Theory of Radiation],” vol. 18, pp. 121–128, 1917.

[14] Buluta, I., Ashhab, S., y Nori, F., “Natural and artificial atoms for quantum computation,” Reports on Progress in Physics, vol. 74, p. 104401, 2011, doi:10.1088/0034-4885/74/10/104401.

[15] Doherty, M. W., Manson, N. B., Delaney, P., Jelezko, F., Wrachtrup, J., y Hollenberg, L. C., “The nitrogen-vacancy colour centre in diamond,” Physics Reports, vol. 528, no. 1, pp. 1–45, 2013, doi:10.1016/j.physrep.2013.02.001.

[16] Gerry, C. y Knight, P., Introductory Quantum Optics. Cambridge University Press, 2004, doi:10.1017/cbo9780511791239.

[17] Loudon, R., The Quantum Theory of Light. Oxford: Clarendon Press, second ed., 1983.

[18] Carmichael, H., Statistical Methods in Quantum Optics. Springer, 1999.

[19] Breuer, H.-P. y Petruccione, F., The Theory of Open Quantum Systems. Oxford University PressOxford, 2007, doi:10.1093/acprof:oso/9780199213900.001.0001.

[20] Ávila, C. y Alejandro, J., “New quantum technologies: batteries with qubits and electromagnetic resonators,” 2021.

[21] Wiegand, E., Quantum optics and waveguide quantum electrodynamics in superconducting circuits. 2021.

[22] Gardiner, C. W. y Zoller, P., Quantum noise. Springer Series in Synergetics, Berlin, Germany: Springer, 3 ed., 2004.

[23] Meystre, P. y Sargent, M., Elements of quantum optics. Springer, 2007.

[24] MAIMAN, T. H., “Stimulated optical radiation in ruby,” Nature, vol. 187, no. 4736, pp. 493–494, 1960, doi:10.1038/187493a0.

[25] “The nobel prize in physics 2018.”, https://www.nobelprize.org/prizes/physics/2018/summary/.

[26] “The nobel prize in physics 2017.”, https://www.nobelprize.org/prizes/physics/2017/press-release/.

[27] Xu, M., Theory of Steady-State Superradiance. Ph.D. thesis, University of Colorado, 2011.

[28] D. A. Tieri, Minghui Xu, D. M., “Theory of the crossover from lasing to steady state superradiance.” e-print arXiv:1702.04830, 2017.

[29] Haken, H., Laser Theory. Springer Berlin Heidelberg, 1983, doi:10.1007/978-3-642-45556-8.

[30] Open Quantum System with Applications to Precision Measurements. Ph.D. thesis, University of Colorado, 2015.

[31] Haken, H., “Theory of intensity and phase fluctuations of a homogeneously broadened laser,” Zeitschrift fur Physik, vol. 190, no. 3, pp. 327–356, 1966, doi:10.1007/bf01333598.

[32] Schawlow, A. L. y Townes, C. H., “Infrared and optical masers,” Physical Review, vol. 112, no. 6, pp. 1940–1949, 1958, doi:10.1103/physrev.112.1940.

[33] Meiser, D., Ye, J., Carlson, D., y Holland, M., “Prospect for a millihertz-linewidth laser,” Physical Review Letter, 2009.

[34] Kimble, H. J., “The quantum internet,” Nature, vol. 453, no. 7198, pp. 1023–1030, 2008, doi:10.1038/nature07127.

[35] Ritter, S., Nölleke, C., Hahn, C., Reiserer, A., Neuzner, A., Uphoff, M., Mücke, M., Figueroa, E., Bochmann, J., y Rempe, G., “An elementary quantum network of single atoms in optical cavities,” Nature, vol. 484, no. 7393, pp. 195–200, 2012, doi:10.1038/nature11023.

[36] Paulisch, V., Kimble, H. J., y González-Tudela, A., “Universal quantum computation in waveguide QED using decoherence free subspaces,” New Journal of Physics, vol. 18, no. 4, p. 043041, 2016, doi:10.1088/1367-2630/18/4/043041.

[37] Thoma, A., Schnauber, P., Gschrey, M., Seifried, M., Wolters, J., Schulze, J.-H., Strittmatter, A., Rodt, S., Carmele, A., Knorr, A., Heindel, T., y Reitzenstein, S., “Exploring dephasing of a solid-state quantum emitter via time- and temperature-dependent hong-ou-mandel experiments,” Physical Review Letters, vol. 116, no. 3, 2016, doi:10.1103/physrevlett.116.033601.

[38] Kannan, B., Ruckriegel, M. J., Campbell, D. L., Kockum, A. F., Braumüller, J., Kim, D. K., Kjaergaard, M., Krantz, P., Melville, A., Niedzielski, B. M., Vepsäläinen, A., Winik, R., Yoder, J. L., Nori, F., Orlando, T. P., Gustavsson, S., y Oliver, W. D., “Waveguide quantum electrodynamics with superconducting artificial giant atoms,” Nature, vol. 583, no. 7818, pp. 775–779, 2020, doi:10.1038/s41586-020-2529-9.

[39] Gieβen, H., Berger, J. D., Mohs, G., Meystre, P., y Yelin, S. F., “Cavity-modified spontaneous emission: From rabi oscillations to exponential decay,” Phys. Rev. A, vol. 53, pp. 2816–2821, 1996, doi:10.1103/PhysRevA.53.2816.

[40] Saff, E. y Snider, A., Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics, pp.–563. 2003.

[41] Kubo, R., Toda, M., y Hashitsume, N., Statistical Physics II. Springer Berlin Heidelberg, 1991, doi:10.1007/978-3-642-58244-8.

[42] Khintchine, A., “Korrelationstheorie der stationären stochastischen prozesse,” Mathematische Annalen, vol. 109, pp. 604–615, 1934, http://eudml.org/doc/159698.

[43] Bohnet, J. G., Chen, Z., Weiner, J. M., Meiser, D., Holland, M. J., y Thompson, J. K., “A steady-state superradiant laser with less than one intracavity photon,” Nature, vol. 484, no. 7392, pp. 78–81, 2012, doi:10.1038/nature10920.

[44] Gardiner, C. W. y Collett, M. J., “Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation,” Phys. Rev. A, vol. 31, pp. 3761–3774, 1985, doi:10.1103/PhysRevA.31.3761.

Annex A

Derivation of the Phase Fluctuations

$$\left\langle (\hat{\phi}_k(t) - \phi_k(0))^2 \right\rangle = \left\langle \left(\Delta_k t - \int_0^t Im\left[i \frac{g_k^* N}{a_{ko}} \tilde{F} \right] dt' \right) \left(\Delta_k t - \int_0^t Im\left[i \frac{g_k^* N}{a_{ko}} \tilde{F} \right] dt'' \right) \right\rangle \tag{A.1}$$

$$\left\langle (\hat{\phi}_k(t) - \phi_k(0))^2 \right\rangle = \left\langle \Delta_k^2 t^2 \right\rangle + \left\langle \left(\int_0^t Im\left[i \frac{g_k^* N}{a_{ko}} \tilde{F} \right] dt' \right) \left(\int_0^t Im\left[i \frac{g_k^* N}{a_{ko}} \tilde{F} \right] dt'' \right) \right\rangle \tag{A.2}$$

$$\left\langle (\hat{\phi}_k(t) - \phi_k(0))^2 \right\rangle = \left\langle \Delta_k^2 t^2 \right\rangle + \frac{1}{4} \int dt' \int dt'' \left(\left\langle \left[i \frac{g_k^* N}{a_{ko}} \tilde{F} \right] \left[i \frac{g_k^* N}{a_{ko}} \tilde{F} \right]^\dagger \right\rangle + \left\langle \left[i \frac{g_k^* N}{a_{ko}} \tilde{F} \right]^\dagger \left[i \frac{g_k^* N}{a_{ko}} \tilde{F} \right] \right\rangle \right) \tag{A.3}$$

$$\left\langle (\hat{\phi}_k(t) - \phi_k(0))^2 \right\rangle = \left\langle \Delta_k^2 t^2 \right\rangle + \frac{1}{2} \frac{|g_k|^2 N^2}{|a_{ko}|^2} \int dt' \int dt'' \left(\left\langle \tilde{F}^- \tilde{F}^+ \right\rangle + \left\langle \tilde{F}^+ \tilde{F}^- \right\rangle \right) \tag{A.4}$$

$$\left\langle (\hat{\phi}_k(t) - \phi_k(0))^2 \right\rangle = \left\langle \Delta_k^2 t^2 \right\rangle + \frac{1}{2} \frac{|g_k|^2 N^2}{|a_{ko}|^2} \int dt' \int dt'' \left(\frac{\gamma}{2} + \frac{w}{2} \right) \delta(t' - t'') \tag{A.5}$$

$$\left\langle (\hat{\phi}_k(t) - \phi_k(0))^2 \right\rangle = \Delta_k^2 t^2 + \frac{1}{2} \frac{|g_k|^2 N^2}{|a_{ko}|^2} \left(\frac{\gamma}{2} + \frac{w}{2} \right) t \tag{A.6}$$

Annex B

Equations of motion of Atomics Opertors

B.1. Equation of Motion of Coherence Operator

This results are shown in sections F.2.2.1 and F.2.2.2 for both reservoir

$$N\dot{S}^{-} = \frac{-i}{\hbar}\sum_{j}\left[\sigma_j^{-}, H_I\right] = \frac{-i}{\hbar}\left[NS^{-}, H_I^{R} + H_I^{\mathcal{P}} + H_I^{1D}\right] \tag{B.1}$$

$$\dot{S}^{-}_{(1D)} \quad = \frac{i}{N}\sum_{j,k} g_k \sigma_j^{z} a_k e^{-i(\omega_k-\omega_A)t} \tag{B.2}$$

$$\dot{S}^{-}_{(R)} = \tilde{\mathcal{F}}^{R}_{(-)} - \frac{1}{2}\gamma^{(R)}S^{-} + i\Omega^{(R)}_{(-)}S^{-} \tag{B.3}$$

$$\dot{S}^{-}_{(\mathcal{P})} = \tilde{\mathcal{F}}^{\mathcal{P}}_{(-)} - \frac{1}{2}\gamma^{(\mathcal{P})}S^{-} + i\Omega^{(\mathcal{P})}_{(-)}S^{-} \tag{B.4}$$

$$\dot{S}^{-} = i\sum_{k} g_k S^{z} a_k e^{-i(\omega_k-\omega_A)t} + \tilde{\mathcal{F}}_{(-)} - \frac{1}{2}\left(\gamma^{(R)} + \gamma^{(\mathcal{P})}\right)S^{-} + i\Omega_{(-)}S^{-} \tag{B.5}$$

Where $\Omega_{(-)} = \Omega^{(\mathcal{P})}_{(-)} + \Omega^{(R)}_{(-)}$, is the shift generated by the reservoirs R and $\mathcal{P}$.

B.2. Equation of Motion of Inversion Operator

This results are shown in sections F.2.1.1 and F.2.1.2 for both reservoir

$$N\dot{S}^{z} = \frac{-i}{\hbar}\sum_{j}\left[\sigma_j^{z}, H_I\right] = \frac{-i}{\hbar}\left[NS^{z}, H_I^{R} + H_I^{\mathcal{P}} + H_I^{1D}\right] \tag{B.6}$$

$$\dot{S}^{z}_{(1D)} \quad = \frac{2i}{N}\sum_{j,k}\left(g_k^{*} a_k^{\dagger}\sigma_j^{-} e^{i(\omega_k-\omega_A)t} - g_k\sigma_j^{+} a_k e^{-i(\omega_k-\omega_A)t}\right) \tag{B.7}$$

$$\dot{S}^z_{(R)} = \frac{-i}{\hbar}\left[S^z, H_I^R\right] = \quad \tilde{\mathcal{F}}^R - \gamma^{(R)}\left[S^z + 1\right] + i\Omega^{(R)}\left[S^z + 1\right] \tag{B.8}$$

$$\dot{S}^z_{(\mathcal{P})} = \frac{-i}{\hbar}\left[S^z, H_I^{\mathcal{P}}\right] = \quad \tilde{\mathcal{F}}^{\mathcal{P}} - \gamma^{(\mathcal{P})}\left[S^z - 1\right] + i\Omega^{(\mathcal{P})}\left[S^z - 1\right] \tag{B.9}$$

$$\begin{aligned}\dot{S}^z = \quad & \frac{2i}{N}\sum_{j,k}\left(g_k^* a_k^\dagger \sigma_j^- e^{i(\omega_k-\omega_A)t} - g_k \sigma_j^+ a_k e^{-i(\omega_k-\omega_A)t}\right) - \gamma^{(R)}\left[S^z+1\right] + i\Omega^{(R)}\left[S^z+1\right] \\ & + \tilde{\mathcal{F}}^R + \tilde{\mathcal{F}}^{\mathcal{P}} - \gamma^{(\mathcal{P})}\left[S^z-1\right] + i\Omega^{(\mathcal{P})}\left[S^z-1\right]\end{aligned} \tag{B.10}$$

$$\begin{aligned}\dot{S}^z = \quad & 2i\sum_{k}\left(g_k^* a_k^\dagger S^- e^{i(\omega_k-\omega_A)t} - g_k S^+ a_k e^{-i(\omega_k-\omega_A)t}\right) - \left(\gamma^{(R)}+\gamma^{(\mathcal{P})}\right)S^z + i\left(\Omega^{(R)}+\Omega^{(\mathcal{P})}\right)S^z \\ & - \left(\gamma^{(R)}-\gamma^{(\mathcal{P})}\right) + i\left(\Omega^{(R)}-\Omega^{(\mathcal{P})}\right) + \tilde{\mathcal{F}}_z\end{aligned} \tag{B.11}$$

$$\begin{aligned}\dot{S}^z = \quad & 2i\sum_{k}\left(g_k^* a_k^\dagger S^- e^{i(\omega_k-\omega_A)t} - g_k S^+ a_k e^{-i(\omega_k-\omega_A)t}\right) - \left(\gamma^{(R)}+\gamma^{(\mathcal{P})}\right)\left[S^z - d_0\right] \\ & + i\Omega\left[S^z - \Delta_0\right] + \tilde{\mathcal{F}}_z\end{aligned} \tag{B.12}$$

Where $d_0 = \frac{\gamma^{(R)}-\gamma^{(\mathcal{P})}}{\gamma^{(R)}+\gamma^{(\mathcal{P})}}$, $\Delta_0 = \frac{\Omega^{(R)}-\Omega^{(\mathcal{P})}}{\Omega}$ and $\tilde{\mathcal{F}}_z = \tilde{\mathcal{F}}^R + \tilde{\mathcal{F}}^{\mathcal{P}}$.

Annex C

Rotating frame variable change

From the equation of motion of the bosonic operator (??,??,C.3), and considering the change of variables $\tilde{a}_k = a_k e^{-i(\omega_k-\omega)t}$ and $\tilde{S}^- = S^- e^{-i(\omega_a-\omega)t}$:

$$\begin{aligned}
\dot{\tilde{a}}_k &= \dot{a}_k e^{-i(\omega_k-\omega)t} - i(\omega_k-\omega)a_k e^{-i(\omega_k-\omega)t} \\
&= \left(-ig_k^* N S^- e^{i(\omega_k-\omega_A)t}\right) e^{-i(\omega_k-\omega)t} - i(\omega_k-\omega)a_k e^{-i(\omega_k-\omega)t} \\
&= -ig_k^* N S^- e^{-i(\omega_A-\omega)t} - i(\omega_k-\omega)\tilde{a}_k \\
&= -ig_k^* N \tilde{S}^- - i(\omega_k-\omega)\tilde{a}_k
\end{aligned} \tag{C.1}$$

$$\begin{aligned}
\dot{\tilde{S}}^- =&\quad \dot{S}^- e^{-i(\omega_A-\omega)t} - i(\omega_A-\omega)S^- e^{-i(\omega_A-\omega)t} \\
=&\quad \left(\tilde{\mathcal{F}}_{(-)} - \frac{1}{2}(\gamma_R+\gamma_{\mathcal{P}})S^- + i\sum_k g_k S^z a_k e^{-i(\omega_k-\omega_A)t}\right) e^{-i(\omega_A-\omega)t} - i(\omega_A-\omega)S^- e^{-i(\omega_A-\omega)t} \\
=&\quad \tilde{\mathcal{F}}_{(-)} e^{-i(\omega_A-\omega)t} - \frac{1}{2}(\gamma_R+\gamma_{\mathcal{P}})S^- e^{-i(\omega_A-\omega)t} + i\sum_k \left(g_k S^z a_k e^{-i(\omega_k-\omega_A)t}\right) e^{-i(\omega_A-\omega)t} \\
&\quad -i(\omega_A-\omega)S^- e^{-i(\omega_A-\omega)t} \\
=&\quad \bar{\mathcal{F}}_{(-)} - \frac{1}{2}(\gamma_R+\gamma_{\mathcal{P}})\tilde{S}^- + i\sum_k \left(g_k S^z a_k e^{-i(\omega_k-\omega_A)t}\right) e^{-i(\omega_A-\omega)t} - i(\omega_A-\omega)\tilde{S}^- \\
=&\quad \bar{\mathcal{F}}_{(-)} - \frac{1}{2}\left[\gamma_R+\gamma_{\mathcal{P}} - 2i(\omega_A-\omega)\right]\tilde{S}^- + i\sum_k \left(g_k S^z a_k e^{-i(\omega_k-\omega)t}\right) \\
=&\quad \bar{\mathcal{F}}_{(-)} - \frac{1}{2}\left[\gamma_R+\gamma_{\mathcal{P}} - 2i(\omega_A-\omega)\right]\tilde{S}^- + i\sum_k g_k S^z \tilde{a}_k
\end{aligned} \tag{C.2}$$

$$
\begin{aligned}
\dot{S}^z = \quad & \tilde{\mathcal{F}}_{(z)} - (\gamma_R + \gamma_{\mathcal{P}})(S^z - d_0) + 2i \sum_k \left(g_k^* a_k^\dagger S^- e^{i(\omega_k - \omega_A)t} - g_k S^+ a_k e^{-i(\omega_k - \omega_A)t} \right) \\
= \quad & \tilde{\mathcal{F}}_{(z)} - (\gamma_R + \gamma_{\mathcal{P}})(S^z - d_0) \\
& + 2i \sum_k \left(g_k^* (\tilde{a}_k^\dagger e^{-i(\omega_k - \omega)t})(\tilde{S}^- e^{i(\omega_A - \omega)t}) e^{i(\omega_k - \omega_A)t} - g_k (\tilde{S}^+ e^{-i(\omega_A - \omega)t})(a_k e^{i(\omega_k - \omega)t}) e^{-i(\omega_k - \omega_A)t} \right) \\
= \quad & \tilde{\mathcal{F}}_{(z)} - (\gamma_R + \gamma_{\mathcal{P}})(S^z - d_0) + 2i \sum_k \left(g_k^* \tilde{a}_k^\dagger \tilde{S}^- - g_k \tilde{S}^+ \tilde{a}_k \right)
\end{aligned}
\tag{C.3}
$$

Annex D

Schrödinger, Heisenberg and Interaction Pictures

Considering an operator A_S on the Schrödinger Picture, we could change the reference frame of this operator througth a unitary transformation.

On the Heinsenberg Picture, the operator A_S will have the form

$$A_H = e^{\frac{iH_s t}{\hbar}} A_s e^{\frac{-iH_s t}{\hbar}} \tag{D.1}$$

On the Interaction Picture, the operator A_S will have the form

$$A_I = e^{\frac{iH_0 t}{\hbar}} A_s e^{\frac{-iH_0 t}{\hbar}} \tag{D.2}$$

On this section we will study the Hamiltonian in those different pictures.

D.1. Interaction Picture

$$\begin{aligned} H_{\mathcal{I}} = \quad & \sum_j \frac{\hbar\omega_A}{2}\sigma_j + \int d\omega_k \hbar\omega_k a^\dagger(\omega_k)a(\omega_k) \\ & \sum_j \int d\omega_k \hbar \left(g^*(\omega_k)a^\dagger(\omega_k)\sigma_j^- e^{i(\omega_k-\omega_A)t} + g(\omega_k)\sigma_j^+ a(\omega_k)e^{-i(\omega_k-\omega_A)t} \right) \end{aligned} \tag{D.3}$$

D.2. Schrödinger Picture

$$H_{\mathcal{S}} = \sum_j \frac{\hbar\omega_A}{2}\sigma_j + \int d\omega_k \hbar\omega_k a^\dagger(\omega_k)a(\omega_k) + \sum_j \int d\omega_k \hbar \left(g^*(\omega_k)a^\dagger(\omega_k)\sigma_j^- + g(\omega_k)\sigma_j^+ a(\omega_k) \right) \tag{D.4}$$

D.3. Heisenberg Picture

$$H_{\mathcal{H}} = \sum_j \frac{\hbar\omega_A}{2}\sigma_j + \int d\omega_k \hbar\omega_k a^\dagger(\omega_k)a(\omega_k) + \sum_j \int d\omega_k \hbar \left(g^*(\omega_k)a^\dagger(\omega_k)\sigma_j^- + g(\omega_k)\sigma_j^+ a(\omega_k)\right) \tag{D.5}$$

Note that from the equation (D.1), we have that

$$H_H = e^{\frac{iH_s t}{\hbar}} H_s e^{\frac{-iH_s t}{\hbar}} \tag{D.6}$$

But H_s commute with itself, so we get that $H_H = H_S$.

D.4. Electric Field

Now, we will study the Electric Field Operator on different Pictures. First, we will consider the Interaction Picture

$$E_I^+ = \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} a(\omega_k) e^{i\omega_k t} \tag{D.7}$$

To change the Picture to the Schrödinger Picture, we need to apply the transformation

$$A_s = e^{\frac{-iH_0 t}{\hbar}} A_I e^{\frac{iH_0 t}{\hbar}} \tag{D.8}$$

Then, in the case of the Electric Field Operator

$$\begin{aligned} E_s^+ \quad &= e^{\frac{-iH_0 t}{\hbar}} E_I^+ e^{\frac{iH_0 t}{\hbar}} \\ &= E_I^+ + \frac{i(-t)}{\hbar}\left[H_0, E_I^+\right] + \frac{i^2(-t)^2}{2!\hbar^2}\left[H_0, \left[H_0, E_I^+\right]\right] + \frac{i^3(-t)^3}{3!\hbar^3}\left[H_0, \left[H_0, \left[H_0, E_I^+\right]\right]\right] ... \end{aligned} \tag{D.9}$$

Tthe Electric Field operator on the Schrödinger Picture

$$E_S^+ = \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} a(\omega_k) \tag{D.10}$$

Now, we want to obtain the expression of the Electric Field Operator in the Heisenberg Picture. To do this, we will apply the unitary transformation to the Electric Field opertator in the Schrödinger Picture.

$$
\begin{aligned}
E_H^+ &= e^{\frac{iH_S t}{\hbar}} E_S^+ e^{\frac{-iH_S t}{\hbar}} \\
&= E_I^+ + \frac{it}{\hbar}\left[H_S, E_S^+\right] + \frac{i^2t^2}{2!\hbar^2}\left[H_S,\left[H_S, E_S^+\right]\right] + \frac{i^3t^3}{3!\hbar^3}\left[H_S,\left[H_S,\left[H_S, E_S^+\right]\right]\right]...
\end{aligned} \tag{D.11}
$$

Studying the commutator

$$
\begin{aligned}
\left[H_S, E_S^+\right] = \quad & \left[\frac{\hbar\omega_A}{2} N S^z, \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} a(\omega_k)\right] \\
& + \left[\int d\omega_k' \hbar\omega_k' a^\dagger(\omega_k') a(\omega_k'), \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} a(\omega_k)\right] \\
& + \left[\int d\omega_k' N\hbar \left(g^*(\omega_k') a^\dagger(\omega_k') S^- + g(\omega_k') S^+ a(\omega_k')\right), \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} a(\omega_k)\right]
\end{aligned} \tag{D.12}
$$

$$
\begin{aligned}
\left[H_S, E_S^+\right] = \quad & \int d\omega_k' \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} \hbar\omega_k' \left[a^\dagger(\omega_k') a(\omega_k'), a(\omega_k)\right] \\
& + \int d\omega_k' \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} N\hbar \left[\left(g^*(\omega_k') a^\dagger(\omega_k') S^- + g(\omega_k') S^+ a(\omega_k')\right), a(\omega_k)\right]
\end{aligned} \tag{D.13}
$$

$$
\left[H_S, E_S^+\right] = \quad - \int d\omega_k' \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} \hbar\omega_k' \delta(\omega_k - \omega_k') a(\omega_k) - \int d\omega_k' \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} N\hbar g^*(\omega_k') S^- \delta(\omega_k -
$$

(D

$$
\left[H_S, E_S^+\right] = \quad - \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} \hbar\omega_k a(\omega_k) - \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} N\hbar g^*(\omega_k) S^- \tag{D.15}
$$

Now, we will study the second commutator term

$$\left[H_S,\left[H_S,E_S^+\right]\right] = \left[\frac{\hbar\omega_A}{2}NS^z, -\int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}\hbar\omega_k a(\omega_k) - \int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}N\hbar g^*(\omega_k)S^-\right]$$
$$+\left[\int d\omega_k'\hbar\omega_k' a^\dagger(\omega_k')a(\omega_k'), -\int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}\hbar\omega_k a(\omega_k) - \int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}N\hbar g^*(\omega_k)S\right.$$
$$+\left[\int d\omega_k' N\hbar\left(g^*(\omega_k')a^\dagger(\omega_k')S^- + g(\omega_k')S^+a(\omega_k')\right), -\int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}\hbar\omega_k a(\omega_k)\right]$$
$$+\left[\int d\omega_k' N\hbar\left(g^*(\omega_k')a^\dagger(\omega_k')S^- + g(\omega_k')S^+a(\omega_k')\right), -\int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}N\hbar g^*(\omega_k)S^-\right] \quad \text{(D.}$$

$$\left[H_S,\left[H_S,E_S^+\right]\right] = -\int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}N\hbar g^*(\omega_k)\frac{\hbar\omega_A}{2}N\left[S^z,S^-\right]$$
$$-\int d\omega_k'\int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}\hbar\omega_k\hbar\omega_k'\left[a^\dagger(\omega_k')a(\omega_k'),a(\omega_k)\right]$$
$$-\int d\omega_k'\int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}\hbar\omega_k N\hbar g^*(\omega_k')\left[a^\dagger(\omega_k')S^-,a(\omega_k)\right]$$
$$-\int d\omega_k'\int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}N\hbar g^*(\omega_k)N\hbar g(\omega_k')\left[S^+a(\omega_k'),S^-\right] \quad \text{(D.17)}$$

$$\left[H_S,\left[H_S,E_S^+\right]\right] = -\int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}N\hbar g^*(\omega_k)\frac{\hbar\omega_A}{2}N(-2S^-)$$
$$\int d\omega_k'\int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}\hbar\omega_k\hbar\omega_k'\delta(\omega_k-\omega_k')a(\omega_k)$$
$$\int d\omega_k'\int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}\hbar\omega_k N\hbar g^*(\omega_k')\delta(\omega_k-\omega_k')S^-$$
$$-\int d\omega_k'\int d\omega_k\sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}}N\hbar g^*(\omega_k)N\hbar g(\omega_k')S^z a(\omega_k') \quad \text{(D.18)}$$

$$
\begin{aligned}
\Big[H_S, \Big[H_S, E_S^+\Big]\Big] = \quad & \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} N^2\hbar^2\omega_A g^*(\omega_k)S^- \\
& + \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} \hbar^2\omega_k^2 a(\omega_k) \\
& + \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} \hbar^2\omega_k N g^*(\omega_k)S^- \\
& - \int d\omega_k' \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} N^2\hbar^2 g^*(\omega_k)g(\omega_k')S^z a(\omega_k')
\end{aligned}
\tag{D.19}
$$

$$
\begin{aligned}
\Big[H_S, \Big[H_S, E_S^+\Big]\Big] = \quad & \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} \Big(N^2\hbar^2\omega_A g^*(\omega_k)S^- + \hbar^2\omega_k^2 a(\omega_k) + \hbar^2\omega_k N g^*(\omega_k)S^-\Big) \\
& - \int d\omega_k' \int d\omega_k \sqrt{\frac{\hbar\omega_k}{4\pi\epsilon_0 cA}} N^2\hbar^2 g^*(\omega_k)g(\omega_k')S^z a(\omega_k')
\end{aligned}
\tag{D.20}
$$

Annex E

Noise Operator Two times Correlation

From the correlation function expression

$$g^{(1)}(\tau) = \frac{\langle E^{-}(t)E^{+}(t+\tau)\rangle}{\langle E^{-}(t)E^{+}(t)\rangle} \tag{E.1}$$

$$g^{(1)}(\tau) = \frac{\int d\omega_k \int d\omega_k' a_0^{\dagger}(\omega_k) a_0(\omega_k') \left\langle e^{i(\phi_k(t)-\phi_{k'}(t+\tau))}\right\rangle}{\int d\omega_k \int d\omega_k' a_0^{\dagger}(\omega_k) a_0(\omega_k') \left\langle e^{i(\phi_k(t)-\phi_{k'}(t))}\right\rangle} \tag{E.2}$$

$$g^{(1)}(\tau) = \frac{\int d\omega_k \int d\omega_k' a_0^{\dagger}(\omega_k) a_0(\omega_k') e^{-\frac{1}{2}\left\langle(\phi_k(\tau)-\phi_{k'}(0))^2\right\rangle}}{\int d\omega_k \int d\omega_k' a_0^{\dagger}(\omega_k) a_0(\omega_k') e^{-\frac{1}{2}\langle(\phi_k(0)-\phi_{k'}(0))^2\rangle}} \tag{E.3}$$

E.1. Gaussian Noise

$$\begin{aligned}
\left\langle e^{i(\phi_k(t)-\phi_k(0))}\right\rangle &= \left\langle \sum_n \frac{i^n(\phi_k(t)-\phi_k(0))^n}{n!}\right\rangle \\
&= \sum_n \frac{i^n}{n!}\langle(\phi_k(t)-\phi_k(0))^n\rangle \\
&= 1 + i\langle(\phi_k(t)-\phi_k(0))\rangle + \frac{(-1)}{2}\left\langle(\phi_k(t)-\phi_k(0))^2\right\rangle + \frac{-i}{3!}\left\langle(\phi_k(t)-\phi_k(0))^3\right\rangle + ... \\
&= 1 + \frac{(-1)}{2}\left\langle(\phi_k(t)-\phi_k(0))^2\right\rangle ... \\
&= \sum_n \frac{(-1)^n(\frac{1}{2})^n}{n!}\left\langle(\phi_k(t)-\phi_k(0))^{2n}\right\rangle \\
&= e^{-\frac{1}{2}\left\langle(\phi_k(t)-\phi_k(0))^2\right\rangle}
\end{aligned} \tag{E.4}$$

$$\begin{aligned}\left\langle e^{i(\phi_k(t)-\phi_{k'}(0))}\right\rangle &= \left\langle \sum_n \frac{i^n(\phi_k(t)-\phi_{k'}(0))^n}{n!}\right\rangle \\ &= \sum_n \frac{i^n}{n!}\left\langle(\phi_k(t)-\phi_{k'}(0))^n\right\rangle \\ &= 1+i\left\langle(\phi_k(t)-\phi_{k'}(0))\right\rangle+\frac{(-1)}{2}\left\langle(\phi_k(t)-\phi_{k'}(0))^2\right\rangle\end{aligned} \tag{E.5}$$

Studying the second term of the RHS

$$\left\langle(\phi_k(t)-\phi_{k'}(0))\right\rangle = 0 \tag{E.6}$$

The second term of the RHS

$$\begin{aligned}\left\langle(\phi_k(t)-\phi_{k'}(0))^2\right\rangle &= \left\langle(\phi_k(t)-\phi_{k'}(0))^\dagger(\phi_k(t)-\phi_{k'}(0))\right\rangle \\ &= \left\langle(\phi_k(t)^\dagger-\phi_{k'}(0)^\dagger)(\phi_k(t)-\phi_{k'}(0))\right\rangle \\ &= \left\langle\phi_k^\dagger(t)\phi_k(t)\right\rangle-\left\langle\phi_k^\dagger(t)\phi_{k'}(0)\right\rangle-\left\langle\phi_{k'}^\dagger(0)\phi_k(t)\right\rangle+\left\langle\phi_{k'}^\dagger(0)\phi_{k'}(0)\right\rangle\end{aligned} \tag{E.7}$$

Considering that

$$\phi_k(t)-\phi_k(0)=\frac{2}{a_0(\omega_k)(\Gamma+\kappa)}\int dt' Im\left[\mathcal{F}(t)\right]-\int dt'\frac{\Gamma}{(\Gamma+\kappa)}(\omega_k-\omega_A) \tag{E.8}$$

Then

$$\phi_k(t)=\frac{2}{a_0(\omega_k)(\Gamma+\kappa)}\int dt' Im\left[\mathcal{F}(t)\right]-\int dt'\frac{\Gamma}{(\Gamma+\kappa)}(\omega_k-\omega_A)+\phi_k(0) \tag{E.9}$$

The first term on the RHS of the equation (E.7)

$$\begin{aligned}\left\langle\phi_k^\dagger(t)\phi_k(t)\right\rangle = &\frac{4}{|a_0(\omega_k)|^2(\Gamma+\kappa)^2}\left\langle\left(\int dt'\frac{1}{2}(\mathcal{F}(t')-\mathcal{F}^\dagger(t'))\right)^\dagger\left(\int dt''\frac{1}{2}(\mathcal{F}(t'')-\mathcal{F}^\dagger(t''))\right)\right\rangle \\ &+\frac{2}{a_0(\omega_k)(\Gamma+\kappa)^2}\left\langle\left(\int dt'\frac{1}{2}(\mathcal{F}(t')-\mathcal{F})^\dagger(t')\right)\phi_k(0)\right\rangle \\ &+\frac{2}{a_0(\omega_k)(\Gamma+\kappa)^2}\left\langle\phi_k(0)\left(\int dt'\frac{1}{2}(\mathcal{F}(t')-\mathcal{F})^\dagger(t')\right)\right\rangle \\ &+\left\langle\int dt'\int dt''\frac{\Gamma^2}{(\Gamma+\kappa)^2}(\omega_k-\omega_A)^2\right\rangle+\left\langle\phi_k^\dagger(0)\phi_k(0)\right\rangle\end{aligned} \tag{E.10}$$

The second term

$$\left\langle \phi_k^\dagger(t)\phi_{k'}(0)\right\rangle = \frac{2}{a_0^*(\omega_k)(\Gamma+\kappa)}\left\langle \int dt'\frac{1}{2}(\mathcal{F}(t')-\mathcal{F}^\dagger(t'))\phi_{k'}(0)\right\rangle - \int dt\left(\frac{\Gamma}{\Gamma+\kappa}\right)(\omega_k-\omega_A)\left\langle \phi_k(0)\right\rangle + \left\langle \phi_k^\dagger(0)\phi_{k'}(0)\right\rangle$$
$$= \frac{2}{a_0^*(\omega_k)(\Gamma+\kappa)}\left\langle \int dt'\frac{1}{2}(\mathcal{F}(t')-\mathcal{F}^\dagger(t'))\phi_{k'}(0)\right\rangle + \left\langle \phi_k^\dagger(0)\phi_{k'}(0)\right\rangle \tag{E.11}$$

$$\left\langle \phi_{k'}^\dagger(0)\phi_k(t)\right\rangle = \frac{2}{a_0(\omega_k)(\Gamma+\kappa)}\left\langle \phi_{k'}^\dagger(0)\int dt'\frac{1}{2}(\mathcal{F}(t')-\mathcal{F}^\dagger(t'))\right\rangle + \left\langle \phi_{k'}^\dagger(0)\phi_k(0)\right\rangle \tag{E.12}$$

Replacing in the equation (E.7)

$$\left\langle (\phi_k(t)-\phi_{k'}(0))^2\right\rangle = \left\langle \phi_k^\dagger(t)\phi_k(t)\right\rangle - \left\langle \phi_k^\dagger(t)\phi_{k'}(0)\right\rangle - \left\langle \phi_{k'}^\dagger(0)\phi_k(t)\right\rangle + \left\langle \phi_{k'}^\dagger(0)\phi_{k'}(0)\right\rangle \tag{E.13}$$

$$\begin{aligned}\left\langle (\phi_k(t)-\phi_{k'}(0))^2\right\rangle = {} & \frac{4}{|a_0(\omega_k)|^2(\Gamma+\kappa)^2}\left\langle \left(\int dt'\frac{1}{2}(\mathcal{F}(t')-\mathcal{F}^\dagger(t'))\right)^\dagger \left(\int dt''\frac{1}{2}(\mathcal{F}(t'')-\mathcal{F}^\dagger(t''))\right)\right\rangle \\ & +\frac{2}{a_0(\omega_k)(\Gamma+\kappa)^2}\left\langle \left(\int dt'\frac{1}{2}(\mathcal{F}(t')-\mathcal{F})^\dagger(t')\right)\phi_k(0)\right\rangle \\ & +\frac{2}{a_0(\omega_k)(\Gamma+\kappa)^2}\left\langle \phi_k(0)\left(\int dt'\frac{1}{2}(\mathcal{F}(t')-\mathcal{F})^\dagger(t')\right)\right\rangle \\ & +\left\langle \int dt'\int dt''\frac{\Gamma^2}{(\Gamma+\kappa)^2}(\omega_k-\omega_A)^2\right\rangle + \left\langle \phi_k^\dagger(0)\phi_k(0)\right\rangle \\ & -\frac{2}{a_0^*(\omega_k)(\Gamma+\kappa)}\left\langle \int dt'\frac{1}{2}(\mathcal{F}(t')-\mathcal{F}^\dagger(t'))\phi_{k'}(0)\right\rangle - \left\langle \phi_k^\dagger(0)\phi_{k'}(0)\right\rangle \\ & -\frac{2}{a_0(\omega_k)(\Gamma+\kappa)}\left\langle \phi_{k'}^\dagger(0)\int dt'\frac{1}{2}(\mathcal{F}(t')-\mathcal{F}^\dagger(t'))\right\rangle - \left\langle \phi_{k'}^\dagger(0)\phi_k(0)\right\rangle \\ & +\left\langle \phi_{k'}^\dagger(0)\phi_{k'}(0)\right\rangle\end{aligned} \tag{E.14}$$

If $\left\langle \left(\int dt'\frac{1}{2}(\mathcal{F}(t')-\mathcal{F})^\dagger(t')\right)\phi_k(0)\right\rangle = \left\langle \int dt'\frac{1}{2}(\mathcal{F}(t')-\mathcal{F}^\dagger(t'))\phi_{k'}(0)\right\rangle = 0$, there is no correlation at $t=0$, then

$$\left\langle (\phi_k(t) - \phi_{k'}(0))^2 \right\rangle = \frac{4}{|a_0(\omega_k)|^2(\Gamma+\kappa)^2} \left\langle \left(\int dt' \frac{1}{2}(\mathcal{F}(t') - \mathcal{F}^\dagger(t')) \right)^\dagger \left(\int dt'' \frac{1}{2}(\mathcal{F}(t'') - \mathcal{F}^\dagger(t'')) \right) \right\rangle$$
$$+ \left\langle \int dt' \int dt'' \frac{\Gamma^2}{(\Gamma+\kappa)^2}(\omega_k - \omega_A)^2 \right\rangle + \left\langle \phi_k^\dagger(0)\phi_k(0) \right\rangle$$
$$- \left\langle \phi_k^\dagger(0)\phi_{k'}(0) \right\rangle - \left\langle \phi_{k'}^\dagger(0)\phi_k(0) \right\rangle + \left\langle \phi_{k'}^\dagger(0)\phi_{k'}(0) \right\rangle \quad \text{(E.15)}$$

If we assume that $\left\langle \phi_k^\dagger(0)\phi_{k'}(0) \right\rangle = \left\langle \phi_{k'}^\dagger(0)\phi_k(0) \right\rangle$ and that $\left\langle \phi_{k'}^\dagger(0)\phi_{k'}(0) \right\rangle + \left\langle \phi_k^\dagger(0)\phi_k(0) \right\rangle$, the we have

$$\left\langle (\phi_k(t) - \phi_{k'}(0))^2 \right\rangle = \frac{4}{|a_0(\omega_k)|^2(\Gamma+\kappa)^2} \left\langle \left(\int dt' \frac{1}{2}(\mathcal{F}(t') - \mathcal{F}^\dagger(t')) \right)^\dagger \left(\int dt'' \frac{1}{2}(\mathcal{F}(t'') - \mathcal{F}^\dagger(t'')) \right) \right\rangle$$
$$+ \left\langle \int dt' \int dt'' \frac{\Gamma^2}{(\Gamma+\kappa)^2}(\omega_k - \omega_A)^2 \right\rangle + 2\left\langle \phi_k^\dagger(0)\phi_k(0) \right\rangle \quad \text{(E.16)}$$

$$\left\langle (\phi_k(t) - \phi_{k'}(0))^2 \right\rangle = \frac{1}{|a_0(\omega_k)|^2(\Gamma+\kappa)^2} \left\langle \left(\int dt' (\mathcal{F}(t') - \mathcal{F}^\dagger(t')) \right)^\dagger \left(\int dt'' (\mathcal{F}(t'') - \mathcal{F}^\dagger(t'')) \right) \right\rangle$$
$$+ \frac{\Gamma^2}{(\Gamma+\kappa)^2}(\omega_k - \omega_A)^2 t^2 + 2\left\langle \phi_k^\dagger(0)\phi_k(0) \right\rangle \quad \text{(E.17)}$$

E.2. Fluctuation Operator two times correlation

Let's consider a Fluctuation Operator of the form:

$$\mathcal{F}(t) = e^{i\phi_k(t)} \left[\dot{\mathcal{F}}_\kappa - ig^*(\omega_k)N\mathcal{F}_{(-)} - \frac{\Gamma}{2}\mathcal{F}_\kappa \right], \quad \text{(E.18)}$$

where $\mathcal{F}_\kappa$ and $\mathcal{F}_{(-)}$ are the noise operators of equations of the bosonic and coherence equations of motion 3.36.

Then, the two times correlation of the noise operator

$$\int dt' \int dt'' \left\langle \mathcal{F}^\dagger(t')\mathcal{F}(t'') \right\rangle = \int dt' \int dt'' \left\langle \left(\dot{\mathcal{F}}_\kappa - ig^*(\omega_k)N\mathcal{F}_{(-)} - \frac{\Gamma}{2}\mathcal{F}_\kappa \right)^\dagger \left(\dot{\mathcal{F}}_\kappa - ig^*(\omega_k)N\mathcal{F}_{(-)} - \frac{\Gamma}{2}\mathcal{F}_\kappa \right) \right\rangle \quad \text{(E}$$

$$\int dt' \int dt'' \left\langle \mathcal{F}^\dagger(t')\mathcal{F}(t'')\right\rangle = \int dt' \int dt'' \left\langle \left(ig(\omega_k)N\mathcal{F}_{(+)} - \frac{\Gamma}{2}\mathcal{F}_\kappa^\dagger \right) \left(-ig^*(\omega_k)N\mathcal{F}_{(-)} - \frac{\Gamma}{2}\mathcal{F}_\kappa \right) \right\rangle \tag{E.20}$$

$$\int dt' \int dt'' \left\langle \mathcal{F}^\dagger(t')\mathcal{F}(t'')\right\rangle = \int dt' \int dt'' \left[|g(\omega_k)|^2 N^2 \left\langle \mathcal{F}_{(+)}(t')\mathcal{F}_{(-)}(t'')\right\rangle + \frac{\Gamma^2}{4}\left\langle \mathcal{F}_\kappa^\dagger(t')\mathcal{F}_\kappa(t'')\right\rangle \right] \tag{E.21}$$

E.2.1. Coherence Noise Operator correlation

$$\begin{aligned} \mathcal{F}_{(-)} = &\ \mathcal{F}_{(-)}^{\mathcal{R}} + \mathcal{F}_{(-)}^{\mathcal{P}} \\ = &\ -i\int d\omega_\mu \kappa(\omega_\mu) N S^z r_\mu(t_0) e^{i(\omega_\mu - \omega_A)t} - i\int d\omega_\beta \tilde{\kappa}(\omega_\beta)\mathcal{P}_\beta(t_0) N S^z e^{-i(\omega_\beta-\omega_A)t} \end{aligned} \tag{E.22}$$

Then, see the document New Phi Correlation Fuction

$$\begin{aligned} \left\langle \mathcal{F}_{(+)}(t')\mathcal{F}_{(-)}(t'')\right\rangle = &\ \int d\omega_\mu \int \omega_{\mu'} \kappa^*(\omega_\mu)\kappa(\omega_{k'}) \left\langle r_{t_0}^\dagger(\omega_\mu)\omega_{t_0}(\omega_{\mu'})\right\rangle e^{i(\omega_\mu-\omega_A)t} e^{-i(\omega_{\mu'}-\omega_A)t} \\ &+ \int d\omega_\beta \int d\omega_{\beta'} \tilde{\kappa}(\omega_\beta)\tilde{\kappa}(\omega_{\beta'}) \left\langle \left[\frac{1}{2}(\mathbb{I}_\beta - \mathcal{P}_\beta^z) \right] \right\rangle e^{-i(\omega_\beta-\omega_A)t'} e^{i(\omega_{\beta'}-\omega_A)t''} \\ = &\ \int d\omega_\mu \int \omega_{\mu'} \kappa^*(\omega_\mu)\kappa(\omega_{k'})\delta(\omega_\mu - \omega_{\mu'})\bar{n}(\omega_\mu) e^{i(\omega_\mu-\omega_A)t'} e^{-i(\omega_{\mu'}-\omega_A)t''} \\ &+ \int d\omega_\beta \int d\omega_{\beta'} \tilde{\kappa}(\omega_\beta)\tilde{\kappa}(\omega_{\beta'}) \frac{1}{2}\delta(\omega_\beta - \omega_{\beta'}) e^{-i(\omega_\beta-\omega_A)t} e^{i(\omega_{\beta'}-\omega_A)t} \\ = &\ \int d\omega_\mu |\kappa(\omega_\mu)|^2 \bar{n}(\omega_\mu) e^{i(\omega_\mu-\omega_A)(t'-t'')} + \frac{1}{2}\int d\omega_\beta |\tilde{\kappa}(\omega_\beta)|^2 e^{i(\omega_\beta-\omega_A)(t'-t'')} \end{aligned} \tag{E.23}$$

The term $\bar{n}(\omega_\mu) = 0$ in the optical regime.

$$\left\langle \mathcal{F}_{(+)}(t')\mathcal{F}_{(-)}(t'')\right\rangle = \frac{1}{2}\int d\omega_\beta |\tilde{\kappa}(\omega_\beta)|^2 e^{-i(\omega_\beta-\omega_A)(t'-t'')} \tag{E.24}$$

Meanwhile the conjugate

$$\left\langle \mathcal{F}_{(-)}(t')\mathcal{F}_{(+)}(t'')\right\rangle = \int d\omega_\mu |\kappa(\omega_\mu)|^2 \left(\bar{n}(\omega_\mu)+1\right) e^{i(\omega_\mu-\omega_A)(t'-t'')} + \frac{1}{2}\int d\omega_\beta |\tilde{\kappa}(\omega_\beta)|^2 e^{-i(\omega_\beta-\omega_A)(t'-t'')} \tag{E.25}$$

$$\left\langle \mathcal{F}_{(-)}(t')\mathcal{F}_{(+)}(t'')\right\rangle = \int d\omega_\mu |\kappa(\omega_\mu)|^2 e^{i(\omega_\mu-\omega_A)(t'-t'')} + \frac{1}{2}\int d\omega_\beta |\tilde{\kappa}(\omega_\beta)|^2 e^{-i(\omega_\beta-\omega_A)(t'-t'')} \tag{E.26}$$

E.2.2. Bosonic Noise Operator Correlation

Now, we will study the bosonic noise operator

$$\mathcal{F}_\kappa(\omega_k) = -i\int d\omega_\mu \zeta(\omega_k,\omega_\mu) r_{t_0}(\omega_\mu) e^{i(\omega_\mu-\omega_k)t} \tag{E.27}$$

Then, the two times correlations

$$\left\langle \mathcal{F}_\kappa^\dagger(t')\mathcal{F}_\kappa(t'')\right\rangle = \int d\omega_\mu \int d\omega_{\mu'} \zeta^*(\omega_k,\omega_\mu)\zeta(\omega_k,\omega_{\mu'}) \left\langle r_{t_0}^\dagger(\omega_\mu) r_{t_0}(\omega_{\mu'})\right\rangle e^{-i(\omega_\mu-\omega_k)t'} e^{i(\omega_{\mu'}-\omega_k)t''} \tag{E.28}$$

$$\left\langle \mathcal{F}_\kappa^\dagger(t')\mathcal{F}_\kappa(t'')\right\rangle = \int d\omega_\mu \int d\omega_{\mu'} \zeta^*(\omega_k,\omega_\mu)\zeta(\omega_k,\omega_{\mu'}) \bar{n}(\omega_\mu)\delta(\omega_\mu-\omega_{\mu'}) e^{-i(\omega_\mu-\omega_k)t'} e^{i(\omega_{\mu'}-\omega_k)t''} = 0 \tag{E.29}$$

While the conjugate

$$\left\langle \mathcal{F}_\kappa(t')\mathcal{F}_\kappa^\dagger(t'')\right\rangle = \int d\omega_\mu \int d\omega_{\mu'} \zeta^*(\omega_k,\omega_\mu)\zeta(\omega_k,\omega_{\mu'}) \left\langle r_{t_0}(\omega_\mu) r_{t_0}^\dagger(\omega_{\mu'})\right\rangle e^{-i(\omega_\mu-\omega_k)t'} e^{i(\omega_{\mu'}-\omega_k)t''} \tag{E.30}$$

$$\begin{aligned} \left\langle \mathcal{F}_\kappa^\dagger(t')\mathcal{F}_\kappa(t'')\right\rangle &= \int d\omega_\mu \int d\omega_{\mu'} \zeta^*(\omega_k,\omega_\mu)\zeta(\omega_k,\omega_{\mu'}) \left(\bar{n}(\omega_\mu)+1\right)\delta(\omega_\mu-\omega_{\mu'}) e^{-i(\omega_\mu-\omega_k)t'} e^{i(\omega_{\mu'}-\omega_k)t''} \\ &= \int d\omega_\mu |\zeta(\omega_k,\omega_\mu)|^2 e^{-i(\omega_\mu-\omega_k)(t'-t'')} \end{aligned} \tag{E.31}$$

$$\left\langle \mathcal{F}_\kappa^\dagger(t')\mathcal{F}_\kappa(t'')\right\rangle = \int d\omega_\mu |\zeta(\omega_k,\omega_\mu)|^2 e^{-i(\omega_\mu-\omega_k)(t'-t'')} \tag{E.32}$$

E.3. Summary of Correlations

$$\left\langle \mathcal{F}_\kappa^\dagger(t')\mathcal{F}_\kappa(t'')\right\rangle = \int d\omega_\mu |\zeta(\omega_k,\omega_\mu)|^2 e^{-i(\omega_\mu-\omega_k)(t'-t'')} \tag{E.33}$$

$$\left\langle \mathcal{F}_{\kappa}^{\dagger}(t')\mathcal{F}_{\kappa}(t'')\right\rangle = 0 \tag{E.34}$$

$$\left\langle \mathcal{F}_{(-)}(t')\mathcal{F}_{(+)}(t'')\right\rangle = \int d\omega_{\mu}|\kappa(\omega_{\mu})|^2 e^{i(\omega_{\mu}-\omega_A)(t'-t'')} + \frac{1}{2}\int d\omega_{\beta}|\tilde{\kappa}(\omega_{\beta})|^2 e^{-i(\omega_{\beta}-\omega_A)(t'-t'')} \tag{E.35}$$

$$\left\langle \mathcal{F}_{(+)}(t')\mathcal{F}_{(-)}(t'')\right\rangle = \frac{1}{2}\int d\omega_{\beta}|\tilde{\kappa}(\omega_{\beta})|^2 e^{-i(\omega_{\beta}-\omega_A)(t'-t'')} \tag{E.36}$$

Replacing in the expression (E.21)

$$\begin{aligned}\int dt'\int dt''\left\langle \mathcal{F}^{\dagger}(t')\mathcal{F}(t'')\right\rangle = & \int dt'\int dt''\left[|g(\omega_k)|^2 N^2\frac{1}{2}\int d\omega_{\beta}|\tilde{\kappa}(\omega_{\beta})|^2 e^{-i(\omega_{\beta}-\omega_A)(t'-t'')}\right] \\ & +\int dt'\int dt''\left[\frac{\Gamma^2}{4}\left\langle \mathcal{F}_{\kappa}^{\dagger}(t')\mathcal{F}_{\kappa}(t'')\right\rangle\right]\end{aligned} \tag{E.37}$$

$$\begin{aligned}\int dt'\int dt''\left\langle \mathcal{F}^{\dagger}(t')\mathcal{F}(t'')\right\rangle = & \frac{1}{2}|g(\omega_k)|^2 N^2\int dt'\int dt''\left[\int d\omega_{\beta}|\tilde{\kappa}(\omega_{\beta})|^2 e^{-i(\omega_{\beta}-\omega_A)(t'-t'')}\right] \\ = & \frac{1}{2}|g(\omega_k)|^2 N^2\int dt''\left[\int d\omega_{\beta}|\tilde{\kappa}(\omega_{\beta})|^2\int dt' e^{-i(\omega_{\beta}-\omega_A)(t'-t'')}\right] \\ = & \frac{1}{2}|g(\omega_k)|^2 N^2\int dt''\left[\int d\omega_{\beta}|\tilde{\kappa}(\omega_{\beta})|^2\left(\pi\delta(\omega_{\beta}-\omega_A)\right)e^{i(\omega_{\beta}-\omega_A)t''}\right] \\ = & \frac{\pi}{2}|g(\omega_k)|^2 N^2\int dt''|\tilde{\kappa}(\omega_A)|^2 \\ = & \frac{\pi}{2}|g(\omega_k)|^2 N^2|\tilde{\kappa}(\omega_A)|^2 t \\ \equiv & |g(\omega_k)|^2 N\frac{w}{2}t\end{aligned} \tag{E.38}$$

And

$$
\begin{aligned}
\int dt' \int dt'' \left\langle \mathcal{F}(t')\mathcal{F}^\dagger(t'')\right\rangle = \quad & \int dt' \int dt'' \left[|g(\omega_k)|^2 N^2 \left\langle \mathcal{F}_{(-)}(t')\mathcal{F}_{(+)}(t'')\right\rangle\right] \\
& + \int dt' \int dt'' \left[\frac{\Gamma^2}{4}\left\langle \mathcal{F}_\kappa(t')\mathcal{F}_\kappa^\dagger(t'')\right\rangle\right] \\
= \quad & \int dt' \int dt'' \left[|g(\omega_k)|^2 N^2 \left(\int d\omega_\mu |\kappa(\omega_\mu)|^2 e^{i(\omega_\mu-\omega_A)(t'-t'')}\right)\right] \\
& \int dt' \int dt'' \left[|g(\omega_k)|^2 N^2 \left(\frac{1}{2}\int d\omega_\beta |\tilde{\kappa}(\omega_\beta)|^2 e^{-i(\omega_\beta-\omega_A)(t'-t'')}\right)\right] \\
& + \int dt' \int dt'' \left[\frac{\Gamma^2}{4}\int d\omega_\mu |\bar{\kappa}(\omega_k,\omega_\mu)|^2 e^{-i(\omega_\mu-\omega_k)(t'-t'')}\right] \\
= \quad & |g(\omega_k)|^2 N\gamma t + g(\omega_k)|^2 N\frac{w}{2}t \\
& + \int dt'' \left[\frac{\Gamma^2}{4}\int d\omega_\mu |\bar{\kappa}(\omega_k,\omega_\mu)|^2 \int dt' e^{-i(\omega_\mu-\omega_k)(t'-t'')}\right] \\
= \quad & |g(\omega_k)|^2 N\left(\gamma + \frac{w}{2}\right) t \\
& + \int dt'' \left[\frac{\Gamma^2}{4}\int d\omega_\mu |\bar{\kappa}(\omega_k,\omega_\mu)|^2 \delta(\omega_k-\omega_\mu) e^{i(\omega_\mu-\omega_k)t''}\right] \\
= \quad & |g(\omega_k)|^2 N\left(\gamma + \frac{w}{2}\right) + t\int dt'' \left[\frac{\Gamma^2}{4}|\bar{\kappa}(\omega_k,\omega_\mu)|^2\right] \\
= \quad & |g(\omega_k)|^2 N\left(\gamma + \frac{w}{2}\right) t + \frac{\Gamma^2}{4}\pi|\bar{\kappa}(\omega_k,\omega_k)|^2 t
\end{aligned}
\tag{E.39}
$$

$$
\int dt' \int dt'' \left\langle \mathcal{F}(t')\mathcal{F}^\dagger(t'')\right\rangle = \quad |g(\omega_k)|^2 N\left(\gamma + \frac{w}{2}\right) t + \frac{\Gamma^2}{4}\pi|\bar{\kappa}(\omega_k,\omega_k)|^2 t \tag{E.40}
$$

Annex F

Heisenberg-Langevin Equations for the reservoir operators in a Laser on a waveguide QED

F.1. Heisenberg Equations of the Reservoir Operators

The evolution equation of the bosonic and the reservoir operators[18]:

$$\dot{a}_k = -i \sum_i g_k^* \sigma_i^- e^{i(\omega_k - \omega_A)t} \tag{F.1}$$

$$\dot{r}_\alpha = -i \sum_i \kappa_{\alpha,i}^* \sigma_i^- e^{i(\omega_\alpha - \omega_A)t} \tag{F.2}$$

The Heisenberg evolution of the raising and lowering pumping operators:

$$\dot{\mathcal{P}}_\mu^- = -i\tilde{\kappa}_{\mu,j} \mathcal{P}_\mu^z \sigma_j^+ e^{-i(\omega_\mu - \omega_A)t} \tag{F.3a}$$

$$\dot{\mathcal{P}}_\mu^+ = i\tilde{\kappa}_{\mu,j} \sigma_j^- \mathcal{P}_\mu^z e^{i(\omega_\mu - \omega_A)t} \tag{F.3b}$$

Finally the inversion evolution equation

$$\dot{\mathcal{P}}_\mu^z = 2i \left(\tilde{\kappa}_{\mu,j}^* \mathcal{P}_\mu^+ \sigma_j^+ e^{-i(\omega_\mu - \omega_A)t} - \tilde{\kappa}_{\mu,j} \sigma_j^- \mathcal{P}_\mu^- e^{i(\omega_\mu - \omega_A)t} \right) \tag{F.4}$$

F.1.1. Solutions of Heisenberg Equations

The solution of equation (F.2):

$$\dot{r}_\alpha = r_\alpha(t_0) - i \int_{t_0}^t \sum_i \kappa_{\alpha,i}^* \sigma_i^-(t') e^{i(\omega_\alpha - \omega_A)t} dt' \tag{F.5}$$

The evolution of the bosonic operators:

$$\dot{a}_k(t) = a_k(t_0) - i g_k^* \int_{t_0}^t \sum_i \sigma_i^-(t') e^{i(\omega_k - \omega_A)t'} dt' \tag{F.6}$$

The solutions of equations (F.3):

$$\mathcal{P}_\mu^-(t) = \mathcal{P}_\mu^-(t_0) - i\int_{t_0}^{t} \kappa_{\mu,j}^* \mathcal{P}_\mu^z(t')\sigma_j^+(t')e^{-i(\omega_\mu-\omega_A)t'}dt' \tag{F.7a}$$

$$\mathcal{P}_\mu^+(t) = \mathcal{P}_\mu^+(t_0) + i\int_{t_0}^{t} \sum_j \kappa_{\mu,j}\sigma_j^-(t')\mathcal{P}_\mu^z(t')e^{i(\omega_\mu-\omega_A)t'}dt' \tag{F.7b}$$

And the evolutions of the inversion:

$$\mathcal{P}_{\mu,j}^z(t) = \mathcal{P}_{\mu,j}^z(t_0) + 2i\int_{t_0}^{t} \sum_j \tilde{\kappa}_{\mu,j}^* \mathcal{P}_{\mu,j}^+(t')\sigma_j^+ e^{-i(\omega_{\mu,j}-\omega_A)t'}dt' - 2i\int_{t_0}^{t} \sum_j \tilde{\kappa}_{\mu,j}\sigma_j^- \mathcal{P}_{\mu,j}^-(t')e^{i(\omega_{\mu,j}-\omega_A)t'}dt' \tag{F.8}$$

F.2. Derivation of the Equations of Motion of the Atomic Operators

F.2.1. Equation of Motion of Inversion

This section calculations are base on [23, 44]

F.2.1.1. Evolution generated by the reservoir R

$$\dot{\sigma_j^z} = \frac{-i}{\hbar}\left[\sigma_j^z, H_I\right] = \frac{-i}{\hbar}\left[\sigma_j^z, H_I^R + H_I^{\mathcal{P}} + H_I^{1D}\right] \tag{F.9}$$

$$\begin{aligned}
\frac{-i}{\hbar}\left[\sigma_j^z, H_I^R\right] = &\quad \frac{-i}{\hbar}\sum_{\mu,i}\hbar\left(\kappa_{\mu,i}^* r_\mu^\dagger\left[\sigma_j^z,\sigma_i^-\right]e^{i(\omega_\mu-\omega_A)t} + \kappa_{\mu,i}\left[\sigma_j^z,\sigma_i^+\right]r_\mu e^{-i(\omega_\mu-\omega_A)t}\right)\\
= &\quad -i\sum_{\mu,i}\left(\kappa_{\mu,i}^* r_\mu^\dagger(-2\delta_{i,j}\sigma_i^-)e^{i(\omega_\mu-\omega_A)t} + \kappa_{\mu,i}(2\delta_{i,j}\sigma_i^+)r_\mu e^{-i(\omega_\mu-\omega_A)t}\right)\\
= &\quad 2i\sum_\mu\left(\kappa_{\mu,j}^* r_\mu^\dagger\sigma_j^- e^{i(\omega_\mu-\omega_A)t} - \kappa_{\mu,j}\sigma_j^+ r_\mu e^{-i(\omega_\mu-\omega_A)t}\right)
\end{aligned} \tag{F.10}$$

Replacing r_μ and $r_\mu^\dagger$ by the solution of the equation of motion:

$$\begin{aligned}
\frac{-i}{\hbar}\left[\sigma_j^z, H_I^R\right] = &\quad 2i\sum_\mu\left[\kappa_{\mu,j}^*\left(r_\mu^\dagger(t_0) + i\int_{t_0}^{t}\sum_i \kappa_{\mu,i}\sigma_i^+(t')e^{-i(\omega_\mu-\omega_A)t}dt'\right)\sigma_j^- e^{i(\omega_\mu-\omega_A)t}\right]\\
&-2i\sum_\mu\left[\kappa_{\mu,j}\sigma_j^+\left(r_\mu(t_0) - i\int_{t_0}^{t}\sum_i \kappa_{\mu,i}^*\sigma_i^-(t')e^{i(\omega_\mu-\omega_A)t}dt'\right)e^{-i(\omega_\mu-\omega_A)t}\right]
\end{aligned} \tag{F.11}$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[\sigma_j^z, H_I^R\right] = \quad & 2i\sum_{\mu}\left(\kappa_{\mu,j}^* r_\mu^\dagger \sigma_j^- e^{i(\omega_\mu-\omega_A)t} - \kappa_{\mu,j}\sigma_j^+ r_\mu e^{-i(\omega_\mu-\omega_A)t}\right) \\
& +2i\sum_{\mu}\left[\kappa_{\mu,j}^*\left(i\int_{t_0}^{t}\sum_i \kappa_{\mu,i}\sigma_i^+(t')e^{-i(\omega_\mu-\omega_A)t}dt'\right)\sigma_j^- e^{i(\omega_\mu-\omega_A)t}\right] \\
& -2i\sum_{\mu}\left[\kappa_{\mu,j}\sigma_j^+\left(-i\int_{t_0}^{t}\sum_i \kappa_{\mu,i}^*\sigma_i^-(t')e^{i(\omega_\mu-\omega_A)t}dt'\right)e^{-i(\omega_\mu-\omega_A)t}\right]
\end{aligned}
\tag{F.12}
$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[\sigma_j^z, H_I^R\right] = \quad & 2i\sum_{\mu}\left(\kappa_{\mu,j}^* r_\mu^\dagger(t_0) \sigma_j^- e^{i(\omega_\mu-\omega_A)t} - \kappa_{\mu,j}\sigma_j^+ r_\mu(t_0) e^{-i(\omega_\mu-\omega_A)t}\right) \\
& -2\sum_{\mu}\left[\kappa_{\mu,j}^*\left(\int_{t_0}^{t}\sum_i \kappa_{\mu,i}\sigma_i^+(t')e^{-i(\omega_\mu-\omega_A)t'}dt'\right)\sigma_j^- e^{i(\omega_\mu-\omega_A)t}\right] \\
& -2\sum_{\mu}\left[\kappa_{\mu,j}\sigma_j^+\left(\int_{t_0}^{t}\sum_i \kappa_{\mu,i}^*\sigma_i^-(t')e^{i(\omega_\mu-\omega_A)t'}dt'\right)e^{-i(\omega_\mu-\omega_A)t}\right]
\end{aligned}
\tag{F.13}
$$

Definig the noise operator gererated by the reservoir R.

$$
\mathcal{F}^R = \quad 2i\textstyle\sum_{\mu}\left(\kappa_{\mu,j}^* r_\mu^\dagger(t_0) \sigma_j^- e^{i(\omega_\mu-\omega_A)t} - \kappa_{\mu,j}\sigma_j^+ r_\mu(t_0) e^{-i(\omega_\mu-\omega_A)t}\right)
\tag{F.14}
$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[\sigma_j^z, H_I^R\right] = \quad & \mathcal{F}^R - 2\sum_{\mu,i}\left[\kappa_{\mu,j}^*\kappa_{\mu,i}\left(\int_{t_0}^{t}\sigma_i^+(t')e^{-i(\omega_\mu-\omega_A)t}dt'\right)\sigma_j^- e^{i(\omega_\mu-\omega_A)t}\right] \\
& -2\sum_{\mu,i}\left[\kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_j^+\left(\int_{t_0}^{t}\sigma_i^-(t')e^{i(\omega_\mu-\omega_A)t}dt'\right)e^{-i(\omega_\mu-\omega_A)t}\right]
\end{aligned}
\tag{F.15}
$$

If $\sigma_i^{\pm}(t') \rightarrow \sigma_i^{\pm}(t)$:

$$
\begin{aligned}
\frac{-i}{\hbar}\left[\sigma_j^z, H_I^R\right] = \quad & \mathcal{F}^R - 2\sum_{\mu,i}\left[\kappa^*_{\mu,j}\kappa_{\mu,i}\left(\int_{t_0}^t \sigma_i^+(t')e^{-i(\omega_\mu-\omega_A)t}dt'\right)\sigma_j^- e^{i(\omega_\mu-\omega_A)t}\right] \\
& -2\sum_{\mu,i}\left[\kappa_{\mu,j}\kappa^*_{\mu,i}\sigma_j^+\left(\int_{t_0}^t \sigma_i^-(t')e^{i(\omega_\mu-\omega_A)t}dt'\right)e^{-i(\omega_\mu-\omega_A)t}\right] \\
= \quad & \mathcal{F}^R - 2\sum_{\mu,i}\left[\kappa^*_{\mu,j}\kappa_{\mu,i}\sigma_i^+(t)\left(\int_{t_0}^t e^{-i(\omega_\mu-\omega_A)t}dt'\right)\sigma_j^- e^{i(\omega_\mu-\omega_A)t}\right] \\
& -2\sum_{\mu,i}\left[\kappa_{\mu,j}\kappa^*_{\mu,i}\sigma_j^+\sigma_i^-(t)\left(\int_{t_0}^t e^{i(\omega_\mu-\omega_A)t}dt'\right)e^{-i(\omega_\mu-\omega_A)t}\right]
\end{aligned}
\tag{F.16}
$$

Where, if $t - t_0 \to \infty$:

$$
\lim_{t\to\infty}\int_0^t dt' e^{i(\omega-\omega_0)t'} = \pi\delta(\omega-\omega_0) + i\frac{P}{\omega_0-\omega}
\tag{F.17}
$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[\sigma_j^z, H_I^R\right] = \quad & \mathcal{F}^R - 2\sum_{\mu,i}\left[\kappa^*_{\mu,j}\kappa_{\mu,i}\sigma_i^+(t)\left(\pi\delta(\omega_\mu-\omega_A) - i\frac{P}{\omega_\mu-\omega_A}\right)\sigma_j^- e^{i(\omega_\mu-\omega_A)t}\right] \\
& -2\sum_{\mu,i}\left[\kappa_{\mu,j}\kappa^*_{\mu,i}\sigma_j^+\sigma_i^-(t)\left(\pi\delta(\omega_\mu-\omega_A) + i\frac{P}{\omega_\mu-\omega_A}\right)e^{-i(\omega_\mu-\omega_A)t}\right]
\end{aligned}
\tag{F.18}
$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[\sigma_j^z, H_I^R\right] = \quad & \mathcal{F}^R - 2\sum_{\mu,i}\left[\kappa^*_{\mu,j}\kappa_{\mu,i}\sigma_i^+(t)\left(\pi\delta(\omega_\mu-\omega_A) - i\frac{P}{\omega_\mu-\omega_A}\right)\sigma_j^- e^{i(\omega_\mu-\omega_A)t}\right] \\
& -2\sum_{\mu,i}\left[\kappa_{\mu,j}\kappa^*_{\mu,i}\sigma_j^+\sigma_i^-(t)\left(\pi\delta(\omega_\mu-\omega_A) + i\frac{P}{\omega_\mu-\omega_A}\right)e^{-i(\omega_\mu-\omega_A)t}\right] \\
= \quad & \mathcal{F}^R - 2\sum_{\mu,i}\left[\kappa^*_{\mu,j}\kappa_{\mu,i}\sigma_i^+(t)\left(\pi\delta(\omega_\mu-\omega_A)\right)\sigma_j^- e^{i(\omega_\mu-\omega_A)t}\right] \\
& -2\sum_{\mu,i}\left[\kappa_{\mu,j}\kappa^*_{\mu,i}\sigma_j^+\sigma_i^-(t)\left(\pi\delta(\omega_\mu-\omega_A)\right)e^{-i(\omega_\mu-\omega_A)t}\right] \\
& -2\sum_{\mu,i}\left[\kappa^*_{\mu,j}\kappa_{\mu,i}\sigma_i^+(t)\left(-i\frac{P}{\omega_\mu-\omega_A}\right)\sigma_j^- e^{i(\omega_\mu-\omega_A)t}\right] \\
& -2\sum_{\mu,i}\left[\kappa_{\mu,j}\kappa^*_{\mu,i}\sigma_j^+\sigma_i^-(t)\left(i\frac{P}{\omega_\mu-\omega_A}\right)e^{-i(\omega_\mu-\omega_A)t}\right]
\end{aligned}
\tag{F.19}
$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[\sigma_j^z, H_I^R\right] = \quad & \mathcal{F}^R - 2\sum_{\mu,i}\left[\kappa_{\mu,j}^*\kappa_{\mu,i}\sigma_i^+(t)\left(\pi\delta(\omega_\mu-\omega_A)\right)\sigma_j^- e^{i(\omega_\mu-\omega_A)t}\right] \\
& -2\sum_{\mu,i}\left[\kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_j^+\sigma_i^-(t)\left(\pi\delta(\omega_\mu-\omega_A)\right)e^{-i(\omega_\mu-\omega_A)t}\right] \\
& -2\sum_{\mu,i}\left[\kappa_{\mu,j}^*\kappa_{\mu,i}\sigma_i^+(t)\left(-i\frac{P}{\omega_\mu-\omega_A}\right)\sigma_j^- e^{i(\omega_\mu-\omega_A)t}\right] \\
& -2\sum_{\mu,i}\left[\kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_j^+\sigma_i^-(t)\left(i\frac{P}{\omega_\mu-\omega_A}\right)e^{-i(\omega_\mu-\omega_A)t}\right]
\end{aligned}
\tag{F.20}
$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[\sigma_j^z, H_I^R\right] = \quad & \mathcal{F}^R - 2\pi\sum_{\mu,i}\delta(\omega_\mu-\omega_A)\left[\kappa_{\mu,j}^*\kappa_{\mu,i}\sigma_i^+\sigma_j^- e^{i(\omega_\mu-\omega_A)t} + \kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_j^+\sigma_i^- e^{-i(\omega_\mu-\omega_A)t}\right] \\
& +2\sum_{\mu,i}\left(i\frac{P}{\omega_\mu-\omega_A}\right)\left[\kappa_{\mu,j}^*\kappa_{\mu,i}\sigma_i^+\sigma_j^- e^{i(\omega_\mu-\omega_A)t} + \kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_j^+\sigma_i^- e^{-i(\omega_\mu-\omega_A)t}\right]
\end{aligned}
\tag{F.21}
$$

Defining the collective operator:

$$
S^z = \frac{1}{N}\sum_j \sigma_j^z \tag{F.22}
$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[NS^z, H_I^R\right] = \quad & \frac{-i}{\hbar}\sum_j\left[\sigma_j^z, H_I^R\right] \\
= \quad & \tilde{\mathcal{F}}^R - 2\pi\sum_{\mu,i,j}\delta(\omega_\mu-\omega_A)\left[\kappa_{\mu,j}^*\kappa_{\mu,i}\sigma_i^+\sigma_j^- e^{i(\omega_\mu-\omega_A)t} + \kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_i^+\sigma_j^- e^{-i(\omega_\mu-\omega_A)t}\right] \\
& +2i\sum_{\mu,i,j}\left(\frac{P}{\omega_\mu-\omega_A}\right)\left[\kappa_{\mu,j}^*\kappa_{\mu,i}\sigma_i^+\sigma_j^- e^{i(\omega_\mu-\omega_A)t} - \kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_i^+\sigma_j^- e^{-i(\omega_\mu-\omega_A)t}\right]
\end{aligned}
\tag{F.23}
$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[NS^z, H_I^R\right] = \quad & \tilde{\mathcal{F}}^R - 2\pi\sum_{\mu}\sum_{i,j}\delta(\omega_\mu-\omega_A)\left[\kappa_{\mu,j}^*\kappa_{\mu,i}\sigma_i^+\sigma_j^- e^{i(\omega_\mu-\omega_A)t} + \kappa_{\mu,j}^*\kappa_{\mu,i}\sigma_j^+\sigma_i^- e^{-i(\omega_\mu-\omega_A)t}\right] \\
& +2i\sum_{\mu}\sum_{i,j}\left(\frac{P}{\omega_\mu-\omega_A}\right)\left[\kappa_{\mu,j}^*\kappa_{\mu,i}\sigma_i^+\sigma_j^- e^{i(\omega_\mu-\omega_A)t} - \kappa_{\mu,j}^*\kappa_{\mu,i}\sigma_i^+\sigma_j^- e^{-i(\omega_\mu-\omega_A)t}\right]
\end{aligned}
\tag{F.24}
$$

$$\frac{-i}{\hbar}\left[NS^z, H_I^R\right] = \quad \tilde{\mathcal{F}}^R - 2\pi\sum_\mu\sum_{i,j}\delta(\omega_\mu-\omega_A)\kappa^*_{\mu,j}\kappa_{\mu,i}\left[e^{i(\omega_\mu-\omega_A)t}+e^{-i(\omega_\mu-\omega_A)t}\right]\sigma_i^+\sigma_j^-$$
$$+2i\sum_\mu\sum_{i,j}\left(\frac{P}{\omega_\mu-\omega_A}\right)\kappa^*_{\mu,j}\kappa_{\mu,i}\left[e^{i(\omega_\mu-\omega_A)t}-e^{-i(\omega_\mu-\omega_A)t}\right]\sigma_i^+\sigma_j^- \tag{F.25}$$

If we consider $\sum_\mu \to \sum_m \int d\omega_\mu$

$$\frac{-i}{\hbar}\left[NS^z, H_I^R\right] = \quad \tilde{\mathcal{F}}^R - 2\pi\sum_m\int d\omega_\mu\sum_{i,j}\delta(\omega_\mu-\omega_A)\kappa^*_{\mu,j}\kappa_{\mu,i}\left[e^{i(\omega_\mu-\omega_A)t}+e^{-i(\omega_\mu-\omega_A)t}\right]\sigma_i^+\sigma_j^-$$
$$+2i\sum_m\int d\omega_\mu\sum_{i,j}\left(\frac{P}{\omega_\mu-\omega_A}\right)\kappa^*_{\mu,j}\kappa_{\mu,i}\left[e^{i(\omega_\mu-\omega_A)t}-e^{-i(\omega_\mu-\omega_A)t}\right]\sigma_i^+\sigma_j^- \tag{F.26}$$

$$\frac{-i}{\hbar}\left[NS^z, H_I^R\right] = \quad \tilde{\mathcal{F}}^R - 4\pi\sum_m\sum_{i,j}\kappa^*_{m,j}(\omega_A)\kappa_{m,i}(\omega_A)\sigma_i^+\sigma_j^-$$
$$+2i\sum_m\int d\omega_\mu\sum_{i,j}\left(\frac{P}{\omega_\mu-\omega_A}\right)\kappa^*_{m,j}\kappa_{m,i}\left[e^{i(\omega_\mu-\omega_A)t}-e^{-i(\omega_\mu-\omega_A)t}\right]\sigma_i^+\sigma_j^- \tag{F.27}$$

Definig $\gamma_{i,j}^{(R)} = 2\pi\sum_m \kappa^*_{m,j}(\omega_a)\kappa_{m,i}(\omega_a)$ and $\Omega_{i,j}^{(R)}$the shift generated by the reservoir R:

$$\Omega_{i,j}^{(R)} = \quad \sum_m\int d\omega_\mu\left(\frac{P}{\omega_\mu-\omega_A}\right)\kappa^*_{m,j}\kappa_{m,i}\left[e^{i(\omega_\mu-\omega_A)t}-e^{-i(\omega_\mu-\omega_A)t}\right] \tag{F.28}$$

Then the evolution generated by the reservoir R:

$$\frac{-i}{\hbar}\left[NS^z, H_I^R\right] = \quad \tilde{\mathcal{F}}^R - 2\sum_{i,j}\gamma_{i,j}^{(R)}\sigma_i^+\sigma_j^- + 2i\sum_{i,j}\Omega_{i,j}^{(R)}\sigma_i^+\sigma_j^- \tag{F.29}$$

But studing the term $\sigma_i^+\sigma_j^-$

$$\sigma_i^+\sigma_j^- = \delta_{i,j}\left[\frac{1}{2}\left(\sigma_j^z+\mathcal{I}_j\right)\right] \tag{F.30}$$

Replacing in the equation (F.29)

$$\begin{aligned}
\frac{-i}{\hbar}\left[NS^z, H_I^R\right] = &\quad \tilde{\mathcal{F}}^R - 2\sum_{i,j}\gamma_{i,j}^{(R)}\sigma_i^+\sigma_j^- + 2i\sum_{i,j}\Omega_{i,j}^{(R)}\sigma_i^+\sigma_j^- \\
= &\quad \tilde{\mathcal{F}}^R - 2\sum_{i,j}\gamma_{i,j}^{(R)}\delta_{i,j}\left[\frac{1}{2}\left(\sigma_j^z + \mathcal{I}_j\right)\right] + 2i\sum_{i,j}\Omega_{i,j}^{(R)}\delta_{i,j}\left[\frac{1}{2}\left(\sigma_j^z + \mathcal{I}_j\right)\right] \\
= &\quad \tilde{\mathcal{F}}^R - \sum_{j}\gamma_{j,j}^{(R)}\left[\sigma_j^z + \mathcal{I}_j\right] + i\sum_{j}\Omega_{j,j}^{(R)}\left[\sigma_j^z + \mathcal{I}_j\right]
\end{aligned} \tag{F.31}$$

$$\frac{-i}{\hbar}\left[NS^z, H_I^R\right] = \quad \tilde{\mathcal{F}}^R - \sum_{j}\gamma_{j,j}^{(R)}\left[\sigma_j^z + \mathcal{I}_j\right] + i\sum_{j}\Omega_{j,j}^{(R)}\left[\sigma_j^z + \mathcal{I}_j\right] \tag{F.32}$$

Defining $\gamma^{(R)} = \gamma_{j,j}^{(R)}$ and $\Omega^{(R)} = \Omega_{j,j}^{(R)}$for all j:

$$\begin{aligned}
\frac{-i}{\hbar}\left[NS^z, H_I^R\right] = &\quad \tilde{\mathcal{F}}^R - \gamma^{(R)}\sum_{j}\left[\sigma_j^z + \mathcal{I}_j\right] + i\Omega^{(R)}\sum_{j}\left[\sigma_j^z + \mathcal{I}_j\right] \\
= &\quad \tilde{\mathcal{F}}^R - \gamma^{(R)}\left[NS^z + N\right] + i\Omega^{(R)}\left[NS^z + N\right]
\end{aligned} \tag{F.33}$$

$$\dot{S}^z_{(R)} = \frac{-i}{\hbar}\left[S^z, H_I^R\right] = \quad \tilde{\mathcal{F}}^R - \gamma^{(R)}\left[S^z + 1\right] + i\Omega^{(R)}\left[S^z + 1\right] \tag{F.34}$$

F.2.1.2. Evolution generated by the reservoir $\mathcal{P}$

$$\dot{\sigma_j}^z = \frac{-i}{\hbar}\left[\sigma_j^z, H_I\right] = \frac{-i}{\hbar}\left[\sigma_j^z, H_I^R + H_I^{\mathcal{P}} + H_I^{1D}\right] \tag{F.35}$$

$$\begin{aligned}
\frac{-i}{\hbar}\left[\sigma_j^z, H_I^{\mathcal{P}}\right] = &\quad \frac{-i}{\hbar}\sum_{\beta,i}\hbar\left(\tilde{\kappa}_{\beta,i}^*\mathcal{P}_\beta^+\left[\sigma_j^z, \sigma_i^+\right]e^{-i(\omega_\beta-\omega_A)t} + \tilde{\kappa}_{\beta,i}\left[\sigma_j^z, \sigma_i^-\right]\mathcal{P}_\beta^- e^{i(\omega_\beta-\omega_A)t}\right) \\
= &\quad -i\sum_{\beta,i}\left(\tilde{\kappa}_{\beta,i}^*\mathcal{P}_\beta^-(2\delta_{i,j}\sigma_i^+)e^{-i(\omega_\beta-\omega_A)t} + \tilde{\kappa}_{\beta,i}(-2\delta_{i,j}\sigma_i^-)\mathcal{P}_\beta^+ e^{i(\omega_\beta-\omega_A)t)t}\right) \\
= &\quad -2i\sum_{\beta}\left(\tilde{\kappa}_{\beta,j}^*\mathcal{P}_\beta^+\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t} - \tilde{\kappa}_{\beta,j}\sigma_j^-\mathcal{P}_\beta^- e^{+i(\omega_\beta-\omega_A)t}\right)
\end{aligned} \tag{F.36}$$

Replacing $\mathcal{P}_\beta^+$ and $\mathcal{P}_\beta^-$ by the solution of the equation of motion:

$$\frac{-i}{\hbar}\left[\sigma_j^z, H_I^{\mathcal{P}}\right] = \quad -2i\sum_\beta\left(\tilde{\kappa}^*_{\beta,j}\left(\mathcal{P}^+_\beta(t_0) + i\int_{t_0}^t\sum_i \kappa_{\beta,i}\sigma_i^-(t')\mathcal{P}^z_\beta(t')e^{i(\omega_\beta-\omega_A)t'}dt'\right)\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t}\right)$$
$$+2i\sum_\beta\left(\tilde{\kappa}_{\beta,j}\sigma_j^-\left(\mathcal{P}^-_\beta(t_0) - i\int_{t_0}^t\sum_i \kappa^*_{\beta,i}\mathcal{P}^z_\beta(t')\sigma_i^+(t')e^{-i(\omega_\beta-\omega_A)t'}dt'\right)e^{+i(\omega_\beta-\omega_A)t}\right) \tag{F.37}$$

$$\frac{-i}{\hbar}\left[\sigma_j^z, H_I^{\mathcal{P}}\right] = \quad -2i\sum_\beta\left(\tilde{\kappa}^*_{\beta,j}\mathcal{P}^+_\beta(t_0)\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t} - \tilde{\kappa}_{\beta,j}\sigma_j^-\mathcal{P}^-_\beta(t_0)e^{+i(\omega_\beta-\omega_A)t}\right)$$
$$-2i\sum_\beta\left(\tilde{\kappa}^*_{\beta,j}\left(i\int_{t_0}^t\sum_i \kappa_{\beta,i}\sigma_i^-(t')\mathcal{P}^z_\beta(t')e^{i(\omega_\beta-\omega_A)t'}dt'\right)\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t}\right)$$
$$+2i\sum_\beta\left(\tilde{\kappa}_{\beta,j}\sigma_j^-\left(-i\int_{t_0}^t\sum_i \kappa^*_{\beta,i}\mathcal{P}^z_\beta(t')\sigma_i^+(t')e^{-i(\omega_\beta-\omega_A)t'}dt'\right)e^{i(\omega_\beta-\omega_A)t}\right) \tag{F.38}$$

$$\frac{-i}{\hbar}\left[\sigma_j^z, H_I^{\mathcal{P}}\right] = \quad -2i\sum_\beta\left(\tilde{\kappa}^*_{\beta,j}\mathcal{P}^+_\beta(t_0)\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t} - \tilde{\kappa}_{\beta,j}\sigma_j^-\mathcal{P}^-_\beta(t_0)e^{+i(\omega_\beta-\omega_A)t}\right)$$
$$+2\sum_{\beta,i}\left(\tilde{\kappa}^*_{\beta,j}\tilde{\kappa}_{\beta,i}\left(\int_{t_0}^t \sigma_i^-(t')\mathcal{P}^z_\beta(t')e^{i(\omega_\beta-\omega_A)t'}dt'\right)\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t}\right)$$
$$+2\sum_{\beta,i}\left(\tilde{\kappa}_{\beta,j}\tilde{\kappa}^*_{\beta,i}\sigma_j^-\left(\int_{t_0}^t \mathcal{P}^z_\beta(t')\sigma_i^+(t')e^{-i(\omega_\beta-\omega_A)t'}dt'\right)e^{i(\omega_\beta-\omega_A)t}\right) \tag{F.39}$$

Definig the noise operator gererated by the reservoir $\mathcal{P}$.

$$\mathcal{F}^{\mathcal{P}} = \quad -2i\sum_\beta\left(\tilde{\kappa}^*_{\beta,j}\mathcal{P}^+_\beta(t_0)\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t} - \tilde{\kappa}_{\beta,j}\sigma_j^-\mathcal{P}^-_\beta(t_0)e^{+i(\omega_\beta-\omega_A)t}\right) \tag{F.40}$$

$$\frac{-i}{\hbar}\left[\sigma_j^z, H_I^{\mathcal{P}}\right] = \quad \mathcal{F}^{\mathcal{P}} + 2\sum_{\beta,i}\left(\tilde{\kappa}^*_{\beta,j}\tilde{\kappa}_{\beta,i}\left(\int_{t_0}^t \sigma_i^-(t')\mathcal{P}^z_\beta(t')e^{i(\omega_\beta-\omega_A)t'}dt'\right)\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t}\right)$$
$$+2\sum_{\beta,i}\left(\tilde{\kappa}_{\beta,j}\tilde{\kappa}^*_{\beta,i}\sigma_j^-\left(\int_{t_0}^t \mathcal{P}^z_\beta(t')\sigma_i^+(t')e^{-i(\omega_\beta-\omega_A)t'}dt'\right)e^{i(\omega_\beta-\omega_A)t}\right) \tag{F.41}$$

If $\sigma_i^\pm(t') \to \sigma_i^\pm(t)$ and $\mathcal{P}^z_{\beta,j}(t') \to \mathcal{P}^z_{\beta,j}(t)$:

$$\frac{-i}{\hbar}\left[\sigma_j^z, H_I^{\mathcal{P}}\right] = \quad \mathcal{F}^{\mathcal{P}} + 2\sum_{\beta,i}\left(\tilde{\kappa}^*_{\beta,j}\tilde{\kappa}_{\beta,i}\left(\int_{t_0}^{t} e^{i(\omega_\beta-\omega_A)t'}dt'\right)\mathcal{P}^z_\beta\sigma_i^-\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t}\right)$$
$$+2\sum_{\beta,i}\left(\tilde{\kappa}_{\beta,j}\tilde{\kappa}^*_{\beta,i}\mathcal{P}^z_\beta\sigma_j^-\sigma_i^+\left(\int_{t_0}^{t} e^{-i(\omega_\beta-\omega_A)t'}dt'\right)e^{i(\omega_\beta-\omega_A)t}\right) \tag{F.42}$$

Where, if $t - t_0 \to \infty$:

$$\lim_{t\to\infty}\int_0^t dt' e^{i(\omega-\omega_0)t'} = \pi\delta(\omega-\omega_0) + i\frac{P}{\omega_0-\omega} \tag{F.43}$$

$$\frac{-i}{\hbar}\left[\sigma_j^z, H_I^{\mathcal{P}}\right] = \quad \mathcal{F}^{\mathcal{P}} + 2\sum_{\beta,i}\left(\tilde{\kappa}^*_{\beta,j}\tilde{\kappa}_{\beta,i}\left(\pi\delta(\omega_\beta-\omega_A) + i\frac{P}{\omega_\beta-\omega_A}\right)\mathcal{P}^z_\beta\sigma_i^-\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t}\right)$$
$$+2\sum_{\beta,i}\left(\tilde{\kappa}_{\beta,j}\tilde{\kappa}^*_{\beta,i}\mathcal{P}^z_\beta\sigma_j^-\sigma_i^+\left(\pi\delta(\omega_\beta-\omega_A) - i\frac{P}{\omega_\beta-\omega_A}\right)e^{i(\omega_\beta-\omega_A)t}\right) \tag{F.44}$$

$$\frac{-i}{\hbar}\left[\sigma_j^z, H_I^{\mathcal{P}}\right] = \quad \mathcal{F}^{\mathcal{P}} + 2\sum_{\beta,i}\left(\tilde{\kappa}^*_{\beta,j}\tilde{\kappa}_{\beta,i}\left(\pi\delta(\omega_\beta-\omega_A)\right)\mathcal{P}^z_\beta\sigma_i^-\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t}\right)$$
$$+2\sum_{\beta,i}\left(\tilde{\kappa}_{\beta,j}\tilde{\kappa}^*_{\beta,i}\mathcal{P}^z_\beta\sigma_j^-\sigma_i^+\left(\pi\delta(\omega_\beta-\omega_A)\right)e^{i(\omega_\beta-\omega_A)t}\right)$$
$$+2\sum_{\beta,i}\left(\tilde{\kappa}^*_{\beta,j}\tilde{\kappa}_{\beta,i}\left(+i\frac{P}{\omega_\beta-\omega_A}\right)\mathcal{P}^z_\beta\sigma_i^-\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t}\right)$$
$$+2\sum_{\beta,i}\left(\tilde{\kappa}_{\beta,j}\tilde{\kappa}^*_{\beta,i}\mathcal{P}^z_\beta\sigma_j^-\sigma_i^+\left(-i\frac{P}{\omega_\beta-\omega_A}\right)e^{i(\omega_\beta-\omega_A)t}\right) \tag{F.45}$$

$$\frac{-i}{\hbar}\left[\sigma_j^z, H_I^{\mathcal{P}}\right] = \quad \mathcal{F}^{\mathcal{P}} + 2\sum_{\beta,i}\left(\pi\delta(\omega_\beta-\omega_A)\right)\left(\tilde{\kappa}^*_{\beta,j}\tilde{\kappa}_{\beta,i}\mathcal{P}^z_\beta\sigma_i^-\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t} + \tilde{\kappa}_{\beta,j}\tilde{\kappa}^*_{\beta,i}\mathcal{P}^z_\beta\sigma_j^-\sigma_i^+ e^{i(\omega_\beta-\omega_A)t}\right)$$
$$+2\sum_{\beta,i}\left(i\frac{P}{\omega_\beta-\omega_A}\right)\left(\tilde{\kappa}^*_{\beta,j}\tilde{\kappa}_{\beta,i}\mathcal{P}^z_\beta\sigma_i^-\sigma_j^+ e^{-i(\omega_\beta-\omega_A)t} - \tilde{\kappa}_{\beta,j}\tilde{\kappa}^*_{\beta,i}\mathcal{P}^z_\beta\sigma_j^-\sigma_i^+ e^{i(\omega_\beta-\omega_A)t}\right) \tag{F.46}$$

Defining the collective operator:

$$S^z = \frac{1}{N}\sum_j \sigma_j^z \tag{F.47}$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[NS^z, H_I^{\mathcal{P}}\right] = & \frac{-i}{\hbar}\sum_j \left[\sigma_j^z, H_I^{\mathcal{P}}\right] \\
= & \mathcal{F}^{\mathcal{P}} + 2\pi \sum_{\beta,i,j} \delta(\omega_\beta - \omega_A)\left(\tilde{\kappa}^*_{\beta,j}\tilde{\kappa}_{\beta,i}\mathcal{P}^z_\beta \sigma_i^- \sigma_j^+ e^{-i(\omega_\beta-\omega_A)t} + \tilde{\kappa}_{\beta,j}\tilde{\kappa}^*_{\beta,i}\mathcal{P}^z_\beta \sigma_j^- \sigma_i^+ e^{i(\omega_\beta-\omega_A)t}\right) \\
& + 2\sum_{\beta,i,j}\left(i\frac{P}{\omega_\beta-\omega_A}\right)\left(\tilde{\kappa}^*_{\beta,j}\tilde{\kappa}_{\beta,i}\mathcal{P}^z_\beta \sigma_i^- \sigma_j^+ e^{-i(\omega_\beta-\omega_A)t} - \tilde{\kappa}_{\beta,j}\tilde{\kappa}^*_{\beta,i}\mathcal{P}^z_\beta \sigma_j^- \sigma_i^+ e^{i(\omega_\beta-\omega_A)t}\right)
\end{aligned} \tag{F.48}
$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[NS^z, H_I^{\mathcal{P}}\right] = & \mathcal{F}^{\mathcal{P}} + 2\pi \sum_\beta \sum_{i,j} \delta(\omega_\beta - \omega_A)\tilde{\kappa}^*_{\beta,j}\tilde{\kappa}_{\beta,i}\mathcal{P}^z_\beta \left(e^{-i(\omega_\beta-\omega_A)t} + e^{i(\omega_\beta-\omega_A)t}\right)\sigma_i^-\sigma_j^+ \\
& + 2\sum_\beta\sum_{i,j}\left(i\frac{P}{\omega_\beta-\omega_A}\right)\tilde{\kappa}^*_{\beta,j}\tilde{\kappa}_{\beta,i}\mathcal{P}^z_\beta \left(e^{-i(\omega_\beta-\omega_A)t} - e^{i(\omega_\beta-\omega_A)t}\right)\sigma_i^-\sigma_j^+
\end{aligned} \tag{F.49}
$$

If we consider $\sum_\beta \to \sum_l \int d\omega_\beta$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[NS^z, H_I^{\mathcal{P}}\right] = & \mathcal{F}^{\mathcal{P}} + 2\pi \sum_l \int d\omega_\beta \sum_{i,j} \delta(\omega_\beta - \omega_A)\tilde{\kappa}^*_{l,j}(\omega_\beta)\tilde{\kappa}_{l,i}(\omega_\beta)\mathcal{P}^z_l \left(e^{-i(\omega_\beta-\omega_A)t} + e^{i(\omega_\beta-\omega_A)t}\right)\sigma_i^-\sigma_j^+ \\
& + 2\sum_l \int d\omega_\beta \sum_{i,j}\left(i\frac{P}{\omega_\beta-\omega_A}\right)\tilde{\kappa}^*_{l,j}(\omega_\beta)\tilde{\kappa}_{l,i}(\omega_\beta)\mathcal{P}^z_l \left(e^{-i(\omega_\beta-\omega_A)t} - e^{i(\omega_\beta-\omega_A)t}\right)\sigma_i^-\sigma_j^+
\end{aligned} \tag{F.50}
$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[NS^z, H_I^{\mathcal{P}}\right] = & \mathcal{F}^{\mathcal{P}} + 4\pi \sum_l \sum_{i,j} \tilde{\kappa}^*_{l,j}(\omega_A)\tilde{\kappa}_{l,i}(\omega_A)\mathcal{P}^z_l \sigma_i^-\sigma_j^+ \\
& + 2i\sum_l \int d\omega_\beta \sum_{i,j}\left(\frac{P}{\omega_\beta-\omega_A}\right)\tilde{\kappa}^*_{l,j}(\omega_\beta)\tilde{\kappa}_{l,i}(\omega_\beta)\mathcal{P}^z_l \left(e^{-i(\omega_\beta-\omega_A)t} - e^{i(\omega_\beta-\omega_A)t}\right)\sigma_i^-\sigma_j^+
\end{aligned} \tag{F.51}
$$

Definig $\gamma_{i,j}^{(\mathcal{P})}$ and $\Omega_{i,j}^{(\mathcal{P})}$ the decay rate and the shift generated by the reservoir R, respectively:

$$
\gamma_{i,j}^{(\mathcal{P})} = 2\pi \sum_l \tilde{\kappa}^*_{l,j}(\omega_A)\tilde{\kappa}_{l,i}(\omega_A)\mathcal{P}^z_l \tag{F.52}
$$

$$
\Omega_{i,j}^{(\mathcal{P})} = \quad i\sum_l \int d\omega_\beta \left(\frac{P}{\omega_\beta-\omega_A}\right)\tilde{\kappa}^*_{l,j}(\omega_\beta)\tilde{\kappa}_{l,i}(\omega_\beta)\mathcal{P}^z_l \left(e^{-i(\omega_\beta-\omega_A)t} - e^{i(\omega_\beta-\omega_A)t}\right) \tag{F.53}
$$

Then the evolution generated by the reservoir $\mathcal{P}$:

$$\frac{-i}{\hbar}\left[NS^z, H_I^{\mathcal{P}}\right] = \qquad \tilde{\mathcal{F}}^{\mathcal{P}} + 2\sum_{i,j}\gamma_{i,j}^{(\mathcal{P})}\sigma_i^-\sigma_j^+ + 2i\sum_{i,j}\Omega_{i,j}^{(\mathcal{P})}\sigma_i^-\sigma_j^+ \tag{F.54}$$

But studing the terms $\sigma_i^-\sigma_j^+$ and $\sigma_i^+\sigma_j^-$

$$\sigma_i^-\sigma_j^+ = \delta_{i,j}\left[\frac{1}{2}\left(\mathcal{I}_j - \sigma_j^z\right)\right] \tag{F.55}$$

$$\sigma_i^+\sigma_j^- = \delta_{i,j}\left[\frac{1}{2}\left(\sigma_j^z + \mathcal{I}_j\right)\right] \tag{F.56}$$

Replacing in the equation (F.54)

$$\begin{aligned}\frac{-i}{\hbar}\left[NS^z, H_I^{\mathcal{P}}\right] = &\qquad \tilde{\mathcal{F}}^{\mathcal{P}} + 2\sum_{i,j}\gamma_{i,j}^{(\mathcal{P})}\delta_{i,j}\left[\frac{1}{2}\left(\mathcal{I}_j - \sigma_j^z\right)\right] - 2i\sum_{i,j}\Omega_{i,j}^{(\mathcal{P})}\delta_{i,j}\left[\frac{1}{2}\left(\mathcal{I}_j - \sigma_j^z\right)\right] \\ = &\qquad \tilde{\mathcal{F}}^{\mathcal{P}} + \sum_{j}\gamma_{j,j}^{(\mathcal{P})}\left[\mathcal{I}_j - \sigma_j^z\right] - i\sum_{j}\Omega_{j,j}^{(\mathcal{P})}\left[\mathcal{I}_j - \sigma_j^z\right]\end{aligned} \tag{F.57}$$

Defining $\gamma^{(\mathcal{P})} = \gamma_{j,j}^{(\mathcal{P})}$ and $\Omega^{(\mathcal{P})} = \Omega_{j,j}^{(\mathcal{P})}$ for all j:

$$\begin{aligned}\frac{-i}{\hbar}\left[NS^z, H_I^{\mathcal{P}}\right] = &\qquad \tilde{\mathcal{F}}^{\mathcal{P}} + \gamma^{(\mathcal{P})}\sum_{j}\left[\mathcal{I}_j - \sigma_j^z\right] - i\Omega^{(\mathcal{P})}\sum_{j}\left[\mathcal{I}_j - \sigma_j^z\right] \\ = &\qquad \tilde{\mathcal{F}}^{\mathcal{P}} + \gamma^{(\mathcal{P})}\left[N - NS^z\right] - i\Omega^{(\mathcal{P})}\left[N - NS^z\right]\end{aligned} \tag{F.58}$$

$$\dot{S}^z_{(\mathcal{P})} = \frac{-i}{\hbar}\left[S^z, H_I^{\mathcal{P}}\right] = \qquad \tilde{\mathcal{F}}^{\mathcal{P}} + 2\gamma^{(\mathcal{P})}\left[1 - S^z\right] - i\Omega^{(\mathcal{P})}\left[1 - S^z\right] \tag{F.59}$$

F.2.2. Equation of Motion of Coherence

This section calculations are base on [23, 44].

F.2.2.1. Evolution generated by the reservoir R

$$\begin{aligned}\frac{-i}{\hbar}\left[\sigma_j^-, H_I^R\right] = & \quad \frac{-i}{\hbar}\sum_{\mu,i}\hbar\left(\kappa_{\mu,i}^* r_\mu^\dagger\left[\sigma_j^-,\sigma_i^-\right]e^{i(\omega_\mu-\omega_A)t}+\kappa_{\mu,i}\left[\sigma_j^-,\sigma_i^+\right]r_\mu e^{-i(\omega_\mu-\omega_A)t}\right)\\ = & \quad -i\sum_{\mu,i}\left(\kappa_{\mu,i}(-\delta_{i,j}\sigma_i^z)r_\mu e^{-i(\omega_\mu-\omega_A)t}\right)\\ = & \quad i\sum_{\mu}\left(\kappa_{\mu,j}\sigma_j^z r_\mu e^{-i(\omega_\mu-\omega_A)t}\right)\end{aligned} \tag{F.60}$$

Replacing r_μ and $r_\mu^\dagger$ by the solution of the equation of motion:

$$\begin{aligned}\frac{-i}{\hbar}\left[\sigma_j^-, H_I^R\right] = & \quad i\sum_{\mu}\left(\kappa_{\mu,j}\sigma_j^z\left(r_\mu(t_0)-i\int_{t_0}^t\sum_i\kappa_{\mu,i}^*\sigma_i^-(t')e^{i(\omega_\mu-\omega_A)t}dt'\right)e^{-i(\omega_\mu-\omega_A)t}\right)\\ = & \quad i\sum_{\mu}\left(\kappa_{\mu,j}\sigma_j^z r_\mu(t_0)e^{-i(\omega_\mu-\omega_A)t}-i\left(\int_{t_0}^t\sum_i\kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_j^z\sigma_i^-(t')e^{i(\omega_\mu-\omega_A)t'}dt'\right)e^{-i(\omega_\mu-\omega_A)t}\right)\end{aligned} \tag{F.61}$$

Definig the noise operator gererated by the reservoir R.

$$\mathcal{F}_{(-)}^R = \quad i\textstyle\sum_\mu\left(\kappa_{\mu,j}\sigma_j^z r_\mu(t_0)e^{-i(\omega_\mu-\omega_A)t}\right) \tag{F.62}$$

$$\frac{-i}{\hbar}\left[\sigma_j^-, H_I^R\right] = \quad \mathcal{F}_{(-)}^R+\sum_{\mu,i}\left(\int_{t_0}^t\kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_j^z\sigma_i^-(t')e^{i(\omega_\mu-\omega_A)t'}dt'\right)e^{-i(\omega_\mu-\omega_A)t} \tag{F.63}$$

If $\sigma_i^\pm(t')\to\sigma_i^\pm(t)$:

$$\frac{-i}{\hbar}\left[\sigma_j^-, H_I^R\right] = \quad \mathcal{F}_{(-)}^R+\sum_{\mu,i}\kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_j^z\sigma_i^-\left(\int_{t_0}^t e^{i(\omega_\mu-\omega_A)t'}dt'\right)e^{-i(\omega_\mu-\omega_A)t} \tag{F.64}$$

Where, if $t-t_0\to\infty$:

$$\lim_{t\to\infty}\int_0^t dt' e^{i(\omega-\omega_0)t'} = \pi\delta(\omega-\omega_0)+i\frac{P}{\omega_0-\omega} \tag{F.65}$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[\sigma_j^-, H_I^R\right] = \quad & \mathcal{F}_{(-)}^R + \sum_{\mu,i} \kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_j^z\sigma_i^- \left(\pi\delta(\omega_\mu - \omega_A) - i\frac{P}{\omega_\mu - \omega_A}\right) e^{-i(\omega_\mu - \omega_A)t} \\
= \quad & \mathcal{F}_{(-)}^R + \sum_{\mu,i} \kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_j^z\sigma_i^- \left(\pi\delta(\omega_\mu - \omega_A)\right) e^{-i(\omega_\mu - \omega_A)t} \\
& + \sum_{\mu,i} \kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_j^z\sigma_i^- \left(-i\frac{P}{\omega_\mu - \omega_A}\right) e^{-i(\omega_\mu - \omega_A)t}
\end{aligned}
\tag{F.66}
$$

Defining the collective operator:

$$
S^- = \frac{1}{N}\sum_j \sigma_j^- \tag{F.67}
$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[NS^-, H_I^R\right] = \quad & \frac{-i}{\hbar}\sum_j \left[\sigma_j^z, H_I^R\right] \\
= \quad & \mathcal{F}_{(-)}^R + \sum_{\mu,i,j} \kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_j^z\sigma_i^- \left(\pi\delta(\omega_\mu - \omega_A)\right) e^{-i(\omega_\mu - \omega_A)t} \\
& + \sum_{\mu,i,j} \kappa_{\mu,j}\kappa_{\mu,i}^*\sigma_j^z\sigma_i^- \left(-i\frac{P}{\omega_\mu - \omega_A}\right) e^{-i(\omega_\mu - \omega_A)t}
\end{aligned}
\tag{F.68}
$$

If we consider $\sum_\mu \to \sum_m \int d\omega_\mu$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[NS^-, H_I^R\right] = \quad & \mathcal{F}_{(-)}^R + \sum_m \int d\omega_\mu \sum_{i,j} \kappa_{m,j}(\omega_\mu)\kappa_{\mu,i}^*(\omega_\mu)\sigma_j^z\sigma_i^- \left(\pi\delta(\omega_\mu - \omega_A)\right) e^{-i(\omega_\mu - \omega_A)t} \\
& + \sum_m \int d\omega_\mu \sum_{i,j} \kappa_{m,j}(\omega_\mu)\kappa_{m,i}^*(\omega_\mu)\sigma_j^z\sigma_i^- \left(-i\frac{P}{\omega_\mu - \omega_A}\right) e^{-i(\omega_\mu - \omega_A)t}
\end{aligned}
\tag{F.69}
$$

$$
\begin{aligned}
\frac{-i}{\hbar}\left[NS^-, H_I^R\right] = \quad & \mathcal{F}_{(-)}^R + \pi\sum_{i,j}\sum_m \kappa_{m,j}(\omega_A)\kappa_{\mu,i}^*(\omega_A)\sigma_j^z\sigma_i^- \\
& -i\sum_{i,j}\sum_m \int d\omega_\mu \kappa_{m,j}(\omega_\mu)\kappa_{m,i}^*(\omega_\mu)\left(\frac{P}{\omega_\mu - \omega_A}\right) e^{-i(\omega_\mu - \omega_A)t}\sigma_j^z\sigma_i^-
\end{aligned}
\tag{F.70}
$$

Definig $\gamma_{(-)i,j}^{(R)}$ and $\Omega_{i,j}^{(R)}$ as the decay rate and the shift generated by the reservoir R, respectively:

$$
\gamma_{i,j}^{(R)} = 2\pi\sum_m \kappa_{m,j}(\omega_A)\kappa_{\mu,i}^*(\omega_A) \tag{F.71}
$$

$$\Omega^{(R)}_{(-)i,j} = \quad \sum_m \int d\omega_\mu \kappa_{m,j}(\omega_\mu)\kappa^*_{m,i}(\omega_\mu)\left(\frac{P}{\omega_\mu - \omega_A}\right) e^{-i(\omega_\mu-\omega_A)t} \tag{F.72}$$

$$\frac{-i}{\hbar}\left[NS^-, H_I^R\right] = \quad \mathcal{F}^R_{(-)} + \frac{1}{2}\sum_{i,j}\gamma^{(R)}_{i,j}\sigma^z_j\sigma^-_i - i\sum_{i,j}\Omega^{(R)}_{(-)i,j}\sigma^z_j\sigma^-_i \tag{F.73}$$

But studing the term $\sigma^z_i\sigma^-_j$

$$\sigma^z_i\sigma^-_j = -\delta_{i,j}\left|g\right\rangle_i\left\langle e\right|_j \tag{F.74}$$

$$\begin{aligned}\frac{-i}{\hbar}\left[NS^-, H_I^R\right] = &\quad \mathcal{F}^R_{(-)} + \frac{1}{2}\sum_{i,j}\gamma^{(R)}_{i,j}(-\delta_{i,j}\left|g\right\rangle_i\left\langle e\right|_j) - i\sum_{i,j}\Omega^{(R)}_{(-)i,j}(-\delta_{i,j}\left|g\right\rangle_i\left\langle e\right|_j)\\ = &\quad \mathcal{F}^R_{(-)} - \frac{1}{2}\sum_j\gamma^{(R)}_{j,j}\sigma^-_j + i\sum_j\Omega^{(R)}_{(-)j,j}\sigma^-_j\end{aligned} \tag{F.75}$$

Defining $\gamma^{(R)} = \gamma^{(R)}_{j,j}$ and $\Omega^{(R)} = \Omega^{(R)}_{j,j}$ for all j:

$$\begin{aligned}\frac{-i}{\hbar}\left[NS^-, H_I^R\right] = &\quad \mathcal{F}^R_{(-)} - \frac{1}{2}\sum_j\gamma^{(R)}_{j,j}\sigma^-_j + i\sum_j\Omega^{(R)}_{(-)j,j}\sigma^-_j\\ = &\quad \mathcal{F}^R_{(-)} - \frac{1}{2}\gamma^{(R)}\sum_j\sigma^-_j + i\Omega^{(R)}_{(-)}\sum_j\sigma^-_j\\ = &\quad \mathcal{F}^R_{(-)} - \frac{1}{2}\gamma^{(R)}NS^- + i\Omega^{(R)}_{(-)}NS^-\end{aligned} \tag{F.76}$$

$$\dot{S}^-_{(R)} = \tilde{\mathcal{F}}^R_{(-)} - \frac{1}{2}\gamma^{(R)}S^- + i\Omega^{(R)}S^-$$

F.2.2.2. Evolution generated by the reservoir $\mathcal{P}$

$$\begin{aligned}\frac{-i}{\hbar}\left[\sigma^-_j, H_I^{\mathcal{P}}\right] = &\quad \frac{-i}{\hbar}\sum_{\beta,i}\hbar\left(\tilde{\kappa}^*_{\beta,i}\mathcal{P}^+_\beta\left[\sigma^-_j,\sigma^+_i\right]e^{-i(\omega_\beta-\omega_A)t} + \tilde{\kappa}_{\beta,i}\left[\sigma^-_j,\sigma^-_i\right]\mathcal{P}^-_\beta e^{i(\omega_\beta-\omega_A)t}\right)\\ = &\quad \frac{-i}{\hbar}\sum_{\beta,i}\hbar\left(\tilde{\kappa}^*_{\beta,i}\mathcal{P}^+_\beta(-\delta_{i,j}\sigma^z_i)e^{-i(\omega_\beta-\omega_A)t}\right)\\ = &\quad i\sum_\beta\left(\tilde{\kappa}^*_{\beta,j}\mathcal{P}^+_\beta\sigma^z_j e^{-i(\omega_\beta-\omega_A)t}\right)\end{aligned} \tag{F.77}$$

Replacing $\mathcal{P}_\beta^+$ and $\mathcal{P}_\beta^-$ by the solution of the equation of motion:

$$\begin{aligned}\frac{-i}{\hbar}\left[\sigma_j^-, H_I^{\mathcal{P}}\right] = \quad & i\sum_\beta \left(\tilde{\kappa}_{\beta,j}^* \mathcal{P}_\beta^+ \sigma_j^z e^{-i(\omega_\beta-\omega_A)t}\right) \\ = \quad & i\sum_\beta \left(\tilde{\kappa}_{\beta,j}^* \left(\mathcal{P}_\beta^+(t_0) + i\int_{t_0}^t \sum_i \tilde{\kappa}_{\beta,i}\sigma_i^-(t')\mathcal{P}_\beta^z(t')e^{-i(\omega_\beta-\omega_A)t'}dt'\right)\sigma_j^z e^{-i(\omega_\beta-\omega_A)t}\right)\end{aligned} \tag{F.78}$$

$$\begin{aligned}\frac{-i}{\hbar}\left[\sigma_j^-, H_I^{\mathcal{P}}\right] = \quad & i\sum_\beta \tilde{\kappa}_{\beta,j}^* \mathcal{P}_\beta^+(t_0)\sigma_j^z e^{-i(\omega_\beta-\omega_A)t} \\ & -\sum_\beta\sum_i \left(\tilde{\kappa}_{\beta,j}^*\tilde{\kappa}_{\beta,i}\left(\int_{t_0}^t \sigma_i^-(t')\mathcal{P}_\beta^z(t')e^{-i(\omega_\beta-\omega_A)t'}dt'\right)\sigma_j^z e^{-i(\omega_\beta-\omega_A)t}\right)\end{aligned} \tag{F.79}$$

Definig the noise operator gererated by the reservoir $\mathcal{P}$.

$$\mathcal{F}_{(-)}^{\mathcal{P}} = \quad i\sum_\beta \tilde{\kappa}_{\beta,j}^* \mathcal{P}_\beta^+(t_0)\sigma_j^z e^{-i(\omega_\beta-\omega_A)t} \tag{F.80}$$

$$\frac{-i}{\hbar}\left[\sigma_j^-, H_I^{\mathcal{P}}\right] = \quad \mathcal{F}_{(-)}^{\mathcal{P}} - \sum_\beta\sum_i \left(\tilde{\kappa}_{\beta,j}^*\tilde{\kappa}_{\beta,i}\left(\int_{t_0}^t \sigma_i^-(t')\mathcal{P}_\beta^z(t')e^{-i(\omega_\beta-\omega_A)t'}dt'\right)\sigma_j^z e^{-i(\omega_\beta-\omega_A)t}\right) \tag{F.81}$$

If $\sigma_i^\pm(t') \to \sigma_i^\pm(t)$ and $\mathcal{P}_{\beta,j}^z(t') \to \mathcal{P}_{\beta,j}^z(t)$:

$$\frac{-i}{\hbar}\left[\sigma_j^-, H_I^{\mathcal{P}}\right] = \quad \mathcal{F}_{(-)}^{\mathcal{P}} - \sum_\beta\sum_i \left(\tilde{\kappa}_{\beta,j}^*\tilde{\kappa}_{\beta,i}\left(\int_{t_0}^t e^{-i(\omega_\beta-\omega_A)t'}dt'\right)\mathcal{P}_\beta^z\sigma_i^-\sigma_j^z e^{-i(\omega_\beta-\omega_A)t}\right) \tag{F.82}$$

Where, if $t - t_0 \to \infty$:

$$\lim_{t\to\infty}\int_0^t dt' e^{i(\omega-\omega_0)t'} = \pi\delta(\omega-\omega_0) + i\frac{P}{\omega_0-\omega} \tag{F.83}$$

Then

$$\frac{-i}{\hbar}\left[\sigma_j^-, H_I^{\mathcal{P}}\right] = \quad \mathcal{F}_{(-)}^{\mathcal{P}} - \sum_\beta\sum_i \left(\tilde{\kappa}_{\beta,j}^*\tilde{\kappa}_{\beta,i}\left(\pi\delta(\omega_\beta-\omega_A) - i\frac{P}{\omega_\beta-\omega_A}\right)\mathcal{P}_\beta^z\sigma_i^-\sigma_j^z e^{-i(\omega_\beta-\omega_A)t}\right) \tag{F.84}$$

$$\frac{-i}{\hbar}\left[\sigma_j^-, H_I^{\mathcal{P}}\right] = \mathcal{F}_{(-)}^{\mathcal{P}} - \sum_{\beta,i}\left(\tilde{\kappa}_{\beta,j}^*\tilde{\kappa}_{\beta,i}\left(\pi\delta(\omega_\beta-\omega_A)\right)\mathcal{P}_\beta^z\sigma_i^-\sigma_j^z e^{-i(\omega_\beta-\omega_A)t}\right)$$
$$-\sum_{\beta,i}\left(\tilde{\kappa}_{\beta,j}^*\tilde{\kappa}_{\beta,i}\left(-i\frac{P}{\omega_\beta-\omega_A}\right)\mathcal{P}_\beta^z\sigma_i^-\sigma_j^z e^{-i(\omega_\beta-\omega_A)t}\right) \tag{F.85}$$

Defining the collective operator:

$$S^- = \frac{1}{N}\sum_j \sigma_j^- \tag{F.86}$$

$$\frac{-i}{\hbar}\left[NS^-, H_I^{\mathcal{P}}\right] = \frac{-i}{\hbar}\sum_j\left[\sigma_j^-, H_I^{\mathcal{P}}\right]$$
$$= \tilde{\mathcal{F}}_{(-)}^{\mathcal{P}} - \sum_{\beta,i,j}\left(\tilde{\kappa}_{\beta,j}^*\tilde{\kappa}_{\beta,i}\left(\pi\delta(\omega_\beta-\omega_A)\right)\mathcal{P}_\beta^z\sigma_i^-\sigma_j^z e^{-i(\omega_\beta-\omega_A)t}\right)$$
$$+i\sum_{\beta,i,j}\left(\tilde{\kappa}_{\beta,j}^*\tilde{\kappa}_{\beta,i}\left(\frac{P}{\omega_\beta-\omega_A}\right)\mathcal{P}_\beta^z\sigma_i^-\sigma_j^z e^{-i(\omega_\beta-\omega_A)t}\right) \tag{F.87}$$

If we consider $\sum_\beta \to \sum_l \int d\omega_\beta$

$$\frac{-i}{\hbar}\left[NS^-, H_I^{\mathcal{P}}\right] = \tilde{\mathcal{F}}_{(-)}^{\mathcal{P}} - \sum_{i,j}\sum_l\int d\omega_\beta\left(\tilde{\kappa}_{l,j}^*(\omega_\beta)\tilde{\kappa}_{l,i}(\omega_\beta)\left(\pi\delta(\omega_\beta-\omega_A)\right)\mathcal{P}_l^z\sigma_i^-\sigma_j^z e^{-i(\omega_\beta-\omega_A)t}\right)$$
$$+i\sum_{i,j}\sum_l\int d\omega_\beta\left(\tilde{\kappa}_{l,j}^*(\omega_\beta)\tilde{\kappa}_{l,i}(\omega_\beta)\left(\frac{P}{\omega_\beta-\omega_A}\right)\mathcal{P}_l^z\sigma_i^-\sigma_j^z e^{-i(\omega_\beta-\omega_A)t}\right) \tag{F.88}$$

$$\frac{-i}{\hbar}\left[NS^-, H_I^{\mathcal{P}}\right] = \tilde{\mathcal{F}}_{(-)}^{\mathcal{P}} - \pi\sum_{i,j}\sum_l\left(\tilde{\kappa}_{l,j}^*(\omega_A)\tilde{\kappa}_{l,i}(\omega_A)\mathcal{P}_l^z\right)\sigma_i^-\sigma_j^z$$
$$+i\sum_{i,j}\sum_l\int d\omega_\beta\left(\tilde{\kappa}_{l,j}^*(\omega_\beta)\tilde{\kappa}_{l,i}(\omega_\beta)\left(\frac{P}{\omega_\beta-\omega_A}\right)\mathcal{P}_l^z e^{-i(\omega_\beta-\omega_A)t}\right)\sigma_i^-\sigma_j^z \tag{F.89}$$

Definig $\gamma_{(-)i,j}^{(\mathcal{P})}$ and $\Omega_{(-)i,j}^{(\mathcal{P})}$ the decay rate and the shift generated by the reservoir $\mathcal{P}$, respectively:

$$\gamma_{i,j}^{(\mathcal{P})} = 2\pi\sum_l \kappa_{l,j}^*(\omega_A)\kappa_{l,i}(\omega_A)\mathcal{P}_l^z \tag{F.90}$$

$$\Omega^{(\mathcal{P})}_{(-)i,j} = \sum_l \int d\omega_\beta \left(\tilde{\kappa}_{l,j}(\omega_\beta)\tilde{\kappa}_{l,i}(\omega_\beta)\mathcal{P}^z_l \left(\frac{P}{\omega_\beta - \omega_A} \right) e^{-i(\omega_\beta-\omega_A)t} \right) \tag{F.91}$$

The evolution of the collective coherence operator generated by the pump reservoir:

$$\frac{-i}{\hbar}\left[NS^-, H^{\mathcal{P}}_I\right] = \mathcal{F}^{\mathcal{P}}_{(-)} - \sum_{i,j} \frac{1}{2}\gamma^{(\mathcal{P})}_{i,j}\sigma^-_i\sigma^z_j + i\sum_{i,j}\Omega^{(\mathcal{P})}_{(-)i,j}\sigma^-_i\sigma^z_j \tag{F.92}$$

But studing the term $\sigma^-_i\sigma^z_j$

$$\sigma^-_i\sigma^z_j = \delta_{i,j}\left|g\right\rangle_i \left\langle e\right|_j \tag{F.93}$$

$$\frac{-i}{\hbar}\left[NS^-, H^{\mathcal{P}}_I\right] = \mathcal{F}^{\mathcal{P}}_{(-)} - \frac{1}{2}\sum_{i,j} \gamma^{(\mathcal{P})}_{i,j}\delta_{i,j}\left|g\right\rangle_i \left\langle e\right|_j + i\sum_{i,j}\Omega^{(\mathcal{P})}_{(-)i,j}\delta_{i,j}\left|g\right\rangle_i \left\langle e\right|_j \tag{F.94}$$

$$\frac{-i}{\hbar}\left[NS^-, H^{\mathcal{P}}_I\right] = \mathcal{F}^{\mathcal{P}}_{(-)} - \frac{1}{2}\sum_{j} \gamma^{(\mathcal{P})}_{j,j}\sigma^-_j + i\sum_{j}\Omega^{(\mathcal{P})}_{(-)j,j}\sigma^-_j \tag{F.95}$$

Defining $\gamma^{(\mathcal{P})} = \gamma^{(\mathcal{P})}_{j,j}$ and $\Omega^{(\mathcal{P})} = \Omega^{(\mathcal{P})}_{j,j}$ for all j:

$$\begin{aligned} \frac{-i}{\hbar}\left[NS^-, H^{\mathcal{P}}_I\right] &= \mathcal{F}^{\mathcal{P}}_{(-)} - \frac{1}{2}\gamma^{(\mathcal{P})}\sum_j \sigma^-_j + i\Omega^{(\mathcal{P})}_{(-)}\sum_j \sigma^-_j \\ &= \mathcal{F}^{\mathcal{P}}_{(-)} - \frac{1}{2}\gamma^{(\mathcal{P})}NS^- + i\Omega^{(\mathcal{P})}_{(-)}NS^- \end{aligned} \tag{F.96}$$

$$\frac{-i}{\hbar}\left[S^-, H^{\mathcal{P}}_I\right] = \tilde{\mathcal{F}}^{\mathcal{P}}_{(-)} - \frac{1}{2}\gamma^{(\mathcal{P})}S^- + i\Omega^{(\mathcal{P})}_{(-)}S^- \tag{F.97}$$

$$\dot{S}^-_{(\mathcal{P})} = \tilde{\mathcal{F}}^{\mathcal{P}}_{(-)} - \frac{1}{2}\gamma^{(\mathcal{P})}S^- + i\Omega^{(\mathcal{P})}_{(-)}S^-$$