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CONTRIBUCIÓN AL ESTUDIO DE DINÁMICAS SIMBÓLICAS DE ENTROPÍA CERO: AUTOMORFISMOS, FACTORES Y ESTRUCTURE

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CONTRIBUCIÓN AL ESTUDIO DE DINÁMICAS SIMBÓLICAS DE ENTROPÍA CERO: AUTOMORFISMOS, FACTORES Y ESTRUCTURE

Esta tesis se centra en el estudio de los subshifts minimales a través de secuencias $\mathcal{S}$-ádicas. Primero, investigamos los automorfismos y factores de subshifts minimales generados por secuencias $\mathcal{S}$-ádicas con alfabetos de cardinalidad acotada. Probamos que estos subshifts tienen grupos de automorfismos que son virtualmente $\mathbb{Z}$, que tienen finitos factores simbólicos aperiódicos (salvo conjugación), y damos una descripción fina de sus factores simbólicos. Luego, consideramos la conjetura $\mathcal{S}$-ádica, un viejo problema que pregunta por un teorema de estructura para los subshifts de complejidad lineal. Resolvemos completamente este problema al dar una caracterización $\mathcal{S}$-ádica de esta clase de subshifts. Nuestros métodos se extienden a subshifts de crecimiento no superlineal. Mostramos también cómo esto proporciona un marco unificado y pruebas simplificadas de varios resultados conocidos, incluido el teorema pionero de Cassaigne de 1996.

Abstract

This thesis focuses on the study of minimal subshifts via $\mathcal{S}$-adic sequences. First, we investigate automorphisms and factors of minimal subshifts generated by $\mathcal{S}$-adic sequences with alphabets of bounded cardinality. As a result, we prove that these subshifts have virtually $\mathbb{Z}$ automorphism groups, finitely many infinite symbolic factors (up to conjugacy), and we give a fine description of symbolic factor maps. In the second part, we consider the $\mathcal{S}$-adic conjecture, an old problem asking for a structure theorem for linear-growth complexity subshifts. We completely solve this problem by proving an $\mathcal{S}$-adic characterization of this class of subshifts. Our methods extend to nonsuperlinear-growth subshifts. We show how this provides a unified framework and simplified proofs of several known results, including Cassaigne's Theorem.

Résumé

Cette thèse porte sur l'étude des systèmes symboliques minimaux via des séquences $\mathcal{S}$-adiques. Dans la première partie, nous étudions les automorphismes et les facteurs des systèmes minimaux générés par des séquences $\mathcal{S}$-adiques avec des alphabets de cardinalité bornée. Comme résultat, nous prouvons que les systèmes de cette classe ont des groupes d'automorphismes virtuellement $\mathbb{Z}$, un nombre fini de facteurs symboliques infinis (jusqu'à la conjugaison), et une description fine des facteurs symboliques. Dans la seconde partie, nous considérons la conjecture $\mathcal{S}$-adique, un vieux problème demandant un théorème de structure pour les systèmes symboliques de complexité à croissance linéaire. Nous résolvons complètement ce problème en prouvant une caractérisation $\mathcal{S}$-adique de cette classe de systèmes. Les méthodes s'étendent aux systèmes à croissance non superlinéaire. Nous montrons comment cela fournit un cadre unifié et des preuves simplifiées de plusieurs résultats connus, y compris le théorème de Cassaigne de 1996.

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Introduction

An idea that became unavoidable to study zero entropy symbolic dynamics is that the dynamical properties of a system induce in it a combinatorial structure. The first use of this approach was in the works of Morse and Hedlund [MH38; MH40], where Sturmian sequences were studied based on a structure given by what the authors called *derivative sequences*. As the theory developed, more examples like this one emerged. Relevant ones include substitutive and linearly recurrent subshifts [DHS99], Toeplitz systems [GJ00], (natural codings of) interval exchange transformations [GJ02], dendric sequences [GL22] and general minimal subshifts [HPS92].

In this thesis, we investigate these combinatorial structures under two approaches. The first one assumes a given combinatorial structure and focuses on studying the dynamical properties of the systems supporting such a structure. We will consider *finite topological rank systems*, a class of systems possessing two desirable but opposite properties: It is a large class and contains most of the known zero entropy symbolic systems, yet it presents strong dynamical restrictions. Thus, the finite topological rank class provides a good framework for proving general and interesting theorems. We exploit this idea by studying automorphisms and symbolic factors of finite topological rank subshifts. Several theorems, describing rigidity properties for these objects, are obtained in Chapters 2 and 3.

The second approach consists of finding new combinatorial structures for systems of interest. We study one of the major questions in this direction -the $\mathcal{S}$*-adic conjecture*, which asks for a structure theorem for linear-growth complexity subshifts. In the final chapter, we solve this conjecture and, furthermore, extend it to nonsuperlinear-growth complexity subshifts. An important consequence of our results is that these complexity classes gain access to the $\mathcal{S}$-adic machinery. We show how this provides a unified framework and simplified proofs of several known results, including the pioneering 1996 Cassaigne's Theorem.

We will now discuss the thesis topics in more detail.

Basic terminology

Let us briefly review the modern standard for describing symbolic systems and their structures. An *alphabet* is a finite set $\mathcal{A}$ and a *word* is a finite concatenation of letters, *i.e.*, elements of $\mathcal{A}$. The *full-shift* on $\mathcal{A}$ is the set $\mathcal{A}^{\mathbb{Z}}$ endowed with the product topology of the discrete topology of $\mathcal{A}$. We define the *shift* as the map $S\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by $(x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}$. A *symbolic system* or *subshift* is a closed subset X of $\mathcal{A}^{\mathbb{Z}}$ such that $S(X) = X$. We will mostly consider *minimal* subshifts X, that is, such that $\{S^n x : n \in \mathbb{Z}\}$ is dense in X for all $x \in X$.

A substitution is a map $\tau\colon \mathcal{A}^+ \to \mathcal{B}^+$ that substitutes the letters a_i of a word $w = a_1 \dots a_\ell$ by $\tau(a_i)$. Then, a sequence of substitutions $\boldsymbol{\tau}$ of the form $(\tau_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\geq 0}$ is called an $\mathcal{S}$-adic sequence and generates a subshift $X_{\boldsymbol{\tau}} \subseteq \mathcal{A}_0^{\mathbb{Z}}$ given by requiring that $x \in X_{\boldsymbol{\tau}}$ if and only if, for all $\ell \geq 0$, $x_{[-\ell,\ell)}$ occurs in $\tau_0\tau_1 \dots \tau_{n-1}(a)$ for some $n \geq 1$ and $a \in \mathcal{A}_n$.

Finite topological rank systems

An ordered Bratteli diagram is an infinite directed graph $B = (V, E, \leq)$ such that the vertex set V and the edge set E are partitioned into levels $V = V_0 \cup V_1 \cup \dots$, $E = E_0 \cup \dots$ so that E_n are edges from V_{n+1} to V_n, V_0 is a singleton, each V_n is finite and $\leq$ is a partial order on E such that two edges are comparable if and only if they start at the same vertex. The order $\leq$ can be extended to the set X_B of all infinite paths in B, and the Vershik action V_B on X_B is defined when B has unique minimal and maximal infinite paths with respect to $\leq$. We say that (X_B, V_B) is a BV representation of the Cantor system (X, S) if both are conjugate. Bratteli diagrams are a tool coming from C^*-algebras that, at the beginning of the 90', Herman *et. al.* [HPS92] used to study minimal Cantor systems. Their success at characterizing the strong and weak orbit equivalence for systems of this kind marked a milestone in the theory that motivated many posterior works. Some of these works focused on studying with Bratteli diagrams specific classes of systems and, as a consequence, many of the classical minimal systems have been characterized as Bratteli-Vershik systems with a specific structure. Some examples include odometers as those systems that have a BV representation with one vertex per level, substitutive subshifts as *stationary* BV (all levels are the same) [DHS99], certain Toeplitz sequences as "equal row-sum" BV [GJ00], and (codings of) interval exchanges as BV where the diagram codifies a path in a Rauzy graph [GJ02]. Now, almost all of these examples share certain coarse dynamical behavior: they are subshifts, have finitely many ergodic measures, are not strongly mixing, have zero entropy, and their BV representations have a bounded number of vertices per level, among many others. It turns out that just having a BV representation with a bounded number of vertices per level (or, from now on, having *finite topological rank*) implies the previous properties (see, for example, [BKMS13], [DM08]). In particular, finite topological rank systems are subshifts. Hence, the finite topological rank class arises as a possible framework for studying minimal subshifts and proving general theorems.

This idea has been exploited in many works: Durand *et. al.*, in a series of papers (being [DFM19] the last one), developed techniques from the well-known substitutive case and obtained a criteria for any BV of finite topological rank to decide if a given complex number is a continuous or measurable eigenvalue, Bezugly *et. al.* described in [BKMS13] the simplex of invariant measures together with natural conditions for being uniquely ergodic, Giordano *et. al.* bounded the rational rank of the dimension group by the topological rank ([HPS92]), among other works. It is important to remark that these works were inspired by or first proved in the substitutive case.

Now, since Bratteli-Vershik whose topological rank is at least two are conjugate to a subshift [DM08], it is interesting to try to define them directly as a subshift. This can be done by codifying the levels of the Bratteli diagram as substitutions and then iterate them to obtain a sequence of symbols defining a subshift conjugate to the initial BV system. This procedure also makes sense for arbitrary nested sequences of substitutions (called *directive sequences*), independently from the Bratteli diagram and the various additional properties that its codifying substitutions have. Subshifts obtained in this way are called $\mathcal{S}$-adic (substitution-adic) and may be non-minimal (see for example [BSTY19]).

Although there are some open problems about finite topological rank systems depending directly on the combinatorics of the underlying Bratteli diagrams, others are more naturally stated in the $\mathcal{S}$-adic setting (*e.g.*, when dealing with endomorphisms, it is useful to have the Curtis–Hedlund–Lyndon Theorem) and, hence, there exists an interplay between $\mathcal{S}$-adic subshifts and finite topological rank systems in which theorems and techniques obtained for one of these classes can sometimes be transferred to the other. The question about which is the exact relation between these classes has been recently addressed in [DDMP21] and, in particular, the authors proved:

Theorem 0.1 ([DDMP21]) *A minimal subshift (X, S) has topological rank at most K if and only if it is generated by a proper, primitive and recognizable $\mathcal{S}$-adic sequence of alphabet rank at most K.*

In Chapters 2 and 3, we will use the $\mathcal{S}$-adic formalism to study automorphisms and factors of finite topological rank systems.

Automorphisms

Let X be a subshift. The automorphism group of (X, S), $\mathrm{Aut}(X, S)$, is the set of homeomorphisms from X onto itself that commute with S. The study of the automorphism group of low complexity subshifts (X, S) has attracted a lot of attention in recent years. By *complexity*, we mean the increasing function $p_X : \mathbb{N} \to \mathbb{N}$ which counts the number of words of length $n \in \mathbb{N}$ appearing in points of the subshift (X, S). In contrast to the case of non trivial mixing shifts of finite type or synchronized systems, where the algebraic structure of this group can be very rich [BLR88; KR90; FF96], the automorphism group of low complexity subshifts is expected to present high degrees of rigidity. The most relevant example illustrating this fact are minimal subshifts of non-superlinear complexity, where the automorphism group is virtually $\mathbb{Z}$ [CK15; DDMP16]. Interestingly, in [Sal17] (and then in [DDMP16] in a more general class) the author provides a Toeplitz subshift with complexity $p_X(n) \leq Cn^{1.757}$, whose automorphism group is not finitely generated. So some richness in the algebraic structure of the automorphism groups of low complexity subshifts can arise. Other low complexity subshifts have been considered by Cyr and Kra in a series of works. In [CK16b] they proved that for transitive subshifts, if $\liminf_{n\to+\infty} p_X(n)/n^2 = 0$, then the quotient $\mathrm{Aut}(X, S)/\langle S \rangle$ is a periodic group, where $\langle S \rangle$ is the group spanned by the shift map; and in [CK16a] for a large class of minimal subshifts of subexponential complexity they also proved that the automorphism group is amenable. All these classes and examples show that there is still a lot to be understood on the automorphism groups of low complexity subshifts.

In Chapter 2, we study the automorphism group of *minimal $\mathcal{S}$-adic subshifts of finite or* *bounded alphabet rank*. This class of minimal subshifts is somehow the most natural class containing minimal subshifts of non-superlinear complexity, but it is much broader, as was shown in [DDMP16; DDMP21]. Moreover, this class contains several well studied minimal symbolic systems. Among them, substitution subshifts, linearly recurrent subshifts, symbolic codings of interval exchange transformations, dendric subshifts and some Toeplitz sequences. Thus, this class represents a useful framework for both, proving general theorems in the low complexity world and building subshifts with interesting dynamical behavior. The descriptions made in [BKMS13] of its invariant measures and in [DFM19] of its eigenvalues are examples of the former, and the well-behaved $\mathcal{S}$-adic codings of high dimensional torus translations from [BST20] is an example of the latter.

The main result of Chapter 2 is the following rigidity theorem:

Theorem 0.2 *Let (X, S) be a minimal $\mathcal{S}$-adic subshift given by an everywhere growing directive sequence $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\geq 0}$. Suppose that $\boldsymbol{\tau}$ is of finite alphabet rank,* i.e., $\liminf_{n\to+\infty} \#\mathcal{A}_n < +\infty$. *Then,* $\mathrm{Aut}(X, S)$ *is virtually* $\mathbb{Z}$.

A minimal $\mathcal{S}$-adic subshift of *finite topological rank*, as stated in [DDMP21], is defined as an $\mathcal{S}$-adic subshift in which the defining directive sequence $\boldsymbol{\tau}$ is proper, primitive, recognizable and with finite alphabet rank. In particular, $\boldsymbol{\tau}$ is everywhere growing. Therefore, Theorem 0.2 includes all minimal $\mathcal{S}$-adic subshifts of finite topological rank. Also, in the same paper, the authors prove that minimal subshifts of non-superlinear complexity are $\mathcal{S}$-adic of finite topological rank. Thus, Theorem 0.2 can be seen as a generalization to a much broader class of the already mentioned results from [CK15] and [DDMP16]. Finally, by results stated in [DDMP16], Theorem 0.2 also applies to all level subshifts of minimal Bratteli-Vershik systems of finite topological rank and its symbolic factors.

The proof of Theorem 0.2 follows from a fine combinatorial analysis of asymptotic classes of $\mathcal{S}$-adic subshifts of finite alphabet rank. This idea already appeared in [DDMP16], where the authors prove that the automorphism group of a minimal system is virtually $\mathbb{Z}$ whenever it has finitely many asymptotic classes. The following theorem summarizes this combinatorial analysis.

Theorem 0.3 *Let $\mathcal{W} \subseteq \mathcal{A}^+$ be a set of nonempty words and define $\langle \mathcal{W} \rangle := \min_{w\in\mathcal{W}} \mathrm{length}(w)$. Then, there exists $\mathcal{B} \subseteq \mathcal{A}^{\langle \mathcal{W} \rangle}$ with $\#\mathcal{B} \leq 122(\#\mathcal{W})^7$ such that: if $x, x' \in \mathcal{A}^{\mathbb{Z}}$ are factorizable over $\mathcal{W}$, $x_{(-\infty,0)} = x'_{(-\infty,0)}$ and $x_0 \neq x'_0$, then $x_{[-\langle \mathcal{W} \rangle,0)} \in \mathcal{B}$.*

Here, the important point is that, despite the fact that the length of the elements in $\mathcal{B}$ is $\langle \mathcal{W} \rangle$, the cardinality of $\mathcal{B}$ depends only on $\#\mathcal{W}$, and not on $\langle \mathcal{W} \rangle$.

Finally, we get a bound for the asymptotic classes of an $\mathcal{S}$-adic subshift of finite alphabet rank. This result does not require minimality.

Theorem 0.4 *Let (X, S) be an $\mathcal{S}$-adic subshift (not necessarily minimal) given by an everywhere growing directive sequence of finite alphabet rank K. Then, (X, S) has at most $122K^7$*

asymptotic classes.

Factors

In the context of finite topological rank systems, a fundamental question is the following:

Question 0.1 Are subshift factors of finite topological rank systems of finite topological rank?

Indeed, the topological rank controls various coarse dynamical properties (number of ergodic measures, rational rank of dimension group, among others) which cannot increase after a factor map, and we also know that big subclasses of the finite topological rank class are stable under symbolic factors, such as the linearly recurrent and the non-superlineal complexity classes [DDMP21], so it is expected that this question has an affirmative answer. However, when trying to prove this using Theorem 0.1, we realize that the naturally inherited $\mathcal{S}$-adic structure of finite alphabet rank that a symbolic factor has is never recognizable. Moreover, this last property is crucial for many of the currently known techniques to handle finite topological rank systems (even in the substitutive case it is a deep and fundamental theorem of Mossé), so it is not clear why it would be always possible to obtain this property while keeping the alphabet rank bounded or why recognizability is not connected with a dynamical property of the system. Thus, an answer to this question seems to be fundamental to the understanding of the finite topological rank class.

In Chapter 3, we obtain the optimal answer to Question 0.1 in a more general, non-minimal context:

Theorem 0.5 *Let (X, S) be an $\mathcal{S}$-adic subshift generated by an everywhere growing and proper directive sequence of alphabet rank equal to K, and $\pi\colon (X, S) \to (Y, S)$ be an aperiodic subshift factor. Then, (Y, S) is an $\mathcal{S}$-adic subshift generated by an everywhere growing, proper and recognizable directive sequence of alphabet rank at most K.*

Here, a directive sequence $\boldsymbol{\sigma} = (\sigma_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ is *everywhere growing* if the sequence $\min_{a\in\mathcal{A}_n} |\sigma_0 \dots \sigma_{n-1}(a)|$ diverges as $n \to +\infty$, and a system (X, S) is *aperiodic* if every orbit $\{S^n x : n \in \mathbb{Z}\}$ is infinite. Theorem 0.5 implies that the topological rank cannot increase after a factor map (Corollary 3.19). Theorem 0.5 implies the following sufficient condition for a system to be of finite topological rank:

Corollary 0.6 *Let (X, S) be an aperiodic minimal $\mathcal{S}$-adic subshift generated by an everywhere growing directive sequence of finite alphabet rank. Then, the topological rank of (X, S) is finite.*

An interesting corollary of the underlying construction of the proof of Theorem 0.5 is the coalescence property for this kind of systems, in the following stronger form:

Corollary 0.7 *Let (X, S) be an $\mathcal{S}$-adic subshift generated by an everywhere growing and proper directive sequence of alphabet rank equal to K, and $(X, S) \xrightarrow{\pi_1} (X_1, S) \xrightarrow{\pi_2} \dots \xrightarrow{\pi_L} (X_L, S)$ be a chain of aperiodic subshift factors. If $L > \log_2 K$, then at least one π_j is a conjugacy.*

One of the results in [Dur00] is that factor maps between aperiodic linearly recurrent subshifts are finite-to-one. In particular, they are *almost k-to-1* for some finite k. For finite topological rank subshifts, we prove:

Theorem 0.8 *Let $\pi\colon (X, S) \to (Y, S)$ be a factor map between aperiodic minimal subshifts. Suppose that (X, S) has topological rank equal to K. Then π is almost k-to-1 for some $k \leq K$.*

We use this theorem, in Corollary 3.22, to prove that Cantor factors of finite topological rank subshifts are either odometers or subshifts.

In [Dur00], the author proved that *linearly recurrent* subshifts have finite topological rank, and that this kind of systems have finitely many aperiodic subshifts factors up to conjugacy. Inspired by this result, we use ideas from the proof of Theorem 0.5 to obtain:

Theorem 0.9 *Let (X, S) be a minimal subshift of topological rank K. Then, (X, S) has at most $(3K)^{32K}$ aperiodic subshift factors up to conjugacy.*

Altogether, these results give a rough picture of the set of totally disconnected factors of a given finite topological rank system: they are either equicontinuous or subshifts satisfying the properties in Theorems 0.5, 0.7, 0.9 and 0.8. Now, in a topological sense, totally disconnected factors of a given system (X, S) are "maximal", so, the natural next step in the study of finite topological rank systems is asking about the connected factors. As we have seen, the finite topological rank condition is a rigidity condition. By this reason, we think that the following question has an affirmative answer:

Question 0.2 Let (X, S) be a minimal system of finite topological rank and $\pi\colon (X, S) \to (Y, T)$ be a factor map. Suppose that Y is connected. Is (Y, T) an equicontinuous system?

We remark that the finite topological rank class contains all minimal subshifts of non-superlinear complexity [DDMP21], but even for the much smaller class of linear complexity subshifts the author is not aware of results concerning Question 0.2.

Low complexity subshifts

Structure theorems

Theorems that describe a combinatorial structure of a given class of subshifts are usually an $\mathcal{S}$-adic characterization, namely, of the form: a subshift X belongs to the class $\mathcal{C}$ if and only if X is generated by an $\mathcal{S}$-adic sequence satisfying certain property $\mathcal{P}$. The structure then appears as an infinite desubstitution process for the points of X.

In the context of structure theorems, an interesting intuition is that a subshift of low enough complexity should be very restricted, and thus hide a strong structure. Here, *low complexity* is a vague term referring to a slow growth of the complexity function $p_X(n)$, defined as the number of words of length n that occur in some point of X. This intuition dates back to the 70s, and matured in the 80s and 90s until it was finally established as the following more concrete question.

Question 0.3 Consider the class (L) of linear-growth complexity subshifts, defined by requiring that $p_X(n) \leq dn$ for some $d > 0$. Is there an $\mathcal{S}$-adic characterization of the class (L)?

Question 0.3 is known as the $\mathcal{S}$-adic conjecture. The first time it was explicitly stated was in [Fer96], where the author attributes the idea to B. Host, who, in turn, attributes the idea to the whole Marseille community.

The attempts to solve this conjecture have identified two major difficulties. The first one is that, in contrast to what happens with other structure theorems, there is no clear structure induced by the complexity. For example, in the substitutive case, it was always clear that the substitution itself should produce a self-similar structure; the main obstruction was technical and referred to whether the desubstitution process was properly defined [Mos96]. Similarly, in the Sturmian and IET cases, the known structure came from the geometric counterpart (more precisely, from the Rauzy induction). The second challenge is that the condition $\mathcal{P}$ we are looking for in Question 0.3 is ill-defined. To exemplify this point, observe that a corollary of [Cas11] is the following $\mathcal{S}$-adic characterization of (L): a subshift is in (L) if and only if there exist $\boldsymbol{\tau}$ generating it and such that $X_{\boldsymbol{\tau}}$ is in (L). This tautological answer to Question 0.3 does not provide information. Certain restrictions on Question 0.3 have been proposed to avoid this type of trivial answer, but none of them is considered satisfactory; we refer the reader to [DLR13] for a full discussion.

In Chapter 4, we completely solve the $\mathcal{S}$-adic conjecture for minimal subshifts by proving the following theorem.

Theorem 0.10 *A minimal subshift X has linear-growth complexity,* i.e., *X satisfies*

$$\limsup_{n\to+\infty} p_X(n)/n < +\infty,$$

if and only if there exist $d > 0$ and an $\mathcal{S}$-adic sequence $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ generating X such that, for every $n \geq 0$, the following holds:

($\mathcal{P}_1$) $\#(\mathsf{root}\,\sigma_{[0,n)}(\mathcal{A}_n)) \leq d$ †.

($\mathcal{P}_2$) $|\sigma_{[0,n)}(a)| \leq d \cdot |\sigma_{[0,n)}(b)|$ *for every* $a, b \in \mathcal{A}_n$.

($\mathcal{P}_3$) $|\sigma_{n-1}(a)| \leq d$ *for every* $a \in \mathcal{A}_n$.

† For a word u, $\mathsf{root}\,u$ denotes the shortest prefix v of u such that $u = v^k$ for some k; for a set of words $\mathcal{W}$, $\mathsf{root}\,\mathcal{W} = \{\mathsf{root}\,w : w \in \mathcal{W}\}$.

Our techniques extend to the case of nonsuperlinear complexity subshifts (NSL).

Theorem 0.11 *A minimal subshift X has nonsuperlinear-growth complexity,* i.e., *X satisfies*

$$\liminf_{n\to+\infty} p_X(n)/n < +\infty,$$

if and only if there exist $d > 0$ and an $\mathcal{S}$-adic sequence $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ generating X such that, for every $n \geq 0$, the following holds:

($\mathcal{P}_1$) $\#(\mathsf{root}\,\sigma_{[0,n)}(\mathcal{A}_n)) \leq d$.

($\mathcal{P}_2$) $|\sigma_{[0,n)}(a)| \leq d \cdot |\sigma_{[0,n)}(b)|$ *for every $a, b \in \mathcal{A}_n$.*

The case of non-minimal subshifts does not pose additional intrinsic difficulties and follows from methods similar to those given here. However, we did not include it to avoid over saturating an already technical presentation.

An important consequence of our main results is that the classes (L) and (NSL) gain access to the $\mathcal{S}$-adic machinery. We show in Section 4.10 how this provides a unified framework and simplified proofs of several known results on (L) and (NSL), including Cassaigne's Theorem [Cas95]. Further applications of our main results, which include a new proof of partial rigidity for (NSL) [Cre22] using the technique in [BKMS13, Theorem 7.2], will be presented in a future work.

We prove, in the more specialized Theorems 4.75 and 4.76, that when X is in (L) or in (NSL), then $\boldsymbol{\tau}$ can be assumed to be recognizable. Observe that the conditions ($\mathcal{P}_i$) in Theorems 0.10 and 0.11 are optimal in the sense that if we remove any of them then the corresponding theorem is false. Conditions ($\mathcal{P}_2$) and ($\mathcal{P}_3$) also occur in the positive substitutive case ‡ and in linearly recurrent subshifts, but the behavior in our theorems is very different since we do not impose positiveness.

With regard to ($\mathcal{P}_1$) and ($\mathcal{P}_3$), these were designed on the basis of two conditions that are present in most works that involve $\mathcal{S}$-adic sequences. The first is having bounded alphabets (BA), which requires that $\#\mathcal{A}_n$ is uniformly bounded, and the second is finitariness, which asks for the set $\{\tau_n : n \geq 0\}$ to be finite. Note that finitariness implies both (BA) and Conditions ($\mathcal{P}_1$) and ($\mathcal{P}_3$), that (BA) implies ($\mathcal{P}_1$), and that, under ($\mathcal{P}_3$), finitariness and (BA) are equivalent. There are several papers in which a finitary $\mathcal{S}$-adic sequence is looked for a subshift in (L) (see [Ler14] and the references therein), and $\mathcal{S}$-adic sequences with (BA) have shown to be closely connected with (L) and (NSL) [Fer96; DDMP21]. It is then natural to ask if we can replace, in Theorem 0.10, Conditions ($\mathcal{P}_1$) and ($\mathcal{P}_3$) by finitariness. We show in Theorem 4.77 that this is not possible. More precisely, we build a minimal subshift with linear-growth complexity such that any $\boldsymbol{\tau}$ generating it and satisfying ($\mathcal{P}_1$), ($\mathcal{P}_2$) and ($\mathcal{P}_3$) is not finitary (equivalently, (BA) does not hold). However, in Theorems 4.75 and 4.76 we give a sufficient condition for $\boldsymbol{\tau}$ being finitary. Subshifts satisfying this sufficient condition include substitutive subshifts, codings of IETs and dendric subshifts.

‡A substitution $\sigma \colon \mathcal{A} \to \mathcal{B}^+$ is *positive* if for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, b occurs in $\sigma(a)$

Organization

The first chapter of the thesis is devoted to present the common background on topological and symbolic dynamics that will be used in the rest of the document. We study automorphisms and symbolic factors of finite topological rank systems in Chapters 2 and 3, respectively. Chapter 4 contains our results on the $\mathcal{S}$-adic conjecture. Finally, a discussion of our results and the future work is contained in Chapter 5.

Chapter 1

Background

1.1 Background in topological and symbolic dynamics

All the intervals we will consider consist of integer numbers, *i.e.*, $[a, b] = \{k \in \mathbb{Z} : a \leq k \leq b\}$ with $a, b \in \mathbb{Z}$. For us, the set of natural numbers starts with zero, *i.e.*, $\mathbb{N} = \{0, 1, \dots\}$.

1.1.1 Basics in topological dynamics

A *topological dynamical system* (or just a system) is a pair (X, S), where X is a compact metric space and $S\colon X \to X$ is a *homeomorphism* of X. The orbit of $x \in X$ is the set $\{S^n x : n \in \mathbb{Z}\}$. A point $x \in X$ is *periodic* if its orbit is a finite set and *aperiodic* otherwise. A topological dynamical system is *aperiodic* if any point $x \in X$ is aperiodic and is *minimal* if the orbit of every point is dense in X. We use the letter S to denote the action of a topological dynamical system independently of the base set X.

1.1.2 Basics in symbolic dynamics

Words and subshifts

Let $\mathcal{A}$ be an *alphabet i.e.* a finite set. Elements in $\mathcal{A}$ are called *letters* and concatenations $w = a_1 \dots a_\ell$ of them are called words. The number ℓ is the length of w and it is denoted by $|w|$, the set of all words in $\mathcal{A}$ of length ℓ is $\mathcal{A}^\ell$, and $\mathcal{A}^+ = \bigcup_{\ell \geq 1} \mathcal{A}^\ell$. The word $w \in \mathcal{A}^+$ is $|u|$-*periodic*, with $u \in \mathcal{A}^+$, if w occurs in a word of the form $uu \dots u$. We define $\mathsf{per}(w)$ as the smallest p for which w is p-periodic. We will use notation analogous to the one introduced in this paragraph when dealing with infinite words $x \in \mathcal{A}^\mathbb{N}$ and bi-infinite words $x \in \mathcal{A}^\mathbb{Z}$. The set $\mathcal{A}^+$ equipped with the operation of concatenation can be viewed as the free semigroup on $\mathcal{A}$. It is convenient to introduce the empty word 1, which has length 0 and is a neutral element for the concatenation. In particular, $\mathcal{A}^+ \cup \{1\}$ is the free monoid in $\mathcal{A}$.

Let $\mathcal{W} \subseteq \mathcal{A}^*$ be a set of words and $u \in \mathcal{A}^*$. We write $u\mathcal{W} = \{uw : w \in \mathcal{W}\}$, $\mathcal{W}u = \{wu : w \in \mathcal{W}\}$, and also

$$\langle \mathcal{W} \rangle \coloneqq \min_{w \in \mathcal{W}} |w| \qquad \text{and} \qquad |\mathcal{W}| \coloneqq \max_{w \in \mathcal{W}} |w|.$$

The *shift map* $S\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $S((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. For $x \in \mathcal{A}^{\mathbb{Z}}$ and integers $i < j$, we denote by $x_{[i,j)}$ the word $x_i x_{i+1} \dots x_j$. Analogous notation will be used when dealing with intervals of the form $[i,\infty)$, (i,∞), $(-\infty,i]$ and $(-\infty,i)$. A *subshift* is a topological dynamical system (X,S) where X is a closed and S-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ (we consider the product topology in $\mathcal{A}^{\mathbb{Z}}$) and S is the shift map. Classically one identifies (X,S) with X, so one says that X itself is a subshift. When we say that a sequence in a subshift is periodic (resp. aperiodic), we implicitly mean that this sequence is periodic (resp. aperiodic) for the action of the shift. Therefore, if $x \in \mathcal{A}^{\mathbb{Z}}$ is periodic, then $\mathsf{per}(x)$ is equal to the size of the orbit of x. The *language* of a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is the set $\mathcal{L}(X)$ of all words $w \in \mathcal{A}^+$ that occur in some $x \in X$.

Morphisms and substitutions

Let $\mathcal{A}$ and $\mathcal{B}$ be finite alphabets and $\tau\colon \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism between the free semigroups that they define. Then, τ extends naturally to maps from $\mathcal{A}^{\mathbb{N}}$ to itself and from $\mathcal{A}^{\mathbb{Z}}$ to itself in the obvious way by concatenation (in the case of a twosided sequence we apply τ to positive and negative coordinates separately and we concatenate the results at coordinate zero). We say that τ is *primitive* if for every $a \in \mathcal{A}$, all letters $b \in \mathcal{B}$ occur in $\tau(a)$. The minimum and maximum length of τ are the numbers $\langle\tau\rangle \coloneqq \langle\tau(\mathcal{A})\rangle = \min_{a\in\mathcal{A}} |\tau(a)|$ and $|\tau| \coloneqq |\tau(\mathcal{A})| = \max_{a\in\mathcal{A}} |\tau(a)|$, respectively.

We observe that any map $\tau\colon \mathcal{A} \to \mathcal{B}^+$ can be naturally extended to a morphism (that we also denote by τ) from $\mathcal{A}^+$ to $\mathcal{B}^+$ by concatenation, and we use this convention throughout the document. So, from now on, all maps between finite alphabets are considered to be morphisms between their associated free semigroups.

Factorizations and recognizability

Definition 1.1 Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift and $\sigma\colon \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism. We say that $(k,x) \in \mathbb{Z}\times X$ is a *σ-factorization of $y \in \mathcal{B}^{\mathbb{Z}}$ in X* if $y = S^k\sigma(x)$. If moreover $k \in [0, |\sigma(x_0)|)$, then (k,x) is a *centered σ-factorization in X*.

The pair (X,σ) is *recognizable* if every point $y \in \mathcal{B}^{\mathbb{Z}}$ has at most one centered σ-factorization in X, and *recognizable with constant $r \in \mathbb{N}$* if whenever $y_{[-r,r]} = y'_{[-r,r]}$ and (k,x), (k',x') are centered σ-factorizations of $y, y' \in \mathcal{B}^{\mathbb{Z}}$ in X, respectively, we have $(k,x_0) = (k',x'_0)$.

The *cuts* of (k,x) are defined by

$$c_{\sigma,j}(k,x) = \begin{cases} -k + |\sigma(x_{[0,j)})| & \text{if } j \geq 0, \\ -k - |\sigma(x_{[j,0)})| & \text{if } j < 0. \end{cases}$$

We write $C_\sigma(k,x) = \{c_{\sigma,j}(k,x) : j \in \mathbb{Z}\}$.

Remark 1.1 In the context of the previous definition:

(i) The point $y \in \mathcal{B}^{\mathbb{Z}}$ has a (centered) σ-factorization in X if and only if y belongs to the subshift $Y \coloneqq \bigcup_{n\in\mathbb{Z}} S^n\sigma(X)$. Hence, (X,σ) is recognizable if and only if every $y \in Y$ has a exactly *one* centered σ-factorization in X.

(ii) If (k,x) is a σ-factorization of $y \in \mathcal{B}^{\mathbb{Z}}$ in X, then $(c_{\sigma,j}(k,x), S^j x)$ is a σ-factorization of y in X for any $j \in \mathbb{Z}$. There is exactly one factorization in this class that is centered.

(iii) If (X,σ) is recognizable, then it is recognizable with constant r for some $r \in \mathbb{N}$ [DDMP21].

The behavior of recognizability under composition of morphisms is given by the following lemma.

Lemma 1.1 ([BSTY19], Lemma 3.5) *Let $\sigma\colon \mathcal{A}^+ \to \mathcal{B}^+$ and $\tau\colon \mathcal{B}^+ \to \mathcal{C}^+$ be morphisms, $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift and $Y = \bigcup_{k\in\mathbb{Z}} S^k\sigma(X)$. Then, $(X,\tau\sigma)$ is recognizable if and only if (X,σ) and (Y,τ) are recognizable.*

$\mathcal{S}$-adic subshifts

We recall the definition of an $\mathcal{S}$*-adic subshift* as stated in [BSTY19]. An $\mathcal{S}$*-adic sequence* or *directive sequence* $\boldsymbol{\sigma}$ is a sequence of morphisms having the form $(\sigma_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$. For $0 \le n < N$, we denote by $\sigma_{[n,N)}$ the morphism $\sigma_n \circ \sigma_{n+1} \circ \cdots \circ \sigma_{N-1}$. We say that $\boldsymbol{\sigma}$ is *everywhere growing* if

$$\lim_{N\to+\infty} \langle \sigma_{[0,N)} \rangle = +\infty, \tag{1.1}$$

and *primitive* if for any $n \in \mathbb{N}$ there exists $N > n$ such that $\sigma_{[n,N)}$ is positive. We remark that this notion is slightly different from the usual one used in the context of substitutional dynamical systems. Observe that $\boldsymbol{\sigma}$ is everywhere growing if $\boldsymbol{\sigma}$ is primitive. Let $\mathcal{P}$ be a property for morphisms (*e.g.* proper, letter-onto, etc). We say that $\boldsymbol{\sigma}$ has property $\mathcal{P}$ if σ_n has property $\mathcal{P}$ for every $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, we define

$$X_{\boldsymbol{\sigma}}^{(n)} = \left\{ x \in \mathcal{A}_n^{\mathbb{Z}} :\ \forall \ell \in \mathbb{N},\, x_{[-\ell,\ell]} \text{ occurs in } \sigma_{[n,N)}(a) \text{ for some } N > n, a \in \mathcal{A}_N \right\}.$$

This set clearly defines a subshift that we call the *nth level of the* $\mathcal{S}$*-adic subshift generated by* $\boldsymbol{\sigma}$. We set $X_{\boldsymbol{\sigma}} = X_{\boldsymbol{\sigma}}^{(0)}$ and simply call it the $\mathcal{S}$*-adic subshift* generated by $\boldsymbol{\sigma}$. If $\boldsymbol{\sigma}$ is everywhere growing, then every $X_{\boldsymbol{\sigma}}^{(n)}$, $n \in \mathbb{N}$, is nonempty; if $\boldsymbol{\sigma}$ is primitive, then $X_{\boldsymbol{\sigma}}^{(n)}$ is minimal for every $n \in \mathbb{N}$. There are non-everywhere growing directive sequences that generate minimal subshifts.

The relation between levels of an $\mathcal{S}$-adic subshift is given by the following lemma.

Lemma 1.2 ([BSTY19], Lemma 4.2) *Let $\boldsymbol{\sigma} = (\sigma_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ be a directive sequence of morphisms. If $0 \le n < N$ and $x \in X_{\boldsymbol{\sigma}}^{(n)}$, then there exists a (centered) $\sigma_{[n,N)}$-factorization in $X_{\boldsymbol{\sigma}}^{(N)}$. In particular, $X_{\boldsymbol{\sigma}}^{(n)} = \bigcup_{k\in\mathbb{Z}} S^k \sigma_{[n,N)}(X_{\boldsymbol{\sigma}}^{(N)})$.*

We define the *alphabet rank* of a directive sequence $\boldsymbol{\tau}$ as

$$AR(\boldsymbol{\tau}) = \liminf_{n\to+\infty} \#\mathcal{A}_n.$$

A *contraction* of $\boldsymbol{\tau}$ is a sequence $\tilde{\boldsymbol{\tau}} = (\tau_{[n_k, n_{k+1})} \colon \mathcal{A}_{n_{k+1}}^+ \to \mathcal{A}_{n_k}^+)_{k\in\mathbb{N}}$, where $0 = n_0 < n_1 < n_2 < \dots$. Observe that any contraction of $\boldsymbol{\tau}$ generates the same $\mathcal{S}$-adic subshift $X_{\boldsymbol{\tau}}$. When the context is clear, we will use the same notation to refer to $\boldsymbol{\tau}$ and its contractions. If $\boldsymbol{\tau}$ has finite alphabet rank, then there exists a contraction $\tilde{\boldsymbol{\tau}} = (\tau_{[n_k, n_{k+1})} \colon \mathcal{A}_{n_{k+1}}^+ \to \mathcal{A}_{n_k}^+)_{k\in\mathbb{N}}$ of $\boldsymbol{\tau}$ in which $\mathcal{A}_{n_k}$ has cardinality $AR(\boldsymbol{\tau})$ for every $k \geq 1$.

Chapter 2

Automorphisms

2.1 Introduction

Automorphism groups of low complexity subshifts have gained considerable attention in recent years. Unlike the case of mixing shifts of finite type, where the algebraic structure of this group can be very rich [BLR88; KR90; FF96], the automorphism group of low complexity subshifts has a high degree of rigidity. The most relevant example illustrating this fact is the case of minimal subshifts of nonsuperlinear-growth complexity, in which the automorphism group is virtually $\mathbb{Z}$ [CK15; DDMP16]. In this chapter, we study the automorphism group of minimal $\mathcal{S}$-adic subshifts of finite alphabet rank. This class of subshifts contains all minimal subshifts of nonsuperlinear-growth complexity, but it is much broader, as was shown in [DDMP16; DDMP21].

The main result of this chapter is the following rigidity theorem:

Theorem 2.1 *Let (X,T) be a minimal $\mathcal{S}$-adic subshift generated by an everywhere growing $\mathcal{S}$-adic sequence $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\geq 0}$. Suppose that $\boldsymbol{\tau}$ is of finite alphabet rank,* i.e. $\liminf_{n\to+\infty} \#\mathcal{A}_n < +\infty$. *Then,* $\mathrm{Aut}(X,T)$ *is virtually $\mathbb{Z}$.*

The proof of Theorem 2.1 is a consequence of a fine combinatorial analysis of asymptotic classes of $\mathcal{S}$-adic subshifts of finite alphabet rank, which we summarize in the following theorem.

Theorem 2.2 *Let $\mathcal{W} \subseteq \mathcal{A}^+$ be a set of nonempty words and define $\langle\mathcal{W}\rangle := \min_{w\in\mathcal{W}} \mathrm{length}(w)$. Then, there exists $\mathcal{B} \subseteq \mathcal{A}^{\langle\mathcal{W}\rangle}$ with $\#\mathcal{B} \leq 122(\#\mathcal{W})^7$ such that: if $x, x' \in \mathcal{A}^{\mathbb{Z}}$ are factorizable over $\mathcal{W}$, $x_{(-\infty,0)} = x'_{(-\infty,0)}$ and $x_0 \neq x'_0$, then $x_{[-\langle\mathcal{W}\rangle,0)} \in \mathcal{B}$.*

This chapter was published as a standalone article in [Esp22a].

2.1.1 Organization

The chapter is organized as follows. In the next section we give additional background in topological and symbolic dynamics. In Section 2.3 we introduce some special ingredients allowing to prove the main theorems: the notions of *interpretation* and *reducibility* of sets of words together with its properties and the key Proposition 2.6, whose technical proof is given in Section 2.5. In Section 2.4 we restate our main results and provide complete proofs.

2.2 Additional background

An *automorphism* of the topological dynamical system (X, S) is a homeomorphism $\varphi\colon X \to X$ such that $\varphi \circ S = S \circ \varphi$. We use the notation $\varphi\colon (X, S) \to (X, S)$ to indicate the automorphism. The set of all automorphisms of (X, S) is denoted by $\mathrm{Aut}(X, S)$ and is called the *automorphism group* of (X, S). It has a group structure given by the composition of functions. It is said that $\mathrm{Aut}(X, S)$ is *virtually* $\mathbb{Z}$ if the quotient $\mathrm{Aut}(X, S)/\langle S\rangle$ is finite, where $\langle S\rangle$ is the subgroup generated by S.

We write $\leq_p$ and $\leq_s$ for the relations in $\mathcal{A}^*$ of being prefix and suffix, respectively. We also write $u <_p v$ (resp. $u <_s v$) when $u \leq_p v$ (resp. $u \leq_s v$) and $u \neq v$. When $v = sut$, we say that u *occurs* in v or that u is a *subword* of v. We also use these notions and notations when considering prefixes, suffixes and subwords of infinite sequences.

2.3 Notion of *Interpretation*

In this section we introduce the concepts of *interpretation* and *double interpretation* of a word together with its basic properties. The definitions we provide here are variants of the same notion used seldom in combinatorics of words, see for example [Lot97]. The key Proposition 2.6, where we provide a fundamental upper bound for the number of *irreducible sets of simple double interpretations*, is announced here and proved in the last section of the chapter.

For the rest of this section we fix an alphabet $\mathcal{A}$ and a finite set of nonempty words $\mathcal{W} \subseteq \mathcal{A}^+$. If $u, v, w \in \mathcal{A}^*$ are such that $w = uv$, then we write $u = wv^{-1}$ and $v = u^{-1}w$.

2.3.1 Interpretations and simple double interpretations

Definition 2.1 Let $d \in \mathcal{A}^+$. A $\mathcal{W}$*-interpretation* of d is a sequence of words $I = \mathsf{d}_L, \mathsf{d}_M, \mathsf{d}_R, a$ such that:

(1) $\mathsf{d}_M \in \mathcal{W}^*$ and $a \in \mathcal{A}$;

(2) there exist $\mathsf{u}_L, \mathsf{u}_R \in \mathcal{W}$ such that $1 \neq \mathsf{d}_L \leq_s \mathsf{u}_L$, $\mathsf{d}_R a \leq_p \mathsf{u}_R$;

(3) $d = \mathsf{d}_L \mathsf{d}_M \mathsf{d}_R$.

See Figure 2.1 for an illustration of this definition. Note that d_M and d_R can be the empty word. The extra letter a will be crucial to handle asymptotic pairs and $\mathcal{W}$-interpretations later.

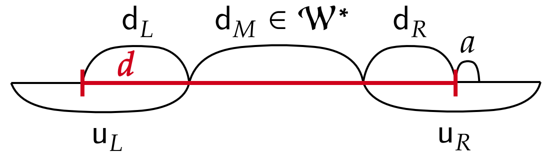


Figure 2.1: Diagram of the $\mathcal{W}$-interpretation $I = \mathsf{d}_L, \mathsf{d}_M, \mathsf{d}_R, a$ of d in Definition 2.1.

If the context is clear, we will say *interpretation* instead of $\mathcal{W}$-interpretation.

Now we make an observation that will be useful when we want to inherit interpretations of a given word to some of its subwords. We state it as a lemma without proof.

Lemma 2.3 *Let $I = \mathsf{d}_L, \mathsf{d}_M, \mathsf{d}_R, a$ be a $\mathcal{W}$-interpretation of $d \in \mathcal{A}^+$. Suppose that $d' \leq_p d$ satisfy $|d'| \geq |\mathsf{d}_L|$. Then, d' has a $\mathcal{W}$-interpretation of the form $I' = \mathsf{d}_L, \mathsf{d}'_M, \mathsf{d}'_R, a'$ such that $d'a' \leq_p da$.*

The proofs of our main theorems are based in a procedure allowing to reduce the so called *double interpretations* (defined below) to a special class called *simple double interpretations*.

Definition 2.2 Let $d \in \mathcal{A}^+$. A $\mathcal{W}$-double interpretation (written for short $\mathcal{W}$-*d.i.*) of d is a tuple $D = (I; I')$, where $I = \mathsf{d}_L, \mathsf{d}_M, \mathsf{d}_R, a$, $I' = \mathsf{d}'_L, \mathsf{d}'_M, \mathsf{d}'_R, a'$ are $\mathcal{W}$-interpretations of d such that $a \neq a'$. We say that D is *simple* if in addition

(1) $\mathsf{d}'_M \mathsf{d}'_R \leq_s \mathsf{d}_R$, and

(2) $\mathsf{d}'_L \in \mathcal{W}$ or $|\mathsf{d}'_L| \geq |\mathsf{u}|$ for some $\mathsf{u} \in \mathcal{W}$ having $\mathsf{d}_R a$ as a prefix.

Again, if there is no ambiguity, we will omit $\mathcal{W}$ and simply say *double interpretation* or *d.i.*

Note that if D is simple, then $D' = (I'; I)$ is a d.i., which is not necessarily simple. Condition (1) in the previous definition says that d'_L, the left-most word of I', "touches" d_R, the right-most word of I; see Figure 2.2 for an illustration of this. Condition (2) is more technical and we will comment about it at the end of the Subsection 2.3.2.

Remark 2.1 From condition ((2)) in previous definition we have that $|\mathsf{d}'_L|, |d| \geq \langle \mathcal{W} \rangle$, whenever D is a simple $\mathcal{W}$-d.i.

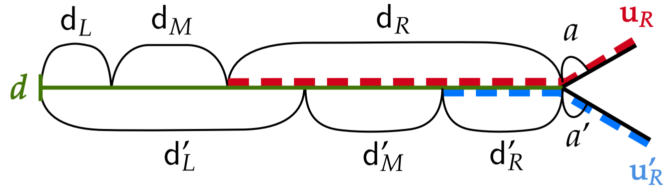


Figure 2.2: Diagram of a d.i. of d satisfying (1) in Definition 2.2. Here, $\mathsf{d}_R a \leq_p \mathsf{u}_R$ and $\mathsf{d}'_R a' \leq_p \mathsf{u}'_R$, where $\mathsf{u}_R, \mathsf{u}'_R$ are the words given in condition (2) of Definition 2.1.

The next lemma will be useful to build a *simple* double interpretation from a word having a double interpretation.

Lemma 2.4 *Let $D = (I = \mathsf{d}_L, \mathsf{d}_M, \mathsf{d}_R, a; I' = \mathsf{d}'_L, \mathsf{d}'_M, \mathsf{d}'_R, a')$ be a double interpretation of a word $d \in \mathcal{A}^+$. Suppose that $\mathsf{d}'_L \in \mathcal{W}$ and $|\mathsf{d}_L| \leq |\mathsf{d}'_L \mathsf{d}'_M|$. Then, there exists $e \leq_s d$ with a simple double interpretation.*

Proof. By considering the shortest suffix of d verifying the hypotheses of the lemma we can assume without loss of generality that this suffix is d itself. We consider three cases.

(1) $\mathsf{d}'_L <_p \mathsf{d}_L$. This condition and the hypotheses of the lemma imply that $\mathsf{d}'_L <_p \mathsf{d}_L \leq_p \mathsf{d}'_L \mathsf{d}'_M$. Therefore, d'_M is not the empty word and we can write $\mathsf{d}'_M = uv$, with $u \in \mathcal{W}$ and $v \in \mathcal{W}^*$. Then, $e \coloneqq \mathsf{d}'_M \mathsf{d}'_R <_s d$ has the interpretations $J = (\mathsf{d}'_L)^{-1}\mathsf{d}_L, \mathsf{d}_M, \mathsf{d}_R, a$ (here we are using that $(\mathsf{d}'_L)^{-1}\mathsf{d}_L \neq 1$) and $J' = u, v, \mathsf{d}'_R, a'$. But $u \in \mathcal{W}$ and $|(\mathsf{d}'_L)^{-1}\mathsf{d}_L| \leq |(\mathsf{d}'_L)^{-1}\mathsf{d}'_L \mathsf{d}'_M| = |uv|$, so e is a strict suffix of d having a d.i. $E \coloneqq (J; J')$ verifying the hypotheses of the lemma, which contradicts the minimality of d. Thus, this case is incompatible with the hypotheses.

(2) $\mathsf{d}_L <_p \mathsf{d}'_L$. If D is not a simple d.i. we have $\mathsf{d}_R <_s \mathsf{d}'_M \mathsf{d}'_R$ since $\mathsf{d}'_L \in \mathcal{W}$ and then $\mathsf{d}_L <_p \mathsf{d}'_L \leq_p \mathsf{d}_L \mathsf{d}_M$. This implies that d_M is not the empty word. Then, we can write $\mathsf{d}_M = uv$ with $u \in \mathcal{W}$ and $v \in \mathcal{W}^*$. We have that $E = (J = \mathsf{d}_L^{-1}\mathsf{d}'_L, \mathsf{d}'_M, \mathsf{d}'_R, a';\ J' = u, v, \mathsf{d}_R, a)$ is a d.i. of $e \coloneqq \mathsf{d}_M \mathsf{d}_R <_s d$ which, in addition, satisfies $u \in \mathcal{W}$ and $|\mathsf{d}_L^{-1}\mathsf{d}'_L| \leq |uv|$. This contradicts the minimality of d and D must be simple.

(3) $\mathsf{d}_L = \mathsf{d}'_L$. If $\mathsf{d}_M = 1$ or $\mathsf{d}'_M = 1$, it follows directly from definition that $D = (I, I')$ or $D' = (I', I)$ are simple d.i. respectively. So we assume $\mathsf{d}_M \neq 1$ and $\mathsf{d}'_M \neq 1$. Therefore, we can write $\mathsf{d}_M = uv$ and $\mathsf{d}'_M = u'v'$, with $u, u' \in \mathcal{W}$ and $v, v' \in \mathcal{W}^*$. Let $e \coloneqq \mathsf{d}_M \mathsf{d}_R = \mathsf{d}'_M \mathsf{d}'_R$, $J = u, v, \mathsf{d}_R, a$ and $J' = u', v', \mathsf{d}'_R, a'$. Observe that when $|u'| \leq |u|$, $E = (J'; J)$ is a d.i. of e satisfying $u \in \mathcal{W}$ and $|u'| \leq |uv|$, and when $|u| \leq |u'|$, $E = (J; J')$ is a d.i. of e satisfying $u' \in \mathcal{W}$ and $|u| \leq |u'v'|$. In both cases we get a contradiction with the minimality of d. Then, in this case either D or D' is a simple d.i. of d.

□

A point $x \in \mathcal{A}^{\mathbb{Z}}$ is *factorizable* over $\mathcal{W}$ if there exist a point $y \in \mathcal{W}^{\mathbb{Z}}$ and $k \in \mathbb{Z}$ such that $x_{[k,\infty)} = y_0 y_1 y_2 \cdots$ and $x_{(-\infty,k)} = \cdots y_{-3} y_{-2} y_{-1}$. For example, if $\boldsymbol{\tau}$ is a directive sequence, $0 \leq n < N$ and $x \in X_{\boldsymbol{\tau}}^{(n)}$, from Lemma 1.2 we see that x is factorizable over $\tau_{[n,N)}(\mathcal{A}_N)$.

The last lemma of this subsection gives the relation between asymptotic pairs that are factorizable over the set of words $\mathcal{W}$ and simple double interpretations over $\mathcal{W}$. This lemma is crucial to reduce our combinatorial studies in next sections to the case of simple double interpretations.

Lemma 2.5 *If $x, x' \in \mathcal{A}^{\mathbb{Z}}$ are factorizable over $\mathcal{W}$, $x_{(-\infty,0)} = x'_{(-\infty,0)}$ and $x_0 \neq x'_0$, then there exists a word $e \leq_s x_{(-\infty,0)}$ having a simple double interpretation over $\mathcal{W}$.*

Proof. Let $l \geq 2|\mathcal{W}|$ and $d := x_{[-l,0)}$. Then d inherits in a natural way interpretations $I = \mathsf{d}_L, \mathsf{d}_M, \mathsf{d}_R, a$ and $I' = \mathsf{d}'_L, \mathsf{d}'_M, \mathsf{d}'_R, a'$ from the factorizations of x and x' respectively. Since $a = x_0 \neq x'_0 = a'$, the tuple $D := (I; I')$ is a d.i. Moreover, by choosing adequately l we can suppose that $d'_L \in \mathcal{W}$. Also, $|\mathsf{d}_L| \leq |\mathcal{W}| \leq l - |\mathsf{d}'_R| = |\mathsf{d}'_L \mathsf{d}'_M|$, so the hypotheses of Lemma 2.4 hold. Thus d (and of course $x_{(-\infty,0)}$) has a suffix e with a simple double interpretation over $\mathcal{W}$. This proves the lemma. □

2.3.2 Reducible and irreducible simple double interpretations

In this section we introduce the notions of *reducible* and *irreducible* sets of simple double interpretations. In Proposition 2.6 we provide an upper bound for the size of irreducible sets of simple d.i. (the proof of this proposition is very technical and is postponed until Section 2.5). Thus, even if in some cases it is not necessary, most of the notions appearing in this section will be considered only for simple d.i.

For the rest of the chapter each time we use a letter D to denote a d.i. on $\mathcal{W}$, then it double interprets the word $d \in \mathcal{A}^+$ and is written $D = (I_D = \mathsf{d}_L, \mathsf{d}_M, \mathsf{d}_R, a_D;\ I'_D = \mathsf{d}'_L, \mathsf{d}'_M, \mathsf{d}'_R, a'_D)$.

Definition 2.3 Given $U = (\mathsf{u}_M, \mathsf{u}_R, \mathsf{u}'_L, \mathsf{u}'_M, \mathsf{u}'_R, \ell) \in \mathcal{W}^5 \times \mathbb{N}$, we define $\mathcal{D}_U$ as the set of simple $\mathcal{W}$-d.i. D such that:

(1) either $\mathsf{d}_M \in \mathcal{W}^* \mathsf{u}_M$ or $\mathsf{d}_M = 1$ and $\mathsf{d}_L \leq_s \mathsf{u}_M$;

(2) $\mathsf{d}_R a_D \leq_p \mathsf{u}_R$ and $|\mathsf{u}_\mathsf{R}| = \min\{|w| : \mathsf{d}_R a_D \leq_p w,\ w \in \mathcal{W}\}$;

(3) $\mathsf{d}'_R a'_D \leq_p \mathsf{u}'_R$, $\mathsf{d}'_L \leq_s \mathsf{u}'_L$ and $|\mathsf{u}'_L| = \min\{|w| : \mathsf{d}'_L \leq_s w,\ w \in \mathcal{W}\}$;

(4) $\mathsf{d}'_M = 1$ or $\mathsf{d}'_M = v_1 \cdots v_n \in \mathcal{W}^+$, $v_1 = \mathsf{u}'_M$ and $\max_{1 \leq j \leq n} |v_j| = \ell$.

It is easy to see that

$$\mathcal{D} := \bigcup_{U \in \mathcal{W}^5 \times \mathbb{N}} \mathcal{D}_U$$

is the set of all simple $\mathcal{W}$-d.i. of words in $\mathcal{A}^+$. Moreover, from ((4)) of Definition 2.3 we have that $\ell \in \{|w| : w \in \mathcal{W}\} \cup \{0\}$ when $\mathcal{D}_U \neq \emptyset$, so $\mathcal{D}$ is the union of no more than $\#\mathcal{W}^5(\#\mathcal{W}+1)$ sets $\mathcal{D}_U$.

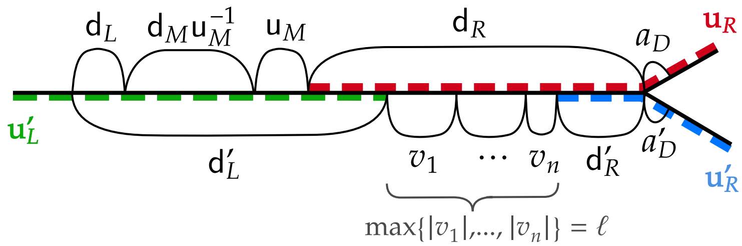


Figure 2.3: Diagram illustrating restrictions in Definition 2.3 for a simple d.i. in the case $\mathsf{d}_M, \mathsf{d}'_M \neq 1$.

Definition 2.4 Let D, E be simple d.i. on $\mathcal{W}$. We say that,

(1) D is equivalent to E, and we write $D \sim E$, if d and e have a common suffix of length at least $\langle \mathcal{W} \rangle$ (this makes sense by Remark 2.1).

(2) D reduces to E, and we write $D \Rightarrow E$, if $e <_s d$.

Observe that, when D and E are simple d.i. on $\mathcal{W}$ with $D \Rightarrow E$, then, by Remark 2.1, $D \sim E$.

Definition 2.5 A subset $\mathcal{D}' \subseteq \mathcal{D}$ of simple d.i. is *reducible* if

(1) there are two different and equivalent elements in $\mathcal{D}'$, or

(2) there exists $D \in \mathcal{D}'$ that reduces to some simple d.i.

If $\mathcal{D}'$ is not reducible, we say that it is *irreducible*.

The main combinatorial result about irreducible sets of simple d.i. is the following proposition, whose proof will be carried out in Section 2.5.

Proposition 2.6 *Let $U \in \mathcal{W}^5 \times \mathbb{N}$. Any irreducible subset of $\mathcal{D}_U$ has at most $61(\#\mathcal{W})$ elements.*

The use of condition (2) of Definition 2.2 appears during the proof of this proposition. This proof consists in directly showing that sets $\mathcal{D}' \subseteq \mathcal{D}_U$ with more than $61(\#\mathcal{W})$ elements are reducible. For this, one finds elements in $\mathcal{D}'$ that are equivalent or can be reduced. In this process, one observes that eliminating condition (2) in the definition of simple d.i. has two opposite effects. On one hand, it should be easier to find a reduction of a given simple d.i., since more d.i. are simple; but on the other hand, without condition (2) being simple means less structure, so it is more difficult to actually find the desired reductions during the proof. Balancing this trade-off is the reason behind the technical condition (2). It is worth mentioning that this condition (2) is only used in the proof of Lemma 2.11.

2.4 Proof of main results

In this section we prove our main results. As we commented in the introduction, the proof of Theorem 2.1 is based on two general steps: first we use a proposition from [DDMP16] relating the number of asymptotic components with the "size" of the automorphism group and secondly we develop a complete combinatorial analysis of the asymptotic classes arising in an $\mathcal{S}$-adic subshift of finite alphabet rank.

Let (X, S) be a topological dynamical system. Two points $x, x' \in X$ are (negatively) *asymptotic* if $\lim_{n\to-\infty} \operatorname{dist}(S^n x, S^n x') = 0$. We define the relation $\sim$ in X as follows: $x \sim x'$ whenever x is asymptotic to $S^k x'$ for some $k \in \mathbb{Z}$. It is easy to see that $\sim$ is an equivalence relation. An equivalence class for $\sim$ that is not the orbit of a single point is called an *asymptotic class*, and we write $\operatorname{Asym}(X, S)$ for the set of asymptotic classes of (X, S). Observe that if (X, S) is a subshift, then $x \sim x'$ if and only if $x_{(-\infty,k)} = x'_{(-\infty,\ell)}$ for some $k, \ell \in \mathbb{Z}$.

The following proposition, which is a direct consequence of Corollary 3.3 in [DDMP16], gives a relation between the number of asymptotic classes and the cardinality of $\operatorname{Aut}(X, S)/\langle S\rangle$ under conditions that any infinite minimal subshift satisfies.

Proposition 2.7 *Let (X, S) be a topological dynamical system. Assume there exists a point $x_0 \in X$ with $\omega(x_0) \coloneqq \bigcap_{n\geq 0} \overline{\{S^k x_0 : k \geq n\}} = X$ that is asymptotic to a different point. Then, $\#\operatorname{Aut}(X, S)/\langle S\rangle \leq \#\operatorname{Asym}(X, S)!$.*

Now we prove our first combinatorial theorem.

Theorem 2.2 Let $\mathcal{W} \subseteq \mathcal{A}^+$ be a set of nonempty words. Then, there exists $\mathcal{B} \subseteq \mathcal{A}^{\langle\mathcal{W}\rangle}$ with $\#\mathcal{B} \leq 122(\#\mathcal{W})^7$ such that: if $x, x' \in \mathcal{A}^{\mathbb{Z}}$ are factorizable over $\mathcal{W}$, $x_{(-\infty,0)} = x'_{(-\infty,0)}$ and $x_0 \neq x'_0$, then $x_{[-\langle\mathcal{W}\rangle,0)} \in \mathcal{B}$.

As will be clear from the proof, the bound "$122(\#\mathcal{W})^7$" is not necessarily optimal. Here, the important point is that, despite the fact that the length of the elements in $\mathcal{B}$ is $\langle\mathcal{W}\rangle$, the cardinality of $\mathcal{B}$ depends only on $\#\mathcal{W}$, and not on $\langle\mathcal{W}\rangle$.

Proof. We start by defining the set $\mathcal{B}$. For each $U = (\mathsf{u}_M, \mathsf{u}_R, \mathsf{u}'_L, \mathsf{u}'_M, \mathsf{u}'_R, \ell) \in \mathcal{W}^5 \times \mathbb{N}$, fix $\mathcal{D}'_U \subseteq \mathcal{D}_U$ an irreducible subset of maximal size (we consider the empty set as an irreducible set, so there always exists such set $\mathcal{D}'_U$). We define

$$\mathcal{B} \coloneqq \big\{w \in \mathcal{A}^{\langle\mathcal{W}\rangle} : \exists\, U \in \mathcal{W}^5 \times \mathbb{N},\ D \in \mathcal{D}'_U,\ w \leq_s d\big\},$$

where in this set $d \in \mathcal{A}^+$ represents the word that is double interpreted by D. We note that this makes sense because $|d| \geq \langle\mathcal{W}\rangle$ for all simple d.i. As we observed previously, we have $\ell \in \{|w| : w \in \mathcal{W}\} \cup \{0\}$ when $\mathcal{D}_U$ is nonempty. Thus, there are no more than $\#\mathcal{W}^5(\#\mathcal{W}+1)$ choices for U such that $\mathcal{D}_U$ is nonempty. Using this and Proposition 2.6 we get:

$$\begin{aligned}\#\mathcal{B} &\leq 61\#\mathcal{W} \cdot \#\{U \in \mathcal{W}^5 \times \mathbb{N} : \mathcal{D}_U \neq \emptyset\},\\ &\leq 61\#\mathcal{W} \cdot \#\mathcal{W}^5(\#\mathcal{W} + 1) \leq 122(\#\mathcal{W})^7.\end{aligned}$$

It rests to prove the main property of the theorem. In this purpose, let $x, x' \in \mathcal{A}^{\mathbb{Z}}$ be factorizable over $\mathcal{W}$ with $x_{(-\infty,0)} = x'_{(-\infty,0)}$ and $x_0 \neq x'_0$. From Lemma 2.5 we can find a simple d.i. D of $d \leq_s x_{(-\infty,0)}$. Let

$$D =: D(0) \Rightarrow D(1) \Rightarrow D(2) \Rightarrow \cdots \Rightarrow D(n)$$

be a sequence of reductions that starts with D (where, possibly, $n = 0$ and D has no reduction). We write, for convenience, $D(j) = (I(j); I'(j))$ and $d(j)$ for the word that is double interpreted by $D(j)$. Since $|d(0)| > |d(1)| > \dots$, any sequence like this ends after a finite number of steps. In particular, we can take (and we *are* taking) this sequence so that n is maximal. This implies that $D(n)$ has no reduction.

Since $\mathcal{D} = \bigcup_{U \in \mathcal{W}^5 \times \mathbb{N}} \mathcal{D}_U$, we can find $U \in \mathcal{W}^5 \times \mathbb{N}$ satisfying $D(n) \in \mathcal{D}_U$. We claim that there is a word e with a simple d.i. $E = (I_E; I'_E) \in \mathcal{D}'_U$ such that $D(n)$ is equivalent to E. Indeed, if $D(n) \in \mathcal{D}'_U$ then, since $D(n)$ is equivalent to itself, we can take $E \coloneqq D(n)$. If $D(n)$ is not in $\mathcal{D}'_U$, then, from the maximality of $\mathcal{D}'_U$ we see that $\mathcal{D}'_U \cup \{D(n)\}$ is reducible. Since $D(n)$ has no reduction and $\mathcal{D}'_U$ is irreducible, there exists $E \in \mathcal{D}'_U$ equivalent to $D(n)$. This proves the claim.

Then, using the definitions of reduction and equivalence of simple d.i., we have that the suffix $w \in \mathcal{A}^{\langle \mathcal{W} \rangle}$ of e satisfies

$$w \leq_s d(n) <_s d(n-1) <_s \cdots <_s d(0) \leq_s x_{(-\infty,0)},$$

and $w \in \mathcal{B}$ since $E \in \mathcal{D}'_U$. This finishes the proof. □

Now we have all the ingredients to compute the number of asymptotic classes in the case of $\mathcal{S}$-adic subshifts of finite alphabet rank.

Theorem 0.4 Let (X, S) be an $\mathcal{S}$-adic subshift given by an everywhere growing directive sequence of alphabet rank K. Then, (X, S) has at most $122K^7$ asymptotic classes.

Proof. Set $K' = 122K^7$. We are going to prove the following stronger result.

Claim 2.1 Let $\mathcal{P}$ be the set of pairs $(x, y) \in X \times X$ such that $x_{(-\infty,0)} = y_{(-\infty,0)}$ and $x_0 \neq y_0$. Then, $\#\{x_{(-\infty,0)} : (x, y) \in \mathcal{P}\} \leq K'$.

First, we show how this claim implies the theorem. Suppose the claim is true and let $C_0, \dots, C_{K'}$ be asymptotic classes for (X, S). For each $j \in \{0, \dots, K'\}$ we choose $(z_j, z'_j) \in C_j$ such that z_j and z'_j do not belong to the same orbit. Then, there exist $m_j, m'_j \in \mathbb{Z}$ such that $x_j \coloneqq S^{m_j} z_j$ and $y_j \coloneqq S^{m'_j} z'_j$ satisfy

$$(x_j)_{(-\infty,0)} = (y_j)_{(-\infty,0)} \quad \text{and} \quad (x_j)_0 \neq (y_j)_0, \quad \forall j \in \{0, \dots, K'\}. \tag{2.1}$$

Thus, $(x_j, y_j) \in \mathcal{P}$ for all $j \in \{0, \dots, K'\}$ and, by the claim and the Pigeonhole Principle, there exist different $j, j' \in \{0, \dots, K'\}$ such that $(x_j)_{(-\infty,0)} = (x_{j'})_{(-\infty,0)}$. This implies $C_j = C_{j'}$ and, thus, that (X, S) has at most K' asymptotic classes.

Now we prove the claim. Let $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\geq 0}$ be an everywhere growing directive sequence of alphabet rank K generating X. By doing a contraction, if required, we can suppose that $\#\mathcal{A}_n = K$ for every $n \geq 1$. For $n \geq 1$ put $\mathcal{W}_n = \tau_{[0,n)}(\mathcal{A}_n)$ and let $\mathcal{B}_n \subseteq \mathcal{A}_0^+$ be the set given by Theorem 2.2 when it is applied to $\mathcal{W}_n$. By hypothesis, $\#\mathcal{W}_n \leq \#\mathcal{A}_n = K$, so $\#\mathcal{B}_n \leq 122(\#\mathcal{W}_n)^7 \leq 122K^7 = K'$.

For $j \in \{0, \dots, K'\}$ let $(x_j, y_j) \in \mathcal{P}$. We have to show that $(x_j)_{(-\infty,0)} = (x_{j'})_{(-\infty,0)}$ for different $j, j' \in \{0, \dots, K'\}$. Since for all $n \geq 1$ and $j \in \{0, \dots, K'\}$ the points x_j and y_j are factorizable over $\mathcal{W}_n$ (Lemma 1.2), from Theorem 2.2 we have that $(x_j)_{[-\langle \mathcal{W}_n \rangle,0)} \in \mathcal{B}_n$. But $\#\mathcal{B}_n \leq K'$ so by the Pigeonhole Principle there exist $j_n, j'_n \in \{0, \dots, K'\}$ with $j_n \neq j'_n$ such that

$$(x_{j_n})_{[-\langle \mathcal{W}_n \rangle,0)} = (x_{j'_n})_{[-\langle \mathcal{W}_n \rangle,0)}. \tag{2.2}$$

Thus, again by the Pigeonhole Principle, we can choose $1 \leq n_1 < n_2 < \dots$ such that $j_{n_1} = j_{n_2} = \dots = j \neq j' = j'_{n_1} = j'_{n_2} = \dots$ By (2.2),

$$(x_j)_{[-\langle \mathcal{W}_{n_i} \rangle,0)} = (x_{j'})_{[-\langle \mathcal{W}_{n_i} \rangle,0)}, \qquad \forall i \geq 1. \tag{2.3}$$

Since $\boldsymbol{\tau}$ is everywhere growing, $\langle \mathcal{W}_n \rangle$ goes to infinity when $n \to +\infty$. Thus, (2.3) implies that $(x_j)_{(-\infty,0)} = (x_{j'})_{(-\infty,0)}$, as desired. This completes the proof. □

We remark again that in previous result we do not assume minimality. This hypothesis is needed in the next proof (of Theorem 2.1) only because we bound the size of the automorphism group by the number of asymptotic classes via Proposition 2.7. Thus, Theorem 2.1 is mainly a consequence of combinatorial facts inherent to $\mathcal{S}$-adic subshifts.

Theorem 2.1 Let (X, S) be a minimal $\mathcal{S}$-adic subshift given by an everywhere growing sequence of finite alphabet rank K. Then, its automorphism group is virtually $\mathbb{Z}$.

Proof. From Proposition 2.7 and Theorem 0.4 we get

$$\#\mathrm{Aut}(X, S)/\langle S \rangle \leq \#\mathrm{Asym}(X, S)! \leq \big(122K^7\big)! < +\infty.$$

This inequality proves that $\mathrm{Aut}(X, S)$ is virtually $\mathbb{Z}$. □

2.5 Proof of Proposition 2.6

In this last section we prove Proposition 2.6. All but one result we need (Lemma 2.4) are presented and proved here, so the section is almost self contained.

We fix, for the rest of this section, a finite set of words $\mathcal{W} \subseteq \mathcal{A}^+$ and a sequence $U = (\mathsf{u}_M, \mathsf{u}_R, \mathsf{u}'_L, \mathsf{u}'_M, \mathsf{u}'_R, \ell) \in \mathcal{W}^5 \times \mathbb{N}$. For $D \in \mathcal{D}_U$, we define:

$$\tilde{\mathsf{d}} \coloneqq \mathsf{d}_R(\mathsf{d}'_M\mathsf{d}'_R)^{-1} = (\mathsf{d}_L\mathsf{d}_M)^{-1}\mathsf{d}'_L.$$

We need a last definition: two words $u, v \in \mathcal{A}^*$ are *prefix dependent* (resp. suffix dependent) if $u \leq_p v$ or $v \leq_p u$ (resp. $u \leq_s v$ or $v \leq_s u$). In this case, u and v share a common prefix (resp. suffix) of length $\min(|u|, |v|)$.

Lemma 2.8 *Consider different elements D, E in $\mathcal{D}_U$. If any of the following conditions holds, then the set $\{D, E\}$ is reducible:*

(i) $\mathsf{d}'_M\mathsf{d}'_R a_D$, $\mathsf{e}'_M\mathsf{e}'_R a_E$ *are prefix dependent;*

(ii) $|\mathsf{d}_R| = |\mathsf{e}_R|$*;*

(iii) $|\tilde{\mathsf{d}}| \leq |\tilde{\mathsf{e}}| \leq |\tilde{\mathsf{d}}\mathsf{d}'_M|$ *or* $|\tilde{\mathsf{e}}| \leq |\tilde{\mathsf{d}}| \leq |\tilde{\mathsf{e}}\mathsf{e}'_M|$.

Proof. We will show that under conditions of the lemma one of the following relations occurs: $D \sim E$, E reduces to a simple d.i. or D reduces to a simple d.i.

(i) Without loss of generality, we can suppose that $\mathsf{d}'_M\mathsf{d}'_R a_D \leq_p \mathsf{e}'_M\mathsf{e}'_R a_E$. We distinguish two cases:

(1) $\mathsf{d}'_M\mathsf{d}'_R a_D = \mathsf{e}'_M\mathsf{e}'_R a_E$. Using item ((3)) of Definition 2.3 we can write $d = \mathsf{d}'_L\mathsf{d}'_M\mathsf{d}'_R \leq_s \mathsf{u}'_L\mathsf{d}'_M\mathsf{d}'_R$. Similarly, $e \leq_s \mathsf{u}'_L\mathsf{e}'_M\mathsf{e}'_R$. This and hypothesis (a) imply that d and e are suffix dependent. But, since D and E are simple d.i., by Remark 2.1 we have that $|d|, |e| \geq \langle\mathcal{W}\rangle$. We conclude that d and e share a suffix of length at least $\min(|d|, |e|) \geq \langle\mathcal{W}\rangle$, which implies $D \sim E$.

(2) $\mathsf{d}'_M\mathsf{d}'_R a_D <_p \mathsf{e}'_M\mathsf{e}'_R a_E$ (so, $\mathsf{d}'_M\mathsf{d}'_R a_D \leq_p \mathsf{e}'_M\mathsf{e}'_R$). We claim that $\ell > 0$ in the definition of U. Suppose that $\ell = 0$. Then, $\mathsf{d}'_M = \mathsf{e}'_M = 1$ and we can write:

$$\mathsf{d}'_R a_D \leq_p \mathsf{e}'_R \leq_p \mathsf{u}'_R.$$

Since by ((3)) of Definition 2.3 we also have $\mathsf{d}'_R a'_D \leq_p \mathsf{u}'_R$, we conclude that $a_D = a'_D$. This contradicts the fact that E is a d.i. Thus, $\ell > 0$.

Now, $\ell > 0$ and ((4)) of Definition 2.3 imply that $v_D := (\mathsf{u}'_M)^{-1}\mathsf{d}'_M \in \mathcal{W}^*$ and $v_E := (\mathsf{u}'_M)^{-1}\mathsf{e}'_M \in \mathcal{W}^*$. Let $w := \mathsf{d}'_M\mathsf{d}'_R$. Observe that $J_D = \mathsf{u}'_M, v_D, \mathsf{d}'_R, a'_D$ is an interpretation of w. Moreover, since $\mathsf{u}'_M \leq_p w <_p \mathsf{u}'_M v_E \mathsf{e}'_R$ by hypothesis (b) and $v_E \in \mathcal{W}^*$, we can obtain, using Lemma 2.3, an interpretation of w of the form $J_E = \mathsf{u}'_M, \mathsf{e}''_M, \mathsf{e}''_R, a''_E$ such that $w a''_E \leq_p \mathsf{u}'_M v_E \mathsf{e}'_R$.

Next, we prove that $F := (J_D; J_E)$ is a d.i. of w. Observe that $v_D\mathsf{d}'_R a_D \leq_p v_E\mathsf{e}'_R$ by hypothesis (b) and $\mathsf{e}''_M\mathsf{e}''_R a''_E \leq_p v_E\mathsf{e}'_R$ by the definition of J_E. But $v_D\mathsf{d}'_R = (\mathsf{u}'_R)^{-1}w = \mathsf{e}''_M\mathsf{e}''_R$, so $a_D = a''_E$. Hence, $a'_D \neq a_D = a''_E$ and F is a d.i. of w.

Finally, we note that since J_D and J_E start with $\mathsf{u}'_M \in \mathcal{W}$, we can use Lemma 2.4 with F to obtain a simple d.i. G of a word g such that $g \leq_s w <_s d$. This corresponds to the fact that D reduces to G.

(ii) Assume $|\mathsf{d}_R| = |\mathsf{e}_R|$. Since, by ((2)) of Definition 2.3, we have that d_R and e_R are prefix of u_R, hypothesis (ii) implies that $\mathsf{d}_R = \mathsf{e}_R$. In addition, from ((1)) of Definition 2.3 we see that $\mathsf{d}_L\mathsf{d}_M$ and $\mathsf{e}_L\mathsf{e}_M$ either share the suffix $\mathsf{u}_M \in \mathcal{W}$ or are suffix dependent. We conclude that $d = \mathsf{d}_L\mathsf{d}_M\mathsf{d}_R$ and $e = \mathsf{e}_L\mathsf{e}_M\mathsf{e}_R$ share a suffix of length at least $\langle\mathcal{W}\rangle$. This is, $D \sim E$.

(iii) We consider the case $|\tilde{\mathsf{d}}| \leq |\tilde{\mathsf{e}}| \leq |\tilde{\mathsf{d}}\mathsf{d}'_M|$, the other one is symmetric.

We start with some simplifications. Observe that condition ((2)) in Definition 2.3 implies

$$\mathsf{d}_R a_D = \tilde{\mathsf{d}}\mathsf{d}'_M\mathsf{d}'_R a_D \leq_p \mathsf{u}_R \quad \text{and} \quad \mathsf{e}_R a_E = \tilde{\mathsf{e}}\mathsf{e}'_M\mathsf{e}'_R a_E \leq_p \mathsf{u}_R. \tag{2.4}$$

Then, if $|\tilde{\mathsf{d}}| = |\tilde{\mathsf{e}}|$, we are in case (i), and if $|\mathsf{d}_R| = |\mathsf{e}_R|$, we are in case (ii). Thus, we can suppose, without loss of generality, that

$$|\tilde{\mathsf{d}}| < |\tilde{\mathsf{e}}|, \tag{2.5}$$

$$|\mathsf{d}_R| \neq |\mathsf{e}_R|. \tag{2.6}$$

The idea of the proof is the following. We are going to define a word w, which is suffix of d or e, and that has a d.i. F satisfying the hypothesis of Lemma 2.4. This would imply that F (and then also D or E) reduces to a simple d.i., as desired.

From (2.5) and hypothesis (iii) we have that $|\tilde{\mathsf{d}}| \neq |\tilde{\mathsf{d}}\mathsf{d}'_M|$ and thus $\ell \neq 0$. In particular, this last fact implies that $v_D := (\mathsf{u}'_M)^{-1}\mathsf{d}'_M \in \mathcal{W}^*$ and $v_E := (\mathsf{u}'_M)^{-1}\mathsf{e}'_M \in \mathcal{W}^*$. Also, from (2.4) and (2.5) we see that it makes sense to define $t := \tilde{\mathsf{d}}^{-1}\tilde{\mathsf{e}} \neq 1$. Then, $J_D = \mathsf{u}'_M, v_D, \mathsf{d}'_R, a'_D$ is an interpretation of $\mathsf{d}'_M\mathsf{d}'_R$ and $J_E = t, \mathsf{e}'_M, \mathsf{e}'_R, a'_E$ is an interpretation of $t\mathsf{e}'_M\mathsf{e}'_R$. Now, using (2.4) and (2.6) we also obtain that either $\mathsf{d}'_M\mathsf{d}'_R <_p t\mathsf{e}'_M\mathsf{e}'_R$ or $t\mathsf{e}'_M\mathsf{e}'_R <_p \mathsf{d}'_M\mathsf{d}'_R$. We analyze these two cases separately:

(1) Assume $\mathsf{d}'_M\mathsf{d}'_R <_p t\mathsf{e}'_M\mathsf{e}'_R$. We define $w = \mathsf{d}'_M\mathsf{d}'_R <_s d$. Note that J_D is an interpretation of w. By hypothesis (iii), we have $t \leq_p w <_p t\mathsf{e}'_M\mathsf{e}'_R$, so we can use Lemma 2.3 with J_E to obtain an interpretation of w having the form $J'_E = t, \mathsf{e}''_M, \mathsf{e}''_R, a$ and satisfying $wa \leq_p \mathsf{e}'_M\mathsf{e}'_R$. We set $F = (J_D, J'_E)$. Since $wa \leq_p t\mathsf{e}'_M\mathsf{e}'_R = \tilde{\mathsf{d}}^{-1}\mathsf{e}_R \leq_p \tilde{\mathsf{d}}^{-1}\mathsf{u}_R$ and $wa_D = \mathsf{d}'_M\mathsf{d}'_R a_D = \tilde{\mathsf{d}}^{-1}\mathsf{d}_R a_D \leq_p \tilde{\mathsf{d}}^{-1}\mathsf{u}_R$, we have $a = a_D$. Being $a_D \neq a'_D$ as D is a d.i., we conclude that $a \neq a'_D$ and that F is a d.i. Recall that $\mathsf{u}'_R \in \mathcal{W}$ and observe that $|t| \leq |\mathsf{d}'_M|$ by hypothesis (iii). Thus, F satisfies the hypothesis of Lemma 2.4. This implies that D is reducible.

(2) Suppose $t\mathsf{e}'_M\mathsf{e}'_R <_p \mathsf{d}'_M\mathsf{d}'_R$. Observe that from ((4)) of Definition 2.3 we know that there exist $n \geq 0$ and, for $j \in \{1, \ldots, n\}$, $v_j \in \mathcal{W}$ with $|v_j| \leq \ell$, such that $v_D = v_1 \cdots v_n$ (we interpret $v_1 \cdots v_n = 1$ when $n = 0$). We define $w = t\mathsf{e}'_M\mathsf{e}'_R <_s e$ and $v_{n+1} = \mathsf{d}'_R$. See Figure 2.4 for an illustration of the definitions so far. Since $|w| \geq |\mathsf{u}'_R|$, we have $\mathsf{u}'_M \leq_p w <_p \mathsf{u}'_M v_1 \cdots v_{n+1}$ by (b), and thus, there exists a least integer $m \in \{1, \ldots, n+1\}$ such that $w \leq_p \mathsf{u}'_M v_1 \cdots v_m$. Being m minimal, we can write $w = \mathsf{u}'_M v_1 \cdots v_{m-1} v'_m$, with $v'_m \leq_p v_m$ and $wa \leq_p \mathsf{d}'_M\mathsf{d}'_R$ for some $a \in \mathcal{A}$. Then, $J'_D := \mathsf{u}'_M, v_1 \cdots v_{m-1}, v'_m, a$ and J_E are interpretations of w.

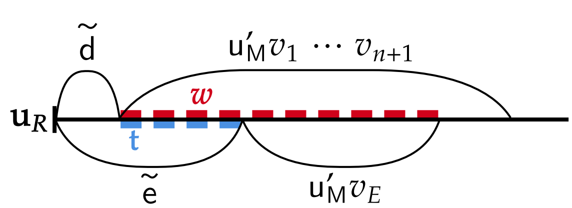


Figure 2.4: Diagram of the construction in Case (b) of the proof of Lemma 2.8. Observe that conditions (b) and (2.4) say that $\tilde{\mathsf{d}} <_p \tilde{\mathsf{e}} \leq_p \tilde{\mathsf{e}}\mathsf{u}'_M v_E \leq_p \tilde{\mathsf{d}}\mathsf{u}'_M v_1 \cdots v_{n+1} \leq_p \mathsf{u}_R$. This and the definitions of w and t are represented in the figure.

We set $F = (J'_D, J_E)$ and claim that F is a d.i. Indeed, on the one hand, the definition of J'_D gives $wa \leq_p \mathsf{d}'_M\mathsf{d}'_R \leq_p \tilde{\mathsf{d}}^{-1}\mathsf{u}_R$. On the other hand, since $w = \tilde{\mathsf{d}}^{-1}\tilde{\mathsf{e}}\mathsf{e}'_M\mathsf{e}'_R = \tilde{\mathsf{d}}^{-1}\mathsf{e}_R$, we have $wa_E \leq_p \tilde{\mathsf{d}}^{-1}\mathsf{u}_R$ by ((2)) of Definition 2.3. We conclude that $a = a_E$. Then, $a \neq a'_E$ (because E is a d.i.) and F is a d.i.

Finally, we prove that F satisfies the hypothesis of Lemma 2.4. Since J'_D starts with $\mathsf{u}'_M \in \mathcal{W}$, we only need to show that $|t| \leq |\mathsf{u}'_M v_1 \cdots v_{m-1}|$. By contradiction, we assume $\mathsf{u}'_M v_1 \cdots v_{m-1} <_p t$. This condition implies two things. First, that we can define $t' = (\mathsf{u}'_M v_1 \cdots v_{m-1})^{-1}t \neq 1$, and then, since $\mathsf{u}'_M v_1 \cdots v_{m-1}v'_m = t\mathsf{e}'_M\mathsf{e}'_R$, that $v'_m = t'\mathsf{e}'_M\mathsf{e}'_R$. In particular, $\ell \leq |\mathsf{e}'_M| < |v'_m|$. The second fact is that $m \leq n$. Indeed, by hypothesis (iii) we have $|\mathsf{u}'_M v_1 \cdots v_{m-1}| < |t| \leq |\mathsf{d}'_M| = |\mathsf{u}'_M v_1 \cdots v_n|$. Hence, $\ell < |v'_m| \leq |v_m| \leq \ell$, which is a contradiction. This proves that Lemma 2.4 can be applied with F, so F (and then also E) reduces to a simple d.i.

□

If $u \in \mathcal{A}^+$, then we write $u^\infty := uuu\cdots$ and ${}^\infty u := \cdots uuu$. Recall that an integer $k \geq 1$ is a period of $w \in \mathcal{A}^+$ if $w \leq_p u^\infty$ (equivalently, $w \leq_s {}^\infty u$) for some $u \in \mathcal{A}^k$. The following result (also known as the Fine and Wilf Lemma) is classical.

Lemma 2.9 (Proposition 1.3.2, [Lot97]) *If $p, p' \geq 1$ are periods of $w \in \mathcal{A}^+$ and $p + p' \leq |w|$, then* $\gcd(p, p')$ *is also a period of w.*

We fix an irreducible subset $\mathcal{D}' \subseteq \mathcal{D}_U$. For $D, E \in \mathcal{D}'$, since $\tilde{\mathsf{d}}, \tilde{\mathsf{e}} \leq_p \mathsf{u}_R$ and $\tilde{\mathsf{d}}, \tilde{\mathsf{e}} \leq_s \mathsf{u}'_L$, we have that $\tilde{\mathsf{d}}$ and $\tilde{\mathsf{e}}$ are both prefix and suffix dependent. So it makes sense to define in $\mathcal{D}'$:

$$D \leq E \quad \text{iff} \quad \tilde{\mathsf{d}} \leq_p \tilde{\mathsf{e}}.$$

Observe that Lemma 2.8 part (iii) implies that $D = E$ if and only if $\tilde{\mathsf{d}} = \tilde{\mathsf{e}}$. Therefore, $\leq$ is a total order. In particular, we can use the notation $D < E$ when $D \leq E$ and $D \neq E$. In this case it is not difficult to prove that $|\tilde{\mathsf{e}}| - |\tilde{\mathsf{d}}|$ is a period of $\tilde{\mathsf{e}}$.

Let $D(1) < \cdots < D(s)$ be all the elements in $\mathcal{D}'$ (deployed in increasing order). We adopt the mnemotechnical notation:

$$D(j) = (\mathsf{d}_L(j), \mathsf{d}_M(j), \mathsf{d}_R(j), a(j);\ \mathsf{d}'_L(j), \mathsf{d}'_M(j), \mathsf{d}'_R(j), a'(j)); \tag{2.7}$$

$$d(j) = \mathsf{d}_L(j)\mathsf{d}_M(j)\mathsf{d}_R(j),\ \tilde{\mathsf{d}}(j) = (\mathsf{d}_L(j)\mathsf{d}_M(j))^{-1}\mathsf{d}'_L(j). \tag{2.8}$$

For $D, E \in \mathcal{D}'$, since $\mathsf{d}_R a_D, \tilde{\mathsf{e}} \leq_p \mathsf{u}_R$, we have that $\mathsf{d}_R a_D \leq_p \tilde{\mathsf{e}}$ if and only if $|\mathsf{d}_R| < |\tilde{\mathsf{e}}|$. Thus, for $j \in \{1, \ldots, s\}$ we can define

$$\mathcal{D}'(j) := \{D \in \mathcal{D}' : \mathsf{d}_R a_D \leq_p \tilde{\mathsf{d}}(j)\} = \{D \in \mathcal{D}' : |\mathsf{d}_R| < |\tilde{\mathsf{d}}(j)|\}$$

and $\mathcal{D}'(s+1) := \mathcal{D}'$. By definition of the total order, this is a nondecreasing sequence. Moreover, $\mathcal{D}'(j) \subseteq \{D(k) : k \in \{1, \ldots, j-1\}\}$ for all $j \in \{1, \ldots, s+1\}$. In particular, $\mathcal{D}'(1) = \emptyset$.

Lemma 2.10 *Let $p \in \{1, \ldots, s+1\}$ be such that $\mathcal{D}'(p)$ is nonempty and let $D(p') := \max \mathcal{D}'(p)$, where the maximum is taken with respect to the total order. Then, $\#(\mathcal{D}'(p) \backslash \mathcal{D}'(p')) \leq 6$.*

Proof. We prove the lemma by contradiction. Suppose $\#(\mathcal{D}'(p) \backslash \mathcal{D}'(p')) \geq 7$ and let $D(j_1) < D(j_2) < \cdots < D(j_7)$ be seven different elements in $\mathcal{D}'(p) \backslash \mathcal{D}'(p')$.

We start by obtaining some relations. First, from part (iii) of Lemma 2.8 and the irreducibility of $\mathcal{D}'$, we get

$$\tilde{\mathsf{d}}\mathsf{d}'_M <_p \tilde{\mathsf{e}} \text{ for all } D, E \in \mathcal{D}'(p) \text{ such that } D < E. \tag{2.9}$$

Thus,

$$\tilde{\mathsf{d}}(j_k) \leq_p \tilde{\mathsf{d}}(j_k)\mathsf{d}'_M(j_k) <_p \tilde{\mathsf{d}}(j_{k+1}) \leq_p \tilde{\mathsf{d}}(j_{k+1})\mathsf{d}'_M(j_{k+1}) \text{ for all } k \in \{1, \ldots, 6\}. \tag{2.10}$$

In Figure 2.5 we illustrate these conditions.

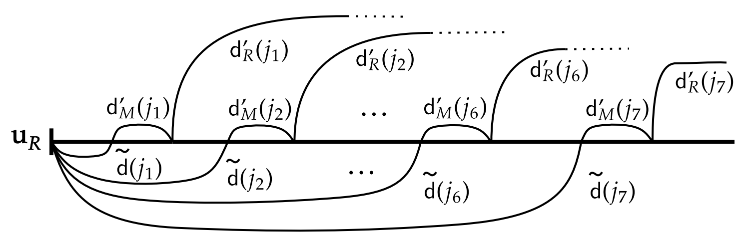


Figure 2.5: Diagram of conditions in equation (2.10). Observe that, since $\tilde{\mathsf{d}}\mathsf{d}'_M\mathsf{d}'_R = \mathsf{d}_R \leq_p \mathsf{u}_R$ for any $D \in \mathcal{D}'$ by ((2)) of Definition 2.3, all the words in the figure occur inside u_R.

We set $v_k = \tilde{\mathsf{d}}(j_k)\mathsf{d}'_M(j_k)$, $k \in \{1, \ldots, 6\}$. By (2.10),

$$v_1 <_p \cdots <_p v_5 <_p \tilde{\mathsf{d}}(j_6) <_p v_6 <_p \tilde{\mathsf{d}}(j_7).$$

Also, observe that for any $D \in \mathcal{D}'(p) \backslash \mathcal{D}'(p')$ we have $D \leq D(p')$ and $D \notin \mathcal{D}'(p')$, which gives

$$\tilde{\mathsf{d}} \leq_p \tilde{\mathsf{d}}(p') \leq_p \mathsf{d}_R \leq_p \mathsf{u}_R. \tag{2.11}$$

Equation (2.10), the first inequality of (2.11) used with $\mathsf{d}(j_7)$ and the second inequality of (2.11) used with $\tilde{\mathsf{d}}(j_k)$ imply that

$$v_k <_p \tilde{\mathsf{d}}(j_7) \leq_p \tilde{\mathsf{d}}(p') \leq_p \mathsf{d}_R(j_k) \text{ for all } k \in \{1, \ldots, 6\}. \tag{2.12}$$

From previous relations we can define the nonempty word $w := v_1^{-1}\tilde{\mathsf{d}}(j_7)$. Let $q \leq_p w$ be such that $|q|$ is the least period of w. We will prove that $|q|$ divides $|v_1^{-1}v_k|$ for all $k \in \{1, \ldots, 5\}$.

On the one hand, the observation made before the proof shows that $|\tilde{\mathsf{d}}(j_6)^{-1}\tilde{\mathsf{d}}(j_7)|$ is a period of $\tilde{\mathsf{d}}(j_7)$, and thus also of w. On the other hand, if $k \in \{1, \ldots, 6\}$, then from (2.12) and the definition of $\tilde{\mathsf{d}}$ we get

$$(v_1^{-1}v_k)^{-1}w = v_k^{-1}\tilde{\mathsf{d}}(j_7) \leq_p v_k^{-1}\mathsf{d}_R(j_k) = \mathsf{d}'_R(j_k) \leq_p \mathsf{u}'_R,$$

being the last step true due to item ((3)) of Definition 2.3. In particular, for $k = 1$ we get $w \leq_p \mathsf{u}'_R$. These inequalities imply $w \leq_p (v_1^{-1}v_k)^\infty$. Consequently, $|v_1^{-1}v_k|$ is a period of w. Since, by (2.10), $v_k^{-1}\tilde{\mathsf{d}}(j_6)$ is defined for all $k \in \{1, \ldots, 5\}$, then for these values of k we can compute

$$|q| + |v_1^{-1}v_k| \leq |\tilde{\mathsf{d}}(j_6)^{-1}\tilde{\mathsf{d}}(j_7)| + |v_1^{-1}v_k| = |w| - |v_k^{-1}\tilde{\mathsf{d}}(j_6)| \leq |w|.$$

Hence, Lemma 2.9 can be applied to get that $\gcd(|q|, |v_1^{-1}v_k|)$ is a period of w for $k \in \{1, \ldots, 5\}$. In particular, $|q| = \gcd(|q|, |v_1^{-1}v_k|)$ and $|q|$ divides $|v_1^{-1}v_k|$ for $k \in \{1, \ldots, 5\}$.

Then, we have $w \leq_p q^\infty$ and, by the claim, for $k \in \{1, \ldots, 5\}$ there exists $n_k \geq 0$ satisfying $v_1^{-1}v_k = q^{n_k}$. Moreover, from the definition of v_k we have $v_k = v_1 q^{n_k}$, which implies

$$\mathsf{d}'_R(j_k)a(j_k) = v_k^{-1}\mathsf{d}_R(j_k)a(j_k) \leq_p v_k^{-1}\mathsf{u}_R = q^{-n_k}v_1^{-1}\mathsf{u}_R$$

and $\mathsf{d}'_R(j_k)a'(j_k) \leq_p \mathsf{u}'_R$. Thus, since $a(j_k) \neq a'(j_k)$, we deduce that $\mathsf{d}'_R(j_k)$ is the maximal common prefix of $q^{-n_k}v_1^{-1}\mathsf{u}_R$ and u'_R.

Now, let $n, n' \geq 0$ and $r, r' <_p q$ be maximal such that $q^n r \leq_p v_1^{-1}\mathsf{u}_R$ and $q^{n'}r' \leq_p \mathsf{u}'_R$. We conclude that

$$\mathsf{d}'_R(j_k) = q^{n-n_k}r \text{ if } n - n_k < n' \text{ and } \mathsf{d}'_R(j_k) = q^{n'}r' \text{ if } n - n_k > n' \tag{2.13}$$

for $k \in \{1, \ldots, 5\}$.

We have all the elements to complete the proof. Since $n_2 < n_3 < n_4 < n_5$, we have $n_2 < n_3 < n - n'$ or $n_5 > n_4 > n - n'$. We are going to show that both cases give a contradiction, proving, thereby, the lemma.

First, suppose that $n_2 < n_3 < n - n'$. Then, for $k \in \{2, 3\}$, we have $n - n_k > n'$, and thus, by (2.13), $\mathsf{d}'_R(j_k) = q^{n'}r'$. If $\ell = 0$, $d(j_k) = \mathsf{d}'_L(j_k)\mathsf{d}'_R(j_k) \leq_s \mathsf{u}'_L q^{n'}r'$. Then, $d(j_2)$ and $d(j_3)$ are suffix dependent, which gives that $D(j_2)$ is equivalent to $D(j_3)$, contradicting the irreducibility of $\mathcal{D}'$. If $\ell > 0$, we have $\mathsf{d}_R(j_k) = v_1(v_1^{-1}v_k)\mathsf{d}'_R(j_k) = v_1 q^{n_k+n'}r'$. Then, using (2.10),

$$|q^{n_k}| = |v_1^{-1}v_k| \geq |v_1^{-1}v_2| \geq |\mathsf{d}'_M(j_2)| \geq |\mathsf{u}'_M| \geq \langle \mathcal{W} \rangle,$$

and hence $d(j_2)$ and $d(j_3)$ share a common suffix of length $\langle \mathcal{W} \rangle$. This is, $D(j_2) \sim D(j_3)$, which is a contradiction.

Finally, assume $n_5 > n_4 > n - n'$. We have, by (2.13), that $\mathsf{d}'_R(j_k) = q^{n-n_k}r$ for $k \in \{4, 5\}$. Hence, $\mathsf{d}_R(j_k) = v_1(v_1^{-1}v_k)\mathsf{d}'_R(j_k) = v_1 q^{n_k}\mathsf{d}'_R(j_k) = v_1 q^n r$. In particular, condition (ii) of Lemma 2.8 holds for $\{D(j_4), D(j_5)\}$, contradicting the irreducibility of $\mathcal{D}'$. This completes the proof. □

Lemma 2.11 *Let $p \in \{1, \ldots, s\}$ be such that $\#\mathcal{D}'(p) \geq 2$ and let $D(p') = \max \mathcal{D}'(p)$, $D(p'') = \max \mathcal{D}'(p) \setminus \{D(p')\}$. Then, there exist $w \in \mathcal{W}$ and $w' \leq_p \tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p'')^{-1}$ such that w and w' are suffix dependent, $|w| \geq |\tilde{\mathsf{d}}(p')|$ and $|w'| > |\tilde{\mathsf{d}}(s)| - |\tilde{\mathsf{d}}(p)|$.*

Proof. Note that $p'' < p' < p$. Before proving the main statement of the lemma, we highlight two useful relations. First, note that

$$\mathsf{d}_L(p'')\mathsf{d}_M(p'')\tilde{\mathsf{d}}(p'') = \mathsf{d}'_L(p'') \tag{2.14}$$

as $D(p'')$ is simple. Second, since u_R and u'_L are, by Definition 2.3, the shortest words in $\mathcal{W}$ satisfying $\mathsf{d}_R(p'')a(p'') \leq_p \mathsf{u}_R$ and $\mathsf{d}'_L(p'') \leq_s \mathsf{u}'_L$, respectively, we have, by condition ((2)) of the definition of simple d.i., that $|\mathsf{d}'_L(p'')| \geq \min(|\mathsf{u}_R|, |\mathsf{u}'_L|) \geq |\tilde{\mathsf{d}}(k)|$ for $k \in \{1, \ldots, s\}$. This and the fact that $\mathsf{d}'_L(p'')$ and $\tilde{\mathsf{d}}(k)$ are both suffix of u'_L imply

$$\tilde{\mathsf{d}}(k) \leq_s \mathsf{d}'_L(p'') \text{ for } k \in \{1, \ldots, s\}. \tag{2.15}$$

Now we are ready to prove the main statement of the lemma. Using (2.15) and $\tilde{\mathsf{d}}(p') \leq_p \tilde{\mathsf{d}}(p)$, we have $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p') \leq_p \mathsf{d}'_L(p'')$. In addition, $\mathsf{d}_L(p'') \leq_p \mathsf{d}'_L(p'')$ by the simplicity of $D(p'')$. Thus, $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p')$ and $\mathsf{d}_L(p'')$ are prefix dependent. In what follows, we split the proof in two cases according to which of these words is prefix of the other.

(a) $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p') \leq_p \mathsf{d}_L(p'')$. Observe that $\tilde{\mathsf{d}}(s) \leq_s \mathsf{u}'_L$ and $\mathsf{d}_M(p'')\tilde{\mathsf{d}}(p'') \leq_s \mathsf{d}'_L(p'') \leq_s \mathsf{u}'_L$, so $\tilde{\mathsf{d}}(s)$ and $\mathsf{d}_M(p'')\tilde{\mathsf{d}}(p'')$ are suffix dependent. In addition, from (2.14) and (a) we get

$$\begin{aligned} |\mathsf{d}_M(p'')\tilde{\mathsf{d}}(p'')| &= |\mathsf{d}'_L(p'')| - |\mathsf{d}_L(p'')| \\ &\leq |\mathsf{d}'_L(p'')| - |(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p')| \\ &= |\tilde{\mathsf{d}}(p)| - |\tilde{\mathsf{d}}(p')| \leq |\tilde{\mathsf{d}}(s)|. \end{aligned} \tag{2.16}$$

We conclude that

$$\mathsf{d}_M(p'')\tilde{\mathsf{d}}(p'') \leq_s \tilde{\mathsf{d}}(s).$$

Thus, it makes sense to define $w' := \tilde{\mathsf{d}}(s)(\mathsf{d}_M(p'')\tilde{\mathsf{d}}(p''))^{-1}$. Clearly, $w' \leq_p \tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p'')^{-1}$. Let $w \in \mathcal{W}$ be a word satisfying $\mathsf{d}_L(p'') \leq_s w$, as in the definition of interpretation. Observe that, by (2.15) and (2.14),

$$w' \leq_s \mathsf{d}'_L(p'')(\mathsf{d}_M(p'')\tilde{\mathsf{d}}(p''))^{-1} = \mathsf{d}_L(p'') \leq_s w,$$

so w and w' are suffix dependent. It is left to prove that $|w'| \geq |\tilde{\mathsf{d}}(s)| - |\tilde{\mathsf{d}}(p)|$ and $|w| \geq |\tilde{\mathsf{d}}(p')|$. For this, we note that in (2.16) it was shown that $|\mathsf{d}_M(p'')\tilde{\mathsf{d}}(p'')| \leq |\tilde{\mathsf{d}}(p)| - |\tilde{\mathsf{d}}(p')|$. Thus,

$$|w'| \geq |\tilde{\mathsf{d}}(s)| - |\tilde{\mathsf{d}}(p)| + |\tilde{\mathsf{d}}(p')| \geq \max(|\tilde{\mathsf{d}}(s)| - |\tilde{\mathsf{d}}(p)|, |\tilde{\mathsf{d}}(p')|).$$

We conclude that $|w'| \geq |\tilde{\mathsf{d}}(s)| - |\tilde{\mathsf{d}}(p)|$ and, since $w' \leq_s w$, $|w| \geq |w'| \geq |\tilde{\mathsf{d}}(p')|$. This completes the proof in case (a).

(b) $\mathsf{d}_L(p'') <_p (\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p')$. We start by claiming that

$$|\tilde{\mathsf{d}}(p'')| + |\tilde{\mathsf{d}}(p')| < |\tilde{\mathsf{d}}(p)|. \tag{2.17}$$

Assume that (2.17) does not hold. Let q be the shortest word satisfying $\tilde{\mathsf{d}}(p) \leq_s {}^{\infty}q$. As we commented before Lemma 2.10, condition $p', p'' < p$ implies that $\tilde{\mathsf{d}}(p')$, as well as $\tilde{\mathsf{d}}(p'')$, are prefixes and suffixes of $\tilde{\mathsf{d}}(p)$. So $|\tilde{\mathsf{d}}(p)| - |\tilde{\mathsf{d}}(p')|$ and $|\tilde{\mathsf{d}}(p)| - |\tilde{\mathsf{d}}(p'')|$ are periods of $\tilde{\mathsf{d}}(p)$. Moreover, since we are assuming (2.17) is not true, we also have that $(|\tilde{\mathsf{d}}(p)| - |\tilde{\mathsf{d}}(p')|) + (|\tilde{\mathsf{d}}(p)| - |\tilde{\mathsf{d}}(p'')|) \leq |\tilde{\mathsf{d}}(p)|$. Then, by Lemma 2.9, we obtain that $|q|$ divides $|\tilde{\mathsf{d}}(p)| - |\tilde{\mathsf{d}}(p')|$ and $|\tilde{\mathsf{d}}(p)| - |\tilde{\mathsf{d}}(p'')|$. Hence, there exists $n', n'' \in \mathbb{N}$ such that $q^{n'} = \tilde{\mathsf{d}}(p')^{-1}\tilde{\mathsf{d}}(p)$ and $q^{n''} = \tilde{\mathsf{d}}(p'')^{-1}\tilde{\mathsf{d}}(p)$. Now, since $p', p'' \in \mathcal{D}'(p)$, we can write $\mathsf{d}'_M(p')\mathsf{d}'_R(p')a(p') = \tilde{\mathsf{d}}(p')^{-1}\mathsf{d}_R(p')a(p') \leq_p \tilde{\mathsf{d}}(p')^{-1}\tilde{\mathsf{d}}(p) = q^{n'} \leq_p q^{\infty}$ and, in a similar way, $\mathsf{d}'_M(p'')\mathsf{d}'_R(p'')a(p'') \leq_p q^{\infty}$. Thus, $\{D(p'), D(p'')\}$ is reducible by part (i) of Lemma 2.8, which contradicts the irreducibility of $\mathcal{D}'$. This proves the claim.

From (2.17) and (2.14) we get

$$|(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p')| = |\mathsf{d}'_L(p'')| - |\tilde{\mathsf{d}}(p)| + |\tilde{\mathsf{d}}(p')| \\ < |\mathsf{d}'_L(p'')| - |\tilde{\mathsf{d}}(p'')| = |\tilde{\mathsf{d}}(p'')^{-1}\mathsf{d}'_L(p'')| = |\mathsf{d}_L(p'')\mathsf{d}_M(p'')|.$$

Then, since

$$(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p') \leq_p (\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p) = \mathsf{d}'_L(p'') = \mathsf{d}_L(p'')\mathsf{d}_M(p'')\tilde{\mathsf{d}}(p''),$$

we obtain that $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p') <_p \mathsf{d}_L(p'')\mathsf{d}_M(p'')$. This and (b) can be written together as

$$\mathsf{d}_L(p'') <_p (\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p') <_p \mathsf{d}_L(p'')\mathsf{d}_M(p''). \tag{2.18}$$

Since $\mathsf{d}_L(p'')\mathsf{d}_M(p'')\tilde{\mathsf{d}}(p'') = \mathsf{d}'_L(p'')$ by (2.14), we can represent the right-hand side of equation (2.18) as in Figure 2.6.

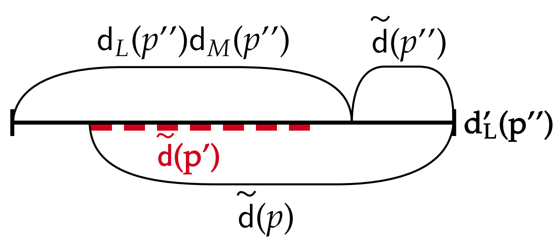


Figure 2.6: Diagram of the right-hand side of equation (2.18).

By (2.18), we can write $\mathsf{d}_L(p'')\mathsf{d}_M(p'') = vwv'$, where $v \in \mathsf{d}_L(p'')\mathcal{W}^*$, $w \in \mathcal{W}$, $v' \in \mathcal{W}^*$ and

$$v <_p (\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p') \leq_p vw. \tag{2.19}$$

The word w is the one we need in the statement of the lemma. To define w', we first note that $\tilde{\mathsf{d}}(s) \leq_s \mathsf{d}'_L(p'')$ and $v'\tilde{\mathsf{d}}(p'') \leq_s \mathsf{d}_L(p'')\mathsf{d}_M(p'')\tilde{\mathsf{d}}(p'') = \mathsf{d}'_L(p'')$, so $\tilde{\mathsf{d}}(s)$ and $v'\tilde{\mathsf{d}}(p'')$ are suffix dependent. Moreover, using (2.19) we get

$$|v'\tilde{\mathsf{d}}(p'')| = |\mathsf{d}'_L(p'')| - |vw| \leq |\mathsf{d}'_L(p'')| - |(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p')| = |\tilde{\mathsf{d}}(p)| - |\tilde{\mathsf{d}}(p')|. \tag{2.20}$$

Then, $|v'\tilde{\mathsf{d}}(p'')| \leq |\tilde{\mathsf{d}}(p)| - |\tilde{\mathsf{d}}(p')| \leq |\tilde{\mathsf{d}}(s)|$ and $v'\tilde{\mathsf{d}}(p'') \leq_s \tilde{\mathsf{d}}(s)$. Now it makes sense to define $w' := \tilde{\mathsf{d}}(s)(v'\tilde{\mathsf{d}}(p''))^{-1}$, which clearly verifies $w' \leq_p \tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p'')^{-1}$. It is also clear that w and w' are suffix dependent. Indeed, from (2.15) and (2.14) we have $w' \leq_s \mathsf{d}'_L(p'')(v'\tilde{\mathsf{d}}(p''))^{-1} = vw$.

Now, from (2.20), $|w'| \geq |\tilde{\mathsf{d}}(s)| - |\tilde{\mathsf{d}}(p)| + |\tilde{\mathsf{d}}(p')| \geq |\tilde{\mathsf{d}}(s)| - |\tilde{\mathsf{d}}(p)|$, proving the desired condition on the length of w'. It only rests to prove that $|w| \geq |\tilde{\mathsf{d}}(p')|$. We argue by contradiction. Assume that

$$|w| < |\tilde{\mathsf{d}}(p')|. \tag{2.21}$$

First, we prove that it makes sense to define the word

$$w'' := ((\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})^{-1}v)^{-1}\mathsf{d}_R(p') \in \mathcal{A}^+. \tag{2.22}$$

From (2.19) and (2.21) we get $|v| \geq |(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p')| - |w| > |\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1}|$. But, $v \leq_p \mathsf{d}_L(p'')\mathsf{d}_M(p'') \leq_p \mathsf{d}'_L(p'')$ and $\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1} \leq_p \mathsf{d}'_L(p'')$, so $\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1} <_p v$ and $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})^{-1}v$ exists and is not the empty word. Hence, by (2.19),

$$(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})^{-1}v <_p \tilde{\mathsf{d}}(p') \leq_p \mathsf{d}_R(p') \tag{2.23}$$

and w'' is well defined.

Now, we have $vw \leq_p \mathsf{d}_L(p'')\mathsf{d}_M(p'') \leq_p \mathsf{d}'_L(p'')$ and, using $p' \in \mathcal{D}'(p)$, that $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\mathsf{d}_R(p') \leq_p (\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p) = \mathsf{d}'_L(p'')$. Thus, vw and $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\mathsf{d}_R(p')$ are prefix dependent. Therefore, there are two cases: vw is prefix of $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\mathsf{d}_R(p')$ and $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\mathsf{d}_R(p')$ is a strict prefix of vw; in each of these cases we will build a reduction for $D(p')$, producing a contradiction.

(b.1) $vw \leq_p (\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\mathsf{d}_R(p')$. We start by building a d.i. of w''. Note that

$$w''a(p') \leq_p wv'\tilde{\mathsf{d}}(p''). \tag{2.24}$$

Indeed, since $D(p') \in \mathcal{D}'(p)$ and $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p) = \mathsf{d}'_L(p'') = vwv'\tilde{\mathsf{d}}(p'')$, we have $\mathsf{d}_R(p')a(p') \leq_p \tilde{\mathsf{d}}(p) = (\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})^{-1}vwv'\tilde{\mathsf{d}}(p'')$, which implies (2.24). Now, since $w \in \mathcal{W}$, $v' \in \mathcal{W}^*$ and $\tilde{\mathsf{d}}(p') <_p \mathsf{u}_R$, the word $wv'\tilde{\mathsf{d}}(p')$ has an interpretation of the form $J = w, v', \tilde{\mathsf{d}}(p'), a$. Moreover, using (b.1) we can get $|w''| = |\mathsf{d}_R(p')| + |\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1}| - |v| \geq |w|$. Hence, by (2.24), Lemma 2.3 can be applied with J to obtain an interpretation of w'' having the form $I' = w, r, r', a(p')$. We need another interpretation of w''. Note that in the middle step of (2.23) we showed that $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})^{-1}v <_p \tilde{\mathsf{d}}(p')$. In particular, the word $((\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})^{-1}v)^{-1}\tilde{\mathsf{d}}(p')$ is nonempty and is a suffix of $\mathsf{u}'_L \in \mathcal{W}$. Then,

$$I \coloneqq ((\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})^{-1}v)^{-1}\tilde{\mathsf{d}}(p'), \mathsf{d}'_M(p'), \mathsf{d}'_R(p'), a'(p'))$$

is an interpretation of w'' (here, we used that $\tilde{\mathsf{d}}(p')\mathsf{d}'_M(p')\mathsf{d}'_R(p') = \mathsf{d}_R(p')$). We set $D = (I, I')$. Since $a(p') \neq a'(p')$, D is a d.i. of w''.

Now we can conclude the proof of this case. From (2.19) we have $|v| \geq |(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\tilde{\mathsf{d}}(p')| - |w|$, which implies $|((\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})^{-1}v)^{-1}\tilde{\mathsf{d}}(p')| \leq |w| \leq |wr|$. This and that $w \in \mathcal{W}$ allow us to use Lemma 2.4 to obtain a simple d.i. E of a word e such that $e \leq_s w''$. Since $w'' <_s \mathsf{d}_R(p') <_s d(p')$, we have that $D(p')$ reduces to E. This is the desired contradiction.

(b.2) $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})\mathsf{d}_R(p') <_p vw$. We are going to build a simple d.i. $D = (I; I')$ of $\mathsf{d}_R(p') <_s d(p')$, proving, thereby, that $D(p')$ has a reduction.

Let $I' = \tilde{\mathsf{d}}(p'), \mathsf{d}'_M(p'), \mathsf{d}'_R(p'), a'(p')$. It is clear that I' is an interpretation of $\mathsf{d}_R(p')$ since $\tilde{\mathsf{d}}(p') \leq_s \mathsf{u}'_L$, $\mathsf{d}'_M(p') \in \mathcal{W}^*$, $\mathsf{d}'_R(p')a'(p') \leq_p \mathsf{u}'_R$ and $|\tilde{\mathsf{d}}(p')| > |\tilde{\mathsf{d}}(p'')| \geq 0$. To define I, observe that in the proof of (2.22) we showed that $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})^{-1}v$ exists and is not the empty word. But, moreover, from $v \in \mathsf{d}_L(p'')\mathcal{W}^*$ we see that we can write $(\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})^{-1}v = rr'$ in such a way that r is a nonempty suffix of some word in $\mathcal{W}$ and $r' \in \mathcal{W}^*$. Since, by definition, $\mathsf{d}_R(p') = rr'w''$, to prove that $I \coloneqq r, r', w'', a(p')$ is an interpretation of $\mathsf{d}_R(p')$ it is enough to show that $w''a(p') \leq_p w$. From (b.2) we get $rr'w'' = \mathsf{d}_R(p') <_p rr'w$, so $w''a' \leq_p w$ for some $a' \in \mathcal{A}$. Then, using that $vw \leq_p vwv'\tilde{\mathsf{d}}(p'') = \mathsf{d}'_L(p'')$, we obtain

$$\begin{aligned}\mathsf{d}_R(p')a' \leq_p rr'w &= (\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})^{-1}vw \\ &\leq_p (\mathsf{d}'_L(p'')\tilde{\mathsf{d}}(p)^{-1})^{-1}\mathsf{d}'_L(p'') = \tilde{\mathsf{d}}(p) \leq_p \mathsf{u}_R.\end{aligned}$$

Since we also have $\mathsf{d}_R(p')a(p') \leq_p \mathsf{u}_R$, we deduce that $a' = a(p')$. Hence, $w''a(p') \leq_p w$ and I is an interpretation of $\mathsf{d}_R(p')$. Being $a(p') \neq a'(p')$, we conclude that $D \coloneqq (I; I')$ is a d.i. of $\mathsf{d}_R(p')$.

Finally, we prove that D is simple. Using the middle step in (2.23) we get $rr' = (\mathsf{d}'_L(p'')^{-1}\tilde{\mathsf{d}}(p))^{-1}v <_p \tilde{\mathsf{d}}(p')$. This implies that $\mathsf{d}'_M(p')\mathsf{d}'_R(p') = \tilde{\mathsf{d}}(p')^{-1}\mathsf{d}_R(p') \leq_s (rr')^{-1}\mathsf{d}_R(p') = w''$, which is the first condition in Definition 2.2. Since $w''a(p') \leq_p w$ and, by (2.21), $|\tilde{\mathsf{d}}(p')| \geq |w|$, the second condition also holds. Hence, D is simple and $D(p')$ reduces to it.

□

Remark that in the last paragraph it was the first time that in a proof we build a reduction to a simple d.i. satisfying the *second* condition of ((2)) in Definition 2.2.

2.5.1 Proof of Proposition 2.6

Proposition [2.6] Any irreducible subset of $\mathcal{D}_U$ has at most $61(\#\mathcal{W})$ elements.

Proof. Let $\mathcal{D}'$ be an irreducible subset of $\mathcal{D}_U$. Recall that, with the notation introduced above, $D(1) < \cdots < D(s)$ are the elements of $\mathcal{D}'$ deployed in increasing order, $\mathcal{D}'(s+1) = \mathcal{D}'$ and $\mathcal{D}'(j) = \{D \in \mathcal{D}' : \mathsf{d}_R a_D \leq_p \tilde{\mathsf{d}}(j)\} = \{D \in \mathcal{D}' : |\mathsf{d}_R| < |\tilde{\mathsf{d}}(j)|\}$ for $j \in \{1, \ldots, s\}$.

We define recursively a finite decreasing sequence $(p_i)_{i=0}^{t+1}$. We start with $p_0 = s+1$. Then, for $i \geq 0$: a) if $\#\mathcal{D}'(p_i) \leq 1$ we put $p_{i+1} = 1$ and the procedure stops; b) if $\#\mathcal{D}'(p_i) > 1$, set $D(p_{i+1}) = \max \mathcal{D}'(p_i)$. Observe that $\mathcal{D}'(p_{i+1}) \subsetneq \mathcal{D}'(p_i)$. Let $t \geq 0$ be the first integer for which $\#\mathcal{D}'(p_t) \leq 1$, so that $\mathcal{D}'(p_{t+1}) = \mathcal{D}'(1) = \emptyset$. This construction gives

$$\mathcal{D}' = \bigcup_{i=0}^{t} \mathcal{D}'(p_i) \backslash \mathcal{D}'(p_{i+1}).$$

From Lemma 2.10 we get that $\#\mathcal{D}' \leq 6t + 1$. To complete the proof we are going to show that $t \leq 8\#\mathcal{W} + 2$.

We proceed by contradiction, so we suppose $t > 8\#\mathcal{W} + 2$. This will imply that $\mathcal{D}'$ is reducible, which contradicts our hypothesis.

Let $1 \leq i \leq t-1$. Since $p_i \neq s+1$ and $\#\mathcal{D}'(p_i) > 1$, we can define $D(p''_i) = \max \mathcal{D}'(p_i) \setminus \{D(p_{i+1})\}$ and use Lemma 2.11 with $\mathcal{D}'(p_i)$ to obtain suffix dependent words $w_i \in \mathcal{W}$ and $w'_i \in \mathcal{A}^*$ such that

$$\text{(i) } |w_i| > |\tilde{\mathsf{d}}(p_{i+1})|, \quad \text{(ii) } |w'_i| \geq |\tilde{\mathsf{d}}(s)| - |\tilde{\mathsf{d}}(p_i)|, \quad \text{(iii) } w'_i \leq_p \tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p''_i)^{-1}. \tag{2.25}$$

Then, by the Pigeonhole Principle, we can find $1 \leq i_5 < \cdots < i_1 \leq t-1$ such that

$$\text{(a) } w \coloneqq w_{i_1} = \cdots = w_{i_5} \text{ and (b) } i_{k+1} + 2 \leq i_k \text{ for any } k \in \{1, \ldots, 4\}.$$

Using (a) and (b) we are going to obtain relations (2.26) and (2.27) below.

First, we use (b) to prove that

$$\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_{k+1}})^{-1} <_p w'_{i_{k+1}} \leq_p \tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_k})^{-1} <_p w'_{i_k} \quad \text{for any } k \in \{1, \ldots, 4\}. \tag{2.26}$$

Let $k \in \{1, \ldots, 4\}$. By (b), we have $i_{k+1} \leq i_{k+1} + 1 < i_{k+1} + 2 \leq t - 1$. Thus, $D(p_{i_{k+1}+2}) < D(p_{i_{k+1}+1})$ and $D(p_{i_{k+1}+1}), D(p_{i_{k+1}+2}) \in \mathcal{D}'(p_{i_{k+1}})$, which implies that $p''_{i_{k+1}} \geq p_{i_{k+1}+2}$ by the definition of $p''_{i_{k+1}}$. Being $p_{i_{k+1}+2} \geq p_{i_k}$ by (b), we obtain $p''_{i_{k+1}} \geq p_{i_k}$. This and (iii) of (2.25) imply $w'_{i_{k+1}} \leq_p \tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p''_{i_{k+1}})^{-1} \leq_p \tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_k})^{-1}$. This proves the middle inequality of (2.26). Let $k \in \{1, \dots, 5\}$. Since $w'_{i_k} \leq_p \tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p''_{i_k})^{-1} \leq_p \tilde{\mathsf{d}}(s)$ by (iii) of (2.25) and $\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_k})^{-1} \leq_p \tilde{\mathsf{d}}(s)$, we have that w'_{i_k} and $\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_k})^{-1}$ are prefix dependent. Moreover, $|w'_{i_k}| > |\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_k})^{-1}|$ by (ii) of (2.25), so $\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_k})^{-1} <_p w'_{i_k}$. This proves the first and last inequality of (2.26), completing the proof.

Thanks to (2.26), the word $(\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_k})^{-1})^{-1}w'_{i_{k'}}$ exists for any $1 \leq k' \leq k \leq 5$. We will use this fact freely through the proof.

Next, we want to obtain from (a) that

$$(\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_4})^{-1})^{-1}w'_{i_k} \leq_s w \text{ for } k \in \{1, \dots, 4\}. \tag{2.27}$$

By (a) and (i) of (2.25), we have $|\tilde{\mathsf{d}}(p_{i_4})| \leq |\tilde{\mathsf{d}}(p_{i_5+1})| \leq |w|$. This and (iii) imply

$$|(\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_4})^{-1})^{-1}w'_{i_k}| \leq |\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p''_{i_k})^{-1}| - |\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_4})^{-1}| \leq |\tilde{\mathsf{d}}(p_{i_4})| \leq |w|.$$

But, being w and $(\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_4})^{-1})^{-1}w'_{i_k}$ suffix dependent since w and w'_{i_k} have the same property and $(\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_4})^{-1})^{-1}w'_{i_k} \leq_s w'_{i_k}$, we obtain that $(\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_4})^{-1})^{-1}w'_{i_k} \leq_s w$, as desired.

Now we use relations (2.26) and (2.27) to obtain restrictions on the smallest period of $v := (\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_4})^{-1})^{-1}w'_{i_1}$. More precisely, we claim that if $q \in \mathcal{A}^+$ is the shortest word satisfying $v \leq_p q^\infty$, then $|q|$ divides $|\tilde{\mathsf{d}}(p_{i_4})| - |\tilde{\mathsf{d}}(p_{i_k})|$ for $k \in \{2, 3\}$.

Fix $k \in \{2, 3\}$. First, observe that $v \leq_s w$ and $v((w'_{i_2})^{-1}w'_{i_1})^{-1} = (\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_4})^{-1})^{-1}w'_{i_2} \leq_s w$ by (2.27). Being $(w'_{i_2})^{-1}w'_{i_1} \neq 1$ by (2.25), we deduce that $v \leq_s {}^{\infty}((w'_{i_2})^{-1}w'_{i_1})$. This implies that $|q| \leq |(w'_{i_2})^{-1}w'_{i_1}|$. Thus,

$$\begin{aligned} |q| + |\tilde{\mathsf{d}}(p_{i_4})\tilde{\mathsf{d}}(p_{i_k})^{-1}| &\leq |(w'_{i_2})^{-1}w'_{i_1}| + |\tilde{\mathsf{d}}(p_{i_4})\tilde{\mathsf{d}}(p_{i_k})^{-1}| \\ &= |v| + |(\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_k})^{-1})^{-1}w'_{i_2}| \leq |v|, \end{aligned} \tag{2.28}$$

where $(\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_k})^{-1})^{-1}w'_{i_2}$ exists because $k \geq 2$.

Second, since $w'_{i_1} \leq_p \tilde{\mathsf{d}}(s)$ by (iii) of (2.25), we have that $v = (\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_4})^{-1})^{-1}w'_{i_1} \leq_p \tilde{\mathsf{d}}(p_{i_4}) \leq_p \mathsf{u}_R$ and $(\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_k})^{-1})^{-1}w'_{i_1} \leq_p \tilde{\mathsf{d}}(p_{i_k}) \leq_p \mathsf{u}_R$. Therefore,

$$v \leq_p \mathsf{u}_R \text{ and } (\tilde{\mathsf{d}}(p_{i_4})\tilde{\mathsf{d}}(p_{i_k})^{-1})^{-1}v = (\tilde{\mathsf{d}}(s)\tilde{\mathsf{d}}(p_{i_k})^{-1})^{-1}w'_{i_1} \leq_p \mathsf{u}_R.$$

This and the fact that, by (2.25), $(\tilde{\mathsf{d}}(p_{i_4})\tilde{\mathsf{d}}(p_{i_k})^{-1}) \neq 1$ imply that $v \leq_p (\tilde{\mathsf{d}}(p_{i_4})\tilde{\mathsf{d}}(p_{i_k})^{-1})^\infty$. Hence,

$$|\tilde{\mathsf{d}}(p_{i_4})\tilde{\mathsf{d}}(p_{i_k})^{-1}| \text{ is a period of } v. \tag{2.29}$$

Then, from (2.28) and (2.29), we can use Lemma 2.9 with v to deduce that $|q|$ divides $|\tilde{\mathsf{d}}(p_{i_4})\tilde{\mathsf{d}}(p_{i_k})^{-1}|$, proving the claim.

Let now $\tilde{q} \in \mathcal{A}^+$ be the shortest word such that $\tilde{\mathsf{d}}(p_{i_4}) \leq_p \tilde{q}^\infty$. From the last claim, we have for $k \in \{2, 3\}$ that $\tilde{\mathsf{d}}(p_{i_4})\tilde{\mathsf{d}}(p_{i_k})^{-1} = q^{n_k}$ for some $n_k \geq 1$. Then, since $|\tilde{\mathsf{d}}(p_{i_4})\tilde{\mathsf{d}}(p_{i_k})^{-1}|$ is a period of $\tilde{\mathsf{d}}(p_{i_4})$ as $p_{i_k} < p_{i_4}$, we obtain $\tilde{\mathsf{d}}(p_{i_4}) \leq_p (\tilde{\mathsf{d}}(p_{i_4})\tilde{\mathsf{d}}(p_{i_k})^{-1})^\infty = q^\infty$ and $\tilde{q} \leq_p q$. Since, $v \leq_p \tilde{\mathsf{d}}(p_{i_4}) \leq_p \tilde{q}^\infty$, we also have $q \leq_p \tilde{q}$. Therefore, $\tilde{q} = q$.

Now we can finish the proof of the proposition. Since $\tilde{\mathsf{d}}(p_{i_4}) \leq_p q^\infty$, there are $n \geq 0$ and $r <_p q$ such that $\tilde{\mathsf{d}}(p_{i_4}) = q^n r$. Then, for $k \in \{2,3\}$, we have $\tilde{\mathsf{d}}(p_{i_k}) = q^{-n_k}\tilde{\mathsf{d}}(p_{i_4}) = q^{n-n_k}r$. Being $p_{i_2}, p_{i_3} \in \mathcal{D}'(p_{i_4})$, we get

$$\tilde{\mathsf{d}}'_M(p_{i_k})\tilde{\mathsf{d}}'_R(p_{i_k})a(p_{i_k}) = \tilde{\mathsf{d}}(p_{i_k})^{-1}\tilde{\mathsf{d}}_R(p_{i_k})a(p_{i_k}) \leq_p \tilde{\mathsf{d}}(p_{i_k})^{-1}\tilde{\mathsf{d}}(p_{i_4}) = r^{-1}q^{n_k} \leq_p r^{-1}q^\infty.$$

Thus, condition (i) of Lemma 2.8 holds, which implies that $\{D(p_{i_2}), D(p_{i_3})\}$ is reducible, contradicting our hypothesis.

□

Chapter 3

Symbolic factors

3.1 Introduction

The class of finite topological rank subshifts have shown to be both a broad class of symbolic systems [DDMP16; DDMP21], containing many of the most studied types of subshifts, and to present high degrees of rigidity [BKMS13; BDM10; EM21]. Hence, it arises as a possible framework for studying minimal subshifts and proving general theorems.

In this direction, a fundamental question is the following:

Question 3.1 Is the finite topological rank class closed under symbolic factors?

Indeed, the topological rank aims to measure how complex is the system, so an affirmative answer is expected to this question. However, symbolic factors inherit a natural yet non-recognizable $\mathcal{S}$-adic structure with finite alphabet rank from their extensions, and thus it is not clear if a structure that is, in addition, recognizable can always be obtained. Thus, an answer to this question seems to be fundamental to the understanding of finite topological rank systems.

In this chapter, we obtain the optimal answer to Question 3.1 in a more general, non-minimal context:

Theorem 3.1 *Let (X, S) be an $\mathcal{S}$-adic subshift generated by an everywhere growing and proper directive sequence of alphabet rank equal to K, and $\pi\colon (X, S) \to (Y, S)$ be an aperiodic subshift factor. Then, (Y, S) is an $\mathcal{S}$-adic subshift generated by an everywhere growing, proper and recognizable directive sequence of alphabet rank at most K.*

Theorem 3.1 implies that the topological rank cannot increase after a factor map (Corollary 3.19).

We are also able to prove the following theorems, which give a finer description of symbolic factors.

Corollary 3.2 *Let (X, S) be an $\mathcal{S}$-adic subshift generated by an everywhere growing and proper directive sequence of alphabet rank equal to K, and $(X, S) \xrightarrow{\pi_1} (X_1, S) \xrightarrow{\pi_2} \dots \xrightarrow{\pi_L} (X_L, S)$ be a chain of aperiodic subshift factors. If $L > \log_2 K$, then at least one π_j is a conjugacy.*

Theorem 3.3 *Let $\pi\colon (X, S) \to (Y, S)$ be a factor map between aperiodic minimal subshifts. Suppose that (X, S) has topological rank equal to K. Then π is almost k-to-1 for some $k \leq K$.*

Theorem 3.4 *Let (X, S) be a minimal subshift of topological rank K. Then, (X, S) has at most $(3K)^{32K}$ aperiodic subshift factors up to conjugacy.*

This chapter was published as a standalone article in [Esp22a].

3.1.1 Organization

This chapter consists of 6 sections. In the first one, we give the additional needed background in topological and symbolic dynamics. Section 3.3 is devoted to prove some technical combinatorial lemmas. The main results about the topological rank of factors are stated and proved in Section 3.4. Next, in Section 3.5, we prove Theorem 3.3. In Section 3.6, we study the problem about the number of symbolic factors and prove Theorem 3.4. The last section contains a combinatorial proof of Proposition 3.15.

3.2 Preliminaries

The *hyperspace* of (X, S) is the system $(2^X, S)$, where 2^X is the set of all closed subsets of X with the topology generated by the Hausdorff metric $d_H(A, B) = \max(\sup_{x \in A} d(x, A),\ \sup_{y \in B} d(y, A))$, and S the action $A \mapsto S(A)$.

A *factor* between the topological dynamical systems (X, S) and (Y, T) is a continuous function π from X onto Y such that $\pi \circ S = T \circ \pi$. We use the notation $\pi\colon (X, S) \to (Y, T)$ to indicate the factor. A factor map $\pi\colon (X, S) \to (Y, T)$ is *almost K-to-1* if $\#\pi^{-1}(y) = K$ for all y in a residual subset of Y. We say that π is *distal* if whenever $\pi(x) = \pi(x')$ and $x \neq x'$, we have $\inf_{k \in \mathbb{Z}} \operatorname{dist}(S^k x, S^k x') > 0$.

Given a system (X, S), the *Ellis semigroup* $E(X, S)$ associated to (X, S) is defined as the closure of $\{x \mapsto S^n x : n \in \mathbb{Z}\} \subseteq X^X$ in the product topology, where the semi-group operation is given by the composition of functions. On X we may consider the $E(X, S)$-action given by $x \mapsto ux$. Then, the closure of the orbit under S of a point $x \in X$ is equal to the orbit of x under $E(X, S)$. If $\pi\colon (X, S) \to (Y, T)$ is a factor between minimal systems, then π induces a surjective map $\pi^*\colon E(X, S) \to E(Y, T)$ which is characterized by the formula

$$\pi(ux) = \pi^*(u)\pi(x) \quad \text{for all } u \in E(X, S) \text{ and } x \in X.$$

If the context is clear, we will not distinguish between u and $\pi^*(u)$. When $u \in E(2^X, S)$, we write $u \circ A$ instead of uA, the last symbol being reserved to mean $uA = \{ux : x \in A\}$. We can describe more explicitly $u \circ A$ as follows: it is the set of all $x \in X$ for which we can find nets $x_\lambda \in A$ and $m_\lambda \in \mathbb{Z}$ such that $\lim_\lambda S^{m_\lambda} x_\lambda = x$ and $\lim_\lambda S^{m_\lambda} = u$. Finally, we identify

X with $\{\{x\} \subseteq 2^X : x \in X\}$, so that the restriction map $E(2^X, S) \to E(X, S)$ which sends $u \in E(2^X, S)$ to the restriction $u|_X : X \to X$ is an onto morphism of semigroups. As above, we will not distinguish between $u \in 2^X$ and $u|_X$.

3.2.1 Basics in symbolic dynamics

Words and subshifts

The pair $(x, \tilde{x}) \in \mathcal{A}^{\mathbb{Z}} \times \mathcal{A}^{\mathbb{Z}}$ is *right asymptotic* if there exist $k \in \mathbb{Z}$ satisfying $x_{(k,\infty)} = \tilde{x}_{(k,\infty)}$ and $x_k \neq \tilde{x}_k$. If moreover $k = 0$, $(x, \tilde{x})$ is a *centered right asymptotic*. A *right asymptotic tail* is an element $x_{(0,\infty)}$, where $(x, \tilde{x})$ is a centered right asymptotic pair. We make similar definitions for left asymptotic pairs and tails.

Morphisms and substitutions

We say that τ is *positive* if for every $a \in \mathcal{A}$, all letters $b \in \mathcal{B}$ occur in $\tau(a)$, is r-*proper*, with $r \geq 1$, if there exist $u, v \in \mathcal{B}^r$ such that $\tau(a)$ starts with u and ends with v for any $a \in \mathcal{A}$, is *proper* when is 1-proper, and is *letter-onto* if for every $b \in \mathcal{B}$ there exists $a \in \mathcal{A}$ such that b occurs in a. The minimum and maximum length of τ are, respectively, the numbers $\langle \tau \rangle \coloneqq \langle \tau(\mathcal{A}) \rangle = \min_{a \in \mathcal{A}} |\tau(a)|$ and $|\tau| \coloneqq |\tau(\mathcal{A})| = \max_{a \in \mathcal{A}} |\tau(a)|$.

Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and $Z \subseteq \mathcal{C}^{\mathbb{Z}}$ be subshifts and $\pi : (X, S) \to (Z, S)$ a factor map. The classic Curtis–Hedlund–Lyndon Theorem asserts that π has a *local code*, this is, a function $\psi : \mathcal{A}^{2r+1} \to \mathcal{C}$, where $r \in \mathbb{N}$, such that $\pi(x) = (\psi(x_{[i-r,i+r]}))_{i \in \mathbb{Z}}$ for all $x \in X$. The integer r is called the a radius of π. The following lemma relates the local code of a factor map to proper morphisms.

Lemma 3.5 *Let $\sigma : \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism, $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and $Z \subseteq \mathcal{C}^{\mathbb{Z}}$ be subshifts, and $Y = \bigcup_{k \in \mathbb{Z}} S^k \sigma(X)$. Suppose that $\pi : (Y, S) \to (Z, S)$ is a factor map of radius r and that σ is r-proper. Then, there exists a proper morphism $\tau : \mathcal{A}^+ \to \mathcal{C}^+$ such that $|\tau(a)| = |\sigma(a)|$ for any $a \in \mathcal{A}$, $Z = \bigcup_{k \in \mathbb{Z}} S^k \tau(X)$ and the following diagram commutes:*

$$\begin{array}{ccc} X & & \\ {\scriptstyle\sigma}\downarrow & \searrow^{\tau} & \\ Y & \xrightarrow{\ \pi\ } & Z \end{array} \tag{3.1}$$

Proof. Let $\psi : \mathcal{A}^{2r+1} \to \mathcal{B}$ be a local code of radius r for π and $u, v \in \mathcal{B}^r$ be such that $\sigma(a)$ starts with u and ends with v for all $a \in \mathcal{A}$. We define $\tau : \mathcal{A} \to \mathcal{C}^+$ by $\tau(a) = \psi(v\sigma(a)u)$. Then, since σ is r-proper, τ is proper and we have $\pi(\sigma(x)) = \tau(x)$ for all $x \in X$ (this is, Diagram (3.1) commutes). In particular,

$$\bigcup_{k \in \mathbb{Z}} S^k \tau(X) = \bigcup_{k \in \mathbb{Z}} S^k \pi(\sigma(X)) = \pi(Y) = Z.$$

□

$\mathcal{S}$-adic subshifts

The levels $X_{\boldsymbol{\sigma}}^{(n)}$ can be described in an alternative way if $\boldsymbol{\sigma}$ satisfies the correct hypothesis.

Lemma 3.6 *Let $\boldsymbol{\sigma} = (\sigma_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ be an everywhere growing and proper directive sequence. Then, for every $n \in \mathbb{N}$,*

$$X_{\boldsymbol{\sigma}}^{(n)} = \bigcap_{N>n} \bigcup_{k\in\mathbb{Z}} S^k \sigma_{[n,N)}(\mathcal{A}_N^{\mathbb{Z}}) \tag{3.2}$$

Proof. Let Z be the set in the right-hand side of (3.2). Since, by Lemma 1.2, $X_{\boldsymbol{\sigma}}^{(n)} = \bigcup_{k\in\mathbb{Z}} S^k\sigma_{[n,N)}(X_{\boldsymbol{\sigma}}^{(N)})$ for any $N > n$, we have that $X_{\boldsymbol{\sigma}}^{(n)}$ included in Z.

Conversely, let $x \in Z$ and $\ell \in \mathbb{N}$. We have to show that $x_{[-\ell,\ell)}$ occurs in $\sigma_{[n,N)}(a)$ for some $N > n$ and $a \in \mathcal{A}_N$. Let $N > n$ be big enough so that $\sigma_{[n,N)}$ is ℓ-proper. Then, by the definition of Z, there exists $y \in \mathcal{A}_N^{\mathbb{Z}}$ such that $x_{[-\ell,\ell)}$ occurs in $\sigma_{[n,N)}(y)$. Since $\langle\sigma_{[n,N)}\rangle \geq \ell$ (as $\sigma_{[n,N)}$ is ℓ-proper), we deduce that

$$x_{[-\ell,\ell)} \text{ occurs in } \sigma_{[n,N)}(ab) \text{ for some word } ab \text{ of length 2 occurring in } y. \tag{3.3}$$

Hence, by denoting by u and v the suffix and prefix of length ℓ of $\tau_{[n,N)}(a)$ and $\tau_{[n,N)}(b)$, respectively, we have that $x_{[-\ell,\ell)}$ occurs in $\sigma_{[n,N)}(a)$, in $\tau_{[n,N)}(b)$, or in uv. In the first two cases, we are done. In the last case, we observe that since $\sigma_{[n,N)}$ is ℓ-proper, the following is true: for every $M > N$ such that $\langle\sigma_{[N,M)}\rangle \geq 2$, $vu \sqsubseteq \sigma_{[n,M)}(c)$ for any $c \in \mathcal{A}_M$. In particular, $x_{[-\ell,\ell)} \sqsubseteq \tau_{[n,M)}(c)$ for such M and c. We have proved that $x \in X_{\boldsymbol{\sigma}}^{(n)}$. □

Finite alphabet rank $\mathcal{S}$-adic subshifts are *eventually recognizable*:

Theorem 3.7 ([DDMP21], Theorem 3.7) *Let $\boldsymbol{\sigma}$ be an everywhere growing directive sequence of alphabet rank equal to K. Suppose that $X_{\boldsymbol{\sigma}}$ is aperiodic. Then, at most $\log_2 K$ levels $(X_{\boldsymbol{\sigma}}^{(n)}, \sigma_n)$ are not recognizable.*

We will also need the following property.

Theorem 3.8 ([EM21], Theorem 3.3) *Let (X, S) be an $\mathcal{S}$-adic subshift generated by an everywhere growing directive sequence of alphabet rank K. Then, X has at most $144K^7$ right (resp. left) asymptotic tails.*

Proof. In the proof of Theorem 3.3 in [EM21] the authors show the following: the set consisting of pairs $(x, y) \in X \times X$ such that $x_{(-\infty,0)} = y_{(-\infty,0)}$ and $x_0 \neq y_0$ has at most $144K^7$ elements. In our language, this is equivalent to saying that X has at most $144K^7$ left asymptotic tails. Since this is valid for any $\mathcal{S}$-adic subshift generated by an everywhere growing directive sequence of alphabet rank K, $144K^7$ is also an upper bound for right asymptotic tails. □

3.3 Combinatorics on words lemmas

In this section we present several combinatorial lemmas that will be used throughout the chapter.

3.3.1 Lowering the rank

Let $\sigma\colon \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism. Following ideas from [RS97], we define the *rank* of σ as the least cardinality of a set of words $\mathcal{D} \subseteq \mathcal{B}^+$ such that $\sigma(\mathcal{A}^+) \subseteq \mathcal{D}^+$. Equivalently, the rank is the minimum cardinality of an alphabet $\mathcal{C}$ in a decomposition into morphisms $\mathcal{A}^+ \xrightarrow{q} \mathcal{C}^+ \xrightarrow{p} \mathcal{B}^+$ such that $\sigma = pq$. In this subsection we prove Lemma 3.12, which states that in certain technical situation, the rank of the morphism σ under consideration is small and its decomposition $\sigma = pq$ satisfies additional properties.

We start by defining some morphisms that will be used in the proofs of this subsection. If $a \neq b \in \mathcal{A}$ are different letters and $\tilde{a}$ is a letter not in $\mathcal{A}$, then we define $\phi_{a,b}\colon \mathcal{A}^+ \to (\mathcal{A}\backslash\{b\})^+$, $\psi_{a,b}\colon \mathcal{A}^+ \to \mathcal{A}^+$ and $\theta_{a,\tilde{a}}\colon \mathcal{A}^+ \to (\mathcal{A} \cup \{\tilde{a}\})^+$ by

$$\phi_{a,b}(c) = \begin{cases} c & \text{if } c \neq b, \\ a & \text{if } c = b. \end{cases} \qquad \psi_{a,b}(c) = \begin{cases} c & \text{if } c \neq b, \\ ab & \text{if } c = b. \end{cases} \qquad \theta_{a,\tilde{a}}(c) = \begin{cases} c & \text{if } c \neq a, \\ \tilde{a}a & \text{if } c = a. \end{cases}$$

Observe that these morphisms are letter-onto. Before stating the basic properties of these morphisms, we need one more set of definitions.

For a morphism $\sigma\colon \mathcal{A}^+ \to \mathcal{B}^+$, we define $|\sigma|_1 = \sum_{a\in\mathcal{A}} |\sigma(a)|$. When $u, v, w \in \mathcal{A}^+$ satisfy $w = uv$, we say that u is a prefix of w and that v a suffix of w. Recall that 1 stands for the empty word.

Lemma 3.9 *Let $\sigma\colon \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism.*

(i) If $\sigma(a) = \sigma(b)$ for some $a \neq b \in \mathcal{A}$, then $\sigma = \sigma'\phi_{a,b}$, where $\sigma'\colon (\mathcal{A} \setminus \{b\})^+ \to \mathcal{B}^+$ is the restriction of σ to $(\mathcal{A} \setminus \{b\})^+$.

(ii) If $\sigma(a)$ is a prefix of $\sigma(b)$ and $\sigma(b) = \sigma(a)t$ for some nonempty $t \in \mathcal{B}^+$, then $\sigma = \sigma'\psi_{a,b}$, where $\sigma'\colon \mathcal{A}^+ \to \mathcal{B}^+$ is defined by

$$\sigma'(c) = \begin{cases} \sigma(c) & \text{if } c \neq b, \\ t & \text{if } c = b. \end{cases} \tag{3.4}$$

(iii) If $\sigma(a) = st$ for some $s, t \in \mathcal{B}^+$ and $a \in \mathcal{A}$, then $\sigma = \sigma'\theta_{a,\tilde{a}}$, where $\sigma'\colon (\mathcal{A}\cup\{\tilde{a}\})^+ \to \mathcal{B}^+$ is defined by

$$\sigma'(c) = \begin{cases} \sigma(c) & \text{if } c \neq a, \tilde{a}, \\ s & \text{if } c = \tilde{a}, \\ t & \text{if } c = a. \end{cases} \tag{3.5}$$

Proof. The lemma follows from unraveling the definitions. For instance, in case (ii), we have $\sigma'(\psi_{a,b}(a)) = \sigma'(a) = \sigma(a)$, $\sigma'(\psi_{a,b}(b)) = \sigma'(ab) = \sigma(a)t = \sigma(b)$, and $\sigma'(\psi_{a,b}(c)) = \sigma'(c) = \sigma(c)$ for all $c \neq a, b$, which shows that $\sigma'\psi_{a,b} = \sigma$. □

Lemma 3.10 *Let $\{\sigma_j\colon \mathcal{A}^+ \to \mathcal{B}_j^+\}_{j\in J}$ be a set of morphisms such that*

$$\text{for every fixed } a \in \mathcal{A},\ \ell_a := |\sigma_j(a)| \text{ is constant for any chosen } j \in J, \tag{3.6}$$

and $u, v \in \mathcal{A}^+$, with u of length at least $\ell := \sum_{a\in\mathcal{A}} \ell_a$. Assume that u and v start with different letters and that $\sigma_j(u)$ is a prefix of $\sigma_j(v)$ for every $j \in J$.

Then, there exist a letter-onto morphism $q\colon \mathcal{A}^+ \to \mathcal{C}^+$, with $\#\mathcal{C} < \#\mathcal{A}$, and morphisms $\{p_j\colon \mathcal{C}^+ \to \mathcal{B}_j^+\}_{j\in J}$ satisfying a condition analogous to (3.6) and such that $\sigma_j = p_j q$.

Remark 3.1 If in the previous lemma we change the last hypothesis to "u and v *end* with different letters and $\sigma_j(u)$ is a *suffix* of $\sigma_j(v)$ for every $j \in J$", then the same conclusion holds. This observation will be used in the proof of Lemma 3.31.

Proof (of Lemma 3.10). By contradiction, we assume that u, v and $\{\sigma_j\}_{j\in J}$, are counterexamples for the lemma. Moreover, we suppose that ℓ is as small as possible.

Let us write $u = au'$ and $v = bv'$, where $a, b \in \mathcal{A}$. Since $\sigma_j(u)$ is a prefix of $\sigma_j(v)$, we have that for every $j \in J$,

$$\text{one of the words in } \{\sigma_j(a), \sigma_j(b)\} \text{ is a prefix of the other.} \tag{3.7}$$

We consider two cases. First, we suppose that $\ell_a = \ell_b$. In this case, (3.7) implies that $\sigma_j(a) = \sigma_j(b)$ for every $j \in J$. Hence, we can use (1) of Lemma 3.9 to decompose each σ_j as $\sigma'_j\phi_{a,b}$, where σ'_j is the restriction of σ_j to $(\mathcal{A}\setminus\{b\})^+$. Since $\phi_{a,b}$ is letter-onto and $\ell_c = |\sigma'_j(c)|$ for every $j \in J$, $c \in \mathcal{A}\setminus\{b\}$, the conclusion of the lemma holds, contrary to our assumptions.

It rests to consider the case in which $\ell_a \neq \ell_b$. We only do the case $\ell_a < \ell_b$ as the other is similar. Then, by (3.7), for every $j \in J$ there exists a nonempty word $t_j \in \mathcal{B}_j^{\ell_b-\ell_a}$ of length $\ell_b - \ell_a$ such that $\sigma_j(b) = \sigma_j(a)t_j$. Thus, we can use (2) of Lemma 3.9 to write, for any $j \in J$, $\sigma_j = \sigma'_j\psi_{a,b}$, where σ'_j is defined as in (3.4).

Let $\tilde{u} = \psi_{a,b}(u')$ and $\tilde{v} = b\psi_{a,b}(v')$. We want now to prove that $\tilde{u}$, $\tilde{v}$ and $\{\sigma'_j : j \in J\}$ satisfy the hypothesis of the lemma. First, we observe that for every $j \in J$,

$$\text{if } c \neq b, \text{ then } |\sigma'_j(c)| = \ell_c, \text{ and } |\sigma'_j(b)| = |t_j| = \ell_b - \ell_a. \tag{3.8}$$

Therefore, $\{\sigma'_j\}_{j\in J}$ satisfy condition (3.6). Also, since $\psi_{a,b}(c)$ never starts with b, we have that

$$\tilde{u},\ \tilde{v} \text{ start with different letters.} \tag{3.9}$$

Furthermore, by using the symbol $\leq_p$ to denote the prefix relation, we can compute:

$$\sigma_j(a)\sigma'_j(\tilde{u}) = \sigma_j(a)\sigma_j(u') = \sigma_j(u) \leq_p \sigma_j(v) = \sigma'_j(\psi_{a,b}(v)) = \sigma'_j(a)\sigma'_j(\tilde{v}).$$

This and the fact that $\sigma_j(a)$ is equal to $\sigma'_j(a)$ imply that

$$\sigma'_j(\tilde{u}) \text{ is a prefix of } \sigma'_j(\tilde{v}) \text{ for every } j \in J. \tag{3.10}$$

Finally, we note

$$|\tilde{u}| \geq |u| - 1 \geq \sum_{c\in\mathcal{A}} \ell_c - \ell_a =: \ell'. \tag{3.11}$$

We conclude from equations (3.8), (3.9), (3.10) and (3.11) that $\tilde{u}$, $\tilde{v}$ and $\{\sigma'_j : j \in J\}$ satisfy the hypothesis of this lemma. Since $\ell' < \ell$, the minimality of ℓ implies that there exist a letter-onto morphism $q' \colon \mathcal{A}^+ \to \mathcal{C}^+$, with $\#\mathcal{C} < \#\mathcal{A}$, and morphisms $\{p_j \colon \mathcal{C}^+ \to \mathcal{B}_j^+\}_{j\in J}$ satisfying $\sigma'_j = p_j q'$ and a property analogous to (3.6). But then $q := q'\psi_{a,b}$ is also letter-onto and the morphisms $\{p_j\}_{j\in J}$ satisfy $\sigma_j = p_j q$ and a property analogous to (3.6). Thus, the conclusion of the lemma holds for $\{\sigma_j\}_{j\in J}$, contrary our assumptions. □

Lemma 3.11 *Let $\sigma \colon \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism, $u, v \in \mathcal{A}^+$, a, b be the first letters of u, v, respectively, and $\sigma(a) = st$ be a decomposition of $\sigma(a)$ in which t is nonempty. Assume that $\sigma(u)$ is a prefix of $s\sigma(v)$, $|u| \geq |\sigma|_1 + |s|$, and either that $s = 1$ and $a \neq b$ or that $s \neq 1$.*

Then, there exist morphisms $q \colon \mathcal{A}^+ \to \mathcal{C}^+$ and $p \colon \mathcal{C}^+ \to \mathcal{B}^+$ such that $\#\mathcal{C} \leq \#\mathcal{A}$, q is letter-onto, $|p|_1 < |\sigma|_1$, and $\sigma = pq$.

Remark 3.2 As in Lemma 3.10, there are symmetric hypothesis for the previous lemma that involve suffixes instead of prefixes and which give the same conclusion. We will use this in the proof of Lemma 3.12.

Proof (of Lemma 3.11). Let us write $u = au'$ and $v = bv'$. We first consider the case in which $s = 1$. In this situation, u and v start with different letters, so Lemma 3.10 can be applied (with the index set J chosen as a singleton) to obtain a decomposition $\mathcal{A}^+ \xrightarrow{q} \mathcal{C}^+ \xrightarrow{p} \mathcal{B}^+$ such that q is letter-onto, $\#\mathcal{C} < \#\mathcal{A}$, and $\sigma = pq$. Since $\mathcal{C}$ has strictly fewer elements than $\mathcal{A}$, we have $|p|_1 < |\sigma|_1$. Hence, the conclusion of the lemma holds in this case.

We now assume that $s \neq 1$. In this case, t and s are nonempty, so we can use (3) of Lemma 3.9 to factorize $\sigma = \sigma'\theta_{a,\tilde{a}}$, where $\tilde{a}$ is a letter not in $\mathcal{A}$ and σ' is defined as in (3.5). We set $\tilde{u} = a\theta_{a,\tilde{a}}(u')$ and $\tilde{v} = \theta_{a,\tilde{a}}(v)$. Our plan is to use Lemma 3.10 with $\tilde{u}$, $\tilde{v}$ and σ'.

Observe that $\theta_{a,\tilde{a}}(c)$ never starts with a, so

$$\tilde{u},\ \tilde{v} \text{ start with different letters.} \tag{3.12}$$

Also, by using, as in the previous proof, the symbol $\leq_p$ to denote the prefix relation, we can write:

$$s\sigma'(\tilde{u}) = s\sigma'(a)\sigma'(\theta_{a,\tilde{a}}(u')) = st\sigma(u') = \sigma(u) \leq_p s\sigma(v) = s\sigma'(\theta_{a,\tilde{a}}(v)) = s\sigma'(\tilde{v}),$$

which implies that

$$\sigma'(\tilde{u}) \text{ is a prefix of } \sigma'(\tilde{v}). \tag{3.13}$$

Finally, we use (3.5) to compute:

$$|\tilde{u}| \geq |u| - 1 \geq |\sigma|_1 + |s| - 1 \geq |\sigma|_1 = |\sigma'|_1. \tag{3.14}$$

We conclude, by equations (3.12), (3.13) and (3.14), that Lemma 3.10 can be applied with $\tilde{u}$, $\tilde{v}$ and σ' (and J as a singleton). Thus, there exist morphisms $q' \colon (\mathcal{A} \cup \{\tilde{a}\})^+ \to \mathcal{C}^+$ and

$p\colon \mathcal{C}^+ \to \mathcal{B}^+$ such that $\#\mathcal{C} < \#(\mathcal{A} \cup \{\tilde{a}\})$, q' is letter-onto and $\sigma' = pq'$. Then, $\#\mathcal{C} \leq \#\mathcal{A}$, $q \coloneqq q'\theta_{a,\tilde{a}}$ is letter-onto and $\sigma = pq'\theta_{a,\tilde{a}} = pq$. Moreover, since $\theta_{a,\tilde{a}}$ is not the identity function, we have $|p|_1 < |\sigma|_1$. □

The next lemma is the main result of this subsection. To state it, we introduce additional notation. For an alphabet $\mathcal{A}$, let $\mathcal{A}^{++}$ be the set of words $w \in \mathcal{A}^+$ in which all letters occur. Observe that $\sigma\colon \mathcal{A}^+ \to \mathcal{B}^+$ is letter-onto if and only if $\sigma(\mathcal{A}^{++}) \subseteq \mathcal{B}^{++}$.

Lemma 3.12 *Let $\phi\colon \mathcal{A}^+ \to \mathcal{C}^+$, $\tau\colon \mathcal{B}^+ \to \mathcal{C}^+$ be morphisms such that τ is ℓ-proper, with $\ell \geq |\phi|_1^4$, and $\phi(\mathcal{A}^+) \cap \tau(\mathcal{B}^{++}) \neq \emptyset$. Then, there exist $\mathcal{B}^+ \xrightarrow{q} \mathcal{D}^+ \xrightarrow{p} \mathcal{C}^+$ such that*

(i) $\#\mathcal{D} \leq \#\mathcal{A}$, (ii) $\tau = pq$, (iii) q is letter-onto and proper.

Proof. By contradiction, we suppose that the lemma does not hold for ϕ and τ and, moreover, that $|\phi|_1$ as small as possible.

That $\phi(\mathcal{A})^+ \cap \tau(\mathcal{B}^{++})$ is nonempty means that there exist $u = u_1 \cdots u_n \in \mathcal{A}^+$ and $w = w_1 \cdots w_m \in \mathcal{B}^{++}$ with $\phi(u) = \tau(w)$. If $m = 1$, then, since $w \in \mathcal{B}^{++}$, we have $\#\mathcal{B} = \{v_1\}$ and the conclusion of the lemma trivially holds for $\mathcal{D} = \{a \in \mathcal{C} : a \text{ occurs in } \tau(w_1)\}$, $q\colon \mathcal{B}^+ \to \mathcal{D}^+$, $w_1 \mapsto \tau(w_1)$, and $p\colon \mathcal{D}^+ \to \mathcal{C}^+$ the inclusion map, contradicting our initial assumption. Therefore, $m \geq 2$ and $\{1, \ldots, m-1\}$ is nonempty.

Let $k \in \{1, \ldots, m-1\}$. We define i_k as the smallest number in $\{1, \ldots, n\}$ for which $|\tau(w_1 \cdots w_k)| < |\phi(u_1 \cdots u_{i_k})|$ holds. Since $|\phi(u_1)| \leq |\phi|_1 \leq \ell \leq |\tau(w_1 \cdots w_k)|$, i_k is at least 2 and, thus, $|\phi(u_1 \cdots u_{i_k - 1})| \leq |\tau(w_1 \cdots w_k)|$ by minimality of i_k. Hence, there exists a decomposition $\phi(u_{i_k}) = s_k t_k$ such that t_k is nonempty and

$$t_k \phi(u_{i_k+1} \ldots u_n) = \tau(w_{k+1} \ldots w_m). \tag{3.15}$$

Our next objective is to use Lemma 3.11 to prove that s_k and u_k have a very particular form:

Claim 3.1 For every $k \in \{1, \ldots, m-1\}$, $s_k = 1$ and $u_1 = u_{i_k}$.

Proof. To prove this, we suppose that it is not true, this is, that there exists $k \in \{1, \ldots, m-1\}$ such that

$$s_k \neq 1 \text{ or } u_1 \neq u_{i_k}. \tag{3.16}$$

Let $\tilde{u} \coloneqq u_{i_k} \ldots u_{i_k + |\phi|_1^2 - 1}$ and $\tilde{v} \coloneqq u_1 \ldots u_{|\phi|_1^3}$. We are going to check the hypothesis of Lemma 3.11 for $\tilde{u}$, $\tilde{v}$ and ϕ.

First, we observe that, since $\phi(u) = \tau(v)$, we have that $\phi(\tilde{v})$ is a prefix of $\tau(v)$. Moreover, given that $|\phi(\tilde{v})| \leq |\phi|_1^4 \leq \ell$ and that τ is ℓ-proper, $\phi(\tilde{v})$ is a prefix of $\tau(b)$ for every $b \in \mathcal{B}$. In particular,

$$\phi(\tilde{v}) \text{ is a prefix of } \tau(w_k). \tag{3.17}$$

Second, from (3.15) and the inequalities $|t_k \phi(u_{i_k+1} \ldots u_{i_k + |\phi|_1^2 - 1})| \leq |\phi|_1^3 \leq \ell \leq |\tau(w_k)|$ we deduce that $t_k \phi(u_{i_k+1} \ldots u_{i_k + |\phi|_1^2 - 1})$ is a prefix of $\tau(w_k)$. Therefore,

$$\phi(\tilde{u}) = s_k t_k \phi(u_{i_k+1} \ldots u_{i_k + |\phi|_1^2 - 1}) \text{ is a prefix of } s_k \tau(w_k). \tag{3.18}$$

We conclude from (3.17), (3.18) and the inequality $|\phi(\tilde{u})| \leq |\phi|_1^3 = |\tilde{v}| \leq |s_k\phi(\tilde{v})|$ that

$$\phi(\tilde{u}) \text{ is a prefix of } s_k\phi(\tilde{v}).$$

This, the inequality $|\tilde{u}| \geq |\phi|_1 + |s_k|$ and (3.16) allow us to use Lemma 3.11 with $\tilde{u}$, $\tilde{v}$ and ϕ and obtain morphisms $\mathcal{A}^+ \xrightarrow{\tilde{q}} \tilde{\mathcal{A}}^+ \xrightarrow{\tilde{\phi}} \mathcal{C}^+$ such that $\#\tilde{\mathcal{A}} \leq \#\mathcal{A}$, $\phi = \tilde{\phi}\tilde{q}$ and $|\tilde{\phi}|_1 < |\phi|_1$. Then, $\ell \geq |\phi|_1^4 > |\tilde{\phi}|_1^4$ and $\tilde{\phi}(\tilde{\mathcal{A}}^+) \cap \tau(\mathcal{B}^{++})$ contains the element $\tilde{\phi}(\tilde{q}(u)) = \tau(w)$, and so τ and $\tilde{\phi}$ satisfy the hypothesis of this lemma. Therefore, by the minimality of $|\phi|_1$, there exists a decomposition $\mathcal{B}^+ \xrightarrow{q} \mathcal{D}^+ \xrightarrow{p} \mathcal{C}^+$ of τ satisfying (i-iii) of this lemma, contrary to our assumptions. □

An argument similar to the one used in the proof of the previous claim gives us that

$$u_n = u_{i_k - 1} \text{ for every } k \in \{1, \ldots, m-1\}. \tag{3.19}$$

We refer the reader to Remark 3.2 for further details.

Now we can finish the proof. First, from (3.15) and the first part of the claim we get that $\tau(w_k) = \phi(u_{i_{k-1}} \cdots u_{i_k - 1})$ for $k \in \{2, \ldots, m-1\}$, $\tau(w_1) = \phi(u_1 \cdots u_{i_1 - 1})$ and $\tau(w_m) = \phi(u_{i_{m-1}} \cdots u_n)$. Being $w \in \mathcal{B}^{++}$, these equations imply that each $\tau(b)$, $b \in \mathcal{B}$, can be written as a concatenation $x_1 \cdots x_N$, with $x_j \in \phi(\mathcal{A})$. Moreover, by the second part of the claim and (3.19), we can choose this decomposition so that $x_1 = u_1$ and $x_N = u_n$. This defines (maybe non-unique) morphisms $\mathcal{B}^+ \xrightarrow{q} \mathcal{D}_1^+ \xrightarrow{p_1} \mathcal{C}^+$ such that $\tau = p_1 q$, $\#\mathcal{D}_1 \leq \#\{\phi(u_1), \ldots, \phi(u_n)\} \leq \#\mathcal{A}$ and q is proper. If we define $\mathcal{D}$ as the set of letters $d \in \mathcal{D}_1$ that occur in some $w \in q(\mathcal{B})$, and p as the restriction of p_1 to $\mathcal{D}$, then we obtain a decomposition $\mathcal{B}^+ \xrightarrow{q} \mathcal{D}^+ \xrightarrow{p} \mathcal{C}^+$ that still satisfies the previous properties, but in which q is letter-onto. Hence, p and q met conditions (i), (ii) and (iii). □

3.3.2 Periodicity lemmas

We will also need classic results from combinatorics on words. We follow the presentation of [RS97, Chapter 6].

Let $w \in \mathcal{A}^*$ be a nonempty word. We say that p is a *local period* of w at the position $|u|$ if $w = uv$, with $u, v \neq 1$, and there exists a word z, with $|z| = p$, such that one of the following conditions holds for some words u' and v':

$$\begin{cases} (i) & u = u'z \text{ and } v = zv'; \\ (ii) & z = u'u \text{ and } v = zv'; \\ (iii) & u = u'z \text{ and } z = vv'; \\ (iv) & z = u'u = vv'. \end{cases} \tag{3.20}$$

Further, the *local period of* w *at the position* $|u|$, in symbols $\mathsf{per}(w, u)$, is defined as the smallest local period of w at the position u. It follows directly from (3.20) that $\mathsf{per}(w, u) \leq \mathsf{per}(w)$.

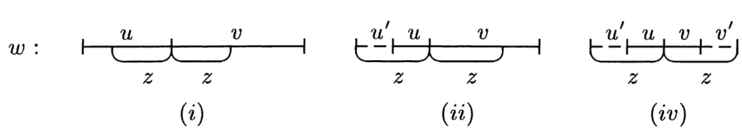


Figure 3.1: The illustration of a local period.

The following result is known as the Critical Factorization Theorem.

Theorem 3.13 (Theorem 6.2, Chapter 6, [RS97]) *Each nonempty word $w \in \mathcal{A}^*$, with $|w| \geq 2$, possesses at least one factorization $w = uv$, with $u, v \neq 1$, which is* critical*, i.e.,* $\mathsf{per}(w) = \mathsf{per}(w, u)$.

3.4 Rank of symbolic factors

In this section we prove Theorem 3.1. We start by introducing the concept of *factor* between directive sequences and, in Proposition 3.15, its relation with factor maps between $\mathcal{S}$-adic subshifts. These ideas are the $\mathcal{S}$-adic analogs of the concept of *premorphism* between ordered Bratteli diagrams from [AEG15] and their Proposition 4.6. Although Proposition 3.15 can be deduced from Proposition 4.6 in [AEG15] by passing from directive sequences to ordered Bratteli diagrams and backwards, we consider this a little bit artificial since it is possible to provide a direct combinatorial proof; this is done in the Appendix. It is interesting to note that our proof is constructive (in contrast of the existential proof in [AEG15]) and shows some additional features that are consequence of the combinatorics on words analysis made.

Next, we prove Theorem 3.1. We apply these results, in Corollary 3.19, to answer affirmatively Question 3.1 and, in Theorem 3.2, to prove a strong coalescence property for the class of systems considered in Theorem 3.1. It is worth noting that this last result is only possible due the bound in Theorem 3.1 being optimal. We end this section by proving that Cantor factors of finite topological rank systems are either subshifts of odometers.

3.4.1 Rank of factors of directive sequences

The following is the $\mathcal{S}$-adic analog of the notion of *premorphism* between ordered Bratteli diagrams in [AEG15].

Definition 3.1 Let $\boldsymbol{\sigma} = (\mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$, $\boldsymbol{\tau} = (\mathcal{B}_{n+1}^+ \to \mathcal{B}_n^+)_{n\in\mathbb{N}}$ be directive sequences. A *factor* $\boldsymbol{\phi}\colon \boldsymbol{\sigma} \to \boldsymbol{\tau}$ is a sequence of morphisms $\boldsymbol{\phi} = (\phi_n)_{n\in\mathbb{N}}$, where $\phi_0\colon \mathcal{A}_1^+ \to \mathcal{B}_0^+$ and $\phi_n\colon \mathcal{A}_n^+ \to \mathcal{B}_n^+$ for $n \geq 1$, such that $\phi_0 = \tau_0\phi_1$ and $\phi_n\sigma_n = \tau_n\phi_{n+1}$ and for every $n \geq 1$.

We say that $\boldsymbol{\phi}$ is *proper* (resp. *letter-onto*) if ϕ_n is proper (resp. *letter-onto*) for every $n \in \mathbb{N}$.

Remark 3.3 Factors are not affected by contractions. More precisely, if $0 = n_0 < n_1 < n_2 < \dots$, then $\boldsymbol{\phi'} = (\phi_{n_k})_{k\in\mathbb{N}}$ is a factor from $\boldsymbol{\sigma'} = (\sigma_{[n_k,n_{k+1})})_{k\in\mathbb{N}}$ to $\boldsymbol{\tau'} = (\tau_{[n_k,n_{k+1})})_{k\in\mathbb{N}}$.

The next lemma will be needed at the end of this section.

Lemma 3.14 *Let $\boldsymbol{\phi} = (\phi_n)_{n\geq 1}\colon \boldsymbol{\sigma} \to \boldsymbol{\tau}$ be a factor. Assume that $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are everywhere growing and proper and that $\boldsymbol{\phi}$ is letter-onto. Then, $X_{\boldsymbol{\tau}} = \bigcup_{k\in\mathbb{Z}} S^k\phi_0(X_{\boldsymbol{\sigma}}^{(1)})$ and $X_{\boldsymbol{\tau}}^{(n)} = \bigcup_{k\in\mathbb{Z}} S^k\phi_n(X_{\boldsymbol{\sigma}}^{(n)})$ for every $n \geq 1$.*

Proof. We start by proving that $X_{\boldsymbol{\tau}}^{(n)} \subseteq \bigcup_{k\in\mathbb{Z}} S^k\phi_n(X_{\boldsymbol{\sigma}}^{(n)})$. Let $y \in X_{\boldsymbol{\tau}}^{(n)}$ and $\ell \in \mathbb{N}$. There exist $N > n$ and $b \in \mathcal{B}_n$ such that $y_{[-\ell,\ell]}$ occurs in $\tau_{[n,N)}(b)$. In addition, since ϕ_N is letter-onto, there exists $a \in \mathcal{A}_N$ for which b occurs in $\phi_N(a)$. Then, $y_{[-\ell,\ell]}$ occurs in $\tau_{[n,N)}\phi_N(b)$ and, consequently, also in $\phi_n\sigma_{[n,N)}(b)$ as $\tau_{[n,N)}\phi_N = \phi_n\sigma_{[n,N)}$. Hence, by taking the limit $\ell \to \infty$ we can find $(k', x) \in \mathbb{Z} \times X_{\boldsymbol{\sigma}}^{(n)}$ such that $y = S^{k'}\phi_n(x)$. Therefore, $y \in \bigcup_{k\in\mathbb{Z}} S^k\phi_n(X_{\boldsymbol{\sigma}}^{(n)})$. To prove the other inclusion, we use Lemma 3.6 to compute:

$$
\begin{aligned}
\phi_n(X_{\boldsymbol{\sigma}}^{(n)}) &= \bigcap_{N>n}\bigcup_{k\in\mathbb{Z}} S^k\phi_n\sigma_{[n,N)}(\mathcal{A}_N^{\mathbb{Z}}) = \bigcap_{N>n}\bigcup_{k\in\mathbb{Z}} S^k\tau_{[n,N)}\phi_N(\mathcal{A}_N^{\mathbb{Z}}) \\
&\subseteq \bigcap_{N>n}\bigcup_{k\in\mathbb{Z}} S^k\tau_{[n,N)}(\mathcal{B}_N^{\mathbb{Z}}) = X_{\boldsymbol{\tau}}^{(n)}.
\end{aligned}
$$

□

As we mentioned before, the following proposition is consequence of the main result in [AEG15]. We provide a combinatorial proof in the Appendix.

Proposition 3.15 *Let $\boldsymbol{\sigma}$ be a letter-onto, everywhere growing and proper directive sequence. Suppose that $X_{\boldsymbol{\sigma}}$ is aperiodic. Then, there exist a contraction $\boldsymbol{\sigma}' = (\sigma'_n)_{n\in\mathbb{N}}$, a letter-onto, everywhere growing, proper and recognizable $\boldsymbol{\tau} = (\tau_n)_{n\in\mathbb{N}}$ generating $X_{\boldsymbol{\sigma}}$, and a letter-onto factor $\boldsymbol{\phi}\colon \boldsymbol{\sigma}' \to \boldsymbol{\tau}$, $\boldsymbol{\phi} = (\phi_n)_{n\in\mathbb{N}}$, such that $\phi_0 = \sigma'_0$.*

The next proposition is the main technical result of this section. To state it, it is convenient to introduce the following concept. The directive sequences $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are *equivalent* if $\boldsymbol{\sigma} = \boldsymbol{\nu}'$, $\boldsymbol{\tau} = \boldsymbol{\nu}''$ for some contractions $\boldsymbol{\nu}'$, $\boldsymbol{\nu}''$ of a directive sequence $\boldsymbol{\nu}$. Observe that equivalent directive sequences generate the same $\mathcal{S}$-adic subshift.

Proposition 3.16 *Let $\boldsymbol{\phi}\colon \boldsymbol{\sigma} \to \boldsymbol{\tau}$ be a letter-onto factor between the everywhere growing and proper directive sequences. Then, there exist a letter-onto and proper factor $\boldsymbol{\psi}\colon \boldsymbol{\sigma}' \to \boldsymbol{\nu}$, where*

(1) $\boldsymbol{\sigma}'$ is a contraction of $\boldsymbol{\sigma}$;

(2) $\boldsymbol{\nu}$ is letter-onto, everywhere growing, proper, equivalent to $\boldsymbol{\tau}$, $\mathrm{AR}(\boldsymbol{\nu}) \leq \mathrm{AR}(\boldsymbol{\sigma})$, and the first coordinate of $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ coincide;

(3) if $\boldsymbol{\tau}$ is recognizable, then $\boldsymbol{\nu}$ is recognizable.

Proof. Let us write $\boldsymbol{\sigma} = (\mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ and $\boldsymbol{\tau} = (\mathcal{B}_{n+1}^+ \to \mathcal{B}_n^+)_{n\in\mathbb{N}}$. Up to contractions, we can suppose that for every $n \geq 1$, $\#\mathcal{A}_n = \mathrm{AR}(\boldsymbol{\sigma})$ and that τ_n is $|\phi_n|_1^4$-proper (for the last property we used that $\boldsymbol{\tau}$ is everywhere growing and proper).

Using that ϕ_{n+1} is letter-onto, we can compute:

$$\tau_n(\mathcal{B}_{n+1}^{++}) \supseteq \tau_n(\phi_{n+1}(\mathcal{A}_{n+1}^{++})) = \phi_n(\sigma_n(\mathcal{A}_{n+1}^{++})) \subseteq \phi_n(\mathcal{A}_n^+),$$

where in the middle step we used the commutativity property of $\boldsymbol{\phi}$. We deduce that

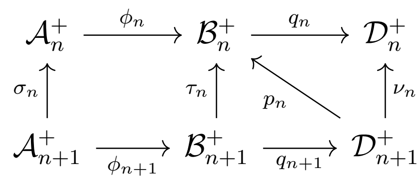
$$\tau_n(\mathcal{B}_{n+1}^{++}) \cap \phi_n(\mathcal{A}_n^+) \neq \emptyset \text{ for every } n \in \mathbb{N}.$$

This and the fact that τ_n is a $|\phi_n|_1^4$-proper morphism allow us to use Lemma 3.12 to find morphisms $\mathcal{B}_{n+1}^+ \xrightarrow{q_{n+1}} \mathcal{D}_{n+1}^+ \xrightarrow{p_n} \mathcal{B}_n^+$ such that

$$\text{(i) } \#\mathcal{D}_{n+1} \leq \#\mathcal{A}_n, \text{ (ii) } \tau_n = p_n q_{n+1}, \text{ (iii) } q_{n+1} \text{ is letter-onto and proper.}$$

We define $\nu_0 \coloneqq p_0$, the morphisms $\nu_n \coloneqq q_n p_n \colon \mathcal{D}_{n+1}^+ \to \mathcal{D}_n^+$ and $\psi_n \coloneqq q_n \phi_n \colon \mathcal{A}_n^+ \to \mathcal{D}_n^+$, $n \geq 1$, and the sequences $\boldsymbol{\nu} = (\nu_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\psi} = (\psi_n)_{n\in\mathbb{N}}$, where $\psi_0 \coloneqq \phi_0$. We are going to show that these objects satisfy the conclusion of the proposition.

We start by observing that it follows from the definitions that the diagram below commutes for all $n \geq 1$:



In particular, $\nu_n \nu_{n+1} = q_n \tau_n p_{n+1}$, so $\langle \nu_{[n,n+1]} \rangle \geq \langle \tau_n \rangle$. Being $\boldsymbol{\tau}$ everywhere growing, this implies that $\boldsymbol{\nu}$ has the same property. We also observe that (iii) implies that $\nu_n = q_n p_n$ is letter-onto and proper. Altogether, these arguments prove that, up to contracting the first levels, $\boldsymbol{\nu}$ is everywhere growing and proper.

Next, we note that $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are equivalent as both are contractions of $(p_0, q_1, p_1, q_2, \dots)$. This implies, by Lemma 1.1, that $\boldsymbol{\nu}$ is recognizable if $\boldsymbol{\tau}$ is recognizable. Further, by (i), $\boldsymbol{\nu}$ has alphabet rank at most $\mathrm{AR}(\boldsymbol{\sigma})$.

It is only left to prove that $\boldsymbol{\psi}$ is a letter-onto and proper factor. By unraveling the definitions we can compute:

$$\psi_0 = \phi_0 = \tau_0 \phi_1 = p_0 q_1 \phi_1 = \nu_0 \psi_1,$$

and from the diagram we have $\sigma_n \psi_n = \psi_{n+1} \tau_n$ for all $n \geq 1$. Therefore, $\boldsymbol{\psi}$ is a factor. Finally, since q_n is letter-onto and proper by (iii) and $\boldsymbol{\phi}$ was assumed to be letter-onto, $\psi_n = q_n \phi_n$ is letter-onto and proper. □

3.4.2 Rank of factors of $\mathcal{S}$-adic subshifts

In this section, we will prove Theorem 3.1 and its consequences. We start with a technical lemma.

The next lemma will allow us to assume without loss of generality that our directive sequences are letter-onto.

Lemma 3.17 *Let $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ be an everywhere growing and proper directive sequence. If $\tilde{\mathcal{A}}_n = \mathcal{A}_n \cap \mathcal{L}(X_{\boldsymbol{\sigma}}^{(n)})$, $\tilde{\tau}_n$ is the restriction of τ_n to $\tilde{\mathcal{A}}_{n+1}$ and $\tilde{\boldsymbol{\tau}} = (\tilde{\tau}_0, \tilde{\tau}_1, \dots)$, then $\tilde{\boldsymbol{\tau}}$ is letter-onto and $X_{\tilde{\boldsymbol{\tau}}}^{(n)} = X_{\boldsymbol{\tau}}^{(n)}$ for every $n \in \mathbb{N}$. Conversely, if $\boldsymbol{\tau}$ is letter-onto, then $\mathcal{A}_n \subseteq \mathcal{L}(X_{\boldsymbol{\tau}}^{(n)})$ for every $n \in \mathbb{N}$.*

Proof. By Lemma 1.2, $\tilde{\tau}_n$ is letter-onto mapping $\tilde{\mathcal{A}}_{n+1}^+$ into $\tilde{\mathcal{A}}_n$. Moreover, that lemma also gives that for every $x \in X_{\boldsymbol{\tau}}^{(n)}$ and $N > n$, there exists a $\tau_{[n,N)}$-factorization (k', x') of x in

$X_{\boldsymbol{\tau}}^{(N)}$. This together with the inclusion $X_{\boldsymbol{\tau}}^{(N)} \subseteq \tilde{\mathcal{A}}_N^{\mathbb{Z}}$ imply that

$$Z \coloneqq \bigcap_{N>n} \bigcup_{k\in\mathbb{Z}} S^k \tau_{[n,N)}(\tilde{\mathcal{A}}_N^{\mathbb{Z}}) \supseteq X_{\boldsymbol{\tau}}^{(n)}$$

Now, $\tilde{\boldsymbol{\tau}}$ is everywhere growing and proper, so we can apply Lemma 3.6 to obtain that $X_{\tilde{\boldsymbol{\tau}}}^{(n)} = Z \supseteq X_{\boldsymbol{\tau}}^{(n)}$. Since it is clear that $X_{\tilde{\boldsymbol{\tau}}}^{(n)} \subseteq X_{\boldsymbol{\tau}}^{(n)}$ as $\tilde{\mathcal{A}}_N \subseteq \mathcal{A}_N$ for every N, we conclude that $X_{\tilde{\boldsymbol{\tau}}}^{(n)} = X_{\boldsymbol{\tau}}^{(n)}$.

If $\boldsymbol{\tau}$ is letter-onto, then $\mathcal{A}_n \subseteq \mathcal{L}(\bigcup_{k\in\mathbb{Z}} S^k \tau_{[n,N)}(\mathcal{A}_N^{\mathbb{Z}}))$ for every $N > n$, and hence, by the formula in Lemma 3.6, $\mathcal{A}_n \subseteq \mathcal{L}(X_{\boldsymbol{\tau}}^{(n)})$. □

Now we are ready to prove Theorem 3.1. We re-state it in a more precise way.

Theorem 3.1 Let $\pi\colon (X,S) \to (Y,S)$ be a factor map between aperiodic subshifts. Suppose that X is generated by the everywhere growing and proper directive sequence $\boldsymbol{\sigma} = (\sigma_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ of alphabet rank K. Then, Y is generated by a letter-onto, everywhere growing, proper and recognizable directive sequence $\boldsymbol{\tau}$ of alphabet rank at most K.

Moreover, if $\boldsymbol{\sigma}$ is letter-onto, then, up to contracting the sequences, there exists a proper factor $\boldsymbol{\phi}\colon \boldsymbol{\sigma} \to \boldsymbol{\tau}$ such that $\pi(\sigma_0(x)) = \phi_0(x)$ for all $x \in X_{\boldsymbol{\sigma}}^{(1)}$ and $|\sigma_0(a)| = |\phi_0(a)|$ for all $a \in \mathcal{A}_1$

Proof. Thanks to Lemma 3.17, we can assume without loss of generality that $\boldsymbol{\sigma}$ is letter-onto. Moreover, in this case we have:

$$\mathcal{A}_n \subseteq \mathcal{L}(X_{\boldsymbol{\sigma}}^{(n)}) \text{ for every } n \in \mathbb{N}. \tag{3.21}$$

Let us write $\boldsymbol{\sigma} = (\sigma_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$. By contracting $\boldsymbol{\sigma}$, we can further assume that σ_0 is r-proper and π has radius r. Then, Lemma 3.5 gives us a proper morphism $\tau\colon \mathcal{A}_1^+ \to \mathcal{B}^+$, where $\mathcal{B}$ is the alphabet of Y, such that

$$\pi(\sigma_0(x)) = \tau(x) \text{ for all } x \in X_{\boldsymbol{\sigma}}^{(1)} \text{ and } |\sigma_0(a)| = |\tau(a)| \text{ for every } a \in \mathcal{A}_1. \tag{3.22}$$

In particular, $\pi(\sigma_{[0,n)}(x)) = \tau\sigma_{[1,n)}(x)$ and $|\sigma_{[0,n)}(a)| = |\tau\sigma_{[1,n)}(a)|$ for all $n \in \mathbb{N}$, $x \in X_{\boldsymbol{\sigma}}^{(n)}$ and $a \in \mathcal{A}_n$, so (3.22) holds for any contraction of $\boldsymbol{\sigma}$.

We define $\tilde{\boldsymbol{\sigma}} = (\tau, \sigma_1, \sigma_2, \dots)$ and observe this is a letter-onto, everywhere growing and proper sequence generating Y. This and that Y is aperiodic allow us to use Proposition 3.15 and obtain, after a contraction, a letter-onto factor $\tilde{\boldsymbol{\phi}}\colon \tilde{\boldsymbol{\sigma}} \to \tilde{\boldsymbol{\tau}}$, where $\tilde{\phi}_0 = \tilde{\sigma}_0 = \tau$ and $\tilde{\boldsymbol{\tau}}$ is a letter-onto, everywhere growing, proper and recognizable directive sequence generating Y. The sequence $\tilde{\boldsymbol{\tau}}$ has all the properties required by the theorem but having alphabet rank bounded by K. To overcome this, we use Proposition 3.16 with $\tilde{\boldsymbol{\phi}}$ and do more contractions to obtain a letter-onto and proper factor $\boldsymbol{\phi}\colon \tilde{\boldsymbol{\sigma}} \to \boldsymbol{\tau}$ such that $\phi_0 = \tilde{\phi}_0 = \tau$ and $\boldsymbol{\tau}$ is a letter-onto, everywhere growing, proper and recognizable directive sequence generating Y and satisfying $\mathrm{AR}(\boldsymbol{\tau}) \leq \mathrm{AR}(\tilde{\boldsymbol{\sigma}}) = \mathrm{AR}(\boldsymbol{\sigma})$.

It is left to prove the last part of the theorem. Observe that since $\tilde{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}$ differ only at their first coordinate, $\boldsymbol{\phi}$ is also a factor from $\boldsymbol{\sigma}$ to $\boldsymbol{\tau}$. Further, by equation (3.22) and the fact that $\phi_0 = \tau$, we have $\pi(\sigma_0(x)) = \tau(x) = \phi_0(x)$ and $|\sigma_0(a)| = |\phi_0(a)|$ for every $x \in X_{\boldsymbol{\sigma}}^{(1)}$ and $a \in \mathcal{A}_1$. □

Corollary 3.18 *Let (X, S) be an aperiodic minimal subshift of generated by an everywhere growing and proper directive sequence of alphabet rank K. Then, the topological rank of X is at most K.*

Proof. We can use Theorem 3.1 to obtain an everywhere growing, proper and recognizable directive sequence $\boldsymbol{\tau} = (\tau_n : \mathcal{B}_{n+1}^+ \to \mathcal{B}_n^+)_{n\in\mathbb{N}}$ generating X and having of alphabet rank at most K. Due to Lemma 3.17, we can assume that $\boldsymbol{\tau}$ is letter-onto. In particular, $\mathcal{B}_n \subseteq \mathcal{L}(X_{\boldsymbol{\tau}}^{(n)})$ for every $n \in \mathbb{N}$.

We claim that $X_{\boldsymbol{\tau}}^{(n)}$ is minimal. Indeed, if $Y \subseteq X_{\boldsymbol{\tau}}^{(n)}$ is a subshift, then $\tau_{[0,n)}(Y)$ is closed (as $\tau_{[0,n)} : X_{\boldsymbol{\tau}}^{(n)} \to X_{\boldsymbol{\tau}}$ is continuous), so $\bigcup_{k\in\mathbb{Z}} S^k\tau_{[0,n)}(Y) = \bigcup_{|k|\le|\tau_{[0,n)}|} S^k\tau_{[0,n)}(Y)$ is a subshift in $X_{\boldsymbol{\tau}}$ which, by minimality, is equal to it. Thus, any point $x \in X_{\boldsymbol{\tau}}^{(n)}$ has a $\tau_{[0,n)}$-factorization (k, z) with $z \in Y$. The recognizability property of $(X_{\boldsymbol{\tau}}^{(n)}, \tau_{[0,n)})$ then implies that $Y = X_{\boldsymbol{\tau}}^{(n)}$.

Now, we prove that for any $n \in \mathbb{N}$ there exists $N > n$ such that $\tau_{[n,N)}$ is positive. This would imply that the topological rank of X is at most K and hence would complete the proof. Let $n \in \mathbb{N}$ and R be a constant of recognizability for $(X_{\boldsymbol{\tau}}^{(n)}, \tau_{[0,n)})$. Since $X_{\boldsymbol{\tau}}^{(n)}$ is minimal, there exists a constant $L \ge 1$ such that two consecutive occurrences of a word $w \in \mathcal{L}(X_{\boldsymbol{\tau}}^{(n)}) \cap \mathcal{B}_n^{2R+1}$ in a point $x \in X_{\boldsymbol{\tau}}^{(n)}$ are separated by at most L. Let $N > n$ be big enough so that $\langle\tau_{[0,N)}\rangle \ge L+2R$. Then, for all $a \in \mathcal{B}_N \subseteq \mathcal{L}(X_{\boldsymbol{\tau}}^{(N)})$ and $w \in \mathcal{L}(X_{\boldsymbol{\tau}}^{(n)}) \cap \mathcal{B}_n^{2R+1}$, w occurs at a position $i \in \{R, R+1, \dots, |\tau_{[0,N)}(a)| - R\}$ of $\tau_{[0,N)}(a)$. Since R is a recognizability constant for $(X_{\boldsymbol{\tau}}^{(n)}, \tau_{[0,n)})$, we deduce that for all $a \in \mathcal{B}_N$ and $b \in \mathcal{B}_n$, b occurs in $\tau_{[n,N)}(a)$. Thus, $\tau_{[n,N)}$ is positive. □

We can now prove Corollary 0.6.

Corollary 0.6 Let (X, S) be an aperiodic minimal subshift generated by an everywhere growing directive sequence of finite alphabet rank. Then, the topological rank of (X, S) is finite.

Proof. We are going to prove that X is generated by an everywhere growing and proper directive sequence $\boldsymbol{\tau}$ of finite alphabet rank. This would imply, by Corollary 3.18, that the topological rank of X is finite. Let $\boldsymbol{\sigma} = (\sigma_n : \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ be an everywhere growing directive sequence of finite alphabet rank generating X. We contract $\boldsymbol{\tau}$ in a way such that $\#\mathcal{A}_n \le K$ for every $n \ge 1$.

We are going to inductively define subshifts X_n, $n \in \mathbb{N}$. We start with $X_0 := X$. We now assume that X_n is defined for some $n \in \mathbb{N}$. Then the set $X'_{n+1} = \{x \in X_{\boldsymbol{\sigma}}^{(n+1} : \sigma_n(x) \in X_n\}$ is a subshift. We define X_{n+1} as any minimal subshift contained in X'_{n+1}. It follows from the definition of X_{n+1} that $\bigcup_{k\in\mathbb{Z}} S^k\sigma_n(X_{n+1}) \subseteq X_n$. Being X_n minimal, we have

$$\bigcup_{k\in\mathbb{Z}} S^k\sigma_n(X_{n+1}) = X_n. \tag{3.23}$$

Let $\tilde{\mathcal{A}}_n = \mathcal{A}_n \cap \mathcal{L}(X_n)$. Equation (3.23) and the fact that $\boldsymbol{\sigma}$ is everywhere growing allow us to assume without loss of generality that, after a contraction of $\boldsymbol{\sigma}$, the following holds for every $n \in \mathbb{N}$:

$$\text{if } a \in \tilde{\mathcal{A}}_{n+1} \text{ and } w \in \mathcal{L}(X_n) \text{ has length 3, then } w \text{ occurs twice in } \sigma_n(a). \tag{3.24}$$

Let us fix a word $w_n = a_n b_n c_b \in \mathcal{L}(X_n)$ of length 3. Then, by (3.24), we can decompose $\sigma_n(a) = u_n(a)v_n(a)$ in a way such that

$$u_n(a) \text{ ends with } a_n,\ v_n(a) \text{ starts with } b_n c_n \text{ and } |v_n(a)| \geq 2. \tag{3.25}$$

To define $\boldsymbol{\tau}$, we need to introduce additional notation first. Let $\mathcal{B}_n$ be the alphabet consisting of tuples $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ such that $ab \in \mathcal{L}(X_n)$. Also, if $w = w_1 \dots w_{|w|} \in \mathcal{L}(X_n)$ has length $|w| \geq 2$, then $\chi_n(w) := \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} w_2 \\ w_3 \end{smallmatrix}\right] \dots \left[\begin{smallmatrix} w_{|w|-1} \\ w_{|w|} \end{smallmatrix}\right] \in \mathcal{B}_n^+$, and if $w' = \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] \dots \left[\begin{smallmatrix} w_{|w|-1} \\ w_{|w|} \end{smallmatrix}\right] \in \mathcal{B}_0^+$, then $\eta(w') := w_1 \dots w_{|w|-1} \in \mathcal{A}_0^+$. Observe that $\eta\colon \mathcal{B}_0^+ \to \mathcal{A}_0^+$ is a morphism.

We now define $\boldsymbol{\tau}$. Let $\tau_n\colon \mathcal{B}_{n+1}^+ \to \mathcal{B}_n^+$ be the unique morphism such that $\tau_n(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]) = \chi_n(v_n(a)u_n(a)b_n)$ for every $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathcal{B}_{n+1}$. Observe that since $v_n(a)u_n(a)b_n \in \mathcal{L}(X_n)$, it is indeed the case that $\tau_n(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]) \in \mathcal{B}_n^+$. We set $\boldsymbol{\tau} = (\eta\tau_0, \tau_1, \tau_2, \dots)$.

It follows from (3.25) that for every $n \in \mathbb{N}$ and $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathcal{B}_{n+1}$, $\tau_n(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right])$ starts with $\left[\begin{smallmatrix} b_n \\ c_n \end{smallmatrix}\right]$ and ends with $\left[\begin{smallmatrix} a_n \\ b_n \end{smallmatrix}\right]$. Thus, $\boldsymbol{\tau}$ is proper. Moreover, since $|v_n(a)| \geq 2$, we have $|v_n(a)u_n(a)b_n| \geq 3$ and thus $|\tau_n(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right])| \geq 2$. Therefore, $\langle \tau_n \rangle \geq 2$ and $\boldsymbol{\tau}$ is everywhere growing. Also, $\#\mathcal{B}_n \leq \#\mathcal{A}_n^2 \leq K^2$ for every $n \in \mathbb{N}$, so the alphabet rank of $\boldsymbol{\tau}$ is finite.

It remains to prove that $X = X_{\boldsymbol{\tau}}$. By minimality, it is enough to prove that $X \supseteq X_{\boldsymbol{\tau}}$. Observe that since $\tau_n\chi_{n+1}(ab) = \chi_n(v_n(a)u_n(b)b_n)$, the word $\tau_n\chi_{n+1}(ab)$ occurs in $\chi_n\sigma_n(ab)$. Moreover, for every $w = w_1 \dots w_{|w|} \in \mathcal{L}(X_{\boldsymbol{\sigma}}^{(n)})$, $\tau_n\chi_{n+1}(w)$ occurs in $\chi_n\sigma_n(w)$. Then, by using the symbol $\sqsubseteq$ to denote the "subword" relation, we can write for every $n \in \mathbb{N}$ and $ab \in \mathcal{L}(X_{\boldsymbol{\sigma}}^{(n)})$:

$$\begin{aligned} \tau_{[0,n)}\chi_n(ab) &\sqsubseteq \tau_{[0,n-1)}\chi_{n-1}\sigma_{n-1}(ab) \\ &\sqsubseteq \tau_{[0,n-2)}\chi_{n-2}\sigma_{[n-2,n)}(ab) \sqsubseteq \dots \sqsubseteq \chi_0\sigma_{[0,n)}(ab) \end{aligned}$$

Hence, $\eta\tau_{[0,n)}(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]) \sqsubseteq \eta\chi_0\sigma_{[0,n)}(ab) \sqsubseteq \sigma_{[0,n)}(ab)$. We conclude that $X_{\boldsymbol{\tau}} \subseteq X_{\boldsymbol{\sigma}} = X$. □

Corollary 3.19 *Let (X,S) be a minimal subshift of topological rank K and $\pi\colon (X,S) \to (Y,S)$ a factor map, where Y is an aperiodic subshift. Then, the topological rank of Y is at most K.*

Proof. By Theorem 0.1, (X,S) is generated by a proper and primitive directive sequence $\boldsymbol{\sigma}$ of alphabet rank equal to K. In particular, $\boldsymbol{\sigma}$ is everywhere growing and proper, so we can use Theorem 3.1 to obtain an everywhere growing, proper and recognizable directive sequence $\boldsymbol{\tau} = (\tau_n\colon \mathcal{B}_{n+1}^+ \to \mathcal{B}_n^+)_{n\geq 0}$ generating (Y,S) and having of alphabet rank at most K. Then, the hypothesis of Corollary 3.18 hold for (Y,S), and thus the topological rank of (Y,S) is at most K. $\square$

The following notion will be used in the proof of the theorem below: $\boldsymbol{\sigma} = (\sigma_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n)_{n\geq 0}$ has *exact alphabet rank at most* K if $\#\mathcal{A}_n \leq K$ for all $n \geq 1$.

Corollary 3.2 Let (X,S) be an $\mathcal{S}$-adic subshift generated by an everywhere growing and proper sequence of alphabet rank K, and $\pi_j\colon (X_{j+1},S) \to (X_j,S)$, $j = 0,\dots,L$, be a chain of aperiodic symbolic factors, with $X_L = X$. Suppose that $L > \log_2(K)$. Then π_j is a conjugacy for some j.

Proof. We start by using Theorem 3.1 with the identity function $\mathrm{id}\colon (X,S) \to (X,S)$ to obtain a letter-onto, everywhere growing, proper and recognizable directive sequence $\boldsymbol{\sigma_L}$ of alphabet rank at most K generating X. By doing a contraction, we can assume that $\boldsymbol{\sigma_L}$ has exact alphabet rank at most K.

By Theorem 3.1 applied to π_{L-1} and $\boldsymbol{\sigma_L}$, there exists, after a contraction of $\boldsymbol{\sigma_L}$, a letter-onto factor $\boldsymbol{\phi_{L-1}}\colon \boldsymbol{\sigma_L} \to \boldsymbol{\sigma_{L-1}}$, where $\boldsymbol{\sigma_{L-1}}$ is letter-onto, everywhere growing, proper, recognizable, has alphabet rank at most K, generates X_{L-1}, and, if $\phi_{L-1,0}$ and $\sigma_{L,0}$ are the first coordinates of $\boldsymbol{\phi_{L-1}}$ and $\boldsymbol{\sigma_L}$, respectively, then $\pi_{L-1}(\sigma_{L,0}(x)) = \phi_{L-1,0}(x)$ for every $x \in X_{\boldsymbol{\sigma_L}}^{(1)}$ and $|\sigma_{L,0}(a)| = |\phi_{L-1,0}(a)|$ for every letter a in the domain of $\sigma_{L,0}$. By contracting these sequences, we can also suppose that $\boldsymbol{\sigma_{L-1}}$ has exact alphabet rank at most K. The same procedure applies to π_{L-2} and $\boldsymbol{\sigma_{L-1}}$. Thus, by continuing in this way we obtain for every $j = 0,\dots,L-1$ a letter-onto factor $\boldsymbol{\phi_j}\colon \boldsymbol{\sigma_{j+1}} \to \boldsymbol{\sigma_j}$ such that

- $\boldsymbol{\sigma_j}$ is letter-onto, everywhere growing, proper, recognizable, has exact alphabet rank at most K, generates X_j, $\pi_j(\sigma_{j+1,0}(x)) = \phi_{j,0}(x)$ for every $x \in X_{\boldsymbol{\sigma_{j+1}}}^{(1)}$, and $|\sigma_{j+1,0}(a)| = |\phi_{j,0}(a)|$ for every $a \in \mathcal{A}_{j+1,1}$.

Here, we are using the notation $\boldsymbol{\sigma_j} = (\sigma_{j,n}\colon \mathcal{A}_{j,n+1}^+ \to \mathcal{A}_{j,n}^+)_{n\in\mathbb{N}}$, $\boldsymbol{\phi_j} = (\phi_{j,n}\colon \mathcal{A}_{j+1,n}^+ \to \mathcal{A}_{j,n}^+)_{n\in\mathbb{N}}$ and $X_j^{(n)} = X_{\boldsymbol{\sigma_j}}^{(n)}$. We note that

($\triangle_1$) for every $x \in X_{j+1}^{(1)}$, $\pi_j(\sigma_{j+1,0}(x)) = \phi_{j,0}(x) = \sigma_{j,0}\phi_{j,1}(x)$ since $\phi_{j,0} = \sigma_{j,0}\phi_{j,1}$;

($\triangle_2$) $X_j^{(1)} = \bigcup_{k\in\mathbb{Z}} S^k \phi_{j,1}(X_{j+1}^{(1)})$ by Lemma 3.14.

Hence, the following diagram commutes:

$$\begin{array}{ccccccccc}
X_0^{(1)} & \xleftarrow{\phi_{0,1}} & \cdots & X_j^{(1)} & \xleftarrow{\phi_{j,1}} & X_{j+1}^{(1)} & \cdots & \xleftarrow{\phi_{L-1,1}} & X_L^{(1)} \\
\big\downarrow \sigma_{0,0} & & & \big\downarrow \sigma_{j,0} & & \big\downarrow \sigma_{j+1,0} & & & \big\downarrow \sigma_{L,0} \\
X_0^{(0)} & \xleftarrow[\pi_0]{} & \cdots & X_j^{(0)} & \xleftarrow[\pi_j]{} & X_{j+1}^{(0)} & \cdots & \xleftarrow[\pi_{L-1}]{} & X_L^{(0)}
\end{array}$$

Claim 3.2 If $(X^{(1)}_{j+1}, \phi_{j,1})$ is recognizable, then π_j is a conjugacy.

Proof. Let us assume that $(X^{(1)}_{j+1}, \phi_{j,1})$ is recognizable and let, for $i = 0, 1$, $x^i \in X^{(1)}_{j+1}$ such that $y = \pi_j(x^0) = \pi_j(x^1)$. We have to show that $x^0 = x^1$. First, we use Lemma 1.2 to find a centered $\sigma_{j+1,0}$-factorization (k^i, z^i) of x^i in $X^{(1)}_{j+1}$. Then, equation $\triangle_1$ allows us to compute:

$$S^{k^0}\sigma_{j,0}\phi_{j,1}(z^0) = S^{k^0}\pi_j(\sigma_{j+1,0}(z^0)) = \pi_j(x^0) = \pi_j(x^1) = S^{k^1}\sigma_{j,0}\phi_{j,1}(z^1).$$

This implies that (k^i, z^i) is a $\sigma_{j,0}\phi_{j,1}$-factorization of y in $X^{(1)}_{j+1}$ for $i = 0, 1$. Moreover, these are centered factorizations as, by •, $|\sigma_{j,0}\phi_{j,1}(a)| = |\sigma_{j+1,0}(a)|$ for all $a \in \mathcal{A}_{j+1,1}$. Now, being $(X^{(1)}_j, \sigma_{0,j})$ and $(X^{(1)}_{j+1}, \phi_{j,1})$ recognizable, Lemma 1.1 gives that $(X^{(1)}_{j+1}, \sigma_{j,1}\phi_{j,1})$ is recognizable, and thus we have that $(k^0, z^0) = (k^1, z^1)$. Therefore, $x^0 = x^1$ and π is a conjugacy. ☐

Now we can finish the proof. We assume, by contradiction, that π_j is not a conjugacy for all j. Then, by the claim,

$$(X^{(1)}_j, \phi_{1,j}) \text{ is not recognizable for every } j \in \{0, \dots, L-1\}. \tag{3.26}$$

Let

$$\boldsymbol{\nu} = (\phi_{0,1}, \phi_{1,1}, \phi_{2,1}, \dots, \phi_{L-1,1}, \sigma_{L,1}, \sigma_{L,2}, \sigma_{L,3}, \dots).$$

The idea is to use Theorem 3.7 with $\boldsymbol{\nu}$ to obtain a contradiction. To do so, we first note that, since $\boldsymbol{\nu}$ and $\boldsymbol{\sigma}^{(L)}$ have the same "tail", $X^{(m+L)}_{\boldsymbol{\nu}} = X^{(m+1)}_L$ for all $m \in \mathbb{N}$. Moreover, $\triangle_2$ and the previous relation imply that

$$\begin{aligned} X^{(j)}_{\boldsymbol{\nu}} &= \bigcup_{k\in\mathbb{Z}} S^k\phi_{j,1}(X^{(j+1)}_{\boldsymbol{\nu}}) = \dots = \bigcup_{k\in\mathbb{Z}} S^k\phi_{j,1}\dots\phi_{L-1,1}(X^{(L)}_{\boldsymbol{\nu}}) \\ &= \bigcup_{k\in\mathbb{Z}} S^k\phi_{j,1}\dots\phi_{L-1,1}(X^{(1)}_L) = \bigcup_{k\in\mathbb{Z}} S^k\phi_{j,1}\dots\phi_{L-2,1}(X^{(1)}_{L-1}) = \dots = X^{(1)}_j. \end{aligned}$$

This and (3.26) imply that for every $j \in \{1, \dots, L-1\}$, the level $(X^{(j)}_{\boldsymbol{\nu}}, \phi_{j,1})$ of $\boldsymbol{\nu}$ is not recognizable. Being $\boldsymbol{\nu}$ everywhere growing as $\boldsymbol{\sigma_L}$ has this property, we conclude that Theorem 3.7 can be applied and, therefore, that $X^{(1)}_0 = X_{\boldsymbol{\nu}}$ is periodic. But then $X_0 = \bigcup_{k\in\mathbb{Z}} S^k\sigma_{0,0}(X^{(1)}_0)$ is periodic, contrary to our assumptions. ☐

A system (X, S) is *coalescent* if every endomorphism $\pi\colon (X, S) \to (X, S)$ is an automorphism. This notion has been relevant in the context of topological dynamics; see for example [Dow97].

Corollary 3.20 *Let (X, S) be an $\mathcal{S}$-adic subshift generated by an everywhere growing and proper directive sequence of finite alphabet rank. Then, (X, S) is coalescent.*

Remark 3.4 A linearly recurrent subshift of constant C is generated by a primitive and proper directive sequence of alphabet rank at most $C(C+1)^2$ ([Dur00], Proposition 6). In [DHS99], the authors proved the following

Theorem 3.21 ([DHS99], Theorem 3) For a linearly recurrent subshift X of constant C, in any chain of factors $\pi_j\colon (X_j, S) \to (X_{j+1}, S)$, $j = 0, \dots, L$, with $X_0 = X$ and $L \geq (2C(2C+1)^2)^{4C^3(2C+1)^2}$ there is at least one π_j which is a conjugacy.

Thus, Theorem 3.2 is not only a generalization of this result to a much larger class of systems, but also improves the previous super-exponential constant to a logarithmic one.

In Proposition 28 of [DHS99], the authors proved that Cantor factors of linearly recurrent systems are either subshifts or odometers. Their proof only uses that this kind of systems satisfy the strong coalescence property that we proved in Corollary 3.20 for finite topological rank systems. Therefore, by the same proof, we have:

Corollary 3.22 *Let $\pi\colon (X,S) \to (Y,T)$ be a factor map between minimal systems. Assume that (X,S) has finite topological rank and that (Y,T) is a Cantor system. Then, (Y,T) is either a subshift or an odometer.*

Proof. We sketch the proof from [DHS99] that we mentioned above.

Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be a sequence of clopen partitions of Y such that $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ and their union generates the topology of Y. Also, let Y_n be the subshift obtained by codifying the orbits of (Y,T) by using the atoms of $\mathcal{P}_n$. Then, the fact that $\mathcal{P}_n$ is a clopen partition induces a factor map $\pi_n\colon (Y,T) \to (Y_n,S)$. Moreover, since $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$, there exists a factor map $\xi_n\colon (Y_{n+1},S) \to (Y_n,S)$ such that $\xi_n\pi_{n+1} = \pi_n$. Hence, we have the following chain of factors:

$$(X,S) \xrightarrow{\pi} (Y,T) \xrightarrow{\pi_n} (Y_n,S) \xrightarrow{\xi_{n-1}} (Y_{n-1},S) \xrightarrow{\xi_{n-2}} \dots \xrightarrow{\xi_1} (Y_0,S).$$

We conclude, by also using the fact that the partitions $\mathcal{P}_n$ generate the topology of Y, that (Y,S) is conjugate to the inverse limit $\overleftarrow{\lim}_{n\to\infty}(Y_n;\xi_n)$.

Now we consider two cases. If Y_n is periodic for every $n \in \mathbb{N}$, then Y is the inverse limit of periodic system, and hence an odometer. In the other case, we have, by Corollary 3.2, that ξ_n is a conjugacy for all big enough $n \in \mathbb{N}$, and thus that (Y,S) is conjugate to one of the subshifts Y_n. □

3.5 Fibers of symbolic factors

The objective of this section is to prove Theorem 3.3, which states that factor maps $\pi\colon (X,S) \to (Y,S)$ between $\mathcal{S}$-adic subshifts of finite topological rank are always *almost* k*-to*-1 for some k bounded by the topological rank of X. We start with some lemmas from topological dynamics.

Lemma 3.23 ([Aus88]) *Let $\pi\colon X \to Y$ be a continuous map between compact metric spaces. Then $\pi^{-1}\colon Y \to 2^X$ is continuous at every point of a residual subset of Y.*

Next lemma gives a sufficient condition for a factor map π to be almost k-to-1. Recall that $E(X,S)$ stands for the Ellis semigroup of (X,S).

Lemma 3.24 *Let $\pi\colon (X,S) \to (Y,T)$ be a factor map between topological dynamical systems,* *with (Y,T) minimal, and $K \geq 1$ an integer. Suppose that for every $y \in Y$ there exists $u \in E(2^X, S)$ such that $\#u \circ \pi^{-1}(y) \leq K$. Then, π is almost k-to-1 for some $k \leq K$.*

Proof. First, we observe that by the description of $u \circ A$ in terms of nets at the end of Subsection 3.2, we have

$$\#u \circ A \leq \#A, \ \forall u \in E(2^X, S), \ A \in 2^X. \tag{3.27}$$

Now, by previous lemma, there exists a residual set $\tilde{Y} \subseteq Y$ of continuity points for π^{-1}. Let $y, y' \in \tilde{Y}$ be arbitrary. Since Y is minimal, there exists a sequence $(n_\ell)_\ell$ such that $\lim_\ell T^{n_\ell} y = y'$. If $w \in E(2^X, S)$ is the limit of a convergent subnet of $(S^{n_\ell})_\ell$, then $wy = y'$. By the continuity of π^{-1} at y' and (3.27), we have

$$\#\pi^{-1}(y') = \#\pi^{-1}(wy) = \#w \circ \pi^{-1}(y) \leq \#\pi^{-1}(y).$$

We deduce, by symmetry, that $\#\pi^{-1}(y') = \#\pi^{-1}(y)$. Hence, $k \coloneqq \pi^{-1}(y)$ does not depend on the chosen $y \in \tilde{Y}$. To end the proof, we have to show that $k \leq K$. We fix $y \in \tilde{Y}$ and take, using the hypothesis, $u \in E(2^X, S)$ such that $\#u \circ \pi^{-1}(y) \leq K$. As above, by minimality, there exists $v \in E(2^X, S)$ such that $vuy = y$. Then, by the continuity of π^{-1} at y,

$$\pi^{-1}(y) = \pi^{-1}(vuy) = (vu) \circ \pi^{-1}(y) = v \circ (u \circ \pi^{-1}(y)).$$

This and (3.27) imply that $k = \#\pi^{-1}(y) \leq \#u \circ \pi^{-1}(y) \leq K$. □

Let $\sigma\colon \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism, (k,x) a centered σ-factorization of $y \in \mathcal{B}^{\mathbb{Z}}$ in $\mathcal{A}^{\mathbb{Z}}$ and $\ell \in \mathbb{Z}$. Note that there exists a unique $j \in \mathbb{Z}$ such that $\ell \in [c_{\sigma,j}(k,x), c_{\sigma,j+1}(k,x))$ (recall the notion of *cut* from Definition 4.2). In this context, we say that $(c_{\sigma,j}(k,x), x_j)$ is *the symbol of* (k,x) *covering position* ℓ *of* y.

Theorem 3.3 Let $\pi\colon (X,S) \to (Y,S)$ be a factor between subshifts, with (Y,S) minimal and aperiodic. Suppose that X is generated by a proper and everywhere growing directive sequence $\boldsymbol{\sigma}$ of alphabet rank K. Then, π is almost k-to-1 for some $k \leq K$.

Proof. Let $\boldsymbol{\sigma} = (\sigma_n\colon \mathcal{A}_{n+1} \to \mathcal{A}_n)_{n\geq 0}$ be a proper and everywhere growing directive sequence of alphabet rank at most K generating X. Due the possibility of contracting $\boldsymbol{\sigma}$, we can assume without loss of generality that $\#\mathcal{A}_n \leq K$ for every $n \geq 1$ and that σ_0 is r-proper, where r is the radius of π. Then, by Lemma 3.5, Y is generated by an everywhere growing directive sequence of the form $\boldsymbol{\tau} = (\tau, \sigma_1, \sigma_2, \dots)$, where $\tau\colon \mathcal{A}_1^+ \to \mathcal{B}^+$ is such that $\tau(x) = \pi(\sigma_0(x))$ for every $x \in X_{\boldsymbol{\tau}}^{(1)} = X_{\boldsymbol{\sigma}}^{(1)}$. We will use the notation $\tau_{[0,n)} = \tau\sigma_{[1,n)}$. Further, for $y \in Y$ and $n \geq 1$, we write $F_n(y)$ to denote the set of $\tau_{[0,n)}$-factorizations of y in $Y_{\boldsymbol{\tau}}^{(n)}$.

Before continuing, we prove the following claim.

Claim 3.3 There exist $\ell_n \in \mathbb{Z}$ and $G_n \subseteq \mathbb{Z} \times \mathcal{B}_{n+1}$ with at most K elements such that if $(k,x) \in F_n(y)$, then the symbol of (k,x) covering position ℓ_n of y is in G_n.

First, since Y is aperiodic, there exists $L \in \mathbb{N}$ such that

$$\text{all words } w \in \mathcal{L}(Y) \text{ of length } \geq L \text{ have least period greater than } |\tau_{[0,n)}|. \tag{3.28}$$

We assume, by contradiction, that the claim does not hold. In particular, for every $\ell \in [0, L)$ there exist $K+1$ $\tau_{[0,n)}$-factorizations (x, k) of y in $Y_{\boldsymbol{\tau}}^{(n)}$ such that their symbols covering position ℓ of y are all different. Now, since $\#\tau_{[0,n)}(\mathcal{A}_{n+1}) \leq K$, we can use the Pigeonhole Principle to find two of such factorizations, say (k, x) and (k', x'), such that if (c, a) and (c', a') are their symbols covering position ℓ of y then $a = a'$ and $c < c'$. Then,

$$y_{(c,c+|\tau_{[0,n)}(a)|]} = \tau_{[0,n)}(a) = y_{(c',c'+|\tau_{[0,n)}(a)|]}$$

and, thus, $y_{(c,c'+|\tau_{[0,n)}(a)|]}$ is $(c'-c)$-periodic. Being $\ell \in (c', c+|\tau_{[0,n)}(a)|)$, we deduce that the local period of $y_{[0,L)}$ at ℓ is at most $c'-c \leq |\tau_{[0,n)}|$. Since this true for every $\ell \in [0, L)$ and since, by Theorem 3.13, $\mathsf{per}(y_{[0,L)}) = \mathsf{per}(y_{[0,L)}, y_{[0,\ell)})$ for some $\ell \in [0, L)$, we conclude that $\mathsf{per}(y_{[0,L)}) \leq |\tau_{[0,n)}|$. This contradicts (3.28) and proves thereby the claim.

Now we prove the theorem. It is enough to show that the hypothesis of Lemma 3.24 hold. Let $y \in Y$ and $\tilde{F}_n(y) \subseteq F_n(y)$ be such that $\#\tilde{F}_n(y) = \#G_n$ and the set consisting of all the symbols of factorizations $(k, x) \in \tilde{F}_n(y)$ covering position ℓ_n of y is equal to G_n. Let $z \in \pi^{-1}(y)$ and (k, x) be a $\sigma_{[0,n)}$-factorization of z in $X_{\boldsymbol{\sigma}}^{(n)}$. Then, $S^k\tau_{[0,n)}(x) = S^k\pi(\sigma_{[0,n)}(x)) = \pi(z) = y$ and (k, x) is a $\tau_{[0,n)}$-factorization of y in $Y_{\boldsymbol{\tau}}^{(n)}$. Thus, we can find $(k', x') \in \tilde{F}_n(y)$ such that the symbols of (k, x) and (k', x') covering position ℓ_n of y are the same; let (m, a) be this common symbol. Since $\boldsymbol{\sigma}$ is proper, we have

$$z_{[m-\langle\sigma_{[0,n-1)}\rangle, m+|\sigma_{[0,n)}(a)|+\langle\sigma_{[0,n-1)}\rangle]} = z'_{[m-\langle\sigma_{[0,n-1)}\rangle, m+|\sigma_{[0,n)}(a)|+\langle\sigma_{[0,n-1)}\rangle]},$$

where $z' = S^{k'}\sigma_{[0,n)}(x') \in X$ is the point that (k', x') factorizes in $(X_{\boldsymbol{\sigma}}^{(n)}, \sigma_{[0,n)})$. Then, as $\ell_n \in (m, m+|\sigma_{[0,n)}(a)|]$,

$$z_{(\ell_n-\langle\sigma_{[0,n-1)}\rangle, \ell_n+\langle\sigma_{[0,n-1)}\rangle]} = z'_{(\ell_n-\langle\sigma_{[0,n-1)}\rangle, \ell_n+\langle\sigma_{[0,n-1)}\rangle]}.$$

Thus, $\operatorname{dist}(S^{\ell_n}z, S^{\ell_n}P_n(y)) \leq \exp(-\langle\sigma_{[0,n-1)}\rangle)$, where $P_n(y) \subseteq \pi^{-1}(y)$ is the set of all points $S^{k''}\sigma_{[0,n)}(x'') \in X$ such that $(k'', x'') \in \tilde{F}_n(y)$. Since this holds for every $n \geq 1$, we obtain that $d_{\mathrm{H}}(S^{\ell_n}\pi^{-1}(y), S^{\ell_n}P_n(y))$ converges to zero as n goes to infinity (where, we recall, d_{H} is the Hausdorff distance). By taking an appropriate convergent subnet $u \in E(2^X, S)$ of $(S^{\ell_n})_{n\in\mathbb{N}}$ we obtain $\#u \circ \pi^{-1}(y) \leq \sup_{n\in\mathbb{N}} \#P_n = \sup_{n\in\mathbb{N}} \#G_n \leq K$. This proves that the hypothesis of Lemma 3.24 holds. Therefore, π is almost k-to-1 for some $k \leq K$. □

3.6 Number of symbolic factors

In this section we prove Theorem 3.4. In order to do this, we split the proof into 3 subsections. First, in Lemma 3.27 of subsection 3.6.1, we deal with the case of Theorem 3.4 in which the factor maps are distal. Next, we show in Lemma 3.31 from Subsection 3.6.2 that in certain technical situation -which will arise when we consider non-distal factor maps- it is possible to reduce the problem to a similar one, but where the alphabet are smaller. Then, we prove Theorem 3.4 in subsection 3.6.3 by a repeated application of the previous lemmas.

3.6.1 Distal factor maps

We start with some definitions. If (X, S) is a system, then we always give X^k the diagonal action $S^{[k]} := S \times \cdots \times S$. If $\pi\colon (X, S) \to (Y, T)$ is a factor map and $k \geq 1$, then we define $R_\pi^k = \{(x^1, \dots, x^k) \in X^k : \pi(x^1) = \cdots = \pi(x^k)\}$. Observe that R_π^k is a closed $S^{[k]}$-invariant subset of X^k.

Next lemma follows from classical ideas from topological dynamics. See, for example, Theorem 6 in Chapter 10 of [Aus88].

Lemma 3.25 *Let $\pi\colon (X, S) \to (Y, T)$ be a distal almost k-to-1 factor between minimal systems, $z = (z^1, \dots, z^k) \in R_\pi^k$ and $Z = \overline{\mathrm{orb}}_{S^{[k]}}(z)$. Then, π is k-to-1 and Z is minimal*

We will also need the following lemma:

Lemma 3.26 ([Dur00], Lemma 21) *Let $\pi_i\colon (X, S) \to (Y_i, T_i)$, $i = 0, 1$, be two factors between aperiodic minimal systems. Suppose that π_0 is finite-to-one. If $x, y \in X$ are such that $\pi_0(x) = \pi_0(y)$ and $\pi_1(x) = T_1^p \pi_1(y)$, then $p = 0$.*

Lemma 3.27 *Let (X, S) be an infinite minimal subshift of topological rank K and J an index set of cardinality $\#J > K(144K^7)^K$. Suppose that for every $j \in J$ there exists a distal symbolic factor $\pi_j\colon (X, S) \to (Y_j, S)$. Then, there are $i \neq j \in J$ such that (Y_i, S) is conjugate to (Y_j, S).*

Proof. We start by introducing the necessary objects for the proof and doing some general observations about them. First, thanks to Theorem 3.3, we know that π_j is almost k_j-to-1 for some $k_j \leq K$, so, by the Pigeonhole Principle, there exist $J_1 \subseteq J$ and $k \leq K$ such that $\#J_1 \geq \#J/K > (144K^7)^K$ and $k_j = k$ for every $j \in J_1$. For $j \in J_1$, we fix $z^j = (z_1^j, \dots, z_k^j) \in R_{\pi_j}^k$ with $z_n^j \neq z_m^j$ for all $n \neq m$. Let $Z_j = \overline{\mathrm{orb}}_{S^{[k]}}(z^j)$ and $\rho\colon X^k \to X$ be the factor map that projects onto the first coordinate. By Lemma 3.25, π_j is k-to-1 and Z_j minimal. This imply that if $x = (x_1, \dots, x_k) \in Z_j$, then

$$\{x_1, \dots, x_k\} = \pi_j^{-1}(\pi_j(x_n)) \text{ for all } n \in \{1, \dots, k\}, \tag{3.29}$$

$$x_n \neq x_m \text{ for all } n, m \in \{1, \dots, k\}. \tag{3.30}$$

Indeed, since Z_j is minimal, $(S^{[k]})^{n_\ell} z \to x$ for some sequence $(n_\ell)_\ell$, so,

$$\inf_{n \neq m} \mathrm{dist}(x_n, x_m) \geq \inf_{n \neq m, l \in \mathbb{Z}} \mathrm{dist}(S^l z_n, S^l z_m) > 0,$$

where in the last step is due the fact that π_j is distal. This gives (3.30). For (3.29) we first note that $\{x_1, \dots, x_k\} \subseteq \pi_j^{-1}(\pi_j(x_n))$ as $x \in R_{\pi_j}$, and then that the equality must hold since $\#\pi_j^{-1}(\pi_j(x_n)) = k = \#\{x_1, \dots, x_k\}$ by (3.30).

The next step is to prove that asymptotic pairs in Z_j are well-behaved:

Claim 3.4 Let $j \in J_1$ and $(x^j = (x_1^j, \dots, x_k^j), \tilde{x}^j = (\tilde{x}_1^j, \dots, \tilde{x}_k^j))$ be a right asymptotic pair in Z_j, this is,

$$\lim_{n\to-\infty} \operatorname{dist}((S^{[k]})^n x^j, S^{[k]}\tilde{x}^j) = 0 \text{ and } x^j \neq \tilde{x}^j. \tag{3.31}$$

Then, $(x^j_n, \tilde{x}^j_n)$ is right asymptotic for every $n \in \{1, \dots, k\}$.

Proof. Suppose, with the aim to obtain a contradiction, that $(x^j_n, \tilde{x}^j_n)$ is not right asymptotic for some $n \in \{1, \dots, k\}$. Observe that (3.31) implies that

$$\text{for every } m \in \{1, \dots, k\}, \text{ either } (x^j_m, \tilde{x}^j_m) \text{ is right asymptotic or } x^j_n = \tilde{x}^j_n. \tag{3.32}$$

Therefore, $x^j_n = \tilde{x}^j_n$. Using this and that $x^j, \tilde{x}^j \in R^k_{\pi_j}$ we can compute:

$$\pi_j(x^j_m) = \pi_j(x^j_n) = \pi_j(\tilde{x}^j_n) = \pi_j(\tilde{x}^j_l) \text{ for all } m, l \in \{1, \dots, k\},$$

and thus, by (3.29),

$$\{x^j_1, \dots, x^j_k\} = \pi_j^{-1}(\pi_j(x^j_n)) = \pi_j^{-1}(\pi_j(\tilde{x}^j_n)) = \{\tilde{x}^j_1, \dots, \tilde{x}^j_k\}.$$

The last equation, (3.30) and that $x^j \neq \tilde{x}^j$ imply that there exist $m \neq l \in \{1, \dots, k\}$ such that $\tilde{x}^j_l = x^j_m$. This last equality and (3.32) tell us that x^j_m and x^j_l are either asymptotic or equal. But in both cases a contradiction occurs: in the first one with the distality of π and in the second one with equation (3.30). □

Let $j \in J_1$. Since Y_j is infinite, Z_j is a infinite subshift. It is a well-known fact from symbolic dynamics that this implies that there exists a right asymptotic pair $(x^j = (x^j_1, \dots, x^j_k), \tilde{x}^j = (\tilde{x}^j_1, \dots, \tilde{x}^j_k))$ in Z_j. We are now going to use Theorem 3.8 to prove the following:

Claim 3.5 There exists $i, j \in J_1$, $i \neq j$, such that $Z_i = Z_j$.

Proof. On one hand, by the previous claim, $(x^j_n, \tilde{x}^j_n) \in X^2$ is right asymptotic for every $n \in \{1, \dots, k\}$ and $j \in J_1$. Let $p^j_n \in \mathbb{Z}$ be such that $(S^{p^j_n} x^j_n, S^{p^j_n}\tilde{x}^j_n)$ is centered right asymptotic. On the other hand, Theorem 3.8 asserts that the set

$$\{x_{(0,\infty)} : (x, \tilde{x}) \text{ is centered right asymptotic in } X\}$$

has at most $144K^7$ elements. Since $\#J_1 > (144K^7)^K$, we conclude, by the Pigeonhole principle, that there exist $i, j \in J_1$, $i \neq j$, such that

$$S^{p^i_n} x^i_n \text{ and } S^{p^j_n} x^j_n \text{ agree on } (0, \infty) \text{ for every } n \in \{1, \dots, k\}. \tag{3.33}$$

We are going to show that $Z_i = Z_j$.

Using (3.33), we can find $u \in E(X, S)$ such that $uS^{p^i_n} x^i_n = uS^{p^j_n} x^j_n$ for every n. Then, by putting $y^i_n = ux^i_n$, $y^j_n = ux^j_n$ and $q_n = p^j_n - p^i_n$, we have

$$y^i := (y^i_1, \dots, y^i_k) \in Z_i,\ y^j := (y^j_1, \dots, y^j_k) \in Z_j \text{ and } y^i_n = S^{q_n} y^j_n.$$

Hence, $\pi(y^i_n) = S^{q_n}\pi(y^j_n)$ and Lemma 3.26 can be applied to deduce that $q := q_n$ has the same value for every n. We conclude that $y^i = S^q y^j \in S^q Z_j = Z_j$, that $Z_i \cap Z_j$ is not empty and, therefore, that $Z_i = Z_j$ as these are minimal systems. □

We can now finish the proof. Let $i \neq j \in J_1$ be the elements given by the previous claim, so that $Z := Z_i = Z_j$. Let $y \in Y_i$ and $x = (x_1, \ldots, x_k) \in \rho^{-1}\pi_i^{-1}(y) \cap Z$. Then, by (3.29), $\pi_i^{-1}(y) = \{x_1, \ldots, x_k\} = \pi_j^{-1}(\pi_j(x_1))$, and so $\pi_j\pi_i^{-1}(y)$ contains exactly one element, which is $\pi_j(x_1)$. We define $\psi\colon Y_i \to Y_j$ by $\psi(y) = \pi_j(x_1)$.

Observe that $\pi_i^{-1}\colon Y_i \to 2^X$ is continuous (as π_i is distal, hence open) and commutes with S. Being π_j a factor map, ψ is continuous and commutes with S. Therefore, $\psi\colon (Y_i, S) \to (Y_j, S)$ is a factor map. A similar construction gives a factor map $\phi\colon Y_j \to Y_i$ which is the inverse function of ψ. We conclude that ψ is a conjugacy and, thus, that Y_i and Y_j are conjugate. □

3.6.2 Non-distal factor maps

To deal with non-factor maps, we study asymptotic pairs belonging to fibers of this kind of factors. The starting point is the following lemma.

Lemma 3.28 *Let $\pi\colon (X, S) \to (Y, S)$ be a factor between minimal subshifts. Then, either π is distal or there exists a fiber $\pi^{-1}(y)$ containing a pair of right or left asymptotic points.*

Proof. Assume that π is not distal. Then, we can find a fiber $\pi^{-1}(y)$ and proximal points $x, x' \in \pi^{-1}(y)$, with $x \neq x'$. This implies that for every $k \in \mathbb{N}$ there exist a (maybe infinite) interval $I_k = (a_k, b_k) \subseteq \mathbb{Z}$, with $b_k - a_k \geq k$, for which x and x' coincide on I and I_k is maximal (with respect to the inclusion) with this property. Since $x \neq x'$, then $a_k > -\infty$ or $b_k < \infty$. Hence, there exists an infinite set $E \subseteq \mathbb{N}$ such that $a_k > -\infty$ for every $k \in E$ or $b_k < \infty$ for every $k \in E$. In the first case, we have that $(S^{b_k}(x, x'))_{k \in E}$ has a left asymptotic pair (z, z') as an accumulation point, while in the second case it is a right asymptotic pair (z, z') who is an accumulation point of $(S^{a_k}(x, x'))_{k \in E}$. In both cases we have that $(z, z') \in R_\pi^2$ since $(S^{b_k}(x, x'))_{k \in E}$ and $(S^{a_k}(x, x'))_{k \in E}$ are contained in R_π^2 and R_π^2 is closed. Therefore, the fiber $\pi^{-1}(\pi(z))$ contains a pair z, z' of asymptotic points. □

The next lemma allows us to pass from morphisms $\sigma\colon X \to Y$ to factors $\pi\colon X' \to Y$ in such a way that X' is defined on the same alphabet as X and has the "same" asymptotic pairs. We remark that its proof is simple, but tedious.

Lemma 3.29 *Let $X \subseteq \mathcal{A}^+$ be an aperiodic subshift, $\sigma\colon \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism and $Y = \bigcup_{k \in \mathbb{Z}} S^k\sigma(X)$. Define the morphism $i_\sigma\colon \mathcal{A}^+ \to \mathcal{A}^+$ by $i_\sigma(a) = a^{|\sigma(a)|}$, $a \in \mathcal{A}$, and $X' = \bigcup_{k \in \mathbb{Z}} S^k i_\sigma(X)$. Then, centered asymptotic pairs in X' are of the form $(i_\sigma(x), i_\sigma(\tilde{x}))$, where $(x, \tilde{x})$ is a centered asymptotic pair in X, and there exists a factor map $\pi\colon (X', S) \to (Y, S)$ such that $\pi(i_\sigma(x)) = \tau(x)$ for all $x \in X$.*

Proof. Our first objective is to prove that (X, i_σ) is recognizable. We start by observing that

$$\text{if } (k, x),\ (\tilde{k}, \tilde{x}) \text{ are centered } i_\sigma\text{-factorizations of } y \in X', \text{ then } x_0 = \tilde{x}_0. \tag{3.34}$$

Indeed, since the factorization are centered, we have $x_0 = i_\sigma(x_0)_k = y_0 = i_\sigma(\tilde{x}_0)_{\tilde{k}} = \tilde{x}_0$.

Let Λ be the set of tuples $(k, x, \tilde{k}, \tilde{x})$ such that $(k, x), (\tilde{k}, \tilde{x})$ are centered i_σ-factorizations of the same point. Moreover, for $\mathcal{R} \in \{=, >\}$, let $\Lambda_{\mathcal{R}}$ be the set of those $(k, x, \tilde{k}, \tilde{x}) \in \Lambda$ satisfying $k \mathcal{R} \tilde{k}$.

Claim 3.6 If $(k, x, \tilde{k}, \tilde{x}) \in \Lambda_=$, then $(0, Sx, 0, S\tilde{x}) \in \Lambda_=$, and if $(k, x, \tilde{k}, \tilde{x}) \in \Lambda_>$, then $(|i_\sigma(x_0)| - k + \tilde{k}, \tilde{x}, 0, Sx) \in \Lambda_>$.

Proof. If $(k, x, \tilde{k}, \tilde{x}) \in \Lambda_=$, then, since $x_0 = \tilde{x}_0$ by (3.34), we can write $i_\sigma(Sx) = S^k i_\sigma(x) = S^{\tilde{k}} i_\sigma(\tilde{x}) = i_\sigma(S\tilde{x})$. Thus, $(0, Sx, 0, S\tilde{x}) \in \Lambda_=$. Let now $(k, x, \tilde{k}, \tilde{x}) \in \Lambda_>$ and $y \coloneqq S^k i_\sigma(x) = S^{\tilde{k}} i_\sigma(\tilde{x})$. We note that

$$S^{|i_\sigma(x_0)|-k+\tilde{k}} i_\sigma(\tilde{x}) = S^{|i_\sigma(x_0)|-k} y = S^{|i_\sigma(x_0)|} i_\sigma(x) = i_\sigma(Sx),$$

so $(|i_\sigma(x_0)| - k + \tilde{k}, \tilde{x})$ and $(0, Sx)$ are i_σ-factorization of the same point. Now, since $x_0 = \tilde{x}_0$ (by (3.34)) and $(k, x), (\tilde{k}, \tilde{x})$ are centered, we have $k, \tilde{k} \in [0, |i_\sigma(x_0)|)$. This and and the fact that $k > \tilde{k}$ imply that $k - \tilde{k} \in (0, |i_\sigma(x_0)|)$. Therefore, $|i_\sigma(x_0)| - k + \tilde{k} \in (0, |i_\sigma(x_0)|)$ and, consequently, $(|i_\sigma(x_0)| - k + \tilde{k}, \tilde{x}, 0, Sx) \in \Lambda_>$. ☐

We prove now that (X, i_σ) is recognizable. Let $(k, x, \tilde{k}, \tilde{x}) \in \Lambda$. We have to show that $(k, x) = (\tilde{k}, \tilde{x})$. First, we consider the case in which $k = \tilde{k}$. In this situation, the previous claim implies that $(0, Sx, 0, S\tilde{x}) \in \Lambda_=$. We use again the claim, but with $(0, Sx, 0, S\tilde{x})$, to obtain that $(0, S^2x, 0, S^2\tilde{x}) \in \Lambda_=$. By continuing in this way, we get $(0, S^n x, 0, S^n \tilde{x}) \in \Lambda_=$ for any $n \geq 0$. Then, (3.34) implies that $x_n = \tilde{x}_n$ for all $n \geq 0$. A similar argument shows that $x_n = \tilde{x}_n$ for any $n \leq 0$, and so $(k, x) = (\tilde{k}, \tilde{x})$. We now do the case $k > \tilde{k}$. Another application of the claim gives us $(p_1, \tilde{x}, 0, Sx) \in \Lambda_>$ for some $p_1 \in \mathbb{Z}$. As before, we iterate this procedure to obtain that $(p_2, Sx, 0, S\tilde{x}) \in \Lambda_>$, $(p_3, S\tilde{x}, 0, S^2x) \in \Lambda_>$ and so on. From these relations and (3.34) we deduce that $x_0 = \tilde{x}_0$, $\tilde{x}_0 = (Sx)_0 = x_1$, $x_1 = (Sx)_0 = (S\tilde{x})_0 = \tilde{x}_1$, $\tilde{x}_1 = (S\tilde{x})_0 = (S^2x)_0 = x_2$, etc. We conclude that $x_n = \tilde{x}_n = x_0$ for any $n \geq 0$. Then, by compacity, the periodic point $\cdots x_0.x_0x_0 \cdots$ belongs to X, contrary to our aperiodicity hypothesis on X. Thus, the case $k > \tilde{k}$ does not occurs. This proves that (X, i_σ) is recognizable.

Using the property we just proved, we can define the factor map $\pi\colon X' \to Y$ as follows: if $x' \in X'$, then we set $\pi(x') = S^k \tau(x) \in Y$, where (k, x) is the unique centered i_σ-factorization of x' in X. To show that π is indeed a factor map, we first observe that since

$$|\tau(a)| = |i_\sigma(a)| \text{ for all } a \in \mathcal{A}, \tag{3.35}$$

π commutes with S. Moreover, thanks to (iii) in Remark 1.1, π is continuous. Finally, if $y \in Y$, then by the definition of Y there exist a centered (k, x) τ-factorization of y in X. Thus, by (3.35), (k, x) is a centered i_σ factorization of $x' \coloneqq S^k i_\sigma(x)$. Therefore, $\pi(x') = y$ and π is onto. Altogether, these arguments show that π is a factor map. That $\pi(i_\sigma(x)) = \tau(x)$ for every $x \in X$ follows directly from the definition of π.

It is left to prove the property about the asymptotic pairs. We only prove it for left asymptotic pairs since the other case is similar. We will use the following notation: if Z is a subshift, then $A(Z)$ denotes the set of centered left asymptotic pairs. To start, we observe that $(i_\sigma(x), i_\sigma(x')) \in A(X')$ for every $(x, \tilde{x}) \in A(X)$. Let now $(z, \tilde{z}) \in A(X')$, and (k, x) and $(\tilde{k}, \tilde{x})$ be the unique centered i_σ-factorizations of z and $\tilde{z}$ in X, respectively. We have to show that

$k = \tilde{k} = 0$ and that $(x, \tilde{x}) \in A(X)$. Due to (iii) in Remark 1.1, (X, i_σ) has a recognizability constant. This and the fact that $(z, \tilde{z})$ is centered left asymptotic imply that (k, x) and $(\tilde{k}, \tilde{x})$ have a common cut in $(-\infty, 0]$, this is, that there exist $p, q \leq 0$ such that

$$m \coloneqq -k - |i_\sigma(x_{[p,0)})| = -\tilde{k} - |i_\sigma(\tilde{x}_{[q,0)})| \in (-\infty, 0].$$

We take m as big as possible with this property. Then, $x_p \neq \tilde{x}_q$. Moreover, being $z_m = x_p$ and $\tilde{z}_m = \tilde{x}_p$ by the definition of i_σ, we have that $z_m \neq \tilde{z}_m$ and consequently, by also using that $(z, \tilde{z})$ is centered left asymptotic, that $m \geq 0$. We conclude that $m = 0$, this is, that $k + |i_\sigma(x_{[p,0)})| = \tilde{k} + |i_\sigma(\tilde{x}_{[q,0)})| = 0$. Hence, $k = \tilde{k} = p = q = 0$. Now, it is clear that $x_{(-\infty,p]} = \tilde{x}_{(-\infty,q]}$, so from the last equations we obtain that $(x, \tilde{x}) \in A(X)$. This completes the proof. □

We will also need the following lemma to slightly strengthen Proposition 3.8.

Lemma 3.30 *Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be an aperiodic subshift with L asymptotic tails. Then, (X, S) has at most $2L^2 \cdot \#\mathcal{A}^2$ centered asymptotic pairs.*

Proof. Let $\mathcal{P}_r$ be the set of centered right asymptotic pairs in X and $\mathcal{T}_r = \{x_{(0,\infty)} : (x, \tilde{x}) \in \Lambda\} \subseteq \mathcal{A}^{\mathbb{N}_{\geq 1}}$ be the set of right asymptotic tails, where $\mathbb{N}_{\geq 1} = \{1, 2, \dots\}$. We are going to prove that

$$\#\mathcal{P}_r \leq \#\mathcal{T}_r^2 \cdot \#\mathcal{A}^2. \tag{3.36}$$

Once this is done, we will have by symmetry the same relation for the centered left asymptotic pairs $\mathcal{P}_l$, and thus we are going to be able to conclude that the number of centered asymptotic pairs in X is at most $(\#\mathcal{T}_r^2 + \#\mathcal{T}_l^2) \cdot \#\mathcal{A}^2 \leq 2L^2 \cdot \#\mathcal{A}^2$, completing the proof.

Let $(x, \tilde{x}) \in \mathcal{P}_r$ and $\mathcal{R}_x = \{k \leq 0 : x_{(k,\infty)} \in \mathcal{T}_r\}$. We claim that $\#\mathcal{R}_x \leq \#\mathcal{T}_r$. Indeed, if this is not the case, then, by the Pigeonhole principle, we can find $k' < k$ and $w \in \mathcal{T}_r$ such that $w = x_{(k,\infty)} = x_{(k',\infty)}$. But this implies that w has period $k - k'$, and so X contains a point of period $k - k'$, contrary to the aperiodicity hypothesis. Thus, $\mathcal{R}_x$ is finite and, since $\mathcal{R}_x$ is nonempty as it contains $x_{(0,\infty)}$, $k_x \coloneqq \min \mathcal{R}_x$ is a well-defined non-positive integer.

Let now $\phi \colon \mathcal{P}_r \to \mathcal{T}_r^2 \times \mathcal{A}^2$ be the function defined by

$$\phi(x, \tilde{x}) = (x_{(k_x,\infty)}, \tilde{x}_{(k_{\tilde{x}},\infty)}, x_{k_x}, \tilde{x}_{k_{\tilde{x}}})$$

If ϕ is injective, then (3.36) follows. Let us then prove that ϕ is injective.

We argue by contradiction and assume that there exist $(x, \tilde{x}) \neq (y, \tilde{y})$ such that $\phi(x, \tilde{x}) = \phi(y, \tilde{y}) = (z, \tilde{z}, a, \tilde{a})$. Without loss of generality, we may assume that $x \neq y$. Then, $x_{(k_x,\infty)} = z = y_{(k_y,\infty)}$ and $x_{k_x} = a = y_{k_y}$. Being $x \neq y$, this implies that (x, y) is asymptotic. Furthermore, it implies that there exist $p < k$ and $q < \ell$ such that $(S^p x, S^q y)$ is centered right asymptotic. In particular, $x_{(p,\infty)} \in \mathcal{T}_r$ and $p < k_x$, contrary to the definition of k_x. We conclude that ϕ is injective and thereby complete the proof of the lemma. □

Lemma 3.31 *Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift of topological rank K, J be an index set and, for $j \in J$, let $\tau_j \colon \mathcal{A}^+ \to \mathcal{B}_j^+$ be a morphism. Suppose that for every $j \in J$*

(I) $Y_j = \bigcup_{k\in\mathbb{Z}} S^k\tau_j(X)$ *is aperiodic;*

(II) for every fixed $a \in \mathcal{A}$, $|\tau_j(a)|$ *is equal to a constant* ℓ_a *independent of* $j \in J$.

Then, one of the following situations occur:

(1) There exist $i, j \in J$, $i \neq j$, *such that* (Y_i, S) *is conjugate to* (Y_j, S).

(2) There exist $\phi\colon \mathcal{A}^+ \to \mathcal{A}_1^+$ *with* $\#\mathcal{A}_1 < \#\mathcal{A}$, *a set* $J_1 \subseteq J$ *having at least* $\#J/2\#\mathcal{A}^2(144K^7)^2 - K(144K^7)^K$ *elements, and morphisms* $\tau'_j\colon \mathcal{C}_1^+ \to \mathcal{B}_j$, $j \in J_1$, *such that* $\tau_j = \tau'_j\phi$. *In particular, the hypothesis of this lemma hold for* $X_1 := \bigcup_{k\in\mathbb{Z}} S^k\phi(X)$ *and* τ'_j, $j \in J_1$.

Proof. Let $\mathfrak{i}\colon \mathcal{A}^+ \to \mathcal{A}^+$ be the morphism defined by $\mathfrak{i}(a) = a^{\ell_a}$, $a \in \mathcal{A}$, and $X' = \bigcup_{k\in\mathbb{Z}} S^k\mathfrak{i}(X)$. We use Lemma 3.29 with X and τ_j to obtain a factor map $\pi_j\colon (X', S) \to (Y_j, S)$ such that

$$\pi(\mathfrak{i}(x)) = \tau_j(x) \text{ for every } x \in X. \tag{3.37}$$

If π_j is distal for $K(144K^7)^K + 1$ different values of $j \in J$, then by Lemma 3.27 we can find i, j such that (Y_i, S) is conjugate to (Y_j, S). Therefore, we can suppose that there exists $J' \subseteq J$ such that

$$\#J' \geq \#J - K(144K^7)^K \text{ and } \pi_j \text{ is not distal for every } j \in J'. \tag{3.38}$$

From this and Lemma 3.28 we obtain, for every $j \in J'$, a centered asymptotic pair $(x^{(j)}, \tilde{x}^{(j)})$ in X' such that $\pi_j(x^{(j)}) = \pi_j(\tilde{x}^{(j)})$. This and (3.37) imply that

$$\tau_j(x^{(j)}) = \pi_j(x^{(j)}) = \pi_j(\tilde{x}^{(j)}) = \tau_j(\tilde{x}^{(j)}). \tag{3.39}$$

Now, by Lemma 3.30, X has at most $2\#\mathcal{A}^2(144K^7)^2$ centered asymptotic pairs and thus, thanks to Lemma 3.29, the same bound holds for X'. Therefore, by the Pigeonhole principle, there exist $J_1 \subseteq J$ satisfying $\#J_1 \geq \#J'/2\#\mathcal{A}^2(144K^7)^2 \geq \#J/2\#\mathcal{A}^2(144K^7)^2 - K(144K^7)^K$ and a centered asymptotic pair $(x, \tilde{x})$ in X' such that $(x, \tilde{x}) = (x^{(j)}, \tilde{x}^{(j)})$ for every $j \in J_1$.

We assume that $(x, \tilde{x})$ is right asymptotic as the other case is similar. Then, equation (3.39) implies that if $\ell = \sum_{a\in\mathcal{A}} \ell_a$, then, for every $j \in J_1$,

$$\text{one of the words in } \{\tau_j(x_{[0,\ell)}), \tau_j(\tilde{x}_{[0,\ell)})\} \text{ is a prefix of the other.} \tag{3.40}$$

This, hypothesis (II) and the fact that, since $(x, \tilde{x})$ a centered asymptotic pair, $x_0 \neq \tilde{x}_0$ allow us to use Lemma 3.10 with $u := x_{[0,\ell)}$, $v := \tilde{x}_{[0,\ell)}$, $J := J_1$ and $w^j := \tau_j(x_{[0,\infty)})_{[0,\ell)}$ and obtain morphisms $\phi\colon \mathcal{A}^+ \to \mathcal{A}_1^+$ and $\tau'_j\colon \mathcal{A}_1^+ \to \mathcal{B}_j^+$, $j \in J_1$, such that $\#\mathcal{A}_1 < \#\mathcal{A}$, $\tau_j = \tau'_j\phi$ and

$$\text{for every } a \in \mathcal{A}_1,\ \ell'_a := |\tau'_j(c)| \text{ does not depend on the chosen } j \in J. \tag{3.41}$$

Finally, we observe that X_1 and τ'_j, $j \in J_1$, satisfy the hypothesis of the lemma: condition (I) holds since, by the relation $\tau_j = \tau'_j\phi$, the subshift $X_1 := \bigcup_{k\in\mathbb{Z}} S^k\phi(X)$ satisfies that $\bigcup_{k\in\mathbb{Z}} S^k\tau'_j(X_1) = Y_j$ is aperiodic; condition (II) is given by (3.41). □

3.6.3 Proof of main result

We now prove Theorem 3.4. We restate it for convenience.

Theorem 3.4 Let (X, S) be an minimal subshift of topological rank K. Then, (X, S) has at most $(3K)^{32K}$ aperiodic symbolic factors up to conjugacy.

Proof. We set $R = (3K)^{32K}$. We prove the theorem by contradiction: assume that there exist $X \subseteq \mathcal{A}^{\mathbb{Z}}$ of topological rank K and, for $j \in \{0, \dots, R\}$, factor maps $\pi_j \colon (X, S) \to (Y_j, S)$ such that (Y_i, S) is not conjugate to (Y_j, S) for every $i \neq j \in \{0, \dots, R\}$. We remark that X must be infinite as, otherwise, it would not have any aperiodic factor.

To start, we build $\mathcal{S}$-representations for the subshifts X and Y_j. Let $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ be the primitive and proper directive sequence of alphabet rank K generating X given by Theorem 0.1. Let $r \in \mathbb{N}$ be such that every π_j has a radius r and let $\mathcal{B}_j$ the alphabet of Y_j. By contracting $\boldsymbol{\sigma}$, we can assume that σ_0 is r-proper and $\#\mathcal{A}_n = K$ for all $n \geq 1$. Then, we can use Lemma 3.5 to find morphisms $\tau_j \colon \mathcal{A}_1^+ \to \mathcal{B}_j^+$ such that

$$\pi_j(\sigma_1(x)) = \tau_j(x) \text{ for all } x \in X_{\boldsymbol{\sigma}}^{(1)} \text{ and } |\tau_j(a)| = |\sigma_0(a)| \text{ for all } a \in \mathcal{A}_1. \tag{3.42}$$

Next, we inductively define subshifts $X_n \subseteq \mathcal{C}_n^{\mathbb{Z}}$ and morphisms $\{\tau_{n,j} \colon \mathcal{C}_n^+ \to \mathcal{B}_j : j \in J_n\}$ such that

(i) X_n has topological rank at most K;

(ii) $Y_j = \bigcup_{k\in\mathbb{Z}} \tau_{n,j}(X_n)$;

(iii) for every $c \in \mathcal{C}_n$, $\ell_{n,a} := |\tau_{n,j}(c)|$ does not depend on the chosen $j \in J_n$.

First, we set $X_0 = X_{\boldsymbol{\sigma}}^{(1)}$, $\mathcal{C}_0 = \mathcal{A}_1$, $J_0 = J$ and, for $j \in J_0$, $\tau_{0,j} = \tau_j$, and note that by the hypothesis and (3.42), they satisfy (i), (ii) and (iii). Let now $n \geq 0$ and suppose that $X_n \subseteq \mathcal{C}_n^{\mathbb{Z}}$ and $\tau_{n,j}$, $j \in J_n$, has been defined in a way such that (i), (ii) and (iii) hold. If $\#J_n/2\#\mathcal{A}^2(144K^7)^2 - K(144K^7)^K \leq 1$, then the procedure stops. Otherwise, we define step $n+1$ as follows. Thanks to (i), (ii), (iii) we can use Lemma 3.31, and since there are no two conjugate (Y_i, S), this lemma gives us a morphism $\phi \colon \mathcal{C}_n^+ \to \mathcal{C}_{n+1}^+$, a set $J_{n+1} \subseteq J_n$ and morphisms $\{\tau_{n+1,j} \colon \mathcal{C}_{n+1}^+ \to \mathcal{B}_j^+ : j \in J_{n+1}\}$ such that

$$\#\mathcal{C}_{n+1} < \#\mathcal{C}_n,\ \#J_{n+1} \geq \#J_n/2\#\mathcal{C}_n^2(144K^7)^2 - K(144K^7)^K \text{ and } \tau_{n,j} = \tau_{n+1,j}\phi_n.$$

Furthermore, $X_{n+1} := \bigcup_{k\in\mathbb{Z}} S^k\phi_n(X_n)$ and $\tau_{n+1,j}$ satisfy the hypothesis of that lemma, that is, conditions (ii) and (iii) above. Since $(\phi_n \dots \phi_0\sigma_1, \sigma_2, \sigma_3, \dots)$ is a primitive and proper sequence of alphabet rank K generating X_{n+1}, Theorem 3.1 implies that condition (i) is met as well.

Since $\#\mathcal{C}_0 > \#\mathcal{C}_1 > \dots$, there is a last $\mathcal{C}_N$ defined. Our next objective is to prove that $N \geq K$. Observe that $\#\mathcal{C}_n \leq K$, so

$$\#J_{n+1} \geq \#J_n/2K^2(144K^7)^2 - K(144K^7)^K \text{ for any } n \in \{0, \dots, N-1\}.$$

Using this recurrence and the inequalities $\#J_0 > (3K)^{32K}$ and $K \geq 2$, it is routine to verify that the following bound holds for every $n \in \{0, \ldots, K-1\}$ such that the nth step is defined:

$$\#J_n/2\#\mathcal{C}_n^2(144K^7)^2 - K(144K^7)^K > 1$$

Therefore, $N \geq K$. We conclude that $\#\mathcal{C}_N \leq \#\mathcal{C}_0 - K = 0$, which is a contradiction. □

Remark 3.5 In Theorem 1 of [Dur00], the author proved that linearly recurrent subshifts have finitely many aperiodic symbolic factors up to conjugacy. Since this kind of systems have finite topological rank (see Remark 3.4), Theorem 3.4 generalizes the theorem of [Dur00] to the much larger class of minimal finite topological rank subshifts.

3.7 Appendix

To prove Proposition 3.15, we start with some lemmas concerning how to construct recognizable pairs (Z, τ) for a fixed subshift $Y = \bigcup_{k \in \mathbb{Z}} S^k \tau(Z)$.

3.7.1 Codings of subshifts

If $Y \subseteq \mathcal{B}^{\mathbb{Z}}$ is a subshift, $U \subseteq Y$ and $y \in Y$, we denote by $\mathcal{R}_U(y)$ the set of *return times* of y to U, this is, $\mathcal{R}_U(y) = \{k \in \mathbb{Z} : S^k y \in U\}$. We recall that the set $C_\tau(k, z)$ in the lemma below corresponds to the cuts of (k, z) (see Definition 4.2 for further details).

Lemma 3.32 *Let $Y \subseteq \mathcal{B}^{\mathbb{Z}}$ be an aperiodic subshift, with $\mathcal{B} \subseteq \mathcal{L}(Y)$. Suppose that $U \subseteq Y$ is*

(I) d-syndetic: for every $y \in Y$ there exists $k \in [0, d-1]$ with $S^k y \in U$,

(II) of radius r: U is a union of sets of the form $[u.v]$, with $u, v \in \mathcal{A}^r$,

(III) ℓ-proper: $U \subseteq [u.v]$ for some $u, v \in \mathcal{A}^\ell$,

(IV) ρ-separated: $U, SU, \ldots, S^{\rho-1}U$ are disjoint.

Then, there exist a letter-onto morphism $\tau \colon \mathcal{C}^+ \to \mathcal{B}^+$ and a subshift $Z \subseteq \mathcal{C}^{\mathbb{Z}}$ such that

(1) $Y = \bigcup_{n \in \mathbb{Z}} S^n \tau(Z)$ and $\mathcal{C} \subseteq \mathcal{L}(Y)$,

(2) (Z, τ) is recognizable with constant $r + d$,

(3) $|\tau| \leq d$, $\langle \tau \rangle \geq \rho$ and τ is $\min(\rho, \ell)$-proper,

(4) $C_\tau(k, z) = \mathcal{R}_U(y)$ for all $y \in Y$ and τ-factorization (k, z) of y in Z.

Remark 3.6 If $U \subseteq Y$ satisfies (III), then U is $\rho := \min \mathsf{per}(\mathcal{L}_\ell(Y))$-separated. Indeed, if $U \cap S^k U \neq \emptyset$ for some $k > 0$, then $[v] \cap S^k[v] \neq \emptyset$, where $v \in \mathcal{A}^\ell$ is such that $U \subseteq [v]$. Hence, v is k periodic and $k \geq \rho$.

Proof. Let $y \in Y$. By (I), the sets $\mathcal{R}_U(y) \cap [0,\infty)$, $\mathcal{R}_U(y) \cap (-\infty, 0]$ are infinite. Thus, we can write $\mathcal{R}_U(y) = \{\dots k_{-1}(y) < k_0(y) < k_1(y) \dots\}$, with $\min\{i \in \mathbb{Z} : k_i(y) > 0\} = 1$. Let $\mathcal{W} = \{y_{[k_i(y),k_{i+1}(y))} : y \in Y,\ i \in \mathbb{Z}\} \subseteq \mathcal{B}^+$. By (I), $\mathcal{W}$ is finite, so we can write $\mathcal{C} := \{1, \dots, \#\mathcal{W}\}$ and choose a bijection $\phi \colon \mathcal{C} \to \mathcal{W}$. Then, ϕ extends to a morphism $\tau \colon \mathcal{C}^+ \to \mathcal{B}^+$. As $\mathcal{B} \subseteq \mathcal{L}(Y)$, ϕ is letter-onto. We define $\psi \colon Y \to \mathcal{C}^{\mathbb{Z}}$ by $\psi(y) = (\phi^{-1}(y_{[k_i(y),k_{i+1}(y))}))_{i\in\mathbb{Z}}$ and set $Z = \psi(Y)$. We are going to prove that τ and Z satisfy (1-4).

Claim 3.7

(i) If $y_{[-d-r,d+r]} = y'_{[-d-r,d+r]}$, then $\psi(y)_0 = \psi(y')_0$,

(ii) $\tau(\psi(y)) = S^{k_0(y)}y$,

(iii) $S^j\psi(y) = \psi(S^k y)$ for $j \in \mathbb{Z}$ and $k \in [k_j(y), k_{j+1}(y))$.

Proof. Let $y, y' \in Y$ such that $y_{[-d-r,d+r]} = y'_{[-d-r,d+r]}$. By (I), we have $k_{i+1}(y) - k_i(y) \le d$ for all $i \in \mathbb{Z}$ and, thus, $|k_0(y)|, |k_1(y)| \le d$. Since U has radius r and $y_{[-d-r,d+r]} = y'_{[-d-r,d+r]}$, we deduce that $k_0(y) = k_0(y')$ and $k_1(y) = k_0(y')$. Hence, $\psi(y)_0 = \phi^{-1}(y_{[k_0(y),k_1(y))}) = \phi^{-1}(y'_{[k_0(y'),k_1(y'))}) = \psi(y')_0$. To prove (ii) we compute:

$$\begin{aligned}\tau(\psi(y)) &= \tau(\cdots \phi^{-1}(y_{[k_{-1}(y),k_0(y))}).\phi^{-1}(y_{[k_0(y),k_1(y))}) \cdots) \\ &= \cdots y_{[k_{-1}(y),k_0(y))}.y_{[k_0(y),k_1(y))} \cdots = S^{k_0}y.\end{aligned}$$

Finally, for (iii) we write, for $k \in [k_j(y), k_{j+1}(y))$,

$$S^j\psi(y) = \dots \phi^{-1}(y_{[k_{j-1}(y),k_j(y))}).\phi^{-1}(y_{[k_j(y),k_{j+1}(y))}) \cdots = \psi(S^k y).$$

□

Now we prove the desired properties of τ and Z.

(1) From (i), we see that ψ is continuous and, therefore, Z is closed. By (iii), Z is also shift-invariant and, then, a subshift. By (ii), $Y = \bigcup_{n\in\mathbb{Z}} S^n\tau(Z)$. The condition $\mathcal{C} \subseteq \mathcal{L}(Y)$ follows from the definition of $\mathcal{W}$ and τ.

(2) We claim that the only centered τ-interpretation in Z of a point $y \in Y$ is $(-k_0(y), \psi(y))$. Indeed, this pair is a τ-interpretation in Z by (ii), and it is centered because $k_0(y) \le 0 < k_1(y)$ implies $-k_0(y) \in [0, k_1(y) - k_0(y)) = [0, |\psi(y)_0|)$. Let (n, z) be another centered τ-interpretation of y in Z. By the definition of Z, there exists $y' \in Y$ with $z = \psi(y')$. Then, by (ii),

$$S^{n+k_0(y')}y' = S^n\tau(\psi(y')) = S^n\tau(z) = y. \tag{3.43}$$

Now, on one hand, we have $|\tau(z_0)| = |\tau(\psi(y')_0)| = k_1(y') - k_0(y')$. On the other hand, that $(n, \psi(y'))$ is centered gives that $n \in [0, |\tau(z_0)|)$. Therefore, $n + k_0(y') \in (k_0(y'), k_1(y')]$. We conclude from this, (iii) and (3.43) that $\psi(y') = \psi(y)$. Hence, $y = S^n\tau\psi(y') = S^n\tau\psi(y) = S^{n+k_0(y)}y$, which implies that $n = -k_0(y)$ as Y is aperiodic. This proves that $(-k_0(y), \psi(y))$ is the only τ-interpretation of y in Z. From this and (i) we deduce property (2).

(3) Since U is d-syndetic, $|\tau(\psi(y)_i)| = |y_{[k_i(y),k_{i+1}(y))}| = k_{i+1}(y) - k_i(y) \leq d$ for $y \in Y$ and $i \in \mathbb{Z}$, so $|\tau| \leq d$. Similarly, we can obtain $\langle \tau \rangle \geq \rho$ using that U is ρ-separated. Let $u, v \in \mathcal{B}^\ell$ satisfying $U \subseteq [u.v]$. Since $k_i, k_{i+1} \in \mathcal{R}_U(y)$, we have that $u = y_{[k_i(y),k_i(y)+|u|)}$, $v = y_{[k_{i+1}(y)-|v|,k_{i+1}(y))}$ and, thus, that τ is $\min(\ell, \langle \tau \rangle)$-proper. In particular, it is $\min(\ell, \rho)$-proper.

(4) This follows directly from the definition of τ and $\mathcal{R}_U(y)$.

□

Lemma 3.33 *For $j \in \{0,1\}$, let $\sigma_j \colon \mathcal{A}_j^+ \to \mathcal{B}^+$ be a morphism and $X_j \subseteq \mathcal{A}_j^{\mathbb{Z}}$ be a subshift such that $Y := \bigcup_{n\in\mathbb{Z}} S^n \sigma_j(X_j)$ and $\mathcal{A}_j \subseteq \mathcal{L}(X_j)$ for every $j \in \{0,1\}$. Suppose that:*

(1) (X_0, σ_0) is recognizable with constant ℓ,

(2) σ_1 is ℓ-proper,

(3) $C_{\sigma_0}(k^0, x^0)(y) \supseteq C_{\sigma_1}(k^1, x^1)(y)$ for all $y \in Y$ and σ_j-factorizations (k^j, x^j) of y in X_j, $j = 0, 1$.

Then, there exist a letter-onto and proper morphism $\nu \colon \mathcal{A}_1^+ \to \mathcal{A}_0^+$ such that $\sigma_1 = \sigma_0 \nu$ and $X_0 = \bigcup_{k\in\mathbb{Z}} S^k \nu(X_1)$.

Proof. Since σ_1 is ℓ-proper, we can find $u, v \in \mathcal{B}^\ell$ such that $\sigma_1(a)$ starts with u and ends with v for every $a \in \mathcal{A}_1$. We define ν as follows. Let $a \in \mathcal{A}_1$ and $x \in X_1$ such that $a = x_0$. Since σ_1 is ℓ-proper, the word $v.\sigma_1(a)u$ occurs in $\sigma_1(x) \in Y$ at position 0. By (3), we can find $w \in \mathcal{L}(X_0)$ with $\sigma_1(x_0) = \sigma_0(w)$. We set $\nu(a) = w$. Since (X_0, σ_0) is recognizable with constant ℓ and u, v have length ℓ, w uniquely determined by $v.\sigma_1(a)u$ and, therefore, ν is well defined. Moreover, the recognizability implies that the first letter of $\nu(a)$ depends only on $v.u$, so ν is left-proper. A symmetric argument shows that ν is right-proper and, in conclusion, that it is proper. We also note that ν is letter-onto as $\mathcal{A}_0 \subseteq \mathcal{L}(X_0)$. It follows from the definition of ν that $\sigma_1 = \sigma_0\nu$. Now, let $x \in X_1$ and (k, x') be a centered σ_0-factorization of $\sigma_1(x)$ in X_0. By (3), $k = 0$ and $\sigma_1(x_j) = \sigma_0(x'_{[k_j,k_{j+1})})$ for some sequence $... < k_{-1} < k_0 < ...$ Hence, by the definition of ν, $\nu(x) = x' \in X_0$. This argument shows that $X'_0 := \bigcup_{n\in\mathbb{Z}} S^n \nu(X_1) \subseteq X_0$. Then, $\bigcup_{n\in\mathbb{Z}} S^n \sigma_0(X'_0) = \bigcup_{n\in\mathbb{Z}} S^n \sigma_0 \nu(X_1) = Y$, where in the last step we used that $\sigma_0\nu = \sigma_1$. Since the points in Y have exactly one σ_0-factorization, we must have $X'_0 = X_0$. This ends the proof. □

3.7.2 Factors of $\mathcal{S}$-adic sequences

Now we are ready to prove Proposition 3.15. For convenience, we repeat its statement.

Proposition 3.34 *Let $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_n \to \mathcal{A}_{n-1})_{n\geq 0}$ be a letter-onto, everywhere growing and proper directive sequence. Suppose that $X_{\boldsymbol{\sigma}}$ is aperiodic. Then, there exists a contraction $\boldsymbol{\sigma}' = (\sigma_{n_k})_{k\in\mathbb{N}}$ and a letter-onto and proper factor $\boldsymbol{\phi} \colon \boldsymbol{\sigma}' \to \boldsymbol{\tau}$, where $\boldsymbol{\tau}$ is letter-onto, everywhere growing, proper, recognizable and generates $X_{\boldsymbol{\sigma}}$.*

Proof. We start by observing that from Lemma 3.17 we can get that

$$\mathcal{A}_n \subseteq \mathcal{L}(X_{\boldsymbol{\sigma}}^{(n)}) \text{ for every } n \in \mathbb{N}. \tag{3.44}$$

Let $p_n = \min\{\mathsf{per}(\sigma_{[0,n)}(a)) : a \in \mathcal{A}_n\}$. Since $\boldsymbol{\sigma}$ is everywhere growing and $X_{\boldsymbol{\sigma}}$ is aperiodic, $\lim_{n\to\infty} p_n = \infty$. Hence, we can contract $\boldsymbol{\sigma}$ in a way such that, for every $n \geq 2$,

$$(\mathrm{I}_n)\ p_n \geq 3|\sigma_{[0,n-1)}|, \quad (\mathrm{II}_n)\ \sigma_{[0,n)} \text{ is } 3|\sigma_{[0,n-1)}|\text{-proper},$$

For $n \geq 2$, let $U_n = \bigcup_{u,v\in\mathcal{A}_n^2}[\sigma_{[0,n)}(u.v)]$. Observe that U_n is $|\sigma_{[0,n)}|$-syndetic, has radius $2|\sigma_{[0,n)}|$, is $3|\sigma_{[0,n-1)}|$-proper and, by Remark 3.6, is p_n-separated. Thus, by (I_n), U is $3|\sigma_{[0,n-1)}|$-separated. We can then use Lemma 3.32 with $(X_{\boldsymbol{\sigma}}^{(n)}, \sigma_{[0,n)})$ to obtain a letter-onto morphism $\nu_n\colon \mathcal{B}_n^+ \to \mathcal{A}_0^+$ and a subshift $Y_n \subseteq \mathcal{B}_n^{\mathbb{Z}}$ such that

(P_n^1) $X_{\boldsymbol{\sigma}} = \bigcup_{k\in\mathbb{Z}} S^k\nu_n(Y_n)$ and $\mathcal{B}_n \subseteq \mathcal{L}(Y_n)$,

(P_n^2) (Y_n, ν_n) is recognizable with constant $3|\sigma_{[0,n)}|$,

(P_n^3) $|\nu_n| \leq |\sigma_{[0,n)}|$, $\langle\nu_n\rangle \geq 3|\sigma_{[0,n-1)}|$, and ν_n is $3|\sigma_{[0,n-1)}|$-proper,

(P_n^4) $C_{\nu_n}(k,y) = \mathcal{R}_{U_n}(x)$ for all $x \in X_{\boldsymbol{\sigma}}$ and ν_n-factorization (k,y) of x in Y_n.

We write $C_{\nu_n}(x) \coloneqq C_{\nu_n}(k,y)$ if $x \in X_{\boldsymbol{\sigma}}$ and (k,y) is the unique ν_n-factorization of x in Y_n. Observe that $U_{n+1} \subseteq U_n$ for $n \geq 2$. Thus, $C_{\nu_{n+1}}(x) = \mathcal{R}_{U_{n+1}}(x) \subseteq \mathcal{R}_{U_n}(x) = C_{\nu_n}(x)$ for all $x \in X_{\boldsymbol{\sigma}}$. This, (P_n^2) and (P_{n+1}^3) allow us to use Lemma 3.33 with (Y_{n+1}, ν_{n+1}) and (Y_n, ν_n) and find a letter-onto and proper morphism $\tau_n\colon \mathcal{B}_{n+1}^+ \to \mathcal{B}_n^+$ such that $\nu_n\tau_n = \nu_{n+1}$ and $Y_n = \bigcup_{k\in\mathbb{Z}} S^k\tau_n(Y_{n+1})$.

Next, we claim that $C_{\nu_n}(x) \supseteq C_{\sigma_{[0,n+1)}}(k,z)$ for all $x \in X_{\boldsymbol{\sigma}}$ and $\sigma_{[0,n+1)}$-factorization (k,z) of x in $X_{\boldsymbol{\sigma}}^{(n+1)}$. Indeed, if $j \in \mathbb{Z}$, then $S^{c_{\sigma_{[0,n+1)},j}(k,z)}x \in [\sigma_{[0,n+1)}(z_{j-1}.z_jz_{j+1})] \subseteq [\sigma_{[0,n)}(a.bc)] \subseteq U_n$, where a is the last letter of $\sigma_n(z_{j-1})$ and bc the first two letters of $\sigma_n(z_jz_{j+1})$, so $c_{\sigma_{[0,n+1)},j}(k,z) \in \mathcal{R}_{U_n}(x) = C_{\nu_n}(x)$, as desired.

Thanks to the claim, (P_n^2), (I_{n+1}) and (3.44), we can use Lemma 3.33 with (Y_n, ν_n) and $(X_{\boldsymbol{\sigma}}^{(n+1)}, \sigma_{[0,n+1)})$ to obtain a proper morphism $\phi_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{B}_n^+$ such that $\sigma_{[0,n+1)} = \nu_n\phi_n$ and $Y_n = \bigcup_{k\in\mathbb{Z}} S^k\phi_n(X_{\boldsymbol{\sigma}}^{(n+1)})$.

Now we can define the morphisms $\tau_1 \coloneqq \nu_2$ and $\phi_1 \coloneqq \nu_2\phi_2$ and the sequences:

$$\boldsymbol{\phi} = (\phi_n)_{n\geq 1},\ \boldsymbol{\tau} = (\tau_n)_{n\geq 1} \text{ and } \boldsymbol{\sigma'} = (\sigma_{[0,2)}, \sigma_2, \sigma_3, \dots)_{n\geq 2}.$$

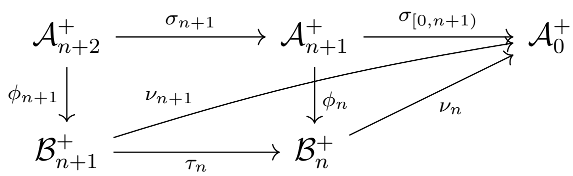
We are going to prove that $\boldsymbol{\phi}$, $\boldsymbol{\sigma'}$, and $\boldsymbol{\tau}$ are the objects that satisfy the conclusion of the Proposition.

These sequences are letter-onto as each ν_n and each ϕ_n is letter-onto. Next, we show that $\boldsymbol{\phi}$ is a factor. The relation $\phi_1 = \tau_1\phi_2$ follows from the definitions. To prove the other relations, we observe that from the commutative relations for τ_n and ϕ_n, we have that

$$\nu_n\phi_n\sigma_{n+1} = \sigma_{[0,n+1)}\sigma_{n+1} = \sigma_{[0,n+2)} = \nu_{n+1}\phi_{n+1} = \nu_n\tau_n\phi_{n+1}. \tag{3.45}$$

In particular, $\nu_n\phi_n\sigma_{n+1}(x) = \nu_n\tau_n\phi_{n+1}(x)$ for any $x \in X_{\boldsymbol{\sigma}}^{(n+2)}$. Since $\phi_n\sigma_{n+1}(x)$ and $\tau_n\phi_{n+1}(x)$ are both elements of Y_n and (Y_n, ν_n) is recognizable, we deduce that $\phi_n\sigma_{n+1}(x) = \tau_n\phi_{n+1}(x)$ for any $x \in X_{\boldsymbol{\sigma}}^{(n+2)}$. Thus, one of the words in $\{\phi_n\sigma_{n+1}(x_0), \tau_n\phi_{n+1}(x_0)\}$ is a prefix of the other. Since $\mathcal{A}_{n+2} \subseteq \mathcal{L}(X_{\boldsymbol{\sigma}}^{(n+2)})$, we deduce that, for any $a \in \mathcal{A}_{n+2}$, one of the words in $\{\tau_n\phi_{n+1}(a), \nu_n\phi_n\sigma_{n+1}(a)\}$ is a prefix of the other. But, by (3.45), the words $\nu_n\tau_n\phi_{n+1}(a)$ and $\nu_n\phi_n\sigma_{n+1}(a)$ have the same length, so $\phi_n\sigma_{n+1}(a)$ must be equal to $\tau_n\phi_{n+1}(a)$ for every $n \geq 2$. This proves that $\phi_n\sigma_{n+1} = \tau_n\phi_{n+1}$ for every $n \geq 2$ and that $\boldsymbol{\phi}\colon \boldsymbol{\sigma'} \to \boldsymbol{\tau}$ is a factor.

The following commutative diagram, valid for all $n \geq 2$, summarizes the construction so far:



As shown in the diagram, we have that $\nu_n\tau_n = \nu_{n+1}$ for $n \geq 2$. Thus, $\tau_1\tau_2\cdots\tau_n = \nu_{n+1}$, and hence $\langle\tau_1\tau_2\cdots\tau_n\rangle \geq \langle\nu_{n+1}\rangle \geq p_n \to_{n\to\infty} \infty$. Therefore, $\boldsymbol{\tau}$ is everywhere growing. Also, by using Lemma 1.1 with $(Y_n, \nu_n) = (Y_n, \tau_1\tau_2\cdots\tau_{n-1})$, we deduce that (Y_n, τ_{n-1}) is recognizable for every $n \geq 2$, which implies that $\boldsymbol{\tau}$ is recognizable. Finally, as each τ_n is proper, $\boldsymbol{\tau}$ is proper. $\square$

Chapter 4

A solution to the $\mathcal{S}$-adic conjecture

A interesting intuition in symbolic dynamics of zero entropy is that a subshift of low enough complexity should be very restricted, and thus hide a strong structure. This idea dates back to the 70s, and matured in the 80s and 90s until it was finally established as the following more concrete question.

Question 4.1 Consider the class (L) of linear-growth complexity subshifts, defined by requiring that $p_X(n) \leq dn$ for some $d > 0$. Is there an $\mathcal{S}$-adic characterization of the class (L)?

Question 4.1 is known as the $\mathcal{S}$-adic conjecture.

In this chapter, we completely solve the $\mathcal{S}$-adic conjecture for minimal subshifts by proving the following theorem.

Theorem 4.1 *A minimal subshift X has linear-growth complexity,* i.e., *X satisfies*

$$\limsup_{n \to +\infty} p_X(n)/n < +\infty,$$

if and only if there exist $d > 0$ and an $\mathcal{S}$-adic sequence $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n \geq 0}$ generating X such that, for every $n \geq 0$, the following holds:

$(\mathcal{P}_1)$ $\#(\mathsf{root}\, \sigma_{[0,n)}(\mathcal{A}_n)) \leq d$ †.

$(\mathcal{P}_2)$ $|\sigma_{[0,n)}(a)| \leq d \cdot |\sigma_{[0,n)}(b)|$ *for every* $a, b \in \mathcal{A}_n$.

$(\mathcal{P}_3)$ $|\sigma_{n-1}(a)| \leq d$ *for every* $a \in \mathcal{A}_n$.

We are able to give a similar structure for nonsuperlinear complexity subshifts (NSL).

†For a word u, $\mathsf{root}\, u$ denotes the shortest prefix v of u such that $u = v^k$ for some k; for a set of words $\mathcal{W}$, $\mathsf{root}\, \mathcal{W} = \{\mathsf{root}\, w : w \in \mathcal{W}\}$.

Theorem 4.2 *A minimal subshift X has nonsuperlinear-growth complexity,* i.e., *X satisfies*

$$\liminf_{n\to+\infty} p_X(n)/n < +\infty,$$

if and only if there exist $d > 0$ and an $\mathcal{S}$-adic sequence $\boldsymbol{\sigma} = (\sigma_n : \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ generating X such that, for every $n \geq 0$, the following holds:

($\mathcal{P}_1$) $\#(\mathsf{root}\,\sigma_{[0,n)}(\mathcal{A}_n)) \leq d$.

($\mathcal{P}_2$) $|\sigma_{[0,n)}(a)| \leq d \cdot |\sigma_{[0,n)}(b)|$ *for every $a, b \in \mathcal{A}_n$.*

We show in Section 4.10 how these theorems provide a unified framework and simplified proofs of several known results on (L) and (NSL), including Cassaigne's Theorem [Cas95]. We also prove, in Theorem 4.77, that Condition ($\mathcal{P}_1$) in Theorems 4.1 and 4.2 cannot be improved to a uniform bound on the cardinalities of the alphabets.

This chapter was published as a standalone article in [Esp22b].

Strategy of the proof

The hard part of the proofs of Theorems 4.1 and 4.2 is constructing an $\mathcal{S}$-adic sequence satisfying properties ($\mathcal{P}_i$) from the complexity hypothesis. We detail here the strategy for doing so in the case of Theorem 4.1; the proof of Theorem 4.2 is similar.

It is convenient to introduce the following terminology: a *coding* of a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a pair (Z, σ), where $Z \subseteq \mathcal{C}^{\mathbb{Z}}$ is a subshift and $\sigma : \mathcal{C} \to \mathcal{A}^+$ a substitution such that $X = \bigcup_{k\in\mathbb{Z}} S^k\sigma(Z)$. It is a standard fact that if $\boldsymbol{\tau}$ is an $\mathcal{S}$-adic sequence then there are subshifts $X_{\boldsymbol{\tau}}^{(n)}$, with $X_{\boldsymbol{\tau}}^{(0)} = X$, such that $(X_{\boldsymbol{\tau}}^{(m)}, \tau_{[n,m)})$ is a coding of $X_{\boldsymbol{\tau}}^{(n)}$ for any $n < m$, where $\tau_{[n,m)} = \tau_n\tau_{n+1}\dots\tau_{m-1}$.

Let X be a linear-growth complexity subshift and $d = \sup_{n\geq 1} p_X(n)/n$. The typical method for building an $\mathcal{S}$-adic sequence for a subshift X is an inductive process: First, $X_0 \coloneqq X$; then, a coding (X_{i+1}, σ_{i+1}) of X_i is defined. In this way, $\boldsymbol{\sigma} \coloneqq (\sigma_n)_{n\geq 0}$ is an $\mathcal{S}$-adic sequence that, under mild conditions, generates X. We, instead, take a more direct approach, similar to that in [DDMP21, Theorem 4.3] and [Esp22a, Corollary 1.4], but with additional technical details. We consider an increasing sequence of positive integers $(\ell_n)_{n\geq 0}$ with adequate growth and build codings $(X_n \subseteq \mathcal{C}_n^{\mathbb{Z}}, \sigma_n : \mathcal{C}_n \to \mathcal{A}^+)$ of $X \subseteq \mathcal{A}^{\mathbb{Z}}$ satisfying ($\mathcal{P}_1$), $\frac{1}{d'}\ell_n \leq |\sigma(a)| \leq d'\ell_n$ for all letters a and with d' depending on d, and such that certain technical properties hold. These technical properties allow us to define *connecting* substitutions $\tau_n : \mathcal{C}_{n+1} \to \mathcal{C}_n^+$ in such a way that $\sigma_n\tau_n(x)$ is, up to a shift, equal to $\sigma_{n+1}(x)$, for all $x \in X_{n+1}$. Then, we can prove that $\boldsymbol{\tau} = (\sigma_0, \tau_0, \tau_1, \tau_2, \dots)$ generates X and satisfies all the properties in Theorem 4.1.

The main idea for constructing the codings (X_n, σ_n) is that, thanks to a modification of the technique from [Fer96, Proposition 5], we can build a coding (X'_n, σ'_n) of X (which is described in Proposition 4.28) in such a way that the words $\sigma'(a)$ are either strongly aperiodic or strongly periodic. The aperiodic words greatly contribute to the complexity, so we can efficiently control them using d. For controlling the periodic words, we rely on tricks from combinatorics on words. These two ideas are used to obtain, in Sections 4.6 to 4.7, two variations of (X'_n, σ'_n), with increasingly better properties, and where the last one is (X_n, σ_n).

Organization

The chapter has three parts. The first one consists of Sections 4.1, 4.2 and 4.3 and provide the necessary background and some lemmas for handling periodic words. Then, in Sections 4.4 to 4.8, we carry out the proofs of Theorems 4.1 and 4.2. Finally, we prove Theorem 4.77 and present applications of our main results in Sections 4.9 and 4.10.

4.1 Preliminaries

The word $w \in \mathcal{A}^+$ is $|u|$-*periodic*, with $u \in \mathcal{A}^+$, if w occurs in u^n for some $n \geq 1$. We denote by $\mathsf{per}(w)$ the least p for which w is p-periodic.

In order to describe more precisely the periodicity properties of w, we use the notion of *root*, which will play a key role throughout the chapter.

Definition 4.1 The *minimal root*, or just *root* for short, of $w \in \mathcal{A}^*$ is the shortest prefix u of w for which $w = u^k$ for some $k \geq 1$, and it is denoted by $\mathsf{root}\, w$.

We remark that $\mathsf{per}(w)$ is an integer but that $\mathsf{root}\, w$ is a word, and that $\mathsf{per}(w)$ is in general different from $|\mathsf{root}\, w|$.

Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift and $v \in \mathcal{A}^+$. We will use the notation $v^\infty = vvv\cdots \in \mathcal{A}^{\mathbb{N}}$ and $b^{\mathbb{Z}} = \ldots vv.vv\cdots \in \mathcal{A}^{\mathbb{Z}}$. We denote by $\mathrm{Pow}_X(v)$ the set of words v^k, where $k \geq 1$, for which there exist $u, w \in \mathcal{A}^+ \setminus \{v\}$ of length $|v|$ such that $uv^kw \in \mathcal{L}(X)$. The *power complexity* of X is the number $\mathsf{pow\text{-}com}(X) = \sup_{v\in\mathcal{A}^+} \#\mathrm{Pow}_X(v)$. Remark that $\mathsf{pow\text{-}com}(X)$ may be infinite. Examples with finite power complexity include linearly recurrent subshifts and subshifts in which the extension graph of every long enough word is acyclic (in particular, Sturmian subshifts and codings of minimal interval exchange transformations).

4.1.1 Morphisms and codings

We say that τ is *positive* if for every $a \in \mathcal{A}$, all letters $b \in \mathcal{B}$ occur in $\tau(a)$, that τ is *proper* if there exist letters $a, b \in \mathcal{B}$ such that $\tau(c)$ starts with a and ends with b for any $c \in \mathcal{A}$, and that τ is *injective on letters* if for all $a, b \in \mathcal{B}$, $\tau(a) = \tau(b)$ implies $a = b$.

Factorizations and recognizability

We now introduce *factorizations*, the *recognizability property* and the associated notation.

Definition 4.2 Let $Y \subseteq \mathcal{B}^{\mathbb{Z}}$ be a subshift and $\tau\colon \mathcal{B}^+ \to \mathcal{A}^+$ be a morphism. We say that $(k, y) \in \mathbb{Z} \times Y$ is a τ-*factorization of* $x \in \mathcal{A}^{\mathbb{Z}}$ *in* Y if $x = S^k\tau(y)$ and $0 \leq k < |\tau(y_0)|$.

The pair (Y, τ) is *recognizable* if every point $x \in \mathcal{A}^{\mathbb{Z}}$ has at most one τ-factorization in Y. We say that (Y, τ) is d-*recognizable*, with $d \geq 1$, if whenever (k, y) and $(\tilde{k}, \tilde{y})$ are τ-factorizations of $x, \tilde{x} \in \mathcal{A}^{\mathbb{Z}}$ in Y, respectively, and $x_{[-d,d)} = \tilde{x}_{[-d,d)}$, we have that $k = \tilde{k}$ and $y_0 = \tilde{y}_0$.

The *cut function* $c\colon \mathbb{Z} \to \mathbb{Z}$ of the τ-factorization (k, y) of x in Y is defined by

$$c_j = \begin{cases} -k + |\tau(y_{[0,j)})| & \text{if } j \geq 0, \\ -k - |\tau(y_{[j,0)})| & \text{if } j < 0. \end{cases} \tag{4.1}$$

When (Y, τ) is recognizable, we write $(c, y) = \mathbf{F}_{(Y,\tau)}(x)$ and $(c_0, y_0) = \mathbf{F}^0_{(Y,\tau)}(x)$.

Remark 4.1 In the context of the previous definition:

(1) If (Y, τ) is recognizable, then a compacity argument shows that it is d-recognizable for some $d \geq 1$.

(2) Suppose that (Y, τ) is recognizable. Let $x \in X$, $(c, y) = \mathbf{F}_{(Y,\tau)}(x)$ and $i \in \mathbb{Z}$. Then, there exists a unique $j \in \mathbb{Z}$ such that $i \in [c_j, c_{j+1})$. Note that the last condition is equivalent to $\mathbf{F}^0_{(Y,\tau)}(S^i x) = (c_j - i, y_j)$.

Lemma 4.3 *Let $\sigma\colon \mathcal{C} \to \mathcal{B}^+$ and $\tau\colon \mathcal{B} \to \mathcal{C}^+$ be morphisms and $Z \subseteq \mathcal{C}^\mathbb{Z}$ be a subshift. We set $Y = \bigcup_{k\in\mathbb{Z}} S^k\sigma(Z)$ and $X = \bigcup_{k\in\mathbb{Z}} S^k\tau(Y)$. Suppose that $(Z, \tau\sigma)$ is recognizable. Let $x \in X$, (k, Y) be a τ-factorization of x in Y and (ℓ, z) be a $\tau\sigma$-factorization of x in Z. Then, there exists $m \in [0, |\sigma(z_0)|)$ such that $y = S^m\sigma(z)$ and $k = |\sigma(z_{[-m,0)})| + \ell$.*

Proof. Being (ℓ, z) a $\tau\sigma$-factorization of x, we have that $\ell \in [0, |\tau(\sigma(z_0))|)$. Hence, there exists $m \in [0, |\sigma(z_0)|)$ such that

$$|\sigma(z)_{[0,m)}| \leq \ell < |\sigma(z)_{[0,m]}|. \tag{4.2}$$

Therefore, as (ℓ, z) is a $\tau\sigma$-factorization of x, we can write

$$S^{\ell - |\tau(\sigma(z)_{[0,m)})|}\tau(S^m\sigma(z)) = S^\ell\tau\sigma(z) = x.$$

This and (4.2) ensure that $(\ell - |\tau(\sigma(z)_{[0,m)})|, S^m\sigma(z))$ is a τ-factorization of x in Y. We conclude, using that (Y, τ) is recognizable by Lemma 1.1, that $\ell - |\tau(\sigma(z)_{[0,m)})| = k$ and $S^m\sigma(z) = y$. □

Codings of a subshift

We fix a subshift $X \subseteq \mathcal{A}^\mathbb{Z}$. A *coding* of X is a pair (Y, τ), where $Y \subseteq \mathcal{B}^\mathbb{Z}$ is a subshift and $\tau\colon \mathcal{B}^+ \to \mathcal{A}^+$ a morphism satisfying $X = \bigcup_{k\in\mathbb{Z}} S^k\tau(Y)$. We present in Proposition 4.4 a general method for building recognizable codings of a subshift X. This idea occurs commonly in the literature under several different names and with different degrees of generality. Our Proposition 4.4 is inspired by the coding based on return words from [Dur98].

Let $U \subseteq X$ be a clopen (*i.e.*, open and closed) set. We say that U is

(1) *ℓ-syndetic* if for all $x \in X$ there exists $k \in [0, \ell)$ such that $S^k x \in U$;

(2) of *radius* r if U is an union of sets of the form $\{x \in X : x_{[-|u|,|v|)} = uv\}$, where $u, v \in \mathcal{A}^r$.

Remark that, in a minimal X, any nonempty clopen set U is ℓ-syndetic and of radius r for some ℓ and r.

Proposition 4.4 *Let $U \subseteq X$ be a nonempty clopen set. There exists a recognizable coding $(Y \subseteq \mathcal{B}^{\mathbb{Z}}, \sigma \colon \mathcal{B} \to \mathcal{A}^+)$ of X, with σ injective on letters, such that if $x \in X$, $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$ and $i \in \mathbb{Z}$, then $S^i x \in U$ if and only if $i = c_j$ for some $j \in \mathbb{Z}$.*

If U is ℓ-syndetic and of radius r, then (Y, σ) additionally satisfies that:

(1) $|\sigma(a)| \leq \ell$ for all $a \in \mathcal{B}$.

(2) (Y, σ) is $(\ell + r)$-recognizable.

4.1.2 The complexity function

The *complexity function* $p_X \colon \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ of a subshift X is defined by $p_X(n) = \#\mathcal{L}(X) \cap \mathcal{A}^n$. Equivalently, $p_X(n)$ counts the number of words of length n that occur in at least one $x \in X$.

Definition 4.3 We say that X has

(1) *linear-growth complexity* if there exists $d > 0$ such that $p_X(n) \leq dn$ for all $n \geq 1$;

(2) *nonsuperlinear-growth complexity* if there exists $d > 0$ such that $p_X(n) \leq dn$ for infinitely many $n \geq 1$.

Remark 4.2 When X is infinite, then a classic theorem of Morse and Hedlund [MH38] ensures that $p_X(n) \geq n + 1$ for all $n \geq 0$. Thus, an infinite subshift of linear-growth complexity satisfies $n \leq p_X(n) \leq dn$, and so p_X grows linearly.

The following theorem is classic.

Theorem 4.5 ([Cas95]) *Let X be a transitive linear-growth complexity subshift. Then, $p_X(n+1) - p_X(n)$ is uniformly bounded.*

For the proof of Theorems 4.75 and 4.76 in Section 4.8, we will need only the following two weaker versions of Cassaigne's Theorem.

Lemma 4.6 *Let X be a subshift and $d \geq 1$ be such that $p_X(n) \leq dn$ for all $n \geq 1$. Then, for every $n \geq 1$ there exists $m \in [n, 2n)$ such that $p_X(m+1) - p_X(m) \leq 2d$.*

Proof. Let $n \geq 1$. We observe that the average of $p_X(m+1) - p_X(m)$ for $m \in [n, 2n)$ can be bounded as follows by using that $p_X(2n) \leq 2dn$:

$$\frac{1}{n}\sum_{m=n}^{2n-1} p_X(m+1) - p_X(m) = \frac{1}{n}(p_X(2n) - p(n)) \leq 2d.$$

Thus, there exists $m \in [n, 2n)$ satisfying $p_X(m+1) - p_X(m) \leq 2d$. □

Lemma 4.7 *Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift and $d \geq 1$ be such that $p_X(n) \leq dn$ for infinitely many $n \geq 1$. Then, there are infinitely many m such that $p_X(m) \leq 3dm$ and $p_X(m+1) - p_X(m) \leq 2d$.*

Proof. Let $n \geq 1$ be arbitrary. The hypothesis permits to find $k \geq 2n$ such that $p_X(k) \leq dk$. We now observe that

$$\frac{1}{\lceil k/2 \rceil}\sum_{m=\lfloor k/2 \rfloor}^{k} p_X(m+1) - p_X(m) \leq \frac{1}{\lceil k/2 \rceil} p_X(k) \leq 2d.$$

Therefore, there exists m such that $\lfloor k/2 \rfloor \leq m \leq k$ and $p_X(m+1) - p_X(m) \leq 2d$. The first condition ensures that $m \geq n$ and $p_X(m) \leq p_X(k) \leq dk \leq 3dm$. □

4.2 Some combinatorial lemmas

In order to prove our main results, we will need to extensively deal with strongly periodic words. The objective of this section is to give the necessary tools for doing so.

A basic result on periodicity of words is the *Fine and Wilf Theorem*, which we state below.

Theorem 4.8 *Let $u, v, w \in \mathcal{A}^+$ and suppose that w is a prefix of u^∞ and v^∞. If $|w| \geq |u| + |v| - 1$, then there exists $t \in \mathcal{A}^+$ such that u and v are powers of t.*

A proof of Theorem 4.8 can be found in [RS97, Chapter 6, Theorem 6.1].

Lemma 4.9 *Let u be a word such that $|u| \geq 2|\mathsf{root}\, u|$. Then, $|\mathsf{root}\, u| = \mathsf{per}(u)$.*

Proof. Note that u is a prefix of $(\mathsf{root}\, u)^{\mathbb{Z}}$ and thus that $\mathsf{per}(u) \leq |\mathsf{root}\, u|$. It is left to prove the other inequality.

Let t be the prefix of u of length $\mathsf{per}(u)$. Then u is a prefix of both t^∞ and $(\mathsf{root}\, u)^\infty$. We deduce, as $|u| \geq 2\,\mathsf{per}(u) \geq |t| + |\mathsf{root}\, u|$, that the hypothesis of Lemma 4.8 is complied. Hence, t and $\mathsf{root}\, u$ are powers of a common word r. In particular, u is a power of r, so we must have that $\mathsf{root}\, u = r$. Therefore, $|\mathsf{root}\, u| = |r| \leq |t| = \mathsf{per}(u)$. □

Remark 4.3 The previous lemma ensures that if u is a word and $k \geq 1$, then $\mathsf{root}\, u^k = \mathsf{root}\, u$. In particular, if u and v are powers of a common word, then they have the same root. These basic relations will be freely used throughout the chapter.

The next proposition will allow us to synchronize occurrences of strongly periodic words.

Proposition 4.10 *Let $t, s \in \mathcal{A}^+$.*

(1) Suppose that $\ell \geq |s|+|t|-1$ and $i, j \in \mathbb{Z}$ are such that $t^{\mathbb{Z}}_{[i,i+\ell)} = s^{\mathbb{Z}}_{[j,j+\ell)}$ Then, $S^i t = S^j s$.

(2) An integer i satisfies $S^i t^{\mathbb{Z}} = t^{\mathbb{Z}}$ if and only if $i = 0 \pmod{|\mathsf{root}\, t|}$.

Proof. We first prove Item (1). Let $t_0 = t^{\mathbb{Z}}_{[i,i+|t|)}$, $s_0 = s^{\mathbb{Z}}_{[j,j+|s|)}$ and $w = t^{\mathbb{Z}}_{[i,i+\ell)} = s^{\mathbb{Z}}_{[j,j+\ell)}$. Then, w is a prefix of both t_0^∞ and s_0^∞. Since $|w| = \ell \geq |s|+|t|-1 = |s_0|+|t_0|-1$, we can use Theorem 4.8 to deduce that s_0 and t_0 are powers of a common word r. We then have $S^i s^{\mathbb{Z}} = s_0^{\mathbb{Z}} = r^{\mathbb{Z}} = t_0^{\mathbb{Z}} = S^j t^{\mathbb{Z}}$.

We now prove Item (2). It is clear that if $i = 0 \pmod{|\mathsf{root}\, t|}$ then $S^i t^{\mathbb{Z}} = t^{\mathbb{Z}}$. Let us suppose that $S^i t^{\mathbb{Z}} = t^{\mathbb{Z}}$. We argue by contradiction and assume that $i \neq 0 \pmod{|\mathsf{root}\, t|}$. We write $\mathsf{root}\, t = ss'$, where $|s| = i \pmod{|\mathsf{root}\, t|}$. Then, $(s's)^{\mathbb{Z}} = S^i t^{\mathbb{Z}} = t^{\mathbb{Z}} = (ss')^{\mathbb{Z}}$, so Theorem 4.8 implies that $s's$ and ss' are powers of a common word r. In particular, $\mathsf{root}(s's) = \mathsf{root}(ss') = \mathsf{root}\, r$. This implies that

$$|\mathsf{root}(s's)| = |\mathsf{root}(ss')| = |\mathsf{root}\,\mathsf{root}\, t| = |\mathsf{root}\, t| = |ss'| = |s's|,$$

so $\mathsf{root}(s's) = s's$. Hence, $s's = ss'$. Now, since $i \neq 0 \pmod{|\mathsf{root}\, t|}$, s and s' are not the empty word. This and the condition $s's = ss'$ imply that $s's$ is a prefix of s^∞ and of s'^∞. We can then use Theorem 4.8 to deduce that s and s' are powers of a common word r'. Therefore, as $\mathsf{root} = ss'$, $ss' = \mathsf{root}\, t = \mathsf{root}\, s = \mathsf{root}\, s'$. This is possible only if $s = 1$ or $s' = 1$. Consequently, $|s| \in \{0, |\mathsf{root}\, t|\}$ and $i = |s| = 0 \pmod{|\mathsf{root}\, t|}$, contradicting our assumptions. ☐

The rest of the section is devoted to prove Propositions 4.12 and 4.13. These results describe situations in which information about the global period of a word can be retrieved from small subwords of it. We remark that Propositions 4.12 and 4.13 can be obtained as a direct consequence of the Critical Factorization Theorem, a fundamental result in combinatorics on words; here we give proofs that depend only on Theorem 4.8 in order to maintain our presentation as self-contained as possible.

Lemma 4.11 *Let u, v, w, s and t be words in $\mathcal{A}$.*

(1) Suppose that uv occurs in t^∞ and that vw occurs in s^∞. If, $|v| \geq |t|+|s|-1$, then uvw occurs both in t^∞ and in s^∞.

(2) Suppose that uv is a prefix of t^∞ and that vw is a suffix of t^∞. If $|v| \geq 2|t|$, then uvw is a power of $\mathsf{root}\, t$.

(3) If $|v| \geq \mathsf{per}(uv) + \mathsf{per}(vw)$, then $\mathsf{per}(uvw) = \mathsf{per}(uv) = \mathsf{per}(vw)$.

Proof. Assume that the hypothesis of Item (1) holds. Then, $uv = t^{\mathbb{Z}}_{[i,i+|uv|)}$ and $vw = s^{\mathbb{Z}}_{[j,j+|vw|)}$ for some $i, j \in \mathbb{Z}$. Hence, $t^{\mathbb{Z}}_{[i+|u|,i+|uv|)} = s^{\mathbb{Z}}_{[j,j+|v|)}$. This and the inequality $|v| \geq$

$|t| + |s| - 1$ allows us to use Item ((1)) in Proposition 4.10 to get that $S^{i+|u|}t^{\mathbb{Z}} = S^j s^{\mathbb{Z}}$. We conclude that

$$uvw = t^{\mathbb{Z}}_{[i,i+|uv|)} s^{\mathbb{Z}}_{[j+|v|,j+|vw|)} = t^{\mathbb{Z}}_{[i,i+|uv|)} t^{\mathbb{Z}}_{[i+|u|+|v|,i+|u|+|vw|)} = t^{\mathbb{Z}}_{[i,i+|uvw|)},$$

and that uvw occurs in t^∞. Similarly, uvw occurs in s^∞.

We now assume that the hypothesis of Item (2) holds. Let $t_0 = \mathsf{root}\, t$ Then, $uv = (t_0^{\mathbb{Z}})_{[0,|uv|)}$ and $vw = (t_0^{\mathbb{Z}})_{[-|vw|,0)}$. This implies that $(t_0^{\mathbb{Z}})_{[|u|,|uv|)} = (t_0^{\mathbb{Z}})_{[-|vw|,-|w|)}$, and then, since $|v| \geq 2|t| \geq 2|t_0|$, Item ((1)) in Proposition 4.10 ensures that

$$S^{|u|}t_0^{\mathbb{Z}} = S^{-|vw|}t_0^{\mathbb{Z}} \text{ and } uvw = (t_0^{\mathbb{Z}})_{[0,|uvw|)}. \tag{4.3}$$

Now, from the first equation in (4.3) and Item ((2)) in Proposition 4.10 we get that $|u| = -|vw| \pmod{|t_0|}$, that is, $|uvw| = 0 \pmod{|t_0|}$. This and the second equation in (4.3) give that $uvw = (t_0^{\mathbb{Z}})_{[0,|uvw|)}$ is a power of $t_0 = \mathsf{root}\, t$.

We finally prove Item (3). Clearly, $\mathsf{per}(uv) \leq \mathsf{per}(uvw)$ and $\mathsf{per}(vw) \leq \mathsf{per}(uvw)$. Let t_0 be the prefix of uv of length $\mathsf{per}(uv)$ and s_0 be the prefix of vw of length $\mathsf{per}(vw)$. Then, uv occurs in t_0 and vw occurs in s_0. This and the inequality $|v| \geq |u| + |v| \geq |t_0| + |s_0|$ allow us to use Item (1) of this lemma to deduce that uvw occurs in $t_0^{\mathbb{Z}}$ and $s_0^{\mathbb{Z}}$. We deduce that $\mathsf{per}(uvw) \leq |t_0| = \mathsf{per}(uv)$ and $\mathsf{per}(uvw) \leq |s_0| = \mathsf{per}(vw)$. Therefore, $\mathsf{per}(uvw) = \mathsf{per}(uv) = \mathsf{per}(vw)$. □

Proposition 4.12 *Let $\mathcal{V} \subseteq \mathcal{A}^+$ and $u \in \mathcal{A}^+$ be such that $|u| \geq 2|\mathcal{V}|$. Suppose that for any subword v of u with length $|v| = 2|\mathcal{V}|$ there exists $w_v \in \mathcal{V}$ such that v occurs in $w_v^{\mathbb{Z}}$. Then, for any such word v, u occurs in $w_v^{\mathbb{Z}}$. In particular, $\mathsf{per}(u) \leq |\mathcal{V}|$.*

Proof. The case $|u| = 2|\mathcal{V}|$ follows directly from the hypothesis. Suppose the lemma is true for words u' of length $2|\mathcal{V}| \leq |u'| < |u|$. Let v be a subword of u with length $|v| = 2|\mathcal{V}|$. We have to prove that w occurs in $w_v^{\mathbb{Z}}$. Let us write $u = au' = u''b$ for certain letters a, b and words u', u''. There is no loss of generality in assuming that v occurs in u'. Since $|u| > 2|\mathcal{V}|$, we can take a subword v' of u'' with length $|v'| = 2|\mathcal{V}|$. Then, the inductive hypothesis can be used to deduce that u' occurs in $w_v^{\mathbb{Z}}$ and that u'' occurs in $w_{v'}^{\mathbb{Z}}$. Now, u' and u'' have a common subword of length $|u| - 2 \geq 2|\mathcal{W}| - 1 \geq |w_v| + |w_{v'}| - 1$. Therefore, Item ((1)) of Lemma 4.11 can be applied and we deduce that w occurs in $w_v^{\mathbb{Z}}$. This proves the inductive step and thereby the proposition. □

Proposition 4.13 *Let u be a word.*

(1) If t is a word occurring in u and $|t| \geq 2\,\mathsf{per}(u)$, then $\mathsf{per}(t) = \mathsf{per}(u)$.

(2) Let $k \geq 1$. If $|u| \geq 2k$ and $\mathsf{per}(u) > k$, then there exists t occurring in u with $|t| = 2k$ and $\mathsf{per}(t) > k$.

Proof. We start with Item (1). Note that $\mathsf{per}(t) \leq \mathsf{per}(u)$, so we only have to prove the other inequality. Let s (resp. s') be the prefix of t of length $\mathsf{per}(t)$ (resp. $\mathsf{per}(u)$). Then, t occurs in $s^{\mathbb{Z}}$ and $s'^{\mathbb{Z}}$. Being $|t| \geq 2\,\mathsf{per}(u) \geq |s| + |s'|$, we can use Item ((1)) in Proposition 4.10 to deduce that $s^{\mathbb{Z}} = S^{\ell} s'^{\mathbb{Z}}$ for some $\ell \in \mathbb{Z}$. This implies, as u occurs in $s'^{\mathbb{Z}}$, that u occurs in $s^{\mathbb{Z}}$. In particular, $\mathsf{per}(u) \leq |s| = \mathsf{per}(t)$.

Next, we prove Item (2) by contradiction. Assume that $k \geq 1$ and u are such that $|u| \geq 2k$ and $\mathsf{per}(u) > k$, but that for all word t occurring in u of length $|t| = 2k$ we have that $\mathsf{per}(t) \leq k$. We define, for all such t, s_t as the prefix of t of length $\mathsf{per}(t)$, and note that t occurs in $s_t^{\mathbb{Z}}$ and that $|s_t| \leq k$. Therefore, the set $\mathcal{V}$ consisting of the words s_t and the word u comply with the hypothesis of Proposition 4.12. We conclude that $\mathsf{per}(u) \leq |\mathcal{V}| \leq k$. $\square$

4.3 The classic coding based on special words

The notion of *right-special word* is an important concept for studying linear-growth complexity subshifts. In this section, we present basic results on right-special words and the coding associated to them. Most of these ideas are common to many works on the $\mathcal{S}$-adic conjecture and related problems. One of the new ingredients of our work is Proposition 4.16.

Definition 4.4 Let X be a subshift. A word $w \in \mathcal{L}(X)$ is called *right-special* if there exist two different letters a and b such that $wa, wb \in \mathcal{L}(X)$. We denote by $\mathsf{RS}_n(X)$ the set of all right special words of X having length n.

Remark 4.4 We can also define *left-special* words, which together with right-special words form the set of *special words* of X. In our work, we will only use right-special words.

The next proposition summarizes the facts about $\mathsf{RS}_n(X)$ and its relation to the complexity of X that are important for us. A *return word* to a clopen set U is an element $w \in \mathcal{A}^+$ such that there exists $x \in X$ satisfying $S^k x \in U$ if $k \in \{0, |w|\}$ and $S^k x \notin U$ if $k \in (0, |w|)$.

Proposition 4.14 *Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be an aperiodic subshift and U the clopen set $U = \{x \in X : x_{[0,n)} \in \mathsf{RS}_n(X)\}$.*

(1) We have the following bounds on the number of right-special words:

$$\frac{1}{\#\mathcal{A}}(p_X(n+1) - p_X(n)) \leq \#\,\mathsf{RS}_n(X) \leq p_X(n+1) - p_X(n).$$

(2) The set U is $(p_X(n) + n)$-recurrent in X.

(3) The number of return words to U is at most $\#\mathcal{A} \cdot \#\,\mathsf{RS}_n(X)$.

Proof. A proof of Items (1), (2) and (3) can be found, with a different notation, in [LR13]. $\square$

We can combine Propositions 4.14 and 4.4 to obtain the following proposition.

Proposition 4.15 *Suppose that $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is an aperiodic subshift and let d be the maximum of $\lceil p_X(n)/n \rceil$, $p_X(n+1) - p_X(n)$ and $\#\mathcal{A}$. Let $(Z \subseteq \mathcal{C}^{\mathbb{Z}}, \tau\colon \mathcal{C}^+ \to \mathcal{A}^+)$ be the coding obtained from Proposition 4.4 with $U = \{x \in X : x_{[0,n)} \in \mathsf{RS}_n(X)\}$. Then:*

(1) $\#\mathcal{C} \leq d^3$.

(2) $|\tau(a)| \leq (d+1)n$ for all $a \in \mathcal{C}$.

(3) (Z, τ) is $(d+2)n$-recognizable.

(4) If $x \in X$, $(c, z) = \mathbf{F}_{(Z,\tau)}(x)$ and $i \in \mathbb{Z}$, then $i = c_j$ for some $j \in \mathbb{Z}$ if and only if $x_{[i,i+n)} \in \mathsf{RS}(X)$.

Proposition 4.15 is the starting point of other works on the $\mathcal{S}$-adic conjecture; see for example [Fer96; Ler12].

Proposition 4.16 *Let (Z, τ) be the coding in Proposition 4.15. Let $x \in X$ and $(c, z) = \mathbf{F}_{(Z,\tau)}(x)$ and suppose that $i, j \in \mathbb{Z}$ satisfy $i + d < j$ and $\ell := \max\{|\tau(z_k)| : k \in [i,j)\} \leq n/6d$. Then, $\mathsf{per}(x_{[c_i - n/3, c_{j-d})}) \leq d\ell$.*

Proof. We start by noticing that, since $x_{[c_m - n, c_m)} \in \mathsf{RS}_n(X)$ for all $m \in \mathbb{Z}$ and since $\# \mathsf{RS}_n(X) \leq d$, we can use the Pigeonhole principle to obtain, for each $k \in [i, j-d)$, integers $p_k, q_k \in [k, k+d)$ such that $p_k < q_k$ and $x_{[c_{p_k} - n, c_{p_k})} = x_{[c_{q_k} - n, c_{q_k})}$. These conditions imply that $\mathsf{per}(x_{[c_{p_k} - n, c_{q_k})}) \leq c_{q_k} - c_{p_k} \leq d\ell$. Therefore, as $c_{p_k} - n \leq c_k + d\ell - n \leq c_k - 2n/3$ and $c_{q_k} \geq c_{k+1}$,

$$x_{[c_k - 2n/3, c_{k+1})} \text{ for all } k \in [i, j-d). \tag{4.4}$$

We will use (4.4) to prove the lemma by contradiction. Assume that $\mathsf{per}(x_{[c_i - n/3, c_{j-d})}) > d\ell$. Then, by Item ((2)) in Lemma 4.13, there exists $m \in [c_i - n/3 + 2d\ell, c_{j-d})$ such that $\mathsf{per}(x_{[m-2d\ell, m)}) > d\ell$. Now, the condition $m \in [c_i - n/3 + 2d\ell, c_{j-d})$ allows us to find $k \in [i, j-d)$ such that $m \in [c_k - n/3, c_{k+1})$. Hence, as $2d\ell \leq n/3$, $x_{[m-2d\ell, m)}$ occurs in $x_{[c_k - 2n/3, c_{k+1})}$, which yields $\mathsf{per}(x_{[c_k - n/3, c_{k+1})}) \geq \mathsf{per}(x_{[m-2d\ell, m)}) > \varepsilon$. This contradicts (4.4) and completes the proof. □

4.4 The first coding

In this section, we begin the proof of the main results: Theorems 4.75 and 4.76. We start by constructing the codings described in Proposition 4.17. Then, in Sections 4.5, 4.6, and 4.7, we will modify these codings to obtain new versions of them, each with better properties than the previous one. We will show in Subsection 4.7.2 that the final codings can be connected with morphisms, and we will use this fact in Section 4.8 to complete the proof of the main results.

Proposition 4.17 *Let X be a minimal infinite subshift, $n \geq 1$ and let d be the maximum of $\lceil p_X(n)/n \rceil$, $p_X(n+1) - p_X(n)$, $\#\mathcal{A}$ and 10^4. Then, there exist a coding $(Z \subseteq \mathcal{C}^{\mathbb{Z}}, \tau\colon \mathcal{C} \to \mathcal{A}^+)$ of X and $\varepsilon \in [n/d^{2d^3+4}, n/d)$ satisfying the following conditions:*

(1) $\mathcal{C}$ has at most d^3 elements.

(2) $|\tau(a)| \leq 3dn$ for all $a \in \mathcal{C}$.

(3) (Z, τ) is $3dn$-recognizable.

(4) The periodicity properties in Proposition 4.18 are satisfied.

Proposition 4.18 *Consider the coding described in Proposition 4.38. Let $z \in Z$, $x = \tau(z)$ and $(c, z) = \mathbf{F}_{(Z,\tau)}(x)$. We define $\mathrm{Q}_{\mathsf{p}}(z)$ as the set of integers $j \in \mathbb{Z}$ such that $|\,\mathsf{root}\,\tau(z_j)| \leq \varepsilon$ and $x_{[c_j-99\varepsilon, c_{j+1}+99\varepsilon)} = (\mathsf{root}\,\tau(z_j))^{\mathbb{Z}}_{[-99\varepsilon, |\tau(z_j)|+99\varepsilon)}$.*

(1) $0 \notin \mathrm{Q}_{\mathsf{p}}(z)$ and $|\tau(z_0)| > 401\varepsilon$ implies that $\mathsf{per}(x_{[c_0+97\varepsilon, c_1-97\varepsilon)}) > \varepsilon$.

(2) Suppose that $0 \notin \mathrm{Q}_{\mathsf{p}}(z)$ and $|\tau(z_0)| \leq 401\varepsilon$. If $-1 \in \mathrm{Q}_{\mathsf{p}}(z)$ or $1 \in \mathrm{Q}_{\mathsf{p}}(x)$, then $\mathsf{per}(x_{[c_0+97\varepsilon, c_1-97\varepsilon)}) > \varepsilon$.

(3) If $k > d$ and $|\tau(z_j)| \leq 401\varepsilon$ for all $j \in [0, k)$, then $[0, k) \subseteq \mathrm{Q}_{\mathsf{p}}(z)$.

(4) Let $z' \in Z$ and assume that $0 \in \mathrm{Q}_{\mathsf{p}}(z)$, $0 \in \mathrm{Q}_{\mathsf{p}}(z')$ and that $\mathsf{root}\,\tau(z_0)$ is conjugate to $\mathsf{root}\,\tau(z'_0)$. Then $\mathsf{root}\,\tau(z_0) = \mathsf{root}\,\tau(z'_0)$.

We fix, for the rest of the section, the following notation. Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a minimal infinite subshift, $n \geq 0$ and d be the maximum of $p_X(n)/n$, $p_X(n+1) - p_X(n)$, $\#\mathcal{A}$ and 10^4. We denote by $(Y \subseteq \mathcal{B}^{\mathbb{Z}}, \sigma\colon \mathcal{B} \to \mathcal{A}^+)$ the coding given by Proposition 4.15 when it is used with X and n.

4.4.1 Construction of the first coding

Lemma 4.19 *Let $\mathcal{W}$ be a finite set of words. Then, there exists $\varepsilon \in [|\mathcal{W}|/d^{2\#\mathcal{W}+4}, |\mathcal{W}|/d)$ such that for all $w \in \mathcal{W}$, either $|w| > 10^4\varepsilon$ or $|w| \leq \varepsilon/d$.*

Proof. Let $d_0 = 10^4 d$ and, for $\ell \in [1, \#\mathcal{W}+1]$, $\mathcal{W}_\ell = \{w \in \mathcal{W} : |\mathcal{W}|/d_0^{\ell+1} < |w| \leq |\mathcal{W}|/d_0^{\ell}\}$. The Pigeonhole principle ensures that $\mathcal{W}_\ell$ is empty for some $\ell \in [1, \#\mathcal{W}+1]$. We set $\varepsilon = \lfloor d|\mathcal{W}|/d_0^{\ell+1} \rfloor$ and note that for any $w \in \mathcal{W}$, either $w \in \cup_{\ell'<\ell}\mathcal{W}_{\ell'}$ and $|w| > 10^4\varepsilon$, or $w \in \cup_{\ell'>\ell}\mathcal{W}_{\ell'}$ and $|w| \leq \varepsilon/d$. Also, since $\ell \in [1, \#\mathcal{W}+1]$, we have that $\varepsilon \in [|\mathcal{W}|/d^{2\#\mathcal{W}+4}, |\mathcal{W}|/d)$. □

We use Lemma 4.19 with the set $\sigma(\mathcal{B})$ to obtain $\varepsilon \in [n/d^{2\#\mathcal{W}+4}, n/d)$ such that

$$\text{for all } a \in \mathcal{B}\text{, either } |\sigma(a)| > 10^4\varepsilon \text{ or } |\sigma(a)| \leq \varepsilon/d. \tag{4.5}$$

Note that $\varepsilon \in [n/d^{2d^3+4}, n/d)$ as $d^3 \geq \#\sigma(\mathcal{B})$ by Item ((1)) in Proposition 4.15.

We now define a set $\mathcal{W}_\varepsilon \subseteq \mathcal{A}^+$ that will be important for controlling the periodicity properties in Proposition 4.17. We start by introducing classic notions related to periodicity of words. Recall that two words $u, v \in \mathcal{A}^+$ are conjugate if $ur = rv$ for some $r \in \mathcal{A}^*$. The relation

$u \sim_R v$ iff u and v are conjugate is an equivalence relation, and a $\sim_R$-equivalence class is called a *rotation class*. A word $u \in \mathcal{A}^+$ is *primitive* if $u = \mathsf{root}\, u$ [‡]. We fix a set $\mathcal{W}_\varepsilon \subseteq \mathcal{A}^+$ consisting of one element of the rotation class of each primitive word $w \in \mathcal{A}^+$ such that $|w| \leq \varepsilon$.

Lemma 4.20 *Let $t \in \mathcal{A}^+$ be such that $\mathsf{per}(t) \leq \varepsilon$ and $|t| \geq 198\varepsilon + \mathsf{per}(t)$. Then, for some $s \in \mathcal{W}_\varepsilon$, $s^{\mathbb{Z}}_{[-99\varepsilon,99\varepsilon)}$ occurs in t.*

Proof. Let u be the prefix of t of length $\mathsf{per}(t)$. Note that u is primitive as otherwise $\mathsf{per}(t) \leq |\mathsf{root}\, u| < |u| = \mathsf{per}(t)$, which is a contradiction. The primitiveness of u and the inequality $|u| = \mathsf{per}(t) \leq \varepsilon$ imply that there exist $s \in \mathcal{W}_\varepsilon$ and a suffix u' of s such that $|s| = |u|$ and $u's$ is a prefix of uu. Being $\mathsf{per}(t) = |u| = |s|$, we then have that

$$t \text{ is a prefix of } u's^\infty. \tag{4.6}$$

We set $k = \left\lceil \frac{99\varepsilon - |u'|}{|s|} \right\rceil$. Observe that, since $|s| \leq \varepsilon$.

$$|u's^k| = |u'| + k|s| \leq \mathsf{per}(t) + \left\lceil \frac{99\varepsilon}{\varepsilon} \right\rceil \varepsilon = 99\varepsilon + \mathsf{per}(t).$$

Hence, $|u's^k| + 99\varepsilon \leq |t|$. From this and Equation (4.6) we deduce that if v is the prefix of s^∞ of length 99ε, then $u's^kv$ is a prefix of t. Now, we have the bound

$$|u's^k| = |u'| + \left\lceil \frac{99\varepsilon - |u'|}{|s|} \right\rceil |s| \geq 99\varepsilon.$$

Hence, $s^{\mathbb{Z}}_{[-99\varepsilon,99\varepsilon)}$ is a suffix of $u's^kv$. We conclude that $s^{\mathbb{Z}}_{[-99\varepsilon,99\varepsilon)}$ occurs in t. □

Lemma 4.21 *Let w be a word of length n. Then, there exists a decomposition $w = vuu'v'$ satisfying one of the following sets of conditions.*

(*a*) *$|u| = |u'| = 99\varepsilon$, $|v|, |v'| \geq n/2 - 500\varepsilon$, and $uu' = s^{\mathbb{Z}}_{[-99\varepsilon,99\varepsilon)}$ for some $s \in \mathcal{W}_\varepsilon$.*

(*b*) *$|u| = |u'| = 500\varepsilon$, $|v| = \lfloor n/2 - 500\varepsilon \rfloor$, $|v'| \geq n/2 - 500\varepsilon$, and $s^{\mathbb{Z}}_{[-99\varepsilon,99\varepsilon)}$ does not occur in uu' for all $s \in \mathcal{W}_\varepsilon$.*

Proof. Since $|w| \geq 2 \cdot 500\varepsilon$, there is a decomposition $w = v_0tv_0'$, where $|v_0| = \lfloor n/2 - 500\varepsilon \rfloor$, $|v_0'| \geq n/2 - 500\varepsilon$ and $|t| = 2 \cdot 500\varepsilon$. There are two cases:

(i) $s^{\mathbb{Z}}_{[-99\varepsilon,99\varepsilon)}$ occurs in t for some $s \in \mathcal{W}_\varepsilon$.

(ii) $s^{\mathbb{Z}}_{[-99\varepsilon,99\varepsilon)}$ does not occur in t for all $s \in \mathcal{W}_\varepsilon$.

Suppose first that case (i) occurs. It is then possible to write $t = v_1uu'v_1'$, where $uu' = s^{\mathbb{Z}}_{[-99\varepsilon,99\varepsilon)}$ and $|u| = |u'| = 99\varepsilon$. We set $v = v_0v_1$ and $v' = v_1'v_0'$ and note that $w = vuu'v'$ satisfies Condition (a).

[‡]We recall the reader that $\mathsf{root}\, u$ is the shortest prefix v of u such that $u = v^k$ for some $k \geq 1$

We now assume that (ii) holds. Being the length of t equal to $2\cdot 500\varepsilon$, we can write $w_0 = uu'$, where $|u| = |u'| = 500\varepsilon$. Then, the decomposition $w = v_0 u u v_0'$ satisfies Condition (b). $\square$

We now can define (Z, τ).

Definition 4.5 For $w \in \mathsf{RS}_n(X)$, we use Lemma 4.21 to fix a decomposition $w = v_w u_w u_w' v_w'$ satisfying one of the following conditions:

($\mathcal{P}_a$) $|u_w| = |u_w'| = 99\varepsilon$, $|v_w|, |v_w'| \geq n/2 - 500\varepsilon$, and $u_w u_w' = s^{\mathbb{Z}}_{[-99\varepsilon, 99\varepsilon)}$ for some $s \in \mathcal{W}_\varepsilon$.

($\mathcal{P}_b$) $|u_w| = |u_w'| = 500\varepsilon$, $|v_w| = \lfloor n/2 - 500\varepsilon \rfloor$, $|v_w'| \geq n/2 - 500\varepsilon$, and $s^{\mathbb{Z}}_{[-99\varepsilon, 99\varepsilon)}$ does not occur in $u_w u_w'$ for all $s \in \mathcal{W}_\varepsilon$.

Moreover, we choose this decomposition so that $|v_w u_w|$ is as small as possible. We define $(Z \subseteq \mathcal{C}^{\mathbb{Z}}, \tau \colon \mathcal{C} \to \mathcal{A}^+)$ as the coding of X obtained from Proposition 4.4 and the clopen set $U = \{x \in X : \exists w \in \mathsf{RS}_n(X), x_{[-|v_w u_w|, |u_w' v_w'|)} = w\}$.

4.4.2 Basic properties of the first coding

Lemma 4.22 *Let $x \in X$ and $i, j \in \mathbb{Z}$ with $i < j$. Suppose that $x_{[i-|v_w u_w|, i+|u_w' v_w'|)} = w$ and $x_{[j-|v_{\tilde{w}} u_{\tilde{w}}|, j+|u_{\tilde{w}}' v_{\tilde{w}}'|)} = \tilde{w}$ for some $w, \tilde{w} \in \mathsf{RS}_n(X)$. Then, $i + |u_w' v_w'| < j + |u_{\tilde{w}}' v_{\tilde{w}}'|$.*

Proof. We assume, with the aim of obtaining a contradiction, that $i + |u_w' v_w'| \geq j + |u_{\tilde{w}}' v_{\tilde{w}}'|$.

First, we consider the case $i + |u_w' v_w'| = j + |u_{\tilde{w}}' v_{\tilde{w}}'|$. Then,

$$w = x_{[i+|u_w' v_w'|-n, i+|u_w' v_w'|)} = x_{[j+|u_{\tilde{w}}' v_{\tilde{w}}'|-n, j+|u_{\tilde{w}}' v_{\tilde{w}}'|)} = \tilde{w}.$$

Hence, $u_w' v_w' = u_{\tilde{w}}' v_{\tilde{w}}'$, and therefore

$$i = (i + |u_w' v_w'|) - |u_w' v_w'| = (j + |u_{\tilde{w}}' v_{\tilde{w}}'|) - |u_{\tilde{w}}' v_{\tilde{w}}'| = j.$$

This contradicts that $i < j$.

Next, we assume that

$$i + |u_w' v_w'| > j + |u_{\tilde{w}}' v_{\tilde{w}}'|. \tag{4.7}$$

Note that this is equivalent to $i - |v_w u_w| > j - |v_{\tilde{w}} u_{\tilde{w}}|$. This fact will be freely used through the proof.

We consider the following two cases:

(i) $i + |u_w'| < j + |u_{\tilde{w}}'|$.

(ii) $i + |u_w'| \geq j + |u_{\tilde{w}}'|$.

Suppose first that case (i) occurs. We are going to define a decomposition $\tilde{w} = v u u' v'$ as the one in Definition 4.5 and such that $|v u_w| < |v_{\tilde{w}} u_{\tilde{w}}|$. This would contradict the minimality of $|v_{\tilde{w}} u_{\tilde{w}}|$.

We start by noting that, thanks to (4.7), if we set $v = x_{[j-|v_{\tilde{w}}u_{\tilde{w}}|,i-|u_w|)}$ and $v' = x_{[i+|u'_w|,j+|u'_{\tilde{w}}v'_{\tilde{w}}|)}$, then $\tilde{w} = vu_wu'_wv'$. Note that from (4.7) we have that

$$|v| = |x_{[j-|v_{\tilde{w}}u_{\tilde{w}}|,i-|v_wu_w|)}| + |v_w| \geq n/2 - 500\varepsilon.$$

Also, (i) implies that

$$|v'| = |x_{[i+|u'_w|,j+|u'_{\tilde{w}}v'_{\tilde{w}}|)}| \geq |x_{[j+|u'_{\tilde{w}}|,j+|u'_{\tilde{w}}v'_{\tilde{w}}|)}| = |v'_{\tilde{w}}| \geq n/2 - 500\varepsilon.$$

We conclude, as $w = v_wu_wu'_wv'_w$ satisfies Condition $(\mathcal{P}_a)$ or $(\mathcal{P}_b)$ in Definition 4.5, that $w = vu_wu'_wv'$ satisfies $(\mathcal{P}_a)$ or $(\mathcal{P}_b)$. Moreover, since $|vu_w| = |x_{[j-|u_{\tilde{w}}v_{\tilde{w}}|,i)}|$ and $|v_{\tilde{w}}u_{\tilde{w}}| = |x_{[j-|v_{\tilde{w}}u_{\tilde{w}}|,j)}|$, we have that

$$|v_{\tilde{w}}u_{\tilde{w}}| = |vu_w| + j - i > |vu_w|.$$

Thus, $w = vu_wu'_wv'$ satisfies $(\mathcal{P}_a)$ or $(\mathcal{P}_b)$, and $|vu_w|$ is strictly smaller than $|v_{\tilde{w}}u_{\tilde{w}}|$. This contradicts the minimality of $|v_{\tilde{w}}u_{\tilde{w}}|$.

Next, we assume that $i+|u'_w| \geq j+|u'_{\tilde{w}}|$. Then, as $i < j$, we have that $[j, j+|u'_{\tilde{w}}|) \subsetneq [i, i+|u'_w|)$. This implies two things. First, since $|u_{\tilde{w}}| = |u'_{\tilde{w}}|$ and $|u_w| = |u'_w|$, that

$$[j - |u_{\tilde{w}}|, j + |u'_{\tilde{w}}|) \subsetneq [i - |u_w|, i + |u'_w|). \tag{4.8}$$

Second, that $|u'_{\tilde{w}}| < |u'_w|$. Being $|u'_w|, |u'_{\tilde{w}}| \in \{99\varepsilon, 500\varepsilon\}$, the last relation is possible only if

$$|u'_{\tilde{w}}| = 99\varepsilon \text{ and } |u'_w| = 500\varepsilon. \tag{4.9}$$

Therefore, Condition $(\mathcal{P}_a)$ holds for $\tilde{w} = v_{\tilde{w}}u_{\tilde{w}}u'_{\tilde{w}}v'_{\tilde{w}}$ and Condition $(\mathcal{P}_b)$ holds for $w = v_wu_wu'_wv'_w$. In particular, we can find $s \in \mathcal{V}_\varepsilon$ such that $u_{\tilde{w}}u'_{\tilde{w}} = s^{\mathbb{Z}}_{[-99\varepsilon,99\varepsilon)}$. This implies, by (4.8), that $s^{\mathbb{Z}}_{[-99\varepsilon,99\varepsilon)} = u_{\tilde{w}}u'_{\tilde{w}}$ occurs in $u_wu'_w$. But then Condition $(\mathcal{P}_b)$ cannot hold for $w = v_wu_wu'_wv'_w$, contradicting our assumptions. □

It is convenient to introduce some notation. Let $x \in X$, $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$ and $(f, z) = \mathbf{F}_{(Z,\tau)}(x)$. For $j \in \mathbb{Z}$, we define $w_j(x) = x_{[c_j-n,c_j)} \in \mathsf{RS}_n(X)$, $v_j(x) = v_{w_j(x)}$, $u_j(x) = u_{w_j(x)}$, $u'_j(x) = u'_{w_j(x)}$ and $v'_j(x) = v'_{w_j(x)}$. Then,

$$x_{[c_j-n,c_j)} = w_j(x) = v_j(x)u_j(x)u'_j(x)v'_j(x).$$

Observe that if $j \in \mathbb{Z}$ then $x_{[f_j-|v_wu_w|,f_j+|u'_wv'_w|)} = w$ for some $w \in \mathsf{RS}_n(X)$, so there exists $i \in \mathbb{Z}$ such that $f_j + |u'_i(x)v'_i(x)| = c_i$. We define $\phi_x(j)$ as the smallest integer such that

$$f_j + |u'_{\phi_x(j)}(x)v'_{\phi_x(j)}(x)| = c_{\phi_x(j)}. \tag{4.10}$$

Then, by Lemma 4.22,

$$\phi_x(i) < \phi_x(j) \text{ for all } x \in X \text{ and } i < j. \tag{4.11}$$

Lemma 4.23 *Let $x \in X$, $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$ and $(f, z) = \mathbf{F}_{(Z,\tau)}(x)$. If $i \in \mathbb{Z}$ and $k \in [\phi_x(i), \phi_x(i+1))$, then $f_i + |u'_k(x)v'_k(x)| = c_k$.*

Proof. Observe that, since $x_{[c_k-n,c_k)} = w_k(x)$, there exists $j \in \mathbb{Z}$ such that $f_j = c_k - |u'_k(x)v'_k(x)|$. We are going to prove that $j = i$.

First, we note that, since $k \in [\phi_x(i), \phi_x(i+1))$ and $c_k = f_j + |u'_k(x)v'_k(x)|$,

$$\begin{aligned} f_i + |u'_{\phi_x(i)}(x)v'_{\phi_x(i)}(x)| = c_{\phi_x(i)} \leq f_j + |u'_k(x)v'_k(x)| \\ < c_{\phi_x(i+1)} = f_{i+1} + |u'_{\phi_x(i+1)}(x)v'_{\phi_x(i+1)}(x)|. \end{aligned} \tag{4.12}$$

This implies, by Lemma (4.22), that $i \leq j \leq i+1$. Now, if $j = i+1$, then Equation (4.12) ensures that $f_{i+1} + |u'_k(x)v'_k(x)|$ is strictly smaller than $f_{i+1} + |u'_{\phi_x(i+1)}(x)v'_{\phi_x(i+1)}(x)|$, which contradicts the minimality of $\phi_x(i+1)$. We conclude that $j = i$. □

Lemma 4.24 *The set $\mathcal{C}$ has at most d^3 elements.*

Proof. Let $x \in X$, $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$ and $(f, z) = \mathbf{F}_{(Z,\tau)}(x)$. We drop the dependency on x in ϕ_x and just write ϕ. The lemma follows from the following claim.

(•) For $j \in \mathbb{Z}$, let $\zeta(j) = (w_{\phi(j+1)-1}, x_{c_{\phi(j+1)-1}}) \in \mathsf{RS}_n(X) \times \mathcal{A}$. Then, $\zeta(i) = \zeta(j)$ implies that $x_{[f_i,f_{i+1})} = x_{[f_j,f_{j+1})}$.

Indeed, being Z minimal (as X is minimal and (Z, τ) is recognizable), (•) implies that

$$\#\tau(\mathcal{C}) = \#\{x_{[f_j,f_{j+1})} : j \in \mathbb{Z}\} \leq \#\,\mathsf{RS}_n(X) \cdot \#\mathcal{A} \leq d^3,$$

where we used that $\#\,\mathsf{RS}_n(X) \leq \#\mathcal{A} \cdot (p_X(n+1) - p_X(n))$ by Item ((1)) in Proposition 4.14. This implies, as τ is injective on letters by Proposition 4.4, that $\#\mathcal{C} = \#\tau(\mathcal{C}) \leq d^3$.

Let us prove the claim. Suppose that $i, j \in \mathbb{Z}$ satisfy $\zeta(i) = \zeta(j) = (w, a)$. We start with some observations. First, the condition $\zeta(i) = \zeta(j) = (w, a)$ implies that

(i) $w = x_{[c_{\phi(j+1)-1}-n,c_{\phi(j+1)-1})} = x_{[c_{\phi(i+1)-1}-n,c_{\phi(i+1)-1})}$; and

(ii) $a = x_{c_{\phi(j+1)-1}} = x_{c_{\phi(i+1)-1}}$.

Also, Equation (4.11) ensures that $\phi(i) < \phi(i+1)$ and $\phi(j) < \phi(j+1)$, so

$$\phi(i) \leq \phi(i+1) - 1 < \phi(i+1) \text{ and } \phi(j) \leq \phi(j+1) - 1 < \phi(j+1). \tag{4.13}$$

We now prove the claim (•). The definition of $c_{\phi(j+1)-1}$ and $c_{\phi(j+1)}$ guarantees that the words $x_{[k-n,k)}$, $k \in (c_{\phi(j+1)-1}, c_{\phi(j+1)})$, are not right-special. Thus, $x_{[c_{\phi(j+1)-1},c_{\phi(j+1)})}$ is determined by $x_{[c_{\phi(j+1)-1}-n,c_{\phi(j+1)-1})}$ and $x_{c_{\phi(j+1)-1}}$. A similar observation holds for $x_{[c_{\phi(i+1)-1},c_{\phi(i+1)})}$. Combining these two things with (i) and (ii) yields that

$$x_{[c_{\phi(j+1)-1},c_{\phi(j+1)})} = x_{[c_{\phi(i+1)-1},c_{\phi(i+1)})}. \tag{4.14}$$

Then, by (i),

$$w_{\phi(j+1)}(x) = x_{[c_{\phi(j+1)}-n,c_{\phi(j+1)})} = x_{[c_{\phi(i+1)}-n,c_{\phi(i+1)})} = w_{\phi(i+1)}(x).$$

Let us write $\tilde{w} = w_{\phi(j+1)}(x) = w_{\phi(i+1)}(x)$. With this notation, we have, by (4.10), that

$$x_{[f_{j+1},c_{\phi(j+1)})} = x_{[f_{i+1},c_{\phi(i+1)})} = u'_{\tilde{w}}v'_{\tilde{w}}. \tag{4.15}$$

Now, Equation (4.13) allows us to use Lemma 4.23 with $\phi(i+1)-1$ and $\phi(j+1)-1$; we deduce, as $w = w_{\phi(j+1)-1} = w_{\phi(i+1)-1}$, that

$$f_j + |u'_w v'_w| = c_{\phi(j+1)-1} \text{ and } f_i + |u'_w v'_w| = c_{\phi(i+1)-1}.$$

In particular,

$$x_{[f_j,c_{\phi(j+1)-1})} = x_{[f_i,c_{\phi(i+1)-1})} = u'_w v'_w.$$

This and Equation (4.14) then give that

$$x_{[f_j,c_{\phi(j+1)})} = x_{[f_i,c_{\phi(i+1)})}.$$

We conclude using (4.15) that $x_{[f_j,f_{j+1})} = x_{[f_i,f_{i+1})}$. This completes the proof of the claim and thereby the proof of the lemma. □

Lemma 4.25 *Let $x \in X$, $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$ and $(f, z) = \mathbf{F}_{(Z,\tau)}(x)$. Then:*

(1) $|\sigma(y_k)| \leq |\tau(z_0)| + 2 \cdot 401\varepsilon$ for any $k \in [\phi_x(0), \phi_x(1))$.

(2) $|\tau(z_0)| \leq |\sigma(y_{\phi(1)-1})| + 2 \cdot 401\varepsilon$.

Proof. We write, for simplicity, $\phi = \phi_x$. Let $k \in [\phi(0), \phi(1))$. Then, by Lemma 4.23, $f_0 + |u'_k(x)v'_k(x)| = c_k$. Hence,

$$\begin{aligned}\tau(z_0) \cdot u'_{\phi(1)}(x)v'_{\phi(1)}(x) &= x_{[f_0,f_1)} \cdot x_{[f_1,c_{\phi(1)})} \\ &= x_{[f_0,c_k)} \cdot x_{[c_k,c_{\phi(1)})} = u'_k(x)v'_k(x) \cdot \sigma(y_{[k,\phi(1))}).\end{aligned}$$

In particular,

$$\big||\tau(z_0)| - |\sigma(y_{[k,\phi(1))})|\big| = \big||u'_{\phi(1)}(x)v'_{\phi(1)}(x)| - |u'_k(x)v'_k(x)|\big|. \tag{4.16}$$

Now, Conditions $(\mathcal{P}_a)$ and $(\mathcal{P}_b)$ in Definition 4.5 ensure that for any $w \in \mathsf{RS}_n(X)$ the inequalities $n/2 - 401\varepsilon \leq |u'_w v'_w| \leq n/2 + 401\varepsilon$ hold. Putting this in (4.16) produces

$$\big||\tau(z_0)| - |\sigma(y_{[k,\phi(1))})|\big| \leq 2 \cdot 401\varepsilon \text{ for all } k \in [\phi(0), \phi(1)). \tag{4.17}$$

Item (1) of this lemma follows. Moreover, since $y_{\phi(1)-1} = y_{[\phi(1)-1,\phi(1))}$, Item (2) is also a consequence of (4.17). □

4.4.3 Proof of Propositions 4.17 and 4.18

We now prove Proposition 4.17.

Proof of Proposition 4.17. Item (1) follows directly from Lemma 4.24.

Let us prove Items (2) and (3). We define $U' = \{x \in X : x_{[0,n)} \in \mathsf{RS}_n(X)\}$ and $U = \{x \in X : \exists w \in \mathsf{RS}_n(X), x_{[-|u_w v_w|, |u'_w v'_w|)} = w\}$. First, we recall that (Z, τ) is defined as the coding of X obtained from U as in Proposition 4.4. Observe that, since $|v_w u_w| \leq |w| = n$ and $|u'_w v'_w| \leq |w| = n$ for all $w \in \mathsf{RS}_n(X)$, U has radius n. Also, Item ((2)) in Proposition 4.14 ensures that U' is $(d+1)n$-syndetic, and thus that U is $(d+3)n$-syndetic. Therefore, Proposition 4.4 ensures that $|\tau(a)| \leq (d+3)n$ for all $a \in \mathcal{C}$ and that (Z, τ) is $(d+4)n$-recognizable. Since $d \geq \#\mathcal{A} \geq 2$, Items (2) and (3) follow. □

The rest of the section is devoted to prove Proposition 4.18.

Lemma 4.26 *Let $x \in X$ and $(c, z) = \mathbf{F}_{(Z,\tau)}(x)$. We use the notation $w = w_{\phi_x(0)}$ and $\tilde{w} = w_{\phi_x(1)}$. Then, the following are equivalent:*

(1) $|\mathsf{root}\,\tau(z_0)| \leq \varepsilon$ and $x_{[c_0-99\varepsilon, c_1+99\varepsilon)} = (\mathsf{root}\,\tau(z_0))^{\mathbb{Z}}_{[-99\varepsilon, |\tau(z_0)|+99\varepsilon)}$.

(2) The decompositions $w = v_w u_w u'_w v'_w$ and $\tilde{w} = v_{\tilde{w}} u_{\tilde{w}} u'_{\tilde{w}} v'_{\tilde{w}}$ satisfy Condition $(\mathcal{P}_a)$ in Definition 4.5 and $\mathsf{per}(x_{[c_0+97\varepsilon, c_1-97\varepsilon)}) \leq \varepsilon$.

Moreover, if any of the previous condition holds, then $\mathsf{root}\,\tau(z_0) \in \mathcal{W}_\varepsilon$.

Proof. We assume that Item (1) holds. Let $s = \mathsf{root}\,\tau(z_0)$ and note that Item (1) ensures that

$$|s| \leq \varepsilon \text{ and } x_{[c_0-99\varepsilon, c_1+99\varepsilon)} = s^{\mathbb{Z}}_{[-99\varepsilon, |\tau(z_0)|+99\varepsilon)}. \tag{4.18}$$

This allows us to use Lemma 4.20 with $x_{[c_0-99\varepsilon, c_0+|s|+99\varepsilon)}$ and find $t \in \mathcal{W}_\varepsilon$ such that $t^{\mathbb{Z}}_{[-99\varepsilon, 99\varepsilon)}$ occurs in $x_{[c_0-99\varepsilon, c_0+|s|+99\varepsilon)}$. Since $|s| \leq \varepsilon$, we have in particular that $t^{\mathbb{Z}}_{[-99\varepsilon, 99\varepsilon)}$ occurs in $x_{[c_0-500\varepsilon, c_0+9\varepsilon)}$. This is incompatible with the decomposition $w = v_w u_w u'_w v'_w$ satisfying Condition $(\mathcal{P}_b)$ in Definition 4.5; therefore, $w = v_w u_w u'_w v'_w$ satisfies Condition $(\mathcal{P}_a)$. A similar argument shows that $\tilde{w} = v_{\tilde{w}} u_{\tilde{w}} u'_{\tilde{w}} v'_{\tilde{w}}$ also satisfies Condition $(\mathcal{P}_a)$. Finally, it follows from (4.18) that $\mathsf{per}(x_{[c_0+97\varepsilon, c_1-97\varepsilon)}) \leq |s| \leq \varepsilon$.

We assume that Item (2) holds. Then, by Definition 4.5, there exist $s, \tilde{s} \in \mathcal{W}_\varepsilon$ such that

$$s^{\mathbb{Z}}_{[-99\varepsilon, 99\varepsilon)} = u_w u'_w = x_{[c_0-99\varepsilon, c_0+99\varepsilon)} \text{ and } \tilde{s}^{\mathbb{Z}}_{[-99\varepsilon, 99\varepsilon)} = u_{\tilde{w}} u'_{\tilde{w}} = x_{[c_1-99\varepsilon, c_1+99\varepsilon)}. \tag{4.19}$$

We claim that

$$\mathsf{per}(x_{[c_0-99\varepsilon, c_1+99\varepsilon)}) \leq \varepsilon. \tag{4.20}$$

Assume, with the objective of obtaining a contradiction, that (4.20) is not satisfied. Then, Item ((2)) in Proposition 4.13 gives $i \in [c_0 - 98\varepsilon, c_1 + 98\varepsilon)$ such that $\mathsf{per}(x_{[i-\varepsilon, i+\varepsilon)}) > \varepsilon$. We consider three cases. If $i \in [c_0 - 98\varepsilon, c_0 + 98\varepsilon)$, then $x_{[i-\varepsilon, i+\varepsilon)}$ occurs in $u_w u'_w$. Thus, by (4.19) and since $s \in \mathcal{W}_\varepsilon$ implies that $|s| \leq \varepsilon$, $\mathsf{per}(x_{[i-\varepsilon, i+\varepsilon)}) \leq |s| \leq \varepsilon$. This contradicts our assumptions. In the case $i \in [c_1 - 98\varepsilon, c_1 + 98\varepsilon)$, a similar argument gives a contradiction. Finally, if $i \in [c_0 + 98\varepsilon, c_1 - 98\varepsilon)$, then $x_{[i-\varepsilon, i+\varepsilon)}$ occurs in $x_{[c_0+97\varepsilon, c_1-97\varepsilon)}$ and thus, by the hypothesis, $\mathsf{per}(x_{[i-\varepsilon, i+\varepsilon)}) \leq \varepsilon$. This proves (4.20).

Our next objective is to use the claim for proving that

$$|s| = |\tilde{s}| = \mathsf{per}(x_{[c_0-99\varepsilon, c_1+99\varepsilon)}). \tag{4.21}$$

We note that $|s| \leq \varepsilon$ as $s \in \mathcal{W}_\varepsilon$. Hence, Equations (4.19) and (4.20) allows us to use Item ((1)) of Proposition 4.13 and deduce that $\mathsf{per}(x_{[c_0-99\varepsilon,c_1+99\varepsilon)})$ is equal to $\mathsf{per}(x_{[c_0-99\varepsilon,c_0+99\varepsilon)})$. Also, (4.20) ensures that $\mathsf{per}(x_{[c_0-99\varepsilon,c_0+99\varepsilon)}) \leq |s|$, so by Item ((1)) in Proposition 4.13 we have that $\mathsf{per}(x_{[c_0-99\varepsilon,c_0+99\varepsilon)})$ is equal to $\mathsf{per}(s^2)$. Moreover, by Lemma 4.9, $\mathsf{per}(s^2) = |\,\mathsf{root}\, s^2| = |s|$. Combining all these relations produces

$$\mathsf{per}(x_{[c_0-99\varepsilon,c_1+99\varepsilon)}) = \mathsf{per}(x_{[c_0-99\varepsilon,c_0+99\varepsilon)}) = \mathsf{per}(s^2) = |s|.$$

Similarly, $\mathsf{per}(x_{[c_0-99\varepsilon,c_1+99\varepsilon)}) = |\tilde{s}|$. Equation (4.21) follows.

We combine (4.21) with (4.19) to obtain that

$$s^{\mathbb{Z}}_{[-99\varepsilon,|\tau(z_0)|+99\varepsilon)} = \tilde{s}^{\mathbb{Z}}_{[-|\tau(z_0)|-99\varepsilon,99\varepsilon)} = x_{[c_0-99\varepsilon,c_1+99\varepsilon)}. \tag{4.22}$$

Being $|s|$ equal to $|\tilde{s}|$, we get that s and $\tilde{s}$ are conjugate. Moreover, since $s, \tilde{s} \in \mathcal{W}_\varepsilon$ and since $\mathcal{W}_\varepsilon$ contains at most one element of a given rotational class, we have that

$$s = \tilde{s}. \tag{4.23}$$

We use (4.23) to prove that Item (1) of the lemma holds. Observe that Equation (4.19) and (4.23) imply that $s^{\mathbb{Z}}_{[-99\varepsilon,99\varepsilon)} = (S^{|\tau(z_0)|}s)_{[-99\varepsilon,99\varepsilon)}$. This and the fact that $|s| \leq \varepsilon$ (as $s \in \mathcal{W}_\varepsilon$) allow us to use Item ((1)) in Proposition 4.10 and deduce that $s^{\mathbb{Z}} = S^{|\tau(z_0)|}s^{\mathbb{Z}}$. Item ((2)) of Proposition 4.10 then gives that $|\tau(z_0)| = 0 \pmod{|s|}$. We conclude that $x_{[c_0,c_1)}$ is a power of s and that $\mathsf{root}\,\tau(z_0) = \mathsf{root}\, s = s$. Item (1) of this lemma is a consequence of the last relation and (4.22). This also shows that if Item (2) of the lemma holds, then $\mathsf{root}\,\tau(z_0) \in \mathcal{W}_\varepsilon$. □

Lemma 4.27 *Let $x \in X$, $(f, z) = \mathbf{F}_{(Z,\tau)}(x)$ and $i, j \in \mathbb{Z}$ with $j > i + d$. Suppose that $|\tau(z_k)| \leq 401\varepsilon$ for all $k \in [i, j)$. Then:*

(1) $\mathsf{root}\,\tau(z_k) = \mathsf{root}\,\tau(z_i)$ *for all* $k \in [i, j)$ *and* $|\,\mathsf{root}\,\tau(z_i)| \leq \varepsilon$.

(2) $x_{[f_i-99\varepsilon,f_j+99\varepsilon)} = (\mathsf{root}\,\tau(z_i))^{\mathbb{Z}}_{[-99\varepsilon,|\tau(z_{[i,j)})|+99\varepsilon)}$.

Proof. Let $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$. We will use Lemma 4.16 with y and $[\phi_x(i), \phi_x(j))$ to prove the following:

$$\mathsf{per}(x_{[f_i-500\varepsilon,f_j+500\varepsilon)}) \text{ is at most } \varepsilon. \tag{4.24}$$

Let us check the hypothesis of Lemma 4.16. Let $k \in [\phi_x(i), \phi_x(j))$ be arbitrary. There exists $\ell \in [i, j)$ such that $k \in [\phi_x(\ell), \phi_x(\ell + 1))$. Putting the hypothesis $|\tau(z_k)| \leq 401\varepsilon$ in the inequality of Lemma 4.25 produces the bound $|\sigma(y_k)| \leq |\tau(z_\ell)| + 2 \cdot 401\varepsilon \leq 10^4\varepsilon$. Hence, by (4.5),

$$|\sigma(y_k)| < \varepsilon/d \text{ for all } k \in [\phi_x(i), \phi_x(j)). \tag{4.25}$$

Since $\varepsilon \leq n/10^4$, Equation (4.25) and the inequalities $\phi_x(j) \geq (j - i) + \phi_x(i) > d + \phi_x(i)$ allow us to use Lemma 4.16 and deduce that $\mathsf{per}(x_{[c_{\phi_x(i)}-n/3,c_{\phi_x(j)-d})}) \leq \varepsilon$. Now, observe that, for any $k \in \mathbb{Z}$,

$$c_{\phi_x(k)} - f_k = |u'_{\phi_x(k)}v'_{\phi_x(k)}| \in [n/2 - 401\varepsilon, n/2 + 401\varepsilon)$$

Hence, as (4.25) ensures that $c_{\phi_x(j)-d} \geq c_{\phi(j)} - \varepsilon$ and since $\varepsilon \leq n/10^4$, we have that $x_{[f_i-500\varepsilon, f_j+500\varepsilon)}$ occurs in $x_{[c_{\phi_x(i)}-n/3, c_{\phi_x(j)-d})}$. Therefore, (4.24) holds.

Next, we use (4.24) to prove the following:

$$\forall k \in [i,j], \text{ the decomposition } w_{\phi_x(k)} = v_{\phi_x(k)} u_{\phi_x(k)} u'_{\phi_x(k)} v'_{\phi_x(k)} \text{ satisfies } (\mathcal{P}_a) \text{ in Definition 4.5.} \quad (4.26)$$

Let $k \in [i,j]$. We note that (4.24) implies that $\mathsf{per}(x_{[c_k-500\varepsilon, c_k+500\varepsilon)}) \leq \varepsilon$. Thus, by Lemma 4.20, there exists $s \in \mathcal{W}_\varepsilon$ such that $s^{\mathbb{Z}}_{[-99\varepsilon, 99\varepsilon)}$ occurs in $x_{[c_k-500\varepsilon, c_k+500\varepsilon)}$. This implies that if $(\mathcal{P}_b)$ in Definition 4.5 holds for the decomposition $w_{\phi_x(k)} = v_{\phi_x(k)} u_{\phi_x(k)} u'_{\phi_x(k)} v'_{\phi_x(k)}$, then $s^{\mathbb{Z}}_{[-99\varepsilon, 99\varepsilon)}$ occurs in $u_{\phi_x(k)} u'_{\phi_x(k)} = x_{[c_k-500\varepsilon, c_k+500\varepsilon)}$, contradicting $(\mathcal{P}_b)$. Therefore, $w_{\phi_x(k)} = v_{\phi_x(k)} u_{\phi_x(k)} u'_{\phi_x(k)} v'_{\phi_x(k)}$ satisfies $(\mathcal{P}_a)$ and (4.26) is proved.

We now prove the properties in the statement of the lemma. Let $k \in [i,j)$. Then, Equations (4.24) and (4.26) imply that Item (1) in Lemma 4.26 is satisfied. Hence, for all $k \in [i,j)$,

$$|\operatorname{\mathsf{root}} \tau(z_k)| \leq \varepsilon \text{ and } x_{[c_k-99\varepsilon, c_{k+1}+99\varepsilon)} = (\operatorname{\mathsf{root}} \tau(z_k))^{\mathbb{Z}}_{[-99\varepsilon, |\tau(z_k)|+99\varepsilon)}. \quad (4.27)$$

In particular, we have for every $k \in [i, j-1)$ that

$$(\operatorname{\mathsf{root}} \tau(z_k))^{\mathbb{Z}}_{[0,99\varepsilon)} = x_{[c_k, c_k+99\varepsilon)} = x_{[c_{k+1}, c_{k+1}+99\varepsilon)} = (\operatorname{\mathsf{root}} \tau(z_{k+1}))^{\mathbb{Z}}_{[0,99\varepsilon)}.$$

This and the inequalities $|\operatorname{\mathsf{root}} \tau(z_k)| \leq \varepsilon$ and $|\operatorname{\mathsf{root}} \tau(z_{k+1})| \leq \varepsilon$ allow us to use Theorem 4.8 to deduce that $\operatorname{\mathsf{root}} \tau(z_k)$ and $\operatorname{\mathsf{root}} \tau(z_{k+1})$ are powers of a common word, and thus that $\operatorname{\mathsf{root}} \tau(z_k) = \operatorname{\mathsf{root}} \tau(z_{k+1})$. And inductive argument then yields Item (1) of this lemma, and therefore, by (4.27), that Item (2) holds as well. □

We have all the necessary elements to prove Proposition 4.18.

Proof of Proposition 4.18. We prove Item (1) by contradiction, Suppose that $0 \notin \mathrm{Q}_{\mathsf{p}}(z)$, $|\tau(z_0)| > 401\varepsilon$ and that $\mathsf{per}(x_{[c_0+97\varepsilon, c_1-97\varepsilon)})$ is at most ε. Let us write $w = w_{\phi_x(0)}$ and $\tilde{w} = w_{\phi_x(1)}$. Then, the condition $0 \notin \mathrm{Q}_{\mathsf{p}}(z)$ ensures that Item (1) in Lemma 4.26 does not hold. Hence, Item (2) does not hold either. This implies, as $\mathsf{per}(x_{[c_0+97\varepsilon, c_1-97\varepsilon)})$ at most ε, that one of the decompositions $w = v_w u_w u'_w v'_w$ or $\tilde{w} = v_{\tilde{w}} u_{\tilde{w}} u'_{\tilde{w}} v'_{\tilde{w}}$ satisfies $(\mathcal{P}_b)$ in Definition 4.5. We assume, without loss of generality, that $w = v_w u_w u'_w v'_w$ satisfies $(\mathcal{P}_b)$. Then, for any $s \in \mathcal{W}_\varepsilon$, $s^{\mathbb{Z}}_{[-99\varepsilon, 99\varepsilon)}$ does not occur in $u_w u'_w = x_{[c_0-401\varepsilon, c_0+401\varepsilon)}$. In particular, $s^{\mathbb{Z}}_{[-99\varepsilon, 99\varepsilon)}$ does not occur in $x_{[c_0+97\varepsilon, c_0+304\varepsilon)}$. Being this valid for all $s \in \mathcal{W}_\varepsilon$ and since $x_{[c_0+97\varepsilon, c_0+304\varepsilon)}$ has length at least 2ε, we deduce from Lemma 4.20 that $\mathsf{per}(x_{[c_0+97\varepsilon, c_0+304\varepsilon)}) > \varepsilon$. But $c_1 - c_0 = |\tau(z_0)| > 401\varepsilon$ so $x_{[c_0+97\varepsilon, c_0+304\varepsilon)}$ occurs in $x_{[c_0+97\varepsilon, c_1-97\varepsilon)}$ and thus $\mathsf{per}(x_{[c_0+97\varepsilon, c_1-97\varepsilon)}) > \varepsilon$. This contradicts our assumptions and thereby proves Item (1).

We continue with Item (2). The proof is by contradiction. We assume that the hypothesis of Item (2) holds and that $\mathsf{per}(x_{[c_0+97\varepsilon, c_1-97\varepsilon)}) \leq \varepsilon$. Let us further assume, without losing generality, that $1 \in \mathrm{Q}_{\mathsf{p}}(z)$. We will use the notation $w = w_{\phi_x(0)}$ and $\tilde{w} = w_{\phi_x(1)}$. Then, the condition $1 \in \mathrm{Q}_{\mathsf{p}}(z)$ is equivalent to Item (1) of Lemma 4.26 being satisfied by Sz;

hence, Item (2) of that lemma holds with Sz. In particular, $\tilde{w} = v_{\tilde{w}} u_{\tilde{w}} u'_{\tilde{w}} v'_{\tilde{w}}$ satisfies $(\mathcal{P}_a)$ in Definition 4.5, that is,

$$x_{[c_1-99\varepsilon,c_1+99\varepsilon)} = u_{\tilde{w}} u'_{\tilde{w}} = s^{\mathbb{Z}}_{[-99\varepsilon,99\varepsilon)} \text{ for some } s \in \mathcal{W}_\varepsilon.$$

Now, the condition $0 \notin \mathsf{Q}_\mathsf{p}(z)$ implies, by Lemma 4.26, that Item (2) of that lemma is not satisfied by z. This implies, since $\tilde{w} = v_{\tilde{w}} u_{\tilde{w}} u'_{\tilde{w}} v'_{\tilde{w}}$ satisfies $(\mathcal{P}_a)$ and since we assumed that $\mathsf{per}(x_{[c_0+97\varepsilon,c_1-97\varepsilon)}) \leq \varepsilon$, that $w = v_w u_w u'_w v'_w$ satisfies $(\mathcal{P}_b)$. Therefore,

$$x_{[c_1-99\varepsilon,c_1+99\varepsilon)} = s^{\mathbb{Z}}_{[-99\varepsilon,99\varepsilon)} \text{ does not occur in } x_{[c_0-500\varepsilon,c_0+500\varepsilon)}. \tag{4.28}$$

But, since $c_1 - c_0 = |\tau(z_0)| \leq 401\varepsilon$, we have that $[c_1 - 99\varepsilon, c_1 + 99\varepsilon)$ is contained in $[c_0 - 500\varepsilon, c_1 + 500\varepsilon)$, and thus that $x_{[c_1-99\varepsilon,c_1+99\varepsilon)}$ occurs in $x_{[c_0-500\varepsilon,c_1+500\varepsilon)}$. This contradicts (4.28), finishing the proof of Item (2).

Next, we consider Item (3). Assume that $k > d$ and that $|\tau(z_j)| \leq 401\varepsilon$ for all $j \in [0, k)$. Then, we can use Lemma 4.27 to deduce that

(1) $\mathsf{root}\,\tau(z_j) = \mathsf{root}\,\tau(z_0)$ for all $j \in [0, k)$ and $|\mathsf{root}\,\tau(z_0)| \leq \varepsilon$;

(2) $x_{[f_0-99\varepsilon,f_k+99\varepsilon)} = (\mathsf{root}\,\tau(z_0))^{\mathbb{Z}}_{[-99\varepsilon,|\tau(z_{[0,k)})|+99\varepsilon)}$.

In particular, $x_{[c_j-99\varepsilon,c_{j+1}+99\varepsilon)} = (\mathsf{root}\,\tau(z_j))^{\mathbb{Z}}_{[-99\varepsilon,|\tau(z_j)|+99\varepsilon)}$ and $|\mathsf{root}\,\tau(z_j)| \leq \varepsilon$ for all $j \in [0, k)$. We conclude that $[0, k) \subseteq \mathsf{Q}_\mathsf{p}(z)$.

Finally, we prove Item (4). Let $z' \in Z$ and assume that $0 \in \mathsf{Q}_\mathsf{p}(z) \cap \mathsf{Q}_\mathsf{p}(z')$ and that $\mathsf{root}\,\tau(z_0)$ is conjugate to $\mathsf{root}\,\tau(z'_0)$. The condition $0 \in \mathsf{Q}_\mathsf{p}(z) \cap \mathsf{Q}_\mathsf{p}(z')$ permits to use Lemma 4.26 to deduce that $\mathsf{root}\,\tau(z_0)$ and $\mathsf{root}\,\tau(z'_0)$ belong to $\mathcal{W}_\varepsilon$. Since $\mathsf{root}\,\tau(z_0)$ conjugate to $\mathsf{root}\,\tau(z'_0)$, the definition of $\mathcal{W}_\varepsilon$ ensures that $\mathsf{root}\,\tau(z_0) = \mathsf{root}\,\tau(z'_0)$. □

4.5 The second coding

We continue the proof of the main theorems. The main result of this section is Proposition 4.28, which describes a modification of the coding in Proposition 4.4. The principal new element in Proposition 4.28 is a period dichotomy for the words $\tau(a)$. This property is shared by the codings constructed in Sections 4.6 and 4.7, so we introduce it as a definition.

Definition 4.6 Let $(Z \subseteq \mathcal{C}^{\mathbb{Z}}, \tau \colon \mathcal{C} \to \mathcal{A}^+)$ be a recognizable coding of the subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, $\mathcal{C}_\mathsf{ap} \cup \mathcal{C}_\mathsf{p}$ be a partition of $\mathcal{C}$, and $\varepsilon \geq 1$. We say that (Z, τ) has *dichotomous periods* w.r.t. $(\mathcal{C}_\mathsf{ap}, \mathcal{C}_\mathsf{p})$ and ε if for $x \in X$ and $(c, z) = \mathbf{F}_{(Z,\tau)}(x)$ the following holds:

(1) $z_0 \in \mathcal{C}_\mathsf{ap}$ implies that $\mathsf{per}(x_{[c_0+\varepsilon,c_1-\varepsilon)}) > \varepsilon$.

(2) $z_0 \in \mathcal{C}_\mathsf{p}$ implies that $|\mathsf{root}\,\tau(z_0)| \leq \varepsilon$ and that $x_{[c_0-\varepsilon,c_1+\varepsilon)}$ is equal to $(\mathsf{root}\,\tau(z_0))^{\mathbb{Z}}_{[-\varepsilon,|\tau(z_0)|+\varepsilon)}$.

(3) If $a \in \mathcal{C}_\mathsf{p}$ and $\mathsf{root}\,\tau(z_0)$ is conjugate to $\mathsf{root}\,\tau(a)$, then $\mathsf{root}\,\tau(z_0) = \mathsf{root}\,\tau(a)$.

Proposition 4.28 *Let X be a minimal infinite subshift, $n \geq 0$ and let d be the maximum of $\lceil p_X(n)/n \rceil$, $p_X(n+1) - p_X(n)$, $\#\mathcal{A}$ and 10^4. There exist a recognizable coding $(Z \subseteq \mathcal{C}^{\mathbb{Z}}, \tau \colon \mathcal{C} \to \mathcal{A}^+)$ of X, a partition $\mathcal{C} = \mathcal{C}_{\mathsf{ap}} \cup \mathcal{C}_{\mathsf{p}}$, and $\varepsilon \in [n/d^{2d^3+4}, n/d)$ such that:*

(1) $\#\mathcal{C}_{\mathsf{ap}} \leq 2d^{3d+6}$, $\#\mathcal{C}_{\mathsf{p}} \leq 2d^{3d+9} \,\mathsf{pow\text{-}com}(X)$ and $\# \,\mathsf{root}\, \tau(\mathcal{C}) \leq 3d^{3d+9}$.

(2) $|\tau(a)| \leq 10d^2 n$ for $a \in \mathcal{C}_{\mathsf{ap}}$ and $|\tau(a)| \geq 80\varepsilon$ for $a \in \mathcal{C}$.

(3) (Z, τ) satisfies the recognizability property in Proposition 4.29.

(4) (Z, τ) has dichotomous periods w.r.t. $(\mathcal{C}_{\mathsf{ap}}, \mathcal{C}_{\mathsf{p}})$ and 8ε.

(5) The set $\mathcal{C}_{\mathsf{p}}$ satisfies the following: if $z \in Z$, then z_0 and z_1 does not simultaneously belong to $\mathcal{C}_{\mathsf{p}}$.

Proposition 4.29 *Consider the coding described in Proposition 4.38. Let $x, \tilde{x} \in X$ be such that $\mathsf{per}(x_{[-\varepsilon,\varepsilon)}) > \varepsilon$ and $x_{[-7d^2n,7d^2n)} = \tilde{x}_{[-7d^2n,7d^2n)}$. Then, $\mathbf{F}^0_{(Z,\tau)}(x)$ is equal to $\mathbf{F}^0_{(Z,\tau)}(\tilde{x})$.*

The strategy for proving Proposition 4.28 is as follows. We consider the coding (Y, σ) given by Proposition 4.17 and, for a point $y \in Y$, we glue together letters y_i to form words y_I, where I is an interval, in such a way that y_I corresponds either to a maximal periodic part of $\sigma(y)$ or to an aperiodic part of $\sigma(y)$ of controlled length. This will produce a new coding where the letters are in correspondence with the words y_I and that satisfies all the properties in Proposition 4.28 except for the lower bound in Item ((2)) for the letters associated to periodic parts y_I. We solve this by slightly moving the edges of the words $\sigma(y_I)$.

We start, in Subsection 4.5.1, by defining stable intervals, which correspond to the intervals I described in the last paragraph. The definition of the coding of Proposition 4.28 is given in Subsection 4.5.2, together with the proof of its basic properties. In the final subsection, we prove Propositions 4.28 and 4.29.

We fix the following notation for the rest of the section. Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a minimal infinite subshift, $n \geq 0$ and let d be the maximum of $\lceil p_X(n)/n \rceil$, $p_X(n+1) - p_X(n)$, $\#\mathcal{A}$ and 10^4. Then, Proposition 4.17 applied to X and n gives a recognizable coding $(Y \subseteq \mathcal{B}^{\mathbb{Z}}, \sigma \colon \mathcal{B} \to \mathcal{A}^+)$ of X and an integer $\varepsilon \in [n/d^{2d^3+4}, n/d)$.

4.5.1 Stable intervals

Let $y \in Y$, $x = \sigma(y)$ and $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$. We define

$$\mathrm{Q}_{\mathsf{short}}(y) = \{i \in \mathbb{Z} : |\sigma(y_i)| \leq 401\varepsilon\} \text{ and } \mathrm{Q}_{\mathsf{long}}(y) = \mathbb{Z} \setminus \mathrm{Q}_{\mathsf{long}}.$$

Let $\mathrm{Q}_{\mathsf{p}}(y)$ be the set of integers $i \in \mathbb{Z}$ such that

$$|\,\mathsf{root}\, \sigma(y_i)| \leq \varepsilon \text{ and } x_{[c_i - 99\varepsilon, c_{i+1}+99\varepsilon)} = (\mathsf{root}\, \sigma(y_i)^{\mathbb{Z}})_{[-99\varepsilon, |\tau(y_i)|+99\varepsilon)}. \tag{4.29}$$

We set $\mathrm{Q}_{\mathsf{ap}}(y) = \mathbb{Z} \setminus \mathrm{Q}_{\mathsf{p}}(y)$. Remark that the definition of $\mathrm{Q}_{\mathsf{p}}(y)$ coincides with the one in Proposition 4.18.

Definition 4.7 A *stable interval* for y is a finite interval $I = [i, j) \subseteq \mathbb{Z}$ satisfying one of the following conditions.

(1) $I \subseteq \mathsf{Q}_{\mathsf{p}}$.

(2) $I \subseteq \mathsf{Q}_{\mathsf{ap}}$, $\#(I \cap \mathsf{Q}_{\mathsf{long}}) \leq 1$, and if $i \in \mathsf{Q}_{\mathsf{short}}$ then $i - 1 \in \mathsf{Q}_{\mathsf{p}}$.

The interval I is of *periodic type* if it satisfies Item (1) of this definition, and of *aperiodic type* if it satisfies Item (2). We say that I is a *maximal stable interval* if for all stable interval $I' \supseteq I$ we have that $I' = I$.

Remark 4.5 We stress on the fact that the previous definition does not depend just on y, but also on σ and ε.

Lemma 4.30 *Let $I = [i, j)$ be a stable interval set for y of periodic type. Then:*

(1) $\mathsf{root}\,\sigma(y_{[i,j)}) = \mathsf{root}\,\sigma(y_k)$ *for all $k \in [i, j)$ and $x_{[c_i-99\varepsilon, c_j+99\varepsilon)}$ is equal to* $(\mathsf{root}\,\sigma(y_i))^{\mathbb{Z}}_{[-99\varepsilon, |\sigma(y_{[i,j)})|+99\varepsilon)}$.

(2) If I is maximal, $I' = [i', j')$ is a stable interval and either $i' = j$ or $j' = i$, then I' is of aperiodic type.

(3) If $y' \in Y$, $0 \in \mathsf{Q}_{\mathsf{p}}(y')$ and $\mathsf{root}\,\sigma(y_i)$ *is conjugate to* $\mathsf{root}\,\sigma(y'_0)$*, then* $\mathsf{root}\,\sigma(y_i) = \mathsf{root}\,\sigma(y'_0)$.

Proof. We first prove Item (1). Let $s_k = \mathsf{root}\,\sigma(y_k)$. Being I of periodic type, we have by Definition 4.7 that

$$|s_k| \leq \varepsilon \text{ and } x_{[c_k-99\varepsilon, c_{k+1}+99\varepsilon)} = (s_k^{\mathbb{Z}})_{[-99\varepsilon, |\sigma(y_k)|+99\varepsilon)} \text{ for all } k \in [i, j). \tag{4.30}$$

Then, for any $k \in [i, j-1)$,

$$(s_k^{\mathbb{Z}})_{[0,99\varepsilon)} = x_{[c_k, c_k+99\varepsilon)} = x_{[c_{k+1}, c_{k+1}+99\varepsilon)} = (s_{k+1}^{\mathbb{Z}})_{[0,99\varepsilon)}.$$

Combining this with the inequalities $|s_k|, |s_{k+1}| \leq \varepsilon$ and Theorem 4.8 produces a word t such that s_k and s_{k+1} are powers of t. Hence, as s_k and s_{k+1} are roots of a word, $s_k = s_{k+1} = t$ for any $k \in [i, j-1)$. Item (1) of the lemma follows from this and (4.30).

For Item (2), we note that if I' is of periodic type then $I \cup I'$ is an interval contained in $\mathsf{Q}_{\mathsf{p}}(y)$, and so $I \cup I'$ is a stable interval for y. This would contradict the maximality of I; therefore, I' is of aperiodic type.

Let us now assume that the hypothesis of Item (3) holds. Then, since $i \in I \subseteq \mathsf{Q}_{\mathsf{p}}(y)$, $0 \in \mathsf{Q}_{\mathsf{p}}(y')$ and $\mathsf{root}\,\sigma(y_i)$ is conjugate to $\mathsf{root}\,\sigma(y'_0)$, the points $S^i y$ and y' comply with the hypothesis of Item ((4)) of Proposition 4.18. We conclude tat $\mathsf{root}\,\sigma(y_i) = \mathsf{root}\,\sigma(y'_0)$. □

Lemma 4.31 *Let $y \in Y$ and $I = [i, j)$ be a stable interval for y of aperiodic type. Then,*

(1) $\mathsf{per}(x_{[c_i+97\varepsilon, c_j-97\varepsilon)}) > \varepsilon$*;*

(2) I has length at most $2d+1$;

(3) $195\varepsilon \le |\sigma(y_I)| \le 9d^2 n$.

Proof. We prove Item (1) by considering two cases. Suppose that $i \in \mathsf{Q}_{\mathsf{long}}(y)$. Then, $i \in \mathsf{Q}_{\mathsf{long}}(y) \cap \mathsf{Q}_{\mathsf{ap}}(y)$ and we can use Item ((1)) of Proposition 4.18 to obtain that $\mathsf{per}(x_{[c_i+97\varepsilon, c_{i+1}-97\varepsilon)}) > \varepsilon$. Assume now that $i \in \mathsf{Q}_{\mathsf{short}}(y)$. Then, Definition 4.7 ensures that $i-1 \in \mathsf{Q}_{\mathsf{p}}(y)$. Hence, Item ((2)) of Proposition 4.18 applies and so $\mathsf{per}(x_{[c_i+97\varepsilon, c_{i+1}-97\varepsilon)}) > \varepsilon$. In particular, Item (1) holds.

We prove Item (2) by contradiction. Assume that $\#I > 2d+1$. Then, it follows from Definition 4.7 that there exists $I' \subseteq I$ such that $\#I' > d$ and $I' \subseteq \mathsf{Q}_{\mathsf{short}}$. These conditions allow us to use Item ((3)) in Proposition 4.18 and deduce that $I' \subseteq \mathsf{Q}_{\mathsf{p}}(y)$, contradicting the fact that I is of aperiodic type.

Finally, we consider Item (3). Item (2) of this lemma and Item ((2)) produce that $|\sigma(y_I)| \le (2d+1)\cdot 3dn$, from which the upper bound in Item (3) follows. To prove the lower bound, we consider two cases. If there exists $k \in I \cap \mathsf{Q}_{\mathsf{long}}(y)$, then $|\sigma(y_I)| \ge |\sigma(y_k)| \ge 401\varepsilon$. Assume now that $I \cap \mathsf{Q}_{\mathsf{long}}(y)$ is empty. Then, $i \in \mathsf{Q}_{\mathsf{short}}(y)$, and so Definition 4.7 indicates that $i-1 \in \mathsf{Q}_{\mathsf{p}}(y)$. This allows us to use Item ((2)) of Proposition 4.18 to obtain that $\mathsf{per}(x_{[c_i+97\varepsilon, c_{i+1}-97\varepsilon)}) > \varepsilon$. In particular, $|x_{[c_i+97\varepsilon, c_{i+1}-97\varepsilon)}| > \varepsilon$; hence,

$$|\sigma(y_I)| \ge |\sigma(y_i)| = |x_{[c_i+97\varepsilon, c_{i+1}-97\varepsilon)}| + 2\cdot 97\varepsilon > 195\varepsilon.$$

□

Lemma 4.32 *There exists a constant C depending only on X such that for any $y \in Y$ and stable interval I for Y, we have that $\#I \le C$. In particular, any stable interval is contained in a maximal stable interval.*

Proof. Let C_0 be the length of the longest word w that occurs in some $x \in X$ such that $\mathsf{per}(w) \le \varepsilon$. We remark that C_0 is finite as X is assumed to be minimal and infinite. Let $C = \max\{C_0, 2d+1\}$. We claim that for any $y \in Y$, any stable interval I for y has length at most C. Indeed, if I is of aperiodic type, then Item (3) of Lemma 4.31 implies that $\#I \le 2d+1 \le C$, and if I is of periodic type, then Item (1) of Lemma 4.30 ensures that $\mathsf{per}(\sigma(y_I)) \le \varepsilon$, and thus that $\#I \le |\sigma(y_I)| \le C_0 \le C$. □

Lemma 4.33 *Let $y \in Y$. Then, the set of all maximal stable intervals for y is a partition of $\mathbb{Z}$.*

Proof. We first prove that any $k \in \mathbb{Z}$ is contained in a stable interval. This would imply that any k is contained in a maximal stable interval by Lemma 4.32.

Let $k \in \mathbb{Z}$ be arbitrary. We consider two cases. If $k \in \mathsf{Q}_{\mathsf{p}}(y)$ or $k \in \mathsf{Q}_{\mathsf{ap}}(y) \cap \mathsf{Q}_{\mathsf{long}}(y)$, then $\{k\}$ is stable interval and we are finished. Suppose now that $k \in \mathsf{Q}_{\mathsf{ap}}(y) \cap \mathsf{Q}_{\mathsf{short}}(y)$. Let $i < k$ be the biggest integer such that $i \notin \mathsf{Q}_{\mathsf{ap}}(y) \cap \mathsf{Q}_{\mathsf{short}}(y)$. Note that $[i+1, k] \subseteq \mathsf{Q}_{\mathsf{ap}}(y) \cap \mathsf{Q}_{\mathsf{short}}(y)$.

Hence, if $i \in \mathsf{Q}_{\mathsf{ap}}(y) \cap \mathsf{Q}_{\mathsf{long}}(y)$, then $[i, k]$ is stable interval of aperiodic type, and if $i \in \mathsf{Q}_{\mathsf{p}}$, then $[i+1, k]$ is stable interval of aperiodic type. These are the only cases as $i \notin \mathsf{Q}_{\mathsf{ap}}(y) \cap \mathsf{Q}_{\mathsf{short}}(y)$, and so we conclude that i belongs to a stable interval.

Next, we prove that for any maximal stable intervals I, I', either $I = I'$ or $I \cap I' = \emptyset$. The lemma follows from this and the fact that any k is contained in a maximal stable interval.

Let $I = [i, j)$ and $I' = [i', j')$ be maximal stable intervals with nonempty intersection. There is no loss of generality in assuming that $i \leq i' < j \leq j'$. Note that if $i = i'$ or $j = j'$, then $I \cup I' \in \{I, I'\}$, so $I = I' = I \cup I'$ by maximality. Hence, we may assume that $i < i' < j < j'$. Remark that this implies that $j - 1 \in I \cap I'$.

In order to continue, we consider three cases.

(1) If $j - 1 \in \mathsf{Q}_{\mathsf{p}}(y)$, then, as $j - 1 \in I \cap I'$, Definition 4.7 implies that I and I' are of periodic type. It then follows from Definition 4.7 that $I \cup I'$ is stable interval of periodic type, and so $I = I' = I \cup I'$ by maximality.

(2) If $j - 1 \in \mathsf{Q}_{\mathsf{ap}}(y) \cap \mathsf{Q}_{\mathsf{long}}$. Then, since $j - 1 \in I \cap I'$, Definition 4.7 ensures that $[i, j-1) \cup [j+1, j') \subseteq \mathsf{Q}_{\mathsf{ap}}(y) \cap \mathsf{Q}_{\mathsf{short}}(y)$. Hence, $[i, j') = I \cup I'$ is a stable interval of aperiodic type, which implies that $I = I' = I \cup I'$ by maximality.

(3) If $j - 1 \in \mathsf{Q}_{\mathsf{ap}}(y) \cap \mathsf{Q}_{\mathsf{short}}(y)$. Then, as $j - 1 \in I \cap I'$, Definition 4.7 guarantees that I and I' are of aperiodic type. In particular, as $i' - 1 \in I$, $i' - 1 \in \mathsf{Q}_{\mathsf{ap}}(y)$ and therefore, by Definition 4.7, $i' \in \mathsf{Q}_{\mathsf{long}}(y)$. We conclude, using Definition 4.7, that $[i, i') \subseteq \mathsf{Q}_{\mathsf{ap}}(y) \cap \mathsf{Q}_{\mathsf{short}}(y)$, $i' \in \mathsf{Q}_{\mathsf{ap}}(y) \cap \mathsf{Q}_{\mathsf{long}}(y)$ and that $[i'+1, j') \subseteq \mathsf{Q}_{\mathsf{ap}}(y) \cap \mathsf{Q}_{\mathsf{short}}(y)$. Hence, $[i, j') = I \cup I'$ is a stable interval of aperiodic type and $I = I' = I \cup I'$ by maximality.

□

4.5.2 Construction of the second coding

The coding (Z, τ) is obtained by modifying the cut function c in $\mathbf{F}_{(Y,\sigma)}(x)$ of the points $x \in X$. We give the construction of the modified cut function as the proof of the following lemma, and we define (Z, τ) right after.

Lemma 4.34 *Let $x \in X$ and set $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$. There exist unique increasing sequences of integers satisfying $(k_x(j))_{j \in \mathbb{Z}}$ and $(r_x(j))_{j \in \mathbb{Z}}$ satisfying the following conditions.*

(1) $\{[k_x(j), k_x(j+1)) : j \in \mathbb{Z}\}$ is the set of all maximal stable intervals of y.

(2) For any $j \in \mathbb{Z}$,

(1) if $[k_x(j), k_x(j+1))$ is of aperiodic type, then $r_x(j) = c_{k_x(j)}$.

(2) if $[k_x(j), k_x(j+1))$ is of periodic type, then $r_x(j) = c_{k_x(j)} - |s^\ell|$, where $s = \mathsf{root}\,\sigma(y_{[k_x(j), k_x(j+1))})$ and $\ell = \lceil 80\varepsilon / |s| \rceil$

(3) 0 belongs to $[r_x(0), r_x(1))$.

Moreover, in this case, $r_x(j+1) \geq r_x(j) + 80\varepsilon$ for all $j \in \mathbb{Z}$.

Proof. Lemma 4.33 ensures that the set of all maximal stable intervals of y can be described as $\{[k(j), k(j+1)) : j \in \mathbb{Z}\}$ for some increasing sequence of integers $(k_j)_{j\in\mathbb{Z}}$. The sequence $(k_j)_{j\in\mathbb{Z}}$ is unique up to an index shift.

We define $r(j)$ as follows:

(i) if $[k(j), k(j+1))$ is of aperiodic type, then $r(j) = c_{k(j)}$.

(ii) if $[k(j), k(j+1))$ is of periodic type, then $r(j) = c_{k(j)} - |s^\ell|$, where $s = \mathsf{root}\,\sigma(y_{[k(j),k(j+1))})$ and $\ell = \lceil 80\varepsilon/|s| \rceil$

It is important to remark that, in case (ii), Lemma 4.30 ensures that $|s| \leq \varepsilon$.

We claim that

$$r(j+1) \geq r(j) + 80\varepsilon \text{ for all } j \in \mathbb{Z}. \tag{4.31}$$

First, we note that the definition of $r(j)$ and $r(j+1)$ guarantees that

$$k(j) - 81\varepsilon < r(j) \leq k(j) \text{ and } k(j+1) - 81\varepsilon < r(j+1) \leq k(j+1). \tag{4.32}$$

We now consider two cases. If $[k(j), k(j+1))$ is of aperiodic type, then, by Lemma 4.31, $|\sigma(y_{[k(j),k(j+1))})|$ is at least 195ε. Combining this with (4.32) yields

$$|x_{[r(j),r(j+1))}| \geq |x_{[c_{k(j)},c_{k(j+1)})}| - 81\varepsilon = |\sigma(y_{[k(j),k(j+1))})| - 81\varepsilon \geq 80\varepsilon$$

Assume now that $[k(j), k(j+1))$ is of periodic type. Then, $[k(j+1), k(j+2))$ is of aperiodic type by Lemma 4.30. In particular, $r(j+1) = c_{k(j+1)}$ by (i). Also, since $[k(j), k(j+1))$ is of periodic type, (ii) ensures that $r(j) \leq k(j) - 80\varepsilon$. These two things imply that $|x_{[r(j),r(j+1))}| \geq |x_{[c_{k(j)}-80\varepsilon, c_{k(j+1)})}| \geq 80\varepsilon$, completing the proof of the claim.

Equation (4.31) implies that $(r(j))_{j\in\mathbb{Z}}$ is increasing. Thus, there exists a unique $\ell \in \mathbb{Z}$ such that $0 \in [r(\ell), r(\ell+1))$. We define $k_x(j) = k(j+\ell)$ and $r_x(j) = r(j+\ell)$. Then, $(k_x(j))_{j\in\mathbb{Z}}$ and $(r_x(j))_{j\in\mathbb{Z}}$ satisfy Items (1), (2) and (3) of the lemma. Moreover, being $(r(j))_{j\in\mathbb{Z}}$ increasing, it is clear ℓ (and then also $(k_x(j))_{j\in\mathbb{Z}}$ and $(r_x(j))_{j\in\mathbb{Z}}$) is unique. □

We now define (Z, τ). It follows from the recognizability property of (Y, σ) and Lemma 4.32 that the map $x \mapsto r_x(0)$ is continuous. In particular, $U = \{x \in X : r_x(0) = 0\}$ is clopen (and nonempty). We define $(Z \subseteq \mathcal{C}^{\mathbb{Z}}, \tau\colon \mathcal{C} \to \mathcal{A}^+)$ as the recognizable coding of X obtained from U as in Proposition 4.4.

4.5.3 Basic properties of the second coding

We fix, for the rest of the section, the following notation. Let x denote an element of X, $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$ and $(f, z) = \mathbf{F}_{(Z,\tau)}(x)$. We also define $(k_x(j))_{j\in\mathbb{Z}}$ and $(r_x(j))_{j\in\mathbb{Z}}$ as the sequences given by Lemma 4.34.

Lemma 4.35 *We have that $r_x(j) = f_j$ for all $j \in \mathbb{Z}$.*

Proof. Note that $\{k \in \mathbb{Z} : S^k x \in U\}$ is equal to $\{r_x(j) : j \in \mathbb{Z}\}$. Then, by Item (1) in Proposition 4.4, there exists a bijective map $g\colon \mathbb{Z} \to \mathbb{Z}$ such that $r_j(x) = f_{g(j)}$ for all $j \in \mathbb{Z}$. Now, since Lemma 4.34 states that $r_x(j) < r_x(j+1)$ for all $j \in \mathbb{Z}$, the map g is increasing. As it is also bijective, we conclude that there exists $\ell \in \mathbb{Z}$ satisfying $g(j) = j + \ell$ for all $j \in \mathbb{Z}$. Finally, by Item (3) in Lemma 4.34 and the definition of f, we have that $0 \in [f_0, f_1)$ and $0 \in [r_x(0), r_x(1)) = [f_{g^{-1}(0)}, f_{g^{-1}(0)+1})$. Hence, $g^{-1}(0) = 0$, $\ell = 0$ and the lemma follows. □

The last lemma allows us to drop the notation $r_x(j)$ and use only f_j. In particular, Items (2) and (3) of Lemma 4.34 hold with f_j.

We define a partition $\mathcal{C}_{\mathsf{a}} \cup \mathcal{C}_{\mathsf{ap}}$ of $\mathcal{C}$ as follows:

$$\mathcal{C}_{\mathsf{ap}} = \{a \in \mathcal{C} : \mathsf{per}(\tau(a)) > \varepsilon\} \text{ and } \mathcal{C}_{\mathsf{p}} = \{a \in \mathcal{C} : \mathsf{per}(\tau(a)) \leq \varepsilon\}.$$

Lemma 4.36 *Let $j \in \mathbb{Z}$. The following are equivalent:*

(1.a) $z_j \in \mathcal{C}_{\mathsf{p}}$.

(1.b) $[k_x(j), k_x(j+1))$ *is of periodic type for* y.

(1.c) Let $s = \mathsf{root}\,\tau(z_j)$. *Then,* $|s| \leq \varepsilon$, $x_{[f_j - 8\varepsilon, f_{j+1}+8\varepsilon)} = s^{\mathbb{Z}}_{[-8\varepsilon, |\tau(z_j)|+8\varepsilon)}$ *and* $s = \mathsf{root}\,\sigma(y_{[k_x(j), k_x(j+1))})$.

The following are also equivalent:

(2.a) $z_j \in \mathcal{C}_{\mathsf{ap}}$.

(2.b) $[k_x(j), k_x(j+1))$ *is of aperiodic type for* y.

(2.c) $\mathsf{per}(x_{[f_j+8\varepsilon, f_{j+1}-8\varepsilon)}) > \varepsilon$.

Proof. We start with a general observation. Let us write $k(j) = k_x(j)$. Then, Item ((2)) in Lemma 4.34 ensures that

$$c_{k(j)} - 81\varepsilon < f_j \leq c_{k(j)} \text{ and } c_{k(j+1)} - 81\varepsilon < f_{j+1} \leq c_{k(j+1)}.$$

Hence,

$$\emptyset \neq [c_{k(j)}, c_{k(j+1)} - 81\varepsilon) \subseteq [f_j, f_{j+1}) \subseteq [c_{k(j)} - 81\varepsilon, c_{k(j+1)} + 81\varepsilon) \tag{4.33}$$

We now prove the lemma. Let us assume that (1.a) holds. Then, (4.33) implies that $\mathsf{per}(x_{[c_{k(j)}+97\varepsilon, c_{k(j+1)}-97\varepsilon)}) \leq \mathsf{per}(x_{[f_j, f_{j-1})}) \leq \varepsilon$. Hence, by Lemma 4.31, $[k(j), k(j+1))$ is not of aperiodic type, that is, $[k(j), k(j+1))$ is of periodic type.

Assume next (1.b). Then, Lemma 4.30 states that $s = \mathsf{root}\,\sigma(y_{[k(j),k(j+1))})$ satisfies $|s| \leq \varepsilon$, $s = \mathsf{root}\,\sigma(y_k)$ for all $k \in [k(j), k(j+1))$ and

$$x_{[c_{k(j)}-99\varepsilon, c_{k(j+1)}+99\varepsilon)} = s^{\mathbb{Z}}_{[-99\varepsilon, c_{k(j+1)}-c_{k(j)}+99\varepsilon)}. \tag{4.34}$$

Moreover, Item (2) in Lemma 4.30 guarantees that $[k(j+1), k(j+2))$ is of aperiodic type. Hence, by Item ((2)) in Lemma 4.34, $f_j = c_{k(j)} - |s^\ell|$ and $f_{j+1} = c_{k(j+1)}$, where $\ell = \lceil 80\varepsilon/|s| \rceil$. We can then compute, thanks to (4.34) and (4.33),

$$x_{[f_j-8\varepsilon, f_{j+1}+8\varepsilon)} = s^{\mathbb{Z}}_{[-|s^\ell|-8\varepsilon, c_{k(j+1)}-c_{k(j+1)}+8\varepsilon)}$$
$$= s^{\mathbb{Z}}_{[-8\varepsilon, |s^\ell|+c_{k(j+1)}-c_{k(j+1)}+8\varepsilon)} = s^{\mathbb{Z}}_{[-8\varepsilon, f_{j+1}-f_j+8\varepsilon)}.$$

Note that the last computation also shows that $\tau(z_j) = x_{[f_j, f_{j+1})}$ is equal to $s^{\mathbb{Z}}_{[-|s^\ell|, c_{k(j+1)}-c_{k(j+1)})} = s^{\mathbb{Z}}_{[-|s^\ell|, 0)}\sigma(y_{[k(j),k(j+1))})$. Hence, $\mathsf{root}\,\tau(z_j) = \mathsf{root}\,s = s$ and we have proved (1.c).

Observe that if (1.c) holds, then $\mathsf{per}(\tau(z_j)) \leq |s| \leq \varepsilon$ and $z_j \in \mathcal{C}_\mathsf{p}$ by the definition of $\mathcal{C}_\mathsf{p}$.

We now assume (2.a). Then, Equation (4.33) implies that $\mathsf{per}(x_{[f_j, f_{j+1})}) \geq \mathsf{per}(x_{[c_{k(j)}, c_{k(j+1)}-81\varepsilon)}) > \varepsilon$. Hence, by Lemma 4.30, $[k(j), k(j+1))$ is not of periodic type, that is, $[k(j), k(j+1))$ is of aperiodic type.

Let us suppose that (2.b) holds. In this case, Lemma 4.31 and (4.33) allows us to compute

$$\mathsf{per}(x_{[f_j+8\varepsilon, f_{j+1}-8\varepsilon)}) \geq \mathsf{per}(x_{[c_{k(j)}+97\varepsilon, c_{k(j+1)}-97\varepsilon)}) > \varepsilon.$$

Finally, if (2.c) is satisfied, then $\mathsf{per}(\tau(z_j)) \geq \mathsf{per}(x_{[f_j+8\varepsilon, f_{j+1}-8\varepsilon)}) > \varepsilon$. □

Lemma 4.37 *Suppose that $z_{-1}z_0z_1 \in \mathcal{C}_\mathsf{ap}\mathcal{C}_\mathsf{p}\mathcal{C}_\mathsf{ap}$. Then, there exists a decomposition $\tau(z_{-1}z_0z_1) = us^m u'$ such that:*

(1) $\varepsilon \leq |u| \leq |\tau(z_{-1})| - 2\varepsilon$ and $\varepsilon \leq |u'| \leq |\tau(z_1)| - 2\varepsilon$.

(2) $s \in \mathsf{root}\,\sigma(\mathcal{B})$.

(3) s is not a suffix of u and is not a prefix of u'.

Proof. Let us denote $k(j) = k_x(j)$. We define $s = \mathsf{root}\,\tau(z_0)$. Then, as $z_0 \in \mathcal{C}_\mathsf{p}$, Lemma 4.36 ensures that $|s| \leq \varepsilon$, $x_{[f_0-8\varepsilon, f_1+8\varepsilon)} = s^{\mathbb{Z}}_{[-8\varepsilon, f_1-f_0+8\varepsilon)}$, $s = \mathsf{root}\,\sigma(y_{[k(0),k(1))})$ and that $[k(0), k(1))$ is of periodic type in y. Thus, by Lemma 4.30, $s = \mathsf{root}\,\sigma(y_{k(0)}) \in \sigma(\mathcal{B})$. In particular, s satisfies Item (2) of this lemma.

Now, we can find an interval $I = [i, j)$ containing $[f_0 - 8\varepsilon, f_1 + 8\varepsilon)$ such that $x_I = s^{\mathbb{Z}}_{[i-f_0, j-f_0)}$ and that no other interval strictly containing I satisfies the same properties. We observe that $i \geq f_{-1} + 8\varepsilon$ as, otherwise, $\mathsf{per}(x_{[f_{-1}+8\varepsilon, f_0-8\varepsilon)}) \leq \varepsilon$, contradicting the fact that, since $z_{-1} \in \mathcal{C}_\mathsf{ap}$, Item ((2.c)) Lemma 4.36 holds. Similarly, $j \leq f_2 - 8\varepsilon$. From these two things and the fact that I contains $[f_0 - 8\varepsilon, f_1 + 8\varepsilon)$ we obtain that

$$f_{-1} + 8\varepsilon \leq i \leq f_0 - 8\varepsilon \text{ and } f_1 + 8\varepsilon \leq j \leq f_2 - 8\varepsilon. \tag{4.35}$$

This allows us to write $x_{[f_{-1}, f_1)} = us^m v$, where $|u| \in [i - f_{-1}, i - f_{-1} + |s|)$, $m \geq 0$ and $|v| \in [f_2 - j, f_2 - j + |s|)$.

We have from (4.35) that $|u| \geq i - f_{-1} \geq 8\varepsilon$ and $|v| \geq f_2 - j \geq 8\varepsilon$. Moreover, as $|s| \leq \varepsilon$, $|u| \leq f_0 - f_{-1} - 7\varepsilon = |\tau(a)| - 7\varepsilon$ and $|v| \leq f_2 - f_1 - 7\varepsilon = |\tau(a')| - 7\varepsilon$. This proves that Item (1) of the lemma holds. Item (3) follows from the fact that $|s| \leq \varepsilon$, (4.35) and the maximality of I. □

4.5.4 Proof of Propositions 4.28 and 4.29

We are ready to prove the main results of this section.

Proof of Proposition 4.29. Let $x, \tilde{x} \in X$ be such that $\mathsf{per}(x_{[-\varepsilon,\varepsilon)}) > \varepsilon$ and $x_{[-7d^2n,7d^2n)} = \tilde{x}_{[-7d^2n,7d^2n)}$. We define $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$, $(\tilde{c}, \tilde{y}) = \mathbf{F}_{(Y,\sigma)}(\tilde{x})$, $(f, z) = \mathbf{F}_{(Z,\tau)}(x)$ and $(\tilde{f}, \tilde{z}) = \mathbf{F}_{(Z,\tau)}(\tilde{x})$. Let $k(j) = k_x(j)$ and $\tilde{k}(j) = k_{\tilde{x}}(j)$ be the sequences from Lemma 4.34. With this notation, we have to prove that $f_0 = \tilde{f}_0$ and $z_0 = \tilde{z}_0$.

Note that $\mathsf{per}(x_{[f_0-8\varepsilon,f_1+8\varepsilon)}) \geq \mathsf{per}(x_{[-\varepsilon,\varepsilon)}) > \varepsilon$. Thus, by Lemma 4.36, $[k(0), k(1))$ is of aperiodic type in y.

We claim that

$$[c_{k(0)-1} - 3dn, c_{k(1)+1} + 3dn) \text{ is contained in } [-7d^2n, 7d^2n) \tag{4.36}$$

Note that, by Items ((2)) and ((3)) in Lemma 4.34, $c_{k(0)} \leq f_0 + 81\varepsilon \leq 81\varepsilon$ and $c_{k(1)} \geq f_1 \geq 0$. Hence, $c_{k(1)+1} \leq c_{k(0)} + (k(1) - k(0) + 1)|\sigma| \leq 81\varepsilon + (k(1) - k(0) + 1)|\sigma|$ and $c_{k(0)-1} \geq c_{k(1)} - (k(1) - k(0) + 1)|\sigma| \geq -(k(1) - k(0) + 1)|\sigma|$. Since, by Lemma 4.31, $[k(0), k(1))$ has at most $2d + 1$ elements, and since $|\sigma| \leq 3dn$ by Item ((2)) in Proposition 4.17, we obtain that $c_{k(1)+1} + 3dn \leq 7d^2n$ and $c_{k(0)-1} - 3dn \geq -7d^2n$. This shows (4.36).

Thanks to (4.36), we can use the fact that (Y, σ) is $3dn$-recognizable (Item ((3)) of Proposition 4.17) to deduce that

$$c_k = \tilde{c}_k \text{ and } y_k = \tilde{y}_k \text{ for all } k \in [k(0) - 1, k(1) + 1). \tag{4.37}$$

We now observe that (4.36) and the hypothesis guarantees that

$$x_{[c_k-99\varepsilon,c_{k+1}+99\varepsilon)} = \tilde{x}_{[\tilde{c}_k-99\varepsilon,\tilde{c}_{k+1}+99\varepsilon)} \text{ for every } k \in [k(0) - 1, k(1) + 1). \tag{4.38}$$

Thus, for any such k, $k \in \mathsf{Q}_{\mathsf{p}}(y)$ if and only if $\mathsf{Q}_{\mathsf{p}}(\tilde{y})$. It is then not difficult to verify, using the definition of stable interval, that if $i \in \{0, 1\}$ then

(I) $k(i) = \tilde{k}(i)$;

(II) the type of $[k(i), k(i + 1))$ in y and the type of $[\tilde{k}(i), \tilde{k}(i + 1))$ in $\tilde{y}$ are equal.

Then, (I) and (4.38) imply that

$$x_{[k(0)-99\varepsilon,k(1)+99\varepsilon)} = \tilde{x}_{[\tilde{k}(0)-99\varepsilon,\tilde{k}(1)+99\varepsilon)}. \tag{4.39}$$

We claim that

$$f_0 = \tilde{f}_0 \text{ and } f_1 = \tilde{f}_1. \tag{4.40}$$

Let $i \in \{0,1\}$. We consider two cases. First, we assume that $[k(i), k(i+1))$ is of aperiodic type in y. Then, then by (II), $[\tilde{k}(i), \tilde{k}(i+1))$ is of aperiodic type in $\tilde{y}$. Hence, by Item ((2)) in Lemma 4.34, $f_i = c_{k(i)}$ and $\tilde{f}_i = \tilde{c}_{\tilde{k}(i)}$. This gives $f_i = \tilde{f}_i$ by (I) and (4.37).

Next, we assume that $[k(i), k(i+1))$ is of periodic type in y. Then, $[\tilde{k}(i), \tilde{k}(i+1))$ is of periodic type in $\tilde{y}$ by (II). Hence, by Lemma 4.30, $s = \mathsf{root}\,\sigma(y_{[k(i),k(i+1))})$ and $\tilde{s} = \mathsf{root}\,\sigma(\tilde{y}_{[\tilde{k}(i),\tilde{k}(i+1))})$ satisfy $|s|, |\tilde{s}| \leq \varepsilon$, $x_{[c_{k(i)}, c_{k(i)}+2\varepsilon)}$ is equal to $s^{\mathbb{Z}}_{[0,2\varepsilon)}$ and $\tilde{x}_{[\tilde{c}_{\tilde{k}(i)}, \tilde{c}_{\tilde{k}(i)}+2\varepsilon)}$ is equal to $\tilde{s}^{\mathbb{Z}}_{[0,2\varepsilon)}$. In this situation, (I) and (4.38) ensures that

$$s^{\mathbb{Z}}_{[0,2\varepsilon)} = x_{[c_{k(i)}, c_{k(i)}+2\varepsilon)} = \tilde{x}_{[\tilde{c}_{\tilde{k}(i)}, \tilde{c}_{\tilde{k}(i)}+2\varepsilon)} = \tilde{s}^{\mathbb{Z}}_{[0,2\varepsilon)}.$$

Since $|s|, |\tilde{s}| \leq \varepsilon$, this allows us the use of Theorem 4.8 and deduce that $s = \tilde{s}$. Putting this and the fact that $[k(i), k(i+1))$ and $[\tilde{k}(i), \tilde{k}(i+1))$ are of periodic type in Item ((2)) of Lemma 4.34 produces $f_i = c_{k(i)} - |s^\ell|$ and $\tilde{f}_i = \tilde{c}_{\tilde{k}(i)} - |s^\ell|$, where $\ell = \lceil 80\varepsilon/|s| \rceil$. Therefore, as $c_{k(i)} = \tilde{c}_{\tilde{k}(i)}$ by (I) and (4.37), $f_i = \tilde{f}_i$. This completes the proof of (4.40).

Finally, we show that $z_0 = \tilde{z}_0$. Item (2) in Lemma 4.34 gives that $|f_i - c_{k(i)}| \leq 81\varepsilon$. Hence, by (4.39) and (4.40), $x_{[f_0,f_1)} = \tilde{x}_{[f_0,f_1)} = \tilde{x}_{[\tilde{f}_0,\tilde{f}_1)}$. We conclude that $\tau(z_0) = \tau(\tilde{z}_0)$, and therefore that $z_0 = \tilde{z}_0$ as τ is injective on letters by Proposition 4.4. □

We end this section with the proof of Proposition 4.28.

Proof of Proposition 4.28. Let $x \in X$, $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$ and $(f, z) = \mathbf{F}_{(Z,\tau)}(x)$. Let $k(j) = k_x(j)$ be the sequence from Lemma 4.34.

We start with Item (ii). Let $a \in \mathcal{C}_{\mathsf{ap}}$. By minimality, there exists $j \in \mathbb{Z}$ such that $z_j = a$. We compute as follows:

$$|\tau(z_j)| = |f_{j+1} - f_j| \leq |f_{j+1} - c_{k(j+1)}| + (k(j+1) - k(j))|\sigma| + |f_j - c_{k(j)}|. \tag{4.41}$$

On one hand, we have by Item (2) in Lemma 4.34 that $|f_{j+1} - c_{k(j+1)}|$ and $|f_j - c_{k(j)}|$ are at most 81ε. On the other hand, since $z_j \in \mathcal{C}_{\mathsf{ap}}$, Lemma 4.36 ensures that $[k(j), k(j+1))$ is of aperiodic type in y. Hence, by Lemma 4.31, $\#[k(j), k(j+1)) \leq 2d+1$. Putting these two things in (4.41) yields $|\tau(z_j)| \leq 2 \cdot 81\varepsilon + (2d+1)|\sigma| \leq 10d^2 n$.

Let now $a \in \mathcal{C}$ and $j \in \mathbb{Z}$ be such that $z_j = a$. Then, by Lemma 4.34, $|\tau(a)| = f_{j+1} - f_j \geq 80\varepsilon$.

Next, we consider Item (i) and the inequality $\#\mathcal{C}_{\mathsf{ap}} \leq 2d^{3d+6}$. Observe that Lemmas 4.30 and 4.31 ensure that

$$|\sigma(y_{[k(j),k(j+1))})| \geq p\varepsilon \text{ for all } j \in \mathbb{Z}. \tag{4.42}$$

This allows us to define u_j as the prefix of $\sigma(y_{[k(j),k(j+1))})$ of length 2ε.

We claim that

if $[k(j), k(j+1))$ is of periodic type, then

$$\mathsf{root}\,\sigma(y_{[k(j),k(j+1))}) \text{ is the prefix of } u_j \text{ of length } \mathsf{per}(u_j). \tag{4.43}$$

Let us suppose that $[k(j), k(j+1))$ is of periodic type. Then, Lemma 4.30 states that $s = \mathsf{root}\,\sigma(y_{[k(j),k(j+1))})$ satisfies $|s| \le \varepsilon$ and $x_{[c_{k(j)}-99\varepsilon, c_{k(j+1)}+99\varepsilon)} = s^{\mathbb{Z}}_{[-99\varepsilon, |\sigma(y_{[k(j),k(j+1))})|+99\varepsilon)}$. In particular, since $u_j = x_{[c_{k(j)}, c_{k(j)}+2\varepsilon)}$, $s^{\mathbb{Z}}_{[0,2\varepsilon)} = u_j$. Being $|s| \le \varepsilon$, we obtain that s^2 is a prefix of u_j and that $\mathsf{per}(u_j) \le |s|$. This permits to use Item ((1)) in Proposition 4.13 to obtain that $\mathsf{per}(s^2) = \mathsf{per}(u_j)$. Moreover, $|s| = \mathsf{per}(s^2)$ by Lemma 4.9; therefore, $|s| = \mathsf{per}(s^2) = \mathsf{per}(u_j)$. Since $s^{\mathbb{Z}}_{[0,2\varepsilon)} = u_j$, this shows that s is the prefix of u_j of length $\mathsf{per}(u_j)$, completing the proof of the claim.

We now use (4.43) to prove the following:

$$\text{if } z_j \in \mathcal{C}_{\mathsf{ap}}, \text{ then } \tau(z_j) \text{ is uniquely determined by } \sigma(y_{[k(j),k(j+1))}) \text{ and wheter } z_{j+1} \text{ belongs to } \mathcal{C}_{\mathsf{ap}}. \quad (4.44)$$

Suppose that $z_j \in \mathcal{C}_{\mathsf{ap}}$. We consider two cases. If $z_{j+1} \in \mathcal{C}_{\mathsf{ap}}$, then Lemma 4.36 ensures that $[k(j), k(j+1))$ and $[k(j+1), k(j+2))$ are of aperiodic type in y, and so, by Item ((2)) in Lemma 4.34, that $\tau(z_j) = \sigma(y_{[k(j),k(j+1))})$. If $z_{j+1} \in \mathcal{C}_{\mathsf{ap}}$, then Lemma 4.36 ensures that $[k(j), k(j+1))$ is of aperiodic type and that $[k(j+1), k(j+2))$ is of periodic type. Hence, by ((2)) in Lemma 4.34, $\tau(z_j) = x_{[c_{k(j)}, c_{k(j+1)}-|s^\ell|)}$, where $s = \mathsf{root}\,\sigma(y_{[k(j+1),k(j+2))})$ and $\ell = \lceil 80\varepsilon/|s| \rceil$. Now, (4.43) says that s is determined by $\sigma(y_{[k(j),k(j+1))})$, and the definition of ℓ depends only on s. Therefore, $\tau(z_j) = x_{[c_{k(j)}, c_{k(j+1)}-|s^\ell|)}$ is determined $\sigma(y_{[k(j),k(j+1))})$. The proof of (4.44) is complete.

Finally, we bound $\mathcal{C}_{\mathsf{ap}}$. Condition (4.44) implies that $\#\tau(\mathcal{C}_{\mathsf{ap}})$ is at most 2 times the number of words of the form $\sigma(y_{[k(j),k(j+1))})$, where $j \in \mathbb{Z}$ is such that $z_j \in \mathcal{C}_{\mathsf{ap}}$. Note that if $z_j \in \mathcal{C}_{\mathsf{ap}}$ then Lemma 4.36 gives that $[k(j), k(j+1))$ is of aperiodic type, and thus, by Lemma 4.31, we have that the length of $[k(j), k(j+1))$ is at most $2d+1$. Hence, there are at most $\#\mathcal{B}^{d+2}$ words $\sigma(y_{[k(j),k(j+1))})$ such that $z_j \in \mathcal{C}_{\mathsf{ap}}$. We conclude that $\#\tau(\mathcal{C}_{\mathsf{ap}}) \le 2 \cdot \#\mathcal{B}^{d+2}$, and therefore that $\#\mathcal{C}_{\mathsf{ap}} \le 2d^{3d+6}$ by Item ((2)) in Proposition 4.17 and the fact that τ is injective on letters

Next, we prove that $\#\,\mathsf{root}\,\tau(\mathcal{C}) \le 3d^{3d+6}$. Since $\#\,\mathsf{root}\,\tau(\mathcal{C}_{\mathsf{ap}}) \le \#\mathcal{C}_{\mathsf{ap}} \le 2d^{3d+6}$ by what we just proved and since $\#\,\mathsf{root}\,\sigma(\mathcal{B}) \le \#\mathcal{B} \le d^3$ by Item ((2)) in Proposition 4.17, it is enough to show that

$$\mathsf{root}\,\tau(\mathcal{C}_{\mathsf{ap}}) \subseteq \{\mathsf{root}\,\sigma(y_0) : y \in Y, 0 \in \mathrm{Q}_{\mathsf{p}}(y)\}. \quad (4.45)$$

Let $a \in \mathcal{C}_{\mathsf{p}}$ and $j \in \mathbb{Z}$ be such that $z_j = a$. Thanks to Lemma 4.36, we have that $\mathsf{root}\,\tau(z_j) = \mathsf{root}\,\sigma(y_{[k(j),k(j+1))})$ and that $[k(j), k(j+1))$ is of periodic type in y. Hence, by Lemma 4.30, $\mathsf{root}\,\tau(z_j) = \mathsf{root}\,\sigma(y_{k(j)})$. This proves (4.45) and thereby that $\#\,\mathsf{root}\,\tau(\mathcal{C}) \le 3d^{3d+6}$.

We now prove that $\#\mathcal{C}_{\mathsf{p}} \le 2d^{3d+9}\,\mathsf{pow\text{-}com}(X)$ using Lemma 4.37. Let $\mathcal{U} = \{z_{j-1}z_jz_{j+1} : j \in \mathbb{Z}, z_j \in \mathcal{C}_{\mathsf{p}}\}$. We define the map $\pi\colon \mathcal{U} \to \cup_{s \in \mathsf{root}\,\sigma(\mathcal{B})} \mathsf{Pow}_X(s)$ as follows. For $aba' \in \mathcal{U}$, Lemma 4.37 gives a decomposition $\tau(aba') = us^mu'$. We set $\pi(aba') = s^m$. Observe that Item (3) in Lemma 4.37 ensures that $s^m \in \mathsf{Pow}_X(s)$, and, by Item (2) of the same lemma, $s \in \mathsf{root}\,\sigma(\mathcal{B})$.

We claim that

$$\text{if } aba', a\tilde{b}a' \in \mathcal{U} \text{ and } \pi(aba') = \pi(a\tilde{b}a'), \text{ then } b = \tilde{b}. \quad (4.46)$$

Let $\tau(aba') = us^mu'$ and $\tau(a\tilde{b}a') = \tilde{u}\tilde{s}^{\tilde{m}}\tilde{u}'$ be the decompositions from the definition of π. With this notation, the hypothesis $\pi(aba') = \pi(a\tilde{b}a')$ is equivalent to $s^m = \tilde{s}^{\tilde{m}}$. Then, as $s = \mathsf{root}\,s$ and $\tilde{s} = \mathsf{root}\,\tilde{s}$, $s = \tilde{s}$ and $m = \tilde{m}$.

We now prove that $u = \tilde{u}$. First, we assume without loss of generality that $|\tilde{u}| \leq |u|$. Then, Lemma 4.37 ensures that

$$\tau(a) \text{ is prefix of both } us^m \text{ and } \tilde{u}s^m \text{ and that } |\tilde{u}| \leq |u| \leq |\tau(a)| - 2\varepsilon. \tag{4.47}$$

This implies that $s^{\mathbb{Z}}_{[0,|\tau(a)|-|u|)} = (S^{|u|-|\tilde{u}|}s^{\mathbb{Z}})_{[0,|\tau(a)|-|u|)}$. Combining this with the bound $|\tau(a)| - |u| \geq 2\varepsilon \geq 2|s|$ given by (4.47) allow us to use Item ((1)) in Proposition 4.10 and conclude that $s^{\mathbb{Z}} = S^{|u|-|\tilde{u}|}s^{\mathbb{Z}}$. Then, by Item ((2)) of the same proposition, $|u| = |\tilde{u}| \pmod{|s|}$. From this and (4.47) we deduce that $u = \tilde{u}s^{\ell}$ for some $\ell \geq 0$. But since, by Item (3) in Lemma 4.37, s is not a suffix of u, we must have that $\ell = 0$. Therefore, $u = \tilde{u}$.

We can show, in a similar fashion, that $u' = \tilde{u}'$. This allows us to conclude that $\tau(aba') = \tau(a\tilde{b}a') = us^m u'$, and thus that $\tau(b) = \tau(\tilde{b})$. Being τ injective on letters by Proposition 4.4, $b = b'$ and the claim is proved.

Condition (4.46) implies that $\#\mathcal{C}_{\mathsf{p}} \leq \#\mathcal{C}_{\mathsf{ap}}^2 \cdot \#(\cup_{s \in \mathsf{root}\,\sigma(\mathcal{B})} \mathsf{Pow}_X(s))$. Hence, $\#\mathcal{C}_{\mathsf{p}} \leq \#\mathcal{C}_{\mathsf{ap}}^2 \cdot \#\mathcal{B} \cdot \mathsf{pow\text{-}com}(X)$. Since $\#\mathcal{C}_{\mathsf{ap}} \leq 2d^{3d+6}$ and since $\#\mathcal{B} \leq d^3$ by Item ((2)) in Proposition 4.17, it follows that $\mathcal{C}_{\mathsf{p}} \leq 2d^{3d+9}\,\mathsf{pow\text{-}com}(X)$.

Item ((3)) is a direct consequence of Proposition 4.29.

Let us prove Item ((4)). Lemma 4.36 ensures that (Z, τ) satisfies Items ((1)) and ((2)) of Definition 4.6. Let now $x \in X$, $(c, z) = \mathbf{F}_{(Z,\tau)}(x)$ and $a \in \mathcal{C}_{\mathsf{p}}$ be such that $\mathsf{root}\,\tau(z_0)$ is conjugate to $\mathsf{root}\,\tau(a)$. We note that, by Lemma 4.36, $|\,\mathsf{root}\,\tau(z_0)| = |\,\mathsf{root}\,\tau(a)| \leq \varepsilon$. Hence, $\mathsf{per}(x_{[c_0+8\varepsilon, c_1-8\varepsilon)}) \leq |\,\mathsf{root}\,\tau(z_0)| \leq \varepsilon$. This implies, by Lemma 4.36, that $z_0 \in \mathcal{C}_{\mathsf{ap}}$. We can then use (4.45) to get that $\mathsf{root}\,\tau(z_0) = \mathsf{root}\,\sigma(y_0)$ and $\mathsf{root}\,\tau(a) = \mathsf{root}\,\sigma(y'_0)$ for certain $y, y' \in Y$ such that $0 \in \mathrm{Q}_{\mathsf{p}}(y) \cap \mathrm{Q}_{\mathsf{p}}(y')$. We remark that, since $\mathsf{root}\,\tau(z_0)$ is conjugate to $\mathsf{root}\,\tau(a)$, the words $\mathsf{root}\,\sigma(y_0)$ and $\mathsf{root}\,\sigma(y'_0)$ are conjugate. Therefore, y and y' satisfy the hypothesis of Item (3) of Lemma 4.30. We conclude that $\mathsf{root}\,\tau(z_0) = \mathsf{root}\,\sigma(y_0) = \mathsf{root}\,\sigma(y'_0)\,\mathsf{root}\,\tau(a)$.

It is left to prove Item ((5)). Let $j \in \mathbb{Z}$. We have, from Lemma 4.30, that $[k(j), k(j+1))$ or $[k(j+1), k(j+2))$ is of aperiodic type. Hence, by Lemma 4.36, z_j or z_{j+1} belongs to $\mathcal{C}_{\mathsf{ap}}$. □

4.6 The third coding

We continue refining the codings. The main addition to this version is that the words $\tau(a)$ have controlled lengths. The properties of the new coding are summarized in Proposition 4.38.

Proposition 4.38 *Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a minimal infinite subshift, $n \geq 1$ and let d be the maximum of $\lceil p_X(n)/n \rceil$, $p_X(n+1) - p_X(n)$, $\#\mathcal{A}$ and 10^4. There exist a recognizable coding $(Z \subseteq \mathcal{C}^{\mathbb{Z}}, \tau\colon \mathcal{C} \to \mathcal{A}^+)$ of X, a partition $\mathcal{C}_{\mathsf{ap}} \cup \mathcal{C}_{\mathsf{sp}} \cup \mathcal{C}_{\mathsf{wp}}$ of $\mathcal{C}$ and $\varepsilon \in [n/d^{2d^3+4}, n/d)$ such that:*

(1) $\#\mathcal{C}_{\mathsf{ap}} \leq 2d^{3d+6}$, $\#\mathcal{C}_{\mathsf{sp}} \leq 3d^{3d+6}$, $\#\mathcal{C}_{\mathsf{wp}} \leq 2d^{3d+9}\,\mathsf{pow\text{-}com}(X)^4$ and $\#\,\mathsf{root}\,\tau(\mathcal{C}) \leq 5d^{3d+6}$.

(2) $20\varepsilon \leq |\tau(a)| \leq 10d^2 n$ for all $a \in \mathcal{C}$.

(3) (Z, τ) satisfies the recognizability property described in Proposition 4.39.

(4) (Z,τ) *has dichotomous periods w.r.t.* $(\mathcal{C}_{\mathsf{ap}}, \mathcal{C}_{\mathsf{sp}} \cup \mathcal{C}_{\mathsf{wp}})$ *and* 8ε.

(5) *The set* $\mathcal{C}_{\mathsf{sp}}$ *satisfies the property described in Proposition 4.40.*

Remark 4.6 Assume the notation of Proposition 4.38. Then, $a \in \mathcal{C} \setminus \mathcal{C}_{\mathsf{ap}}$ implies that $|\operatorname{root}\tau(a)| = \operatorname{per}(\tau(a))$. Indeed, Item ((4)) ensures that $|\operatorname{root}\tau(a)| \leq \varepsilon$, and Item ((2)) that $|\tau(a)| \geq 2\varepsilon$, therefore, by Lemma 4.9, $|\operatorname{root}\tau(a)| = \operatorname{per}(\tau(a))$.

Proposition 4.39 *Consider the coding described in Proposition 4.38. For any* $x, \tilde{x} \in X$, *we have that:*

(1) *If* $\operatorname{per}(x_{[-\varepsilon,\varepsilon)}) > \varepsilon$ *and* $x_{[-7d^2n,7d^2n)} = \tilde{x}_{[-7d^2n,7d^2n)}$, *then* $\mathbf{F}^0_{(Z,\tau)}(x)$ *is equal to* $\mathbf{F}_{(Z,\tau)}(\tilde{x})$.

(2) *If* $k \geq 0$, $x_{[-50d^2n,k+50d^2n)} = \tilde{x}_{[-50d^2n,k+50d^2n)}$ *and* $\mathbf{F}^0_{(Z,\tau)}(x)$ *is equal to* $\mathbf{F}_{(Z,\tau)}(\tilde{x})$, *then* $\mathbf{F}^0_{(Z,\tau)}(S^k x)$ *is equal to* $\mathbf{F}_{(Z,\tau)}(S^k\tilde{x})$.

Proposition 4.40 *The coding of Proposition 4.38 satisfies the following.*

(1) *If* $z \in Z$ *and* $i < j$ *are integers such that* $z_k \in \mathcal{C} \setminus \mathcal{C}_{\mathsf{ap}}$ *for all* $k \in [i, j+1)$, *then* $z_k = z_i \in \mathcal{C}_{\mathsf{sp}}$ *for all* $k \in [i,j)$ *and* $\operatorname{root}\tau(z_k) = \operatorname{root}\tau(z_i)$ *for all* $k \in [i, j+1)$.

(2) *If* $a \in \mathcal{C}_{\mathsf{sp}}$, *then* $\tau(a) = (\operatorname{root}\tau(a))^{2^r}$, *where* r *is the unique integer for which* $2^r|\operatorname{root}\tau(a)|$ *belongs to* $[20\varepsilon, 40\varepsilon)$.

We now introduce the notation that will be used in this section. Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a minimal infinite subshift, $n \geq 1$ and let d be the maximum of $\lceil p_X(n)/n \rceil$, $p_X(n+1) - p_X(n)$, $\#\mathcal{A}$ and 10^4. Then, Proposition 4.28 applied to X and n gives a recognizable coding $(Y \subseteq \mathcal{C}_Y^{\mathbb{Z}}, \sigma\colon \mathcal{C} \to \mathcal{A}^+)$ of X, a partition $\mathcal{C}_Y = \mathcal{C}_{\mathsf{p}} \cup \mathcal{C}_{\mathsf{ap}}$, and an integer $\varepsilon \in [n/d^{2d^3+4}, n/d)$ satisfying the properties described in Proposition 4.28.

The strategy to prove the main proposition of this section is the following. The coding (Z,τ) will be obtained from (Y,σ) by splitting the words in $\sigma(\mathcal{C}_{\mathsf{p}})$ into subwords of carefully chosen lengths. This will maintain most of the properties of (Y,σ) at the same time that we gain control on the lengths of all the words $\tau(a)$. A delicate part involves defining the splittings of the words in $\sigma(\mathcal{C}_{\mathsf{p}})$ in such a way that (Z,τ) has the recognizability properties in Proposition 4.39.

4.6.1 Construction of the third coding

For $s \in \operatorname{root}\sigma(\mathcal{C}_{\mathsf{p}})$, we define $\zeta(s)$ as the unique power of two such that $\zeta(s)\cdot|s|$ lies in $[20\varepsilon, 40\varepsilon)$. Note that, by Item ((4)) of Proposition 4.28, we have that $\zeta(s) \geq 1$. Then, for $a \in \mathcal{C}_{\mathsf{p}}$, we can define p_a and q_a as the unique integers satisfying

$$p_a \cdot \zeta(\operatorname{root}\sigma(a)) + q_a = \frac{|\sigma(a)|}{|\operatorname{root}\sigma(a)|} \text{ and } 0 < q_a \leq \zeta(\operatorname{root}\sigma(a)). \tag{4.48}$$

It is important to remark that Item ((2)) in Proposition 4.28 ensures that $p_a \geq 2$.

For $a \in \mathcal{C}_{\mathsf{p}}$, let

$$\psi_{\mathsf{sp}}(a) = (\mathsf{root}\,\sigma(a))^{\zeta(\mathsf{root}\,\sigma(a))} \text{ and } \psi_{\mathsf{wp}}(a) = (\mathsf{root}\,\sigma(a))^{\zeta(\mathsf{root}\,\sigma(a))+q_a}.$$

Note that

$$\sigma(a) = \psi_{\mathsf{sp}}(a)^{p_a-1}\psi_{\mathsf{wp}}(a) \text{ for all } a \in \mathcal{C}_{\mathsf{p}}.$$

We also choose bijections

$$\phi_{\mathsf{sp}} \colon \mathcal{C}_{\mathsf{sp}} \to \psi_{\mathsf{sp}}(\mathcal{C}_{\mathsf{p}}) \text{ and } \phi_{\mathsf{wp}} \colon \mathcal{C}_{\mathsf{wp}} \to \psi_{\mathsf{wp}}(\mathcal{C}_{\mathsf{p}}),$$

where $\mathcal{C}_{\mathsf{sp}}$, $\mathcal{C}_{\mathsf{wp}}$ and $\mathcal{C}_{\mathsf{ap}}$ are pairwise disjoint. Then, we define for $a \in \mathcal{C}_Y$,

$$\eta(a) = \begin{cases} a & \text{if } a \in \mathcal{C}_{\mathsf{ap}} \\ \phi_{\mathsf{sp}}^{-1}(\psi_{\mathsf{sp}}(a))^{p_a-1}\phi_{\mathsf{wp}}^{-1}(\psi_{\mathsf{wp}}(a)) & \text{if } a \in \mathcal{C}_{\mathsf{wp}} \end{cases} \tag{4.49}$$

Let $\mathcal{C}_Z = \mathcal{C}_{\mathsf{ap}} \cup \mathcal{C}_{\mathsf{sp}} \cup \mathcal{C}_{\mathsf{wp}}$ and, for $a \in \mathcal{C}_Z$, we set

$$\tau(a) = \begin{cases} \sigma(a) & \text{if } a \in \mathcal{C}_{\mathsf{ap}} \\ \phi_{\mathsf{sp}}(a) & \text{if } a \in \mathcal{C}_{\mathsf{sp}} \\ \phi_{\mathsf{wp}}(a) & \text{if } a \in \mathcal{C}_{\mathsf{wp}} \end{cases} \tag{4.50}$$

It then follows that

$$\sigma = \tau\eta. \tag{4.51}$$

Finally, we set $Z = \bigcup_{k\in\mathbb{Z}} S^k\eta(Y)$.

Let us comment on the definition of τ. Equation (4.51) says that $\sigma(a) = \tau(a)$ if $a \in \mathcal{C}_{\mathsf{ap}}$ and that $\sigma(a) = \tau(b)^{p_a-1}\tau(c)$ if $a \in \mathcal{C}_{\mathsf{p}}$, $b = \phi_{\mathsf{sp}}^{-1}(\psi_{\mathsf{sp}}(a))$ and $c = \phi_{\mathsf{wp}}^{-1}(\psi_{\mathsf{wp}}(c))$. In other words, τ is obtained from σ by slicing the words $\sigma(a)$.

4.6.2 Proof of Propositions 4.38, 4.39 and 4.40

Proof of Proposition 4.38. We start with Item (1). Item ((1)) in Proposition 4.28 gives the bound $\#\mathcal{C}_{\mathsf{ap}} \leq 2d^{3d+6}$. Also, it follows from the definitions and Item ((1)) in Proposition 4.28 that

$$\#\mathcal{C}_{\mathsf{sp}} = \#\psi_{\mathsf{sp}}(\mathcal{C}_{\mathsf{p}}) = \#\,\mathsf{root}\,\sigma(\mathcal{C}_{\mathsf{p}}) \leq 3d^{3d+6} \text{ and}$$
$$\#\mathcal{C}_{\mathsf{wp}} = \#\psi_{\mathsf{wp}}(\mathcal{C}_{\mathsf{p}}) \leq \#\mathcal{C}_{\mathsf{p}} \leq 2d^{3d+9}\cdot\mathsf{pow\text{-}com}(X).$$

Now, using 4.50 yields $\mathsf{root}\,\tau(\mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{ap}}) = \mathsf{root}\,\sigma(\mathcal{C}_{\mathsf{p}})$. Therefore, $\#\,\mathsf{root}\,\tau(\mathcal{C}_Z) \leq \#\mathcal{C}_{\mathsf{ap}} + \#\,\mathsf{root}\,\sigma(\mathcal{C}_{\mathsf{p}})$, which gives $\#\,\mathsf{root}\,\tau(\mathcal{C}_Z) \leq 4d^{3d+6}$ if we use the bounds $\#\mathcal{C}_{\mathsf{ap}} \leq 2d^{3d+6}$ and $\#\,\mathsf{root}\,\sigma(\mathcal{C}_Y) \leq 3d^{3d+6}$, where the last bound is given by Item ((1)) in Proposition 4.28.

We now consider Item (2). Let $a \in \mathcal{C}_Z$. If $a \in \mathcal{C}_{\mathsf{ap}}$, then by Item ((2)) in Proposition 4.28 we have that $20\varepsilon \leq |\tau(a)| = |\sigma(a)| \leq 10d^2n$. If $a \in \mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{ap}}$ and $s = \mathsf{root}\,\tau(a)$, then (4.50) implies that

$$|\tau(a)| = \begin{cases} \zeta(s)|s| & \text{if } a \in \mathcal{C}_{\mathsf{sp}} \\ (\zeta(s)+q_a)|s| & \text{if } a \in \mathcal{C}_{\mathsf{wp}} \end{cases}$$

Since $20\varepsilon \le \zeta(s)|s| < 40\varepsilon$ and $0 < q_a \le |\zeta(s)|$, we obtain that $20\varepsilon \le |\tau(a)| \le 80\varepsilon \le 10d^2n$.

Next, we prove Item (3). Let $x \in X$, $(c, y) = \mathbf{F}_{(Y,\sigma)}$ and $(f, z) = \mathbf{F}_{(Z,\tau)}(x)$. We use Lemma 4.3 to obtain $m \in [0, |\eta(y_0)|)$ such that

$$\eta(y_0) = z_{[-m,-m+|\eta(z_0)|)} \text{ and } c_0 = f_0 - |\tau(z_{[-m,0)})|. \tag{4.52}$$

We first assume that $z_0 \in \mathcal{C}_{\mathsf{ap}}$. Then, Equation (4.52) implies that z_0 occurs in $\eta(y_0)$, and thus, since $z_0 \in \mathcal{C}_{\mathsf{ap}}$, Equation (4.49) ensures that $\eta(y_0) = y_0 \in \mathcal{C}_{\mathsf{ap}}$. Hence, $m = 0$, $z_0 = y_0$, $c_0 = f_0$ and $c_1 = c_0 + |\sigma(y_0)| = f_0 + |\tau(z_0)| = f_1$. Using this and Item ((4)) of Proposition 4.28 with x and (c, y) produces

$$\mathsf{per}(x_{[f_0+8\varepsilon,f_1-8\varepsilon)}) = \mathsf{per}(x_{[c_0+8\varepsilon,c_1-8\varepsilon)}) > \varepsilon.$$

Let us now assume that $z_0 \in \mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{ap}}$. This condition and Equation (4.49) imply that $y_0 \in \mathcal{C}_{\mathsf{p}}$. Hence, we can use Item ((4)) in Proposition 4.28 to obtain that $\mathsf{per}(x_{[c_0-8\varepsilon,c_1+8\varepsilon)}) \le \varepsilon$. We conclude, since (4.49) guarantees that $[f_0, f_1)$ is contained in $[c_0, c_1)$, that $\mathsf{per}(x_{[f_0-8\varepsilon,f_1+8\varepsilon)}) \le \mathsf{per}(x_{[c_0-8\varepsilon,c_1+8\varepsilon)}) \le \varepsilon$.

It rests to prove that (Z, τ) satisfies Item ((3)) of Definition 4.6. Let $a \in \mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{ap}}$ be such that $\mathsf{root}\,\tau(z_0)$ is conjugate to $\mathsf{root}\,\tau(a)$. We have from (4.52) that z_0 occurs in $\eta(y_0)$, so, by (4.49), $\mathsf{root}\,\tau(z_0) = \mathsf{root}\,\sigma(y_0)$. Similarly, a occurs in $\eta(b)$ and $\mathsf{root}\,\tau(a) = \mathsf{root}\,\sigma(b)$ for some $b \in \mathcal{C}_Y$. The first condition and (4.49) imply, as $a \in \mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{ap}}$, that $b \in \mathcal{C}_{\mathsf{p}}$. Now, the hypothesis ensures that $\mathsf{root}\,\sigma(y_0)$ is conjugate to $\mathsf{root}\,\sigma(b)$. Therefore, as (Y, σ) has dichotomous periods w.r.t. $(\mathcal{C}_{\mathsf{ap}}, \mathcal{C}_{\mathsf{p}})$, we can use Item (3) of Definition 4.6 to obtain that $\mathsf{root}\,\sigma(y_0) = \mathsf{root}\,\sigma(b)$. We conclude that $\mathsf{root}\,\tau(z_0) = \mathsf{root}\,\tau(a)$, completing the proof of Item (3).

Finally, for Items (4) and (5), we present the proofs of Propositions 4.39 and 4.40 hereafter. □

Lemma 4.41 *Let $\mathcal{C}_{\mathsf{block}}$ be the set of words $a^\ell b$, where $a \in \mathcal{C}_{\mathsf{sp}}$, $b \in \mathcal{C}_{\mathsf{wp}}$, $\ell \ge 1$ and $\mathsf{root}\,\tau(a) = \mathsf{root}\,\tau(b)$. Then, any $z \in Z$ can be written as $z = \dots w_{-1}w_0w_1\dots$, where $w_j \in \mathcal{C}_{\mathsf{block}}$ or $w_j \in \mathcal{C}_{\mathsf{ap}}$ and $w_jw_{j+1} \notin \mathcal{C}_{\mathsf{block}}^2$.*

Proof. Let $z \in Z$ and $(c, y) = \mathbf{F}_{(Y,\sigma)}(\tau(z))$. We set $w_j = \eta(y_j)$. The definition of η in (4.49) ensures that $w_j \in \mathcal{C}_{\mathsf{block}} \cup \mathcal{C}_{\mathsf{ap}}$. Moreover, Item ((5)) in Propositions 4.28 says, in this context, that $w_jw_{j+1} \notin \mathcal{C}_{\mathsf{block}}^2$ for all $j \in \mathbb{Z}$. Finally, by Lemma 4.3 we have that $z = S^\ell\eta(y)$ for some $\ell \in \mathbb{Z}$, and thus that $z = \dots w_{-1}w_0w_1\dots$ □

We can now present the proof of Proposition 4.40.

Proof of Proposition 4.40. Item ((2)) directly follows from the definition of τ in (4.50). Let us prove Item ((1)). Let $z \in Z$ be such that $z_{[i,j+1)} \subseteq (\mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{ap}})^+$. We write $z = \dots w_{-1}w_0w_1\dots$ as in Lemma 4.41 and let $\mathcal{C}_{\mathsf{block}}$ be the set defined in Lemma 4.41. Then, the hypothesis $z_{[i,j+1)} \in (\mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{ap}})^+$ and the condition $w_kw_{k+1} \notin \mathcal{C}_{\mathsf{block}}$ imply that $z_{[i,j+1)}$ occurs in w_k for certain $k \in \mathbb{Z}$ such that $w_k \in \mathcal{C}_{\mathsf{block}}$. Hence, $z_\ell = z_i \in \mathcal{C}_{\mathsf{sp}}$ for all $\ell \in [i, j)$ and $\mathsf{root}\,\tau(z_k) = \mathsf{root}\,\tau(z_i)$ for all $k \in [i, j+1)$. □

Lemma 4.42 *Let $z, \tilde{z} \in Z$ and $\ell \geq 1$ be such that $\tau(\tilde{z}_{[0,\ell)})$ is a prefix of $\tau(z_0)$. Then, $z_0 \in \mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{ap}}$ implies that $\tilde{z}_i \in \mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{ap}}$ and $\mathsf{root}\,\tau(\tilde{z}_i) = \mathsf{root}\,\tau(z_0)$ for all $i \in [0,\ell)$.*

Proof. First, we note that, by Item ((4)) in Proposition 4.38, $\mathsf{per}(\tau(\tilde{z}_i)) \leq \mathsf{per}(\tau(z_0)) \leq \varepsilon$ for all $i \in [0,\ell)$. Thus, by Item ((4)) in Proposition 4.38, $\tilde{z}_i \in \mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{ap}}$ for all $i \in [0,\ell)$.

Let $s = \mathsf{root}\,\tau(z_0)$. In order to continue, we claim that

$$\text{if } a \in \mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{sp}} \text{ and } \tau(a) \text{ is a prefix of } s^\infty, \text{ then } \mathsf{root}\,\tau(a) = s.$$

First, we note $\tau(a)$ is a prefix of $(\mathsf{root}\,\tau(a))^\infty$. Also, Item ((4)) in Proposition 4.38 guarantees that $|\mathsf{root}\,\tau(a)| \leq \varepsilon$ and that $|s| \leq \varepsilon$. Hence, as $|\tau(a)| \geq 2\varepsilon$ by Item ((2)) in Proposition 4.38, we can use Theorem 4.8 to deduce that $\mathsf{root}\,\tau(a)$ and s are powers of a common word r. This implies that $s = \mathsf{root}\,\tau(a) = \mathsf{root}\,r$.

We now prove that $\mathsf{root}\,\tau(\tilde{z}_i) = s$ for $i \in [0,\ell)$ by induction on i. If $i = 0$, then we have from the hypothesis that $\tau(\tilde{z}_0)$ is a prefix of $\tau(z_0)$ and that $\tilde{z}_0 \in \mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{ap}}$. Thus, $\mathsf{root}\,\tau(\tilde{z}_0) = s$ by the claim. Let us assume now that $0 < i < \ell$ and that $\mathsf{root}\,\tau(\tilde{z}_j) = s$ for $j \in [0,i)$. Then, $\tau(\tilde{z}_{[0,i)})$ is a power of s. Being $\tau(\tilde{z}_{[0,i]})$ a prefix of $\tau(z_0)^\infty = s^\infty$, we deduce that $\tau(\tilde{z}_i)$ is a prefix of s^∞. This allows us to use the claim and obtain that $\mathsf{root}\,\tau(\tilde{z}_i) = s$. This finishes the inductive step and the proof of the lemma. □

Finally, we prove Proposition 4.39

Proof of Proposition 4.39. We fix the following notation for this proof. Let $x, \tilde{x} \in X$, $(f,z) = \mathbf{F}_{(Z,\tau)}(x)$, $(\tilde{f},\tilde{z}) = \mathbf{F}_{(Z,\tau)}(\tilde{x})$, $(c,y) = \mathbf{F}_{(Y,\sigma)}(x)$ and $(\tilde{c},\tilde{y}) = \mathbf{F}_{(Y,\sigma)}(\tilde{x})$.

We start by proving Item (1). Assume that $\mathsf{per}(x_{[-\varepsilon,\varepsilon)}) > \varepsilon$ and that $x_{[-7d^2n,7d^2n)} = x'_{[-7d^2n,7d^2n)}$. We have to show that $f_0 = \tilde{f}_0$ and $z_0 = \tilde{z}_0$.

We claim that

(i) $c_0 = \tilde{c}_0$ and $y_0 = \tilde{y}_0$;

(ii) $y_0 \in \mathcal{C}_{\mathsf{ap}}$ and $\tilde{y}_0 \in \mathcal{C}_{\mathsf{ap}}$;

(iii) $z_0 = y_0$, $f_0 = c_0$, $\tilde{z}_0 = \tilde{y}_0$ and $\tilde{f}_0 = \tilde{c}_0$.

Item (i) follows from the fact that the current hypothesis allows us to use Item ((3)) of Proposition 4.28 to get that $\mathbf{F}^0_{(Y,\sigma)}(x) = \mathbf{F}^0_{(Y,\sigma)}(\tilde{x})$, which is equivalent to (i). For Item (ii), we note that if $y_0 \in \mathcal{C}_Y \setminus \mathcal{C}_{\mathsf{ap}} = \mathcal{C}_{\mathsf{p}}$, then Item ((4)) in Proposition 4.28 implies that $\mathsf{per}(x_{[c_0-8\varepsilon,c_1+8\varepsilon)}) \leq \varepsilon$. Hence, as $[-\varepsilon,\varepsilon)$ is included in $[c_0-8\varepsilon,c_1+8\varepsilon)$, $\mathsf{per}(x_{[-\varepsilon,\varepsilon)}) \leq \varepsilon$, contradicting our hypothesis. Therefore, $y_0 = \tilde{y}_0 \in \mathcal{C}_{\mathsf{ap}}$.

To prove Item (iii), we first note that Lemma 4.3 gives an integer $m \in [0,|\tau(z_0)|)$ such that

$$y_{[-m,-m+|\eta(z_0)|)} = \eta(z_0) \text{ and } -c_0 = -f_0 + |\eta(z)_{[0,m)}|. \tag{4.53}$$

In particular, y_0 occurs in $\eta(z_0)$. Since Item (ii) ensures that $y_0 \in \mathcal{C}_{\mathsf{ap}}$, it follows from the definition of η in (4.49) that $\eta(y_0) = y_0 = z_0$. Putting this in (4.53) gives that $m = 0$ and

$c_0 = f_0$. A similar argument shows that $\tilde{z}_0 = \tilde{y}_0$ and $\tilde{f}_0 = \tilde{c}_0$ as well. This completes the proof of the claim.

Items (i) and (iii) of the claim imply that $(f_0, z_0) = (\tilde{f}_0, \tilde{z}_0)$, proving Item (1) of the proposition.

Before proving Item (2), we claim that

$$\text{if } z_0 \in \mathcal{C}_{\mathsf{ap}} \text{ and } x_{[-17d^2n,17d^2n)} = \tilde{x}_{[-17d^2n,17d^2n)}, \text{ then } \mathbf{F}^0_{(Z,\tau)}(x) = \mathbf{F}^0_{(Z,\tau)}(\tilde{x}). \tag{4.54}$$

To prove (4.54), we start by using Item ((4)) in Proposition 4.38 to obtain that $\mathsf{per}(x_{[f_0+8\varepsilon,f_1-8\varepsilon)}) > \varepsilon$. Thus, by Item ((2)) in Proposition 4.13, there exists $j \in [f_0 + 8\varepsilon, f_1 - 8\varepsilon)$ satisfying

$$\mathsf{per}(x_{[j-\varepsilon,j+\varepsilon)}) > \varepsilon. \tag{4.55}$$

Now, since $j \in [f_0+8\varepsilon, f_1-8\varepsilon)$ and $|\tau| \leq 10d^2n$, we have that $j \in [-10d^2n, 10d^2n)$. Therefore, by the hypothesis $x_{[-17d^2n,17d^2n)} = \tilde{x}_{[-17d^2n,17d^2n)}$, $\tilde{x}_{[j-7d^2n,j+7d^2n)} = x_{[j-7d^2n,j+7d^2n)}$. Combining this with (4.55) allows us to use Item (1) of this proposition and deduce that

$$\mathbf{F}^0_{(Z,\tau)}(S^j x) = \mathbf{F}^0_{(Z,\tau)}(S^j \tilde{x}). \tag{4.56}$$

Observe that the condition $j \in [f_0 + 8\varepsilon, f_1 - 8\varepsilon)$ implies that $\mathbf{F}^0_{(Z,\tau)}(S^j x) = (f_0 - j, z_0)$. Let i be the integer satisfying $\tilde{f}_i \leq j < \tilde{f}_{i+1}$ and note that $\mathbf{F}^0_{(Z,\tau)}(S^j \tilde{x}) = (\tilde{f}_i - j, \tilde{z}_i)$. Then, by (4.56), $f_0 = \tilde{f}_i$ and $z_0 = \tilde{z}_i$. In particular, $\tilde{f}_i = f_0 \leq 0 < f_1 = \tilde{f}_{i+1}$, so $i = 0$. We conclude that $\mathbf{F}^0_{(Z,\tau)}(x) = (f_0, z_0) = (\tilde{f}_0, \tilde{z}_0) = \mathbf{F}^0_{(Z,\tau)}(\tilde{x})$.

We now prove Item (2). Assume that $\mathbf{F}^0_{(Z,\tau)}(x) = \mathbf{F}^0_{(Z,\tau)}(\tilde{x})$. The is equivalent to $z_0 = \tilde{z}_0$ and $f_0 = \tilde{f}_0$, so $f_1 = \tilde{f}_1$ as well. Hence,

$$\mathbf{F}^0_{(Z,\tau)}(S^i x) = \mathbf{F}^0_{(Z,\tau)}(S^i \tilde{x}) \text{ for all } i \in [f_0, f_1). \tag{4.57}$$

We are going to prove that

$$\text{if } x_{[-50d^2n,50d^2n)} = \tilde{x}_{[-50d^2n,50d^2n)}, \text{ then } \mathbf{F}^0_{(Z,\tau)}(S^{f_1} x) = \mathbf{F}^0_{(Z,\tau)}(S^{f_1} \tilde{x}). \tag{4.58}$$

The lemma then follows from an inductive argument on k that uses Equations (4.57) and (4.58).

Let us assume that $x_{[-50d^2n,50d^2n)} = \tilde{x}_{[-50d^2n,50d^2n)}$. We consider two cases. First, we assume that z_1 or $\tilde{z}_1$ belongs to $\mathcal{C}_{\mathsf{ap}}$. There is no loss of generality in assuming that z_1 is the one belonging to $\mathcal{C}_{\mathsf{ap}}$. Observe that the hypothesis and that $|\tau| \leq 10d^2n$ ensure that $x_{[f_1-7d^2n,f_1+7d^2n)} = \tilde{x}_{[f_1-7d^2n,f_1+7d^2n)}$. This allows us to use (4.54) and deduce that $\mathbf{F}^0_{(Z,\tau)}(S^{f_1} x) = \mathbf{F}^0_{(Z,\tau)}(S^{f_1} \tilde{x})$.

We now consider the case in which z_1 and $\tilde{z}_1$ belong to $\mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{ap}}$. Observe that, since $f_1 = \tilde{f}_1$, we have that $\mathbf{F}^0_{(Z,\tau)}(S^{f_1} x) = (0, z_1)$ and $\mathbf{F}^0_{(Z,\tau)}(S^{f_1} \tilde{x}) = (0, \tilde{z}_1)$. Thus, it is enough to show that $z_1 = \tilde{z}_1$.

We assume without loss of generality that $|\tau(z_1)| \geq |\tau(\tilde{z}_1)|$. Let ℓ be the integer satisfying $\tilde{f}_\ell \leq f_2 < \tilde{f}_{\ell+1}$. Remark that $\ell \geq 2$ as $\tilde{f}_2 = \tilde{f}_1 + |\tau(\tilde{z}_1)| \leq f_1 + |\tau(z_1)| = f_2$. Being $\tilde{f}_1 = f_1$ and $\tilde{f}_\ell \leq f_2$, the hypothesis $x_{[-50d^2n,50d^2n)} = \tilde{x}_{[-50d^2n,50d^2n)}$ and the bound $|\tau| \leq 7d^2n$ ensure that $\tau(\tilde{z}_{[1,\ell)})$ is a prefix of $\tau(z_1)$. Hence, since we assumed that $z_1 \in \mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{sp}}$, Lemma 4.42 yields that

$$\tilde{z}_i \in \mathcal{C}_Z \setminus \mathcal{C}_{\mathsf{ap}} \text{ and } \mathsf{root}\,\tau(\tilde{z}_i) = \mathsf{root}\,\tau(z_1) \text{ for all } i \in [1,\ell). \tag{4.59}$$

We set $s = \mathsf{root}\,\tau(z_1)$. It then follows from 4.59 and 4.50 that for any $i \in [1,\ell)$

$$z_1 = \begin{cases} s^{\zeta(s)} & \text{if } z_1 \in \mathcal{C}_{\mathsf{sp}} \\ s^{\zeta(s)+q_{z_1}} & \text{if } z_1 \in \mathcal{C}_{\mathsf{wp}} \end{cases} \qquad \tilde{z}_i = \begin{cases} s^{\zeta(s)} & \text{if } \tilde{z}_i \in \mathcal{C}_{\mathsf{sp}} \\ s^{\zeta(s)+q_{\tilde{z}_i}} & \text{if } \tilde{z}_i \in \mathcal{C}_{\mathsf{wp}} \end{cases} \tag{4.60}$$

We can use this to prove that $|\tau(z_1)| = |\tau(\tilde{z}_1)|$ implies that $z_1 = \tilde{z}_1$. Indeed, in the case $z_1 \in \mathcal{C}_{\mathsf{sp}}$, it follows from (4.60) and the fact that $q_a > 0$ for all $a \in \mathcal{C}_{\mathsf{wp}}$ that $\tau(\tilde{z}_1) = \tau(z_1) = s^{\zeta(s)}$, and thus that $\tilde{z}_1 = z_1 = \psi_{\mathsf{sp}}^{-1}(s^{\zeta(s)})$. Similarly, if $z_1 \in \mathcal{C}_{\mathsf{wp}}$, then (4.48) and the equation $\mathsf{root}\,\tau(z_1) = \mathsf{root}\,\tau(\tilde{z}_1)$ ensure that $q_{z_1} = q_{\tilde{z}_1}$, and thus from (4.60) we get that $\tau(\tilde{z}_1) = \tau(z_1) = s^{\zeta(s)+q_{z_1}}$. In particular, $\tilde{z}_1 = z_1 = \psi_{\mathsf{wp}}^{-1}(s^{\zeta(s)+q_{z_1}})$.

It is left to consider the case $|\tau(z_1)| > |\tau(\tilde{z}_1)|$, so let us assume that this condition is satisfied. Then, by (4.60) and the fact that $q_a > 0$ for all $a \in \mathcal{C}_{\mathsf{wp}}$,

$$z_1 \in \mathcal{C}_{\mathsf{wp}} \text{ and } \tilde{z}_1 \in \mathcal{C}_{\mathsf{sp}}. \tag{4.61}$$

In this situation, Item ((1)) in Proposition 4.40 ensures that $z_2 \in \mathcal{C}_{\mathsf{ap}}$. Now, observe that $f_3 \leq 3|\tau| \leq 30d^2n$ and $f_2 \geq 0$; so the hypothesis $x_{[-50d^2n,50d^2n)} = \tilde{x}_{[-50d^2n,50d^2n)}$ gives that $x_{[f_2-7d^2n,f_3+7d^2n)} = \tilde{x}_{[f_2-7d^2n,f_3+7d^2n)}$. Hence, we can use (4.54) to obtain that $\mathbf{F}^0_{(Z,\tau)}(S^{f_2}x)$ is equal to $\mathbf{F}^0_{(Z,\tau)}(S^{f_2}\tilde{x})$. More precisely, since $\tilde{f}_\ell \leq f_2 < \tilde{f}_{\ell+1}$, we can write

$$(\tilde{f}_\ell - f_2, \tilde{z}_\ell) = \mathbf{F}^0_{(Z,\tau)}(S^{f_2}\tilde{x}) = \mathbf{F}^0_{(Z,\tau)}(S^{f_2}x) = (0, z_2).$$

Hence, the equation $\mathbf{F}^0_{(Z,\tau)}(S^{f_2}x) = \mathbf{F}^0_{(Z,\tau)}(S^{f_2}\tilde{x})$ is equivalent to

$$\tilde{f}_\ell = f_2 \text{ and } \tilde{z}_\ell = z_2. \tag{4.62}$$

In particular, $\tau(\tilde{z}_{[1,\ell)}) = \tau(z_1)$.

Now, since $z_1 \in \mathcal{C}_{\mathsf{wp}}$, we have by Item ((1)) in Proposition 4.40 that $\tilde{z}_\ell = z_2 \in \mathcal{C}_{\mathsf{ap}}$. Therefore, Item ((1)) in Proposition 4.40 guarantees that $\tilde{z}_i \in \mathcal{C}_{\mathsf{sp}}$ for all $i \in [0, \ell-1)$ and $\tilde{z}_{\ell-1} \in \mathcal{C}_{\mathsf{sp}}$. Combining this with (4.60) and the equality $\tau(\tilde{z}_{[1,\ell)}) = \tau(z_1)$ produces that

$$(\zeta(s) + q_{z_1})|s| = |\tau(z_1)| = |\tau(\tilde{z}_{[0,\ell)})| = (\ell\zeta(s) + q_{\tilde{z}_{\ell-1}})|s|.$$

Since $q_{z_1} \leq \zeta(s)$ and $q_{\tilde{z}_{\ell-1}} > 0$, we conclude that $\ell \leq 1$. But then $\tau(z_1) = \tau(\tilde{z}_{[1,\ell)})$ is the empty word, which contradicts the definition of τ. Therefore, the case $|\tau(z_1)| > |\tau(\tilde{z}_1)|$ does not occur and the proof is complete. □

4.7 The fourth coding

In this section, we give the final versions of the codings needed in the proof of the main theorems. The new element of these codings is that it is possible to connect them using morphisms.

The section has two parts. In the first one, we construct the new codings, using Proposition 4.38 and a modified higher block construction, and present their basic properties. Then, in the second one, we show how we can connect two of these codings using the morphism described in Subsection 4.7.2.

4.7.1 Construction of the fourth coding

Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be an infinite minimal subshift, $n \geq 0$ and let d be the maximum of $\lceil p_X(n)/n \rceil$, $p_X(n+1) - p_X(n)$, $\#\mathcal{A}$ and 10^4. We use Proposition 4.38 with X and n to obtain a recognizable coding $(Y \subseteq \mathcal{B}^{\mathbb{Z}}, \sigma \colon \mathcal{B} \to \mathcal{A}^+)$ of X, a partition $\mathcal{B} = \mathcal{B}_{\mathsf{ap}} \cup \mathcal{B}_{\mathsf{sp}} \cup \mathcal{B}_{\mathsf{wp}}$ and an integer $\varepsilon \in [n/d^{2d^3+4}, n/d)$ satisfying Items (1) to (5) of Proposition 4.38.

We start with the following observation. Since (Y, σ) is a recognizable coding of a minimal subshift, Y is minimal; thus, for all $y \in Y$ there exists $k < 0$ such that $y_k \in \mathcal{B}_{\mathsf{ap}}$. This observation allows us to define the map $\mathfrak{L}(y) = \max\{k < 0 : y_k \in \mathcal{B}_{\mathsf{ap}}\}$ that returns the index of the first-to-the-left symbol in $\mathcal{B}_{\mathsf{ap}}$.

Let $\psi_0 \colon Y \to \mathcal{B}^4$ be the map $y \mapsto y_{\mathfrak{L}(y)} y_{-1} y_0 y_1$ and $\psi(y) = (\psi_0(S^j y))_{j \in \mathbb{Z}}$. We treat $\psi(y)$ as a sequence over the alphabet $\mathcal{B}^4$ and define $Z = \psi(Y) \subseteq (\mathcal{B}^4)^{\mathbb{Z}}$. We set

$$\mathcal{C} = \{z_0 : z \in Z\} \subseteq \mathcal{B}^4. \tag{4.63}$$

Let $\theta(a a_{-1} a_0 a_1) = a_0$ for $a a_{-1} a_0 a_1 \in \mathcal{C}$ and $\tau = \sigma\theta$. Remark that

$$\theta\psi(y) = y \text{ for any } y \in Y. \tag{4.64}$$

We abuse a bit of the notation and define $\mathfrak{L}(z) = \max\{k < 0 : z_k \in \mathcal{C}_{\mathsf{ap}}\}$ for $z \in Z$. Note that $\mathfrak{L}(\psi(y)) = \mathfrak{L}(y)$ for $y \in Y$.

Basic properties of the fourth coding

We present here the basic properties of (Z, τ).

Lemma 4.43 *The pair (Z, τ) is a recognizable coding of X.*

Proof. It follows from the definitions that ψ commutes with the shift and that it is continuous; hence $Z = \psi(Y)$ is a subshift. Also, by (4.64), we have that $\theta\psi(y) = y$ for any $y \in Y$, so $Y = \theta\psi(Y) = \theta(Z)$. Therefore, (Z, θ) is a coding of Y. It is easy to see from the definition of Z that (Z, θ) is recognizable. Hence, as (Y, σ) is recognizable as well, Lemma 1.1 tells us that $(Z, \sigma\theta)$ is recognizable. Being $\tau = \sigma\theta$, we conclude that (Z, τ) is recognizable. □

Thanks to the last lemma, $\mathbf{F}_{(Z,\tau)}(x)$ and $\mathbf{F}^0_{(Z,\tau)}(x)$ are defined for every $x \in X$.

Lemma 4.44 *Let $x \in X$, $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$ and $(f, z) = \mathbf{F}_{(Z,\tau)}(x)$. Then, $c_k = f_k$ and $\psi_0(S^k y) = z_k$ for all $k \in \mathbb{Z}$.*

Proof. On one hand, (4.64) implies that $\tau\psi(y) = \sigma\theta\psi(y) = \sigma(y)$, and thus that $S^{-c_0}\tau(\psi(y))$ is equal to $S^{-c_0}\sigma(y) = x$. On the other hand, $\sigma(y_0)$ is equal to $\tau(\psi_0(y)) = \tau(\psi(y)_0)$; so, as $(-c_0, y)$ is a σ-factorization, $[0, |\sigma(y_0)|) = [0, |\tau(\psi(y)_0)|)$ contains $-c_0$. From these two things, we conclude that $(-c_0, \psi(y))$ is a τ-factorization of x. Then, since (Z, τ) is recognizable by Lemma 4.43, $(-c_0, \psi(y))$ and $(-f_0, z)$ are the same τ-factorization, that is, $c_0 = f_0$ and $\psi(y) = z$. We use this to compute, for $j \geq 0$,

$$f_j = -f_0 + |\tau(z_{[0,j)})| = -c_0 + |\tau(\psi(y)_{[0,j)})| = -c_0 + |\sigma(y_{[0,j)})| = c_j,$$

where in the last step we used that $\tau(\psi(y)) = \sigma(y)$. A similar computation shows that $f_j = c_j$ for $j < 0$ as well. □

The last lemma has the following important consequence. For any $x \in X$, the cut functions of its σ-factorization in Y and of its τ-factorization in Z are the same. Therefore, we can simply write $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$ and $(c, z) = \mathbf{F}_{(Z,\tau)}(x)$. This will be tacitly used in this subsection.

Lemma 4.45 *Let $x, \tilde{x} \in X$, $(c, z) = \mathbf{F}_{(Z,\tau)}(X)$ and $(\tilde{c}, \tilde{z}) = \mathbf{F}_{(Z,\tau)}(\tilde{x})$. If $z_0 = \tilde{z}_0$, then $x_{[c_{\mathfrak{L}(z)}, c_{\mathfrak{L}(z)+1})} = \tilde{x}_{[\tilde{c}_{\mathfrak{L}(\tilde{z})}, \tilde{c}_{\mathfrak{L}(\tilde{z})+1})}$ and $x_{[c_j, c_{j+1})} = \tilde{x}_{[\tilde{c}_j, \tilde{c}_{j+1})}$ for $j \in [-1, 1]$.*

Proof. Let $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$ and $(\tilde{c}, \tilde{y}) = \mathbf{F}_{(Y,\sigma)}(\tilde{x})$. Then, by Lemma 4.44 and the hypothesis

$$y_{\mathfrak{L}(y)} y_{-1} y_0 y_1 = \psi_0(y) = z_0 = \tilde{z}_0 = \psi_0(\tilde{y}) = \tilde{y}_{\mathfrak{L}(\tilde{y})} \tilde{y}_{-1} \tilde{y}_0 \tilde{y}_1.$$

Hence,

$$x_{[c_{\mathfrak{L}(z)}, c_{\mathfrak{L}(z)+1})} = \sigma(y_{\mathfrak{L}(y)}) = \sigma(\tilde{y}_{\mathfrak{L}(\tilde{y})}) = \tilde{x}_{[\tilde{c}_{\mathfrak{L}(\tilde{z})}, \tilde{c}_{\mathfrak{L}(\tilde{z})+1})}.$$

Similarly, $x_{[c_{-1}, c_2)} = \sigma(y_{-1} y_0 y_1)$ is equal to $\tilde{x}_{[\tilde{c}_{-1}, \tilde{c}_2)} = \sigma(\tilde{y}_{-1} \tilde{y}_0 \tilde{y}_1)$. □

Let

$$\mathcal{C}_{\mathsf{ap}} = \theta^{-1}(\mathcal{B}_{\mathsf{ap}}),\ \mathcal{C}_{\mathsf{wp}} = \theta^{-1}(\mathcal{B}_{\mathsf{wp}}) \text{ and } \mathcal{C}_{\mathsf{sp}} = \theta^{-1}(\mathcal{B}_{\mathsf{sp}}). \tag{4.65}$$

Note that, since $\mathcal{B}_{\mathsf{ap}} \cup \mathcal{B}_{\mathsf{wp}} \cup \mathcal{B}_{\mathsf{sp}}$ is a partition of $\mathcal{B}$, the sets $\mathcal{C}_{\mathsf{ap}}$, $\mathcal{C}_{\mathsf{wp}}$ and $\mathcal{C}_{\mathsf{sp}}$ form a partition of $\mathcal{C} = \theta^{-1}(\mathcal{B})$.

Proposition 4.46 *The following conditions hold:*

(1) $20\varepsilon \leq |\tau(a)| \leq 10d^2 n$ for all $a \in \mathcal{C}$.

(2) $\#\tau(\mathcal{C}_{\mathsf{ap}}) \leq 2d^{3d+6}$, $\#(\mathsf{root}\,\tau(\mathcal{C})) \leq 5d^{3d+6}$ and $\#\mathcal{C} \leq 7^4 d^{12d+36}\,\mathsf{pow\text{-}com}(X)^4$.

(3) (Z, τ) has dichotomous periods w.r.t. $(\mathcal{C}_{\mathsf{ap}}, \mathcal{C}_{\mathsf{sp}} \cup \mathcal{C}_{\mathsf{wp}})$ and 8ε.

Proof. We start with the proof of Items ((2)) and ((1)). We have, from the equation $\tau = \sigma\theta$ and (4.65), that

$$\tau(\mathcal{C}_{\mathsf{ap}}) = \sigma(\mathcal{B}_{\mathsf{ap}}) \text{ and } \tau(\mathcal{C}) = \mathcal{B}.$$

Thus, by Item ((1)) in Proposition 4.38, $\#\tau(\mathcal{C}_{\mathsf{ap}}) = \#\sigma(\mathcal{B}_{\mathsf{ap}}) \leq 2d^{3d+6}$ and $\#(\mathsf{root}\,\tau(\mathcal{C})) = \#(\mathsf{root}\,\sigma(\mathcal{B})) \leq 5d^{3d+6}$. Also, from the definition of $\mathcal{C}$ we get that $\#\mathcal{C} \leq \#\mathcal{B}^4$. Putting the bounds from Item ((1)) of Proposition 4.38 in this inequality gives that $\#\mathcal{C} \leq 7^4 d^{12d+36}\,\mathsf{pow\text{-}com}(X)^4$. Item ((1)) follows from the equation $\tau(\mathcal{C}) = \mathcal{B}$ and Item ((2)) of Proposition 4.38.

We now prove Item ((3)). Let $x \in X$ be arbitrary and define $(c, z) = \mathbf{F}_{(Z,\tau)}(x)$ and $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$. Equation (4.65) ensures that $y_0 \in \mathcal{B}_{\mathsf{ap}}$ if and only if $z_0 \in \mathcal{C}_{\mathsf{ap}}$. We also note that, by Lemma 4.44, $\mathsf{root}\,\tau(z_0) = \mathsf{root}\,\sigma(y_0)$. Therefore, Item ((3)) of this proposition follows Item ((4)) in Proposition 4.38. ☐

Remark 4.7 As was similarly observed in Remark 4.6, a consequence of Items ((3)) and ((1)) in Proposition 4.46 is that, for all $a \in \mathcal{C} \setminus \mathcal{C}_{\mathsf{ap}}$, $|\mathsf{root}\,\tau(a)| = \mathsf{per}(\tau(a))$.

Proposition 4.47 *Let $z \in \mathbb{Z}$.*

(1) If $i < j$ are integers such that $z_k \in \mathcal{C} \setminus \mathcal{C}_{\mathsf{ap}}$ for all $k \in [i, j)$, then $\mathsf{root}\,\tau(z_k) = \mathsf{root}\,\tau(z_i)$ if $k \in [i, j)$, $z_k \in \mathcal{C}_{\mathsf{sp}}$ if $k \in [i, j-1)$, and $z_k = z_{i+1}$ for all $i \in [i+1, j-1)$.

(2) If $z_0 \in \mathcal{C}_{\mathsf{sp}}$, then $\tau(a) = (\mathsf{root}\,\tau(a))^{2^r}$, where r is the unique integer for which $2^r|\,\mathsf{root}\,\tau(a)|$ belongs to $[20\varepsilon, 40\varepsilon)$.

Proof. Suppose that $i < j$ satisfy $z_k \in \mathcal{C} \setminus \mathcal{C}_{\mathsf{ap}}$ for all $k \in [i, j+1)$. Let us denote $\mathbf{F}_{(Y,\sigma)}(\tau(z))$ by (c, y). Then, by Lemma 4.44, $y_k = \theta(z_k) \in \mathcal{B} \setminus \mathcal{B}_{\mathsf{ap}}$ for all $k \in [i, j+1)$. In this context, Item ((1)) of Proposition 4.40 ensures that $\mathsf{root}\,\sigma(y_k) = \mathsf{root}\,\sigma(y_i)$ if $k \in [i, j+1)$ and $y_k = y_i \in \mathcal{B}_{\mathsf{sp}}$ if $k \in [i, j)$ We deduce, as $\tau(z_k) = \sigma(y_k)$, that $\mathsf{root}\,\tau(z_k) = \mathsf{root}\,\tau(z_i)$ for all $k \in [i, j+1)$. Also, for any $k \in [i, j)$, we have that $z_k \in \theta^{-1}(y_k) \in \mathcal{B}_{\mathsf{sp}}$. Now, since $y_k \in \mathcal{C} \setminus \mathcal{C}_{\mathsf{ap}}$ if $[i, j+1)$, we have that $y_{\mathfrak{L}(S^k y)} = y_{\mathfrak{L}(S^i y)}$ for all $k \in [i, j+1)$. Combining this with the fact that $y_k = y_i$ if $k \in [i, j)$ yields

$$\psi_0(S^k y) = y_{\mathfrak{L}(S^k y)} y_{k-1} y_k y_{k+1} = y_{\mathfrak{L}(S^i y)} y_i y_i y_i \text{ for all } k \in [i+1, j-1).$$

We conclude, using Lemma 4.44, that $z_k = z_{i+1}$ for $k \in [i+1, j-1)$. ☐

Lemmas 4.48, 4.50 and 4.49 will use the following notation:

$$E = 50d^2 n.$$

Observe that $|\sigma| \leq E$, $|\tau| \leq E$ and that E is bigger than or equal to the constants $7d^2$ and $50d^2 n$ appearing in Proposition 4.39.

Lemma 4.48 *Let $x, \tilde{x} \in X$ be such that $x_{[-3E,3E)} = \tilde{x}_{[-3E,3E)}$ and $\mathsf{per}(x_{[-\varepsilon,\varepsilon)}) > \varepsilon$. Let $(c, z) = \mathbf{F}_{(Z,\tau)}(x)$ and $(\tilde{c}, \tilde{z}) = \mathbf{F}_{(Z,\tau)}(\tilde{x})$. Then, $\tau(z_0) = \tau(\tilde{z}_0)$ and $\mathbf{F}^0_{(Z,\tau)}(S^{c_1} x) = \mathbf{F}^0_{(Z,\tau)}(S^{c_1} \tilde{x})$. In particular, $c_0 = \tilde{c}_0$ and $c_1 = \tilde{c}_1$.*

Proof. We use the notation $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$ and $(\tilde{c}, \tilde{y}) = \mathbf{F}_{(Y,\sigma)}(\tilde{x})$. Observe that the hypothesis $x_{[-3E,3E)} = \tilde{x}_{[-3E,3E)}$ allows us to use Item ((3)) in Proposition 4.38 to obtain that

$$\mathbf{F}^0_{(Y,\sigma)}(S^i x) = \mathbf{F}^0_{(Y,\sigma)}(S^i \tilde{x}) \text{ for all } i \in [0, 2E]. \tag{4.66}$$

In particular, $c_0 = \tilde{c}_0$ and $y_0 = \tilde{y}_0$. Thus, by Lemma 4.44, $\tau(z_0) = \sigma(y_0) = \sigma(\tilde{y}_0) = \tau(\tilde{z}_0)$. Also, for $j \in \{1, 2\}$ we have that $0 \leq c_j \leq c_2 \leq 2E$, so (4.66) can be applied to deduce that $\mathbf{F}^0_{(Y,\sigma)}(S^{c_j}x) = \mathbf{F}^0_{(Y,\sigma)}(S^{c_j}\tilde{x})$. Then,

$$y_{[0,3)} = \tilde{y}_{[0,3)} \text{ and } c_j = \tilde{c}_j \text{ for all } j \in [0, 3). \tag{4.67}$$

Before continuing, we prove that

$$y_0, \tilde{y}_0 \in \mathcal{C}_{\mathsf{ap}}. \tag{4.68}$$

We note that if $y_0 \in \mathcal{C} \setminus \mathcal{C}_{\mathsf{ap}}$, then Item ((3)) in Proposition 4.46 gives that $\mathsf{per}(x_{[c_0-8\varepsilon,c_1+9\varepsilon)}) \leq \varepsilon$, which is impossible since we assumed that $\mathsf{per}(x_{[-\varepsilon,\varepsilon)}) > \varepsilon$. Thus, $y_0 \in \mathcal{C}_{\mathsf{ap}}$. Similarly, $\tilde{y}_0 \in \mathcal{C}_{\mathsf{ap}}$ as $\tilde{x}_{[-\varepsilon,\varepsilon)} = x_{[-\varepsilon,\varepsilon)}$.

We can now finish the proof. The condition $c_1 = \tilde{c}_1$ follows from (4.67). Also, we have from Lemma 4.44 that $z_1 = \psi_0(Sy) = \mathfrak{L}(Sy)y_0y_1y_2$ and $\tilde{z}_1 = \psi_0(Sy) = \mathfrak{L}(S\tilde{y})\tilde{y}_0\tilde{y}_1\tilde{y}_2$. Now, Equation (4.68) guarantees that $\mathfrak{L}(Sy) = y_0$ and $\mathfrak{L}(S\tilde{y}) = \tilde{y}_0$. Therefore, by Equation (4.67), $z_1 = \tilde{z}_1$. We conclude, using that $c_1 = \tilde{c}_1$, that

$$\mathbf{F}^0_{(Z,\tau)}(S^{c_1}x) = (0, z_1) = (0, \tilde{z}_1) = \mathbf{F}^0_{(Z,\tau)}(S^{c_1}\tilde{x}). \tag{4.69}$$

□

Lemma 4.49 *Let $x, \tilde{x} \in X$, $(c, z) = \mathbf{F}_{(Z,\tau)}(x)$ and $(\tilde{c}, \tilde{z}) = \mathbf{F}_{(Z,\tau)}(\tilde{x})$. Suppose that $x_{[-4E,4E)} = \tilde{x}_{[-4E,4E)}$ and $z_0 \in \mathcal{C}_{\mathsf{ap}}$. Then, $\tau(z_0) = \tau(\tilde{z}_0)$ and $\mathbf{F}^0_{(Z,\tau)}(S^{c_1}x) = \mathbf{F}^0_{(Z,\tau)}(S^{c_1}\tilde{x})$. In particular, $c_0 = \tilde{c}_0$ and $c_1 = \tilde{c}_1$.*

Proof. The condition $z_0 \in \mathcal{C}_{\mathsf{ap}}$ implies, by Item ((3)) in Proposition 4.46, that $\mathsf{per}(x_{[c_0,c_1)}) > \varepsilon$. Thus, by Item ((2)) in Lemma 4.13, there is $i \in [c_0, c_1)$ such that $\mathsf{per}(x_{[i-\varepsilon,i+\varepsilon)}) > \varepsilon$. Now, since $|\tau| \leq E$, the hypothesis ensures that $(S^i x)_{[i-\varepsilon,i+\varepsilon)}$ is equal to $(S^i\tilde{x})_{[i-\varepsilon,i+\varepsilon)}$. Therefore, we can use Lemma 4.48 and conclude that $\tau(z_0) = \tau(\tilde{z}_0)$ and $\mathbf{F}^0_{(Z,\tau)}(S^{c_1}x) = \mathbf{F}^0_{(Z,\tau)}(S^{c_1}\tilde{x})$. □

Lemma 4.50 *Let $x, \tilde{x} \in X$ and $k \geq 0$ be an integer. Suppose that $\mathbf{F}^0_{(Z,\tau)}(x) = \mathbf{F}^0_{(Z,\tau)}(\tilde{x})$ and that $x_{[-3E,k+3E)} = \tilde{x}_{[-3E,k+3E)}$. Then, $\mathbf{F}^0_{(Z,\tau)}(S^i x) = \mathbf{F}^0_{(Z,\tau)}(S^i\tilde{x})$ for all $i \in [0, k]$.*

Proof. We only prove that $\mathbf{F}^0_{(Z,\tau)}(Sx) = \mathbf{F}^0_{(Z,\tau)}(S\tilde{x})$ if $\mathbf{F}^0_{(Z,\tau)}(x) = \mathbf{F}^0_{(Z,\tau)}(\tilde{x})$ and that $x_{[-3E,1+3E)} = \tilde{x}_{[-3E,1+3E)}$, as then an inductive argument on i gives the lemma.

Let us write $(c, y) = \mathbf{F}_{(Y,\sigma)}(x)$ and $(\tilde{c}, \tilde{y}) = \mathbf{F}_{(Y,\sigma)}(x)$. Then, by Lemma 4.44, $z = \psi(y)$ and $\tilde{z} = \psi(\tilde{y})$ satisfy $(c, z) = \mathbf{F}_{(Z,\tau)}(x)$ and $(\tilde{c}, \tilde{z}) = \mathbf{F}_{(Z,\tau)}(x)$. Hence, the hypothesis $\mathbf{F}^0_{(Z,\tau)}(x) = \mathbf{F}^0_{(Z,\tau)}(\tilde{x})$ is equivalent to $z_0 = \tilde{z}_0$ and $c_0 = \tilde{c}_0$. This implies two things:

(i) $y_{\mathfrak{L}(y)}y_{-1}y_0y_1 = \psi_0(y) = z_0 = \tilde{z}_0 = \psi_0(\tilde{y}) = \tilde{y}_{\mathfrak{L}(\tilde{y})}\tilde{y}_0\tilde{y}_1\tilde{y}_2$.

(ii) $c_1 = c_0 + |\sigma(y_0)| = \tilde{c}_1 + |\sigma(\tilde{y}_0)| = \tilde{c}_1$ and, similarly, $c_2 = \tilde{c}_2$.

We deduce that, for any $i \in [0, c_1)$,

$$\mathbf{F}^0_{(Z,\tau)}(S^i x) = (c_0 - i, z_0) = (\tilde{c}_0 - i, \tilde{z}_0) = \mathbf{F}^0_{(Z,\tau)}(S^i\tilde{x}).$$

In particular, if $c_1 > 0$ then $\mathbf{F}^0_{(Z,\tau)}(Sx) = (c_0 - 1, z_0)$ is equal to $\mathbf{F}^0_{(Z,\tau)}(S\tilde{x}) = (\tilde{c}_0 - 1, \tilde{z}_0)$ and the proof is complete.

We now assume that $c_1 = 1$ (so $\tilde{c}_1 = 1$ as well by (ii)). In this case, $\mathbf{F}^0_{(Z,\tau)}(Sx) = (0, z_1)$ and $\mathbf{F}^0_{(Z,\tau)}(S\tilde{x}) = (0, \tilde{z}_1)$; thus, it is enough to prove that $z_1 = \tilde{z}_1$.

We observe that, since $z_1 = \psi_0(Sy)$ and $\tilde{z}_1 = \psi_0(S\tilde{y})$,

(1) $z_1 = y_0 y_0 y_1 y_2$ if $y_0 \in \mathcal{B}_{\mathsf{ap}}$ and $z_1 = y_{\mathfrak{L}(y)} y_0 y_1 y_2$ if $y_0 \notin \mathcal{B}_{\mathsf{ap}}$, and

(2) $\tilde{z}_1 = \tilde{y}_0 \tilde{y}_0 \tilde{y}_1 \tilde{y}_2$ if $\tilde{y}_0 \in \mathcal{B}_{\mathsf{ap}}$ and $\tilde{z}_1 = \tilde{y}_{\mathfrak{L}(\tilde{y})} \tilde{y}_0 \tilde{y}_1 \tilde{y}_2$ if $\tilde{y}_0 \notin \mathcal{B}_{\mathsf{ap}}$.

From these relations and (i) we deduce that

$$z_1 = \tilde{z}_1 \text{ if and only if } y_2 = \tilde{y}_2.$$

Now, since $\mathbf{F}^0_{(Y,\sigma)}(x) = (c_0, y_0) = (\tilde{c}_0, \tilde{y}_0) = \mathbf{F}^0_{(Y,\sigma)}(\tilde{x})$ and since we assumed that $x_{[-3E,1+3E)} = \tilde{x}_{[-3E,1+3E)}$, we can use Item ((3)) in Proposition 4.38 to obtain that $\mathbf{F}^0_{(Y,\sigma)}(S^i x)$ is equal to $\mathbf{F}^0_{(Y,\sigma)}(S^i \tilde{x})$ for any $i \in [0, 2E]$. In particular, since c_2 satisfies $0 \leq c_2 \leq c_0 + 2|\sigma| \leq 2E$, we have that

$$(0, y_2) = \mathbf{F}^0_{(Y,\sigma)}(S^{c_2} x) = \mathbf{F}^0_{(Y,\sigma)}(S^{c_2} \tilde{x}) = \mathbf{F}^0_{(Y,\sigma)}(S^{\tilde{c}_2} \tilde{x}) = (0, \tilde{y}_2),$$

where we used that $c_2 = \tilde{c}_2$ by (ii). It follows that $y_2 = \tilde{y}_2$ and thus that $\mathbf{F}^0_{(Z,\tau)}(Sx)$ is equal to $\mathbf{F}^0_{(Z,\tau)}(S\tilde{x})$. □

4.7.2 Connecting two levels

In this subsection, we consider two of the codings constructed in Subsection 4.7.1 and prove several lemmas that relate them. We start by fixing the necessary notation.

Let X be a minimal infinite subshift, $n, n' \geq 1$ be integers and let d be the maximum of $\lceil p_X(n)/n \rceil$, $\lceil p_X(n')/n' \rceil$, $p_X(n+1) - p_X(n)$, $p_X(n'+1) - p_X(n')$, $\#\mathcal{A}$ and 10^4. Let $E = 50d^2 n$ and $E' = 50d^2 n'$. We will assume throughout the subsection that

$$n' \geq d^{2d^3+4} \cdot 500d^2 n. \tag{4.70}$$

We consider the recognizable codings $(Z \subseteq \mathcal{C}^{\mathbb{Z}}, \tau \colon \mathcal{C} \to \mathcal{A}^+)$ and $(Z' \subseteq \mathcal{C}'^{\mathbb{Z}}, \tau' \colon \mathcal{C}' \to \mathcal{A}^+)$ of X obtained from n and n' as in Subsection 4.7.1, respectively. Let also $\varepsilon' \in [n'/d^{2d^3+4}, n'/d)$ and $\varepsilon \in [n/d^{2d^3+4}, n/d)$ be the constants defined in Subsection 4.7.1, and let us denote by $\mathcal{C}_{\mathsf{ap}} \cup \mathcal{C}_{\mathsf{sp}} \cup \mathcal{C}_{\mathsf{wp}}$ and $\mathcal{C}'_{\mathsf{ap}} \cup \mathcal{C}'_{\mathsf{sp}} \cup \mathcal{C}'_{\mathsf{wp}}$ the partitions of $\mathcal{C}$ and $\mathcal{C}'$ defined there. Let $\mathfrak{L}(z) = \max\{k < 0 : z_k \in \mathcal{C}_{\mathsf{ap}}\}$ and $\mathfrak{L}'(z') = \{k < 0 : z'_k \in \mathcal{C}'_{\mathsf{ap}}\}$ for $z \in Z$ and $z' \in Z'$.

The crucial relation between (Z', τ') and (Z, τ) is the following inequality, which is a consequence of (4.70):

$$\varepsilon' \geq 10E. \tag{4.71}$$

Preliminary lemmas

We fix, for the rest of Subsection 4.7.2, a point $x \in X$ and the notation $(c, z) = \mathbf{F}_{(Z,\tau)}(x)$ and $(c', z') = \mathbf{F}_{(Z',\tau')}(x)$.

Lemma 4.51 *Suppose that* $\mathsf{per}(\tau'(z'_0)) \le \varepsilon$. *Let* $i \in [c'_0 - 7\varepsilon', c'_1 + 7\varepsilon')$ *and* j *be the integer satisfying* $i \in [c_j, c_{j+1})$. *Then,* $z_j = z_0 \in \mathcal{C}_{\mathsf{sp}}$, $|\operatorname{\mathsf{root}} \tau(z_j)| = |\operatorname{\mathsf{root}} \tau'(z'_0)|$ *and* $c_j = c_0 \pmod{|\operatorname{\mathsf{root}} \tau'(z'_0)|}$.

Proof. The condition $\mathsf{per}(\tau'(z'_0)) \le \varepsilon \le \varepsilon'$ implies that $z'_0 \in \mathcal{C}' \setminus \mathcal{C}'_{\mathsf{ap}}$. Hence, by Item ((3)) in Proposition 4.46,

$$x_{[c'_0-8\varepsilon', c'_1+8\varepsilon')} = (\operatorname{\mathsf{root}} \tau'(z'_0))^{\mathbb{Z}}_{[-8\varepsilon', |\tau'(z'_0)|+8\varepsilon')} \tag{4.72}$$

and

$$|\operatorname{\mathsf{root}} \tau'(z'_0)| = \mathsf{per}(\tau'(z'_0)) \le \varepsilon. \tag{4.73}$$

Equation (4.72) implies the following: If k is the integer satisfying $c'_0 - 7\varepsilon' \in [c_k, c_{k+1})$ and ℓ is the integer satisfying $c'_0 - 7\varepsilon' \in [c_\ell, c_{\ell+1})$, then $\tau(z_j) \in \mathcal{C} \setminus \mathcal{C}_{\mathsf{ap}}$ for all $j \in [k, \ell)$. Indeed, for any such j, we have that $\tau(z_j) = x_{[c_j, c_{j+1})}$ occurs in $x_{[c'_0-7\varepsilon'-3E, c'_1+7\varepsilon'+3E)}$, and so $\mathsf{per}(\tau(z_j)) \le \varepsilon$, which implies, by Item ((3)) in Proposition 4.46, that $z_j \in \mathcal{C} \setminus \mathcal{C}_{\mathsf{ap}}$. We can then use Item ((1)) in Proposition 4.47 to get that

$$z_j = z_{k+1} \in \mathcal{C}_{\mathsf{sp}} \text{ and } c_j = c_{k+1} \pmod{|\operatorname{\mathsf{root}} \tau(z_{k+1})|} \text{ for all } j \in [k+1, \ell-1).$$

Since $\varepsilon' \ge |\tau|$, $c_{k+1} \le c_0 \le c_{\ell-1}$, so we in particular have that

$$z_j = z_0 \in \mathcal{C}_{\mathsf{sp}} \text{ and } c_j = c_0 \pmod{|\operatorname{\mathsf{root}} \tau(z_0)|}. \tag{4.74}$$

We are now going to prove that $|\operatorname{\mathsf{root}} \tau(z_0)|$ is equal to $|\operatorname{\mathsf{root}} \tau'(z'_0)|$. The lemma would follow from this and (4.74).

Note that $\tau'(z'_0)$ and $\tau(z_0)$ occur in $x_{[c'_0-8\varepsilon', c'_1+8\varepsilon')}$. Also, Item ((1)) in Proposition 4.46 ensures that $\tau'(z'_0)$ and $\tau(z_0)$ have length at least 2ε. Then, as $\mathsf{per}(x_{[c'_0-8\varepsilon', c'_1+8\varepsilon')}) \le \varepsilon$ by (4.72) and (4.73), we can use Item ((1)) of Proposition 4.13 to deduce that

$$\mathsf{per}(\tau(z_0)) = \mathsf{per}(x_{[c'_0-8\varepsilon', c'_1+8\varepsilon')}) = \mathsf{per}(\tau'(z'_0)) \le \varepsilon.$$

In this situation, Item ((3)) in Proposition 4.46 guarantees that $z_0 \in \mathcal{C} \setminus \mathcal{C}_{\mathsf{ap}}$ and $|\operatorname{\mathsf{root}} \tau(z_0)| = \mathsf{per}(\tau(z_0)) = \mathsf{per}(\tau'(z'_0))$. Equation (4.73) then yields $|\operatorname{\mathsf{root}} \tau(z_0)| = |\operatorname{\mathsf{root}} \tau'(z'_0)|$. □

Lemma 4.52 *Assume that* $z'_0 \in \mathcal{C}' \setminus \mathcal{C}'_{\mathsf{ap}}$ *and that* $\mathsf{per}(\tau'(z'_0)) > \varepsilon$. *Let* $\tilde{x} \in X$ *and suppose that* $x_{[-3\varepsilon', 3\varepsilon')} = \tilde{x}_{[-3\varepsilon', 3\varepsilon')}$. *Then,* $\mathbf{F}^0_{(Z,\tau)}(x) = \mathbf{F}^0_{(Z,\tau)}(\tilde{x})$.

Proof. The hypothesis gives that $\mathsf{per}(x_{[c'_0, c'_1)}) = \mathsf{per}(\tau'(z'_0)) > \varepsilon$, and Item ((1)) in Proposition 4.46 that $|x_{[c'_0, c'_1)}| \ge 2\varepsilon$. Hence, we can use Item ((2)) of Proposition 4.13 to obtain $i_0 \in [c'_0, c'_1)$ such that

$$\mathsf{per}(x_{[i_0-\varepsilon, i_0+\varepsilon)}) > \varepsilon.$$

Now, since $z'_0 \in \mathcal{C}' \setminus \mathcal{C}'_{\mathsf{ap}}$, Item ((3)) of Proposition 4.46 applies, so

$$\mathsf{per}(x_{[c'_0-8\varepsilon', c'_1+8\varepsilon')}) \le \varepsilon'.$$

This implies, as $\varepsilon \le \varepsilon'$, that there exists $i \in [-2\varepsilon', -\varepsilon')$ such that $x_{[i-\varepsilon, i+\varepsilon)} = x_{[i_0-\varepsilon, i_0+\varepsilon)}$.

Our plan is to derive the lemma using Lemma 4.48 with $S^i x$ and $S^i \tilde{x}$. First, we note that

$$\mathsf{per}((S^i x)_{[-\varepsilon,\varepsilon)}) = \mathsf{per}(x_{[i-\varepsilon,i+\varepsilon)}) = \mathsf{per}(x_{[i_0-\varepsilon,i_0+\varepsilon)}) > \varepsilon. \tag{4.75}$$

Also, since $4E \leq \varepsilon'$ and $i \in [-2\varepsilon', -\varepsilon')$, we have that $[i-4E, i+3\varepsilon'+4E)$ is contained in $[-3\varepsilon', 3\varepsilon')$. Thus, by the hypothesis $x_{[-3\varepsilon',3\varepsilon')} = \tilde{x}_{[-3\varepsilon',3\varepsilon')}$,

$$\begin{aligned}(S^{i+E}x)_{[-4E,3\varepsilon'+3E)} &= x_{[i-3E,i+3\varepsilon'+4E)} \\ &= \tilde{x}_{[i-3E,i+3\varepsilon'+4E)} = (S^{i+E}\tilde{x})_{[-4E,3\varepsilon'+3E)}.\end{aligned} \tag{4.76}$$

In particular,

$$(S^i x)_{[-3E,3E)} = (S^i \tilde{x})_{[-3E,3E)}. \tag{4.77}$$

Equations (4.75) and (4.77) allow us to use Lemma 4.48 and deduce that

$$\mathbf{F}^0_{(Z,\tau)}(S^{i+E}x) = \mathbf{F}^0_{(Z,\tau)}(S^{i+E}\tilde{x}).$$

Furthermore, the last equation and (4.76) are the hypothesis of Lemma 4.50; hence,

$$\mathbf{F}^0_{(Z,\tau)}(S^{i+E+k}x) = \mathbf{F}^0_{(Z,\tau)}(S^{i+E+k}\tilde{x}) \text{ for any } k \in [0, 3\varepsilon'). \tag{4.78}$$

Since $i \in [-2\varepsilon', -\varepsilon')$ and $E \leq \varepsilon'$, $k := -(i+E)$ belongs to $[0, 3\varepsilon')$, so (4.78) gives that $\mathbf{F}^0_{(Z,\tau)}(x) = \mathbf{F}^0_{(Z,\tau)}(\tilde{x})$. □

Lemma 4.53 *Suppose that $z'_0 \in \mathcal{C}'_{\mathsf{sp}}$. Then,*

$$\mathbf{F}^0_{(Z,\tau)}(S^{c'_0+i}x) = \mathbf{F}^0_{(Z,\tau)}(S^{c'_1+i}x) \textit{ for any } i \in [-5\varepsilon', 5\varepsilon').$$

Proof. We consider two cases. First, we assume that $\mathsf{per}(\tau'(z'_0)) \leq \varepsilon$. This allows us to use Lemma 4.51 and obtain that, if $i \in [c'_0 - 7\varepsilon, c'_1 + 7\varepsilon)$ and j is the integer satisfying $i \in [c_j, c_{j+1})$, then

$$\begin{aligned}&c_j = c_0 \pmod{|\,\mathsf{root}\,\tau'(z'_0)|},\ z_j = z_0 \in \mathcal{C}_{\mathsf{sp}} \text{ and} \\ &\qquad |\,\mathsf{root}\,\tau(z_j)| = |\,\mathsf{root}\,\tau'(z'_0)|.\end{aligned} \tag{4.79}$$

Let $i \in [-5\varepsilon', 5\varepsilon')$ be arbitrary and denote by k and ℓ the integers satisfying $c'_0 + i \in [c_k, c_{k+1})$ and $c'_1 + i \in [c_\ell, c_{\ell+1})$. With this notation, $\mathbf{F}^0_{(Z,\tau)}(S^{c'_0+i}x) = (c_k - c'_0 - i, z_p)$ and $\mathbf{F}^0_{(Z,\tau)}(S^{c'_1+i}x) = (c_\ell - c'_1 - i, z_q)$, so we have to prove that $z_k = z_\ell$ and $c_k - c'_0 - i = c_\ell - c'_1 - i$.

We have, by (4.79), that $z_k = z_\ell$. Thus, it only rests to prove that $c_k - c'_0 - i = c_\ell - c'_1 - i$.

We note that the definition of k and ℓ ensures that

(i) $c_k \leq c'_0 + i < c_{k+1} = c_k + |\tau(z_k)|$; and

(ii) $c_\ell \leq c'_1 + i < c_{\ell+1} = c_\ell + |\tau(z_\ell)|$.

If we use the equality $c_\ell = c_k + (\ell - k)|\tau(z_0)|$, which is a consequence of (4.79), and that $c'_1 = c'_0 + |\tau'(z'_0)|$ to replace c_ℓ and c'_1 in (ii), we get that

$$c_k \leq c'_0 + i + (|\tau'(z'_0)| - (\ell - k)|\tau(z_0)|) < c_k + |\tau(z_0)|.$$

This and (i) yield

$$||\tau'(z'_0)| - (\ell - k)|\tau(z_0)|| < |\tau(z_0)|. \tag{4.80}$$

Now, since $z_0 \in \mathcal{C}_{\mathsf{sp}}$ and $|\operatorname{\mathsf{root}}\tau(z_k)| = |\operatorname{\mathsf{root}}\tau'(z'_0)|$ by (4.79) and since $z'_0 \in \mathcal{C}'_{\mathsf{sp}}$ by the hypothesis, the definition of $\mathcal{C}_{\mathsf{sp}}$ and $\mathcal{C}'_{\mathsf{sp}}$ in Proposition 4.46 guarantees that $|\tau(z_0)|$ divides $|\tau'(z'_0)|$. Therefore, the inequality in (4.80) is possible only if $|\tau'(z'_0)| = (\ell - k)|\tau(z_0)|$. We conclude, as (4.79) implies that $c_\ell = c_k + (\ell - k)|\tau(z_0)|$, that

$$c_\ell = c_k + (\ell - k)|\tau(z_0)| = c_k + |\tau'(z'_0)| = c_k + c'_1 - c'_0.$$

Hence, $c_k - c'_0 - i = c_\ell - c'_1 - i$ and the proof of the first case is complete.

Next, we assume $\mathsf{per}(\tau'(z'_0)) > \varepsilon$. Observe that the condition $z'_0 \in \mathcal{C}'_{\mathsf{sp}}$ implies that $z'_0 \in \mathcal{C}' \backslash \mathcal{C}'_{\mathsf{ap}}$. Hence, by Item ((3)) in Proposition 4.46, $x_{[c'_0 - 8\varepsilon', c'_1 + 8\varepsilon')} = (\operatorname{\mathsf{root}}\tau'(z'_0))^{\mathbb{Z}}_{[-8\varepsilon', |\tau'(z'_0)| + 8\varepsilon')}$ and $|\operatorname{\mathsf{root}}\tau'(z'_0)| = \mathsf{per}(\tau'(z'_0)) \leq \varepsilon$. In particular, if $i \in [-5\varepsilon', 5\varepsilon')$, then $x_{[c'_0 + i - 3\varepsilon', c'_0 + 3\varepsilon')}$ is equal to $x_{[c'_1 + i - 3\varepsilon', c'_1 + 3\varepsilon')}$. Then, the hypothesis of Lemma 4.52 is satisfied for $S^{c'_0 + i}x$ and $S^{c'_1 + i}x$, and thus we obtain that $\mathbf{F}^0_{(Z,\tau)}(S^{c'_0 + i}x) = \mathbf{F}^0_{(Z,\tau)}(S^{c'_1 + i}x)$ for any $i \in [-5\varepsilon', 5\varepsilon')$. □

Lemma 4.54 *Let $\tilde{x} \in X$ and $(\tilde{c}, \tilde{z}) = \mathbf{F}_{(Z,\tau)}(\tilde{x})$. Suppose that $z'_0 \in \mathcal{C}_{\mathsf{ap}}$, $k \geq 1$, and that $x_{[c_0, c_1 + k + 8\varepsilon')} = \tilde{x}_{[\tilde{c}_0, \tilde{c}_1 + k + 8\varepsilon')}$. Then,*

$$\mathbf{F}_{(Z,\tau)}(S^{c'_1 + i}x) = \mathbf{F}_{(Z,\tau)}(S^{\tilde{c}'_1 + i}\tilde{x}) \text{ for all } i \in [-7\varepsilon', k + 7\varepsilon'). \tag{4.81}$$

Proof. First, Item ((3)) in Proposition 4.46 ensures that $\mathsf{per}(x_{[c'_0 + 8\varepsilon', c'_1 - 8\varepsilon')}) > \varepsilon' \geq \varepsilon$. Thus, Item ((2)) in Proposition 4.13 ensures that there exists an integer m such that

$$m \in [c'_0 + 8\varepsilon', c'_1 - 8\varepsilon') \text{ and } \mathsf{per}(x_{[m-\varepsilon, m+\varepsilon)}) > \varepsilon. \tag{4.82}$$

Using that $4E \leq \varepsilon'$ and $m \in [c'_0 + 8\varepsilon', c'_1 - 8\varepsilon')$ it can be checked that $[m - 3E, c'_1 + 7\varepsilon' + 3E)$ is contained in $[c'_0, c'_1 + 8\varepsilon')$. Hence, by the hypothesis $x_{[c'_0, c'_1 + 8\varepsilon')} = \tilde{x}_{[\tilde{c}'_0, \tilde{c}'_1 + 8\varepsilon')}$,

$$\begin{aligned}(S^{m+E}x)_{[-4E, c'_1 + 7\varepsilon' - m + 3E)} &= x_{[m-3E, c'_1 + 7\varepsilon' + 4E)} \\ &= \tilde{x}_{[m - 3E + \tilde{c}'_0 - c'_0, c'_1 + 7\varepsilon' + 4E + \tilde{c}'_0 - c'_0)} \\ &= (S^{m + \tilde{c}'_0 - c'_0 + E}\tilde{x})_{[-4E, c'_1 + 7\varepsilon' - m + 3E)}.\end{aligned} \tag{4.83}$$

In particular, as $c'_1 - m \geq 0$ by (4.82),

$$(S^m x)_{[-3E, 3E)} = (S^{m + \tilde{c}'_0 - c'_0}\tilde{x})_{[-3E, 3E)}.$$

This and (4.82) allow us to use Lemma 4.48 and deduce that

$$\mathbf{F}^0_{(Z,\tau)}(S^{m+E}x) = \mathbf{F}^0_{(Z,\tau)}(S^{m + \tilde{c}'_0 - c'_0 + E}\tilde{x}).$$

The last equation and (4.83) imply that the hypothesis of Lemma 4.50 holds; therefore, for every $j \in [0, c'_1 + 7\varepsilon' - m)$, $\mathbf{F}^0_{(Z,\tau)}(S^{m+E+j}x) = \mathbf{F}^0_{(Z,\tau)}(S^{m-c'_0+\tilde{c}'_0+E+j}\tilde{x})$. Equivalently,

$$\mathbf{F}^0_{(Z,\tau)}(S^j x) = \mathbf{F}^0_{(Z,\tau)}(S^{-c'_0+\tilde{c}'_0+j}\tilde{x}) \text{ for all } j \in [m+E, c'_1 + (r-1)\varepsilon'). \tag{4.84}$$

We will derive (4.81) from this.

Let $i \in [-7\varepsilon', 7\varepsilon')$ be arbitrary. Then, (4.82) and the inequality $E \leq \varepsilon'$ imply that $j \coloneqq c'_1 + i \in [m+E, c'_1 + 7\varepsilon')$. Hence, Equation (4.84) gives that

$$\mathbf{F}^0_{(Z,\tau)}(S^{c'_1+i}x) = \mathbf{F}^0_{(Z,\tau)}(S^{i+c'_1-c'_0+\tilde{c}'_0}\tilde{x}). \tag{4.85}$$

Now, we observe that

$$c'_1 - c'_0 = |\tau(z'_0)| = |\tau(\tilde{z}'_0)| = \tilde{c}'_1 - \tilde{c}'_0,$$

so $i + c'_1 - c'_0 + \tilde{c}'_0 = i + \tilde{c}_1$. Therefore, the lemma follows from (4.85). □

Proposition 4.55 *Let $x, \tilde{x} \in X$ and suppose that $\mathbf{F}^0_{(Z',\tau')}(x) = \mathbf{F}^0_{(Z',\tau')}(\tilde{x})$. Then,*

$$\mathbf{F}^0_{(Z,\tau)}(S^i x) = \mathbf{F}^0_{(Z,\tau)}(S^i\tilde{x}) \text{ for all } i \in [c'_0 - 4\varepsilon', c'_2 - \varepsilon'). \tag{4.86}$$

Proof. The hypothesis implies that $c'_0 = \tilde{c}'_0$ and $z'_0 = \tilde{z}'_0$. Combining this with Lemma 4.45 yields

$$x_{[c'_{\mathfrak{L}'(z')}, c'_{\mathfrak{L}'(z')+1})} = \tilde{x}_{[\tilde{c}'_{\mathfrak{L}'(\tilde{z}')}, c_{\mathfrak{L}'(\tilde{z}')+1})}, \quad x_{[c'_j, c'_{j+1})} = \tilde{x}_{[c'_j, c'_{j+1})}$$
$$\text{and } c'_j = \tilde{c}'_j \text{ for } j \in [-1, 1]. \tag{4.87}$$

This and the lower bound in Item ((1)) of Proposition 4.46 ensure that

$$x_{[c'_0 - 8\varepsilon', c'_1 + 8\varepsilon')} = \tilde{x}_{[\tilde{c}'_0 - 8\varepsilon', \tilde{c}'_1 + 8\varepsilon')}. \tag{4.88}$$

Next, we show that the following facts hold.

(i) $(S^{c_k}x)_{[-8\varepsilon', 8\varepsilon')} = (S^{c_l}x)_{[-8\varepsilon', 8\varepsilon')}$ for all $\mathfrak{L}'(y') < k, l \leq 0$.

(ii) $(S^{\tilde{c}_k}\tilde{x})_{[-8\varepsilon', 8\varepsilon')} = (S^{\tilde{c}_l}\tilde{x})_{[-8\varepsilon', 8\varepsilon')}$ for all $\mathfrak{L}'(\tilde{y}') < k, l \leq 0$.

(iii) $x_{[c'_{\mathfrak{L}'(y')}, c'_{\mathfrak{L}'(y')+1} + 8\varepsilon')} = \tilde{x}_{[\tilde{c}'_{\mathfrak{L}'(\tilde{y}')}, \tilde{c}'_{\mathfrak{L}'(\tilde{y}')+1} + 8\varepsilon')}$.

We start with Item (i). If $\mathfrak{L}'(y') = -1$, then (i) is vacuously true. We assume that $\mathfrak{L}'(y') < -1$. Let k be such that $\mathfrak{L}'(y') < k < 0$. The definition of $\mathfrak{L}'$ ensures that $z'_k \in \mathcal{C}' \setminus \mathcal{C}'_{\mathsf{ap}}$. So, by Item ((3)) in Proposition 4.46, $s \coloneqq \mathsf{root}\,\tau'(z'_k)$ satisfies $x_{[c_k - 8\varepsilon', c_{k+1} + 8\varepsilon')} = s^{\mathbb{Z}}_{[-8\varepsilon', |\tau'(z'_k)| + 8\varepsilon')}$. In particular, as $|\tau'(z'_0)| = 0 \pmod{|s|}$,

$$x_{[c_k - 8\varepsilon', c_k + 8\varepsilon')} = s^{\mathbb{Z}}_{[-8\varepsilon', 8\varepsilon')} = (S^{|\tau'(z'_k)|}s^{\mathbb{Z}})_{[-8\varepsilon', 8\varepsilon')} = x_{[c_{k+1} - 8\varepsilon', c_{k+1} + 8\varepsilon')}.$$

Being this valid for all $k \in [\mathfrak{L}'(z') + 1, 0)$, an inductive argument gives (i). Fact (ii) follows analogously. For (iii), we use (4.88), (i) and (ii) to deduce that

$$x_{[c'_{\mathfrak{L}'(z')+1}, c'_{\mathfrak{L}'(z')+1} + 8\varepsilon')} = x_{[c_0 - 8\varepsilon', c_0 + 8\varepsilon')} = \tilde{x}_{[\tilde{c}_0 - 8\varepsilon', \tilde{c}_0 + 8\varepsilon')} = \tilde{x}_{[\tilde{c}'_{\mathfrak{L}'(\tilde{z}')+1}, \tilde{c}'_{\mathfrak{L}'(\tilde{z}')+1} + 8\varepsilon')}.$$

Fact (iii) follows from this and the first equality in (4.87).

Now, since $z'_{\mathfrak{L}'(z')} \in \mathcal{C}'_{\mathsf{ap}}$ and since (iii) holds, the hypothesis of Lemma 4.54 are satisfied. Therefore,

$$\mathbf{F}^0_{(Z,\tau)}(S^{c'_{\mathfrak{L}'(z')+1}+i}x) = \mathbf{F}^0_{(Z,\tau)}(S^{\tilde{c}'_{\mathfrak{L}'(\tilde{z}')+1}+i}\tilde{x}) \text{ for all } i \in [-7\varepsilon', 7\varepsilon'). \tag{4.89}$$

In order to continue, we need to consider two cases. We first assume that $\mathfrak{L}'(z') = -1$. Then, $z'_{-1} \in \mathcal{C}_{\mathsf{ap}}$, so (4.87) and Item ((3)) in Proposition 4.46 give

$$\mathsf{per}(\tilde{x}_{[\tilde{c}'_{-1}+8\varepsilon', \tilde{c}'_0-8\varepsilon')}) = \mathsf{per}(x_{[c'_{-1}+8\varepsilon', c'_0-8\varepsilon')}) > \varepsilon'.$$

This implies, by Item ((3)) in Proposition 4.46, that $\tilde{z}_{-1} \in \mathcal{C}_{\mathsf{ap}}$. Hence, $\mathfrak{L}'(\tilde{z}) = -1$ and then (4.86) follows from (4.89).

Next, we assume that $\mathfrak{L}'(z') \leq -2$. In this case, we first prove the following.

(a) $\mathbf{F}^0_{(Z,\tau)}(S^{c'_{\mathfrak{L}'(y')+1}+i}x) = \mathbf{F}^0_{(Z,\tau)}(S^{c'_{-1}+i}x)$ for all $i \in [-5\varepsilon', 5\varepsilon')$.

(b) $\mathbf{F}^0_{(Z,\tau)}(S^{\tilde{c}'_{\mathfrak{L}'(\tilde{y}')+1}+i}\tilde{x}) = \mathbf{F}^0_{(Z,\tau)}(S^{\tilde{c}'_{-1}+i}\tilde{x})$ for all $i \in [-5\varepsilon', 5\varepsilon')$.

We only prove (a) as (b) follows from an analogous argument. If $\mathfrak{L}'(z') = -2$, then (a) is trivially true. Assume then that $\mathfrak{L}'(z') \leq -3$. The definition of $\mathfrak{L}'$ ensures that $z'_j \in \mathcal{C}' \setminus \mathcal{C}_{\mathsf{ap}}$ for all $\mathfrak{L}'(z') + 1 \leq j \leq -1$. Thus, by Item ((1)) in Proposition 4.47, $z'_j \in \mathcal{C}_{\mathsf{sp}}$ for all $\mathfrak{L}'(z') + 1 \leq j \leq -2$. This allows us to inductively apply Lemma 4.53 and deduce that, for any $i \in [-5\varepsilon', 5\varepsilon')$,

$$\mathbf{F}^0_{(Z,\tau)}(S^{c'_{\mathfrak{L}'(z')+1}+i}x) = \mathbf{F}^0_{(Z,\tau)}(S^{c'_{\mathfrak{L}'(z')+2}+i}x) = \cdots = \mathbf{F}^0_{(Z,\tau)}(S^{c'_{-1}+i}x).$$

This shows (a).

Now, combining Equation (4.89), (a) and (b) produces

$$\mathbf{F}^0_{(Z,\tau)}(S^{c'_{-1}+i}x) = \mathbf{F}^0_{(Z,\tau)}(S^{\tilde{c}'_{-1}+i}\tilde{x}) \text{ for all } i \in [-5\varepsilon', 5\varepsilon'). \tag{4.90}$$

We are going to derive (4.86) from this and (4.87).

Let $i \in [c'_0 - 4\varepsilon', c'_2 - \varepsilon')$ be arbitrary. We note that (4.90) gives, in particular, that $x_{[c_{-1}-5\varepsilon', c_{-1}+5\varepsilon')} = \tilde{x}_{[\tilde{c}_{-1}-5\varepsilon', \tilde{c}_{-1}+5\varepsilon')}$. From this, (4.87) we get that

$$x_{[c_{-1}-5\varepsilon', c_2)} = \tilde{x}_{[\tilde{c}_{-1}-5\varepsilon', \tilde{c}_2)}. \tag{4.91}$$

In view of Equations (4.90) and (4.91) and of $3E \leq \varepsilon'$, the hypothesis of Lemma 4.50 holds; hence,

$$\mathbf{F}^0_{(Z,\tau)}(S^{c'_{-1}+j}x) = \mathbf{F}^0_{(Z,\tau)}(S^{\tilde{c}'_{-1}+j}\tilde{x}) \text{ for all } j \in [-4\varepsilon', c'_2 - c'_{-1} - \varepsilon').$$

We set $j = i - c'_{-1}$ and note that $j \in [-4\varepsilon', c'_2 - c'_{-1} - \varepsilon')$. Therefore, the last equation can be used to obtain that

$$\mathbf{F}^0_{(Z,\tau)}(S^i x) = \mathbf{F}^0_{(Z,\tau)}(S^{c'_{-1}+j}x) = \mathbf{F}^0_{(Z,\tau)}(S^{\tilde{c}'_{-1}+j}\tilde{x}) = \mathbf{F}^0_{(Z,\tau)}(S^{i+\tilde{c}'_{-1}-c'_{-1}}\tilde{x}).$$

Being $\tilde{c}'_{-1} = c'_{-1}$ by (4.87), we deduce that (4.86) holds. □

The connecting morphism

In this subsection, we build a morphism γ that connects (Z', τ') with (Z, τ). We start by introducing the auxiliary map $r\colon Z' \to \mathbb{Z}$ and proving some properties for it. The crucial Proposition 4.57 will allow us to define the connecting morphism γ. We finish the section with Propositions 4.60, 4.61 and 4.62, which will be crucial for proving $(\mathcal{P}_1)$ in Theorems 4.75 and 4.76.

For $z' \in Z'$ and $(c, z) = \mathbf{F}_{(Z,\tau)}(\tau'(z'))$, let

$$r(z') = \begin{cases} 0 & \text{if } \mathsf{per}(\tau'(z'_0)) \leq \varepsilon \\ \min\{i \geq 0 : z_i \in \mathcal{C}_{\mathsf{ap}}\} & \text{if } \mathsf{per}(\tau'(z'_0)) > \varepsilon \end{cases} \tag{4.92}$$

Lemma 4.56 *Let $z' \in Z'$ and $(c, z) = \mathbf{F}_{(Z,\tau)}(\tau'(z'))$.*

(1) If $z'_0 \in \mathcal{C}'_{\mathsf{ap}}$, then $c_{r(z')} \in [-\varepsilon', |\tau'(z'_0)| - 8\varepsilon')$.

(2) If $\varepsilon < |\operatorname{\mathsf{root}} \tau'(z'_0)| \leq \varepsilon'$ and i is the integer satisfying $|\tau'(z'_0)| \in [c_i, c_{i+1})$, then $c_{r(z')} \in [-\varepsilon', \varepsilon')$ and $c_{i+r(Sz')} \in [|\tau'(z'_0)| - \varepsilon', |\tau'(z'_0)| + \varepsilon')$.

(3) If $|\operatorname{\mathsf{root}} \tau'(z'_0)| \leq \varepsilon$, then $c_{r(z')} \in [-\varepsilon', \varepsilon')$.

Proof. We start with Item ((1)). Being $r(z')$ nonnegative by the definition of r, we have that $c_{r(z')} \geq c_0$. Hence, $c_{r(z')} \geq -|\tau| \geq -\varepsilon'$. To prove the other inequality, we note that the condition $z'_0 \in \mathcal{C}'_{\mathsf{ap}}$ implies, by Item ((3)) in Proposition 4.46, that $\mathsf{per}(\tau'(z')_{[8\varepsilon', |\tau'(z'_0)|-8\varepsilon')}) > \varepsilon$. Using Item (2) of Lemma 4.13, we get $k \in [8\varepsilon', |\tau'(z'_0)| - 8\varepsilon')$ satisfying $\mathsf{per}(\tau'(z')_{[k-\varepsilon', k+\varepsilon')}) > \varepsilon$. Let j be the integer satisfying $k \in [c_j, c_{j+1})$. Then, $\mathsf{per}(\tau'(z')_{[c_j-8\varepsilon, c_{j+1}+8\varepsilon)}) \geq \mathsf{per}(\tau'(z')_{[k-\varepsilon, k+\varepsilon)}) > \varepsilon$, so $z_j \in \mathcal{C}_{\mathsf{ap}}$ by Item ((3)) in Proposition 4.46. Also, since $c_{j+1} \geq k \geq 0$, we have that $j \geq 0$. We conclude, by the minimality condition in the definition of r, that $r(z') \leq j$. Therefore, $c_{r(z')} \leq c_j \leq k \leq |\tau'(z'_0)| - 8\varepsilon'$.

We now consider Item ((2)). Let $s = \operatorname{\mathsf{root}} \tau'(z'_0)$. Since $|s| \leq \varepsilon'$, Item ((3)) in Proposition 4.46 ensures that $\mathsf{per}(\tau'(z'_0)) = |s| \in (\varepsilon, \varepsilon']$. Thus, by Item (2) of Lemma 4.13, there is $k \in [0, |\tau'(z'_0)|)$ such that $\mathsf{per}(\tau'(z')_{[k-\varepsilon', k+\varepsilon')}) > \varepsilon$. Moreover, the condition $|s| \leq \varepsilon'$ implies, by Item ((3)) in Proposition 4.46, that $\mathsf{per}(\tau'(z')_{[-8\varepsilon', |\tau'(z'_0)|+8\varepsilon')})$ is at most ε'. Therefore, we can find $k_0 \in [0, \varepsilon')$ and $k_1 \in [|\tau'(z'_0)|, |\tau'(z'_0)| + \varepsilon')$ satisfying $\tau'(z')_{[k-\varepsilon, k+\varepsilon)} = \tau'(z')_{[k_0-\varepsilon, k_0+\varepsilon)} = \tau'(z')_{[k_1-\varepsilon, k_1+\varepsilon)}$. In particular,

$$\mathsf{per}(\tau'(z')_{[k-\varepsilon, k+\varepsilon)}) = \mathsf{per}(\tau'(z')_{[k_0-\varepsilon, k_0+\varepsilon)}) = \mathsf{per}(\tau'(z')_{[k_1-\varepsilon, k_1+\varepsilon)}) > \varepsilon. \tag{4.93}$$

Let $j_0, j_1 \in \mathbb{Z}$ be the integers satisfying $k_0 \in [c_{j_0}, c_{j_0+1})$ and $k_1 \in [c_{j_1}, c_{j_1+1})$. Observe that, by (4.93) and Item ((3)) in Proposition 4.46, z_{j_0} and z_{j_1} belong to $\mathcal{C}_{\mathsf{ap}}$. Also, since $c_{j_0+1} \geq k_0 \geq 0$ and $c_{j_1+1} \geq k_1 \geq |\tau'(z'_0)|$ (where i is the element defined in the statement of the lemma), we have that $j_0 \geq 0$ and $j_1 \geq i$. We conclude, from the definition of r, that $r(z') \leq j_0$ and $i + r(Sz') \leq j_1$. Therefore, $c_{r(z')} \leq c_{j_0} \leq k_0 \leq \varepsilon'$ and $c_{i+r(Sz')} \leq c_{j_1} \leq |\tau'(z'_0)| + \varepsilon'$. Finally, (4.92) ensures that $r(z') \geq 0$ and $i + r(Sz') \geq i$, so $c_{r(z')} \geq -|\tau| \geq -\varepsilon'$ and $c_{i+r(Sz')} \geq |\tau'(z'_0)| \geq |\tau'(z'_0)| - \varepsilon'$. This completes the proof of Item ((2)).

For Item ((3)), we note that the condition $|s| \leq \varepsilon$ implies that $\mathsf{per}(\tau'(z'_0))$. Hence, $r(z') = 0$ and $c_{r(z')} \in [-|\tau|, 0] \subseteq [-\varepsilon', \varepsilon')$. □

Proposition 4.57 *Let $z', \tilde{z}' \in Z'$, $(c, z) = \mathbf{F}_{(Z,\tau)}(\tau'(z'))$ and $(\tilde{c}, \tilde{z}) = \mathbf{F}_{(Z,\tau)}(\tau'(\tilde{z}'))$. We define i and j as the integers satisfying $|\tau'(z'_0)| \in [c_i, c_{i+1})$ and $|\tau'(\tilde{z}'_0)| \in [\tilde{c}_j, \tilde{c}_{j+1})$. If $z'_0 = \tilde{z}'_0$, then $c_{r(z')} = \tilde{c}_{r(\tilde{z}')}$, $c_{i+r(Sz')} = \tilde{c}_{j+r(S\tilde{z}')}$ and $z_{[r(z'),i+r(Sz'))}$ is equal to $\tilde{z}_{[r(\tilde{z}'),j+r(S\tilde{z}'))}$.*

Proof. We start with some observations that will be used throughout the proof. Since $z'_0 = \tilde{z}'_0$, Lemma 4.45 gives that

$$\tau'(z'_{-1}) = \tau'(\tilde{z}'_{-1}), \tau'(z'_0) = \tau'(\tilde{z}'_0) \text{ and } \tau'(z'_1) = \tau'(\tilde{z}'_1).$$

In particular,

$$\tau'(z'_0)_{[-|\tau'(z'_{-1})|, |\tau'(z'_0 z'_1)|)} = \tau'(\tilde{z}'_0)_{[-|\tau'(z'_{-1})|, |\tau'(z'_0 z'_1)|)}. \tag{4.94}$$

Also, since $z'_0 = \tilde{z}'_0$, we have from Proposition 4.55 that

$$\mathbf{F}^0_{(Z,\tau)}(S^k \tau'(z')) = \mathbf{F}^0_{(Z,\tau)}(S^k \tau'(\tilde{z}')) \text{ for all } k \in [-4\varepsilon', |\tau'(z'_0)| + 4\varepsilon'). \tag{4.95}$$

We now prove that

$$c_{r(z')} = \tilde{c}_{r(\tilde{z}')} \text{ and } z_{r(z')} = \tilde{z}_{r(\tilde{z}')}. \tag{4.96}$$

Note that Lemma (4.56) ensures that

$$c_{r(z')}, \tilde{c}_{r(\tilde{z}')} \in [-\varepsilon', |\tau'(z'_0)| - 8\varepsilon'). \tag{4.97}$$

Hence, from (4.95) we get that $\mathbf{F}^0_{(Z,\tau)}(S^{c_{r(z')}} \tau'(z')) = \mathbf{F}^0_{(Z,\tau)}(S^{c_{r(z')}} \tau'(\tilde{z}'))$. This implies the following: If ℓ is the integer satisfying $c_{r(z')} \in [\tilde{c}_\ell, \tilde{c}_{\ell+1})$, then

$$c_{r(z')} = \tilde{c}_\ell \text{ and } z_{r(z')} = \tilde{z}_\ell. \tag{4.98}$$

Note that $\ell \geq 0$ (as $\tilde{c}_{\ell+1} = c_{r(z')+1} \geq 0$). Being $\tau'(z'_0) = \tau'(\tilde{z}'_0)$, we get, from (4.92), that $r(\tilde{z}') \leq \ell$. In particular, $\tilde{c}_{r(\tilde{z}')} \leq \tilde{c}_\ell = c_{r(z')}$. A symmetric argument shows that $c_{r(z')} \leq \tilde{c}_{r(\tilde{z}')}$, which allows us to conclude that $\tilde{c}_{r(\tilde{z})} = c_{r(z')}$. Then, it follows from (4.98) that $\tilde{c}_{r(\tilde{z})} = \tilde{c}_\ell$. Therefore, $r(\tilde{z}) = \ell$, and thus $z_{r(z')} = \tilde{z}_\ell = \tilde{z}_{r(\tilde{z})}$ by (4.98). This proves (4.96).

Observe that (4.96) implies that $\mathbf{F}_{(Z,\tau)}(S^{c_{r(z')}} \tau'(z'))$ is equal to $\mathbf{F}_{(Z,\tau)}(S^{c_{r(z')}} \tau'(\tilde{z}'))$. This, (4.94) and (4.97) permit to use Lemma 4.50 and obtain that

$$\mathbf{F}^0_{(Z,\tau)}(S^k \tau'(z')) = \mathbf{F}^0_{(Z,\tau)}(S^k \tau'(\tilde{z}')) \text{ for all } k \in [c_{r(z')}, |\tau'(z'_0 z'_1)| - \varepsilon'). \tag{4.99}$$

Then, since $c_{r(z')} = \tilde{c}_{r(\tilde{z}')}$, we have, for any $k \in \mathbb{Z}$ such that $c_k \in [c_{r(z')}, |\tau'(z'_0 z'_1)| - \varepsilon')$, that

$$c_k = \tilde{c}_{k-r(z')+r(\tilde{z}')} \text{ and } z_k = \tilde{z}_{k-r(z')+r(\tilde{z}')}. \tag{4.100}$$

To continue, we consider two cases. Assume that $\mathsf{per}(\tau'(z'_1)) \leq \varepsilon$. We note that, since $\tau'(z'_1) = \tau'(\tilde{z}'_1)$, $\mathsf{per}(\tau'(\tilde{z}'_1)) \leq \varepsilon$. Hence, by (4.92), $r(Sz') = r(S\tilde{z}') = 0$. Now, the definition of i and j and (4.100) imply that $c_i = \tilde{c}_j$ and $z_i = \tilde{z}_j$. Therefore, $c_{i+r(Sz')} = \tilde{c}_{j+r(S\tilde{z}')}$ and $z_{i+r(Sz')} = \tilde{z}_{j+r(S\tilde{z}')}$. This completes the proof in this case.

Let us now assume that $\mathsf{per}(\tau'(z_1')) > \varepsilon$. We are going to argue as in the proof of (4.96). Being $\tau'(z_1') = \tau'(\tilde{z}_1')$, we have that $\mathsf{per}(\tau'(\tilde{z}_1')) > \varepsilon$. Hence, by the definition of r, $z_{i+r(Sz')}$ and $\tilde{z}_{j+r(S\tilde{z}')}$ belong to $\mathcal{C}_{\mathsf{ap}}$. Then, since $c_{i+r(Sz')} \in [|\tau'(z_0') - \varepsilon', |\tau'(z_0'z_1')| - 8\varepsilon')$ by Lemma 4.56, it follows from (4.99) that

$$\mathbf{F}^0_{(Z,\tau)}(S^{c_{i+r(Sz')}}\tau'(z')) = \mathbf{F}^0_{(Z,\tau)}(S^{c_{i+r(Sz')}}\tau'(\tilde{z}')).$$

Therefore, if k is the integer satisfying $c_{i+r(Sz')} \in [\tilde{c}_k, \tilde{c}_{k+1})$, then $c_{i+r(Sz')} = \tilde{c}_k$ and $z_{i+r(Sz')} = \tilde{z}_k$. As $\mathsf{per}(\tau'(z_1')) > \varepsilon$, we have that $\tilde{z}_k = z_{i+r(Sz')} \in \mathcal{C}_{\mathsf{ap}}$. Also, since $|\tau'(z_0')| = |\tau'(\tilde{z}_0')|$, $\tilde{c}_{k+1} = c_{i+r(Sz')+1} \geq |\tau'(\tilde{z}_0')|$, so $k \geq j$. The last two things imply, by the definition of $r(S\tilde{z}')$, that $\tilde{c}_{j+r(S\tilde{z}')} \leq \tilde{c}_k = c_{i+r(Sz')}$. Similarly, $c_{i+r(Sz')} \leq \tilde{c}_{j+r(S\tilde{z}')}$. We conclude that $k = j+r(S\tilde{z}')$, $\tilde{c}_{j+r(S\tilde{z}')} = c_{i+r(Sz')}$ and that $\tilde{z}_{j+r(S\tilde{z}')} = z_{i+r(Sz')}$. □

Definition 4.8 The last proposition allows us to define $\gamma\colon \mathcal{C}' \to \mathcal{C}^+$ in such a way that, if $z' \in Z'$, $(c,z) = \mathbf{F}_{(Z,\tau)}(\tau'(z'))$ and i is the integer satisfying $|\tau'(z_0')| \in [c_i, c_{i+1})$, then

$$\gamma(z_0') = z_{[r(z'),i+r(Sz'))}. \tag{4.101}$$

We call γ the *connecting morphism* from (Z',τ') to (Z,τ).

Remark 4.8 Let $z' \in Z'$ and $(c,z) = \mathbf{F}_{(Z,\tau)}(\tau'(z'))$. Then, (4.101) ensures that $r(z') + |\gamma(z_0')| = i + r(Sz')$, where i is the integer satisfying $|\tau'(z_0')| \in [c_i, c_{i+1})$. This relation will be freely used throughout this subsection.

The rest of this section is devoted to prove the main properties of γ. We first introduce some notation. Let $\rho(a') = \tau'(a')$ if $a' \in \mathcal{C}'_{\mathsf{ap}}$ and $\rho(a') = \mathsf{root}\,\tau'(a')$ if $a' \in \mathcal{C}' \setminus \mathcal{C}'_{\mathsf{ap}}$. We define $\psi(z') = (\rho(z'_{-1}), \tau'(z_0'), \rho(z_1'))$ if $z' \in Z'$. Let $\rho(a)$ and $\psi(z)$ be defined analogously for $a \in \mathcal{C}$ and $z \in Z$.

We fix, for the rest of the section, points $z', \tilde{z}' \in Z'$ and the notation $(c,z) = \mathbf{F}_{(Z,\tau)}(\tau'(z'))$ and $(\tilde{c},\tilde{z}) = \mathbf{F}_{(Z,\tau)}(\tau'(\tilde{z}'))$.

Lemma 4.58 *Let $x, \tilde{x} \in X$, $(c,z) = \mathbf{F}_{(Z,\tau)}(x)$ and $(\tilde{c},\tilde{z}) = \mathbf{F}_{(Z,\tau)}(\tilde{x})$. If $x_{[-\varepsilon',\varepsilon')} = \tilde{x}_{[-\varepsilon',\varepsilon')}$ and $z_0 \in \mathcal{C}_{\mathsf{ap}}$, then $\psi(z) = \psi(\tilde{z})$.*

Proof. The hypothesis implies that $(S^{c_0}x)_{[-E,E)} = (S^{c_0}\tilde{x})_{[-E,E)}$. Then, as $z_0 \in \mathcal{C}_{\mathsf{ap}}$, we can use Lemma 4.49 to deduce that $c_0 = \tilde{c}_0$ and $\tau(z_0) = \tau(\tilde{z}_0)$. It is left to show that $\rho(z_{-1}) = \rho(\tilde{z}_{-1})$ and $\rho(z_1) = \rho(\tilde{z}_1)$. We will only prove the first equality as the other follows from a similar argument.

There are three cases. Assume first that $z_{-1} \in \mathcal{C}_{\mathsf{ap}}$. Then, the hypothesis ensures that $(S^{c_0-1}x)_{[-E,E)} = (S^{c_0-1}\tilde{x})_{[-E,E)}$. Since $z_{-1} \in \mathcal{C}_{\mathsf{ap}}$, this permits using Lemma 4.49 with $S^{c_0-1}x$ and $S^{c_0-1}\tilde{x}$ to deduce that $\tau(z_{-1}) = \tau(\tilde{z}_{-1})$. The case $\tilde{z}_{-1} \in \mathcal{C}_{\mathsf{ap}}$ is analogous.

Let us now assume that $z_{-1}, \tilde{z}_{-1} \in \mathcal{C} \setminus \mathcal{C}_{\mathsf{ap}}$. We define $s = \mathsf{root}\,\tau(z_{-1})$ and $\tilde{s} = \mathsf{root}\,\tau(\tilde{z}_{-1})$. We have to prove that $s = \tilde{s}$. Observe that, by Item (3) in Proposition 4.46, $s^{\mathbb{Z}}_{[-8\varepsilon,0)} = x_{[c_0-8\varepsilon,c_0)}$ and $\tilde{s}^{\mathbb{Z}}_{[-8\varepsilon,0)} = \tilde{x}_{[\tilde{c}_0-8\varepsilon,\tilde{c}_0)}$. Being c_0 equal to $\tilde{c}_0$ and since $x_{[-\varepsilon',\varepsilon')} = \tilde{x}_{[-\varepsilon',\varepsilon')}$, we deduce that

$s^{\mathbb{Z}}_{[-8\varepsilon',0)} = \tilde{s}^{\mathbb{Z}}_{[-8\varepsilon',0)}$. Then, by Theorem 4.8, s and $\tilde{s}$ are power of a common word, which implies that $s = \tilde{s}$. □

Lemma 4.59 *Suppose that $\rho(z'_0) = \rho(\tilde{z}'_0)$ and that $\mathsf{per}(\tau'(z'_0)) > \varepsilon$. Assume that $\rho(z'_{-1}) = \rho(\tilde{z}'_{-1})$ or that $\mathsf{per}(\tau'(z'_0)) \le \varepsilon'$. Then, $c_{r(z')} = \tilde{c}_{r(\tilde{z}')}$ and $\psi(S^{r(z')}z) = \psi(S^{r(\tilde{z}')}\tilde{z})$.*

Proof. We first prove the lemma in the case $\mathsf{per}(\tau'(z'_0)) \le \varepsilon'$. Note that the condition $\rho(z'_0) = \rho(\tilde{z}'_0)$ and Items ((1)) and ((3)) in Proposition 4.46 guarantee that $\mathsf{per}(\tau'(\tilde{z}'_0)) = \mathsf{per}(\tau'(z'_0)) \le \varepsilon'$ and $z'_0, \tilde{z}'_0 \in \mathcal{C}' \setminus \mathcal{C}'_{\mathsf{ap}}$. The first thing and Lemma 4.56 give

$$c_{r(z')}, \tilde{c}_{r(\tilde{z})} \in [-\varepsilon', \varepsilon'). \tag{4.102}$$

The second thing and the hypothesis $\rho(z'_0) = \rho(\tilde{z}'_0)$ imply that

$$\tau'(z')_{[-8\varepsilon',8\varepsilon')} = \tau'(\tilde{z}')_{[-8\varepsilon',8\varepsilon')}. \tag{4.103}$$

Now, we know from the definition of r and the condition $\mathsf{per}(\tau'(z'_0)) = \mathsf{per}(\tau'(\tilde{z}'_0)) > \varepsilon$ that $z_{r(z')}, \tilde{z}_{r(\tilde{z}')} \in \mathcal{C}_{\mathsf{ap}}$. Hence, by Equations (4.102) and (4.103), we can use Lemma 4.49 and deduce the following: If i and j are the integers satisfying $c_{r(z')} \in [\tilde{c}_i, \tilde{c}_{i+1})$ and $\tilde{c}_{r(\tilde{z}')} \in [c_j, c_{j+1})$, then $c_{r(z')} = \tilde{c}_i$ and $\tilde{c}_{r(\tilde{z}')} = c_j$. Therefore, by the definition of r, that $\tilde{c}_{r(\tilde{z}')} \le \tilde{c}_i = c_{r(z')}$ and $c_{r(z')} \le c_j = \tilde{c}_{r(\tilde{z}')}$. We conclude that $c_{r(z')} = \tilde{c}_{r(\tilde{z}')}$. This and Equations (4.102) and (4.103) allow us to use Lemma 4.58, yielding $\psi(S^{r(z')}z) = \psi(S^{r(\tilde{z}')}\tilde{z})$.

We now assume that $\rho(z'_{-1}) = \rho(\tilde{z}'_{-1})$ and that $\mathsf{per}(\tau'(z'_0)) > \varepsilon'$. Then, $z'_0 \in \mathcal{C}'_{\mathsf{ap}}$, so, since $\rho(z'_0) = \rho(\tilde{z}'_0)$, we have that $\tau'(z'_0) = \tau'(\tilde{z}'_0)$. Combining this with the equation $\rho(z'_{-1}) = \rho(\tilde{z}'_{-1})$ and Item ((3)) of Proposition 4.46 produces

$$\tau'(z')_{[-8\varepsilon',|\tau'(z'_0)|+8\varepsilon')} = \tau'(\tilde{z}')_{[-8\varepsilon',|\tau'(z'_0)|+8\varepsilon')}. \tag{4.104}$$

Now, by Lemma 4.56,

$$c_{r(z')}, \tilde{c}_{r(\tilde{z}')} \in [-\varepsilon', |\tau'(z'_0)| - 8\varepsilon'). \tag{4.105}$$

Equations (4.104) and (4.105) imply that

$$(S^{c_{r(z')}}\tau'(z'))_{[-\varepsilon',\varepsilon')} = (S^{c_{r(z')}}\tau'(\tilde{z}'))_{[-\varepsilon',\varepsilon')} \\ \text{and } (S^{\tilde{c}_{r(\tilde{z}')}}\tau'(z'))_{[-\varepsilon',\varepsilon')} = (S^{\tilde{c}_{r(\tilde{z}')}}\tau'(\tilde{z}'))_{[-\varepsilon',\varepsilon')}. \tag{4.106}$$

Since $z_{r(z')}, \tilde{z}_{r(\tilde{z}')} \in \mathcal{C}_{\mathsf{ap}}$ by (4.92), we can use Lemma 4.49 to deduce the following: If i and j are the integers satisfying $c_{r(z')} \in [\tilde{c}_i, \tilde{c}_{i+1})$ and $\tilde{c}_{r(\tilde{z})} \in [c_j, c_{j+1})$, then $c_{r(z')} = \tilde{c}_i$ and $\tilde{c}_{r(\tilde{z})} = c_j$. We can then argue as in the first case to conclude that $c_{r(z')} = \tilde{c}_{r(\tilde{z}')}$ and $\psi(S^{r(z')}z) = \psi(S^{r(\tilde{z}')}\tilde{z})$. □

Proposition 4.60 *Suppose that $z'_0, \tilde{z}'_0 \in \mathcal{C}'_{\mathsf{ap}}$ and $\psi(z') = \psi(\tilde{z}')$. Then:*

(1) $c_{r(z')} = \tilde{c}_{r(\tilde{z}')}$ and $c_{r(z')+|\gamma(z'_0)|} = \tilde{c}_{r(\tilde{z}')+|\gamma(\tilde{z}'_0)|}$.

(2) $\psi(S^{r(z')}z) = \psi(S^{r(\tilde{z}')}\tilde{z})$ and $\psi(S^{r(z')+|\gamma(z'_0)|}z) = \psi(S^{r(\tilde{z}')+|\gamma(\tilde{z}'_0)|}\tilde{z})$.

PROOF. We have, from the condition $\psi(z') = \psi(\tilde{z}')$, that $\rho(z'_{-1}) = \rho(\tilde{z}'_{-1})$ and $\rho(z'_0) = \rho(\tilde{z}'_0)$. Also, since $z'_0 \in \mathcal{C}'_{\mathsf{ap}}$, we have that $\mathsf{per}(\tau'(z'_0)) > \varepsilon$. Hence, we can use Lemma 4.59 to deduce that

$$c_{r(z')} = \tilde{c}_{r(\tilde{z}')} \text{ and } \psi(S^{r(z')}z) = \psi(S^{r(\tilde{z}')}\tilde{z}). \tag{4.107}$$

Now, from Lemma 4.56 we have that $c_{r(z')} \in [-\varepsilon', |\tau'(z'_0)| - 8\varepsilon')$. Also, the hypothesis and Item ((1)) in Proposition 4.46 give

$$\tau'(z')_{[-8\varepsilon',|\tau'(z'_0)|+8\varepsilon')} = \tau'(\tilde{z}')_{[-8\varepsilon',|\tau'(z'_0)|+8\varepsilon')} \tag{4.108}$$

These two things, together with the fact that $z_{r(z')} \in \mathcal{C}_{\mathsf{ap}}$, allow us to use Lemma 4.49 and deduce that

$$\mathbf{F}^0_{(Z,\tau)}(S^{c_{r(z')+1}}\tau'(z')) = \mathbf{F}^0_{(Z,\tau)}(S^{c_{r(z')+1}}\tau'(z')). \tag{4.109}$$

In particular, $c_{r(z')+1} = \tilde{c}_{r(\tilde{z}')+1}$.

To continue, we have to consider two cases. We first assume that $z'_1 \in \mathcal{C}'_{\mathsf{ap}}$. Then, since $\psi(z') = \psi(\tilde{z}')$, we can use Lemma 4.59 to obtain that $c_{r(z')+|\gamma(z'_0)|} = \tilde{c}_{r(\tilde{z}')+|\gamma(\tilde{z}'_0)|}$ and $\psi(S^{r(z')+|\gamma(z'_0)|}z) = \psi(S^{r(\tilde{z}')+|\gamma(\tilde{z}'_0)|}\tilde{z})$.

It rests to consider the case $z'_1 \in \mathcal{C}' \setminus \mathcal{C}'_{\mathsf{ap}}$. Equations (4.108) and (4.109) enable us to use Lemma 4.50 and deduce that $\mathbf{F}^0_{(Z,\tau)}(S^k\tau'(z'))$ is equal to $\mathbf{F}^0_{(Z,\tau)}(S^k\tau'(\tilde{z}'))$ for all $k \in [c_{r(z')+1}, |\tau'(z'_0)| + 7\varepsilon')$. Since $c_{r(z')+1} \leq |\tau'(z'_0)| - 7\varepsilon'$, we in particular have that

$$\mathbf{F}^0_{(Z,\tau)}(S^k\tau'(z')) = \mathbf{F}^0_{(Z,\tau)}(S^k\tau'(\tilde{z}')) \text{ for all } k \in [|\tau'(z'_0)| - 7\varepsilon', |\tau'(z'_0)| + 7\varepsilon'). \tag{4.110}$$

Now, the condition $\psi(z') = \psi(\tilde{z}')$ implies that $\tilde{z}_1 \in \mathcal{C}' \setminus \mathcal{C}'_{\mathsf{ap}}$. Thus, by Lemma 4.56,

$$c_{r(z')+|\gamma(z'_0)|}, \tilde{c}_{r(\tilde{z}')+|\gamma(\tilde{z}'_0)|} \in [|\tau'(z'_0)| - \varepsilon', |\tau'(z'_0)| + \varepsilon').$$

We conclude, using (4.110), that $c_{r(z')+|\gamma(z'_0)|} = \tilde{c}_{r(\tilde{z}')+|\gamma(\tilde{z}'_0)}$ and $z_{r(z')+|\gamma(z'_0)|} = \tilde{z}_{r(\tilde{z}')+|\gamma(\tilde{z}'_0)}$. The lemma follows. □

Proposition 4.61 *Suppose that $\rho(z^n_0) = \rho(\tilde{z}^n_0)$ and $\varepsilon < |\operatorname{root}\tau'(z'_0)| \leq \varepsilon'$. Then:*

(1) $c_{r(z')} = c_{r_1(z')+|\gamma(z'_0)|} - |\tau'(z'_0)| = \tilde{c}_{r(\tilde{z}')} = \tilde{c}_{r(\tilde{z}')+|\gamma(\tilde{z}'_0)|} - |\tau'(\tilde{z}'_0)|$.

(2) $z_{r(z')} = z_{r(z')+|\gamma(z'_0)|} = \tilde{z}_{r(\tilde{z}')} = \tilde{z}_{r(\tilde{z}')+|\gamma(\tilde{z}'_0)|}$.

PROOF. Note that, by Item (3) in Proposition 4.46, $\mathsf{per}(\tau'(z'_0)) = |\operatorname{root}\tau'(z'_0)| \in (\varepsilon, \varepsilon']$. This and the condition $\rho(z^n_0) = \rho(\tilde{z}^n_0)$ permit to use Lemma 4.59 to obtain that $c_{r(z')} = \tilde{c}_{r(\tilde{z}')}$. Then, since $z'_0 \in \mathcal{C}' \setminus \mathcal{C}'_{\mathsf{ap}}$, from Lemma 4.56 we have that

$$c_{r(z')} = \tilde{c}_{r(\tilde{z}')} \in [-\varepsilon', \varepsilon').$$

Now, the hypothesis allows us to use Lemma 4.52 and deduce that $\mathbf{F}^0_{(Z,\tau)}(S^{c_{r(z')}}\tau'(z')) = \mathbf{F}^0_{(Z,\tau)}(S^{c_{r(z')}}\tau'(\tilde{z}'))$. Since $c_{r(z')} = \tilde{c}_{r(\tilde{z}')}$, we get that

$$z_{r(z')} = \tilde{z}_{r(\tilde{z}')}.$$

We use again Lemma 4.52 to obtain that $\mathbf{F}^0_{(Z,\tau)}(S^k\tau'(z')) = \mathbf{F}^0_{(Z,\tau)}(S^{k+|\tau'(z'_0)|}\tau'(\tilde{z}'))$ for all $k \in [-4\varepsilon', 4\varepsilon')$. This implies, since $c_{r(z')} \in [-\varepsilon', \varepsilon')$, that

$$c_{r(z')+|\gamma(z'_0)|} = c_{r(z')} + |\tau'(z'_0)|.$$

Similarly, $\tilde{c}_{r(\tilde{z}')+|\gamma(\tilde{z}'_0)|} = \tilde{c}_{r(\tilde{z}')} + |\tau'(z'_0)|$. □

Proposition 4.62 *Let $k, \ell \geq 1$ and s be such that $|s| \leq \varepsilon$ and $s = \mathsf{root}\,\tau'(z'_i) = \mathsf{root}\,\tau'(\tilde{z}'_j)$ for all $i \in [0,k)$ and $j \in [0,\ell)$. Then:*

(1) There is t such that $|t| = |s|$ and $t = \mathsf{root}\,\tau(z_i) = \mathsf{root}\,\tau(\tilde{z}_j)$ for all $i \in [r(z'), r(z') + |\gamma(z'_{[0,k)})|)$ and $j \in [r(\tilde{z}'), r(\tilde{z}') + |\gamma(\tilde{z}'_{[0,\ell)})|)$.

(2) $c_{r(z')} = \tilde{c}_{r(\tilde{z}')} = c_{r(z')+|\gamma(z'_{[0,k)})|} = \tilde{c}_{r(\tilde{z}')+|\gamma(\tilde{z}'_{[0,\ell)})} \pmod{|s|}$.

(3) $\psi(S^{r(z')}z) = \psi(S^{r(\tilde{z}')}\tilde{z})$ and, if $\rho(z'_k) = \rho(\tilde{z}'_\ell)$, then $\psi(S^{r(z')+|\gamma(z'_{[0,k)})|}z) = \psi(S^{r(\tilde{z}')+|\gamma(\tilde{z}'_{[0,\ell)})|}\tilde{z})$.

Proof. We note that, since $|s| \leq \varepsilon$, Lemma 4.56 implies that

$$c_{r(z')}, \tilde{c}_{r(\tilde{z}')} \in [-\varepsilon', \varepsilon'). \tag{4.111}$$

Hence, by Lemma 4.51, every $i \in \mathbb{Z}$ such that $c_i \in [-4\varepsilon', |\tau'(z'_{[0,k)})| + 4\varepsilon')$ satisfies

$$t :== \mathsf{root}\,\tau(z_{r(z')}) = \mathsf{root}\,\tau(z_i), |t| = |s| \text{ and } c_i = c_{r(z')} \pmod{|s|} \tag{4.112}$$

Similarly, for all $j \in \mathbb{Z}$ such that $\tilde{c}_j \in [-4\varepsilon', |\tau'(\tilde{z}'_{[0,\ell)})| + 4\varepsilon')$,

$$\tilde{t} :== \mathsf{root}\,\tau(\tilde{z}_{r(\tilde{z}')}) = \mathsf{root}\,\tau(\tilde{z}_j), |\tilde{t}| = |s| \text{ and } \tilde{c}_j = \tilde{c}_{r(\tilde{z}')} \pmod{|s|} \tag{4.113}$$

We will use these relations to prove the following:

$$c_{r(z')} = \tilde{c}_{r(\tilde{z}')} \pmod{|s|} \text{ and } t = \tilde{t}. \tag{4.114}$$

Since $|s| \leq \varepsilon \leq \varepsilon'$, we can use Item ((3)) in Proposition 4.46 and (4.111) to get that $s^{\mathbb{Z}}_{[c_{r(z')}, c_{r(z')}+8\varepsilon)} = \tau'(z')_{[c_{r(z')}, c_{r(z')}+8\varepsilon)} = t^{\mathbb{Z}}_{[0,8\varepsilon)}$. As $|s| = |t| \leq \varepsilon$, Item ((1)) of Lemma 4.10 gives that $S^{c_{r(z')}}s^{\mathbb{Z}} = t^{\mathbb{Z}}$. Similarly, $S^{\tilde{c}_{r(\tilde{z}')}}s^{\mathbb{Z}} = \tilde{t}^{\mathbb{Z}}$. We conclude that

$$S^{-c_{r(z')}}t^{\mathbb{Z}} = S^{-\tilde{c}_{r(\tilde{z}')}}\tilde{t}^{\mathbb{Z}}. \tag{4.115}$$

Since $|\,\mathsf{root}\,\tau(z_{r(z')})| = |\,\mathsf{root}\,\tau(\tilde{z}_{r(\tilde{z}')})| = |s|$, we deduce that t and $\tilde{t}$ are conjugate. Therefore, by Item ((3)) in Proposition 4.46, $t = \tilde{t}$. Putting this in (4.115) and then using Item ((2)) of Lemma 4.10 yields $c_{r(z')} = \tilde{c}_{r(\tilde{z}')} \pmod{|s|}$. This completes the proof of (4.114).

Let α be the integer satisfying $|\tau'(z'_{[0,k)})| \in [c_\alpha, c_{\alpha+1})$. We have, by (4.112), that $z_i \in \mathcal{C} \setminus \mathcal{C}_{\mathsf{ap}}$ for all $i \in [r(z'), \alpha)$. Also, by the definition of r, we have that $z_i \in \mathcal{C} \setminus \mathcal{C}_{\mathsf{ap}}$ for all $i \in [\alpha, \alpha + r(S^k z'))$. Hence, by ((1)) in Proposition 4.47, $\mathsf{root}\,\tau(z_i) = \mathsf{root}\,\tau(z_{r(z')}) = t$ for every $i \in [r(z'), \alpha + r(S^k z'))$. In particular, $c_{\alpha+r(S^k z')} = c_{r(z')} \pmod{|s|}$. Since $\alpha + r(S^k z') = r(z') + |\gamma(z'_{[0,k)})|$, we get that $c_{r(z')+|\gamma(z'_{[0,k)})|} = c_{r(z')} \pmod{|s|}$ and that $\mathsf{root}\,\tau(z_i) = t$ for every $i \in [r(z'), r(z') + |\gamma(z'_{[0,k)})|)$. We can prove in a similar way that $\tilde{c}_{r(\tilde{z}')+|\gamma(\tilde{z}'_{[0,\ell)})|} = \tilde{c}_{r(\tilde{z}')}$ (mod $|s|$) and that $\mathsf{root}\,\tau(\tilde{z}_j) = \tilde{t}$ for every $j \in [r(\tilde{z}'), r(\tilde{z}') + |\gamma(\tilde{z}'_{[0,\ell)})|)$. Being t equal to $\tilde{t}$, we obtain Item (1). Moreover, since $c_{r(z')} = \tilde{c}_{r(\tilde{z}')}$ (mod $|s$), we also have Item (2).

It is left to prove Item (3). We note that, since $t = \tilde{t}$, Equations (4.112) and (4.113) imply that $\psi(S^{r(z')}z) = \psi(S^{r(\tilde{z}')}\tilde{z}) = (t,t,t)$. Let us now assume that $\rho(z'_k) = \rho(\tilde{z}'_\ell)$. There are two cases. First, we assume that $|\,\mathsf{root}\,\tau'(z'_k)| \leq \varepsilon$. Then, by Lemma 4.56, $c_{r(z')+|\gamma(z'_{[0,k)})|} \in [|\tau'(z'_{[0,k)})| - \varepsilon', |\tau'(z'_{[0,k)})| + \varepsilon')$. We get, using (4.112), that $\psi(S^{r(z')+|\gamma(z'_{[0,k)})|}z) = (t,t,t)$. Now, since $\rho(z'_k) = \rho(\tilde{z}'_\ell)$, we have that $|\,\mathsf{root}\,\tau'(\tilde{z}'_\ell)| \leq \varepsilon$. Hence, a similar argument shows that $\psi(S^{r(\tilde{z})+|\gamma(\tilde{z}'_{[0,\ell)})|}\tilde{z}) = (t,t,t) = \psi(S^{r(z')+|\gamma(z'_{[0,k)})|}z)$. Next, we assume that $|\,\mathsf{root}\,\tau'(z'_k)| > \varepsilon$. Then, as $\rho(z'_k) = \rho(\tilde{z}'_\ell)$ and $\rho(z'_{k-1}) = \rho(\tilde{z}'_{\ell-1}) = s$, we can use Lemma 4.59 with z'_k and $\tilde{z}'_\ell$ to deduce that $\psi(S^{r(\tilde{z})+|\gamma(\tilde{z}'_{[0,\ell)})|}\tilde{z}) = \psi(S^{r(z')+|\gamma(z'_{[0,k)})|}z)$. □

4.8 Main Theorems

We now complete the proofs of Theorems 4.75 and 4.76. The part of the proof in which we have to obtain a complexity restriction from the $\mathcal{S}$-adic structures can be done without difficulties with Lemma 4.74. For the other part, we first present in Theorem 4.63 sufficient condition under which an $\mathcal{S}$-adic structure as the ones in Theorems 4.75 and 4.76 can be obtained. Then, we check that linear-growth and nonsuperlinear-growth complexity subshifts satisfy these conditions using Lemmas 4.6 and 4.7.

4.8.1 A set of sufficient conditions

This subsection is devoted to prove the following theorem.

Theorem 4.63 *Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be an infinite minimal subshift. Let $(\ell_n)_{n\geq 0}$ be an increasing sequence of positive integers and $d \geq \max\{10^4, \#\mathcal{A}\}$. Suppose that for every $n \geq 0$*

$$p_X(\ell_n) \leq d,\ p_X(\ell_n + 1) - p_X(\ell_n) \leq d, \text{ and } \frac{\ell_{n+1}}{\ell_n} \geq 10^4 d^{2d^3+6}. \tag{4.116}$$

Then, there exists a recognizable $\mathcal{S}$-adic sequence $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ generating X such that for all $n \geq 1$:

($\mathcal{P}_1$) $\#(\mathsf{root}\,\sigma_{[0,n)}(\mathcal{A}_n)) \leq 35 d^{12d+24}$ *and* $\#\mathcal{A}_n \leq 7^4 d^{12d+36} \cdot \mathsf{pow\text{-}com}(X)^4$.

($\mathcal{P}_2$) $|\sigma_{[0,n)}(a)| \leq 4d^{d^3+6} \cdot |\sigma_{[0,n)}(b)|$ *for every $a, b \in \mathcal{A}_n$.*

($\mathcal{P}_3$) $|\sigma_{n-1}(a)| \leq 40 d^{2d^3+8} \cdot \frac{\ell_n}{\ell_{n-1}}$ *for every $a \in \mathcal{A}_n$.*

The proof is presented as a series of lemmas.

We fix an infinite minimal subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, an increasing sequence $(\ell_n)_{n\geq 0}$ of positive integers and $d \geq \max\{10^4, \#\mathcal{A}\}$ such that (4.116) holds for every $n \geq 0$.

We start by defining $\boldsymbol{\sigma}$. Let $(Z_n \subseteq \mathcal{C}_n^{\mathbb{Z}}, \tau_n : \mathcal{C}_n \to \mathcal{A}^+)$ be the coding constructed in Subsection 4.7.1 using ℓ_n, and let $\varepsilon_n \in [\ell_n/d^{2d^3+4}, \ell_n/d)$ and $\mathcal{C}_n = \mathcal{C}_{n,\mathrm{ap}} \cup \mathcal{C}_{n,\mathrm{wp}} \cup \mathcal{C}_{n,\mathrm{sp}}$ be the constant and the partition that appear in this construction. In this context, Proposition 4.46 states the following:

(i) $(\# \,\mathsf{root}\, \tau_n(\mathcal{C}_n)) \leq 5d^{3d+6}$, $\#\mathcal{C}_n \leq 7^4 d^{12d+36}\, \mathsf{pow\text{-}com}(X)^4$ and $\#\tau(\mathcal{C}_{n,\mathrm{ap}}) \leq 2d^{3d+6}$.

(ii) $2\varepsilon_n \leq |\tau_n(a)| \leq 10d^2\ell_n$ for all $a \in \mathcal{C}_n$.

Also, the definition of $\mathcal{C}_n$ in (4.63) guarantees that

(iii) For all $a \in \mathcal{C}_n$ there is $z \in Z_n$ such that $z_0 = a$.

Moreover, (4.116) implies that $500d^2\ell_n \leq \ell_{n+1}/d^{2d^3+4}$, so the results from Subsection 4.7.2 can be used with (Z_{n+1}, ℓ_{n+1}) and (Z_n, τ_n). In particular,

(iv) Propositions 4.57, 4.60, 4.61 and 4.62 can be used with (Z_{n+1}, ℓ_{n+1}) and (Z_n, τ_n).

We define the map r_n as follows. If $z' \in Z_{n+1}$ and $(c, z) = \mathbf{F}_{(Z_n,\tau_n)}(\tau_{n+1}(z'))$, then

$$r_n(z') = \begin{cases} 0 & \text{if } \mathsf{per}(\tau_{n+1}(z'_0)) \leq \varepsilon \\ \min\{i \geq 0 : z_i \in \mathcal{C}_{n,\mathrm{ap}}\} & \text{if } \mathsf{per}(\tau_{n+1}(z'_0)) > \varepsilon \end{cases} \tag{4.117}$$

Note that this is analogous to the definition of r in (4.92). Therefore, Proposition 4.57 ensures that the connecting morphism $\sigma_n : \mathcal{C}_{n+1} \to \mathcal{C}_n^+$ described in Definition 4.8 is well-defined. The morphism σ_n satisfies the following: If $z' \in Z_{n+1}$, $(c, z) = \mathbf{F}_{(Z_n,\tau_n)}(\tau_{n+1}(z'))$ and i is the integer satisfying $|\tau_{n+1}(z'_0)| \in [c_i, c_{i+1})$, then

$$\sigma_n(z'_0) = z_{[r_n(z'), i+r_n(Sz'))}. \tag{4.118}$$

We set $\sigma_0 = \tau_0$ and $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$.

Next, we describe $\sigma_{[0,n)}(z'_0)$ in terms of $\tau_n(z'_0)$ and the auxiliary functions $q_{j,n}$ that we now define. For $z' \in Z_n$, we set $q_{n,n}(z') = 0$ and then inductively define, for $0 \leq j < n$,

$$q_{j,n}(z') = q_{j+1,n}(z') + c_{r_j(z)}, \tag{4.119}$$

where $(c, z) = \mathbf{F}_{(Z_{j+1},\tau_{j+1})}(S^{q_{j+1,n}(z')}\tau_n(z'))$. An inductive use of (4.119) yields the formula

$$\sigma_{[0,n)}(z'_0) = \tau_n(z')_{[q_{0,n}(z'), |\tau_n(z'_0)|+q_{0,n}(Sz'))}. \tag{4.120}$$

In particular,

$$\sigma_{[0,n)}(z') = S^{q_{0,n}(z')}\tau_n(z') \text{ for all } n \geq 1 \text{ and } z' \in Z_n. \tag{4.121}$$

We now prove that $\boldsymbol{\sigma}$ satisfies all the conditions in Theorem 4.63.

Lemma 4.64 *Let $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ be an $\mathcal{S}$-adic sequence. Suppose there are subshifts $Z_n \subseteq \mathcal{A}_n^{\mathbb{Z}}$ satisfying $\mathcal{A}_n \subseteq \mathcal{L}(Z_n)$. Then, for every $x \in X_{\boldsymbol{\tau}}$ there are sequences $(n_\ell)_{\ell\geq 0}$ and $x_\ell \in \cup_{k\in\mathbb{Z}} S^k\tau_{[0,n_\ell)}(Z_{n_\ell})$ such that x is the limit of $(x_\ell)_{\ell\geq 0}$.*

Proof. Let $x \in X_{\boldsymbol{\tau}}$. Then, for all $\ell \geq 0$ there exist $n_\ell \geq 0$ and $a_\ell \in \mathcal{A}_{n_\ell}$ for which $x_{[-\ell,\ell)}$ occurs in $\tau_{[0,n_\ell)}(a_\ell)$. The hypothesis permits to find $z_\ell \in Z_{n_\ell}$ such that $(z_\ell)_0 = a_\ell$. Being $x_{[-\ell,\ell)}$ a subword of $\tau_{[0,n_\ell)}((z_\ell)_0)$, there is a point of the form $x_\ell = S^{k_\ell}\tau_{[0,n_\ell)}(z_\ell)$ satisfying $(x_\ell)_{[-\ell,\ell)} = x_{[-\ell,\ell)}$. Then, x is the limit of $(x_\ell)_{\ell\geq 0}$. The lemma follows. □

Lemma 4.65 *The $\mathcal{S}$-adic sequence $\boldsymbol{\sigma}$ is recognizable and generates X.*

Proof. First, we show that $\boldsymbol{\sigma}$ generates X. Note that (4.121) ensures that

$$\bigcup_{k\in\mathbb{Z}} S^k\sigma_{[0,n)}(Z_n) \subseteq \bigcup_{k\in\mathbb{Z}} S^k\tau_n(Z_n) = X.$$

Now, thanks to Condition (iii), we can use Lemma 4.64, so any $x \in X_{\boldsymbol{\sigma}}$ is an adherent point of a sequence $x_n \in \cup_{k\in\mathbb{Z}} S^k\sigma_{[0,n)}(Z_n) = X$. Therefore, $X_{\boldsymbol{\sigma}} \subseteq X$. We conclude that $X_{\boldsymbol{\sigma}} = X$ by the minimality of X.

It rests to prove that $(Z_n, \sigma_{[0,n)})$ is recognizable. Let (k, z) and $(\tilde{k}, \tilde{z})$ be two $\sigma_{[0,n)}$-factorizations in Z_n of $x \in X$. Then, Equation (4.121) implies that

$$S^{k+q_{0,n}(z)}\tau_n(z) = S^k\sigma_{[0,n)}(z) = S^{\tilde{k}}\sigma_{[0,n)}(\tilde{z}) = S^{\tilde{k}+q_{0,n}(\tilde{z})}\tau_n(\tilde{z}).$$

In particular, $S^\ell\tau_n(z) = \tau_n(\tilde{z})$ where $\ell = k + q_{0,n}(z) - \tilde{k} - q_{0,n}(\tilde{z})$. Without loss of generality, we assume that $\ell \geq 0$. We can find $i \geq 0$ such that $(\ell - |\tau_n(z_{[0,i)})|, S^i z)$ is a τ_n-factorization of $S^\ell\tau_n(z)$ in Z_n. Then, as $(0, \tilde{z})$ is a τ_n-factorization of $S^\ell\tau_n(z)$ in Z_n, we deduce from the recognizability property of (Z_n, τ_n) that

$$\ell = k + q_{0,n}(z) - \tilde{k} - q_{0,n}(\tilde{z}) = |\tau_n(z_{[0,i)})| \text{ and } S^i z = \tilde{z} \tag{4.122}$$

Using this and the fact that (k, z) and $(\tilde{k}, \tilde{z})$ are $\sigma_{[0,n)}$-factorizations of x, we can write

$$\begin{aligned}\sigma_{[0,n)}(z) = S^{\tilde{k}-k}\sigma_{[0,n)}(\tilde{z}) = S^{\tilde{k}-k}\sigma_{[0,n)}(S^i z)\\ = S^{q_{0,n}(z)-q_{0,n}(\tilde{z})-|\tau_n(z_{[0,i)})|+|\sigma_{[0,n)}(z_{[0,i)})|}\sigma_{[0,n)}(z).\end{aligned}$$

Being Z_n aperiodic (as X is aperiodic), we get that $q_{0,n}(z) - q_{0,n}(\tilde{z}) + |\sigma_{[0,n)}(z_{[0,i)})|$ is equal to $|\tau_n(z_{[0,i)})|$. Putting this in (4.122) produces $|\sigma_{[0,n)}(z_{[0,i)})| = k - \tilde{k}$. Since $k \in [0, |\sigma_{[0,n)}(z_0)|)$ and $\tilde{k} \in [0, |\sigma_{[0,n)}(\tilde{z}_0)|)$, we obtain that

$$|\sigma_{[0,n)}(z_{[0,i)})| \leq k < |\sigma_{[0,n)}(z_0)|.$$

We deduce that $i = 0$, and then, from (4.122), that $z = \tilde{z}$ and $k = \tilde{k}$. □

Before continuing, we give some bounds for $q_{0,n}$.

Lemma 4.66 *Let $n \geq 1$ and $z' \in Z_n$. Then,*

$$-2\varepsilon_n \leq q_{0,n}(z') \leq |\tau_n(z'_0)| - 7\varepsilon_n. \tag{4.123}$$

Moreover, if $z'_0 \in \mathcal{C}_n \setminus \mathcal{C}_{n,\mathrm{ap}}$, then

$$-2\varepsilon_n \leq q_{0,n}(z') \leq 2\varepsilon_n. \tag{4.124}$$

Proof. Lemma (4.56) gives the bound $-\varepsilon_j \le c_{r(z)} \le |\tau_{j+1}(z_0)| - 8\varepsilon_j$ for all $0 \le j < n$ and $z \in Z_{j+1}$. Thus, from (4.120), $-\varepsilon_n \le q_{n-1,n}(z') \le |\tau_n(z'_0)| - 8\varepsilon_n$ and $q_{j+1,n}(z') - |\tau_{j+1}| \le q_{j,n}(z') \le q_{j+1,n}(z') + |\tau_{j+1}|$.. We obtain that

$$-\varepsilon_n - \sum_{j=0}^{n-2} |\tau_{j+1}| \le q_{0,n}(z') \le |\tau_n(z'_0)| - 8\varepsilon_n + \sum_{j=0}^{n-2} |\tau_{j+1}|. \tag{4.125}$$

Now, since $|\tau_j| \le 10d^2\ell_j$ and $500d^{2d^3+6} \cdot \ell_j \le \ell_{j+1}$, we have the bound $d^{n-2-j}|\tau_j| \le \ell_{n-1}$ for every $j \in [0, n-1)$. Therefore,

$$\sum_{j=0}^{n-2} |\tau_{j+1}| \le \sum_{j=0}^{n-2} \frac{1}{d^{n-2-j}} \ell_{n-1} \le 2d\ell_{n-1} \le \varepsilon_n.$$

Putting this in (4.125) yields (4.123). Moreover, if $z'_0 \in \mathcal{C}_n \setminus \mathcal{C}_{n,\mathrm{ap}}$, then Lemma 4.56 gives that $q_{n-1,n}(z') \in [-\varepsilon_n, \varepsilon_n)$. So, the previous argument shows, in this case, that $q_{0,n}(z') \in [-2\varepsilon_n, 2\varepsilon_n)$. ☐

Lemma 4.67 *For every $n \ge 1$ and $z' \in Z_n$,*

$$\frac{5}{d^{d^3+4}} \ell_n \le |\sigma_{[0,n)}(z'_0)| \le 20d^2\ell_n. \tag{4.126}$$

In particular, $\boldsymbol{\sigma}$ satisfies Items $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$ of Theorem 4.63.

Proof. We first show that (4.126) implies that $\boldsymbol{\sigma}$ satisfies Items $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ of Theorem 4.63. Observe that, by Condition (iii), (4.126) gives, for every $n \ge 1$ and $a, b \in \mathcal{C}_n$, that

$$|\sigma_{[0,n)}(a)| \le 20d^2\ell_n \le 4d^{d^3+6} \cdot |\sigma_{[0,n)}(b)|. \tag{4.127}$$

Thus, $(\mathcal{P}_2)$ is satisfied. For Item $(\mathcal{P}_3)$, we note that, for any pair of morphisms ξ and ξ' such that $\xi\xi'$ is defined, we have that $|\xi\xi'| \ge \langle \xi \rangle |\xi'|$. Therefore,

$$|\sigma_{[0,n+1)}| \ge \langle \sigma_{[0,n)} \rangle |\sigma_n|.$$

Then, by Item (4.127),

$$|\sigma_n| \le \frac{|\sigma_{[0,n+1)}|}{\langle \sigma_{[0,n)} \rangle} \le \frac{10d^2\ell_{n+1}}{1/4d^{2d^3+6}\ell_n} = 40d^{2d^3+8}\frac{\ell_{n+1}}{\ell_n}.$$

We now prove (4.126). Let $n \ge 0$ and $z' \in Z_n$ be arbitrary. On one hand, from (4.120) we have that

$$|\sigma_{[0,n)}(z'_0)| = |\tau_n(z'_0)| - q_{0,n}(z') + q_{0,n}(Sz') \text{ for any } z' \in Z_n.$$

Hence, by (4.123) and Condition (ii),

$$|\sigma_{[0,n)}(z'_0)| = |\tau_n(z'_0)| - q_{0,n}(z') + q_{0,n}(Sz') \le |\tau_n(z'_0)| + |\tau_n(z'_1)| - 5\varepsilon_n \le 20d^2\ell_n.$$

Similarly,

$$|\sigma_{[0,n)}(z'_0)| = |\tau_n(z'_0)| - q_{0,n}(z') + q_{0,n}(Sz') \ge 5\varepsilon_n \ge \frac{5}{d^{d^3+4}}\ell_n.$$

☐

We now introduce some notation. Let $\rho(a) = a$ if a belongs to $\mathcal{C}_{n,\mathrm{ap}}$ for some $n \geq 1$ and let $\rho(a) = \mathsf{root}\,\tau_n(a)$ if $n \geq 1$ and $a \in \mathcal{C}_n \setminus \mathcal{C}_{n,\mathrm{ap}}$. We set $\psi(z) = (\rho(z_{-1}), \rho(z_0), \rho(z_1))$ for $n \geq 1$ and $z \in Z_n$. Note that these definitions are consistent with the ones in Subsection 4.7.2.

The proof of the following lemma will be postponed until the end of the subsection.

Lemma 4.68 *Let $z, \tilde{z} \in Z_n$ be such that $\psi(z) = \psi(\tilde{z})$. Then,* $\mathsf{root}\,\sigma_{[0,n)}(z_0) = \mathsf{root}\,\sigma_{[0,n)}(\tilde{z}_0)$.

Lemma 4.69 *Item $(\mathcal{P}_1)$ of Theorem 4.63 is satisfied by $\boldsymbol{\sigma}$.*

Proof. The inequality $\#\mathcal{C}_n \leq 7^4 d^{12d+36} \mathsf{pow\text{-}com}(X)^4$ in Item $(\mathcal{P}_1)$ follows from Condition (i). To prove the other inequality, we note that Lemma 4.68 implies that

$$\#\,\mathsf{root}\,\sigma_{[0,n)}(\mathcal{C}_n) \leq \#\psi(Z_n) \cdot \#\,\mathsf{root}\,\tau_n(\mathcal{C}_n).$$

Now, it follows from the definition of ψ that $\#\psi(Z_n)$ is at most $(\#\,\mathsf{root}\,\tau_n(\mathcal{C}_n) + \#\tau_n(\mathcal{C}_{n,\mathrm{ap}}))^3$. Combining this with the bounds given by Condition (i) yields

$$\begin{aligned}\#\,\mathsf{root}\,\sigma_{[0,n)}(\mathcal{C}_n) &\leq \#\psi(Z_n) \cdot \#\,\mathsf{root}\,\tau_n(\mathcal{C}_n) \\ &\leq (5d^{3d+6} + 2d^{3d+6})^3 \cdot 5d^{3d+6} \leq 35d^{12d+24}.\end{aligned}$$

□

It only rests to prove Lemma 4.68. We start by fixing some notation. Let $z^n, \tilde{z}^n \in Z_n$ be such that $\psi(z^n) = \psi(\tilde{z}^n)$. We set $s = \mathsf{root}\,\tau_n(z^n_0) = \mathsf{root}\,\tau_n(\tilde{z}^n_0)$. For $j \in [0, n)$, we inductively define $z^j = \sigma_j(z^{j+1})$ and $\tilde{z}^j = \sigma_j(\tilde{z}^{j+1})$. Let $(c^j, y^j) = \mathbf{F}_{(Z_j,\tau_j)}(\tau_{j+1}(z^{j+1}))$ and $(\tilde{c}^j, \tilde{y}^j) = \mathbf{F}_{(Z_j,\tau_j)}(\tau_{j+1}(\tilde{z}^{j+1}))$

With the notation introduced, we have, for every $j \in [0, n)$, that

$$S^{r_j(z^{j+1})} y^j = z^j \tag{4.128}$$

and that

$$z^j = \sigma_{[j,n)}(z^n). \tag{4.129}$$

We can also write, thanks to (4.119),

$$\begin{aligned} q_{j,n}(z^n) &= q_{j+1,n}(z^n) + c^j_{r_j(z^{j+1})} \\ &\text{and } q_{j,n}(Sz^n) = q_{j+1,n}(Sz^n) + c^j_{r_j(z^{j+1}) + |\sigma_{[j,n)}(z^n_0)|} \end{aligned} \tag{4.130}$$

for every $j \in [0, n)$. Similar relations hold for $\tilde{z}^n$.

The next three lemmas are the core of the proof of Lemma 4.68.

Lemma 4.70 *Suppose that $\psi(z^n) = \psi(\tilde{z}^n)$ and that $\varepsilon_n < |s|$. Then, for every $j \in [0, n]$, the following holds:*

(a) $q_{j,n}(z^n) = q_{j,n}(\tilde{z}^n)$ *and* $q_{j,n}(Sz^n) = q_{j,n}(S\tilde{z}^n)$.

(b) $\psi(z^j) = \psi(\tilde{z}^j)$ *and* $\psi(S^{|\sigma_{[j,n)}(z^n)|}z^j) = \psi(S^{|\sigma_{[j,n)}(\tilde{z}^n)|}\tilde{z}^j)$.

(c) $z_0^j, \tilde{z}_0^j, z^j_{|\sigma_{[j,n)}(z^n)|}, \tilde{z}^j_{|\sigma_{[j,n)}(\tilde{z}^n)|} \in \mathcal{C}_{j,\mathrm{ap}}$.

Proof. We prove the claim by induction on j. The case $j = n$ is a direct consequence of the hypothesis. Assume that $j \in [0, n)$ and that the claim is true for $j + 1$. The inductive hypothesis gives that z_0^{j+1} and $\tilde{z}_0^{j+1}$ belong to $\mathcal{C}_{j+1,\mathrm{ap}}$. In particular, $\mathsf{per}(\tau_{j+1}(z_0^{j+1}))$ and $\mathsf{per}(\tau_{j+1}(\tilde{z}_0^{j+1}))$ are greater than ε_{j+1}. Hence, by the definition of r_j and (4.128), $z_0^j = y^j_{r_j(z^{j+1})} \in \mathcal{C}_{j,\mathrm{ap}}$ and $\tilde{z}_0^j \in \mathcal{C}_{j,\mathrm{ap}}$.

Next, by Items (b) and (c) of the induction hypothesis, z^j and $\tilde{z}^j$ satisfy the hypothesis of Lemma 4.60. Hence, by (4.128),

(1) $c^j_{r_j(z^{j+1})} = \tilde{c}^j_{r_j(\tilde{z}^{j+1})}$, and

(2) $\psi(z^j) = \psi(S^{r_j(z^{j+1})}y^j) = \psi(S^{r_j(\tilde{y}^{j+1})}\tilde{z}^j) = \psi(\tilde{z}^j)$.

Putting the first equation and Item (a) of the induction hypothesis in the definition of $q_{j,n}$ yields

$$q_{j,n}(z^n) = q_{j+1,n}(z^n) + c^j_{r_j(z^{j+1})} = q_{j+1,n}(\tilde{z}^n) + \tilde{c}^j_{r_j(\tilde{z}^{j+1})} = q_{j,n}(\tilde{z}^n).$$

The rest of the inductive step follows from similar arguments. □

Lemma 4.71 *Suppose that* $\rho(z_0^n) = \rho(\tilde{z}_0^n)$ *and* $\varepsilon_{n-1} < |s| \leq \varepsilon_n$. *Then, for every* $j \in [0, n)$, *the following holds:*

(a) $q_{j,n}(z^n) = q_{j,n}(\tilde{z}^n) = q_{j,n}(Sz^n) = q_{j,n}(S\tilde{z}^n)$.

(b) $\psi(z^j) = \psi(\tilde{z}^j) = \psi(S^{|\sigma_{[j,n)}(z^n)|}z^j) = \psi(S^{|\sigma_{[j,n)}(\tilde{z}^n)|}\tilde{z}^j)$.

(c) z_0^j, $\tilde{z}_0^j$, $z^j_{|\sigma_{[j,n)}(z^n)|}$ *and* $\tilde{z}^j_{|\sigma_{[j,n)}(\tilde{z}^n)|}$ *belong to* $\mathcal{C}_{j,\mathrm{ap}}$.

Proof. We first assume that $j = n-1$. Let us write $r = r_{n-1}$, $c_j = c_j^{n-1}$, $z = z^{n-1}$, $y = y^{n-1}$, etc. Since $\varepsilon_{n-1} < |s| \leq \varepsilon_n$ and $\rho(z_0^n) = \rho(\tilde{z}_0^n)$, we can use Lemma 4.61 with z_0^n and $\tilde{z}_0^n$ to deduce the following:

(a') $c_{r(z^n)} = \tilde{c}_{r(\tilde{z}^n)} = c_{r(z^n)+|\sigma_{n-1}(z_0^n)|} - |\tau_n(z_0^n)| = \tilde{c}_{r(\tilde{z}^n)+|\sigma_{n-1}(\tilde{z}_0^n)|} - |\tau_n(\tilde{z}_0^n)|$.

(b') $y_{r(z^n)} = \tilde{y}_{r(\tilde{z}^n)} = y_{r(z^n)+|\sigma_{n-1}(z_0^n)|} = \tilde{y}_{r(\tilde{z}^n)+|\sigma_{n-1}(\tilde{z}_0^n)|}$.

Item (b') implies, by (4.128), that Item (b) of the claim holds for $j = n-1$. Also, since $|s| > \varepsilon_{n-1}$, the definition of r ensures that $z_0 = y_{r(z^n)} \in \mathcal{C}_{n-1,\mathrm{ap}}$, so Item (c) of the claim holds. For Item (a), we note that, since $q_{n,n} \equiv 0$, the definition of $q_{n-1,n}$ ensures that $q_{n-1,n}(z^n) = c_{r(z^n)}$, $q_{n-1,n}(\tilde{z}^n) = \tilde{c}_{r(\tilde{c}^n)}$, $q_{n-1,n}(Sz^n) = c_{r(z^n)+|\sigma_{n-1}(z_0^n)|} - |\tau_n(z_0^n)|$ and $q_{n-1,n}(S\tilde{z}^n) = \tilde{c}_{r(\tilde{z}^n)+|\sigma_{n-1}(\tilde{z}_0^n)|} - |\tau_n(\tilde{z}_0^n)|$. Therefore, Item (a) of the claim follows from Item (a').

We now assume that $j \in [0, n-1)$ and that the claim holds for $j+1$. Item (c') of the induction hypothesis ensures that

$$z_0^{j+1}, \tilde{z}_0^{j+1} \in \mathcal{C}_{j+1,\mathrm{ap}} \text{ and } \psi(z^{j+1}) = \psi(\tilde{z}^{j+1}). \tag{4.131}$$

Then, by the definition of r_j and (4.128), $z_0^j = y^j_{r_j(z^{j+1})} \in \mathcal{C}_{j,\mathrm{ap}}$ and $\tilde{z}_0^j \in \mathcal{C}_{j,\mathrm{ap}}$. Equation (4.131) also allows us to use Lemma 4.60 with z^{j+1} and $\tilde{z}^{j+1}$ and deduce the following:

(a') $c^j_{r_j(z^{j+1})} = \tilde{c}^j_{r_j(\tilde{z}^{j+1})}$.

(b') $\psi(S^{r_j(z^{j+1})} y^j) = \psi(S^{r_j(\tilde{z}^{j+1})} \tilde{y}^j)$.

Equation (4.128) ensures that Item (b') is equivalent to $\psi(z^j) = \psi(\tilde{z}^j)$. Now, putting Item (a') and Item (a) of the induction hypothesis in the definition of $q_{j,n}$ yields

$$q_{j,n}(z^n) = q_{j+1,n}(z_0^n) + c^j_{r_j(z^{j+1})} = q_{j+1,n}(\tilde{z}_0^n) + \tilde{c}^j_{r_j(\tilde{z}^{j+1})} = q_{j,n}(\tilde{z}^n).$$

Similar arguments, which rely on using Lemma 4.60 with $S^{|\sigma_{[j+1,n)}(z_0^n)|} z^{j+1}$ and $S^{|\sigma_{[j+1,n)}(\tilde{z}_0^n)|} \tilde{z}^{j+1}$, show that $z^j_{|\sigma_{[j,n)}(z^n)|}, \tilde{z}^j_{|\sigma_{[j,n)}(\tilde{z}^n)|} \in \mathcal{C}_{j,\mathrm{ap}}$, $\psi(S^{|\sigma_{[j,n)}(z^n)|} z^j) = \psi(S^{|\sigma_{[j,n)}(\tilde{z}^n)|} \tilde{z}^j)$ and $q_{j,n}(Sz^n) = q_{j,n}(S\tilde{z}^n)$.

To complete the proof, it is enough to show that $\psi(z^j) = \psi(S^{|\sigma_{[j,n)}(z^n)|} z^j)$ and $q_{j,n}(z^n) = q_{j,n}(Sz^n)$. We observe that Item (b) of the inductive hypothesis guarantees that $\psi(z^{j+1}) = \psi(S^{|\sigma_{[j+1,n)}(z_0^n)|} z^{j+1})$. Since we know that z_0^{j+1} and $z^{j+1}_{|\sigma_{[j+1,n)}(z_0^n)|}$ belong to $\mathcal{C}_{j+1,\mathrm{ap}}$, we can use Lemma 4.60 with z^{j+1} and $S^{|\sigma_{[j+1,n)}(z_0^n)|} z^{j+1}$ to obtain the following:

(a") $c^j_{r_j(z^{j+1})} = c^j_{r_j(z^{j+1}) + |\sigma_{[j,n)}(z_0^n)|} - |\tau_{j+1} \sigma_{[j+1,n)}(z_0^n)|$.

(b") $\psi(S^{r_j(z^{j+1})} y^j) = \psi(S^{r_j(z^{j+1}) + |\sigma_{[j,n)}(z_0^n)|} y^j)$.

Item (b") implies, by (4.128), that $\psi(z^j) = \psi(S^{|\sigma_{[j,n)}(z_0^n)|} z^j)$. Also, using the definition of $q_{j+1,n}$ and Item (a") we can write

$$\begin{aligned} q_{j,n}(Sz^n) - q_{j+1,n}(Sz^n) &= c^j_{r_j(z^{j+1}) + |\sigma_{[j,n)}(z_0^n)|} - |\tau_{j+1} \sigma_{[j+1,n)}(z_0^n)| \\ &= c^j_{r_j(z^{j+1})} = q_{j,n}(z^n) - q_{j+1,n}(z^n). \end{aligned}$$

This and Item (a) of the induction hypothesis gives that $q_{j,n}(Sz^n) = q_{j,n}(z^n)$. □

Lemma 4.72 *Suppose that $\rho(z_0^n) = \rho(\tilde{z}_0^n)$ and that $|s| \leq \varepsilon_{n-1}$. Let $j_0 \in [0, n)$ be the least element satisfying $|s| \leq \varepsilon_{j_0}$. Then, for every $j \in [j_0, n]$, the following holds:*

(a) There is s_j such that $|s_j| = |s|$ and $s_j = \mathsf{root}\, \tau_j(z_k^j) = \mathsf{root}\, \tau_j(\tilde{z}_\ell^j)$ for all $k \in [0, |\sigma_{[j,n)}(z_0^n)|)$ and $\ell \in [0, |\sigma_{[j,n)}(\tilde{z}_0^n)|)$.

(b) $q_{j,n}(z^n) = q_{j,n}(\tilde{z}^n) = q_{j,n}(Sz^n) = q_{j,n}(S\tilde{z}^n) \pmod{|s|}$.

(c) $\psi(z^j) = \psi(\tilde{z}^j)$ and $\psi(S^{|\sigma_{[j,n)}(z^n)|} z^j) = \psi(S^{|\sigma_{[j,n)}(\tilde{z}^n)|} \tilde{z}^j)$.

Proof. The case $j = n$ follows directly from the hypothesis. Assume that $j \in [j_0, n)$ and that the claim holds for $j+1$. We observe that, by Items (a) and (c) of the induction hypothesis, $z^{j+1}_{[0,|\sigma_{[j,n)}(z^n_0)|)}$ and $\tilde{z}^{j+1}_{[0,|\sigma_{[j,n)}(\tilde{z}^n_0)|)}$ comply with the hypothesis of Lemma 4.62. Therefore, by (4.128), Items (a) and (c) hold for j. Moreover, we have that

$$c^j_{r_j(y^{j+1})} = \tilde{c}^j_{r_j(\tilde{y}^{j+1})} = c^j_{r_j(y^{j+1})+|\sigma_{[j,n)}(z^n_0)|} = \tilde{c}^j_{r_j(\tilde{y}^{j+1})+|\sigma_{[j,n)}(\tilde{z}^n_0)|} \pmod{|s|}. \tag{4.132}$$

Now, Item (b) of the induction hypothesis gives that $q_{j+1,n}(z^n) = q_{j+1,n}(\tilde{z}^n) = q_{j+1,n}(Sz^n) = q_{j+1,n}(S\tilde{z}^n) \pmod{|s|}$. Hence, by the definition of $q_{j,n}$,

$$q_{j,n}(z^n) = q_{j+1,n}(z^n) + c^j_{r_j(y^{j+1})} = q_{j+1,n}(\tilde{z}^n) + c^j_{r_j(\tilde{y}^{j+1})} = q_{j,n}(\tilde{z}^n) \pmod{|s|}.$$

We note that, since Item (a) holds for j, we have that $|\tau(z^j_{[0,|\sigma_{[j,n)}(z^n_0)|)})| = 0 \pmod{|s|}$. Hence, by the definition of $q_{j,n}$,

$$\begin{aligned} q_{j,n}(Sz^n) - q_{j+1,n}(Sz^n) &= c^j_{r_j(y^{j+1})+|\sigma_{[j,n)}(z^n_0)|} - |\sigma_{[j,n)}(z^n_0)| \\ &= c^j_{r_j(y^{j+1})+|\sigma_{[j,n)}(z^n_0)|} \pmod{|s|} \end{aligned}$$

Thus, by (4.132) and Item (b) of the induction hypothesis, $q_{j,n}(Sz^n) = q_{j,n}(z^n)$. Similarly, $q_{j,n}(S\tilde{z}^n) = q_{j,n}(\tilde{z}^n)$. We conclude that Item (b) holds for j. □

The last ingredient for the proof of Lemma 4.68 is the following lemma.

Lemma 4.73 *Suppose that $\psi(z^n) = \psi(\tilde{z}^n)$.*

(1) If $z^n_0 \in \mathcal{C}_{n,\mathrm{ap}}$, $q_{0,n}(z^n) = q_{0,n}(\tilde{z}^n)$ and $q_{0,n}(Sz^n) = q_{0,n}(S\tilde{z}^n)$, then $\sigma_{[0,n)}(z^n_0) = \sigma_{[0,n)}(\tilde{z}^n_0)$.

(2) Let $s = \mathsf{root}\,\tau_n(z^n_0)$ and suppose that $|s| \leq \varepsilon_n$ and $q_{0,n}(z^n) = q_{0,n}(\tilde{z}^n) = q_{0,n}(z^n) = q_{0,n}(\tilde{z}^n) \pmod{|s|}$. Then, $\mathsf{root}\,\sigma_{[0,n)}(z^n_0) = \mathsf{root}\,\sigma_{[0,n)}(\tilde{z}^n_0)$.

Proof. Assume that the hypothesis of Item (1) holds. We also assume, without loss of generality, that $|\tau_n(z^n_1)| \leq |\tau_n(\tilde{z}^n_1)|$. We start by noticing that, since $z^n_0 \in \mathcal{C}_{n,\mathrm{ap}}$ and $\psi(z^n) = \psi(\tilde{z}^n)$, we have that $\tau_n(z^n_0)$ is equal to $\tau_n(\tilde{z}^n_0)$. Furthermore, by Condition (ii) we have that

$$\tau_n(z^n)_{[-8\varepsilon_n,|\tau_n(z^n_0 z^n_1)|)} = \tau_n(\tilde{z}^n)_{[-8\varepsilon_n,|\tau_n(z^n_0 z^n_1)|)}. \tag{4.133}$$

Now, from Lemma 4.66 and the hypothesis we get that

$$\begin{aligned} q_{0,n}(\tilde{z}^n) = q_{0,n}(z^n) &\in [-2\varepsilon_n, |\tau_n(z^n_0)| - 7\varepsilon_n) \\ \text{and } q_{0,n}(S\tilde{z}^n) = q_{0,n}(Sz^n) &\in [|\tau_n(z^n_0)| - 2\varepsilon_n, |\tau_n(z^n_0 z^n_1)| - 7\varepsilon_n). \end{aligned}$$

We conclude, using (4.133), that

$$\begin{aligned} \sigma_{[0,n)}(z^n_0) &= \tau_n(z^n)_{[q_{0,n}(z^n),|\tau_n(z^n_0)|+q_{0,n}(Sz^n))} \\ &= \tau_n(\tilde{z}^n)_{[q_{0,n}(\tilde{z}^n),|\tau_n(\tilde{z}^n_0)|+q_{0,n}(S\tilde{z}^n))} = \sigma_{[0,n)}(\tilde{z}^n_0). \end{aligned}$$

Next, we assume that the hypothesis of Item (2) holds. The condition $|s| \le \varepsilon_n$ enables us to use (4.124) from Lemma 4.66, so

$$q_{0,n}(z^n) \in [-2\varepsilon_n, 2\varepsilon_n).$$

Also, since $|s| \le \varepsilon_n$, Item ((3)) in Proposition 4.46 guarantees that

$$\tau_n(z^n)_{[-8\varepsilon_n, |\tau_n(z_0^n)|+8\varepsilon_n)} = s^{\mathbb{Z}}_{[-8\varepsilon_n, |\tau_n(z_0^n)|+8\varepsilon_n)}.$$

Therefore,

$$\sigma_{[0,n)}(z_0^n) = s^{\mathbb{Z}}_{[q_{0,n}(z^n), |\tau_n(z_0^n)|+q_{0,n}(Sz^n))}. \tag{4.134}$$

Now, by the hypothesis, there is $k \in \mathbb{Z}$ such that $k = q_{0,n}(z^n) = q_{0,n}(Sz^n) \pmod{|s|}$. We deduce from (4.134) that

$$\mathsf{root}\,\sigma_{[0,n)}(z_0^n) = s_{[k,k+|s|)}.$$

Being $\psi(z^n)$ equal to $\psi(\tilde{z}^n)$, we have that $s = \mathsf{root}\,\tau_n(\tilde{z}_0^n)$. Hence, we can give similar arguments to prove that $\mathsf{root}\,\sigma_{[0,n)}(\tilde{z}_0^n) = s_{[\tilde{k},\tilde{k}+|s|)}$, where $\tilde{k} = q_{0,n}(\tilde{z}^n) = q_{0,n}(S\tilde{z}^n) \pmod{|s|}$. We conclude, as the hypothesis ensures that $k = \tilde{k} \pmod{|s|}$, that $\mathsf{root}\,\sigma_{[0,n)}(z_0^n) = \mathsf{root}\,\sigma_{[0,n)}(\tilde{z}_0^n)$. □

We have all the necessary elements to prove Lemma 4.68.

Proof of Lemma 4.68. Let $z', \tilde{z}' \in Z_n$ be such that $\psi(z') = \psi(\tilde{z}')$ and let $s = \mathsf{root}\,\tau(z'_0) = \mathsf{root}\,\tau(\tilde{z}'_0)$. We split the proof into two cases. Let us first assume that $|s| > \varepsilon_n$. Then, we can use Lemma 4.70 and deduce that $q_{0,n}(z^n) = q_{0,n}(\tilde{z}^n)$ and $q_{0,n}(Sz^n) = q_{0,n}(S\tilde{z}^n)$. Thus, by Lemma 4.73, $\sigma_{[0,n)}(z_0^n) = \sigma_{[0,n)}(\tilde{z}_0^n)$, which implies that $\mathsf{root}\,\sigma_{[0,n)}(z_0^n) = \mathsf{root}\,\sigma_{[0,n)}(\tilde{z}_0^n)$.

Next, we assume that $|s| \le \varepsilon_n$. Let $j \in [0, n]$ be the least element satisfying $|s| \le \varepsilon_j$. We claim that the following is true:

(a) If $j < n$, then $\mathsf{root}\,\tau_j(z^j)$ is equal to $\mathsf{root}\,\tau_j(S^{|\sigma_{[j,n)}(z^n)|-1}z^j)$ and has length $|s|$.

(b) $q_{j,n}(z^n) = q_{j,n}(\tilde{z}^n) = q_{j,n}(Sz^n) = q_{j,n}(S\tilde{z}^n) \pmod{|s|}$.

(c) $\psi(z^j) = \psi(\tilde{z}^j)$ and $\psi(S^{|\sigma_{[j,n)}(z^n)|}z^j) = \psi(S^{|\sigma_{[j,n)}(\tilde{z}^n)|}\tilde{z}^j)$.

If $j = n$, then the claim is equivalent to the hypothesis $\psi(z') = \psi(\tilde{z}')$. We assume that $j < n$. Then, $|s| \le \varepsilon_{n-1}$, which permits to use Lemma 4.72 and conclude that Items (b) and (c) of the claim hold. Moreover, Lemma 4.72 also states that there is t such that $|t| = |s|$ and $t = \mathsf{root}\,\tau_j(z_k^j) = \mathsf{root}\,\tau_j(\tilde{z}_\ell^j)$ for all $k \in [0, |\sigma_{[j,n)}(z_0^n)|)$ and $\ell \in [0, |\sigma_{[j,n)}(\tilde{z}_0^n)|)$. In particular, Item (a) holds. This completes the proof of the claim.

Next, we now prove that

$$q_{0,n}(z^n) = q_{0,n}(\tilde{z}^n) = q_{0,n}(Sz^n) = q_{0,n}(S\tilde{z}^n) \pmod{|s|} \tag{4.135}$$

If $j = 0$, then (4.135) follows from the claim. Let us assume that $j > 0$. Then, $\varepsilon_{j-1} < |s| \le \varepsilon_j$. This and Items (a) and (c) of the claim allow us to use Lemma 4.71 twice, first with z^j and $\tilde{z}^j$, and then with $S^{|\sigma_{[j,n)}(z^n)|-1}z^j$ and $S^{|\sigma_{[j,n)}(\tilde{z}^n)|-1}\tilde{z}^j$. We get

$$q_{0,j}(z^j) = q_{0,j}(\tilde{z}^j). \text{ and } q_{0,j}(S^{|\sigma_{[j,n)}(z^n)|}z^j) = q_{0,j}(S^{|\sigma_{[j,n)}(\tilde{z}^n)|}\tilde{z}^j).$$

Moreover, since $\mathsf{root}\,\tau_j(z^j)$ is equal to $\mathsf{root}\,\tau_j(S^{|\sigma_{[j,n)}(z^n)|-1}z^j)$ and has length $|s| \leq \varepsilon_j$, we can use Lemma 4.71 with z^j and $S^{|\sigma_{[j,n)}(z^n)|-1}z^j$ to obtain that $q_{0,j}(z^j) = q_{0,j}(S^{|\sigma_{[j,n)}(z^n)|}z^j)$. Therefore,

$$q_{0,j}(z^j) = q_{0,j}(\tilde{z}^j) = q_{0,j}(S^{|\sigma_{[j,n)}(z^n)|}z^j) = q_{0,j}(S^{|\sigma_{[j,n)}(\tilde{z}^n)|}\tilde{z}^j). \tag{4.136}$$

Now, from the definition of $q_{0,n}$ we have that $q_{0,n}(z^n) = q_{j,n}(z^n) + q_{0,j}(z^j)$ and $q_{0,n}(\tilde{z}^n) = q_{j,n}(\tilde{z}^n) + q_{0,j}(\tilde{z}^j)$. Putting (4.136) and Item (b) of the claim in this relation produces $q_{0,n}(z^n) = q_{0,n}(\tilde{z}^n) \pmod{|s|}$. The rest of the equalities in (4.135) follow from (4.136) and Item (b) of the claim in the same way. The proof of (4.135) is complete.

We recall that we assumed that $\psi(z^n) = \psi(\tilde{z}^n)$ and $|s| \leq \varepsilon_n$. These two things and (4.135) permit to use Lemma 4.73 and conclude that $\mathsf{root}\,\sigma_{[0,n)}(z^n_0) = \mathsf{root}\,\sigma_{[0,n)}(\tilde{z}^n_0)$. □

4.8.2 Proof of the main theorems

Lemma 4.74 *Let X be a subshift and $\mathcal{W}$ a set of words such that $X \subseteq \bigcup_{k\in\mathbb{Z}} S^k\mathcal{W}^{\mathbb{Z}}$. Then, $p_X(\langle\mathcal{W}\rangle) \leq |\mathcal{W}| \cdot \#(\mathsf{root}\,\mathcal{W})^2$.*

Proof. The hypothesis implies that any w of length $\langle\mathcal{W}\rangle$ occurring in some $x \in X$ occurs in a word of the form uv, where $u, v \in \mathcal{W}$. In particular, w occurs in $(\mathsf{root}\,u)^{|\mathcal{W}|}(\mathsf{root}\,v)^{|\mathcal{W}|}$. There are at most $|\mathcal{W}| \cdot \#(\mathsf{root}\,\mathcal{W})^2$ words satisfying this condition, so $p_X(\langle\mathcal{W}\rangle) \leq |\mathcal{W}| \cdot \#(\mathsf{root}\,\mathcal{W})^2$. □

Theorem 4.75 *A minimal subshift X has linear-growth complexity* i.e.

$$\limsup_{n\to+\infty} p_X(n)/n < +\infty,$$

if and only if there exist $d \geq 1$ and an $\mathcal{S}$-adic sequence $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ such that for every $n \geq 0$:

$(\mathcal{P}_1)$ *$\#(\mathsf{root}\,\sigma_{[0,n)}(\mathcal{A}_n)) \leq d$.*

$(\mathcal{P}_2)$ *$|\sigma_{[0,n)}(a)| \leq d \cdot |\sigma_{[0,n)}(b)|$ for every $a, b \in \mathcal{A}_n$.*

$(\mathcal{P}_3)$ *$|\sigma_{n-1}(a)| \leq d$ for every $a \in \mathcal{A}_n$.*

If X is infinite and has linear-growth complexity, then $\boldsymbol{\sigma}$ can be chosen to be recognizable and satisfying $\#\mathcal{A}_n \leq d \cdot \mathsf{pow\text{-}com}(X)^4$ for all $n \geq 0$.

Theorem 4.76 *A minimal subshift X has nonsuperlinear-growth complexity* i.e.

$$\liminf_{n\to+\infty} p_X(n)/n < +\infty,$$

if and only if there exists $d \geq 1$ and an $\mathcal{S}$-adic sequence $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ such that for every $n \geq 0$

$(\mathcal{P}_1)$ *$\#(\mathsf{root}\,\sigma_{[0,n)}(\mathcal{A}_n)) \leq d$.*

($\mathcal{P}_2$) $|\sigma_{[0,n)}(a)| \leq d \cdot |\sigma_{[0,n)}(b)|$ *for every* $a, b \in \mathcal{A}_n$.

If X *is infinite and has nonsuperlinear-growth complexity, then* $\boldsymbol{\sigma}$ *can be chosen to be recognizable and satisfying* $\#\mathcal{A}_n \leq d \cdot \mathsf{pow\text{-}com}(X)^4$ *for all* $n \geq 0$.

We prove Theorems 4.75 and 4.76 simultaneously.

Proof of Theorems 4.75 and 4.76. Let $d = \liminf_{k\to+\infty} p_X(k)/k$ and $d' = \sup_{k\geq 0} p_X(k)/k$.

We first assume that p_X has nonsuperlinear- or linear-growth and show that there exists an $\mathcal{S}$-adic sequence as the ones in Theorems 4.75 and 4.76, respectively.

If p_X has nonsuperlinear-growth, then d is finite and so, using Lemma 4.7, we obtain a sequence $(\ell_n)_{n\geq 0}$ such that for all $n \geq 0$

$$\ell_{n+1} \geq d\ell_n,\ p_X(\ell_n) \leq d\ell_n \text{ and } p_X(\ell_n + 1) - p_X(\ell_n) \leq d. \tag{4.137}$$

If p_X has linear growth, then d' is finite and using Lemma 4.6 we get a sequence $(\ell_n)_{n\geq 0}$ that satisfies (4.137) and

$$\ell_{n+1} \leq d'\ell_n \text{ for every } n \geq 0. \tag{4.138}$$

We use Theorem 4.63 with the sequence $(\ell_n)_{n\geq 0}$. This produces a recognizable $\mathcal{S}$-adic sequence $\boldsymbol{\sigma} = (\sigma_n\colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ generating X such that for every $n \geq 1$:

($\mathcal{P}'_1$) $\#(\mathsf{root}\,\sigma_{[0,n)}(\mathcal{A}_n)) \leq d$ and $\#\mathcal{A}_n \leq \mathsf{pow\text{-}com}(X)$.

($\mathcal{P}'_2$) $|\sigma_{[0,n)}(a)| \leq d \cdot |\sigma_{[0,n)}(b)|$ for every $a, b \in \mathcal{A}_n$.

($\mathcal{P}'_3$) $|\sigma_{n-1}(a)| \leq d\ell_n/\ell_{n-1}$ for every $a \in \mathcal{A}_n$.

In particular, the conclusion of Theorem 4.76 holds. Moreover, if p_X has linear growth, then Equation (4.138) holds, so we also have the bound $|\sigma_{n-1}(a)| \leq dd'$ for every $n \geq 1$ and $a \in \mathcal{A}_n$. Therefore, in this case, σ satisfies the conclusion of Theorem 4.75.

We now assume that there exists an $\mathcal{S}$-adic sequence $\boldsymbol{\sigma} = (\sigma_n\colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ satisfying the conclusion of Theorem 4.76 or the one of Theorem 4.75.

Note that since $\boldsymbol{\sigma}$ generates X, we have that

$$X \subseteq \bigcup_{k\in\mathbb{Z}} S^k \sigma_{[0,n)}(\mathcal{A}_n^{\mathbb{Z}})$$

for any $n \geq 1$. Thus, by Lemma 4.74, $p_X(\langle\sigma_{[0,n)}\rangle)$ is at most $|\sigma_{[0,n)}| \cdot \#(\mathsf{root}\,\sigma_{[0,n)}(\mathcal{A}_n))^2$. Items ($\mathcal{P}_1$) and ($\mathcal{P}_2$) of Theorems 4.76 and 4.75 then imply that

$$p_X(\langle\sigma_{[0,n)}\rangle) \leq d^3\langle\sigma_{[0,n)}\rangle \text{ for all } n \geq 1. \tag{4.139}$$

This proves that X has nonsuperlinear-growth complexity.

It rests to prove that X has linear-growth complexity when the conclusion of Theorem 4.75 holds. We assume that $\boldsymbol{\sigma}$ satisfies Items ($\mathcal{P}_1$), ($\mathcal{P}_2$) and ($\mathcal{P}_3$) of Theorem 4.75. Let $k \geq 1$ be arbitrary and let $n \geq 1$ be the biggest integer such that $\langle \sigma_{[0,n)} \rangle \leq k$. Then, by the maximality of n and Items $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$ of Theorem 4.75, we can compute

$$k < \langle \sigma_{[0,n+1)} \rangle \leq |\sigma_{[0,n)}| \cdot |\sigma_n| \leq d^2 \langle \sigma_{[0,n)} \rangle \leq d^2 k.$$

Combining this with (4.139) yields $p_X(k) \leq d^5 k$. This proves that X has linear-growth complexity. □

4.9 Bounded alphabet structures

Theorems 4.75 and 4.76 provide $\mathcal{S}$-adic structures for linear-growth complexity subshifts and nonsuperlinear-growth complexity subshifts. These are not the only representations known for these classes: for instance, in [DDMP21] it is proved that if X has nonsuperlinear-growth complexity, then X is generated by a recognizable, proper and primitive $\mathcal{S}$-adic sequence $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ such that $\#\mathcal{A}_n$ is uniformly bounded. The last condition is known as the *bounded alphabet property* and it is considered natural in the low complexity setting; see [Fer96; DLR13; Esp22a]. Note that the representations given by Theorems 4.75 and 4.76 do not necessarily satisfy this property. In fact, our construction gives a bonded alphabet $\mathcal{S}$-adic sequence if and only if the subshift has finite power complexity. Thus, it is natural to ask whether it is possible to modify Theorems 4.75 and 4.76 so that they give bounded alphabet $\mathcal{S}$-adic sequences. In this section, we show that such a strengthening is not possible for Theorem 4.75. More precisely, we prove the following:

Theorem 4.77 *There exists a minimal subshift X such that:*

(1) X has linear-growth complexity.

(2) If $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ is an $\mathcal{S}$-adic sequence generating X and satisfying Items (1), (2) and (3) of Theorem 4.75, then $\sup_{n\geq 1} \#\mathcal{A}_n = +\infty$.

We were not able to obtain an analogous result for Theorem 4.76, so we leave this as an open question.

It is interesting to compare Theorem 4.77 with the main result of [Ler14], which describes bounded alphabet $\mathcal{S}$-adic representations of minimal subshifts whose complexity function satisfies $p_X(n+1) - p_X(n) \leq 2$ for all $n \geq 1$. We are led to ask the following.

Question 4.2 How small can $\sup_{n\geq 1} p_X(n+1) - p_X(n)$ be made in Theorem 4.77?

We now turn into proving Theorem 4.77. We start with some technical lemmas.

Let $n, n_0, d, \ell \geq 1$. We define $P(n, n_0, \ell)$ as the set of integer sequences $(p_1, \dots, p_\ell)$ such that $p_j n_0 \in [8^j n, 2 \cdot 8^j n)$. Let $K(n, d, \ell)$ be the set of integer sequences $(k_1, \dots, k_\ell) \in P(n, n_0, \ell)$ for which there exists $E \subseteq [d^{-1}n, dn)$, with at most d elements, such that every k_j can be written as $\sum_{e\in E} \alpha_e e$, where $\alpha_e \in \mathbb{Z}_{\geq 0}$.

Lemma 4.78 *Suppose that* $n > n_0^3$, $n > \ell^{18\ell^2}$ *and* $\ell > 3d > 8$. *Then,* $P(n, n_0, \ell) \setminus K(n, d, \ell)$ *is nonempty.*

Proof. We will show that $\#P(n, n_0, \ell) > K(n, d, \ell)$, which implies that $P(n, n_0, \ell) \setminus K(n, d, \ell)$ is nonempty. We first estimate $\#P(n, n_0, \ell)$. Note that there are at least $(2 \cdot 8^j n - 8^j n)/n_0$ ways of choosing p_j in a sequence $(p_1, \ldots, p_\ell) \in P(n, n_0, \ell)$. Thus,

$$\#P(n, n_0, \ell) \geq \prod_{j=1}^{\ell} 8^j n/n_0 \geq (n/n_0)^\ell. \tag{4.140}$$

Next, we estimate $\#K(n, d, \ell)$. Choosing a set $E \subseteq [d^{-1}n, dn)$ with at most d elements can be done in no more than $(dn - d^{-1}n)^d \leq (dn)^d$ different ways. For each set E and $j \in [1, d]$, there are at most $(2 \cdot 8^j n/d^{-1}n)^d$ numbers $\sum_{e \in E} \alpha_e e$ in $[8^j n, 2 \cdot 8^j n)$. Thus, each E generates at most $(2d \cdot 8^1)^d \cdot (2d \cdot 8^2)^d \ldots (2d \cdot 8^\ell)^d$ sequences $(k_1, \ldots, k_\ell) \in K(n, d, \ell)$. Therefore,

$$\begin{aligned}\#K(n, d, \ell) &\leq (dn)^d \cdot (2d \cdot 8^1) \cdot (2d \cdot 8^2) \ldots (2d \cdot 8^\ell) \\ &\leq d^{2d} n^d 2^\ell 8^{(\ell+2)^2} \leq n^d \ell^{6\ell^2},\end{aligned} \tag{4.141}$$

where we used that $\ell > 3d > 8$.

Now, since we assumed that $n > n_0^3$, we have that $(n/n_0)^\ell > n^{2\ell/3}$. Hence, as the hypothesis ensures that $\ell > 3d$ and $n > \ell^{18\ell^2}$,

$$(n/n_0)^\ell > n^d n^{\ell/3} > n^d \ell^{6\ell^2}.$$

This and Equations (4.140) and (4.141) imply that $P(n, n_0, \ell) \setminus K(n, d, \ell)$ is nonempty. □

Lemma 4.79 *Let* $(\ell_n)_{n \geq 0}$ *be a sequence of positive integers. We consider, for each* $n \geq 0$, $k_n \geq 1$ *and a sequence* $(p_1^n, \ldots, p_{\ell_n}^n)$ *such that* $p_j^n \in [8^j k_n, 2 \cdot 8^j k_n)$. *For* $a \in \{0, 1\}$ *and* $\bar{a} = 1 - a$, *we define*

$$\tau_n(a) = a^{p_1^n} \bar{a}^{p_1^n} a^{p_2^n} \bar{a}^{p_2^n} \ldots a^{p_{\ell_n}^n} \bar{a}^{p_{\ell_n}^n}. \tag{4.142}$$

and $\boldsymbol{\tau} = (\tau_n)_{n \geq 0}$. *Then:*

(1) $X_{\boldsymbol{\tau}}$ *is infinite, minimal and with linear-growth complexity.*

(2) $|\tau_{[0,n)}(0)| = |\tau_{[0,n)}(1)|$.

(3) $(X_{\boldsymbol{\tau}}^{(n)}, \tau_{[0,n)})$ *is* $|\tau_{[0,n)}|$*-recognizable.*

(4) $10^{p_j^n} 1 \in \mathcal{L}(X_{\boldsymbol{\tau}}^{(n)})$ *for all* $j \in [1, \ell_n]$.

Proof. Let $\mathcal{A} = \{0, 1\}$. For $a \in \mathcal{A}$, $n \geq 0$ and $j \in [1, \ell_n]$, we use the notation $w_{n,j}(a) = \tau_{[0,n)}(a)^{p_j^n}$ and

$$W_{n,j}(a) = w_{n,1}(a) w_{n,1}(\bar{a}) \ldots w_{n,2}(a) w_{n,2}(\bar{a}) \ldots w_{n,j}(a) w_{n,j}(\bar{a}). \tag{4.143}$$

Remark that $W_{n,\ell_n}(a) = \tau_{[0,n+1)}(a)$.

We start by proving the following properties of the morphisms τ_n.

(i) $|w_{n,j}(0)| = |w_{n,j}(1)|$ and $|W_{n,j}(0)| = |W_{n,j}(1)|$ for all $n \geq 0$ and $j \in [1, \ell_n]$.

(ii) $8^j k_n |\tau_{[0,n)}| \leq |w_{n,j}(a)| \leq 2 \cdot 8^j k_n |\tau_{[0,n)}|$ and $2 \cdot 8^j k_n |\tau_{[0,n)}| \leq |W_{n,j}(a)| < 8^{j+1} k_n |\tau_{[0,n)}|$.

(iii) If $n \geq 0$, $a, b \in \mathcal{A}$ and t is a word such that $|t| \geq |\tau_n|/2$, t is a prefix of $\tau_n(a)$ and t is a suffix of $\tau_n(b)$, then $\tau_n(a) = \tau_n(b) = t$.

Item (i) directly follows from (4.142). In Item (ii), the first inequality is a consequence of the equality $|w_{n,j}(a)| = p_j^n |\tau_{[0,n)}|$ and that, by the hypothesis, $p_j^n \in [8^j k_n, 2 \cdot 8^j k_n)$. We can use this to compute $|W_{n,j}(a)| \geq 2|w_{n,j}(a)| \geq 2 \cdot 8^j k_n |\tau_{[0,n)}|$ and

$$|W_{n,j}(a)| = \sum_{i=1}^{j} |w_{n,j}(a) w_{n,j}(\bar{a})| \leq 2 \sum_{i=1}^{j} 2 \cdot 8^i k_n |\tau_{[0,n)}| < 8^{j+1} k_n |\tau_{[0,n)}|,$$

which shows the second inequality in (ii). Finally, we prove Item (iii). We note that (4.143) implies that $|\tau_{[0,n+1)}| = |W_{n,\ell_n}(0)| \geq 2|w_{n,\ell_n}(0)|$. Hence, as (i) ensures that $|\tau_{[0,n)}(0)| = |\tau_{[0,n)}(1)|$,

$$|\tau_n| = |\tau_{[0,n+1)}|/|\tau_{[0,n)}| \geq 2|w_{n,\ell_n}(a)|/|\tau_{[0,n)}| = 2|a^{p^n_{\ell_n}}|,$$

which allows us to bound $|t| \geq |\tau_n|/2 \geq |a^{p^n_{\ell_n}}|$. Being t a suffix of $\tau_n(a)$, this implies that $a^{p^n_{\ell_n}}$ is a suffix of t. Moreover, since t is a prefix of $\tau_n(b)$, $a^{p^n_{\ell_n}}$ occurs in $\tau_n(b)$. But (4.142) guarantees that $a^{p^n_{\ell_n}}$ occurs in $\tau_n(b)$ only as a suffix, so we must have that $t = \tau_n(b)$. Therefore, $\tau_n(a) = \tau_n(b) = t$.

We now prove that $\boldsymbol{\tau}$ satisfies the properties of the lemma. The morphisms τ_n are positive, so X is minimal. It follows from (4.142) that $0^{p_1^n}1$ and $0^{p_0^n}0$ belong to $\mathcal{L}(X^{(n)}_{\boldsymbol{\tau}})$ for all $n \geq 0$, so $\tau_{[0,n)}(0)1$ and $\tau_{[0,n)}(0)0$ are elements of $\mathcal{L}(X)$. This shows that X has infinitely many right-special words, and thus that X is infinite. To prove that X has linear-growth complexity, we will show that $p_X(k) \leq 1024k$ for all $k \geq 1$. Let $k \geq 1$ be arbitrary. We take $n \geq 0$ such that $|\tau_{[0,n)}| \leq k < |\tau_{[0,n+1)}|$. We consider three cases. Assume first that $k < |w_{n,1}(0)|$. Then, from (4.142) we have that any $w \in \mathcal{L}(X) \cap \mathcal{A}^k$ occurs in a word of the form $w_{n,1}(a) w_{n,1}(b)$ for some $a, b \in \mathcal{A}$. This implies, since $|w| \geq |\tau_{[0,n)}(a)|$ and $w_{n,1}(a) = \tau_{[0,n)}(a)^{p_1^n}$ for any $a \in \mathcal{A}$, that $p_X(k) \leq \#\mathcal{A}^2 \cdot k = 4k$.

Let us now assume that $|w_{n,1}(0)| \leq k < |W_{n,\ell_n - 1}(0)|$. Let $j \in [1, \ell_n - 1]$ be the least integer satisfying $k < |W_{n,j}(0)|$. Then, by (ii),

$$k \leq |w_{i,n}(a)| = |\tau_{[0,n)}(0)^{p_i^n}| \text{ for all } i \in [j+1, \ell_n] \text{ and } a \in \mathcal{A}.$$

Using this, (4.142) and the definition of $w_{n,i}(a)$ we deduce that any $w \in \mathcal{L}(X) \cap \mathcal{A}^k$ occurs in a word having either the form $W_{n,j}(a)\tau_{[0,n)}(b)^{p^n_{j+1}}$ or the form $\tau_{[0,n)}(a)^{p^n_{j+1}} \tau_{[0,n)}(b)^{p^n_{j+1}}$, where $a, b \in \mathcal{A}$. Therefore,

$$p_X(k) \leq \#\mathcal{A}^2 \cdot (|W_{n,j}(0)\tau_{[0,n)}(0)^{p^n_{j+1}}| + |\tau_{[0,n)}(0)^{p^n_{j+1}} \tau_{[0,n)}(0)^{p^n_{j+1}}|).$$

Putting that $|W_{n,j}(0)| \leq |\tau_{[0,n)}(0)^{p^n_{j+1}}|$ in the last inequality yields

$$p_X(k) \leq 16 p^n_{j+1} |\tau_{[0,n)}(0)| = 16 |w_{n,j+1}(0)|.$$

Now, if $j = 1$, then $16|w_{n,2}(0)| \leq 16 \cdot 16|w_{n,1}(0)| \leq 256k$ by (ii), and so $p_X(k) \leq 256k$. If $j > 1$, then the minimality of j implies that $k \geq |W_{n,j-1}(0)|$. Hence, by (ii), $16|w_{n,j+1}(0)| \leq 16 \cdot 8^2|W_{n,j}(0)| \leq 1024k$, and therefore $p_X(k) \leq 1024k$.

Finally, we assume that $k \geq |W_{n,\ell_n-1}(0)|$. Since $k < |\tau_{[0,n+1)}|$, we have that any $w \in \mathcal{L}(X) \cap \mathcal{A}^k$ occurs in a word of the form $\tau_{[0,n+1)}(a)\tau_{[0,n+1)}(b)$, where $a, b \in \mathcal{A}$. Thus, $p_X(k) \leq 4|\tau_{[0,n+1)}|$. Moreover, being $|W_{n,\ell_n-1}(0)|$ at most k, we have from (ii) that $4|\tau_{[0,n+1)}| = 4|W_{n,\ell_n}(0)| \leq 8^2|W_{n,\ell_n-1}(0)| \leq 8^2k$; therefore, $p_X(k) \leq 64k$.

We conclude that $p_X(k) \leq 1024k$ for every $k \geq 1$ and that X has linear-growth complexity.

Items (2) and (4) of the lemma follow from (4.142). Thus, it only left to prove Item (3). We note that, since $|\tau_{[0,n)}| = |\tau_0| \cdots |\tau_{n-1}|$, it is enough, by Lemma 1.1, to prove that $(\mathcal{A}^{\mathbb{Z}}, \tau_n)$ is $|\tau_n|$-recognizable for all $n \geq 0$. Let $x, \tilde{x} \in \mathcal{A}_n$ and $(k, y), (\tilde{k}, \tilde{y})$ be τ_n-factorizations of $x, \tilde{x}$ in $\mathcal{A}^{\mathbb{Z}}$, respectively, and assume that $x_{[-|\tau_n|,|\tau_n|)}$ is equal to $\tilde{x}_{[-|\tau_n|,|\tau_n|)}$. We assume with no loss of generality that $k \leq \tilde{k}$. There are two cases. If $\tilde{k} - k \leq |\tau_n|/2$, then $x_{[-k,-\tilde{k}+|\tau_n|)}$ has length at least $|\tau_n|/2$ and is a suffix of $\tau_n(y_0)$. As $x_{[-|\tau_n|,|\tau_n|)} = \tilde{x}_{[-|\tau_n|,|\tau_n|)}$, we also have that $x_{[-k,-\tilde{k}+|\tau_n|)}$ is a prefix of $\tau_n(\tilde{y}_1)$. We deduce, using (iii), that $\tau_n(y_0) = \tau_n(\tilde{y}_0) = x_{[-k,-\tilde{k}+|\tau_n|)}$, and thus that $y_0 = \tilde{y}_0$. Moreover, since $k, \tilde{k} \in [0, |\tau_n|)$, $k = \tilde{k}$. Let us now suppose that $\tilde{k} - k \geq |\tau_n|/2$. Then, $|x_{[-\tilde{k},-k)}| \geq |\tau_n|/2$ and $x_{[-\tilde{k},-k)}$ is both a suffix of $\tau_n(\tilde{y}_{-1})$ and a prefix of $\tau_n(y_0)$. Hence, by (iii), $\tau_n(\tilde{y}_{-1}) = \tau_n(y_0) = x_{[-k,-\tilde{k})}$, which is impossible as $k, \tilde{k} \in [0, |\tau_n|)$. We conclude that $(X_{\boldsymbol{\tau}}^{(n)}, \tau_n)$ is $|\tau_n|$-recognizable. □

We can now prove Theorem 4.77.

Proof of Theorem 4.77. Let $(\ell_n)_{n\geq 0}$ and $(d_n)_{n\geq 0}$ be nondecreasing diverging sequences of integers with $\ell_n > 3d_n > 8$. We inductively define M_n, m_n, $(p_1^n, \ldots, p_{\ell_n}^n)$ and τ_n as follows. Let $m_0 = 1$ and M_0 be such that $M_0 > m_0^3$ and $M_0 > \ell_0^{18\ell_0^2}$. Then, we can use Lemma 4.78 to find

$$(p_1^0, p_2^0, \ldots, p_{\ell_0}^0) \in P(M_0, m_0, \ell_0) \setminus K(M_0, d_0, \ell_0).$$

We define τ_0 using $(p_1^0, \ldots, p_{\ell_0}^0)$ as in (4.142). Suppose now that M_n, m_n, $(p_1^n, \ldots, p_{\ell_n}^n)$ and τ_n are defined. We set $m_{n+1} = |\tau_{[0,n+1)}|$ and take M_{n+1} so that $M_{n+1} > m_{n+1}^3$ and $M_{n+1} > \ell_{n+1}^{18\ell_{n+1}^2}$. Then, we can use Lemma (4.142) to find

$$(p_0^{n+1}, p_1^{n+1}, \ldots, p_{\ell_n}^{n+1})_{j=1}^{\ell_n} \in P(M_{n+1}, m_{n+1}, \ell_{n+1}) \setminus K(M_{n+1}, d_{n+1}, \ell_{n+1}) \tag{4.144}$$

and define τ_{n+1} using $(p_1^{n+1}, \ldots, p_{\ell_{n+1}}^{n+1})$ as in (4.142).

We set $\boldsymbol{\tau} = (\tau_n)_{n\geq 0}$. Then, Items (1) to (4) in Lemma 4.79 hold. In particular, $X_{\boldsymbol{\tau}}$ is minimal and has linear-growth complexity. We prove that $X_{\boldsymbol{\tau}}$ satisfies the conclusion of the theorem by contradiction. Suppose that there exist d and $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ satisfying Items (1), (2) and (3) of Theorem 4.75 and $\#\mathcal{A}_n \leq d$ for all $n \geq 1$.

We claim that there exists $n, n' \geq 0$ such that

$$d_n \geq 2d^6 + d,\ 2m_n \leq \frac{1}{6d^2}M_n \leq \langle \sigma_{[0,n')} \rangle \text{ and } |\sigma_{[0,n')}| \leq \frac{1}{6}M_n. \tag{4.145}$$

We take $n \geq 0$ big enough so that $d_n \geq 12d^4 + d$ and $m_n \geq 12d^2$. Let $n' \geq 0$ be the integer satisfying $|\sigma_{[0,n')}| \leq \frac{1}{6}M_n < |\sigma_{[0,n'+1)}|$. Then, by Items (2) and (3) in Theorem 4.75,

$$\frac{1}{6}M_n < |\sigma_{[0,n'+1)}| \leq d^2\langle\sigma_{[0,n')}\rangle.$$

Also, since we chose M_n and n so that $M_n > m_n^3$ and $m_n \geq 12d^2$, we have that $m_n \leq \frac{1}{12d^2}M_n$. This completes the proof of the claim.

Let $w_a = \tau_{[0,n)}(a)$ for $a \in \{0,1\}$. Then, by Item (4) in Lemma 4.79, $10^{p_j^n}1 \in \mathcal{L}(X_{\boldsymbol{\tau}}^{(n)})$ for every $j \in [1, p_{\ell_n}^n]$. Being $X_{\boldsymbol{\tau}}$ generated by $\boldsymbol{\sigma}$, there exist $u_j \in \mathcal{A}^+$ such that $w_1 w_0^{p_j^n} w_1$ occurs in $\sigma_{[0,n')}(u_j)$. Moreover, we can take u_j so that the following condition holds: If a_j is the first letter of u_j and b_j is the last letter of u_j, then there exists a prefix s_j of $\sigma_{[0,n')}(a_j)$ and a suffix t_j of $\sigma_{[0,n')}(b_j)$ such that $s_j w_1 w_0^{p_j^n} w_1 t_j = \sigma_{[0,n')}(u_j)$. Observe that $|\sigma_{[0,n')}(u_j)| \geq |w_0^{p_1^n}| = M_n$, so

$$|u_j| \geq |\sigma_{[0,n')}(u_j)|/|\sigma_{[0,n')}| \geq M_n/|\sigma_{[0,n')}| \geq 6. \tag{4.146}$$

We define $a_j a_j' a_j''$ as the first three letters of u_j and $b_j'' b_j' b_j$ as the last three letters of u_j.

We claim that

$$\text{if } i,j \in [1,\ell_n] \text{ and } a_i a_i' a_i'' b_i'' b_i' b_i = a_j a_j' a_j'' b_j'' b_j' b_j, \text{ then } s_i = s_j \text{ and } t_i = t_j. \tag{4.147}$$

The inequality $|w_1| = m_n \leq |\sigma_{[0,n')}|$ and (4.146) ensure that $\sigma_{[0,n)}(a_i)$ is a prefix of $s_i w_1 w_0^\infty$. Hence, as s_i is a prefix of $\sigma_{[0,n)}(a_i)$, we can write $\sigma_{[0,n)}(a_i a_i') = s_i w_1 w_0^{q_i} r_i$, where $q_i \geq 1$ and r_i is a prefix of w_0 different from w_0. Similarly, $\sigma_{[0,n)}(a_j a_j') = s_j w_1 w_0^{q_j} r_j$ for some $q_j \geq 2$ and prefix r_j of w_0 different from w_0. Then, as the hypothesis implies that $\sigma_{[0,n)}(a_i a_i' a_i'') = \sigma_{[0,n)}(a_j a_j' a_j'')$, Item (4) in Lemma 4.79 can be used to obtain $r_i = r_j$. We obtain that

$$s_i w_1 w_0^{q_i} r_i = \sigma_{[0,n)}(a_i a_i') = \sigma_{[0,n)}(a_j a_j') = s_j w_1 w_0^{q_j} r_j = s_j w_1 w_0^{q_j} r_i.$$

Therefore, if $q_i \neq q_j$ then $w_0 = w_1$, which contradicts the fact that X is infinite. We conclude that $q_i = q_j$, and thus that $s_i = s_j$. A similar argument shows that $t_i = t_j$, and the claim follows.

Thanks to the claim, we have that the set

$$S = \{|\sigma_{[0,n')}(a_k a_k')| - |s_k w_1|, |\sigma_{[0,n')}(b_k' b_k)| - |w_1 t_k| : k \in [1,\ell_n]\} \\ \cup \{|\sigma_{[0,n')}(a)| : a \in \mathcal{A}_{n'}\}$$

has no more than $2\#\mathcal{A}_{n'}^6 + \#\mathcal{A}_{n'}$ elements. Thus, by the choice of d_n,

$$\#S \leq 2d^6 + d \leq d_n. \tag{4.148}$$

Also, by (4.145), $\max S \leq 2|\sigma_{[0,n')}| \leq M_n$. Moreover, as $|s_j w_1| \leq |\sigma_{[0,n')}(a_j)| + m_n$ and $|w_1 t_j| \leq |\sigma_{[0,n')}(b_j)| + m_n$, we have that $\min S \geq \langle\sigma_{[0,n')}\rangle - m_n \geq \frac{1}{12d^2}M_n$, where in the last step we used (4.145). Therefore, as $d_n \geq 2d^6 + d$,

$$S \subseteq [\frac{1}{d_n}M_n, d_n M_n]. \tag{4.149}$$

Let us write $u_j = a_j a'_j v_j b'_j b_j$ for certain v_j. Then, the equation $\sigma_{[0,n)}(u_j) = s_i w_1 w_0^{q_i} r_i$ implies that

$$p_j^n m_n = |w_0^{p_j^n}| = (\sigma_{[0,n')}(a_j a'_j) - |s_j w_1|) + |\sigma_{[0,n')}(v_j)| + (\sigma_{[0,n')}(b'_j b_j) - |w_1 t_j|).$$

This shows that

$$\text{every } p_j^n m_n \text{ can be written as } \sum_{e \in S} \gamma_e e \text{ for certain } \gamma_e \in \mathbb{Z}_{\geq 0}. \tag{4.150}$$

We conclude, from Equations (4.148), (4.149) and (4.150) that $(p_1^n, \dots, p_{\ell_n}^n)$ belongs to $K(M_n, d_n, \ell_n)$. This contradicts (4.144). □

4.10 Applications

We present in this section new and simpler proofs, based on Theorems 4.75 and 4.76, of known results about linear-growth and nonsuperlinear-growth complexity subshifts.

4.10.1 Cassaigne's Theorem

A classic result on linear-growth complexity subshift is Cassaigne's Theorem [Cas95], which states that, for any transitive subshift X in this complexity class, $p_X(n+1) - p_X(n)$ is uniformly bounded. We show in this subsection how to use Theorem 4.75 to give a different proof of this result, in the case in which X is minimal.

We start with a lemma containing the technical core of our approach.

Lemma 4.80 *Let $x, y \in \mathcal{A}^{\mathbb{Z}}$, $p_1, \dots, p_n \in \mathbb{Z}$ be a collection of different integers and $\ell_1, \dots, \ell_n \geq 1$. Suppose that:*

(1) $x_{[p_j, p_j+\ell_j)} = y_{[0,\ell_j)}$ for all $j \in [1, n]$.

(2) $|p_j - p_i| \leq \frac{1}{2}\ell_k$ for all $i, j, k \in [1, n]$.

Then, there exists $w \in \mathcal{A}^+$ such that, for all $i, j \in [1, n]$, $x_{[p_i, p_j)}$ is a power of w and $x_{[p_i, p_j+\min(\ell_i,\ell_j))}$ is a prefix of w^∞.

Proof. Being the p_j different, there is no loss of generality in assuming that $p_1 < p_2 < \cdots < p_n$. We define, for $i, j \in [1, n]$ with $i < j$, $w_{i,j} = \mathsf{root}\, x_{[p_i, p_j)}$ and $\ell_{i,j} = \min(\ell_i, \ell_j)$. Then, Item (1) in the statement of the lemma ensures that $x_{[p_i, p_i+\ell_{i,j})} = x_{[p_j, p_j+\ell_{i,j})}$, and thus that $x_{[p_i, p_j+\ell_{i,j})}$ is a prefix of $w_{i,j}^\infty$. In particular, as $p_i < p_j$ and $w_{i,j} = \mathsf{root}\, x_{[p_i, j_j)}$, we have that for all $i, j \in [1, n]$ with $i < j$,

$$x_{[p_i, p_i+\ell_{i,j}]} \text{ and } x_{[p_j, p_j+\ell_{i,j})} \text{ are prefixes of } w_{i,j}^\infty. \tag{4.151}$$

Therefore, it is enough to find w such that $w = w_{i,j}$ for all $i < j$.

First, we show that

$$w_{i,k} = w_{j,k} \text{ for all } i, j, k \in [1, n] \text{ with } i, j < k. \tag{4.152}$$

Observe that, in this situation, we have from (4.151) that if $m = \min(\ell_{i,k}, \ell_{j,k})$, then $x_{[p_k, p_k+m)}$ is a prefix of $w_{i,k}^{\infty}$ and $x_{[p_k,p_k+m)}$ is a prefix of $w_{j,k}^{\infty}$. Thus,

$$(w_{i,k}^{\mathbb{Z}})_{[0,m)} = (w_{j,k}^{\mathbb{Z}})_{[0,m)}. \tag{4.153}$$

Now, by Item (2) in the statement of the lemma and the definition of $w_{i,k}$ and $w_{j,k}$, $w_{i,k}$ and $w_{j,k}$ have length at most $m/2$. This and (4.153) permits to use Theorem 4.8 and obtain that $w_{i,k}$ and $w_{j,k}$ are powers of a common word. This implies, since $w_{i,k}$ and $w_{j,k}$ are defined as roots, that $w_{i,k} = w_{j,k}$.

We now note that if $i, j \in [1, n]$ and $i < j$, then (4.152) ensures that $w_{1,j} = w_{i,j}$. Hence, as $x_{[p_1,p_j)} = x_{[p_1,p_i)}x_{[p_i,p_j)}$, $w_{1,i} = w_{1,j} = w_{i,j}$. Being i, j arbitrary, this implies that $w_{1,2} = w_{1,j} = w_{i,j}$. Therefore, the lemma follows from defining $w \coloneqq w_{1,2}$. □

The proposition below uses Lemma 4.80 to give a bound for $p_X(n+1) - p_X(n)$ in a very general context.

Proposition 4.81 *Let $\mathcal{W} \subseteq \mathcal{A}^+$ and $X \subseteq \bigcup_{k\in\mathbb{Z}} S^k\mathcal{W}^{\mathbb{Z}}$. Then, for any $\ell < \langle\mathcal{W}\rangle$,*

$$p_X(\ell+1) - p_X(\ell) \leq 256\#\mathcal{A} \cdot \#(\mathsf{root}\,\mathcal{W})^2|\mathcal{W}|^2/\ell^2.$$

Proof. We prove the proposition by contradiction. Suppose that $\ell < \langle\mathcal{W}\rangle$ and that $p_X(\ell + 1) - p_X(\ell) \geq 256\#\mathcal{A} \cdot \#(\mathsf{root}\,\mathcal{W})^2|\mathcal{W}|^2/\ell^2$. Then, by Proposition 4.14, we can find at least $256\#(\mathsf{root}\,\mathcal{W})^2|\mathcal{W}|^2/\ell^2$ right-special words $\{u_i : i \in I\}$ of length ℓ in X. Let $u_i a_{i,0}$ and $u_i a_{i,1}$ be two different right extensions for u_i in X. We are going to prove that $a_{i,0} = a_{i,1}$ for some $i \in I$, contradicting the fact that $u_i a_{i,0}$ and $u_i a_{i,1}$ are different.

Let $X' = \{\dots vvv.v'v'v'\cdots : v, v' \in \mathsf{root}\,\mathcal{W}\} \subseteq \mathcal{A}^{\mathbb{Z}}$. Then, it is not difficult to check that:

(a) every $w \in \mathcal{L}(X)$ of length at most $\langle\mathcal{W}\rangle$ occurs in some $x \in X'$.

(b) $\#X' \leq \#(\mathsf{root}\,\mathcal{W})^2$.

In particular, each $u_i a_{i,j}$ occurs in some $x_{i,j} = \dots v_{i,j}.v'_{i,j}\cdots \in X'$, so

$$u_i a_{i,j} = (x_{i,j})_{[\beta_{i,j},\beta_{i,j}+\ell]} \text{ for some } \beta_{i,j} \in [-|\mathcal{W}|, |\mathcal{W}|). \tag{4.154}$$

We use (b) and the Pigeonhole principle to obtain a set $I' \subseteq I$ and $x_0, x_1 \in X'$ such that $\#I' \geq 256|\mathcal{W}|^2/\ell^2$ and $x_j = x_{i,j}$ for all $i \in I'$ and $j \in \{0, 1\}$. We use again the Pigeonhole principle to find $I'' \subseteq I'$ satisfying $\#I'' \geq \#I'/(8|\mathcal{W}|/\ell)^2 \geq 4$ and

$$|\beta_{i,j} - \beta_{i',j}| \leq 2|\mathcal{W}|/(8|\mathcal{W}|/\ell) = \ell/4 \text{ for all } i, i' \in I'' \text{ and } j \in \{0, 1\}. \tag{4.155}$$

Let $\beta = \max\{\beta_{i,1} : i \in I''\}$ and, for $i \in I''$, $\gamma_i = \beta_{i,0} - \beta_{i,1} + \beta$. We claim that for all $i \in I''$,

(i) $\beta_{i,1} \leq \beta \leq \beta_{i,1} + \ell/4$ and $\beta_{i,0} \leq \gamma_i \leq \beta_{i,0} + \ell/4$;

(ii) u_i has a suffix u'_i of length at least $\frac{3}{4}\ell$ such that $u'_i a_{i,0} = (x_0)_{[\gamma_i, \gamma_i+|u'_i|]}$ and $u'_i a_{i,1} = (x_1)_{[\beta,\beta+|u'_i|]}$;

(iii) if $p, q \in I''$ are different, then $\gamma_p + |u'_p| \neq \gamma_q + |u'_q|$ and $\gamma_p \neq \gamma_q$.

Item (i) follows from (4.155). For Item (ii), we first note that the definition of γ_i ensures that $m_i := \ell + \beta_{i,1} - \beta = \ell + \beta_{i,0} - \gamma_i$, and that (i) gives $\frac{3}{4}\ell \leq m_i \leq \ell$. Thus, u_i has a suffix u'_i of length $m_i \geq \frac{3}{4}\ell$ such that, by (4.154), satisfies Item (ii). It is left to prove (iii). Assume that $p, q \in I''$ and $\gamma_p + |u'_p| = \gamma_q + |u'_q|$. Then, $\beta_{p,0} = \gamma_p + |u'_p| - \ell = \gamma_q + |u'_q| - \ell = \beta_{q,0}$ and hence $u_p = u_q$, which implies that $p = q$. Let us now suppose that $\gamma_p = \gamma_q$. Note that if $|u'_p| = |u'_q|$ then $\gamma_p + |u'_p| = \gamma_q + |u'_q|$, and so $p = q$ by what we just proved. Thus, there is no loss of generality in assuming that $|u'_p| < |u'_q|$. Then, (ii) allows us to write $u'_p a_{p,0} = (x_0)_{[\gamma_p,\gamma_p+|u'_p|]}$ and $u'_q = (x_0)_{[\gamma_p,\gamma_p+|u'_q|]}$. In particular, $u'_p a_{p,0}$ is a prefix of u'_q. Similarly, (ii) implies that $u'_p a_{p,1} = (x_1)_{[\beta,\beta+|u'_p|]}$ is a prefix of $u'_q = (x_1)_{[\beta,\beta+|u'_q|)}$. Therefore, $u'_p a_{p,0} = u'_p a_{p,1}$, which contradicts the definition of $a_{p,0}$ and $a_{p,1}$. This shows that the case $|u'_p| < |u'_q|$ does not occur, so $|u'_p| = |u'_q|$ and $p = q$. This completes the proof of the claim.

Thanks to the claim, we have that $(x_0)_{[\gamma_i,\gamma_i+|u'_i|)} = (S^\beta x_1)_{[0,|u'_i|)}$ and $|\gamma_i - \gamma_{i'}| \leq \frac{1}{2}|u'_i|$ for all $i, i' \in I''$. Moreover, all the γ_i are different by (iii). Therefore, we can use Lemma 4.80 and deduce that there exists $w \in \mathcal{A}^+$ such that for any $p, q \in I''$,

$$(x_0)_{[\gamma_p,\gamma_q)} \text{ is a power of } w \quad \text{and } (x_0)_{[\gamma_p,\gamma_p+\min(|u'_p|,|u'_q|))} \text{ is a prefix of } w^\infty. \tag{4.156}$$

We use Item (iii) of the claim and that $\#I'' \geq 4$ to find $p, q \in I''$ such that $|u'_t| < |u'_p| < |u'_q|$ for all $t \in I'' \setminus \{p, q\}$. Furthermore, (iii) allows us to find $r, s \in I'' \setminus \{p, q\}$ such that $\gamma_r + |u'_r| < \gamma_s + |u'_s|$.

We observe that, since $|u'_s| < |u'_p|$, the second part of (4.156) ensures that that $u'_s = (x_0)_{[\gamma_s,\gamma_s+|u'_s|)}$ is a prefix of w^∞. Then, by the first part of (4.156), $(x_0)_{[\gamma_r,\gamma_s+|u'_s|)}$ is a prefix of w^∞. Since $\gamma_r + |u'_r| < \gamma_s + |u'_s|$, we get that

$$u'_r a_{r,0} = (x_0)_{[\gamma_r,\gamma_r+|u'_r|]} \text{ is a prefix of } w^\infty. \tag{4.157}$$

Now, the definition of r and p guarantees that $|u'_r| < |u'_p|$, so, by (4.154), $u'_r a_{r,1} = (x_1)_{[\beta,\beta+|u'_r|])}$ is a prefix of $(x_0)_{[\gamma_p,\gamma_p+|u'_p|)} = (x_1)_{[\beta,\beta+|u'_p|)}$. Moreover, as $|u'_p| < |u'_q|$, the second part of (4.156) gives that $(x_0)_{[\gamma_p,\gamma_p+|u'_p|)}$ is a prefix of w^∞. We conclude that $u'_r a_{r,1}$ is a prefix of w^∞. But then (4.157) implies that $u'_r a_{r,1} = u'_r a_{r,0}$, contradicting our assumptions. □

Theorem 4.82 ([Cas95]) *Let X be a minimal linear-growth complexity subshift. Then, $p_X(\ell + 1) - p_X(\ell)$ is uniformly bounded.*

Proof. Let $\ell \geq 1$ be arbitrary. The theorem is trivial if X is finite, so we assume that X is infinite. Then, we can use Theorem 4.75 to obtain an $\mathcal{S}$-adic sequence $\boldsymbol{\sigma} = (\sigma_n : \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\geq 1}$ generating X and $d \geq 1$ such that the conditions (1), (2) and (3) of Theorem 4.75 hold.

Let $n \geq 1$ be the least integer such that $\langle\sigma_{[0,n)}\rangle > \ell$. Then, X is a subset of $\bigcup_{k\in\mathbb{Z}} S^k\sigma_{[0,n)}(\mathcal{A}_n^{\mathbb{Z}})$, so Proposition 4.81 and the conditions in Theorem 4.75 give the bounds:

$$\begin{aligned} p_X(\ell+1) - p_X(\ell) &\leq 256\#\mathcal{A}_0\#(\mathsf{root}\,\sigma_{[0,n)}(\mathcal{A}_n))^2|\sigma_{[0,n)}|^2/\ell^2 \\ &\leq 256\#\mathcal{A}_0 \cdot d^2 \cdot |\sigma_{[0,n)}|^2/\ell^2. \end{aligned}$$

Now, the minimality of n ensures that $\langle\sigma_{[0,n-1)}\rangle \leq \ell$, so by Items (2) and (3) in Theorem 4.75,

$$|\sigma_{[0,n)}| \leq |\sigma_{n-1}| \cdot |\sigma_{[0,n-1)}| \leq d^2\langle\sigma_{[0,n-1)}\rangle \leq d^2\ell.$$

Therefore, $p_X(\ell+1) - p_X(\ell) \leq 256\#\mathcal{A}_0 \cdot d^6$ and $p_X(\ell+1) - p_X(\ell)$ is uniformly bounded. □

4.10.2 A theorem of Cassaigne, Frid, Puzynina and Zamboni

The following result was proven in [CFPZ18].

Theorem 4.83 *Let $x \in \mathcal{A}^{\mathbb{N}}$ be an infinite sequence. The following conditions are equivalent:*

(1) x has linear-word complexity.

(2) There exists $S \subseteq \mathcal{A}^*$ such that $S^2 \supseteq \mathcal{L}(x)$ and $\sup_{n\geq 1} p_S(n) < +\infty$.

In this subsection, we give a different proof of Theorem 4.83 for the case of minimal subshifts. We start by proving the following corollary of Theorem 4.75.

Proposition 4.84 *Let X be an infinite minimal subshift of linear-growth complexity. There exists $d \geq 1$ such that for any $d' \geq 2$ we can find $\boldsymbol{\tau} = (\tau_n\colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ generating X such that:*

(1) $\#(\mathsf{root}\,\tau_{[0,n)}(\mathcal{A}_n)) \leq d$.

(2) $|\tau_{[0,n)}(a)| \leq d \cdot |\tau_{[0,n)}(b)|$ for all $a, b \in \mathcal{A}_n$.

(3) $d' \leq |\tau_{n-1}(a)| \leq d^{6\log_2 d'}$ for all $a \in \mathcal{A}_n$.

Proof. Let $\boldsymbol{\tau}' = (\tau_n\colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ and d be the elements given by Theorem 4.75 when it is applied with X, and let $d' \geq 1$ be arbitrary. We will construct $\boldsymbol{\tau}$ by carefully contracting $\boldsymbol{\tau}'$.

Let $n_0 = 0$ and inductively define n_{k+1} as the smallest integer such that $n_{k+1} > n_k$ and $\langle\tau_{[n_k,n_{k+1})}\rangle \geq 2$. We observe that, since $\langle\tau_{[n_k,n_{k+1}-1)}\rangle = 1$ by the minimality of n_{k+1}, we have that $\langle\tau_{[0,n_{k+1}-1)}\rangle \leq |\tau_{[0,n_k)}|$. Then, by Items (2) and (3) in Theorem 4.75, we can bound

$$|\tau_{[0,n_{k+1})}| \leq |\tau_{[0,n_{k+1}-1)}||\tau_{n_{k+1}-1}| \leq d\langle\tau_{[0,n_{k+1}-1)}\rangle \cdot d \leq d^2|\tau_{[0,n_k)}| \leq d^3\langle\tau_{[0,n_k)}\rangle.$$

We note now that for any pair of morphisms σ and σ' for which $\sigma\sigma'$ is defined we have that $|\sigma\sigma'| \geq \langle\sigma\rangle|\sigma'|$. Therefore,

$$|\tau_{[n_k,n_{k+1})}| \leq \frac{|\tau_{[0,n_{k+1})}|}{\langle\tau_{[0,n_k)}\rangle} \leq d^3. \tag{4.158}$$

We set $\ell = \lceil \log_2 d' \rceil$ and consider the contraction $\boldsymbol{\tau} = (\tau_{[n_{\ell k}, n_{\ell(k+1)})})_{k \geq 0}$. It follows from the definition of n_k that $\langle \tau_{[n_{\ell k}, n_{\ell(k+1)})} \rangle \geq 2^\ell \geq d'$, and, from (4.158), that $|\tau_{[n_{\ell k}, n_{\ell(k+1)})}| \leq d^{3\ell} \leq d^{6 \log_2 d'}$. Thus, $\boldsymbol{\tau}$ satisfies Item (3) of this proposition. Moreover, since $\boldsymbol{\tau}'$ satisfies Item (1) and (2) in Theorem 4.75 and since $\boldsymbol{\tau}$ is a contraction of $\boldsymbol{\tau}'$, Items (1) and (2) of this proposition hold. □

Lemma 4.85 *Let $w \in \mathcal{A}^+$ and $\ell \leq |w|$. There exists a set of words $\mathcal{V}$ such that:*

(1) $\#\mathcal{V} \cap \mathcal{A}^n \leq 2^5 |w|/\ell$ for all $n \geq 1$.

(2) $\langle \mathcal{V} \rangle \geq \ell$, $|\mathcal{V}| \leq |w|$.

(3) For any u occurring in w of length $|u| \geq 2^6 \ell$ we have that $u \in \mathcal{V}^2$.

Proof. For $i \geq 0$ and $j \in [0,7]$, let $w = u_{i,j}(1) u_{i,j}(2) \dots u_{i,j}(2^i)$ be the (unique) decomposition of w into 2^i words $u_{i,j}(k) \in \mathcal{A}^*$ such that $|u_{i,j}(1) \dots u_{i,j}(k)| = \lfloor (8k+j)|w|/2^{i+3} \rfloor$ for all $k \in [1, 2^i]$. We define $\mathcal{V}_i$ as the set of words that are a prefix or a suffix of length at least ℓ of some $u_{i,j}(k)$. Set $\mathcal{V} \coloneqq \cup_{0 \leq i < \log_2(|w|/\ell)} \mathcal{V}_i$.

It follows from the definition of $\mathcal{V}$ that $\langle \mathcal{V} \rangle \geq \ell$ and that $|\mathcal{V}| \leq |w|$, so Item (2) holds. For Item (1), we note that if $n \geq 1$, then each $u_{i,j}(k)$ has at most one prefix of length n and at most one suffix of length n. Hence, $\#\mathcal{V}_i \cap \mathcal{A}^n$ is bounded by above by $2 \cdot 8 \cdot 2^i = 2^{i+4}$. Therefore, $\#\mathcal{V} \cap \mathcal{A}^n \leq \sum_{0 \leq i < \log_2 |w|/\ell} 2^{i+4} \leq 2^5 |w|/\ell$.

We now prove Item (3). Let u be a word of length $|u| \geq 2^6 \ell$ that occurs in w. Let us write $w = tus$, where $t, s \in \mathcal{A}^*$, and take $i \geq 0$ such that $|w|/2^{i+1} < |u| \leq |w|/2^i$. Remark that $i < \log_2(|w|/\ell)$ as $|u| \geq 2^6 \ell$. We also consider the unique pair $(k,j) \in [1, 2^i] \times [0,7]$ such that

$$(8k+j)|w|/2^{i+3} \leq |t| + |w|/2 < (8k+j+1)|w|/2^{i+3}.$$

Then, we can write $u = u'u''$ in such a way that $|tu'| = \lfloor (8k+j)|w|/2^{i+3} \rfloor$. It is not difficult to check that u' is a suffix of $u_{i,j}(k)$ and that u'' is a prefix of $u_{i,j}(k+1)$. Moreover, $|u'|, |u''| \geq |w|/2^i \geq \ell$, so $u', u'' \in \mathcal{V}_i$ and $u \in \mathcal{V}^2$. □

Theorem 4.86 *Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be an minimal subshift. The following conditions are equivalent:*

(1) X has linear-word complexity.

(2) There exists $S \subseteq \mathcal{A}^*$ such that $S^2 \supseteq \mathcal{L}(X)$ and $\sup_{n \geq 1} p_S(n) < +\infty$.

Proof. We first suppose that X satisfies Condition (2) and define $d = \sup_{n \geq 1} p_S(n)$. Then $\mathcal{L}(X) \cap \mathcal{A}^n \subseteq \{ss' : s, s' \in S, |s| = n - |s'|\}$. Hence,

$$p_X(n) \leq \sum_{k=0}^{n} p_S(k) p_S(n-k) \leq (n+1) d^2$$

and p_X has linear growth.

Let us now suppose that X has linear-growth complexity. The case in which X is finite is trivial; hence, we assume that X is infinite. Using Proposition 4.84 with $d' = 2$, we can find a constant d and an $\mathcal{S}$-adic sequence $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ generating X such that Items (1), (2) and (3) in Proposition 4.84 hold.

We define S as follows. Let $n \geq 1$. For $u \in \mathsf{root}\,\tau_{[0,n)}(\mathcal{A}_n)$, we take $p_{n,u} \geq 1$ such that $|\tau_{[0,n)}| \leq |u^{p_{n,u}}| < 2|\tau_{[0,n)}|$. We define $\mathcal{W}_n = \{u^{p_{n,u}} v^{p_{n,v}} : u, v \in \mathsf{root}\,\tau_{[0,n)}(\mathcal{A}_n)\}$. For each $w \in \mathcal{W}_n$, we use Lemma 4.85 with $\ell = \langle \tau_{[0,n-1)} \rangle / 2^6$ to obtain a set $\mathcal{V}_{n,w}$ satisfying the following:

(a) $\#\mathcal{V}_{n,w} \cap \mathcal{A}^k \leq 2^{11}|w|/\langle \tau_{[0,n-1)} \rangle$ for all $k \geq 1$.

(b) $\langle \mathcal{V}_{n,w} \rangle \geq \langle \tau_{[0,n-1)} \rangle / 2^6$, $|\mathcal{V}_{n,w}| \leq |w|$.

(c) if u occurs in w and $|u| \geq \langle \tau_{[0,n-1)} \rangle$, then $w \in \mathcal{V}_{n,w}^2$.

We set $S = \cup_{n\geq 1} \cup_{w \in \mathcal{W}_n} \mathcal{V}_{n,w}$.

Before continuing, we make some observations about the definitions. It follows from the definition of $\mathcal{W}_n$ and Item (1) in Proposition 4.84 that

$$\#\mathcal{W}_n \leq \#(\mathsf{root}\,\tau_{[0,n)}(\mathcal{A}_n))^2 \leq d^2. \tag{4.159}$$

We also have that

$$\text{if } a, b \in \mathcal{A}_n \text{ and } v \text{ occurs in } \tau_{[0,n)}(ab), \text{ then } v \text{ occurs in some } w \in \mathcal{W}_n. \tag{4.160}$$

Note that since $|w| \leq 2|\tau_{[0,n)}|$ for all $w \in \mathcal{W}_n$, (a) and (b) imply that

$$|\mathcal{V}_{n,w}| \leq 2|\tau_{[0,n)}| \text{ and } \#\mathcal{V}_{n,w} \leq 2^{12}|\tau_{[0,n)}|/\langle \tau_{[0,n-1)} \rangle \leq 2^{12} d^2, \tag{4.161}$$

where in the last step we used Items (2) and (3) of Proposition 4.84.

We now prove that S satisfies the desired properties. Let us start by showing that $\mathcal{L}(X) \subseteq S^2$. Let $u \in \mathcal{L}(X)$ and let $n \geq 1$ be the biggest integer such that $|u| \geq \langle \tau_{[0,n)} \rangle$. Then, $|u| < \langle \tau_{[0,n+1)} \rangle$ and, thus, as $\boldsymbol{\tau}$ generates X, there exists $a, b \in \mathcal{A}_{n+1}$ such that u occurs in $\tau_{[0,n+1)}(ab)$. Hence, by (4.160), u occurs in some $w \in \mathcal{W}_{n+1}$, which implies, by (c), that $u \in \mathcal{V}_{n+1,w}^2 \subseteq S^2$.

It remains to prove that p_S is uniformly bounded. Let $S_n = \cup_{w \in \mathcal{W}_n} \mathcal{V}_{n,w}$. We claim the following:

(i) $\#p_{S_n}(k) \leq 2^{12} d^4$ for all $n \geq 0$ and $k \geq 0$.

(ii) for any $k \geq 0$, there are at most $\log_2(d) + 7$ integers n such that $S_n \cap \mathcal{A}^k$ is not empty.

Observe that (i) and (ii) allow us to write

$$p_S(k) \leq \sum_{n : S_n \cap \mathcal{A}^k \neq \emptyset} p_{S_n}(k) \leq (\log_2 d + 7) \cdot 2^{12} d^4,$$

which would show that p_S is uniformly bounded and would complete the proof.

Let us first prove (i). The definition of S_n ensures that $p_{S_n}(k) \leq \#\mathcal{W}_n \cdot \max\{\#\mathcal{V}_{n,w} \cap \mathcal{A}^k : w \in \mathcal{W}_n\}$. Hence, by (4.159) and (4.161), $p_{S_n}(k) \leq d^2 \cdot 2^{12} d^2$.

Next, we prove (ii) by contradiction. Assume that there are more than $\log_2 d + 7$ integers n such that $S_n \cap \mathcal{A}^k \neq \emptyset$. Then, we can find n and m such that $S_n \cap \mathcal{A}^k \neq \emptyset$, $S_m \cap \mathcal{A}^k \neq \emptyset$ and $m > n + \log_2 d + 7$. We have, on one hand, that the definition of S_m ensures that $k \geq \min_{w \in \mathcal{W}_n} \langle \mathcal{V}_{n,w} \rangle$. Hence, by (b) and Item (3) in Proposition 4.84,

$$k \geq \langle \tau_{[0,m-1)} \rangle / 2^6 \geq 2^{m-n-7} \langle \tau_{[0,n)} \rangle. \tag{4.162}$$

On the other hand, the definition of S_n guarantees that $k \leq \max_{w \in \mathcal{W}_n} |\mathcal{V}_{n,w}|$. Combining this with (4.161) and Item (2) in Proposition 4.84 produces

$$k \leq 2|\tau_{[0,n)}| \leq 2d \langle \tau_{[0,n)} \rangle. \tag{4.163}$$

Equations (4.162) and (4.163) are incompatible as $m - n - 7 > \log_2 d$. This contradiction proves (ii) and completes the proof of the theorem. □

4.10.3 Topological rank

The topological rank of a minimal subshift X is the least element $k \in [1, +\infty]$ such that there exists a recognizable $\mathcal{S}$-adic sequence $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n \geq 0}$ satisfying, for every $n \geq 1$, that $\#\mathcal{A}_n \leq k$ and that τ_n is positive and proper. The class of finite topological rank subshifts satisfies several rigidity properties, and many tools have been developed to handle it; a non-exhaustive list includes [BKMS13; DFM19; Esp22a; EM21; DM08; BSTY19; HPS92].

It was proved in [DDMP21] that a minimal subshift of nonsuperlinear-growth complexity has finite topological rank, and thus that the aforementioned rigidity properties hold for this class. We present in this subsection a new proof of this fact based in Theorem 4.76.

Theorem 4.87 ([DDMP21], Theorem 5.5) *Let X be a minimal subshift of nonsuperlinear-growth complexity. Then, X has finite topological rank.*

Proof. The case in which X is finite is trivial, and so we may assume that X is infinite. Then, Theorem 4.76 gives d and a recognizable $\mathcal{S}$-adic sequence $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n \geq 0}$ generating X such that Items (1) and (2) of Theorem 4.76 hold. In particular,

$$X \subseteq \bigcup_{k \in \mathbb{Z}} S^k(\mathsf{root}\, \sigma_{[0,n)}(\mathcal{A}_n))^{\mathbb{Z}} \text{ for all } n \geq 1. \tag{4.164}$$

Now, since $|\sigma_{[0,n)}|$ goes to $+\infty$ as $n \to +\infty$ and $d|\sigma_{[0,n)}| \leq \langle \sigma_{[0,n)} \rangle$ by Item (2) in Theorem 4.76, we have that $\langle \sigma_{[0,n)} \rangle$ diverges to $+\infty$ as $n \to +\infty$. Hence, since X is aperiodic,

$$\lim_{n \to +\infty} \langle \mathsf{root}\, \sigma_{[0,n)}(\mathcal{A}_n) \rangle = +\infty.$$

This and (4.164) allow us to use [Esp22a, Corollary 1.4] or [DDMP21, Theorem 4.3] to conclude that X has finite topological rank. □

Chapter 5

Perspectives and future work

In this final chapter, we will present some open questions and comments that have emerged from the thesis work. These ideas correspond to a future research plan.

5.1 More on symbolic factors

The theorems in Chapter 3 provide a fine description of the symbolic factors for a general class of subshifts: those having finite topological rank. It is then natural to search for applications within the finite topological rank class. We now describe two ideas for doing this.

5.1.1 Symbolic factors of eventually dendric shifts

The class of minimal dendric subshifts was introduced in [Ber+14] (under the name of *tree shifts*) and are a generalization of Arnoux-Rauzy subshifts and (the natural coding of) interval exchanges. This class presents interesting rigidity properties, such as that any set of return words is a basis of the free group of a fixed cardinality or that the complexity function is an affine function [Ber+14]. Moreover, the closely related class of eventually dendric subshifts was independently discovered in [DF22] while generalizing a theorem on the number of ergodic measures of interval exchanges. Due to this, dendric and eventually dendric shifts have gained attention, and, in particular, the question about their symbolic factors has become relevant.

Problem 5.1 Describe the symbolic factors of (eventually) dendric shifts.

There are examples of dendric subshifts with non-dendric symbolic factors. However, all known such factors are eventually dendric. This has led to the following conjecture.

Conjecture 5.1 Are all symbolic factors of a given eventually dendric shift eventually dendric?

Interestingly, a finite topological rank structure for minimal eventually dendric subshift was recently obtained [GL22]. Therefore, the methods developed in Chapter 3 can be applied to this case and, by doing so, we may be able to shed some light on the conjecture.

5.1.2 Symbolic factors of interval exchanges

The following is an old question regarding interval exchange transformations:

Question 5.1 Let $\mathcal{F}$ be the set of interval exchange transformations that do not have non-trivial measure-theoretic factors. Is $\mathcal{F}$ generic?

Observe that an affirmative answer to this question has, as a particular case, the Avila-Forni Theorem, so it is probably a difficult problem. We consider instead a topological version of it.

Question 5.2 Let $\mathcal{F}_{\text{top}}$ be the set of interval exchange transformations whose natural coding does not have non-trivial symbolic factors. Is $\mathcal{F}_{\text{top}}$ generic?

In a work in progress with Vincent Delecroix, we have outlined a strategy, using the ideas of Chapter 3, for giving an affirmative answer to Question 5.2. This would represent progress towards Question 5.1.

5.2 More on the $\mathcal{S}$-adic conjecture

Our work on the $\mathcal{S}$-adic conjecture opened at least two new directions of research, which we now discuss.

5.2.1 Applications of the structure theorems

The $\mathcal{S}$-adic characterization obtained in Chapter 4 permit the use of the $\mathcal{S}$-adic machinery to study linear- and nonsuperlinear-growth complexity subshifts. Some cases in which this idea produces interesting results were presented in Section 4.10 of Chapter 4. We plan on continuing investigating in this direction. In particular, it seems that the absence of the strong mixing property and the partial rigidity (with respect to an ergodic measure) may be better understood using the methods in [BKMS13]. More generally, any of the currently known techniques for handling $\mathcal{S}$-adic sequences can now be applied to linear- and nonsuperlinear-growth complexity subshifts, see [HPS92; BKMS13; DFM19; Ber+21]. In some cases, non-proper variations of those techniques must be developed first.

5.2.2 Finite alphabet rank structures

Let (L) and (NSL) be the classes of linear- and nonsuperlinear-growth complexity subshifts, respectively. We showed in Theorem 4.77 that the structure provided we obtained for (L) must have, in some cases, infinite alphabet rank †. Now, most of the techniques for handling $\mathcal{S}$-adic sequences are designed for finite alphabet rank sequences. Although some of them can be adapted to our case, the following question seems natural:

†The *alphabet rank* of $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\geq 0}$ is $\liminf_{n\to+\infty} \#\mathcal{A}_n$.

Question 5.3 Let $\mathcal{C}$ be (L) or (NSL). Is there a finite alphabet rank $\mathcal{S}$-adic characterization of $\mathcal{C}$?

This question is sometimes called the *strong $\mathcal{S}$-adic conjecture*. Observe that this question is ill-defined in the same sense as the $\mathcal{S}$-adic conjecture is.

In the direction of Question 5.3, a close inspection of the proof of Theorem 4.77 shows that, in some cases, the sets $\mathrm{Pow}_X(w)$ encode certain long-range information that seems to be incompatible with finite alphabet rank $\mathcal{S}$-adic sequences. Therefore, we suspect that Question 5.3 has a negative answer.

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