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A stability result for the identification of a permeability parameter on Navier–Stokes equations

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Abstract

In this work, we present a stability result for the inverse problem of recovering a smooth scalar permeability parameter given by the Brinkman's law applied to the steady Navier–Stokes equations from local observations of the fluid velocity on a fixed domain. In comparison with (Choulli *et al* 2013 *Appl. Anal.* **92** 2127–43), we prove a logarithmic estimate under weaker assumptions, since our proof is based in a strategy that does not require pressure observations. This kind or result are useful for inverse problems in soft tissue elastography (see Honarvar *et al* 2012 *Phys. Med. Biol.* **57** 5909–27). Finally, we present some numerical tests that validate our theoretical results.

Keywords: Navier–Stokes equations, Carleman inequalities, stability estimate, inverse problems

(Some figures may appear in colour only in the online journal)

1. Introduction and main model

Let us a non-empty bounded domain $\Omega \subseteq \mathbb{R}^3$. The Lebesgue measure of Ω is denoted by $|\Omega|$, which extends to lesser dimension spaces. The norm and seminorms for Sobolev spaces $W^{m,p}(\Omega)$ are denoted by $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$, respectively. For $p = 2$, the norm, seminorms

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and inner product of the space $W^{m,2}(\Omega) = H^m(\Omega)$ are denoted by $\|\cdot\|_{m,\Omega}$, $|\cdot|_{m,\Omega}$ and $(\cdot,\cdot)_{m,\Omega}$, respectively. Also, $\mathcal{C}^m(\Omega)$ and $\mathcal{C}^\infty(\Omega)$ denote the spaces of functions with m continuous derivatives and all continuous derivatives, respectively. Given Ω_1 and Ω_2 two open subsets of $\mathbb{R}^3$, we denote by $\Omega_1 \Subset \Omega_2$ if there exists a compact set K such that $\Omega_1 \subseteq K \subseteq \Omega_2$.

The spaces $\boldsymbol{H}^m(\Omega)$, $\boldsymbol{W}^{m,p}(\Omega)$, $\boldsymbol{C}^m(\Omega)$ and $\boldsymbol{C}^\infty(\Omega)$ are defined by $\boldsymbol{H}^m(\Omega) = [H^m(\Omega)]^3$, $\boldsymbol{W}^{m,p}(\Omega) = [W^{m,p}(\Omega)]^3$, $\boldsymbol{C}^m(\Omega) = [\mathcal{C}^m(\Omega)]^3$ and $\boldsymbol{C}^\infty(\Omega) = [\mathcal{C}^\infty(\Omega)]^3$. The notation for norms, seminorms and inner products of those spaces will be extended from $\boldsymbol{H}^m(\Omega)$, $\boldsymbol{W}^{m,p}(\Omega)$, $\boldsymbol{C}^m(\Omega)$ and $\boldsymbol{C}^\infty(\Omega)$, respectively. Given $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, $[\boldsymbol{a}]_j$ denotes the $j-$th component of vector $\boldsymbol{a}$, $\boldsymbol{a}^{\mathrm{T}}$ denotes the transpose vector of $\boldsymbol{a}$ and $\boldsymbol{a} \times \boldsymbol{b}$ denotes the cross product given by

$$\boldsymbol{a} \times \boldsymbol{b} = ([\boldsymbol{a}]_2[\boldsymbol{b}]_3 - [\boldsymbol{a}]_3[\boldsymbol{b}]_2, [\boldsymbol{a}]_3[\boldsymbol{b}]_1 - [\boldsymbol{a}]_1[\boldsymbol{b}]_3, [\boldsymbol{a}]_1[\boldsymbol{b}]_2 - [\boldsymbol{a}]_2[\boldsymbol{b}]_1)^{\mathrm{T}}.$$

Also, $\nabla \times \boldsymbol{u}$ (or $\operatorname{curl} \boldsymbol{u}$) denotes the curl of $\boldsymbol{u}$, given by

$$\nabla \times \boldsymbol{u} = \left(\frac{\partial[\boldsymbol{u}]_3}{\partial x_2} - \frac{\partial[\boldsymbol{u}]_2}{\partial x_3}, \frac{\partial[\boldsymbol{u}]_1}{\partial x_3} - \frac{\partial[\boldsymbol{u}]_3}{\partial x_1}, \frac{\partial[\boldsymbol{u}]_2}{\partial x_1} - \frac{\partial[\boldsymbol{u}]_1}{\partial x_2} \right)^{\mathrm{T}}.$$

All the results presented in this article are also valid when $\Omega \subseteq \mathbb{R}^2$, adapting the definitions of cross product and curl to the two dimensional case. Let Ω be a $\mathcal{C}^2$-bounded domain with boundary $\partial\Omega$ and outer normal vector $\boldsymbol{n}$, $\nu \in \mathbb{R}$ with $\nu > 0$, $M \in \mathbb{R}$ with $M > 0$, $\gamma_j \in H^1(\Omega)$ such that $0 \leqslant \gamma_j \leqslant M$ for $j \in \{1, 2\}$ and $\boldsymbol{u}_D \in \boldsymbol{H}^{3/2}(\partial\Omega)$. The model problem

$$\begin{aligned} -\nu \triangle \boldsymbol{u}_j + (\nabla \boldsymbol{u}_j)\, \boldsymbol{u}_j + \nabla p_j + \gamma_j \boldsymbol{u}_j &= \boldsymbol{0} \quad \text{in } \Omega \\ \operatorname{div} \boldsymbol{u}_j &= 0 \quad \text{in } \Omega \\ \boldsymbol{u}_j &= \boldsymbol{u}_D \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

admits an unique solution $(\boldsymbol{u}_j, p_j) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ with $(p_j, 1)_\Omega = 0$. For $\varepsilon > 0$, we define

$$\Omega_\varepsilon = \{\boldsymbol{x} \in \Omega | d(\boldsymbol{x}, \partial\Omega) \geqslant \varepsilon\}.$$

We suppose

$$\gamma_j \in H(\Omega) = \left\{ f \in H_0^1(\Omega) \,|\, f = 0 \quad \text{in } \Omega \backslash \Omega_\varepsilon \right\}.$$

Then, there exists a constant $c_1 > 0$ only dependent on Ω and M such that

$$\|\boldsymbol{u}_j\|_{2,\Omega}^2 + \|p_j\|_{1,\Omega}^2 \leqslant c_1 \|\boldsymbol{u}_D\|_{1/2,\partial\Omega}^2.$$

Defining $\gamma = \gamma_1 - \gamma_2$, $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $p = p_1 - p_2$, $(\boldsymbol{u}, p)$ is the unique solution of the Oseen equations given by

$$\begin{aligned} -\nu \triangle \boldsymbol{u} + (\nabla \boldsymbol{u})\, \boldsymbol{u}_1 + (\nabla \boldsymbol{u}_2)\, \boldsymbol{u} + \nabla p + \gamma_1 \boldsymbol{u} &= -\boldsymbol{f} \quad \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $\boldsymbol{f} = \gamma \boldsymbol{u}_2$. In this case, we have a constant $c_{\mathrm{NS}} > 0$ such that

$$\|\boldsymbol{u}\|_{2,\Omega}^2 + \|p\|_{1,\Omega}^2 \leqslant c_{\mathrm{NS}} \|\gamma\|_{0,\Omega}^2.$$

We pose the following assumptions:

(a) There exists a constant $K > 0$ such that $\|\gamma\|_{1,\Omega} \leqslant K$.
(b) There exist constants $M_2 > 0$ and $c_{\mathrm{NS}} > 0$ such that $\|\boldsymbol{u}\|_{2,\Omega}^2 + \|p\|_{1,\Omega}^2 \leqslant M_2^2 \leqslant c_{\mathrm{NS}} K^2$.
(c) There exists a constant $M_3 > 0$ such that $\|\boldsymbol{u}\|_{3,\Omega} \leqslant M_3$
(d) The velocity $\boldsymbol{u}_2$ verifies $\operatorname{curl} \boldsymbol{u}_2 \in \boldsymbol{L}^{\infty}(\Omega)$.

Remark. Assumption (a) is usual in problems where the permeability coefficient is studied. Assumptions (b) and (c) are similar to the one used in [6]. Finally, assumption (d) is similar to smoothness assumptions in [6, 7].

In order to avoid an analysis of the gradient of the pressure p we eliminate this variable using the curl operator. If we define $\boldsymbol{z} = \operatorname{curl} \boldsymbol{u}$, where $\boldsymbol{u}$ is the solution of equation (2), then $\boldsymbol{u}$ verifies the following second-order elliptical equation:

$$
\begin{aligned}
-\triangle \boldsymbol{u} &= \operatorname{curl}(\operatorname{curl} \boldsymbol{u}) - \nabla(\operatorname{div} \boldsymbol{u}) = \operatorname{curl} \boldsymbol{z} && \text{in } \Omega \\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega.
\end{aligned}
\tag{3}
$$

Thanks to $\operatorname{curl}(\nabla p) = \boldsymbol{0}$, the vector field $\boldsymbol{z}$ verifies the following equations

$$
\begin{aligned}
-\nu \triangle \boldsymbol{z} + (\nabla \boldsymbol{z})\boldsymbol{u}_1 + \gamma_1 \boldsymbol{z} &= -(\operatorname{curl} \boldsymbol{f} + \boldsymbol{h}) && \text{in } \Omega \\
\operatorname{div} \boldsymbol{z} &= 0 && \text{in } \Omega \\
\boldsymbol{z} &= \operatorname{curl} \boldsymbol{u} && \text{on } \partial\Omega,
\end{aligned}
\tag{4}
$$

where $\boldsymbol{h} \in \boldsymbol{L}^2(\Omega)$ is defined by

$$
\boldsymbol{h} = \nabla\gamma_1 \times \boldsymbol{u} + (\nabla \operatorname{curl}(\boldsymbol{u}_2))\boldsymbol{u} + \sum_{j=1}^{3} \nabla[\boldsymbol{u}_1]_j \times \frac{\partial \boldsymbol{u}}{\partial x_j} + \nabla[\boldsymbol{u}]_j \times \frac{\partial \boldsymbol{u}_2}{\partial x_j}.
$$

From an open connected non-empty subset $\omega \subseteq \Omega$, the inverse problem we studied here is to determinate $\gamma = \gamma_1 - \gamma_2$ in Ω from the observation data $\boldsymbol{u}|_{\omega} = (\boldsymbol{u}_1 - \boldsymbol{u}_2)|_{\omega}$, where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ verify the Navier–Stokes equations given in 1 for γ_1 and γ_2, respectively.

The main objective of this work is to obtain a stability result for this parameter identification problem, where we search a permeability parameter γ from a reference velocity $\boldsymbol{u}$. This problem was already studied in [3] by minimizing a quadratic functional for a model based in Oseen equations, so this article validates the strategy by ensuring the uniqueness of the quadratic functional minimizer in those cases where the hypotheses of this article are verified. A first approach to this problem is given in [6, 7]. In [6], the authors describe Carleman inequalities for steady Oseen equations applied to find stability results for Navier and Robin boundary coefficients in a compact subset $K \subseteq \partial\Omega$ such that $|\boldsymbol{u}_2| \geqslant m$ in K, where $m > 0$ is a constant. That estimates need an analysis of pressure to be computed. In [7], the authors obtain a Lipschitz stability result for the right-hand side of a unsteady linearized Navier–Stokes equation, recovering a source scalar term f using a global observation of $\boldsymbol{u}$ and $\operatorname{curl} \boldsymbol{u}$ in a fixed time and local observations of $\boldsymbol{u}$ in a time interval. The source term represent the density of external force with a form $f\boldsymbol{R}$, where $\boldsymbol{R}$ is a vector field that verifies a non-degeneracy condition. Both ideas can be adapted to this new problem, obtaining a Carleman inequality and a stability result with no observation data of p.

This article is structured as follows. In section 2, we have adapted the technique used to prove theorem 2.3 in [6] to obtain an improved version of a Carleman inequality for weak solutions of equation (2). In section 3, we present a modified version of the non-degeneracy condition introduced in [7] that allow us to prove a logarithmic stability result using a Carleman estimate obtained in section 2 and a similar Carleman estimate obtained for strong solutions of equation (3). Finally, in section 4 we present two numerical test that validates the main result recovering a smooth and a discontinuous parameter solving a minimization problem. The second test, inspired in [3], is not covered by our main result. However, we add an adaptive refinement algorithm that improves the numerical results.

2. A Carleman estimate

This first result allows us to define the Carleman weights for our estimates.

Lemma 1. *Let* $c_0 \geqslant 0$. *There exists a function* $\varphi \in \mathcal{C}^{\infty}\left(\overline{\Omega}\right)$ *such that* $\varphi = c_0$ *on* $\partial\Omega$, $\varphi > c_0$ *in* Ω *and* $\nabla\varphi \neq \mathbf{0}$ *in* $\overline{\Omega\backslash\omega}$.

Proof. See lemma 1.1 in [12]. □

The following lemma is the Carleman inequality for weak solutions of linear second-order elliptic PDE with homogenous Dirichlet boundary conditions.

Lemma 2. *Let* $f \in L^2(\Omega)$, $\boldsymbol{F} \in \boldsymbol{H}^1(\Omega)$, $\nu \in \mathbb{R}$ *with* $\nu > 0$, $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{L}^{\infty}(\Omega)$, $c \in L^{\infty}(\Omega)$ *and* $u \in H^2(\Omega)$ *solution of*

$$\begin{aligned} -\nu\triangle u + \boldsymbol{a}\cdot\nabla u + \operatorname{div}(u\boldsymbol{b}) + cu &= f + \operatorname{div}\boldsymbol{F} && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Then, there exist $C > 0$, $\tilde{\lambda} > 1$ *and* $\tilde{s} > 1$, *independent on* u, *such that for all* $k \in \{0, 1\}$, $\lambda \geqslant \tilde{\lambda}$ *and* $s \geqslant \tilde{s}$,

$$\begin{aligned} &\int_\Omega \left(\mathrm{e}^{(k-1)\lambda\varphi}|\nabla u|^2 + s^2\lambda^2\,\mathrm{e}^{(k+1)\lambda\varphi}|u|^2\right)\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \\ &\quad\leqslant C\left(\int_\Omega \frac{1}{s\lambda^2}\,\mathrm{e}^{(k-2)\lambda\varphi}|f|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x + \int_\Omega s\mathrm{e}^{k\lambda\varphi}|\boldsymbol{F}|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x + \int_\omega s^2\lambda^2\mathrm{e}^{(k+1)\lambda\varphi}|u|^2\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x\right). \end{aligned}$$

Proof. For $k = 1$, the result is given by theorem A.1 in [13]. When $k = 0$, define $z = \mathrm{e}^{-\lambda\varphi/2}u$ and use the result for $k = 1$. □

To determine a Carleman estimate for our problem, a first step is to analyze the equation (3). Each component of $\operatorname{curl}\boldsymbol{z}$ can be written as a divergence of a vector field resulting from a linear transformation of $\boldsymbol{u}$. Applying lemma 2 with $k = 1$ in each component, we can obtain that there exist $C > 0$, $\tilde{\lambda} > 1$ and $\tilde{s} > 1$, such that for all $\lambda \geqslant \tilde{\lambda}$ and $s \geqslant \tilde{s}$,

$$\int_\Omega \left(|\nabla\boldsymbol{u}|^2 + s^2\lambda^2\,\mathrm{e}^{2\lambda\varphi}|\boldsymbol{u}|^2\right)\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \leqslant C\left(\int_\Omega s\mathrm{e}^{\lambda\varphi}|\boldsymbol{z}|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x + \int_\omega s^2\lambda^2\,\mathrm{e}^{2\lambda\varphi}|\boldsymbol{u}|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x\right). \tag{5}$$

Later, it is clear that we need an upper bound for the first term on the right-hand side of equation (3) in terms of $\boldsymbol{u}$. A second step is to establish a similar result from equation (4), rewriting each component of the right-hand side of that equation to the form $f + \operatorname{div} \boldsymbol{F}$. Because of the counts are analogous, we only show the analysis of the first component of curl $\boldsymbol{f} + \boldsymbol{h}$

$$
\begin{aligned}
h_1 = {} & \frac{\partial \gamma_1}{\partial x_2}[\boldsymbol{u}]_3 - \frac{\partial \gamma_1}{\partial x_3}[\boldsymbol{u}]_2 + \sum_{j=1}^{3} \frac{\partial [\boldsymbol{u}_1]_j}{\partial x_2}\frac{\partial [\boldsymbol{u}]_3}{\partial x_j} - \frac{\partial [\boldsymbol{u}_1]_j}{\partial x_3}\frac{\partial [\boldsymbol{u}]_2}{\partial x_j} + \sum_{j=1}^{3} \frac{\partial [\boldsymbol{u}]_j}{\partial x_2}\frac{\partial [\boldsymbol{u}_2]_3}{\partial x_j} \\
& - \frac{\partial [\boldsymbol{u}]_j}{\partial x_3}\frac{\partial [\boldsymbol{u}_2]_2}{\partial x_j} + \frac{\partial [\boldsymbol{f}]_3}{\partial x_2} - \frac{\partial [\boldsymbol{f}]_2}{\partial x_3} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(\frac{\partial [\boldsymbol{u}_2]_3}{\partial x_2} - \frac{\partial [\boldsymbol{u}_2]_2}{\partial x_3}\right)[\boldsymbol{u}]_j \\
= {} & \frac{\partial \gamma_1}{\partial x_2}[\boldsymbol{u}]_3 - \frac{\partial \gamma_1}{\partial x_3}[\boldsymbol{u}]_2 + \frac{\partial [\boldsymbol{f}]_3}{\partial x_2} - \frac{\partial [\boldsymbol{f}]_2}{\partial x_3} \\
& + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left([\boldsymbol{u}]_3 \frac{\partial [\boldsymbol{u}_1]_j}{\partial x_2} - [\boldsymbol{u}]_2 \frac{\partial [\boldsymbol{u}_1]_j}{\partial x_3}\right) + \frac{\partial}{\partial x_2}\left([\boldsymbol{u}]_j \frac{\partial [\boldsymbol{u}_2]_3}{\partial x_j}\right) - \frac{\partial}{\partial x_3}\left([\boldsymbol{u}]_j \frac{\partial [\boldsymbol{u}_2]_2}{\partial x_j}\right) \\
& - \sum_{j=1}^{3}\left([\boldsymbol{u}]_3 \frac{\partial^2 [\boldsymbol{u}_1]_j}{\partial x_j \partial x_2} - [\boldsymbol{u}]_2 \frac{\partial^2 [\boldsymbol{u}_1]_j}{\partial x_j \partial x_3} + [\boldsymbol{u}]_j \frac{\partial^2 [\boldsymbol{u}_2]_3}{\partial x_j \partial x_2} - [\boldsymbol{u}]_j \frac{\partial^2 [\boldsymbol{u}_2]_2}{\partial x_j \partial x_3}\right) \\
= {} & \frac{\partial \gamma_1}{\partial x_2}[\boldsymbol{u}]_3 - \frac{\partial \gamma_1}{\partial x_3}[\boldsymbol{u}]_2 - \sum_{j=1}^{3}\left([\boldsymbol{u}]_3 \frac{\partial^2 [\boldsymbol{u}_1]_j}{\partial x_j \partial x_2} - [\boldsymbol{u}]_2 \frac{\partial^2 [\boldsymbol{u}_1]_j}{\partial x_j \partial x_3} + [\boldsymbol{u}]_j \frac{\partial^2 [\boldsymbol{u}_2]_3}{\partial x_j \partial x_2} - [\boldsymbol{u}]_j \frac{\partial^2 [\boldsymbol{u}_2]_2}{\partial x_j \partial x_3}\right) \\
& + \frac{\partial}{\partial x_1}\left([\boldsymbol{u}]_3 \frac{\partial [\boldsymbol{u}_1]_1}{\partial x_2} - [\boldsymbol{u}]_2 \frac{\partial [\boldsymbol{u}_1]_1}{\partial x_3}\right) \\
& + \frac{\partial}{\partial x_2}\left([\boldsymbol{u}]_3 \frac{\partial [\boldsymbol{u}_1]_2}{\partial x_2} - [\boldsymbol{u}]_2 \frac{\partial [\boldsymbol{u}_1]_2}{\partial x_3} + [\boldsymbol{f}]_3 + \sum_{j=1}^{3} [\boldsymbol{u}]_j \frac{\partial [\boldsymbol{u}_2]_3}{\partial x_j}\right) \\
& + \frac{\partial}{\partial x_3}\left([\boldsymbol{u}]_3 \frac{\partial [\boldsymbol{u}_1]_3}{\partial x_2} - [\boldsymbol{u}]_2 \frac{\partial [\boldsymbol{u}_1]_3}{\partial x_3} - [\boldsymbol{f}]_2 - \sum_{j=1}^{3} [\boldsymbol{u}]_j \frac{\partial [\boldsymbol{u}_2]_2}{\partial x_j}\right).
\end{aligned}
$$

However, equation (4) does not have homogenous Dirichlet boundary conditions. A Carleman inequality in the case of non-homogeneous boundary data can be obtained following the same arguments that in section 2.2 in [6]. In the following, we consider a function $\varphi \in \mathcal{C}^\infty\left(\overline{\Omega}\right)$ that verifies lemma 1 for a constant $c_0 > 0$.

Definition 3. We define the space $\boldsymbol{H}_0^2(\Omega) = \left\{\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)\,|\boldsymbol{u} = \boldsymbol{0} \text{ and } \nabla \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega\right\}$.

In order to simplify the proof of our Carleman inequality, we present the following technical result. We present a similar proof to the one for theorem 2.2 in [6] for the sake of self-containedness.

Lemma 4. *Let* $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^2(\Omega_0) \times H_0^1(\Omega_0)$ *solutions of* (2). *There exist* $C > 0$, $\tilde{s} > 1$ *and* $\tilde{\lambda} > 1$ *such that for every* $s \geqslant \tilde{s}$ *and* $\lambda \geqslant \tilde{\lambda}$

$$
\begin{aligned}
& \int_\Omega \left(s\mathrm{e}^{\lambda\varphi}|\operatorname{curl} \boldsymbol{u}|^2 + |\nabla \boldsymbol{u}|^2 + s^2\lambda^2\,\mathrm{e}^{2\lambda\varphi}|\boldsymbol{u}|^2\right)\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \\
& \qquad \leqslant C\left(\int_\Omega \frac{1}{\lambda^2}|-\nu\triangle \boldsymbol{u} + \nabla p|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x + \int_\omega s^3\lambda^2\,\mathrm{e}^{3\lambda\varphi}|\boldsymbol{u}|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x\right).
\end{aligned}
$$

Proof. First, we define $\boldsymbol{g} = -\nu\triangle\boldsymbol{u} + \nabla p$. Then, we have

$$-\nu\triangle\,(\text{curl }\boldsymbol{u}) + \text{curl}(\nabla p) = \text{curl }\boldsymbol{g}.$$

Let $\omega_0 \Subset \omega$ a non-empty open subset. Applying lemma 2 with $k = 0$, there exist $C_1 > 0, \tilde{s} > 1$ and $\tilde{\lambda} > 1$ such that for every, $s \geqslant \tilde{s}$ and $\lambda \geqslant \tilde{\lambda}$, we obtain

$$\begin{aligned}
&\int_\Omega \left(\mathrm{e}^{-\lambda\varphi}|\nabla\,\text{curl }\boldsymbol{u}|^2 + s^2\lambda^2\,\mathrm{e}^{\lambda\varphi}|\text{curl }\boldsymbol{u}|^2\right)\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \\
&\qquad\leqslant C_1\left(\int_\Omega s|\boldsymbol{g}|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x + \int_{\omega_0} s^2\lambda^2\,\mathrm{e}^{\lambda\varphi}\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}|\text{curl }\boldsymbol{u}|^2\,\mathrm{d}x\right) \\
&\int_\Omega \left(\frac{\mathrm{e}^{-\lambda\varphi}}{s\lambda^2}|\nabla\,\text{curl }\boldsymbol{u}|^2 + s\mathrm{e}^{\lambda\varphi}|\text{curl }\boldsymbol{u}|^2\right)\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \\
&\qquad\leqslant C_1\left(\int_\Omega \frac{1}{\lambda^2}|\boldsymbol{g}|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x + \int_{\omega_0} s\mathrm{e}^{\lambda\varphi}\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}|\text{curl }\boldsymbol{u}|^2\,\mathrm{d}x\right).
\end{aligned}$$

Let $\rho \in \mathcal{C}_0^\infty(\omega)$ such that $0 \leqslant \rho \leqslant 1$ and $\rho = 1$ in ω_0. Then, for all $s > 0$, we obtain

$$\int_{\omega_0} s\mathrm{e}^{\lambda\varphi}\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}|\text{curl }\boldsymbol{u}|^2\,\mathrm{d}x = \int_{\omega_0} s\mathrm{e}^{\lambda\varphi}\rho|\text{curl }\boldsymbol{u}|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \leqslant \int_\omega s\mathrm{e}^{\lambda\varphi}\rho|\text{curl }\boldsymbol{u}|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x.$$

Applying integration by parts and the triangular inequality, there exists a constant $C_2 > 0$ only dependent on ρ such that

$$\begin{aligned}
\int_\omega s\mathrm{e}^{\lambda\varphi}\rho|\text{curl }\boldsymbol{u}|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x &= \int_\omega \left(s\mathrm{e}^{\lambda\varphi}\rho\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\text{curl }\boldsymbol{u}\right)\cdot\text{curl }\boldsymbol{u}\,\,\mathrm{d}x \\
&= \int_\omega \boldsymbol{u}\cdot\text{curl}\left(s\mathrm{e}^{\lambda\varphi}\rho\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\text{curl }\boldsymbol{u}\right)\mathrm{d}x - \int_{\partial\omega} s\mathrm{e}^{\lambda\varphi}\rho\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\text{curl }\boldsymbol{u}\times\boldsymbol{u}\,\mathrm{d}x \\
&= \int_\omega \boldsymbol{u}\cdot\text{curl}\left(s\mathrm{e}^{\lambda\varphi}\rho\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\text{curl }\boldsymbol{u}\right)\mathrm{d}x \\
&\leqslant C_2\left(\int_\omega s^2\lambda\mathrm{e}^{2\lambda\varphi}\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,|\text{curl }\boldsymbol{u}|\,|\boldsymbol{u}|\,\,\mathrm{d}x + \int_\omega s\mathrm{e}^{\lambda\varphi}\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,|\nabla\,\text{curl }\boldsymbol{u}|\,|\boldsymbol{u}|\,\,\mathrm{d}x\right).
\end{aligned} \tag{6}$$

Using Hölder inequality, there exists a constant $C_3 > 0$ independent on $\boldsymbol{u}$ such that for all $\varepsilon > 0$,

$$\begin{aligned}
\int_{\omega_0} s\mathrm{e}^{\lambda\varphi}|\text{curl }\boldsymbol{u}|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x &\leqslant C_2\left(\int_\omega s^2\lambda\,\mathrm{e}^{2\lambda\varphi}\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,|\text{curl }\boldsymbol{u}|\,|\boldsymbol{u}|\,\,\mathrm{d}x + \int_\omega s\,\mathrm{e}^{\lambda\varphi}\,\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\,|\nabla\,\text{curl }\boldsymbol{u}|\,|\boldsymbol{u}|\,\,\mathrm{d}x\right) \\
&\leqslant \varepsilon\left(s\int_{\Omega_0} \mathrm{e}^{\lambda\varphi}\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}|\text{curl }\boldsymbol{u}|^2\,\mathrm{d}x + \frac{1}{s\lambda^2}\int_{\Omega_0} \mathrm{e}^{-\lambda\varphi}\,\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}|\nabla\,\text{curl }\boldsymbol{u}|^2\,\mathrm{d}x\right) \\
&\quad + \frac{C_3}{\varepsilon}\int_\omega s^3\lambda^2\,\mathrm{e}^{3\lambda\varphi}\,\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}|\boldsymbol{u}|^2\,\mathrm{d}x.
\end{aligned}$$

Then,

$$\int_{\Omega}\left(\frac{\mathrm{e}^{-\lambda\varphi}}{s\lambda^2}|\nabla\,\mathrm{curl}\,\boldsymbol{u}|^2+s\,\mathrm{e}^{\lambda\varphi}|\mathrm{curl}\,\boldsymbol{u}|^2\right)\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x$$
$$\leqslant C_3\left(\int_{\Omega}\frac{1}{\lambda^2}|\boldsymbol{g}|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x+\frac{1}{\varepsilon}\int_{\omega}s^3\lambda^2\mathrm{e}^{3\lambda\varphi}\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}|\boldsymbol{u}|^2\,\mathrm{d}x\right.$$
$$\left.+\frac{\varepsilon}{C_3}\left(s\int_{\Omega_0}\mathrm{e}^{\lambda\varphi}\,\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}|\mathrm{curl}\,\boldsymbol{u}|^2\,\mathrm{d}x+\frac{1}{s\lambda^2}\int_{\Omega_0}\mathrm{e}^{-\lambda\varphi}\,\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}|\nabla\,\mathrm{curl}\,\boldsymbol{u}|^2\,\mathrm{d}x\right)\right).$$

Choosing $\varepsilon > 0$ small enough, we can absorb the first term of the right-hand side with the terms of the left-hand side. Thus, there exists $C_4 > 0$ such that

$$\int_{\Omega}\left(\frac{1}{s\lambda^2}\,\mathrm{e}^{-\lambda\varphi}|\nabla\,\mathrm{curl}\,\boldsymbol{u}|^2+s\mathrm{e}^{\lambda\varphi}|\mathrm{curl}\,\boldsymbol{u}|^2\right)\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x$$
$$\leqslant C_4\left(\int_{\Omega}\frac{1}{\lambda^2}|\boldsymbol{g}|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x+\int_{\omega}s^3\lambda^2\mathrm{e}^{3\lambda\varphi}\,\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}|\boldsymbol{u}|^2\,\mathrm{d}x\right)$$
$$\int_{\Omega}s\mathrm{e}^{\lambda\varphi}|\mathrm{curl}\,\boldsymbol{u}|^2\,\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x\leqslant C_4\left(\int_{\Omega}\frac{1}{\lambda^2}|\boldsymbol{g}|^2\,\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x+\int_{\omega}s^3\lambda^2\,\mathrm{e}^{3\lambda\varphi}\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}|\boldsymbol{u}|^2\,\mathrm{d}x\right).$$

Applying this result to (5), we finally obtain

$$\int_{\Omega}\left(s\mathrm{e}^{\lambda\varphi}|\mathrm{curl}\,\boldsymbol{u}|^2+|\nabla\boldsymbol{u}|^2+s^2\lambda^2\,\mathrm{e}^{2\lambda\varphi}|\boldsymbol{u}|^2\right)\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x$$
$$\leqslant C\left(\int_{\Omega}\frac{1}{\lambda^2}|\boldsymbol{g}|^2\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x+\int_{\omega}s^3\lambda^2\,\mathrm{e}^{3\lambda\varphi}|\boldsymbol{u}|^2\,\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x\right),$$

proving this lemma. □

Now we can formulate our new Carleman estimate.

Theorem 5. *There exist $C > 0$, $\tilde{s} > 1$ and $\tilde{\lambda} > 1$ such that for every $s \geqslant \tilde{s}$ and $\lambda \geqslant \tilde{\lambda}$*

$$\int_{\Omega}\left(s\mathrm{e}^{\lambda\varphi}|\boldsymbol{z}|^2+|\nabla\boldsymbol{u}|^2+s^2\lambda^2\,\mathrm{e}^{2\lambda\varphi}|\boldsymbol{u}|^2\right)\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x$$
$$\leqslant C\left(\frac{\mathrm{e}^{2s\,\mathrm{e}^{\lambda c_0}}}{\lambda^2}\left(\|\boldsymbol{u}\|^2_{2,\Omega}+\|p\|^2_{1,\Omega}\right)+\int_{\Omega}\frac{1}{\lambda^2}|\boldsymbol{f}|^2\,\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x+\int_{\omega}s^3\,\mathrm{e}^{3\lambda\varphi}\,\mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}|\boldsymbol{u}|^2\,\mathrm{d}x\right),$$

where $(\boldsymbol{u},p)$ are the solutions of equation (2) and $\boldsymbol{z} = \mathrm{curl}\,\boldsymbol{u}$.

Proof. The proof is similar to the proof of theorem 2.3 from [6] with some modifications due to the permeability term. Let $\Omega_0 \subseteq \mathbb{R}^3$ be a bounded domain with a $\mathcal{C}^2$ boundary $\partial\Omega_0$ such that $\Omega \Subset \Omega_0$. We can extend φ to Ω_0 (keeping the same name) such that $\varphi \in \mathcal{C}^2\left(\overline{\Omega_0}\right)$,

$$\varphi > 0 \quad \text{in}\,\Omega_0 \quad \varphi = 0 \quad \text{on}\,\partial\Omega_0 \quad \varphi = c_0 \quad \text{on}\,\partial\Omega$$
$$0 < \varphi < c_0 \quad \text{in}\,\Omega_0\backslash\overline{\Omega} \quad \varphi > c_0 \quad \text{in}\,\Omega \quad \nabla\varphi \neq \mathbf{0} \quad \text{in}\,\overline{\Omega_0\backslash\omega}.$$

It is easy to see that this extension exists thanks to the regularity of the domain and lemma 1. Taking the extension operator $\boldsymbol{A} : \boldsymbol{H}^2(\Omega) \times H^1(\Omega) \to \boldsymbol{H}^2_0(\Omega_0) \times H^1_0(\Omega_0)$ given by the Stein's theorem (see [2]) such that $\boldsymbol{A}(\boldsymbol{u},p) = (\boldsymbol{u},p)$ in Ω, we define $(\tilde{\boldsymbol{u}},\tilde{p}) = \boldsymbol{A}(\boldsymbol{u},p)$. We also denote by

$\tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2, \tilde{\gamma}_1$ and $\tilde{\boldsymbol{f}}$ the continuous extensions of $\boldsymbol{u}_1, \boldsymbol{u}_2, \gamma$ and $\boldsymbol{f}$ in the $\boldsymbol{H}^2(\Omega_0)$, $L^\infty(\Omega_0) \cap \boldsymbol{H}_0^1(\Omega_0)$ and $\boldsymbol{L}^2(\Omega_0)$ spaces, respectively, where γ is extended by 0 in $\Omega_0 \backslash \Omega$. Then, $(\tilde{\boldsymbol{u}}, \tilde{p})$ is solution to the system

$$\begin{aligned} -\nu \triangle \tilde{\boldsymbol{u}} + (\nabla \tilde{\boldsymbol{u}})\tilde{\boldsymbol{u}}_1 + (\nabla \tilde{\boldsymbol{u}}_2)\tilde{\boldsymbol{u}} + \nabla \tilde{p} + \tilde{\gamma}_1 \tilde{\boldsymbol{u}} &= \tilde{\boldsymbol{f}} \quad \text{in } \Omega_0 \\ \operatorname{div} \tilde{\boldsymbol{u}} &= 0 \quad \text{in } \Omega_0 \\ \tilde{\boldsymbol{u}} &= \boldsymbol{0} \quad \text{on } \partial\Omega_0 \\ \frac{\partial \tilde{\boldsymbol{u}}}{\partial \boldsymbol{n}} &= \boldsymbol{0} \quad \text{on } \partial\Omega_0 \\ \tilde{p} &= 0 \quad \text{on } \partial\Omega_0, \end{aligned} \tag{7}$$

where $\tilde{\boldsymbol{f}} \in \boldsymbol{L}^2(\Omega_0)$ is given by

$$\tilde{\boldsymbol{f}} = \begin{cases} -\boldsymbol{f} & \text{in } \Omega \\ -\nu \triangle \tilde{\boldsymbol{u}} + (\nabla \tilde{\boldsymbol{u}})\tilde{\boldsymbol{u}}_1 + (\nabla \tilde{\boldsymbol{u}}_2)\tilde{\boldsymbol{u}} + \nabla \tilde{p} + \tilde{\gamma}_1 \tilde{\boldsymbol{u}} & \text{in } \Omega_0 \backslash \overline{\Omega} \end{cases}$$

Using the continuity of $\boldsymbol{A}$, there exists a constant $C_1 > 0$ depending only on $\tilde{\boldsymbol{u}}_1$, $\tilde{\boldsymbol{u}}_2$, $\tilde{\gamma}_1$, ν and the continuity constant of $\boldsymbol{A}$ such that

$$\left\| \tilde{\boldsymbol{f}} \right\|_{0,\Omega_0}^2 \leqslant C_1 \left(\|\boldsymbol{u}\|_{2,\Omega}^2 + \|p\|_{1,\Omega}^2 \right).$$

Now, taking $\tilde{\boldsymbol{z}} = \operatorname{curl} \tilde{\boldsymbol{u}}$ and applying lemma 4, we obtain

$$\begin{aligned} &\int_{\Omega_0} \left(s\, \mathrm{e}^{\lambda\varphi} |\tilde{\boldsymbol{z}}|^2 + |\nabla \tilde{\boldsymbol{u}}|^2 + s^2 \lambda^2 \mathrm{e}^{2\lambda\varphi} |\tilde{\boldsymbol{u}}|^2 \right) \mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x \\ &\quad \leqslant C_2 \left(\int_{\Omega_0} \frac{1}{\lambda^2} |-\nu \triangle \tilde{\boldsymbol{u}} + \nabla \tilde{p}|^2 \mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x + \int_\omega \left(s^3 \lambda^2 \mathrm{e}^{3\lambda\varphi} |\boldsymbol{u}|^2 \right) \mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x \right) \\ &\quad \leqslant C_2 \left(\int_{\Omega_0} \frac{1}{\lambda^2} \left| \tilde{\boldsymbol{f}} - ((\nabla \tilde{\boldsymbol{u}})\tilde{\boldsymbol{u}}_1 + (\nabla \tilde{\boldsymbol{u}}_2)\tilde{\boldsymbol{u}} + \tilde{\gamma}_1 \tilde{\boldsymbol{u}}) \right|^2 \mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x + \int_\omega \left(s^3 \lambda^2 \mathrm{e}^{3\lambda\varphi} |\boldsymbol{u}|^2 \right) \mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x \right) \end{aligned} \tag{8}$$

for all $\lambda \geqslant \tilde{\lambda}$ and $s \geqslant \tilde{s}$, where $C_2 > 0$, $\tilde{\lambda} > 1$ and $\tilde{s} > 1$ are independent on u. Applying the Sobolev embedding theorem, we can see that $\tilde{\boldsymbol{u}}_1 \in \boldsymbol{L}^\infty(\Omega_0)$. Since $\tilde{\gamma} \in L^\infty(\Omega_0)$, we have

$$\int_{\Omega_0} |(\nabla \tilde{\boldsymbol{u}})\tilde{\boldsymbol{u}}_1|^2 \mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x \leqslant \|\tilde{\boldsymbol{u}}_1\|_{0,\infty,\Omega_0}^2 \int_{\Omega_0} |(\nabla \tilde{\boldsymbol{u}})|^2 \mathrm{e}^{2s\,\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x.$$

Now, applying again Sobolev embedding theorem and Hölder inequality, there exist a constant $C_3 > 0$ independent on $\boldsymbol{u}$ such that

$$\begin{aligned} \int_{\Omega_0} |(\nabla \tilde{\boldsymbol{u}}_2)\tilde{\boldsymbol{u}}|^2 \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x &\leqslant \int_{\Omega_0} |(\nabla \tilde{\boldsymbol{u}}_2)|^2 \left| \mathrm{e}^{s\mathrm{e}^{\lambda\varphi}} \tilde{\boldsymbol{u}} \right|^2 \mathrm{d}x \\ &\leqslant \|\tilde{\boldsymbol{u}}_2\|_{1,3,\Omega_0}^2 \left\| \mathrm{e}^{s\mathrm{e}^{\lambda\varphi}} \tilde{\boldsymbol{u}} \right\|_{0,6,\Omega_0}^2 \\ &\leqslant \|\tilde{\boldsymbol{u}}_2\|_{1,3,\Omega_0}^2 \left\| \nabla \left(\mathrm{e}^{s\mathrm{e}^{\lambda\varphi}} \tilde{\boldsymbol{u}} \right) \right\|_{0,2,\Omega_0}^2 \\ &\leqslant C_3 \|\tilde{\boldsymbol{u}}_2\|_{1,3,\Omega_0}^2 \int \left(|\nabla \tilde{\boldsymbol{u}}|^2 + s^2 \lambda^2 \mathrm{e}^{2\lambda\varphi} |\boldsymbol{u}|^2 \right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x. \end{aligned}$$

Analogously, we have

$$\int_{\Omega_0} |\tilde{\gamma}_1 \tilde{\boldsymbol{u}}|^2 \, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x \leqslant C_4 \|\tilde{\gamma}_1\|^2_{0,3,\Omega_0} \int \left(|\nabla \tilde{\boldsymbol{u}}|^2 + s^2 \lambda^2 \, \mathrm{e}^{2\lambda\varphi} |\boldsymbol{u}|^2 \right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \mathrm{d}x$$

for a constant $C_4 > 0$ independent on $\boldsymbol{u}$. Hence, there is a constant $C_5 > 0$ independent on $\boldsymbol{u}$ such that the integral term with $\tilde{\boldsymbol{f}}$ verifies

$$\begin{aligned} &\int_{\Omega_0} \frac{1}{\lambda^2} \left| \tilde{\boldsymbol{f}} - ((\nabla \tilde{\boldsymbol{u}}) \, \tilde{\boldsymbol{u}}_1 + (\nabla \tilde{\boldsymbol{u}}_2) \, \tilde{\boldsymbol{u}} + \tilde{\gamma}_1 \tilde{\boldsymbol{u}}) \right|^2 \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x \\ &\quad \leqslant \int_{\Omega_0} \frac{1}{\lambda^2} \left(\left| \tilde{\boldsymbol{f}} \right|^2 + C_5 \left(|\nabla \tilde{\boldsymbol{u}}|^2 + s^2 \lambda^2 \, \mathrm{e}^{2\lambda\varphi} |\boldsymbol{u}|^2 \right) \right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x, \end{aligned}$$

where the last terms can be absorbed by the left-hand side of inequality 8 for $\lambda \geqslant \lambda_2$, with λ_2 large enough and independent on $\boldsymbol{u}$. Then, there exists a constant $C_6 > 0$ such that

$$\begin{aligned} &\int_{\Omega_0} \left(s\mathrm{e}^{\lambda\varphi} |\tilde{z}|^2 + |\nabla \tilde{\boldsymbol{u}}|^2 + s^2 \lambda^2 \, \mathrm{e}^{2\lambda\varphi} |\tilde{\boldsymbol{u}}|^2 \right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x \\ &\quad \leqslant C_6 \left(\int_{\Omega_0} \frac{1}{\lambda^2} \left| \tilde{\boldsymbol{f}} \right|^2 \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x + \int_{\omega} \left(s^3 \lambda^2 \mathrm{e}^{3\lambda\varphi} |\boldsymbol{u}|^2 \right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x \right). \end{aligned}$$

Finally, we have

$$\begin{aligned} \int_{\Omega_0} \frac{1}{\lambda^2} \left| \tilde{\boldsymbol{f}} \right|^2 \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x &= \int_{\Omega} \frac{1}{\lambda^2} |\boldsymbol{f}|^2 \, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x + \int_{\Omega_0 \setminus \Omega} \frac{1}{\lambda^2} \left| \tilde{\boldsymbol{f}} \right|^2 \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x \\ &\leqslant \int_{\Omega} \frac{1}{\lambda^2} |\boldsymbol{f}|^2 \, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x + C_1 \frac{\mathrm{e}^{2s\mathrm{e}^{\lambda c_0}}}{\lambda^2} \left(\|\boldsymbol{u}\|^2_{2,\Omega} + \|p\|^2_{1,\Omega} \right), \end{aligned}$$

proving that there exist $C > 0$, $\tilde{s} > 1$ and $\tilde{\lambda} > 1$ such that for every $s \geqslant \tilde{s}$ and $\lambda \geqslant \tilde{\lambda}$

$$\begin{aligned} &\int_{\Omega} \left(s\mathrm{e}^{\lambda\varphi} |\boldsymbol{z}|^2 + |\nabla \boldsymbol{u}|^2 + s^2 \lambda^2 \, \mathrm{e}^{2\lambda\varphi} |\boldsymbol{u}|^2 \right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x \leqslant \int_{\Omega_0} \left(s\mathrm{e}^{\lambda\varphi} |\tilde{z}|^2 + |\nabla \tilde{\boldsymbol{u}}|^2 + s^2 \lambda^2 \, \mathrm{e}^{2\lambda\varphi} |\tilde{\boldsymbol{u}}|^2 \right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x \\ &\quad \leqslant C_6 \left(\int_{\Omega} \frac{1}{\lambda^2} |\boldsymbol{f}|^2 \, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x + \frac{\mathrm{e}^{2s\mathrm{e}^{\lambda c_0}}}{\lambda^2} \left(\|\boldsymbol{u}\|^2_{2,\Omega} + \|p\|^2_{1,\Omega} \right) + \int_{\omega} \left(s^3 \lambda^2 \, \mathrm{e}^{3\lambda\varphi} |\boldsymbol{u}|^2 \right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \, \mathrm{d}x \right), \end{aligned}$$

concluding the proof of the theorem. □

3. Main result

In this section, we present a logarithmic local stability result for our inverse problem. Unlike [6], the right-hand side is more complex and requires a special treatment. We need to prove the following result as a preliminary step.

Theorem 6. *Let $\beta \in \mathcal{C}^2\left(\overline{\Omega}\right)$, $\boldsymbol{a} \in \boldsymbol{C}^1\left(\overline{\Omega}\right)$. There exist $s_0 > 1$ and $C > 0$ such that for all $g \in H(\Omega)$, $\lambda > \lambda_0$ and $s > s_0$*

$$s^2 \int_{\Omega} \left(|\boldsymbol{a} \cdot \nabla \beta|^2 - \frac{C}{s} \right) |g|^2 \, \mathrm{e}^{2s\beta} \, \mathrm{d}x \leqslant C \int_{\Omega} |\boldsymbol{a} \cdot \nabla g|^2 \, \mathrm{e}^{2s\beta} \, \mathrm{d}x.$$

Proof. Let us consider $s > 0$ and define $w = e^{s\beta} g$. Then,

$$e^{s\beta} \boldsymbol{a} \cdot \nabla g = e^{s\beta} \boldsymbol{a} \cdot \nabla \left(e^{-s\beta} w\right) = \boldsymbol{a} \cdot \nabla w - sw \left(\boldsymbol{a} \cdot \nabla\beta\right).$$

Later,

$$\begin{aligned}\int_\Omega e^{2s\beta} \left|\boldsymbol{a} \cdot \nabla g\right| \, \mathrm{d}x &= \int_\Omega |\boldsymbol{a} \cdot \nabla w|^2 \, \mathrm{d}x + s^2 \int_\Omega e^{2s\beta} |g|^2 |\boldsymbol{a} \cdot \nabla\beta|^2 \, \mathrm{d}x - 2s \int_\Omega w \left(\boldsymbol{a} \cdot \nabla\beta\right) \left(\boldsymbol{a} \cdot \nabla w\right) \mathrm{d}x \\ &\geqslant s^2 \int_\Omega e^{2s\beta} |g|^2 |\boldsymbol{a} \cdot \nabla\beta|^2 \, \mathrm{d}x - 2s \int_\Omega w \left(\boldsymbol{a} \cdot \nabla\beta\right) \left(\boldsymbol{a} \cdot \nabla w\right) \mathrm{d}x.\end{aligned}$$

Integrating by parts,

$$\begin{aligned}2s \int_\Omega w \left(\boldsymbol{a} \cdot \nabla\beta\right) \left(\boldsymbol{a} \cdot \nabla w\right) \mathrm{d}x &= s \int_\Omega \left(\boldsymbol{a} \cdot \nabla\beta\right) \left(\boldsymbol{a} \cdot \nabla \left(w^2\right)\right) \mathrm{d}x \\ &= s \int_{\partial\Omega} e^{2s\beta} g \left(\boldsymbol{a} \cdot \nabla\beta\right) \left(\boldsymbol{a} \cdot \boldsymbol{n}\right) \mathrm{d}x - s \int_\Omega w^2 \, \mathrm{div} \left(\left(\boldsymbol{a} \cdot \nabla\beta\right) \boldsymbol{a}\right) \mathrm{d}x \\ &= -s \int_\Omega w^2 \, \mathrm{div} \left(\left(\boldsymbol{a} \cdot \nabla\beta\right) \boldsymbol{a}\right) \mathrm{d}x \\ &= -s \int_\Omega e^{2s\beta} |g|^2 \, \mathrm{div} \left(\left(\boldsymbol{a} \cdot \nabla\beta\right) \boldsymbol{a}\right) \mathrm{d}x,\end{aligned}$$

because $g = 0$ on $\partial\Omega$. Also, $\mathrm{div} \left(\left(\boldsymbol{a} \cdot \nabla\beta\right) \boldsymbol{a}\right)$ is bounded in $\overline{\Omega}$. Then, there exists a constant $C_1 > 0$ only dependent on $\boldsymbol{a}$, β and $\overline{\Omega}$ such that

$$2s \int_\Omega w \left(\boldsymbol{a} \cdot \nabla\beta\right) \left(\boldsymbol{a} \cdot \nabla w\right) \mathrm{d}x \geqslant -C_1 s \int_\Omega e^{2s\beta} |g|^2 \, \mathrm{d}x.$$

Thus, there exist $s_0 > 1$ and $C > 0$ such that for all $s > s_0$

$$C \int_\Omega e^{2s\beta} \left|\boldsymbol{a} \cdot \nabla g\right| \, \mathrm{d}x \geqslant s^2 \int_\Omega e^{2s\beta} \left(|\boldsymbol{a} \cdot \nabla\beta|^2 - \frac{C}{s}\right) |g|^2 \, e^{2s\beta} \, \mathrm{d}x,$$

proving the theorem. □

The previous theorem reduces the study of ***curlf*** recovering a non-degeneracy condition very similar to Theorem 1 in [7] given by $|\nabla\varphi \times \boldsymbol{u}_2| \neq 0$ in $\bar{\Omega}$.

Remark. Despite the fact that $\nabla\varphi$ vanishes at some points of ω, we can always consider two regions of the observation zone, a small open subset included in $\Omega \setminus \bar{\Omega}_\varepsilon$ containing the critical points of φ and another open subset of Ω_ε with absence of them. Velocity and vorticity measurements are required in both sets.

Now, we have

$$\mathrm{curl}\,(\boldsymbol{f}) = \gamma \, \mathrm{curl} \, \boldsymbol{u}_2 + \nabla\gamma \times \boldsymbol{u}_2.$$

Taking $\boldsymbol{a}_1 = (0, [\boldsymbol{u}_2]_3, -[\boldsymbol{u}_2]_2)^{\mathrm{T}}$, $g = \gamma$ and $\beta = e^{\lambda\varphi}$ in theorem 6, we obtain

$$s^2 \int_\Omega \left(\lambda^2 e^{2\lambda\varphi} |\boldsymbol{a}_1 \cdot \nabla\varphi|^2 - \frac{C_1}{s}\right) |\gamma|^2 \, e^{2se^{\lambda\varphi}} \, \mathrm{d}x \leqslant C \int_\Omega |\boldsymbol{a}_1 \cdot \nabla\gamma|^2 \, e^{2se^{\lambda\varphi}} \, \mathrm{d}x.$$

We can repeat this with $\boldsymbol{a}_2 = (-[\boldsymbol{u}_2]_3, 0, [\boldsymbol{u}_2]_1)^{\mathrm{T}}$ and $\boldsymbol{a}_3 = ([\boldsymbol{u}_2]_2, -[\boldsymbol{u}_2]_2, 0)^{\mathrm{T}}$ obtaining

$$s^2 \int_\Omega \left(\lambda^2\,\mathrm{e}^{2\lambda\varphi}|\boldsymbol{a}_2\cdot\nabla\varphi|^2 - \frac{C_1}{s}\right)|\gamma|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \leqslant C_1 \int_\Omega |\boldsymbol{a}_2\cdot\nabla\gamma|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x$$
$$s^2 \int_\Omega \left(\lambda^2\,\mathrm{e}^{2\lambda\varphi}|\boldsymbol{a}_3\cdot\nabla\varphi|^2 - \frac{C_1}{s}\right)|\gamma|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \leqslant C_1 \int_\Omega |\boldsymbol{a}_3\cdot\nabla\gamma|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x,$$

where

$$(\boldsymbol{a}_1\cdot\nabla\varphi, \boldsymbol{a}_2\cdot\nabla\varphi, \boldsymbol{a}_3\cdot\nabla\varphi)^{\mathrm{T}} = \nabla\varphi\times\boldsymbol{u}_2$$
$$(\boldsymbol{a}_1\cdot\nabla\gamma, \boldsymbol{a}_2\cdot\nabla\gamma, \boldsymbol{a}_3\cdot\nabla\gamma)^{\mathrm{T}} = \nabla\gamma\times\boldsymbol{u}_2.$$

In conclusion, adding the three inequalities, we can obtain that

$$\begin{aligned} s^2 \int_\Omega \left(\lambda^2\,\mathrm{e}^{2\lambda\varphi}|\nabla\varphi\times\boldsymbol{u}_2|^2 - \frac{3C_1}{s}\right)|\gamma|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x &\leqslant C_1 \int_\Omega |\nabla\gamma\times\boldsymbol{u}_2|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \\ &\leqslant C_1 \int_\Omega \left(|\mathrm{curl}\,(\boldsymbol{f})|^2 + |\gamma\,\mathrm{curl}\ \boldsymbol{u}_2|^2\right)\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x, \end{aligned} \tag{9}$$

recovering the term $|\nabla\varphi\times\boldsymbol{u}_2|$ on the left-hand side of this inequality. Furthermore, the left-hand side of this inequality can be simplified thanks to the following lemma.

Lemma 7. *Let $f \in \mathcal{C}\left(\overline{\Omega}\right)$ such that $f(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x} \in \Omega$. There exist constants $R > 0$, $\lambda_1 > 0$ and $s_1 > 0$ such that for all $g \in L^2(\Omega)$, $s > s_1$ and $\lambda > \lambda_1$,*

$$\int_\Omega (\lambda^2|f(x)|^2 - \frac{1}{s})|g(x)|^2\,\mathrm{d}x \geqslant R\lambda^2 \int_\Omega |g(x)|^2\,\mathrm{d}x.$$

Proof. The property is fulfilled when $\|g\|_{0,\Omega} = 0$. Then, we suppose that $\|g\|_{0,\Omega} \neq 0$. Since $f \in \mathcal{C}\left(\overline{\Omega}\right)$ and $f \neq 0$, there exists $R_1 > 0$ such that $|f(\boldsymbol{x})| \geqslant R_1$ for all $\boldsymbol{x} \in \Omega$. Then,

$$\int_\Omega \left(\lambda^2|f(x)|^2 - \frac{1}{s}\right)|g(x)|^2\,\mathrm{d}x \geqslant \left(\lambda^2 R_1 - \frac{1}{s}\right)\int_\Omega |g(x)|^2\,\mathrm{d}x.$$

Now, choosing $\lambda_1 = 1$, $s_1 = \frac{2}{R_1}$ and $R = \frac{R_1}{2}$, we obtain that for all $s > s_1$ and $\lambda > \lambda_1$,

$$\begin{aligned} \int_\Omega \left(\lambda^2|f(x)|^2 - \frac{1}{s}\right)|g(x)|^2\,\mathrm{d}x &\geqslant \left(R_1\lambda^2 - \frac{1}{s}\right)\int_\Omega |g(x)|^2\,\mathrm{d}x \\ &\geqslant \left(R_1\lambda^2 - \frac{R_1}{2}\right)\int_\Omega |g(x)|^2\,\mathrm{d}x \\ &\geqslant \frac{R_1}{2}\lambda^2 \int_\Omega |g(x)|^2\,\mathrm{d}x = R\lambda^2 \int_\Omega |g(x)|^2\,\mathrm{d}x, \end{aligned}$$

proving the lemma. □

Remark. Note that there exist $R > 0$, $s_1 > 1$ and $\lambda_1 > 1$ such that, for all $s > s_1$ and $\lambda > \lambda_1$,

$$\int_{\Omega_\varepsilon} \left(\lambda^2\,\mathrm{e}^{2\lambda\varphi}|\nabla\varphi\times\boldsymbol{u}_2|^2 - \frac{3C_1}{s}\right)|\gamma|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \geqslant R\int_{\Omega_\varepsilon} \lambda^2|\gamma|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x. \tag{10}$$

Since $\boldsymbol{u}_2 \in \boldsymbol{H}^2(\Omega)$, theorem 5.8.4 in [11] states that $\boldsymbol{u}_2 \in \boldsymbol{C}(\Omega)$. Then, $|\nabla\varphi \times \boldsymbol{u}_2| \in \mathcal{C}(\Omega)$ and $|\nabla\varphi \times \boldsymbol{u}_2| \neq 0$ in $\overline{\Omega_\varepsilon \setminus \omega}$ almost everywhere. Then, choosing $f(x) = \mathrm{e}^{\lambda\varphi}|\nabla\varphi(x) \times \boldsymbol{u}_2(x)|$ and $g(x) = \mathrm{e}^{s\mathrm{e}^{\lambda\varphi}}|\gamma(x)|$ on lemma 7, we can deduce the inequality 10.

It is possible to prove, similar to lemma 2 and theorem 5 in section 2, the following Carleman estimates for strong solutions of linear second-order elliptic PDE with homogenous and nonhomogeneous Dirichlet boundary conditions.

Lemma 8. *Let $f \in L^2(\Omega)$, $\nu \in \mathbb{R}$ with $\nu > 0$, $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{L}^\infty(\Omega)$, $c \in L^\infty(\Omega)$ and $u \in H^2(\Omega)$ solution of*

$$-\nu \triangle u + \boldsymbol{a} \cdot \nabla u + \mathrm{div}(u\boldsymbol{b}) + cu = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Then, there exist $C > 0$, $\tilde{\lambda} > 1$ and $\tilde{s} > 1$, independent on u, such that for all $k \in \mathbb{N} \cup \{0\}$, $\lambda \geqslant \tilde{\lambda}$ and $s \geqslant \tilde{s}$,

$$\int_\Omega \left(\left(s\lambda\,\mathrm{e}^{\lambda\varphi}\right)^{k-2} |\triangle u|^2 + \left(s\lambda\,\mathrm{e}^{\lambda\varphi}\right)^{k} |\nabla u|^2 + \left(s\lambda\,\mathrm{e}^{\lambda\varphi}\right)^{k+2} |u|^2 \right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x$$
$$\leqslant C \left(\int_\Omega \frac{1}{\lambda} \left(s\lambda\mathrm{e}^{\lambda\varphi}\right)^{k-1} |f|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x + \int_\omega \left(s\lambda\,\mathrm{e}^{\lambda\varphi}\right)^{k+2} |u|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \right). \tag{11}$$

Proof. See theorem A.1 in [13]. □

Lemma 9. *Let $f \in L^2(\Omega)$, $\boldsymbol{F} \in \boldsymbol{L}^2(\Omega)$, $\nu \in \mathbb{R}$ with $\nu > 0$, $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{L}^\infty(\Omega)$, $c \in L^\infty(\Omega)$, $g \in H^{3/2}(\partial\Omega)$ and $u \in H^2(\Omega)$ solution of*

$$-\nu\triangle u + \boldsymbol{a} \cdot \nabla u + \mathrm{div}(u\boldsymbol{b}) + cu = f + \mathrm{div}\,\boldsymbol{F} \qquad \text{in } \Omega$$
$$u = 0 \qquad \text{on } \partial\Omega.$$

Then, there exist $C > 0$, $\tilde{\lambda} > 1$ and $\tilde{s} > 1$, independent on u, such that for all $k \in \mathbb{N} \cup \{0\}$, $\lambda \geqslant \tilde{\lambda}$ and $s \geqslant \tilde{s}$,

$$\int_\Omega \left(\left(s\lambda\mathrm{e}^{\lambda\varphi}\right)^{k-2} |\triangle u|^2 + \left(s\lambda\,\mathrm{e}^{\lambda\varphi}\right)^{k} |\nabla u|^2 + \left(s\lambda\,\mathrm{e}^{\lambda\varphi}\right)^{k+2} |u|^2 \right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x$$
$$\leqslant C \left(\frac{\mathrm{e}^{2s\mathrm{e}^{\lambda c_0}}}{\lambda} \left(s\lambda\,\mathrm{e}^{\lambda c_0}\right)^{k-1} \|u\|_{2,\Omega}^2 + \int_\Omega \frac{1}{\lambda} \left(s\lambda\,\mathrm{e}^{\lambda\varphi}\right)^{k-1} |f|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \right. \tag{12}$$
$$\left. + \int_\Omega \frac{1}{\lambda} \left(s\lambda\,\mathrm{e}^{\lambda\varphi}\right)^{k+1} |\boldsymbol{F}|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x + \int_\omega \left(s\lambda\,\mathrm{e}^{\lambda\varphi}\right)^{k+2} |u|^2\,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \right).$$

Proof. See theorem 2.2 in [13]. □

Then, we can present our local stability result.

Theorem 10. *Consider an non-empty open subset $\omega \subseteq \Omega$, $\boldsymbol{u}_2 \in \boldsymbol{C}^1\left(\overline{\Omega}\right)$ and $|\nabla\varphi(\boldsymbol{x}) \times \boldsymbol{u}_2(\boldsymbol{x})| \neq 0$ for all $\boldsymbol{x} \in \overline{\Omega_\varepsilon}$. Then, defining a constant $M_3' = (M_3^2 + K^2)^{1/2} > 0$, there exists a constant $C > 0$ independent on $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ such that*

$$\|\,\gamma\,\|_{0,\Omega_\varepsilon} \leqslant \frac{CM_3'}{\left[\ln\left(1 + \frac{M_3'}{\|\boldsymbol{u}\|_{0,\omega} + \|\mathrm{curl}\,\boldsymbol{u}\|_{0,\omega}}\right)\right]^{1/2}}.$$

Proof. Let us consider $c = \|\varphi\|_{0,\infty,\Omega}$ and $\boldsymbol{z} = \operatorname{curl} \boldsymbol{u}$. From the equation

$$-\nu \triangle \boldsymbol{z} + (\nabla \boldsymbol{z})\, \boldsymbol{u}_1 + \gamma_1 \boldsymbol{z} = -\left(\operatorname{curl} \boldsymbol{f} + \boldsymbol{h}\right),$$

there exists a constant $C_2 > 0$ that only depends of $\boldsymbol{u}_1, \boldsymbol{u}_2$ and M such that

$$|\operatorname{curl} \boldsymbol{f}|^2\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \leqslant C_2 \left(|\triangle \boldsymbol{z}|^2 + |\nabla \boldsymbol{z}|^2 + |\boldsymbol{z}|^2 + |\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}|^2\right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}.$$

Applying theorem 9 with $k = 2$, then there exists constants $C_3 > 0$, $\tilde{\lambda} \geqslant 1$ and $\tilde{s} \geqslant 1$ such that for all $\lambda \geqslant \tilde{\lambda}$ and $s \geqslant \tilde{s}$

$$\begin{aligned}
&\int_\Omega \left(|\triangle \boldsymbol{z}|^2 + |\nabla \boldsymbol{z}|^2\right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x \\
&\leqslant C_3 \left(s\, \mathrm{e}^{\lambda c_0}\, \mathrm{e}^{2s\mathrm{e}^{\lambda c_0}} \|\, \boldsymbol{z}\, \|_{2,\Omega}^2 + \int_\Omega s\mathrm{e}^{\lambda\varphi} |(\operatorname{curl} \boldsymbol{f} + \boldsymbol{h})|^2\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x + \int_\omega (s\lambda\, \mathrm{e}^{\lambda\varphi})^4 |\boldsymbol{z}|^2\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x\right) \\
&\leqslant C_3 \left(s\, \mathrm{e}^{\lambda c_0}\, \mathrm{e}^{2s\mathrm{e}^{\lambda c_0}} \|\, \boldsymbol{u}\, \|_{3,\Omega}^2 + \int_\Omega s\, \mathrm{e}^{\lambda\varphi} \left(|\operatorname{curl} \boldsymbol{f}|^2 + |\boldsymbol{u}|^2 + |\nabla \boldsymbol{u}|^2\right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x + \int_\omega (s\lambda\, \mathrm{e}^{\lambda\varphi})^4 |\boldsymbol{z}|^2\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x\right) \\
&\leqslant C_3 \left(s\, \mathrm{e}^{\lambda c_0}\, \mathrm{e}^{2s\mathrm{e}^{\lambda c_0}} M_3^2 + \int_\Omega s\, \mathrm{e}^{\lambda\varphi} \left(|\gamma|^2 + |\nabla\gamma|^2 + |\boldsymbol{u}|^2 + |\nabla \boldsymbol{u}|^2\right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x + \int_\omega (s\lambda\, \mathrm{e}^{\lambda\varphi})^4 |\boldsymbol{z}|^2\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x\right) \\
&\leqslant C_3 \left(s\, \mathrm{e}^{\lambda c_0}\, \mathrm{e}^{2s\mathrm{e}^{\lambda c_0}} M_3^2 + s\, \mathrm{e}^{\lambda c}\, \mathrm{e}^{2s\mathrm{e}^{\lambda c}} K^2 + \int_\Omega s\, \mathrm{e}^{\lambda\varphi} \left(|\gamma|^2 + |\boldsymbol{u}|^2 + |\nabla \boldsymbol{u}|^2\right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x + \int_\omega (s\lambda\, \mathrm{e}^{\lambda\varphi})^4 |\boldsymbol{z}|^2\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x\right),
\end{aligned}$$

where we use that $\boldsymbol{u}_1, \boldsymbol{u}_2, \operatorname{curl}\, \boldsymbol{u}_2 \in \boldsymbol{L}^\infty(\Omega)$ and the assumption (a). Now, applying theorem 5, there exists a constant $C_4 > 0$ such that for all $\lambda \geqslant \tilde{\lambda}$ and $s \geqslant \tilde{s}$

$$\begin{aligned}
\int_\Omega \left(|\boldsymbol{z}|^2 + |\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}|^2\right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x &\leqslant \int_\Omega \left(s\mathrm{e}^{\lambda\varphi} |\boldsymbol{z}|^2 + |\nabla \boldsymbol{u}|^2 + \left(s\lambda\, \mathrm{e}^{\lambda\varphi}\right)^2 |\boldsymbol{u}|^2\right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x \\
&\leqslant C_4 \left(\frac{\mathrm{e}^{2s\mathrm{e}^{\lambda c_0}}}{\lambda^2} M_2^2 + \int_\Omega \frac{1}{\lambda^2} |\gamma|^2\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x + \int_\omega s^3\, \mathrm{e}^{3\lambda\varphi}\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} |\boldsymbol{u}|^2\, \mathrm{d}x\right).
\end{aligned}$$

Then, there exists a constant $C_5 > 0$ such that for all $\lambda \geqslant \tilde{\lambda}$ and $s \geqslant \tilde{s}$

$$\begin{aligned}
\int_\Omega |\operatorname{curl} \boldsymbol{f}|^2\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x &\leqslant C_2 \int_\Omega \left(|\triangle \boldsymbol{z}|^2 + |\nabla \boldsymbol{z}|^2 + |\boldsymbol{z}|^2 + |\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}|^2\right) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x \\
&\leqslant C_5 \left(s\, \mathrm{e}^{\lambda c_0}\, \mathrm{e}^{2s\mathrm{e}^{\lambda c_0}} M_3^2 + \int_\Omega s\, \mathrm{e}^{\lambda\varphi} |\gamma|^2\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x + \int_\omega \left(s\lambda\, \mathrm{e}^{\lambda\varphi}\right)^4 \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} \left(|\boldsymbol{u}|^2 + |\boldsymbol{z}|^2\right)\, \mathrm{d}x\right).
\end{aligned}$$

Replacing this in inequality (9) and using that $\operatorname{curl}\, \boldsymbol{u}_2 \in \boldsymbol{L}^\infty(\Omega)$, we deduce that there exists a constant $C_6 > 0$ such that

$$\begin{aligned}
&Rs^2 \int_{\Omega_\varepsilon} |\gamma|^2\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x \leqslant s^2 \int_{\Omega_\varepsilon} (\lambda^2\, \mathrm{e}^{2\lambda\varphi} |\nabla\varphi \times \boldsymbol{u}_2|^2 - \frac{3C_1}{s}) |\gamma|^2\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x \\
&\quad \leqslant C_1 \int_\Omega (|\operatorname{curl} \boldsymbol{f}|^2 + |\gamma \operatorname{curl} \boldsymbol{u}_2|^2) \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x \\
&\quad \leqslant C_6 (s\mathrm{e}^{\lambda c}\, \mathrm{e}^{2s\mathrm{e}^{\lambda c}} (M_3^2 + K^2) + \int_\Omega s\mathrm{e}^{\lambda\varphi} |\gamma|^2\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\, \mathrm{d}x + \int_\omega (s\lambda\, \mathrm{e}^{\lambda\varphi})^4\, \mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}} (|\boldsymbol{u}|^2 + |\boldsymbol{z}|^2) \mathrm{d}x),
\end{aligned}$$

where $|\nabla\varphi \times \boldsymbol{u}_2| \in \boldsymbol{C}(\Omega)$ and $|\nabla\varphi \times \boldsymbol{u}_2| \neq 0$. Then, taking $s > 0$ sufficiently large and fixing $\lambda = \tilde{\lambda}$, we can absorb the second term of the right-hand side by the left-hand side. Thus, there exist constants $C_7 > 0$, $\hat{s} > 0$, $L = (M_3^2 + K^2)^{1/2} > 0$ and $c^* > 1$ such that for all $s \geqslant \hat{s}$

$$s^2 \int_{\Omega_\varepsilon} |\gamma|^2 \,\mathrm{e}^{2s\mathrm{e}^{\lambda\varphi}}\,\mathrm{d}x \leqslant C_7 (s\,\mathrm{e}^{\tilde{\lambda}c}\,\mathrm{e}^{2s\mathrm{e}^{\tilde{\lambda}c}}(M_3')^2 + \int_\omega (s\tilde{\lambda}\,\mathrm{e}^{\tilde{\lambda}\varphi})^4\,\mathrm{e}^{2s\mathrm{e}^{\tilde{\lambda}\varphi}}(|\boldsymbol{u}|^2 + |\boldsymbol{z}|^2)\mathrm{d}x)$$

$$\| \gamma \|_{0,\Omega_\varepsilon}^2 \leqslant \frac{(M_3')^2}{s} + \mathrm{e}^{2sc^*}(\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega})^2. \tag{13}$$

If $\|\boldsymbol{u}\|_{0,\omega} + \|\boldsymbol{z}\|_{0,\omega} = 0$, then for all $s \geqslant \hat{s}$ we have $\| \gamma \|_{0,\Omega_\varepsilon} \leqslant \frac{M_3'}{s^{1/2}}$ for all $s \geqslant \hat{s}$. Later, $\|\gamma\|_{0,\Omega_\varepsilon} = 0$.

Now, we assume that $\|\boldsymbol{u}\|_{0,\omega} + \|\boldsymbol{z}\|_{0,\omega} \neq 0$. We have two cases. In the first case, if we suppose that

$$\frac{1}{2c^*} \ln \left(1 + \frac{(M_3'}{\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega}} \right) \leqslant \hat{s}.$$

Later,

$$M_3' \leqslant \mathrm{e}^{\widehat{2sc^*}} \left(\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega} \right).$$

Then, taking $s = \hat{s}$ in inequality (13), and using $\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega} \leqslant M'{}_3$ and $\frac{1}{x} \leqslant \frac{1}{\ln(x+1)}$ for all $x > 0$, we obtain

$$\begin{aligned}
\| \gamma \|_{0,\Omega_\varepsilon}^2 &\leqslant \frac{(M_3')^2}{\hat{s}} + \mathrm{e}^{2\hat{s}c^*}\left(\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega}\right)^2 \\
&\leqslant \frac{\mathrm{e}^{4\hat{s}c^*}}{\hat{s}}\left(\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega}\right)^2 + \mathrm{e}^{4\hat{s}c^*}\left(\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega}\right)^2 \\
&\leqslant 2\,\mathrm{e}^{4\hat{s}c^*}\left(\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega}\right)^2 \\
&\leqslant 2\,\mathrm{e}^{4\hat{s}c^*} M_3' \left(\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega}\right) \\
&\leqslant 2\,\mathrm{e}^{4\hat{s}c^*}(M_3')^2 \frac{\left(\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega}\right)}{M_3'} \leqslant \frac{2\,\mathrm{e}^{4\hat{s}c^*}(M_3')^2}{\left[\ln\left(1 + \frac{M_3'}{\|\boldsymbol{u}\|_{0,\omega} + \|\boldsymbol{z}\|_{0,\omega}}\right)\right]}.
\end{aligned} \tag{14}$$

In the second case, if we suppose that

$$\frac{1}{2c^*} \ln \left(1 + \frac{M_3'}{\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega}} \right) \geqslant \hat{s}.$$

Taking $s = \frac{1}{2c^*} \ln \left(1 + \frac{M_3'}{\|\boldsymbol{u}\|_{0,\omega} + \|\boldsymbol{z}\|_{0,\omega}} \right)$ in inequality (13), using $\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega} \leqslant M_3 \leqslant M_3'$ and

$$\frac{(x+1)\ln(1+x)}{x^2} \leqslant 2$$

for all $x > 1$, we obtain

$$\mathrm{e}^{2sc^*} = 1 + \frac{M_3'}{\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega}}$$

and

$$\begin{aligned}\| \gamma \|_{0,\Omega_\varepsilon}^2 &\leqslant \frac{(M_3')^2}{s} + \mathrm{e}^{2sc^*}\left(\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega}\right)^2 \\ &\leqslant \frac{(M_3')^2}{s}\left(1 + \left[\frac{1}{2c^*}\ln\left(1 + \frac{M_3'}{\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega}}\right)\right]\left(1 + \frac{M_3'}{\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega}}\right)\frac{(\| \boldsymbol{u} \|_{0,\omega} + \| \boldsymbol{z} \|_{0,\omega})^2}{(M_3')^2}\right) \\ &\leqslant \frac{(M_3')^2}{s}\left(1 + \frac{1}{c^*}\right) = \frac{(2c^* + 1)(M_3')^2}{\ln\left(1 + \frac{M_3'}{\|\boldsymbol{u}\|_{0,\omega} + \|\boldsymbol{z}\|_{0,\omega}}\right)}. \end{aligned} \tag{15}$$

From (14) and (15), we can deduce that there exists a constant $C > 0$ such that

$$\| \gamma \|_{0,\Omega_\varepsilon} \leqslant \frac{CM_3'}{\left[\ln\left(1 + \frac{L}{\|\boldsymbol{u}\|_{0,\omega} + \|\mathrm{curl}\ \boldsymbol{u}\|_{0,\omega}}\right)\right]^{1/2}},$$

proving the main result. □

4. Numerical results

In this section, we present two numerical tests that support the stability result proved in theorem 10. We perform numerical experiments in 2D, but the theory of the previous sections is valid both in two and three dimensions. In both examples will use the sames 2D domain $\Omega = [-1, 1]^2$, subset $\Omega_\varepsilon = [-0.9, 0.9]^2$ (with a $\varepsilon = 0.1$), and observation region $\omega = [-0.5, 0.5]^2$, similar Dirichlet boundary conditions, and a different function γ_R (R for reference) such that $\gamma_R = 0$ in $\Omega\backslash\Omega_\varepsilon$. We obtain numerical approximations for the solutions $(\boldsymbol{u}_R, p_R)$ of equation (1) with $\gamma = \gamma_R$ using the finite element method with Taylor–Hood elements ($\mathbb{P}_2$ for the velocity $\boldsymbol{u}_R$ and $\mathbb{P}_1$ for the pressure p_R) on an unstructured hyperfine triangular mesh (with mesh size $h = 0.01$). We recover the coefficient γ_R as the solution of the following minimization problem

$$\text{minimize} \quad J(\gamma, \boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{u}_R\|_{0,\omega}^2 + \frac{1}{2}\|\mathrm{curl}\ \boldsymbol{u} - \mathrm{curl}\ \boldsymbol{u}_R\|_{0,\omega}^2 \tag{16}$$

$$\begin{aligned} \text{subject to} \quad -\nu\triangle \boldsymbol{u} + (\nabla \boldsymbol{u})\,\boldsymbol{u} + \nabla p + \gamma \boldsymbol{u} &= \boldsymbol{0} \quad \text{in } \Omega \\ \mathrm{div}\ \boldsymbol{u} &= 0 \quad \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{u}_D \quad \text{on } \partial\Omega \\ \boldsymbol{u} \in \boldsymbol{H}^1(\Omega)\, \gamma &\in H^1(\Omega) \\ 0 \leqslant \gamma \leqslant M &\text{ a.e. in } \Omega, \end{aligned} \tag{17}$$

where $M = \max_{\boldsymbol{x}\in\Omega} \gamma_R(\boldsymbol{x})$. The functional J was chosen because it is differentiable with respect to γ and is equivalent to $\|\boldsymbol{u} - \boldsymbol{u}_R\|_{0,\omega} + \|\mathrm{curl}\ \boldsymbol{u} - \mathrm{curl}\ \boldsymbol{u}_R\|_{0,\omega}$. This problem is numerically solved approximating the Navier–Stokes equations with finite element method using stable pairs of spaces (in terms of the inf-sup condition, see section 3.6 in [14]) in a coarse structured mesh, where γ is approximated by $\mathbb{P}_1$ elements. It should be noted that the second example is based on recovering a discontinuous coefficient γ_R, which is not covered by theorem 10, but it complements the work of Aguayo *et al* [3]. The FEM solver is implemented using the finite element library FEniCS 2019.1.0 [5] with the default configuration. The nonlinear problems were solved using the Newton method with a relaxation parameter $\alpha \in [0.9999, 1]$. The dolfin-adjoint library [15] were used to numerically solve the optimization problems with the L-BFGS-B algorithm (see section 4.3 in [10]). Furthermore, we explain more details of the parameters, domains, meshes, numerical methods and tolerances used on each example.

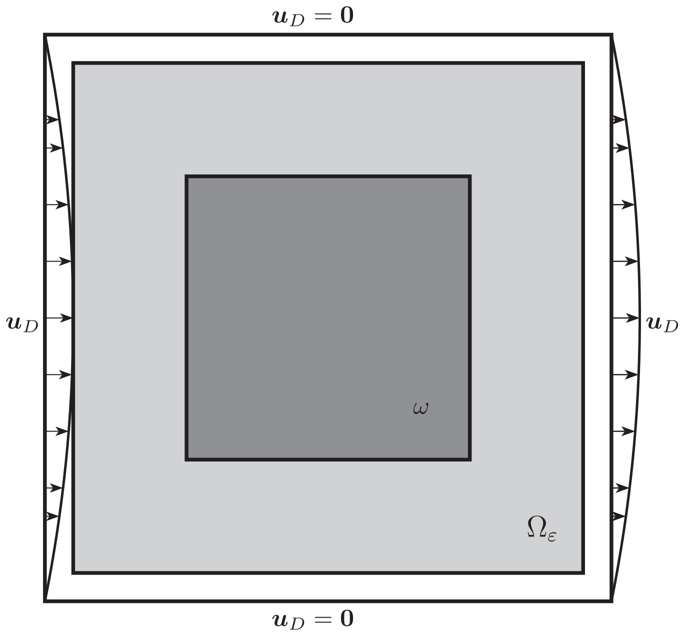

Figure 1. Domain Ω for the numerical tests and boundary conditions.

4.1. Recovering a smooth function

In this first test, we consider $\nu = 1$, $\boldsymbol{f} = \mathbf{0}$, Dirichlet boundary conditions given by a function $\boldsymbol{u}_D$ such that

$$\boldsymbol{u}_D(\boldsymbol{x}) = \boldsymbol{u}_D(x, y) = \begin{cases} \left(5\left(1-y^2\right), 0\right)^{\mathrm{T}} & \text{if } x \in \{-1, 1\} \\ 0 & \text{if } y \in \{-1, 1\} \end{cases}$$

and a function $\gamma_R \in H(\Omega)$ such that

$$\gamma_R(\boldsymbol{x}) = \gamma_R(x, y) = \begin{cases} \dfrac{100}{16}\left(1+\cos\left(\dfrac{\pi x}{0.9}\right)\right)^2\left(1+\cos\left(\dfrac{\pi y}{0.9}\right)\right)^2 & \text{if } \boldsymbol{x} = (x, y) \in \Omega_\varepsilon \\ 0 & \text{if } \boldsymbol{x} = (x, y) \in \Omega \backslash \Omega_\varepsilon \end{cases}.$$

The reference solutions are the numerical solutions $(\boldsymbol{u}_R, p_R)$ of equation (1), obtained by finite element method with Taylor–Hood elements ($\mathbb{P}_2$ for the velocity $\boldsymbol{u}$ and $\mathbb{P}_1$ for the pressure p) in a hyperfine unstructured triangular mesh (mesh size $h = 0.01$, 53 649 nodes and 107 296 elements), using the function γ_R defined previously (figure 1).

The optimization problem was discretized with the same FEM scheme for a coarse structured triangular mesh. The function γ was discretized using $\mathbb{P}_1$ elements. The discretized Navier–Stokes equations were solved using Newton method with a tolerance of 10^{-7} for the discrete ℓ_2 residual norm. The tolerance criterion for the L-BFGS-B algorithm was 5×10^{-9} for consecutive evaluations of functional J or approximations of the Riesz representant of ∇J, the Fréchet derivative of J, in norms $L^2(\Omega)$ or ℓ_2. We used $\gamma_0 = 0$ as a initial guess for the L-BFGS-B algorithm. If we denote γ_k and $\boldsymbol{u}_k$ as the optimal control and their respective state on the k iteration of the L-BFGS-B algorithm, we can define the errors $\gamma_{E,k} = \gamma_k - \gamma_R$ and $\boldsymbol{u}_{E,k} = \boldsymbol{u}_k - \boldsymbol{u}_R$.

Table 1. Evolution of L-BFGS-B algorithm.

k	$J(\gamma_k)$	$\Vert\nabla J(\gamma_k)\Vert_{0,\Omega}$	$\Vert\boldsymbol{u}_{E,k}\Vert_{0,\omega} + \Vert\mathrm{curl}\ \boldsymbol{u}_{E,k}\Vert_{0,\omega}$	$\Vert\gamma_{E,k}\Vert_{0,\Omega_\varepsilon}$
0	4.8156×10^{1}	1.7378×10^{-2}	1.1536×10^{1}	4.9219×10^{1}
3	8.9908×10^{0}	4.6005×10^{-3}	5.0507×10^{0}	3.6516×10^{1}
5	1.3860×10^{0}	1.1846×10^{-3}	2.0324×10^{0}	2.5782×10^{1}
8	1.5234×10^{-1}	1.8093×10^{-4}	6.9552×10^{-1}	1.6227×10^{1}
14	1.0747×10^{-2}	1.1264×10^{-5}	2.0450×10^{-1}	1.1485×10^{1}
41	1.0969×10^{-3}	2.0214×10^{-6}	6.1340×10^{-2}	6.1686×10^{0}
68	1.0095×10^{-4}	4.0138×10^{-7}	1.9318×10^{-2}	5.0834×10^{0}
196	5.0897×10^{-5}	3.3451×10^{-8}	1.0601×10^{-2}	4.6065×10^{0}

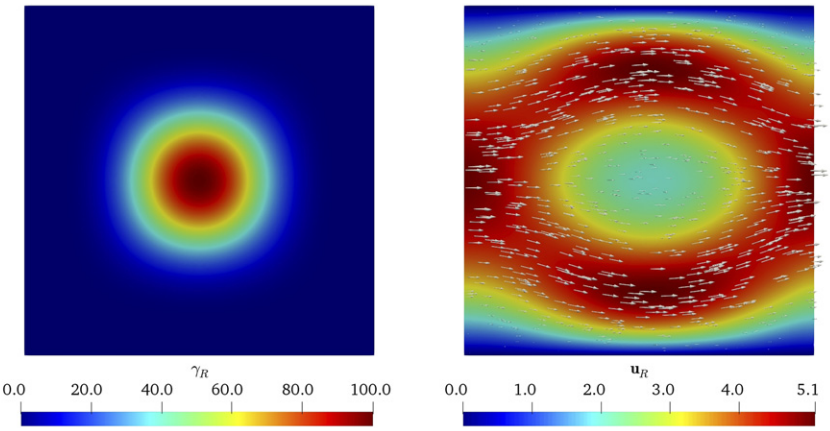

Figure 2. Plots of the Brinkman's law reference permeability parameter γ_R and the correspondent reference isovalues and flow $\boldsymbol{u}_R$.

Also we define γ^* as the optimal function obtained by the L-BFGS-B algorithm and $(\boldsymbol{u}^*, p^*)$ as the optimal states. Table 1 and figures 3 and 4 summarize the numerical results obtained.

Comparing figures 2 and 3, we can see that there is a fast convergence of the velocity at the optimal $\boldsymbol{u}_R$, both in the measurement region ω and in the rest of Ω. However, the convergence rate of γ is low, according to the theory. In the measurement region ω, γ^* has a similar shape to γ_R, with differences of less than 4% in L^∞ norm. Outside the measurement region ω, γ^* presents some noise, as can be seen in figures 3 and 4, which is mainly associated with the measurement region, the chosen objective function J and the finite element approximation.

4.2. Recovering a non-smooth function

Unlike the previous test, in this one we are looking for recovering a function $\gamma_R \in L^2(\Omega)$ with $\gamma_R = 0$ in $\Omega\backslash\Omega_\varepsilon$ such that $\gamma_R \notin H^1(\Omega)$. Then, in this test we do not expect to recover the theoretical results, since the hypothesis of the main theorem is not fulfilled, but rather to present a strategy that allows recovering a discontinuous coefficient γ. This example is motivated by [3],

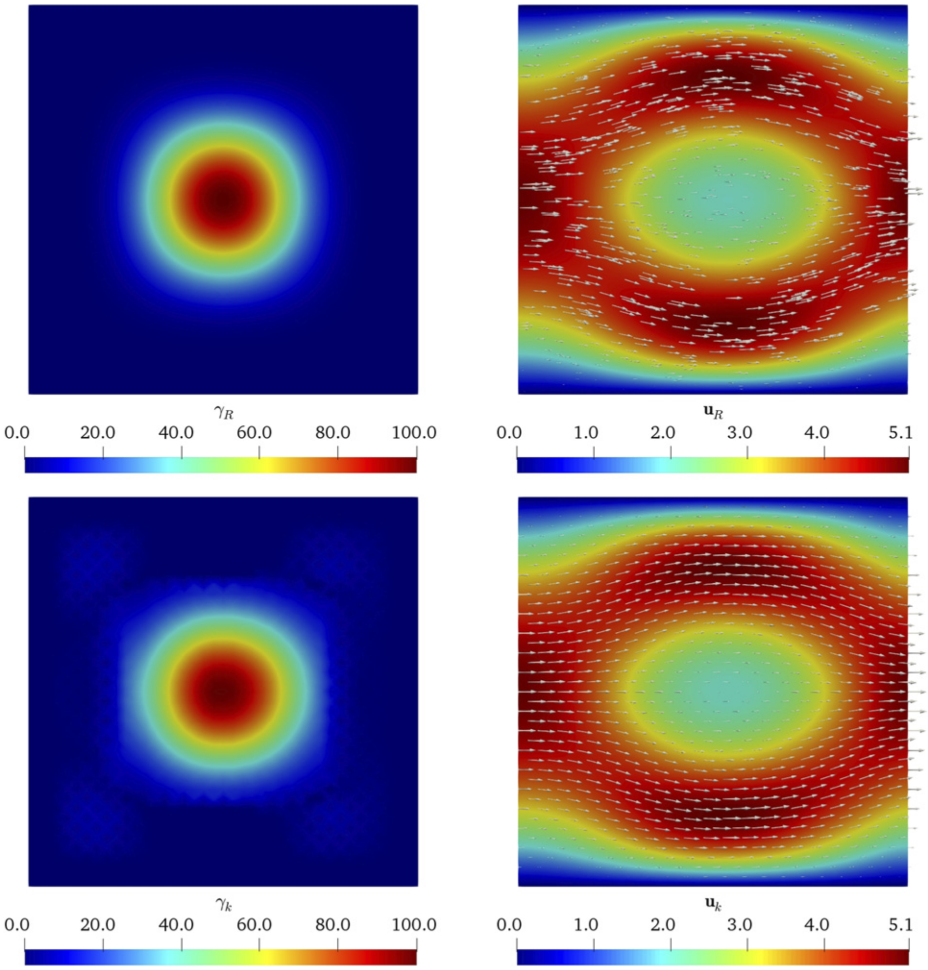

Figure 3. Plots of reference parameters (top) and recovered permeability parameter γ_k and velocity $\boldsymbol{u}_k$ on iteration 198 (bottom).

where the authors solved numerically an inverse problem to recover a discontinuous coefficient that represent an obstacle.

We consider the sames domains Ω and Ω_ε as in the first test and the same the parameters $\nu = 1$ and $\boldsymbol{f} = \mathbf{0}$. The Dirichlet boundary condition is given by a function $\boldsymbol{u}_D$ such that

$$\boldsymbol{u}_D(\boldsymbol{x}) = \boldsymbol{u}_D(x, y) = \begin{cases} \left(30\left(1 - y^2\right), 0\right)^{\mathrm{T}} & \text{if } x \in \{-1, 1\} \\ 0 & \text{if } y \in \{-1, 1\} \end{cases}$$

and a function $\gamma_R \in H(\Omega)$ such that

$$\gamma_R(\boldsymbol{x}) = \gamma_R(x, y) = \begin{cases} 10\,000 & \text{if } (x, y) \in B \\ 0 & \text{if } (x, y) \in \Omega \backslash B \end{cases}$$

where $B = \left\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leqslant 0.2^2\right\}$.

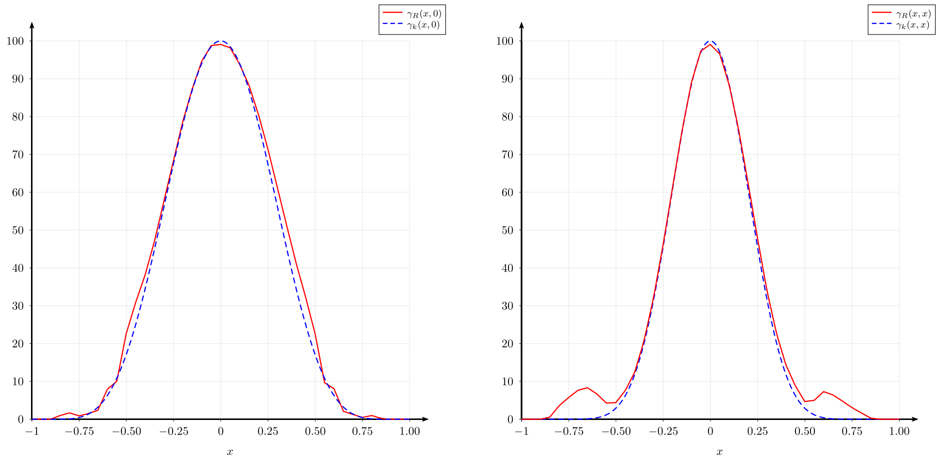

Figure 4. Cut lines of the recovered γ_k on iteration 198 and γ_R on $y = 0$ (left) and $y = x$ (right).

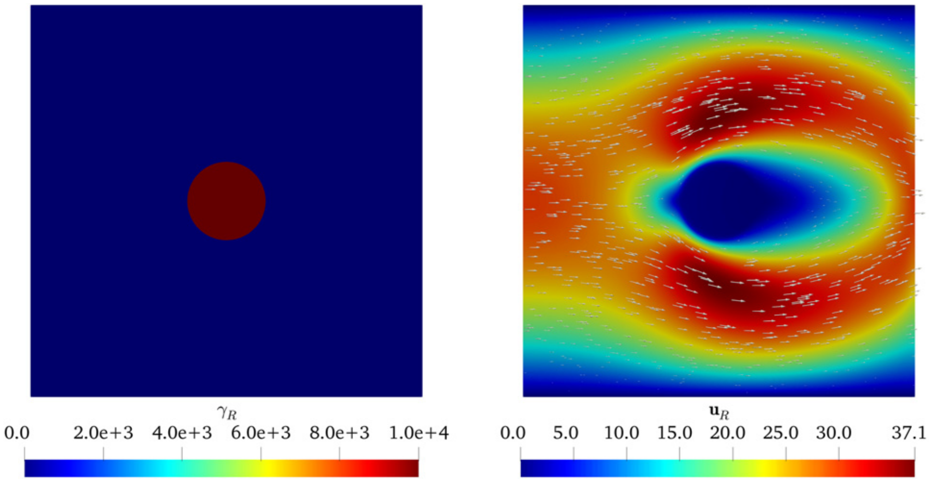

Figure 5. Plots of γ_R and $\boldsymbol{u}_R$.

The reference solutions are the numerical solutions $(\boldsymbol{u_R}, p_R)$ of equation (1), obtained by finite element method with the Taylor–Hood elements ($\mathbb{P}_2$ for the velocity $\boldsymbol{u_R}$ and $\mathbb{P}_1$ for the pressure p_R) in a hyperfine unstructured triangular mesh (mesh size $h = 0.01$, 53 649 nodes and 106 496 elements), using the function γ_R defined previously (figure 5).

The optimization problem were discretized with the MINI element ($\mathbb{P}_1 \oplus V_{\text{bub}}$ for the velocity $\boldsymbol{u}$ and $\mathbb{P}_1$ for the pressure p, where V_{bub} is the space of the bubble functions, see section 3.6.1 in [14]) for a coarse structured triangular mesh. The discretized Navier–Stokes equations were solved using Newton method with a tolerance of 10^{-7} for the discrete ℓ_2 residual norm. Thanks to we can recover a discontinuous L^2 function, we decided to use $\mathbb{P}_1$ elements for γ discretization combined with a new algorithm for this optimization problem based in adaptive refinement. If $\mathcal{T}_h$ is a triangulation for $\overline{\Omega}$, we denote by T the elements of $\mathcal{T}_h$ and by $\mathcal{E}_h$ the set of all edges $\mathcal{T}_h$. Also $\mathcal{E}_h = \mathcal{E}_\Omega \cup \mathcal{E}_\partial$, where $\mathcal{E}_\Omega$ and $\mathcal{E}_\partial$ are the sets of edges lying in the interior and the boundary of $\overline{\Omega}$, respectively. We use h_T as the diameter of T and $h_F = |F|$ for each

Algorithm 1. Algorithm of each adaptive refinement stage.

Require: A coarse mesh $\mathcal{T}_h$, $N, \Delta \in \mathbb{N}$, $\gamma = 0$

1: Run N iterations of the L-BFGS-B algorithm for the problem 16 on the current mesh

2: For each $T \in \mathcal{T}_h$, compute the estimators $\eta_{\gamma,T}$ and η_T using the optimal function and the optimal states

3: Given $T \in \mathcal{T}_h$ such that $\eta_{\gamma,T} \geqslant 0.8 \max\limits_{K\in\mathcal{T}_h} \eta_{\gamma,K}$ or $\eta_T \geqslant 0.5 \max\limits_{K\in\mathcal{T}_h} \eta_K$, mark T and generate a new mesh $\mathcal{T}_h$ refining the marked elements

4: If the stop criterion is not satisfied, choose γ as the Lagrange interpolation of the optimal control in the new finite element space obtained in the step 1, increase N to $N + \Delta$ and go to the step 1

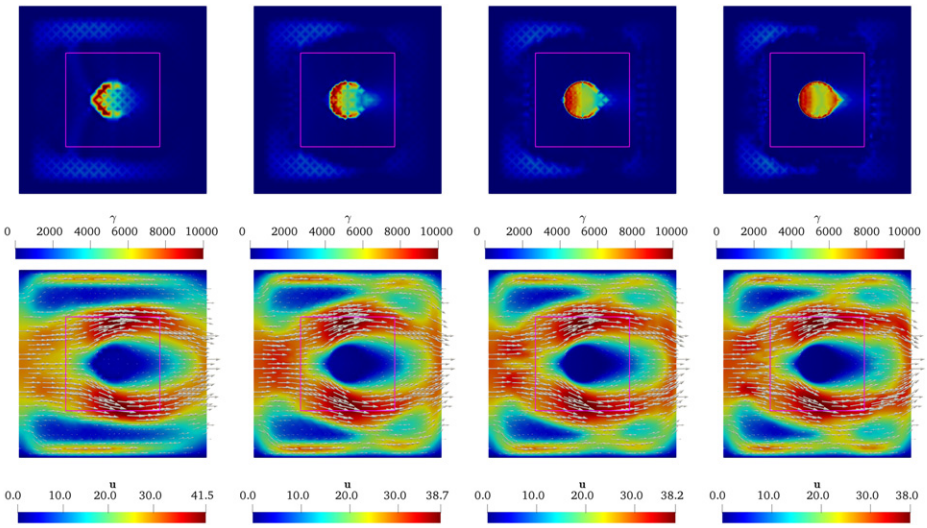

Figure 6. Plots of γ_k (first row) and $\boldsymbol{u}_k$ (second row) after stages 1, 3, 7 and 13 (from left to right).

$F \in \mathcal{E}_\Omega$. Then, we define

$$\eta_{\gamma,T} = \left(\sum_{F\in\partial T\cap\mathcal{E}_\Omega} h_F \|[[\nabla\gamma_F]]_F\|^2_{0,F} \right)^{1/2}$$

$$\eta_T = \left(h_T^2 \| -\nu\triangle\boldsymbol{u} + (\nabla\boldsymbol{u})\boldsymbol{u} + \gamma\boldsymbol{u} + \nabla p\|^2_{0,T} + \|\operatorname{div}\,\boldsymbol{u}\|^2_{0,T} + \sum_{F\in\partial T\cap\mathcal{E}_\Omega} h_F \left\| \left[\!\left[\nu\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}} - p\boldsymbol{n} \right]\!\right] \right\|^2_{0,F} \right)^{1/2},$$

where $[\![\cdot]\!]_F$ denotes the vectorial jump operator on the edge $F \in \mathcal{E}_\Omega$. The term $\eta_{\gamma,T}$ corresponds to a quantification of the jumps of γ for the element T, which we want to reduce in order to obtain a better resolution. The expression η_T is the a-posteriori error estimator presented by Verfürth calculated in the element T (see section 8 in [16]).

At each stage of the algorithm, we partially solve the optimization problem until reaching a maximum number of iterations or a convergence criterion of the L-BFGS-B algorithm. Next, we quantify the error estimators, mark some elements and refine the marked elements following

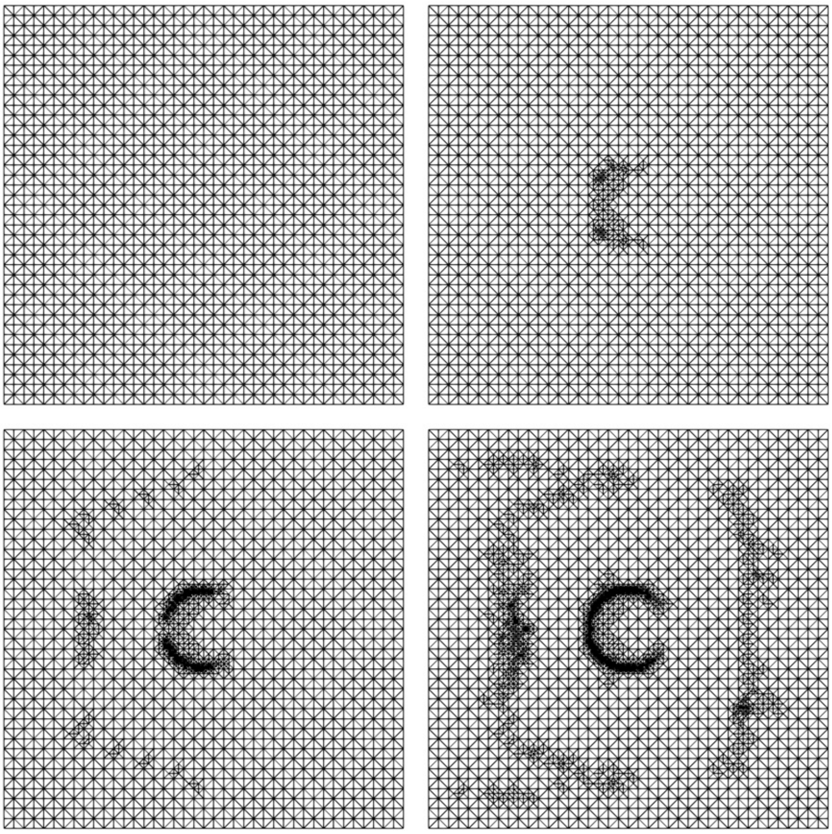

Figure 7. Plots of refined meshes on stages 1 (top left), 3 (top right), 7 (bottom left) and 13 (bottom right).

the algorithm. The next stage uses the Lagrange interpolation of the optimal control obtained in the last stage. Also we increment the maximum number of iterations for L-BFGS-B algorithm for the next stage because the descent directions of that algorithm are not compatible with the discrete spaces obtained after the adaptive refinement (algorithm 1).

The tolerance criterion for the L-BFGS-B algorithm were 2×10^{-5} for consecutive evaluations of functional J or approximations of the Riesz representant of ∇J, the Fréchet derivative of J, in norms $L^2(\Omega)$ or ℓ_2. We used $\gamma_0 = 0$ as a initial prediction for the L-BFGS-B algorithm. We choose $N = 60$ as the maximum number of iterations for the first stage, with increments of 30 iterations for the following stages. If we denote γ_k and $\boldsymbol{u}_k$ as the optimal control and their respective state on the k stage of the refinement algorithm, we define the errors $\gamma_{E,k} = \gamma_k - \gamma_R$ and $\boldsymbol{u}_{E,k} = \boldsymbol{u}_k - \boldsymbol{u}_R$, and γ^* as the optimal function obtained by the L-BFGS-B algorithm withe the optimal state $(\boldsymbol{u}^*, p^*)$ as in the previous test. Figures 6 and 7, and table 2 summarize the numerical results obtained.

We appreciate that the convergence of the numerical solution to the real solution is slow, similar to the previous test, which is benefited by the adaptive refinement strategy. The adaptive refinement strategy allows to recover smoothly the boundary of the set B, where $\gamma_R = 10\,000$. However, we can obtain numerical noise in the boundary of ω, drawn with magenta lines in figure 6. Indeed, we can see that the prediction of γ is not accurate outside ω due to the same

Table 2. Evolution of the adaptive refinement algorithm.

k	It	$J(\gamma_k)$	$\|\nabla J(\gamma_k)\|_{0,\Omega}$	$\|\boldsymbol{u}_{E,k}\|_{0,\omega} + \|\mathrm{curl}\ \boldsymbol{u}_{E,k}\|_{0,\omega}$	$\|\gamma_{E,k}\|_{0,\Omega_\varepsilon}$
0	0	7.3218×10^3	6.8677×10^{-1}	1.3714×10^2	3.4662×10^3
1	60	2.4691×10^2	1.0367×10^{-3}	2.5022×10^1	2.1659×10^3
3	270	8.4010×10^1	2.6439×10^{-4}	1.4178×10^1	1.8784×10^3
5	600	5.0553×10^1	8.4971×10^{-5}	1.0996×10^1	1.7416×10^3
7	1050	2.9497×10^1	4.3242×10^{-5}	8.4704×10^0	1.6257×10^3
9	1620	2.5874×10^1	3.6794×10^{-5}	7.9048×10^0	1.5348×10^3
11	2310	2.0731×10^1	2.4171×10^{-5}	7.0901×10^0	1.5229×10^3
13	2848	1.9714×10^1	1.9335×10^{-5}	6.9145×10^0	1.5304×10^3

explanations of the previous test: the measurement region, the chosen objective function J and the finite element approximation. Furthermore, the values of the numerical noise are sufficient to significantly modify the magnitude of $\boldsymbol{u}$ outside ω with respect to the reference $\boldsymbol{u}_R$, but that noise is slightly attenuated by the effect of the optimization solver and the adaptive refinement algorithm.

5. Conclusions

We have presented a new stability result for the inverse problem of recovering a smooth scalar permeability parameter for a steady Navier–Stokes equations with permeability modeled by Brinkman's law from local observations of the fluid velocity and vorticity in a fixed subdomain. Our main result is a logarithmic estimate obtained from H^1 and H^2 global Carleman inequalities for second-order elliptical equations. We followed similar extension technique used as the one used in [6] under an analogous non-degeneracy condition as the one introduced in [7]. The approach of eliminating the pressure and measuring only velocity $\boldsymbol{u}$ is useful not only for fluids, but also in some problems appearing in elastography (see [8]).

Our numerical test for recovering a smooth parameter shows a slow convergence of the optimization solver, which is directly related to our stability result. Likewise, the numerical test that recovers a discontinuous coefficient with an adaptive refinement strategy follows a similar behavior to the first test, which allows us to observe that we could relax the regularity hypotheses of our main theorem. Also, one of the problems was the numerical noise generated by the discrete scheme. An alternative is to consider the vorticity $\mathrm{curl}\,\boldsymbol{u}$ as a new unknown in the finite element system.

In [4], authors describe that an obstacle immersed in a fluid can be represented asymptotically by a discontinuous permeability coefficient. The adaptive refinement strategy is effective to recover discontinuous coefficients with greater precision, facilitating the detection of obstacles with better resolution. However, the use of error estimators may not be appropriate. Then, one of the future improvements for this work is to apply new techniques, for example the one explained in [9], as a new strategy of mesh adaptivity.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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