Operations Research Letters 50 (2022) 99–106

Contents lists available at ScienceDirect

Operations Research Letters

www.elsevier.com/locate/orl

Negative prices in network pricing games

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ARTICLE INFO

Article history:
Received 7 October 2020
Received in revised form 22 December 2021
Accepted 3 January 2022
Available online 10 January 2022

Keywords:
Stackelberg games
Negative prices
Bundling
Road tolling

ABSTRACT

In a Stackelberg network pricing game a leader sets prices for a given subset of edges so as to maximize profit, after which one or multiple followers choose a shortest path. Our main result shows that the profit when allowing for negative prices can be a factor $\Theta(\log(m \cdot \bar{k}))$ larger than the maximum profit with only positive prices, where m is the number of priceable edges and $\bar{k} \le 2^m$ the number of followers. In particular, this factor cannot be bounded for a single follower.

1. Introduction

Bundle pricing is a very common business strategy to increase the profit. In this paper we study how a very simple form of bundle pricing can increase the profit in Stackelberg network pricing games. These models are typically used for road tolling problems and were first introduced by [25].

In Stackelberg network pricing games a leader moves first by setting prices on edges he owns, after which each follower decides on a path of minimum cost between her source and sink. The objective of the leader is to maximize profit. It is therefore a natural and common assumption in the literature that prices are non-negative, see, e.g., [3,6,7,23,34]. However, [25] gave an example of a Stackelberg single follower shortest path pricing game in which the profit is maximized by using negative prices. We call this phenomenon the *negative price paradox*. The use of negative prices in that example can be seen as a bundle pricing strategy of the leader that guarantees that the follower uses multiple edges owned by the leader. The main question we want to answer is how much more profit the leader can earn by using such bundle pricing strategies. Since calculating optimal bundle prices is rather intractable as a price has to be determined for each of the exponentially many bundles of resources, we restrict our analysis to single item pricing. We model bundle pricing by allowing the leader to set negative prices.

For multiple followers, it might not be surprising that the profit can be arbitrarily larger with negative prices compared to restricting to positive prices. However, our main result shows that the same is true for a single follower. Our contribution is two-fold. From a practical perspective, we show that a very simple bundle pricing mechanism, allowing for negative single item prices, can achieve arbitrarily higher profits than single item pricing with only positive prices. From a theoretical perspective, we show that a seemingly innocent assumption, non-negative prices, might have a big impact on the outcome of the game.

1.1. Contribution

We start by studying Stackelberg network pricing games in which the followers choose a shortest path from source to sink. Our main goal is to quantify the loss in profit due to the assumption that prices are non-negative. For this purpose, we define the *Price of Positivity* (PoP), which is the ratio between the profit of the leader when he is allowed to use negative prices and when he is not. Theorem 3.3 proves that the PoP can be of order $\Theta(\log m \cdot \bar{k})$, where m is the number of priceable edges and $\bar{k} \le 2^m$ the number of followers. We prove that this bound is asymptotically tight by means of two different classes of instances. First, Theorem 3.5 shows that the price of positivity can be arbitrarily large even with a single follower. To prove this, we use the class of generalized Braess graphs [32]. Second, Theorem 3.8 shows that the price of positivity can be arbitrarily large in series-parallel graphs given that there are sufficiently many followers.

Then we turn to the question of which network topologies are immune to the negative price paradox. A network is *immune* to

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https://doi.org/10.1016/j.orl.2022.01.001
0167-6377/

the negative price paradox if for all instances within that network the negative price paradox cannot occur. We show that for a single follower series-parallel graphs are exactly the class of networks that are immune to the negative price paradox. So in a market in which goods are either perfect complements or perfect substitutes, this type of bundle pricing will not improve the leader's profit. As a side result, we conclude that Stackelberg network pricing games in series-parallel graphs with a single follower are polynomial-time solvable.

We lastly consider a different variant of Stackelberg network pricing games. We consider the setting in which followers, instead of an s,t-path, choose a basis of a given matroid, for instance, a spanning tree in a graph. We prove that in this setting the negative price paradox cannot occur.

1.2. Related literature

Stackelberg competition was first introduced by [35] and is now commonly used to describe leader-follower models. Stackelberg network pricing games gained attention due to [25], who used the game to model road tolling problems. They showed that the Stackelberg shortest path problem is NP-hard for a single follower and prices that have lower bounds. [31] proved the more general result that the problem is also NP-hard when prices are unrestricted. [23] and [6] showed that the problem is even APX-hard. For a more detailed survey on this problem, see [34]. Recently, also different combinatorial problems were studied in a Stackelberg setting. For example, minimum spanning trees [12], shortest path trees [3,11], packing problems [10], matroids [9] and knapsack problems [30,29] have been considered.

[2] and [7] considered single price strategies. They both show independently that this very simple pricing strategy provides a logarithmic approximation algorithm. [8] extended the analysis of this simple algorithm beyond the combinatorial setting to arbitrary continuous cost functions.

Using negative prices can be seen as a bundling strategy. The performance of selling optimal bundles is widely studied in the mechanism design literature. There, it is well-known that revenue maximization with more than one good is a difficult problem. For some results on the performance of selling optimal bundles, see, e.g., [26], and [21], and the references therein.

A more common application of pricing in road tolling problems is to restore inefficiency in models with congestion externalities. [4] showed that marginal tolls induce efficient flows when considering the model introduced by [37]. Optimal tolls also exist when users are heterogeneous with respect to the trade-off between time and money. See, e.g., [15,18,24,36]. With selfish leaders, efficiency can be attained with a monopolist [1,22] or with competition regulation [20,16].

A seemingly related paradox is Braess's paradox [5]. It describes the phenomenon in which the increase of resources, like building a new road in a network, may in fact lead to larger costs for the users. [27] derived a characterization that shows that for undirected single-commodity networks, series-parallel graphs are the largest class of graphs for which Braess's paradox does not occur. This result has been generalized by [14] and [13] to directed graphs and multi-commodity instances. Roughgarden [32] investigated how to improve the performance of a network when it is allowed to remove edges. [19] proved that the matroid property is the maximal property that guarantees the absence of the Braess's paradox.

2. Model

A Stackelberg network pricing game is given by a tuple $\mathcal{M} = (G, (c_e)_{e\in E}, E_p, K, (s^k, t^k, R^k)_{k\in K})$, where $G = (V, E)$ is a directed

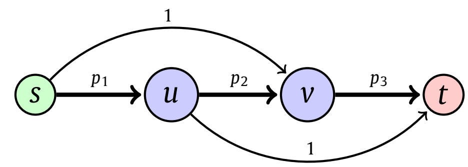

Fig. 1. The negative price paradox for $R = 3$.

multigraph, $c_e \in \mathbb{R}_+$ is the *fixed cost* of edge $e \in E$, $E_p \subseteq E$ is the set of *priceable edges*, $K = \{1, \dots, \bar{k}\}$ is the set of *followers*, $(s^k, t^k) \in V \times V$ with $s^k \neq t^k$ is the *source-sink pair* and $R^k \in \mathbb{R}_+$ is the *reservation value* of follower k for each $k \in K$. For each $k \in K$, let $\mathcal{P}^k$ denote the set of simple s^k, t^k-paths.

A Stackelberg network pricing game contains two types of players: one leader, and one or more followers. For each priceable edge $e \in E_p$, the leader specifies a price $p_e \in \mathbb{R}$. Let $p = (p_e)_{e\in E_p}$ denote a vector of prices. Given a vector of prices $p \in \mathbb{R}^{E_p}$, the total cost of a simple path $P \in \mathcal{P}^k$ for follower $k \in K$ is defined by

$$c_P = \sum_{e\in P} c_e + \sum_{e\in P\cap E_p} p_e,$$

and we define $P^k(p) = \{P \in \mathcal{P}^k \mid c_P \leq R^k \text{ and } c_P \leq c_{P'} \text{ for all } P' \in \mathcal{P}^k\}$. For each $p \in \mathbb{R}^{E_p}$, each follower chooses a simple path $P \in P^k(p)$, and if no such path exists, chooses $P = \emptyset$.

For each $p \in \mathbb{R}^{E_p}$, the profit of the leader is equal to

$$\pi(p) = \sum_{k\in K} \sum_{e\in P\cap E_p} p_e.$$

We assume that the leader wants to maximize his profit. To this end, we call a price vector $p \in \mathbb{R}^{E_p}$ *optimal* if for all $p' \in \mathbb{R}^{E_p}$, $\pi(p) \geq \pi(p')$. We denote an optimal strategy by p^*. We call a price vector $p \in \mathbb{R}_+^{E_p}$ *optimal for non-negative prices* if for all $p' \in \mathbb{R}_+^{E_p}$, $\pi(p) \geq \pi(p')$. We denote an optimal strategy for non-negative prices by p_+^*. For a given model $\mathcal{M}$, we define the *price of positivity* by

$$PoP(\mathcal{M}) = \frac{\pi(p^*)}{\pi(p_+^*)}.$$

We make the following two assumptions. First, we assume that when the followers face multiple optimal solutions, ties are broken in favor of the leader. Second, we assume that the graph is irredundant, i.e., each edge is contained in at least one s^k, t^k-path for some $k \in K$. Edges that are on no such path are not relevant for our problem and can be deleted.

We first give an example that illustrates the phenomenon we want to study. A similar example was given by [25].

Example 2.1. Consider a game with one follower that chooses a shortest path from s to t in the network of Fig. 1. For ease of notation, we omit the superscripts of the follower when we consider a single follower. The priceable edges are depicted as thicker arrows, and right above each edge is its cost or price. Whenever we do not write a fixed cost of a priceable edge in a figure, we assume the fixed cost to be zero. The reservation value is $R = 3$. Let P_1, P_2 and P_3 denote the paths defined by the edges that join the sequences of vertices (s, u, v, t), (s, u, t) and (s, v, t), respectively.

Suppose that $p_1, p_2, p_3 \geq 0$. If the leader wants to induce P_1 as a shortest path for the follower, then necessarily $p_1 + p_2 \leq 1$ and $p_2 + p_3 \leq 1$ and thus $\pi(p) = p_1 + p_2 + p_3 \leq p_1 + 2p_2 + p_3 \leq 2$. If the leader wants the follower to choose path P_2, then $p_1 + 1 \leq 3$ and thus $\pi(p) = p_1 \leq 2$. Similarly for path P_3. Combining these three statements implies $\pi(p_+^*) \leq 2$. Price vector $p_+^* = (p_1, p_2, p_3) = (1, 0, 1)$ yields $\pi(p_+^*) = 2$.

Now, if the leader sets prices $p^* = (3, -3, 3)$, then P_1 has a cost of 3, and P_2 and P_3 have a cost of 4, and thus the follower will choose path P_1. Hence $\pi(p^*) = 3$ and $PoP(\mathcal{M}) = 3/2$.

Let $\ell_v^k(p)$ denote the cost of a shortest simple s^k-v path of follower $k \in K$ for a given vector of prices $p \in \mathbb{R}^{E_p}$. Observe that for all $\mathcal{M}$ we have that

$$\pi(p^*) \le \sum_{k \in K} \min\{R^k, \ell_{t^k}^k(\infty)\} - \ell_{t^k}^k(0) \tag{1}$$

and the right-hand side represents the *surplus* that can be extracted from the followers. [7] proved the existence of a logarithmic approximation algorithm that in polynomial time calculates a non-negative *single price strategy*, i.e., a vector of prices that sets the same price on all priceable edges for Stackelberg network pricing games. This result is later used in our analysis, so for completeness we formally state it below. Let $H_n = \sum_{i=1}^n \frac{1}{i}$ denote the n-th harmonic number.

Theorem 2.2 *([7]). For all $\epsilon > 0$, there exists a non-negative single price strategy that yields a profit of at least*

$$\frac{\sum_{k \in K} \min\{R^k, \ell_{t^k}^k(\infty)\} - \ell_{t^k}^k(0)}{(1+\epsilon) \cdot H_{m\bar{k}}}.$$

Remark 2.3. [7] assumes that priceable edges do not have a fixed cost. An instance of our model can be transformed to an instance of their model by splitting the priceable edge into two edges: one priceable edge and one fixed cost edge. Such a transformation does not change the profit of the leader.

3. Optimal profit

3.1. Warmup

Our first result formalizes an observation from Example 2.1 for a single follower. If the leader owns a shortest path of the follower when $p_e = 0$ for all $e \in E_p$, then the leader can extract all surplus from the follower by using negative prices. In particular, these single follower problems are solvable in polynomial time as they can be solved by means of a linear program.

Proposition 3.1. *Let $\bar{k} = 1$ and $P_0 \in P(0)$. If $P_0 \subseteq E_p$, then $\pi(p^*) = \min\{R, \ell_t(\infty)\} - \ell_t(0)$.*

Proof. Assume that $P_0 \subseteq E_p$. We will show that the leader can extract all surplus from the follower while inducing path P_0 as a shortest path. Set $p_e = \infty$ for all $e \in E_P \setminus P_0$. The profit maximization problem of the leader under the constraint that P_0 is a shortest path is

$$\begin{array}{lll} \max\limits_{(p_e)_{e\in E_p},(\ell_v(p))_{v\in V}} \sum_{e\in P_0} p_e & & \\ \ell_v(p) - \ell_u(p) - p_e & = c_e & \forall e=(u,v) \in P_0 \\ \ell_v(p) - \ell_u(p) & \le c_e & \forall e=(u,v) \in E \setminus E_p \\ \ell_s(p) & = 0 & \\ \ell_t(p) & \le R. & \end{array}$$

The above constraints guarantee that P_0 is a shortest path for any feasible price vector p. Notice the problem is feasible and bounded. In fact, by definition of P_0, a feasible solution is to set $p_e = 0$ for all $e \in P_0$, and the optimal value is at most R.

Define $E' = P_0 \cup E \setminus E_P$, $\delta^-(v) = \{(u,v) \in E'$ for some $u \in V\}$ and $\delta^+(v) = \{(v,w) \in E'$ for some $w \in V\}$. The dual of the above linear program is

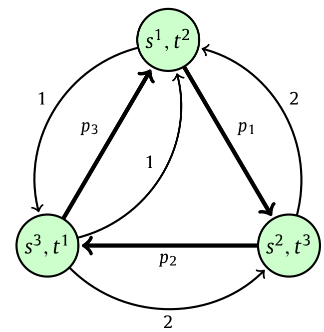

Fig. 2. A triangle network.

$$\begin{array}{lll} \min\limits_{(y_e)_{e\in E'},y_s,y_t} \sum_{e\in E'} y_e \cdot c_e + y_t \cdot R & & \\ \sum_{e\in\delta^-(v)} y_e - \sum_{e\in\delta^+(v)} y_e & = 0 & \forall v \in V\setminus\{s,t\} \\ \sum_{e\in\delta^-(s)} y_e - y_s - \sum_{e\in\delta^+(s)} y_e & = 0 & \\ \sum_{e\in\delta^-(t)} y_e + y_t - \sum_{e\in\delta^+(t)} y_e & = 0 & \\ y_e & = -1 & \forall e \in P_0 \\ y_e & \ge 0 & \forall e \in E \setminus E_p \\ y_t & \ge 0, & \end{array}$$

which is equivalent to

$$\begin{array}{ll} \min\limits_{(y_e)_{e\in E\setminus E_p},y_s,y_t} \sum_{e\in E\setminus E_p} y_e \cdot c_e + y_t \cdot R - \sum_{e\in P_0} c_e & \\ \forall v \in V\setminus\{s,t\}: \sum_{e\in\delta^-(v)\cap(E\setminus E_p)} y_e - \sum_{e\in\delta^+(v)\cap(E\setminus E_p)} y_e & = 0 \\ \sum_{e\in\delta^-(s)\cap(E\setminus E_p)} y_e - y_s - \sum_{e\in\delta^+(s)\cap(E\setminus E_p)} y_e & = -1 \\ \sum_{e\in\delta^-(t)\cap(E\setminus E_p)} y_e + y_t - \sum_{e\in\delta^+(t)\cap(E\setminus E_p)} y_e & = 1 \\ \forall e \in E\setminus E_p : y_e & \ge 0 \\ y_t & \ge 0. \end{array}$$

If we remove the additive term of $-\sum_{e\in P_0} c_e = -\ell_t(0)$, this linear program corresponds to the problem of finding the shortest s,t-path in the network that results from deleting all edges in E_p and adding an edge from s to t with fixed cost R. Therefore, by definition of $\ell_t(\infty)$, we conclude that the optimal value is exactly $\min\{R, \ell_t(\infty)\} - \ell_t(0)$. □

The result in Proposition 3.1 only applies to a single follower, as the following example shows.

Example 3.2. Consider the network in Fig. 2, where $K = \{1,2,3\}$. Let $R^1 = 1$ and $R^2 = R^3 = 2$. Observe that the leader owns a shortest path from source to sink when $p_e = 0$ for all $e \in E$ for each follower $k \in K$.

The total surplus of all followers equals $\sum_{k\in K} R^k - \ell_t^k(0) = 1 + 2 + 2 = 5$. However, the following set is empty and thus the leader cannot extract all surplus:

$$\{(p_1,p_2,p_3) \in \mathbb{R}^3 \mid p_1 + p_2 = 1,\ p_2 + p_3 = 2,\ p_1 + p_3 = 2,\ p_3 \le 1\}.$$

3.2. Main results

We now prove the main theorem. We characterize the loss in profit due to assuming that prices are non-negative. Let $\mathcal{F}(m,\bar{k})$ be the family of Stackelberg network pricing games with $|E_p| = m$ and $|K| = \bar{k} \le 2^m$.

Theorem 3.3.

$$\sup_{\mathcal{M}\in\mathcal{F}(m,\bar{k})} PoP(\mathcal{M}) \in \Theta\left(\log(m \cdot \bar{k})\right).$$

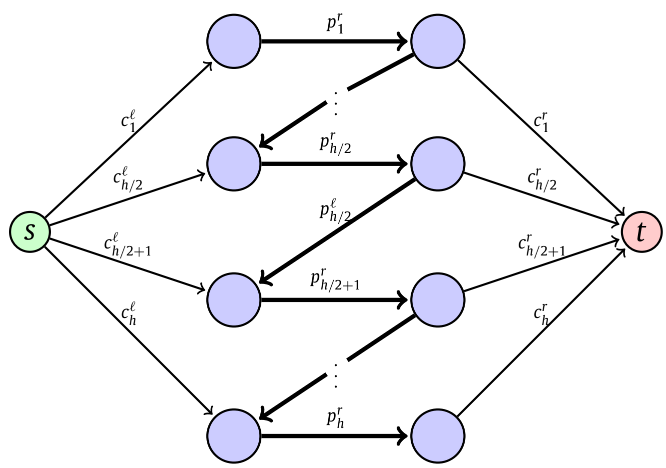

Fig. 3. The h-th network in the proof of Theorem 3.4.

We separate the result into three distinct theorems: one for the upper bound, one for a lower bound with $\bar{k}=1$ and another for a lower bound with $\bar{k}=2^m-1$.

Theorem 3.4. *Let $\mathcal{M}\in\mathcal{F}(m,\bar{k})$. Then*

$$PoP(\mathcal{M})\le H_{m\bar{k}}.$$

Proof. Let $p^*\in\mathbb{R}^{E_p}$, $p^*_+\in\mathbb{R}^{E_p}_+$ denote an optimal and an optimal positive strategy of the leader, respectively. Given that Theorem 2.2 is true for every $\epsilon>0$, we have that

$$\pi(p^*_+)\ge\frac{\sum_{k\in K}\min\{R^k,\ell^k_{t^k}(\infty)\}-\ell^k_{t^k}(0)}{H_{m\bar{k}}}. \tag{2}$$

Combining (1) and (2) yields the result. $\square$

The next result shows that the price of positivity can be arbitrarily large, even if there is only a single follower.

Theorem 3.5. *For all $n\in\mathbb{N}$, there exists a model $\mathcal{M}$ with $m=2^{2n-1}-1$ and $\bar{k}=1$ such that*

$$PoP(\mathcal{M})\ge n.$$

Proof. Assume that $\bar{k}=1$ with a reservation value of R. Consider the network of Fig. 3, where $h\ge1$. Besides s and t, there are h left vertices $v^\ell_1,v^\ell_2,\dots,v^\ell_h$ and h right vertices $v^r_1,v^r_2,\dots,v^r_h$. There is a fixed cost edge from s to each left vertex v^ℓ_i, with cost c^ℓ_i, for each $i\in\{1,\dots,h\}$. There is a fixed cost edge from each right vertex v^r_i to t, with cost c^r_i, for each $i\in\{1,\dots,h\}$. There is a priceable edge from v^ℓ_i to v^r_i for each $i\in\{1,\dots,h\}$ and a priceable edge from v^r_i to v^ℓ_{i+1} for each $i\in\{1,\dots,h-1\}$, each having a fixed cost of 0. Denote their prices by p^r_i and p^ℓ_i, respectively. Assume that $c^\ell_1=c^r_h=0$, and $c^\ell_i,c^r_i\ge0$ for all $i\in\{1,\dots,h\}$.

Note first that every path in the graph is completely defined by an edge leaving s and an edge entering t. For $1\le i\le j\le h$, we define P_{ij} as the path starting with edge (s,v^ℓ_i) and ending with edge (v^r_j,t). Let $\pi^*_+(P_{ij})$ denote the maximum profit when we impose that path P_{ij} is chosen by the follower and prices are all non-negative. Observe that the leader can always delete a priceable edge by setting its price to be ∞. Notice that path P_{ij} might never be chosen: in this case, define $\pi^*_+(P_{ij})=0$.

In order to bound the optimal profit with positive prices, we need the following two lemmas. We bound $\pi^*_+(P_{ij})$ first for the case $i=j$ and then for the case $i<j$.

Lemma 3.6. *Let $1\le i\le h$. If $c^\ell_i+c^r_i>R$, then $\pi^*_+(P_{ii})=0$. If $c^\ell_i+c^r_i\le R$, then*

$$\pi^*_+(P_{ii})\le R-c^\ell_i-c^r_i.$$

Proof. The fixed cost of path P_{ii} equals $c^\ell_i+c^r_i$. Thus, if $c^\ell_i+c^r_i>R$, then P_{ii} is never chosen by the follower when prices are non-negative and therefore $\pi^*_+(P_{ii})=0$. Otherwise, the cost of P_{ii} for the follower is $p^r_i+c^\ell_i+c^r_i$, so if it is less than the reservation value R, we must have that $\pi^*_+(P_{ii})=p^r_i\le R-c^\ell_i-c^r_i$. $\square$

Lemma 3.7. *Let $1\le i\le j'<i'\le j\le h$. If $c^\ell_i>c^\ell_{i'}$, then $\pi^*_+(P_{ij})=0$. If $c^r_j>c^r_{j'}$, then $\pi^*_+(P_{ij})=0$. If $c^\ell_i\le c^\ell_{i'}$ and $c^r_j\le c^r_{j'}$, then*

$$\pi^*_+(P_{ij})\le c^\ell_{i'}-c^\ell_i+c^r_{j'}-c^r_j.$$

Proof. If $c^\ell_i>c^\ell_{i'}$, then $P_{i'j}$ has a lower cost than P_{ij} and thus P_{ij} is never a shortest path. If $c^r_j>c^r_{j'}$, then $P_{ij'}$ has a lower cost than P_{ij} and thus P_{ij} is never a shortest path.

If $c^\ell_i\le c^\ell_{i'}$ and $c^r_j\le c^r_{j'}$, then we have to make sure that the total cost of P_{ij} is at most the cost of $P_{i'j}$ and $P_{ij'}$. In other words, we have to guarantee that the cost of reaching $v^\ell_{i'}$ following P_{ij} is lower than the edge $(s,v^\ell_{i'})$, and that the cost of the path from $v^r_{j'}$ to t following P_{ij} is lower than the edge $(v^r_{j'},t)$. This means that $c^\ell_i+\sum_{m=i}^{i'-1}p^r_m+\sum_{m=i}^{i'-1}p^\ell_m\le c^\ell_{i'}$ and $c^r_j+\sum_{m=j'+1}^{j}p^r_m+\sum_{m=j'}^{j-1}p^\ell_m\le c^r_{j'}$, and thus $\pi^*_+(P_{ij})=\sum_{m=i}^{j}p^r_m+\sum_{m=i}^{j-1}p^\ell_m\le c^\ell_{i'}-c^\ell_i+c^r_{j'}-c^r_j$. $\square$

Let $n\in\mathbb{N}$, with $n\ge2$, and $h=4^{n-1}$. We define fixed costs for the h-th network such that the maximum profit using non-negative prices is 2 while the maximum profit when allowing negative prices is $2n$. Let $R=2n$. We will set fixed costs that satisfy

$$c^\ell_i=2n-2-c^r_i=c^r_{h/2+1-i}=2n-2-c^\ell_{h/2+1-i}, \tag{3}$$

for all $1\le i\le h/2$. Thus, we only need to specify $c^\ell_{h/2+i}$ for $1\le i\le h/2$.

We start by letting $c^\ell_{h/2+1}=1$. We define $2n-3$ sets of edges S_i recursively, where each set S_i with $i=1,\dots,2n-3$ consists of 2^{i-1} edges. For each set S_i with $i=1,\dots,2n-3$, define the fixed costs of the next 2^{i-1} edges as the fixed costs of all previously defined 2^{i-1} edges plus 1. More precisely, for each S_i with $i=1,\dots,2n-3$, we take $c^\ell_{h/2+2^{i-1}+j}=c^\ell_{h/2+j}+1$ for all $j=1,\dots,2^{i-1}$. In other words, we define the following sequence:

$$(1);\,(2);\,(2,3);\,(2,3,3,4);\,(2,3,3,4,3,4,4,5);\dots;$$
$$(2,3,\dots,2n-2).$$

Now, we will prove that given these fixed costs, $\pi^*_+(P_{ij})\le2$ for all $1\le i\le j\le h$. Firstly, by condition (3), $c^\ell_i+c^r_i=2n-2$ for all $1\le i\le h$, and thus by Lemma 3.6, $\pi^*_+(P_{ii})\le2$.

Secondly, for each path P_{ij} with $i\le h/2$ and $j\ge h/2+1$, Lemma 3.7 with $i'=h/2+1$ and $j'=h/2$ implies $\pi^*_+(P_{ij})\le 2-c^\ell_i-c^r_j\le2$. Then by symmetry of the constructed graph, we can restrict ourselves to paths P_{ij} with $i\ge h/2+1$.

Thirdly, consider a path P_{ij} with $i\ge h/2+1$ and suppose that

$$\frac{h}{2}+1\le i\le\frac{h}{2}+2^{i''-1}\text{ and }\frac{h}{2}+2^{i''-1}+1\le j\le\frac{h}{2}+2^{i''} \tag{4}$$

for some $1\le i''\le2n-3$. Necessarily $c^\ell_i\ge1$ and $c^r_j=2n-2-c^\ell_j\ge2n-2-(i''+1)$ from the definition of the costs. If we take $i'=$

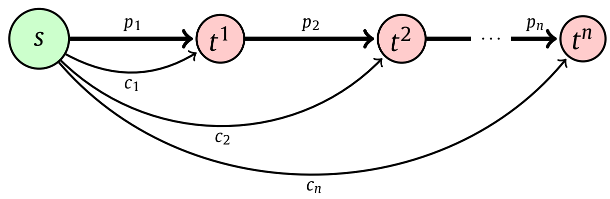

Fig. 4. The n-th path graph.

$h/2+2^{i''-1}+1$ and $j'=h/2+2^{i''-1}$, then $c^\ell_{i'}=2$ and $c^r_{j'}=2n-2-i''$. So Lemma 3.7 implies that $\pi^*_+(P_{ij}) \le c^\ell_{i'} - c^\ell_i + c^r_{j'} - c^r_j \le 2$. If condition (4) does not hold, then $h/2+2^{i''-1}+1 \le i \le j \le h/2+2^{i''}$, for some $1 \le i'' \le 2n-3$. Therefore, by Lemma 3.7, for any $i \le j' < i' \le j$, we have that $\pi^*_+(P_{ij}) \le c^\ell_{i'} - c^\ell_i + c^r_{j'} - c^r_j$. But from the definition of the costs, $c^\ell_{i'} - c^\ell_i + c^r_{j'} - c^r_j = c^\ell_{i'-2^{i''-1}} - c^\ell_{i-2^{i''-1}} + c^r_{j'-2^{i''-1}} - c^r_{j-2^{i''-1}}$. So any bound that we can derive by applying Lemma 3.7 for $P_{i-2^{i''-1},j-2^{i''-1}}$, also holds for P_{ij}. If $i<j$ we can iterate this procedure until condition (4) holds, so we conclude that $\pi^*_+(P_{ij}) \le 2$.

Combining the above three steps yields $\pi^*_+(P_{ij}) \le 2$ for all $1 \le i \le j \le h$. Notice that with the prices $p^r_i = R$ for $i=1,\dots,h$ and $p^\ell_i = -R$ for $i=1,\dots,h-1$, every s,t-path costs at least R and path P_{1h} yields a profit of R. Thus, $PoP(\mathcal{M}) = \frac{2n}{2} = n$ for each $n \in \mathbb{N}$. $\square$

The last result of this section proves that the price of positivity can be arbitrarily large for series-parallel networks, as long as there are sufficiently many followers.

Theorem 3.8. *For all $n \in \mathbb{N}$, there exists a model $\mathcal{M}$ with $m=n$ and $\bar{k} = 2^n-1$ followers such that*

$$PoP(\mathcal{M}) \ge n/2.$$

Proof. Consider the network of Fig. 4, where $n \ge 1$. Let $c_i = 2^{n-i+1}$ for $i=1,\dots,n$. For each $i=1,\dots,n$, there are 2^{i-1} followers with source-sink pair (s,t^i) and reservation value c_i.

In order to bound the optimal profit with positive prices, we claim that if a follower with source-sink pair (s,t^i) chooses the path of priceable edges, then all source-sink pairs (s,t^j) with $j=1,\dots,i$ also choose the path of priceable edges. This follows because the reservation value of follower $j<i$ is larger than i, but the cost of the path of priceable edges is at most as large. So in order to maximize profit, the leader can restrict himself to choosing $p_1 \in \{c_1,\dots,c_n\}$ and $p_i=0$ for all $i=2,\dots,n$. Setting $p_1=c_i$ yields a profit of $2^{n-i+1}\sum_{j=1}^{i} 2^{j-1} = 2^{n+1}(1-2^{-i}) \le 2^{n+1}$ for all $i=2,\dots,n$.

The price vector $p^*_i = c_i - c_{i-1}$, where $c_0=0$, for all $i=1,\dots,n$ yields a profit of $\sum_{i=1}^{n} 2^n = n\cdot 2^n$. Hence $PoP(\mathcal{M}) \ge n/2$. $\square$

4. Immune structures to the negative price paradox

We divide the following section into two parts. In the first part we restrict ourselves to instances with a single follower and characterize the class of topologies that are immune to the negative price paradox, i.e., for which the price of positivity is 1. In the second part we consider a different variant of the model in which followers choose bases of a matroid, instead of paths in a network. We prove that in this variant the price of positivity is always 1.

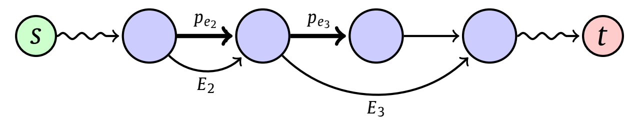

Fig. 5. A series-parallel graph is immune to the negative price paradox.

4.1. Shortest path follower

In this section we assume $K=\{1\}$. Let G_{st} denote a graph $G=(V,E)$ with a fixed source-sink pair (s,t). We say that a graph G_{st} is *immune to the negative price paradox* if for all models $\mathcal{M} = (G_{st},(c_e)_{e\in E},E_p,K=\{1\},(s,t,R))$, $PoP(\mathcal{M})=1$. A directed graph is *series-parallel* with source s and sink t, if it either consists of a single edge (s,t), or is obtained from two series-parallel graphs with source-sink pairs (s_1,t_1) and (s_2,t_2) composed either in series or in parallel. In both types of compositions we take the disjoint union of the sets of edges, but merge the source-sink pairs in different ways. In a *series composition*, t_1 is identified with s_2, s_1 becomes s and t_2 becomes t. In a *parallel composition*, s_1 is identified with s_2 and becomes s, and t_1 is identified with t_2 and becomes t.

The main result in this section shows that the leader can extract all surplus from the follower in series-parallel graphs by using non-negative prices, where the source-sink pair of the graph is the same as the source-sink pair of the follower.

Theorem 4.1. *Let $|K|=1$. If G is series-parallel, then $\pi(p^*_+) = \min\{R, \ell_t(\infty)\} - \ell_t(0)$.*

Proof. Let us define some notation first. An open ear is a directed simple path and we describe an open ear either by the set of edges it is composed of, or by the sequence of vertices it visits. The start and end vertex of an open ear are the first and last vertex it visits, respectively. We refer to all other vertices it visits as its internal vertices.

An *open ear decomposition* of $G=(V,E)$ is a partition of E into open ears $\{E_1,\dots,E_{\bar{h}}\}$ such that for all $2 \le i \le \bar{h}$, the start and end vertex of E_i are visited by some other open ears E_j and $E_{j'}$, with $j \le j' < i$, and no internal vertex of E_i is visited by an open ear $E_{j''}$ with $j'' < i$. We say an open ear decomposition $\{E_1,\dots,E_{\bar{h}}\}$ is *nested* if (i) for all $2 \le i \le \bar{h}$, both the start and end vertex of E_i are visited by the same open ear E_j with $j<i$; and (ii) for all $j<i<i'$, if the start and end vertices of both E_i and $E_{i'}$ are visited by E_j, then their *nest intervals* in E_j are either disjoint or one a subset of the other, where the nest interval of E_i in E_j is the subset of edges of E_j between the start and end vertices of E_i. Notice that an open ear decomposition of a series-parallel network can be constructed starting with an arbitrary s,t-path and iteratively adding open ears until all edges are covered. Eppstein [17] proved that every open ear decomposition of a series-parallel graph is nested.

In this proof we can assume w.l.o.g. that $\ell_t(\infty) \le R$. If this was not the case, we could add a fixed-price edge from s to t with a fixed cost of R. With this addition the graph would remain series-parallel and since there is only one follower, $\pi(p^*_+)$ and $\pi(p^*)$ would remain unchanged.

Let $P_0 \in P(0)$. Denote by p' the vector of non-negative prices that yields the maximum profit under the restriction that $P_0 \in P(p')$. This maximization problem is well defined and has at least one solution (all prices equal to 0). We will prove that in fact $\pi(p') = \ell_t(\infty) - \ell_t(0)$.

Assume by contradiction that $\pi(p') < \ell_t(\infty) - \ell_t(0)$. We prove that there exists a path $\hat{P} \subseteq E \setminus E_p$ that costs strictly less than $\ell_t(\infty)$, which contradicts the definition of $\ell_t(\infty)$. We will construct an open ear decomposition of G, starting with $E_1 = P_0$. First, create the vector p'' starting with p' and setting the price of all priceable edges outside P_0 to ∞. Clearly $\pi(p'') = \pi(p')$. Second, take any $e_2 \in P_0 \cap E_p$. See Fig. 5. If no such e_2 exists, $\ell_t(\infty) = \ell_t(0)$ and thus $\pi(p') < 0$ which is a contradiction as the leader can always guarantee a profit of $\pi(0) = 0$. There must exist an open ear that we denote $E_2 \subseteq E \setminus E_p$ with start and end vertex in $E_1 = P_0$ such that e_2 is in its nest interval and the cost of E_2 equals the cost of its nest interval under the pricing p''. If such an open ear did not exist, the leader could increase the price of e_2 without changing the shortest path of the follower, increasing his profit.

Assume inductively that we have constructed a sequence of disjoint open ears $\{E_1, E_2, \dots, E_i\}$, with $i \geq 2$, that forms an open ear decomposition of $\cup_{j=1}^{i} E_j$ that is nested. To be more precise, assume for each $j > j' \geq 2$ that the nest intervals of E_j and $E_{j'}$ are contained in E_1 and are either disjoint or one a subset of the other. For each $2 \leq j \leq i$, denote by $\hat{E}_j$ the nest interval of E_j in E_1. Assume there exists $e_{i+1} \in P_0 \cap E_p \setminus (\cup_{j=2}^{i} \hat{E}_j)$. Analogous to the case of e_2, there must be an open ear E_{i+1} with its start and end vertices in P_0, its nest interval $\hat{E}_{i+1}$ containing e_{i+1}, and the cost of E_{i+1} equal to the cost of $\hat{E}_{i+1}$ under the pricing p''. Necessarily the internal vertices of E_{i+1} are not visited by previous open ears, as otherwise, we obtain an open ear decomposition that is not nested, which would contradict the assumption that G is series-parallel. At some point, we have covered all priceable edges of P_0, and at this point e_{i+1} does not exist. Now, since the obtained decomposition is nested, we can construct a path $\hat{P}$ that avoids all priceable edges in P_0, using open ears with disjoint nest intervals, that has the same cost as P_0 under p''. We conclude noting that the cost of P_0 under p'' is at most $\pi(p') + \ell_t(0)$ which by our assumption is strictly less than $\ell_t(\infty)$, which is a contradiction. $\square$

Corollary 4.2. *Stackelberg network pricing games in series-parallel graphs with a single follower are polynomial-time solvable.*

Proof. Theorem 4.1 shows that the leader can extract all surplus from the follower. In order to do so, the follower must choose a path $P_0 \in P(0)$. For a given choice of path of the follower, the problem of the leader can be written as a linear program (see the proof of Proposition 3.1). In fact, [25] even gives a combinatorial algorithm to solve this problem in polynomial time. $\square$

Theorem 4.3. *Let $|K| = 1$. A graph G_{st} is immune to the negative price paradox if and only if G_{st} is a series-parallel graph.*

Proof. First, assume that G_{st} is a series-parallel graph. It follows from (1) and Theorem 4.1 that G_{st} is immune to the negative price paradox.

Second, assume that G_{st} is not a series-parallel graph. We show that there is a model $\mathcal{M} = (G_{st}, (c_e)_{e\in E}, E_p, R)$ such that $PoP(\mathcal{M}) > 1$.

For a path P, denote by $V(P)$ the set of vertices it visits. We call a subgraph G' of G an s,t-paradox if $G' = P_1 \cup P_2 \cup P_3$ is the union of three paths P_1, P_2, P_3 with the following properties:

(i) P_1 is an s,t-path going through distinct vertices a, u, v, b such that $s \preceq_{P_1} a \prec_{P_1} u \prec_{P_1} v \prec_{P_1} b \preceq_{P_1} t$, where $(\prec_{P_1})$ denotes the order in which P_1 visits the vertices.
(ii) P_2 is an $a - v$ path with $V(P_2) \cap V(P_1) = \{a, v\}$.
(iii) P_3 is a $u - b$ path with $V(P_3) \cap V(P_1) = \{u, b\}$ and $V(P_3) \cap V(P_2) = \emptyset$.

[14] proved that if G_{st} is not series-parallel, then G_{st} has a subgraph that is an s,t-paradox. Let $G'_{st} = (V', E') = P_1 \cup P_2 \cup P_3$ be an s,t-paradox contained in G_{st}. Let e_1, e_2 be the two outgoing edges from a with $e_1 \in P_2$ and $e_2 \in P_1$, let e_3 be an outgoing edge from u with $e_3 \in P_1$, and let e_4, e_5 be the two incoming edges to b with $e_4 \in P_1$ and $e_5 \in P_3$. Define $\mathcal{M}$ as follows: $c_e = 1$ if $e \in \{e_1, e_5\}$, $c_e = 0$ if $e \in E' \setminus \{e_1, e_5\}$ and $c_e = \infty$ if $e \in E \setminus E'$, $E_p = \{e_2, e_3, e_4\}$, and $R = 3$. See Fig. 6 for an illustration.

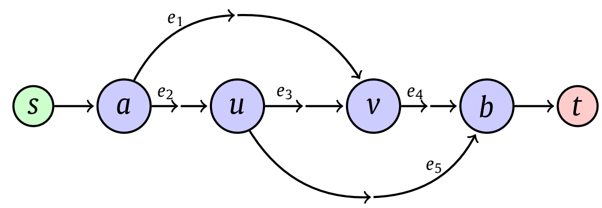

Fig. 6. An s,t-paradox.

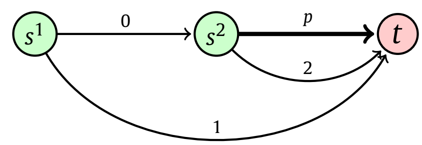

Fig. 7. A series-parallel graph.

Suppose that $p \in \mathbb{R}_{+}^{E_p}$. No matter which path is selected by the follower, $\pi(p) \leq 2$. Now, suppose that $p^* = (p_{e_2}, p_{e_3}, p_{e_4}) = (3, -3, 3)$, then $\pi(p^*) = 3$. Hence $PoP(\mathcal{M}) = \frac{3}{2} > 1$. $\square$

We stress that there is no analogous version of Theorem 4.1 for multiple followers. In the following example we present an instance with two followers in a series-parallel graph where it is not possible to extract all surplus from both followers simultaneously.

Example 4.4. Consider the network of Fig. 7. Each follower $k = 1, 2$ is defined by (s^k, t), and $R^1 = 1$ and $R^2 = 2$.

The profit $\pi(p)$ is $2p$ if $p \in [0, 1]$ and p if $p \in (1, 2]$, whereas the sum of the surplus of the followers is 3.

4.2. Matroid followers

A more general description of the model we have been looking at is the following. A Stackelberg pricing game is a tuple $\mathcal{M} = \big(E, (c_e)_{e\in E}, E_p, K, (\mathcal{S}^k)_{k\in K}, (R^k)_{k\in K}\big)$, where E is the set of resources, $c_e \in \mathbb{R}$ is a fixed cost for each $e \in E$, $E_p \subseteq E$ is the set of priceable resources, $K = \{1, \dots, \bar{k}\}$ is the set of followers, $\mathcal{S}^k \subseteq 2^E$ is the set of strategies and $R^k \in \mathbb{R}_+$ is the reservation value for each $k \in K$.

Given a vector of prices $p \in \mathbb{R}^{E_p}$, the total cost of a set $S \in \mathcal{S}^k$ for follower $k \in K$ is defined by

$$c_S = \sum_{e\in S} c_e + \sum_{e\in S\cap E_p} p_e.$$

For each p, each follower chooses a set $S^k(p) \in \mathcal{S}^k$ with $c_{S^k(p)} \leq R^k$ so as to minimize total costs, and if no such set exists, chooses $S^k(p) = \emptyset$.

So far we have studied the case where E is the set of edges in a network and $\mathcal{S}^k$ is the set of s^k, t^k-paths for each $k \in K$. Now, we assume that set system $(E, \mathcal{S}^k)$ is a *matroid* for each $k \in K$. A *matroid* is a tuple $M = (E, \mathcal{I})$, where E is a finite set, called the ground set, and $\mathcal{I} \subseteq 2^E$ is a non-empty family of subsets of E, called independent sets, such that: (1) $\emptyset \in \mathcal{I}$, (2) if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$, and (3) if $X, Y \in \mathcal{I}$ with $|X| > |Y|$, then there exists an $e \in X \setminus Y$ such that $Y \cup \{e\} \in \mathcal{I}$ (this property is called the augmentation property). The inclusion-wise maximal independent sets of $\mathcal{I}$ are called bases of matroid M. See [28,33,38] for more information on matroids.

We now consider the case in which instead of a path, each follower k chooses a minimum cost basis of a given matroid $M^k = (E, \mathcal{I}^k)$. This is a generalization of the setting considered in [12], in which followers choose spanning trees of a given graph. We prove that in this case there is no need for negative prices.

Theorem 4.5. *Let $\mathcal{M}$ be a Stackelberg game where $\mathcal{S}^k$ is the base set of a matroid M^k for each follower $k \in K$. Then $PoP(\mathcal{M}) = 1$.*

We first prove a lemma that is standard for matroids and then proceed to prove Theorem 4.5. These proofs strongly rely on the fact that matroids are exactly those structures for which the greedy algorithm finds the optimal solution. Greedy algorithms are very simple algorithms in which in each step a local optimal choice is selected. This means that we can assume that for finding a minimum cost base, each follower considers sequentially the next cheapest resource. If adding this resource to the set is feasible, then the resource is selected. If not, the resource is not selected.

Lemma 4.6. *Let $M = (E, \mathcal{I})$ be a matroid with weights $w : E \to \mathbb{R}$, and A a minimum weight base. If w' is obtained by increasing the weight of an element $e \in E$, and if A is not optimal under w', then there is an element $f \in E$ such that $A - e + f$ is an optimal base under w'.*

Proof. Assume A is not optimal under w', and that we run in parallel the greedy algorithm on both instances. Call G1 the greedy algorithm running with weights w and G2 the greedy algorithm running with weight w'. When looking at the ordered lists, it is clear that both lists are equal up to element e. So the partial solutions up to element e are the same. Now, when G1 finds e, it adds it to the solution, but G2 sees the next element. Since A is not optimal for w', G2 must find an element $f \notin A$ and add it to the partial solution before it reaches e in the list induced by w'. Besides e and f, the two algorithms have added the same resources so far. Therefore, the two partial solutions have the same size, and then, when one of the two algorithms can add a new item, the other can as well, because of the augmentation property, whereas if a new item cannot be added for one algorithm, it also cannot be added for the other algorithm, again by the augmentation property. With this observation we can conclude that, at the end, the only difference between the two solutions is elements e and f. □

Proof of Theorem 4.5. Assume by contradiction that in p^*, the optimal strategy of the leader, there is a negative price $p^*_e < 0$ for some resource $e \in E_p$. For each $k \in K$, let $S^k(p^*)$ be the optimal strategy for agent k, given p^*, obtained using the greedy algorithm. Let now p' be equal to p^* in every resource but e, in which $p'_e = 0$.

If $S^k(p^*)$ is not optimal under p', by Lemma 4.6 there is an element $f \in E$ such that $S^k(p^*) - e + f$ is optimal for the follower k. If $f \notin E_p$, then the profit for the leader increases because he was losing value in e, and if $f \in E_p$, the optimality of $S^k(p^*)$ implies that $p^*_e \le p^*_f$ and then the leader also increases his profit. We conclude that there is a positive solution p^*_+ such that $\pi(p^*) \le \pi(p^*_+)$. □

5. Open problems

Several natural questions remain open. One important open question is whether the lower bound on the price of positivity obtained in Theorem 3.8 is asymptotically tight in the number of priceable edges. From [7], we know that a quadratic upper bound applies, whereas the lower bound only grows linearly. A second open question is whether there is a simple characterization of a maximal class of structures that are immune to the negative price paradox when having a single follower. Note that such a class of structures should contain paths in series-parallel graphs and bases of matroids.

A potential generalization is to allow for congestion externalities. We have an example that shows that the price of positivity can already be larger than one in non-atomic congestion games on series-parallel graphs. This direction would also open up research towards models with multiple leaders.

Acknowledgements

We would like to thank the participants of the AGCO-seminar, José Correa, Britta Peis and Oliver Schaudt for helpful discussions. This work was partially supported by the Millennium Nucleus Information and Coordination in Networks, ICM/FIC RC130003, by the Millennium Institute for Research in Market Imperfections and Public Policy, MIPP, IS130002, and by the Chilean Science Council under CONICYT-PFCHA/Doctorado Nacional/2018-21180347.

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