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CONTRIBUCIONES AL ESTUDIO DEL PROBLEMA DE KATZNELSON PARA ACCIONES DE GRUPOS GENERALES.

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RESUMEN DE TESIS PARA OPTAR AL GRADO
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Y MEMORIA PARA OPTAR AL TÍTULO DE
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CONTRIBUCIONES AL ESTUDIO DEL PROBLEMA DE KATZNELSON PARA ACCIONES DE GRUPOS GENERALES.

Propiedades de recurrencia en sistemas dinámicos y los conjuntos de enteros asociados son conceptos clásicos en dinámica topológica. En esta área, la pregunta de Katznelson surge como una pregunta abierta de larga data sobre recurrencia, con fuertes vínculos históricos y matemáticos con problemas abiertos en combinatoria y análisis armónico. Más precisamente, Katznelson pregunta si los conjuntos de Bohr recurrencia (recurrencia para rotaciones en toros de dimensión finita) son conjuntos de recurrencia para $\mathbb{Z}$-acciones minimales.

El objetivo de este trabajo es estudiar la pregunta de Katznelson para acciones de grupos generales. Los resultados conocidos en este problema son escasos y pareciera no haber consenso entre los expertos sobre una respuesta esperada. Sin embargo, este problema ha sido intensamente estudiado en la clase de los $\mathbb{Z}$-nilsistemas, una clase de sistemas de origen algebraico que generaliza las rotaciones en el toro. Host, Kra y Maass probaron en 2016 [1] que los conjuntos de $\mathbb{Z}$-Bohr recurrencia son conjuntos de recurrencia para cada $\mathbb{Z}$-nilsistema minimal. En el mismo artículo, ellos también prueban que las extensiones proximales levantan la $\mathbb{Z}$-Bohr recurrencia.

En esta tesis generalizamos parte del progreso en este problema para acciones de grupos más generales que $\mathbb{Z}$. Primeramente, nos concentramos en recurrencia para acciones G en un espacio métrico compacto X, donde G es un grupo abeliano localmente compacto de homeomorfismos de X. En este contexto planteamos la pregunta de Katznelson presentando algunas preguntas equivalentes y demostrando que los límites inversos y las extensiones proximales levantan la recurrencia, generalizando los resultados presentados en [1] para $\mathbb{Z}$-acciones. Luego, exploramos esta pregunta en $\mathbb{Z}^d$-nilsistemas, donde describimos el principal problema al intentar generalizar la demostración de [1] para $\mathbb{Z}$-nilsistemas, y resolvemos tal problema en dos casos. En el primero, estudiamos la relación entre Bohr recurrencia y los cubos dinámicos introducidos en [2] para $\mathbb{Z}^d$-sistemas, dando caracterizaciones de los conjuntos de Bohr recurrencia y definimos la noción de fuerte propiedad de cierre, demostrando que los conjuntos de Bohr recurrencia son conjuntos de recurrencia para $\mathbb{Z}^d$-nilsistemas con esta propiedad. En el segundo, abordamos el problema en la familia de $\mathbb{Z}^d$-nilsistemas donde la componente conexa de la identidad en G es abeliana. Para esto, introducimos la noción de correlaciones de Bohr, con la cual desarrollamos técnicas de clasificación de conjuntos de $\mathbb{Z}^d$-Bohr recurrencia, reduciendo el problema al contexto más apropiado, y demostramos que los conjuntos de $\mathbb{Z}^d$-Bohr recurrencia son conjuntos de recurrencia en esta familia.

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CONTRIBUTIONS TO THE STUDY OF THE KATZNELSON'S PROBLEM FOR GENERAL GROUP ACTIONS.

Recurrence properties of dynamical systems and associated sets of integers that suffice for recurrence are classical objects in topological dynamics. In this area, Katznelson's Question arises as a long-standing open question concerning recurrence in topological dynamics, with strong historical and mathematical ties to open problems in combinatorics and harmonic analysis. More precisely, Katznelson's Question wonders if sets of Bohr recurrence (recurrence for all finite dimensional torus translations) are sets of recurrence for minimal $\mathbb{Z}$-actions.

The objective of this work is to study Katznelson's Question for general group actions. The results known in this problem are quite scarce, and there seems to be no consensus among scholars on an expected answer. However, this problem has been intensively studied for $\mathbb{Z}$-actions in the class of nilsystems, a class of systems of algebraic origin that generalizes rotations on tori. Host, Kra, and Maass proved in 2016 [1] that sets of $\mathbb{Z}$-Bohr recurrence are sets of recurrence for every minimal $\mathbb{Z}$-nilsystem. In the same article, they also proved that proximal extensions lift up $\mathbb{Z}$-Bohr recurrence.

In this thesis, we generalize part of the progress on this problem to more general group actions than $\mathbb{Z}$. Our study focuses firstly on recurrence for actions G on a compact metric space X, where G is a locally compact abelian group of homeomorphisms of X. We state Katznelson's Question in this context, giving some equivalences and proving that inverse limits and proximal extensions lift up recurrence, generalizing the result presented in [1] for $\mathbb{Z}$-actions. Then we explore such question in the family of $\mathbb{Z}^d$-nilsystems, for which we prove several properties, and we described the main problem when trying to generalize the aforementioned result from [1] to $\mathbb{Z}^d$-nilsystem. We overcome such problem for two subfamilies of $\mathbb{Z}^d$-nilsystems. First, we study the relationship between Bohr recurrence and the dynamical cubes introduced in [2] for $\mathbb{Z}^d$-systems, provide characterizations of sets of Bohr recurrence in such context, and define the notion of strong closing property, proving that sets of Bohr recurrence are sets of recurrence in $\mathbb{Z}^d$-nilsystems with such property. Second, we study the problem in the family of $\mathbb{Z}^d$-nilsystems, for which the connected component of the identity in G is abelian. This family enrolls for example all $\mathbb{Z}^d$-affine nilsystems. For this, we introduce the notion of Bohr correlations, with which we develop techniques to classify sets of $\mathbb{Z}^d$-Bohr recurrence, narrowing the problem to the most appropriate context, and proving that sets of $\mathbb{Z}^d$-Bohr recurrence are set of recurrence in such family.

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Introduction

Recurrence in dynamical systems and ergodic theory is a topic that has been highly studied for many years in mathematics. It concerns the classical question of how and when a point or set recurs to its initial position. Its origin can be traced back to the works of H. Poincaré in the late 19th century, when he was studying the three-body problem, which involves the motion of three celestial bodies under their mutual gravitational attraction. Poincaré found that in certain dynamical systems, trajectories tend to return to their initial positions after a certain amount of time, even when the system may be chaotic or unpredictable. This led to his famous Poincaré Recurrence Theorem (PRT), which is stated as follows:

Theorem 0.1 (PRT, ([3], Ch. 26, Vol. 3)) *If T is a measure-preserving transformation of a probability space $(\Omega, \mathcal{B}, \mu)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$, then there exists a measurable subset $A_0 \subseteq A$ with $\mu(A_0) = \mu(A)$ such that for any $x \in A_0$ there exists an infinite sequence $(n_i)_{i=1}^{\infty}$ such that $T^{n_i}x \in A_0$ for all $i \in \mathbb{N}$.*

Another breakthrough in the theory was the proof by Furstenberg and Weiss [4] of Van der Waerden's classic theorem, which states the existence of arbitrarily long monochromatic progressions in any finite coloring of the integers [5]. This proof established a deep connection between the fields of dynamical systems and additive combinatorics, with recurrence properties of systems and associated sets of recurrence being central to the intersection of both areas. In this context, the class of nilsystems plays an important role. This is evident, for example, in the ergodic context of multiple convergence along arithmetic progressions [6], or in the topological context, where nilsystems have been used to construct explicit examples of sets of multiple recurrence in [7] and [8].

In particular, we are interested in sets of Bohr recurrence, which are sets of recurrence for all rotations, the simplest type of nilsystem. Since rotations comprise a narrow subclass of topological dynamical systems, one may expect that sets of topological recurrence comprise a narrow subclass of the sets of Bohr recurrence. The extent to which this is true remains an important unsolved problem, popularized in the dynamics community by Katznelson [9] for $\mathbb{Z}$-actions.

Katznelson's Question: *Is every set of Bohr recurrence a set of topological recurrence?*

Katznelson's Question is a long-standing open problem concerning recurrence in topological dynamics, with strong historical and mathematical ties to open problems in combinatorics and harmonic analysis. Additionally, there seems to be no consensus among experts regarding a positive or negative answer. For a negative answer, there are very few concrete potential examples, such as sets that are known to be sets of Bohr recurrence but whose other dynamical recurrence properties are unknown (see Grivaux and Roginskaya [10] and Frantzikinakis and McCutcheon [[11], Future Directions]).

On the other hand, for a positive answer to Katznelson's Question, some of the principal ideas consist of decomposing minimal systems into chains of factors, in which we can prove recurrence. The idea is to prove that the extensions described by these factors preserve the property of sets of Bohr recurrence being sets of recurrence. For example, it is known that every minimal system arises from a proximal extension, which is a weakly mixing extension of a strictly PI system. The latter is a system obtained by taking consecutive (possibly infinite) proximal and equicontinuous extensions from the trivial system, as shown in Fig. 3.1 (for specific definitions, see Chapter 3, and for background on this topic, see [12] Chapter 14 or [13] Chapter 6).

$$\begin{array}{ccccccccccc} (Y,G) & \xrightarrow[\text{weakly mixing}]{} & (Y_{PI},G) & \xrightarrow[\text{proximal}]{} & \cdots & \xrightarrow[\text{equicontinuous}]{} & (Y_2,G) & \xrightarrow[\text{proximal}]{} & (Y_1,G) & \xrightarrow[\text{equicontinuous}]{} & (\{1\},G) \\ & \searrow^{\text{proximal}} & & & & & & & & & \\ & & (X,G) & & & & & & & & \end{array}$$

Figura 0.1: PI Chain of Factors.

Another example of the idea of decomposing a minimal system into a chain of consecutive factors comes from the fact that any system can be obtained by a sequence of extensions, passing through every maximal d-step nilsystem (including $d = \infty$) and its maximal distal factor, as shown in Fig. 3.2 (for specific definitions, see Chapter 3, and for background on this topic, see [14] and [15]).

$$(X,G) \xrightarrow[\text{weakly mixing}]{} (X_d,G) \xrightarrow[\text{distal}]{} (\mathcal{Z}_\infty,G) \xrightarrow[\text{inverrse limit}]{} \cdots \xrightarrow[\text{g.e.}]{} (\mathcal{Z}_2,G) \xrightarrow[\text{g.e.}]{} (\mathcal{Z}_1,G)$$

Figura 0.2: Nilsystems' Chain of Factors.

In where "g.e." stands for group extension. For $\mathbb{Z}$-actions, Host, Kra, and Maass proved in [1] that Bohr recurrence can be lifted up to $(\mathcal{Z}_\infty, G)$ in this chain. In other words, sets of Bohr recurrence are sets of recurrence for arbitrary minimal $\mathbb{Z}$-pronilsystems.

The main purpose of this thesis is to study Katznelson's question outside the context of $\mathbb{Z}$-actions. Initially, we aimed to find easier counterexamples for more complex group actions, but ultimately, we did the opposite. For general group actions, we prove that inverse limits and proximal extensions lift recurrence (a result that was already proved for $G = \mathbb{Z}$ by Host, Kra, and Maass in [1]). We generalize the theorem of Host, Kra, and Maass on minimal $\mathbb{Z}$-nilsystems to minimal $\mathbb{Z}^d$-nilsystems with a stronger form of the closing property introduced by Cabezas, Donoso, and Maass in [2] for $\mathbb{Z}^d$-actions. We then prove the theorem for $\mathbb{Z}^d$-quasi-affine nilsystems, which are nilsystems $(X = G/\Gamma, T_1, \ldots, T_d)$ whose connected component of the identity in G is abelian, developing some especial techniques over sets of Bohr recurrence in $\mathbb{Z}^d$.

The structure of this thesis is divided into five chapters:

- ➜ Chapter 1 is dedicated to introduce most of the basic tools that will be used in this work. Section 1.1 introduces the essential definitions and properties of dynamical systems and measurable systems. Meanwhile Section 1.2 defines rotations, nilmanifolds, and nilsystems and their properties as dynamical systems. Finally, in Section 1.3 we introduce all the necessary tools in the framework of dynamical cubes, including the closing property.

- ➜ Chapter 2 is devoted to the introduction of recurrence for general group actions. Specifically, in Section 2.1 we provide definitions for recurrence in systems and families of systems, and establish some fundamental properties of recurrence. Moving forward to Section 2.2, we utilize duality in locally compact abelian groups to define Bohr neighborhoods of zero and sets of Bohr recurrence. We demonstrate many important properties of these sets, in particular, showing that Bohr recurrence can be characterized as recurrence in equicontinuous systems. In Section 2.2.2, we establish the connection between the aforementioned concepts and (Bohr) almost periodicity. We prove classical properties usually stated for $\mathbb{Z}$-actions and present examples of nontrivial Bohr neighborhoods of 0. Lastly, in Section 2.2.3 we prove several important properties of $\mathbb{Z}^d$-sets of Bohr recurrence that will be utilized in subsequent chapters, or which are worth to state.

- ➜ Chapter 3 focuses on studying Katznelson's Question for general group actions. In Section 3.1 we present Katznelson's Question for general group actions, along with some equivalencies. In Section 3.2, we discuss some current strategies for obtaining a positive answer to the question. Lastly, in Section 3.3 we prove that Bohr recurrence can be lifted through inverse limits and proximal extensions.

- ➜ Chapter 4 aims to study whether sets of Bohr recurrence are also sets of recurrence for minimal $\mathbb{Z}^d$-nilsystems. We begin in Section 4.1 by introducing some reductions in nilsystems and discussing properties related to connectedness. Then, in Section 4.1.2 we discuss the main problem when trying to generalize the method from Host, Kra, and Maass in [1] from $\mathbb{Z}$-nilsystems to $\mathbb{Z}^d$-nilsystems, using affine systems to identify this difficulty. Thereafter, in Section 4.2.1 we characterize sets of $\mathbb{Z}^d$-Bohr recurrence through products of sets of returns of $\mathbb{Z}$-nilsystems. Then, in Section 4.2.2 we introduce the notion of strong closing property, which allows to prove that sets of Bohr recurrence are sets of recurrence for minimal $\mathbb{Z}^d$-nilsystems with such property. In Section 4.1.2 we prove that the problem can also be solved for the family of $\mathbb{Z}^d$-quasi-affine nilsystems. To accomplish this, in Section 4.3.1 we identify patterns in sets of $\mathbb{Z}^d$-Bohr recurrence, introducing the notion of Bohr correlation, and establishing several properties of this new concept. Then in Section 4.3.2 we show that the problem can be reduced to the case in which Bohr correlations are either 0 or rationally independent. We take advantage of such reduction to prove several properties in $\mathbb{Z}^d$-nilsystems that will allow us to prove the result for $\mathbb{Z}^d$-affine nilsystems.

- ➜ Chapter 5 summarizes the most significant results obtained in this thesis. We also discuss open questions in the field of recurrence in dynamical systems and combinatorics, as well as some potential avenues for future research in this area.

Chapter 1

Preliminaries

This chapter provides an introduction to topological dynamical systems, measurable dynamical systems, nilsystems, and dynamical cubes. We begin by defining a dynamical system and key concepts such as transitivity, minimality, almost periodicity, equicontinuity, proximality, distality, factors, joinings, inverse limits, and group extensions, which are fundamental to our study. We also define what a measurable preserving system is and what ergodicity means in our context. Next, we focus on the most basic type of dynamical systems for our analysis, namely rotations. We then introduce Lie groups, nilmanifolds, and nilsystems, highlighting their essential properties for our study. Finally, we define the basic notions on dynamical cubes, and the properties known for $\mathbb{Z}^d$-groups actions. Overall, this chapter serves as a foundation for our exploration of various types of dynamical systems and their properties.

1.1. Basic Definitions and Properties

1.1.1. Topological Dynamical Systems

A topological dynamical system (t.d.s.) is a pair (X, G), where X is a compact metric space and G is a locally compact abelian group of homeomorphisms of the space X into itself. Although we will prove some theorems with general actions, later on we will primarily focus in the case that G is spanned by $d \geq 1$ commuting homeomorphisms $T_1, \dots, T_d : X \to X$, case in which we write $(X, T_1, \dots, T_d)$, and we call this kind of system $\mathbb{Z}^d$-systems.

A topological dynamical system (X, G) is *transitive* if for every nonempty open sets $U, V \subseteq X$, there is $g \in G$ such that $U \cap g^{-1}V \neq \emptyset$. Given a topological dynamic system (X, G) and a point $x \in X$ we denote the orbit of x by $\mathcal{O}_G(x) = \{gx : g \in G\}$ (or simply $\mathcal{O}(x)$ if the dynamic is implicitly understood). We say that a subset $Y \subseteq X$ is G-invariant if $GY = Y$.

The following theorem establishes different equivalences of transitivity.

Theorem 1.1 ([12], Ch. 1) *Let (X, G) be a t.d.s.. The following are equivalent:*

1. *(X, G) is transitive.*

2. *There is $x \in X$ such that $\mathcal{O}(x)$ is dense in X.*

3. *The set* $\{x \in X \mid \overline{\mathcal{O}(x)} = X\}$ *is a* G_δ*-dense*[1] *set.*

4. *If* $U \subseteq X$ *is an* G*-invariant open set, then* U *is dense.*

A stronger form of transitivity is minimality. A t.d.s. (X, G) is *minimal* if every point on X is transitive. Similarly with respect to the previous theorem, we have some alternative ways to define minimality.

Theorem 1.2 ([12], Ch. 1) *Let* (X, G) *be a t.d.s.. The following are equivalent:*

1. *The system* (X, G) *is minimal.*

2. *There is no nonempty closed and* G*-invariant proper sets in* X*.*

3. *For all nonempty open set* $U \subseteq X$*, we have that* $\bigcup_{g\in G} gU = X$*.*

4. *For all nonempty open set* $U \subseteq X$*, there exist* $N \in \mathbb{N}$ *and* $g_1, \cdots, g_N \in G$ *such that* $\bigcup_{i=1}^{N} g_i U = X$*.*

We say that (Y, G) is a subsystem of (X, G) if Y is a nonempty closed G-invariant set of X. A well-known relevant fact is that every dynamical system has a minimal subsystem.

Theorem 1.3 ([12], Ch. 1 Theorem 4) *Let* (X, G) *be a topological dynamical system. Then* X *contains a minimal subsystem.*

We now define the following form of minimality, which has strong relationship with recurrence in dynamical systems.

Definition 1.1 *Let* (X, G) *a topological dynamical system. A point* $x \in X$ *is almost periodic if* $(\overline{\mathcal{O}_G(x)}, G)$ *is minimal. In addition, we say that* (X, G) *is pointwise almost periodic if every point in* X *is almost periodic,*

Remark 1 All minimal dynamical systems are pointwise almost periodic, but the converse is not true. For instance, consider $X = \{1, 2\}$ and $G = \{id\}$ the trivial action on X. As every point is a fixed point of the action, the system is pointwise almost periodic. However, it is not minimal for the same reason.

Definition 1.2 *Let* (X, G) *be a topological dynamic system. Then* (X, G) *is equicontinuous if* $\forall \epsilon > 0$*, exists* $\delta > 0$ *such that for every* $x, y \in X$*,*

$$d_X(x, y) < \delta \Longrightarrow d_X(g \cdot x, g \cdot y) < \epsilon, \ \forall g \in G.$$

It is easy to see that every isometric action in a dynamical system is equicontinuous. Furthermore, the converse is also true if the system's metric is replaced with an equivalent one, as shown in the following proposition.

Proposition 1.1 *Let* (X, G) *be a topological dynamical system. Then* (X, G) *is equicontinuous if and only if* (X, G) *can be endowed with a metric* d *such that the action of* G *is isometric in* (X, d)*.*

[1] A G_δ set is a subset of a topological space that is a countable intersection of open sets.

Proof. If the metric d_X is an isometry for the system (X, G), we have that $d_X(gx, gy) = d_X(x, y)$, for all $x, y \in X$ and $g \in G$, from which follows that (X, G) is equicontinuous.

For the converse direction, suppose that (X, G) is an equicontinuous system equipped with a metric d_X. We define $\overline{d_X} : X \times X \to \mathbb{R}_+$ as

$$\overline{d_X}(x, y) = \sup_{g \in G} d_X(gx, gy).$$

Notice that $\overline{d_X}$ is well defined, as $d_X : X \times X \to \mathbb{R}_+$ is continuous and X is compact. To show that $\overline{d_X}$ defines a metric on X, we first note that as d_X is positive and symmetric, so is $\overline{d_X}$. Also, note that $\overline{d_X}(x, y) = 0$ implies that $d_X(x, y) = 0$ and, therefore, $x = y$, and conversely $\overline{d_X}(x, x) = \sup_{g \in G} d_X(gx, gx) = 0$. Finally, for $x, y, z \in X$, we have that

$$\overline{d_X}(x, y) = \sup_{g \in G} d_X(gx, gy) \leq \sup_{g \in G} d_X(gx, gz) + d_X(gz, gy) \leq \overline{d_X}(x, z) + \overline{d_X}(z, y),$$

so we conclude that $\overline{d_X}$ defines a metric on X.

Now we see that $\overline{d_X}$ is compatible with the topology of (X, G). In fact, first it is easy to see that $d_X(x, y) \leq \overline{d_X}(x, y)$, $\forall x, y \in X$. Second, note that for $\epsilon > 0$, we can always find $\delta > 0$ such that

$$\forall x, y \in X, d_X(x, y) < \delta \Longrightarrow d_X(gx, gy) < \epsilon, \quad \forall g \in G,$$

so $d_X(x, y) < \delta \Longrightarrow \overline{d_X}(x, y) < \epsilon$, and we derive that $\overline{d_X}$ induces the same topology as d_X. Finally, as for all $x, y \in X$ and $g \in G$, $\overline{d_X}(gx, gy) = \overline{d_X}(x, y)$, we conclude that the action of G is isometric in $(X, \overline{d_X})$. □

A pair $(x, y) \in X \times X$ is called *proximal* if there exists a sequence $(t_n)_n \subseteq G$ such that

$$\lim_{n \to \infty} d(t_n x, t_n y) = 0,$$

otherwise, the pair (x, y) is said to be *distal*. We will denote by $\mathbf{P}(X)$ the set of proximal pairs of a t.d.s. (X, G). Whenever the context is clear, we will just write $\mathbf{P}$ instead of $\mathbf{P}(X)$ to denote the set of proximal pairs. Note that we always have the inclusion

$$\Delta(X) := \{(x, x) \mid x \in X\} \subseteq P(X),$$

as every point $x \in X$ is proximal to itself.

Definition 1.3 *Let (X, G) be a topological dynamic system. Then (X, G) is distal if there are no nontrivial proximal pairs (i.e. $\boldsymbol{P}(X) = \Delta$).*

A factor map between topological dynamical systems (X, G) and (Y, G) is an onto continuous map $\pi : Y \to X$ satisfying $\pi \circ g = g \circ \pi$, $\forall g \in G$. In this case, Y is called an extension of X or equivalently, X is a factor of Y. If π is also an injection, we say that (Y, G) and (X, G) are isomorphic or conjugated, and π is an isomorphism or a conjugation respectively.

The following theorem comprises some of the most basic properties of distal systems.

Theorem 1.4 ([12], Chapters 5 and 7)

1. *The Cartesian product of distal systems is distal.*
2. *Distality is preserved by taking factors and subsystems.*
3. *A distal system is minimal if and only if it is transitive, that is, a distal system is pointwise almost periodic.*
4. *If (X, G) is distal and G' is a subgroup of G, then (X, G') is distal.*
5. *Factor maps between distal systems are open maps.*

Let $(X_1, G), \ldots, (X_n, G)$ be n topological dynamical systems. A *joining* of $(X_1, G), \ldots, (X_n, G)$ is a closed subset $Z \subseteq X_1 \times \cdots \times X_n$ which is invariant under the diagonal action of G (namely, the action given by $\{g \times \cdots \times g \mid g \in G\}$) and projects onto each factor. When $J \neq X_1 \times \cdots \times X_n$, we say that J is a nontrivial joining.

It is worth noting that an equivalence relation $R \subseteq X \times X$ which is closed and G-invariant defines a factor $\pi_R : (X, G) \to (X/R, G)$. Conversely, every factor $\pi : X \to Y$ defines a closed and G-invariant equivalence relation $R_\pi := \{(x, x') : \pi(x) = \pi(x')\}$.

Now we give an example of a factor that characterizes completely the equicontinuous behavior of a dynamical system, and which has motivated many other generalizations that have contributed to recent developments in dynamical systems and ergodic theory. Let $\pi : X \to Y$ be a factor between dynamical systems (X, G) and (Y, G). We say that (Y, G) is an equicontinuous factor if (Y, G) is equicontinuous.

Theorem 1.5 ([12], Ch. 9 Theorem 1) *Let (X, G) be a t.d.s.. Then, there is a maximal equicontinuous factor, this is, an equicontinuous factor (Y, G) such that if (Z, G) is another equicontinuous factor of (X, G), then (Z, G) is also a factor of (Y, G).*

We can construct this maximal equicontinuous factor explicitly. In fact, let (X, G) be a t.d.s. and $x, y \in X$. We say that x, y are *regionally proximal* if for all $\delta > 0$ and $\epsilon > 0$ there are $\overline{x} \in B(x, \delta), \overline{y} \in B(y, \delta)$ such that $d(g\overline{x}, g\overline{y}) < \epsilon$. We write $\mathbf{RP}(X)$ the set of regionally proximal pairs. It is a classic result that in a minimal t.d.s. (X, G), $\mathbf{RP}(X)$ is a equivalence relation.

Theorem 1.6 ([12], Ch. 9 Theorem 3) *Let (X, G) be a t.d.s., and let S_{eq} the smallest closed and G-invariant equivalence relation which contains $\boldsymbol{RP}(X)$. Then $(X/S_{eq}, G)$ is the equicontinuous maximal factor.*

The following notion is a classical concept in dynamical systems and will be essential in subsequent chapters.

Definition 1.4 *Consider a sequence of topological dynamical systems $\{(X_m, G)\}_{m\in\mathbb{N}}$ where $\pi_m : (X_m, G) \to (X_{m-1}, G)$ are factor maps for $m \geq 2$. We define the inverse limit of $\{(X_m, G)\}_{m\in\mathbb{N}}$ by*

$$\varprojlim (X_m, G) := \{(x_m) \in \prod_{m\in\mathbb{N}} X_m \ : \ \pi_{m+1}(x_{m+1}) = x_m \ \textit{ for } m \geq 1\},$$

which is equipped with the product topology and the diagonal action of G, given by $g(x_m)_{m\in\mathbb{N}} = (gx_m)_{m\in\mathbb{N}}$ for $g \in G$.

Lastly, we define group extensions in topological dynamics. Let (X, G) be a topological system. If K is a compact abelian group, then a *free right action of* K on (X, G) by automorphisms is a map $(h, x) \to V_h x$ for $h \in K$ and $x \in X$ such that

- the map $(h, x) \to V_h x$ from $K \times X$ to X is continuous;
- for all $h \in K$, the map $V_h : X \to X$ is an automorphism of (X, G);
- the identities agree, meaning that $V_{e_K} = Id_X$;
- for all $u, h \in K$, we have $V_{uh} = V_h \circ V_u$;
- if $u \in K$ and $x \in X$ satisfy $V_u x = x$, then $u = e_K$.

Given such an action of K on (X, G), let Y be the quotient of X under the action of K and let $\pi : X \to Y$ be the associated quotient map. Define the quotient distance on Y, meaning the distance

$$d_Y(y, y') = \inf\{d_X(x, x') : x, x' \in X, \pi(x) = y \text{ and } \pi(x') = y'\}.$$

Endowed with this distance, Y is a compact metric space. In particular, this implies that for every $y \in Y$, the group K acts freely and transitively on the fiber $\pi^{-1}(\{y\})$, and this fiber is homeomorphic to K.

The group action of G in X induces a group action in Y, and $\pi : (X, G) \to (Y, G)$ is a factor map. In this case, we say that (X, T) is a *group extension of* (Y, G) *by the compact abelian group* K.

1.1.2. Measurable Dynamical Systems

A measure preserving system (m.p.s.) (X, μ, G) is a probability space $(X, \mathcal{X}, \mu)$ endowed with an action G which is $\mathcal{X} - \mathcal{X}$-measurable and preserves the measure, i.e. $\forall A \in \mathcal{X}$ and $\forall g \in G$, $\mu(gA) = \mu(A)$.

Let (X, μ, G) be a m.p.s.. A set $A \in \mathcal{X}$ is *invariant* if $\mu(A \Delta gA) = 0$, $\forall g \in G$. A m.p.s. (X, μ, G) is *ergodic* if there are no nontrivial invariant sets, i.e. if $A \in \mathcal{X}$ and $\mu(A \Delta gA) = 0$, $\forall g \in G$, then $\mu(A) \in \{0, 1\}$.

For a m.p.s. (X, μ, G) we will use the word *factor* to mean two things: A G-invariant sub-σ-algebra $\mathcal{Y}$ of $\mathcal{X}$, or a m.p.s. (Y, ν, G) and a function $\pi : X' \subseteq X \to Y' \subseteq Y$ measurable with $\mu(X') = \nu(Y') = 1$ and such that $\pi\mu = \nu$ and $g \circ \pi = \pi \circ g$, $\forall g \in G$. These two uses of the same word is not a problem since it is always possible to identify the σ-algebra $\mathcal{Y}$ of Y with the invariant sub-σ-algebra $\pi^{-1}(\mathcal{Y})$ of $\mathcal{X}$, and the other way around: It is always possible to define a factor $\pi : X' \subseteq X \to Y' \subseteq Y$ from an invariant sub-σ-algebra $\mathcal{Y}$ of $\mathcal{X}$ (see, for example, Chapter 5 of [16]).

We will need the following known functional characterization for ergodicity, which proof can be found in [17].

Proposition 1.2 ([17], Prop. 2.7) *Let (X, μ, G) be a measure preserving dynamical system. Then (X, μ, G) is ergodic if and only if $\forall f \in L^2(X)$, if f is G-invariant (meaning that $\forall g \in G$, $\forall x \in X$, $f(x) = f(gx)$) then f is constant μ-c.s..*

1.2. Nilsystems

This section aims to provide a definition of nilsystems and explore their basics properties for our study. We will begin with the fundamental type of nilsystems, namely rotations, and then proceed to the general definition of nilsystems, which involves Lie groups and nilmanifolds.

Definition 1.5 *Let (X, G) be a dynamical system. We say that (X, G) is a rotation if X is a compact abelian group, and the action of G on X is induced by a homomorphism $\varphi : G \to X$ such that $t \cdot x = \varphi(t) + x$, $\forall t \in G$ and $x \in X$, in where the action of X is written in additive notation.*

A very important fact that we will explode is that rotations are distal systems.

Lemma 1.1 *If (X, G) is a rotation, then (X, G) is distal.*

The proof is direct from the fact that in rotations there is always an invariant distance. However, as later we will cite a more general statement, we will omit the proof.

We also can characterize minimal rotations by their property of being equicontinuous, as shown in the following theorem.

Theorem 1.7 ([18], Theorem 2.4) *Let (X, G) be a minimal dynamical system. Then (X, G) is equicontinuous if and only if (X, G) is conjugate to a minimal rotation.*

Last but not least, it is important that in rotations, transitivity and minimality are the same.

Proposition 1.3 *Let (X, G) be a rotation. Then (X, G) is minimal if and only if (X, G) is transitive.*

Proof. We just prove the nontrivial implication. Suppose that (X, G) is a transitive rotation, and let $x \in X$ be a transitive point. Then $\overline{Gx} = X$. Let $y \in X$ be a point. Then

$$\overline{Gy} = \overline{Gx}x^{-1}y = X(x^{-1}y) = X,$$

therefore, (X, G) is minimal. □

1.2.1. Nilmanifolds

In this memory it will appears as a study object the concept of Lie group. In this subsection we define what a Lie group is, give some basic properties and some classic examples. A general bibliography for this topic is [19].

A Lie group $(G, \cdot)$ is a group which is also a differential manifold, where the functions

$$(g, h) \to g \cdot h := gh$$
$$g \to g^{-1}$$

are differentiable functions from $G \times G \to G$ and $G \to G$ respectively.

Example 1 The group $(\mathbb{R}^n, +)$ is a Lie group.

Example 2 Let

$$U_n = \{ \begin{pmatrix} 1 & u_{1,2} & \cdots & \cdots & u_{1,n} \\ 0 & 1 & u_{2,3} & \cdots & u_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & u_{n-1,n} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \mid u_{i,j} \in \mathbb{R} \text{ for } j - i \geq 1\},$$

with the usual matrix product. Then $(U_n, \cdot)$ is a Lie group, which we will call the Heisenberg group of order n.

Let G be a group and $[\cdot, \cdot]$ the commutator operator, i.e. for $a, b \in G$, $[a, b] = aba^{-1}b^{-1}$. Given subgroups $A, B \subseteq G$, we write $[A, B]$ to denote the subgroup of G generated by $\{[a, b] \mid a \in A, b \in B\}$. We define recursively the commutator subgroups G_j of G as:

$$G_1 = G, \quad G_{j+1} = [G, G_j], \quad j \geq 1.$$

Definition 1.6 *We say that a group G is nilpotent of order d if $G_d = \{e_G\}$.*

Example 3

- The group trivial $\{e\}$ is the only group of order 0.
- A group is nilpotent of order 1 if and only if is abelian. Therefore, rotations are precisely the nilgroups or order 1.
- The Heisenberg groups U_n defined previously are $(n-1)$-nilpotent.

Definition 1.7 *Let G be a s-nilpotent Lie group and Γ a discrete cocompact subgruop of G (i.e. Γ is countable and G/Γ is compact). Then we say that the compact nilmanifold $X = G/\Gamma$ is an s-step nilmanifold.*

For a nilmanifold $X = G/\Gamma$, we denote G_0 the connected component of 1_G in G. Then G_0 is an open normal subgroup of G (see, for example, [1], section 4.1).

Lemma 1.2 ([15], Ch. 10 Lemma 11) *Let $X = G/\Gamma$ be a nilmanifold. Then the subgroup $G_0\Gamma$ has finite index in G. Its image X_0 in X is the connected component of e_X. In particular, X is connected if and only if $G = G_0\Gamma$.*

Theorem 1.8 ([15], Ch. 10 Theorem 13) *Any s-step nilmanifold $X = G/\Gamma$ can be represented as $\tilde{G}/\tilde{\Gamma}$ where $\tilde{G}$ is a simply connected s-step nilpotent Lie group, and such that every element of G is represented in $\tilde{G}$.*

1.2.2. Nilsystems and Properties

Let us consider $H \subseteq G$ a subgroup of a s-nilpotent Lie group. We will consider the action of H given by left translation. It can be prove that (X, H) is a t.d.s. Let μ the Haar measure of (X, H) (the unique rotation invariant measure). Then (X, μ, H) is a m.p.s., and we will called both (X, H) and (X, μ, T) nilsystems of order s.

Example 4 The Heisenberg system of order n given by $(U_n/\Gamma_n, G)$ where

$$\Gamma_n = \{ \begin{pmatrix} 1 & m_{1,2} & \cdots & \cdots & u_{1,n} \\ 0 & 1 & m_{2,3} & \cdots & m_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & m_{n-1,n} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \mid m_{i,j} \in \mathbb{Z} \text{ for } j - i \geq 1\},$$

and G is the cyclic group generated by an element $g \in U_n$, is a nilsystem of order $n-1$.

Definition 1.8 *Let (X, G) be a t.d.s., we say that (X, G) is a s-step pronilsystem if (X, G) is an inverse limit of s-step nilsystems. In the case (X, G) is a inverse limit of nilsystem (without fixing the step), we say that (X, G) is a ∞-step nilsystem.*

Remark 2 In general, if a property is holds for nilsystems, we will try to lift it through the inverse limit.

The fact that rotations are distal can be generalized to nilsystems.

Theorem 1.9 ([20], Ch.4, Theorem 3.) *If $(X = G/\Gamma, G)$ is a s-step nilsystem, then X is distal. In particular, if A is a finitely generated amenable group such that there is a homomorphism $\varphi : A \to G$ with A acting on X by translation $(\varphi(u))(x) = \varphi(u)x$, $u \in A, x \in X$, then (X, A) is distal.*

Now we give some important structural properties of nilsystems that will be useful in the subsequent chapters.

Theorem 1.10 (Cf. [21]) *If the action of A is ergodic on X then the action of A is uniquely ergodic on X.*

Theorem 1.11 (Cf. [21]) *Let $(X = G/\Gamma, A)$ be a k-step nilsystem. Then the groups G_j, $j \geq 2$ are connected. In particular, $G_j \subseteq G_0$.*

Theorem 1.12 (Cf. [21]) *Let $N = \langle G_0, \varphi(A) \rangle$ and $Z = X/[N, N]$. Then the action of A is ergodic if and only if X is minimal with respect to the action of A, and if and only if Z is minimal with respect to the action of A.*

Corollary 1.1 ([22], Corollary section 2.18) *Assume that X is connected and consider $T = X/[G_0, G_0]$, the maximal factor-torus of X. Since Z is a factor of T, we have then the action of A is ergodic on X if and only if it is ergodic on T.*

The next theorem is a well-known result in the field of nilsystems. However, we were unable to find a proof of this result for general actions in the existing literature. Most of the proofs available in the literature are restricted to $\mathbb{Z}$-actions. Hence, we provide a proof of this theorem for the sake of completeness.

Theorem 1.13 *For a nilsystem (X, μ, A), the following properties are equivalent:*

1. *The nilsystem (X, μ, A) is ergodic.*
2. *The topological nilsystem (X, A) is uniquely ergodic.*
3. *The topological nilsystem (X, A) is minimal.*
4. *The topological nilsystem (X, A) is topologically transitive.*

PROOF. 1. $\Rightarrow$ 2. comes directly from Theorem 1.10, meanwhile 2. $\Rightarrow$ 1. comes from the uniqueness of the Haar measure (as the only invariant measure under translations).

On the other hand, by Theorem 1.12 we have that 1. $\iff$ 3.. As clearly 3. $\Rightarrow$ 4., the only thing left to prove is 4. $\Rightarrow$ 3.. If (X, A) is topologically transitive, then any factor is topologically transitive as well. Therefore (Z, A) is topologically transitive, and as (Z, A) is a rotation, then it is minimal by Proposition 1.3. Finally, by Theorem 1.12 we conclude that (X, A) is minimal. □

The following proposition shows that orbits are still nilsystems.

Proposition 1.4 *Let (X, A) an s-step nilsystem, and $x \in X$. Then $\overline{\mathcal{O}}_A(x)$ is a nilmanifold. In particular, $(\overline{\mathcal{O}}(x), A)$ is an s-step nilsystem.*

The proof of this proposition can be found in [23], Theorem 1.3. or in [15] Section 3.2.

Remark 3 Let $(X = G/\Gamma, A)$ be a s-step nilsystem and $(Y = G^Y/\Gamma^Y, A)$ an orbit of X. In the proof of Proposition 1.4 in [15] Section 3.2 is shown that G_0^Y is a rational subgroup of G_0.

In the case when $G = \mathbb{Z}^d$ we can define an interest type of nilsystems, which are the affine nilsystems. These systems are the case when $X = \mathbb{T}^r$ for $r \geq 1$ and the transformations $T_1, \cdots, T_d$ are defined by $T_i(x) = A_i x + \alpha_i$ commutes, where the matrices $(A_i)_{i=1}^d$ are unipotent and commute as well.

Frantzikinakis and Kra showed in [24] that $\mathbb{Z}^d$-affine nilsystems can be characterized by the fact that G_0 is abelian in such case.

Proposition 1.5 ([24]) *Let $X = G/\Gamma$ be a connected nilmanifold such that G_0 is abelian. Then any nilrotation $T_a(x) = ax$ defined on X with the Haar measure μ is isomorphic to a unipotent affine transformation on some finite dimensional torus.*

This inspire the following definition.

Definition 1.9 *Let $(X = G/\Gamma, T_1, \ldots, T_d)$ be a s-step $\mathbb{Z}^d$-nilsystem. We will say that $(X = G/\Gamma, T_1, \ldots, T_d)$ is quasi-affine if G_0 is abelian.*

In future Chapters we will see that any s-step $\mathbb{Z}^d$-nilsystems is union of connected s-step $\mathbb{Z}^d$-nilsystems. Therefore, $\mathbb{Z}^d$-quasi-affine nilsystems are basically finite union of $\mathbb{Z}^d$-affine nilsystems.

1.3. $\mathbb{Z}^d$-Dynamical Cubes

In this section, we introduce the classical notions of cubes, which will be useful in subsequence chapters. We will use the same definitions and notations as in [2], we remember all of them, including some theorems that we will use as well.

For a set X we denote $X^{[d]} := X^{2^d}$. A point $x \in X^{[d]}$ can be described in different ways:

$$x = (x_\epsilon : \epsilon \in \{0,1\}^d) = (x_\epsilon : \epsilon \subseteq [d]),$$

in where we see the hypercube $\{0,1\}^d$ in two ways: First as a set of sequences $\epsilon = \epsilon_1 \dots \epsilon_d$ of zeros and ones, and second as subsets of $[d]$, in where a subset $\epsilon \subseteq [d]$ corresponds with a sequence $\epsilon_1 \dots \epsilon_d \in \{0,1\}^d$ such that $\epsilon_i = 1$ if and only if $i \in \epsilon$.

For $x \in X$ we write $x^{[d]} = (x, \cdots, x) \in X^{[d]}$, and we denote $\Delta_X = \{x^{[d]} : x \in X\}$.

We also isolate the first coordinate (associated to $\epsilon = (0, \cdots, 0) \in \{0,1\}^d$ or $\epsilon = \emptyset \subseteq [d]$), writing $X_*^{[d]} = X^{2^d-1}$, so for $x \in X^{[d]}$ we can write $x = (x_\emptyset, x_*)$ with $x_\emptyset \in X$ and $x_* \in X_*^{[d]}$. Additionally, we decompose a point $x \in X^{[d]}$ as $x = (x', x'')$ with $x', x'' \in X^{[d-1]}$ where $x' = (x_{\epsilon 0} : \epsilon \in \{0,1\}^{d-1})$ and $x'' = (x_{\epsilon 1} : \epsilon \in \{0,1\}^{d-1})$.

Definition 1.10 *Let $(X, T_1, \cdots, T_d)$ be a $\mathbb{Z}^d$-system. The set of directional dynamical cubes associated to $(X, T_1, \cdots, T_d)$ is defined by*

$$Q_{T_1,\cdots,T_d}(X) = \overline{\{(T_1^{n_1\epsilon_1} \dots T_d^{n_d\epsilon_d} x)_{\epsilon\in\{0,1\}^d} : x \in X, \vec{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d\}} \subseteq X^{[d]}.$$

Additionally, given $x_0 \in X$, we consider the following restriction of these cubes to $X_*^{[d]}$:

$$K^{x_0}_{T_1,\cdots,T_d}(X) = \overline{\{(T_1^{n_1\epsilon_1} \dots T_d^{n_d\epsilon_d} x_0)_{\epsilon\in\{0,1\}^d\setminus\{\vec{0}\}} : \vec{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d\}} \subseteq X_*^{[d]}$$

We now define the notion of continuity point, which are the points for which $K^{x_0}_{T_1,\cdots,T_d}(X)$ coincides with the fiber of x_0 through the projection of $Q_{T_1,\cdots,T_d}(X)$ on the first coordinate.

Definition 1.11 *Let $(X, T_1, \cdots, T_d)$ be a $\mathbb{Z}^d$-system. We say that $x_0 \in X$ is a continuity point if $K^{x_0}_{T_1,\cdots,T_d}$ coincides with the set of $\boldsymbol{x} \in X_*^{[d]}$ such that $(x_0, \boldsymbol{x}) \in Q_{T_1,\cdots,T_d}(X)$.*

The set of continuity points $x_0 \in X$ is a dense G_δ set of points of X (by [25] Lemma 4.5), so there always exists a continuity point (infinite many indeed).

Now we define a crucial property of a $\mathbb{Z}^d$-system for our purposes.

Definition 1.12 *Let $(X, T_1, \cdots, T_d)$ be a $\mathbb{Z}^d$-system. We say that $(X, T_1, \cdots, T_d)$ has the unique closing parallelepiped property if whenever $x, y \in Q_{T_1,\cdots,T_d}(X)$ have $2^d - 1$ coordinates in common then $x = y$.*

If a system $(X, T_1, \cdots, T_d)$ has the unique closing parallelepiped property, we will just say that $(X, T_1, \cdots, T_d)$ has the closing property.

Definition 1.13 *For $x, y \in X$, $a_* \in X_*^{[d-1]}$ and $j \in [d]$, define $\boldsymbol{z}(x, y, a_*, j) \in X^{[d]}$ with* *coordinates*

$$(\boldsymbol{z}(x,y,a_*,j))_\epsilon = \begin{cases} x & \text{if } \epsilon=\emptyset \\ y & \text{if } \epsilon=\{j\} \\ (a_*)_\eta & \text{if } \epsilon=\Psi_j^0(\eta)\vee\epsilon=\Psi_j^1(\eta) \end{cases},$$

where the maps Ψ_j^0 *and* Ψ_j^1 *denote*

$$\Psi_j^0:\{0,1\}^{d-1}\to\{0,1\}^d,\quad \epsilon\to\Psi_j^0(\epsilon)=\epsilon_1\epsilon_2\cdots\epsilon_{j-1}0\epsilon_j\cdots\epsilon_{d-1},$$

$$\Psi_j^1:\{0,1\}^{d-1}\to\{0,1\}^d,\quad \epsilon\to\Psi_j^1(\epsilon)=\epsilon_1\epsilon_2\cdots\epsilon_{j-1}1\epsilon_j\cdots\epsilon_{d-1}.$$

As we saw before, it is possible to characterize dynamical properties with certain equivalence relations. The case of the unique closing parallelepiped property is not different, and we define its equivalence relation now.

Definition 1.14 *Let* $(X,T_1,\cdots,T_d)$ *be a* $\mathbb{Z}^d$*-system. For each* $j\in[d]$ *we define the* T_j*-regionally proximal relation as*

$$\mathcal{R}_j(X)=\{(x,y)\in X\times X:\exists a_*\in X_*^{[d-1]},\boldsymbol{z}(x,y,a_*,j)\in Q_{T_1,\cdots,T_d}(X)\}.$$

Finally, we define the $(T_1,\cdots,T_d)$*-regionally proximal relation as*

$$\mathcal{R}_{T_1,\cdots,T_d}(X)=\bigcap_{j=1}^d\mathcal{R}_{T_j}(X).$$

The following result is of great relevance because it establishes a very strong structure in systems with the closing property, that allows to separate the dynamics.

Theorem 1.14 *Let* $(X,T_1,\cdots,T_d)$ *be a minimal distal* $\mathbb{Z}^d$*-system. The following statements are equivalent.*

1. *The system* $(X,T_1,\cdots,T_d)$ *has the unique closing parallelepiped property.*
2. *The following equality holds* $\mathcal{R}_{T_1,\cdots,T_d}=\Delta_X$.
3. *The structure of* $(X,T_1,\cdots,T_d)$ *can be described as follows:*
   - *It is a factor of a minimal distal* $\mathbb{Z}^d$*-system* $(Y,T_1,\cdots,T_d)$ *which is a joining of* $\mathbb{Z}^d$*-systems* $(Y_1,T_1,\cdots,T_d),\cdots,(Y_1,T_1,\cdots,T_d)$*, where for each* $i\in\{1,\cdots,d\}$ *the action of* T_i *on* Y_i *is the identity;*
   - *for each* $i,j\in\{1,\cdots,d\}$*,* $i<j$*, there exists a* $\mathbb{Z}^d$*-system* $(Y_{i,j},T_1,\cdots,T_d)$ *which is a common factor of* $(Y_i,T_1,\cdots,T_d)$ *and* $(Y_j,T_1,\cdots,T_d)$ *and where* T_i *and* T_j *act as the identity; and*
   - *(iii) the system* Y *is jointly relatively independent with respect to the systems*

   $$((Y_{i,j},T_1,\cdots,T_d):i,j\in\{1,\cdots,d\},i<j).$$

   more precisely, $Y=K^{x_0}_{T_1,\cdots,T_d}$ *and* $Y_j=K^{x_0}_{T_1,\cdots,T_{j-1},T_{j+1},\cdots,T_d}$ *with* x_0 *being a continuity point.*

Remark 4 Notice that the closing property is preserved by taking factors, by point 3. of Theorem 1.14.

Cabezas, Donoso, and Maass [[2], Lemma 9.2.] proved that for an affine nilsystem $(\mathbb{T}^r, T_1, \cdots, T_d)$, if

$$\prod_{i=1}^{d}(A_i - I) = 0, \text{ and for all } j \in [d], \prod_{\substack{i=1 \\ i \neq j}}^{d}(A_i - I)\alpha_j = 0,$$

then $(\mathbb{T}^r, T_1, \cdots, T_d)$ has the closing property. In particular, rotations have the closing property. However, not all nilsystems have the closing property. Indeed, we now give an example of a 2-step nilsystem that does not have this property. Let $H = \mathbb{R}^3$ be the group with the multiplication given by

$$(a, b, c) \cdot (a', b', c') = (a + a', b + b', c + c + ab').$$

This is the Heisenberg group, introduced in Section 1.2.

Let $H_2 = [H, H]$ the group commutator, by a direct computation we have that $H_2 = \{(0, 0, c) : c \in \mathbb{R}\}$ and thus H_2 is central in H. Therefore H is a 2-step nilpotent Lie group and $\Gamma = \mathbb{Z}^3$ is a cocompact subgroup, meaning that $X_H = H/\Gamma$ is a compact space. Let $\alpha \in \mathbb{R}$ be such that $1, \alpha, \alpha^{-1}$ are linearly independent over $\mathbb{Q}$. Let $s = (\alpha, 0, 0)$ and $t = (0, \alpha^{-1}, \alpha)$. These two elements induce two transformations $S, T : X_H \to X_H$ given by

$$S(h\Gamma) = sh\Gamma, \ T(h\Gamma) = th\Gamma, \ \forall h \in H.$$

By Donoso and Sun [26], we have the following propositions.

Lemma 1.3 *Let X_H, S and T defined as above. Then (X_H, S, T) is a minimal distal system with commuting transformations S and T.*

Proposition 1.6 *On the Heisenberg system (X_H, S, T), we have that*

$$\mathcal{R}_{S,T}(X_H) = \{((a, b, c)\Gamma, (a, b, c')\Gamma) \in X_H \times X_H \mid a, b, c, c' \in \mathbb{R}\}.$$

This yields that the system (X_H, S, T) does not have the unique closing parallelepiped property by Theorem 1.14, since $\mathcal{R}_{S,T}(X_H)$ is not the diagonal.

Chapter 2

Recurrence for General Group Actions

This chapter focuses on the concept of recurrence for systems and families of systems, along with the notion of Bohr recurrence and its essential properties. We begin by introducing the sets of return times of a system, which form the basis for defining recurrence for a specific system. We explore this notion, including its basic properties, connections with minimality, sindeticity, and combinatorics.

Next, we define recurrence for a specific family of systems, with a particular emphasis on the family of rotations, also known as Bohr recurrence. We prove that Bohr recurrence is essentially recurrence for the family of equicontinuous systems. Additionally, we explore its relationship with almost periodicity and provide examples of nontrivial sets of Bohr neighborhoods of zero. Then we devote ourselves to explore $\mathbb{Z}^d$-Bohr recurrence. We establish several properties for sets of $\mathbb{Z}^d$-Bohr recurrence, introducing important concepts such as the Ramsey Property and the notion of essential sets of $\mathbb{Z}^d$-Bohr recurrence, and studying its relationship with the closing property.

2.1. Recurrence in Topological Dynamical Systems

Henceforth, unless otherwise stated, G will refer to a locally compact abelian group. In the following sections, we will introduce all the necessary concepts related to recurrence.

To begin, we define the notion of return times in a topological dynamical system.

Definition 2.1 *For a dynamical system (X, G), and sets $U, V \subseteq X$ we denote*

$$N_G(V, U) = \{g \in G \mid V \cap g^{-1}U \neq \emptyset\}.$$

In the case $U = V$ we just denote $N_G(U) = N_G(U, U)$. In the case $V = \{x\}$, we denote

$$N_G(x, U) = \{g \in G \mid gx \in U\}.$$

We now define the notion of recurrence in a topological dynamical system.

Definition 2.2 *A set $R \subseteq G$ is a set of recurrence for a system (X, G) if there exists $x \in X$ such that for all neighborhood U of x we have that $R \cap N_G(x, U) \neq \emptyset$.*

Remark 5 Notice that $R \subseteq G$ is a set of recurrence for a system (X, G) if and only if $\forall U \subseteq X$ nonempty open set, $R \cap N_G(U) \neq \emptyset$.

Definition 2.3 *Let (X, G) be a dynamical system equipped with a metric d. The set of ϵ-returns of (X, G) is*

$$\mathcal{R}_\epsilon(X, G) = \{g \in G \mid \inf_{x \in X} d(gx, x) < \epsilon\}.$$

If the context allows it, we will only write $\mathcal{R}_\epsilon$, instead of $\mathcal{R}_\epsilon(X, G)$.

Remark 6 Notice that for a t.d.s. (X, G), $R \subseteq G$ is a set of recurrence if and only if for every $\epsilon > 0$ we have $R \cap \mathcal{R}_\epsilon(X, G) \neq \emptyset$.

A stronger form of recurrence can be defined, which allows for the associated condition to hold at any point.

Definition 2.4 *A set $R \subseteq G$ is a set of pointwise recurrence for a system (X, G) if for all $x \in X$ and for all neighborhood U of x we have that $R \cap N_G(x, U) \neq \emptyset$.*

We will usually use the following notation.

Definition 2.5 *For $A, B \subseteq G$ and $g \in G$. Using additive notation for the group operation we define:*

- $A - g = \{h \in G \mid g + h \in A\}$,
- $A - B = \{a - b \mid a \in A, b \in B\}$.

In the case $G = \mathbb{Z}^d$ we also define for $n = (n_1, \cdots, n_d) \in \mathbb{Z}^d$:

- $nA = \{(n_1 m_1, \cdots, n_d m_d) \in \mathbb{Z}^d \mid m \in A\}$,
- $A/n = \{m \in \mathbb{Z}^d \mid (n_1 m_1, \cdots, n_d m_d) \in A\}$.

The following notion has an important connection with both minimality and recurrence.

Definition 2.6 *A subset $S \subseteq G$ is syndetic if there exists a compact set $F \subseteq G$ such that $FS = G$.*

In a minimal system, the return times of a given open set are syndetic, as the following proposition shows.

Proposition 2.1 *Let (X, G) be a topological dynamic system. If (X, G) is minimal then for all nonempty open set $U \subseteq X$, $N_G(U)$ is syndetic.*

Proof. Suppose that (X, G) is minimal, and let $U \subseteq X$ a nonempty open set. Let $K \subseteq G$ be the finite set such that $X = KU$, given by Theorem 1.2. We claim that $KN_G(U) = G$, in fact let $g \in G$, as gU is nonempty we have that there is some $k \in K$ such that $gU \cap kU \neq \emptyset$, or equivalently $k^{-1}gU \cap U \neq \emptyset$. Therefore $k^{-1}g \in N_G(U)$, which yields

$$g = kk^{-1}g \in KN_G(U).$$

□

The following theorem was proven by Host, Kra, and Maass in [1] for the case $G = \mathbb{Z}$, and now we present the proof for a general group action.

Theorem 2.1 *Let G be a countable discrete abelian group. For a set $R \subseteq G$, the following are equivalent:*

1. *R is a set of recurrence.*
2. *For every system (X, G) and every open cover $\mathcal{U} = (U_1, \cdots, U_r)$ of X, there exists $j \in \{1, \cdots, r\}$ and $g \in R$ such that $g \in N_G(U_j)$.*
3. *For every finite partition $G = C_1 \cup \cdots \cup C_r$ of G, there is some cell C_j containing two elements $g_1, g_2 \in C_j$ such that $g_1 g_2^{-1} \in R$.*
4. *Every syndetic subset E of G contains two elements $g_1, g_2 \in E$ such that $g_1 g_2^{-1} \in R$.*
5. *For every system (X, G) and every $\epsilon > 0$, $\mathcal{R}_\epsilon(X, G) \cap R \neq \emptyset$.*
6. *For every system (X, G), there exists a dense $x \in X$*
$$\inf_{g \in R} d(gx, x) = 0.$$
7. *For every minimal system (X, G), there exists a dense G_δ-set $X_0 \subseteq X$ such that for every $x \in X_0$,*
$$\inf_{g \in R} d(gx, x) = 0.$$

Proof. We will follow the order 1. ⇒ 7. ⇒ 6. ⇒ 5. ⇒ 2. ⇒ 3. ⇒ 4. ⇒ 1.

(1. $\Longrightarrow$ 7.) For $\epsilon > 0$, define Ω_ϵ to be
$$\Omega_\epsilon = \{x \in X : \exists g \in R,\ d(gx, x) < \epsilon\}.$$

The set Ω_ϵ is open since it is union of open sets. Let $U \subseteq X$ be an open ball of radius $\delta < \epsilon/2$, by hypothesis we have that $\exists g \in R$ such that $U \cap g^{-1}U \neq \emptyset$. As U has diameter less than ϵ, it follows that $U \cap g^{-1}U \subseteq \Omega_\epsilon$, in particular $U \cap \Omega_\epsilon \neq \emptyset$ so Ω_ϵ is dense in X. Defining $X_0 = \bigcap_{m \in \mathbb{N}} \Omega_{1/m}$, we have by Baire category theorem that X_0 is a G_δ-dense set, and by definition, for all $x \in X_0$, $\forall m \in \mathbb{N}$ we have that exists $g \in R$ such that $d(gx, x) < 1/m$ and therefore
$$\inf_{g \in R} d(gx, x) = 0.$$

(7. $\Longrightarrow$ 6.) Direct from applying 7. to a minimal subset of X.

(6. $\Longrightarrow$ 5.) Direct from the infimum characterization.

(5. $\Longrightarrow$ 2.) Let $\epsilon > 0$ be the Lebesgue number of the cover $\mathcal{U} = (U_1, \cdots, U_r)$. Let $x \in X$ and $g \in \mathcal{R}_\epsilon(X, G) \cap R$ associated to 5., and let $j \in \{1, \cdots, r\}$ such that $B(x, \epsilon) \subseteq U_j$, then $g \in N_G(U_j)$ given that $x, gx \in B(x, \epsilon) \subseteq U_j$.

(2. $\Longrightarrow$ 3.) Given a partition $G = C_1 \cup \cdots \cup C_r$, we can define $x \in \{1, \cdots, r\}^G$ such that $x(g) = i \iff g \in C_i$, and with that we consider the subsystem $(X = \overline{O_G(x)}, G)$ of the Bebutov system[2] $(\{1, \ldots, r\}^G, G)$. In this way, we can define the partition of clopens[3] sets

[2] See [16] Definition 1.6 for a definition of a Bebutov system.

[3] A clopen is a subset of a topological space which is open and close at the same time.

given by $U_i = [i]_{e_G}$. Then applying 2. to (X, G) we have that there exist $j \in \{1, \cdots, r\}$ and $g \in R$ such that $g \in N_G(U_j)$. In this light, $U_j \cap gU_j$ is an open nonempty set, in particular by density we have that there exists $h \in G$ such that $hx \in U_j \cap gU_j$. Thus we have $x(h) = j$ and $x(g^{-1}h) = j$, which yields $h, (g^{-1}h) \in C_j$ are such that $h(g^{-1}h)^{-1} = g \in R$.

(3. $\implies$ 4.) Let E be a syndetic subset of G, and let $F = \{f_1, \cdots, f_r\} \subseteq G$ finite such that $FE = G$, given by syndeticity and the fact that G is discrete. We choose a partition of $G = C_1 \cup \cdots \cup C_r$ such that $C_j \subseteq f_j E$. By hypothesis, there is a cell C_j containing two elements $g_1, g_2 \in C_j$ such that $g_1 g_2^{-1} \in R$, therefore we have that $g_1 f_j^{-1}, g_2 f_j^{-1} \in E$ are such that $(g_1 f_j^{-1})(g_2 f_j^{-1})^{-1} = g_1 g_2^{-1} \in R$, concluding the implication.

(4. $\implies$ 1.) For (X, G) minimal and $U \subseteq X$ an open nonempty set, let $x \in X$ and $E := N_G(x, U)$. As (X, G) is minimal, we have that E is syndetic by Proposition 2.1, and then there exists $g_1, g_2 \in E$ such that $g_1 g_2^{-1} \in R$. Note that $g_2 x \in U$ and $g_1 g_2^{-1}(g_2 x) \in U$ therefore $g_1 g_2^{-1} \in N_G(U) \cap R$, concluding.

□

The notion of recurrence can be extended to a specific family of systems in the following manner.

Definition 2.7 *If $\mathcal{F}$ is a family of G-systems, a set $R \subseteq G$ is a set of recurrence for the family $\mathcal{F}$ if for any minimal system (X, G) in the family $\mathcal{F}$, R is a set of recurrence for (X, G).*

When we say that $R \subseteq G$ is a *set of recurrence* without specifying any particular family, we will interpret that R is a set or recurrence for all G-system.

Proposition 2.2 *A subset $R \subseteq G$ is a set of recurrence if and only if for all system (X, G), we have $R \cap \mathcal{R}_\epsilon \neq \emptyset$, for all $\epsilon > 0$.*

Proof. The direction to the left comes directly from Remark 6. For the other implication, let (X, G) be a t.d.s., and let (Y, G) a minimal subsystem of (X, G). Notice that for $\epsilon > 0$

$$\emptyset \neq R \cap \mathcal{R}_\epsilon(Y, G) \subseteq R \cap \mathcal{R}_\epsilon(X, G),$$

concluding. □

2.2. Bohr Recurrence

In this section, we will introduce the fundamental concepts for studying sets of recurrence in the family of rotations. We will begin by discussing duality in locally compact abelian groups in order to define Bohr recurrence. Then we will study Bohr almost periodicity, which is a notion closely related to Bohr recurrence. Thereafter we will define and prove some useful properties of sets of Bohr recurrence in $\mathbb{Z}^d$, that will be of great importance in subsequence chapters.

2.2.1. Duality in locally compact abelian groups

For the purposes of this discussion, we will denote the unitary circle as $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Definition 2.8 *Let G be a locally compact abelian group. A character of G is a continuous* *homomorphism* $\chi : G \to \mathbb{S}^1$.

Remark 7 The set $\hat{G}$ of characters of G forms an abelian group under pointwise multiplication.

We can endowed $\hat{G}$ of the following topology in order to make it a locally compact abelian topological group.

Theorem 2.2 ([27], Theorem 23.15) *Let G be a locally compact abelian group. For a compact subset $F \subseteq G$ and $\epsilon > 0$ set*

$$P(F, \epsilon) := \{\chi \in \hat{G} \mid |\chi(g) - 1| < \epsilon, \forall g \in F\}.$$

Then, with all sets $P(F, \epsilon)$ taken as an open basis at the identity $1_{\hat{G}}$, $\hat{G}$ is a locally compact abelian topological group.

We have the following theorems, which will be useful later.

Theorem 2.3 ([27], Theorem 24.17) *Let G be a locally compact abelian group. Then, if G is compact, $\hat{G}$ is discrete; and if G is discrete then $\hat{G}$ is compact.*

Theorem 2.4 ([27], Theorem 24.3) *Let G be a compact abelian group. Then G is isomorphic to $\hat{\hat{G}}$ (topologically and algebraically).*

Next we define Bohr neighborhoods of 0 which will allow us to introduce sets of Bohr recurrence with general group actions.

Definition 2.9 *Let G be a locally compact abelian group, and $\chi_1, \ldots, \chi_d \in \hat{G}$. The Bohr neighborhood of 0 in G having rank d and radius $\epsilon > 0$ determined by $\{\chi_1, \ldots, \chi_d\}$ is*

$$Bohr(\chi_1, \ldots, \chi_d; \epsilon) := \{g \in G : |\chi_i(g) - 1| < \epsilon, \quad \forall i \in [d]\}.$$

Remark 8 Theorem 2.4 implies that if G is compact, then the topology generated by the basis of e_G given by the sets $Bohr(\chi_1, \ldots, \chi_d; \epsilon)$, for $\chi_1, \ldots, \chi_d \in \hat{G}$ and $\epsilon > 0$, coincides with the topology in G.

We will now present some classic definitions in the context of families of subsets of a group. These definitions include the notion of dual family, and the properties of being a filter and partition regular.

Lemma 2.1 *Let G be a locally compact abelian group. A family $\mathcal{F}$ of subsets of G is said to be*

- *a filter if it is upward closed and closed under intersections, and*
- *partition regular if for every $A \in \mathcal{F}$, if we partition A into N sets $\{A_n\}_{n=1}^N$, then there is a $n \in [N]$ such that $A_n \in \mathcal{F}$.*

Additionally, we define the dual family $\mathcal{F}^*$ of a family $\mathcal{F}$ of subsets of G as

$$\mathcal{F}^* = \{B \subseteq G \mid B \cap A \neq \emptyset, \ \forall A \in \mathcal{F}\}.$$

Definition 2.10 *Let G be a locally compact abelian group. We say that $V \subset G$ is a Bohr$_0$ set if V contains a Bohr neighborhood of 0. Additionally, we say that W is in Bohr$_0^*$ is for all V Borh$_0$ set, $W \cap V \neq \emptyset$.*

Remark 9 The family of Bohr$_0^*$ sets is the dual family of Bohr$_0$ sets.

We will prove that the family of Bohr$_0$ sets is a filter, meanwhile its dual family is partition regular. To prove this, we first prove the following results.

Lemma 2.2 *Let G be a locally compact abelian group and $\mathcal{F}$ an upward closed family of subsets of G. Then $\mathcal{F} = (\mathcal{F}^*)^*$.*

Proof. First, let $V \in \mathcal{F}$, by definition $\forall W \in \mathcal{F}^*$ we have that $V \cap W \neq \emptyset$, and then $V \in (\mathcal{F}^*)^*$.

Now, suppose that $V \in (\mathcal{F}^*)^*$, therefore $G \setminus V$ is not in $\mathcal{F}^*$ set. In particular, there exists $A \in \mathcal{F}$ such that $A \cap (G \setminus V) = \emptyset$ which is equivalent to $A \subseteq V$, and therefore $V \in \mathcal{F}$ thanks to the fact that $\mathcal{F}$ is upward closed. □

Remark 10 Notice that for a family $\mathcal{F}$, the family $\mathcal{F}^*$ is always upward closed.

Proposition 2.3 *Let G be a locally compact abelian group and $\mathcal{F}$ an upward closed family of subsets of G. Then, $\mathcal{F}$ is a filter if and only if $\mathcal{F}^*$ is partition regular.*

Proof. Suppose that $\mathcal{F}$ is a filter and let $A \in \mathcal{F}^*$ and $\{A_n\}_{n=1}^N$ a partition of A. Suppose by contradiction that for each $n \in [N]$ there is $V_n \in \mathcal{F}$ such that $A_n \cap V_n = \emptyset$. As $\mathcal{F}$ is a filter, then $V = \bigcap_{n=1}^N V_n \in \mathcal{F}$, but

$$A \cap V = \bigcup_{n=1}^{N} A_n \cap V \subseteq \bigcup_{n=1}^{N} A_n \cap V_n = \emptyset,$$

which cannot be. Therefore, there is $n \in [N]$ such that $A_n \in \mathcal{F}^*$.

For the converse direction, by Lemma 2.2 it is enough to prove that if $\mathcal{F}$ is partition regular, then $\mathcal{F}^*$ is a filter. To prove this, let $C_1, C_2 \in \mathcal{F}^*$ and $V \in \mathcal{F}$, as $\mathcal{F}$ is partition regular we have that $V \cap C_1$ is in $\mathcal{F}$ as $(V \setminus C_1) \cap C_1 = \emptyset$. Therefore $(V \cap C_1) \cap C_2 \neq \emptyset$, and as $V \in \mathcal{F}$ was arbitrary, we conclude that $C_1 \cap C_2 \in \mathcal{F}^*$, concluding. □

Now we are able to prove the aforementioned property of the family of Bohr$_0$ sets and its dual family.

Corollary 2.1 *The family of Bohr$_0$ sets is a filter and the family of Bohr$_0^*$ sets is partition regular.*

Proof. First, it is obvious that the family of Bohr$_0$ sets is upward closed as a Bohr$_0$ set is defined by containing a Bohr neighborhood of 0. Now, let $\chi_1, \dots, \chi_d, \chi'_1, \dots, \chi'_p \in \hat{G}$ and $\epsilon, \epsilon' > 0$. Note that

$$Bohr(\chi_1, \dots, \chi_d, \chi'_1, \dots, \chi'_p; \min(\epsilon, \epsilon')) \subseteq Bohr(\chi_1, \dots, \chi_d; \epsilon) \cap Bohr(\chi'_1, \dots, \chi'_p; \epsilon'),$$

which yields that the family of Bohr_0 sets is closed under intersections.

The rest of the corollary follows from Proposition 2.3. $\square$

For $x \in \mathbb{R}$ we denote the maximum integer z such that $z \leq x$ as $\lfloor x \rfloor$, the decimal part of x as $\{x\} = x - \lfloor x \rfloor \in [0,1)$, and the torus norm of x as $\|x\|_{\mathbb{T}} = \min\{\{x\}, 1-\{x\}\}$ which is the distance of x to the nearest integer. We generalize this notation for $x \in \mathbb{R}^r$ as follows:

- $\lfloor x \rfloor := (\lfloor x_i \rfloor)_{i \in [r]}$,
- $\{x\} := (x_i - \lfloor x_i \rfloor)_{i \in [r]} \in [0,1)^r$,
- $\|x\|_{\mathbb{T}^r} := \sum_{i=1}^r \|x_i\|_{\mathbb{T}}$.

When it is well understood, we will simply denote $\|x\|$ instead of $\|x\|_{\mathbb{T}^r}$.

Remark 11 In the case when the action is given by $\mathbb{Z}^d$, it is straightforward to see that $V \subseteq \mathbb{Z}^d$ is a Bohr_0 set if $\exists \epsilon > 0$, $d \in \mathbb{N}$ and $\alpha_1, \cdots, \alpha_d \in \mathbb{T}^d$ such that

$$\{n \in \mathbb{Z}^d \mid \|n_i \alpha_i\| < \epsilon, \forall i \in [d]\} \subseteq V.$$

A classic example of a Bohr_0 set which is not trivial is the following.

Proposition 2.4 *Let (X,G) be a rotation, $x \in X$ a point, and $U \subseteq X$ a neighborhood of x. Then $N(x,U)$ is a $Bohr_0$ set.*

Proof. First, without loss of generality we can assume that $x = e_X$ given that every neighborhood of x is a translation of a neighborhood of e_X. Let $\varphi : G \to X$ be the homomorphism associated to the action of G on e_X. For a open neighborhood U of e_X we have to prove that

$$N(e_X, U) = \{g \in G \mid \varphi(g) \in U\} = \varphi^{-1}(U)$$

is a Bohr_0 set. Notice that suffices to prove that $U \subseteq X$ is a Bohr_0 in X, given that if $\chi_1, \ldots, \chi_d \in \hat{X}$, and $\epsilon > 0$ are such that

$$Bohr(\chi_1, \ldots, \chi_d; \epsilon) \subseteq U,$$

then

$$Bohr(\chi_1 \circ \varphi, \ldots, \chi_d \circ \varphi; \epsilon) \subseteq \varphi^{-1}(U),$$

with $\chi_1 \circ \varphi, \ldots, \chi_d \circ \varphi \in \hat{G}$. In this light, by Remark 8, we have that U is a Bohr_0, concluding. $\square$

As minimal equicontinuous systems are basically minimal rotations, we can extent Proposition 2.4 to equicontinuous systems.

Proposition 2.5 *Let (X,G) be an equicontinuous dynamic system. Then for all $x \in X$ and for all neighborhood U of x, $N_G(x,U)$ is a $Bohr_0$ set.*

Proof. Let $x \in X$. Without loss of generality (X,G) is minimal, since we can take the orbit of x under G, and as (X,G) is distal, such orbit is a minimal equicontinuous system.

From Theorem 1.7 we know that there is an isomorphism of topological dynamical systems $\pi : (X, G) \to (Y, G)$ where (Y, G) is a minimal rotation. Observe that

$$N_G(x, U) = \{g \in G \mid gx \in U\} = \{g \in G \mid g\pi(x) \in \pi(U)\} = N_G(\pi(x), \pi(U)),$$

which is a set of return times in a rotation. By Proposition 2.4, the latter is a Bohr_0 and so is the former. $\square$

We now define a notion which is the uniform version of almost periodicity.

Definition 2.11 *A dynamical system (X, G) is said to be uniformly almost periodic if, for every $x \in X$ and every nonempty open set $U \subseteq X$, the set $N(x, U)$ is syndetic.*

Uniform almost periodicity can be used to characterize equicontinuous dynamical systems as follows.

Theorem 2.5 (see [12, Theorem 2 Chapter 2]) *A dynamical system is equicontinuous if and only if it is uniformly almost periodic.*

To end this section, we show that being a Bohr_0 set is a stronger than being syndetic.

Proposition 2.6 *Let $A \subseteq G$ a $Bohr_0$ set. Then A is syndetic.*

Proof. Let $\chi_1, \ldots, \chi_d \in \hat{G}$ (which we are going to write in additive notation) and $\epsilon > 0$ such that

$$Bohr(\chi_1, \ldots, \chi_d, \epsilon) \subseteq A.$$

Consider the system $(\mathbb{T}^d, G)$ such that for $x \in \mathbb{T}^d$ and $g \in G$ the action of g over x is defined by

$$gx = (\chi_i(g) + x_i)_{i=1}^d.$$

Note that this system is equicontinuous, indeed for $x, y \in \mathbb{T}^d$, note that

$$\|gx - gy\| = \|x - y\|.$$

Therefore, from Theorem 2.5 we deduce that $N(0, B(0, \epsilon))$ is syndetic, and as we have that

$$N(0, B(0, \epsilon)) = \{g \in G \mid \|\chi_i(g)\| < \epsilon, \quad \forall i \in [d]\} = Bohr(\chi_1, \ldots, \chi_d, \epsilon),$$

we conclude that $Bohr(\chi_1, \ldots, \chi_d, \epsilon)$ is syndetic.

$\square$

2.2.2. Bohr Almost Periodicity

In this section we consider G as an abelian countable discrete topological group. We will study the notion of Almost Periodicity, and its relation with Bohr recurrence.

Definition 2.12 *Let $f : G \to \mathbb{R}$ and $\epsilon > 0$. We define the set of ϵ-periods of f as*

$$Per(f, \epsilon) := \{g \in G \mid \sup_{h \in G} d(f(hg), f(h)) < \epsilon\}.$$

We say that f is

- *Almost periodic (A.P.) on G if $\forall \epsilon > 0$, $Per(f, \epsilon)$ is syndetic.*
- *Bohr Almost periodic (B.A.P.) on G if $\forall \epsilon > 0$, $Per(f, \epsilon)$ is a $Bohr_0$ set.*

We begin showing that both notions coincide, and we give some classical equivalences spread in the literature, usually proved for the case $G = \mathbb{Z}$.

Lemma 2.3 *Let $f : G \to \mathbb{R}$. The following are equivalent:*

1. *f is B.A.P.,*
2. *f is A.P.,*
3. *$Gf := \{\sigma_g f \mid g \in G\}$ is precompact in $l^\infty(G)$, where σ_g is the shift by g.*
4. *There exist an equicontinuous system (X, G), a point $x \in X$, and a continuous function $h : X \to \mathbb{R}$ such that $f(g) = h(gx)$.*

Proof. (1. $\Longrightarrow$ 2.) Direct from Proposition 2.6.

(2. $\Longrightarrow$ 3.) Let $\epsilon > 0$ and $F \subseteq G$ be a compact set such that $FPer(f, \epsilon) = G$. We will see that Gf is totally bounded, or more specifically that

$$Gf \subseteq \bigcup_{g \in F} B(\sigma_g f, \epsilon).$$

In fact, let $h \in G$. We take $k \in F$ and $v \in Per(f, \epsilon)$ such that $kv = h$. We have that

$$\sup_{x \in G} d(f(xh), f(xk)) = \sup_{x \in G} d(f(xh), f((xh)v^{-1})) < \epsilon,$$

in where we used that v^{-1} is an ϵ-period of f. In this way,

$$\sigma_h f \in B(\sigma_k f, \epsilon) \subseteq \bigcup_{g \in F} B(\sigma_g f, \epsilon).$$

(3. $\Longrightarrow$ 4.) As $f \in \mathbb{R}^G$, we can always consider the shift associated to the orbit of f under the action of $\{\sigma_g\}_{g \in G}$. By hypothesis $X := \overline{Gf}$ is compact, and if we equip it with the uniform metric ($d(h, k) = \sup_{g \in G} |h(g) - k(g)|$), we have that (X, G) is an equicontinuous system. Define $h : X \to \mathbb{R}$ as $h(x) = x_{e_g}$. This function is clearly continuous (indeed, it is 1-Lipschitz), and note that $f(g) = (\sigma_g f)(e_G) = h(g \cdot f)$, concluding this implication.

(4. $\Longrightarrow$ 1.) We have to see that for all $\epsilon > 0$

$$Per(f, \epsilon) = \{g \in G \mid \sup_{v \in G} d(f(vg), f(v)) < \epsilon\} = \{g \in G \mid \sup_{v \in G} d(h(gvx), h(vx)) < \epsilon\},$$

is Bohr_0. As h is uniformly continuous, it is enough to see that for all $\epsilon > 0$

$$\{g \in G \mid \sup_{v \in G} d_X(gvx, vx) < \epsilon\},$$

is Bohr_0. Note that without loss of generality d_X is an isometry by Proposition 1.1, and then the previous statement is equivalent to see that for all $\epsilon > 0$,

$$\{g \in G \mid d_X(gx, x) < \epsilon\} = N_G(x, B(x, \epsilon)),$$

is Bohr_0. The latter statement follows from the fact that $N_G(x, B(x, \epsilon))$ is Bohr_0 by being the set of return times of a point to one of its neighborhood in an equicontinuous system, by Proposition 2.5. □

The following lemma provides another nontrivial example of a Bohr_0 set. It was proven by D. Glasscock, A. Koutsogiannis, and F. Richter in the case $G = \mathbb{Z}$ in [28, Lemma 2.10]. The proof for arbitrary G is fairly similar, and we write it for completeness.

Lemma 2.4 *Let (X, G) be an equicontinuous system. For all $\epsilon > 0$, the set*

$$\{g \in G \mid \sup_{x \in X} d_X(x, gx) < \epsilon\},$$

is a Bohr_0 set.

Proof. Let $\epsilon > 0$, and let $0 < \delta < \epsilon/3$ be sufficiently small so that for all $x, y \in X$ with $d_X(x, y) < \delta$ and for all $g \in G$, $d_X(gx, gy) < \epsilon/3$. Let Y be a finite δ-dense subset of X. The set

$$A_\epsilon := \bigcap_{y \in Y} N(y, B(y, \epsilon/3))),$$

is a Bohr_0 set, since it is the intersection of finitely many Bohr_0 sets by Proposition 2.5.

We will show that

$$A_\epsilon \subseteq \{g \in G \mid \sup_{x \in X} d_X(x, gx) < \epsilon\}. \tag{2.1}$$

Indeed, let $g \in A_\epsilon$ and $x \in X$. Given that Y is δ-dense, there exists $y \in Y$ such that $d_X(x, y) < \delta$. In this light, we have that $d_X(gx, gy) < \epsilon/3$ and $d_X(y, gy) < \epsilon/3$. By the triangle inequality, we derive that

$$d_X(x, gx) \leq d_X(x, y) + d_X(gx, gy) + d_X(gy, y) < \delta + 2\epsilon/3 \leq \epsilon.$$

The fact that $x \in X$ and $g \in A_\epsilon$ were arbitrary yields Eq. (2.1), and as A_ϵ is a Bohr_0 set, we conclude. □

2.2.3. $\mathbb{Z}^d$-Bohr Recurrence

In the following section we will study some properties of set of $\mathbb{Z}^d$-Bohr recurrence which will be of utility on subsequence chapters. We start showing that Bohr recurrence is equivalent to pointwise recurrence in minimal rotation.

Proposition 2.7 *Let $R \subseteq \mathbb{Z}^d$. Then R is a set of Bohr recurrence if and only if is a set of pointwise recurrence for minimal rotation.*

Proof. The left implication is obvious by the fact that pointwise recurrence is stronger than normal recurrence.

On the other hand, if $R \subseteq \mathbb{Z}^d$ a set of Bohr recurrence, let $(X, \tau_1, \cdots, \tau_d)$ a minimal rotation, $x \in X$ and $\epsilon > 0$. As R is a set of Bohr recurrence, we obtain

$$R \cap N_{T_1,\cdots,T_d}(B(e_X, \epsilon/2)) \neq \emptyset,$$

and then there exist $(n_1, \cdots, n_d) \in R$ and $y \in B(e_X, \epsilon/2)$ such that

$$d(\tau_1^{n_1} \cdots \tau_n^{n_d} y, e_X) < \epsilon/2.$$

In particular

$$d(\tau_1^{n_1} \cdots \tau_n^{n_d} y, y) \leq d(\tau_1^{n_1} \cdots \tau_n^{n_d} y, e_X) + d(y, e_X) < \epsilon/2 + \epsilon/2 = \epsilon.$$

By the right invariance of the distance, we conclude that $d(\tau_1^{n_1} \cdots \tau_n^{n_d} x, x) < \epsilon$, and then $R \cap N(x, B(x, \epsilon)) \neq \emptyset$. $\square$

Another useful fact is that in a set with Bohr recurrence, we can divide each coordinate by a number and still obtain a set with Bohr recurrence.

Proposition 2.8 *Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence and $\vec{k} \in \mathbb{N}^d$. Then the set $R_0 := \{\vec{n} \in \mathbb{Z}^d \mid (k_1 n_1, \cdots, k_d n_d) \in R\}$ is a set of Bohr recurrence.*

Proof. Let $(X, T_1, \cdots, T_d)$ be a minimal equicontinuous system, $x \in X$ and $\epsilon > 0$. Consider the system $(X \times \prod_{i=1}^d \mathbb{Z}_{k_i}, S_1, \cdots, S_d)$, where for $i \in [d]$, S_i is defined by

$$S_i(y, (m_1, \cdots, m_d)) = \begin{cases} (y, (m_1, \cdots, m_{i-1}, m_i + 1, m_{i+1}, \cdots, m_d)) & \text{if } m_i \neq k_i - 1, \\ (T_i y, (m_1, \cdots, m_{i-1}, 0, m_{i+1}, \cdots, m_d)) & \text{if } m_i = k_i - 1. \end{cases},$$

note that $(X \times \prod_{i=1}^d \mathbb{Z}_{k_i}, S_1, \cdots, S_d)$ is dynamical system with d-commuting dynamics, also these dynamics satisfy $S_1^{k_1 n_1} \ldots S_d^{k_d n_d}(x, \vec{m}) = (T_1^{n_1} \ldots T_d^{n_d} x, \vec{m})$, $\forall \vec{n} \in \mathbb{Z}^d$. From the fact that $(X, T_1, \cdots, T_d)$ is minimal, we have that $(X \times \prod_{i=1}^d \mathbb{Z}_{k_i}, S_1, \cdots, S_d)$ is also minimal. Furthermore, it is equicontinuous. In fact, given $(y, \vec{m}) \in X \times \prod_{i=1}^d \mathbb{Z}_{k_i}$ and $\epsilon > 0$, if $\delta > 0$ is obtain using the equicontinuity of X for y and ϵ, then set $\tilde{\delta} = \min(\delta, 1/2)$. Now, using the distance given by the sum of the distances of every system involved in the product for $X \times \prod_{i=1}^d \mathbb{Z}_{k_i}$, if $(z, \vec{n}) \in X \times \prod_{i=1}^d \mathbb{Z}_{k_i}$ is such that

$$d((y, \vec{m}), (z, \vec{n})) < \tilde{\delta},$$

then we have

$$d_X(y, z) < \delta, \text{ and } d(m_i, n_i) < 1/2, \ \forall i \in [d].$$

This yields $\vec{n} = \vec{m}$ and by the equicontinuity of X we have that $\forall l \in \mathbb{Z}^d$:

$$d_X(T_1^{l_1} \cdots T_d^{l_d} y, T_1^{l_1} \cdots T_d^{l_d} z) < \epsilon,$$

from where we deduce that for $p \in \mathbb{Z}^d$

$$d(S_1^{p_1} \cdots S_d^{p_d}(y, \vec{m}), S_1^{p_1} \cdots S_d^{p_d}(z, \vec{n})) = d_X(T_1^{\lfloor \frac{p_1}{k_1} \rfloor} \cdots T_d^{\lfloor \frac{p_d}{k_d} \rfloor} y, T_1^{\lfloor \frac{p_1}{k_1} \rfloor} \cdots T_d^{\lfloor \frac{p_d}{k_d} \rfloor} z) < \epsilon,$$

and therefore, the system is equicontinuous. If we consider $x \in X$, U an open neighborhood of x and the open neighborhood $U \times \{0\}^d$ of $(x, 0, \cdots, 0)$, then we have that as R is a set of Bohr recurrence, there exists $\vec{n} \in R$ such that

$$S_1^{n_1} \cdots S_d^{n_d}(x, 0) \in U \times \{0\}^d.$$

In this way, $\vec{n} = k\vec{m}$ for some $\vec{m} \in \mathbb{N}^d$ such that $T_1^{m_1} \cdots T_d^{m_d} x \in U$, and therefore $m \in N_{T_1,\cdots,T_d}(x, U) \cap R_0$ and we conclude that R_0 is a set Bohr recurrence. $\square$

In what follows, we will generalize to $\mathbb{Z}^d$-recurrence sets the Ramsey property, focusing in sets of Bohr recurrence and following the ideas from Host, Kra, and Maass in [1].

Definition 2.13 (Ramsey Property) *Let $d \in \mathbb{N}$. A property is Ramsey if for any set $R \subseteq \mathbb{Z}^d$ having this property, and any partition $R = A \cup B$, at least one of A or B has this property.*

Proposition 2.9 *The family of sets of $\mathbb{Z}^d$-Bohr recurrence has the Ramsey property.*

Proof. Assume that $R \subseteq \mathbb{Z}^d$ is a set of Bohr recurrence and that $R = A \cup B$ is a partition such that neither A nor B is a set of Bohr recurrence. Hence, there exist two minimal rotations $(X, \tau_1, \cdots, \tau_d)$ and $(Y, s_1, \cdots, s_d)$ and $\epsilon > 0$ such that $N_{\tau_1,\cdots,\tau_d}(e_X, B(e_X, \epsilon)) \cap A = \emptyset$ and $N_{s_1,\cdots,s_d}(e_Y, B(e_Y, \epsilon)) \cap B = \emptyset$. Consider the product system $X \times Y$ equipped with the translations by $\{(\tau_i, s_i)\}_{i=1}^d$ and set $Z = \overline{\mathcal{O}(e_X, e_Y)}$ the closed orbit of the identity in $X \times Y$. We know that Z is a compact abelian subgroup of $X \times Y$, which is minimal by distality. We consider $U = Z \cap (B(e_X, \epsilon) \times B(e_Y, \epsilon))$ an open neighborhood of (e_X, e_Y) in Z. We have that $R \cap N((e_X, e_Y), U) \neq \emptyset$, but this is a contradiction because

$$N((e_X, e_Y), U) \subseteq N_{s_1,\cdots,s_d}(e_Y, B(e_Y, \epsilon)) \cap N_{\tau_1,\cdots,\tau_d}(e_X, B(e_X, \epsilon)),$$

and $R = A \cup B$. $\square$

We introduce the following definition to avoid dealing with some pathological cases.

Definition 2.14 *Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence. We will say that R is essential if for every $n \in R$, we have $\forall j \in [d]$, $n_j \neq 0$.*

The next proposition shows that every relevant set of Bohr recurrence can be reduced to an essential set of Bohr recurrence.

Proposition 2.10 *Let $R \subseteq \mathbb{Z}^d \setminus \{\vec{0}\}$ be a set of Bohr recurrence. Then, there exist $d' \leq d$, a permutation $\pi : \mathbb{Z}^d \to \mathbb{Z}^d$, and an essential set of Bohr recurrence $R' \subseteq \mathbb{Z}^{d'}$ such that $R' \times \{0\}^{d-d'} \subseteq \pi(R)$.*

Proof. For $J \subseteq [d]$ denote

$$R_J := \{(n_j)_{j\in[d]} \in R \mid n_j \neq 0 \iff j \in J\}.$$

Notice that, as $\vec{0} \notin R$, we have that

$$R = \bigcup_{\substack{J\subseteq[d] \\ J\neq\emptyset}} R_J.$$

By the Ramsey property there exists $J \subseteq [d]$ nonempty such that R_J is a set of Bohr recurrence. Consider $d' = |J|$, $\pi : \mathbb{Z}^d \to \mathbb{Z}^d$ one permutation of coordinates taking the coordinates J to $\{1, \cdots, d'\}$ maintaining their order, and the set $R' = p(R_J)$ where $p : \mathbb{Z}^d \to \mathbb{Z}^J$ is the canonical projection. Then $R' \times \{0\}^{d-d'} \subseteq \pi(R)$ and R' is a set of Bohr recurrence which is essential.

□

Remark 12 Notice that if $\mathscr{X} = (X, T_1, \cdots, T_d)$ is a minimal nilsystem, and $R \subseteq \mathbb{Z}^d$ is a set of Bohr recurrence, then:

- If $\vec{0} \in R$, then R is trivially a set of recurrence for $\mathscr{X}$,
- otherwise, by Proposition 2.10 there exist $d' \leq d$, a permutation $\pi : \mathbb{Z}^d \to \mathbb{Z}^d$ and an essential set of Bohr recurrence $R' \subseteq \mathbb{Z}^{d'}$ such that $R' \times \{0\}^{d-d'} \subseteq \pi(R)$. Given that $\mathscr{Y} = (\overline{O_{T_{\pi^{-1}(1)},\ldots,T_{\pi^{-1}(d')}}(e_G)}, T_{\pi^{-1}(1)}, \ldots, T_{\pi^{-1}(d')})$ is a minimal nilsystem and for all $n \in R'$ if $m \in \{n\} \times \{0\}^{d-d'} \subseteq \pi(R)$ we have that
$$T_{\sigma_1}^{n_1} \cdots T_{\sigma_{d'}}^{n_{d'}} = T_1^{\pi^{-1}(m)_1} \cdots T_d^{\pi^{-1}(m)_d},$$
with $\sigma_i = \pi((1, \cdots, d))_i$ for $i \in [d']$. In this way, if R' is a set of Bohr recurrence for $\mathscr{Y}$ then R is a set of Bohr recurrence for $\mathscr{X}$.

Therefore, we can reduce to the case in which R is essential when studying recurrence in the family of nilsystems.

From now on, we will always assume that a set of Bohr recurrence $R \subseteq \mathbb{Z}^d$ is essential, unless we state otherwise, or if the context demands otherwise.

The following property allows us to remove as many bands as we want in an essensial set of recurrence, generalizing the fact that for a set of Bohr recurrence $R \subseteq \mathbb{Z} \setminus \{0\}$, we can remove as many elements as we want.

Proposition 2.11 (The Bands Property) *Let $R \subseteq \mathbb{Z}^d$ be an essential set of Bohr recurrence, $k \in \mathbb{Z} \setminus \{0\}$ and $i \in [d]$. Set $B_k^i = \{\vec{n} \in \mathbb{Z}^d \mid n_i = k\}$, then the set $R_0 = R \setminus B_k^i$ is a set of Bohr recurrence.*

Proof. Let $R \subseteq \mathbb{Z}^d$ a set of Bohr recurrence, $k \in \mathbb{Z}$ and $i \in [d]$. Define

$$B_k^i = \{\vec{n} \in \mathbb{Z}^d \mid n_i = k\}, \text{ and } R_0 = R \setminus B_k^i.$$

We prove first that $(R \cap B_k^i)$ is not a set of Bohr recurrence, indeed we can consider $X = (\mathbb{Z}_{|k|+1})^d$ the d-product of the cyclic groups $\mathbb{Z}_{|k|+1} = \{0, \ldots, |k|\}$, with the dynamic $(T_1, \cdots, T_d)$ where T_i correspond to the translation by the element $\{1_{i=j}\}_{j=1}^d$. Clearly $(\mathbb{Z}_{k+1})^d$ is a compact abelian group by being finite product of compact abelian groups, and the translations $(T_1, \cdots, T_d)$ are minimal. Note that for every $(n_1, \cdots, n_d) \in R \cap B_k^i$ we have that

$$\vec{0} \neq T_1^{n_1} \ldots T_d^{n_d} \vec{0},$$

by the fact that the i-th coordinate of $T_1^{n_1} \dots T_d^{n_d}\vec{0}$ is k mod $(|k|+1)$ which is

$$= \begin{cases} 1 & \text{if } k < 0 \\ k & \text{if } k > 0 \end{cases}.$$

Using the distance $d_{\mathbb{Z}_{|k|+1}}(x,y) = 1_{x=y}$ in $\mathbb{Z}_{|k|+1}$ and the classic distance $d(x,y) = \sum_{i=1}^{d} d_{\mathbb{Z}_{|k|+1}}(x_i, y_i)$ on the product $\mathbb{Z}_{|k|+1}^d$, we have that for every $(n_1, \cdots, n_d) \in R \cap B_k^i$,

$$d(\vec{0}, T_1^{n_1} \dots T_d^{n_d}\vec{0}) > 1/2,$$

and consequently $R \cap B_k^i$ cannot be a set of Bohr recurrence. Writing the partition

$$R = R_0 \cup (R \cap B_k^i),$$

we obtain that $(R \cap B_k^i)$ is not a set of Bohr recurrence. Hence, we conclude by the Ramsey property that R_0 is a set of Bohr recurrence, concluding. □

Now we will focus our attention in investigating some properties in $\mathbb{Z}^d$-Bohr recurrence for minimal distal $\mathbb{Z}^d$-systems, with the idea of relating the closing property with Bohr recurrence.

We say that a set $B \subseteq \mathbb{Z}^d$ contains a set of return times for a $\mathbb{Z}^d$-system if there exists a $\mathbb{Z}^d$-system $(X, T_1, \cdots, T_d)$, $x \in X$ and an open neighborhood U of x such that $N_{T_1,\cdots,T_d}(x, U) \subseteq B$.

Definition 2.15 *Let $B_1, \cdots, B_d \subseteq \mathbb{Z}^{d-1}$ for $d \geq 2$. We define the joining of $B_1, \cdots, B_d$ as the set*

$$\{\vec{n} \in \mathbb{Z}^d \mid (n_1, \cdots, n_{i-1}, n_{i+1}, \cdots, n_d) \in B_i, \ \forall i \in [d]\}.$$

The following theorem allows to characterize when a subset of $\mathbb{Z}^d$ contains a set of return times of a minimal distal system with the unique closing parallelepiped property.

Theorem 2.6 ([2], Theorem 35) *Let $d \geq 2$ be an integer. A subset $B \subseteq \mathbb{Z}^d$ contains a set of return times for a minimal distal $\mathbb{Z}^d$-system with the closing property if and only if B contains a d-joining of sets that are return times of minimal distal $\mathbb{Z}^{d-1}$-systems.*

Remark 13 In the proof of Theorem 2.6 in [2], the aforementioned $\mathbb{Z}^{d-1}$-systems correspond to

$$Y_j = K^{x_0}_{T_1,\cdots,T_{j-1},T_{j+1},\cdots,T_d},$$

with the dynamics given by $(T_j^{[d]})_{j\in[d]}$, where $T_j^{[d]} : X^{[d]} \to X^{[d]}$ is the j-th face transformation defined for every $x \in X^{[d]}$ and $\epsilon \subseteq [d]$ as

$$(T_j^{[d]}x)_\epsilon = \begin{cases} T_j x_\epsilon & \text{if } j \in \epsilon, \\ x_\epsilon & \text{if } j \notin \epsilon. \end{cases}$$

Note that these dynamics are identified with $(T_i)_{i=1}^d$.

We denote by $\mathcal{B}_d$ the family generated by sets of return times arising from minimal distal $\mathbb{Z}^d$-systems with the unique closing parallelepiped property, and by $\mathcal{B}_d^*$ its dual family.

Now we will attempt to obtain similar characterizations for Bohr recurrence as in Theorem 2.6, trying to take advantage of the fact that minimal rotations are also minimal distal systems.

Proposition 2.12 *A set $R \subseteq \mathbb{Z}^d$ is a set of Bohr recurrence if and only if R has nonempty intersection with all d-joining of sets of return times arising from $\mathbb{Z}^{d-1}$-minimal rotations.*

Proof. Let $R \subseteq \mathbb{Z}^d$ be a set with nonempty intersection with all d-joining of set of return times arising from $\mathbb{Z}^{d-1}$-minimal rotations. Let $(X, T_1, \cdots, T_d)$ be a minimal $\mathbb{Z}^d$-rotation with dynamics given by the translation by elements $\tau_i \in X$. Let $x \in X$ be an element of X and U an open neighborhood of x. Given that X has the unique closing parallelepiped property, by Theorem 2.6 we have that $N(x, U)$ contains a d-joining of sets that are return times of $\mathbb{Z}^{d-1}$ minimal distal systems. As we pointed in Remark 13, these systems are

$$Y_j = K^{x_0}_{T_1,\cdots,T_{j-1},T_{j+1},\cdots,T_d},$$

where $x_0 \in X$ is a continuity point. We note that

$$Y_j = \overline{\{\Big(\big(\prod_{i\in[d]\setminus\{j\}} \tau_i^{n_i\epsilon_i}\big)x_0\Big)_{\epsilon\in\{0,1\}^d\setminus\{\vec{0}\}} \mid (n_i)_{i\in[d]\setminus\{j\}} \in \mathbb{Z}^{d-1}\}} \subseteq X_*^{[d]},$$

is a change of base point of the system

$$\hat{Y}_j = \overline{\{\Big(\prod_{i\in[d]\setminus\{j\}} \tau_i^{n_i\epsilon_i}\Big)_{\epsilon\in\{0,1\}^d\setminus\{\vec{0}\}} \mid (n_i)_{i\in[d]\setminus\{j\}} \in \mathbb{Z}^{d-1}\}} = \overline{\langle\{(\tau_i^{\epsilon_i})_{\epsilon\in\{0,1\}^d\setminus\{\vec{0}\}}\}_{i\neq j}\rangle}.$$

Besides, $\hat{Y}_j$ is an abelian compact group because is the closed subgroup of $X_*^{[d]}$ generated by the elements $\{(\tau_i^{\epsilon_i})_{\epsilon\in\{0,1\}^d\setminus\{\vec{0}\}}\}_{i\neq j}$, and it is minimal under the distal dynamic $\{(\tau_i^{\epsilon_i})_{\epsilon\in\{0,1\}^d\setminus\{\vec{0}\}}\}_{i\neq j}$ by being isomorphic to the minimal system Y_j. By hypothesis, we have that R intersect every d-joining of set of return times arising from the $\mathbb{Z}^{d-1}$-minimal rotations given by $(\hat{Y}_1, \cdots, \hat{Y}_d)$, so R has nonempty intersection with $N(x, U)$. Therefore R is a set of Bohr recurrence.

Conversely, let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence. For every $j \in [d]$ let B_j be a set of return times of a minimal $\mathbb{Z}^{d-1}$-rotation $(Y_j, S_{j,1}, \ldots, S_{j,j-1}, S_{j,j+1}, \ldots, S_{j,d})$. Let $y_j \in Y_j$ and let U_j be an open neighborhood of y_j such that

$$N_{S_{j,1},\cdots,S_{j,d-1}}(y_j, U_j) \subseteq B_j.$$

Consider the product system $\prod_{j=1}^d Y_j$ and the action of $(T_1, \cdots, T_d)$ such that for every $x = (x_j)_{j=1}^d \in \prod_{j=1}^d Y_j$, we have

$$T_i x = (S_{j,i}x_j)_{j=1}^d,$$

where we define $S_{j,j} = id_{Y_j}$ for every $j \in [d]$. In other words, T_j acts as the identity in Y_j.

We observe that this system is a rotation since it is an abelian compact group (by being product of abelian compact groups) with translation given by $(T_1 \cdots, T_d)$. In this way, if we consider $y = (y_1, \cdots, y_d)$, then $Y = \overline{\mathcal{O}_{T_1,\cdots,T_d}(y)}$ is a compact subgroup of $\prod_{j=1}^d Y_j$, where the dynamics $(T_1, \cdots, T_d)$ act minimally by distality. Hence, if $U = \prod_{j=1}^d (U_j \cap Y_j)$, then $R \cap N(y, U) \neq \emptyset$, and noting that $N(y, U)$ is a d-joining of the sets $N_{T_1, \cdots, T_{j-1}, T_{j+1}, \cdots, T_d}(y_j, U_j)$, we conclude.

□

Another important lemma that will be useful is the following, which characterizes the relation that defines the closing property through sets of return times of minimal distal $\mathbb{Z}^d$-systems.

Lemma 2.5 ([2], Lemma 37) *Let $(X, T_1, \cdots, T_d)$ be a minimal distal $\mathbb{Z}^d$-system. Then for $x, y \in X$, $(x, y) \in \mathcal{R}_{T_1, \cdots, T_d}(X)$ if and only if $N_{T_1, \cdots, T_d}(x, U) \in \mathcal{B}_d^*$ for any open neighborhood U of y.*

For the next theorem is key to note the following property.

Proposition 2.13 *If $R \in \mathcal{B}_d^*$ then R is a set of Bohr recurrence.*

Proof. If $R \in \mathcal{B}_d^*$ then R has nonempty intersection with all sets of return times arising from minimal distal $\mathbb{Z}^d$-systems with the unique closing parallelepiped property. In particular R has nonempty intersection with all sets of return times arising from minimal rotations. In this light, R is a set pointwise recurrence for minimal rotation, and therefore, a set of Bohr recurrence. □

Theorem 2.7 *Let $(X, T_1, \cdots, T_d)$ a minimal distal $\mathbb{Z}^d$-system, in which every $R \subseteq \mathbb{Z}^d$ set of Bohr recurrence is a set of pointwise recurrence. Then $(X, T_1, \cdots, T_d)$ has the closing property.*

Proof. Suppose by contradiction that X does not have the closing property. By Theorem 1.14, we have that there exists $(x, y) \in \mathcal{R}_{T_1, \cdots, T_d}(X) \setminus \Delta_X$. Let U and V open neighborhoods of x and y respectively, such that $U \cap V = \emptyset$, then by Lemma 2.5, $N(x, V) \in \mathcal{B}_d^*$, in particular $N(x, V)$ is a set of Bohr recurrence by Proposition 2.13, and by hypothesis

$$N(x, U) \cap N(x, V) \neq \emptyset,$$

therefore $U \cap V \neq \emptyset$ which is a contradiction. □

Remark 14 For instance, by Theorem 2.7 and Proposition 1.6, not all sets of Bohr recurrence are sets of pointwise recurrence for the Heisenberg nilsystem.

Chapter 3

Katznelson's Question

In this chapter, we present Katznelson's question regarding general group actions and its equivalencies. We then explore several contemporary strategies for demonstrating a positive response to this question, including the definitions of PI systems, proximal extensions, equicontinuous extensions, and weakly mixing extensions. Finally, we prove that (Bohr) recurrence can be extended through inverse limits and proximal extensions, as demonstrated by Host, Kra, and Maass in [1] for $\mathbb{Z}$-group actions.

3.1. Katznelson Question and Equivalences

In this section, we will describe Katznelson's question and its equivalences with general group actions. By "equivalent questions" we mean questions for which a positive or negative answer is equivalent to a positive or negative answer to Katznelson's question, respectively.

Katznelson's question: Given an action G, is every set of Bohr recurrence a set of (topological) recurrence in G?

In order to state the equivalent questions, we need the following definition.

Definition 3.1 *A system (X, G) has Bohr_0 large returns if for all $\epsilon > 0$, $\mathcal{R}_\epsilon(X, T)$ is a Bohr_0 set.*

Question 1 Do all systems (X, G) have Bohr_0 large returns?

Proposition 3.1 *Question* 1 *is equivalent to Katznelson Question.*

We see that this question is equivalent to Katznelson's question.

Proof. Suppose a positive answer to Question 1, and let (X, G) be a dynamical system and $R \subseteq G$ a set of Bohr recurrence. As $R \in \mathrm{Bohr}_0^*$ we have that for $\epsilon > 0$, $R \cap \mathcal{R}_\epsilon \neq \emptyset$, thus R is a set of recurrence for (X, G) by Remark 6.

For the other direction, let (X, G) be a system and let $\epsilon > 0$. By hypothesis, for every $R \in \mathrm{Bohr}_0^*$, $R \cap \mathcal{R}_\epsilon \neq \emptyset$, therefore $\mathcal{R}_\epsilon \in (\mathrm{Bohr}_0^*)^* = \mathrm{Bohr}_0$ concluding. □

Question 2: If (X, G) is a minimal topological dynamical system, is it true that $\forall U \subseteq X$, $N_G(U)$ is a Bohr_0 set?.

Although it can be proved that Question 2 and Katznelson's question are equivalent directly, the following Lemma explains the relationship between question 1 and question 2.

Lemma 3.1 *Let (X, G) be a minimal system, then $\forall U \subseteq X$ nonempty open set, there exists $\epsilon > 0$ such that*

$$\mathcal{R}_\epsilon(X, G) \subseteq N_G(U) \subseteq \mathcal{R}_{diam(U)}(X, G)$$

Proof. For the second inclusion, let $g \in N(U)$ and $x \in U \cap g^{-1}U$, then $d_X(x, gx) < \text{diam}(U)$, therefore $g \in \mathcal{R}_{\text{diam}(U)}(X, G)$.

For the first inclusion, let $\delta > 0$ be such that U contains a ball $B(u, 2\delta)$ of radius 2δ. Since (X, G) is minimal, $\{gB(u, \delta)\}_{g \in G}$ covers X and by compactness, there exists a finite subset $F \subseteq G$ such that $\forall x \in X, \exists g \in F, gx \in B(u, \delta)$. Let $\epsilon > 0$ be such that for all $x, y \in X$ with $d_X(x, y) < \epsilon$ and $g \in F$, $d_X(gx, gy) < \delta$. Note that if $g \in \mathcal{R}_\epsilon(X, T)$, there exists $x \in X$ such that $d_X(x, gx) < \epsilon$. It follows that there exists $h \in F$ such that $hx \in B(u, \delta)$ and $d_X(hx, hgx) < \delta$. Consequently $hx, g(hx) \in U$, concluding that $g \in N(U)$. □

Proposition 3.2 *Question 1 is equivalent to Question 2.*

Proof. Assume a positive answer to Question 2. Then, Question 1 follows from the fact that any system (X, G) contains a minimal subsystem (Y, G) and

$$N_G(B_Y(y, \epsilon/2)) \subseteq \mathcal{R}_\epsilon(Y, G) \subseteq \mathcal{R}_\epsilon(X, G),$$

where $y \in Y$ is any element, and the first inclusion comes from Lemma 3.1.

Conversely, assume a positive answer to Question 1. Then, for $U \subseteq X$ a nonempty open set, by Lemma 3.1, there is an $\epsilon > 0$ such that

$$\mathcal{R}_\epsilon(X, G) \subseteq N_G(U),$$

and as $\mathcal{R}_\epsilon(X, G)$ is a Bohr_0 set, so is the set $N_G(U)$, concluding. □

3.2. Strategies for a Positive Answer

As mentioned in the introduction, some of the current strategies involve lifting recurrence through a chain of factors, starting from the identity or an equicontinuous factor, where we know that sets of Bohr recurrence are sets of recurrence. In this regard, we will initially define a PI system and the PI chain of factors that are associated with any minimal system. To accomplish this, we require some previous definitions.

Definition 3.2 *Let (X, G) and (Y, G) be topological dynamical systems. Let $\pi : X \to Y$ be a factor map. Then, the extension π is called:*

- *equicontinuous if for every $\epsilon > 0$, exists $\delta > 0$ such that if $d_X(x, y) < \delta$ and $\pi(x_1) = \pi(x_2)$, then $d_X(gx, gy) < \epsilon$, $\forall g \in G$,*
- *proximal if the fiber $\pi^{-1}(\{y_0\})$ of every $y_0 \in Y$ is proximal (meaning that every pair of points in $\pi^{-1}(\{y_0\})$ are proximal), and*

- *weakly mixing if the relation*

$$R(\pi) = \{(x, x') \mid \pi(x) = \pi(x')\},$$

is topologically transitive, i.e. every invariant open set is dense in $R(\pi)$.

A minimal t.d.s. is said to be strictly PI if it can be obtained from the trivial t.d.s. by a (transfinite) succession of proximal and equicontinuous extensions. Namely, there is an ordinal number ν, a collection of minimal systems $\{(W_\alpha, G)\}_{\alpha \leq \nu}$ and factors $\pi_\alpha : W_{\alpha+1} \to W_\alpha$ such that π_α is either proximal or equicontinuous, and for a limit ordinal α, W_α is the inverse limit of $\{W_\beta\}_{\beta<\alpha}$. Additionally, a minimal t.d.s. (X, G) is said to be PI if a proximal extension (X', G) of (X, G) is strictly PI.

We are now in position to state the theorem associated to the PI chain.

Theorem 3.1 ([12], Ch. 14 Theorem 30) *Let (X, G) be a minimal t.d.s.. Then there is a proximal extension of (X, G) which is a weakly mixing extension of a strictly PI t.d.s..*

The idea is that as recurrence is preserved under factors between minimal systems, we just need to prove that sets of Bohr recurrence are sets of recurrence for weakly mixing extensions of strictly PI systems. This clearly is not an easy task to do, and it is a problem which can be divided in many difficult problems. In particular, it is necessary to prove that Bohr recurrence is preserved under proximal, equicontinuous, and weakly mixing extensions. In this direction, some of the progress has been done by Host, Kra, and Maass in [1] who proved that Bohr recurrence can be lifted through proximal extensions of $\mathbb{Z}$-systems. Another progress comes from Glasscock, Koutsogiannis, and Richter [28], who proved that sets of $\mathbb{Z}$-Bohr recurrence are sets of recurrence for skew product extensions of an equicontinuous system by a d-dimensional torus, which encompasses a wide family of equicontinuous extensions.

The following diagram summarizes this strategy. We colored in green extensions and systems in which we do know that sets of Bohr recurrence are sets of recurrence for $\mathbb{Z}$-group actions.

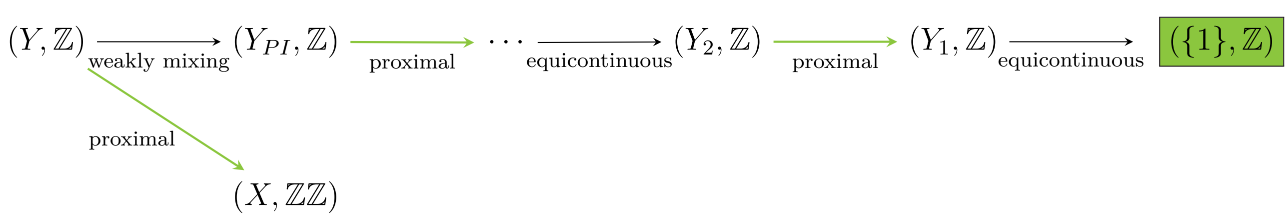

Figura 3.1: $\mathbb{Z}$-PI Chain of Factors Colored.

The second aforementioned strategy has to do with the pronilfactors associated to a system. We define them now.

Definition 3.3 *Let (X, G) be a topological dynamical system. We will denote by $(\mathcal{Z}_k, G)$ the maximal k-step pronilfactor of X, and by $(\mathcal{Z}_\infty, G)$ the maximal ∞-step pronilfactor of X.*

We will not be interested in the existence of these factor. However, some references in this topic are [15], [14], and [29].

In this case, the strategy is to lift Bohr recurrence through the finite pronilfactors first, and then to the maximal ∞-step pronilfactor. From this point, we can try to lift recurrence to the maximal distal factor, and the to the original system through a weakly mixing extension (the extension from the maximal distal factor).

The progress in this strategy comes primarily from Host, Kra, and Maass, who proved in [1] that sets of Bohr recurrence are sets of recurrence for s-step $\mathbb{Z}$-nilsystems. As Bohr recurrence can be lift through $\mathbb{Z}$-inverse limits, their results basically proved that Bohr recurrence can be lifted up to ∞-step $\mathbb{Z}$-pronilsystems.

As before, the following diagram illustrates this strategy. We have highlighted in green the extensions and systems for which we know that sets of Bohr recurrence are sets of recurrence for $\mathbb{Z}$-group actions.

Figura 3.2: $\mathbb{Z}$-Nilsystems' Chain of Factors Colored.

3.3. Bohr Recurrence in Inverse Limits and Proximal Extension

In this section, we will prove that inverse limits and proximal extensions for general group actions lift Bohr recurrence. We begin with the case of inverse limits of minimal G-systems, in where we provide a direct proof.

Theorem 3.2 *Let $\{(X_m, G)\}_{m\in\mathbb{N}}$ be a collection of minimal topological dynamical systems, with factor maps $\pi_m : (X_m, G) \to (X_{m-1}, G)$. Suppose R is a set of recurrence in (X_m, G), $\forall m \in \mathbb{N}$. Then, this also holds for $(X, G) := \varprojlim (X_m, G)$.*

Proof. Let $U \subseteq X$ be a nonempty open set. Without loss of generality, we can assume that there is $d \in \mathbb{N}$ and $U_1, \cdots, U_d$ nonempty open sets of $X_1, \cdots, X_d$ respectively, such that

$$U = \left(\prod_{i=1}^{d} U_i \times \prod_{i=d+1}^{\infty} X_i\right) \cap X.$$

We can further assume that $\pi_i(U_i) \subseteq U_{i-1}$ for $i \in \{2, \cdots, d\}$, indeed, suppose $i \in \{2, \cdots, d\}$ is maximal such that this does not happen, i.e. $\pi_j(U_j) \subseteq U_{j-1}$ for $j < i$, but $\pi_i(U_i) \cap U_{i-1}^c \neq \emptyset$. Notice that $U_i \cap \pi_i^{-1}(U_{i-1}) \neq \emptyset$ given that

$$U = \left(\prod_{i=1}^{d} U_i \times \prod_{i=d+1}^{\infty} X_i\right) \cap X,$$

is nonempty. Substituting U_i by the nonempty open set

$$\tilde{U_i} = U_i \cap \pi_i^{-1}(U_{i-1}),$$

we have that

$$\pi_i(\tilde{U}_i) = \pi(U_i \cap \pi^{-1}(U_{i-1})) \subseteq \pi_i(\pi_i^{-1}(U_{i-1})) \subseteq U_{i-1}.$$

Repeating the process from $i+1$ onwards, we obtain $V_1, \cdots, V_d$ nonempty open sets such that $V_i \subseteq U_i$ and $\pi_i(V_i) \subseteq V_{i-1}$, $\forall i \in \{2, \cdots, d\}$. We replace then U with

$$\Big(\prod_{i=1}^{d} V_i \times \prod_{i=d+1}^{\infty} X_i\Big) \cap X.$$

Now, by hypothesis, there exists $g \in R$ and $x_d \in V_d$ such that $gx_d \in V_d$. We define inductively $x_i := \pi(x_{i+1})$ for $i \in \{1, \cdots, d-1\}$. Notice that if $x_{i+1} \in V_{i+1}$ and $gx_{i+1} \in V_{i+1}$, then $x_i, gx_i \in V_i$, indeed $x_i = \pi(x_{i+1}) \in \pi(V_{i+1}) \subseteq V_i$ and $gx_i = g\pi(x_{i+1}) = \pi(gx_{i+1}) \in \pi(V_{i+1}) \subseteq V_i$. Hence, we have that $x_i, gx_i \in V_i$, $\forall i \in [d]$.

We extent $(x_i)_{i=1}^d$ to a point $x = (x_i)_{i \in \mathbb{N}} \in X$. Given the previous computation, we have that

$$x \in \Big(\prod_{i=1}^{d} V_i \times \prod_{i=d+1}^{\infty} X_i\Big) \cap X \subseteq U, \text{ and } gx \in \Big(\prod_{i=1}^{d} V_i \times \prod_{i=d+1}^{\infty} X_i\Big) \cap X \subseteq U,$$

therefore

$$U \cap g^{-1}U \neq \emptyset,$$

concluding that R is a set of recurrence for X. □

In the following, we will focus on proving that proximal extensions preserve Bohr recurrence. Initially, we will demonstrate that an extension can be established as proximal under weaker assumptions.

Proposition 3.3 *Let $\pi : (X, G) \to (Y, G)$ be a factor map and assume that (Y, G) is minimal and that some $y_0 \in Y$ has a proximal fiber. Then π is a proximal extension.*

Proof. Let $y_0 \in Y$ be a point with proximal fiber. For $x, x' \in X$, let

$$\delta(x, x') = \inf_{g \in G} d(gx, gx').$$

Suppose by contradiction that exists $y \in Y$ and $u, v \in \pi^{-1}(y)$ distal points (meaning that $\delta(u, v) > 0$). We have that by minimality of Y, there exists a sequence $(g_n)_n$ such that $g_n y \to y_0$, also we can suppose without loss of generality that $(g_n u)_n$ and $(g_n v)_n$ converge to $\tilde{u}, \tilde{v} \in X$ respectively. Hence, as $g_n y = \pi(g_n u) = \pi(g_n v)$, and letting $n \to \infty$ we conclude that $y_0 = \pi(\tilde{u}) = \pi(\tilde{v})$. However, thanks to the fact that $\delta(x, x') \leq \delta(gx, gx')$, $\forall x, x' \in X$, $g \in G$, we have that $\delta(u, v) \leq \delta(g_n u, g_n v)$, and taking limsup and noticing that δ is a lower semi-continuous function (since it is the infimum of continuous functions) we have that

$$\delta(u, v) \leq \limsup_n \delta(g_n u, g_n v) \leq \delta(\tilde{u}, \tilde{v}) = 0,$$

in where the last equality comes from the proximality of the fiber $\pi^{-1}(y_0)$. As $\delta(u, v) > 0$, the previous inequality is a contradiction, concluding that π is a proximal extension. □

The following lemma is key to extent recurrence in proximal extensions.

Lemma 3.2 *Let $\pi : (X,G) \to (Y,G)$ be a proximal extension between minimal systems. Then for every $l \geq 1$ and all $x_0, \cdots, x_l$ lying in the same fiber, there exists a sequence $(g_n)_n \subseteq G$ such that each of the sequences $(g_n x_0), \cdots, (g_n x_l)$ converge to x_0.*

PROOF. We proceed by induction on l. Assume that $l = 1$, and let $y_0 \in Y$, $x_0, x_1 \in \pi^{-1}(\{y_0\})$, and $\epsilon > 0$. By proximality and compactness, there exists a sequence $(g_n)_n \subseteq G$ such that the sequences $(g_n x_0)_n$ and $(g_n x_1)_n$ converge to the same point $a \in X$. Also by minimality of (X,G) there exists $h \in G$ such that $d(ha, x_0) < \epsilon/2$, and by continuity of the action, we will have that for n big enough and $k = 0, 1$

$$d(hg_n x_k, x_0) < \epsilon,$$

and the result follows in this case.

Assume that $l > 1$ and that the result holds for $l-1$. Let $y_0 \in Y$, $x_0, \cdots, x_l \in \pi^{-1}(\{y_0\})$, and $\epsilon > 0$. By the induction hypothesis, there exists a sequence $(g_n)_n \subseteq G$ such that $(g_n x_k)_n$ converge to x_0, for $0 \leq k \leq l-1$. Passing to subsequence, we can further assume that the sequence $(g_n x_l)_n$ converges to a point $a \in X$. For every n, we have $\pi(g_n x_l) = \pi(g_n x_0)$ and, passing to the limit $\pi(a) = \pi(x_0) = y_0$. By applying the result for $l = 1$ to the points x_0 and a, we obtain the existence of $h \in G$ with $d(hx_0, x_0) < \epsilon/2$ and $d(ha, x_0) < \epsilon/2$. By continuity of the action, for every sufficiently large n and every $k \in \{0, \ldots, l\}$, we have $d(hg_n x_k, x_0) < \epsilon$, completing the proof.

□

Now we present the extension of Theorem 3.8 from [1].

Theorem 3.3 *Let $\pi : (X,G) \to (Y,G)$ be a proximal extension between minimal systems, $l \geq 1$, and R be a set of recurrence for (Y,G). Then R is a set of recurrence for (X,G).*

PROOF. Let $\epsilon > 0$. The fact that R is a set of recurrence for (Y,G) implies that there exists $y_0 \in Y$ such that

$$\inf_{g \in R} d(gy_0, y_0) = 0,$$

and thus there exists a sequence $(g_n)_n \subseteq R$ such that $g_n y_0 \to y_0$. Let $x_0 \in X$ with $\pi(x_0) = y_0$, without loss of generality we can assume by compactness that $(g_n x_0)_n$ converges in X, and we denote x_1 the limit of this sequence. The points x_0, x_1 belongs to $\pi^{-1}(\{y_0\})$ and this fiber is proximal by hypothesis. Then by Lemma 3.2, there exists a sequence $(h_n)_n \subseteq G$ such that $(h_n x_0)_n$ and $(h_n x_1)_n$ converge to x_0. Choose j such that

$$d(h_j x_k, x_0) < \epsilon, \text{ for } k \in \{0, 1\}.$$

Let $\delta > 0$ be such that

$$d(x, x') < \delta \Longrightarrow d(h_j x, h_j x') < \epsilon,$$

and let i be such that $d(g_i x_0, x_1) < \delta$. We have that $d(h_j g_i x_0, h_j x_1) < \epsilon$ and thus $d(h_j g_i x_0, x_0) < 2\epsilon$.

Letting $z = h_j x_0$, we have that $d(z, x_0) < \epsilon$ and $d(g_i z, x_0) < 2\epsilon$ for $1 \leq k \leq k$, and therefore R is a set of recurrence for X.

□

Chapter 4

Bohr Recurrence in $\mathbb{Z}^d$-Nilsystems

In this chapter, we explore different cases in where sets of Bohr recurrence are sets of recurrence for $\mathbb{Z}^d$-nilsystems. We begin with some preliminaries, in where we establish some reductions, introduce the notion of total ergodicity for $\mathbb{Z}^d$-group actions characterize it in nilsystems via connectedness, and discuss the prime challenge faced in generalizing the result for $\mathbb{Z}$-nilsystems from Host, Kra, and Maass in [1]. Subsequently, we investigate two cases in where this problem can be avoided.

First, we obtain some properties on Bohr recurrence using products of sets of return times of $\mathbb{Z}$-nilsystems, with which we observe that sets of Bohr recurrence serve as sets of recurrence for $\mathbb{Z}^2$-nilsystems with the closing property. To extent this result, we introduce the notion of strong closing property and establish several properties characterizing it through product systems.

Second, in order to prove that sets of Bohr recurrence are sets of recurrence for $\mathbb{Z}^d$-quasi-affine nilsystems, we study the structure of sets of Bohr recurrence, introducing the notion of Bohr correlations. We prove numerous properties on this new concept, in particular, we prove that our problem can be reduced to studying sets of Bohr recurrence with either correlation 0 or an irrational correlation. Thereafter, we use this reduction to prove the aforementioned result in $\mathbb{Z}^d$-quasi-affine nilsystems.

4.1. Preliminaries

4.1.1. Connectedness in Nilsystems

In what follows, [20, 21, 30] are sources of background. Given a s-step $\mathbb{Z}^d$-nilsystem, we can assume G_0 simply connected without loss of generality, by changing the representation of $X = G/\Gamma$. Set $\Gamma_0 = \Gamma \cap G_0$, in this case G_0 can be endowed with a Malcev basis and for this reason we have the following identifications:

- G_0/G_2 can be identify with $\mathbb{R}^p$ for some $p \in \mathbb{N}$, and the subgroup $\Gamma_0/(\Gamma_0 \cap G_2)$ corresponds to $\mathbb{Z}^p$, and therefore, $G_0/G_2\Gamma_0$ can be identify with $\mathbb{T}^p$.
- The abelian group G_s can be identify $\mathbb{R}^r$ for some $r \in \mathbb{N}$ and $\Gamma \cap G_s$ corresponds to $\mathbb{Z}^r$, inducing the identification between $G_s/(\Gamma \cap G_s)$ and $\mathbb{T}^r$.
- Lastly, the group G_{s-1}/G_s is abelian and it is not trivial if X is not an $(s-2)$-nilsystem.

In this case, G_{s-1}/G_s can be identify with $\mathbb{R}^q$ for some $q \in \mathbb{N}$ such that $(\Gamma \cap G_{s-1})/(\Gamma \cap G_s)$ corresponds to $\mathbb{Z}^q$.

Moreover, the distance d_G in G can be chosen such that the identifications are isometries when the quotient groups are endowed with the quotient distances and $\mathbb{R}^p$, $\mathbb{T}^p$, $\mathbb{R}^r$ and $\mathbb{R}^q$ are endowed with the Euclidean distances.

Additionally, for a nilsystem $(X = G/\Gamma, T_1, \cdots, T_d)$ we can always assume that G is spanned by G_0 and the elements $\tau_1, \cdots, \tau_d$ defining the dynamic. Indeed, set $G' = \langle G_0, \tau_1, \cdots, \tau_d \rangle$, then G' is an open subgroup of G, and since $\forall a \in X$ the map $g \to g \cdot a$ is open, the sets $G' \cdot a$, $a \in X$ are open subsets of X that pairwise equal or disjoint. As such sets covers X, they are also closed in X so they are compact. Moreover, given that X is compact, there exist $a_1, \cdots, a_k \in X$ such that $\{X_i := G' \cdot a_i\}_{i=1}^k$ covers X. For $i = 1, \cdots, k$, let Γ_i denote the stabilizer of a_i in G, i.e.

$$\Gamma_i = g_i \Gamma g_i^{-1},$$

where $g_i \in G$ is an element such that $g_i \cdot e_X = a_i$. Note that $\Gamma_i \cap G'$ represents the stabilizer of a_i in G', and X_i can be viewed as the nilmanifold $G'/(\Gamma_i \cap G')$. Since $\tau_1, \cdots, \tau_d \in G'$, X_i is a $(\tau_1, \cdots, \tau_d)$-invariant set and $(X_i, T_1, \cdots, T_d)$ is a nilsystem. In this light, we obtain a partition of X into finitely many nilsystems, and each can be studied separately. Thus, without loss of generality, we can substitute G for G' and assume that G is spanned by G_0 and $(\tau_1, \cdots, \tau_d)$.

In order to characterize connectedness in nilsystems, we will need to characterize first ergodicity in a $\mathbb{Z}^d$-torus rotation.

Theorem 4.1 *Let $(\mathbb{T}^N, \alpha_1, \cdots, \alpha_d)$ a $\mathbb{Z}^d$-torus. Then, the following are equivalent:*

1. *$(\mathbb{T}^N, \alpha_1, \cdots, \alpha_d)$ is ergodic,*
2. *$\forall k \in \mathbb{Z}^N \setminus \{0\}$, $\exists i \in [d]$ such that $k \cdot \alpha_i \notin \mathbb{Z}$,*
3. *$\forall k \in \mathbb{Z}^N \setminus \{0\}$, $\exists (t_i)_{i=1}^d \in \mathbb{Z}^d$, $k \cdot (\sum_{i=1}^d t_i \alpha_i) \notin \mathbb{Z}$.*

Proof. (1. $\implies$ 2.) We prove the contrapositive assertion. Suppose that $\exists k \in \mathbb{Z}^N \setminus \{0\}$ such that $\forall i \in [d]$ $k \cdot \alpha_i \in \mathbb{Z}$. We notice that $f(x) = e^{2\pi i k \cdot x}$ is an invariant function in $L^\infty(\mathbb{T}^N)$ that is not constant, and therefore $(\mathbb{T}^N, \alpha_1, \cdots, \alpha_d)$ is not ergodic.

(2. $\implies$ 3.) Let $k \in \mathbb{Z}^N \setminus \{0\}$ and $i \in [d]$ given by the hypothesis. Define $t_j = 1_{i=j}$, $\forall j \in [d]$, then

$$k \cdot (\sum_{j=1}^d t_j \alpha_j) = k \cdot \alpha_i \notin \mathbb{Z}.$$

(3. $\implies$ 1.) Suppose that $(\mathbb{T}^N, \alpha_1, \cdots, \alpha_d)$ is not ergodic. By Section 2.1 there exists a $\langle \alpha_1, \cdots, \alpha_d \rangle$-invariant function $f \in L^\infty(\mathbb{T}^N)$ which is not constant. Using the fact $L^\infty(\mathbb{T}^N) \subseteq L^2(\mathbb{T}^N)$, we have $\{e^{2\pi i k \cdot \bullet}\}_{k \in \mathbb{Z}^N}$ is a Hilbert basis, and we can write

$$f(x) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k \cdot x}.$$

Using the continuity of the operation $x \to gx$,$\forall g \in \langle \alpha_1, \cdots, \alpha_d \rangle$, we have that

$$f(gx) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k \cdot g} e^{2\pi i k \cdot x}.$$

Note that by invariance we have that $f(x) = f(gx)$ and given that $\{e^{2\pi i k \cdot \bullet}\}_{k \in \mathbb{Z}^N}$ is an orthonormal basis, we conclude that $\forall k \in \mathbb{Z}^N$,

$$c_k = c_k e^{2\pi i k \cdot g}.$$

As f is not constant, we know that exists $k \in \mathbb{Z}^N \setminus \{0\}$ such that $c_k \neq 0$, and therefore $k \cdot g \in \mathbb{Z}$, $\forall g \in \langle \alpha_1, \cdots, \alpha_d \rangle$. In other words, we prove that $\exists k \in \mathbb{Z}^N \setminus \{0\}$ such that $\forall (t_i)_{i=1}^d \in \mathbb{Z}^d$, $k \cdot (\sum_{j=1}^d t_j \alpha_j) \in \mathbb{Z}$ which contradicts the hypothesis.

□

In what follows, we define total ergodicity, and we show that $\mathbb{Z}^d$-torus rotations are totally ergodic.

Definition 4.1 *A dynamical system $(Y, S_1, \cdots, S_d)$ is totally ergodic if for all $m \in \mathbb{N}^d$ the system $(Y, S_1^{m_1}, \cdots, S_d^{m_d})$ is ergodic.*

Theorem 4.1 is not only a good characterization for ergodicity (and minimality) on $\mathbb{Z}^d$-torus rotations, but also shows that ergodicity is preserved for certain subgroup actions, as we see in the following proposition.

Proposition 4.1 *Let $(\mathbb{T}^r, \alpha_1, \cdots, \alpha_d)$ be an ergodic torus action. Then $(\mathbb{T}^r, \alpha_1, \cdots, \alpha_d)$ is totally ergodic.*

Proof. Let $m \in \mathbb{N}^d$. Suppose that $(\mathbb{T}^r, m_1\alpha_1, \cdots, m_d\alpha_d)$ is not ergodic. Then by Theorem 4.1, we have that there is $k \in \mathbb{Z}^r \setminus \{0\}$ such that

$$\forall i \in [d], \quad k \cdot m_i \alpha_i \in \mathbb{Z},$$

in particular

$$\forall i \in [d], \quad (k \prod_{j=1}^d m_j) \cdot \alpha_i \in \mathbb{Z},$$

and again by Theorem 4.1, $(\mathbb{T}^r, \alpha_1, \cdots, \alpha_d)$ is not ergodic, which is a contradiction. □

We will prove that total ergodicity is equivalent to connectedness in nilsystems. To prove this, we previously need the following lemma.

Lemma 4.1 *Let $(X = G/\Gamma, \mu, T_1, \cdots, T_d)$ a s-step $\mathbb{Z}^d$-nilsystem. Then, there are $p_1, \cdots, p_d \in \mathbb{N}$ such that X_0 is a $(T_1^{p_1}, \cdots, T_d^{p_d})$-invariant clopen subset of X, where X_0 is the connected component of e_X.*

Proof. Let μ be the Haar measure on X, which has full support. By the fact that X_0 is open, we have that for each $i \in [d]$, there exists a minimal period $p_i \in \mathbb{N}$ which $\tau_i^{p_i} X_0 = X_0$, otherwise the sets $(\tau_i^k X_0)_{k \in \mathbb{N}}$ are pairwise disjoint and with the same measure $\mu(X_0)$, which is a contradiction with the finitude of μ. Given that $\{\tau_i^j X_0\}_{j=1}^{p_i}$ partitions X, and that

X_0 is open, the set X_0 is also closed. Finally, as $\tau_i^{p_i} X_0 = X_0$, we have the invariance, concluding. $\square$

Remark 15 Notice that $(X_0, T_1^{p_1}, \cdots, T_d^{p_d})$ is a connected $\mathbb{Z}^d$-nilsystem. Besides, the dynamics move cyclically among the connected components of $(X, T_1, \ldots, T_d)$, which all are defined by a nilsystem isomorphic to $(X_0, T_1^{p_1}, \cdots, T_d^{p_d})$. If the original nilsystem is quasi-affine, then all these connected subsystems are quasi-affine (given that $X_0 = G_0\Gamma/\Gamma$), then by Proposition 1.5, $\mathbb{Z}^d$-quasi-affine nilsystems are finite union of $\mathbb{Z}^d$-affine nilsystems.

Finally, we have the following characterization of total ergodicity in a $\mathbb{Z}^d$-nilsystem.

Corollary 4.1 *For an ergodic s-step $\mathbb{Z}^d$-nilsystem $(X = G/\Gamma, \mu, T_1, \cdots, T_d)$ the following are equivalent*

1. *The space X is connected.*
2. *The system is totally ergodic.*
3. *We have $G = G_0\Gamma$.*

Proof. (1. $\Rightarrow$ 2.) This direction comes from Corollary 1.1, given that (X, μ, H) is ergodic if and only if the maximal factor torus (T, H) is ergodic, which is always true given that $(T, \tau_1, \cdots, \tau_d)$ is totally ergodic by Proposition 4.1.

(2. $\Rightarrow$ 3.$\wedge$1.) By Lemma 4.1 we derive that X_0 is an $(T_1^{p_1}, \cdots, T_d^{p_d})$-invariant closed subset of X, but as $(X, T_1^{p_1}, \cdots, T_d^{p_d})$ is ergodic, and thus minimal by Theorem 1.13, we have that $X_0 = X$. In particular, X is connected and by Lemma 1.2, we conclude that $G = G_0\Gamma$.

(3. $\Rightarrow$ 1.) As $X = G_0\Gamma/\Gamma = X_0$ (connected component of $e_X = 1_G\Gamma$), X is connected. $\square$

The following lemma will be useful in subsequence proofs.

Lemma 4.2 *Let $(X, T_1, \cdots, T_d)$ be a s-step $\mathbb{Z}^d$-nilsystem and $K \in \mathbb{Z}^{d\times d}$ a matrix. Then the system $(X, S_1, \ldots, S_d)$ is a s-step $\mathbb{Z}^d$-nilsystem, where $S_i = T_1^{K_{i,1}} \circ \cdots \circ T_d^{K_{i,d}}$, for each $i \in [d]$. In addition, if K is invertible and $(X, T_1, \cdots, T_d)$ is connected and minimal, then $(X, S_1, \ldots, S_d)$ is connected and minimal as well.*

Proof. Given a matrix $K \in \mathbb{Z}^{d\times d}$ and $(X, T_1, \cdots, T_d)$ a s-step $\mathbb{Z}^d$-nilsystem, it is clear that $(X, (T_1^{K_{i,1}} \circ \cdots \circ T_d^{K_{i,d}})_{i=1}^d)$ is still a s-step $\mathbb{Z}^d$-nilsystem, as X is a s-step nilmanifold and the new dynamics $(T_1^{K_{i,1}} \circ \cdots \circ T_d^{K_{i,d}})_{i=1}^d$ are defined by the rotations associated.

For the second part of the statement, suppose that $(X, T_1, \cdots, T_d)$ is connected and minimal, and K is invertible. Let $N \in \mathbb{N}$ be such that $N \cdot K^{-T} \in \mathbb{Z}^{d\times d}$. Then, notice that the action of S_i corresponds to the action generated by the vector $s_i := K^T e_i$ of $\mathbb{Z}^d$, so $N \cdot e_i = N \cdot K^{-T} s_i \in \mathbb{Z}^d$. Hence

$$\langle S_1, \ldots, S_d \rangle = \langle (T_1^{K_{i,1}} \circ \cdots \circ T_d^{K_{i,d}})_{i=1}^d \rangle = \langle T_1^N, \cdots, T_d^N \rangle,$$

and thus the action of $(S_1, \ldots, S_d)$ is minimal, given that the system $(X, T_1, \cdots, T_d)$ is totally ergodic by Corollary 4.1.

$\square$

The following proposition generalizes Lemma 4.3 from [1], and shows some of the advantages of considering X connected.

Proposition 4.2 *Let $(X, T_1, \cdots, T_d)$ be a minimal s-step nilsystem and assume that X is connected and that G_0 is simply connected. Then for every $\epsilon > 0$, there exists $C > 0$ such that for every $w \in G_s$, there exist $h_1, \cdots, h_d \in G_{s-1}$ and $\gamma \in \Gamma \cap G_s$ with*

$$d_G(h_i, 1_G) < C, \quad \forall i \in [d]; \quad d_G([h_1, \tau_1] \cdots [h_d, \tau_d], w\gamma) < \epsilon.$$

PROOF. Since X is connected, it follows that $G = \langle G_0, \Gamma \rangle$ and there exist $\tau_1^0, \cdots, \tau_d^0 \in G_0$ and $\gamma_1^0, \cdots, \gamma_d^0 \in \Gamma$ such that $\tau_i = \tau_i^0 \gamma_i^0$. Given $G = \langle G_0, \tau_1, \cdots, \tau_d \rangle$, we have that $G = \langle G_0, \gamma_1^0, \cdots, \gamma_d^0 \rangle$ and thus $\Gamma = \langle \Gamma_0, \gamma_1^0, \cdots, \gamma_d^0 \rangle$ with $\Gamma_0 = \Gamma \cap G_0$.

We also recall that $Z := G/(G_2\Gamma) = G_0/(G_2\Gamma_0) = \mathbb{T}^p$, and that the image $(\alpha_i)_{i=1}^d$ of $(\tau_i)_{i=1}^d$ in Z induces a minimal dynamic. Let $(\beta_i)_{i=1}^d$ the projection of $(\tau_i^0)_{i=1}^d$ to $G_0/G_2 = \mathbb{R}^p$. Then the projection of $(\beta_i)_{i=1}^d$ in $G_0/(G_2\Gamma_0)$ is equal to the projection α of τ in $G/(G_2\Gamma)$. It follows that $(\beta_l)_{l=1}^d$ satisfy that $\forall N \in \mathbb{Z}^p \setminus \{0\}$, $\exists l \in [d]$ such that $N \cdot \beta_l \notin \mathbb{Z}$ by Theorem 4.1. We will denote $\beta_{l,k}$ the k-th coordinate of β_l, for $l \in [d]$ and $k \in [p]$.

Let $\pi_s : G_s \to G_s/(\Gamma \cap G_s)$ be the quotient map. We claim that the map

$$f : (h_1, \cdots, h_d) \to \pi_s([h_1, \tau_1] \cdots [h_d, \tau_d]),$$

takes $G_{s-1} \times \cdots \times G_{s-1}$ to a dense subset of $G_s/(\Gamma \cap G_s)$.

Assuming the claim, there exists $C > 0$ such that the image under f of

$$(B_G(1_G, C) \times \cdots \times B_G(1_G, C)) \cap (G_{s-1} \times \cdots \times G_{s-1}),$$

is ϵ-dense in $G_s/(\Gamma \cap G_s)$, and this is the statement of the proposition.

Now we will prove the claim. First note that for every $l \in [d]$ the map $g \to [g, \gamma_l^0]$ induces a group homomorphism $F^l : G_{s-1}/G_s \to G_s$. Using additive notation and writing in coordinates,

$$\text{for } 1 \leq i \leq r, \ (F^l(x))_i = \sum_{j=1}^{q} F_{i,j}^l x_j,$$

and since $[G_{s-1} \cap \Gamma, \gamma_l^0] \subseteq G_s \cap \Gamma$, F^l maps $(G_{s-1} \cap \Gamma)/(G_s \cap \Gamma)$ to $G_s \cap \Gamma$, so we have that the coefficients $F_{i,j}^l$ are integers.

The commutator map $G_{s-1} \times G_0 \to G_s$ induces a homomorphism $\Phi : G_{s-1}/G_s \times G_0/G_2 \to G_s$. Using additive notation and writing in coordinates,

$$\text{for } 1 \leq i \leq r, (\Phi(x, y))_i = \sum_{j=1}^{q} \sum_{k=1}^{p} \Phi_{i,j,k} x_j y_k$$

where the coefficients $\Phi_{i,j,k}$ are integers since Φ maps $(G_{s-1} \cap \Gamma) \times \Gamma_0$ to $G_s \cap \Gamma$.

For each $l \in [d]$, the commutator map

$$\begin{aligned} G_{s-1} &\to G_s \\ g &\to [g, \tau_l] \end{aligned}$$

induces a homomorphism $\Psi^l : G_{s-1}/G_s \to G_s$. Using multiplicative notation, we have $\Psi^l(x) = \Phi(x, \tau_l^0)F^l(x)$. In coordinates, with additive notation,

$$\text{for } 1 \le i \le r, (\Psi^l(x))_i = \sum_{j=1}^{q}(F_{i,j}^l + \sum_{k=1}^{p} \Phi_{i,j,k}\beta_{l,k})x_j. \tag{4.1}$$

Note that $f(G_{s-1} \times \cdots \times G_{s-1})$ is the range of $\pi_s \circ (\sum_{l=1}^{d} \Psi^l)$, and if this range is not dense in $\mathbb{T}^r$, then it is included in a proper subtorus, and there exist integers $\lambda_1, \cdots, \lambda_r$ not all equal to 0, such that the range of $\sum_{l=1}^{d} \Psi^l$ is included in the group H define by the relationship

$$z \in H \text{ if and only if } \sum_{i=1}^{r} \lambda_i z_i \in \mathbb{Z}.$$

In coordinates,

$$\text{for every } (x_l)_{l=1}^d \in (\mathbb{R}^q)^d, \quad \sum_{i=1}^{r} \lambda_i \sum_{l=1}^{d} \sum_{j=1}^{q} (F_{i,j}^l + \sum_{k=1}^{p} \Phi_{i,j,k}\beta_{l,k})x_{l,j} \in \mathbb{Z},$$

and thus

$$\text{for } 1 \le j \le q, \quad 1 \le l \le d, \quad \sum_{i=1}^{r}(F_{i,j}^l + \sum_{k=1}^{p} \Phi_{i,j,k}\beta_{l,k})\lambda_i = 0. \tag{4.2}$$

Since the coefficients $F_{i,j}^l$ are integers,

$$\text{for } 1 \le j \le q, \quad 1 \le l \le d, \quad \sum_{k=1}^{p} \Big(\sum_{i=1}^{r} \Phi_{i,j,k}\lambda_i\Big)\beta_{l,k} \in \mathbb{Z},$$

and for each for $1 \le j \le q$, as $\Big(\sum_{i=1}^{r} \Phi_{i,j,k}\lambda_i\Big)_{k=1}^{p} \in \mathbb{Z}^p$, if $\Big(\sum_{i=1}^{r} \Phi_{i,j,k}\lambda_i\Big)_{k=1}^{p} \neq 0$, then by hypothesis we have that always exists $l \in [d]$ that if

$$\Big(\sum_{i=1}^{r} \Phi_{i,j,k}\lambda_i\Big)_{k=1}^{p} \cdot \beta_l \notin \mathbb{Z},$$

which is a contradiction. In this way,

$$\text{for } 1 \le j \le q \text{ and } 1 \le k \le p, \sum_{i=1}^{r} \lambda_i \Phi_{i,j,k} = 0. \tag{4.3}$$

This means that the range of Φ is included in the proper closed subgroup H of $G_s = \mathbb{R}^r$, and thus $[G_0, G_{s-1}] \subseteq H$.
Moreover, inserting Eq. (4.3) into Eq. (4.2), we have that

$$\text{for } 1 \le j \le q, \quad 1 \le l \le d, \quad \sum_{i=1}^{r} F_{i,j}^l \lambda_i = 0.$$

This means that the range of F^l is included in H, in particular $[\gamma_1^0 \cdots \gamma_d^0, G_{s-1}] \subseteq H$.
As $G = \langle \Gamma_0, G_0 \rangle$ and for every $x \in G_{s-1}$ the map $g \to [g, x]$ is a group homomorphism, then

$$G_s = [G, G_{s-1}] = [G_0, G_{s-1}][\gamma_1^0 \cdots \gamma_d^0, G_{s-1}] \subseteq H,$$

which is a contradiction, concluding that the claim is true and finishing the proof.

□

To finish this subsection, we will prove two propositions that will allow us to reduce to the connected case in general.

Proposition 4.3 *Let $(X = G/\Gamma, T_1, \cdots, T_d)$ be a minimal s-step $\mathbb{Z}^d$-nilsystem. Then, there exists an invertible matrix $K \in \mathbb{N}^{d\times d}$ such that $(X_0, (T_1^{K_{i,1}} \circ \cdots \circ T_d^{K_{i,d}})_{i=1}^d)$ is a connected minimal s-step $\mathbb{Z}^d$-nilsystem. Moreover,*

$$(X_0, (T_1^{K_{i,1}} \circ \cdots \circ T_d^{K_{i,d}})_{i=1}^d),$$

is totally ergodic.

Proof. By Lemma 4.1 and Lemma 4.2, there are $p_1, \ldots, p_d \in \mathbb{N}$ such that $(X_0, T_1^{p_1}, \cdots, T_d^{p_d})$ is a nilsystem. However, $(X_0, T_1^{p_1}, \cdots, T_d^{p_d})$ is not necessarily minimal. To fix this, we will define a superior triangular matrix $K \in \mathbb{R}^d \times \mathbb{R}^d$ as follows. For $j \geq i$, we suppose that we have defined $K_{i,l}$ for every $l < j$, we define $K_{i,j}$ as follows

$$K_{i,j} = \min\{k_j \in [p_j] \mid \exists k_j, \cdots k_d \in \mathbb{Z}, \tau_i^{K_{i,i}} \cdots \tau_{j-1}^{K_{i,j-1}} \tau_j^{k_j} \cdots \tau_d^{k_d} X_0 = X_0\}.$$

We claim that $(X_0, (T_1^{K_{i,1}} \circ \cdots \circ T_d^{K_{i,d}})_{i=1}^d)$ is a minimal dynamical nilsystem. As $(X, T_1, \cdots, T_d)$ is minimal nilsystem, we only need to prove that

$$\{\tau_1^{l_1} \cdots \tau_d^{l_d} \mid l_1, \cdots, l_d \in \mathbb{Z}, \tau_1^{l_1} \cdots \tau_d^{l_d} X_0 = X_0\} \subseteq H := \langle (\tau_1^{K_{i,1}} \circ \cdots \circ \tau_d^{K_{i,d}})_{i=1}^d \rangle.$$

We claim that $\forall i \in [d]$, there are $l_{i,i}, l_{i,i+1}, \cdots, l_{i,d} \in \mathbb{Z}$ such that

$$\tau_1^{l_1} \cdots \tau_d^{l_d} \equiv \tau_i^{l_{i,i}} \cdots \tau_d^{l_{i,d}} \mod H.$$

Indeed, we assume that it is true for $i < d$, and let $m \in \mathbb{Z}$ and $r \in \{0, \cdots, K_{i,i} - 1\}$ such that $l_{i,i} = mK_{i,i} + r$. We define $l_{i+1,j} = l_{i,j} - mK_{i,j}$, $\forall j \geq i + 1$, and as $K_{i,j} = 0$, $\forall j < i$ then we have that

$$\tau_i^{l_{i,i}} \cdots \tau_d^{l_{i,d}} \equiv \tau_i^r \tau_{i+1}^{l_{i+1,i+1}} \cdots \tau_{i+1}^{l_{i+1,d}} \mod H.$$

Note that $\tau_i^r \tau_{i+1}^{l_{i+1,i+1}} \cdots \tau_{i+1}^{l_{i+1,d}} X_0 = X_0$, so by the minimality of $K_{i,i}$ we obtain $r = 0$, concluding the induction. In this light

$$\tau_1^{l_1} \cdots \tau_d^{l_d} \equiv 0 \mod H,$$

thus $\tau_1^{l_1} \cdots \tau_d^{l_d} \in H$.

Finally, as X_0 is connected, by Corollary 1.1 and Proposition 4.1, we have that the system

$$(X_0, (T_1^{K_{i,1}} \circ \cdots \circ T_d^{K_{i,d}})_{i=1}^d),$$

is totally ergodic. □

Remark 16 The previous proof is based in the idea that there is a parallelepiped that generates all the mesh associated to the dynamic of $\{T_i\}_{i=1}^d$ in X_0 (that parallelepiped repeats itself periodically in $\mathbb{Z}^d$).

We also will need a generalization of the fact that if $R \subseteq \mathbb{Z}^d$ is a set of Bohr recurrence, then for all $\vec{k} \in \mathbb{N}^d$, $R_0 = \{(n_1, \cdots, n_d) \in \mathbb{Z}^d \mid (k_1 n_1, \cdots, k_d n_d) \in R\}$ is a set of Bohr recurrence.

Proposition 4.4 *Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence, and $M \in \mathbb{Z}^{d\times d}$ an invertible matrix. Then*

$$R_0 = \{n = (n_1, \ldots, n_d)^T \in \mathbb{Z}^d \mid M \cdot n \in R\},$$

is a set of Bohr recurrence.

Proof. Consider a torus rotation $(\mathbb{T}^r, T_1, \ldots, T_d)$ with $T_1, \ldots, T_d$ the rotations associated to elements $\alpha^1, \cdots, \alpha^d \in \mathbb{R}^r$ respectively. Let $A \in \mathbb{Z}^{d\times d}$ and $N \in \mathbb{N}$ be such that $M^{-1} = \frac{1}{N}A$. We take $m \in R$ such that N divides all coordinates of m and such that

$$||m^T \beta|| \leq \epsilon,$$

with $\beta = \frac{1}{N}A^T(\alpha^1, \cdots, \alpha^d)^T \in \mathbb{R}^{d\times r}$. We define $n = \frac{1}{N}Am \in \mathbb{Z}^d$, then we have that

$$m^T\beta = m^T \frac{1}{N} A^T (\alpha^1, \cdots, \alpha^d)^T = n^T (\alpha^1, \cdots, \alpha^d)^T = \sum_{i=1}^{d} n_i \alpha^i.$$

In this way, we found $n \in R_0$ such that

$$||T_1^{n_1} \cdots T_d^{n_d} x - x|| = \left\| \sum_{i=1}^{d} n_i \alpha^i \right\| \leq \epsilon,$$

concluding that R_0 is a set of Bohr recurrence by Proposition 2.7.

□

With all this, it will be possible to reduce to the case in that $(X, T_1, \cdots, T_d)$ is a s-step nilsystem with X connected and G_0 simply connected.

4.1.2. Generalization Difficulties

In 2016 Host, Kra, and Maass proved the following theorem in [1].

Theorem 4.2 ([1], Theorem 4.1) *If $R \subseteq \mathbb{Z}$ is a set of Bohr recurrence then R is a set of recurrence for all minimal $\mathbb{Z}^d$-nilsystems.*

We want an analogous but for $\mathbb{Z}^d$-nilsystems. In this section we make use of affine nilsystems to obtain a better understanding of the problem when trying to generalize Theorem 4.2 to $\mathbb{Z}^d$-actions. We will see that depending on the structure of the sets of Bohr recurrence, this problem can be avoided.

We recall that affine nilsystems are the case when $X = \mathbb{T}^r$ for $r \geq 1$, and the transformations $T_1, \cdots, T_d$ are defined by $T_i(x) = A_i x + \alpha_i$ where the matrices A_i and the elements α_i satisfy

- $\forall i \in [d]$, $\exists p \in \mathbb{N}$, $(A_i - I)^p = 0$,
- $A_i A_j = A_j A_i$, $\forall i, j \in [d]$,
- $(A_i - I)\alpha_j = (A_j - I)\alpha_i$, $\forall i, j \in [d]$.

For an affine nilsystem $(X = \mathbb{T}^r, T_1, \cdots, T_d)$ with dynamics $T_i x = M_i x + \alpha_i$ for $x \in X$, we have that $\mathbb{T}^r = G/\Gamma$ with G is spanned by $\{M_i\}_{i=1}^d$ and the translations $S_\beta(x) = x + \beta$ for $\beta \in \mathbb{T}^r$, and Γ is the discrete cocompact subgroup spanned by $\{M_i\}_{i=1}^d$.

The following lemma will be useful in the subsequence proposition, which allows to understand the commutator groups in an affine nilsystem.

Lemma 4.3 *Let $(\mathbb{T}^r = G/\Gamma, T_1, \cdots, T_d)$ be an s-step $\mathbb{Z}^d$-affine nilsystem. Then, $\forall h \in \mathbb{T}^r, n_1, \cdots, n_d \in \mathbb{Z}$,*

$$(\prod_{i=1}^{d} M_i^{n_i} - I)h \in \langle \{\prod_{i=1}^{d} (M_i - I)^{j_i} \mathbb{T}^r \mid \{j_i\}_{i=1}^d \subseteq \mathbb{N}_0,\ j_1 + \cdots + j_d = 1\} \rangle$$

Proof. Denote

$$S_1 = \langle \{\prod_{i=1}^{d} (M_i - I)^{j_i} \mathbb{T}^r \mid \{j_i\}_{i=1}^d \subseteq \mathbb{N}_0,\ j_1 + \cdots + j_d = 1\} \rangle.$$

We will prove the statement using induction over d. For $d = 1$ and $h \in \mathbb{T}^r$ if $n_1 = 0$ the result is direct, otherwise:

$$\begin{aligned}(M_1^{n_1} - I)h &= (M_1^{|n_1|} - I) sgn(n_1) M_1^{n_1 1_{n_1<0}} h \\ &= (M_1 - I)(M_1^{|n_1|-1} + \cdots + I) sgn(n_1) M_1^{n_1 1_{n_1<0}} h \in (M_1 - I)\mathbb{T}^r \subseteq S_1.\end{aligned}$$

Now suppose that for $l < d$ we have that $\forall h \in \mathbb{T}^r, n_1, \cdots, n_l \in \mathbb{Z}$:

$$(M_1^{n_1} \cdots M_{l-1}^{n_{l-1}} - I)h \in S_1.$$

Let $h \in \mathbb{T}^r, n_1, \cdots, n_l \in \mathbb{Z}$, we obtain that

$$(M_1^{n_1} \cdots M_l^{n_l} - I)h = (M_1^{n_1} \cdots M_{l-1}^{n_{l-1}} - I)M_l^{n_l} h + (M_l^{n_l} - I)h \in S_1 + S_1 = S_1,$$

in where the first term is in S_1 by the induction hypothesis, and the second one by the case $d = 1$, applied with the matrix M_l. In this light, the statement follows by induction. □

Proposition 4.5 *Let $(\mathbb{T}^r, T_1, \ldots, T_d)$ be a s-step $\mathbb{Z}^d$-affine nilsystem. Then $\forall j \geq 2$:*

$$G_j = \langle \{(M_1 - I)^{i_1} \cdots (M_d - I)^{i_d} \mathbb{T}^r \mid i_1, \ldots, i_d \in \{0, \cdots, j-1\}, i_1 + \cdots + i_d = j - 1\} \rangle,$$

in where G_j is the j-th commutator group (i.e. $G_j := [G_{j-1}, G]$ with $G_1 = G$).

Proof. Let $j = 2$. Notice that every element $g \in G$ can be written as a map $x \to M(g)x + \alpha(g)$, where $M(g) = M_1^{m_1} \cdots M_d^{m_d}$ with $m_1, \ldots, m_d \in \mathbb{Z}$ and $\beta(g) \in \mathbb{T}^r$. For $g_1, g_2 \in G$ the commutator $[g_1, g_2]$ is the map

$$x \to x + (M(g_1) - I)\beta(g_2) - (M(g_2) - I)\beta(g_1),$$

and thus is a translation of $\mathbb{T}^r$. Besides, if $g \in G$ and $\beta \in \mathbb{T}^r$, then

$$[g, \beta]x = x + (M(g) - I)\beta.$$

It follows that for $n \geq 2$ and $g_1, \cdots, g_n \in G$ the iterated commutator

$$[\cdots[g_1, g_2], g_3], \cdots, g_k],$$

belongs to $\mathbb{T}^r$ and it is contained in

$$\langle\{(M(h_1) - I) \cdots (M(h_{k-1}) - I)\mathbb{T}^r \mid h_1, \cdots, h_{k-1} \in G\}\rangle.$$

On the other hand, Lemma 4.3 yields that for all $h \in \mathbb{T}^r$, $(M(g_1) - I) \cdots (M(g_{k-1}) - I)h$ is in

$$S_{k-1} := \langle\{(M_1 - I)^{j_1} \cdots (M_d - I)^{j_d}\mathbb{T}^r \mid \{j_i\}_{i=1}^d \subseteq \mathbb{N}_0,\ j_1 + \cdots + j_d = k - 1\}\rangle,$$

and therefore, the inclusion $G_k \subseteq S_{k-1}$ follows for all $k \geq 2$.

For the other inclusion, note that for $\{j_i\}_{i=1}^d \subseteq \mathbb{N}_0$ with $j_1 + \cdots + j_d = k - 1$, and $h \in \mathbb{T}^r$, we have that for all $j \in [d]$:

$$(M_j - I)^{i_j}h = [\cdots[h, T_j], \cdots, T_j],$$

in where the commutator is taken i_j times. Therefore,

$$(M_1 - I)^{i_1} \cdots (M_d - I)^{i_d}h \in G_{k-1},$$

as $i_1 + \cdots + i_d = k - 1$, concluding.

□

Remark 17 In the proof of Theorem 4.2, for $\mathbb{Z}$-affine nilsystems it was crucial the fact that for every $\epsilon > 0$ and $n \in R$ big enough, the set

$$\{n(M - I)y \mid y \in G_{s-1} \cap B(0, \epsilon), n \in R\}$$

is dense in $G_s = \{(M - I)\beta \mid \beta \in G_{s-1}\}$. However, the analog for $\mathbb{Z}^d$-affine nilsystems is not always true. Namely, for $d = 2$, it is not always true that

$$\{n_1(M_1 - I)y + n_2(M_2 - I)y \mid y \in G_{s-1} \cap B(0, \epsilon), (n_1, n_2) \in R\},$$

is dense in $G_s = \{(M_1 - I)\beta_1 + (M_2 - I)\beta_2 \mid \beta_1, \beta_2 \in G_{s-1}\}$.

To provide an example of what is said in Remark 17, we first note the following:

Remark 18 If $R \subseteq \mathbb{Z}$ is a set of Bohr recurrence then $\{(n,n) \mid n \in R\}$ is a set of Bohr recurrence.

As we will prove a general statement later, we will omit the proof of this remark.

Example 5 Consider an affine system $(X = \mathbb{T}^4, T_1, T_2)$, with $T_i x = M_i x + \alpha_i$, for $i \in [2]$. If we choose $M_1 - I = -(M_2 - I)$, for example:

$$M_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $R = \{(n,n) \mid n \in R\}$ the expression $n(M_1 - I)y + n(M_2 - I)y$ is always 0. Meanwhile $(M_1 - I)\beta_1 + (M_2 - I)\beta_2$ for $\beta_1, \beta_2 \in G_{s-1}$ is not 0 necessarily.

Paradoxically, we do have that R is a set of recurrence for (X, T_1, T_2). Indeed, note that $M_1 M_2 - I = (M_1 - I)(M_2 - I)$ so $M_1 M_2$ is a nilpotent matrix. Also notice that

$$T_1 T_2 x = M_1 M_2 x + \tilde{\alpha} \text{ where } \tilde{\alpha} = M_2 \alpha_1 + \alpha_2 = M_1 \alpha_2 + \alpha_1.$$

Then consider $Tx = M_1 M_2 x + \tilde{\alpha}$, and the system (X, T) which could be non-minimal, but if we consider U an open set of X, and $x \in U$ some point, we can always reduce to the system $(Y = \overline{O_T(x)}, T)$ which is a minimal nilsystem. If $\tilde{U} = U \cap Y$, we know that there always exists $n \in R$ such that $\tilde{U} \cap T^{-n}\tilde{U} \neq \emptyset$, and therefore $U \cap T_1^{-n} T_2^{-n} U \neq \emptyset$.

From now on, for a dynamic $T_i x = M_i x + \alpha_i$ we will denote $N_i := (M_i - I)$ and $R_i x := x + \alpha_i$.

A class of nilsystems where the problem presented in Remark 17 does not appear is, for instance, the class of 2-step $\mathbb{Z}^d$-affine nilsystems, as we show in the following theorem.

Theorem 4.3 *Let $(X, T_1, \cdots, T_d)$ be a 2-step $\mathbb{Z}^d$-affine nilsystem and $R \subseteq \mathbb{Z}^d$ a set of Bohr recurrence. Then R is a set of recurrence in $(X, T_1, \cdots, T_d)$.*

Proof. Choose

$$\vec{n} \in R \cap \{(m_1, \cdots, m_d) \in \mathbb{Z}^d \mid \|m_1 \alpha_1\| < \delta, \ldots, \|m_d \alpha_d\| < \delta\},$$

where $\delta < \frac{\epsilon}{d}$ is such that $\|x\| < \delta$ then $N_i(x) < \frac{\epsilon}{d}$, for every $i \in [d]$.

We will prove by induction that

$$T_1^{n_1} \cdots T_d^{n_d} x = R_1^{n_1} \cdots R_d^{n_d}(x) + \sum_{i=1}^{d} N_i(n_i x + \frac{n_i(n_i - 1)}{2}\alpha_i) + \frac{1}{2} \sum_{\substack{i \neq j \\ i,j \in [d]}} n_i n_j N_{i,j}(\alpha_{j,i}), \tag{4.4}$$

where $N_{i,j}(\alpha_{j,i}) := N_i(\alpha_j) = N_j(\alpha_i)$. In fact, the case $d = 1$ follows by induction over n_1: We will show that if $Tx = Mx + \alpha$ then

$$T^n(x) = R^n(x) + N(nx + \frac{n(n-1)}{2}\alpha).$$

Indeed, the case $n = 1$ is derived from the definition. Suppose that we have the claim for $n-1$, then

$$\begin{aligned}
T^n(x) &= (n-1)N(T(x)) + R^{(n-1)}(T(x)) + \frac{(n-1)(n-2)}{2}N(\alpha) \\
&= (n-1)N^2(x) + (n-1)N(x) + (n-1)N(\alpha) + N(x) + R^n(x) + \frac{(n-1)(n-2)}{2}N(\alpha) \\
&= nN(x) + R^n(x) + \frac{n(n-1)}{2}N(\alpha) \\
&= R^n(x) + N(nx + \frac{n(n-1)}{2}\alpha),
\end{aligned}$$

in where we used that $N^2 = 0$ since the step of the system is 2.

Now, assume that Eq. (4.4) holds for $d-1$, we prove the formula for d. In fact, the induction hypothesis yields:

$$T_1^{n_1} \cdots T_d^{n_d} x = T_d^{n_d}\Big(R_1^{n_1} \cdots R_{d-1}^{n_{d-1}}(x) + \sum_{i=1}^{d-1} N_i(n_i x + \frac{n_i(n_i-1)}{2}\alpha_i) + \frac{1}{2}\sum_{\substack{i\neq j \\ i,j\in[d-1]}} n_i n_j N_{i,j}(\alpha_{i,j})\Big),$$

then using the case $d = 1$:

$$\begin{aligned}
T_1^{n_1} \cdots T_d^{n_d} x &= R_1^{n_1} \cdots R_{d-1}^{n_{d-1}} R_d^{n_d}(x) + \sum_{i=1}^{d-1} N_i(n_i x + \frac{n_i(n_i-1)}{2}\alpha_i) + \frac{1}{2}\sum_{\substack{i\neq j \\ i,j\in[d-1]}} n_i n_j N_{i,j}(\alpha_{i,j}) \\
&+ N_d\Bigg(n_d\Big(R_1^{n_1} \cdots R_{d-1}^{n_{d-1}}(x) + \sum_{i=1}^{d-1} N_i(n_i x + \frac{n_i(n_i-1)}{2}\alpha_i) + \frac{1}{2}\sum_{\substack{i\neq j \\ i,j\in[d-1]}} n_i n_j N_{i,j}(\alpha_{i,j})\Big) + \frac{n_d(n_d-1)}{2}\alpha_d\Bigg) \\
&= R_1^{n_1} \cdots R_d^{n_d}(x) + \sum_{i=1}^{d-1} N_i(n_i x + \frac{n_i(n_i-1)}{2}\alpha_i) + \frac{1}{2}\sum_{\substack{i\neq j \\ i,j\in[d-1]}} n_i n_j N_{i,j}(\alpha_{i,j}) + N_d(n_d x + \frac{n_d(n_d-1)}{2}\alpha_d) \\
&+ N_d(n_d \sum_{i=1}^{d-1} n_i \alpha_i) \\
&= R_1^{n_1} \cdots R_d^{n_d}(x) + \sum_{i=1}^{d} N_i(n_i x + \frac{n_i(n_i-1)}{2}\alpha_i) + \frac{1}{2}\sum_{\substack{i\neq j \\ i,j\in[d]}} n_i n_j N_{i,j}(\alpha_{i,j}),
\end{aligned}$$

in where we used the fact that the system $(X, T_1, \cdots, T_d)$ is of step 2, and that $N_d(\alpha_i) = N_{i,d}(\alpha_{d,i})$, $\forall i \leq d-1$. Now, we consider a solution of the form $x = -\sum_{i=1}^{d} n_i \alpha_d / 2 \in B(0, \epsilon)$, with this we note that

$$\|R_1^{n_1} \cdots R_d^{n_d}(x)\| \leq 2\epsilon,$$

and

$$\begin{aligned}
&\sum_{i=1}^{d} N_i\Big(n_i(-\sum_{j=1}^{d} n_j\frac{\alpha_j}{2}) + \frac{n_i(n_i-1)}{2}\alpha_i\Big) + \frac{1}{2}\sum_{\substack{i\neq j\\ i,j\in[d]}} n_in_jN_{i,j}(\alpha_{i,j})\\
&= \sum_{\substack{i\neq j\\ i,j\in[d]}} \frac{-n_in_j}{2}N_i(\alpha_j) + \sum_{i=1}^{d} N_i(\frac{-n_i}{2}\alpha_i) + \frac{1}{2}\sum_{\substack{i\neq j\\ i,j\in[d]}} n_in_jN_{i,j}(\alpha_{i,j})\\
&= \sum_{i=1}^{d} N_i(\frac{-n_i}{2}\alpha_i).
\end{aligned}$$

By the election of $\vec{n}\in R$ we have that $\left\|\sum_{i=1}^{d} N_i(\frac{-n_i}{2}\alpha_i)\right\| \leq \epsilon$.

Therefore

$$\|T_1^{n_1}\cdots T_d^{n_d}x\| \leq 3\epsilon,$$

concluding the proof. □

We cannot expect to have a similar proof to all nilsystems, given the constraints that were presented in Example 5. This suggests that the form that the set of recurrence R takes should be considered. We will explore different cases in which either this difficulty can be skipped, or the form of the set of Bohr recurrence can be changed in order to avoid such problem.

4.2. Bohr Recurrence and the Strong Closing Property

In this section we will analize the case when the nilsystem $(X = G/\Gamma, T_1,\cdots,T_d)$ is factor of a product nilsystem. This is the case of a nilsystem with the closing property for $d = 2$, but for $d \geq 3$ we will need to define a stronger property to have this property. First we give a characterization of sets of $\mathbb{Z}^d$-Bohr recurrence through d-product of sets of return times of $\mathbb{Z}$-nilsystems. Second, we introduce the notion of strong closing property, which we will see characterizes the nilsystems which are factors of product nilsystems. We also give many properties in this new notion, and we prove that for nilsystems with this stronger closing property we can solve the problem presented in section Section 4.1.2.

4.2.1. Bohr Recurrence and Products of Return Times

From now on, we assume $s \geq 2$. For a s-step $\mathbb{Z}^d$-nilsystem $(X, T_1, \dots, T_d)$ denote

$$\tilde{G} := G/G_s, \quad \tilde{\Gamma} = \Gamma/(\Gamma\cap G_s), \quad \text{and} \quad \tilde{X} := \tilde{G}/\tilde{\Gamma}. \tag{4.5}$$

Then $\tilde{G}$ is a $(s-1)$-nilpotent Lie group, $\tilde{\Gamma}$ is a discrete cocompact subgroup, $\tilde{X}$ is a $(s-1)$-nilmanifold and the quotient map $G\to\tilde{G}$ induces a projection $\pi : X\to\tilde{X}$, therefore we can view $\tilde{X}$ as the quotient of X under the action of G_s. Let $\tilde{\tau}_1,\cdots,\tilde{\tau}_d$ the image of $\tau_1,\cdots,\tau_d$ in $\tilde{G}$ and $\tilde{T}_1,\cdots,\tilde{T}_d$ the rotations by $\tilde{\tau}_1,\cdots,\tilde{\tau}_d$ in $\tilde{X}$. Then $(\tilde{X},\tilde{T}_1,\cdots,\tilde{T}_d)$ is a $(s-1)$-nilsystem and $\pi : X\to\tilde{X}$ is a factor map. Note that we can also consider the case $s = 1$, in where $\tilde{X}$ and all associated groups are the trivial group, this is possible if we consider the trivial group as a 0-nilsystem.

We will state a small property that it is also proved in [1] indirectly.

Proposition 4.6 *Consider (X,T) a nilsystem where X is connected and G_0 is simply connected. Denote $\pi : X \to \tilde{X}$ the factor describe right after Eq. (4.5). Suppose that exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\forall k \in \mathbb{N}$,*

$$\pi(B(e_X,\epsilon)) \cap \tilde{T}^{-n_k}\pi(B(e_X,\epsilon)) \neq \emptyset,$$

then exists $K \in \mathbb{N}$ such that $\forall k \geq K$ $B(e_X, 3\epsilon) \cap T^{-n_k}(B(e_X,3\epsilon)) \neq \emptyset$.

We will write the proof only for completeness.

Proof. Let $k \in \mathbb{N}$. From the hypothesis, there are a sequence $(n_k)_k$, $x \in X$ and $v \in G_s$ such that

$$d_X(x, e_X) < \epsilon, \quad \text{and } d_X(T^{n_k}x, v \cdot e_X) < \epsilon.$$

Lifting x to G, we obtain $g \in G_i$ and $\gamma \in \Gamma$ with

$$d_G(g, 1_G) < \epsilon, \quad \text{and } d_G(\tau^{n_k}g, v\gamma) < \epsilon.$$

We claim that it suffices to show that if n_k is sufficiently large, there exists $h \in G_{s-1}$ and $\theta \in G_s \cap \Gamma$ such that

$$d_G(h, 1_G) < \epsilon \text{ and } d_G([h^{-1}, \tau^{n_k}], v^{-1}\theta) < \epsilon. \tag{4.6}$$

In fact, writing $y = h \cdot x$ we have that y is the projection of hg in X and that

$$d_X(y, e_X) \leq d_G(hg, 1_G) \leq d_G(h, 1_G) + d_G(g, 1_G) < 2\epsilon.$$

Furthermore,

$$\begin{aligned} d_X(T^{n_k}y, e_X) \leq d_G(\tau^{n_k}hg, \theta\gamma) &= d_G(h[h^{-1},\tau^{n_k}]\tau^{n_k}g, \theta\gamma) \\ &\leq \epsilon + d_G([h^{-1},\tau^{n_k}]\tau^{n_k}g, \theta\gamma) = \epsilon + d_G(\tau^{n_k}g[h^{-1},\tau^{n_k}], \theta\gamma) \\ &\leq 2\epsilon + d_G(v\gamma[h^{-1},\tau^{n_k}], \theta\gamma) = 2\epsilon + d_G([h^{-1},\tau^{n_k}]v\gamma, \theta\gamma) \\ &= 2\epsilon + d_G([h^{-1},\tau^{n_k}], v^{-1}\theta) < 3\epsilon, \end{aligned}$$

where we used the right invariance of the distance d_G, the fact that $[h^{-1}, \tau^{n_k}] \in G_s$, and that G_s is included in the center of G. This proves the claim.

We are left with finding $h \in G_{s-1}$ and $\theta \in G_s$ satisfying Eq. (4.6). Let C from Proposition 4.2 for the case $d = 1$, applied with ϵ and v^{-1}. There exist $h' \in G_{s-1}$ and $\theta \in G_s \cap \Gamma$ such that

$$d_G(h', 1_G) < C \text{ and } d_G([h', \tau], v^{-1}\theta) < \epsilon.$$

Since G_s is isomorphic to $\mathbb{R}^r$, there exists $h \in G_s$ with $h^{-n_k} = h'$ and

$$d_G(h, 1_G) \leq d_G(h', 1_G)/n_k < C/n_k.$$

We take $K \in \mathbb{N}$ such that for all $k \geq K$, we have $n_k \geq C/\epsilon$. This selection of k yields

$d_G(h, 1_G) < \epsilon$. Since $h \in G_{s-1}$, we have

$$[h^{-1}, \tau^{n_k}] = [h^{-1}, \tau]^{n_k} = [h^{-n_k}, \tau] = [h', \tau],$$

and h satisfies the announced properties.

□

The following Theorem has the same nature that Proposition 2.12, but instead of using joinings of $\mathbb{Z}^{d-1}$-rotations, it uses product of $\mathbb{Z}$-nilsystems.

Theorem 4.4 *If $R \subseteq \mathbb{Z}^d$ is a sets of Bohr recurrence, then it has nonempty intersection with all d-product of set of return times of open sets arising from $\mathbb{Z}$-minimal nilsystems.*

Proof. Notice that is enough to prove the statement in the case R is essential. In fact, by Proposition 2.10 there exist $d' \leq d$, a permutation $\pi : \mathbb{Z}^d \to \mathbb{Z}^d$, and an essential set of Bohr recurrence $R' \subseteq \mathbb{Z}^{d'}$ such that $R' \times \{0\}^{d-d'} \subseteq \pi(R)$. Thus, as 0 is trivially in any set of return times, we can prove the statement changing d for d' and R by R', which is essential.

For minimal nilsystems $(X_1, T_1), \cdots, (X_d, T_d)$ of order $s_1, \cdots, s_d$ respectively, we denote $s = \max\{s_1, \cdots, s_d\}$. We will use induction over s that every set of Bohr recurrence $R \subseteq \mathbb{Z}^d$ intersects all d-product of sets of return times arising from $\mathbb{Z}$-minimal nilsystems of order at most s.

The base case is when $s = 1$, in which all systems are of order 1 and therefore these system correspond to minimal rotation. The result follow by noting that every d-product of sets of return times arising from rotations is a set of return times from the product system, which can be non-minimal, but we can always reduce to the orbit of some point associated to the open set of the set of return times.

Suppose that for all $t < s$ and set of Bohr recurrence $R \subseteq \mathbb{Z}^d$, R has nonempty intersection with all d-Cartesian product of sets of return times of t-step minimal nilsystems. Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence and $(X_1, T_1), \cdots, (X_d, T_d)$ minimal nilsystems of order $s_1, \cdots, s_d$ respectively, such that $s = \max\{s_1, \cdots, s_d\}$. Let $X_{i,0}$ denote the connected component of e_{X_i} in X_i for $i = 1, \ldots, d$ respectively. Then there exists a minimal $\vec{k} \in \mathbb{N}^d$ such that $T_i^{k_i} X_{i,0} = X_{i,0}$ for $i = 1, \ldots, d$. The systems $(X_{i,0}, T_i^{k_i})$ are minimal s-step nilsystem by minimality of the period k_i and by minimality of the original system. On the other hand, the set

$$R_0 = \{\vec{n} \in \mathbb{Z}^d \mid (k_1 n_1, \ldots, k_d n_d) \in R\},$$

is a set of Bohr recurrence by Proposition 2.8. Substituting $X_{i,0}$ for X_i and R_0 for R, we reduce to the case in which X_i is connected. We can further assume without loss of generality that $G_{i,0}$ are simply connected for $i = 1, \cdots, d$.

For $i \in [d]$, let U_i be a nonempty open subset of X_i. We want to show that there exists $\vec{n} \in R$ such $U_i \cap T_i^{-n_i} U_i \neq \emptyset$ for every $i \in [d]$. We can assume without loss of generality that U_i is the open ball $B(e_{X_i}, \epsilon)$ for some $\epsilon > 0$, by minimality.

For $i \in [d]$, let $\pi_i : X_i \to \tilde{X}_i$ be the factor map defined after Eq. (4.5). Since $(\tilde{X}_i, \tilde{T}_i)$ is an $(s-1)$ nilsystem, it follows from the induction hypothesis that there exist $\vec{n} \in R$ such as $\pi_i(B(e_{X_i},\epsilon))\cap \tilde{T}_i^{-n_i}\pi(B(e_X,\epsilon))\neq\emptyset$. Note that we can take $n_1,\dots,n_d$ infinitely large. In fact, if this is not true for n_1 for example, there are finite $n_1^1,\dots,n_1^N\in\mathbb{Z}$ satisfying that for $j=1,\dots,N$ exist $n_2^j,\cdots,n_d^j\in\mathbb{Z}$ with $(n_1^j,\dots,n_d^j)\in R$ such as

$$\pi_i(B(e_{\tilde{X}_i},\epsilon))\cap \tilde{T}_i^{-n_i^j}\pi(B(e_{\tilde{X}_i},\epsilon))\neq\emptyset.$$

Consider $\tilde{R}=R\setminus(\bigcup_{j=1}^N B^1_{n_1^j})$ with $B^1_{n_1^j}$ defined as in Proposition 2.11. We have that $\tilde{R}$ is a set of Bohr recurrence, that does not intersect the product $\prod_{i=1}^d N(B(e_{\tilde{X}_i},\epsilon))$, which is a contradiction with the induction hypothesis.

Then we can find strictly increasing sequences $(n_1^k)_k,\dots,(n_d^k)_k$ such that $\forall k\in\mathbb{N}$,

$$(n_1^k,\dots,n_d^k)\in R,\text{ and for every } i\in[d],\pi_i(B(e_{\tilde{X}_i},\epsilon))\cap\tilde{T}_i^{-n_i^j}\pi(B(e_{\tilde{X}_i},\epsilon))\neq\emptyset.$$

By Proposition 4.6 we can find $K\in\mathbb{N}$ satisfying

$$(n_1^K,\dots,n_d^K)\in R,\text{ and for every } i\in[d],B(e_{X_i},3\epsilon)\cap T_i^{-n_i}B(e_{X_i},3\epsilon)\neq\emptyset,$$

concluding the proof. □

Given that the closing property in $\mathbb{Z}^2$-nilsystems implies being factor of a product system by Theorem 1.14, we have the following result

Theorem 4.5 *Let $R\subseteq\mathbb{Z}^2$ a set of Bohr recurrence. Then R is a set of recurrence for all $\mathbb{Z}^2$-minimal nilsystem with the unique closing parallelepiped property.*

The proof of Theorem 4.5 will be omitted for now, because we will prove a general result in Section 4.2.2, in where we will define the property that characterizes $\mathbb{Z}^d$-systems which are factor of d-product systems for $d\geq 2$.

4.2.2. The Strong Closing Property.

In order to generalize Theorem 4.5 to the case $d\geq 3$, strong conditions over the system are needed. Note that for the case $d=2$, the closing property reduces to the following: Let $x_0\in X$ be a continuity point, if $z_1=\lim_{k\to\infty}T_1^{n_k^1}x_0$ and $z_2=\lim_{k\to\infty}T_2^{n_k^2}x_0$, then

$$\lim_{k\to\infty}T_1^{n_k^1}T_2^{n_k^2}x_0\text{ is uniquely determined.}$$

In other words, we just need to know the values in each "canonical" axis to know every diagonal limit. This notion is stronger than the closing property in the case $d\geq 3$, so we define it now.

Definition 4.2 (Strong Closing Property) *Let $(X,T_1,\cdots,T_d)$ be a dynamical system, with d-commuting dynamics. We say that X has the strong closing property if for $x,y\in Q_{T_1,\cdots,T_d}(X)$ are such that*

$$x_\emptyset=y_\emptyset,\ \textit{and}\ ,x_{e_j}=y_{e_j},\forall j\in[d],$$

then, we have that $x=y$.

The next property comes directly from Definition 1.12.

Proposition 4.7 *Let $(X, T_1, \cdots, T_d)$ a dynamical system, with d-commuting dynamics. If $(X, T_1, \cdots, T_d)$ has the strong closing property, then $(X, T_1, \cdots, T_d)$ has the closing property.*

The following proposition gives sufficient conditions to have the strong closing property through the classical closing property.

Proposition 4.8 *Let $(X, T_1, \cdots, T_d)$ be a distal dynamical system, with d-commuting dynamics. If for every non-empty $I \subseteq [d]$ we have that $(X, (T_i)_{i \in I})$ has the closing property, then $(X, T_1, \cdots, T_d)$ has the strong closing property.*

Proof. We prove the statement by induction. The case $d = 2$ is trivial since both notions coincide. For $d \geq 3$, let $(X, T_1, \cdots, T_d)$ be a dynamical system, with d-commuting dynamics, such that non-empty $\forall I \subseteq [d]$, $(X, (T_i)_{i \in I})$ has the closing property. Let $x, y \in Q_{T_1, \cdots, T_d}(X)$ such that x and y coincide in their canonical coordinates. Let $i \in [d]$, then the restriction of x and y to $Q_{T_1, \cdots, T_{i-1}, T_{i+1}, \cdots, T_d}(X)$ have their canonical coordinates equal, then by induction hypothesis, x and y coincide in all their coordinates $\epsilon \in \{0,1\}^d$ such that $\epsilon_i = 0$. As $i \in [d]$ was arbitrary, we conclude that x and y coincide in all their coordinates $\epsilon \subseteq \{0,1\}^d \setminus \{\vec{1}\}$. Then closing property then yields that x and y are equal.

□

Next we show that product systems have the strong closing property.

Proposition 4.9 *Let (Y_i, S_i) be minimal system. Then the product $(Y = \prod_{i=1}^d Y_i, S_1, \ldots, S_d)$, where S_i acts in the i-th coordinate, have the strong closing property.*

Proof. For every $i \in [d]$ denote $\pi_i : Y \to Y_i$ the canonical factor. Let $x \in Q_{S_1, \cdots, S_d}(Y)$ and let $(y_k)_k \subseteq Y$ and $\{(n_{1,k}, \ldots, n_{d,k})\}_k \subseteq \mathbb{Z}^d$ be a sequences such that for every $\epsilon \subseteq [d]$,

$$S_1^{n_{1,k}\epsilon_1} \cdots S_d^{n_{d,k}\epsilon_d} y_k \to_k x_\epsilon.$$

We observe that

$$\begin{aligned} x_\epsilon &= \lim_{k \to \infty} S_1^{n_{1,k}\epsilon_1} \cdots S_d^{n_{d,k}\epsilon_d} y_k \\ &= \lim_{k \to \infty} (\pi_1(S_1^{n_{1,k}\epsilon_1} y_k), \cdots, \pi_d(S_d^{n_{d,k}\epsilon_d} y_k)) = (\pi_1(x^*_{\epsilon,e_1}), \cdots, \pi_d(x^*_{\epsilon,e_d})), \end{aligned}$$

where

$$x^*_{\epsilon,e_j} = \begin{cases} x_{e_j} & \text{if } \epsilon_j = 1 \\ x_\emptyset & \text{if } \epsilon_j = 0 \end{cases}.$$

In this way, x_ϵ is determined by the canonical coordinates of x, thereby Y has the strong closing property. □

We aim to show that factors of product systems characterize completely the family of minimal distal systems which have the strong closing property. We will require the following proposition.

Proposition 4.10 ([2], Prop. 3.3) *Let $\pi : Y \to X$ be a factor map between the $\mathbb{Z}^d$-systems $(Y, S_1, \cdots, S_d)$ and $(X, T_1, \cdots, T_d)$. Then*

$$\pi^{[d]}(Q_{S_1, \cdots, S_d}(Y)) = Q_{T_1, \cdots, T_d}(X),$$

where $\pi^{[d]} : Y^{[d]} \to X^{[d]}$ is defined from π coordinatewise.

Taking Proposition 4.9 as inspiration, we now characterize the strong closing property.

Proposition 4.11 *Let $(X, T_1, \dots, T_d)$ be a minimal distal $\mathbb{Z}^d$-system. Then $(X, T_1, \dots, T_d)$ has the strong closing property if and only if $(X, T_1, \dots, T_d)$ is factor of a minimal product systems.*

Proof. We prove each implication of the statement.

($\Longrightarrow$) Suppose that X has the strong closing property. Note that X is a factor of the dynamical system given by the product of the systems

$$Y_i := K_{T_i}^{x_0} = \overline{\mathcal{O}_{T_i}(e_X)}x_0, \quad \forall i \in [d],$$

with $x_0 \in X$ a continuity point. Indeed if $z = (z_1, \cdots, z_d) \in \prod_{i=1}^d Y_i$, then we can extend (x_0, z) uniquely to an element $\overline{z} \in Q_{T_1,\cdots,T_d}(X)$, in particular the last coordinate $\overline{z}_{\vec{1}}$ define uniquely an element of X. In this light, the aforementioned function is a factor from $\prod_{i=1}^d Y_i$ to X, since both system are minimal and clearly the function described before is a dynamical morphism.

($\Longleftarrow$) Assume that $(X, T_1, \cdots, T_d)$ is factor of the minimal product system $(\prod_{i=1}^d Y_i, S_1, \cdots, S_d)$ and denote $\pi : \prod_{i=1}^d Y_i \to X$ such factor. Let $x, z \in Q_{T_1,\cdots,T_d}(X)$ such that $x_0 = z_0$ and $x_{e_j} = z_{e_j}$ for each $j \in [d]$. Let $\epsilon \in \{0,1\}^d$, we want to show that $x_\epsilon = z_\epsilon$.

First, for $u \in Q_{S_1,\cdots,S_d}(Y)$ and for every $i \in [d]$, let $(n_i^N)_{N\in\mathbb{N}}$ be a sequence such that

$$S_i^{n_i^N}(u_{\emptyset,j})_{j=1}^d \to_N (u_{\{i\},j})_{j=1}^d.$$

Notice that for $\epsilon \subseteq [d]$ and $i \in [d]$,

$$(\lim_{N\to\infty} S_1^{\epsilon_1 n_1^N} \cdots S_d^{\epsilon_d n_d^N}(u_{\emptyset,j})_{j=1}^d)_i = \begin{cases} u_{\emptyset,i} & \text{if } \epsilon_i = 0 \\ u_{\{i\},i} & \text{if } \epsilon_i = 1 \end{cases},$$

in where we applied that for every $i \in [d]$, S_i acts in the i-th coordinate only. Thus, by Proposition 4.9, we derive that for every $\epsilon \subseteq [d]$ and $i \in [d]$

$$u_{\epsilon,i} = \begin{cases} u_{\emptyset,i} & \text{if } \epsilon_i = 0 \\ u_{\{i\},i} & \text{if } \epsilon_i = 1 \end{cases}.$$

In this light, an element $u \in Q_{S_1,\cdots,S_d}(Y)$ is uniquely determined by their coordinates $\{u_{\emptyset,i}\}_{i=1}^d$ and $\{u_{\{i\},i}\}_{i=1}^d$.

Second, we know from Proposition 4.10 that $\pi^{[d]} : Q_{S_1,\cdots,S_d}(Y) \to Q_{T_1,\cdots,T_d}(X)$ is a factor. Therefore, we have that there exists $y, y' \in Q_{S_1,\cdots,S_d}(Y)$ such that

$$\pi(y) = x, \quad \text{and } \pi(y') = z.$$

For every $i \in [d]$, let $(n_i^N)_{N\in\mathbb{N}}$ be a sequence such that

$$S_i^{n_i^N} y_{\emptyset,i} \to_N y_{\{i\},i}.$$

We can assume without loss of generality that $S_i^{n_i^N} y'_{\emptyset,i}$ converges to some element $y''_{\{i\},i}$. We define $y'' \in Q_{S_1,\cdots,S_d}(Y)$ as the element determined by $(y'_{\emptyset,i})_{i=1}^d$ and $(y''_{\{i\},i})_{i=1}^d$.

Notice that $\pi^{[d]}(y'') = x$, since $\pi((y_{\emptyset,j})_{j=1}^d) = \pi((y'_{\emptyset,j})_{j=1}^d)$, and therefore, for $\epsilon \subseteq [d]$:

$$\begin{aligned}
x_\epsilon &= \pi(y_\epsilon)\\
&= \pi(\lim_{N\to\infty} S_1^{\epsilon_1 n_1^N} \cdots S_d^{\epsilon_d n_d^N} (y_{\emptyset,j})_{j=1}^d)\\
&= \lim_{N\to\infty} T_1^{\epsilon_1 n_1^N} \cdots T_d^{\epsilon_d n_d^N} \pi((y_{\emptyset,j})_{j=1}^d)\\
&= \lim_{N\to\infty} T_1^{\epsilon_1 n_1^N} \cdots T_d^{\epsilon_d n_d^N} \pi((y'_{\emptyset,j})_{j=1}^d)\\
&= \pi(y''_\epsilon).
\end{aligned}$$

As $\pi^{[d]}$ is π acting pointwise, we have that for $i \in [d]$,

$$\pi((y'_{i,j})_{j=0}^d) = z_{e_i} = x_{e_i} = \pi(y'_{i,1}, \cdots, y'_{i,i-1}, y''_{i,i}, y'_{i,i+1}, \cdots, y'_{i,d}).$$

For every $i \in [d]$, by minimality of (Y_i, S_i), we deduce that for

$$(w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_d) \in \prod_{j\neq i} Y_j,$$

we have that

$$\pi(w_{i,1}, \cdots, w_{i-1}, y'_{\{i\},i}, w_{i+1}, \cdots, w_d) = \pi(w_{i,1}, \cdots, w_{i-1}, y''_{\{i\},i}, w_{i+1}, \cdots, w_d). \tag{4.7}$$

Hence, for $\epsilon \in \{0,1\}^d$ we have that

$$\begin{aligned}
x_\epsilon &= \pi(\{y''_{\emptyset,j} \mid j \in [d], \epsilon_j = 0\} \cup \{y''_{\{j\},j} \mid j \in [d], \epsilon_j = 1\})\\
&= \pi(\{y'_{\emptyset,j} \mid j \in [d], \epsilon_j = 0\} \cup \{y''_{\{j\},j} \mid j \in [d], \epsilon_j = 1\})\\
&= \pi(\{y'_{\emptyset,j} \mid j \in [d], \epsilon_j = 0\} \cup \{y'_{\{j\},j} \mid j \in [d], \epsilon_j = 1\}) = z_\epsilon,
\end{aligned}$$

where the second equality comes from the definition of y'', and the third equality comes from applying Eq. (4.7) iteratively. As $\epsilon \in \{0,1\}^d$ was arbitrary, we conclude the proof.

□

Corollary 4.2 *The strong closing property is preserved under factors.*

Minimal distal $\mathbb{Z}^d$-systems that have the strong closing property also exhibit the property whereby the dynamical cubes of these systems depend solely on the cubes of every single transformation.

Proposition 4.12 *Let $(X, T_1, \cdots, T_d)$ a minimal distal $\mathbb{Z}^d$-system with the strong closing property. If $S : X \to X$ is a distal dynamic such that for $i \in [d]$, S commutes with $(T_j)_{j\neq i}$*

and $Q_S = Q_{T_i}$, then we have that

$$Q_{T_1,\cdots,T_d}(X) = Q_{T_1,\cdots,T_{i-1},S,T_{i+1},\cdots,T_d}(X).$$

Proof. First, as X has the strong closing property, X is a factor of the system $(Y = \prod_{i=1}^d Y_j, T_1, \cdots, T_d)$, where $Y_j = \overline{\mathcal{O}_{T_j}(x_0)}$ (with $x_0 \in X$ a continuity point), and T_j only acts in the j-th coordinate of Y. By Proposition 4.10, we have that

$$\pi^{[d]}(Q_{T_1,\cdots,T_d}(Y)) = Q_{T_1,\cdots,T_d}(X). \tag{4.8}$$

Second, we will show that

$$\pi : (Y, T_1, \dots, T_{i-1}, S, T_{i+1}, \dots, T_d) \to (X, T_1, \dots, T_{i-1}, S, T_{i+1}, \dots, T_d),$$

defines a factor (with S only acting in the i-th coordinate). The equality $Q_S(X) = Q_{T_i}(X)$ yields $Y_j = \overline{\mathcal{O}_S(x_0)}$ as x_0 is a continuity point. Thereby, we just need to prove that for $y \in Y$, we have that $\pi(Sy) = S\pi(y)$, moreover, it is enough to show that $\pi(Sx_0^d) = S\pi(x_0^d)$, with $x_0^d = (x_0, \cdots, x_0)$, by minimality of $(Y, T_1, \dots, T_{i-1}, S, T_{i+1}, \dots, T_d)$. Notice that $\pi(x_0^d) = x_0$ by definition of π, and therefore $S\pi(x_0^d) = Sx_0$. Let $(n_k)_k$ such that $T_i^{n^k} x_0 \to Sx_0$, we have that

$$\pi(T_i^{n^k} x_0^d) = T_i^{n^k} \pi(x_0^d) = T_i^{n^k} x_0,$$

the right side goes to Sx_0, meanwhile the left side goes to $\pi(Sx_0^d)$, thereby we have that $\pi(Sx_0^d) = S\pi(x_0^d)$. Therefore, S commutes with π and $\pi : Y \to X$ is still a factor if we replace T_i for S. By Proposition 4.10 we derive

$$\pi^{[d]}(Q_{T_1,\dots,T_{i-1},S,T_{i+1},\dots,T_d}(Y)) = Q_{T_1,\dots,T_{i-1},S,T_{i+1},\dots,T_d}(X). \tag{4.9}$$

Last but not least, we prove that $Q_{T_1,\dots,T_{i-1},S,T_{i+1},\dots,T_d}(Y) = Q_{T_1,\dots,T_d}(Y)$. Let $y \in Q_{T_1,\dots,T_d}(Y)$, then there exist $(z_j)_j \subseteq Y$ and $(n^j)_j \subseteq \mathbb{Z}^d$ such that for every $\epsilon \subseteq [d]$

$$\lim_{j\to\infty} T_1^{\epsilon_1 n_1^j} \cdots T_d^{\epsilon_d n_d^j} z_j = y_\epsilon.$$

Notice that $z_j \to_j y_\emptyset$ and $T_i^{n_i^j} z_j \to_j y_{\{i\}}$. In this light, $(y_{\emptyset,i}, y_{\{i\},i}) \in Q_{T_i}(X) = Q_S(X)$, and as $y_{\emptyset,i}, y_{\{i\},i} \in Y_i = \overline{O}_S(x_0)$ by minimality of S in Y_i, we find a sequence $(m_i)_i$ such that $S^{m_j} y_{\emptyset,i} \to y_{\{i\},i}$. Thereby, defining for $j \in \mathbb{N}$ and $k \in [d]$

$$\tilde{z}_{j,k} = \begin{cases} z_{j,k} & \text{if } k \neq i \\ y_{\emptyset,i} & \text{if } k = i \end{cases},$$

we obtain that for every $\epsilon \subseteq [d]$

$$\lim_{j\to\infty} T_1^{\epsilon_1 n_1^j} \cdots T_{i-1}^{\epsilon_{i-1} n_{i-1}^j} S^{\epsilon_i n_i^j} T_{i+1}^{\epsilon_{i+1} n_{i+1}^j} \cdots T_d^{\epsilon_d n_d^j} \tilde{z}_j = y_\epsilon,$$

which yields $y \in Q_{T_1,\dots,T_{i-1},S,T_{i+1},\dots,T_d}(Y)$. The other inclusion is analogous, thus

$$Q_{T_1,\dots,T_{i-1},S,T_{i+1},\dots,T_d}(Y) = Q_{T_1,\dots,T_d}(Y). \tag{4.10}$$

We conclude by Eq. (4.8), Eq. (4.9), and Eq. (4.10) that

$$Q_{T_1,\cdots,T_d}(X)=\pi^{[d]}(Q_{T_1,\cdots,T_d}(Y))=\pi^{[d]}(Q_{T_1,\ldots,T_{i-1},S,T_{i+1},\ldots,T_d}(Y))=Q_{T_1,\ldots,T_{i-1},S,T_{i+1},\ldots,T_d}(X).$$

□

Question: Let $(X,T_1,\cdots,T_d)$ be a minimal distal $\mathbb{Z}^d$-system such that if for $i\in[d]$, there exists $S:X\to X$ a commuting dynamic (with $\{T_1,\cdots,T_d\}$) such that $Q_S(X)=Q_{T_i}(X)$ then

$$Q_{T_1,\cdots,T_d}(X)=Q_{T_1,\cdots,T_{i-1},S,T_{i+1},\cdots,T_d}(X).$$

Is it true that X has the strong closing property?

To conclude this section, we prove with this stronger version of the closing property that we can generalize Theorem 4.5.

Theorem 4.6 *Let $R\subseteq\mathbb{Z}^d$ a set of Bohr recurrence. Then R is a set of recurrence for all $\mathbb{Z}^d$-minimal nilsystem with the strong unique completeness property.*

Proof. Let $d\geq 2$,$(X,T_1,\cdots,T_d)$ be a minimal nilsystem with the strong closing property, and U a nonempty open subset of X. We want to show that $R\cap N(U)\neq\emptyset$. As we mentioned in the proof of Proposition 4.11, the system X is a factor of the dynamical system given by the product of the systems

$$Y_i:=K_{T_i}^{x_0}=\overline{\mathcal{O}_{T_i}(e_X)}x_0,\quad\forall i\in[d],$$

with $x_0\in X$ a continuity point. We note that $(Y:=\prod_{i=1}^d Y_i,T_1,\cdots,T_d)$ is product of $\mathbb{Z}$-minimal nilsystems, and therefore R is a set of recurrence for Y thanks to Theorem 4.4. Therefore, as X is a factor of Y, we conclude that R is a set of recurrence for X, given that $R\cap N_{T_1,\cdots,T_d}(\pi^{-1}(U))\neq\emptyset$ implies that $R\cap N_{T_1,\cdots,T_d}(U)\neq\emptyset$. □

Remark 19 The main problem to generalize this for a nilsystem $(X,T_1,\cdots,T_d)$ with only the closing property for $d\geq 3$, is that we are unable to ensure that the extension $(Y,T_1,\cdots,T_d)$ described by Theorem 1.14 is the product of the systems $(Y_j,T_1,\cdots,T_d)$. We only can ensure that is a joining of these systems.

4.3. Bohr Recurrence in Affine Nilsystems

This section is dedicated to proving that sets of Bohr recurrence are sets of recurrence for $\mathbb{Z}^d$-quasi-affine nilsystems. Unlike $\mathbb{Z}^d$-nilsystems with the strong closing property, we cannot avoid the problem presented in Section 4.1.2 directly. Instead, it will be of prime importance to modify the form of the set of Bohr recurrence.

4.3.1. The Notion of Bohr Correlations

Consider $R\subseteq\mathbb{Z}^2$ is an essential set of Bohr recurrence. In this case, it makes sense to ask if $|\frac{n_1}{n_2}|$ is tending to some specific value, for $(n_1,n_2)\in R$ big enough. This value obviously lies on $[0,\infty]$, but we will see that if such value is in $\mathbb{Q}\setminus\{0\}$, we can eliminate some coordinates (and lower the dimension of the problem). Otherwise, the value is in $[0,\infty]\setminus\mathbb{Q}$, and we will see that the problem exposed in Remark 17 can be solved in $\mathbb{Z}^d$-quasi-affine nilsystems.

For now, we will provide examples to justify that the cases exposed are valid. In other words, that there exist sets of Bohr recurrence for each case.

Proposition 4.13 *Let $R \subseteq \mathbb{Z}$ a set of Bohr recurrence, then for every $(k_1, k_2) \in \mathbb{Z}^2$ we have that*

$$\tilde{R} = \{(k_1 n, k_2 n) \in \mathbb{Z}^2 \mid n \in R\},$$

is a set of Bohr recurrence.

Proof. Let (G, τ_1, τ_2) an abelian minimal rotation. Notice that $\tau_1^{k_1 n}\tau_2^{k_2 n} = (\tau_1^{k_1}\tau_2^{k_2})^n$ so if we consider the minimal abelian rotation $(\overline{O_{\tau_1^{k_1}\tau_2^{k_2}}(e_G)}, \tau_1^{k_1}\tau_2^{k_2})$ (minimal by distality), then as $R \subseteq \mathbb{N}$ is a set of Bohr recurrence we have that for every $\epsilon > 0$, there exists $n \in R$ such that

$$d((\tau_1^{k_1}\tau_2^{k_2})^n, e_G) = d(\tau_1^{k_1 n}\tau_2^{k_2 n}, e_G) \leq \epsilon.$$

Therefore, we conclude that $\tilde{R} \subseteq \mathbb{N}^2$ is a set of Bohr recurrence.

□

Proposition 4.14 *Let $R \subseteq \mathbb{Z}$ a set of Bohr recurrence, then for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we have that*

$$\tilde{R} = \{(n, \lfloor n\alpha \rfloor) \in \mathbb{Z}^2 \mid n \in R\},$$

is a set of Bohr recurrence.

Proof. Let be $R \subseteq \mathbb{Z}$ a set of Bohr recurrence, and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Define

$$\tilde{R} = \{(n, \lfloor n\alpha \rfloor) \in \mathbb{Z}^2 \mid n \in R\}.$$

Then, $\tilde{R}$ is a set of Bohr recurrence, in fact let $r > 0$ and $\beta, \gamma \in \mathbb{T}^r$. For $\epsilon > 0$, take $n \in R$ such that $\|n\alpha\| \leq \epsilon/2$, $\|n\beta\alpha\| \leq \epsilon/2$, $\|n\gamma\| \leq \epsilon$. This is possible due to the set

$$\{n \in \mathbb{Z} \mid \|n\alpha\| \leq \epsilon/2,\ \|n\beta\alpha\| \leq \epsilon/2,\ \|n\gamma\| \leq \epsilon\},$$

is a Bohr neighborhood of 0, by being the intersection of 3 neighborhoods of 0.

Note that

$$\|\lfloor n\alpha \rfloor \beta\| = \|n\alpha\beta - \{n\alpha\}\beta\| \leq \|\alpha\beta n\| + \|\alpha n\||\beta| \leq \epsilon/2 + \epsilon/2 = \epsilon,$$

therefore

$$\|\gamma n\| \leq \epsilon, \qquad \|\beta\lfloor \alpha n \rfloor\| \leq \epsilon,$$

concluding that $\tilde{R}$ is a set of Bohr recurrence. □

Remark 20 Observe that if

$$\tilde{R} = \{(n, \lfloor n\alpha \rfloor) \in \mathbb{N}^2 \mid n \in R\},$$

and $(n_1, n_2) \in \tilde{R}$, then $n_2/n_1 \approx \alpha$ as $\lim_{n\to\infty} \frac{\lfloor n\alpha \rfloor}{n} = \alpha$.

It is also possible to create sets of Bohr recurrence in $\mathbb{Z}$ from sets of Bohr recurrence in $\mathbb{Z}^d$.

Proposition 4.15 *Let $R \subseteq \mathbb{Z}^d$ a set of Bohr recurrence. Then for all $(k_1, \cdots, k_d) \in \mathbb{N}_0^d$ the set*

$$R_1 = \{\sum_{i=1}^{d} k_i n_i \mid (n_1, \cdots, n_2) \in R\},$$

is a set of Bohr recurrence in $\mathbb{Z}$.

Proof. Let (G, τ) to be an abelian minimal rotation. Then, note that $\tau^{\sum_{i=1}^{d} k_i n_i} = (\tau^{k_1})^{n_1} \cdots (\tau^{k_d})^{n_d}$ so we consider the abelian rotation $(\overline{O_{\tau^{k_1},\cdots,\tau^{k_d}}(e_G)}, \tau^{k_1}, \cdots, \tau^{k_d})$ which is minimal by distality. Hence as $R \subseteq \mathbb{N}^d$ is a set of Bohr recurrence, for every $\epsilon > 0$ we have that there exists $(n_1, \cdots, n_d) \in R$ such that

$$d((\tau^{k_1})^{n_1} \cdots (\tau^{k_d})^{n_d}, e_G) = d(\tau^{\sum_{i=1}^{d} k_i n_i}, e_G) \leq \epsilon,$$

and therefore R_1 is a set of Bohr recurrence. □

Example 6 We have that if $R \subseteq \mathbb{N}^2$ is a set of Bohr recurrence then $\{n_1 \mid (n_1, n_2) \in R\}$ and $\{n_2 \mid (n_1, n_2) \in R\}$ are a sets of Bohr recurrence.

The only case left is when $(n_1, n_2) \in R$ are such that $\frac{n_1}{n_2}$ is tending (in some sense) to 0 (or equivalently, to ∞). We present now a way to construct some of such sets of Bohr recurrence.

Proposition 4.16 *Let $R \subseteq \mathbb{Z}^2$. Denote R_1 the projection of R on the first coordinate, and for $n \in R_1$ we denote*

$$R(n, \bullet) = \{m \in \mathbb{Z} \mid (n, m) \in R\}.$$

Suppose that $R_1 \neq \{0\}$ and that R_1 is a set of Bohr recurrence. For $n \in R_1$ denote L_n the length of the biggest interval contain in $R(n, \bullet)$. We have that if $L_n \to_{|n| \to \infty} \infty$, then R is a set of Bohr recurrence.

Proof. We will prove that R is a set of pointwise recurrence on rotations. Indeed, if (X, g_1, g_2) is a minimal rotation, we can always find a $n_1 \in R_1$ as big as we want such that $d(g_1^{n_1}, e_X) \leq \epsilon/2$. Now, observe that the set $N_{g_2}(e_G, B(e_G, \epsilon/2))$ is syndetic by the fact that the system (X, g_2) is distal so $(\overline{O_{g_2}(e_G)}, g_2)$ is minimal. Taking n_1 such that $L_{n_1} > L$ (with L the syndetic constant associated to $B(e_G, \epsilon/2)$), we have that there exists $n_2 \in R(n_1, \bullet)$ such that $d(g_2^{n_2}, e_X) \leq \epsilon/2$, therefore, there exists $(n_1, n_2) \in R$ such that $d(g_1^{n_1} g_2^{n_2}, e_X) \leq \epsilon$. □

Example 7 The set $R = \{(n_1, n_2) \in \mathbb{Z}^2 \mid n_1^2 \leq |n_2| \leq 2n_1^2\}$ is a set of Bohr recurrence from Proposition 4.16.

We have showed that for $R \subseteq \mathbb{Z}^2$ there are different cases for the values to which $|n_1/n_2|$ is tending for $(n_1, n_2) \in R$. Now we will study this in a general framework, showing that for $R \subseteq \mathbb{Z}^d$ we can always find a vector of correlationships if R fulfills mild conditions. To understand what these conditions are, we begin with the following definitions.

Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence, which coordinates are ordered:

$$\forall n \in R, \quad n_1 \geq \cdots \geq n_d.$$

We will refer to R as an ordered Bohr recurrence set.

Remark 21 Note that we do not lose generality when considering $R \subseteq \mathbb{Z}^d$ ordered. In fact, let $(X, T_1, \ldots, T_d)$ be nilsystem. Consider S_d the symmetric group (the group of permutations of $[d]$)

$$R = \bigcup_{\varphi \in S_d} R \cap \{(n_1, \cdots, n_d) \in \mathbb{Z}^d \mid n_{\varphi(1)} \geq \cdots \geq n_{\varphi(d)}\},$$

an as $|S_d| \leq d!$. By the Ramsey property, there is $\varphi \in S_d$ such that

$$R' = R \cap \{(n_1, \cdots, n_d) \in \mathbb{Z}^d \mid n_{\varphi(1)} \geq \cdots \geq n_{\varphi(d)}\},$$

is a set of Bohr recurrence. Note that recurrence in $(X, T_1, \ldots, T_d)$ with R' is equivalent to recurrence in $(X, T_{\varphi(1)}, \ldots, T_{\varphi(d)})$ with

$$R'' = \{(n_{\varphi(1)}, \ldots, n_{\varphi(d)}) \mid (n_1, \cdots, n_d) \in R'\}.$$

Notice that trivially R'' is an ordered set of Bohr recurrence and $\langle T_1, \ldots, T_d \rangle = \langle T_{\varphi(1)}, \ldots, T_{\varphi(d)} \rangle$, so all the dynamical properties of $(X, T_1, \ldots, T_d)$ are preserved.

Definition 4.3 *Let $R \subseteq \mathbb{Z}^d$ be an essential set of Bohr recurrence. We define a vector of Bohr correlations for R as $P := (P_{i,j})_{j \geq i} \in [0,1]^{d \times (d+1)/2}$ such that $\forall \epsilon > 0$*

$$R_{P,\epsilon} := \{n \in R \ : \ |\frac{n_j}{n_i} - P_{i,j}| \leq \epsilon, \ \forall j \geq i\},$$

is a set of Bohr recurrence. We denote $\mathcal{BC}(R) \subseteq \mathbb{R}^{d \times (d+1)/2}$ the set of Bohr correlations of R.

Proposition 4.17 *If $R \subseteq \mathbb{Z}^d$ is an (essential) ordered Bohr recurrence set, then $\mathcal{BC}(R) \neq \emptyset$.*

Proof. Notice that $(\frac{n_j}{n_i})_{j \geq i} \in [0,1]^{d(d+1)/2}$, $\forall n \in R$. For $m \in \mathbb{N}$, covering $[0,1]^{d(d+1)/2}$ by finite intervals of the form $\prod_{j \geq i}(P_{i,j} - \frac{1}{m}, P_{i,j} + \frac{1}{m})$, for $(P_{i,j})_{j \geq i} \in ([0,1] \cap \mathbb{Q})^{d(d+1)/2}$, and using the Ramsey property, we obtain $P^m := (P^m_{i,j})_{j \geq i} \in ([0,1] \cap \mathbb{Q})^{d(d+1)/2}$ such that

$$R \cap \{n \in \mathbb{Z}^d \ : \ |\frac{n_j}{n_i} - P^m_{i,j}| < \frac{1}{m}, \ \forall j \geq i\},$$

is a set of Bohr recurrence. Passing to a subsequence, we can suppose that $(P^{m_l})_l$ converges to $P \in [0,1]^{d(d+1)/2}$. We claim that

$$R_{P,\epsilon} := R \cap \{n \in \mathbb{Z}^d \ : \ |\frac{n_j}{n_i} - P_{i,j}| < \epsilon, \ \forall j \geq i\},$$

is a set of Bohr recurrence for all $\epsilon > 0$. Indeed, as $P^{m_l} \to_l P$, take l big enough such that

$$(P^{m_l}_{i,j} - \frac{1}{m_l}, P^{m_l}_{i,j} + \frac{1}{m_l}) \subseteq (P_{i,j} - \epsilon, P_{i,j} + \epsilon), \forall j \geq i.$$

Therefore,

$$R \cap \{n \in \mathbb{Z}^d \ : \ |\frac{n_j}{n_i} - P^{m_l}_{i,j}| < \frac{1}{m_l}, \ \forall j \geq i\} \subseteq R \cap \{n \in \mathbb{Z}^d \ : \ |\frac{n_j}{n_i} - P_{i,j}| < \epsilon, \ \forall j \geq i\},$$

as the left-side set is a set of Bohr recurrence, so is the right-side set, concluding that $P \in \mathcal{BC}(R)$. □

Now we prove some properties of Bohr correlations. First, we show that Bohr correlation are consistent, in the following sense.

Lemma 4.4 *Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence, and $P \in \mathcal{BC}(R)$. Then, $\forall i, j, l \in [d]$ with $i \leq j \leq l$*

$$P_{i,l} = P_{i,j}P_{j,l}.$$

Proof. Let $\epsilon > 0$, as $R_{P,\epsilon}$ is a set of Bohr recurrence we have that $\forall n \in R_{P,\epsilon}$

$$\begin{aligned}
|P_{i,l} - P_{i,j}P_{j,l}| &\leq |\frac{n_l}{n_i} - P_{i,l}| + |\frac{n_j}{n_i}\frac{n_l}{n_j} - P_{i,j}P_{j,l}| \\
&\leq \epsilon + |\frac{n_j}{n_i}\frac{n_l}{n_j} - P_{i,j}\frac{n_l}{n_j}| + |P_{i,j}\frac{n_l}{n_j} - P_{i,j}P_{j,l}| \\
&\leq \epsilon + |\frac{n_l}{n_j}|\epsilon + |P_{i,j}|\epsilon \\
&\leq \epsilon + |\frac{n_l}{n_j} - P_{j,l}|\epsilon + |P_{j,l}|\epsilon + |P_{i,j}|\epsilon \\
&\leq \epsilon + \epsilon^2 + |P_{j,l}|\epsilon + |P_{i,j}|\epsilon = \epsilon(1 + |P_{j,l}| + |P_{i,j}|) + \epsilon^2.
\end{aligned}$$

Letting ϵ tend to 0, we have that $P_{i,l} = P_{i,j}P_{j,l}$. □

In what follows, we define the notion of partial vector of Bohr correlations, and then we will show that such partial vector can be extended to an actual vector of Bohr correlations.

Definition 4.4 *Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence and $I \subseteq \{(i,j) \in [d]^2 \mid j \geq i\}$. We say that $P = (P_{i,j})_{(i,j)\in I} \in \mathbb{R}^I$ are I-Bohr correlations if $\forall \epsilon > 0$,*

$$R_{I,\epsilon} = \{n \in R \mid |\frac{n_j}{n_i} - P_{i,j}| \leq \epsilon, \ \forall (i,j) \in I\},$$

is a set of Bohr recurrence.

Lemma 4.5 *Let $R \subseteq \mathbb{Z}^d$ be an ordered set of Bohr recurrence. Consider*

$$I \subseteq \{(i,j) \in [d]^2 \mid j \geq i\},$$

and, $P = (P_{i,j})_{(i,j)\in I} \in [0,1]^I$ I-Bohr correlations. Then, there exists $\tilde{P} \in \mathcal{BC}(R)$ that extents $(P_{i,j})_{(i,j)\in I}$, in the sense that $\tilde{P}_{i,j} = P_{i,j}$, $\forall (i,j) \in I$.

Proof. For every $n \in \mathbb{N}$, take $P^n \in \mathcal{BC}(R_{I,\frac{1}{n}})$. As $(P^n)_n \subseteq [0,1]^{d(d+1)/2}$, we can take a convergent subsequence $(P^{n_l})_l$ to a vector $\tilde{P} \in [0,1]^{d(d+1)/2}$. We claim that $\tilde{P} \in \mathcal{BC}(R)$ and that $\tilde{P}_{i,j} = P_{i,j}$, $\forall (i,j) \in I$. Indeed, let $\epsilon > 0$ and take $l \in \mathbb{N}$ big enough such that

$|P^{n_l}_{i,j} - \tilde{P}_{i,j}| \leq \epsilon/2$, $\forall j \geq i$, and such that $\frac{1}{n_l} \leq \epsilon$. Then we have that $\forall n \in (R_{I,\frac{1}{n_l}})_{P^{n_l},\epsilon/2} \subseteq R$

$$|\frac{n_j}{n_i} - \tilde{P}_{i,j}| \leq |\frac{n_j}{n_i} - P^{n_l}_{i,j}| + |P^{n_l}_{i,j} - \tilde{P}_{i,j}| \leq \epsilon, \quad \forall j \geq i.$$

Then $(R_{I,\frac{1}{n_l}})_{P^{n_l},\epsilon/2} \subseteq R_{\tilde{P},\epsilon}$, and as $(R_{I,\frac{1}{n_l}})_{P^{n_l},\epsilon/2}$ is a set of Bohr recurrence, we conclude $\tilde{P} \in \mathcal{BC}(R)$. In addition, $\forall n \in (R_{I,\frac{1}{n_l}})_{P^{n_l},\epsilon/2} \subseteq R$, $(i,j) \in I$

$$|P_{i,j} - \tilde{P}_{i,j}| \leq |\frac{n_j}{n_i} - P_{i,j}| + |\frac{n_j}{n_i} - \tilde{P}_{i,j}| \leq \frac{3}{2}\epsilon,$$

taking $\epsilon \to 0$ yields $P_{i,j} = \tilde{P}_{i,j}$, $\forall (i,j) \in I$, concluding. □

4.3.2. The Property of Complete Independence

In this section, we define the property of complete independence and show its consequences. Then, we show that we can reduce to the case in which we have this property, with which we are able to generalize Theorem 4.2 in the case of $\mathbb{Z}^d$-quasi-affine nilsystems.

Definition 4.5 *Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence and $P \in \mathcal{BC}(R)$ a Bohr correlation. We say that P has the property of complete independence if for all $i \in [d]$,*

$$\{P_{i,j} \mid j \geq i, P_{i,j} \neq 0\},$$

are rationally independent.

Now, we prove the prime consequence of the property of complete independence for our purposes.

Theorem 4.7 *Let R be an essential set of Bohr recurrence and $P \in \mathcal{BC}(R)$ satisfying the property of complete independence. Then $\forall \epsilon \in (0,1)$ there is $M > 0$ such that $\forall n \in R_{P,\epsilon/M} \cap B(0,M)^c$ the map*

$$\varphi : \mathbb{R}^r \cap B(0,\epsilon) \to (\mathbb{T}^r)^d$$
$$y \to \varphi(y) = (n_1 y, \cdots, n_d y) \ mod\ 1$$

has ϵ-dense image.

Proof. Let $\epsilon > 0$, and define $N \geq r$ big enough such that for all $l \in [d]$, if we define

$$I_l = \{j \in [d] \mid j > l, P_{l,j} \neq 0\},$$

then the N-orbit of $\vec{0}$ in $(\mathbb{T}^{|I_l|}, +(P_{l,j})_{j \in I_l})$ is ϵ-dense (notice that this system is minimal given the fact that $\{P_{l,j} \mid j \geq l, P_{l,j} \neq 0\}$ are rationally independent by the hypothesis of complete independence), in where the N-orbit of a point x in a system (X,T) is $\{T^n x \mid n \in [-N,N]\}$.

Consider $M > N/\epsilon$ and $n \in R_{P,\epsilon/M} \cap B(0,M)^c$. Let $w \in (\mathbb{T}^r)^d$, we want to solve $\|\varphi(y) - w\| \leq \epsilon$. We consider a solution of the form $y = \sum_{i=1}^{d} \frac{y_i}{n_i}$, where we define $\{y_i\}_{i=1}^d$ inductively as follows:

- First we define

$$y_d = w_d,$$

note that $||\frac{y_d}{n_d}||_{\mathbb{R}^r} \leq |\frac{1}{n_d}| \leq \epsilon$. Also note that

$$\begin{aligned}\|\varphi(y) - w\| &= \left\| \sum_{j=1}^{d} e_j \Big(n_j (\sum_{i=1}^{d} \frac{y_i}{n_i}) - w_j \Big) \right\| \\ &= \left\| \sum_{j=1}^{d} n_j e_j (\sum_{i=1}^{d-1} \frac{y_i}{n_i}) + \sum_{j=1}^{d-1} \frac{n_j}{n_d} e_j y_d - \sum_{j=1}^{d-1} e_j w_j \right\|\end{aligned}$$

- Next, suppose we have defined for $1 \leq l < d$: $y_{l+1}, \cdots, y_d \in \mathbb{R}^r \cap B(0, C)$, where $C = N + d + 1$, such that

$$\|\varphi(y) - w\| = \left\| \sum_{j=1}^{d} n_j e_j (\sum_{i=1}^{l} \frac{y_i}{n_i}) + \sum_{i=l+1}^{d} \sum_{j>i} \frac{n_j}{n_i} e_j (y_i) + \sum_{j=1}^{l} \sum_{i=l+1}^{d} \frac{n_j}{n_i} e_j y_i - \sum_{j=1}^{l} e_j w_j \right\|,$$

we define

$$y_l = w_l + k_l - \sum_{i=l+1}^{d} \{\frac{n_l}{n_i} y_i\} \in \mathbb{R}^r \cap B(0, C),$$

where $k_l \in \mathbb{Z} \cap [-N, N]$ is such that for all $j > l$

$$||k_l P_{l,j} + P_{l,j}(w_l - \sum_{i=l+1}^{d} \{\frac{n_l}{n_i} y_i\})|| \leq \epsilon. \tag{4.11}$$

The existence of such k_l in ensure by the definition of N and by the fact that such condition is trivial for $j \notin I_l$.

In this way, we have that

$$\left\| \frac{y_l}{n_l} \right\|_{\mathbb{R}^r} \leq \frac{C}{n_l} \leq (1 + \frac{d+1}{N}) \epsilon \leq (d+2)\epsilon,$$

and we obtain that

$$\begin{aligned}&\left\| \sum_{j=1}^{d} n_j e_j (\sum_{i=1}^{l} \frac{y_i}{n_i}) + \sum_{i=l+1}^{d} \sum_{j>i} \frac{n_j}{n_i} e_j y_i + \sum_{j=1}^{l} \sum_{i=l+1}^{d} \frac{n_j}{n_i} e_j y_i - \sum_{j=1}^{l} w_j \right\| \\ &= \left\| \sum_{j=1}^{d} n_j e_j (\sum_{i=1}^{l-1} \frac{y_i}{n_i}) + \sum_{i=l+1}^{d} \sum_{j>i} \frac{n_j}{n_i} e_j y_i + \sum_{j=1}^{l-1} \sum_{i=l+1}^{d} \frac{n_j}{n_i} e_j y_i - \sum_{j=1}^{l-1} w_j + \sum_{\substack{j=1 \\ j \neq l}}^{d} \frac{n_j}{n_l} e_j y_l \right\|. \\ &= \left\| \sum_{j=1}^{d} n_j e_j (\sum_{i=1}^{l-1} \frac{y_i}{n_i}) + \sum_{i=l}^{d} \sum_{j>i} \frac{n_j}{n_i} e_j y_i + \sum_{j=1}^{l-1} \sum_{i=l}^{d} \frac{n_j}{n_i} e_j y_i - \sum_{j=1}^{l-1} w_j \right\|.\end{aligned}$$

Now we prove some properties of y. First, it is direct that

$$\|\varphi(y)-w\| = \left\|\sum_{i=1}^{d}\sum_{j>i}\frac{n_j}{n_i}e_j y_i\right\|, \tag{4.12}$$

by the election of every y_i, $i \in [d]$.

Secondly, note that $\|P_{i,j}y_i\| \leq s^2\epsilon$, $\forall j > i$. In fact, this comes from the election of every $k_i \in \mathbb{Z}^d$, noting that by (4.11)

$$||P_{i,j}y_i|| = \left\|\Big(P_{i,j}w_l + P_{i,j}k_i - P_{i,j}\sum_{i=l+1}^{d}\{\frac{n_l}{n_i}y_i\}\Big)\right\| \leq \epsilon.$$

Therefore, we have found $y \in G_{s-1} \cap B(0, \epsilon d(d+2))$ such that

$$\begin{aligned}
\|\varphi(y)-w\| &= \left\|\sum_{i=1}^{d}\sum_{j>i}\frac{n_j}{n_i}e_j y_i\right\| \\
&\leq \left\|\sum_{i=1}^{d}\sum_{j>i}P_{i,j}y_i\right\| + \sum_{i=1}^{d}\sum_{j>i}\left\|(\frac{n_j}{n_i} - P_{i,j})e_j y_i\right\| \\
&\leq \left\|\sum_{i=1}^{d}\sum_{j>i}P_{i,j}y_i\right\| + \frac{\epsilon C d^2}{M} \\
&\leq d^2\epsilon + d^2\epsilon^2(d+2) \leq (d+3)^3\epsilon,
\end{aligned}$$

concluding modulus a constant (to recover the exact statement, take $\epsilon = \epsilon'/(d+3)^3$ and notice that

$$R_{P,\epsilon'/M(d+2)^3} \cap B(0, M(d+2)^3)^c \subseteq R_{P,\epsilon'/M(d+2)^3} \cap B(0,M)^c = R_{P,\epsilon/M} \cap B(0,M)^c$$

and that $B(0, \epsilon d(d+2)) \subseteq B(0, \epsilon(d+2)^3) = B(0, \epsilon'))$. $\square$

The next theorem present how we are able to avoid the problem presented in Section 4.1.2 using Theorem 4.7, and asking for certain properties in the system.

Theorem 4.8 *Let $(X, T_1, \cdots, T_d)$ be a minimal s-step $\mathbb{Z}^d$-quasi-affine nilsystem, with X connected and G_0 simply connected, and R a set of Bohr recurrence with $P \in \mathcal{BC}(R)$ satisfying the property of complete independence. Then for all $\epsilon > 0$, there is $C > 0$ and $M > 0$, such that $\forall h_1, \ldots, h_d \in G_{s-1} \cap B(1_G, C)$ and $\forall n \in R_{P,\epsilon/M} \cap B(0,M)^c$ there exist $h \in G_{s-1}$ and $\theta \in G_s \cap \Gamma$ such that*

$$d_G(h, 1_G) < \epsilon, \text{ and } d_G([h, \tau_1^{n_1}]\cdots[h, \tau_d^{n_d}], [h_1, \tau_1]\cdots[h_d, \tau_d]\theta) < \epsilon. \tag{4.13}$$

Proof. Recall the identifications

$$G_s = \mathbb{R}^r, G_s \cap \Gamma = \mathbb{Z}^r, G_{s-1}/G_s = \mathbb{R}^q, \text{ and } \Gamma \cap G_{s-1}/(\Gamma \cap G_s) = \mathbb{Z}^q,$$

which can be taken isometries. We will use the properties and notation developed in Proposition 4.2. Denote $\Psi : G_{s-1}/G_s \times \cdots \times G_{s-1}/G_s \to G_s/(G_s \cap \Gamma)$ define by

$$\Psi(h_1, \ldots, h_d) = \sum_{i=1}^{d} \pi_s \circ \Psi^i(h_i),$$

with $\pi_s : G_s \to G_s/(\Gamma \cap G_s)$ the quotient map and $\Psi^i : G_{s-1}/G_s \to G_s$ the map induced by the map $g \to [g, \tau_i]$. We notice that for every $i \in [d]$ and $g \in G_{s-1}/G_s$,

$$[h, \tau_i] = [h, \tau_i^0] \cdots [h, \gamma_i] = [h, \gamma_i],$$

with $\tau_i = \tau_i^0 \gamma_i$, in where we used that $[G_{s-1}, \tau_i^0] \subseteq [G_0, G_0] = \{e_G\}$ given that G_0 is abelian and Theorem 1.11. This implies that Ψ^i takes

$$(G_{s-1}/G_s)/(G_{s-1} \cap \Gamma / G_s \cap \Gamma),$$

to $G_s \cap \Gamma$. In this way, Ψ is a (linear) map between $(\mathbb{R}^q)^d$ and $\mathbb{T}^r$, which takes

$$\Big((G_{s-1}/G_s)/(G_{s-1} \cap \Gamma / G_s \cap \Gamma)\Big)^d = (\mathbb{Z}^q)^d,$$

to $e_G(\Gamma \cap G_s) = \vec{0} + \mathbb{Z}^r$. Therefore, we have that the restriction $\tilde{\Psi} : (\mathbb{T}^q)^d \to \mathbb{T}^r$ is a continuous morphism. As $(\mathbb{T}^q)^d$ is compact, $\tilde{\Psi}$ is a Lipschitz function.

We note that for $h \in G_{s-1}$,

$$\begin{aligned}\pi_s([h, \tau_1^{n_1}] \cdots [h, \tau_d^{n_d}]) &= \pi_s([h^{n_1}, \tau_1] \cdots [h^{n_d}, \tau_d]) \\ &= \pi_s(\sum_{i=1}^{d} \Psi^i(n_i h)) \\ &= \Psi(n_1 h, \ldots, n_d h),\end{aligned}$$

and similarly:

$$\pi_s([h_1, \tau_1] \cdots [h_d, \tau_d]) = \pi_s(\sum_{i=1}^{d} \Psi^i(h_i)) = \Psi(h_1, \ldots, h_d),$$

in where we used the abuse of notation considering $h, h_1, \cdots, h_d$ as elements of $\mathbb{R}^q = G_{s-1}/G_s$.

Let $\delta \in (0, \epsilon)$ constant of uniform continuity for $\Psi_1, \cdots, \Psi_d$ for ϵ/d. Using Theorem 4.7 with δ, we find $M > 0$ such that for $n \in R_{\delta/M,P} \cap B(0, M)^c$ we find $h \in \mathbb{R}^r \cap B(0, \epsilon)$ satisfying

$$\|(n_1 h, \cdots, n_d h) - (h_1, \cdots, h_d)\|_{\mathbb{T}^q} < \delta.$$

Therefore, we have that

$$\left\|\tilde{\Psi}(n_1 h, \ldots, n_d h) - \tilde{\Psi}(h_1, \ldots, h_d)\right\|_{\mathbb{T}^r} < \epsilon.$$

Lifting to $G_s = \mathbb{R}^r$, we find a $\theta \in \mathbb{Z}^r = \Gamma \cap G_s$ such that

$$d_G([h, \tau_1^{n_1}] \cdots [h, \tau_d^{n_d}], [h_1, \tau_1] \cdots [h_d, \tau_d]\theta) < \epsilon,$$

concluding. $\square$

In what is next, we show that it is always possible to reduce to the case in which R has a vector of Bohr correlations with the property of complete independence. For this, we will need some additional definitions and properties.

The following notion will be important, given that allows to eliminate some irrelevant cases.

Definition 4.6 *Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence. We say that R is redundant is there exist $k \in [d]$ and $\{q_i\}_{i\in[d]\setminus\{k\}} \subseteq \mathbb{Q}$ such that*

$$R' = \{(n_1, \cdots, n_d) \in R \mid n_k + \sum_{i\in[d]\setminus\{k\}} q_i n_i = 0\},$$

is a set of Bohr recurrence.

Remark 22 Notice that $R \subseteq \mathbb{Z}^d$ is redundant if and only if there is $v \in \mathbb{Q}^d \setminus \{\vec{0}\}$ such that

$$R_v = \{n \in R \mid v^T \cdot n = 0\},$$

is a set of Bohr recurrence.

The property of being non-redundant is preserved under the transformation described in Proposition 4.4.

Proposition 4.18 *Let $R \subseteq \mathbb{Z}^d$ be a non-redundant set of Bohr recurrence, and $M \in \mathbb{Z}^{d\times d}$ an invertible matrix. Define*

$$R_0 = \{n = (n_1, \ldots, n_d)^T \in \mathbb{Z}^d \mid M \cdot n \in R\}.$$

Then R_0 is non-redundant as well.

Proof. Suppose that R is non-redundant. First, we prove that

$$R/N = \{n \in \mathbb{Z}^d \mid (Nn_1, \cdots, Nn_d) \in R\},$$

is a non-redundant set of Bohr recurrence. In fact, Proposition 4.4 shows that R/N is a set of Bohr recurrence. If R/N is redundant, then there exists $w \in \mathbb{Q}^d \setminus \{\vec{0}\}$ such that

$$(R/N)' = \{n \in R/N \mid w^T \cdot n = 0\},$$

is a set of Bohr recurrence. In particular, $N(R/N)' \subseteq R$ is a set of Bohr recurrence which is redundant, contradicting the hypothesis.

Second, suppose by contradiction that R_0 is redundant. This means that there exists $v \in \mathbb{Q}^d \setminus \{\vec{0}\}$ such that

$$R_{0,v} = \{n \in R_0 \mid v^T \cdot n = 0\},$$

is a set of Bohr recurrence. In particular, by Proposition 4.4, the set

$$R'_{0,v} = \{n \in \mathbb{Z}^d \mid An \in R_{0,v}\},$$

is a set of Bohr recurrence, which will be redundant given that $\forall n \in R'_{0,v}$,

$$(M^{-T}v)^T n = \frac{1}{N} v^{-T}(An) = 0,$$

and $M^{-T}v \in \mathbb{Q}^d \setminus \{\vec{0}\}$. Furthermore, $R'_{0,v} \subseteq R/N$, indeed, if $n \in R'_{0,v}$ then $An \in R_{0,v} \subseteq R_0$, which implies that $Nn = MAn \in R$. Therefore R/N is redundant, which is a contradiction. In this light, we conclude that R_0 is not redundant. □

We are going to show that if we have a redundant set of Bohr recurrence R, then we can change the system, reducing the amount of dynamics until R is not redundant. For this, following lemma will be needed.

Lemma 4.6 *Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence and let $C \subseteq [d]$, $N \in C$, and $\{q_j\}_{j \in C \setminus \{N\}} \subseteq \mathbb{Q}$. We define $\sigma : R \to \mathbb{Q}$ by*

$$\sigma(n)_j = \begin{cases} n_j & \text{if } j \notin C \\ n_j / q_{j;2} & \text{if } j \in C \setminus \{N\} \\ n_N + \sum_{j \in C \setminus \{N\}} q_j n_j & \text{if } j = N \end{cases}, \tag{4.14}$$

in where $\forall j \in C$, $q_j = \frac{q_{j;1}}{q_{j;2}}$, with $q_{j;1} \in \mathbb{Z}$ and $q_{j;2} \in \mathbb{N}$.

Then $\tilde{R} = \{\sigma(n) \in \mathbb{Z}^d \mid n \in R\}$ is a set of Bohr recurrence.

Proof. Define $M \in \mathbb{Z}^{d \times d}$ by

$$M_{i,j} = \begin{cases} 1 & \text{if } i = j \text{ and } j \notin (C \setminus \{N\}) \\ q_{j;2} & \text{if } i = j \text{ and } j \in C \setminus \{N\} \\ q_{j;1} & \text{if } i = N \text{ and } j \in C \setminus \{N\} \\ 0 & \text{else} \end{cases}.$$

Then $\tilde{R} = \{n \in \mathbb{Z}^d \mid M \cdot n \in R\}$, and the result follows from Proposition 4.4. □

First, we prove that one dynamic is redundant for the recurrence if the set of Bohr recurrence is redundant, so it can be eliminated.

Proposition 4.19 *Let $(X, T_1, \ldots, T_d)$ be a minimal s-step $\mathbb{Z}^d$-nilsystem, and $R \subseteq \mathbb{Z}^d$ a set of Bohr recurrence. If R is redundant, then there are $R' \subseteq \mathbb{Z}^{d-1}$ a set of Bohr recurrence and a minimal s-step $\mathbb{Z}^{d-1}$-nilsystem $(Y, T'_1, \cdots, T'_{d-1})$ such that if R' is a set of recurrence for $(Y, T'_1, \cdots, T'_{d-1})$ then so is R for $(X, T_1, \ldots, T_d)$.*

Proof. Suppose $R \subseteq \mathbb{Z}^d$ is redundant, and let $k \in [d]$ and $\{q_i\}_{i \in [d] \setminus \{k\}} \subseteq \mathbb{Q}$ given by the definition. We assume without loss of generality that $\forall n \in R$,

$$n_k + \sum_{i\in[d]\setminus\{k\}} q_i n_i = 0,$$

replacing R by

$$\{(n_1, \cdots, n_d) \in R \mid n_k + \sum_{i\in[d]\setminus\{k\}} q_i n_i = 0\}.$$

Then, if we take $C = [d]$ and $N = k$ in Lemma 4.6, we will have that

$$\tilde{R} = \{\sigma(n) \in \mathbb{Z}^d \mid n \in R\},$$

is a set of Bohr recurrence, with $\sigma : R \to \mathbb{Q}$ defined by

$$\sigma(n)_i = \begin{cases} n_i/q_{i;2} & \text{if } i \in [d] \setminus \{k\} \\ n_N + \sum_{i\in[d]\setminus\{k\}} q_i n_i & \text{if } i = k \end{cases}, \tag{4.15}$$

in where $\forall i \in [d]$, $q_i = \frac{q_{i;1}}{q_{i;2}}$, with $q_{i;1} \in \mathbb{Z}$ and $q_{i;2} \in \mathbb{N}$. Notice that $\forall n \in \tilde{R}$, $n_k = 0$.

Set $R'' = \{(n_j)_{j\in[d]\setminus\{k\}} \mid n \in \tilde{R}\}$ and $(Y, T_1', \cdots, T_d')$ given by

$$T_i' = \begin{cases} T_i^{q_{i;2}} T_k^{-q_{i;1}} & \text{if } i \in [d] \setminus \{k\} \\ id_Y & \text{if } i = k \end{cases},$$

and

$$Y = \overline{O_{T_1',\ldots,T_d'}(e_X)}.$$

Then clearly $(Y, T_1', \cdots, T_d')$ is a minimal s-step $\mathbb{Z}^d$-nilsystem. Finally, note that $\forall n \in R$ such that $\sigma(n) \in \mathbb{Z}^d$,

$$T_1^{n_1} \cdots T_d^{n_d} = T_1'^{\sigma(n)_1} \cdots T_d'^{\sigma(n)_d},$$

therefore, if we set $(Y, T_1'', \cdots, T_{d-1}'')$ by eliminating T_k' and shifting the indexes of the subsequence dynamics, we conclude that if R'' is a set of recurrence for $(Y, T_1'', \cdots, T_{d-1}'')$ then so is R for $(X, T_1, \ldots, T_d)$. □

Remark 23 If $(X = G/\Gamma, T_1, \ldots, T_d)$ is a $\mathbb{Z}^d$-quasi-affine nilsystem, then $(Y = G^Y/\Gamma^Y, T_1', \cdots, T_{d-1}')$ is a $\mathbb{Z}^d$-quasi-affine nilsystem, given that G_0^Y is abelian as well. This comes from the fact that by Remark 3 we have that G_0^Y is a rational subgroup of G_0.

Corollary 4.3 *Let $(X, T_1, \ldots, T_d)$ be a minimal s-step $\mathbb{Z}^d$-nilsystem, and $R \subseteq \mathbb{Z}^d$ a set of Bohr recurrence. Then there are $d' \leq d$, $R' \subseteq \mathbb{Z}^{d'}$ a non redundant set of Bohr recurrence and $(Y, T_1', \cdots, T_{d'}')$ a minimal s-step $\mathbb{Z}^{d'}$-nilsystem such that if R' is a set of recurrence for $(Y, T_1', \cdots, T_{d'}')$ then so is R for $(X, T_1, \ldots, T_d)$.*

Proof. By applying Proposition 4.19 iteratively, we get $d' \geq 1$ such that there are $R' \subseteq \mathbb{Z}^{d'}$ a non-redundant set of Bohr recurrence and $(Y, T_1', \cdots, T_{d'}')$ a minimal s-step $\mathbb{Z}^{d'}$-nilsystem such that if R' is a set of recurrence for $(Y, T_1', \cdots, T_{d'}')$ then so is R for $(X, T_1, \ldots, T_d)$. This is possible by the fact that the process stops at most in $d' = 1$, in where every set of Bohr recurrence is non-redundant.

□

Remark 24 Given Remark 23, we have that if the system $(X, T_1, \dots, T_d)$ is quasi-affine, then $(Y, T'_1, \cdots, T'_{d'})$ is quasi-affine as well. In particular, by Remark 15 Y is a finite union of affine nilsystems.

Now we present the result which allows us to reduce to the case in which we have a set of Bohr recurrence $R \subseteq \mathbb{Z}^d$ with a Bohr correlation $P \in \mathcal{BC}(R)$ with the property of complete independence.

Theorem 4.9 *Let $R \subseteq \mathbb{Z}^d$ be a non-redundant set of Bohr recurrence and $(X, T_1, \cdots, T_d)$ a connected minimal s-step $\mathbb{Z}^d$-nilsystem. There exist $d' \leq d$, a non-redundant essential ordered set of Bohr recurrence $\tilde{R} \subseteq \mathbb{Z}^{d'}$, a connected minimal s-step $\mathbb{Z}^d$-nilsystem $(X, S_1, \cdots, S_d)$ and $P \in \mathcal{BC}(\tilde{R})$ with the property of complete independence, such that if $\tilde{R}$ is a set of recurrence for $(X, S_1, \cdots, S_d)$, then so is R for $(X, T_1, \cdots, T_d)$.*

Proof. First, without loss of generality, we can assume that R is ordered by Remark 21. Let $P = (1)$, we will follow an induction process, changing $(R, P, (X, T_1, \cdots, T_d))$ iteratively: For $N < d$ suppose we have $(R^N, P^N, (X, T_{1,N}, \cdots, T_{d,N}))$ such that R^N is an (essential) ordered set of Bohr recurrence with its first N coordinates ordered, $P^N \in \mathbb{R}^{N(N+1)/2}$ a partial vector of Bohr correlations, (R^N, P^N) holds that

$$\forall i \in [N], \{P^N_{i,j} \mid P^N_{i,j} \neq 0, \ \forall i \leq j \leq N\},$$

are rationally independent, and if R^N is a set of recurrence for $(X, T_{1,N}, \cdots, T_{d,N})$, then so is R for $(X, T_1, \cdots, T_d)$. We will extent this to $N+1$ as follow:

First, we extent P to a vector in $\mathbb{R}^{(N+1)(N+2)/2}$ using Lemma 4.5 and only including the values of $P_{i,N+1}$, $\forall i \leq N+1$. Then, we have two cases: In the first one $\forall \epsilon > 0$,

$$R' = \{n \in R_{P,\epsilon} \mid |\frac{n_i}{n_{N+1}}| < \epsilon, \ \forall i \in [N]\},$$

is a set of Bohr recurrence. In this case, we define the set R^{N+1} by putting the $N+1$ coordinate of R at the beginning. We also define the system $(X, T_{1,N+1}, \cdots, T_{d,N+1})$, by moving the $(N+1)$-th dynamic to the beginning. In other words, for each $i \in [d]$

$$T_{i,N+1} = \begin{cases} T_{N+1,N} & \text{if } i = 1 \\ T_{i-1,N} & \text{if } 1 < i \leq N+1 \\ T_{i,N} & \text{else} \end{cases}.$$

Lastly, we define $P^{N+1} \in \mathbb{R}^{(N+1)(N+2)/2}$ by setting for $i \leq j \leq N+1$

$$P^{N+1}_{i,j} = \begin{cases} 0 & \text{if } i = 1, j \in \{2, \cdots, N+1\} \\ P_{i-1,j-1} & \text{if } (i,j) \in \{2, \cdots, N+1\}^2 \\ 1 & \text{if } i = j \end{cases}.$$

In the second case, starting with $i = 1$, we will follow the following process:

- If $\forall \epsilon > 0$ the set

$$R' = \{n \in R_{P,\epsilon} \mid |\frac{n_{N+1}}{n_i}| < \epsilon\},$$

is a set of Bohr recurrence, then we continue the process with $i+1$ if $i < N$, but if $i \geq N$ we stop the process and define $R^{N+1} := R^N$, $(X, T_{1,N+1}, \cdots, T_{d,N+1})$ with

$$T_{i,N+1} = T_{i,N}, \forall i \in [d],$$

and, we define P^{N+1} by

$$P^{N+1}_{j,l} = \begin{cases} 0 & \text{if } j \in \{1, \cdots, N\}, l = N+1 \\ P_{j,l} & \text{if } (j,l) \in [N]^2, l \geq j \\ 1 & \text{if } j = l \end{cases}.$$

- Otherwise, we can define

$$I_i := \{j \in [d] \mid P^N_{i,j} \neq 0,\ \forall i \leq j \leq N+1\} \neq \emptyset.$$

If $\{P^N_{i,j}\}_{j \in I_i}$ are rationally independent, then we stop the process, re-arranging the coordinates of I_i in R^N, generating R^{N+1} set of Bohr recurrence with its $N+1$ first coordinates ordered, and $(P^{N+1}, (X, T_{1,N+1}, \cdots, T_{d,N+1}))$ (again, by rearranging the coordinates of P^N and the dynamics of $(X, T_{1,N}, \cdots, T_{d,N})$ accordingly).

In the case such that $\{P^N_{i,j}\}_{j \in I_i}$ are rationally dependent, there are rational numbers $\{q_j\}_{j \in I_i \setminus \{N+1\}} \subset \mathbb{Q}$ satisfying

$$P^N_{i,N+1} + \sum_{j \in I_i \setminus \{N+1\}} q_j P^N_{i,j} = 0.$$

For every $j \in [d]$ fix $q_{j;1} \in \mathbb{Z}$ and $q_{j;2} \in \mathbb{N}$ satisfying $q_j = \frac{q_{j;1}}{q_{j;2}}$. Define $\sigma : R^N \to \mathbb{Q}^d$ such that

$$\sigma(n)_j = \begin{cases} n_j & \text{if } j \notin I_i, \\ n_j / q_{j;2} & \text{if } j \in I_i \setminus \{N+1\} \\ n_{N+1} + \sum_{j \in I_i \setminus \{N+1\}} q_j n_j & \text{if } j = N+1 \end{cases}. \tag{4.16}$$

We also define

$$\tilde{R} = \{\sigma(n) \in \mathbb{Z}^d \mid n \in R^N\}.$$

By Lemma 4.6 we have that $\tilde{R}$ are sets of Bohr recurrence.

Now, notice that $\tilde{R}$ may not be neither essential nor ordered. The only possible coordinate in $\tilde{R}$ that can be 0 is the $(N+1)$-th coordinate, given that R^N is essential. However, the set $R'^N := \{n \in \tilde{R} \mid n_{N+1} \neq 0\}$ is a set of Bohr recurrence, by the fact that R^N is non redundant.

We also redefine $(X, T_{1,N}, \cdots, T_{d,N})$ as $(X, T'_{1,N}, \cdots, T'_{d,N})$ with

$$T'_{j,N} = \begin{cases} T_{j,N}^{q_{j;2}} T_{N+1,N}^{-q_{j;1}} & \text{if } j \notin I_i \setminus \{N\} \\ T_j & \text{else} \end{cases}.$$

Notice that $(X, T'_{1,N}, \cdots, T'_{d,N})$ is still a connected s-step minimal nilsystem, thanks to Lemma 4.2. It also can be proved, by a simple calculation, that

$$T_{1,N}^{n_1} \cdots T_{d,N}^{n_d} = T_{1,N}^{\prime\sigma(n)_1} \cdots T_{d,N}^{\prime\sigma(n)_d}, \forall n \in R.$$

Additionally, we define for $j, l \in [N]$, with $j \leq l$,

$$P'^{N+1}_{j,l} = \begin{cases} \frac{q_{j;2}}{q_{l;2}} P^N_{j,l} & \text{if } (j,l) \in I_i \times I_i \\ P_{j,l} & \text{if } j \notin I_i \vee l \notin I_i \\ 1 & \text{if } j = l \end{cases}.$$

Now, note that R'^N is essential, but not necessarily has its coordinates in I_{N+1} ordered. This problem is solved by rearranging its first $N+1$ coordinates, and passing to a subset if necessary. This is also done accordingly to P'^N and to $(X, T'_{1,N}, \cdots, T'_{d,N})$. Notice that this does not change the property in P'^N of

$$\forall i \in [N], \{P^N_{i,j} \mid P^N_{i,j} \neq 0, \ \forall i \leq j \leq N\},$$

being rationally independent. Besides, if R'^N is a set of recurrence for $(X, T'_{1,N}, \cdots, T'_{d,N})$, then so is R^N for $(X, T_{1,N}, \cdots, T_{d,N})$. So we can replace $(R^N, P^N, X, T_{1,N}, \cdots, T_{d,N}))$ by $(R'^N, P'^N, X, T'_{1,N}, \cdots, T'_{d,N}))$, and we continue the process with $i = \max(I_i \setminus \{N+1\}) + 1$.

This process ends with the conclusion desired.

□

Remark 25 If the original system $(X, T_1, \cdots, T_d)$ in Theorem 4.9 is (quasi-)affine, then the resultant system $(X, S_1, \cdots, S_{d'})$ is still (quasi-)affine, given that G_0 is abelian in both systems and X is connected.

Finally, we are able to prove the main result of this section.

Theorem 4.10 *Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence. Then for every integer $s \geq 1$, R is a set of recurrence for every minimal s-step $\mathbb{Z}^d$-quasi-affine nilsystem.*

PROOF. We proceed by induction on s. If $s = 1$, the result is trivial. Henceforth, we assume that $s \geq 2$, and that the statement holds for $(s-1)$-step $\mathbb{Z}^d$-quasi-affine nilsystems. Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence and let $(X = G/\Gamma, T_1, \ldots, T_d)$ be a minimal s-step $\mathbb{Z}^d$-quasi-affine nilsystem. First, we can assume that R is non-redundant, by Corollary 4.3, an replacing the pair $(R, (X, T_1, \ldots, T_d))$ by another pair $(R', (Y, T'_1, \cdots, T'_{d'}))$, in where $d' < d$, $R' \subseteq \mathbb{Z}^{d'}$ is a set of Bohr recurrence and $(Y, T'_1, \cdots, T'_{d'})$ is a minimal s-step $\mathbb{Z}^{d'}$-quasi-affine nilsystem.

Next, we assume without loss that X is connected and G_0 is simply connected. Indeed, by Proposition 4.3 we have that there exists an invertible matrix $K \subseteq \mathbb{N}^{d \times d}$ such that

$(X_0, (T_1^{K_{i,1}} \circ \cdots \circ T_d^{K_{i,d}})_{i=1}^d)$ is a minimal dynamical (affine) nilsystem. On the other hand, by Proposition 4.4 the set $R_0 = \{n \in \mathbb{Z}^d \mid K \cdot n \in R\}$ is a set of Bohr recurrence which is not redundant. Substituting X_0 for X and R_0 for R, we reduce to the case that X is connected. We can assume without loss that G_0 is simply connected as well.

Thanks to Theorem 4.9, we can assume that R is essential with $P \in \mathcal{BC}(R)$ satisfying the complete independence property, changing the system $(X = G/\Gamma, T_1, \cdots, T_d)$ by another connected minimal s-step $\mathbb{Z}^d$-affine nilsystem.

Let U be a nonempty open subset of X. Without loss, we can assume that U is the open ball $B(e_X, 3\epsilon)$ for some $\epsilon > 0$, using minimality.

Let $\pi : X \to \tilde{X} = X/G_s$ be the canonical factor map and $(\tilde{X}, \tilde{T}_1, \ldots, \tilde{T}_d)$ the $(s-1)$-nilsystem discussed in the beginning of Section 4.3. Since $(\tilde{X}, \tilde{T}_1, \ldots, \tilde{T}_d)$ is an $(s-1)$-step affine nilsystem, it follows from the induction hypothesis and Proposition 2.11 that there exists arbitrarily large $n \in R_{P,\epsilon/M}$ with $\pi(B(e_X, \epsilon)) \cap \tilde{T}_1^{-n_1} \cdots \tilde{T}_d^{-n_d} \pi(B(e_X, \epsilon)) \neq \emptyset$, where $M > 0$ is given by Theorem 4.8. It follows that for these values of $n = (n_1, \cdots, n_d)^T$, there exist $x \in X$ and $v \in G_s$ with $d_X(x, e_X) < \epsilon$ and $d_X(T_1^{n_1} \cdots T_d^{n_d} x, v \cdot e_X) < \epsilon$. Lifting x to G, we obtain $g \in G$ and $\gamma \in \Gamma$ with

$$d_G(g, 1_G) < \epsilon, \text{ and } d_G(\tau_1^{n_1} \cdots \tau_d^{n_d} g, v\gamma) < \epsilon.$$

By Proposition 4.2, we have that there is $C > 0$ such that for $w = v^{-1} \in G_s$ there exist $h_1, \ldots, h_d \in G_{s-1}$ and $\gamma \in \Gamma \cap G_s$ with

$$d_G(h_i, 1_G) < C, \forall i \in [d], \text{ and } d_G([h_1, \tau_1] \cdots [h_d, \tau_d], v^{-1}\gamma) < \epsilon. \tag{4.17}$$

Theorem 4.8 and the fact that $n \in R_{P,\epsilon/M}$ yield that there exist $h \in G_{s-1}$ and $\theta \in G_s \cap \Gamma$ such that

$$d_G(h, 1_G) < \epsilon, \text{ and } d_G([h^{-1}, \tau_1^{n_1}] \cdots [h^{-1}, \tau_d^{n_d}], v^{-1}\theta) < 2\epsilon. \tag{4.18}$$

Write $y = h \cdot x$, we have that y is the projection of hg in X and that

$$d_X(y, e_X) \leq d_G(hg, 1_g) \leq d_G(h, 1_G) + d_G(g, 1_G) < 2\epsilon.$$

Moreover,

$$\begin{aligned} d_X(T_1^{n_1} \cdots T_d^{n_d} y, e_X) &\leq d_G(\tau_1^{n_1} \cdots \tau_d^{n_d} hg, \theta\gamma) \\ &= d_G(h[h^{-1}, \tau_1^{n_1}] \cdots [h^{-1}, \tau_d^{n_d}]\tau_1^{n_1} \cdots \tau_d^{n_d} g, \theta\gamma) \\ &\leq \epsilon + d_G(\tau_1^{n_1} \cdots \tau_d^{n_d} g[h^{-1}, \tau_1^{n_1}] \cdots [h^{-1}, \tau_d^{n_d}], \theta\gamma) \\ &\leq 2\epsilon + d_G([h^{-1}, \tau_1^{n_1}] \cdots [h^{-1}, \tau_d^{n_d}]v\gamma, \theta\gamma) \\ &= 2\epsilon + d_G([h^{-1}, \tau_1^{n_1}] \cdots [h^{-1}, \tau_d^{n_d}], v^{-1}\theta) < 4\epsilon, \end{aligned}$$

where we used the right invariance of the distance d_G and the fact that for $w \in G$, $[G_{s-1}, w] \subseteq G_s$, proving in that way the result.

□

Chapter 5

Conclusions and Open Questions

To conclude this work, we summarize the results obtained and suggest possible future lines of work. First, in Chapter 2, we proved several properties of sets of Bohr recurrence, which are usually proved for $\mathbb{Z}$-actions. Second, in Chapter 3 we stated Katznelson's Question in a general framework and generalized that proximal extensions and inverse limits lift Bohr recurrence. Lastly, in Chapter 4 we generalized to $\mathbb{Z}^d$-nilsystems several properties usually proved for $\mathbb{Z}$-nilsystems, and most importantly, we proved that every set of $\mathbb{Z}^d$-Bohr recurrence is a set of recurrence for the family of $\mathbb{Z}^d$-nilsystems with the strong closing property and for the family of $\mathbb{Z}^d$-quasi-affine nilsystems.

Despite these advances, many open questions remain. First, it is still unclear in the context of $\mathbb{Z}^d$-actions if sets of Bohr recurrence are sets of recurrence for $\mathbb{Z}^d$-nilsystems in general. The problem presented in Section 4.1.2 suggests that a new proof is needed, given that in Theorem 4.8 is crucial the property of the commutators of the dynamics taking $G_{s-1} \cap \Gamma / G_s \cap \Gamma$ to $G_s \cap \Gamma$, which is not always true in an arbitrary $\mathbb{Z}^d$-nilsystem.

Second, Katznelson's Question is still unresolved for $\mathbb{Z}^d$ for each $d \in \mathbb{N}$. Whether searching for a counterexample or searching for more families of dynamical systems in which we can have a positive answer to this question, there is still a lot of work to do on this topic. For a positive answer for $\mathbb{Z}$-group actions, some open problems are related to proving that $\mathbb{Z}$-Bohr recurrence can be lifted through equicontinuous and weakly mixing extensions, in order to complete the chains mentioned in the introduction and in Chapter 3. On the other hand, for a negative answer, a future problem we would like to tackle is to study if we can find a counterexample of the form (X, T, S), where T is a minimal dynamic in X, and S is a commuting dynamic with T, such that not every set of recurrence is a set of recurrence in (X, S). The main difficulty here is that most systems in which sets of Bohr recurrence are not sets of recurrence are pathological, and do not seem to have a minimal dynamic which commutes with the original pathological dynamic.

Third, there are many recurrence related questions still open around $\mathbb{Z}$-nilsystems. For instance, a set $R \subseteq \mathbb{Z}$ is a set of *s-recurrence* for a family $\mathcal{F}$ of $\mathbb{Z}$-systems if for all minimal system $(X, T) \in \mathcal{F}$ and for all nonempty open set $U \subseteq X$ we have that

$$R \cap \{n \in \mathbb{N} \mid U \cap T^{-n}U \cap T^{-2n}U \cap \cdots \cap T^{-sn}U \neq \emptyset\} \neq \emptyset.$$

In [[1], Conjecture 5.4.], Host, Kra, and Maass conjectured that a set R of s-recurrence for s-step $\mathbb{Z}$-nilsystems is a set of s-recurrence for all t-step $\mathbb{Z}$-nilsystems for any $t \geq s$. Even though the authors of the paper have proved the conjecture for the family of $\mathbb{Z}$-affine nilsystems, there is no published work proving the conjecture for $\mathbb{Z}$-nilsystems. The situation is even less clear for $\mathbb{Z}^d$-actions, in where the notion of l-recurrence is not even defined (However, there are works of Ackelsberg, Bergelson, and Shalom for Khintchine-type recurrence for $\mathbb{Z}^2$-actions in [31]).

Lastly, another open problem could be to find a purely measure-theoretical proof for Theorem 4.2 from [1]. In such paper, the authors developed a strategy using measurable arguments, but it only works for 2-step $\mathbb{Z}$-nilsystems and for $\mathbb{Z}$-nilsystem $(X = G/\Gamma, T)$ such that G is connected. Notice that measurable recurrence is equivalent to topological recurrence in nilsystems, given the existence of an invariant measure with full support. Thus, developing such proof for $\mathbb{Z}$-nilsystems could also help to generalize Theorem 4.2 to $\mathbb{Z}^d$-nilsystems.

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