UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

A PROOF OF SLOW-ROLL LOCAL DECAY OF INFLATON FIELDS IN COSMOLOGY AND AXION FIELDS IN COLD DARK MATTER MODELS

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MATEMÁTICAS APLICADAS

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

MATÍAS BENJAMÍN MORALES SALINAS

PROFESOR GUÍA:
CLAUDIO MUÑOZ CERÓN

MIEMBROS DE LA COMISIÓN:
FRANCISCO CORREA SANTANDER
CHRISTOPHER MAULÉN MARCHANT
GONZALO PALMA QUILODRÁN
HANNE VAN DEN BOSCH

Este trabajo ha sido parcialmente financiado por los proyectos:
CMM ANID BASAL FB210005, Fondecyt 1231250,
Fondecyt Exploración 13220060.

SANTIAGO DE CHILE
2023

RESUMEN DE LA TESIS PARA OPTAR AL GRADO
DE MAGÍSTER EN CIENCIAS DE LA INGENIERÍA,
MENCIÓN MATEMÁTICAS APLICADAS
Y MEMORIA PARA OPTAR AL TITULO DE
INGENIERO CIVIL MATEMÁTICO
POR: MATÍAS BENJAMÍN MORALES SALINAS
FECHA: 2023
PROF. GUÍA: CLAUDIO MUÑOZ CERÓN

UNA DEMOSTRACIÓN DE DECAIMIENTO LOCAL DE CAMPOS DE INFLACIÓN EN COSMOLOGÍA Y CAMPOS DE AXION EN MODELOS DE MATERIA OSCURA FRÍA

Este trabajo está basado en el estudio de las propiedades de decaimiento de campos provenientes de la teoría de inflación cósmica y la teoría de axiones en materia oscura fría (CDM). El problema concreto a estudiar se presenta en el Capítulo 1, donde además se dan las motivaciones de este, presentando una breve introducción a la teoría de inflación cósmica y de los modelos a considerar. Así mismo se presentan los resultados principales de esta tesis, los cuales consisten en el decaimiento local de soluciones en el espacio de energía bajo ciertas suposiciones que dependerán de la naturaleza de la no linealidad trabajada y de si es considerada o no la constante cosmológica.

En el Capítulo 2 se introduce la herramienta principal para el estudio de ecuaciones de onda, los espacios de Sobolev, así como también sus propiedades básicas y algunas desigualdades clásicas que serán útiles en el desarrollo de este trabajo. Se presentan también los resultados básicos de existencia y unicidad de soluciones para ecuaciones de onda lineales y semilineales, tanto si se considera o no la constante cosmológica.

En el Capítulo 3 se presentan algunas identidades viriales, las cuales son el punto clave en la demostración de los teoremas presentados en el Capítulo 1. Luego, en el Capítulo 4 se finaliza la demostración de estos resultados.

En el Capítulo 5 se estudian en detalle los modelos presentados en la Introducción, donde se ve que no pueden poseer soluciones estacionarias de energía finita y se muestran los distintos tipos de decaimiento que estos satisfacen.

Finalmente, en el Capítuo 6 se discute sobre los resultados obtenidos, así como las límitaciones de estos y proyecciones de trabajos futuros en esta línea.

RESUMEN DE LA TESIS PARA OPTAR AL GRADO
DE MAGÍSTER EN CIENCIAS DE LA INGENIERÍA,
MENCIÓN MATEMÁTICAS APLICADAS
Y MEMORIA PARA OPTAR AL TITULO DE
INGENIERO CIVIL MATEMÁTICO
POR: MATÍAS BENJAMÍN MORALES SALINAS
FECHA: 2023
PROF. GUÍA: CLAUDIO MUÑOZ CERÓN

A PROOF OF SLOW-ROLL LOCAL DECAY OF INFLATON FIELDS IN COSMOLOGY AND AXION FIELDS IN COLD DARK MATTER MODELS

This work is based on the study of the decay properties of fields from the cosmic inflation theory and the axion theory in cold dark matter (CDM). The specific problem to be studied is presented in Chapter 1, where the motivations for it are also given, presenting a brief introduction to the theory of cosmic inflation and the models to be considered. Likewise, the main results of this thesis are presented, which consist of the local decay of solutions in the energy space under certain assumptions that will depend on the nature of the nonlinearity worked and if it is considered or not the cosmological constant.

In Chapter 2, the main tool for the study of wave equations, Sobolev spaces, are introduced, as well as its basic properties and some classical inequalities that will be useful in the development of this work. The basic results of existence and uniqueness of solutions for linear and semilinear wave equations are also presented, whether or not the cosmological constant is considered.

In Chapter 3 some virial identities are presented, which are the key point in the proof of the theorems presented in Chapter 1. Then, in Chapter 4 the proofs of these results is finished.

In Chapter 5 the models presented in the Introduction are studied in detail, where it is seen that they cannot have stationary solutions of finite energy and the different types of decay that they satisfy are shown.

Finally, in Chapter 6 the results obtained are discussed, as well as their limitations and projections of future work on this line.

Debemos saber,
sabremos.

David Hilbert

Agradecimientos

Primero, quiero agradecer a mi familia por todo el cariño y apoyo que me han dado desde siempre. Quiero agradecer a mi papá por creer en mi, por incentivarme a mejorar y a llegar siempre un poquito más lejos, y por enseñarme desde pequeño el valor de la resposabilidad y del trabajo. Quiero agradecer a mi mamá por su apoyo incondicional y por aguantarme siempre, hasta en mis peores momentos, estando conmigo aunque sea a la distancia en forma de un mensaje o una llamada preocupada. También quiero agradecer a mi hermana Sofía, y a mis sobrinos Martín, Mirko y Valeria por alegrarme los días con sus chistes y sus bromas y recordarme lo lindo de las simples cosas. Gracias a ustedes soy el hombre que soy y he podido llegar hasta aquí.

Así mismo quiero agradecerle a Esteban, mi amigo y compañero de tantas historias y aventuras, que me ha acompañado en las buenas y en las malas, y quien se ha convertido en casi un hermano para mi. Por su puesto también agradezco mucho a su familia, en particular a sus padres, por todo su apoyo durante estos años y por haberme recibido siempre en su casa con el mayor de los cariños.

A mis amigos del DIM: David, Pablo, Vale, Diego, Guti, Arie, Tomy, Cynthia, Nico, Lucho, Tapia, Enzo, etc, gracias por todo su apoyo, por todas esas tardes de estudio y de comida, y por hacer de mi paso por el departamento una experiencia increible. Agradezco también a Jessica y a Nacho, por nuestras conversaciones sobre EDPs y la física que hay detrás de ellas, que fueron fundamentales para este trabajo y me recuerdan lo mucho que me gusta esta área.

Quiero agradecer también a la Anto por todo el cariño y apoyo que me ha dado en el desarrollo de la tesis, por estar para mi en los momentos de frustración cuando las cosas no salían, y recordarme lo capaz que soy.

A mis amigos de primer año: Bruno, Lucas, Dani, Vale, Ale, Bea, Karin, Cami, Tamy, Quina, Roa, y tantos otros, por haberme acompañado durante mi paso por la u, por todos las juntas, los carretes y en general los buenos momentos que pasamos durante estos 6 años.

Finalmente quiero agradecer a quienes me formaron como matemático: a Claudio Muñoz por su excelente guía y su gran apoyo durante el desarrollo de esta tesis, y a los miembros de la comisión por aceptar ser parte de esta y darse el tiempo de ver mi trabajo. Agradezco también a las profesoras Hanne Van Den Bosch y Gabrielle Nornberg, con quienes di mis primeros pasos en el mundo de las ecuaciones en derivadas parciales, y quienes hicieron que me encantara de esta área.

Table of Content

1. Introduction **1**
1.1. FLRW Cosmology 1
1.2. Cosmological Inflation 2
1.3. Slow roll and Cold Dark Matter Models 3
1.3.1. E and T models 3
1.3.2. Natural Inflation and Axion potentials 4
1.3.3. The D-brane and Hilltop models 5
1.3.4. Axion-Monodromy and log potentials 5
1.4. Setting the problem 6
1.5. Main results 6

2. Theory of wave equations **9**
2.1. Notation 9
2.2. Sobolev Spaces 10
2.3. Linear wave equations 12
2.4. Semilinear wave equations 15

3. Virial Identities **19**
3.1. The case $H = 0$ 19
3.1.1. First calculations 19
3.1.2. Choice of weight function 21
3.2. The case $H > 0$ 24

4. Proof of the main results **26**
4.1. Proof of Theorems 1.1 and 1.2 26
4.2. Proof of Theorem 1.3 28
4.2.1. Existence 28
4.2.2. Decay 30

5. Applications **32**
5.1. The E-model and T-model 33
5.2. The non-minimal coupling and Hilltop models 34
5.3. The Natural Inflation and Axion potential 35
5.4. The D-brane model 35
5.5. The Axion-Monodromy and Logarithmic potential 36

6. Conclusions **37**

Bibliography **37**

Chapter 1

Introduction

1.1. FLRW Cosmology

The theory of General Relativity has been one of the most successful theories of modern physics, giving an exceptionally elegant framework where understand the electromagnetism and describe the behavior of our universe at large scales, providing many important predictions which includes the existences of black holes or gravitational waves, objects that continue to be one of the most important topics on the current research lines.

The main concept in the theory, and the one that makes it revolutionary, is the notion of space-time, which is a four dimensional, oriented and time-oriented Lorentzian manifold $(\mathcal{M}, g)$ [22]. The metric g must satisfies the Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

where $R_{\mu\nu}$ is the Ricci tensor, R the Ricci scalar, $T_{\mu\nu}$ the energy-momentum tensor and G is the Einstein's gravitational constant. The main assumption when we use Einstein's equations to study our large-scale universe is that it is homogeneous and isotropic, that is, the metric $g_{\mu\nu}$ has the form

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

This hypothesis is known as cosmological principle, and the corresponding metric as the Friedman-Lemaitre-Robertson-Walker (FLRW) metric. The parameter $k \in \{-1, 0, 1\}$ is related to the spatial curvature of the space time (being $k = 0$ the case of a flat space) meanwhile the scale factor $a(t)$ corresponds to a measure of the expansion or contraction of the universe with respect to time.

As we are assuming the cosmological principle, we can approximate galaxies as points and the universe's contents can be described as a perfect fluid. Consequently, the energy-momentum tensor is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$$

where (u_μ) is the relativistic velocity of the fluid, ρ its density and p its preasure. This assumptions leads us to the following simplified version of Einstein's equations

$$\begin{aligned} H^2 =& \frac{1}{3}\rho - \frac{k}{a^2} \\ \frac{\ddot{a}}{a} =& -\frac{1}{6}(\rho + 3p) \end{aligned}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. The first equations is known as Friedmann equation, and the second is the acceleration equation. The pressure and the density are related through the equation of state

$$p = w\rho$$

where w is a constant known as state parameter. Notice that when $w < -\frac{1}{3}$ we have $\ddot{a} > 0$, i.e. an accelerated expansion, but on the other side, ordinary matter satisfies the condition $\rho + 3p > 0$, that is, generates deceleration.

1.2. Cosmological Inflation

The above description of our universe, although it looks reasonable, induces some theoretical problems such that the horizon problem or the flatness problem. To solve them the idea is make decrease the comoving Hubble radio, i.e.

$$\frac{d}{dt}\left(\frac{1}{Ha}\right) < 0$$

In virtue of the Friedmann and acceleration equations this condition is equivalent to $\ddot{a} > 0$ or to $\rho + 3p < 0$. To do this a new field was introduced: the inflaton. This was done for the first time by Starovinski [25] and Guth [9] independently. The first aiming to obtain a model of the universe that avoid singularities, meanwhile the second as a way to solve the horizon problem.

Since then many authors have proposed new models to describe the mechanism of the inflation, but the basic idea is the following. Inflaton's dynamics is given by the following action

$$\mathcal{S} = \int_{\mathbb{R}^{1+3}} \sqrt{-g}\left(\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - F(\phi)\right),$$

From now on, we will work on a spatially flat space-time, i.e., with $k = 0$. Varying this action with respect to the inflaton we obtain the equation of motion

$$\partial_t^2\phi + 3H\partial_t\phi - \frac{\Delta\phi}{a^2} + f(\phi) = 0 \tag{1.1}$$

where $f = F'$. The energy-momentum tensor for this action is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\lambda\phi\partial_\lambda\phi + F(\phi)\right)$$

If we assume that the metric is FLRW for all times and that ϕ depends only on time we have that the energy-momentum tensor takes the form of a perfect fluid where

$$\rho_\phi = \frac{1}{2}\phi_t^2 + F(\phi)$$
$$p_\phi = \frac{1}{2}\phi_t^2 - F(\phi)$$

Using this we can write the Friedmann and acceleration equations in terms of the inflaton field, obtaining

$$H^2 = \frac{1}{2}\phi_t^2 + F(\phi)$$
$$\frac{\ddot{a}}{a} = -\left(\phi_t^2 - F(\phi)\right)$$

and we see that if the potential energy $F(\phi)$ dominates the kinetic energy ϕ_t^2, that is

$$\phi_t^2 - F(\phi) < 0$$

we have the condition to inflation. These equations, together to the equation of motion of the inflaton

$$\partial_t^2\phi + 3H\partial_t\phi + f(\phi) = 0$$

determine the dynamics of the FLRW metric and the scalar field. Since the inflation must occur in the period of the primordial universe, perturbations of the inflaton must consider quantum effects. Such perturbations must satisfy equation (1.1), and the precisely potential f is not known today.

It is important to mention that, between the lots of models proposed to describe inflation, some of them have better opportunities to describe the actual behavior of our universe. This type of classification for inflation models is based on an statistical analysis of the Cosmic Microwave Backgroud (CMB) [20], a vestige of the decoupling between matter and radiation. Some of these models are going to be considered in the next chapter.

1.3. Slow roll and Cold Dark Matter Models

In this section we describe briefly the models considered in this work.

1.3.1. E and T models

From Planck [20, Table 5] one can access to a selection of slow-roll inflationary models of high interest in order to explain cosmological inflation. Among the most favorable models we highlight the Starobinsky R^2, $f(R)$ modified gravity or E_1 model, represented by the potential

$$F_{1,1}(\phi) := (1 - e^{-\phi})^2. \tag{1.2}$$

Notice that $F_{1,1}$ is a potential exponentially unbounded as $\phi \to -\infty$ (see Fig. 1.1), with some very unpleasant features. Among them, we can find

$$F_{1,1}(0) = F'_{1,1}(0) = 0, \quad \text{but} \quad F''_{1,1}(0) = 2 > 0.$$

This last positive sign makes mathematical treatment of the small data theory not easy. Cosmological theory supposes that the initial configuration starts with $\phi \gg 1$, and slowly decays in time towards the zero field value. This process is called the “slow-roll” dynamics, and the exponential growth of the scaling parameter of the universe ($a(t) \sim e^{Ht}$) is described as the “e-fold” procedure.

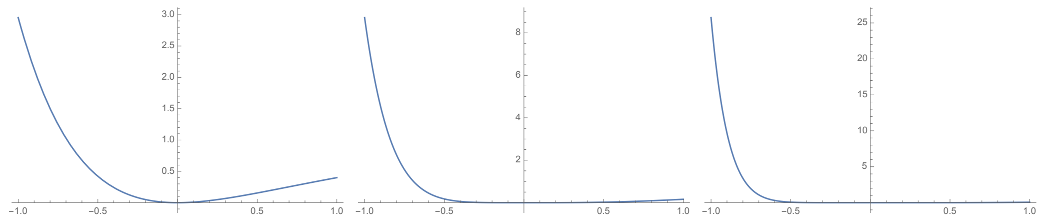

Figure 1.1: The potentials $F_{1,1}$ (1.2), $F_{1,2}$ and $F_{1,3}$ in (1.3).

The $F_{1,1}$ model is part of a family of inflationary potentials that gives rise to the so-called E_n theories:

$$F_{1,n}(\phi) := (1 - e^{-\phi})^{2n}, \quad n \geq 1, \tag{1.3}$$

see Fig. 1.1. For us the most interesting cases are the ones with $n = 1, 2$. Additionally to the E_n models, one has the T_n ones, which are also highly relevant in Planck data analysis. These are given by

$$F_{2,n}(\phi) := \tanh^{2n}(\phi), \quad n \geq 1, \tag{1.4}$$

see Fig. 1.2. The case $n = 1$ is highly favorable in our setting, producing the best result of this work, but $n = 2$ has some drawbacks due to the lack of a sign condition. Only small data will be suitable to prove decay.

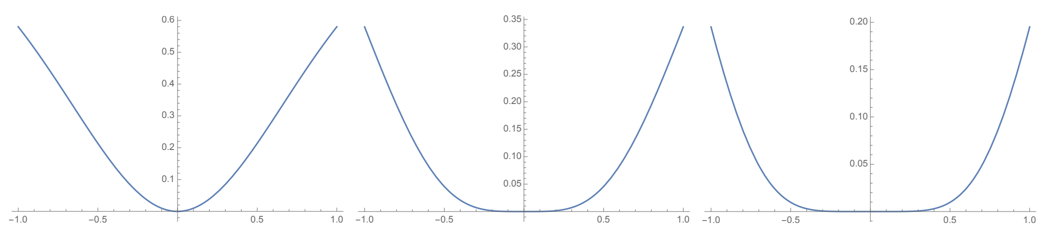

Figure 1.2: The potentials $F_{2,n}$ in (1.4).

1.3.2. Natural Inflation and Axion potentials

There are other models of equal interest and high importance in the quest for the inflaton potential. These are the so-called Natural inflation (+ sign) [20] and Axion potential (- sign) [4, p. 4] (see Fig. 1.3)

$$F_{3,\pm}(\phi) := 1 \pm \cos\phi. \tag{1.5}$$

In both cases it is assumed that the field ϕ is no larger than $\frac{\pi}{2}$ in absolute value. The potential $F_{3,\pm}$ is the classical appearing in 1D sine-Gordon models, making the scalar field model integrable. In 3D the situation is different, and in radial symmetry integrability seem lost. In both cases we are able to give answers to the decay problem.

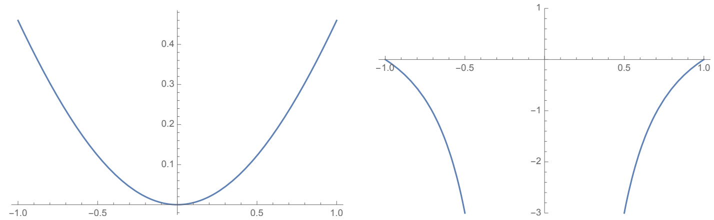

Figure 1.3: Left: The potential $F_{3,-}$ from (1.5). Right: The potentials $F_{4,n}$, $n = 1$ in (1.6). Note that the last potential is singular at $\phi = 0$.

1.3.3. The D-brane and Hilltop models

In addition to the sine-Gordon models, another important model is given by the singular D-brane (Fig. 1.3):

$$F_{4,n}(\phi) := 1 - \frac{1}{\phi^{2n}}, \quad n = 1, 2, \quad \phi \neq 0. \tag{1.6}$$

Notice that the physical problem here is the perturbation of a ϕ large initial state. In this case we shall assume

$$\phi = 1 + v, \quad |v| \ll 1,$$

so that after renormalization (to have finite energy) we will work with the modified potential

$$\tilde{F}_{4,n}(v) := 1 - \frac{1}{(1+v)^{2n}} - 2nv = \frac{(1+v)^{2n} - 1 - 2nv(1+v)^{2n}}{(1+v)^{2n}}. \tag{1.7}$$

Also considered in this work will be the Hilltop models (Fig. 1.4):

$$F_{5,n}(\phi) := -\phi^{2n}, \quad n = 1, 2. \tag{1.8}$$

The case $n = 1$ is exactly linear Klein-Gordon and will not be studied in this work. However, the case $n = 2$ is highly interesting because it behaves as one of the most promising potentials to describe inflation. A closely related model is the so-called Non-minimal coupling model, whose potential is given by

$$F(\phi) = \lambda^2\phi^4 + \beta^2\phi^2$$

We will see in Chapter 5 that, despite the difference of these models, both have a similar asymptotic behavior, at least for small solutions.

1.3.4. Axion-Monodromy and log potentials

The last two examples that we will study here also appear when studying CDM. These are the axion monodromy potential [29]

$$F_{6,q}(\phi) := \frac{1}{q}\left((1+\phi^2)^{q/2} - 1\right), \quad q \in [-1, 1], \quad q \neq 0, \tag{1.9}$$

and the logarithm potential (Fig. 1.4):

$$F_7(\phi) := \frac{1}{2}\log\left(1 + \phi^2\right). \tag{1.10}$$

The potential $F_{6,q}$ formally converges to F_7 as $q \to 0$.

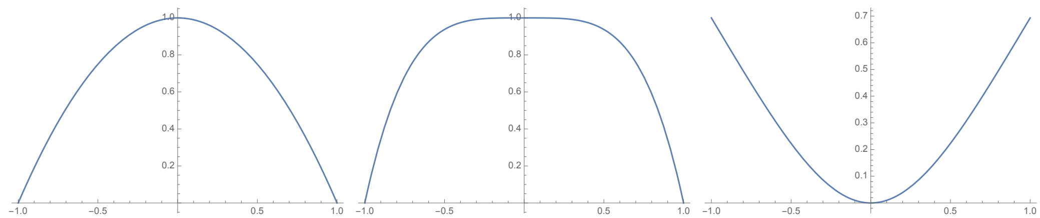

Figure 1.4: The potentials $F_{5,n}$, $n = 1, 2$ in (1.8) and F_7 from (1.10).

These families have their own properties, usually not being part of the standard local and global well-posedness theory appearing in the literature. For each of these models, we will prove local and/or global well-posedness, and provide a proof of decay under suitable assumptions on the initial data.

1.4. Setting the problem

Consider now a perturbation $\phi(t, x)$ of the inflaton field. It must satisfies the equation (1.1). To simplify, we shall assume that the space time is de Sitter for all times, that is, the metric g takes the form

$$ds^2 = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2)$$

for a constant $H \geq 0$. Thus, the equation (1.1) becomes

$$\partial_t^2 \phi + 3H\partial_t \phi - \frac{\Delta\phi}{e^{2Ht}} + f(\phi) = 0 \tag{1.11}$$

Notice that when $H = 0$ we recover the classical wave equation. As we have seen in the previous section, this equation appears not only in the theory of inflation, but also in the theory of Axions as a model of cold dark matter [4], [11],[12], [26]. This is the reason why we are interested in understand the well posedness theory for this type of equations and study their decay properties. Our main objective will be obtain decay results for radial solutions to these equations, under assumptions on the nonlinearity that fit on the inflationary and Axion models.

1.5. Main results

The results presented in this thesis are separated between the case $H = 0$ and the case $H > 0$. This is because the term $3H\partial_t\phi$ in (1.11) plays a very important role in the dynamics of the solution. Before to enunciate the main theorems we need to assume some hypothesis:

- $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function, such that $f(0) = 0$.
- The initial data $(\phi_0, \phi_1) \in H^1 \times L^2$ is radial.

Note that from the well posedness theory we have that our second hypothesis implies that the solution is radial for all time whenever it is well defined.

With this we can enunciate the main results proved in this work. The first theorem gives us a sufficient condition for local decay in the energy space.

Theorem 1.1 *Let f be globally Lipschitz and satisfying the sign conditions $F(\phi) \geq 0$ and $2F(\phi) - \phi f(\phi) \geq 0$. Then the solution $(\phi, \partial_t \phi)$ of (1.11) with $H = 0$ is defined for all times and*

$$\lim_{t \to \infty} \|(\phi, \partial_t \phi)(t)\|_{H^1 \times L^2(B(0,R))} = 0, \tag{1.12}$$

for any $R > 0$.

Note that Theorem 1.1 is valid for any size of the initial data.

The second result concerns the case where the condition $2F(\phi) - \phi f(\phi)$ is not satisfied, still in the case $H = 0$. Several cosmological models are in this class. Our result is now local decay under smallness assumption and growth below a critical power.

Theorem 1.2 *If $f : \mathbb{R} \to \mathbb{R}$ is of class C^1 and satisfies that for some $C, \delta > 0$*

$$0 \leq \phi f(\phi) \leq C\phi^4, \quad \forall \phi \in (-\delta, \delta), \tag{1.13}$$

or

$$2F(\phi) - \phi f(\phi) \geq 0, \quad \forall \phi \in (-\delta, \delta), \tag{1.14}$$

then any global solution $\phi \in C([0, \infty); H^1 \times L^2)$ of (1.11) with $H = 0$ such that

$$\sup_{t \geq 0} \|\phi(t)\|_{H^1 \cap L^\infty} \leq \varepsilon, \tag{1.15}$$

satisfies

$$\lim_{t \to \infty} \|(\phi, \phi_t)(t)\|_{H^1 \times L^2(B(0,R))} = 0, \tag{1.16}$$

for any $R > 0$ provided that $\varepsilon > 0$ is small enough.

Remark Note that the condition $\phi f(\phi) \geq 0$ ensures that $F(\phi) \geq 0$, and the model has defocusing character. Condition (1.13) describes that the model has sufficient flatness at the origin to allow decay, and finally condition (1.14) it is just an application of Theorem 1.1 to the case of small data and needs no proof.

Now we turn into the case of our current universe, assuming $H > 0$. Here we have the following results:

Theorem 1.3 *Consider equation (1.11) with initial data $(u_0, u_1) = (\varepsilon g, \varepsilon h)$, $g, h \in C_0^\infty(\mathbb{R}^3)$. If $H > 0$, $F(x) \geq 0$ for all $x \in \mathbb{R}$ and there exist $\delta, M > 0$ such that*

$$|f'(\phi)| \leq M\phi^2, \quad \forall \phi \in (-\delta, \delta),$$

then there exist $\varepsilon > 0$ such that the solution to (1.11) is global in time. Moreover,

1. *The energy of any global solution decays to zero outside of the forward light cone, that*

is
$$\lim_{t\to\infty}\int_{R(t)}\left(\frac{\phi_t^2}{2}+\frac{|\nabla\phi|^2}{2e^{2Ht}}+F(\phi)\right)=0, \tag{1.17}$$
where $R(t)=\{x\in\mathbb{R}^3 \mid |x|>(1+b)t\}$ *for any* $b>1$.

2. *If* $R>0$ *is fixed,*
$$\lim_{t\to\infty}\int_{B(0,R)}\left(\frac{\phi_t^2}{2}+\frac{|\nabla\phi|^2}{2e^{2Ht}}+F(\phi)\right)=0. \tag{1.18}$$

Estimate (1.18) shows that locally the energy must converge to zero, however, the global energy, although decreasing, may not converge to zero in general. Their main outcome should depend on the existence of moving solitary waves. In the case of radial data, this is strongly unlikely, but solitary rings of finite energy might exists.

Chapter 2

Theory of wave equations

This chapter is devoted to present the main tools needed to prove the results presented in the previous chapter. We begin giving some elementary properties of Sobolev spaces, to then present the basic results about existence and uniqueness for linear wave equations, most of them based on some type of energy estimates. Finally, we will use those estimates to prove existence and uniqueness for the nonlinear case.

2.1. Notation

Along this work $\Omega \subseteq \mathbb{R}^n$ will be an open set. We will use the following notations:

- $L^p(\Omega)$, for $1 \leq p \leq \infty$, is the space of measurable functions $f : \Omega \to \mathbb{R}$ such that $\|f\|_p < \infty$ where

$$\|f\|_p = \begin{cases} \left(\int_\Omega |f(x)|^p dx\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \text{ess sup}_{x\in\Omega} |f(x)| & \text{if } p = \infty \end{cases}$$

- $L^p_{loc}(\Omega)$, is the space of measurable functions such that $f|_K \in L^p(K)$ for every $K \subseteq \Omega$ compact.

- $C_0^\infty(\Omega)$ is the set of smooth functions with compact support contained in Ω.

- For $\alpha \in \mathbb{N}^n$ and $f : \Omega \to \mathbb{R}$ smooth enough we denote

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$

- If $(X, \|\cdot\|)$ is a Banach space and $T > 0$, we denote $L^p([0,T]; X)$ the Banach space of measurable functions $f : [0,T] \to X$ such that the following norm

$$\|f\|_p = \begin{cases} \left(\int_0^T \|f(t)\|^p dt\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \text{ess sup}_{t\in[0,T]} \|f(t)\| & \text{if } p = \infty \end{cases}$$

is finite.

2.2. Sobolev Spaces

Definition 2.1 *Let $f \in L^1_{loc}(\Omega)$ and $\alpha \in \mathbb{N}^n$. We say that $g \in L^1_{loc}(\Omega)$ is the weak derivative of order α of f and we denote it as $\partial^\alpha f = g$ if*

$$\int_\Omega f \partial^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g\varphi, \quad \forall \varphi \in C_0^\infty(\Omega)$$

Definition 2.2 *For $1 \leq p \leq \infty$ and $m \in \mathbb{N}$ we define the Sobolev space $W^{m,p}(\Omega)$ as*

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) \mid \partial^\alpha f \in L^p(\Omega)\ \forall |\alpha| \leq m\}$$

with the norm

$$\|f\|_{m,p} = \sum_{|\alpha|\leq m} \|\partial^\alpha f\|_p$$

When $p = 2$ we denote $W^{m,p}(\Omega) = H^m(\Omega)$.

The following results are classic, and can be found in [5] and [1].

Theorem 2.1 *For every $m \in \mathbb{N}$, $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ is a Banach space, and it is separable when $1 \leq p < \infty$. If $1 < p < \infty$ it is reflexive and if $p = 2$ it is a Hilbert space with the inner product*

$$\langle f, g\rangle_{H^m} = \sum_{|\alpha|\leq m} \int_\Omega \partial^\alpha f(x) \partial^\alpha g(x) dx$$

The corresponding induced norm is equivalent to the one defined in Definition 2.2

Theorem 2.2 *If $1 \leq p < \infty$ then the set $\{f|_\Omega \mid f \in C_0^\infty(\mathbb{R}^n)\}$ is dense in $W^{m,p}(\Omega)$. In particular, $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$.*

Theorem 2.3 (Sobolev's embedding) *For any integer $m \geq 1$ and $p \in [1, \infty]$ we have the following continuous injections:*

$$W^{m,p}(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n) \qquad \text{where } \frac{1}{q} = \frac{1}{p} - \frac{m}{n} \qquad \text{if } \frac{1}{p} - \frac{m}{n} > 0$$

$$W^{m,p}(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n) \qquad \forall q \in [p, \infty) \qquad \text{if } \frac{1}{p} - \frac{m}{n} = 0$$

$$W^{m,p}(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n) \qquad \text{if } \frac{1}{p} - \frac{m}{n} < 0$$

Moreover, when $\frac{1}{p} - \frac{m}{n} < 0$ we define $k = [m - n/p]$ and we have that

$$\|\partial^\alpha f\|_\infty \leq C\|f\|_{m,p}, \quad \forall |\alpha| \leq k, f \in W^{m,p}(\mathbb{R}^n)$$

In particular, $W^{m,p}(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n)$.

Now, we prove some lemmas that will be useful in the following chapters. The first of them concerned with a dense subset of $H^1(\mathbb{R}^n)$, $n \geq 3$, meanwhile the second provides an estimate in the particular case of $n = 3$.

Lemma 2.1 *If $n \geq 3$ then $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ is dense in $H^1(\mathbb{R}^n)$. Moreover, if $u \in H^1(\mathbb{R}^n)$ is radial and $u_k \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ is such that $u_k \to u$ in $H^1(\mathbb{R}^n)$, then we can choose $(u_k)_k$ also radial.*

Proof. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$ is enough to prove that

$$C_0^\infty(\mathbb{R}^n) \subseteq \overline{C_0^\infty(\mathbb{R}^n \setminus \{0\})}^{\|\|_{H^1}}.$$

Let $u \in C_0^\infty(\mathbb{R}^n)$ and $\theta \in C^\infty(\mathbb{R}^n)$ such that

$$\theta(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } |x| > 2. \end{cases}$$

Since $\theta(kx) \to 1$ a.e. we have that $u_k(x) = u(x)\theta(kx) \to u(x)$ in $L^2(\mathbb{R}^n)$ by dominated convergence theorem. For the derivative we have that

$$\partial_i(u(x)\theta(kx)) = \partial_i u(x)\theta(kx) + ku(x)\partial_i\theta(kx)$$

and the first term converges to $\partial_i u(x)$ by the same argument as above. For the second term notice that $\partial_i\theta(kx)$ converges to 0 a.e. and

$$\operatorname{supp}(ku(x)\partial_i\theta(kx)) \subseteq B\left(0, \frac{2}{k}\right) \cap B\left(0, \frac{1}{k}\right)^c.$$

This implies that

$$|ku(x)\partial_i\theta(kx)| \leq \frac{2|u(x)\partial_i\theta(kx)|}{|x|},$$

and thus $ku(x)\partial_i\theta(kx) \to 0$ a.e.. In addition, we have that

$$|ku(x)\partial_i\theta(kx)| \leq \frac{2\|u\|_\infty\|\partial_i\theta\|_\infty}{|x|}\mathbb{1}_{B(0,1)} \in L^2(\mathbb{R}^n),$$

and we conclude by dominated convergence theorem.

When u is radial we can take

$$\varphi(x) = \begin{cases} \dfrac{1}{|x|^2 - 1} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

and define $\varphi_k(x) = Ck^n\varphi(kx)$, where $C > 0$ is a normalization constant. It's well known that $(\varphi_k \star u) \to u$ in $H^1(\mathbb{R}^n)$, and the convolution of radial functions is also radial. Thus, is enough to apply the previous part of the lemma to the sequence $u_k = (\varphi_k \star u)$ to conclude. □

Lemma 2.2 *Suppose that $u \in H^1(\mathbb{R}^3)$ is radial; with abuse of notation we write $u(x) = u(r)$. Then $u(r) \in L^2(0,\infty)$ and $ru(r) \in L^p(0,\infty)$ for all $p \in [2,\infty]$. Moreover, we have the estimate*

$$\sup_{r \geq 0} |ru(r)| \leq C\|u\|_{H^1(\mathbb{R}^3)}.$$

PROOF. Since $u \in H^1$, from Hardy's inequality [21] we have

$$4\pi \int_0^\infty u^2(r)dr = \left\| \frac{u}{|x|} \right\|_{L^2(\mathbb{R}^3)}^2 \leq 4\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 = 16\pi \int_0^\infty r^2 u_r^2(r)dr.$$

Hence $u(r) \in L^2(0,\infty)$ and one has the inequality

$$\int_0^\infty u^2(r)dr \leq 4 \int_0^\infty r^2 u_r^2(r)dr.$$

In order to prove that $ru(r) \in L^\infty(0,\infty)$, notice first that $(ru(r))' = u(r) + ru_r(r)$ and since $u \in H^1(\mathbb{R}^3)$ from the previous part we have that $ru(r) \in H^1(0,\infty)$. By the Sobolev's embedding we conclude that $ru(r)$ is continuous and bounded. Using interpolation between L^2 and L^∞, we conclude $ru(r) \in L^p(0,\infty)$ for all $p \in [2,\infty]$. Moreover, we have from Sobolev's embedding that

$$\begin{aligned}\sup_{r\geq 0} |ru(r)| \leq C\|ru(r)\|_{H^1(0,\infty)} & \\ &\leq C(\|ru(r)\|_{L^2(0,\infty)} + \|u + ru_r\|_{L^2(0,\infty)}) \\ &\leq C(\|u\|_{L^2(\mathbb{R}^3)} + 2\|u_r\|_{L^2(\mathbb{R}^3)}) \leq 2C\|u\|_{H^1(\mathbb{R}^3)},\end{aligned}$$

proving the desired estimate. □

2.3. Linear wave equations

This section is concerned with basic properties of wave equations that will be extensively used through this work. Consider the linear wave equation on $\mathbb{R}^n$

$$\begin{aligned}\partial_t^2\phi - \Delta\phi =& f(t,x) \\ (\phi,\phi_t)(0) =& (g,h)\end{aligned} \tag{2.1}$$

We have the following result

Theorem 2.4 *Let $T > 0$ and $m \in \mathbb{N}$. Suppose that $f \in L^1([0,T];H^m)$ and $(g,h) \in H^{m+1} \times H^m$. Then the equation (2.1) has an unique solution $\phi \in C([0,T];H^{m+1}) \cap C^1([0,T];H^m)$. This solution satisfies the estimate*

$$\|\phi(t)\|_{H^{m+1}} + \|\phi_t(t)\|_{H^m} \leq C(1+t)\left(\|g\|_{H^{m+1}} + \|h\|_{H^m} + \int_0^t \|f(s)\|_{H^m}ds\right) \tag{2.2}$$

for all $t \in [0,T]$.

The factor $(1+t)$ is necessary, taking in mind the norms involved in the above estimate. This is due to the fact that the natural energy space is not $H^1 \times L^2$, but $\dot{H}^1 \times L^2$, where we can define $\dot{H}^1$ as the completion of the Schwartz space under the norm

$$\|f\|_{\dot{H}^1} = \|\nabla f\|_{L^2}$$

However, we choose to work on $H^{m+1} \times H^m$ because in order to get decay properties of the solutions we shall need to have some control on the L^2 norm of those solutions.

Another important point is that in the linear case we are able to find an explicit formula for the solution, which is given in terms of its Fourier transform by

$$\hat{\phi}(t,\xi) = \cos(|\xi|t)\hat{g} + \frac{\sin(|\xi|t)}{|\xi|}\hat{h} + \int_0^t \frac{\sin(|\xi|(t-\tau))}{|\xi|}\hat{f}(\tau,\xi)d\tau$$

In fact, this formula is crucial to prove the existence and uniqueness of the solution, and using the characterization via Fourier transform of Sobolev spaces (see for example [16]) it also gives us its regularity. In the folowing section we also use the estimate (2.2) to prove existence for semilinear wave equations.

Now we consider the general wave equation on $\mathbb{R}^n$

$$\begin{aligned} \sum_{i,j=0}^{n} g_{ij}(t,x)\partial_i\partial_j\phi + \sum_{i=0}^{n} b_i(t,x)\partial_i\phi + a(t,x)\phi =& f(t,x) \\ (\phi,\phi_t)(0) =& (g,h) \end{aligned} \tag{2.3}$$

where all the coefficients are $C^2([0,T]\times\mathbb{R}^n)$ and g_{ij} is symmetric. Here we denote $\partial_0 = \partial_t$ and

$$g^0_{ij} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}$$

the coefficients of the D'Alembertian. With this we have the following theorem:

Theorem 2.5 *Let $T > 0$ and $m \in \mathbb{N}$. Set $r_{ij}(t,x) = g_{ij}(t,x) - g^0_{ij}$. Suppose that*

$$\sum_{i,j=0}^{n} |r_{i,j}(t,x)| \leq \frac{1}{2}, \quad \forall (t,x) \in [0,T]\times\mathbb{R}^n$$

and $f \in L^1([0,T];H^m)$. Then for $(g,h) \in H^{m+1}\times H^m$ the equation (2.3) has an unique solution $\phi \in C([0,T];H^{m+1}) \cap C^1([0,T];H^m)$. This solution satisfies the estimate

$$\|\phi(t)\|_{H^{m+1}} + \|\phi_t(t)\|_{H^m} \leq C_T\left(\|g\|_{H^{m+1}} + \|h\|_{H^m} + \int_0^t \|f(s)\|_{H^m}ds\right) \tag{2.4}$$

for a constant $C \geq 0$ that only depends on n and T.

Note that the constant C_T is non decreasing on T. We shall use this fact in the proof of Theorem 2.6. The proof of this theorem is much more delicate than the case of the D'Alembertian (see for example [24]). This is due in part because we have no longer an explicit formula for the solution, and we need another tools to prove its existence. As in the previous theorem, this estimate will be very important to prove local existence for the semilinear equation associated with equation (2.3).

Notice that the hypothesis on r_{ij} is not sharp in the case that g_{ij} is diagonal and depends only on t. From the proof of Sogge, we see that this hypothesis is used only to ensure that

$$e(\phi) = \sum_{j=0}^{n} g_{0j}\partial_0\phi\partial_j\phi - \frac{1}{2}\sum_{i,j=0}^{n} g_{ij}\partial_i\phi\partial_j\phi$$

satisfies that

$$\frac{1}{4}(|\phi_t|^2 + |\nabla\phi|^2) \leq e(\phi) \leq |\phi_t|^2 + |\nabla\phi|^2$$

that is, that $e(\phi)$ and $|\phi_t|^2 + |\nabla\phi|^2$ define equivalent norms, but when g_{ij} is diagonal we have that $e(\phi)$ simplifies to

$$e(\phi) = \frac{1}{2}g_{00}(\partial_t\phi)^2 - \frac{1}{2}\sum_{j=1}^{n} g_{jj}(\partial_j\phi)^2$$

and in the case which we are interested, which corresponds to $g_{00} = 1$, $g_{jj} = -e^{-2Ht}$, for $H \geq 0$,the last conditions is satisfied, so we can apply the theorem.

Finally, we close this section giving an energy estimate for the linear equation associated with (1.11)

$$\partial_t^2\phi + 3H\partial_t\phi - \frac{\Delta\phi}{e^{2Ht}} = f(t,x) \tag{2.5}$$

For this equation, we define its energy as

$$E_L(t) = \int_{\mathbb{R}^3} \frac{\phi_t^2}{2} + \frac{|\nabla\phi|}{2e^{2Ht}}dx \tag{2.6}$$

With this, we have the following lemma

Lemma 2.3 *Let $T > 0$. If $f \in L^1([0,T], L^2(\mathbb{R}^3))$ then every solution $(\phi, \phi_t) \in H^1 \times L^2$ to* (2.5) *satisfies*

$$E_L(t) \leq e^{-Ht}E_L(0) + \int_0^t e^{-H(t-s)}\|f(s)\|_{L^2}^2 ds$$

Proof. Notice that multiplying the equation (2.5) by ϕ_t we obtain that

$$\partial_t\left(\frac{\phi_t^2}{2} + \frac{|\nabla\phi|^2}{2e^{2Ht}}\right) - \operatorname{div}\left(\frac{\phi_t\nabla\phi}{e^{2Ht}}\right) + 3H\phi_t^2 + H\frac{|\nabla\phi|^2}{e^{2Ht}} = f(t,x)\phi_t$$

Integrating on $\mathbb{R}^3$ we get

$$\begin{aligned}
\partial_t E_L(t) &= -H\int 3\phi_t^2 + \frac{|\nabla\phi|^2}{e^{2Ht}} + \int f(t,x)\phi_t \\
&\leq -H\int 3\phi_t^2 + \frac{|\nabla\phi|^2}{e^{2Ht}} + \int \frac{\beta\phi_t^2}{2} + \int \frac{f(t,x)^2}{2\beta} \\
&\leq -\int\left(3H - \frac{\beta}{2}\right)\phi_t^2 - \int H\frac{|\nabla\phi|^2}{e^{2Ht}} + \frac{\|f(t)\|_{L^2}^2}{2\beta}
\end{aligned}$$

where in the second line we have applied Cauchy Schwarz inequality to $f(t,x)\phi_t = \frac{f(t,x)}{\sqrt{\beta}}\sqrt{\beta}\phi_t$. Taking $\beta = 4H$ and applying Gronwall's inequality we finish the proof. □

2.4. Semilinear wave equations

In this sections we present some basic results on existence of solution for semilinear wave equations. This results are strongly based on the estimates presented on the previous section, and are obtained via an application of Banach fixed point theorem. We start showing an energy estimates that will be useful to prove global existence in Chapter 4. First, for a solution ϕ of (1.11) we define the energy density as

$$e(t,x) = \left(\frac{\phi_t^2}{2} + \frac{|\nabla \phi|}{2e^{2Ht}} + F(\phi)\right)(t,x) \tag{2.7}$$

and its energy

$$E(t) = \int_{R^3} e(t,x)dx \tag{2.8}$$

With this, we have the following elementary result.

Lemma 2.4 *Let $\phi \in C([0,T];H^1) \cap C^1([0,T];L^2)$ be a solution of (1.11). Then we have that*

$$\frac{d}{dt}E(t) = -H \int_{R^3} 3\phi_t^2 + \frac{|\nabla \phi|^2}{e^{2Ht}} \tag{2.9}$$

Now we will show some existence theorems for equation (1.11). We start with the most basic result on this line, where we make the strong assumption that the nonlinearity is globally Lipschitz. Despite this restrictive hypothesis, the proof of this theorem will serve as a model for more complex results.

Theorem 2.6 *If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and $f(0) = 0$ then for initial data in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ equation (1.11) has a unique global solution such that*

1. *$\phi \in C([0,\infty);H^1(\mathbb{R}^n)) \cap C^1([0,\infty);L^2(\mathbb{R}^n))$.*
2. *If the initial datum $(\phi_0, \phi_1) \in H^1 \times L^2$ is radial, then $(\phi, \partial_t\phi)$ is radial for all times.*
3. *Equation (2.9) is satisfied for all times $t \geq 0$.*

Proof. Consider the Banach space

$$X = C([0,T];H^1) \cap C^1([0,T],L^2),$$

with norm

$$\|\phi\|_X = \sup_{t\in[0,T]} (\|\phi(t)\|_{H^1} + \|\phi_t\|_{L^2}),$$

for some $T > 0$ to be chosen later. We define the operator $A : X \to X$ by $Av = \phi$, where ϕ is the unique solution to

$$\begin{aligned} \partial_t^2\phi + 3H\partial_t\phi - \frac{\Delta\phi}{e^{2Ht}} =& f(v), \\ (\phi(0), \phi_t(0)) =& (h,g). \end{aligned}$$

Note that A is well defined because, since f is globally Lipschitz and $f(0) = 0$, we have that $|f(v)| \leq M|v|$ and hence $f(v) \in L^1([0,T],L^2)$. Thus, given $v_1, v_2 \in X$ we have that

$w = Av_1 - Av_2$ satisfies

$$\partial_t^2 w + 3H\partial_t w - \frac{\Delta w}{e^{2Ht}} = f(v_1) - f(v_2)$$
$$(w(0), w_t(0)) = (0, 0),$$

and so we can use the estimate (2.4), obtaining

$$\begin{aligned}\|w(t)\|_{H^1} + \|w_t(t)\|_{L^2} &\le C_T \int_0^t \|f(v_1(s)) - f(v_2(s))\|_{L^2} ds \\ &\le C_T M t \sup_{s\in[0,T]} \|v_1(s) - v_2(s)\|_{L^2},\end{aligned}$$

where $M > 0$ is the Lipschitz constant of f. Thus we have

$$\|w\|_X = \|Av_1 - Av_2\|_X \le C_T M T \|v_1 - v_2\|_X,$$

and taking T small enough such that $C_T M T < 1$ by Banach's fixed point theorem there exist $\phi \in X$ such that

$$\partial_t^2 \phi + 3H\partial_t \phi - \frac{\Delta \phi}{e^{2Ht}} = f(\phi),$$
$$(\phi(0), \phi_t(0)) = (h, g).$$

Since T does not depend on h, g we can extend this solution for all times, concluding the proof. $\square$

Now we make a weaker assumption on the nonlinearity and suppose only that it is of class C^2. This gives us less control on the nonlinear term, and consequently is expected to obtain weaker results. To control the nonlinear term we shall need to suppose more regularity on the initial data. Specifically, we have the following.

Theorem 2.7 *If $f : \mathbb{R} \to \mathbb{R}$ is of class C^2 and $f(0) = f'(0) = 0$, then equation* (1.11) *has a unique maximal solution provided that the initial data $(\phi_0, \phi_1) \in H^2 \times H^1$ is small enough. This solution satisfies*

1. $\phi \in C([0, T); H^2(\mathbb{R}^3)) \cap C^1([0, T); H^1(\mathbb{R}^3))$.
2. *If the initial datum $(\phi_0, \phi_1) \in H^2 \times H^1$ is radial, then $(\phi, \partial_t \phi)$ is radial for all times.*
3. *Equation* (2.9) *is satisfied along $0 \le t < T$.*

Proof. Proceeding as in the previous theorem, consider the Banach space

$$X = C([0, T]; H^2) \cap C^1([0, T]; H^1),$$

with norm

$$\|\phi\|_X = \sup_{t\in[0,T]} (\|\phi(t)\|_{H^2} + \|\phi_t(t)\|_{H^1}),$$

and consider the subset $Y = \{\phi \in X \mid \|\phi\|_X \le R\}$ with the metric induced by the norm of

X. Define the operator $A : Y \to X$ by $Av = \phi$, where ϕ is the unique solution to

$$\partial_t^2 \phi + 3H\partial_t \phi - \frac{\Delta \phi}{e^{2Ht}} = f(v)$$
$$(\phi(0), \phi_t(0)) = (h, g),$$

where $(h, g) \in H^2 \times H^1(\mathbb{R}^3)$. In order to see that $f(v) \in H^1$ notice that from Sobolev's embedding we have that $H^2(\mathbb{R}^3) \hookrightarrow C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, and then $|v(t, x)| \leq M$ for all $(t, x) \in [0, T] \times \mathbb{R}^3$. Note that this constant is independent of v, because $\|v\|_X \leq R$. Since f is of class C^2 (and in particular locally Lipschitz continuous) we have

$$|f(v)| \lesssim |v|,$$

and $f(v) \in L^2$. The same argument allows us to prove that

$$|\nabla f(v)| = |f'(v)\nabla v| \lesssim |\nabla v|,$$

and thus $f(v) \in L^1([0, T]; H^1)$ and A is well defined. Using (2.4) for $n = 1$ we see that

$$\begin{aligned} \|w\|_X &\leq C_T \left(\|h\|_{H^2} + \|g\|_{H^1} + \int_0^T \|f(v(s))\|_{H^1} ds \right) \\ &\leq C_T \left(\|h\|_{H^2} + \|g\|_{H^1} + TM\|v\|_X \right), \end{aligned}$$

where $M > 0$ depends only on f and R. Thus, taking $R \geq C(1+T)\|(h, g)\|_{H^2 \times H^1}$ and $T > 0$ small enough we have that $\|w\|_X \leq R$. Now, for $v_1, v_2 \in Y$ we have that $w = Av_1 - Av_2$ satisfies

$$\partial_t^2 w - \Delta w = f(v_1) - f(v_2)$$
$$(w(0), w_t(0)) = (0, 0),$$

and using (2.4) again we have that

$$\begin{aligned} \|w(t)\|_{H^2} + \|w_t(t)\|_{H^1} &\leq C_T \int_0^t \|f(v_1(s)) - f(v_2(s))\|_{H^1} ds \\ &\leq C_T \int_0^t (\|f(v_1) - f(v_2)\|_{L^2} + \|f'(v_1)\nabla v_1 - f'(v_2)\nabla v_2\|_{L^2}) ds. \end{aligned}$$

Since both f and f' are locally Lipschitz continuous we have that

$$\|f(v_1) - f(v_2)\|_{L^2} \lesssim \|v_1 - v_2\|_{L^2},$$

and

$$\begin{aligned} \|f'(v_1)\nabla v_1 - f'(v_2)\nabla v_2\|_{L^2} &\leq \|f'(v_1)\nabla v_1 - f'(v_1)\nabla v_2\|_{L^2} + \|(f'(v_1) - f'(v_2))\nabla v_2\|_{L^2} \\ &\lesssim \|\nabla v_1 - \nabla v_2\|_{L^2} + \|v_1 - v_2\|_{L^2}. \end{aligned}$$

This implies that

$$\|w\|_X \leq C_T NT \|v_1 - v_2\|_X.$$

and taking T small enough we have that $A : Y \to Y$ is a contraction, and we conclude as in the proof of Theorem 2.6. By standard arguments this solution can be extended to a maximal interval $[0, T)$. $\square$

To finish this chapter, we recall a classical existence theorem for the classical semilinear wave equation (i.e. with $H = 0$). The proof of this result is long and uses many lemmas that we will not use in the following chapters, so for simplicity we only enunciate this theorem. For a detailed proof see for example [24].

Theorem 2.8 *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is of class C^2, $f(0) = f'(0) = 0$ and there exists $C > 0$ such that*

$$|f''(s)| \leq C|s|^p, \quad |s| \leq 1$$

for $p \geq 1$. Then for initial data of the form $(\phi_0, \phi_1) = \varepsilon(g, h)$, where g, h are smooth compactly supported functions, the equation (1.11) for $H = 0$ has a unique global solution $\phi \in C^2(\mathbb{R}^{1+3})$ provided that $\varepsilon > 0$ is small enough.

Chapter 3

Virial Identities

This chapter is devoted to prove virial estimates, which are the base point to prove decay properties for the equations considered in Chapter 1. This approach is very common in the study of nonlinear dispersive equations and one can see it in works of Merle, Martel, Kowalczyk, Alejo and Maulen for example [2], [13], [14]. This chapter is divided into two sections, the first concerned with estimates for the case $H = 0$, while the second with an estimate for the case $H > 0$. This because those cases are enough different qualitatively speaking, and therefore need separate treatment.

3.1. The case $H = 0$

3.1.1. First calculations

For a locally integrable function $\psi : \mathbb{R} \to \mathbb{R}$ to be chosen later we define the following functionals

$$\mathcal{P}(\phi)(t) = \int_0^\infty \psi(r)\phi_r(t,r)\phi_t(t,r)dr, \tag{3.1}$$

$$\mathcal{R}(\phi)(t) = \int_0^\infty \psi'(r)\phi(t,r)\phi_t(t,r)dr, \tag{3.2}$$

$$\mathcal{I}(\phi)(t) = \mathcal{P}(\phi)(t) + \frac{1}{2}\mathcal{R}(\phi)(t), \tag{3.3}$$

Our first result is concerned with the behavior of the time derivative of these functionals.

Lemma 3.1 *If $\phi \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ is a radial solution of (1.11) with $H = 0$ then*

$$\frac{d\mathcal{P}(\phi)}{dt} = \int_0^\infty \frac{2\psi}{r}\phi_r^2 - \int_0^\infty \psi'\left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2} - F(\phi)\right), \tag{3.4}$$

$$\frac{d\mathcal{R}(\phi)}{dt} = \int_0^\infty \psi'\left(\phi_t^2 - \phi_r^2 - \phi f(\phi)\right) + \int_0^\infty \left(\frac{\psi'}{r^2} - \frac{\psi''}{r} + \frac{\psi'''}{2}\right)\phi^2, \tag{3.5}$$

and

$$\frac{d\mathcal{I}(\phi)}{dt} = \int_0^\infty \left(\left(\frac{\psi'}{r^2} - \frac{\psi''}{r} + \frac{\psi'''}{2}\right)\frac{\phi^2}{2} + \left(\frac{2\psi}{r} - \psi'\right)\phi_r^2 + \frac{\psi'}{2}(2F(\phi) - \phi f(\phi))\right). \tag{3.6}$$

Proof. Thanks to Lemma 2.1, it is enough to compute all derivatives assuming data in $C_0^\infty(\mathbb{R}^3\backslash\{0\})$.

Using equation (1.11) and the definition of $F(s) = \int_0^s f(\sigma)d\sigma$, we have in (3.1):

$$\begin{aligned}\frac{d\mathcal{P}(\phi)}{dt} &= \int_0^\infty \psi\left(\phi_{rt}\phi_t + \phi_r\phi_{rr} + \frac{2}{r}\phi_r^2 - \phi_r f(\phi)\right)dr\\ &= \int_0^\infty \psi\partial_r\left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2} - F(\phi)\right) + \frac{2\psi}{r}\phi_r^2\\ &=\psi\left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2} - F(\phi)\right)\Big|_0^\infty + \int_0^\infty \frac{2\psi}{r}\phi_r^2 - \psi'\left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2} - F(\phi)\right).\end{aligned}$$

Thanks to Lemma 2.1, every boundary term at zero and infinity disappear. We get (3.4).

We compute now $\dfrac{d\mathcal{R}(\phi)}{dt}$. We have from (3.2):

$$\begin{aligned}\frac{d\mathcal{R}(\phi)}{dt} &= \int_0^\infty \psi'\left(\phi_t^2 + \phi\phi_{rr} + \frac{2}{r}\phi\phi_r - \phi f(\phi)\right)\\ &= \int_0^\infty \psi'(\phi_t^2 - \phi f(\phi)) + \int_0^\infty \psi'\phi\phi_{rr} + \int_0^\infty \psi'\frac{2}{r}\phi\phi_r =: K_1 + K_2 + K_3.\end{aligned}$$

K_1 is left as it is. We compute K_2 first saving every boundary term:

$$\begin{aligned}K_2 &=\psi'\phi\phi_r\Big|_0^\infty - \int_0^\infty \phi_r(\psi''\phi + \psi'\phi_r)\\ &=\psi'\phi\phi_r\Big|_0^\infty - \int_0^\infty \psi'\phi_r^2 - \int_0^\infty \psi''\partial_r\left(\frac{\phi^2}{2}\right)\\ &=\psi'\phi\phi_r\Big|_0^\infty - \int_0^\infty \psi'\phi_r^2 - \left(\frac{\psi''\phi^2}{2}\Big|_0^\infty - \int_0^\infty \psi'''\frac{\phi^2}{2}\right)\\ &=\psi'\phi\phi_r\Big|_0^\infty - \frac{\psi''\phi^2}{2}\Big|_0^\infty + \int_0^\infty \psi'''\frac{\phi^2}{2} - \int_0^\infty \psi'\phi_r^2.\end{aligned}$$

Similarly,

$$\begin{aligned}K_3 &= \int_0^\infty \frac{\psi'}{r}\partial_r(\phi^2) = \frac{\psi'}{r}\phi^2\Big|_0^\infty - \int_0^\infty \partial_r\left(\frac{\psi'}{r}\right)\phi^2\\ &=\frac{\psi'}{r}\phi^2\Big|_0^\infty - \int_0^\infty\left(\frac{\psi''}{r} - \frac{\psi'}{r^2}\right)\phi^2.\end{aligned}$$

Arranging all previous computations, we conclude that $\mathcal{R}(\phi) = \int_0^\infty \psi'\phi\phi_t dr$ satisfies:

$$\begin{aligned}&\frac{d\mathcal{R}(\phi)}{dt}\\ &= \int_0^\infty \psi'(\phi_t^2 - \phi f(\phi))\\ &\quad + \psi'\phi\phi_r\Big|_0^\infty - \frac{\psi''\phi^2}{2}\Big|_0^\infty + \int_0^\infty \psi'''\frac{\phi^2}{2} - \int_0^\infty \psi'\phi_r^2 + \frac{\psi'}{r}\phi^2\Big|_0^\infty - \int_0^\infty\left(\frac{\psi''}{r} - \frac{\psi'}{r^2}\right)\phi^2.\end{aligned}$$

Again thanks to Lemma 2.1, every boundary term disappears. We obtain

$$\begin{aligned}\frac{d\mathcal{R}(\phi)}{dt} &= \int_0^\infty \psi'(\phi_t^2 - \phi f(\phi)) + \int_0^\infty \psi''' \frac{\phi^2}{2} - \int_0^\infty \psi' \phi_r^2 - \int_0^\infty \left(\frac{\psi''}{r} - \frac{\psi'}{r^2}\right)\phi^2 \\ &= \int_0^\infty \psi' \left(\phi_t^2 - \phi_r^2 - \phi f(\phi)\right) + \int_0^\infty \left(\frac{\psi'}{r^2} - \frac{\psi''}{r} + \frac{\psi'''}{2}\right)\phi^2.\end{aligned}$$

This proves (3.5). Finally, gathering (3.4) and the previous identity in the definition of $\mathcal{I}(t)$ (3.3), we arrive to (3.6). □

3.1.2. Choice of weight function

Here we will use Lemma 3.1 with a particular choice of ψ.

Corollary 3.1 *Consider the weight*

$$\psi(r) := \frac{r^2}{1+r}. \tag{3.7}$$

Then the following are satisfied:

1. *One has that*

$$\mathcal{I}(\phi) = \int_0^\infty \frac{r^2}{1+r}\phi_r\phi_t + \frac{r(r+2)}{2(1+r)^2}\phi\phi_t \tag{3.8}$$

is well-defined and bounded uniformly in time by the energy of the solution:

$$\sup_{t\geq 0} |\mathcal{I}(\phi)(t)| \lesssim E[\phi, \partial_t\phi](t=0).$$

2. *Also,*

$$\frac{d\mathcal{P}(\phi)}{dt} = \int_0^\infty \frac{r(2+3r)}{2(1+r)^2}\phi_r^2 - \frac{r(r+2)}{(1+r)^2}\left(\frac{\phi_t^2}{2} - F(\phi)\right) \tag{3.9}$$

$$\frac{d\mathcal{R}(\phi)}{dt} = \int_0^\infty \frac{r(r+2)}{(1+r)^2}(\phi_t^2 - \phi f(\phi)) + \frac{r(r+4)}{(1+r)^4}\phi^2 - \frac{r(r+2)}{(1+r)^2}\phi_r^2 \tag{3.10}$$

$$\frac{d\mathcal{I}(\phi)}{dt} = \int_0^\infty r^2\left(\frac{1}{(1+r)^2}\phi_r^2 + \frac{r+4}{2r(1+r)^4}\phi^2\right) + \frac{r(r+2)}{2(1+r)^2}(2F(\phi) - \phi f(\phi)). \tag{3.11}$$

Proof. The proof of (3.8), (3.9), (3.10) and (3.11) are direct from Lemma 3.1 and (3.7). We check now that $\mathcal{I}(\phi)$ is well-defined. Indeed,

$$\left|\int_0^\infty \frac{r^2}{(1+r)}\phi_r\phi_t\right| \leq \int_0^\infty r^2 |\phi_t\phi_r| \leq E(\phi, \phi_t).$$

Additionally,

$$\left|\int_0^\infty \frac{r(r+2)}{2(1+r)^2}\phi\phi_t\right| \leq \int_0^\infty r|\phi\phi_t| \leq E(\phi,\phi_t)^{1/2}\left(\int_0^\infty \phi^2 dr\right)^{1/2}.$$

Finally, Lemma 2.2 gives the desired uniform in time proof. $\square$

If we define the weighted norms

$$\|\phi\|_{H^1_w}^2 = \int_0^\infty \frac{r^2}{(1+r)^4}(\phi^2+\phi_r^2), \qquad \|\phi\|_{L^2_w}^2 = \int_0^\infty \frac{r^2}{(1+r)^4}\phi^2$$

we can see that

$$\frac{d\mathcal{I}(\phi)}{dt} \geq \|\phi\|_{H^1_w}^2 + \int_0^\infty \frac{r(r+2)}{2(1+r)^2}(2F(\phi)-\phi f(\phi))$$

On the other hand, if we choose $\psi(r) = -\dfrac{3r^2+3r+1}{3(1+r)^3}$ we have

$$\tilde{\mathcal{R}} = \int_0^\infty \frac{r^2}{(1+r)^4}\phi\phi_t,$$

and

$$\frac{d\tilde{\mathcal{R}}}{dt}(\phi) = \int_0^\infty \frac{r^2}{(1+r)^4}(\phi_t^2-\phi_r^2-\phi f(\phi)) + \frac{2r(3r-2)}{(1+r)^6}\phi^2.$$

This allows us to prove the following propositions:

Proposition 3.1 *Under the hypothesis of Theorem 1.1 the solution* (ϕ,ϕ_t) *of* (1.11) *satisfies*

$$\int_0^\infty (\|\phi\|_{H^1_w}^2 + \|\phi_t\|_{L^2_w}^2)dt < +\infty.$$

Proof. From Corollary 3.1 and the previous calculations we see that

$$\frac{d\mathcal{I}(\phi)}{dt} \geq \|\phi\|_{H^1_w}^2,$$

and then

$$\begin{aligned}\int_0^\infty \|\phi\|_{H^1_w}^2 dt &\leq \lim_{t\to\infty} \mathcal{I}(\phi(t)) - \mathcal{I}(\phi(0)) \\ &\lesssim E(\phi,\phi_t) + |\mathcal{I}(\phi(0))|.\end{aligned}$$

On the other hand, since

$$\frac{d\tilde{\mathcal{R}}}{dt}(\phi) = \int_0^\infty \left(\frac{r^2}{(1+r)^4}(\phi_t^2-\phi_r^2-\phi f(\phi)) + \frac{2r(3r-2)}{(1+r)^6}\phi^2\right),$$

we have

$$\begin{aligned}\|\phi_t\|_{L^2_w}^2 &= \frac{d\tilde{\mathcal{R}}}{dt} + \|\phi_r\|_{L^2_w}^2 + \int_0^\infty \left(\frac{r^2}{(1+r)^4}\phi f(\phi) + \frac{2r(2-3r)}{(1+r)^6}\phi^2\right) \\ &\leq \frac{d\tilde{\mathcal{R}}}{dt} + \|\phi_r\|_{L^2_w}^2 + \int_0^\infty \left(\frac{Mr^2}{(1+r)^4} + \frac{2r(2-3r)}{(1+r)^6}\right)\phi^2 \\ &\leq \frac{d\tilde{\mathcal{R}}}{dt} + \|\phi_r\|_{L^2_w}^2 + C(M)\int_0^\infty \frac{r(4+r)}{(1+r)^4}\phi^2 \\ &\leq \frac{d\tilde{\mathcal{R}}}{dt} + \|\phi_r\|_{L^2_w}^2 + C(M)\frac{d\mathcal{I}}{dt},\end{aligned}$$

where M is the Lipschitz constant of f, and we have used (3.11) and the previous observations. We note that

$$|\tilde{R}(\phi)| \leq \int_0^\infty \frac{r^2}{1+r}|\phi\phi_t| \leq \int_0^\infty r|\phi\phi_t|,$$

and from the proof of Corollary 3.1 we see that $\tilde{R}(\phi)$ is uniformly bounded in time by the energy of the solution. Integrating the last inequality the result follows. □

Proposition 3.2 *Under the hypothesis of Theorem 1.2 the solution* (ϕ, ϕ_t) *of* (1.11) *satisfies*

$$\int_0^\infty (\|\phi\|^2_{H^1_w} + \|\phi_t\|^2_{L^2_w})dt < +\infty.$$

Proof. From (1.13) we see that $F(\phi) \geq 0$ an using the inequality (1.13) we obtain that

$$\frac{d\mathcal{I}(\phi)}{dt} \geq \int_0^\infty r^2 \left(\frac{1}{(1+r)^2}\phi_r^2 + \frac{r+4}{2r(1+r)^4}\phi^2 \right) - C\frac{r(r+2)}{2(1+r)^2}\phi^4.$$

Since we have supposed that $\sup_{t\geq 0} \|\phi(t)\|_{H^1\cap L^\infty} \leq \varepsilon$ we have

$$\begin{aligned} |\phi(t,r)| &\leq \varepsilon \quad \forall t, r \geq 0, \\ \|\phi(t)\|_{H^1} &\leq \varepsilon \quad \forall t \geq 0, \end{aligned}$$

and from Lemma 2.2 we have that

$$r|\phi(r)| \leq C\|\phi\|_{H^1}.$$

Gathering both inequalities we obtain

$$\phi(r)^2 \leq \frac{(1+C)\varepsilon^2}{(1+r)^2},$$

and hence

$$\begin{aligned} \frac{d\mathcal{I}(\phi)}{dt} &\geq \int_0^\infty r^2 \left(\frac{1}{(1+r)^2}\phi_r^2 + \frac{r+4}{2r(1+r)^4}\phi^2 \right) - (1+C)\frac{r(r+2)}{2(1+r)^4}\varepsilon^2\phi^2 \\ &\geq \int_0^\infty \frac{r^2}{(1+r)^2}\phi_r^2 + \frac{r(r+2)}{2(1+r)^4}(1-(1+C)\varepsilon^2)\phi^2 \\ &\geq \|\phi\|^2_{H^1_w}, \end{aligned}$$

provided that ε is small enough. Notice that, since $\phi(t,r)$ is uniformly bounded and f is C^1 there existe a constant $M > 0$ such that

$$|f(\phi(t))| \leq M|\phi(t)|, \quad \forall t \geq 0,$$

which implies that $\phi f(\phi) \leq M\phi^2$, and we conclude as in the previous proposition. □

3.2. The case $H > 0$

In this section we present virial identities for equation (1.11) with $H>0$. As we mentioned at the beginning of the chapter, this estimates are a bit different to the presented in the last section. In fact, for a locally integrable function $\varphi(t,r)$ we define

$$\mathcal{J}(\phi)(t) = \int_0^\infty r^2\varphi(t,r)\left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2e^{2Ht}} + F(\phi)\right). \tag{3.12}$$

and analogously to the previous section we have the following lemma.

Lemma 3.2 *If $\phi \in C([0,\infty);H^1)\cap C^1([0,\infty);L^2)$ is a globally defined radial solution of* (1.11)*, then*

$$\begin{aligned}\frac{d\mathcal{J}(\phi)}{dt} &= \int_0^\infty r^2\varphi_t(t,r)\left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2e^{2Ht}} + F(\phi)\right)\\ &\quad - H\int_0^\infty r^2\varphi(t,r)\left(3\phi_t^2 + \frac{\phi_r^2}{e^{2Ht}}\right) - \int_0^\infty r^2\varphi_r(t,r)\frac{\phi_t\phi_r}{e^{2Ht}}.\end{aligned}$$

Proof. As in Lemma 3.1 we shall assume data in $C_0^\infty(\mathbb{R}^3\setminus\{0\})$, and consequently, every boundary term will disappear. Deriving (3.12) we have

$$\begin{aligned}\frac{d\mathcal{J}(\phi)}{dt} &= \int_0^\infty r^2\varphi_t(t,r)\left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2e^{2Ht}} + F(\phi)\right)\\ &\quad + \int_0^\infty r^2\varphi(t,r)\left(\phi_{tt}\phi_t + \frac{\phi_r\phi_{rt}}{e^{2Ht}} - H\frac{\phi_r^2}{e^{2Ht}} + f(\phi)\phi_t\right)\\ &= K_1 + K_2\end{aligned}$$

For the second term, using (1.11) we get

$$\begin{aligned}K_2 &= \int_0^\infty r^2\varphi(t,r)\left(-3H\phi_t^2 + \frac{\phi_t\Delta\phi}{e^{2Ht}} - f(\phi)\phi_t + \frac{\phi_r\phi_{rt}}{e^{2Ht}} - H\frac{\phi_r^2}{e^{2Ht}} + f(\phi)\phi_t\right)\\ &= -H\int_0^\infty r^2\varphi(t,r)\left(3\phi_t^2 + \frac{\phi_r^2}{e^{2Ht}}\right) + \int_0^\infty r^2\varphi\left(\frac{\phi_r\phi_{rt}}{e^{2Ht}} + \frac{\phi_t\Delta\phi}{e^{2Ht}}\right)\\ &= -H\int_0^\infty r^2\varphi(t,r)\left(3\phi_t^2 + \frac{\phi_r^2}{e^{2Ht}}\right) - \int_0^\infty r^2\varphi_r(t,r)\frac{\phi_r\phi_t}{e^{2Ht}},\end{aligned}$$

where in the last equality we have used the Green identity on $\mathbb{R}^3$. Arranging the previous calculations we got

$$\begin{aligned}\frac{d\mathcal{J}(\phi)}{dt} &= \int_0^\infty r^2\varphi_t(t,r)\left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2e^{2Ht}} + F(\phi)\right)\\ &\quad - H\int_0^\infty r^2\varphi(t,r)\left(3\phi_t^2 + \frac{\phi_r^2}{e^{2Ht}}\right) - \int_0^\infty r^2\varphi_r(t,r)\frac{\phi_t\phi_r}{e^{2Ht}}.\end{aligned}$$

$\square$

This allows us to prove the following corollary.

Corollary 3.2 *For $\sigma, b \in \mathbb{R}$ consider the weight*

$$\varphi(t,r) = 1 + \tanh(r + \sigma t + b). \tag{3.13}$$

Then if $F(x) \geq 0$ for all $x \in \mathbb{R}$ we have the estimate

$$\frac{d\mathcal{J}(\phi)}{dt} \leq (1+\sigma) \int_0^\infty r^2 \operatorname{sech}^2(r + \sigma t + b) \left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2e^{2Ht}} + F(\phi) \right). \tag{3.14}$$

Proof. Replacing (3.13) in Lemma 3.2 we get

$$\begin{aligned} \frac{d\mathcal{J}(\phi)}{dt} &= \sigma \int_0^\infty r^2 \operatorname{sech}^2(r + \sigma t + b) \left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2e^{2Ht}} + F(\phi) \right) \\ &\quad - H \int_0^\infty r^2 (1 + \tanh(r + \sigma t + b)) \left(3\phi_t^2 + \frac{\phi_r^2}{e^{2Ht}} \right) - \int_0^\infty r^2 \operatorname{sech}^2(r + \sigma t + b) \frac{\phi_t \phi_r}{e^{2Ht}}. \end{aligned}$$

Noticing that the second term is strictly negative and

$$-\operatorname{sech}^2(r + \sigma t + b) \frac{\phi_r \phi_t}{e^{2Ht}} \leq \operatorname{sech}^2(r + \sigma t + b) \left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2e^{2Ht}} \right),$$

we have that

$$\begin{aligned} \frac{d\mathcal{J}(\phi)}{dt} &\leq \sigma \int_0^\infty r^2 \operatorname{sech}^2(r + \sigma t + b) \left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2e^{2Ht}} + F(\phi) \right) \\ &\quad + \int_0^\infty r^2 \operatorname{sech}^2(r + \sigma t + b) \left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2e^{2Ht}} \right). \end{aligned}$$

Using that $F(\phi) \geq 0$ we obtain the desired estimate (3.14). $\square$

Chapter 4

Proof of the main results

In this chapter we complete the proofs of Theorems 1.1, 1.2 and 1.3 using the identities developed in Chapter 3, and we see that the decay properties are a direct consequence of the integrability on time of the weighted norms of the solution. For the existence part of Theorem 1.3 we shall need a different argument as the presented in Section 2.4. In this case the proof will be based on a bootstrap argument to control the size of the solution. Once we have controlled the norm of the solution we will use that to prove that we can extend the local solution to a global one.

We start with Theorems 1.1 and 1.2

4.1. Proof of Theorems 1.1 and 1.2

For a locally integrable function $\psi : \mathbb{R} \to \mathbb{R}$ let

$$\mathcal{H}(t) = \int_0^\infty \psi(\phi^2 + \phi_r^2 + \phi_t^2),$$

then, we can see that

$$\begin{aligned}
\frac{d}{dt}\mathcal{H}(t) &= \int_0^\infty 2\psi(\phi\phi_t + \phi_r\phi_{rt} + \phi_t\phi_{tt}) \\
&= \int_0^\infty 2\psi\left(\phi\phi_t + \phi_r\phi_{rt} + \phi_t\phi_{rr} + \frac{2}{r}\phi_t\phi_r - \phi_t f(\phi)\right) \\
&= \int_0^\infty 2\psi\left(\phi\phi_t + \phi_r\phi_{rt} + \frac{2}{r}\phi_t\phi_r - \phi_t f(\phi)\right) dr + \int_0^\infty 2\psi\phi_t\phi_{rr}.
\end{aligned}$$

Using Lemma 2.1 we have

$$\int_0^\infty 2\psi\phi_t\phi_{rr} = -\int_0^\infty 2\phi_r(\psi\phi_{tr} + \psi'\phi_t),$$

and hence

$$\frac{d}{dt}\mathcal{H}(t) = 2\int_0^\infty \psi(\phi\phi_t - \phi_t f(\phi)) + \left(\frac{2\psi}{r} - \psi'\right)\phi_t\phi_r$$

Since $\psi(r) = \dfrac{r^2}{(1+r)^4}$, we have that

$$\mathcal{H}(t) = \int_0^\infty \frac{r^2}{(1+r)^4}(\phi^2 + \phi_r^2 + \phi_t^2) = \|\phi\|_{H^1_w}^2 + \|\phi_t\|_{L^2_w}^2,$$

and

$$\frac{d}{dt}\mathcal{H}(t) = 2\int_0^\infty \frac{r^2}{(1+r)^4}(\phi\phi_t - \phi_t f(\phi)) + \frac{4r^2}{(1+r)^5}\phi_r\phi_t.$$

As we see, whether f is globally Lipschitz or f is C^1 and $\|\phi(t)\|_{L^\infty}$ is uniformly bounded in time then we have

$$|f(u)| \lesssim |u|.$$

Thus, we see that

$$\begin{aligned}\left|\frac{d}{dt}\mathcal{H}(t)\right| &\lesssim \int_0^\infty \frac{r^2}{(1+r)^4}(|\phi||\phi_t| + |\phi_t||f(\phi)|) + \frac{4r^2}{(1+r)^5}\phi_r\phi_t \\ &\lesssim \int_0^\infty \frac{r^2}{(1+r)^4}(\phi^2 + \phi_t^2 + \phi^2) + \frac{4r^2}{(1+r)^4}(\phi_r^2 + \phi_t^2) \\ &\lesssim \mathcal{H}(t).\end{aligned}$$

From Proposition 3.1 and 3.2 there exists a sequence $t_n \to \infty$ such that $\mathcal{H}(t_n) \to 0$. Integrating the inequality above on $[t, t_n]$we see that

$$\begin{aligned}|\mathcal{H}(t_n) - \mathcal{H}(t)| &= \left|\int_t^{t_n} \frac{d}{dt}\mathcal{H}(s)ds\right| \\ &\le \int_t^{t_n} \left|\frac{d}{dt}\mathcal{H}(s)ds\right| \lesssim \int_t^{t_n} \mathcal{H}(s)ds,\end{aligned}$$

and passing to limit as $n \to \infty$ we have

$$\mathcal{H}(t) \le \int_t^\infty \mathcal{H}(s)ds,$$

and hence $\lim_{t\to\infty} \mathcal{H}(t) = \lim_{t\to\infty} \left(\|\phi\|_{H^1_w}^2 + \|\phi_t\|_{L^2_w}^2\right) = 0$. To conclude the proof is enough to note that for any $R > 0$ we have

$$\begin{aligned}\|(\phi, \phi_t)\|_{H^1\times L^2(B(0,R))}^2 &= \int_{B(0,R)} (\phi^2 + \phi_r^2 + \phi_t^2)dx \\ &= 4\pi \int_0^R r^2(\phi^2 + \phi_r^2 + \phi_t^2)dr \\ &\le 4\pi(1+R)^4 \int_0^R \frac{r^2}{(1+r)^4}(\phi^2 + \phi_r^2 + \phi_t^2)dr \\ &\le 4\pi(1+R)^4\mathcal{H}(t),\end{aligned}$$

and the result follows.

4.2. Proof of Theorem 1.3

4.2.1. Existence

Let $\phi(t,x)$ be the local solution to (1.11) on $[0,T)\times\mathbb{R}^3$ given by Theorem 2.7. We shall prove that for $\varepsilon>0$ small enough we can extend this solution smoothly to $[0,T]\times\mathbb{R}^3$. First notice that we have the identity

$$\partial_t\left(\frac{\phi_t^2}{2}+\frac{|\nabla\phi|^2}{2e^{2Ht}}+F(\phi)\right)-\operatorname{div}\left(\frac{\phi_t\nabla\phi}{e^{2Ht}}\right)=-3H\phi_t^2-H\frac{|\nabla\phi|^2}{e^{2Ht}}, \tag{4.1}$$

where the divergence is taken in the spatial variables. We denote

$$K(t_0,x_0)=\{(t,x)\in\mathbb{R}^4\mid t\le t_0,\quad H|x-x_0|\le e^{-Ht}-e^{-Ht_0}\},$$

the backward light cone and

$$M(t_0,x_0)=\{(t,x)\in\mathbb{R}^4\mid t\le t_0,\quad H|x-x_0|= e^{-Ht}-e^{-Ht_0}\},$$

its lateral boundary. We want to integrate on

$$K_s^t(t_0,x_0)=\{(t,x)\in\mathbb{R}^4\mid H|x-x_0|\le e^{-Ht}-e^{-Ht_0}\}\cap[s,t]\times\mathbb{R}^3;$$

To do this, we note that the normal to the lateral boundary of $K_s^t(t_0,x_0)$ is

$$\hat{n}=\left(e^{-Ht},\frac{x-x_0}{|x-x_0|}\right)$$

If we define

$$G(t,x)=\left(\frac{\phi_t^2}{2}+\frac{|\nabla\phi|^2}{2e^{2Ht}}+F(\phi),-\frac{\phi_t\nabla\phi}{e^{2Ht}}\right)$$

we see that

$$\begin{aligned}\hat{n}\cdot G(t,x) &= e^{-Ht}\left(\frac{\phi_t^2}{2}+\frac{|\nabla\phi|^2}{2e^{2Ht}}+F(\phi)\right)-\frac{\phi_t\nabla\phi}{e^{2Ht}}\cdot\frac{x-x_0}{|x-x_0|}\\ &= e^{-Ht}\left[F(\phi)+\frac{1}{2}\left(\phi_t^2+\frac{|\nabla\phi|^2}{e^{2Ht}}-2\frac{\phi_t\nabla\phi}{e^{Ht}}\cdot\frac{x-x_0}{|x-x_0|}\right)\right]\\ &= e^{-Ht}\left[F(\phi)+\frac{\left|\phi_t\frac{x-x_0}{|x-x_0|}-e^{-Ht}\nabla\phi\right|^2}{2}\right]\end{aligned}$$

Noticing $|\hat{n}|=\sqrt{1+e^{-2Ht}}$ we have that, integrating (4.1) on $K_s^t(t_0,x_0)$ and applying the divergence theorem

$$\int_{B(x_0,R(t))}e(t)dx+\int_{M_s^t}\frac{1}{\sqrt{1+e^{2Ht}}}\left(\frac{\left|\phi_t\frac{x-x_0}{|x-x_0|}-e^{-Ht}\nabla\phi\right|^2}{2}+F(\phi)\right)dS=$$
$$\int_{B(x_0,R(s))}e(s)dx-\int_{K_s^t}3H\phi_t^2+H\frac{|\nabla\phi|^2}{e^{2Ht}},$$

where $R(t) = \frac{e^{-Ht} - e^{-Ht_0}}{H}$, M_s^t is the lateral boundary of the truncated cone K_s^t and

$$e(t) = \left(\frac{\phi_t^2}{2} + \frac{|\nabla \phi|^2}{2e^{2Ht}} + F(\phi) \right)(t, x)$$

is the energy density. Since $F(x) \geq 0$, the last identity implies that if $\phi = 0$ on $B(x_0, R(s))$ then $\phi = 0$ on $B(x_0, R(t))$ for every $t \in [s, t_0]$, that is, ϕ have finite speed of propagation and then, if the initial data have compact support then ϕ has it too for every time where it is defined. From the well posedness theory we have in addition that the solution is smooth in the spatial variables for every time. Note also that in the limit when $H \to 0$ the backward light cone $K(t_0, x_0)$ coincide with the usual one for the wave equation, and we recover the classical finite speed of propagation property.

Now we must to show that

$$\sup_{t \in [0,T)} \|\phi(t)\|_\infty < \infty.$$

To do this we shall estimate the H^2 norm of ϕ. Notice that from the hypothesis on the initial data there exist some constant such that

$$\|(\phi, \phi_t)(0)\|_{H^2 \times H^1} \leq \frac{C_0 \varepsilon}{4}.$$

To estimate $\|\phi(t)\|_{H^2}$ suppose that

$$\sup_{t \in [0,T)} \|\phi(t)\|_{H^2} \leq C_0 \varepsilon$$

for the same constant as above. We will show that we can improve this estimate to obtain that

$$\sup_{t \in [0,T)} \|\phi(t)\|_{H^2} \leq \frac{C_0 \varepsilon}{2}.$$

For this we note that from (2.9) if $\varepsilon > 0$ is small enough we have that

$$\frac{\|\phi_t\|_{L^2}^2}{2} + \frac{\|\nabla \phi\|_{L^2}^2}{2e^{2Ht}} \leq \varepsilon^2 \left(\frac{\|h\|_{L^2}^2}{2} + \frac{\|\nabla g\|_{L^2}^2}{2e^{2Ht}} \right) + \varepsilon^4 \int g^4,$$

and in consequence

$$\sup_{t \in [0,T)} \|\phi(t)\|_{H^1} \leq (1 + T) \frac{C_0(\varepsilon + \varepsilon^2)}{4}.$$

To estimate the H^2 norm note that $u_i = \partial_{x_i} \phi$ satisfies the equation

$$\partial_t^2 u_i + 3H \partial_t u_i - \frac{\Delta u_i}{e^{2Ht}} + f'(\phi) u_i = 0,$$

and from Lemma 2.3 we have

$$\begin{aligned}\frac{\|u_{it}\|_{L^2}^2}{2}+\frac{\|\nabla u_i\|_{L^2}^2}{2e^{2Ht}} \leq & \varepsilon^2 e^{-Ht}\left(\frac{\|\partial_{x_i}h\|_{L^2}^2}{2}+\frac{\|\partial_{x_i}\nabla g\|_{L^2}^2}{2e^{2Ht}}\right)+\int_0^t e^{-H(t-s)}\|f'(\phi)u_i\|_{L^2}^2 ds \\ \leq & \varepsilon^2 e^{-Ht}\left(\frac{\|\partial_{x_i}h\|_{L^2}^2}{2}+\frac{\|\partial_{x_i}\nabla g\|_{L^2}^2}{2e^{2Ht}}\right)+\varepsilon^4\int_0^t e^{-H(t-s)}\|\phi^2 u_i\|_{L^2}^2 ds,\end{aligned}$$

and in the same way as before we obtain that

$$\sup_{t\in[0,T)}\|\partial_{x_i}\phi(t)\|_{H^1} \leq C(T)\frac{C_0(\varepsilon+\varepsilon^2)}{4}.$$

This implies, for $\varepsilon > 0$ small enough, that

$$\sup_{t\in[0,T)}\|\phi(t)\|_{H^2} \leq \frac{C_0\varepsilon}{2},$$

as we wanted. Note that applying the same argument as above, in addition to Lemma 2.3 we obtain that all the spatial derivatives of ϕ are bounded in time. Additionally, since $u = \phi_t$ satisfies

$$\partial_t^2 u + 3H\partial_t u - \frac{\Delta u}{e^{2Ht}} + 2H\frac{\Delta\phi}{e^{2Ht}} + f'(\phi)u = 0$$

we can apply the same idea to prove that the H^2 norm of ϕ_t is uniformly bounded in time, because ϕ is uniformly bounded in H^2, and then $\Delta\phi \in L^1([0,T];L^2)$. This allow us to extend ϕ to a smooth function on $[0,T]$, and from the finite speed of propagation property we have that $\phi(T,\cdot)$ is a smooth function with compact support. In consequence we can extend the solution globally in time.

4.2.2. Decay

We first prove (1.17). For $t_0 > 0$ $\sigma < -1$ and $b = 1+\sigma$ in Corollary 3.2, which yields

$$\mathcal{J}(\phi)(t) = \int_0^\infty r^2(1+\tanh(r+\sigma t+(1+\sigma)t_0))\left(\frac{\phi_t^2}{2}+\frac{\phi_r^2}{2e^{2Ht}}+F(\phi)\right),$$

and

$$\frac{d\mathcal{J}(\phi)}{dt}(t) \leq (1+\sigma)\int_0^\infty r^2\operatorname{sech}^2(r+\sigma t+(1+\sigma)t_0)\left(\frac{\phi_t^2}{2}+\frac{\phi_r^2}{2e^{2Ht}}+F(\phi)\right).$$

Since $\sigma < -1$ we have that $\mathcal{J}(\phi)$ is decreasing on $[2,t_0]$. In particular

$$\begin{aligned}&\int_0^\infty r^2(1+\tanh(r+(1+2\sigma)t_0))\left(\frac{\phi_t^2}{2}+\frac{\phi_r^2}{2e^{2Ht}}+F(\phi)\right)(t_0,r) \\ &\leq \int_0^\infty r^2(1+\tanh(r+2\sigma+(1+\sigma)t_0))\left(\frac{\phi_t^2}{2}+\frac{\phi_r^2}{2e^{2Ht}}+F(\phi)\right)(2,r).\end{aligned}$$

Since $1+\sigma < 0$ we have that

$$\lim_{t_0\to\infty}(1+\tanh(r+2\sigma+(1+\sigma)t_0)) = 0.$$

By dominated convergence theorem we have that the right hand side converges to 0 as $t_0 \to \infty$, and consequently we get

$$\lim_{t\to\infty} \int_0^\infty r^2(1+\tanh(r+(1+2\sigma)t)) \left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2e^{2Ht}} + F(\phi) \right)(t,r) = 0.$$

To conclude it is enough to note that if $r > -(1+2\sigma)t$ then

$$1+\tanh(r+(1+2\sigma)t) \geq 1,$$

and then

$$\begin{aligned}\int_{R(t)} \left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2e^{2Ht}} + F(\phi) \right) &\leq \int_{-(1+2\sigma)t}^\infty r^2(1+\tanh(r+(1+2\sigma)t)) \left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2e^{2Ht}} + F(\phi) \right) \\ &\leq \int_0^\infty r^2(1+\tanh(r+(1+2\sigma)t)) \left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2e^{2Ht}} + F(\phi) \right).\end{aligned}$$

Since we have proved that the right hand side converges to 0 we got the desired decay (1.17).

Now we prove (1.18). Let $R > 0$ fixed. Lemma 3.2 with $\varphi(t,r) = \varphi(r) \geq 0$ yields for some $C > 0$

$$\begin{aligned}\frac{d\mathcal{J}(\phi)}{dt} =& -H\int_0^\infty r^2\varphi(r)\left(3\phi_t^2 + \frac{\phi_r^2}{e^{2Ht}}\right) - \int_0^\infty r^2\varphi'(r)\frac{\phi_t\phi_r}{e^{2Ht}} \\ \leq& -3H\int_0^\infty r^2\phi_t^2\left(\varphi(r) - Ce^{-Ht}\varphi'(r)\right) - H\int_0^\infty r^2\frac{\phi_r^2}{e^{2Ht}}\left(\varphi(r) - \varphi'(r)\right).\end{aligned}$$

Choosing $\varphi_0(r), \varphi(r)$ satisfying

$$\varphi_0(r) = 1, \quad r \leq 1, \quad \varphi_0' \leq 0, \quad \varphi_0(r) = e^{-r}, \quad r \geq 2, \quad \varphi(r) = \varphi_0(r/R),$$

If R and t are large,

$$\frac{d\mathcal{J}(\phi)}{dt} \leq -\frac{3}{2}H\int_0^\infty r^2\varphi(r)\phi_t^2 - \frac{1}{2}H\int_0^\infty r^2\varphi(r)\frac{\phi_r^2}{e^{2Ht}}.$$

Consequently, for some $t_0 > 0$ large,

$$H\int_{t_0}^\infty \int_0^\infty r^2\varphi(r)\left(\phi_t^2 + \frac{\phi_r^2}{e^{2Ht}}\right)dt < +\infty.$$

Again, we have now

$$\frac{d\mathcal{J}(\phi)}{dt} \lesssim H\int_0^\infty r^2\varphi(r)\left(3\phi_t^2 + \frac{\phi_r^2}{e^{2Ht}}\right),$$

following the same ideas as in previous proofs, we get the desired result.

Chapter 5

Applications

In this chapter we shall see how the theorems proved in this work can be used to study space-time dynamics of the models introduced in Chapter 1. Recall the notation introduced in Section 1.3

$$\begin{aligned} F_{1,n}(s) &= (1-e^{-s})^{2n}, \quad n = 1, 2, 3, \ldots \\ F_{2,1}(s) &= \tanh^2(s), \quad s \in \mathbb{R}, \\ F_{6,q}(s) &= \frac{1}{q}[(1+s^2)^{q/2} - 1], \quad q \in [-1, 1], \quad q \neq 0. \end{aligned}$$

To simplify the exposition, we are going to prove the following lemma.

Lemma 5.1 *There exists a constant $C > 0$ such that*

$$\begin{aligned} sf_{1,n}(s) \leq Cs^4, \quad s \in [-1, 1], \quad n \geq 2, \\ 2F_{2,1}(s) - sf_{2,1}(s) \geq 0, \quad s \in \mathbb{R}, \\ 2F_{6,q}(s) - sf_{6,q}(s) \geq 0, \quad q \in [-1, 1], s \in \mathbb{R}. \end{aligned}$$

Proof. For $F_{2,1}$ by the symmetry of the function is enough to prove the inequality for $s \geq 0$. We have

$$\begin{aligned} 2F_{2,1}(s) - sf_{2,1}(s) &= 2\tanh^2(s) - 2s\tanh(s)\,\mathrm{sech}^2(s) \\ &= 2\tanh(s)(\tanh(s) - s\,\mathrm{sech}^2(s)), \end{aligned}$$

and since $|\tanh(s)| \leq |s|$ we have the inequality. For $F_{1,n}$ notice that for $s \in [-1, 1]$

$$\begin{aligned} sf_{1,n}(s) &= 2ns(1-e^{-s})^{2n-1}e^{-s} \\ &\leq \left(\sup_{y\in[-1,1]} 2n(1-e^{-y})^{2n-4}\right) s(1-e^{-s})^3, \end{aligned}$$

and therefore, from the classic inequality $1 + s \leq e^s$ we see that for $s \in [0, 1]$ we have

$$sf_{1,n}(s) \leq \left(\sup_{y\in[-1,1]} 2n(1-e^{-y})^{2n-4}\right) s^4.$$

For $s \in [-1, 0]$ we define

$$g_a(s) = s(1-e^{-s})^3 - as^4,$$

and we prove that there exists some $a > 0$ such that this $g_a(s) \leq 0$ on $[-1, 0]$. Indeeed, a straightforward calculation gives us that

$$g_a(0) = g'_a(0) = g''_a(0) = g'''_a(0) = 0,$$

and

$$g_a^{(iv)}(s) = h(s) - 24a,$$

for some function $h(s)$. We can take $a > 0$ as large as we need to guarantee that $g^{(iv)}(s) \leq 0$ on $[-1, 0]$, and this implies that $g_a(s) \leq 0$ on $[-1, 0]$, giving the desired inequality.

For $F_{6,q}$ we have that

$$g(s) = 2F_{6,q}(s) - sf_{6,q}(s) = (1 + s^2)^{q/2-1}\left[\frac{2}{q} + s^2\left(\frac{2}{q} - 1\right)\right] - \frac{2}{q},$$

and we notice that $g(0) = 0$. By the symmetry of the function it is sufficient to prove that $g'(s) \geq 0$ for $s \geq 0$

$$\begin{aligned} g'(s) =& f_{6,q}(s) - sf'_{6,q}(s) \\ =& (2 - q)\, s^3(1 + s^2)^{q/2-2}, \end{aligned}$$

and since $q \in [-1, 1]$ we conclude. □

We also recall the following classical result for elliptic equation, [28]

Theorem 5.1 (Pohozaev's identity) *Let $f \in C^1(\mathbb{R})$ be such that $f(0) = 0$ and let $u \in H^1(\mathbb{R}^n)$ be a finite energy solution of*

$$-\Delta u = f(u)$$

Then we have

$$\frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla u|^2 = n\int_{\mathbb{R}^n} F(u)$$

Note that this theorem assures us that there are not finite energy stationary solutions for equation (1.11) when the potential is positive.

5.1. The E-model and T-model

For the T-models we have that, since

$$f_{2,n}(\phi) = 2n\tanh^{2n-1}(\phi)\,\mathrm{sech}^2(\phi)$$

is globally Lipschitz for any initial data in $H^1 \times L^2$ there exist a global solution. When $n = 1$ Theorem 1.1 gives us the local decay of such solution. For $n \geq 2$ we have the following.

Lemma 5.2 *We have $sf_{2,n}(s) \leq 2ns^4$, $s \in \mathbb{R}$.*

Proof. We have

$$\begin{aligned} sf_{2,n}(s) &= 2ns\tanh^{2n-1}(s)\,\mathrm{sech}^2(s) \\ &= 2ns\tanh^3(s)\tanh^{2n-4}(s)\,\mathrm{sech}^2(s) \leq 2ns^4. \end{aligned}$$

$\square$

This allows us to apply Theorem 1.2 to conclude the same decay as long as the solution satisfies (1.15). The case of E-models however is not so easy. For $n \geq 2$ Lemma 5.1 allows us to apply Theorem 1.2, but for $n = 1$ we cannot apply neither Theorem 1.1 nor 1.2 for $F_{1,1}$ because in this case we have no longer the required sign condition and $f'_{1,1}$ is not well behaved near the origin.

However, note that for $n \geq 2$ we can apply Theorem 2.8 to guarantee the existence of global solutions for the E-model given smooth and small enough data. Both T-model and E-model (for $n \geq 2$) satisfy the hypothesis of Theorem 1.3, and consequently these fields decay when $H > 0$.

5.2. The non-minimal coupling and Hilltop models

For the non-minimal coupling model

$$F(\phi) = \lambda^2\phi^4 + \beta^2\phi^2$$

we give a slightly modified argument to the presented in Theorem 1.2 to conclude the decay of small enough solutions. Since $2F(\phi) - \phi f(\phi) = -2\lambda^2\phi^4$ we have in (3.11) that

$$\frac{d\mathcal{I}(\phi)}{dt} = \int_0^\infty r^2\left(\frac{1}{(1+r)^2}\phi_r^2 + \frac{r+4}{2r(1+r)^4}\phi^2\right) - 2\lambda^2\frac{r(r+2)}{2(1+r)^2}\phi^4$$

Supposing that $\sup_{t\geq 0}\|\phi(t)\|_{H^1\cap L^\infty} < \varepsilon$ and using that

$$\phi(r)^2 \leq \frac{(1+C)\varepsilon^2}{(1+r)^2},$$

we have

$$\begin{aligned}
\frac{d\mathcal{I}(\phi)}{dt} &\geq \int_0^\infty r^2\left(\frac{1}{(1+r)^2}\phi_r^2 + \frac{r+4}{2r(1+r)^4}\phi^2\right) - (1+C)\lambda^2\varepsilon^2\frac{r(r+2)}{2(1+r)^4}\phi^2 \\
&\geq \int_0^\infty \left(\frac{r^2}{(1+r)^2}\phi_r^2 + \frac{r(r+2)}{2(1+r)^4}\phi^2(1-(1+C)\lambda^2\varepsilon^2)\right) \\
&\geq \|\phi\|_{H^1_w}^2,
\end{aligned}$$

provided that ε is small enough. Thus, we can conclude as in Theorem 1.2. Notice that the hypothesis (1.15) is also needed to prove that

$$\int_0^\infty \|\phi_t\|_{L^2_w} dt < \infty$$

as in Proposition 3.2. For the Hilltop model with $n=2$ we have a simpler situation. We have that $2F(\phi)-\phi f(\phi)=2\phi^4$ and therefore

$$
\begin{aligned}
\frac{d\mathcal{I}(\phi)}{dt} &= \int_0^\infty r^2\left(\frac{1}{(1+r)^2}\phi_r^2+\frac{r+4}{2r(1+r)^4}\phi^2\right)+2\frac{r(r+2)}{2(1+r)^2}\phi^4\\
&\geq \int_0^\infty r^2\left(\frac{1}{(1+r)^2}\phi_r^2+\frac{r+4}{2r(1+r)^4}\phi^2\right)\\
&\geq \|\phi\|_{H^1_w}.
\end{aligned}
$$

Supposing (1.15) we can conclude as in Theorem 1.2.

5.3. The Natural Inflation and Axion potential

Assume small data ϕ. Observe that in this case $F_{3,\pm}\geq 0$ in (1.5), and

$$2F_{3,-}(\phi)-\phi f_{3,-}(\phi)=\frac{1}{12}\phi^4+O(\phi^6).$$

In the other case, we must take out the nonzero value of the potential at infinity to get finite energy, considering $F_{3,+}(\phi)=\cos\phi-1$. We get

$$2F_{3,+}(\phi)-\phi f_{3,+}(\phi)=-\frac{1}{12}\phi^4+O(\phi^6).$$

In the former case, natural inflation, by virtue of (3.11) we get the desired result applying the same ideas as in the previous subsection. The case of Axion potential remains an interesting open problem.

5.4. The D-brane model

For the D-brane model $F_{4,n}$ in (1.6), since the potential and his derivative are singular in the origin we look for solutions of the form $\phi=1+v$, where we suppose that $v(t)\in H^2$ and

$$\|v(t)\|_{H^2}<1,$$

for all times. Notice that this bound ensures control of the L^∞ norm of v. In this case we have that the function v satisfies (see (1.7))

$$\partial_t^2 v-\partial_r^2 v-\frac{2}{r}\partial_r v+2n\left(\frac{1}{(1+v)^{2n+1}}-1\right)=0.$$

We conclude that

$$2\tilde{F}_{4,n}(v)-v\tilde{f}_{4,n}(v)=\begin{cases}\frac{-2v^3(v+2)}{(1+v)^3} & n=1\\ -\frac{2v^3(10+15v+9v^2+2v^3)}{(1+v)^5} & n=2.\end{cases}$$

Consequently, Theorems 1.1 and 1.2 remain inconclusive in this setting.

5.5. The Axion-Monodromy and Logarithmic potential

From Lemma 5.1 we have that the Axion-Monodromy potential satisfies $2F_{6,q}(s) - sf_{6,q}(s) \geq 0$. For the Logarithmic potential we have that from the well known inequality

$$1 - \frac{1}{x} \leq \log(x)$$

that

$$2F_7(s) - sf_7(s) \geq 0$$

Since in both cases the derivative of the potential is clearly Lipschitz continuous, we can apply directly Theorem 1.1 and 1.3 to conclude that global solutions always exists and decay locally in the energy space.

Chapter 6

Conclusions

We finish this work we some brief remarks:

- First, we see in this work that local decay is a common denominator between cosmological models, mainly due to their defocusing character given by the positivity of the potentials involved in the dynamics. When $H = 0$ it is not expected that the solutions decay globally. This because of the conservation of energy of the solutions; under reasonable hypothesis decay of the solutions would imply that the energy must be 0, and then the solution is trivial. On the other hand, when $H > 0$ global decay is an expected feature of these models, because the term $\partial_t \phi$ acts as a friction for the dynamics as we can see in equation (2.9). However, to be able to obtain a global decay of the solutions we must get a finer control on the nonlinearity in terms of the field.

- The question of when there exist global solutions for the field equations considered in this work, despite of being very natural, it is so far from being completely understood. For the case $H = 0$ Theorem 2.8 gives a satisfactory answer. When $H > 0$ we have a similar result; Theorem 1.3, which gives us global existence for small enough smooth initial data, but given the good properties exhibited by these models (given again mainly by its defocusing character) we would expect to be able to lift the regularity assumption on the initial data and suppose, for example, data in $H^1 \times L^2$. As far as we know, this is an unsolved problem.

- General relativity from a mathematical point of view is a very active research topic today, and there are many open questions in the subject. This work only consider the dynamics of fields on a de Sitter space-time, but it would be interesting to consider another type of space-times, as well as consider the full dynamic of the inflaton, that is, allowing that the field modify the metric of the space-time along its evolution.

Bibliography

[1] Adams, Robert A.; Fournier, John J. F. *Sobolev Spaces*, London: Academic Press, 2003.

[2] M. A. Alejo, and C. Maulén, *Decay for Skyrme wave maps*, Lett. Math. Phys. 112 (2022), no. 5, Paper No. 90, 33 pp.

[3] M. A. Amin, R. Easther, H. Finkel, R. Flauger, and M. P. Hertzberg, *Oscillons after Inflation*, Phys. Rev. Lett. 108, 241302 – Published 14 June 2012.

[4] E. Braaten, and H. Zhang, *Axion stars*, preprint arXiv https://arxiv.org/abs/1810.11473 (2018).

[5] Brezis, Haim. *Function Analysis, Sobolev Spaces and Partial Differential Equations*, Springer-Verlag, 2010, 10.1007/978-0-387-70914-7.

[6] F. Chadha-Day; J. Ellis; D. J. E. Marsh, *Axion dark matter: What is it and why now?*. Science Advances. 8 (8): eabj3618 (23 February 2022), https://arxiv.org/abs/2105.01406.

[7] L. C. Evans, *Partial Differential Equations*, Second Edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010. xxii+749 pp. ISBN: 978-0-8218-4974-3.

[8] E. Di Valentino, O. Mena, S. Pan, L. Visinelli, W. Yang, A. Melchiorri, D. F. Mota, A. G. Riess, J. Silk, *In the Realm of the Hubble tension–a Review of Solutions*, preprint 2021 https://arxiv.org/pdf/2103.01183.pdf.

[9] A. H. Guth, *Inflationary universe: A possible solution to the horizon and flatness problems*, Phys. Rev. D, vol. 23, pp. 347–356, (1981).

[10] E. Hubble, *A relation between distance and radial velocity among extra-galactic nebulae*, PNAS vol. 15 pp. 168–173, 1929.

[11] Hong-Yi Zhang et al JCAP07 (2020) 055.

[12] M. Kawasaki, N. Kitajima, and T. T. Yanagida, *Primordial black hole formation from an axionlike curvaton model*, Phys. Rev. D, vol. 87, no. 6, p. 063519, 2013.

[13] M. Kowalczyk, Y. Martel, and C. Muñoz, *Kink dynamics in the u^4 model: asymptotic stability for odd perturbations in the energy space*, J. Amer. Math. Soc. 30 (2017), 769–798.

[14] M. Kowalczyk, Y. Martel, and C. Muñoz, *Nonexistence of small, odd breathers for a class of nonlinear wave equations*, Letters in Mathematical Physics, (2017) Vol. 107, Issue 5, 921–931.

[15] M. Kowalczyk, Y. Martel, and C. Muñoz, *On asymptotic stability of nonlinear waves*, Laurent Schwartz seminar notes (2017), see url at http://slsedp.cedram.org/slsedp-bin/fitem?id=SLSEDP_2016-2017______A18_0.

[16] F. Linares, and G. Ponce, *Introduction to nonlinear dispersive equations*. Second edition.

Universitext. Springer, New York, 2015. xiv+301 pp.

[17] D. H. Lyth, *Axions and Inflation: Vaccuum Fluctuations*, Phys. Rev. D. Vol. 45 no. 10, May 15 1992, p. 3394, https://doi.org/10.1103/PhysRevD.45.3394.

[18] G. Palma, S. Sypsas, and C. Zenteno, *Seeding Primordial Black Holes in Multifield Inflation*, PRL 125, 121301 (2020).

[19] A. A. Penzias and R. W. Wilson, *A Measurement of excess antenna temperature at 4080-Mc/s,* Astrophys. J., vol. 142, pp. 419–421, 1965.

[20] Planck Collaboration, *Planck 2018 results. X. Constraints on inflation*, A&A Volume 641, September 2020.

[21] Ruzhansky, Michael; Suragan, Durvudkhan. *Hardy Inequalities on Homogeneous Groups: 100 Years of Hardy Inequalities.*, Birkhäuser Basel, 2019, ISBN 978-3-030-02894-7.

[22] Sachs, R. K. & Wu, Hung-hsi. *General relativity for mathematicians*, New York : Springer-Verlag, 1977

[23] N. Schöneberg, G. F. Abellán, A. Pérez Sánchez, S. J. Witte, V. Poulin, J. Lesgourgues, *The $H0$ Olympics: A fair ranking of proposed models*, Physics Reports Volume 984, 26 October 2022, Pages 1-55 https://arxiv.org/abs/2107.10291.

[24] C. D. Sogge, *Lectures on Non-Linear Wave Equations*, Second edition. International Press, Boston, MA, 2008. x+205 pp. ISBN: 978-1-57146-173-5.

[25] A. A. Starobinsky, *A New Type of Isotropic Cosmological Models Without Singularity*, Phys. Lett. B 91, 99 (1980).

[26] P. Svrček, and E. Witten, *Axions in string theory.* J. High Energy Phys. 2006, no. 6, 051, 52 pp.

[27] K. Tsutaya, Y. Wakasugi, *Blow up of solutions of semilinear wave equations related to nonlinear waves in de Sitter spacetime*, Partial Differential Equations and Applications 3, Article number: 6 (2022).

[28] Mi. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996. x+162 pp. ISBN: 0-8176-3913-6.

[29] H. Zhang, M. A. Amin, E. J. Copeland, P. M. Saffin and K. D. Lozanov, *Classical decay rates of oscillons* Journal of Cosmology and Astroparticle Physics (2020)