Equivalence of Group Actions on Riemann Surfaces

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para la mama y mi tata.....

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Contents

symbol	means
$\|G\|$	Order of the group G.
$[G:H]$	Index of the subgroup H in G.
G_p	Stabilizer subgroup of p.
$G(p)$	Orbit of p under action of G.
$H \rtimes K$	Semidirect product of H by K ($H \trianglelefteq H \rtimes K$)
$\widehat{\mathbb{C}}$	The Riemann sphere, this is $\mathbb{C} \cup \{\infty\}$
Δ	Unit disk, this is $\{z \in \mathbb{C} : \|z\| < 1\}$
$\partial\Delta$	$\{z \in \mathbb{C} : \|z\| = 1\}$
S/G	Quotient space given by the actions of G on S.
$\Omega^{1,0}(S)$	The set of holomorphic forms on S.
$\operatorname{Tr} M$	The Trace of a square matrix M.
$GL(n,\mathbb{C})$	The set of $n \times n$ invertible matrices with entries from $\mathbb{C}$.

Abstract

This work was intended as a contribution to the problem of equivalent group actions on Riemann surfaces.
The main result of this thesis is the constructions of 1–parameter families of Riemann surfaces admitting automorphism groups with two cyclic subgroups H_1 and H_2 that are conjugate in the group of orientation–preserving homeomorphisms of the corresponding Riemann surfaces, but not conjugate in the group of conformal automorphisms.
This property is interesting because it implies that the subvariety $\mathcal{M}_g(H_1)$ of the moduli space $\mathcal{M}_g$ consisting of the points representing the Riemann surfaces of genus g admitting a group of automorphisms topologically conjugate to H_1 (equivalently to H_2) is not a normal subvariety.
This construction is done for subgroups of order 2^n for each natural number $n \geq 3$ and, in particular, for arbitrarily large genus.
The main tool is the theory of Fuchsian groups. A key result of this thesis is the generalization of a theorem by Harvey that establishes a relation between cyclic coverings of order n of the Riemann sphere and epimorphisms of certaing Fuchsian groups to the cyclic groups of order n, by means of the rotation angles for the automorphism defining the cyclic group.

Abstract

Este trabajo nace como una contribución al problema de equivalencia de aciones de grupos en superficies de Riemann.
El principal resultado de la tesis es la construción de familias 1–paramétricas de superficies de Riemann que admiten grupos de automorfismos con dos subgrupos cíclicos H_1 y H_2 que son conjugados en el grupo de homeomorfismos que preservan orientación de las correspondientes superficies, pero no son conjugados en el grupo de automorfismos conformes.
Esta propiedad es interesante porque implica que la subvariedad $\mathcal{M}_g(H_1)$ de el espacio de moduli $\mathcal{M}_g$ cuyos elementos representan superficies de Riemann de género g que admiten un grupo de automorfismos topológicamente conjugado a H_1 (equivalentemente a H_2) no es un subvariedad normal.
Esta construción es hecha para subgrupos de orden 2^n para cada número natural $n \geq 3$ y, en particular, para géneros arbitrariamente grandes.
La principal herramienta usada es la teoria de grupos Fuchsianos. Un resultado clave de esta tesis es la generalización de un teorema de Harvey que establece una relación entre los cubrimientos cíclicos de la esfera de Riemann de orden n y los epimorfismos de ciertos grupos Fuchsianos a grupos cíclicos de orden n, por medio de los ángulos de rotación de los automorfismos que definen el grupo cíclico.

Introduction

When we consider a group G and say that G *acts on a Riemann surface* S, we are saying that there exists a group monomorphism from G to $\text{Aut}(S)$, where $\text{Aut}(S)$ is the group consisting of the self–maps of S (automorphism or bi–holomorphic map) which preserve the complex structure.

In the study of Riemann surfaces, the classification of actions on compact Riemann surfaces is an interesting problem. The classification of finite group actions, up to topological equivalence, on a surface of low genus is studied by A. Broughton in [3]. In the case where the group is cyclic, a relationship between the local structure for the automorphisms with fixed points and the epimorphism associated to the action is given by W. Harvey in [10]. Continuing with the cyclic case, in particular for a group of prime order, a relationship between two topologically equivalent actions for the generating vectors is given by J. Gilman in [9]. In [8], G. González–Diez and R. Hidalgo give an example of two actions of $\mathbb{Z}/8\mathbb{Z}$ on a family of compact Riemann surfaces of genus 9 that are directly topologically, but not conformally, equivalent, except for finitely many cases.

Studying the classification of actions contributes to the understanding of the properties of the moduli space $\mathcal{M}_g$.
For a compact Riemann surface S_0 of genus g, consider the subgroup $H_0 \leq \text{Aut}(S_0)$, the set

$$X(S_0, H_0) = \left\{(S, H) : \exists t \in \text{ Homeo }^+(S_0, S)\, , tH_0t^{-1} = H\right\}$$

and the equivalence relation: $(S_1, H_1) \sim (S_2, H_2)$ if and only if there is $\phi \in \text{Isom}(S_1, S_2)$ so that $\phi H_1 \phi^{-1} = H_2$.
We denote by $\widetilde{\mathcal{M}}_g(H_0)$ the quotient space defined by the above relation. This turns out to be a normal space.
Consider $\mathcal{M}_g$ the moduli space associated to S_0, that is, a model of moduli space of genus g.
Let $\mathcal{M}_g(H_0) = \{[S] \in \mathcal{M}_g : \exists t \in \text{ Homeo }^+(S_0, S)\, , tH_0t^{-1} < \text{Aut}(S)\}$.

The forgetful map is defined by

$$\begin{array}{rcl} p:\widetilde{\mathcal{M}}_g(H_0) & \longrightarrow & \mathcal{M}_g(H_0) \\ [(S,H)] & \rightsquigarrow & [S] \end{array}$$

As is well known, $\widetilde{\mathcal{M}}_g(H_0)$ is the normalization of $\mathcal{M}_g(H_0)$. Moreover, p is not bijective if only if there exists a compact Riemann surface S of genus g admitting two groups of automorphisms H_1 and H_2 which are directly topologically, but not conformally, conjugate to H_0. For further details, see [7].

The Thesis is organized as follows:

Chapter 1 contains an overview of definitions and relevant results about automorphisms of Riemann surfaces and Fuchsian groups.

Chapter 2 contains some our contribution to the problem of the classification of actions. For cyclic groups, Theorem 13 gives a condition on the generating vectors under which two actions are directly topologically equivalent. Also, we generalize a result due to Harvey [10, Theorem 7].

In Chapter 3, inspired by the paper of G. González–Diez and R. Hidalgo [8], we produce for each $n \in \mathbb{N}$ the families $\mathfrak{S}_1$ and $\mathfrak{S}_2$. By definition, $\mathfrak{S}_i$ with $i = 1, 2$, consists of the Riemann surfaces of genus $3(2^n - 1)$ defined by

$$f_{a,\lambda}(x, y) = y^{2^n} - x^a \left(x^2 - 1\right)^a \left(x^2 - \lambda^2\right) \left(x^2 - \lambda^{-2}\right)$$

When $i = 1$ (resp. $i = 2$), the automorphism group for the elements of $\mathfrak{S}_1$ (resp. $\mathfrak{S}_2$) is $\mathbb{Z}/2^{n+1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (resp. $\mathbb{Z}/2^{n+1}\mathbb{Z} \rtimes_h \mathbb{Z}/2\mathbb{Z}$). In both cases, there exist two cyclic subgroups which define directly topologically, but not conformally, equivalent actions.

Chapter 1

Preliminary Results

In this chapter we recall several definitions and results. They may be found in [17], [12], [16] and [6].

1.1 Little Group Method

In this section we give a Theorem for computing the irreducible representation over $\mathbb{C}$ of a finite group G, such that

$$G = K \rtimes H$$

where K is a abelian group.
This method is know as *Little Group Method.* For more detail see [21].

Proposition 1. *Let $G = K \rtimes H$ be a semidirect product of finite groups. Assume that K is an abelian group. Then the irreducible representations of G, $\{\beta_{i,j}\}_{i,j}$, are given by the induced representations of*

$$\alpha_{i,j} = \overline{\rho}_j \otimes \overline{\sigma}_{i,j}$$

where $\{\rho_j\}_j$ is the irreducible representations set of K and the $\sigma_{i,j}$ are determined by the ρ_j.

We will see the definitions for the representations ρ_j and $\sigma_{i,j}$.
Consider $\mathcal{R}_K$ the set of all the irreducible representations for K.
$\mathcal{R}_K$ is a finite set, in fact $|\mathcal{R}_K| = |K|$. We will use ρ_j to denote the elements in $\mathcal{R}_K$.

Consider the action of G on $\mathcal{R}_K$, given by

$$g \bullet \rho_j = \rho_j^g : \qquad \rho_j^g(k) = \rho_j(gkg^{-1})$$

where $g \in G, \rho_j \in \mathcal{R}_K$.

We denote by $\mathcal{R}_K^H$ the set of the representatives of the orbits under the action of H.

For each $\rho_j \in \mathcal{R}_K^H$ we consider

$$H_j = \{h \in H : h \bullet \rho_j = \rho_j\}$$
$$G_j = \{g \in G : g \bullet \rho_j = \rho_j\}$$

Now we have $G_j = K \rtimes H_j$ and thus we may extend ρ_j to G_j. The extension is given by

$$\overline{\rho}_j(g) = \left\{ \begin{array}{ll} \rho_j(1) & , g \in H_j \\ \rho_j(g) & , g \in K \end{array} \right\}$$

Finally we consider the irreducible representations for H_j. We denote these representations by $\{\sigma_{i,j}\}_i$.

Now also we may extend $\sigma_{i,j}$ to G_j. The extension is given by

$$\overline{\sigma}_{i,j}(g) = \left\{ \begin{array}{ll} \sigma_{i,j}(1) & , g \in K \\ \sigma_{i,j}(g) & , g \in H_j \end{array} \right\}$$

1.2 Automorphisms of Riemann Surfaces

A *Riemann surface* is a connected Hausdorff topological space S endowed with a complex structure. We say that S has a *complex structure* if for all $p \in S$, there exist U_p an open neighborhood of p and a homeomorphism $\phi_p : U_p \longrightarrow V \subset \mathbb{C}$ (the pair (U_p, ϕ_p) is called *chart on* S) and if for any two charts (U_1, ϕ_1) and (U_2, ϕ_2) such that $U_1 \cap U_2 \neq \emptyset$, we have the diagram

$$\begin{array}{ccc} & U_1 \cap U_2 & \\ {}^{\phi_2}\swarrow & & \searrow^{\phi_1} \\ \phi_2(U_1 \cap U_2) & \xrightarrow[\phi_1 \circ \phi_2^{-1}]{} & \phi_1(U_1 \cap U_2) \end{array} \tag{1.1}$$

where $T = \phi_1 \circ \phi_2^{-1}$ is a holomorphic function.
When $p \in U$ is such that $\phi(p) = 0$ we say (U, ϕ) is a *chart centered at p*.

Since for any two charts, diagram (1.1) holds, we have that T is an invertible holomorphic function.

We may suppose that $\phi_p(p) = 0 = \phi_q(q)$, and then we have $T(0) = 0$, and T locally at zero is of the form

$$T(z) = a_1 z + a_2 z^2 + \cdots = \sum_{m \geq 1} a_m z^m ,$$

(its Taylor series form) where $a_1 \neq 0$ since T is invertible ($R = T^{-1}$) and thus

$$1 = (R \circ T)'(z) = R'(T(z))T'(z) \tag{1.2}$$

(chain rule).

Further it is clear that $T'(0) = a_1$ and thus by the equation (1.2) we have

$$R'(0) = \frac{1}{T'(0)} .$$

Definition 1. Let $\sigma : S_1 \longrightarrow S_2$ be a map between Riemann surfaces. We say σ is a *holomorphic map at* $p \in S_1$ if for each (U, ϕ) chart centered at p and each (V, ψ) chart centered at $q = \sigma(p) \in S_2$, we have the diagram

$$\begin{array}{ccc} U \cap \sigma^{-1}(V) \subset S_1 & \xrightarrow{\sigma} & S_2 \supset V \cap \sigma(U) \\ \phi \downarrow & & \downarrow \psi \\ \phi\left(U \cap \sigma^{-1}(V)\right) & \xrightarrow[\psi \circ \sigma \circ \phi^{-1}]{} & \psi\left(V \cap \sigma(U)\right) \end{array}$$

where $\psi \circ \sigma \circ \phi^{-1}$ is a holomorphic function.

We say σ is a *holomorphic map* if σ is a holomorphic map at p for each $p \in S_1$.

We may take the expansion series at 0, and we have

$$\psi \circ \sigma \circ \phi^{-1}(z) = \sum_{m \geq 1} c_m z^m .$$

Now we may consider

$$m_0 = \min\{m : c_m \neq 0\} .$$

The number m_0 is called the *multiplicity* at p .
When $m_0 > 1$ we say p is a *ramification point of* σ with multiplicity m_0.
The point $q = \sigma(p)$ is called *branch point of* σ with multiplicity m_0.

For a bijective holomorphic map, $\sigma : S_1 \longrightarrow S_2$, we say σ is a *biholomorphic map* or a *isomorphism* between Riemann surfaces.
When $S_1 = S_2$ we say σ is an *automorphism of* S .
From now on, $\mathrm{Isom}(S_1, S_2)$ (respectively $\mathrm{Aut}(S)$) denotes the isomorphisms set between S_1 and S_2 (respectively automorphism of S).

As σ^{-1} is a holomorphic map, we see that $\psi \circ \sigma \circ \phi^{-1}$ is an invertible holomorphic function, then $c_1 \neq 0$, where

$$\psi \circ \sigma \circ \phi^{-1}(z) = \sum_{m \geq 1} c_m z^m ,$$

then the automorphisms of Riemann surfaces do not have ramification points, in fact since for each $p \in S$ we have $c_1 \neq 0$, the multiplicity of p is 1.

Consider S a compact Riemann surface of genus $g \geq 2$.
By the Riemann Hurwitz's theorem [17, pag. 82] we have

$$|\,\mathrm{Aut}(S)| \leq 84(g-1) .$$

Now for each $p \in S$, consider the subgroup of $\mathrm{Aut}(S)$ given by

$$\mathrm{Aut}(S)_p = \{\sigma \in \mathrm{Aut}(S) : \sigma(p) = p\} ,$$

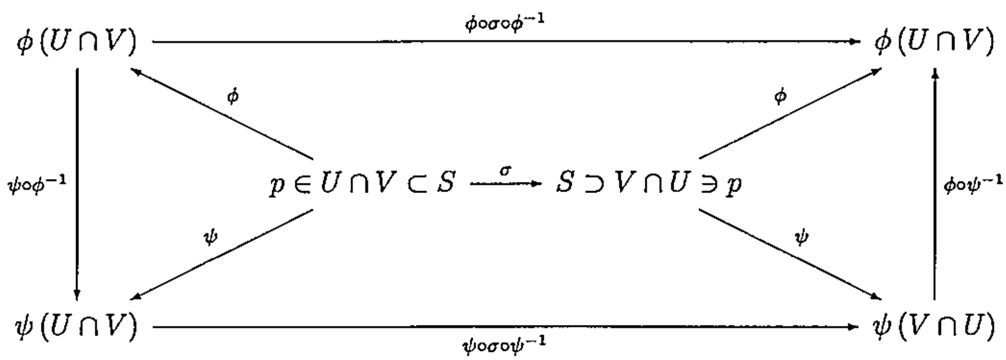
called *stabilizer subgroup*. Now let (U, ϕ) be a chart centered at p, and $\sigma \in \mathrm{Aut}(S)_p$ then we have

$$\phi \circ \sigma \circ \phi^{-1}(z) = \sum_{m \geq 1} c_m(\sigma) z^m$$

and we define

$$\begin{array}{rcl} \delta_p : \mathrm{Aut}(S)_p & \longrightarrow & \mathbb{C}^* \\ \sigma & \rightsquigarrow & c_1(\sigma) \end{array}$$

first we will see that this map is well defined, in fact if we take (V, ψ) another chart centered at p, then we have the following diagram:



explicitly we have

$$\begin{aligned}
\psi \circ \sigma \circ \psi^{-1}(w) &= \psi \circ \phi^{-1} \circ (\phi \circ \sigma \circ \phi^{-1}) \circ \phi \circ \psi^{-1}(w) \\
&= \psi \circ \phi^{-1} \circ (\phi \circ \sigma \circ \phi^{-1}) \left(\sum_{m \geq 1} b_m w^m \right) \\
&= \psi \circ \phi^{-1} \left(\sum_{k \geq 1} c_k(\sigma) \left(\sum_{m \geq 1} b_m w^m \right)^k \right) \\
&= \sum_{j \geq 1} a_j \left(\sum_{k \geq 1} c_k(\sigma) \left(\sum_{m \geq 1} b_m w^m \right)^k \right)^j \\
&= a_1 c_1(\sigma) b_1 z + \text{ higher terms} \\
&= c_1(\sigma) z + \text{ higher terms}
\end{aligned}$$

Theorem 1. *The map δ_p is a group monomorphism. Further,* $\mathrm{Aut}(S)_p$ *is a cyclic finite subgroup of* $\mathrm{Aut}(S)$.

Proof. Now we will see that δ_p is a group homomorphism . Let $\sigma, \tau \in$

$\mathrm{Aut}(S)_p$,

$$\begin{aligned}\phi\circ\sigma\tau\circ\phi^{-1}(z) &= \phi\circ\sigma\circ\phi^{-1}\circ\phi\circ\tau\circ\phi^{-1}(z)\\ &= \phi\circ\sigma\circ\phi^{-1}\left(\sum_{k\geq 1} c_k(\tau)z^k\right)\\ &= \sum_{m\geq 1} c_m(\sigma)\left(\sum_{k\geq 1} c_k(\tau)z^k\right)^m\\ &= c_1(\sigma)c_1(\tau)z + \text{ higher terms}\end{aligned}$$

Furthermore δ_p is a monomorphism, because if we let $\sigma \in \ker\delta_p$

$$\phi\circ\sigma\circ\phi^{-1}(z) = z + c_m(\sigma)z^m + \cdots$$

where $m \geq 2$ such that $c = c_m(\sigma) \neq 0$ then we have

$$\begin{aligned}\phi\circ\sigma\circ\phi^{-1}(z) &\equiv z + cz^m \mod z^{m+1}\\ \Rightarrow \phi\circ\sigma^2\circ\phi^{-1}(z) &\equiv z + 2cz^m \mod z^{m+1}\\ \text{by induction } &\vdots\\ \phi\circ\sigma^k\circ\phi^{-1}(z) &\equiv z + kcz^m \mod z^{m+1}\end{aligned}$$

but the group $\mathrm{Aut}(S)_p$ is finite and therefore σ must have finite order, and for some k, kc must be zero, forcing $c = 0$, therefore $\sigma(z) = z$.

If the order of $\sigma \in \mathrm{Aut}(S)_p$ is n, then $\delta_p(\sigma)$ is a nth root of unity, furthermore as δ_p is a monomorphism $\delta_p(\sigma)$ is a primitive nth root of unity. Therefore the group $\mathrm{Aut}(S)_p$ is isomorphic to a finite subgroup of S^1, thus $\mathrm{Aut}(S)_p$ is a cyclic group.

Alternative proof
Since $\mathrm{Aut}_p(S)$ is a finite group we may find a neighborhood U of p which is invariant under $\mathrm{Aut}_p(S)$. Without loss of generality we can assume U is contained inside a chart neighborhood and also that this simply–connected. Consider a chart $\phi : U \to V$. By the Riemann mapping theorem, we may assume V the unit disc and that $\phi(p) = 0$. The result it follows apply Schwarz's lemma.

□

1.3 Fuchsian Groups

Definition 2. Consider the unit disk

$$\Delta = \{z \in \mathbb{C} : | z | < 1\} .$$

We say a subgroup $\Gamma < \text{Aut}(\Delta)$ is a *Fuchsian Group* if Γ is a discrete subgroup of $\text{Aut}(\Delta)$.

Consider the group

$$PSL(2,\mathbb{C}) = \left\{ z \to \frac{az+b}{cz+d} : ad - bc \neq 0, \quad a,b,c,d \in \mathbb{C} \right\} .$$

We have that $PSL(2,\mathbb{C})$ is a topological space endowed with the quotient topology given by

$$\begin{array}{rcl} \mathbb{C}^4 - \{(a,b,c,d) : ad - bc = 0\} & \longrightarrow & PSL(2,\mathbb{C}) \\ (a,b,c,d) & \rightsquigarrow & \left(z \to \dfrac{az+b}{cz+d} \right) . \end{array}$$

Since the group $\text{Aut}(\Delta) < PSL(2,\mathbb{C})$, saying that Γ is a discrete group means that Γ is a discrete set with this topology. It can be noted that this topology is the pointwise convergent one. As each Möbius transformation is uniquely determined by its action at three different points, then it coincides with the local uniform convergence; and as the Riemann sphere is compact, then it coincides with the uniform convergence topology.

Let Υ be a subgroup of $PSL(2,\mathbb{C})$, and $z_0 \in \widehat{\mathbb{C}}$. We say that Υ acts *properly discontinuously* at z_0 provided that:

i) The stabilizer subgroup

$$\Upsilon_{z_0} = \{T \in \Upsilon : T(z_0) = z_0\}$$

is finite, and

ii) There exists a neighborhood U of z_0 such that

$$\begin{array}{rcll} T(U) & = & U & ,\forall T \in \Upsilon_{z_0} \quad \text{and} \\ U \cap T(U) & = & \emptyset & ,\forall T \in \Upsilon - \Upsilon_{z_0} \end{array}$$

We denote the *region of discontinuity* of Υ by $\Omega(\Upsilon)$. In other words

$$\Omega(\Upsilon) = \{z \in \widehat{\mathbb{C}} : \Upsilon \text{ acts properly discontinuously at } z\}.$$

The set

$$\Lambda(\Upsilon) = \widehat{\mathbb{C}} - \Omega(\Upsilon)$$

is called the *limit set* of Υ.

A Fuchsian group Γ (acting on the unit disk Δ) must satisfies that $\Delta \subset \Omega(\Gamma)$.

Theorem 2. *Let K be a Fuchsian group such that K is torsion-free and $\Lambda(K) = \partial(\Delta)$. Then the quotient space $S = \Delta/K$ is a Riemann surface so that $\Pi_1(S) \cong K$. If K is a finitely generated and has no parabolic elements, then S is a compact Riemann surface of genus $g \geq 2$.*

Proof. We may find the proof of this Theorem in [6]. □

Theorem 3. *Let K, K' be Fuchsian groups such that K, K' are torsion-free. Then $S = \Delta/K$ and $S' = \Delta/K'$ are bi-holomorphic Riemann surfaces if only if there exists $T \in \mathrm{Aut}(\Delta)$ such that*

$$K' = TKT^{-1}.$$

Remark 1. Let Γ be a Fuchsian group. Since Γ acts on Δ, we may consider the natural projection

$$\pi_\Gamma : \Delta \longrightarrow \Delta/\Gamma$$

$\mathcal{O} = \Delta/\Gamma$ has a *Riemann orbifold structure*, that is,

(i) an underlying Riemann surface structure $\mathcal{O}$ so that $\pi_\Gamma : \Delta \to \mathcal{O}$ is a holomorphic map;

(ii) a discrete collection of cone points (branch values of π); and

(iii) at each cone point p a cone order; this being the order of the stabilizer cyclic subgroup of any point q so that $\pi(q) = p$.

If Γ is finitely generated, without parabolic transformations and $\Lambda(\Gamma) = \partial(\Delta)$, then $\mathcal{O}$ is a compact Riemann surface of some genus γ and there are a finite set of cone points.

Definition 3. . If Γ is finitely generated, without parabolic transformations and $\Lambda(\Gamma) = \partial(\Delta)$, whose underlying Riemann surface has genus γ and the cone orders are $m_1, ..., m_r$, then we define its *signature* (for both, Γ and $\mathcal{O}$) as the tuple $(\gamma; m_1, ..., m_r)$.

The holomorphic map π_Γ is called a *branched covering* of type $(\gamma; m_1, ..., m_r)$.

When a Fuchsian group Γ has signature $(\gamma; m_1, m_2, \cdots, m_r)$, there is a presentation associated for the group Γ, this is, there exist $a_1, b_1, \cdots, a_\gamma, b_\gamma, x_1, \cdots, x_r \in \Gamma$ such that Γ has a presentation:

$$\Gamma = \left\langle a_1, b_1, \cdots, a_\gamma, b_\gamma, x_1, \cdots, x_r : x_1^{m_1} = \cdots = x_r^{m_r} = \prod_{i=1}^{\gamma} [a_i, b_i] \prod_{j=1}^{r} x_j = 1 \right\rangle$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$.
Further, we have that for each j, the subgroup generate for $< x_j >$ is a maximal finite cyclic subgroup. Moreover, this subgroup is the $\Gamma-$ stabilizer of a unique point in Δ, and each element of finite order in Γ is conjugate to a power of some x_j.
When x_j is conjugate (in $\mathrm{Aut}(\Delta)$) to the rotation $R(z) = \exp\left(\frac{2\pi i}{m_j}\right) z$ (respectively $R(z) = \exp\left(-\frac{2\pi i}{m_j}\right) z$), say x_j is a *positive minimal rotation* (respectively *non-positive minimal rotation*).

We remark that for every signature $(\gamma; m_1, m_2, \cdots, m_r)$ such that

$$2\gamma - 2 + \sum \left(1 - \frac{1}{m_j}\right) > 0$$

there exists Γ a Fuchsian group, uniquely determined up conjugation in $\mathrm{Aut}(\Delta)$, with this signature.
For more details see [15], [6], [12] and [16].

Definition 4. Consider two Fuchsian groups Γ_1, Γ_2 . We say that Γ_1 is *geometrically isomorphic* to Γ_2 if there exist a self–homeomorphism of Δ, say $T \in$ Homeo (Δ), and a group isomorphism $\chi : \Gamma_1 \to \Gamma_2$ such that for all $x \in \Gamma_1$ the following holds

$$\chi(x) = T \circ x \circ T^{-1}.$$

We also say that the group isomorphism $\chi : \Gamma_1 \to \Gamma_2$ can be *realized geometrically* if there exists $T \in$ Homeo (Δ) such that the previous condition is true.

For more details see [15].

In the paper of Macbeath [15] we find the following example:

Example 1. *Consider the group*

$$\Gamma = \left\langle \rho_5 : z \rightsquigarrow \exp\left(\frac{2\pi i}{5}\right) z = \omega_5 z \right\rangle \simeq \mathbb{Z}/5\mathbb{Z}$$

and let χ denote the automorphism of Γ given by

$$\chi(\rho_5) = \rho_5^2 .$$

This isomorphism can not be realized geometrically.

Note that ρ_5 is a minimal rotation and $\chi(\rho_5) = \rho_5^2$ is not a minimal rotation. Conjugation of a minimal rotation,by any homeomorphism, still a minimal rotation. Orientation-preserving homeomorphisms conjugates positive minimal rotations to positive minimal rotations.

Theorem 4. *Let $\chi : \Gamma_1 \longrightarrow \Gamma_2$ be an isomorphism between finitely generated Fuchsian groups with $\Lambda(\Gamma_j) = \partial(\Delta)$, for $j = 1, 2$. Assume that*

(1) $x \in \Gamma_1$ is parabolic if only if $\chi(x) \in \Gamma_2$ is parabolic; and

(2) $x \in \Gamma_1$ is a minimal rotation if and only if $\chi(x) \in \Gamma_2$ is a minimal rotation.

Then χ is geometric.

See [20].

Corollary 1. *Let $\chi : \Gamma_1 \longrightarrow \Gamma_2$ be an isomorphism between finitely generated Fuchsian groups, both without parabolic elements , with $\Lambda(\Gamma_j) = \partial(\Delta)$, for $j = 1, 2$. If*

(1) $x \in \Gamma_1$ is a minimal rotation if and only if $\chi(x) \in \Gamma_2$ is a minimal rotation.

Then χ is geometric.

It was proved by Macbeath that in the above Corollary we may delete the assumed condition on rotation, that is, the following holds.

Theorem 5. *Let $\chi : \Gamma_1 \longrightarrow \Gamma_2$ be an isomorphism between finitely generated Fuchsian group, both without parabolic elements, with $\Lambda(\Gamma_j) = \partial(\Delta)$, for $j = 1, 2$. Then χ is geometric.*

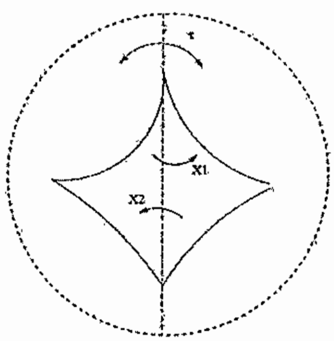
The preceding theorem holds at the level of Non Euclidean plane Crystallographic groups (NEC groups), that is, finitely generated discrete subgroup of isometries of the hyperbolic disc containing no parabolic elements. See [15, pag. 1201].

At this point it is important to note that two homeomorphism, say $F_1, F_2 : \Delta \longrightarrow \Delta$, defining the same isomorphism $\chi : \Gamma_1 \longrightarrow \Gamma_2$, must have the same orientability type. In fact, the homeomorphism $F_2^{-1} \circ F_1 : \Delta \longrightarrow \Delta$ defines the identity automorphism of Γ_1. It can be proved that, in this case, $F_2^{-1} \circ F_1$ is homotopic to the identity.

Remark 2. Consider the following example. Let

$$\Gamma = < x_1, x_2, x_3 : x_1^5 = x_2^5 = x_3^5 = x_1 x_2 x_3 = 1 >$$

whose fundamental polygon is the following



Consider the isomorphism (induced by τ) given by

$$\begin{array}{rcl} \chi : \Gamma & \longrightarrow & \Gamma \\ x_1 & \rightsquigarrow & x_1^{-1} \\ x_2 & \rightsquigarrow & x_2^{-1} \\ x_3 & \rightsquigarrow & x_2 x_1 \end{array}$$

since $x_3 = x_2^{-1} x_1^{-1}$ and it has order 5, then $x_1 x_2$ has order 5. Hence $x_2 x_1$ has order 5, in fact

$$(x_2 x_1)^5 = x_2 (x_1 x_2)^4 x_1 = x_2 (x_2^{-1} x_1^{-1}) x_1 = 1$$

Note that our isomorphism sends positive minimal rotations to non-positive minimal rotations.
By the preceding theorem the isomorphism χ is geometric. Assume that $F : \Delta \longrightarrow \Delta$ is a homeomorphism inducing χ. It is clear that F must fix the fixed points of x_1 and x_2 and must permute the fixed point of x_2x_1 with the fixed point of $(x_1x_2)^{-1}$ (the same fixed point of x_1x_2). Now, it follows that the homeomorphism $g = \tau \circ F$ fixes the fixed points of x_1, x_2, x_2x_1 and x_1x_2. These four points are the vertices of the hyperbolic polygon. Now it is easy to note that, up homotopy, we may assume that g induces the identity isomorphism to see that g is the identity homeomorphism of Δ.

Now, all the above permits to see that in general we may not assume the homeomorphism that realizes the isomorphism should be orientation preserving.

Chapter 2

Equivalence of group actions

2.1 Actions on Riemann Surfaces

We say that a group G *acts on a Riemann Surface* S, if there exists a monomorphism

$$\varepsilon : G \longrightarrow \mathrm{Aut}(S)\,,$$

where $\mathrm{Aut}(S)$ denotes the automorphisms group of S.
We call to the monomorphism ε an *action* of G on S.

When we consider a compact Riemann Surface of genus $g \geq 2$, and G a group acting on S then necessarily G is a finite group, by the Hurwitz's Theorem.

Remark 3. We may consider for each $p \in S$ the *stabilizer subgroup* of p, this is

$$G_p = \{D \in G : \varepsilon(D)(p) = p\}\ .$$

Since $\varepsilon(G_p) \leq \mathrm{Aut}(S)_p$ then G_p is a cyclic group.

Given an action $\varepsilon : G \to \mathrm{Aut}(S)$ of G on S, we have the natural projection

$$\pi : S \longrightarrow S/\varepsilon(G)\,.$$

As before we have $S/\varepsilon(G)$ is a Riemann surface so that π is a holomorphic map. Since S is a compact Riemann surfaces then $S/\varepsilon(G)$ is a compact Riemann surfaces of genus γ.
Further we have that $p \in S$ is a ramification point of π if only if $G_p \neq \{Id\}$, furthermore the multiplicity of p is $|G_p|$. Then π is a smooth covering (unbranched covering) on the complement of a finite set, the ramification

points set.
We called to π a *branched covering* , and we say that π has a *signature*

$$(\gamma; m_1, m_2, .., m_r),$$

where m_j are the multiplicity of the ramification points, and r is the number of the branch points of π. Sometimes also we will say G acts on S with signature $(\gamma; m_1, m_2, .., m_r)$.

We consider the numbers m_j with the following order

$$m_1 \geq \cdots \geq m_r.$$

Theorem 6. *Let Γ be a Fuchsian group with signature $(\gamma; m_1, m_2, \cdots, m_r)$ and let $\theta : \Gamma \longrightarrow G$ be an epimorphism of groups with $K = \ker\theta$ torsion–free Fuchsian group and $\Lambda(K) = \partial(\Delta)$.*
Then G acts on the Riemann surface $S = \Delta/K$ with signature $(\gamma; m_1, m_2, \cdots, m_r)$. Furthermore, S has genus g given by

$$g = |G|(\gamma - 1) + 1 + \frac{|G|}{2}\sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right),$$

where $g \geq 2$.

Proof. First by the Theorem 2 we have that $S = \Delta/K$ is a compact Riemann surfaces of genus at least 2, then $g \geq 2$.
For each $x \in \Gamma$, since K is a normal subgroup of Γ, then

$$xKx^{-1} = K.$$

Thus we have the following diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{x} & \Delta \\ \rho\downarrow & & \downarrow\rho \\ S & \xrightarrow{\widetilde{x}} & S \end{array}$$

where $\widetilde{x}(Kz) = Kx(z)$.

Since $x \in \mathrm{Aut}(\Delta)$ then $\widetilde{x} \in \mathrm{Aut}(S)$.

Hence for each $D \in G$, we may take $x \in \Gamma$ such that $\theta(x) = D$. Further, if for $y \in \Gamma$ we have $\theta(y) = D = \theta(x)$ then

$$xy^{-1} \in K\,,$$

therefore $\widetilde{x} = \widetilde{y}$.

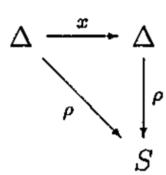
Now we may define

$$\begin{array}{rcll} \varepsilon_\theta : G & \longrightarrow & \mathrm{Aut}(S) & \\ D & \rightsquigarrow & \widetilde{x} & ,\theta(x) = D \end{array}$$

ε_θ is well defined because $\ker\theta = K$.
Since θ is a group homomorphism, then ε_θ is a group homomorphism.
We may see that ε_θ is a monomorphism , in fact if $\varepsilon_\theta(D) =$ Id , then for $x \in \Gamma$, such that $\theta(x) = D$, we have

$$\widetilde{x} = \text{ Id }.$$

Thus we have the following diagram



therefore $x \in K$, and thus $D = \theta(x) = 1$.

We will compute the signature for the action:
Since Γ has signature $(\gamma; m_1, \cdots, m_r)$ then there exist $a_i, b_i, x_j \in \Gamma$ such that

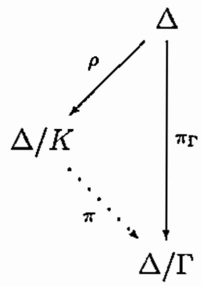
$$\Gamma = \left\langle a_1, b_1, \cdots, a_\gamma, b_\gamma, x_1, \cdots, x_r \,:\, x_1^{m_1} = \cdots = x_r^{m_r} = \prod_{i=1}^{\gamma}[a_i, b_i]\prod_{j=1}^{r} x_j = 1 \right\rangle \tag{2.1}$$

where $[a_i, b_i] = a_i^{-1}b_i^{-1}a_ib_i$.
Further, we have Δ/Γ is a compact surface of genus γ and r branch points with multiplicity $m_1, \cdots, m_r$. In other words the natural projection

$$\pi_\Gamma : \Delta \longrightarrow \Delta/\Gamma$$

is a branched covering with r branch points.

Now we have the following diagram

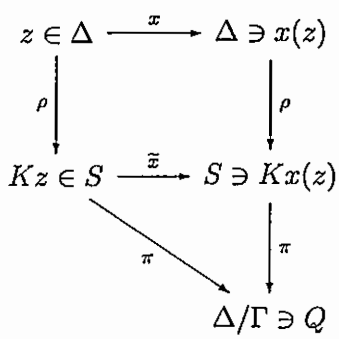


Since ρ, π_Γ are holomorphic maps, then π is a holomorphic map.

In general we have for each $Q \in \Delta/\Gamma$

$$\pi^{-1}(Q) = \{\widetilde{x}(Kz) : x \in \Gamma\} = \{\varepsilon_\theta(D)(Kz) : D \in G\},$$

where $\pi_\Gamma(z) = Q$, in fact



Hence the covering π is induced by the action ε_θ. Since π_Γ is a branched covering, and ρ is a regular covering, we have that π is a branched covering with r branch point in Δ/Γ, say $\{Q_1, \cdots, Q_r\}$.

We recall that Γ has a presentation as (2.1) where we may choose x_j such that for each j the subgroup $< x_j >$ is the stabilizer of a single point in Δ, say z_j.
Then for $P_j = Kz_j$, we have that P_j is a fixed point in S and

$$\mathrm{Stab}_{P_j}(\varepsilon(G)) = < \widetilde{x}_j >,$$

as K is a torsion free, we have $\widetilde{x}_j$ has order m_j.

Now if P is a fixed point by the action ε_θ, then $P = Kz$ where z is a fixed point by the action Γ. Since the fixed point for Γ belongs $\pi_\Gamma^{-1}(Q_j)$ for some j, then

$$P \in \pi^{-1}(Q_j) = \{\widetilde{x}(Kz_j) : x \in \Gamma\}.$$

The multiplicity for these points is m_j, and

$$|\pi^{-1}(Q_j)| = \frac{|G|}{m_j}.$$

By the Hurwitz formula we have the genus of S is

$$g = |G|(\gamma - 1) + 1 + \frac{|G|}{2}\sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right).$$

□

Theorem 7. *Let S be a compact Riemann surface of genus $g \geq 2$ and let G be a finite group. There is an action ε of G on S with signature $(\gamma; m_1, \cdots, m_r)$ if and only if there are a Fuchsian group Γ with signature $(\gamma; m_1, \cdots, m_r)$, an epimorphism $\theta_\varepsilon : \Gamma \to G$ such that $K = \ker(\theta_\varepsilon)$ is torsion–free Fuchsian group and Δ/K, the quotient space induced by the action of K on Δ, is a Riemann surface bi-holomorphic to S.*

Proof. First we will prove the implication $\Rightarrow$.
By the Uniformization Theorem (see [6, pag. 191,192]) and the Existence theorem branched covering we know that there is Γ a Fuchsian group with signature $(\gamma; m_1, \cdots, m_r)$ and $K \lhd \Gamma$ a torsion free Fuchsian group such that

$$\begin{array}{ccc}
\Delta & \xrightarrow{F} & X \\
\downarrow & & \downarrow \\
\Delta/K & \xrightarrow{f} & S \\
\downarrow & & \downarrow \\
\Delta/\Gamma & \longrightarrow & S/\varepsilon(G)
\end{array}$$

where each level is an isomorphism of Riemann surfaces, and X is the universal covering for S.
Now for each $x \in \Gamma$, we consider

$$\begin{array}{ccc} \Delta & \xrightarrow{x} & \Delta \\ \downarrow & & \downarrow \\ \Delta/K & \xrightarrow{\widetilde{x}} & \Delta/K \end{array}$$

Since $x \in \mathrm{Aut}(\Delta)$ then $\widetilde{x} \in \mathrm{Aut}(\Delta/K)$.

For $\widetilde{x}$, we have

$$f \circ \widetilde{x} \circ f^{-1} \in \mathrm{Aut}(S)$$

then we have the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{F^{-1}} & \Delta & \xrightarrow{x} & \Delta & \xrightarrow{F} & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \xrightarrow{f^{-1}} & \Delta/K & \xrightarrow{\widetilde{x}} & \Delta/K & \xrightarrow{f} & S \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S/\varepsilon(G) & \longrightarrow & \Delta/\Gamma & \xrightarrow{\mathrm{Id}} & \Delta/\Gamma & \longrightarrow & S/\varepsilon(G) \end{array}$$

therefore $f \circ \widetilde{x} \circ f^{-1} \in \varepsilon(G)$.

Now we may define

$$\begin{array}{rcl} \theta_\varepsilon : \Gamma & \longrightarrow & \varepsilon(G) \\ x & \rightsquigarrow & f \circ \widetilde{x} \circ f^{-1} \end{array}$$

Using the diagram it easy check that θ is a group homomorphism and $\ker_\varepsilon(\theta) = K$.
Further for $\tau \in \varepsilon(G)$ as X is the universal covering for S, we may take a lift

of τ, Ξ, this is

$$\begin{array}{ccc} X & \xrightarrow{\Xi} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{\tau} & S \end{array}$$

Then $F^{-1} \circ \Xi \circ F \in \Gamma$ and

$$\theta(F^{-1} \circ \Xi \circ F) = \tau\,.$$

The converse was proven in the Theorem 6. □

2.1.1 Topological Equivalence

Definition 5. Let S_j be a Riemann Surface for $j = 1, 2$, and G be a group. The actions ε_1, ε_2 of G on S_1 and S_2 respectively, are called *topologically equivalent* if there exist $\Phi \in \mathrm{Aut}\,(G)$ and $t \in$ Homeo $(S_1 \to S_2)$ such that the following diagram is commutative

$$\begin{array}{ccc} G & \xrightarrow{\varepsilon_1} & \varepsilon_1(G) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Psi_t} \\ G & \xrightarrow{\varepsilon_2} & \varepsilon_2(G) \end{array}$$

where $\Psi_t(\tau) := t \circ \tau \circ t^{-1}$ and Homeo $(S_1 \to S_2)$ is the group of homeomorphisms of S_1 on S_2.

If $t \in$ Homeo $^+(S_1 \to S_2)$ then we say that ε_1, ε_2 are *directly topologically equivalent*, where Homeo $^+(S_1 \to S_2)$ is the group of orientation preserving homeomorphisms of S_1 on S_2.

Remark 4. Let ε_1, ε_2 be topologically equivalent actions as before.
For $t \in$ Homeo $(S_1 \to S_2)$, according to the notation of the Theorem 7, we may lift $f_2^{-1} \circ t \circ f_1$ to $T \in$ Homeo (Δ), this is

$$\begin{array}{ccccccc} \Delta & \xrightarrow{F_1} & X_1 & \xrightarrow{F_2 \circ T \circ F_1^{-1}} & X_2 & \xrightarrow{F_2^{-1}} & \Delta \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Delta/K_1 & \xrightarrow{f_1} & S_1 & \xrightarrow{t} & S_2 & \xrightarrow{f_2^{-1}} & \Delta/K_2 \end{array}$$

Let x be an element in $\Gamma_1, \theta_1(x) \in \varepsilon_1(G)$ then $\Psi_t(\theta_1(x)) \in \varepsilon_2(G)$. In the diagram we have:

$$\begin{CD}
\Delta @>{T^{-1}}>> \Delta @>{x}>> \Delta @>{T}>> \Delta \\
@VVV @VVV @VVV @VVV \\
\Delta/K_2 @>{f_1^{-1}t^{-1}f_2}>> \Delta/K_1 @>{f_1^{-1}\theta_1(x)f_1}>> \Delta/K_1 @>{f_2^{-1}tf_1}>> \Delta/K_2
\end{CD}$$

then $T \circ x \circ T^{-1} \in \Gamma_2$ and

$$\begin{aligned}
\theta_2(T \circ x \circ T^{-1}) &= f_2 \circ (f_2^{-1} t f_1 \circ f_1^{-1}\theta_1(x) f_1 \circ f_1^{-1} t^{-1} f_2) \circ f_2^{-1} \\
&= t \circ \theta_1(x) \circ t^{-1} \\
&= \Psi_t(\theta_1(x))
\end{aligned}$$

Therefore we have the following commutative diagram

$$\begin{CD}
\Gamma_1 @>{\theta_1}>> \varepsilon_1(G) \\
@V{\chi_T}VV @VV{\Psi_t}V \\
\Gamma_2 @>{\theta_2}>> \varepsilon_2(G)
\end{CD}$$

where $\chi_T(x) = T \circ x \circ T^{-1}$.

Note that if $t \in$ Homeo $^+(S_1 \to S_2)$ then $T \in$ Homeo $^+(\Delta)$.

Theorem 8. *Let ε_j be an action of G on S_j, for $j = 1, 2$.*
Let $\theta_j : \Gamma_j \to \varepsilon_j(G)$ be the epimorphism associated to ε_j given by the Theorem 7 for $j = 1, 2$.
Then ε_1 is topologically equivalent to ε_2 if only if there exists a group isomorphism $\chi : \Gamma_1 \to \Gamma_2$ and a group isomorphism $\Phi : \varepsilon_1(G) \to \varepsilon_2(G)$ such that the following diagram is commutative

$$\begin{CD}
\Gamma_1 @>{\theta_1}>> \varepsilon_1(G) \\
@V{\chi}VV @VV{\Phi}V \\
\Gamma_2 @>{\theta_2}>> \varepsilon_2(G)
\end{CD}$$

Proof. The implication $\Rightarrow$ is the preceding remark.
We will prove the converse.
Since the diagram is commutative, we have $\chi(K_1) = K_2$ where $K_j = \ker\theta_j$ for $j = 1, 2$.
Since Γ_1, Γ_2 are Fuchsian groups with compact quotient, then by the Theorem 5 the isomorphism $\chi : \Gamma_1 \longrightarrow \Gamma_2$ is geometrically realized, this is there exists $T \in$ Homeo (Δ) such that

$$\chi(x) = T \circ x \circ T^{-1}$$

therefore $\Gamma_2 = T\Gamma_1T^{-1}$ and $K_2 = TK_1T^{-1}$.

We have the following diagram:

$$\begin{array}{ccc} \Delta & \xrightarrow{\ T\ } & \Delta \\ \downarrow & & \downarrow \\ \Delta/K_1 & \xrightarrow{f_2^{-1}tf_1} & \Delta/K_2 \end{array}$$

Claim: We have the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\ \varepsilon_1\ } & \varepsilon_1(G) \\ {\scriptstyle \varepsilon_1}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ \varepsilon_1(G) & \xrightarrow{\ \mathrm{Id}\ } & \varepsilon_1(G) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Psi_t} \\ \varepsilon_2(G) & \xrightarrow{\ \mathrm{Id}\ } & \varepsilon_2(G) \\ {\scriptstyle \varepsilon_2}\uparrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ G & \xrightarrow{\ \varepsilon_2\ } & \varepsilon_2(G) \end{array}$$

where $\Psi_t(\tau) = t \circ \tau \circ t^{-1}$

In fact for $g \in G$, $\varepsilon_1(g) = \theta_1(x)$ for some $x \in \Gamma_1$, then

$$\begin{aligned}\Psi_t(\varepsilon_1(g)) &= t \circ \theta_1(x) \circ t^{-1} \\ &= \theta_2(T \circ x \circ T^{-1}) \\ &= \theta_2(\chi(x)) \\ &= \Phi(\theta_1(x)) = \Phi(\varepsilon_1(g))\end{aligned}$$

Recalling the Definition 5, by the commutative diagram then the actions are topologically equivalent. □

2.1.2 Conformal Equivalence

Definition 6. Let S_j be a Riemann Surface for $j = 1, 2$, and G be a group. The actions ε_1, ε_2 of G on S_1 and S_2 respectively, are called *conformally equivalent* actions if there exist $\Phi \in \mathrm{Aut}(G)$ and $t \in \mathrm{Isom}(S_1, S_2)$ such that the following diagram is commutative

$$\begin{array}{ccc} G & \xrightarrow{\varepsilon_2} & \varepsilon_2(G) \\ {\scriptstyle\Phi}\downarrow & & \uparrow{\scriptstyle\Psi_t} \\ G & \xrightarrow{\varepsilon_1} & \varepsilon_1(G) \end{array}$$

where $\Psi_t(\tau) := t \circ \tau \circ t^{-1}$.

Theorem 9. *Let ε_j be an action of G on S_j, for $j = 1, 2$ and $\theta_j : \Gamma_j \to \varepsilon_j(G)$ be the epimorphism associated to ε_j given by the Theorem 7 for $j = 1, 2$. Then ε_1 is conformally equivalent to ε_2 if only if there exists $T \in \mathrm{Aut}(\Delta)$ and a group isomorphism $\Phi : \varepsilon_1(G) \to \varepsilon_2(G)$ such that the following diagram is commutative*

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\theta_1} & \varepsilon_1(G) \\ {\scriptstyle\chi_T}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \Gamma_2 & \xrightarrow{\theta_2} & \varepsilon_2(G) \end{array}$$

where $\chi_T(x) = T \circ x \circ T^{-1}$.

Proof. By the preceding theorem we have that the actions are topologically equivalent.
When the action are conformally equivalent, we know that there exists $t \in \mathrm{Isom}(S_1, S_2)$, then we may lift $f_2^{-1} t f_1$ (using the previous notations) to an automorphism T of Δ, this is

$$\begin{array}{ccc} \Delta & \xrightarrow{T} & \Delta \\ \downarrow & & \downarrow \\ \Delta/K_1 & \xrightarrow{f_2^{-1} t f_1} & \Delta/K_2 \end{array}$$

Then T define a group isomorphism between Γ_1 and Γ_2.

For the proof of the converse, we have that since $T \in \mathrm{Aut}(\Delta)$, and by the commutative diagram $K_2 = TK_1T^{-1}$, then using the Theorem 3 we have T induce an isomorphism, t, between S_1 and S_2 according to the previous diagram.

□

Remark 5. If two actions are conformally equivalent, then they are directly topologically equivalent.

Corollary 2. *ε_1 and ε_2 are actions on S conformally equivalent if only if $\varepsilon_1(G)$ and $\varepsilon_2(G)$ are conjugate groups in* $\mathrm{Aut}(S)$.

Proof. By the remark 4 we have $\Phi = \Psi_t$, and since in this case $T \in \mathrm{Aut}(\Delta)$ then $t \in \mathrm{Aut}(S)$. □

Remark 6. According to the Theorem 7, for each action ε, we have associated an epimorphism θ_ε and for each epimorphism θ we have associated an action ε_θ.
Claim: the actions ε and $\varepsilon_{\theta_\varepsilon}$ are conformally equivalent.
The corresponding diagram is

$$\begin{array}{ccc} G & \xrightarrow{\varepsilon} & \mathrm{Aut}\,(S) \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\Psi_f} \\ \varepsilon(G) & \xrightarrow{\varepsilon_{\theta_\varepsilon}} & \mathrm{Aut}\,(\Delta/K) \end{array}$$

where $\Psi_f(\tau) = f^{-1} \circ \tau \circ f$ and f is given by

$$\begin{array}{ccc} \Delta & \xrightarrow{F} & X \\ \downarrow & & \downarrow \\ \Delta/K & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ \Delta/\Gamma & \longrightarrow & S/\varepsilon(G) \end{array}$$

such that each level is an isomorphism of Riemann surfaces.
We recall the definition of $\varepsilon_{\theta_\varepsilon}$

$$\varepsilon_{\theta_\varepsilon}(\tau) = \widetilde{x}\,,$$

where $\theta_\varepsilon(x) = \tau = f \circ \widetilde{x} \circ f^{-1}$ and $\widetilde{x}$ is determined by the following diagram:

$$\begin{array}{ccc}
\Delta & \xrightarrow{x} & \Delta \\
\downarrow & & \downarrow \\
\Delta/K & \xrightarrow{\widetilde{x}} & \Delta/K
\end{array}$$

therefore

$$\varepsilon_{\theta_\varepsilon}(\tau) = \Psi_f(\tau)\,.$$

Remark 7. For ε and ε' actions of a group G on S, such that

$$\varepsilon(G) = \varepsilon'(G)\,,$$

then we have that ε and ε' are actions conformally equivalent.
To prove this, we recall the theorem 7, then we have the following diagram

$$\begin{array}{ccccc}
\Delta & \xrightarrow{F'} & X & \xrightarrow{F^{-1}} & \Delta \\
\downarrow & & \downarrow & & \downarrow \\
\Delta/K & \xrightarrow{f'} & S & \xrightarrow{f^{-1}} & \Delta/K \\
\downarrow & & \downarrow & & \downarrow \\
\Delta/\Gamma' & \longrightarrow & S/\varepsilon(G) & \longrightarrow & \Delta/\Gamma
\end{array}$$

therefore the groups Γ and Γ' are conjugates by $T = F^{-1} \circ F' \in \mathrm{Aut}(\Delta)$.
We use this T for to find the commutative diagram.

2.2 Properties of δ

Recall the definition of δ in the Theorem 1.
First we will present an example of the computation of δ_p.

Example 2. *We consider*

$$S = \{[X, Y, Z] \in \mathbb{P}^2\mathbb{C} : X^4 + Y^4 + Z^4 = 0\}$$

S is a compact Riemann surface of genus 3.

Let $\sigma[X, Y, Z] = [iX, Y, Z]$ be an automorphism of S.
It is easy to check that the order of σ is 4.
The fixed points for σ are:

$$\begin{array}{rc} & \sigma[X,Y,Z] = [X,Y,Z] \\ \Leftrightarrow & [iX,Y,Z] = [X,Y,Z] \\ \Rightarrow \textit{fixed points on } S: & \{[0,\omega_8,1],[0,i\omega_8,1],[0,-\omega_8,1],[0,-i\omega_8,1]\} \end{array}$$

where $\omega_8 = \exp\left(\frac{2\pi i}{8}\right)$.

We have

$$\begin{array}{rcl} U_3 = \{[X,Y,Z] \in \mathbb{P}^2\mathbb{C} : Z \neq 0\} & \leftrightsquigarrow & \mathbb{C}^2 \\ [X,Y,Z] & \leftrightsquigarrow & \left(\frac{X}{Z}, \frac{Y}{Z}\right) \\ P = [0,\omega_8,1] & \leftrightsquigarrow & p = (0,\omega_8) \end{array}$$

Obviously the point p is a zero of

$$f(x, y) = x^4 + y^4 + 1$$

as $\frac{\partial f}{\partial y}(p) \neq 0$, *then by the Implicit function's theorem, we have that there exist an open neighborhood U of p , a holomorphic function, h, and $t > 0$, such that $h(0) = \omega_8$ and*

$$U = \{(x, h(x)) : x \in B(0,t)\} \subset \{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\},$$

thus a chart for $P = [0, \omega_8, 1]$ is

$$\begin{array}{rccl} \phi: & \widetilde{U} & \longrightarrow & \mathbb{C} \\ & [x, h(x), 1] & \rightsquigarrow & x \end{array}$$

Now we calculate the Rotation constant for σ at P

$$\begin{aligned}\phi\circ\sigma\circ\phi^{-1}(x) &= \phi\circ\sigma[x,h(x),1]\\ &= \phi[ix,h(x),1]\\ &= ix\end{aligned}$$

then $\delta_P(\sigma)=i$.

In the literature we may find the concept of *Rotation Constant* .
For $p\in S$, and $\sigma\in \mathrm{Aut}(S)_p$ we say that the Rotation constant of σ at p is

$$\delta_p(\sigma^{-1})=\delta_p(\sigma)^{-1}.$$

The following lemma we give a relationship between the local structure for the points in the same orbit, under action of G.

Lemma 1. *Let σ,τ be in* $\mathrm{Aut}(S)$*, and suppose* $\sigma\in\mathrm{Aut}(S)_p$*, then*

$$\delta_q(\tau\sigma\tau^{-1})=\delta_p(\sigma),$$

where $q=\tau(p)$.

Proof. Let (ψ,V) be a chart centered at q, then

$$\begin{array}{ccccccc} q\in V\subset S & \xrightarrow{\tau^{-1}} & S & \xrightarrow{\sigma} & S & \xrightarrow{\tau} & S\supset V\ni q\\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi}\\ \psi(V) & \xrightarrow[\phi\circ\tau^{-1}\circ\psi^{-1}]{} & \phi(U) & \xrightarrow[\phi\circ\sigma\circ\phi^{-1}]{} & \phi(U) & \xrightarrow[\psi\circ\tau\circ\phi^{-1}]{} & \psi(U)\end{array}$$

thus we have

$$\begin{aligned}\psi\circ\tau\circ\sigma\circ\tau^{-1}\circ\psi^{-1}(w) &= \psi\circ\tau\circ\phi^{-1}\circ\left(\phi\circ\sigma\circ\phi^{-1}\right)\circ\phi\circ\tau^{-1}\circ\psi^{-1}(w)\\ &= \psi\circ\tau\circ\phi^{-1}\left(\sum_{k\geq 1}c_k(\sigma)\left(\sum_{m\geq 1}a_m w^m\right)^k\right)\\ &= c_1(\sigma)w+\text{ higher terms}\\ &= \delta_p(\sigma)w+\text{ higher terms}\end{aligned}$$

recall $(\psi\circ\tau\circ\phi^{-1})'(0)$ is the multiplicative inverse of $(\phi\circ\tau^{-1}\circ\psi^{-1})'(0)$. □

Now we will see an important theorem providing a relationship between the local structure and the lifts of an automorphism of S.
First recall that for S we have there exist K a Fuchsian group that uniformising S, this is

$$\begin{array}{ccc} \Delta & \xrightarrow{F} & X \\ \rho\downarrow & & \downarrow \\ \Delta/K & \xrightarrow{f} & S \end{array}$$

where each level is an isomorphism of Riemann surfaces, and X is the universal covering for S.

Theorem 10. *Let* $\sigma \in \mathrm{Aut}(S)_p \leq \mathrm{Aut}(S)$ *of order* n*, and let*

$$\mathcal{L} = \left\{T \in \mathrm{Aut}(\Delta) : \exists z_0 \in \Delta\,, f \circ \rho(z_0) = p\,, T(z_0) = z_0 \text{ and } (f^{-1} \circ \sigma \circ f) \circ \rho = \rho \circ T\right\}$$

Then there is unique primitive complex n*th root of unity* ζ *such that for all* $T \in \mathcal{L}$ *we have that* T *is conjugate to multiplication by* ζ*,* $R(z) = \zeta z$*, in* $\mathrm{Aut}(\Delta)$.
Furthermore $\zeta = \delta_p(\sigma)$.

The proof of this theorem may to find in [2].

Since ζ is a primitive complex nth root of unity, we may write $\zeta = \omega_n^j$ where $\omega_n = \exp\left(\dfrac{2\pi i}{n}\right)$ and the numbers $j\,, n$ are relative primes ($(j,n) = 1$).
We call $\dfrac{2\pi j i}{n}$ the *rotation angle* for σ at p.

2.2.1 Analytic Representation

Let $\Omega^{1,0}(S)$ be the set of holomorphic forms on S. We know that $\Omega^{1,0}(S)$ is a complex vector space of dimension g, where g is the genus of S.

Let $\Sigma : G \longrightarrow \Omega^{1,0}(S)$ be the group representation of G given by:

$$\Sigma(\sigma)(\omega) = \sigma * \omega = \omega(\sigma^{-1})\,.$$

This representation is called the *Analytic representation* for G, and its degree is g.

Theorem 11 (Eichler trace formula). *Let S be a compact Riemann surface of genus at least 2, and let σ be an automorphism of S of order $n > 1$. Suppose $\Sigma(\sigma)$ is a matrix via its isomorphism between $\Omega^{1,0}(S)$ and $GL(g, \mathbb{C})$. Then*

$$\operatorname{Tr} \Sigma(\sigma) = 1 + \sum_{\sigma(p)=p} \frac{\delta_p(\sigma^{-1})}{1 - \delta_p(\sigma^{-1})}$$

For more detail see [6, pag. 254]

2.2.2 Epimorphisms and Local structure

As we have seen, for an action, ε, of a finite group G on a compact Riemann surface S of genus at least 2, according to Theorem 7 we have an epimorphism

$$\theta_\varepsilon : \Gamma \longrightarrow \varepsilon(G)\,,$$

where $K = \ker(\theta_\varepsilon)$ is a torsion-free Fuchsian group and Γ is a Fuchsian group with the same signature that the action, and S is a bi-holomorphic to Δ/K. We recall the definition of θ_ε, for $x \in \Gamma$ we have

$$\begin{array}{ccc} \Delta & \xrightarrow{\;x\;} & \Delta \\ \downarrow & & \downarrow \\ \Delta/K & \xrightarrow{\;\widetilde{x}\;} & \Delta/K \end{array}$$

thus we define

$$\theta_\varepsilon(x) = f \circ \widetilde{x} \circ f^{-1}\,,$$

where $f : \Delta/K \to S$ is an isomorphism between Δ/K and S.

We will give a relation between the epimorphism θ, and the homomorphism δ_P , for P fixed point of the action.
We have that if $x_0 \in \Gamma$ is an automorphism of Δ with fixed point,say z_0, then x_0 is conjugate in $\operatorname{Aut}(\Delta)$ to a rotation, $R(z) = \omega z$, therefore

$$\delta_P(\theta_\varepsilon(x_0)) = \omega\,,$$

where $P = f(Kz_0)$.

When ε has signature $(0; m_1, \cdots, m_r)$, Γ has a presentation:

$$\Gamma = \langle x_1.., x_r : x_1^{m_1} = \cdots = x_r^{m_r} = 1, x_1 \cdots x_r = 1 \rangle \ ,$$

where for each $j = 1, .., r-1$, we may choose x_j as a counterclockwise rotation about z_j through angle $\dfrac{2\pi}{m_j}$.
For more detail see [1].

Hence we have that for each $j = 1, .., r-1$, $\theta_\varepsilon(x_j)$ is an automorphism of S with a fixed point $P_j = f(Kz_j)$ and

$$\delta_{P_j}(\theta_\varepsilon(x_j)) = \omega_{m_j} = \exp\left(\frac{2\pi i}{m_j}\right) .$$

2.3 Equivalence of Cyclic Groups

Let $G = \mathbb{Z}/n\mathbb{Z}$ be a cyclic group of order n. We consider G as the integers module n, this is

$$G = \{0, 1, \cdots, n-1\}.$$

Consider Γ a Fuchsian group with signature $(\gamma; m_1, \cdots, m_r)$ and presentation

$$\Gamma = \left\langle a_1, b_1, \cdots, a_\gamma, b_\gamma, x_1, \cdots, x_r \,:\, x_1^{m_1} = \cdots = x_r^{m_r} = \prod_{i=1}^{\gamma}[a_i, b_i] \prod_{j=1}^{r} x_j = 1 \right\rangle.$$

The following lemma was given by Kuribayashi (see [14, Lemma 3.1]).

Lemma 2. *Let $\theta : \Gamma \longrightarrow G$ be a group epimorphism and assume that g is a generator of G. Then for any permutation p of $\{1, \cdots, r\}$ with $m_{p(j)} = m_j$ $(j = 1, \cdots, r)$, there exists an automorphism χ of Γ such that*

(i) $\theta \circ \chi(a_i) = \theta \circ \chi(b_i) = g$ *with* $i = 1, \cdots, \gamma$;

(ii) $\chi(x_j) = D_j x_{p(j)} D_j^{-1}$, *for some* $D_j \in \Gamma$ *with* $j = 1, \cdots, r$.

Remark 8. The following list corresponds to the automorphism of Γ used to compute the automorphism χ of the preceding lemma.

$\chi_{(1)} : a_1 \to a_1 b_1$

$\chi_{(2)} : a_1 \to a_1 b_1 a_1^{-1}$; $b_1 \to a_1^{-1}$

$\chi_{(3)} : x_j \to a_2 x_j a_2^{-1}$, $(j = 1, .., r)$; $a_1 \to a_2 a_1$; $a_2 \to b_1 a_2 b_1^{-1}$;
$b_2 \to a_2 b_2 a_2^{-1} b_1^{-1}; a_i \to a_2 a_i a_2^{-1}$; $b_i \to a_2 b_i a_2^{-1}$, $(i = 3, .., \gamma)$

$\chi_i : a_i \to a_{i+1}$; $b_i \to b_{i+1}$; $a_{i+1} \to C_{i+1}^{-1} a_i C_{i+1}$; $b_{i+1} \to C_{i+1}^{-1} b_i C_{i+1}$ $,i = 1, .., \gamma - 1$
where $C_{i+1} = [a_{i+1}, b_{i+1}]$

$\widetilde{\chi}_l : x_j \to a_1^{-1} x_j a_1$, $(j = l, .., r)$; $a_1 \to \left[a_1^{-1}, (x_l \cdots x_r)^{-1}\right] a_1$; $,l = 1, .., r$
$b_1 \to b_1 a_1^{-1} (x_l \cdots x_r) a_1$

$\widehat{\chi}_k : x_k \to x_{k+1}$; $x_{k+1} \to x_{k+1}^{-1} x_k x_{k+1}$ $,k = 1, .., r-1$
where $m_k = m_{k+1}$

Note that theses automorphisms are geometric induced by orientation preserving homeomorphisms.

We have the following relations

$$\begin{aligned}
\chi_{(1)}([a_1,b_1]) &= [a_1,b_1] \\
\chi_{(2)}([a_1,b_1]) &= [a_1,b_1] \\
\chi_{(3)}([a_1,b_1][a_2,b_2]) &= a_2[a_1,b_1][a_2,b_2]a_2^{-1} \\
\chi_{(3)}([a_i,b_i]) &= a_2[a_i,b_i]a_2^{-1}, \qquad i = 3,..,\gamma
\end{aligned}$$

Now for $\chi_i, \widetilde{\chi_l}, \widehat{\chi_k}$ we have the following relations

$$\begin{aligned}
\chi_i([a_i,b_i]) &= C_{i+1} = [a_{i+1},b_{i+1}] \\
\chi_i([a_{i+1},b_{i+1}]) &= C_{i+1}^{-1}[a_i,b_i]C_{i+1} \\
\chi_i([a_i,b_i][a_{i+1},b_{i+1}]) &= [a_i,b_i][a_{i+1},b_{i+1}] \\
\widetilde{\chi_l}([a_1,b_1]) &= [a_1^{-1},(x_l\cdots x_r)^{-1}][a_1,b_1] \\
\widehat{\chi_k}(x_k x_{k+1}) &= x_k x_{k+1}
\end{aligned}$$

Now on, no loss of generality consider the group epimorphism θ given by

$$\begin{aligned}
\theta : \Gamma &\longrightarrow G \\
a_i &\rightsquigarrow 1 \quad , \text{ with } i = 1,..,\gamma \\
b_i &\rightsquigarrow 1 \quad , \text{ with } i = 1,..,\gamma \\
x_j &\rightsquigarrow \xi_j \quad , \text{ with } j = 1,..,r
\end{aligned}$$

where $K = \ker\theta$ is a torsion free group.

Theorem 12. *Let χ be an automorphism of the list in Remark 8. There exists a $\Phi \in \mathrm{Aut}(G)$ such that diagram (2.2) commutes*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \Gamma & \xrightarrow{\theta} & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \chi|_K} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \Phi} & & \\
0 & \longrightarrow & K & \longrightarrow & \Gamma & \xrightarrow{\theta} & G & \longrightarrow & 0
\end{array} \qquad (2.2)$$

if only if there exist $1 \le k < r$ such that

$$\begin{aligned}
s^2 &\equiv 1 \quad \mod m_k \\
s &\equiv 1 \quad \mod m_j, \quad \textit{with } j \neq k, k+1
\end{aligned}$$

where $\Phi(1) = s$.

Proof. Let $\Phi \in \mathrm{Aut}(G)$ be such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \Gamma & \xrightarrow{\theta} & G & \longrightarrow & 0 \\
 & & \vdots & & \downarrow \chi & & \downarrow \Phi & & \\
0 & \longrightarrow & K & \longrightarrow & \Gamma & \xrightarrow{\theta} & G & \longrightarrow & 0
\end{array}$$

Then for $x \in K$ we have

$$\theta\chi(x) = \Phi\left(\theta(x)\right) = \Phi(0) = 0$$
$$\therefore \chi(x) \in K$$

Therefore, $\chi|_K : K \to K$ is an automorphism of K.

First we will prove the implication $\Rightarrow$.
It follows from the list of Remark 8 that

$$\begin{array}{rcccl}
\theta \circ \chi_{(1)}(x_j) = & \theta(x_j) & = & \xi_j & ,\forall j \in \{1, \cdots, r\} \\
\theta \circ \chi_{(2)}(x_j) = & \theta(x_j) & = & \xi_j & ,\forall j \in \{1, \cdots, r\} \\
\theta \circ \chi_{(3)}(x_j) = & \theta(a_2 x_j a_2^{-1}) & = & \xi_j & ,\forall j \in \{1, \cdots, r\} \\
\theta \circ \chi_i(x_j) = & \theta(x_j) & = & \xi_j & ,\forall j \in \{1, \cdots, r\} \\
\theta \circ \widetilde{\chi}_l(x_j) = & \theta(a_1^{-1} x_j a_1) & = & \xi_j & ,\forall j \in \{1, \cdots, r\}
\end{array}$$
$$\theta \circ \widehat{\chi}_k(x_j) = \left\{ \begin{array}{ccccc} \theta(x_{k+1}) & = & \xi_{k+1} & , & \text{for } j = k \\ \theta(x_{k+1}^{-1} x_k x_{k+1}) & = & \xi_k & , & \text{for } j = k+1 \\ \theta(x_j) & = & \xi_j & , & \text{otherwise} \end{array} \right\}.$$

We have that $\Phi(1) = s = 1$ for the first five equations, and we obtain the commutative diagram (2.2).

For the last equation, we have the following three conditions:

$$\begin{array}{rcl}
\xi_{k+1} & = & \theta(x_{k+1}) = \theta(\chi(x_k)) = \Phi(\theta(x_k)) = \Phi(\xi_k) = s\xi_k \\
\xi_k & = & \theta(x_{k+1}^{-1} x_k x_{k+1}) = \theta(\chi(x_{k+1})) = \Phi(\theta(x_{k+1})) = \Phi(\xi_{k+1}) = s\xi_{k+1} \\
\xi_j & = & \theta(x_j) = \theta(\chi(x_j)) = \Phi(\theta(x_j)) = \Phi(\xi_j) = s\xi_j \quad j \neq k, k+1 .
\end{array}$$

Then

$$\begin{array}{rcll}
s^2\xi_k & \equiv & \xi_k & \\
s^2\xi_{k+1} & \equiv & \xi_{k+1} & \\
s\xi_j & \equiv & \xi_j & j \neq k, k+1
\end{array} \tag{2.3}$$

Recall that the each element $x_j \in \Gamma$ has order m_j. Since K is a torsion free group, we have that $\theta(x_j) = \xi_j$ has order m_j. Recall also that the elements of order m in $G = \mathbb{Z}/n\mathbb{Z}$ are given by

$$\left\{ \frac{nt}{m} : \quad (m,t) = 1, \quad 0 \leq t < m \right\} .$$

Thus for each j we have

$$\xi_j = \frac{nt_j}{m_j} .$$

Since t_j and m_j are relative primes, there exists $\widetilde{t_j}$ such that

$$t_j \widetilde{t_j} \equiv 1, \qquad \bmod m_j .$$

Hence

$$\xi_j \widetilde{t_j} = \frac{nt_j \widetilde{t_j}}{m_j} = \frac{n(1+m_j)}{m_j} \equiv \frac{n}{m_j}, \qquad \bmod n .$$

According to equations (2.3), we have

$$\begin{array}{lcccll} & s^2 \xi_k & \equiv & \xi_k & \bmod n & | \cdot \widetilde{t_k} \\ & s^2 \dfrac{n}{m_k} & \equiv & \dfrac{n}{m_k} & \bmod n & \\ \Leftrightarrow & s^2 \dfrac{n}{m_k} & = & \dfrac{n}{m_k} + nt & & | \cdot \dfrac{m_k}{n} \\ \Leftrightarrow & s^2 & = & 1 + m_k t & & \\ \Leftrightarrow & s^2 & \equiv & 1 & \bmod m_k & \end{array}$$

Proceeding for $k+1$, just as we did for k, yields the same condition for s. Again, proceeding for $j \neq k, k+1$, just as we did for k, yields

$$s \equiv 1 \quad \bmod m_j \qquad j \neq k, k+1 .$$

Using the condition for s, we see that the converse is clear. In fact, suppose $1 \leq s < n$. Then if $s = 1$, we may take $\chi = \chi_{(1)}$. Now if $s \neq 1$, we may take $\chi = \widehat{\chi}_k$. □

Topological equivalence

Consider Γ Fuchsian group with signature $(\gamma; m_1, .., m_r)$.

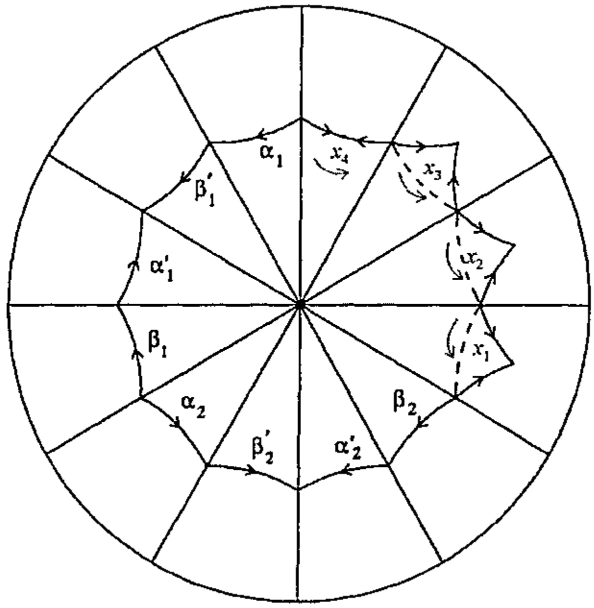
According to the paper of L.Keen [13] (also see [12, Theorem 4.3.2]) we may construct for Γ a hyperbolic polygon with $4\gamma + 2r$ sides. The polygon has r external isosceles hyperbolic triangles such that the angles between the equal sides of the triangles are $\frac{2\pi}{m_1}, \cdots, \frac{2\pi}{m_r}$.

Hence Γ has a presentation according to the hyperbolic polygon associated, this is

$$\Gamma = \left\langle a_1, b_1.., a_\gamma, b_\gamma, x_1.., x_r \,:\, x_1^{m_1} = \cdots = x_r^{m_r} = \prod_{i=1}^{\gamma}[a_i, b_i]\prod_{j=1}^{r} x_j = 1 \right\rangle .$$

where x_j is a positive minimal rotation, in other words x_j is a counterclockwise rotation about a some point through angle $\frac{2\pi}{m_j}$.

The following figure is the hyperbolic polygon for $\gamma = 2\,, r = 3\,,$ $m_1, m_2, m_3 > 2$ and $m_4 = 2$.



In this hyperbolic polygon it has $a_i(\alpha_i') = \alpha_i$ and $b_j(\beta_i') = \beta_i$ for $i = 1, \cdots, \gamma$.

Remark 9. We may find a system of generators, with the same presentation, but with non-positive minimal rotations. Take,

$$y_1 = x_r^{-1}\,, y_2 = x_{r-1}^{-1}\,, ..., \, y_r = x_1^{-1}\,,$$

$$\widetilde{a}_1 = b_\gamma, \widetilde{b}_1 = a_\gamma, \dots, \widetilde{a}_\gamma = b_1, \widetilde{b}_\gamma = a_1.$$

Now we consider two Fuchsian group Γ, Γ' with signature $(\gamma; m_1, .., m_r)$ and presentations according to your hyperbolic polygons associated

$$\Gamma = \left\langle a_1, b_1 .., a_\gamma, b_\gamma, x_1 .., x_r \, : \, x_1^{m_1} = \cdots = x_r^{m_r} = \prod [a_i, b_i] \prod x_j = 1 \right\rangle .$$

$$\Gamma' = \left\langle a'_1, b'_1 .., a'_\gamma, b'_\gamma, x'_1 .., x'_r \, : \, x_1'^{m_1} = \cdots = x_r'^{m_r} = \prod [a'_i, b'_i] \prod x'_j = 1 \right\rangle .$$

Theorem 13. *Let Γ, Γ' be Fuchsian groups both with signature $(\gamma; m_1, .., m_r)$ and presentations according to your hyperbolic polygons associated.*
Let θ, θ' be epimorphisms given by

$$\begin{array}{rcll} \theta : \Gamma & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \\ a_i, b_i & \rightsquigarrow & 1 & , i = 1, .., \gamma \\ x_j & \rightsquigarrow & \xi_j & , j = 1, .., r \end{array}$$

$$\begin{array}{rcll} \theta' : \Gamma' & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \\ a'_i, b'_i & \rightsquigarrow & 1 & , i = 1, .., \gamma \\ x'_j & \rightsquigarrow & \xi'_j & , j = 1, .., r \end{array}$$

where $K = \ker \theta$ and $K' = \ker \theta'$ are torsion free groups.

If there exists an $s \in \mathbb{Z}$ with $(s, n) = 1$, such that

$$(\xi'_1, \dots, \xi'_r) \equiv s(\xi_1, \dots, \xi_r), \quad \bmod n \tag{2.4}$$

then the actions induced by θ on Δ/K and by θ' on Δ/K' are directly topologically equivalent.

Proof. Suppose that we have the equation (2.4), then we may define the homomorphism

$$\Phi : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z},$$

induced by $\Phi(1) = s$, then for each j we have

$$\Phi(\xi_j) = s\xi_j = \xi'_j .$$

Since s and n are relative primes, Φ is an automorphism.
We consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \Gamma & \xrightarrow{\theta} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\ & & \vdots & & \vdots\, \chi & & \downarrow \Phi & & \\ 0 & \longrightarrow & K' & \longrightarrow & \Gamma' & \xrightarrow{\theta'} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \end{array}$$

hence we may define

$$\chi(a_j) = a'_i, \qquad \chi(b_i) = b'_j, \qquad \chi(x_j) = x'_j$$

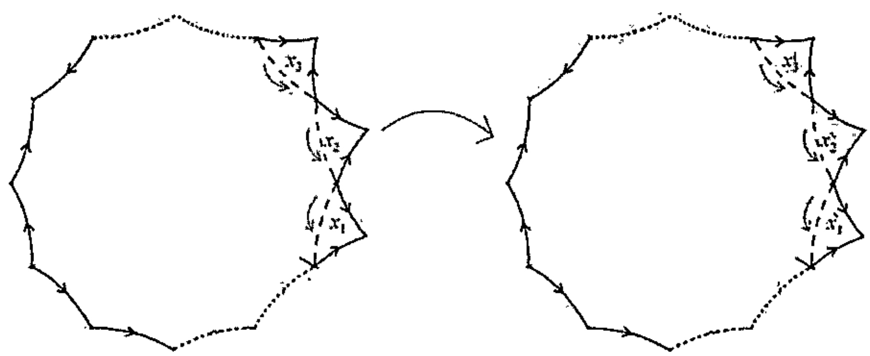
and then we have that the diagram is commutative.
Since χ maps generators on generators, and these elements satisfy the same relation, we have χ is an isomorphism.

Furthermore $\chi(K) = K'$ since for $x \in K$, we have

$$\begin{aligned} \theta'\chi(x) &= \Phi\theta(x) = \Phi(0) = 0 \\ &\text{therefore } \chi(x) \in K' \end{aligned}$$

therefore $\chi|_K : K \to K'$ is an isomorphism.
It follow of Theorem 8 that the actions induced by θ and by θ' are topologically equivalent.

Our next claim is that the actions induced by θ and θ' are directly topologically equivalent. We have to construct according to [13] the hyperbolic polygons associated Γ and Γ'.



Since x_j and x'_j are positive minimal rotation in the same angle, we have the automorphism χ is induced by a $f \in$ Homeo $^+(\Delta)$. $\square$

Remark 10. Let ε and ε' be actions of $\mathbb{Z}/n\mathbb{Z}$ on S and S', respectively, such that the signature for the actions is $(\gamma; m_1, \cdots, m_r)$. Suppose ε and ε' are topologically equivalent actions. Then by Theorem 8 we obtain the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\theta_\varepsilon} & \varepsilon(\mathbb{Z}/n\mathbb{Z}) \\ {\scriptstyle \chi_T}\downarrow & & \downarrow{\scriptstyle \Phi_t} \\ \Gamma' & \xrightarrow{\theta_{\varepsilon'}} & \varepsilon'(\mathbb{Z}/n\mathbb{Z}) \end{array}$$

where χ_T, Φ_t are group isomorphism.
Since ε is a monomorphism, then $\varepsilon(\mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$, the same for ε', thus associated to the previous diagram we have

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\theta_\varepsilon} & \varepsilon(\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{(\varepsilon|)^{-1}} & \mathbb{Z}/n\mathbb{Z} \\ {\scriptstyle \chi_T}\downarrow & & \downarrow{\scriptstyle \Phi_t} & & \vdots{\scriptstyle \widetilde{\Phi}} \\ \Gamma' & \xrightarrow{\theta_{\varepsilon'}} & \varepsilon'(\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{(\varepsilon'|)^{-1}} & \mathbb{Z}/n\mathbb{Z} \end{array}$$

then $\widetilde{\Phi}$ is an automorphism of $\mathbb{Z}/n\mathbb{Z}$.

We recall Γ is a Fuchsian groups with signature $(\gamma; m_1 \cdots m_r)$ then we have

$$\Gamma = \left\langle a_1, b_1.., a_\gamma, b_\gamma, x_1.., x_r : x_j^{m_j} = 1, \prod [a_i, b_i] \prod x_j = 1 \right\rangle ,$$

Since the kernel of θ_ε is torsion free, we have that for each j

$$\operatorname{ord}(\theta_\varepsilon(x_j)) = m_j .$$

Now if we call

$$\theta = (\varepsilon|)^{-1} \circ \theta_\varepsilon ,$$
$$\theta' = (\varepsilon'|)^{-1} \circ \theta_{\varepsilon'} ,$$

we have for each j

$$\widetilde{\Phi}\theta(x_j) = \theta'(\chi_T(x_j)) ,$$

and since $\widetilde{\Phi}$ is an automorphism, we have that there exists $1 \leq s \leq n-1$, $(s,n) = 1$, such that $\widetilde{\Phi}(1) = s$, then

$$\theta'(\chi_T(x_j)) = s \cdot \theta(x_j) .$$

By the preceding remark it follow the Theorem 14. Observe that this result is a generalization of a result of J. Gilman in which n is a prime number (See [9, Lemma 2, p. 54].)

Theorem 14. *If ε and ε' are topologically equivalent actions, then we have the following commutative diagram*

$$\begin{array}{ccc} \Gamma & \xrightarrow{\theta} & \mathbb{Z}/n\mathbb{Z} \\ {\scriptstyle \chi_T}\downarrow & & \downarrow{\scriptstyle \widetilde{\Phi}_t} \\ \Gamma' & \xrightarrow{\theta'} & \mathbb{Z}/n\mathbb{Z} \end{array}$$

such that, for each j, we have

$$\theta'(\chi_T(x_j)) = s \cdot \theta(x_j)$$

where $\widetilde{\Phi}(1) = s$.

2.4 Generalization of Harvey's result

In this section we generalize a result due to Harvey. In [10] Harvey gives a relationship between cyclic covering of Riemann sphere and the epimorphisms of Fuchsian group using the rotation angles. Our result gives a relationship for any covering of Riemann sphere.

Let S be a compact Riemann surface of genus at least 2.
Let G be a subgroup of $\mathrm{Aut}(S)$, where G acts on S with signature $(0; m_1, \cdots, m_r)$. Then there exists a Fuchsian group K uniformising S and Γ a Fuchsian group with signature $(0; m_1, \cdots, m_r)$, such that we have the following diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{F} & X \\ \downarrow & & \downarrow \\ \Delta/K & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ \Delta/\Gamma & \longrightarrow & S/G \end{array}$$

where in each level we have isomorphisms of Riemann surfaces.

Then we have the following exact sequence

$$1 \to K \to \Gamma \xrightarrow{\theta} G \to 1$$

where $\theta(x) = f \circ \widetilde{x} \circ f^{-1}$ and $\widetilde{x}$ is given by the following diagram:

$$\begin{array}{ccc} \Delta & \xrightarrow{x} & \Delta \\ \downarrow & & \downarrow \\ \Delta/K & \xrightarrow{\widetilde{x}} & \Delta/K \end{array}$$

We remark that for $x_1, \cdots, x_r \in \Gamma$ such that

- $x_1 \cdots x_r = 1$,

- $\Gamma = < x_1, \cdots, x_r >$
- for each $j = 1, .., r-1$, we have x_j is a counterclockwise rotation about z_j through angle $\frac{2\pi}{m_j}$,

then we know

$$\delta_{P_j}(\theta(x_j)) = \omega_{m_j} = \exp\left(\frac{2\pi i}{m_j}\right), \quad \text{with } P_j = f(Kz_j).$$

Theorem 15. *The epimorphism θ is determined by the fixed points of the action and their stabilizer groups. In other words, if we consider x_j and P_j as before, then*

$$\theta(x_j) = \tau_j^{\xi_j}$$

where $< \tau_j > = G_{P_j}$, and where the number ξ_j is determined by the equations

$$\begin{aligned} \delta_{P_j}(\tau_j) &= \omega_{m_j}^{\eta_j}, \qquad 1 \le \eta_j < m_j, (\eta_j, m_j) = 1, \\ \eta_j \cdot \xi_j &\equiv 1 \mod m_j \end{aligned}$$

where $1 \le \xi_j < m_j$, with $(\xi_j, m_j) = 1$.

Proof. Let τ_j be a generator of the group G_{P_j}.
As $\theta(x_j) \in G_{P_j}$, then

$$\theta(x_j) = \tau_j^{\xi_j},$$

for some $0 < \xi_j < m_j$.

We recall $\delta_{P_j}(\theta(x_j)) = \omega_{m_j}$, but as δ_{P_j} is a group monomorphism there exists a unique ξ_j, with $1 \le \xi_j < m_j, (\xi_j, m_j) = 1$ and $\delta_{P_j}(\tau_j^{\xi_j}) = \omega_{m_j}$.
Now we will calculate ξ_j. If

$$\delta_{P_j}(\tau_j) = \omega_{m_j}^{\eta_j}$$

then

$$\delta_{P_j}(\tau_j^{\xi_j}) = \omega_{m_j}^{\eta_j \cdot \xi_j}$$

therefore

$$\eta_j \cdot \xi_j \equiv 1 \mod m_j$$

If we take another generator of G_{P_j}, say $\widehat{\tau}_j$, then we may to do the same calculations, hence

$$\theta(x_j) = \widehat{\tau}_j^{\widehat{\xi}_j}$$

where

$$\begin{aligned} \delta_{P_j}(\widehat{\tau}_j) &= \omega_{m_j}^{\widehat{\eta}_j} \\ \widehat{\eta}_j \cdot \widehat{\xi}_j &\equiv 1 \mod m_j \end{aligned}$$

As τ_j is a generator of G_{P_j}, we have there exists $0 < t < m_j$ such that

$$\widehat{\tau}_j = \tau_j^t$$

thus we have

$$\omega_{m_j}^{\widehat{\eta}_j} = \delta_{P_j}(\widehat{\tau}_j) = \delta_{P_j}(\tau_j^t) = \omega_{m_j}^{\eta_j \cdot t}$$

then

$$\begin{aligned} \widehat{\eta}_j &\equiv \eta_j \cdot t \mod m_j \qquad | \cdot \xi_j \\ \widehat{\eta}_j \cdot \xi_j &\equiv \eta_j \cdot \xi_j t \equiv t \mod m_j \qquad | \cdot \widehat{\xi}_j \\ \xi_j \equiv \widehat{\xi}_j \cdot \widehat{\eta}_j \cdot \xi_j &\equiv t \cdot \widehat{\xi}_j \mod m_j \end{aligned}$$

therefore

$$\tau_j^{\xi_j} = \tau_j^{t \cdot \widehat{\xi}_j} = \widehat{\tau}_j^{\widehat{\xi}_j}$$

□

Remark 11. If we take $\sigma \in G$, then $\sigma(P_j) \in G(P_j)$, where $G(P_j)$ is the orbit of P_j by the action of G.
Now we may consider $(\sigma \circ f) : \Delta/K \to S$, and we have the following diagram

$$\begin{array}{ccccc} \Delta & \xrightarrow{F} & X & \xrightarrow{\overline{\sigma}} & X \\ \downarrow & & \downarrow & & \downarrow \\ \Delta/K & \xrightarrow{f} & S & \xrightarrow{\sigma} & S \end{array}$$

then we may define an epimorphism from Γ to G associated to $\sigma \circ f$:

$$\widetilde{\theta}(x_j) = \sigma_j^{\xi_j}$$

where $<\sigma_j>= G_{\sigma(P_j)} =<\sigma\circ\tau_j\circ\sigma^{-1}>$ hence by the previous

$$\widetilde{\theta}(x_j)=\sigma\circ\tau_j^{\xi_j}\circ\sigma^{-1}=\sigma\circ\theta(x_j)\circ\sigma^{-1}$$

therefore we have the following diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\theta} & G \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \psi_\sigma} \\ \Gamma & \xrightarrow{\widetilde{\theta}} & G \end{array}$$

where $\psi_\sigma(\tau)=\sigma\circ\tau\circ\sigma^{-1}$.

Remark 12. Take the automorphism $\chi=\widehat{\chi}_k$. We have that

$$\begin{aligned} \theta(\chi(x_k)) &= \theta(x_{k+1})=\tau_{k+1}^{\xi_{k+1}} \\ \theta(\chi(x_{k+1})) &= \theta(x_{k+1}^{-1}x_kx_{k+1})=\tau_{k+1}^{-\xi_{k+1}}\tau_k^{\xi_k}\tau_{k+1}^{\xi_{k+1}}. \end{aligned}$$

However, we cannot conclude whether the actions are equivalent, or in other words we do not know, in general, whether there exists a $\Phi\in\mathrm{Aut}(G)$ so that the following diagram is commutative

$$\begin{array}{ccc} \Gamma & \xrightarrow{\theta} & G \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \Gamma & \xrightarrow{\theta\circ\chi} & G \end{array}$$

Let S be a compact Riemann surface of genus $g\geq 2$.

Definition 7. Let G be a subgroup of $\mathrm{Aut}(S)$ such that G acts on S with signature $(0;m_1,\cdots,m_r)$.
We say that the $r-$tuple $(\sigma_1,\cdots,\sigma_r)\in G^r$ is a *generating vector* for G of type $(0;m_1,\cdots,m_r)$ if

1. $G=<\sigma_1,\cdots,\sigma_r>$.
2. $\mathrm{ord}(\sigma_j)=m_j$, for each $j=1,..,r$.
3. $\sigma_1\cdots\sigma_r=1$.

For the action of G on S, we have that for each $j = 1, .., r-1$, there is $x_j \in \Gamma$ a counterclockwise rotation about z_j through angle $\frac{2\pi}{m_j}$.
By definition,

$$\theta(x_j) = \tau_j^{\xi_j}$$

where $K = \ker(\theta)$, $P_j = f(Kz_j)$, $\tau_j \in G_{P_j}$ and $\delta_{P_j}(\tau_j^{\xi_j}) = \omega_{m_j}$.

According to [19], the tuple

$$(\theta(x_1)^{-1}, \cdots, \theta(x_r)^{-1}),$$

determines the decomposition of the Analytic Representation into complex irreducible representations of G by the following formula.

Proposition 2 (Chevalley-Weil formula). *Consider $\{\rho_1, .., \rho_q\}$ the complex irreducible representation of G. Then the multiplicity n_t for ρ_t in the analytic representation is given by*

$$n_t = -\deg(\rho_t) + \sum_{j=1}^{r} \sum_{k=0}^{m_j-1} N_{tjk} \left\langle -\frac{k}{m_j} \right\rangle + \vartheta$$

where

- $\vartheta = 1$ *if ρ_t is the trivial representation; otherwise $\vartheta = 0$.*
- $\langle k \rangle = k - [k]$*, where $[k]$ is the integer part of the number k.*
- N_{tjk} *is the cardinality of the eigenvalues of $\rho_t\left(\theta(x_j^{-1})\right)$ which are equal to $\omega_{m_j}^k$, where $\omega_{m_j} = \exp\left(\frac{2i\pi}{m_j}\right)$.*

Chapter 3

Families of Riemann Surfaces with equivalent actions

In this chapter we produce for each n, two families of Riemann surfaces of genus $3(2^n - 1)$ with group of automorphisms of order 2^{n+2} and signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2)$. For one of the families the group we will be abelian, and for the other it will be a semidirect product. In both cases, we will have that there exist two cyclic subgroups which define directly topologically, but not conformally, equivalent actions.

3.1 Structure of the groups

3.1.1 The group $G_1 = \mathbb{Z}/2^{n+1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Consider

$$\mathbb{Z}/2^{n+1}\mathbb{Z} =< A_1 : A_1^{2^{n+1}} = 1 >$$

and

$$\mathbb{Z}/2\mathbb{Z} =< C_1 : C_1^2 = 1 > .$$

Since G_1 is an abelian group, it is not difficult to verify that the elements of G_1 are

$$G_1 = \left\{1, A_1, A_1^2, \cdots, A_1^{2^{n+1}-1}, C_1, A_1C_1, \cdots, A_1^{2^{n+1}-1}C_1\right\}$$

Proposition 3. *G_1 has a presentation of the form*

$$\left\langle g_1, g_2 \, : \, g_1^{2^{n+1}} = g_2^{2^{n+1}} = 1 \, , g_1^2 = g_2^2 \, , g_1g_2 = g_2g_1 \right\rangle .$$

Proof. We first take
$$g_1 = A_1, \qquad g_2 = A_1^{1-2^n} C_1$$
Since A_1 has order 2^{n+1} and C_1 has order 2 it follows that g_1 and g_2 holds the relationship.

We claim that g_1 and g_2 generating G_1. In fact
$$A_1 = g_1, \quad C_1 = g_1^{2^n-1} g_2$$
□

Elements on G_1

For now on we call $B_1 = g_2$.
Using the relationships above, we may compute the elements of $\widetilde{G}_1$ as follows:

elements	exponent	number of this type
A_1^j	$0 \leq j \leq 2^{n+1} - 1$	2^{n+1}
B_1^j	j is an odd numer	2^n
$A_1^j B_1$	j is an odd number	2^n
		2^{n+2} elements

In addition, for each element in $\widetilde{G}_1$ we may compute its order as follows. Computing
$$\left(A_1^j B_1\right)^2 = A_1^{2(j+1)},$$
for $j = 2^k t - 1$ we have
$$\operatorname{ord}(A_1^j B_1) = \left\{ \begin{array}{cc} 2^{n-k+1} & , k \leq n, t \equiv 1 \mod 2 \\ 2 & , \text{otherwise} \end{array} \right\}$$

Group Representations

Since G_1 is an abelian group, the number of irreducible representations over $\mathbb{C}$ is equal to the order of G_1. Thus the number of irreducible representations is 2^{n+2}.

Now for $0 \leq j < 2^{n+1}$, the irreducible representations are
$$\begin{array}{rcl} \rho_{j,0} : \widetilde{G}_1 & \longrightarrow & \mathbb{C} \\ A_1 & \rightsquigarrow & \omega_{2^{n+1}}^j \\ B_1 & \rightsquigarrow & \omega_{2^{n+1}}^j \end{array}$$

$$\begin{array}{rcl}\rho_{j,1} : \widetilde{G}_1 & \longrightarrow & \mathbb{C} \\ A_1 & \rightsquigarrow & \omega_{2n+1}^{j} \\ B_1 & \rightsquigarrow & -\omega_{2n+1}^{j} \,.\end{array}$$

Here $\omega_m = \exp\left(\dfrac{2\pi i}{m}\right)$.

3.1.2 The group $G_2 = \mathbb{Z}/2^{n+1}\mathbb{Z} \rtimes_h \mathbb{Z}/2\mathbb{Z}$

Consider

$$\mathbb{Z}/2^{n+1}\mathbb{Z} =< A_2 : A_2^{2^{n+1}} = 1 >$$

and

$$\mathbb{Z}/2\mathbb{Z} =< C_2 : C_2^2 = 1 > \; .$$

We define the map

$$\begin{array}{rcl} h :< C_2 > & \longrightarrow & \mathrm{Aut}(< A_2 >) \\ C_2 & \rightsquigarrow & h(C_2) : A_2 \to C_2A_2C_2 = A_2^{2^n+1} \,. \end{array}$$

Thus $G_2 =< A_2 > \rtimes_h < C_2 >$.

Proposition 4. *G_2 has a presentation of the form*

$$\left\langle g_1, g_2 \,:\, g_1^{2^{n+1}} = g_2^{2^{n+1}} = 1 \,, g_1^2 = g_2^2 \,, g_2g_1 = g_1^{2^n+1}g_2 \right\rangle$$

Proof. We first take

$$g_1 = A_2, \qquad g_2 = A_2^{1-2^{n-1}}C_2$$

Since A_2 has order 2^{n+1} and C_2 has order 2 it follows g_1 and g_2 holds the relationship. In fact

$$\begin{array}{rcl} g_2^2 & = & A_2^{1-2^{n-1}}C_2A_2^{1-2^{n-1}}C_2 = A_2^{1-2^{n-1}}A_2^{(1-2^{n-1})(2^n+1)} = A_2^2 = g_1^2 \\ g_2g_1 & = & A_2^{1-2^{n-1}}C_2A_2 = A_2^{1-2^{n-1}}A_2^{2^n+1}C_2 = A_2^{2^{n+1}}g_2 = g_1^{2^n+1}g_2 \end{array}$$

It follows immediately that g_1 and g_2 generating G_2. □

Elements of G_2

We consider G_2 with presentation (4). For now on we call $B_2 = g_2$.

G_2 is a group of order 2^{n+2}, we may calculate all elements on G_2 and its orders, we have

$$\left(A_2^j B_2\right)^2 = A_2^{2(2^{n-1}+j+1)}$$

and thus for $j = 2^k t - 2^{n-1} - 1$

$$\operatorname{ord}(A_2^j B_2) = \left\{ \begin{array}{cc} 2^{n-k+1} & ,k \le n, t \equiv 1 \mod 2 \\ 2 & ,\text{in other case} \end{array} \right\}$$

then we have the following table of elements on G_2

elements	exponent	number of this type
A_2^j	$0 \le j \le 2^{n+1} - 1$	2^{n+1}
B_2^j	j is an odd numer	2^n
$A_2^j B_2$	j is an odd number	2^n
		2^{n+2} elements

Furthermore we have

g	exponent	gA_2g^{-1}
A_2^j	$0 \le j \le 2^{n+1} - 1$	A_2
B_2^j	j is an odd numer	$A_2^{2^n+1}$
$A_2^j B_2$	j is an odd number	$A_2^{2^n+1}$

Group Representations

Using the Little Group Method (see Proposition 1), we may compute the complex irreducible representations of G_2, which we explain in the next Proposition.

Proposition 5. *Let G_2 be a finite group of order 2^{n+2} with presentation as in (4). Then G_2 has $3 \cdot 2^{n-1}$ irreducible representations, where 2^{n+1} of them have degree 1, and 2^{n-1} of them have degree 2.*

Proof. We use the Little Group Method.

We have

$$\begin{aligned} \mathcal{R}_{A_2} &= \{\text{irreducible representations of } < A_2 >\} \\ &= \{\rho_j : A_2 \to \omega_{2^{n+1}}^j : 0 \le j < 2^{n+1}\} \end{aligned}$$

where $\omega_{2^{n+1}} = \exp\left(\dfrac{2\pi i}{2^{n+1}}\right)$.

G_2 acts on $\mathcal{R}_{A_2}$, the actions is given by: for each $g \in G_2$, $\rho_j \in \mathcal{R}_{A_2}$ we define

$$g \bullet \rho_j = \rho_j^g : \qquad \rho_j^g(A_2) = \rho_j(gA_2g^{-1})$$

Then we have

$$\begin{aligned} \rho_j^g &= \rho_j \\ \Leftrightarrow \rho_j^g(A_2) &= \rho_j(A_2) \\ \Leftrightarrow \rho_j(gA_2g^{-1}) &= \rho_j(A_2) \\ \Leftrightarrow A_2^j &= \begin{cases} A_2^j & , g \in < A_2 > \\ A_2^{(2^n+1)j} & , \text{another case} \end{cases} \\ \Leftrightarrow j &\in 2\mathbb{Z} \end{aligned}$$

Now we calculate the orbits for the action, for example for $j = 1$ we have

$$\begin{aligned} \rho_1^g(A_2) &= \rho_1(gA_2g^{-1}) \\ &= \begin{cases} \rho_1(A_2) & , g \in < A_2 > \\ \rho_1(A_2^{2^n+1}) & , \text{another case} \end{cases} \\ &= \begin{cases} A_2 & , g \in < A_2 > \\ A_2^{2^n+1} & , \text{another case} \end{cases} \end{aligned}$$

therefore the orbit for ρ_1 is

$$[\rho_1] = \{\rho_1, \rho_{2^n+1}\}$$

Using the same ideas we have

$$[\rho_j] = \begin{cases} \{\rho_j\} & , j \in 2\mathbb{Z} \\ \{\rho_j, \rho_{(2^n+1)j}\} = \{\rho_j, \rho_{2^n+j}\} & , \text{another case} \end{cases}$$

where $\rho_{(2^n+1)j} = \rho_{2^n+j}$, since for $j = 2k + 1$ we have

$$\begin{aligned} (2^n + 1)j &= 2^n j + j \\ &= 2^n(2k + 1) + 2k + 1 \\ &= 2^{n+1}k + 2^n + 2k + 1 \\ &= 2^{n+1}k + 2^n + j \end{aligned}$$

Thus we call $\mathcal{R}_{A_2}^K$ the orbits space under the action of K, more precisely if $\rho_{j_1}, \rho_{j_2} \in \mathcal{R}_{A_2}^K$ then ρ_{j_2} are not same orbit that ρ_{j_1}. Then we have

$$\mathcal{R}_{A_2}^{G_2} = \left\{ \rho_j : \begin{array}{ll} j \text{ is an odd number} & , 0 \le j < 2^n \\ j \text{ is an even number} & , 0 \le j < 2^{n+1} \end{array} \right\}.$$

Now we consider $H =< A_2^{2^{n-1}-1}B >$. Recalling the previous calculations we see that

$$\mathcal{R}_{A_2}^{H} = \mathcal{R}_{A_2}^{G_2}$$

and the cardinality of $\mathcal{R}_A^{G_2}$ is $2^{n-1} + 2^n$.

For each $\rho_j \in \mathcal{R}_{A_2}^{G_2}$ we consider

$$K_j = \{k \in K < G_2 : k \bullet \rho_j = \rho_j\}$$

and then we have

$$H_j = \left\{ \begin{array}{ll} H & ,j \text{ is an even number} \\ \{1\} & ,j \text{ is an odd number} \end{array} \right\}, \qquad G^{(j)} = \left\{ \begin{array}{ll} G_2 & ,j \text{ is an even number} \\ < A_2 > & ,j \text{ is an odd number} \end{array} \right\},$$

The process continue extending the action ρ_j to $G^{(j)}$.
When j is an odd number, this is trivial, since we have $G^{(j)} =< A_2 >$.
Now we extend when j is an even number, this is

$$\overline{\rho}_j = \left\{ \begin{array}{ll} \rho_j(1) & ,g \in H \\ \rho_j(g) & ,g \in < A_2 > \end{array} \right\}$$

For H we have two irreducible representation, if we call $C_2 = A_2^{2^{n-1}-1}B_2$, they are

$$\{\sigma_0 : C_2 \to 1, \qquad \sigma_1 : C_2 \to -1\}$$

Now we may extend the representation, this is

$$\overline{\sigma}_i = \left\{ \begin{array}{ll} \sigma_i(g) & ,g \in H \\ \sigma_i(1) & ,g \in < A_2 > \end{array} \right\}$$

When j is an even number and $0 \leq j < 2^{n+1}$, we may define

$$\alpha_{j,i} = \overline{\rho}_j \otimes \overline{\sigma_i}$$

where $i = 0, 1$.
Therefore we have 2^{n+1} irreducible representations of degree 1.

When j is an odd number the before process is not produce representations for G_2.
Now we will calculate the induced representation for ρ_j, that we denote by β_j, with $0 \leq j < 2^n$.
We know $G_2 =< A_2 > \quad \cup \quad C_2 < A_2 >$ then

$$\deg(\beta_j) = |G_2 :< A_2 > | \deg(\rho_j) = 2$$

Therefore we have 2^{n-1} irreducible representations of degree 2. □

Remark 13. We now compute the representations.

When j is an even number we have:

$$\alpha_{i,j}(A_2) = \overline{\rho}_j(A_2) \otimes \overline{\sigma_i}(1) = \omega_{2^n+1}^j$$

$$\alpha_{i,j}(C_2) = \overline{\rho}_j(1) \otimes \overline{\sigma_i}(C_2) = \begin{cases} 1 & ,i=0 \\ -1 & ,i=1 \end{cases}$$

For $B_2 = A_2^{1-2^{n-1}} C_2$ we have

$$\alpha_{i,j}(B_2) = \overline{\rho}_j(A_2^{1-2^{n-1}}) \otimes \overline{\sigma_i}(C_2) = \begin{cases} \omega_{2^n+1}^{(1-2^{n-1})j} & ,i=0 \\ -\omega_{2^n+1}^{(1-2^{n-1})j} & ,i=1 \end{cases}$$

When j is an odd number, we consider $V = \mathbb{C} \oplus C_2\mathbb{C}$. Thus

$$\begin{aligned} A_2 \cdot 1 &= A_2 \\ A_2 \cdot C_2 &= A_2^{2^{n-1}} B_2 \\ &= A_2^{2^{n-1}-1} A_2 B_2 \\ &= A_2^{2^{n-1}-1} B_2 A_2^{2^n+1} \\ &= C_2 A_2^{2^n+1} \end{aligned}$$

Then we have

$$A_2 \cdot (z_1 \oplus c z_2) = A_2 z_1 \oplus C_2 A_2^{2^n+1} z_2$$

and

$$\begin{aligned} \beta_j(A_2) &= \rho_j(A_2) z_1 \oplus C_2 \rho_j(A_2^{2^n+1}) z_2 \\ &= \omega_{2^n+1}^j z_1 \oplus C_2 \omega_{2^n+1}^{(2^n+1)j} z_2 \\ &= \omega_{2^n+1}^j z_1 \oplus -C_2 \omega_{2^n+1}^j z_2 \end{aligned}$$

therefore the matrix associated to $\beta_j(A_2)$ in the canonical basis is

$$[\beta_j(A_2)] = \begin{pmatrix} \omega_{2^n+1}^j & 0 \\ 0 & -\omega_{2^n+1}^j \end{pmatrix}.$$

Now for C_2, we have

$$C_2 \cdot (z_1 \oplus C_2 z_2) = C_2 z_1 \oplus z_2$$

therefore the matrix associated to $\beta_j(C_2)$ in the canonical basis is

$$[\beta_j(C_2)] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Finally, we compute the matrix for $\beta_j(B_2)$. Since $B_2 = A_2^{1-2^{n-1}} C_2$, the matrix in the canonical basis is

$$[\beta_j(B_2)] = \begin{pmatrix} 0 & -\omega_{2^n+1}^{(2^{n-1}+1)j} \\ \omega_{2^n+1}^{(2^{n-1}+1)j} & 0 \end{pmatrix}.$$

3.2 Families of Riemann Surfaces

In this section we define Riemann surfaces of genus $3(2^n-1)$. This construction is adapted from [8].

Theorem 16. *Let $f_{a,\lambda}$ be the polynomial given by*

$$f_{a,\lambda}(x,y) = y^{2^n} - x^a\left(x^2-1\right)^a\left(x^2-\lambda^2\right)\left(x^2-\lambda^{-2}\right)$$

where $n, a \in \mathbb{N}$, $\lambda \in \mathbb{C}$, and $\lambda^4 \neq 1, 0$.
Then, for each odd number a and each λ, we have that $f_{a,\lambda}$ defines a Riemann surface $S_{a,\lambda}$ of genus $3(2^n-1)$.
Furthermore, the possible singular points for the homogeneous polynomial associated to $f_{a,\lambda}$ are

case	*sign*(2^n-3a-4)	a	*condition*	*singular points*
1	$+$	$\neq 1$	$2^n-3a-5\neq 0$	$\{[0,0,1],[1,0,0],[1,0,1],[-1,0,1]\}$
2	$+$	$\neq 1$	$2^n-3a-5=0$	$\{[0,0,1],[1,0,1],[-1,0,1]\}$
3	$+$	1	$2^n-3a-5\neq 0$	$\{[1,0,0]\}$
4	$+$	1	$n=3$	$\emptyset$
5	$-$	$\neq 1$		$\{[0,0,1],[0,1,0],[1,0,1],[-1,0,1]\}$
6	$-$	1	$n=1,2$	$\{[0,1,0]\}$

Remark 14. Observe that, for the conditions given in the above Table, we have the following.

1. If $2^n-3a-4=0$ (hence $3a=2^n-4$), then a is an even number.

2. We have $2^n-5=3a$, for some odd number a, if and only if n is an odd number. In fact, we use the induction process. For $n=1$ it is clear. Now suppose 2^n+5 is multiple of 3. Then

$$2^{n+2}+5 = 2^n(4)+5 = 2^n(3+1)+5 = 2^n\cdot 3+2^n+5\,.$$

   Moreover if $a=1$, then $2^n=8$. Hence $n=3$.

3. The equations $2^n-3a-3=0$ has not solutions.

Proof. We have two cases for the homogeneous polynomial associated to $f_{a,\lambda}$.

1. Case $2^n - 3a - 4 > 0$.

$$F_1(X,Y,Z) = Y^{2^n} - X^a Z^{2^n-3a-4}\left(X^2 - Z^2\right)^a \left(X^2 - \lambda^2 Z^2\right)\left(X^2 - \lambda^{-2} Z^2\right)$$

$$\begin{aligned}
\frac{\partial F_1}{\partial X} &= -X^{a-1} Z^{2^n-3a-4}\left(X^2 - Z^2\right)^{a-1}[a\left(X^2 - Z^2\right)\left(X^2 - \lambda^2 Z^2\right)\left(X^2 - \lambda^{-2} Z^2\right) \\
&\quad +2aX^2\left(X^2 - \lambda^2 Z^2\right)\left(X^2 - \lambda^{-2} Z^2\right) \\
&\quad +2X^2\left(X^2 - Z^2\right)\left(2X^2 - \left(\lambda^2 + \lambda^{-2}\right)Z^2\right)] \\
\frac{\partial F_1}{\partial Y} &= 2^n Y^{2^n-1} \\
\frac{\partial F_1}{\partial Z} &= -X^a Z^{2^n-3a-5}\left(X^2 - Z^2\right)^{a-1}[\\
&\quad \left(2^n - 3a - 4\right)\left(X^2 - Z^2\right)\left(X^2 - \lambda^2 Z^2\right)\left(X^2 - \lambda^{-2} Z^2\right) \\
&\quad +2aZ^2\left(X^2 - \lambda^2 Z^2\right)\left(X^2 - \lambda^{-2} Z^2\right) \\
&\quad -2Z^2(X^2 - Z^2)\left(\left(\lambda^2 + \lambda^{-2}\right)X^2 - 2Z^2\right)]
\end{aligned}$$

Recall that $P \neq 0$ is a singular point if only if

$$F_1(P) = \frac{\partial F_1}{\partial X}(P) = \frac{\partial F_1}{\partial Y}(P) = \frac{\partial F_1}{\partial Z}(P) = 0\,.$$

then we have $Y = 0$

Further if

- $a \neq 1, 2^n - 3a - 5 \neq 0$, we have 4 singular points, they are

$$\{[0,0,1],[1,0,0],[1,0,1],[-1,0,1]\}\ .$$

- $a \neq 1, 2^n - 3a - 5 = 0$, then n is an odd number and the singular points are

$$\{[0,0,1],[1,0,1],[-1,0,1]\}\ .$$

- $a = 1, 2^n - 3a - 5 \neq 0$, then $n > 3$, since

$$2^n - 7 > 0 \text{ and } 2^n - 8 \neq 0\,.$$

  Then the singular points are

$$\{[1,0,0]\}$$

- $a = 1, 2^n - 3a - 5 = 0$, then $n = 3$, since

$$2^n - 7 > 0 \text{ and } 2^n - 8 = 0$$

Then in this case F_1 has not singular points.

Charts for the singular points:

(a) $P_1 = [0, 0, 1]$
We consider the local coordinates:

$$(x, y) = \left(\frac{X}{Z}, \frac{Y}{Z}\right),$$

thus the objects on $S_{a,\lambda}$ near at P_1 are determinate by the following polynomial

$$f(x, y) = y^{2^n} - x^a \left(x^2 - 1\right)^a \left(x^2 - \lambda^2\right) \left(x^2 - \lambda^{-2}\right).$$

This is an irreducible polynomial on $\mathbb{C}[y]\{x\}$ if only if a is an odd number.
For $x = s^{2^n}$, we have:

$$\begin{aligned} f\left(s^{2^n}, y\right) &= y^{2^n} - s^{a2^n} \left(s^{2^n} - 1\right)^a \left(s^{2^n} - \lambda^2\right) \left(s^{2^n} - \lambda^2\right) \\ &= \prod_{j=1}^{2^n} (y - s^a h_j(s)) \end{aligned}$$

where $(h_j(s))^{2^n} = \left(s^{2^n} - 1\right)^a \left(s^{2^n} - \lambda^2\right) \left(s^{2^n} - \lambda^2\right)$.

Thus h_j is determinate by some of these roots, we denote h_0 such that $h_0(0) = 1$.

Therefore by the Normalization process, we have a chart for P_1 given by

$$s \leftrightsquigarrow [s^{2^n}, s^a h_0(s), 1].$$

(b) For the points $P_2 = [1, 0, 1]$ and $P_3 = [-1, 0, 1]$ we may define charts using the same previous process.
Thus the chart is given by

$$s \leftrightsquigarrow [s^{2^n} + 1, s^a h_1(s), 1],$$

where $h_1(s)$ is a root 2^nth of

$$\left(s^{2^n}+1\right)^a\left(s^{2^n}+2\right)^a\left(\left(s^{2^n}+1\right)^2-\lambda^2\right)\left(\left(s^{2^n}+1\right)^2-\lambda^{-2}\right).$$

(c) $P_4=[1,0,0]$
We consider the local coordinates

$$(u,v)=\left(\frac{Y}{X},\frac{Z}{X}\right),$$

thus the objects on $S_{a,\lambda}$ near at P_4 are determinate by the following polynomial

$$f_\infty(u,v)=u^{2^n}+v^{2^n-3a-4}(1-v^2)^a(1-\lambda^2v^2)(1-\lambda^{-2}v^2).$$

Then we have f_∞ is irreducible on $\mathbb{C}[u]v$ if only if 2^n-3a-4 is an odd number, and this is equivalent to that a is an odd number too.
For $v=t^{2^n}$ we have

$$\begin{aligned}f_\infty\left(u,t^{2^n}\right) &= u^{2^n}+t^{2^n(2^n-3a-4)}(1-t^{2^{n+1}})^a(\lambda^2-t^{2^{n+1}})(\lambda^{-2}-t^{2^{n+1}})\\ &= \prod_{j=1}^{2^n}\left(u-t^{2^n-3a-4}h_j(t)\right)\end{aligned}$$

where $(h_j(t))^{2^n}=\left(1-t^{2^{n+1}}\right)^a\left(\lambda^2-t^{2^{n+1}}\right)\left(\lambda^{-2}-t^{2^{n+1}}\right)$.
Then h_j is determinate by some of these roots, we denote h_∞ such that $h_\infty(0)=1$.

Therefore by the Normalization process a chart for $P_4=[1,0,0]$ is given by

$$t\leadsto[1,t^{2^n-3a-4}h_\infty(t),t^{2^n}]$$

2. Case $2^n-3a-4<0$.

$$F_2(X,Y,Z)=Y^{2^n}Z^{3a+4-2^n}-X^a\left(X^2-Z^2\right)^a\left(X^2-\lambda^2Z^2\right)\left(X^2-\lambda^{-2}Z^2\right).$$

$$
\begin{aligned}
\frac{\partial F_2}{\partial X} &= -aX^{a-1}\left(X^2-Z^2\right)^{a-1}\left[a\left(X^2-Z^2\right)\left(X^2-\lambda^2Z^2\right)\left(X^2-\lambda^{-2}Z^2\right)\right. \\
&\quad +2aX^2\left(X^2-\lambda^2Z^2\right)\left(X^2-\lambda^{-2}Z^2\right) \\
&\quad \left.+2X^2\left(X^2-Z^2\right)\left(2X^2-\left(\lambda^2+\lambda^{-2}\right)Z^2\right)\right] \\
\frac{\partial F_2}{\partial Y} &= 2^nY^{2^n-1}Z^{3a+4-2^n} \\
\frac{\partial F_2}{\partial Z} &= \left(3a+4-2^n\right)Z^{3a+3-2^n}Y^{2^n}-X^a\left(X^2-Z^2\right)^{a-1}Z[\\
&\quad 2a\left(X^2-\lambda^2Z^2\right)\left(X^2-\lambda^{-2}Z^2\right) \\
&\quad +\left(X^2-Z^2\right)\left(\left(\lambda^2+\lambda^{-2}\right)X^2-2Z^2\right)]
\end{aligned}
$$

Then we have $P \neq 0$ is a singular points if only if

$$F_2(P)=\frac{\partial F_2}{\partial X}(P)=\frac{\partial F_2}{\partial Y}(P)=\frac{\partial F_2}{\partial Z}(P)=0\,.$$

If $a \neq 1$, we have that the singular points with $Y=0$ are

$$\{[0,0,1],[1,0,1],[-1,0,1]\}\ .$$

Further for $Z=0$, we have $[0,1,0]$ is a singular point , since for all n we have $3a+3-2^n \neq 0$.
Now when $a=1$ as $2^n-3a-4<0$ then we have $n \leq 2$. These cases are studied by G. González-Diez and R. Hidalgo in [8].

Charts for the singular points:

(a) $Q_1=[0,0,1]$
We consider locally coordinates

$$(x,y)=\left(\frac{X}{Z},\frac{Y}{Z}\right),$$

thus the objects on $S_{a,\lambda}$ near at Q_1 are determinate by the following polynomial

$$f(x,y)=y^{2^n}-x^a\left(x^2-1\right)^a\left(x^2-\lambda^2\right)\left(x^2-\lambda^{-2}\right)\ .$$

This is an irreducible polynomial on $\mathbb{C}[y]\{x\}$ if only if a is an odd number.

For $x = s^{2^n}$, we have:

$$\begin{aligned} f\left(s^{2^n}, y\right) &= y^{2^n} - s^{a2^n}\left(s^{2^n}-1\right)^a\left(s^{2^n}-\lambda^2\right)\left(s^{2^n}-\lambda^2\right) \\ &= \prod_{j=1}^{2^n}\left(y - s^a g_j(s)\right) \end{aligned}$$

where $\left(g_j(s)\right)^{2^n} = \left(s^{2^n}-1\right)^a\left(s^{2^n}-\lambda^2\right)\left(s^{2^n}-\lambda^2\right)$.
Thus we have g_j is determinate for some of these roots, we denote g_0 such that $g_0(0) = 1$.

Therefore by the Normalization process a chart for $Q_1 = [0,0,1]$ is given by

$$s \leftrightsquigarrow \left[s^{2^n}, s^a g_0(s), 1\right].$$

(b) For the points $Q_2 = [1,0,1]$ and $Q_3 = [-1,0,1]$ we may define charts using the same previous process. Then a chart for Q_2 is

$$s \leftrightsquigarrow \left[s^{2^n}+1, s^a g_1(s), 1\right],$$

where $g_1(s)$ is a root 2^nth of

$$\left(s^{2^n}+1\right)^a\left(s^{2^n}+2\right)^a\left(\left(s^{2^n}+1\right)^2-\lambda^2\right)\left(\left(s^{2^n}+1\right)^2-\lambda^{-2}\right).$$

(c) $Q_4 = [0,1,0]$
We consider the local coordinates

$$(r,q) = \left(\frac{X}{Y}, \frac{Z}{Y}\right),$$

thus the objects on $S_{a,\lambda}$ near at Q_4 are determinate by the following polynomial

$$f_\infty(r,q) = q^{3a+4-2^n} + r^a(r^2-q^2)^a(r^2-\lambda^2q^2)(r^2-\lambda^{-2}q^2).$$

Thus we have that the Taylor expansion series for f_∞ as sum of homogeneous polynomial is

$$f_\infty(r,q) = r^{3a+4} + \cdots + q^{3a+4-2^n}.$$

Then for $(0,0)$ there is a tangent but this has multiplicity $3a + 4 - 2^n$.

Therefore by the Normalization process a chart for $Q_4 = [0,1,0]$ is given by

$$t \leftrightsquigarrow [t^{3a+4-2^n} g_\infty(t), 1, t^{3a+4}],$$

where $g_\infty(0) = 1$.

Now using the Normalization process, in either case and as a is an odd number and $\lambda^4 \neq 0, 1$, we get a Riemann Surfaces of genus $3(2^n - 1)$, since we may define

$$\begin{aligned} \pi : S_{a,\lambda} &\longrightarrow \widehat{\mathbb{C}} \\ (x,y) &\rightsquigarrow x \end{aligned}$$

This map is a holomorphic map of Riemann surfaces.
Further for each $x_0 \in \widehat{\mathbb{C}} - B$ where

$$B = \{0, \infty, \pm\lambda, \pm\lambda^{-1}, \pm 1\}$$

we have that the cardinality of the set $\pi^{-1}(x_0)$ is 2^n. Then the map has degree 2^n.

Now according to the homogeneous polynomials, we have that the points in the following sets, either

$$\{[0,0,1], [1,0,0], [\pm 1, 0, 1], [\pm\lambda, 0, 1], [\pm\lambda^{-1}, 0, 1]\},$$

or

$$\{[0,0,1], [0,1,0], [\pm 1, 0, 1], [\pm\lambda, 0, 1], [\pm\lambda^{-1}, 0, 1]\},$$

they are Ramification points for π, and they have multiplicity 2^n.
Now by the Riemann–Hurwitz formula we have that the genus of $S_{a,\lambda}$ is

$$\begin{aligned} g &= 2^n(0-1) - 1 - \frac{1}{2}(8(2^n - 1)) \\ &= -2^n + 1 + 4(2^n - 1) \\ &= (2^n - 1)(4 - 1) \\ &= 3(2^n - 1) \end{aligned}$$

□

Remark 15. For each point in $S_{a,\lambda}$, its chart we will be one of the following list according to the homogeneous polynomial associated to $f_{a,\lambda}$

For F_1 we have ($2^n - 3a - 5 \neq 0$ and $a \neq 1$):

singular point	local coordinate	
$[0,0,1]$	$s \rightsquigarrow [s^{2^n}, s^a h_0(s), 1]$	$h_0(0) \neq 0$
$[1,0,1]$	$s \rightsquigarrow [s^{2^n}+1, s^a h_1(s), 1]$	$h_1(0) \neq 0$
$[-1,0,1]$	$s \rightsquigarrow [s^{2^n}-1, s^a h_{-1}(s), 1]$	$h_{-1}(0) \neq 0$
$[1,0,0]$	$t \rightsquigarrow [1, t^{2^n-3a-4} h_\infty(t), t^{2^n}]$	$h_\infty(0) \neq 0$

And for the non–singular points the charts are

- If $\dfrac{\partial F_1}{\partial X}(P) \neq 0$, where $P = [x_0, y_0, 1]$, then

$$[h_P(y), y, 1] \rightsquigarrow y \qquad , h_P(y_0) = x_0 .$$

- If $\dfrac{\partial F_1}{\partial Z}(Q) \neq 0$, where $Q = [1, u_0, v_0]$, then

$$[1, u, h_Q(u)] \rightsquigarrow u \qquad , h_Q(u_0) = v_0 .$$

For F_2 we have :

point	local coordinate	
$[0,0,1]$	$s \rightsquigarrow [s^{2^n}, s^a g_0(s), 1]$	$g_0(0) = 1$
$[1,0,1]$	$s \rightsquigarrow [s^{2^n}+1, s^a g_1(s), 1]$	$g_1(0) \neq 0$
$[-1,0,1]$	$s \rightsquigarrow [s^{2^n}-1, s^a g_{-1}(s), 1]$	$g_{-1}(0) \neq 0$
$[0,1,0]$	$t \rightsquigarrow [t^{3a+4-2^n} g_\infty(t), 1, t^{3a+4}]$	$g_\infty(0) = 1$

And for the points non singular the charts are
If $\dfrac{\partial F_2}{\partial X}(P) \neq 0$, where $P = [x_0, y_0, 1]$ then

$$[g_P(y), y, 1] \rightsquigarrow y \qquad , g_P(y_0) = x_0 .$$

Remark 16. When a is even number the situation is different.
For example for $a = 2$, $n = 2$

$$f(x,y) = y^4 - x^2 \left(x^2 - 1\right)^2 \left(x^2 - \lambda^2\right) \left(x^2 - \lambda^{-2}\right) .$$

Then we have the homogeneous polynomial associated to f is

$$F(X,Y,Z) = Y^4 Z^6 - X^2 \left(X^2 - Z^2\right)^2 \left(X^2 - \lambda^2 Z^2\right) \left(X^2 - \lambda^{-2} Z^2\right) ,$$

thus the singularities are

$$\{[0,0,1],[0,1,0],[1,0,1],[-1,0,1]\},$$

then we have

sing. point	local equation	normal. points	charts
$[0,0,1]$	$(y^2 - xh_0(x))(y^2 + xh_0(x))$	$\{P_1^0, P_2^0\}$	$t \rightsquigarrow (t^2, tg_i^0(t))$
$[1,0,1]$	$(y^2 - uh_1(u))(y^2 + uh_1(u))$	$\{P_1^1, P_2^1\}$	$t \rightsquigarrow (t^2+1, tg_i^1(t))$
$[-1,0,1]$	$(y^2 - uh_{-1}(u))(y^2 + uh_{-1}(u))$	$\{P_1^{-1}, P_2^{-1}\}$	$t \rightsquigarrow (t^2-1, tg_i^{-1}(t))$
$[0,1,0]$	$-s^{10} + \cdots + v^6$		

where $(s,v) = \left(\frac{X}{Y}, \frac{Z}{Y}\right)$.
Using $\pi : S_\lambda \to \widehat{\mathbb{C}}$ defined by

$$(x,y) \rightsquigarrow x\,,$$

if we suppose $[0,1,0]$ has two points in the Normalization, then the genus of $S_{a,\lambda}$ is 6.

3.3 The actions

The following theorem yields information about the automorphisms group of $S_{a,\lambda}$. The interest of the theorem is in the assertion that for each Riemann surface $S_{a,\lambda}$ the automorphisms group is not trivial.

Theorem 17. *Let τ_1, τ_2 be the self-maps of $S_{a,\lambda}$ defined by*

$$\begin{aligned} \tau_1(x,y) &= (-x, \omega_{2^{n+1}} y) \\ \tau_2(x,y) &= \left(\frac{1}{x}, \frac{\omega_{2^{n+1}} y}{x^c}\right) \end{aligned}$$

where c is a natural number determined by the equation

$$c \cdot 2^{n-1} = 2a + 2 \,.$$

Then $\tau_1, \tau_2 \in \mathrm{Aut}(S_{a,\lambda})$ and they have order 2^{n+1} each.
Furthermore depending on the values for a we have:

1. *If c is an even number, then $G_1 =< \tau_1, \tau_2 >$ is an abelian group, isomorphic to $\mathbb{Z}/2^{n+1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*
   We call $\mathfrak{S}_1$ the corresponding family of surfaces.

2. *If c is an odd number, then $G_2 =< \tau_1, \tau_2 >$ is a group isomorphic to $\mathbb{Z}/2^{n+1}\mathbb{Z} \rtimes_h \mathbb{Z}/2\mathbb{Z}$, where $\tau_2 \tau_1 = \tau_1^{2^n+1} \tau_2$.*
   We call $\mathfrak{S}_2$ the corresponding family of surfaces.

Proof. First we will see that τ_1, τ_2 are self-maps of $S_{a,\lambda}$.

For τ_1 it is easy check that if $(x,y) \in \mathbb{C}^2$, such that $f_{a,\lambda}(x,y) = 0$ we have

$$f_{a,\lambda}(\tau_1(x,y)) = 0 \,.$$

For τ_2 we have

$$\begin{aligned} f_{a,\lambda}(\tau_2(x,y)) &= -\frac{y^{2^n}}{x^{c2^n}} - \frac{1}{x^a}\left(\frac{1}{x^2} - 1\right)^a \left(\frac{1}{x^2} - \lambda^2\right)\left(\frac{1}{x^2} - \lambda^{-2}\right) \\ &= -\frac{y^{2^n}}{x^{c2^n}} + \frac{(-1)^{a+1}}{x^{3a+4}} \left(x^2 - 1\right)^a \left(x^2 - \lambda^2\right)\left(x^2 - \lambda^{-2}\right) \\ &= \frac{1}{x^{c2^n}}\left(-y^{2^n} + (-1)^{a+1} x^{c2^n - 3a - 4}\left(x^2 - 1\right)^a \left(x^2 - \lambda^2\right)\left(x^2 - \lambda^{-2}\right)\right) \end{aligned}$$

then we have τ_2 makes sense if a is an odd number and $c2^n - 3a - 4 = a$, this is

$$c2^{n-1} = 2a + 2$$

We may calculate τ_1^2, τ_2^2, this is:

$$\begin{aligned}
\tau_1^2(x,y) &= \tau_1(-x, \omega_{2^{n+1}}y) \\
&= (x, \omega_{2^{n+1}}^2 y) \\
\tau_2^2(x,y) &= \tau_2\left(\frac{1}{x}, \frac{\omega_{2^{n+1}}y}{x^c}\right) \\
&= \left(x, \frac{\omega_{2^{n+1}}y}{x^c}x^c\omega_{2^{n+1}}\right) \\
&= (x, \omega_{2^{n+1}}^2 y)
\end{aligned}$$

then we have $\tau_1^2 = \tau_2^2$.
Further we may see that the order of τ_1^2 is 2^n, since $\omega_{2^{n+1}}^2$ is a 2^n-th primitive root of unity, then τ_1 and τ_2 has order 2^{n+1}.

Now we consider τ_1, τ_2 self–maps which are given by

$$\begin{aligned}
\tau_1[X,Y,Z] &= [-X, \omega_{2^{n+1}}Y, Z] \\
\tau_2[X,Y,Z] &= [ZX^{c-1}, \omega_{2^{n+1}}YZ^{c-1}, X^c].
\end{aligned}$$

To prove that $\tau_1, \tau_2 \in \mathrm{Aut}(S_{a,\lambda})$, we must prove that for any charts $(U,\varphi), (V,\phi)$, on $S_{a,\lambda}$, such that $\tau_j(V) \cap U \neq \emptyset$, the map

$$\varphi \circ \tau_j \circ \phi^{-1},$$

is a holomorphic function (on some subset of $\mathbb{C}$).
We will do the calculations for τ_2 at $[0,0,1]$ and suppose $2^n - 3a - 4 > 0$
Recall that a chart at $[0,0,1]$ is given by

$$\phi^{-1} : s \rightsquigarrow [s^{2^n}, s^a h_0(s), 1].$$

Using the homogeneous coordinates for τ_2 we have

$$\begin{aligned}
\tau_2[s^{2^n}, s^a h_0(s), 1] &= [s^{2^n(c-1)}, \omega_{2^{n+1}}s^a h_0(s), s^{2^n c}] \\
&= [s^{4a+4-2^n}, \omega_{2^{n+1}}s^a h_0(s), s^{4a+4}] \\
&= [s^{3a+4-2^n}, \omega_{2^{n+1}}h_0(s), s^{3a+4}] \\
&= \left[1, s^{2^n-3a-4}\omega_{2^{n+1}}h_0(s), s^{2^n}\right].
\end{aligned}$$

Evaluating at $s = 0$ we have $\tau_2[0,0,1] = [1,0,0]$.
Now a chart at $[1,0,0]$ is given by

$$\varphi^{-1} : t \rightsquigarrow [1, t^{2^n-3a-4}h_\infty(t), t^{2^n}].$$

By the intersection condition

$$[1, s^{2^n-3a-4}\omega_{2^{n+1}}h_0(s), s^{2^n}] = [1, t^{2^n-3a-4}h_\infty(t), t^{2^n}].$$

Thus

$$\begin{aligned} \varphi \circ \tau_2 \circ \phi^{-1}(s) &= \varphi \circ \tau_2[s^{2^n}, s^a h_0(s), 1] \\ &= \varphi[1, s^{2^n-3a-4}\omega_{2^{n+1}}h_0(s), s^{2^n}] \\ &= \omega s \end{aligned}$$

where ω is a 2^n-th root of unity (we recall that $t^{2^n} = s^{2^n}$). Therefore, the map is a holomorphic function.
It is not dificult to verify the preceding for the other charts.

Recalling that $2^{n-1}c = 2a+2$, we have

$$\tau_1\tau_2[X,Y,Z] = [-ZX^{c-1}, \omega_{2^{n+1}}^2 Z^{c-1}Y, X^c]$$

$$\begin{aligned} \tau_2\tau_1[X,Y,Z] &= [(-1)^{c-1}ZX^{c-1}, \omega_{2^{n+1}}^2 Z^{c-1}Y, (-1)^c X^c] \\ &= \begin{cases} [-ZX^{c-1}, \omega_{2^{n+1}}^2 Z^{c-1}Y, X^c] & , c \text{ is an even number} \\ [-ZX^{c-1}, \omega_{2^{n+1}}^{2^n+2} Z^{c-1}Y, X^c] & , c \text{ is an odd number} \end{cases} \\ &= \begin{cases} \tau_1\tau_2[X,Y,Z] & , c \text{ is an even number} \\ \tau_1^{2^n+1}\tau_2[X,Y,Z] & , c \text{ is an odd number} \end{cases}. \end{aligned}$$

Further for each odd number j we have

$(\tau_1^j\tau_2)^2 = \tau_1^{2(j+1)}$	c is an even number
$(\tau_1^j\tau_2)^2 = \tau_1^{2(j+1)+2^n}$	c is an odd number

Then

$j = 2^k t - 1$	$\operatorname{ord}(\tau_1^j\tau_2) = \begin{cases} 2^{n-k+1} & , k \leq n, t \equiv 1 \mod 2 \\ 2 & , \text{in other case} \end{cases}$	c even number
$j = 2^k t - 2^{n-1} - 1$	$\operatorname{ord}(\tau_1^j\tau_2) = \begin{cases} 2^{n-k+1} & , k \leq n, t \equiv 1 \mod 2 \\ 2 & , \text{in other case} \end{cases}$	c odd number

Thus we may write, for both cases when either c is an even number or c is an odd number, the following complete list of elements on $< \tau_1, \tau_2 >$

elements	exponent	cardinality of these type of elements
τ_1^j	$0 \leq j \leq 2^{n+1} - 1$	2^{n+1}
τ_2^j	j is an odd number	2^n
$\tau_1^j \tau_2$	j is an odd number	2^n
		2^{n+2} elements

Now when c is an even number, then the group is abelian non cyclic and it has elements of order 2^{n+1} , therefore it is isomorphic to

$$G_1 = \mathbb{Z}/2^{n+1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\,.$$

Since τ_1 and τ_2 holds the relationships for the presentation give in Proposition 3, we may define the following group isomorphism

$$\begin{array}{rcl} < \tau_1, \tau_2 > & \longrightarrow & G_1 \\ \tau_1 & \rightsquigarrow & A_1 \\ \tau_2 & \rightsquigarrow & B_1 \end{array}$$

Now when c is an odd number, we have that τ_1 and τ_2 holds the relationships for the presentation give in Proposition 4. Then we have the following group isomorphism

$$\begin{array}{rcl} < \tau_1, \tau_2 > & \longrightarrow & G_2 = \mathbb{Z}/2^{n+1}\mathbb{Z} \rtimes_h \mathbb{Z}/2\mathbb{Z} \\ \tau_1 & \leftrightsquigarrow & A_2 \\ \tau_2 & \leftrightsquigarrow & B_2 \\ \tau_1^{2^{n-1}-1}\tau_2 & \leftrightsquigarrow & C_2 \end{array}$$

Recall $h(C_2)(A_2) = A_2^{2^n+1}$.

□

Remark 17. We consider the particular cases.

1. When $a = 1$,
   The previous equation for c in this case it is the following

$$c2^{n-2} = 2\,,$$

thus as $c \in \mathbb{Z}$, then $n \leq 3$.
For $n < 3$ then c is an even number and these cases were studied [8].
For $n = 3$ then $c = 1$.
Further we conclude that the case (3) in the Theorem 16 is empty, this is the Riemann surfaces defined by these condition has no automorphisms of type τ_j for $j = 1, 2$.

2. Case $a > 1$, $2^n - 3a - 5 = 0$ and $2^n - 3a - 4 > 0$ (in this case the homogeneous polynomial associated to $f_{a,\lambda}$ is F_1).
We recall that this case, corresponds in the Theorem 16 to case (2).
Further we have that $2^n - 3a - 5 = 0$ if only if n is an odd number.
First we note that as $2^n - 3a - 5 = 0$, then the condition $a \neq 1$ is equivalent to $n \neq 3$.
Since $c2^{n-1} = 2a + 2$, then

$$3c2^{n-1} = 6a + 6\,,$$

now replacing $3a = 2^n - 5$ we have

$$3c2^{n-1} = 2^{n+1} - 10 + 6 = 2^{n+1} - 4\,.$$

For $n = 1$, we have $3c = 0$, then $c = 0$ and $a = -1$, then this case has not sense.
For $n = 2$ has not sense because n most be odd number.
For $n \geq 3$, we may multiply by 2^{-2}, then

$$\begin{aligned} 3c2^{n-3} &= 2^{n-1} - 1 \\ \therefore n &= 3 \end{aligned}$$

and this is a contradiction.
Thus we have that this case is empty, in other words for $n > 3$ we have not automorphism.

Theorem 18. *Let c be an even number. Consider $G_1 =< \tau_1, \tau_2 >$ the group defined by this c.*
Then for each $S_1 \in \mathfrak{S}_1$, the cyclic subgroups of G_1 acting with fixed points (different) are given as follows:

1. *$H_1 =< \tau_1 >$ subgroup of order 2^{n+1}, acting on S_1 with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$.*

2. *$H_2 =< \tau_2 >$ subgroup of order 2^{n+1}, acting on S_1 with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$.*

3. *$H_3 =< \tau_1^2 >$ subgroup of order 2^n, acting on S_1 with signature $(0; 2^n, 2^n, 2^n, 2^n, 2^n, 2^n, 2^n, 2^n)$.*

4. *$H_4 =< \tau_1^{2^{n-1}c-1}\tau_2 >$ subgroup of order 2, acting on S_1 with signature $(2^n - 1; 2^{n+1} \cdot 2)$ ($\tau_1^{2^{n-1}c-1}\tau_2$ has 2^{n+1} fixed points).*

Furthermore, we have that the group G_1 acts on S_1 with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2)$ and that $G_1 = \mathrm{Aut}(S_1)$, except for finitely many $S_1 \in \mathfrak{S}_1$.

We remark that $2^{n-1}c - 1 \equiv 2^n - 1 \mod 2^{n+1}$, since c is a non trivial even number, then we have

$$2^{n-1}c - 1 - 2^n - 1 = 2^{n-1}(c-2) \equiv 0 \mod 2^{n+1}.$$

Proof. We will compute the fixed points.
First we calculate for $\tau_1[X, Y, Z] = [-X, \omega_{2^{n+1}}Y, Z]$,

$$\begin{array}{ll} & \tau_1[X,Y,Z] = [X,Y,Z] \\ \Leftrightarrow & (-X, \omega_{2^{n+1}}Y, Z) = (tX, tY, tZ) \\ \Leftrightarrow & X(t+1) = 0 \qquad Z(t-1) = 0 \qquad Y(t - \omega_{2^{n+1}}) = 0 \\ \Rightarrow & \text{the fixed point on } S_1 : \left\{ \begin{array}{ll} \{[0,0,1],[1,0,0]\} & ,2^n - 3a - 4 > 0 \\ \{[0,0,1],[0,1,0]\} & ,2^n - 3a - 4 < 0 \end{array} \right\} \end{array}$$

Now for $\tau_2[X, Y, Z] = [ZX^{c-1}, \omega_{2^{n+1}}Z^{c-1}Y, X^c]$, we have

$$\begin{array}{ll} & \tau_2[X,Y,Z] = [X,Y,Z] \\ \Leftrightarrow & t(ZX^{c-1}, \omega_{2^{n+1}}Z^{c-1}Y, X^c) = (X,Y,Z) \\ & X = t^2X^{2c-1}, \qquad X(1 - t^2X^{2c-2}) = 0 \\ \Rightarrow & [X, 0, tX^c] = [X, 0, \pm X^{c-c+1}] = [1, 0, \pm 1] \\ \therefore & \text{fixed point on } S_1 : \{[1,0,1],[-1,0,1]\} \end{array}$$

We may see that according to the homogeneous polynomial, we have either $\tau_2[0,0,1] = [1,0,0]$ or $\tau_2[0,0,1] = [0,1,0]$, in fact we recall that $2c^{2^{n-1}} = 2a + 2$ then

$$\begin{array}{rcl} \tau_2[s^{2^n}, s^a h_0(s), 1] & = & [s^{2^n(c-1)}, \omega_{2^{n+1}} s^a h_0(s), s^{2^n c}] \\ & = & [s^{4a+4-2^n}, \omega_{2^{n+1}} s^a h_0(s), s^{4a+4}] \\ & = & \left\{ \begin{array}{ll} [1, \omega_{2^{n+1}} s^{2^n - 3a - 4} h_0(s), s^{2^n}] & ,2^n - 3a - 4 > 0 \\ [s^{3a+4-2^n}, \omega_{2^{n+1}} h_0(s), s^{3a+4}] & ,2^n - 3a - 4 < 0 \end{array} \right. \end{array}$$

When j is an even number, we have $\tau_1^j[X,Y,Z] = [X, \omega_{2^n+1}^j Y, Z]$ then

$$
\begin{array}{ll}
 & \tau_1^j[X,Y,Z] = [X,Y,Z] \\
\Leftrightarrow & (X, \omega_{2^n+1}^j Y, Z) = (tX, tY, tZ) \\
\Leftrightarrow & X(t-1) = 0 \qquad Z(t-1) = 0 \qquad Y(t - \omega_{2^n+1}^j) = 0 \\
\Rightarrow & \text{the fixed point on} S_1 :
\end{array}
$$

$$
\begin{cases}
\{[0,0,1],[1,0,0],[\pm 1,0,1],[\pm\lambda,0,1],[\pm\lambda^{-1},0,1]\} & ,2^n - 3a - 4 > 0 \\
\{[0,0,1],[0,1,0],[\pm 1,0,1],[\pm\lambda,0,1],[\pm\lambda^{-1},0,1]\} & ,2^n - 3a - 4 < 0
\end{cases}
$$

For an odd number j, we consider $\tau_1^j\tau_2[X,Y,Z] = [-ZX^{c-1}, \omega_{2^n+1}^{j+1} YZ^{c-1}, X^c]$, we recall c is an even number, then we have the following equation:

$$
\begin{array}{rcll}
\tau_1^j\tau_2[X,Y,Z] & = & [X,Y,Z] & \\
(-ZX^{c-1}, \omega_{2^n+1}^{j+1} YZ^{c-1}, X^c) & = & (tX, tY, tZ) & \\
-ZX^{c-1} & = & tX & (1) \\
\omega_{2^n+1}^{j+1} YZ^{c-1} & = & tY & (2) \\
X^c & = & tZ & (3)
\end{array}
$$

thus we have :

$$
t(X^2 + Z^2) = 0, \qquad Y(\omega_{2^n+1}^{j+1} Z^{c-1} - t) = 0
$$

Now we may see if $X = Z = 0$, the point [0,1,0] is not fixed point, and as $t \in \mathbb{C}^*$, we have

$$
\frac{X}{Z} = \pm i, \qquad t = \omega_{2^n+1}^{j+1} Z^{c-1}
$$

Evaluating on the equation (1) we have

$$
\begin{array}{rcll}
-ZX^{c-1} & = & \omega_{2^n+1}^{j+1} Z^{c-1} X & \quad | \cdot Z \\
-Z^2X^{c-1} & = & \omega_{2^n+1}^{j+1} Z^c X & \\
X^{c+1} & = & \omega_{2^n+1}^{j+1} Z^c X &
\end{array}
$$

Evaluating on the equation (3) we have the same equation. Thus when $\dfrac{X}{Z} = i = \omega_{2^n+1}^{2^{n-1}}$ we have

$$
\omega_{2^n+1}^{2^{n-1}c} = \omega_{2^n+1}^{j+1}
$$

therefore $j = 2^{n-1}c - 1$.

And when $\dfrac{X}{Z} = -i = \omega_{2^n+1}^{3\cdot 2^{n-1}}$ we have

$$
\omega_{2^n+1}^{3\cdot 2^{n-1}c} = \omega_{2^n+1}^{j+1}
$$

therefore $j = 3 \cdot 2^{n-1}c - 1$.

Finally as c is an even number we have

$$3 \cdot 2^{n-1}c - 1 \equiv j = 2^{n-1}c - 1 \mod 2^{n+1}$$

Then we have $\tau_1^{2^{n-1}c-1}\tau_2$ has 2^{n+1} fixed points, they are:

$$\left\{ \begin{array}{ll} [i,p,1]: & p^{2^n} = -(2i)^a(1+\lambda^2)(1+\lambda^{-2}) \\ [-i,q,1]: & q^{2^n} = (2i)^a(1+\lambda^2)(1+\lambda^{-2}) \end{array} \right\}$$

Now for each H we calculate the signature.
For this we use the Riemann Hurwitz formula.

- $H_4 =< \tau_1^{2^{n-1}c-1}\tau_2 >$,
  $|H_4|$=2,
  Fixed points: $\left\{ \begin{array}{ll} [i,p,1]: & p^{2^n} = -(2i)^a(1+\lambda^2)(1+\lambda^{-2}) \\ [-i,q,1]: & q^{2^n} = (2i)^a(1+\lambda^2)(1+\lambda^{-2}) \end{array} \right\}$
  Riemann Hurwitz formula:

$$\begin{aligned} 3(2^n-1) &= 2(\gamma-1)+1+\frac{2^{n+1}}{2}(2-1) \\ \gamma &= 2^n-1 \end{aligned}$$

  Therefore the signature for H_4 is

$$(2^n-1; 2, \cdots, 2) = (2^n-1; 2^{n+1} \cdot 2).$$

- $H_3 =< \tau_1^2 >$,
  $|H_3| = 2^n$,
  Fixed points:

$$\left\{ \begin{array}{ll} \{[0,0,1],[1,0,0],[\pm 1,0,1],[\pm\lambda,0,1],[\pm\lambda^{-1},0,1]\} & , \text{ if } 2^n-3a-4>0 \\ \{[0,0,1],[0,1,0],[\pm 1,0,1],[\pm\lambda,0,1],[\pm\lambda^{-1},0,1]\} & , \text{ if } 2^n-3a-4<0 \end{array} \right.$$

  Riemann Hurwitz formula:

$$\begin{aligned} 3(2^n-1) &= 2^n(\gamma-1)+1+\frac{8}{2}(2^n-1) \\ \gamma &= 0 \end{aligned}$$

  Therefore the signature for H_3 is

$$(0; 2^n, 2^n, 2^n, 2^n, 2^n, 2^n, 2^n, 2^n).$$

- $H_1 = < \tau_1 >$,
  $|H_1| = 2^{n+1}$,
  Fixed points τ_1: $\begin{cases} \{[0,0,1],[1,0,0]\} & , \text{ if } 2^n - 3a - 4 > 0 \\ \{[0,0,1],[0,1,0]\} & , \text{ if } 2^n - 3a - 4 < 0 \end{cases}$

  Fixed points τ_1^2: $\{[\pm 1, 0, 1], [\pm\lambda, 0, 1], [\pm\lambda^{-1}, 0, 1]\}$
  Riemann Hurwitz formula:

  $$\begin{aligned} 3(2^n - 1) &= 2^{n+1}(\gamma - 1) + 1 + \frac{1}{2}\left(2(2^{n+1} - 1) + 6(2^n - 1)\right) \\ \gamma &= 0 \end{aligned}$$

  Therefore the signature for H_1 is

  $$(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n).$$

- $H_2 = < \tau_2 >$
  The calculations for τ_2 are equal to those for τ_1, since H_2 has 2 fixed points with multiplicity 2^{n+1} and 6 points with multiplicity 2^n, then the signature is

  $$(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n).$$

- $G_1 = < \tau_1, \tau_2 >$.
  By the previous data, using Riemann Hurwitz formula, we have

  $$\begin{aligned} 3(2^n - 1) &= 2^{n+2}(\gamma - 1) + 1 + \frac{1}{2}\left(4(2^{n+1} - 1) + 4(2^n - 1) + 2^{n+1}(2 - 1)\right) \\ \gamma &= 0 \end{aligned}$$

  Therefore the signature for G_1 is

  $$(0; 2^{n+1}, 2^{n+1}, 2^n, 2).$$

Let Γ be a Fuchsian group with the above signature. By Singerman [22], there is no other Fuchsian group with signature of the form $(0; a, b, c, d)$ that contains it strictly. It follows that it may only be contained in a triangular signature. Hence by dimension arguments, Γ cannot be contained strictly in other subgroup as finite index subgroup except for a finite number of possibilities (up to conjugation by Möebius transformations). Therefore, the family of Riemann surfaces does not have any other automorphisms than those of G_1, except for finitely many $S_1 \in \mathfrak{S}_1$.

□

We can now state the analogue of the preceding Theorem for the case no abelian.

Theorem 19. *Let c be an odd number. Consider* $G_2 =< \tau_1, \tau_2 >$ *defined by this c.*
Then for each $S_2 \in \mathfrak{S}_2$*, the cyclic subgroups of* G_2 *acting with fixed points (different) are given as follows:*

1. $H_1 =< \tau_1 >$ *subgroup of order* 2^{n+1}*, acting on* S_2 *with signature* $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$.

2. $H_2 =< \tau_2 >$ *subgroup of order* 2^{n+1}*, acting on* S_2 *with signature* $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$.

3. $H_3 =< \tau_1^2 >$ *subgroup of order* 2^n*, acting on* S_2 *with signature* $(0; 2^n, 2^n, 2^n, 2^n, 2^n, 2^n, 2^n, 2^n)$.

4. $H_4 =< \tau_1^{3\cdot 2^{n-1}c-1}\tau_2 >$ *subgroup of order* 2*, acting on* S_2 *with signature* $(5 \cdot 2^{n-2} - 1; 2^n \cdot 2)$ *(*$\tau_1^{3\cdot 2^{n-1}c-1}\tau_2$ *has* 2^n *fixed points).*

5. $H_5 =< \tau_1^{2^{n-1}c-1}\tau_2 >$ *subgroup of order* 2*, acting on* S_2 *with signature* $(5 \cdot 2^{n-2} - 1; 2^n \cdot 2)$ *(*$\tau_1^{2^{n-1}c-1}\tau_2$ *has* 2^n *fixed points).*
   H_5 *is a subgroup conjugate to* H_4.

Furthermore, we have that the group G_2 *acts on* S_2 *with signature* $(0; 2^{n+1}, 2^{n+1}, 2^n, 2)$ *and that* $G_2 = \mathrm{Aut}(S_2)$*, except for finitely many* $S_2 \in \mathfrak{S}_2$.

Proof. The proof is similar to that of Theorem 18. In that proof, the only instance we used the fact that c was an even number occurred when we showed

$$\tau_1^{2^{n-1}c-1}\tau_2 = \tau_1^{3\cdot 2^{n-1}c-1}.$$

This does not happen here. We thus have 2^n fixed points for $\tau_1^{2^{n-1}c-1}\tau_2$, which are

$$\{[i, p, 1] : p^{2^n} = -(2i)^a(1 + \lambda^2)(1 + \lambda^{-2})\},$$

and also we have 2^n fixed points for $\tau_1^{3\cdot 2^{n-1}c-1}\tau_2$ they are

$$\{[-i, q, 1] : q^{2^n} = (2i)^a(1 + \lambda^2)(1 + \lambda^{-2})\}.$$

We remark that the fixed points of H_4 and H_5 are in the same orbit under action of G_2, since let $[i, p, 1]$ be a fixed point for H_5,

$$\tau_1[i, p, 1] = [-i, \omega_{2^{n+1}}p, 1],$$

and we have

$$(\omega_{2^{n+1}}p)^{2^n} = -p^{2^n} = (2i)^a(1+\lambda^2)(1+\lambda^{-2}),$$

then $\tau_1[i,p,1]$ is a fixed point for H_4.
Therefore the subgroups are conjugates.

For finish the proof we calculate the signature using the Riemann Hurwitz formula.
$H_4 =< \tau_1^{3\cdot 2^{n-1}c-1} >$
$|H_4| = 2$,
Fixed points: $\{[-i,q,1] : q^{2^n} = (2i)^a(1+\lambda^2)(1+\lambda^{-2})\}$
Riemann Hurwitz formula:

$$\begin{aligned} 3(2^n-1) &= 2(\gamma-1)+1+\frac{2^n}{2}(2-1) \\ \gamma &= 5\cdot 2^{n-2}-1 \end{aligned}$$

Therefore the signature for H_4 is

$$(5\cdot 2^{n-2}-1; 2\cdots 2) = (5\cdot 2^{n-2}-1; 2^n\cdot 2).$$

The signature for H_5 is equal to the signature for H_4, since they are conjugates.
For the another subgroups we have the same calculations that in the previous theorem.

□

It follows from Theorems 18 and 19 (item 3)) that the form of affine algebraic equations that define $S_{a,\lambda}$ is of type

$$y^{2^n} = x^{d_1}(x-1)^{d_2}(x-\lambda_3)^{d_3}(x-\lambda_4)^{d_4}(x-\lambda_5)^{d_5}(x-\lambda_6)^{d_6}(x-\lambda_7)^{d_7}$$

where in this case $d_1 = a$, $d_2 = d_3 = a$, $d_4 = d_5 = d_6 = d_7 = 1$, $\lambda_3 = -1$, $\lambda_4 = \lambda$, $\lambda_5 = -\lambda$, $\lambda_6 = \lambda^{-1}$ and $\lambda_7 = -\lambda^{-1}$.

Our next objective is to determine the local structure for each automorphism with fixed points.

Remark 18. We give the details for the computations in the case $2^n-3a-4>0$, $a>1$, $2^n-3a-5\neq 0$ where the homogeneous polynomial associated to $f_{a,\lambda}$ is F_1. These computations hold for the elements in both $\mathfrak{S}_1$ and $\mathfrak{S}_2$.

We first recall the charts for the fixed points of Theorem 16.

1. $P_1 = [0,0,1]$.
   Chart for P_1:
$$\phi_1^{-1} : s \rightsquigarrow [s^{2^n}, s^a h_0(s), 1] .$$
$$\begin{aligned}\phi_1 \circ \tau_1 \circ \phi_1^{-1}(s) &= \phi_1[-s^{2^n}, \omega_{2^{n+1}} s^a h_0(s), 1] \\ &= \omega_{2^{n+1}}^{j_0} s\end{aligned}$$
   where
$$\omega_{2^{n+1}}^{j_0 2^n} = -1, \qquad \omega_{2^{n+1}}^{j_0 a} = \omega_{2^{n+1}} ,$$
   then we have j_0 is an odd number, and $j_0 a \equiv 1 \mod 2^{n+1}$.
   Therefore $\delta_{P_1}(\tau_1) = \omega_{2^{n+1}}^{j_0}$. Furthermore since δ_{P_1} is a group homomorphism we have
$$\delta_{P_1}(\tau_1^a) = (\delta_{P_1}(\tau_1))^a = \omega_{2^{n+1}}^{j_0 a} = \omega_{2^{n+1}} .$$

2. $P_4 = [1,0,0]$.
   Chart for P_4:
$$\phi_4^{-1} : t \rightsquigarrow [1, t^{2^n - 3a - 4} h_\infty(t), t^{2^n}] .$$
$$\begin{aligned}\phi_4 \circ \tau_1 \circ \phi_4^{-1}(t) &= \phi_4[1, \omega_{2^{n+1}}^{2^n+1} t^{2^n - 3a - 4} h_\infty(t), -t^{2^n}] \\ &= \omega_{2^{n+1}}^{j_\infty} t\end{aligned}$$
   where
$$\omega_{2^{n+1}}^{j_\infty 2^n} = -1, \qquad \omega_{2^{n+1}}^{j_\infty(2^n - 3a - 4)} = \omega_{2^{n+1}}^{2^n+1} ,$$
   then we have j_∞ is an odd number, and $j_\infty(2^n - 3a - 4) \equiv 2^n + 1 \mod 2^{n+1}$, and this is equivalent to
$$j_\infty(-3a-4) \equiv 1 \mod 2^{n+1} ,$$
   since j is an odd number then $2^n(j_0 - 1) \equiv 0 \mod 2^{n+1}$.

3. $P_2 = [1,0,1]$.
   Chart for P_2:
$$\phi_2^{-1} : s \rightsquigarrow [s^{2^n} + 1, s^a h_1(s), 1] .$$
$$\begin{aligned}\phi_2 \circ \tau_2 \circ \phi_2^{-1}(s) &= \phi_2 \left[\frac{1}{s^{2^n}+1}, \frac{\omega_{2^{n+1}} s^a h_1(s)}{(s^{2^n}+1)^c}, 1 \right] \\ &= \phi_2 \left[\sum_{k \geq 0} (-s^{2^n})^k, \omega_{2^{n+1}} s^a h_1(s) \left(\sum_{k \geq 0} (-s^{2^n})^k \right)^c, 1 \right] \\ &= \phi_2[1 - s^{2^n} + \cdots, \omega_{2^{n+1}} s^a h_1(s) + \cdots, 1] \\ &= \omega_{2^{n+1}}^{j_1} s\end{aligned}$$

where

$$\omega_{2^{n+1}}^{j_1 2^n} = -1, \qquad \omega_{2^{n+1}}^{j_1 a} = \omega_{2^{n+1}} ,$$

then we have j_1 is an odd number, and $j_1 a \equiv 1 \mod 2^{n+1}$.

4. $P_3 = [-1, 0, 1]$.
   Chart for P_3:

$$\phi_3^{-1} : s \rightsquigarrow [s^{2^n} - 1, s^a h_{-1}(s), 1] .$$

$$\begin{aligned}
\phi_3 \circ \tau_2 \circ \phi_4^{-1}(s) &= \phi_3 \left[\frac{1}{s^{2^n} - 1}, \frac{\omega_{2^{n+1}} s^a h_{-1}(s)}{(s^{2^n} - 1)^c}, 1 \right] \\
&= \phi_3 \left[- \sum_{k \geq 0} (s^{2^n})^k, (-1)^c \omega_{2^{n+1}} s^a h_{-1}(s) \left(\sum_{k \geq 0} (s^{2^n})^k \right)^c, 1 \right] \\
&= \begin{cases} \phi_3[-1 - s^{2^n} + \cdots, \omega_{2^{n+1}} s^a h_{-1}(s) + \cdots, 1] & , c = \dot{2} \\ \phi_3[-1 - s^{2^n} + \cdots, \omega_{2^{n+1}}^{2^n+1} s^a h_{-1}(s) + \cdots, 1] & , c = \dot{2} + 1 \end{cases} \\
&= \begin{cases} \omega_{2^{n+1}}^{j_{-1}} s & , j_{-1} a \equiv 1 \mod 2^{n+1} & , c = \dot{2} \\ \omega_{2^{n+1}}^{j_{-1}} s & , j_{-1} a \equiv 2^n + 1 \mod 2^{n+1} & , c = \dot{2} + 1 \end{cases}
\end{aligned}$$

where j_{-1} is an odd number.

When c is an odd number, we have $\delta_{P_3}(\tau_2^a) = \omega_{2^{n+1}}^{2^n+1}$.
If we take $j = 2^n + 1$ then $\delta_{P_3}(\tau_2^{aj}) = \omega_{2^{n+1}}$, since

$$(\omega_{2^{n+1}}^{2^n+1})^{2^n+1} = (-\omega_{2^{n+1}})^{2^n+1} = -\omega_{2^{n+1}}^{2^n+1} = \omega_{2^{n+1}} .$$

The previous calculus we summarize in the following table:

order	τ	P	local auto.	$\delta_P(\tau)$
2^{n+1}	τ_1	$[0,0,1]$	$s \rightsquigarrow \omega_{2^{n+1}}^{j_0} s$ $j_0 a \equiv 1 \mod 2^{n+1}$ j_0 is an odd number	$\omega_{2^{n+1}}^{j_0}$
2^{n+1}	τ_1^a	$[0,0,1]$		$\omega_{2^{n+1}}$
2^{n+1}	τ_1	$[1,0,0]$	$t \rightsquigarrow \omega_{2^{n+1}}^{j_\infty} t$ $j_\infty(-3a-4) \equiv 1 \mod 2^{n+1}$ j_∞ is an odd number	$\omega_{2^{n+1}}^{j_\infty}$
2^{n+1}	τ_1^{-3a-4}	$[1,0,0]$		$\omega_{2^{n+1}}$
2^{n+1}	τ_2	$[1,0,1]$	$s \rightsquigarrow \omega_{2^{n+1}}^{j_1} s$ $j_1 a \equiv 1 \mod 2^{n+1}$ j_1 is an odd number	$\omega_{2^{n+1}}^{j_1}$
2^{n+1}	τ_2^a	$[1,0,1]$		$\omega_{2^{n+1}}$
2^{n+1}	τ_2	$[-1,0,1]$	$s \rightsquigarrow \omega_{2^{n+1}}^{j_{-1}} s$ $j_{-1} a \equiv 1 \mod 2^{n+1}$ j_{-1} is an odd number	$\omega_{2^{n+1}}^{j_{-1}}$ c is an even number
2^{n+1}	τ_2^a	$[-1,0,1]$		$\omega_{2^{n+1}}$ c is an even number
2^{n+1}	τ_2	$[-1,0,1]$	$s \rightsquigarrow \omega_{2^{n+1}}^{j_{-1}} s$ $j_{-1} a \equiv 2^n + 1 \mod 2^{n+1}$ j_{-1} is an odd number	$\omega_{2^{n+1}}^{j_{-1}}$ c is an odd number
2^{n+1}	τ_2^a	$[-1,0,1]$		$\omega_{2^{n+1}}^{2^n+1}$ c is an odd number
2^{n+1}	$\tau_2^{a(2^n+1)}$	$[-1,0,1]$		$\omega_{2^{n+1}}$ c is an odd number
2^n	τ_1^2	$[\lambda,0,1]$	$y \rightsquigarrow \omega_{2^{n+1}}^2 y$	ω_{2^n}
2^n	τ_1^2	$[-\lambda,0,1]$	$y \rightsquigarrow \omega_{2^{n+1}}^2 y$	ω_{2^n}
2^n	τ_1^2	$[\lambda^{-1},0,1]$	$y \rightsquigarrow \omega_{2^{n+1}}^2 y$	ω_{2^n}
2^n	τ_1^2	$[-\lambda^{-1},0,1]$	$y \rightsquigarrow \omega_{2^{n+1}}^2 y$	ω_{2^n}

Remark 19. Case $n = 3$, $a = 1$, $c = 1$.
In this case we have that there is not singular point.
When $Y = 0$, we have the following points

$$\{[0,0,1], [\pm 1,0,1], [\pm\lambda,0,1], [\pm\lambda^{-1},0,1]\}.$$

Then the charts for these points are given by

$$[h_p(y), y, 1] \rightsquigarrow y\,,$$

where h_p is holomorphic function defined in a neighborhood at 0, such that

$$h_p(0) = p\,,$$

where $p \in \{0, \pm 1, \pm\lambda, \pm\lambda^{-1}\}$.

Now $[1, 0, 0]$ has locally coordinates

$$f_\infty(u, v) = u^8 + v(v^2 - 1)(1 - \lambda^2 v^2)(1 - \lambda^{-2} v^2)\,,$$

where $(u, v) = (\frac{Y}{X}, \frac{Z}{X})$.
Then a chart in this point is given by

$$[1, u, h_\infty(u)] \rightsquigarrow u\,,$$

where h_∞ is holomorphic function defined in a neighborhood at 0, with $h_\infty(0) = 0$.

Now we may calculate for each point δ_P. Then we have

order	τ	P	$\delta_P(\tau)$
16	τ_1	$[0, 0, 1]$	ω_{16}
16	τ_1	$[1, 0, 0]$	ω_{16}^9
16	τ_1^9	$[1, 0, 0]$	ω_{16}
16	τ_2	$[1, 0, 1]$	ω_{16}
16	τ_2	$[-1, 0, 1]$	ω_{16}^9
16	τ_2^9	$[-1, 0, 1]$	ω_{16}
8	τ_1^2	$[\lambda, 0, 1]$	ω_8
8	τ_1^2	$[-\lambda, 0, 1]$	ω_8
8	τ_1^2	$[\lambda^{-1}, 0, 1]$	ω_8
8	τ_1^2	$[-\lambda^{-1}, 0, 1]$	ω_8

Remark 20. Now we give the details for the computations in the case $2^n - 3a - 4 < 0$, $a > 1$, where the homogeneous polynomial associated to $f_{a,\lambda}$ is F_2. These computations hold for the elements in both $\mathfrak{S}_1$ and $\mathfrak{S}_2$.

We note the charts for Q_1, Q_2, Q_3 are the same that for P_1, P_2, P_3.
We must calculate for $Q_4 = [0, 1, 0]$
A chart for Q_4 is:

$$\phi_4^{-1} : t \rightsquigarrow [t^{3a+4-2^n} g_\infty(t), 1, t^{3a+4}]\,.$$

Thus we have

$$\begin{aligned} \phi_4 \circ \tau_1 \circ \phi_4^{-1}(t) &= \phi_4[\omega_{2^n+1}^{2^n-1} t^{3a+4-2^n} g_\infty(t), 1, \omega_{2^n+1}^{-1} t^{3a+4}] \\ &= \omega_{2^n+1}^{j_\infty} t \end{aligned}$$

where
$$\omega_{2^{n+1}}^{j_\infty(3a+4-2^n)} = \omega_{2^{n+1}}^{2^n-1}, \quad \omega_{2^{n+1}}^{j_\infty(3a+4)} = \omega_{2^{n+1}}^{-1}.$$

Then j_∞ is an odd number and $j_\infty(3a+4) \equiv -1 \mod 2^{n+1}$. Therefore $\delta_{Q_4}(\tau_1^{-3a-4}) = \omega_{2^{n+1}}$.
Now we may summarize this case in the following table

order	τ	P	$\delta_P(\tau)$
2^{n+1}	τ_1	$[0,0,1]$	$\omega_{2^{n+1}}^{j_0}$
2^{n+1}	τ_1^a	$[0,0,1]$	$\omega_{2^{n+1}}$
2^{n+1}	τ_1	$[0,1,0]$	$\omega_{2^{n+1}}^{j_\infty}$
2^{n+1}	τ_1^{-3a-4}	$[0,1,0]$	$\omega_{2^{n+1}}$
2^{n+1}	τ_2	$[1,0,1]$	$\omega_{2^{n+1}}^{j_1}$
2^{n+1}	τ_2^a	$[1,0,1]$	$\omega_{2^{n+1}}$
2^{n+1}	τ_2	$[-1,0,1]$	$\omega_{2_{n+1}}^{j_{-1}}$ c is an even number
2^{n+1}	τ_2^a	$[-1,0,1]$	$\omega_{2_{n+1}}$ c is an even number
2^{n+1}	τ_2	$[-1,0,1]$	$\omega_{2^{n+1}}^{j_{-1}}$ c is an odd number
2^{n+1}	τ_2^a	$[-1,0,1]$	$\omega_{2_{n+1}}^{2^n+1}$ c is an odd number
2^{n+1}	$\tau_2^{a(2^n+1)}$	$[-1,0,1]$	$\omega_{2_{n+1}}$ c is an odd number
2^n	τ_1^2	$[\lambda,0,1]$	ω_{2^n}
2^n	τ_1^2	$[-\lambda,0,1]$	ω_{2^n}
2^n	τ_1^2	$[\lambda^{-1},0,1]$	ω_{2^n}
2^n	τ_1^2	$[-\lambda^{-1},0,1]$	ω_{2^n}

3.3.1 Geometric Presentation

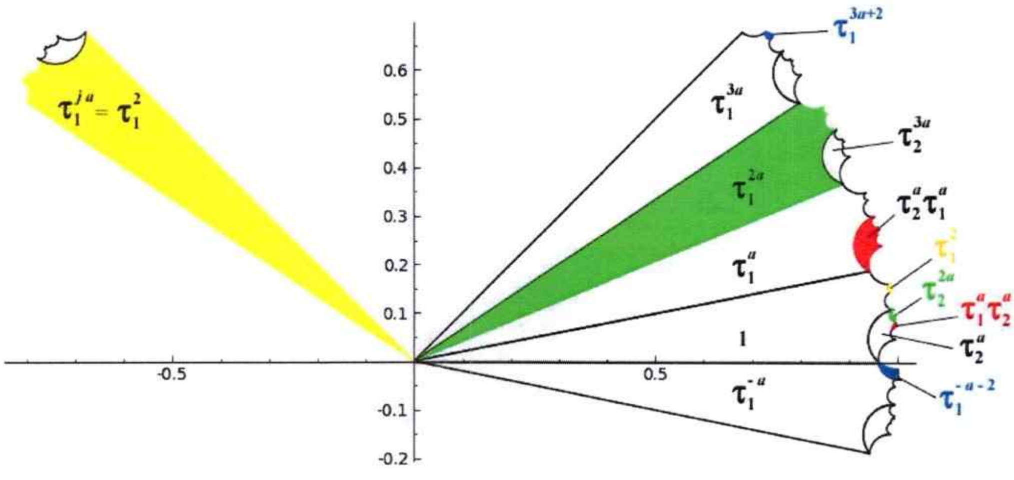
We have showed that, for each $S_1 \in \mathfrak{S}_1$, the group

$$G_1 = \mathbb{Z}/2^{n+1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

acts on S_1 with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2)$. For this action, we have that the 4–tuple

$$(\tau_1^a, \tau_2^a, \tau_1^2, \tau_1^{2^{n-1}c-1}\tau_2)$$

is a generating vector. Using the polygon method [1], according to geometric data, we find a presentation for G_1. We thus obtain the polygon



We call D_1, D_2, D_3 the elements associated respectively to τ_1^a, τ_2^a, τ_1^2. From the picture we conclude the relationships

$$\begin{aligned} D_3 D_1^4 D_3 &= 1, && \text{blue} \\ D_2 D_1 D_2^{-1} D_1^{-1} &= 1, && \text{red} \\ D_1^2 D_2^{-2} &= 1, && \text{green} \\ D_1^j D_3^{-1} &= 1, && \text{yellow} \end{aligned}$$

The reader should keep in mind that we are in the case where c is an even number and $2^{n-1}c = 2a + 2$. It is not difficult to verify that $j = 2^{n-1}c - 2$. Indeed, since a is an odd number, we have

$$j \cdot a = 2^{n-1}ca - 2a = 2^{n-1}ca - 2^{n-1}c + 2 = 2^{n-1}c(a-1) + 2 \equiv 2 \mod 2^{n+1}.$$

The following is a consecuence.

Proposition 6. *G_1 has a presentation of the form*

$$\left\langle D_1, D_2, D_3, D_4 : \begin{array}{c} D_1^{2^{n+1}} = D_2^{2^{n+1}} = D_3^{2^n} = D_4^2 = 1, \\ D_1D_2D_3D_4 = 1, \quad D_1^2D_2^{-2} = 1, \quad D_1^{c2^{n-1}-2}D_3^{-1} = 1 \end{array} \right\rangle, \tag{3.1}$$

where c is an even number.

Proof. Let

$$\widetilde{G}_1 = \left\langle D_1, D_2, D_3, D_4 : \begin{array}{c} D_1^{2^{n+1}} = D_2^{2^{n+1}} = D_3^{2^n} = D_4^2 = 1, \\ D_1D_2D_3D_4 = 1, \quad D_1^2D_2^{-2} = 1, \quad D_1^{c2^{n-1}-2}D_3^{-1} = 1 \end{array} \right\rangle.$$

First we may see that $D_1D_2 = D_2D_1$, since $D_1^2 = D_2^2$ and $D_3 = D_1^{c2^{n-1}-2}$ thus $D_4 = D_1^{c2^{n-1}-1}D_2$, then since D_4 has order 2

$$\begin{aligned} 1 = D_4^2 &= D_1^{c2^{n-1}-1}D_2D_1^{c2^{n-1}-1}D_2 \\ &= D_1^{2c2^{n-1}-3}D_2D_1D_2 \\ &= D_1^{c2^n-3}D_2D_1D_2 \\ &= D_1^{-3}D_2D_1D_2 \\ &= D_1^{-1}D_2^{-2+1}D_1D_2 \\ \therefore D_2D_1 &= D_1D_2 \end{aligned}$$

Now we have

$$\widetilde{G}_1 = \{D_1^jD_2^i : 0 \le j < 2^{n+1}, 0 \le i \le 1\},$$

then $\widetilde{G}_1$ is an abelian group of order 2^{n+2} and we have the following isomorphism

$$\begin{array}{ccc} \widetilde{G}_1 & \longrightarrow & G_1 \\ D_1 & \rightsquigarrow & A_1 \\ D_2 & \rightsquigarrow & B_1 \end{array}$$

□

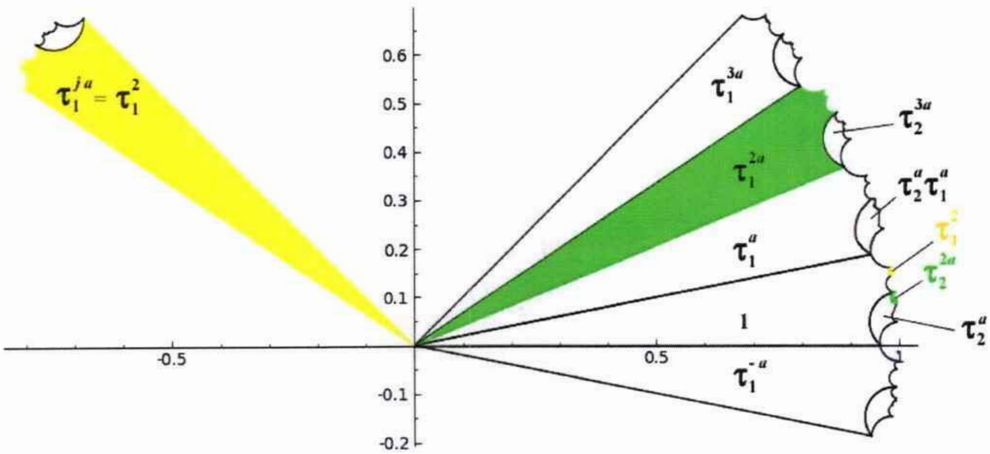
Also we have showed that, for each $S_2 \in \mathfrak{S}_2$, the group

$$G_2 = \mathbb{Z}/2^{n+1}\mathbb{Z} \rtimes_h \mathbb{Z}/2\mathbb{Z}$$

acts on S_2 with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2)$. For this action, we have that the 4–tuple

$$(\tau_1^a, \tau_2^a, \tau_1^2, \tau_1^{2^{n-1}c-1}\tau_2)$$

is a generating vector. As before, using the polygon method [1], according to geometric data, we find a presentation for G_2. We thus obtain the polygon



We call D_1, D_2, D_3 the elements associated respectively to τ_1^a, τ_2^a, τ_1^2. From the picture we conclude the relationships

$$\begin{aligned} D_1^2 D_2^{-2} &= 1, \qquad \text{green} \\ D_1^j D_3^{-1} &= 1, \qquad \text{yellow} \end{aligned}$$

The reader should keep in mind that we are in the case where c is an odd number and $2^{n-1}c = 2a + 2$. It is not difficult to verify that $j = 3 \cdot 2^{n-1}c - 2$, for $n > 3$, in fact

$$\begin{aligned} j \cdot a &= 3 \cdot 2^{n-1}ca - 2a \\ &= 3ca2^{n-1} - 2^{n-1}c + 2 \\ &= 2^{n-1}c(3a - 1) + 2 \\ &= 2^{n-1}c(3a + 3 - 4) + 2 \\ &= 2^{n-1}c(32^{n-2}c - 4) + 2 \\ &\equiv 3c^2 2^{n+1} 2^{n-4} + 2 \qquad \bmod 2^{n+1} \\ &\equiv 2 \qquad \bmod 2^{n+1} \end{aligned}$$

For $n = 3$ we have $a = 1$ hence $j = 2$.

The following is a consecuence.

Proposition 7. *For $n > 3$, G_2 has a presentation of the form*

$$\left\langle D_1, D_2, D_3, D_4 : \begin{array}{c} D_1^{2^{n+1}} = D_2^{2^{n+1}} = D_3^{2^n} = D_4^2 = 1, \\ D_1D_2D_3D_4 = 1, \quad D_1^2D_2^{-2} = 1, \quad D_1^{3\cdot 2^{n-1}c-2}D_3^{-1} = 1 \end{array} \right\rangle . \tag{3.2}$$

Proof. Let

$$\widetilde{G}_2 = \left\langle D_1, D_2, D_3, D_4 : \begin{array}{c} D_1^{2^{n+1}} = D_2^{2^{n+1}} = D_3^{2^n} = D_4^2 = 1, \\ D_1D_2D_3D_4 = 1, \quad D_1^2D_2^{-2} = 1, \quad D_1^{3\cdot 2^{n-1}c-2}D_3^{-1} = 1 \end{array} \right\rangle .$$

First we will see that $D_2D_1 = D_1^{2^n+1}D_2$, in fact as D_4 has order 2, $D_4 = D_1D_2D_3$ and $D_3 = D_1^{3\cdot 2^{n-1}c-2}$ y $D_1^2 = D_2^2$ we have

$$\begin{aligned} 1 &= D_1D_2D_3D_1D_2D_3 \\ &= D_1^{3\cdot 2^{n-1}c-1}D_2D_1^{3\cdot 2^{n-1}c-1}D_2 \\ &= D_1^{6\cdot 2^{n-1}c-3}D_2D_1D_2 \\ \therefore D_2D_1 &= D_1^{3-3\cdot 2^n c}D_2^{-1} = D_1^{1-3\cdot 2^n c}D_2 \end{aligned}$$

but as c is an odd number we have

$$1 - 3\cdot 2^n c \equiv 2^n + 1 \qquad \mod 2^{n+1}.$$

Then we have the following isomorphism

$$\begin{array}{ccc} \widetilde{G}_2 & \longrightarrow & G_2 \\ D_1 & \rightsquigarrow & A_2 \\ D_2 & \rightsquigarrow & B_2 \end{array}$$

□

3.4 Classification of actions on $\mathfrak{S}_1$ and $\mathfrak{S}_2$

In this section we will study the epimorphisms associated to the actions defined in the previous section.
From now on we consider the notations as in Theorem 7.
We recall that in any case (abelian and no abelian) $H_1 =< \tau_1 >$ is a subgroup of $\text{Aut}(S_i)$ for $i = 1, 2$, and acts with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$, then we have, according to the Theorem 7, epimorphisms $\theta_{1,i}$ associated to the action.
By the Theorem 7, there exists a Fuchsian group $\Gamma_{1,i}$ with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$, then

$$\Gamma_{1,i} = \left\langle x_{1,i}, x_{2,i}, x_{3,i}, x_{4,i}, x_{5,i} : x_{1,i}^{2^{n+1}} = x_{2,i}^{2^{n+1}} = x_{3,i}^{2^n} = x_{4,i}^{2^n} = x_{5,i}^{2^n} = 1 = x_{1,i}x_{2,i}x_{3,i}x_{4,i}x_{5,i} \right\rangle$$

where we may choose for each $j = 1, 2, 3, 4$, $x_{j,i}$ as counterclockwise rotation about $z_{j,i}$ through angle $\dfrac{2\pi}{m_j}$, for

$$m_1 = m_2 = 2^{n+1}, \quad m_3 = m_4 = 2^n .$$

Hence we have

$$\begin{array}{rcl} \theta_{1,i} : \Gamma_{1,i} & \longrightarrow & H_1 \\ x_{j,i} & \rightsquigarrow & \theta_{1,i}(x_{j,i}) \end{array}$$

Further we recall that there exists isomorphisms $f_{1,i} : \Delta/K_{1,i} \to S_i$, where $K_{1,i} = \ker(\theta_{1,i})$.
We call $P_{j,i} = f_{1,i}(K_{1,i}z_{j,i})$ then we have

$$\begin{array}{rcl} \delta_{P_{1,i}}(\theta_{1,i}(x_{1,i})) & = & \omega_{2^{n+1}} \\ \delta_{P_{2,i}}(\theta_{1,i}(x_{2,i})) & = & \omega_{2^{n+1}} \\ \delta_{P_{3,i}}(\theta_{1,i}(x_{3,i})) & = & \omega_{2^n} \\ \delta_{P_{4,i}}(\theta_{1,i}(x_{4,i})) & = & \omega_{2^n} \end{array}$$

We may choose

$$P_{1,i} = [0, 0, 1], \quad P_{2,i} = \tau_2(P_{1,i}), \quad P_{3,i} = [1, 0, 1], \quad P_{4,i} = [\lambda, 0, 1] ,$$

thus according to the tables of the previous section we have

$$\begin{array}{rcl} \theta_{1,i}(x_{1,i}) & = & \tau_1^a \\ \theta_{1,i}(x_{2,i}) & = & \tau_1^{-3a-4} \\ \theta_{1,i}(x_{3,i}) & = & \tau_1^{2a} \\ \theta_{1,i}(x_{4,i}) & = & \tau_1^2 \\ \theta_{1,i}(x_{5,i}) & = & \tau_1^2 \end{array}$$

We may do the same process for $H_2 =< \tau_2 >$.
H_2 is a subgroup of $\mathrm{Aut}(S_i)$ and acts on S_i with the same signature that H_1.
By Theorem 7 there exists a Fuchsian group $\Gamma_{2,i}$ with signature $(0; 2^{n+1}, 2^{n+1}, 2^n, 2^n, 2^n)$, then

$$\Gamma_{2,i} = \left\langle y_{1,i}, y_{2,i}, y_{3,i}, y_{4,i}, y_{5,i} : y_{1,i}^{2^{n+1}} = y_{2,i}^{2^{n+1}} = y_{3,i}^{2^n} = y_{4,i}^{2^n} = y_{5,i}^{2^n} = 1 = y_{1,i}y_{2,i}y_{3,i}y_{4,i}y_{5,i} \right\rangle$$

where we may choose for each $j = 1, 2, 3, 4$, $\quad y_{j,i}$ as counterclockwise rotation about $w_{j,i}$ through angle $\dfrac{2\pi}{m_j}$, with m_j as before.

Hence we have epimorphisms

$$\theta_{2,i} : \Gamma_{2,i} \longrightarrow H_2 .$$

Further we have the isomorphism associated $f_{2,i} : \Delta/K_{2,i} \to S_{i,\lambda}$ where $K_{2,i} = \ker(\theta_{2,i})$.

We call $Q_{j,i} = f_{2,i}(K_{2,i}w_{j,i})$, and may choose

$$Q_{1,i} = [1, 0, 1], \quad Q_{2,i} = [-1, 0, 1], \quad Q_{3,i} = [0, 0, 1], \quad Q_{4,i} = [\lambda, 0, 1] .$$

According to the tables of the preceding section, we have

1. If c is an even number, then

$$\begin{aligned} \theta_{2,1}(y_{1,1}) &= \tau_2^a \\ \theta_{2,1}(y_{2,1}) &= \tau_2^a \\ \theta_{2,1}(y_{3,1}) &= \tau_2^{2a} \\ \theta_{2,1}(y_{4,1}) &= \tau_2^2 \\ \theta_{2,1}(y_{5,1}) &= \tau_2^2 \end{aligned}$$

   We remark that since $4a + 4 = 2^n c$ and c is an even number, then $4a + 4 \equiv 0 \mod 2^{n+1}$, therefore

$$\theta_{2,1}(y_{1,1})\theta_{2,1}(y_{2,1})\theta_{2,1}(y_{3,1})\theta_{2,1}(y_{4,1})\theta_{2,1}(y_{5,1}) = 1 .$$

2. If c is an odd number, then

$$\begin{aligned} \theta_{2,2}(y_{1,2}) &= \tau_2^a \\ \theta_{2,2}(y_{2,2}) &= \tau_2^{a(2^n+1)} \\ \theta_{2,2}(y_{3,2}) &= \tau_2^{2a} \\ \theta_{2,2}(y_{4,2}) &= \tau_2^2 \\ \theta_{2,2}(y_{5,2}) &= \tau_2^2 \end{aligned}$$

We remark that since $4a+4+a2^n = 2^n c + 2^n a = 2^n(a+c)$ and c, a are odd number, then $4a+4+a2^n \equiv 0 \mod 2^{n+1}$, therefore

$$\theta_{2,2}(y_{1,2})\theta_{2,2}(y_{2,2})\theta_{2,2}(y_{3,2})\theta_{2,2}(y_{4,2})\theta_{2,2}(y_{5,2}) = 1\,.$$

Theorem 20. *Assume that $2^n - 3a - 5 \neq 0$ and $a > 1$. Then the actions induced by H_1 and H_2 on $S_i \in \mathfrak{S}_i$ for each $i = 1, 2$, are topologically, but not conformally, equivalent, except for S_i defined by $\lambda = \pm 1 \pm \sqrt{2}$.*

Proof. By Theorem 8 it is enough to prove that there exists a commutative diagram for the corresponding epimorphisms.

Using the previous computations, we may define an isomorphism between $\Gamma_{1,i}$ and $\Gamma_{2,i}$, by

$$\begin{array}{rcl} \chi : \Gamma_{1,i} & \longrightarrow & \Gamma_{2,i} \\ x_{j,i} & \rightsquigarrow & y_{j,i} \end{array}$$

Since $x_{j,i}$ and $y_{j,i}$ satisfy the same relations, it is clear that χ is an isomorphism.
Then we have the following diagram:

$$\begin{array}{ccc} \Gamma_{1,i} & \xrightarrow{\theta_{1,i}} & H_1 \\ \chi \big\downarrow & & \big\downarrow \Phi \\ \Gamma_{2,i} & \xrightarrow{\theta_{2,i}} & H_2 \end{array} \tag{3.3}$$

where Φ is the isomorphism given by

$$\Phi(\tau_1) = \tau_2\,.$$

We now prove this claim.
When c is an even number, we have the following generating vectors associated to $\theta_{1,1}$

$$(\tau_1^a, \tau_1^{-3a-4}, \tau_1^{2a}, \tau_1^2, \tau_1^2)\,,$$

and to $\theta_{2,1}$

$$(\tau_2^a, \tau_2^a, \tau_2^{2a}, \tau_2^2, \tau_2^2)\,.$$

Since c is an even number, we have

$$-3a - 4 \equiv a \mod 2^{n+1}$$

(recall that $4a+4=2^n c$).
Therefore we have the commutative diagram (3.3).

When c is an odd number, we have the following generating vectors associated to $\theta_{1,1}$

$$(\tau_1^a, \tau_1^{-3a-4}, \tau_1^{2a}, \tau_1^2, \tau_1^2)\,,$$

and to $\theta_{2,2}$

$$(\tau_2^a, \tau_2^{a(2^n+1)}, \tau_2^{2a}, \tau_2^2, \tau_2^2)\,,$$

Now we have

$$-3a-4 \equiv a(2^n+1) \mod 2^{n+1}\,,$$

in fact since $4a+4=2^n c$ and a, c are odd number then

$$a2^n + 4a + 4 = 2^n(a+c) \equiv 0 \mod 2^{n+1}\,.$$

Therefore we have the commutative diagram (3.3).

Alternative proof for topological equivalence.
We consider the isomorphism between H_j and $\mathbb{Z}/2^{n+1}\mathbb{Z}$ given by

$$\begin{array}{ccc} H_j & \longrightarrow & \mathbb{Z}/2^{n+1}\mathbb{Z} \\ \tau_j & \rightsquigarrow & 1 \end{array}$$

With this isomorphism we may associate to each generating vector a 5-tuple of elements in $\mathbb{Z}/2^{n+1}\mathbb{Z}$.
Thus we have

- If c is an even number, then

$$(a, a, 2a, 2, 2) \equiv (a, -3a-4, 2a, 2, 2) \mod 2^{n+1}\,.$$

- If c is an odd number, then

$$(a, a(2^n+1), 2a, 2, 2) \equiv (a, -3a-4, 2a, 2, 2) \mod 2^{n+1}\,.$$

By Theorem 13 we have in these cases $s=1$, then the actions are directly topologically equivalent.

Now we will prove that for S_i defined by $\lambda \neq \pm 1 \pm \sqrt{2}$ the actions on S_i are not conformally equivalent.

We will follow the idea of the proof given by G.González-Diez and R.Hidalgo [8] in the case $n = 2$ with $a = 1$ and $c = 2$.
By contradiction we suppose that are conformally equivalent; that is, there exists σ on $\mathrm{Aut}(S_i)$ such that

$$\sigma\tau_1 = \tau_2^j\sigma\,,$$

where j is an odd number.

Now we consider the holomorphic branched covering associated to the action of $H_3 =< \tau_1^2 >$ on S_i, this is

$$\begin{array}{rccc} \pi: & S_i & \longrightarrow & \mathbb{C} \\ & [X,Y,Z] & \rightsquigarrow & \dfrac{X}{Z} \end{array}$$

,since $\tau_1^2(x,y) = (x, \omega_{2^n}y)$ then for $P = (x_0, y_0)$,

$$H_3(P) = \{\tau(P) : \tau \in H_3\} = \{(x_0, \omega_{2^n}^j y_0) : 0 \leq j < 2^n\}\,.$$

Hence we have the following diagram

$$\begin{array}{ccc} S_i & \xrightarrow{\quad\sigma\quad} & S_i \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi} \\ S_i/H_3 = \widehat{\mathbb{C}} & \overset{T}{\dashrightarrow} & \widehat{\mathbb{C}} = S_i/H_3 \end{array}$$

where T is given by ,

$$T(x) := \pi\left(\sigma(x,y)\right)\,,$$

for $x \in \widehat{\mathbb{C}}$, where $(x,y) \in \pi^{-1}(x)$.

Since $\sigma\tau_1 = \tau_2^j\sigma$, then

$$\pi\sigma\tau_1^2(x,y) = \pi\tau_2^{2j}\sigma(x,y) = \pi\sigma(x,y)\,,$$

and T is well defined.

We have that T is a holomorphic map, because σ is holomorphic.

For $P_1 = [0,0,1]$ let

$$P_4 = \tau_2(P_1) = \begin{cases} [1,0,0] & 2^n - 3a - 4 > 0 \\ [0,1,0] & 2^n - 3a - 4 < 0 \end{cases}$$

Now we have the following properties

1. Consider the set of fixed points of H_3:

$$B = \left\{P_1, P_4, [\pm 1, 0, 1], [\pm\lambda, 0, 1], [\lambda^{-1}, 0, 1]\right\} .$$

   Each point P in B, is a fixed point of τ_1^2 (Theorems 18,19), hence we have

$$\sigma(P) = \sigma(\tau_1^2(P)) = \tau_2^{2j}(\sigma(P)) = \tau_1^{2j}(\sigma(P)),$$

   and therefore $\sigma(P) \in B$.

   It is easy to verify that

$$\pi(B) = \{0, \infty, \pm 1, \pm\lambda, \pm\lambda^{-1}\};$$

   since $\sigma(B) = B$ then

$$T(\pi(B)) = \pi(B).$$

   Furthermore $T(0), T(\infty) \in \{1, -1\}$, because if $\tau_1(P) = P$, then

$$\sigma(P) = \sigma(\tau_1(P)) = \tau_2^j(\sigma(P));$$

   therefore $\sigma(P) \in \{[1, 0, 1], [-1, 0, 1]\}$, and $\pi(P) \in \{0, \infty\}$.

2. T is a bijective function:
   Since π is surjective then T is surjective.
   If $T(x_1) = T(x_2)$ we have

$$\begin{aligned} \pi\sigma(x_1, y_1) &= \pi\sigma(x_2, y_2) \\ \sigma(x_1, y_1) &= \tau_2^{2j}\sigma(x_2, y_2) \\ \sigma(x_1, y_1) &= \sigma\tau_1^{2k}(x_2, y_2) \\ \therefore (x_1, y_1) &= \tau_1^{2k}(x_2, y_2) \\ \therefore x_1 &= x_2 \end{aligned}$$

3. T is a Möbius transformation, since T is a bijective and holomorphic map of $\widehat{\mathbb{C}}$.

Now consider the coverings associated to the subgroups H_1 and H_2. Let $\pi_i : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be coverings, for $i = 1, 2$, given by

$$\begin{aligned} \pi_1(x) &= x^2 \\ \pi_2(x) &= x + \frac{1}{x} \end{aligned}$$

Then we have the following commutative diagram

$$
\begin{array}{ccc}
S_i & \xrightarrow{\ \sigma\ } & S_i \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\widehat{\mathbb{C}} & \xrightarrow{\ T\ } & \widehat{\mathbb{C}} \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\
S_i/H_1 = \widehat{\mathbb{C}} & \overset{R}{\cdots\to} & \widehat{\mathbb{C}} = S_i/H_2
\end{array}
$$

where $R(x) = \pi_2 T(x_0)$, with $\pi_1(x_0) = x$.

Let us verify that R is well defined.
First we remark that if we consider $T_j : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ for $j = 1, 2$, given by

$$
\begin{aligned}
T_1(x) &= -x \\
T_2(x) &= \frac{1}{x}
\end{aligned}
$$

then we have

$$T_j \pi = \pi \tau_j \qquad , j = 1, 2 .$$

Further it is easy to check that for each j we have $T_j^2 = T_j$.

Now we have

$$
\begin{aligned}
T(-x) &= T\pi(-x, y) \\
&= \pi\sigma(-x, y) \\
&= \pi\sigma\tau_1(x, y') \\
&= \pi\tau_2^j\sigma(x, y') \\
&= T_2\pi\sigma(x, y') \\
&= T_2 T(x) \\
&= \frac{1}{T(x)}
\end{aligned}
$$

Thus if $\pi_1(x_1) = \pi_1(x_2)$ then

$$x_2 \in \{x_1, -x_1\} .$$

Using the previous calculations we have

$$T(x_2) \in \left\{ T(x_1), \frac{1}{T(x_1)} \right\},$$

then

$$\pi_2(T(x_1)) = T(x_1) + \frac{1}{T(x_1)} = \pi(T(x_2)),$$

and therefore R is well defined.

We remark that for each j, π_j is the covering associated to the action T_j.

R has the following properties:

1. R is a Möbius transformation, in fact
   R is a holomorphic map because T is a holomorphic map.
   Further since π_1 and π_2 are surjective, then R is surjective.
   Now if $R(x) = R(z)$, for $x_0^2 = x$ and $z_0^2 = z$ we have

$$\begin{array}{rc} & T(x_0) + \frac{1}{T(x_0)} = T(z_0) + \frac{1}{T(z_0})\\ \Leftrightarrow & (T(x_0) - T(z_0))\,(T(x_0)T(z_0) - 1) = 0 \\ \Rightarrow & T(z_0) \in \left\{ T(x_0), \frac{1}{T(x_0)} \right\} \\ \therefore & z_0 \in \{x_0, -x_0\} \\ \therefore & z = z_0^2 = x_0^2 = x \end{array}$$

2. We recall that

$$T(\pi(B)) = \pi(B) = \{0, \infty, 1, \lambda^2, \lambda^{-2}\}.$$

   and $T(\{0, \infty\}) = \{1, -1\}$.

   Then we have

$$R\left(\{0, \infty, 1, \lambda^2, \lambda^{-2}\}\right) = \{\infty, \pm 2, \pm\left(\lambda + \lambda^{-1}\right)\},$$

   and $R(0), R(\infty) \in \{2, -2\}$.

3. We recall that a Möbius transformation is determined by its values at 3 points. First we suppose $R(0) = 2$ and $R(\infty) = -2$,

(a) If $R(1) = \infty$, we have

$$R(z) = \frac{2z+2}{-z+1}$$

thus $R(\lambda^2) = \dfrac{2(\lambda^2+1)}{1-\lambda^2}$, and then

$$\begin{aligned} R(\lambda^2) &= \lambda + \lambda^{-1} \Leftrightarrow \lambda = -1 \pm \sqrt{2}, \text{ or} \\ R(\lambda^2) &= -\lambda - \lambda^{-1} \Leftrightarrow \lambda = 1 \pm \sqrt{2} \end{aligned}$$

(b) If $R(\lambda^2) = \infty$, we have

$$R(z) = \frac{2z+2\lambda^2}{-z+\lambda^2}$$

thus

$$\begin{aligned} R(1) &= \lambda + \lambda^{-1} \Leftrightarrow \lambda = 1 \pm \sqrt{2}, \text{ or} \\ R(1) &= -\lambda - \lambda^{-1} \Leftrightarrow \lambda = -1 \pm \sqrt{2} \end{aligned}$$

(c) If $R(\lambda^{-2}) = \infty$, we have

$$R(z) = \frac{2\lambda^2 z + 2\lambda^2}{-\lambda^2 z + 1}$$

then

$$\begin{aligned} R(1) &= \lambda + \lambda^{-1} \Leftrightarrow \lambda = -1 \pm \sqrt{2}, \text{ or} \\ R(1) &= -\lambda - \lambda^{-1} \Leftrightarrow \lambda = 1 \pm \sqrt{2} \end{aligned}$$

In the other case, when $R(0) = -2$ and $R(\infty) = 2$, we obtain the same results.

Therefore the Möbius transformation R exists if only if

$$\lambda \in \{1 \pm \sqrt{2}, -1 \pm \sqrt{-2}\}.$$

□

Theorem 21. *For $n = 3, a = 1$ and $c = 1$, the actions induced by H_1 and H_2 are directly topologically, but not conformally, equivalent, except for $\lambda = \pm 1 \pm \sqrt{2}$.*

Proof. Using remark 19, the generating vectors associated to the epimorphisms $\theta_{j,2}$ are:

$$H_1 : \ (\tau_1 \ , \tau_1^9 \ , \tau_1^2 \ , \tau_1^2 \ , \tau_1^2)$$

$$H_2 : \ (\tau_2 \ , \tau_2^9 \ , \tau_2^2 \ , \tau_2^2 \ , \tau_2^2)$$

Now by Theorem 13 we have that the actions H_1 and H_2 are directly topologically equivalents, we take $s = 1$.
The proof that the actions are not conformally equivalent is the same as that for Theorem 20. □

Remark 21. Now we will see some examples in the case $n = 3, a = 1$.
We consider for $n = 3$ a group $\widetilde{G}$ as in Proposition 7; that is,

$$\begin{aligned} \widetilde{G} &= \left\langle A_1, A_2, A_3, A_4 : \begin{array}{c} A_1^{16} = A_2^{16} = A_3^8 = A_4^2 = 1, \\ A_1A_2A_3A_4 = 1, \quad A_1^{-2}A_2^2 = 1, \quad A_1^{-2}A_3 = 1 \end{array} \right\rangle \\ &\simeq \mathbb{Z}/16\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

We may define an action for $\widetilde{G}$ on S_λ, by

$$\begin{array}{rcl} \varepsilon : \widetilde{G} & \longrightarrow & \mathrm{Aut}(S_\lambda) \\ A_1 & \rightsquigarrow & \tau_1 \\ A_2 & \rightsquigarrow & \tau_2 \\ A_3 & \rightsquigarrow & \tau_1^2 \\ A_4 & \rightsquigarrow & \tau_1^3\tau_2 \end{array}$$

Since $\varepsilon(A_j)$ satisfy the same relations as A_j, we have ε is a monomorphism.
Now using the Eichler trace formula, see Theorem 11, we may calculate the character associated to the Analytic representation induced by ε, and we obtain the following table

elements	Character of analytic Representation
1	21
A_1	$-\omega_8 - \omega_8^2 - \omega_8^3$
A_2	$-\omega_8 - \omega_8^2 - \omega_8^3$
A_1^2	$-3 - 4\omega_8 - 4\omega_8^2 - 4\omega_8^3$
A_1^4	$-3 - 4\omega_4$
A_1^8	-3
$A_1^9 A_2$	1
A_1^3	$-\omega_8 + \omega_8^2 - \omega_8^3$
A_1^5	$\omega_8 - \omega_8^2 + \omega_8^3$
A_2^3	$-\omega_8 + \omega_8^2 - \omega_8^3$
A_2^5	$\omega_8 - \omega_8^2 + \omega_8^3$
A_1^6	$-3 - 4\omega_8 + 4\omega_8^2 - 4\omega_8^3$
A_1^{10}	$-3 + 4\omega_8 - 4\omega_8^2 + 4\omega_8^3$
A_1^{12}	$-3 + 4\omega_4$
$A_1^{11} A_2$	-3
$A_1^{13} A_2$	1
A_1^7	$\omega_8 + \omega_8^2 + \omega_8^3$
A_2^7	$\omega_8 + \omega_8^2 + \omega_8^3$
A_1^{14}	$-3 + 4\omega_8 + 4\omega_8^2 + 4\omega_8^3$
$A_1^{15} A_2$	1

Now we consider the irreducible representations for $\widetilde{G}$ according to section 1 of this chapter.

$\alpha_{0,0}: \; A_1 \to 1, \; A_2 \to 1$	$\alpha_{1,0}: \; A_1 \to 1, \; A_2 \to -1$
$\alpha_{0,2}: \; A_1 \to \omega_8, \; A_2 \to -\omega_8$	$\alpha_{1,2}: \; A_1 \to \omega_8, \; A_2 \to \omega_8$
$\alpha_{0,4}: \; \tau_1 \to \omega_4, \; \tau_2 \to \omega_4$	$\alpha_{1,4}: \; A_1 \to \omega_4, \; A_2 \to -\omega_4$
$\alpha_{0,6}: \; A_1 \to \omega_8^3, \; A_2 \to \omega_8^3$	$\alpha_{1,6}: \; A_1 \to \omega_8^3, \; A_2 \to -\omega_8^3$
$\alpha_{0,8}: \; A_1 \to -1, \; A_2 \to -1$	$\alpha_{1,8}: \; A_1 \to -1, \; A_2 \to 1$
$\alpha_{0,10}: \; A_1 \to -\omega_8, \; A_2 \to -\omega_8$	$\alpha_{1,10}: \; A_1 \to -\omega_8, \; A_2 \to \omega_8$
$\alpha_{0,12}: \; A_1 \to -\omega_4, \; A_2 \to -\omega_4$	$\alpha_{1,12}: \; A_1 \to -\omega_4, \; A_2 \to \omega_4$
$\alpha_{0,14}: \; A_1 \to -\omega_8^3, \; A_2 \to -\omega_8^3$	$\alpha_{1,14}: \; A_1 \to -\omega_8^3, \; A_2 \to \omega_8^3$
$\beta_1: \; A_1 \to \begin{bmatrix} \omega_{16} & 0 \\ 0 & -\omega_{16} \end{bmatrix}, \; A_2 \to \begin{bmatrix} 0 & -\omega_{16}^5 \\ \omega_{16}^5 & 0 \end{bmatrix}$	$\beta_3: \; A_1 \to \begin{bmatrix} \omega_{16}^3 & 0 \\ 0 & -\omega_{16}^3 \end{bmatrix}, \; A_2 \to \begin{bmatrix} 0 & -\omega_{16}^7 \\ -\omega_{16}^7 & 0 \end{bmatrix}$
$\beta_5: \; A_1 \to \begin{bmatrix} \omega_{16}^5 & 0 \\ 0 & -\omega_{16}^5 \end{bmatrix}, \; A_2 \to \begin{bmatrix} 0 & \omega_{16} \\ -\omega_{16} & 0 \end{bmatrix}$	$\beta_7: \; A_1 \to \begin{bmatrix} \omega_{16}^7 & 0 \\ 0 & -\omega_{16}^7 \end{bmatrix}, \; A_2 \to \begin{bmatrix} 0 & -\omega_{16}^3 \\ \omega_{16}^3 & 0 \end{bmatrix}$

If we call Σ the Analytic Representation induced by ε, we have

$$\Sigma = \alpha_{1,4} \oplus \alpha_{0,6} \oplus \alpha_{1,6} \oplus \alpha_{0,10} \oplus \alpha_{0,12} \oplus \alpha_{1,12} \oplus 2\alpha_{0,14} \oplus \alpha_{1,14} \oplus \beta_3 \oplus 2\beta_5 \oplus 3\beta_7 \quad (3.4)$$

Consider $\varepsilon(A_j^{-1})$, for each j, then we have the following 4-tuple

$$(\tau_1^{15}, \tau_2^{15}, \tau_1^{14}, \tau_1^3\tau_2)\,.$$

Now using Chevalley Weil formula, for this tuple, we may compute, the decomposition for the analytic representation in to irreducible representations, where denote by $n_{i,j}$ the multiplicity associated to $\alpha_{i,j}$ and by n_j the multiplicity associated to the representations β_j

- $n_{0,0} = 0$
- $n_{1,0} = -1 + \left(1 - \frac{8}{16}\right) + \frac{1}{2} = 0$
- $n_{0,2} = -1 + \left(1 - \frac{14}{16}\right) + \left(1 - \frac{6}{16}\right) + \left(1 - \frac{6}{8}\right) = 0$
- $n_{1,2} = -1 + 2\left(1 - \frac{14}{16}\right) + \left(1 - \frac{6}{8}\right) + \frac{1}{2} = 0$
- $n_{0,4} = -1 + 2\left(1 - \frac{12}{16}\right) + \frac{1}{2} = 0$
- $n_{1,4} = -1 + \left(1 - \frac{12}{16}\right) + \left(1 - \frac{4}{16}\right) + \frac{1}{2} + \frac{1}{2} = 1$
- $n_{0,6} = -1 + 2\left(1 - \frac{10}{16}\right) + \left(1 - \frac{2}{8}\right) + \frac{1}{2} = 1$
- $n_{1,6} = -1 + \left(1 - \frac{2}{16}\right) + \left(1 - \frac{10}{16}\right) + \left(1 - \frac{2}{8}\right) = 1$
- $n_{0,8} = -1 + 2\left(1 - \frac{8}{16}\right) = 0$
- $n_{1,8} = -1 + \left(1 - \frac{8}{16}\right) + \frac{1}{2} = 0$
- $n_{0,10} = -1 + 2\left(1 - \frac{6}{16}\right) + \left(1 - \frac{6}{8}\right) + \frac{1}{2} = 1$
- $n_{1,10} = -1 + \left(1 - \frac{6}{16}\right) + \left(1 - \frac{14}{16}\right) + \left(1 - \frac{6}{8}\right) = 0$

- $n_{0,12} = -1 + 2\left(1 - \frac{4}{16}\right) + \left(1 - \frac{4}{8}\right) = 1$

- $n_{1,12} = -1 + \left(1 - \frac{4}{16}\right) + \left(1 - \frac{12}{16}\right) + \frac{1}{2} + \frac{1}{2} = 1$

- $n_{0,14} = -1 + 2\left(1 - \frac{2}{16}\right) + \left(1 - \frac{2}{8}\right) + \frac{1}{2} = 2$

- $n_{1,14} = -1 + \left(1 - \frac{2}{16}\right) + \left(1 - \frac{10}{16}\right) + \left(1 - \frac{2}{8}\right) = 1$

- $n_1 = -2 + \frac{5}{4} + \frac{1}{4} + \frac{1}{2} = 0$

- $n_3 = -2 + \frac{7}{4} + \frac{3}{4} + \frac{1}{2} = 1$

- $n_5 = -2 + \frac{9}{4} + \frac{5}{4} + \frac{1}{2} = 2$

- $n_7 = -2 + \frac{11}{4} + \frac{7}{4} + \frac{1}{2} = 3$

Then the decomposition coincides with our previous calculations in (3.4).

3.4.1 Especial case $\lambda = 1 + \sqrt{2}$

Theorem 22. *For $\lambda_0 = 1 + \sqrt{2}$, consider S_{i,λ_0} the Riemann surface in $\mathfrak{S}_i$ determined by λ_0, with $i = 1, 2$. Then for each $i = 1, 2$, there exists a finite group of order 2^{n+3} which acts on S_{i,λ_0} with signature $(0; 2^{n+1}, 2^{n+1}, 4)$. We have the following two possibilities:*

1. *If $i = 1$, then the group is*

$$G_{0,1} = \mathbb{Z}/4\mathbb{Z} \rtimes_{h_0} \mathbb{Z}/2^{n+1}\mathbb{Z}$$

where $h_0 : \mathbb{Z}/2^{n+1}\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/4\mathbb{Z})$.
Furthermore, $G_1 = \mathbb{Z}/2^{n+1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is a subgroup of $G_{0,1}$.

2. *If $i = 2$, then the group, say $G_{0,2}$, is an extension of $\mathbb{Z}/2^n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by a group $\mathbb{Z}/4\mathbb{Z}$, that is,*

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow G_{0,2} \longrightarrow \mathbb{Z}/2^n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow 0\,.$$

Furthermore, $G_2 = \mathbb{Z}/2^{n+1}\mathbb{Z} \rtimes_h \mathbb{Z}/2\mathbb{Z}$ is a subgroup of $G_{0,2}$.

Proof. We recall the proof of Theorem 20. We had Möbius transformation R and T; now using $\lambda_0 = 1 + \sqrt{2}$ we may explicitely calculate R and T:

$$R(x) = \frac{2x + 2}{1 - x}$$
$$T(x) = \frac{1 - x}{1 + x}$$

In the proof we had the following commutative diagram

$$\begin{array}{ccc} S_{i,\lambda_0} & \xrightarrow{\sigma} & S_{i,\lambda_0} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \widehat{\mathbb{C}} & \xrightarrow{T} & \widehat{\mathbb{C}} \end{array}$$

thus for $(x,y) \in S_{i,\lambda_0}$ we have

$$\begin{aligned}
0 = f(\sigma(x,y)) &= f(T(x), \sigma_2(x,y)) \\
&= \sigma_2^{2^n}(x,y) - T(x)^a \left(T(x)^2 - 1\right)^a \left(T(x)^2 - \lambda_0^2\right)\left(T(x)^2 - \lambda_0^{-2}\right) \\
&= \sigma_2^{2^n}(x,y) + \frac{2^{2a+2}(x-1)^a}{(x+1)^{3a+4}} x^a (x^2+2x-1)(x^2-2x-1) \\
&= \sigma_2^{2^n}(x,y) + \frac{2^{2a+2}}{(x+1)^{4a+4}} x^a \left(x^2-1\right)^a \left(x^2-\lambda_0^2\right)\left(x^2-\lambda_0^{-2}\right) \\
&= \sigma_2^{2^n}(x,y) + \frac{2^{2a+2}y^{2^n}}{(x+1)^{4a+4}}
\end{aligned}$$

Now choosing a root of $\frac{2^{2a+2}y^{2^n}}{(x+1)^{4a+4}}$ we have

$$\sigma(x,y) = \left(\frac{1-x}{1+x}, \frac{\sqrt{2^c}\omega_{2^{n+1}}y}{(x+1)^c}\right).$$

Note that

$$\begin{aligned}
\sigma\left(\sigma(x,y)\right) &= \sigma\left(\frac{1-x}{1+x}, \frac{\sqrt{2^c}\omega_{2^{n+1}}y}{(x+1)^c}\right) \\
&= \left(\frac{x-1-1+x}{x+1}\frac{x+1}{x+1+1-x}, \frac{\sqrt{2^{2c}}\omega_{2^{n+1}}^2 y}{(x+1)^c}\frac{(x+1)^c}{(1-x+x+1)^c}\right) \\
&= (x, \omega_{2^n}y) \\
&= \tau_1^2(x,y)
\end{aligned}$$

and therefore σ has order 2^{n+1}.

Further we have $\sigma\tau_1 = \tau_2\sigma$:

$$\sigma\tau_1(x,y) = \sigma(-x, \omega_{2^{n+1}}y) = \left(\frac{1+x}{1-x}, \frac{\sqrt{2^c}\omega_{2^n}y}{(1-x)^c}\right).$$

$$\begin{aligned}
\tau_2\sigma(x,y) &= \tau_2\left(\frac{1-x}{1+x}, \frac{\sqrt{2^c}\omega_{2^{n+1}}y}{(x+1)^c}\right) \\
&= \left(\frac{x+1}{1-x}, \frac{\sqrt{2^c}\omega_{2^{n+1}}^2 y}{(x+1)^c}\frac{(x+1)^c}{(1-x)^c}\right) \\
&= \left(\frac{1+x}{1-x}, \frac{\sqrt{2^c}\omega_{2^n}y}{(1-x)^c}\right)
\end{aligned}$$

Structure Description

$$G_{0,i} =< \tau_1, \tau_2, \sigma >$$

We have

$\left(\tau_1^j\sigma\right)^2 = \tau_1^{2j+1}\tau_2$	
$\left(\sigma\tau_1^j\right)^2 = \tau_1^{2j+1}\tau_2$	c is an even number
$\left(\sigma\tau_1^j\right)^2 = \tau_1^{2^n+2j+1}\tau_2$	c is an odd number
$\left(\tau_2\tau_1^j\sigma\right)^2 = \tau_1^{2(j+2)}$	

Thus we have the following elements in the group G_0.

elements	exponent	number this type
τ_1^j	$0 \leq j \leq 2^{n+1}-1$	2^{n+1}
τ_2^j	j is odd numer	2^n
$\tau_1^j\tau_2$	j is odd number	2^n
σ^j	j is odd number	2^n
$\tau_1^j\sigma$	j is odd number	2^n
$\sigma\tau_1^j$	j is odd number	2^n
$\tau_2\tau_1^j\sigma$	j is odd number	2^n
		2^{n+3} elements

- When c is an even number, case $i=1$, it is easy to verify that

$$G_{0,1} \simeq \left\langle A, B, C : \begin{array}{c} A^{2^{n+1}} = B^{2^{n+1}} = C^{2^{n+1}} = 1, \\ A^2 = B^2 = C^2, \quad BA = AB, \quad AC = CB \end{array} \right\rangle .$$

  We will prove that

$$G_{0,1} \simeq \mathbb{Z}/4\mathbb{Z} \rtimes_{h_0} \mathbb{Z}/2^{n+1}\mathbb{Z},$$

  where h_0 defined below.
  We consider the following subgroups of $G_{0,1}$

$$H =< \tau_1^{-1}\sigma >, \quad K =< \tau_2 > .$$

We have H is a normal cyclic subgroup of order 4, in fact

$$\begin{aligned}
(\tau_1^{-1}\sigma)^2 &= \tau_1^{-1}\tau_2 \\
(\tau_1^{-1}\sigma)^3 &= \tau_1^{-1}\tau_2\tau_1^{-1}\sigma = \tau_2^{-1}\sigma = \sigma\tau_1^{-1} \\
(\tau_1^{-1}\sigma)^4 &= \tau_1^{-1}\tau_2\tau_1^{-1}\tau_2 = 1 \\
\sigma(\tau_1^{-1}\sigma)\sigma^{-1} &= \sigma\tau_1^{-1} \\
\tau_1(\tau_1^{-1}\sigma)\tau_1^{-1} &= \sigma\tau_1^{-1} \\
\tau_2^{-1}(\tau_1^{-1}\sigma)\tau_2 &= \sigma\tau_1^{-1}
\end{aligned}$$

(we recall that for c an even number we have $\tau_1\tau_2 = \tau_2\tau_1$.)

Is clear that $H \cap K = \{1\}$.
We may define

$$\begin{aligned}
h_0 : K &\longrightarrow \operatorname{Aut}(H) \\
\tau_2 &\rightarrow h_0(\tau_2) : \tau_1^{-1}\sigma \rightsquigarrow \sigma\tau_1^{-1}
\end{aligned}$$

then $H \rtimes_{h_0} K$ is a subgroup of $G_{0,1}$.
Since the order of H is 4 and the order of K is 2^{n+1} the

$$|H \rtimes_{h_0} K| = 2^{n+3},$$

therefore $H \rtimes_{h_0} K = G_{0,1}$.

- When c is an odd number, case $i = 2$, it is easy to verify that

$$G_{0,2} \simeq \left\langle A, B, C : \begin{array}{c} A^{2^{n+1}} = B^{2^{n+1}} = C^{2^{n+1}} = 1, \\ A^2 = B^2 = C^2, \quad BA = A^{2^n+1}B, \quad AC = CB \end{array} \right\rangle .$$

We will prove that $G_{0,2}$ is an extension of a group $\mathbb{Z}/2^n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by a group $\mathbb{Z}/4\mathbb{Z}$, this is

$$1 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow G_{0,2} \longrightarrow \mathbb{Z}/2^n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow 1\,.$$

Now we consider a subgroup of $G_{0,2}$, given by

$$H = < \tau_1^{2^n-1}\tau_2 > = \{1, \tau_1^{2^n-1}\tau_2, \tau_1^{2^n}, \tau_1^{-1}\tau_2\}\,.$$

Note that H is a normal subgroup of $G_{0,2}$:

$$\begin{aligned}\tau_1^{-1}(\tau_1^{2^n-1}\tau_2)\tau_1 &= \tau_1^{-1}\tau_2\\ \tau_2^{-1}(\tau_1^{2^n-1}\tau_2)\tau_2 &= \tau_1^{-1}\tau_2\\ \sigma^{-1}(\tau_1^{2^n-1}\tau_2)\sigma &= \tau_1^{-1}\tau_2\end{aligned}$$

and that $H \cap <\sigma> = <\tau_1^{2^n}>$.

Now we consider the quotient group $G_{0,2}/H$. Since H has order 4 then $G_{0,2}/H$ has order 2^{n+1}.
We may write the elements in $G_{0,2}/H$

class in $G_{0,2}/H$	number of classes
$[\tau_1^j] = \left\{\tau_1^j, \tau_2^j, \tau_1^{j+2^n}, \tau_2^{j+2^n}\right\}$	2^{n-1}
$[\sigma^j] = \left\{\sigma^j, \tau_2\tau_1^{j-2}\sigma, \sigma^{j+2^n}, \tau_2\tau_1^{2^n+j-2}\sigma\right\}$	2^{n-1}
$[\tau_1^j\sigma] = \left\{\tau_1^j\sigma, \sigma\tau_1^{j+2^n}, \tau_1^{j+2^n}\sigma, \sigma\tau_1^j\right\}$	2^{n-1}
$[\tau_1^j\tau_2] = \left\{\tau_1^j\tau_2, \tau_1^{2^n+1+j}, \tau_1^{j+2^n}\tau_2, \tau_1^{j+1}\right\}$	2^{n-1}
	2^{n+1} classes

Since $[\tau_1][\sigma] = [\sigma][\tau_1]$, $G_{0,2}/H$ is an abelian group of order 2^{n+1}, and its elements has order at most 2^n, then we have the following isomorphism

$$\begin{aligned}G_{0,2}/H &\longrightarrow \mathbb{Z}/2^n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\\ [\tau_1] &\rightsquigarrow A_1\\ [\sigma] &\rightsquigarrow B_1\end{aligned}$$

where we consider for $\mathbb{Z}/2^n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ the presentation given by Proposition 3. Therefore we have:

$$1 \longrightarrow H \longrightarrow G_{0,2} \longrightarrow G_{0,2}/H \longrightarrow 1\,.$$

□

Theorem 23. *For $i = 1, 2$, the group $G_{0,i}$ acts on S_{i,λ_0} with signature* $(0; 2^{n+1}, 2^{n+1}, 4)$.

Proof. If P is a fixed point for some element in $G_{0,i}$, say ς, then for any j, $\varsigma^j(P) = P$.
By the previous tables $\varsigma^2 \in G_i$, then P is a fixed point for the action of G_i. Therefore the fixed points for the action of $G_{0,i}$ are the same that for the action of G_i.

It is easy to verify that $[-\lambda_0, 0, 1], [\lambda_0^{-1}, 0, 1]$ are fixed points for σ. Furthermore $[\lambda_0, 0, 1], [-\lambda_0^{-1}, 0, 1]$ are fixed point for $\tau_2\sigma\tau_2^{-1} = \tau_2\tau_1^{-1}\sigma$:

$$\tau_2\sigma\tau_2^{-1} = \left(\frac{x+1}{x-1}, \frac{\sqrt{2^c}\omega_{2^{n+1}}y}{(x-1)^c}\right).$$

Now we will find elements $\varsigma \in G_{0,i}$, such that

- When c is an even number (case $i = 1$), then

$$\varsigma^2 = \tau_1^{2^{n-1}c-1}\tau_2\,.$$

- When c is an odd number (case $i = 2$), then

$$\varsigma^2 = \tau_1^{2^{n-1}c-1}\tau_2\,,$$

  or

$$\varsigma^2 = \tau_1^{3\cdot 2^{n-1}c-1}\tau_2\,.$$

We will see that $\varsigma \neq \tau_2\tau_1^j\sigma$. We have

$$\tau_2\tau_1^j\sigma = \tau_2\sigma^{j+2}\tau_2^{-1}\,,$$

and the power this elements are conjugate to σ or power of τ_1^2.

Then the only possibility is $\varsigma = \tau_1^j\sigma$ or $\varsigma = \sigma\tau_1^j$.

If $\varsigma = \tau_1^j\sigma$, then by the previous tables

$$\varsigma^2 = \tau_1^{2j+1}\tau_2\,.$$

- If $\varsigma^2 = \tau_1^{2^{n-1}c-1}\tau_2$ then

$$2^{n-1}c - 1 \equiv 2j+1 \mod 2^{n+1}\,,$$

  therefore $j = 2^n + 2^{n-2}c - 1$.

- If $\varsigma^2 = \tau_1^{3\cdot 2^{n-1}c-1}\tau_2$ then

$$3 \cdot 2^{n-1}c - 1 \equiv 2j + 1 \mod 2^{n+1},$$

therefore $j = 2^n + 3 \cdot 2^{n-2}c - 1$.

With these calculations we prove that there exist elements $\varsigma \in G_{0,i}$ such that they have fixed points of type

$$\{[i, p, 1], [-i, q, 1] : p, q\},$$

and as ς^2 has order 2 then ς has order 4. Further we recall these points belongs to the same orbit.

Now we remark that since $\sigma[0,0,1] = [1,0,1]$ then the points

$$\{[0,0,1], \tau_2[0,0,1], [\pm 1, 0, 1]\}$$

are in the same orbit under $G_{0,i}$.

Using the Riemann Hurwitz formula for the branched covering

$$S \longrightarrow S/G_{0,i},$$

we have

$$\begin{aligned} 3(2^n - 1) &= 2^{n+3}(\gamma - 1) + 1 + \frac{1}{2}(4(2^{n+1} - 1) + 4(2^{n+1} - 1) + 2^{n+1}(4 - 1)) \\ \therefore \gamma &= 0 \end{aligned}$$

Therefore the signature for this action is

$$(0; 2^{n+1}, 2^{n+1}, 4)$$

□

Remark 22. In the case when $n = 3, a = 1$ and $c = 1$, we have the following table associated to the action of $G_{0,2}$ on S_{2,λ_0}:

auto.	order	fixed point	$G_{0,2}$–orbit
τ_1	16	$\{[1,0,0],[0,0,1]\}$	$\{[1,0,0],[0,0,1],[\pm 1,0,1]\}$
τ_2	16	$\{[1,0,1],[-1,0,1]\}$	$\{[1,0,0],[0,0,1],[\pm 1,0,1]\}$
σ	16	$\{[-\lambda,0,1],[\lambda^{-1},0,1]\}$	$\{[\pm\lambda,0,1],[\pm\lambda^{-1},0,1]\}$
$\tau_2\tau_1\sigma$	16	$\{[\lambda,0,1],[-\lambda^{-1},0,1]\}$	$\{[\pm\lambda,0,1],[\pm\lambda^{-1},0,1]\}$
τ_1^2	8	-	-
$\tau_1\tau_2$	8	-	-
$\tau_1^{11}\sigma$	8	-	-
$\tau_1^{15}\sigma$	8	-	-
τ_1^4	4	-	-
$\tau_1\sigma$	4	-	-
$\sigma\tau_1$	4	-	-
$\tau_1^5\sigma$	4	$\{[i,p,1] : p^8 = -2i(1+\lambda^2)(1+\lambda^{-2})\}$	$\{[i,p,1]\} \cup \{[-i,q,1]\}$
$\sigma\tau_1^5$	4	$\{[-i,q,1] : q^8 = 2i(1+\lambda^2)(1+\lambda^{-2})\}$	$\{[i,p,1]\} \cup \{[-i,q,1]\}$
$\tau_1^7\tau_2$	4	—	—
τ_1^8	2	—	—
$\tau_1^3\tau_2$	2	-	-
$\tau_1^{11}\tau_2$	2	-	-

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