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DINÁMICA DE KINKS PARA CAMPOS DE YANG-MILLS SOBRE ESPACIO-TIEMPOS HIPERBÓLICOS

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MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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RESUMEN DE TESIS PARA OPTAR AL GRADO
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MENCIÓN MATEMÁTICAS APLICADAS Y
MEMORIA PARA OPTAR AL TÍTULO DE
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DINÁMICA DE KINKS PARA CAMPOS DE YANG-MILLS SOBRE ESPACIO-TIEMPOS HIPERBÓLICOS

Comprender la estructura de la naturaleza desde la física a menudo plantea problemas profundos que desafían las capacidades de las matemáticas. Dos ejemplos emblemáticos son la formulación de la relatividad general y la búsqueda de una teoría cuántica de campos para explicar las interacciones entre partículas. En este contexto, la teoría de Yang-Mills se erige como una formulación pionera para una teoría de gauge no-abeliana, con un fructífero desarrollo y estudio en ambas áreas.

El objetivo principal de esta tesis es estudiar la estabilidad asintótica de ciertas soluciones para un modelo de campo de Yang-Mills sobre $SU(2)$, bajo una geometría determinada por el exterior de un agujero negro tipo Reissner-Nordström.

La tesis está compuesta por 3 capítulos. El primero consiste en una breve y concisa introducción a los tópicos generales en que se enmarca el trabajo realizado, sirviendo para presentar el contexto físico donde surge el problema a estudiar.

En el Capítulo 2 consideramos el modelo de Yang-Mills $SU(2)$ sobre el exterior de un agujero negro extremal Reissner-Nordström, empleando un ansatz de un campo esféricamente simétrico puramente magnético, originalmente propuesto por Bizoń y Kahl [8]. Estudiamos en mayor detalle la dinámica del kink $H(x) = \tanh(x/2)$, siendo la primera solución estática no trivial del modelo. El operador lineal asociado posee un único valor propio negativo, sugiriendo una alta inestabilidad de H. Mediante el uso de identidades viriales siguiendo el espíritu de Kowalczyk, Martel, Muñoz, y Van Den Bosch, probamos estabilidad asintótica condicional para perturbaciones en el espacio de energías $\boldsymbol{E_H}$. En la sección 4 realizamos una primera estimación con un virial a gran escala. Necesitando un mayor control, en la sección 5 estudiamos un segundo virial empleando una dualización del problema, en las secciones 6 y 7 demostramos unas estimaciones técnicas y controlamos terminos no lineales del segundo virial. En la sección 8 obtenemos una estimación del problema dual y por último, en la sección 9 concluimos el teorema principal. A diferencia de trabajos anteriores, nuestro resultado se basa en un estudio detallado del operador L, desarrolado en las secciones 10 y 11, modificaciones a estimaciones técnicas desrrolladas en la sección 6, además de modificaciones de viriales ampliamente empleado en la literatura.

Concluimos en el Capítulo 3 con un repaso del trabajo desarrollado y mención a posibles trabajos futuros.

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KINK DYNAMICS FOR YANG-MILLS FIELDS ON HYPERBOLIC SPACETIMES

Comprehending the structure of nature from physics often raises deep problems that challenge the capabilities of mathematics. Two emblematic examples are the formulation of general relativity and the research for a quantum field theory to explain particle interactions. In this context, the Yang-Mills theory stands as a pioneer formulation for a non-Abelian gauge theory, with fruitful development in both areas.

The main objective of this thesis is to study the asymptotic stability of particular solutions for a Yang-Mills field model on $SU(2)$, under a geometry determined by the exterior of an extremal Reissner-Nordström black hole.

This thesis is composed of 3 chapters. The first consists of a brief and concise introduction to the general topics related to the research, presenting the physical context where the problem to be studied arises.

In Chapter 2, we consider the $SU(2)$ Yang-Mills model on the exterior of a Reissner-Nordström extremal black hole, employing an ansatz of a purely magnetic spherically symmetric field, proposed initially by Bizoń and Kahl [8]. We study in detail the dynamics of the kink $H(x) = \tanh(x/2)$, being the first non-trivial static solution of the model. The associated linear operator has a unique negative eigenvalue, suggesting a high instability of H. In section 4 we perform a first estimation with a large-scale virial. Requiring more control, in section 5 we study a second virial using a dualization of the problem, in sections 6 and 7 we demonstrate some technical estimates and control for non-linear terms of the second virial. In section 8 we obtain an estimate of the dual problem, and finally, in section 9 we conclude the main theorem. In contrast to previous works, our result is based on a detailed study of the linear operator L, developed in sections 10 and 11, modifications to technical estimates developed in section 6, and modifications of virials widely used in the literature to capture the specific features of the model.

We conclude in Chapter 3 with a review of the thesis and a mention of possible future works.

La pensée n'est qu'un écliar au milieu d'une longue nuit.
Mais c'est cet éclair qui est tout.

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Table of Content

Table of Figures

Chapter 1

Introduction

1. Preliminaries

Physics has had great success in explaining various natural phenomena through mathematical formulations. On the one hand, thanks to the appearance of the theory of general relativity at the beginning of the last century, low-energy phenomena have been predicted with great accuracy, and on the other, the appearance of quantum mechanics later achieved a satisfactory description of phenomena at high energy scales. Subsequently, since the mid-twentieth century, with the introduction of the notion of fields, a successful formulation of a quantum field theory for particles at relativistic levels was achieved, such as quantum electrodynamics (QED) or quantum chromodynamics (QCD), establishing the foundations of what is known today as the Standard Model of particles, whose initial predictions have been verified over the decades, the latest achievement being the discovery of the Higgs boson, whose mechanism is responsible for giving particles a certain mass.

An essential component of the Standard Model is the Yang-Mills Theory, conceived by Yang and Mills in the 1950s. This field theory describes the strong nuclear fundamental forces and their interactions. There is great interest in the scientific community in studying this field coupled with other fields of matter [1, 18, 41, 43], more specifically, considering a Yang-Mills-Higgs theory, as well as studying it over a curved spacetime due to gravity [6, 7, 8]. Due to the complexity of these phenomena, it is necessary to use advanced techniques in mathematics.

In this context, the first aim of this thesis is to study a Yang-Mills field on a given geometry in the vicinity of an extremal Reissner-Nordstrom black hole. In particular, we study a non-trivial solution known as *kink*, recently found by Bizoń and Kahl. These types of solutions are of great physical and mathematical relevance and have been extensively studied for various [29, 35] models. Given the dispersive nature of the Yang-Mills equations, we employ virial techniques to study the asymptotic behavior of these solutions, inspired by a wide range of recent works employing similar techniques, but due to the nature of the problem, it will be necessary to apply a series of modifications.

Before describing the Yang-Mills equations for a geometry close to an extremal Reisssner-Nordström black hole considered in this thesis, we shortly recall some important notions to be accounted. We concentrate on giving a description of what are dispersive equations, elements of general relativity, and the main technique employed in this thesis: virial identities.

1.1. Dispersive Equations

In this section, we introduce the study of constant-coefficient linear dispersive PDE, which are the simplest example of dispersive equations. It is essential to have a satisfactory theory of the linear equation before proceeding to the nonlinear because much of the theory of nonlinear PDE, especially for short times or small data, is obtained by perturbation of the linear theory. To simplify the discussion, we consider the spatial domain $\mathbb{R}^d$ and PDE which are first-order in time.

A constant-coefficient linear dispersive PDE first-order in time is an equation of the form

$$\partial_t u(t,x) = Lu(t,x), \quad u(0,x) = u_0(x), \tag{1.1}$$

where the field $u : \mathbb{R} \times \mathbb{R}^d \to V$ takes values in a finite-dimensional Hilbert space V, and L is a skew-adjoint constant coefficient differential operator in space, taking the form

$$Lu(x) := \sum_{|\alpha| \leq k} c_\alpha \partial_x^\alpha u(x),$$

where $k \geq 1$ is an integer (the order of the differential operator), $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{Z}_+^d$ ranges over all multi-indices with $|\alpha| := \alpha_1 + ... + \alpha_d$ less than or equal to k, ∂_x^α is the partial derivative

$$\partial_x^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} ... \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$$

and $c_\alpha \in \mathrm{End}(V)$ are coefficients that do not depend on x, where $\mathrm{End}(V)$ is the set of linear transformations from V to itself. This operator is classically only defined on k-times continuously differentiable functions, but we may extend it to distributions or functions in other function spaces in the usual manner; thus we can talk about classical and weak (distributional) solutions to the equation (1.1).

In order to give a complete notion of what a dispersive partial differential equation is, we consider the one-dimensional case and look for *plane wave* solutions of the form

$$u(t,x) = Ae^{i(kx-\omega t)},$$

where A, k and ω are constants representing the amplitude, wavenumber, and frequency respectively. Hence u will be a solution of (1.1) if and only if

$$\omega(k) + \sum_{\alpha \leq k} c_\alpha i^{\alpha-1} k^\alpha = 0.$$

This equation for w is called the *dispersion relation*, and determines how time oscillations $e^{i\omega t}$ are linked to spatial oscillations e^{ikx} of wave number k. In other words, the dispersion relation is the function for which the plane waves $e^{i(kx-\omega(k)t)}$ solve the equation.

One frequently used criterion for defining dispersive equations is that $\omega(k)$ is a real-valued function of k and $\frac{d^2\omega}{dk^2} \neq 0$. In the physical context, this means that different frequencies in the equation will tend to propagate at different velocities, thus dispersing the solution over time.

An interesting notion to describe the phenomena is the *phase velocity* of the plane waves, which is defined by

$$\nu_p(k) := \frac{\omega}{k}.$$

With this definition, one can re-write the solution as:

$$u(t,x) = e^{ik(x-\nu_p(k)t)} = u(0, x - \nu_p(k)t),$$

and conclude that plane waves travel with velocity $\nu_p(k)$. In particular, large-frequency data travel faster than smaller ones. Another related definition is the group velocity,

$$\nu_g(k) := -\nabla\omega(k),$$

which describes how a frequency-localized bump function around k moves. The group velocity is the more important of the two velocities, as it controls the motion of frequency envelopes and thus of energy and mass, whereas the phase velocity merely controls the apparent motion of crests and troughs, which are of little physical significance.

Under this notion, the transport equation

$$\partial_t u = -v \cdot \nabla u. \quad u(0,x) = u_0(x)$$

and the wave equation

$$\partial_t^2 u - \partial_x^2 u = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x)$$

are not dispersive. The first one has a dispersion equation of $\omega(k) = -v \cdot k$, which means that all frequencies move at the same velocity (a degenerate case of dispersion). The second one has a dispersion equation of $\omega(k) = \pm|k|$, which means that only the direction of propagation is determined by frequency, but not its group velocity.

More interesting examples (many of which arise from physics) can be constructed if one either raises the order of L or makes u vector-valued instead of scalar. Examples include the *free Schrödinger equation*

$$i\partial_t u + \frac{\hbar}{2m}\Delta u = 0, \quad u(0,x) = u_0(x)$$

where $u : \mathbb{R} \times \mathbb{R}^d \to V$ is a complex field and *Planck's constant* $\hbar > 0$ and the mass $m > 0$ are fixed scalars, as well as the second-order in time *Klein-Gordon equation*

$$\Box u - \frac{m^2c^2}{\hbar^2}u = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x)$$

where $u : \mathbb{R} \times \mathbb{R}^d \to V$ is a real field, the mass $m > 0$ and speed of light $c > 0$ are fixed, and $\Box$ is the *d'Alembertian operator*

$$\Box = \partial^\alpha \partial_\alpha = -\frac{1}{c^2}\partial_t^2 + \Delta.$$

If we consider $m = c = \hbar = 1$, the dispersion relations here are $\omega(k) = -|k|^2$ and $\omega(k) = \pm\langle k\rangle$ respectively.

In this thesis, we consider a specific model of dispersive equations, known as the Yang-Mills equations. They arise as the Euler-Lagrange equations of the Yang-Mills action functional, having a linear component of wave type. Now we start by describing this model.

2. $SU(N)$ Yang-Mills Models

In the 50s, R. Mills and C. N. Yang developed (essentially independent of the mathematical literature) a theory of principal bundles and connections to explain the concept of gauge symmetry and gauge invariance as it applies to physical theories [45]. The gauge theories Yang and Mills discovered, now called *Yang-Mills theories*, generalized the classical work of James Maxwell on Maxwell's equations, which can be understood in the language of a $U(1)$ Lie group gauge theory [40]. The novelty of the work of Yang and Mills was to define gauge theories for an arbitrary choice of Lie group G, called the gauge group. This group could be non-Abelian instead of the case $G = U(1)$ corresponding to electromagnetism.

We show a classical formulation of these equations for a $G = SU(N)$ Lie group in the following. We recommend [18] for an introduction to Lie groups, Lie algebras and gauge theory, and [37] for explicit computations of Yang-Mills equations in curved space-time.

Let (M, g) be a $n-$dimensional pseudo-Riemannian manifold. In physics, M is spacetime and g usually has Lorentzian signature. We start establishing some necessary definitions before formulating the Yang-Mills theory.

Definition 2.1 (Canonical volume form)**.** *The metric g together with the orientation of the manifold M define a **canonical volume form** $dvol_g$ on M. If (U, ϕ) is an oriented chart for M with local coordinates x^μ, then*

$$dvol_g = \sqrt{|det(g_{\mu\nu})|}dx^1 \wedge ... \wedge dx^n, \tag{1.2}$$

where $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$ are the components of the 2-form metric.

If $\partial_{\mu_1}, ..., \partial_{\mu_n}$ is an oriented, orthonormal basis of T_pM, the $dvol_g$ is characterized by

$$dvol_g(\partial_{\mu_1}, ..., \partial_{\mu_n}) = +1.$$

We denote by $g^{\mu\nu}$ the entries of the inverse of g. We can raise indices of tensors in the standard way using $g^{\mu\nu}$. For example,

$$T^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}T_{\rho\sigma},$$

where the Einstein summation convention is assumed.

Definition 2.2. *We define the **scalar product of k-forms***

$$\langle\cdot,\cdot\rangle_k : \Omega^k(M) \times \Omega^k(M) \longrightarrow \mathcal{C}^\infty(M)$$

as follows. For real-valued $k - forms$ $\omega, \eta \in \Omega^k(M)$ on M we set

$$\langle \omega, \eta \rangle_k = \sum_{\mu_1 < ... < \mu_k} \omega_{\mu_1 ... \mu_k} \eta^{\mu_1 ... \mu_k} = \frac{1}{k!} \omega_{\mu_1 ... \mu_k} \eta^{\mu_1 ... \mu_k},$$

where $\omega_{\mu_1 ... \mu_k} = \omega(\partial_{\mu_1}, ..., \partial_{\mu_k})$ in a local chart (U, ϕ) of M.

Definition 2.3. *If we denote $\Omega^k(M)$ the space of $k-$forms, we define the Hodge star operator*

$$* : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$$

as the linear map defined for real-valued forms by

$$\langle \omega, \eta \rangle_k dvol_g = \omega \wedge * \eta$$

for all $\omega, \eta \in \Omega^k(M)$. Choosing a local frame, it can be shown that this uniquely defines $*$.

We can obtain an explicit form of the Hodge operator as follows. Suppose $\{\partial_{\mu_1}, ..., \partial_{\mu_n}\}$ is an oriented, orthonormal basis of tangent vectors with $g(\partial_\mu, \partial_\mu) = g_{\mu\mu} = g^{\mu\mu} = \pm 1$. Let $\{d^{\mu_1}, ..., d^{\mu_n}\}$ be the dual basis of 1-forms with $d^\mu(\partial_\nu) = \delta^\mu_\nu$. Then, from (1.2)

$$\mathrm{dvol}_g = d^{\mu_1} \wedge ... \wedge d^{\mu_n}$$

and we have:

Lemma 2.4. *The Hodge star operator is given by*

$$*(d^{m_1} \wedge ... \wedge d^{m_n}) = g^{m_1 m_1} ... g^{m_k m_k} \epsilon_{m_1 ... m_k m_{k+1} ... m_n} d^{m_{k+1}} \wedge ... \wedge d^{m_n}.$$

We will use the usual exterior differential, wich is represented by $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ given by $d = \partial_\mu dx^\mu$.

Let A be a connection 1-form in the gauge group $SU(N)$, defined by

$$A = A_\mu dx^\mu = A^a_\mu t_a dx^\mu.$$

where the A^a_μ are the non-abelian fields associated to $SU(N)$, t_a the generators of $SU(N)$, with $a = 1, ..., N^2 - 1$.

2.1. $SU(N)$ Yang-Mills equations

Formally, the connection A is said to obey the Yang-Mills equation if it is a critical point for the Yang-Mills action

$$S_{YM}(A) = \frac{1}{2} \int_M \langle F^A, F^A \rangle_2 \mathrm{dvol}_g = \int_M \mathcal{L}_{YM}[A] dx^n,$$

where $F^A \in \Omega(M)$ is the 2-form **curvature of the connection** A, defined via the *structure equation*

$$F^A = dA + A \wedge A, \tag{1.3}$$

and $\mathcal{L}_{YM}[A] : M \to \mathbb{R}$ is the **Yang-Mills lagrangian density** for the fixed connection A, defined by

$$\mathcal{L}_{YM}[A] = \frac{1}{4} F^{Aa}_{\mu\nu} F^{A\mu\nu} = \frac{1}{4} F^{Aa}_{\mu\nu} F^{A}_{\alpha\beta} g^{\alpha\mu} g^{\beta\nu}. \tag{1.4}$$

Expressing the curvature using the generators of $SU(N)$ we have that the Yang-Mills action has the following equivalent form

$$S_{YM}(A) = \frac{1}{4} \int_M F^{A}_{\mu\nu} F^{A\mu\nu} \mathrm{dvol}_g = \frac{1}{4} \int_M F^{Aa}_{\mu\nu} F^{Aa\mu\nu} \mathrm{dvol}_g,$$

where we have expressed the curvature as

$$F^A = F^{A}_{\mu\nu} dx^\mu \wedge dx^\nu = F^{Aa}_{\mu\nu} t_a dx^\mu \wedge dx^\nu, \tag{1.5}$$

with $F^a_{\mu\nu}$ representing the non-abelian stress tensor associated to the gauge group $SU(N)$.

Theorem 2.5. *A connection A is a critical point of the Yang-Mills action S_{YM} if and only if A satisfies the* ***Yang-Mills equation***

$$d * F^A + A \wedge *F - *F \wedge A = 0. \tag{1.6}$$

Proof. We refer to [18][5] for a proof of this theorem. □

Remark 2.6. *Recall that any connection A must satisfy the* ***Biachi identity***

$$dF^A + A \wedge F - F \wedge A = 0.$$

Atiyah and Bott [4] proved that the curvature F^A of a connection A that satisfies in addition to the Bianchi identity the Yang-Mills equation (1.6) *can thus be considered as a harmonic form (in a non-linear sense if the Lie group is non-abelian) in $\Omega^2(M)$. Thus, the Yang-Mills equation corresponds to a second-order partial differential equation for the connection A.*

For $M = \mathbb{R}^{d+1}$ with minkowski metric g, the equations (1.6) *has the explicit form [44]*

$$\Box A + \nabla(\partial_\mu A^\mu) = [A, \nabla A] + [A, [A, A]], \tag{1.7}$$

where the equation is under-determined because of the gauge invariance

$$A \to U^{-1} dU + U^{-1} A U$$
$$F \to U^{-1} F U$$

in the equation, where U is an arbitrary function taking values in G. To solve this problem, typically one has to impose a further constraint on the gauge. For example, $A^0 = 0$ (temporal gauge) or $\partial_i A^i = 0$ (coulomb gauge). We can see from (1.7) *that Yang-Mills equations on flat space correspond to wave-type equations.*

Remark 2.7. *Note that the Yang-Mills equation depends through the Hodge star operator on the pseudo-Riemannian metric g on M. If the equation holds for one metric, it does not necessarily hold for another metric.*

2.2. $SU(2)$ Yang-Mills Model on the Extremal Reissner-Nordström Black Hole

Let us consider the Reissner-Nordström metric [11, 26]

$$\hat{g} = -\left(1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1.8)$$

for $r > 0, (\theta, \phi) \in S^2$, which describe a charged black hole of mass M, total electric charge Q, and total magnetic charge P in the Schwarzchild spherical coordinates.

If we consider the case where $M^2 = Q^2 + P^2$, the metric (1.8) assumes the form

$$\hat{g} = -\frac{(r - M)^2}{r^2} dt^2 + \frac{r^2}{(r - M)^2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad \text{for } r > 0, (\theta, \phi) \in S^2, \quad (1.9)$$

known as the extremal Reissner-Nordström black hole metric, with event horizon at $r = M$.

The exterior of the extremal Reisser-Nordström black hole is a globally hyperbolic static space-time $(M, \hat{g})$, where M is defined by the condition $r > M$.

As in [8], we use the change of variables

$$t = \frac{t}{4M}, \quad x = \ln\left(\frac{r}{M} - 1\right).$$

Replacing in (1.9), we obtain

$$\hat{g} = \Omega(x)(-dt^2 + C^4(x)(dx^2 + d\theta^2 + \sin^2\theta d\phi^2) =: \Omega(x)g, \quad (1.10)$$

where

$$\Omega(x) = \frac{16M^2}{(1 + e^{-x})^2}, \qquad C(x) = \cosh\left(\frac{x}{2}\right).$$

Thus, we see that the exterior region in terms of the new variable $x \in (-\infty, +\infty)$ is characterized by a spatial origin $x = 0$ where $C(x)$ is minimized, and corresponding to the classical Schwarzschild radius $r_S = 2M$. In addition, contrary to (M, g), the spacetime (M, g) is geodesically complete.

As remarked in [8], (M, g) has two asymptotically flat ends at $x = \pm\infty$ (see Fig. 1.1). This is evident when we apply a second change of variable $\rho = C^2(x)$ for which we have

$$g = -dt^2 + \left(1 - \frac{1}{\rho}\right)^{-1} d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2).$$

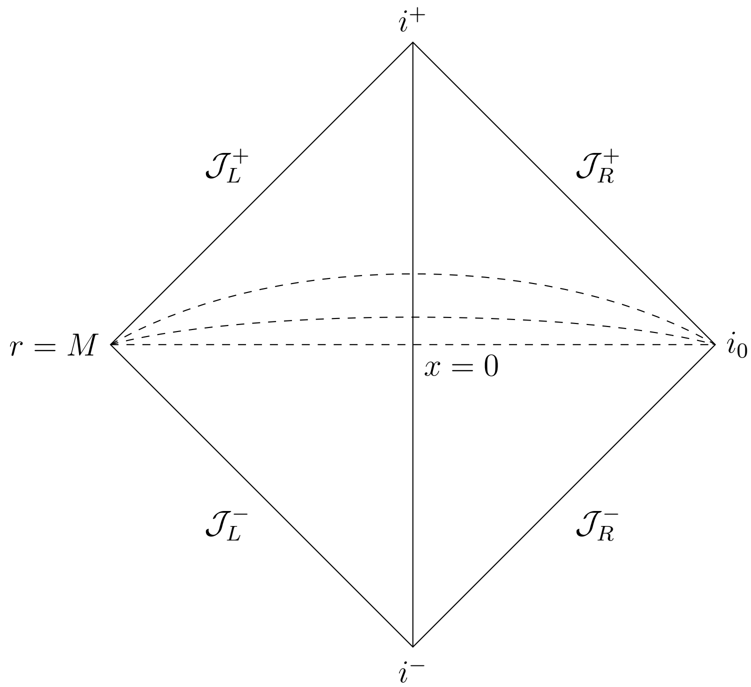

Figure 1.1: Penrose diagram for (M, g).

We consider an $SU(2)$ Yang-Mills field over the spacetime (M, g). Using σ_a with $a = 1, 2, 3$ to denote the Pauli matrices and setting the generators of $\mathfrak{su}(2)$,

$$t_a = \frac{\sigma}{2i}, \quad a = 1, 2, 3.$$

In addition, $\{t_a\}$ satisfies the commutator relation $[t_a, t_b] = \epsilon_{abc} t_c$.

We define the connection or gauge potential $A = A_\mu dx^\mu$, taking values in the Lie algebra $\mathfrak{su}(2)$. Analogously to [8], we choose the connection to be the 1-form spherically symmetric purely magnetic ansatz

$$A = \varphi(t, x)\omega + t_3 \cos\theta d\phi, \qquad \text{with } \omega = t_1 d\theta + t_2 \sin\theta d\phi, \tag{1.11}$$

where $\varphi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a scalar field. Employing the structure equation (1.3), we compute the Yang-Mills curvature field for (1.11) obtaining

$$F = t_1(\partial_t\varphi dt + \partial_x\varphi dx) \wedge d\theta + t_2 \sin\theta(\partial_t\varphi dt + \partial_x\varphi dx) \wedge d\phi + t_3(\varphi^2 - 1)\sin\theta d\theta \wedge d\phi.$$

Identifying $x^0 = t$, $x^1 = x$, $x^2 = \theta$, $x^3 = \phi$ and using (1.5), we obtain the nonzero and independent components of the curvature field

$$F^1_{02} = \frac{1}{2}\partial_t\varphi, \quad F^1_{12} = \frac{1}{2}\partial_x\varphi, \quad F^2_{03} = \frac{1}{2}\partial_t\varphi\sin\theta, \quad F^2_{13} = \frac{1}{2}\partial_x\varphi\sin\theta, \quad F^3_{23} = \frac{1}{2}(\varphi^2 - 1)\sin\theta.$$

Thus, replacing into (1.4), and using that in 4 dimensions, the quantity $g^{\alpha\mu}g^{\beta\nu}\sqrt{|g|}$ is invariant under a conformal transformation (1.10), the Yang-Mills lagrangian density for the ansatz corresponds to

$$\mathcal{L} = \left(-\frac{1}{4} C^2(x)(\partial_t \varphi)^2 + \frac{1}{4C(x)^2}(\partial_x \varphi)^2 + \frac{1}{8C^2(x)}(1-\varphi^2)^2 \right) \sin\theta. \qquad (1.12)$$

We notice that the expression in parenthesis in (1.12) has no dependence on the variables (θ, ϕ). Then, using (1.4), we can define the reduced lagrangian density

$$\mathcal{L}_\varphi = -\frac{1}{2} C^2(x)(\partial_t \varphi)^2 + \frac{1}{2C(x)^2}(\partial_x \varphi)^2 + \frac{1}{2C^2(x)}(1-\varphi^2)^2, \qquad (1.13)$$

and so, the Yang-Mills equations (1.6) for A are equivalent to the Euler-Lagrange equation of (1.13)

$$\partial_t^2 \varphi - C^{-2}(x)\partial_x (C^{-2} \partial_x \varphi) - C^{-4}(x)\varphi(1-\varphi^2) = 0. \qquad (1.14)$$

In this thesis, we are interested in the long-time asymptotic stability of solutions of the Yang-Mills field given by (1.14). This model exhibits time translation invariance, is Hamiltonian, and has the following conserved quantity

$$E[\varphi, \varphi_t](t) = \int \left(C^2(x)(\partial_t \varphi)^2 + C^{-2}(x) \left((\partial_x \varphi)^2 + \frac{(1-\varphi^2)^2}{2} \right) \right) dx. \qquad (1.15)$$

Respecting to the Cauchy problem, (1.14) is globally well-posed for initial data in $\dot{H}^{k+1}_{\text{loc}}(\Sigma)$, where Σ correspond to the Cauchy surface (see [14]).

3. The Virial Technique

We briefly describe one of the main techniques we will use in this thesis, based on Virial identities [34]. In Physics, the well-known Virial Theorem gives a relation between the average total kinetic energy and the total potential energy of the system. Moreover, in elliptic PDEs and mathematical physics, there exists an equivalent identity known as Pokhozhaev's identity, which is applicable to localized solutions to the stationary nonlinear Schrödinger equation or the stationary nonlinear Klein-Gordon equation.

The Virial identities in their modern mathematical form were introduced by Glassey [17] to show blow-up for certain focusing nonlinear Schrödinger equations (NLS). In general, these identities are used to show that a positive quantity involving the solution u has a monotonic behavior in time, and at the same time, they are bounded uniformly in time (e.g., by using conservation laws). Thus, from the fundamental theorem of calculus, we conclude that its time derivative decays to zero as $t \to \pm\infty$, at least in some averaged or weighted sense. This type of long-time decay is especially useful for understanding the asymptotic behavior of the solution.

Monotonic quantities have recently been used in a wide and powerful way in the context of dispersive equations, se [19, 20, 21, 23, 24, 27, 30, 31, 32, 33, 38]. It has allowed describing the behavior of several equations in a wide range of properties, such as the decay of solutions, the existence of blow-ups, and asymptotic stability.

We describe in simple words how Virial works. Unlike conservation laws, which can be systematically generated from symmetries via Noether's theorem, we do not have a fully automated way of producing monotone quantities other than trial and error. The base of the argument is the election of a conserved quantity and modifying it, losing conservation but expecting to obtain monotonicity. This monotonicity relation is narrowly related to the dynamic in time of some particular norm of the solution.

For example, for the following class of non-linear wave equations,

$$\partial_t^2 u - \partial_x^2 u = u - |u|^p, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1.16}$$

if we denote $v = \partial_t u$, one has the equivalent system of first order

$$\begin{cases} \partial_t u = v \\ \partial_t v = \partial_x^2 u + u + |u|^p. \end{cases} \tag{1.17}$$

which is Hamiltonian. Its Hamiltonian character leads to the conservation of energy and momentum, given by

$$\begin{aligned} H(u,v) &= \frac{1}{2}\int (v^2 + (\partial_x u)^2 - u^2) - \frac{1}{p+1}\int |u|^{p+1}, \\ P(u,v) &= \int uv. \end{aligned}$$

which, are well defined in $H^1 \times L^2$. Now, let's define the following functional

$$\mathcal{I}[u](t) := \int \psi(\partial_x u)v + \frac{1}{2}\int \psi' uv$$

where ψ is a smooth, bounded function. In [22], they prove that for a weight function ψ and odd function u solution of (1.16), the following relation is satisfied:

$$\frac{d}{dt}\mathcal{I} \lesssim -\int (|\partial_x u|^2 + u^2)\,\mathrm{sech}(x).$$

This identity has the good sign property (for small solutions) and is used to prove the decay of solutions to 0 in [22].

4. Result in this thesis: Asymptotic Stability of a Yang-Mills Kink

In chapter 2, we are motivated by the long time behavior problem for the kink of (1.14). To our knowledge, this is the first result on the asymptotic stability of the kink for the model considered.

In this Chapter, we will prove that any small perturbation of the static solution in the energy space, under certain orthogonality conditions, is (locally) asymptotically stable. Considering the change of variables $x \to \alpha^{-1}(x)$, where

$$\alpha(x) = \frac{1}{3}(\sinh x + x),$$

and defining $\phi = \varphi \circ \alpha^{-1}$, $\widetilde{H} = H \circ \alpha^{-1}$, this can be summarized in the following statement.

Theorem 4.1 (Main theorem)**.** *There exists $\delta > 0$ such that if a global solution $(\phi, \partial_t \phi)$ of* (2.4) *satisfies for all $t \geq 0$,*

$$\|(\phi, \partial_t \phi)(t) - (\widetilde{H}, 0)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})} < \delta, \tag{1.18}$$

then for any I compact interval in $\mathbb{R}$*,*

$$\|(\phi, \partial_t \phi)(t) - (\widetilde{H}, 0)\|_{H^1(I) \times L^2(\mathbb{R})} = 0\,. \tag{1.19}$$

As far as we understand, this is the first description of the standing wave dynamics in the $SU(2)$ Yang-Mills model over the exterior of a Reissner-Nordsröm black hole, which seems unstable by nature. Clearly, the data under which (1.18) us satisfied is not empty, the kink $(\widetilde{H}, 0)$ being its most important representative.

Chapter 2

Dynamics of a Yang-Mills field in extremal Reissner-Nordström black holes and conditional asymptotic stability of kinks

1. Introduction

In 1954 Chen Ning Yang and Robert Mills presented the first concepts of a gauge theory for non-abelian groups that could give an explanation for strong interactions in physics [45]. This constituted the beginning of the Yang-Mills theory, nowadays present in the foundations of the Standard Model, that attempt to describe the interactions between elementary particles. For our context in the classical Yang-Mills field, there have been several results from a mathematical formalism related to. The global dynamics of a Yang-Mills field propagating in a 4-dimensional Minkowski spacetime is well understood for a smooth initial data [16, 13] as well as the global in time regularity in any globally hyperbolic 4-dimensional curved spacetime [14], however the phase portrait can be richer due to the existence of nontrivial stationary solutions which play the role of unstable attractors.

In this work we are interested in analyzing the evolution and stability of the field for a certain hyperbolic geometry. In a recent paper [8] Bizoń and Kahl studied the static solutions of a Yang-Mills field on the exterior of an extremal Reissner-Nordström black hole (see also [7] for previous work on other black holes). Considering a dimensionless change of variables for the metric of the globally hyperbolic static spacetime, it is obtained the geodesically complete spacetime $(\mathcal{M}, g)$ where the metric g is defined in the following way:

$$g = -dt^2 + C^4(x)(dx^2 + d\theta^2 + \sin^2\theta d\varphi^2),$$

where $C(x) = \cosh(x/2)$. Proposing a spherically symmetric and purely magnetic SU(2) Yang-Mills field propagating in $(\mathcal{M}, g)$ given by

$$A(t, x) = \varphi(t, x)\omega(\tau_1, \tau_2) + \tau_3 \cos\theta d\varphi,$$

where $\omega(\tau_1, \tau_2) = \tau_1 d\theta + \tau_2 \sin\theta d\varphi$ and φ is a real scalar field, Bizon and Kahl obtained the reduced Lagrangian density

$$\mathcal{L}[x, \partial_t\varphi, \partial_x\varphi, \varphi] = -\frac{1}{2}C^2(x)(\partial_t\varphi)^2 + \frac{1}{2}C^{-2}(x)(\partial_x\varphi)^2 + \frac{1}{4}C^{-2}(x)(1-\varphi^2)^2, \tag{2.1}$$

and the associated Euler-Lagrange equation, equivalent to the Yang-Mills equation, for the field

$$\partial_t^2\varphi - \frac{4}{9}\left[Q\partial_x(Q\partial_x\varphi) + Q^2(1-\varphi^2)\varphi\right] = 0, \tag{2.2}$$

where we have denoted

$$Q(x) := \frac{3}{2}\,\mathrm{sech}^2\left(\frac{x}{2}\right). \tag{2.3}$$

In order to simplify the computations, we re-scale (2.2) in time $\tilde{\varphi}(t,x) = \varphi(\frac{3}{2}t, x)$, obtaining the expression

$$\partial_t^2\varphi - Q\partial_x(Q\partial_x\varphi) + Q^2(\varphi^2-1)\varphi = 0. \tag{2.4}$$

This model enjoys time translation invariance, is Hamiltonian, and will be the exact model worked in this paper. Another important property derived from the Lagrangian density (2.1) is the fact that the *energy*

$$E[\varphi, \varphi_t] = \int \left(\frac{1}{2}Q^{-1}(\partial_t\varphi)^2 + \frac{1}{2}Q\left((\partial_x\varphi)^2 + \frac{1}{2}(1-\varphi^2)^2\right)\right)dx$$

is formally conserved along the flow, with the associated continuity equation

$$Q^{-1}\partial_t(\partial_t\varphi)^2 + Q\partial_t\left((\partial_x\varphi)^2 + \frac{1}{2}(1-\varphi^2)^2\right) + \frac{9}{2}\partial_x(Q^{-1}\partial_t\varphi\partial_x\varphi) = 0,$$

while since there is no space translation invariance over the system, there is a lack of conservation for the natural physical momentum

$$P[\varphi, \partial_t\varphi] = \int Q^{-1}\partial_t\varphi\partial_x\varphi dx, \tag{2.5}$$

however, a particular version of this quantity will be essential for the proof of asymptotic stability.

2. Kinks

For the case of a static solution, it is direct that the equation (2.4) is reduced to the expression

$$H'' - \tanh\left(\frac{x}{2}\right)H' + H(1-H^2) = 0, \tag{2.6}$$

whose well-known [8] first non-trivial solution is given by

$$H(x) = \tanh\left(\frac{x}{2}\right). \tag{2.7}$$

We call $\mathbf{H} = (H, 0)$ the *kink* associated with the model. In order to apply virial-type arguments, it is necessary to reformulate the model in the following way.

Following Bizoń and Kahl [8], we introduce the change of variables given by the function

$$\alpha(x) = \frac{1}{3}(\sinh x + x), \tag{2.8}$$

which is strictly monotone and bijective in $\mathbb{R}$ (see Section 10). With this, if φ is a solution of the equation (2.4), then $\phi = \varphi \circ \alpha^{-1}$ is a solution for

$$\partial_t^2 \phi - \partial_x^2 \phi - \widetilde{Q}^2(\phi - \phi^3) = 0. \tag{2.9}$$

Define now the functions

$$\widetilde{Q}(x) = Q(\alpha^{-1}(x)), \qquad \widetilde{H}(x) = H(\alpha^{-1}(x)), \tag{2.10}$$

with Q and H as in (2.3) and (2.7), respectively.

If we formally consider $\boldsymbol{\varphi} = (\varphi, \partial_t \varphi) = (\varphi_1, \varphi_2)$, then (2.4) has the following representation as a 2×2 system:

$$\begin{cases} \partial_t \varphi_1 = \varphi_2 \\ \partial_t \varphi_2 = Q\partial_x(Q\partial_x \varphi_1) + Q^2(1 - \varphi_1^2)\varphi_1. \end{cases} \tag{2.11}$$

It turns out that this model presents many complications that are difficult to solve. Still following Bizoń and Kahl [8], one can transform the system (2.11) into a new simpler model, but with non exponential decay potentials. Indeed, equivalently to (2.11), and in the framework of the nonlinear equation (2.9), if we formally consider the solution $\boldsymbol{\phi} = (\phi, \partial_t \phi) = (\phi_1, \phi_2)$, it defines the following system of equations:

$$\begin{cases} \partial_t \phi_1 = \phi_2 \\ \partial_t \phi_2 = \partial_x^2 \phi_1 + \widetilde{Q}^2(x)(1 - \phi_1^2)\phi_1. \end{cases} \tag{2.12}$$

This will be the exact model worked in this paper. An important point to be considered here is that the term $\widetilde{Q}^2(x)$ destroys any possible shift or scaling symmetry. Nevertheless, the system (2.12) is Hamiltonian, and has the conserved quantity

$$E[\phi_1, \phi_2] = \int \left(\frac{1}{2}\phi_2^2 + \frac{1}{2}(\partial_x \phi_1)^2 + \frac{1}{4}\widetilde{Q}^2(1 - \phi_1^2)^2 \right) dx. \tag{2.13}$$

This law defines the set of functions $\boldsymbol{\phi} \in L^1_{\text{loc}}(\mathbb{R}) \times L^1_{\text{loc}}(\mathbb{R})$ for which the energy is finite

$$\mathbf{E} = \left\{ \boldsymbol{\phi} = (\phi_1, \phi_2) \in L^1_{\text{loc}}(\mathbb{R}) \times L^1_{\text{loc}}(\mathbb{R}) : \ \partial_x \phi_1 \in L^2(\mathbb{R}), \ \widetilde{Q}(1 - \phi_1^2) \in L^2(\mathbb{R}) \ , \ \phi_2 \in L^2(\mathbb{R}) \right\}.$$

Note that the kink now is given by $\widetilde{\boldsymbol{H}} = (\widetilde{H}, 0)$, its energy is finite and thus $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ perturbations of the kink are referred as perturbations in the energy space. By standard fixed-point arguments, the system (2.12) is locally well-posed for arbitrary finite energy data; however, the global existence of solutions for initial data with small energy is not obvious but it can be deduced using the conservation of energy.

Due to the dissipation of energy by dispersion, solutions of the system (2.12) are expected to settle down to critical points of the potential energy. This is why the relevance of studying the stability under the static solutions for the field.

To study the stability of $\mathbf{H}$, we define $\widetilde{\mathbf{H}} = (\widetilde{H}, 0)$ and the subset $\boldsymbol{E_H}$ of $\mathbf{E}$

$$\boldsymbol{E_H} = \left\{ \boldsymbol{\phi} \in \mathbf{E} : \boldsymbol{\phi} - \widetilde{\mathbf{H}} \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \right\}.$$

Following the standard procedure, let us consider a perturbation in (2.12) over $\widetilde{\mathbf{H}}$ of the form $\boldsymbol{\phi} = \widetilde{\mathbf{H}} + \mathbf{w}$. Explicitly

$$\phi_1(t,x) = \widetilde{H}(x) + w_1(t,x), \quad \phi_2(t,x) = w_2(t,x).$$

Then one see that this perturbation satisfies the following system:

$$\text{(KB)} \quad \begin{cases} \partial_t w_1 = w_2 \\ \partial_t w_2 = -L[w_1] - \widetilde{Q}^2(3\widetilde{H}w_1^2 + w_1^3), \end{cases} \tag{2.14}$$

where we have defined the linear operator

$$L[w] = -\partial_x^2 w + V(x)w, \quad \text{with} \quad V(x) = 2\widetilde{Q}^2(1 - \widetilde{Q}). \tag{2.15}$$

This is a Schrödinger operator associated with $\widetilde{Q}$, and from the system (2.14) one has that $\partial_t^2 w_1 = -L[w_1] + O(|w_1|^2)$. We observe that this operator over $D(L) := H^1(\mathbb{R})$ with the associated inner product in $L^2(\mathbb{R})$ denoted by $\langle \cdot, \cdot \rangle$ is self adjoint. Consequently, for the well-understanding of the problem we require to study the second order operator L. In Chapter 10 we will give the properties of this Schrödinger operator. In particular, we will show that L has a unique even eigenfunction $\phi_0(x)$ associated with the first simple and negative eigenvalue $-\mu_0^2 < 0$ of L (found numerically by Bizón and Kahl in [8]), satisfying

$$\langle \phi_0, \phi_0 \rangle = 1, \quad L[\phi_0] = -\mu_0^2 \phi_0, \quad |\phi_0(x)| \lesssim e^{-\frac{\sqrt{2}}{2}\mu_0 x}. \tag{2.16}$$

By the spectral theorem, the operator L is non-negative if ϕ is orthogonal in $L^2(\mathbb{R})$ to ϕ_0. See Section 10 for more details and full proofs of all the previous statements.

The negative eigenvalue of the linearized operator L introduces exponentially stable and unstable modes for the dynamics in the neighborhood of the kink. Indeed, the linear subsystem of (2.14) is given by

$$\begin{cases} \partial_t w_1 = w_2 \\ \partial_t w_2 = -L[w_1], \end{cases} \tag{2.17}$$

which has the exponentially decreasing and growing solution $\mathbf{w}_\pm(t,x) = e^{\pm\mu_0 t}\mathbf{Y}_\pm(x)$, where

$$\mathbf{Y}_\pm = \begin{pmatrix} \phi_0 \\ \pm\mu_0\phi_0 \end{pmatrix}. \tag{2.18}$$

This is an even-even function. i.e., the first and second coordinates are even (see Section 10).

In view of the critical properties of the linearized system (2.17), and in the line of previous results [20, 32], one may therefore only hope to establish a conditional asymptotic stability result for the kink $\mathbf{H}$ under the unstable linear manifold. In what follows, we refers to *global solution* of (2.14) to a function $C([0,\infty);\mathbf{E_H})$ that satisfies (2.14) for all $t \geq 0$.

2.1. Main result

Our main result establishes for perturbations of the kink $\mathbf{H}$ under certain orthogonality condition, that stability in the energy space $\mathbf{E_H}$ implies asymptotic stability in a spatially localized energy norm:

Theorem 2.1 (Main theorem). *There exists $\delta > 0$ such that if a global solution $(\phi, \partial_t\phi)$ of* (2.4) *satisfies for all $t \geq 0$,*

$$\|(\phi, \partial_t\phi)(t) - (\widetilde{H}, 0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} < \delta, \tag{2.19}$$

then for any I compact interval in $\mathbb{R}$,

$$\lim_{t\to\infty} \|(\phi, \partial_t\phi)(t) - (\widetilde{H}, 0)\|_{H^1(I)\times L^2(\mathbb{R})} = 0\,. \tag{2.20}$$

The previous result shows that, even if the dynamics is highly unstable, there is room for the existence of a stable dynamics.

2.2. Previous results

The study of the mathematical structure of kink solutions have achieved an impressive advance during the past years. The physical meaning of kinks and their key importance in high energy physics and general relativity is out of the scope of this paper, the interested reader can consult the monographs [29, 42, 39]. See [23] for a detailed survey on the long-time behavior of waves.

The case of kinks in variable coefficients field models was first studied by Snelson in the ϕ^4 case [38], see also the recent results by Alammari and Snelson [2, 3] for general scalar field models. Compared to their results, the model treated here has additional difficulties appearing from the unstable character of the dynamics, and the slow decay of solutions. In particular, in our case the spectral theory has not been taken front grant, being proved in Section 10, and in contrast to previous studies, we consider a non-linearity with variable coefficients.

3. Preliminaries

We shall start with some basic properties about the function α defined in (2.8), and the modified soliton $\widetilde{Q}$ in (2.10), deeply involved in the spectral analysis of L.

Lemma 3.1. *The function $\alpha(x)$ is strictly monotone, bijective. Moreover, if α^{-1} denotes the inverse of α,*

$$\partial_x\alpha(x) = Q^{-1}(x), \quad \partial_x\alpha^{-1}(x) = \widetilde{Q}(x). \tag{2.21}$$

Proof. By direct computation one has

$$\alpha'(x) = \frac{1}{3}(\cosh x + 1) = \frac{2}{3}\cosh^2\left(\frac{x}{2}\right) = \frac{1}{Q(x)},$$

proving that $\alpha(x)$ is strictly monotone and bijective, since $\alpha'(x)$ grows with x. For the inverse of α we have

$$(\alpha^{-1})'(x) = \frac{1}{\alpha'(\alpha^{-1}(x))} = Q(\alpha^{-1}(x)) = \widetilde{Q}(x).$$

This ends the proof of (2.21) and the lemma. □

The standard $\lesssim$ symbol means that there exists $C > 0$ such that $a(x) \leq Cb(x)$, C independent of x.

Lemma 3.2. *The functions $\alpha^{-1}(x)$ and $\widetilde{Q}(x)$ are odd and even, respectively, and they have the following asymptotic descriptions.*

For $|x| \ll 1$,

$$\alpha^{-1}(x) = \frac{3}{2}x + \mathcal{O}(x^2), \qquad \widetilde{Q}(x) = \frac{3}{2} - \frac{27}{32}x^2 + \mathcal{O}(x^4). \tag{2.22}$$

For $|x| \gg 1$, we have the limit

$$\lim_{x\to\pm\infty} \frac{2|\alpha^{-1}(x)|}{\ln(|x|)} = 1, \qquad \lim_{x\to\pm\infty} (1+|x|)\widetilde{Q}(x) = 1. \tag{2.23}$$

Even more, the integral $\int \widetilde{Q}^{1+\varepsilon} dx$ is finite for any $\varepsilon > 0$.

Proof. Let us first prove (2.23). Recall that $\widetilde{Q}(x) = Q(\alpha^{-1}(x))$. Employing the fact that $x = \alpha(y)$ is continuous bijective, and goes to $\pm\infty$ when $y \to \pm\infty$, as well as (2.8), we have that

$$\begin{aligned}
\lim_{x\to\pm\infty} x\widetilde{Q}(x) &= \lim_{y\to\pm\infty} \alpha(y)Q(y) = \lim_{y\to\pm\infty} \frac{1}{2}\frac{\sinh y + y}{\cosh^2\left(\frac{y}{2}\right)} \\
&= \lim_{y\to\pm\infty} \frac{1}{4}\frac{\cosh y + 1}{\cosh\left(\frac{y}{2}\right)\sinh\left(\frac{y}{2}\right)} = \frac{1}{2}\lim_{y\to\pm\infty} \coth\left(\frac{y}{2}\right) \\
&= \pm\frac{1}{2},
\end{aligned}$$

where in the second line we have used a simple L'Hôpital's rule. On the other hand, using (2.21),

$$\lim_{x\to\pm\infty} \frac{\alpha^{-1}(x)}{\ln(|x|)} = \lim_{x\to\pm\infty} |x|\widetilde{Q}(x) = \pm\frac{1}{2}.$$

This proves the first limit of (2.23) , and $\widetilde{Q} \lesssim |x|^{-1}$.

Now we restrict our analysis of $\widetilde{Q}$, by parity, to the positive real numbers. From definition

(2.8) we obtain for $x>0$,

$$e^x = e^{-x} - 2x + 6\alpha(x).$$

Employing this,

$$\operatorname{sech}^2\left(\frac{x}{2}\right) = \frac{1}{\cosh^2\left(\frac{x}{2}\right)} = \frac{4}{e^x + 2 + e^{-x}} = \frac{4}{3e^{-x} + 2 - 2x + 6\alpha(x)}.$$

Replacing in (2.10), and using that $|\alpha^{-1}| \sim \frac{1}{2}\ln(|x|)$, we have for any $\varepsilon > 0$

$$\widetilde{Q}(x) = \frac{6}{3e^{-\alpha^{-1}(x)} + 2 - 2\alpha^{-1}(x) + 6x} \leq \frac{3}{1 - \alpha^{-1}(x) + 3x}.$$

Analogously,

$$\widetilde{Q}(x) \geq \frac{3}{1 - \alpha^{-1}(x) + 3x} \geq \frac{3}{1+3x}.$$

Therefore,

$$\lim_{x\to+\infty}(1+x)\widetilde{Q}(x) = 1.$$

The case $x \to -\infty$ is obtained by parity, which proves (2.23).

Now we prove (2.22). The proof is based in a simple Taylor expansion in second and fourth order around $x=0$.

$$\alpha^{-1}(x) = \alpha^{-1}(0) + \partial_x\alpha^{-1}(0)x + \mathcal{O}(x^2) = \frac{3}{2}x + \mathcal{O}(x^2).$$

$$\begin{aligned}\widetilde{Q}(x) &= \widetilde{Q}(0) + \widetilde{Q}'(0)x + \frac{1}{2}\widetilde{Q}''(0)x^2 + \frac{1}{6}\widetilde{Q}'''(0)x^3 + \mathcal{O}(x^4)\\ &= \frac{3}{2} - \frac{27}{32}x^2 + \mathcal{O}(x^4),\end{aligned}$$

where we have used that $\widetilde{Q}'(x) = -\widetilde{Q}^2(x)\widetilde{H}(x)$, $\widetilde{Q}''(x) = \widetilde{Q}^2(x)(1-\widetilde{Q}(x))$ and $\widetilde{Q}$ is even. Finally, by (2.21) we have

$$\int \widetilde{Q}^{1+\varepsilon}(x)dx = \int Q^{\varepsilon}(s)ds < +\infty.$$

□

4. Virial estimate at large scale

The first step is to consider a small perturbation of the modified kink $(\widetilde{H},0)$. In what follows we describe this decomposition, introduce some notation, and develop a first virial estimate.

4.1. Decomposition of the solution in a vicinity of the kink

Let $(\phi, \partial_t\phi)$ be a solution of (2.9) satisfying (2.19) for some $\delta > 0$. Let (μ_0, ϕ_0) be given in (2.16). Using $\mathbf{Y}_+$ from (2.18), we decompose $(\phi, \partial_t\phi)$ as follows

$$\begin{cases} \phi(t,x) - \widetilde{H} = a_1(t)\phi_0(x) + u_1(t,x) \\ \quad \partial_t\phi(t,x) = \mu_0 a_2(t)\phi_0(x) + u_2(t,x), \end{cases} \tag{2.24}$$

where we define (see (2.16))

$$a_1(t) = \langle \phi(t) - \widetilde{H}, \phi_0 \rangle = -\frac{1}{\mu_0^2}\langle \phi(t) - \widetilde{H}, L[\phi_0] \rangle,$$

$$a_2(t) = \frac{1}{\mu_0}\langle \partial_t\phi(t), \phi_0 \rangle = -\frac{1}{\mu_0^3}\langle \partial_t\phi(t), L[\phi_0] \rangle,$$

such that

$$\langle u_1(t), \phi_0 \rangle = 0 = \langle u_2(t), \phi_0 \rangle, \tag{2.25}$$

or equivalently,

$$\langle u_1(t), L[\phi_0] \rangle = 0 = \langle u_2(t), L[\phi_0] \rangle.$$

Setting the variables

$$b_+ = \frac{1}{2}(a_1 + a_2), \quad b_- = \frac{1}{2}(a_1 - a_2), \tag{2.26}$$

from the stability hypothesis (2.19) and the decomposition (2.24), we have for all $t \in \mathbb{R}_+$

$$\|\partial_x u_1(t)\|_{L^2} + \|u_1\|_{L^2} + \|u_2(t)\|_{L^2} + |a_1(t)| + |a_2(t)| + |b_+(t)| + |b_-(t)| \le C\delta. \tag{2.27}$$

Moreover, using (2.14), (2.16) and (2.24), we obtain that (a_1, a_2) satisfies the following differential system

$$\begin{cases} \dot{a}_1(t) = \mu_0 a_2(t) \\ \dot{a}_2(t) = \mu_0 a_1(t) - \dfrac{N_0}{\mu_0}, \end{cases} \quad \text{or equivalently} \quad \begin{cases} \dot{b}_+(t) = \mu_0 b_+(t) - \dfrac{N_0}{2\mu_0} \\ \dot{b}_-(t) = -\mu_0 b_-(t) + \dfrac{N_0}{2\mu_0}, \end{cases} \tag{2.28}$$

where

$$N = \widetilde{Q}^2\left(3\widetilde{H}(a_1\phi_0 + u_1)^2 + (a_1\phi_0 + u_1)^3\right), \tag{2.29}$$

and

$$N^\perp = N - N_0\phi_0, \quad \text{and} \quad N_0 = \langle N, \phi_0 \rangle. \tag{2.30}$$

Then, (u_1, u_2) satisfies the following system

$$\begin{cases} \dot{u}_1 = u_2 \\ \dot{u}_2 = -L[u_1] - N^\perp. \end{cases} \tag{2.31}$$

4.2. Notation for virial argument

In this paper, the notation $F \lesssim G$ means that $F \leq CG$ for some constant $C > 0$ independent of F and G. Unless otherwise indicated, the implicit constant $C > 0$ is supposed to be independent of the parameters A, B, γ and δ introduced below. As in [27, 19], it is convenient to define a *modified space* $\mathcal{Y}$ of smooth functions $f : \mathbb{R} \to \mathbb{R}$ with the property that for any $k \geq 0$, there exists a constant $C_k > 0$ such that

$$|f^{(k)}(x)| \leq C_k \widetilde{Q}(x)^3 \quad \text{for all } x \in \mathbb{R}.$$

It is important to stress that $\widetilde{Q}$ and V in (2.15) have only polynomial decay, consequently the definitions of $\mathcal{Y}$ and the virial type functions ζ need some care in our case. Note for example that $\widetilde{Q}, h_0', V \in \mathcal{Y}$.

Let $\chi \in C_c^\infty(\mathbb{R})$ be a smooth even function satisfying

$$\chi(x) = 1 \text{ for } |x| \leq 1, \quad \chi(x) = 0 \text{ for } |x| \geq 2, \quad \chi'(x) \leq 0 \text{ for } x \geq 0. \tag{2.32}$$

For $A > 0$, we define the function ζ_A and ϕ_A as follows

$$\zeta_A^2(x) = \exp\left(-\frac{1}{A}|\alpha^{-1}(x)|(1-\chi(x))\right), \quad \varphi_A(x) = \int_0^x \tilde{Q}\zeta_A^2(y)dy, \ x \in \mathbb{R}. \tag{2.33}$$

Moreover, we introduce the weight function

$$\sigma_A(x) = \operatorname{sech}\left(\frac{1}{A}\alpha^{-1}(x)\right). \tag{2.34}$$

Notice that $\zeta_A \lesssim \sigma_A \lesssim \zeta_A$.

For $B > 0$, we also define

$$\begin{gathered}\zeta_B^2(x) = \exp\left(-\frac{1}{B}|\alpha^{-1}(x)|(1-\chi(x))\right), \quad \varphi_B(x) = \int_0^x \tilde{Q}\zeta_B^2(y)dy, \ x \in \mathbb{R}, \\ \psi_B(x) = \tilde{\chi}_A^2(x)\varphi_B(x), \quad \tilde{\chi}_A(x) = \chi\left(\frac{\alpha^{-1}(x)}{A}\right).\end{gathered} \tag{2.35}$$

These functions will be used in two distinct virial arguments to prove Proposition 4.1 and Proposition 5.1 with different scales

$$1 \ll B \ll A. \tag{2.36}$$

The choice of the switch function φ_A is specifically adapted to the decay rate of the potential of the linear operator in (2.14) and (2.60). We denote by $\sim$ the composition with α^{-1} (i.e., $\widetilde{f}(x) = (f \circ \alpha^{-1})(x)$).

4.3. Virial estimate at large scale

Following [20], and having in mind (2.5) in our new coordinates, we introduce the time dependent virial functional $\mathcal{I}(t)$ defined by

$$\mathcal{I} = \int (\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1) u_2, \tag{2.37}$$

and introduce the variables

$$w_i = \zeta_A u_i, \quad i = 1, 2. \tag{2.38}$$

Here, as in [20], (w_1, w_2) represent a localized version of (u_1, u_2) at scale A.

Proposition 4.1. *There exist $C_0, C > 0$ and $\delta_1 > 0$ such that for any $0 < \delta \le \delta_1$, the following holds. Fix*

$$A = \delta^{-1/4}. \tag{2.39}$$

Assume that for all $t \ge 0$, (2.27) holds. Then for all $t \ge 0$, the functional $\mathcal{I}$ in (2.37) satisfies the estimate

$$\frac{d}{dt}\mathcal{I} \le -\frac{1}{2}C_0 \int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + C\int \widetilde{Q}^6 u_1^2 + C|a_1|^4. \tag{2.40}$$

Remark 4.2. *Estimate (2.40) does not involve any type of spectral analysis. Its purpose is to give a weighted control of $(u_1, \partial_x u_1)$ on a large scale A in terms of a weighted L^2 norm of u_1 with faster decay.*

The rest of this section is devoted to the proof of Proposition 4.1. We start with the following intermediate lemma.

Lemma 4.3. *Let $(u_1, u_2) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ be a solution of (2.31). Consider $\varphi_A = \varphi_A(x)$ a smooth bounded function to be chosen later. Then*

$$\frac{d}{dt}\mathcal{I} = -\int \varphi_A'(\partial_x u_1)^2 + \frac{1}{4}\int \varphi_A''' u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2 - \int \left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right) N^\perp. \tag{2.41}$$

Proof. We define the integrals

$$\mathcal{I}_1 = \int \varphi_A u_2 \partial_x u_1, \qquad \mathcal{I}_2(t) = \frac{1}{2}\int \varphi_A' u_1 u_2.$$

Taking time derivative over $\mathcal{I}_1$ and using (2.31),

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}_1(t) &= \int \varphi_A(\dot{u}_2 \partial_x u_1 + u_2 \partial_x \dot{u}_1)\\
&= -\int \varphi_A \left(L[u_1] + N^\perp\right)\partial_x u_1 + \int \varphi_A u_2 \partial_x u_2\\
\\
&= -\int \varphi_A L[u_1]\partial_x u_1 - \int \varphi_A \partial_x u_1 N^\perp - \frac{1}{2}\int \varphi_A' u_2^2\\
&= -\int \varphi_A L[u_1]\partial_x u_1 - \int \varphi_A \partial_x u_1 N^\perp - \frac{1}{2}\int \varphi_A' u_2^2.
\end{aligned}$$

For the first integral just defined in the RHS,

$$
\begin{aligned}
\int \varphi_A L[u_1]\partial_x u_1 =& \int \varphi_A(-\partial_x^2 u_1 + V u_1)\partial_x u_1 \\
=& -\frac{1}{2}\int \varphi_A \partial_x(\partial_x u_1)^2 + \frac{1}{2}\int \varphi_A V \partial_x u_1^2 \\
=& \frac{1}{2}\int \varphi_A'(\partial_x u_1)^2 - \frac{1}{2}\int \varphi_A' V u_1^2 - \frac{1}{2}\int \varphi_A V' u_1^2.
\end{aligned}
$$

Then, replacing we obtain

$$
\frac{d}{dt}\mathcal{I}_1 = -\frac{1}{2}\int \varphi_A'(\partial_x u_1)^2 + \frac{1}{2}\int \varphi_A' V u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2 - \frac{1}{2}\int \varphi_A' u_2^2 - \int \varphi_A \partial_x u_1 N^{\perp}. \quad (2.42)
$$

Now for the second virial term $\mathcal{I}_2$ analogously we take time derivative and use (2.31):

$$
\begin{aligned}
\frac{d}{dt}\mathcal{I}_2 =& \frac{1}{2}\int \varphi_A'(\dot{u}_1 u_2 + u_1 \dot{u}_2) \\
=& \frac{1}{2}\int \varphi_A' u_2^2 - \frac{1}{2}\int \varphi_A' u_1 \left(L[u_1] + N^{\perp}\right) \\
=& \frac{1}{2}\int \varphi_A' u_2^2 - \frac{1}{2}\int \varphi_A' u_1 L[u_1] - \frac{1}{2}\int \varphi_A' u_1 N^{\perp}.
\end{aligned}
$$

For the integral I_2 defined above

$$
\begin{aligned}
I_2 =& \int \varphi_A' u_1(-\partial_x^2 u_1 + V)u_1) \\
=& \int (\varphi_A' u_1)_x \partial_x u_1 + \int \varphi_A' V u_1^2 \\
=& \int \varphi_A'' u_1 \partial_x u_1 + \int \varphi_A'(\partial_x u_1)^2 + \int \varphi_A' V u_1^2 \\
=& -\frac{1}{2}\int \varphi_A''' u_1^2 + \int \varphi_A'(\partial_x u_1)^2 + \int \varphi_A' V u_1^2.
\end{aligned}
$$

Then replacing we obtain

$$
\frac{d}{dt}\mathcal{I}_2(t) = \frac{1}{2}\int \varphi_A' u_2^2 + \frac{1}{4}\int \varphi_A''' u_1^2 - \frac{1}{2}\int \varphi_A'(\partial_x u_1)^2 - \frac{1}{2}\int \varphi_A' V u_1^2 - \frac{1}{2}\int \varphi_A' u_1 N^{\perp}. \quad (2.43)
$$

Finally, adding (2.42) and (2.43) we arrive to the equation,

$$
\frac{d}{dt}\mathcal{I}(t) = -\int \varphi_A'(\partial_x u_1)^2 + \frac{1}{4}\int \varphi_A''' u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2 - \int \left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right) N^{\perp},
$$

which is nothing but (2.41). □

Now we rewrite the linear part of the virial identity using the new variables (w_1, w_2).

Lemma 4.4. *It holds*

$$\begin{aligned}-\int \varphi_A'(\partial_x u_1)^2 &+ \frac{1}{4}\int \varphi_A''' u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2\\ &= -\int \widetilde{Q}(\partial_x w_1)^2 - \frac{1}{2}\int \left[\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A}\right)^2\right]\widetilde{Q} w_1^2 + \frac{1}{4}\int \widetilde{Q}'' u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2,\end{aligned} \tag{2.44}$$

where

$$\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A}\right)^2 = \frac{1}{A}\left[\chi''|\alpha^{-1}| + 2\chi'\widetilde{Q}\mathrm{sgn}(\alpha^{-1}) + (1-\chi)\widetilde{Q}^2\widetilde{H}\mathrm{sgn}(\alpha^{-1})\right]. \tag{2.45}$$

and there exist $\widetilde{x} > 0$, $C > 0$ *independent of* A *such that*

$$\frac{1}{4}\widetilde{Q}''(x) + \frac{1}{2}\varphi_A(x)V'(x) \leq -C\widetilde{Q}^3(x) \quad \text{for all } |x| \geq \widetilde{x}. \tag{2.46}$$

Finally, one has

$$\left|\frac{\zeta_A'}{\zeta_A}\right| \lesssim \frac{1}{A}\widetilde{Q}\mathbf{1}_{\{|x|\geq 1\}}, \qquad \left|\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A}\right)^2\right| \lesssim \frac{1}{A}\widetilde{Q}^2\mathbf{1}_{\{|x|\geq 1\}}. \tag{2.47}$$

Remark 4.5. *Unlike previous works using this type of virial function, we obtain an expression in terms of* w_1 *with a weight function* $\widetilde{Q}$*, and an extra term* $\frac{1}{4}\int \widetilde{Q}'' u_1^2$*. This is due to the particular definition of* ζ_A *and* φ_A *in* (2.33) *to deal with the specific polynomial decay of the linearized potential. Another relevant feature is the loss of a compact support for the second expression in* (2.47)*, which will have to be controlled by the specific decay from* (2.46)*.*

Proof. Considering $w_1 = \zeta_A u_1$, and $\varphi_A' = \widetilde{Q}\zeta_A^2$, we have,

$$\begin{aligned}\int \varphi_A'(\partial_x u_1)^2 &= \int \widetilde{Q}\left(\partial_x w_1 - \frac{\zeta_A'}{\zeta_A}w_1\right)^2\\ &= \int \widetilde{Q}(\partial_x w_1)^2 - 2\int \widetilde{Q}\frac{\zeta_A'}{\zeta_A}w_1\partial_x w_1 + \int \left(\frac{\zeta_A'}{\zeta_A}\right)^2\widetilde{Q}w_1^2\\ &= \int \widetilde{Q}(\partial_x w_1)^2 + \int \left[\left(\frac{\widetilde{Q}\zeta_A'}{\zeta_A}\right)' + \widetilde{Q}\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right]w_1^2\\ &= \int \widetilde{Q}(\partial_x w_1)^2 + \int \frac{(\widetilde{Q}\zeta_A')'}{\zeta_A}w_1^2,\end{aligned}$$

and

$$\int \varphi_A''' u_1^2 = \int \left[\widetilde{Q}'' + 2\widetilde{Q}'\left(\frac{\zeta_A'}{\zeta_A}\right) + 2\frac{(\widetilde{Q}\zeta_A')'}{\zeta_A}w_1^2 + 2\widetilde{Q}\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right]w_1^2.$$

Then,

$$
\begin{aligned}
-\int \varphi_A'(\partial_x u_1)^2 + \frac{1}{4}\int \varphi_A''' u_1^2 = & -\int \widetilde{Q}(\partial_x w_1)^2 + \frac{1}{4}\int \widetilde{Q}'' w_1^2 \\
& + \int \left[\frac{1}{4}\widetilde{Q}'' + \frac{1}{2}\widetilde{Q}'\left(\frac{\zeta_A'}{\zeta_A}\right) - \frac{1}{2}\frac{(\widetilde{Q}\zeta_A')'}{\zeta_A} w_1^2 + \frac{1}{2}\widetilde{Q}\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right] w_1^2 \\
= & -\int \widetilde{Q}(\partial_x w_1)^2 + \frac{1}{4}\int \widetilde{Q}'' w_1^2 + \frac{1}{2}\int \widetilde{Q}\left[\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A}\right)^2\right] w_1^2.
\end{aligned}
$$

By elementary computations of (2.33), we have

$$
\begin{aligned}
\frac{\zeta_A'}{\zeta_A} &= \frac{1}{A}\left[\chi'|\alpha^{-1}| - \mathrm{sgn}(\alpha^{-1})(\alpha^{-1})'(1-\chi)\right] \\
\frac{\zeta_A''}{\zeta_A} &= \left(\frac{\zeta_A'}{\zeta_A}\right)^2 + \frac{1}{A}\left[\chi''|\alpha^{-1}| + 2\chi'(\alpha^{-1})'\mathrm{sgn}(\alpha^{-1}) - (1-\chi)(\alpha^{-1})''\mathrm{sgn}(\alpha^{-1})\right].
\end{aligned}
$$

Hence, replacing with (2.21), we get (2.45) and the first inequality of (2.47).

Now we describe in more detail the behavior of (2.45), which will differ from previous works on the subject. First, for $1 \le |x| \le 2$, we can see that

$$
\left|\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A}\right)^2\right| \lesssim \frac{1}{A}.
$$

For $|x| \ge 2$, using (2.21)

$$
\left|\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A}\right)^2\right| = \frac{1}{A}\widetilde{Q}^2|\widetilde{H}| \le \frac{1}{A}\widetilde{Q}^2(x).
$$

Then one can see that

$$
\left|\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A}\right)^2\right| \lesssim \frac{\widetilde{Q}^2\mathbf{1}_{\{|x|\ge 1\}}}{A},
$$

which proves the second estimate of (2.47).

Finally, we focus on proving (2.46). By parity we can restrict our analysis to the positive axis. Using the definition of $\widetilde{Q}$ and V, in addition to (2.21), we have for all $x > 0$,

$$
\frac{1}{4}\widetilde{Q}'' + \frac{1}{2}\varphi_A V' = \left[\frac{1}{2} - \frac{5}{12}\widetilde{Q} - \left(2 - 3\widetilde{Q}\right)\varphi_A \widetilde{H}\right]\widetilde{Q}^3. \tag{2.48}
$$

Since by definition $\widetilde{Q} : \mathbb{R}_+ \mapsto [0, \frac{3}{2}]$ is bijective, there exist $x_1 > 0$ such that $\widetilde{Q}(x_1) = \frac{1}{3}$. Even more, since $\widetilde{Q}$ is a decreasing function in the positive axis, we have that

$$
2 - 3\widetilde{Q}(x) \ge 1,
$$

for all $x \geq x_1$. In addition, by definition we have that

$$\varphi_A(x)\widetilde{H}(x) > 0,$$

for all $x > 0$.

Now, if we apply a change of variable in the integral definition of φ_A in (2.33) and properties of χ in (2.32), we have

$$\begin{aligned}\varphi_A &= \int_0^{\alpha^{-1}(x)} e^{-\frac{2}{A}s(1-\chi(\alpha(s)))}ds \geq \int_0^1 ds + \int_1^{\alpha^{-1}(x)} e^{-\frac{2}{A}s}ds \\ &= 1 + \frac{A}{2}\left(e^{-\frac{2}{A}} - e^{-\frac{2}{A}\alpha^{-1}(x)}\right) \geq 1.\end{aligned}$$

for all $x \geq \alpha(1)$.

Collecting these estimates and replacing in (2.48) we obtain for all $x \geq \max\{x_1, \alpha(2)\}$

$$\frac{1}{4}\widetilde{Q}'' + \frac{1}{2}\varphi_A V' \leq \left(\frac{1}{2} - \frac{5}{12}\widetilde{Q} - \varphi_A\widetilde{H}\right)\widetilde{Q}^3 \leq \left(\frac{1}{2} - \frac{5}{12}\widetilde{Q} - \widetilde{H}\right)\widetilde{Q}^3 =: R(\alpha^{-1}(x))\widetilde{Q}^3,$$

where we have defined the auxiliary function $R : \mathbb{R}_+ \to \mathbb{R}$ as

$$R(s) := \frac{1}{2} - \frac{5}{12}Q(s) - H(s).$$

From the bijectivity of α, proving that there exist $s_1 > 0$ such that $R(s) < 0$ for $s > s_1$ is equivalent to prove (2.46), but this is direct using that H is an increasing positive function, and $H(2) \sim 0.76$,

$$R(s) \leq \frac{1}{2} - \frac{5}{12}Q(s) - H(2) \sim -0.26 - \frac{5}{12}Q(s) < 0$$

for all $s \geq 2$.

Taking $\widetilde{x} = \max\{x_1, \alpha(2)\}$ we conclude that there exists some positive constant $C > 0$ such that

$$\frac{1}{4}\widetilde{Q}''(x) + \frac{1}{2}\varphi_A(x)V'(x) \leq -C\widetilde{Q}^3(x)$$

for all $x \geq x_1$. By parity we conclude (2.46). This ends the proof of Lemma 4.4. □

Corollary 4.6. *Let $(u_1, u_2) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ be a solution of (2.31). Then, for A large enough, there exist positive constants $C_0, C' > 0$ such that*

$$-\int \varphi_A'(\partial_x u_1)^2 + \frac{1}{4}\int \varphi_A''' u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2 \leq -C_0\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + C'\int \widetilde{Q}^6 u_1^2. \tag{2.49}$$

Remark 4.7. *From (2.44) and (2.49) we see that the objective must focus on controlling $\int \widetilde{Q}^6 u_1^2$. This term comes from the compact interval where the term associated with the potential is positive. For this purpose we will define a dualized or supersymmetric problem in Section 5. In Section 9 we will show that split the term $\frac{1}{4}\int \widetilde{Q}'' u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2$ into these two positive and negatives regimes will be essential to have enough decay and apply transfer estimates to control it.*

Proof. From (2.44), (2.46) and (2.47) we have that there exist a positive real number $\widetilde{x}$ and constants $C, C' > 0$ such that

$$
\begin{aligned}
-\int \varphi_A'(\partial_x u_1)^2 + \frac{1}{4}\int \varphi_A''' u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2 \\
\leq -\int \widetilde{Q}(\partial_x w_1)^2 + \frac{C'}{A}\int_{|x|\geq 1} \widetilde{Q}^3 w_1^2 - C\int_{|x|\geq \widetilde{x}} \widetilde{Q}^3 u_1^2 + C\int_{|x|\leq \widetilde{x}} \widetilde{Q}^2 u_1^2,
\end{aligned}
$$

where we have used that $|\widetilde{Q}''| + |\varphi_A V'| \lesssim \widetilde{Q}^2$. Even more, using $1 \lesssim \widetilde{Q}$ for $x \in [-\widetilde{x}, \widetilde{x}]$, redefining certain constants and taking A large enough, we conclude that there exist $C_0, C > 0$ such that

$$
-\int \varphi_A'(\partial_x u_1)^2 + \frac{1}{4}\int \varphi_A''' u_1^2 + \frac{1}{2}\int \varphi_A V' u_1^2 \leq -C_0 \int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + C\int \widetilde{Q}^6 u_1^2,
$$

obtaining (2.49). □

Now, we deal with the nonlinear terms N and $N^{\perp}$ introduced in (2.29)-(2.30) with the following result.

Lemma 4.8.

$$
\begin{aligned}
\Big|\int \left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right)\left(\widetilde{Q}^2(3\widetilde{H}(a_1\phi_0 + u_1)^2 + (a_1\phi_0 + u_1)^3) - N_0\phi_0\right)\Big| \\
\lesssim |a_1|^4 + \int \widetilde{Q}^6 u_1^2 + A^2 \|u_1\|_{L^\infty}\left(\int \widetilde{Q}|\partial_x w_1|^2 dx + \int \widetilde{Q}^3 |w_1|^2 dx\right).
\end{aligned}
\tag{2.50}
$$

Proof. We decompose the first integral of (2.50) into several parts and write

$$
\begin{aligned}
&\int \left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right)\widetilde{Q}^2\left(3\widetilde{H}(a_1\phi_0 + u_1)^2 + (a_1\phi_0 + u_1)^3\right) \\
&\quad = a_1^2 \int \widetilde{Q}^2(3\widetilde{H} + a_1\phi_0)\phi_0^2\left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right) \\
&\quad + 3a_1 \int \widetilde{Q}^2(2\widetilde{H} + a_1\phi_0)\phi_0 u_1\left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right) \\
&\qquad + 3\int \widetilde{Q}^2(\widetilde{H} + a_1\phi_0)u_1^2\left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right) + \int \widetilde{Q}^2 u_1^3\left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right) \\
&\quad = I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

For the first term, using integration by parts, the Cauchy-Schwarz inequality, the decay estimates on $\widetilde{Q}$ and ϕ_0, noticing that for all $x \in \mathbb{R}$, $|\varphi_A'(x)| \leq \widetilde{Q}$ and $|\varphi_A(x)| \leq |\alpha^{-1}(x)|$,

$$
\begin{aligned}
|I_1| &\leq a_1^2 \int |\partial_x(\widetilde{Q}^2(3\widetilde{H} + a_1\phi_0)\phi_0^2)\varphi_A u_1| + \frac{1}{2}a_1^2 \int |\widetilde{Q}^2(3\widetilde{H} + a_1\phi_0)\phi_0^2 \varphi_A' u_1| \\
&\lesssim a_1^2 \left(\int \widetilde{Q}^{-2}\phi_0^2|\alpha^{-1}|^2\right)^{\frac{1}{2}}\left(\int \widetilde{Q}^6 u_1^2\right)^{\frac{1}{2}} + a_1^2\left(\int \phi_0^4\right)^{\frac{1}{2}}\left(\int \widetilde{Q}^6 u_1^2\right)^{\frac{1}{2}} \\
&\lesssim a_1^4 + \int \widetilde{Q}^6 u_1^2.
\end{aligned}
\tag{2.51}
$$

For the second integral, by integration by parts, using the exponential decay (2.16),

$\phi_A(x) \lesssim |\alpha^{-1}(x)|$, and in addition $|a_1| < 1$ (see (2.27)), we obtain

$$\begin{aligned} |I_2| &= \frac{3}{2}|a_1| \left| \int \partial_x(\widetilde{Q}^2(2\widetilde{H} + a_1\phi_0)\phi_0)\varphi_A u_1^2 \right| \\ &\lesssim |a_1| \int |\alpha^{-1}(x)|(\phi_0 + \phi_0')\widetilde{Q}^3 u_1^2 \lesssim \int \widetilde{Q}^6 u_1^2. \end{aligned} \tag{2.52}$$

We need a modification of the claim proved in [20] for the non linear terms in u_1.

Claim 4.9. *It holds*

$$\int \widetilde{Q}^3\zeta_A^2|u_1|^{2p+2} = \int \widetilde{Q}^3\zeta_A^{-2p}|w_1|^{2p+2} \lesssim A\|u_1\|_{L^\infty}^{2p} \left(\int \widetilde{Q}(\partial_x w_1)^2 + \int \widetilde{Q}^3 w_1^2 \right). \tag{2.53}$$

Proof of Claim 4.9. The first equality in (2.53) corresponds to the definition of w_1 in (2.38). Next, by integration by parts and standard estimates, we have

$$\begin{aligned} &\int_0^\infty \widetilde{Q}^3 \exp\left(\frac{2p}{A}\alpha^{-1}(x)\right)|w_1|^{2p+2}dx \\ &\quad= \frac{A}{2p}\int_0^\infty \widetilde{Q}^2(x)\partial_x \exp\left(\frac{2p}{A}\alpha^{-1}(x)\right)|w_1|^{2p+2}dx \\ &\quad= -\frac{9A}{8p}|w_1(0)|^{2p+2} - \frac{A}{2p}\int_0^\infty \exp\left(\frac{2p}{A}\alpha^{-1}(x)\right)\partial_x(\widetilde{Q}^2|w_1|^{2p+2})dx \\ &\quad\le \frac{p+1}{p}A\int_0^\infty \widetilde{Q}^2|u_1|^{2p}(\partial_x w_1)w_1 dx + \frac{A}{p}\int_0^\infty \widetilde{Q}^3|u_1|^{2p}w_1^2 dx \\ &\quad\le \frac{p+1}{p}A\|u_1\|_{L^\infty}^{2p}\left(\int_0^\infty \widetilde{Q}(\partial_x w_1)^2 dx + \int_0^\infty \widetilde{Q}^3 w_1^2 dx\right) + \frac{A}{p}\|u_1\|_{L^\infty}^{2p}\int_0^\infty \widetilde{Q}^3 w_1^2 dx. \end{aligned}$$

Thus,

$$\int_0^\infty \widetilde{Q}^3 \exp\left(\frac{2p}{A}\alpha^{-1}(x)\right)|w_1|^{2p+2}dx \le \frac{p+2}{p}A\|u_1\|_{L^\infty}^{2p}\left(\int_0^\infty \widetilde{Q}(\partial_x w_1)^2 dx + \int_0^\infty \widetilde{Q}^3 w_1^2 dx\right),$$

which implies (2.53). □

Using this claim, integrating by parts, employing $\varphi_A \lesssim A$ and the decay estimate (2.16) we have

$$\begin{aligned} |I_3| &\le \frac{1}{2}\int |\widetilde{Q}^2(\widetilde{H} + a_1\phi_0)\varphi_A' u_1^3| + \int |\partial_x(\widetilde{Q}^2(\widetilde{H} + a_1\phi_0))\varphi_A u_1^3| \\ &\lesssim \int \widetilde{Q}^3\zeta_A^2|u_1|^3 + A\int \widetilde{Q}^3|u_1|^3 + A|a_1|\int \widetilde{Q}^2|\phi_0'||u_1|^3 \\ &\lesssim (1 + A|a_1|)\int \widetilde{Q}^3\zeta_A^{-1}|w_1|^3 + A\int \widetilde{Q}^3\zeta_{A/3}^{-1}|w_1|^3 \\ &\lesssim A^2\|u_1\|_{L^\infty}\left(\int \widetilde{Q}(\partial_x w_1)^2 dx + \int \widetilde{Q}^3 w_1^2 dx\right). \end{aligned} \tag{2.54}$$

Additionally,

$$
\begin{aligned}
|I_4| = & \left|\frac{1}{4}\int \widetilde{Q}^2\varphi_A' u_1^4 + \frac{1}{2}\int \widetilde{Q}^3\widetilde{H}\varphi_A u_1^4\right| \lesssim \int \widetilde{Q}^3\zeta_A^{-2}w_1^4 + A\int \widetilde{Q}^3 u_1^4 \\
\lesssim & \int \widetilde{Q}^3\zeta_A^{-2}w_1^4 + A\int \widetilde{Q}^3\zeta_{A/2}^{-2}w_1^4 \qquad (2.55)\\
\lesssim & A^2\|u_1\|_{L^\infty}^2\left(\int \widetilde{Q}(\partial_x w_1)^2dx + \int \widetilde{Q}^3 w_1^2\right).
\end{aligned}
$$

The last term we treat is $-N_0\int \phi_0\left(\varphi_A\partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right)$. By a point-wise estimate in (2.29),

$$|N| \lesssim \widetilde{Q}^2(a_1^2\phi_0^2 + u_1^2),$$

and thus, by decay estimates on $\widetilde{Q}$ and ϕ_0, and by (2.27), $|a_1| \lesssim 1$, $\|u_1\|_{L^\infty} \lesssim \|u_1\|_{H^1} \lesssim 1$, $A \geq 2$, it holds

$$|N_0| \lesssim a_1^2 + \int \widetilde{Q}^2\phi_0 u_1^2 \lesssim a_1^2 + \int \widetilde{Q}^6 u_1^2.$$

Using integration by parts

$$-\int \phi_0\left(\varphi_A\partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right) = \int u_1\left(\varphi_A\phi_0' + \frac{1}{2}\varphi_A'\phi_0\right).$$

Note that from the exponential decay of ϕ_0, ϕ_0', and from the polynomial decay of $\widetilde{Q}$, ζ_A we have

$$|\varphi_A\phi_0' + \varphi_A'\phi_0| \lesssim \alpha^{-1}(x)\phi_0' + \widetilde{Q}\zeta_A^2\phi_0 \lesssim \widetilde{Q}^4.$$

Thus, using the Cauchy-Schwarz inequality and Lemma 3.2,

$$
\begin{aligned}
\left|N_0\int \phi_0\left(\varphi_A\partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right)\right| \lesssim & \left(a_1^2 + \int \widetilde{Q}^6 u_1^2\right)\int \widetilde{Q}^4|u_1| \\
\lesssim & \left(a_1^2 + \int \widetilde{Q}^6 u_1^2\right)\left(\int \widetilde{Q}^6 u_1^2\right)^{\frac{1}{2}}\left(\int \widetilde{Q}^2\right)^{\frac{1}{2}} \qquad (2.56)\\
\lesssim & a_1^4 + \int \widetilde{Q}^6 u_1^2.
\end{aligned}
$$

Finally, collecting the estimates (2.51), (2.52), (2.54), (2.55) and (2.56), we obtain precisely (2.50).

$$
\begin{aligned}
&\left|\int \left(\varphi_A\partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right)\left(\widetilde{Q}^2(3\widetilde{H}(a_1\phi_0 + u_1)^2 + (a_1\phi_0 + u_1)^3) - N_0\phi_0\right)\right| \\
&\qquad \lesssim |a_1|^4 + \int \widetilde{Q}^6 u_1^2 + A^2\|u_1\|_{L^\infty}\left(\int \widetilde{Q}|\partial_x w_1|^2dx + \int \widetilde{Q}^3|w_1|^2dx\right).
\end{aligned}
$$

This ends the proof of Lemma 4.8. □

4.4. End of Proposition 4.1

Applying Lemmas 4.3 and 4.8 with Corollary 4.6, there exist constants $C_0, C > 0$ such that

$$\begin{aligned}\frac{d}{dt}\mathcal{I} = & -\int \widetilde{Q}(\partial_x w_1)^2 - \frac{1}{2}\int \left[\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A}\right)^2\right]\widetilde{Q}w_1^2 + \frac{1}{4}\int \widetilde{Q}''u_1^2 + \int \varphi_A V' u_1^2 \\ & -\int \left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right) N^{\perp} \\ \leq & -C_0 \int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + C\int \widetilde{Q}^6 u_1^2 + C|a_1|^4 \\ & + CA^2 \|u_1\|_{L^\infty}\left(\int \widetilde{Q}(\partial_x w_1)^2 + \int \widetilde{Q}^3 |w_1|^2\right).\end{aligned}$$

Using $A = \delta^{-\frac{1}{4}}$ (from (2.39)) and $\|u_1\|_{L^\infty} \lesssim \delta$ (from (2.27)), for δ_1 small enough, we obtain (2.40).

5. Transformed problem and second virial estimates

5.1. Transformed problem

We refer to [12, Section 3] for more details about factorizations of Schrödinger operators and to [19, 20, 27] for other uses in similar contexts. Recall L from (2.122), and let L_0 be defined as follows:

$$\begin{aligned}L = & -\partial_x^2 + V, \quad \text{with} \quad V := 2\widetilde{Q}^2(1-\widetilde{Q}), \\ L_0 = & -\partial_x^2 + V_0, \quad \text{with} \quad V_0 := 2\left(\frac{\partial_x \phi_0}{\phi_0}\right)^2 - 2\mu_0^2 - V,\end{aligned} \tag{2.57}$$

and

$$U = \phi_0 \cdot \partial_x \cdot \phi_0^{-1}, \quad U^* = -\phi_0^{-1} \cdot \partial_x \cdot \phi_0.$$

An important point to remark here is the unknown character of the terms forming L_0 in (2.57).

Then, the operators L and L_0 rewrite as $L = U^*U - \mu_0^2$, $L_0 = UU^* - \mu_0^2$ and it follows that

$$UL = L_0 U.$$

Let (u_1, u_2) be a solution of the linear part of 2.31, and set $v_1 = Uu_1$, $v_2 = Uu_2$. Then,

$$\begin{cases}\dot{v}_1 = v_2 \\ \dot{v}_2 = -L_0[v_1].\end{cases} \tag{2.58}$$

Our analysis relies in the crucial fact that the potential of L_0 is positive and repulsive. These properties happens to be the only spectral information needed for the proof of Theorem 2.1. See Appendix 11 for more details and the prove of these statements.

With respect to the above heuristic, we must take care of the loss of one derivative due to the operator U, without destroying the spetial algebra described heuristically. Therefore we need a regularization procedure of the functions involved, as in [20]. For this purpose we define the operator $X_\gamma : L^2(\mathbb{R}) \to H^2(\mathbb{R})$, $X_\gamma = (1-\gamma\partial_x^2)^{-1}$ via its Fourier transform representation. For $h \in L^2$,

$$\widehat{X_\gamma h}(\xi) = \frac{\hat{h}(\xi)}{1+\gamma\xi^2}.$$

For $\gamma > 0$ small to be defined later, set

$$\begin{cases} v_1 = (1-\gamma\partial_x^2)^{-1}Uu_1, \\ v_2 = (1-\gamma\partial_x^2)^{-1}Uu_2. \end{cases} \tag{2.59}$$

From the system (2.31) for (u_1, u_2), follows that $(v_1, v_2) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$, and satisfies the system

$$\begin{cases} \dot{v}_1 = v_2 \\ \dot{v}_2 = -(1-\gamma\partial_x^2)^{-1}ULu_1 + (1-\gamma\partial_x^2)^{-1}U(N^\perp). \end{cases}$$

Second, we note that $UL = L_0U$, then

$$\begin{aligned} -(1-\gamma\partial_x^2)^{-1}ULu_1 &= -(1-\gamma\partial_x^2)^{-1}L_0Uu_1 \\ &= -(1-\gamma\partial_x^2)^{-1}L_0[(1-\gamma\partial_x^2)v_1] \\ &= -(1-\gamma\partial_x^2)^{-1}(-\partial_x^2+V_0)(1-\gamma\partial_x^2)v_1 \\ &= \partial_x^2 v_1 - (1-\gamma\partial_x^2)^{-1}[V_0(1-\gamma\partial_x^2)v_1]. \end{aligned}$$

Since

$$\begin{aligned} (1-\gamma\partial_x^2)[V_0v_1] &= V_0v_1 - \gamma(V_0''v_1 + 2V_0'\partial_x v_1 + V_0\partial_x^2 v_1) \\ &= V_0(1-\gamma\partial_x^2)v_1 - \gamma(V_0''v_1 + 2V_0'\partial_x v_1), \end{aligned}$$

we obtain

$$-(1-\gamma\partial_x^2)^{-1}ULu_1 = -L_0v_1 - \gamma(1-\gamma\partial_x^2)^{-1}(V_0''v_1 + 2V_0'\partial_x v_1).$$

Therefore, we have obtained the following system for (v_1, v_2) (compare with (2.58)):

$$\begin{cases} \dot{v}_1 = v_2 \\ \dot{v}_2 = -L_0v_1 - \gamma(1-\gamma\partial_x^2)^{-1}(V_0''v_1 + 2V_0'\partial_x v_1) + (1-\gamma\partial_x^2)^{-1}UN^\perp. \end{cases} \tag{2.60}$$

An important point to be stressed now is that system (2.60), unlike previous systems obtained recently in the field, has unknown function V_0. We do not assume any specific spectral property on V_0, but we will succeed to show the required repulsivity conditions on (2.60) by making interesting computations on its local and global behavior.

5.2. Virial functional for the transformed problem

Recall (v_1, v_2) from (2.59). Set

$$\mathcal{J}(t) = \int \left(\psi_{A,B}(x)\partial_x v_1(t,x) + \frac{1}{2}\psi'_{A,B}(x)v_1(t,x)\right) v_2(t,x)dx \tag{2.61}$$

where we recall that $\psi_{A,B} = \tilde{\chi}_A^2 \varphi_B$, and define the localized version of the function v_1 at scale B as follows

$$z = \tilde{\chi}_A \zeta_B v_1. \tag{2.62}$$

Here z_1 represents a localized version of the variable w_1 at the scale B. This scale is intermediate, and $\mathcal{J}$ involves a cut-off at scale A, which will allow us to obtain an estimate in the same scale than the information obtained in Proposition 4.1, needed to bound some bad error and nonlinear terms; see [20, 24, 33] for similar procedure.

Proposition 5.1. *There exist $C_2 > 0$ and $\delta_2 > 0$ such that for γ small enough and for any $0 < \delta \leq \delta_2$, the following holds. Fix*

$$B = \delta^{-1/8}, \tag{2.63}$$

and assume that for all $t \geq 0$, (2.27) *holds. Then, for all $t \geq 0$, $\mathcal{J}$ in* (2.61) *satisfies*

$$\frac{d}{dt}\mathcal{J} \leq -C_2 \int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] + \delta^{\frac{1}{8}} \int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + \delta^{\frac{3}{4}}|a_1|^2. \tag{2.64}$$

The rest of this section is devoted to the proof of Proposition 5.1, which has been divided in several subsections.

5.3. Proof of Proposition 5.1: first computations

Analogously to the computation of $\dot{\mathcal{I}}$ in the proof of Proposition 4.1, we have from (2.60),

$$\begin{aligned}
\frac{d}{dt}\mathcal{J} &= \int \left(\psi_{A,B}\partial_x v_1 + \frac{1}{2}\psi'_{A,B} v_1\right) \dot{v}_2 \\
&= -\int \left(\psi_{A,B}\partial_x v_1 + \frac{1}{2}\psi'_{A,B} v_1\right) L_0 v_1 \\
&\quad - \gamma \int \left(\psi_{A,B}\partial_x v_1 + \frac{1}{2}\psi'_{A,B} v_1\right) (1-\gamma\partial_x^2)^{-1}(V_0'' v_1 + 2V_0'\partial_x v_1) \\
&\quad + \int \left(\psi_{A,B}\partial_x v_1 + \frac{1}{2}\psi'_{A,B} v_1\right) (1-\gamma\partial_x^2)^{-1} U N^{\perp} \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

First, using the definition of L_0 and integrating by parts such as in the proof of Lemma 4.3, we have

$$J_1 = -\int \psi'_{A,B}(\partial_x v_1)^2 + \frac{1}{4}\int \psi'''_{A,B} v_1^2 - \int \left(\psi_{A,B}\partial_x v_1 + \frac{1}{2}\psi'_{A,B} v_1\right) V_0 v_1.$$

By definition of $\psi_{A,B}$ (see (2.35)), it follows that

$$\begin{aligned}
\psi_{A,B}' &= \widetilde{Q}\tilde{\chi}_A^2\zeta_B^2 + (\tilde{\chi}_A^2)'\varphi_B \\
\psi_{A,B}'' &= \widetilde{Q}'\tilde{\chi}_A^2(\zeta_B^2) + \widetilde{Q}\tilde{\chi}_A^2(\zeta_B^2)' + 2\widetilde{Q}(\tilde{\chi}_A^2)'\zeta_B^2 + (\tilde{\chi}_A^2)''\varphi_B \\
\psi_{A,B}''' &= \widetilde{Q}''\tilde{\chi}_A^2\zeta_B^2 + 3\widetilde{Q}'(\tilde{\chi}_A^2)'\zeta_B^2 + 2\widetilde{Q}'\tilde{\chi}_A^2(\zeta_B^2)' + 3\widetilde{Q}(\tilde{\chi}_A^2)'(\zeta_B^2)' \\
&\quad + 3\widetilde{Q}(\tilde{\chi}_A^2)''\zeta_B^2 + \widetilde{Q}\tilde{\chi}_A^2(\zeta_B^2)'' + (\tilde{\chi}_A^2)'''\varphi_B.
\end{aligned}$$

Thus,

$$\begin{aligned}
-\int \psi_{A,B}'(\partial_x v_1)^2 + \frac{1}{4}\int \psi_{A,B}''' v_1^2 = &-\int \widetilde{Q}\tilde{\chi}_A^2\zeta_B^2(\partial_x v_1)^2 + \frac{1}{4}\int \widetilde{Q}''\tilde{\chi}_A^2\zeta_B^2 v_1^2 + \frac{1}{4}\int \widetilde{Q}\tilde{\chi}_A^2(\zeta_B^2)'' v_1^2 \\
&+ \frac{3}{4}\int \widetilde{Q}'(\tilde{\chi}_A^2)'\zeta_B^2 v_1^2 + \frac{3}{4}\int \widetilde{Q}(\tilde{\chi}_A^2)'(\zeta_B^2)' v_1^2 + \frac{3}{4}\int \widetilde{Q}(\tilde{\chi}_A^2)''\zeta_B^2 v_1^2 \\
&+ \frac{1}{2}\int \widetilde{Q}'\tilde{\chi}_A^2(\zeta_B^2)' v_1^2 - \int (\tilde{\chi}_A^2)'\varphi_B(\partial_x v_1)^2 + \frac{1}{4}\int (\tilde{\chi}_A^2)'''\varphi_B v_1^2.
\end{aligned}$$

For the first term of this integral, by the definition of z in (2.62) and proceeding as in the proof of Lemma 4.3, we have

$$\begin{aligned}
\int \widetilde{Q}\tilde{\chi}_A^2\zeta_B^2(\partial_x v_1)^2 = &\int \widetilde{Q}(\partial_x z)^2 + \int (\widetilde{Q}(\tilde{\chi}_A\zeta_B)')'\tilde{\chi}_A\zeta_B v_1^2 \\
= &\int \widetilde{Q}(\partial_x z)^2 + \int \widetilde{Q}\frac{\zeta_B''}{\zeta_B}z^2 + \int \widetilde{Q}\tilde{\chi}_A''\tilde{\chi}_A\zeta_B^2 v_1^2 + \frac{1}{2}\int \widetilde{Q}(\tilde{\chi}_A^2)'(\zeta_B^2)' v_1^2 \\
&+ \frac{1}{2}\int \widetilde{Q}'(\tilde{\chi}_A^2)'\zeta_B^2 v_1^2 + \frac{1}{2}\int \widetilde{Q}'\tilde{\chi}_A^2(\zeta_B^2)' v_1^2,
\end{aligned}$$

and

$$\frac{1}{4}\int \widetilde{Q}\tilde{\chi}_A^2(\zeta_B^2)'' v_1^2 = \frac{1}{2}\int \widetilde{Q}\left(\frac{\zeta_B''}{\zeta_B} + \frac{(\zeta_B')^2}{\zeta_B^2}\right)z^2.$$

Thus,

$$-\int \psi_{A,B}'(\partial_x v_1)^2 + \frac{1}{4}\int \psi_{A,B}''' v_1^2 = -\left\{\int \widetilde{Q}(\partial_x z)^2 - \frac{1}{4}\int \widetilde{Q}'' z^2 + \frac{1}{2}\int \widetilde{Q}\left(\frac{\zeta_B''}{\zeta_B} - \frac{(\zeta_B')^2}{\zeta_B^2}\right)z^2\right\} + \widetilde{J}_1,$$

where we have set

$$\begin{aligned}
\widetilde{J}_1 = &\frac{1}{4}\int \widetilde{Q}(\tilde{\chi}_A^2)'(\zeta_B^2)' v_1^2 + \frac{1}{4}\int \widetilde{Q}'(\tilde{\chi}_A^2)'\zeta_B^2 v_1^2 + \frac{1}{2}\int \widetilde{Q}[3(\tilde{\chi}_A')^2 + \tilde{\chi}_A''\tilde{\chi}_A]\zeta_B^2 v_1^2 \\
&- \int (\tilde{\chi}_A^2)'\varphi_B(\partial_x v_1)^2 + \frac{1}{4}\int (\tilde{\chi}_A^2)'''\varphi_B v_1^2.
\end{aligned}$$

Recalling (2.62), (2.33), (2.35) and integrating by parts,

$$\int \left(\psi_{A,B}\partial_x v_1 + \frac{1}{2}\psi_{A,B}' v_1\right) V_0 v_1 = \frac{1}{2}\int V_0\partial_x(\psi_{A,B}v_1^2) = -\frac{1}{2}\int \frac{\varphi_B}{\zeta_B^2}V_0' z^2.$$

Therefore, setting the potential

$$V_B = -\frac{1}{4}\widetilde{Q}'' + \frac{1}{2}\widetilde{Q}\left(\frac{\zeta_B''}{\zeta_B} - \frac{(\zeta_B')^2}{\zeta_B^2}\right) - \frac{1}{2}\frac{\varphi_B}{\zeta_B^2}V_0'. \tag{2.65}$$

For convenience, we split this potential into two main parts, given by

$$\begin{aligned} V_B = & \left[\frac{1}{2}\widetilde{Q}\left(\frac{\zeta_B''}{\zeta_B} - \frac{(\zeta_B')^2}{\zeta_B^2}\right) - \frac{1}{10}\frac{\varphi_B}{\zeta_B^2}V_0'\right] + \left[-\frac{1}{4}\widetilde{Q}'' - \frac{2}{5}\frac{\varphi_B}{\zeta_B^2}V_0'\right] \\ =: & \, V_B^I + V_B^{II}. \end{aligned} \tag{2.66}$$

Thus, the main part of the virial term can be written as

$$J_1 = -\int \left[\widetilde{Q}(\partial_x z)^2 + V_B^I z^2 + V_B^{II} z^2\right] + \widetilde{J}_1,$$

with V_B^I, V_B^{II} in (2.66). The following result simplifies the use of V_B^I in some extent.

Lemma 5.2. *There exists $B_0 > 0$ such that for all $B \geq B_0$, $V_B^I \geq 0$ on $\mathbb{R}$. More precisely, there exists $C_1' > 0$ such that*

$$V_B^I \geq V_1 \quad \textit{where} \quad V_1 = C_1'\widetilde{Q}^3(x)\mathbf{1}_{|x|\geq 1}(x), \tag{2.67}$$

for all $x \in \mathbb{R}$.

Proof. First, from (2.47) (with A replaced by B), it holds

$$\left|\frac{\zeta_B''}{\zeta_B} - \left(\frac{\zeta_B'}{\zeta_B}\right)^2\right| \leq \frac{C}{B}\widetilde{Q}^2(x)\mathbf{1}_{|x|\geq 1}(x),$$

for some $C > 0$.

Second, since for $x \in [0, +\infty) \to \zeta_B(x)$ is non-increasing, applying a change of variables, we have for $x \geq 0$,

$$\frac{\varphi_B}{\zeta_B^2} = \frac{1}{\zeta_B^2}\int_0^{\alpha^{-1}(x)} \zeta_B^2(\alpha(s))ds \geq \alpha^{-1}(x). \tag{2.68}$$

Now we will need some technical results about decay, positivity and repulsivity of V_0 that will be proved in Section 11. From Lemma 11.13 we have that $V_0' \leq 0$ for all $x \geq 0$. Using the above inequalities and decomposing,

$$\begin{aligned} V_B^I(x) \geq & \frac{1}{10}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\widetilde{Q}^3(x)\mathbf{1}_{|x|\geq 1}(x) \\ \geq & \left(\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\widetilde{Q}^3(x)\right)\mathbf{1}_{1\leq x\leq x_{2,2}}(x) \\ & + \left(\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\widetilde{Q}^3(x)\right)\mathbf{1}_{x\geq x_{2,2}}(x) \\ & + \frac{1}{20}\alpha^{-1}(x)|V_0'(x)|, \end{aligned} \tag{2.69}$$

where $x_{2,2} > 1$ is the second positive root of V'' (see Lemma 11.4).

For $x \in (1, x_{2,2})$, since by Lemma 11.13 we know $|V_0'(x)| > 0$, we have that there exist $\widetilde{C} > 0$ such that

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| \geq \widetilde{C}.$$

Then, taking $B_1 = \frac{27}{4}\frac{C}{\widetilde{C}}$ we obtain

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\widetilde{Q}^3 \geq \widetilde{C} - \frac{27}{8}\frac{C}{B} \geq \frac{1}{2}\widetilde{C} > 0,$$

for all $B \geq B_1$.

For $x \in (x_{2,2}, \infty)$, using Lemma 11.14, the definition of V and Lemma 3.2, we have that

$$\widetilde{Q}^3 \lesssim |V_0'| \lesssim \widetilde{Q}^3.$$

In particular, there exists $C' > 0$ such that $C'\widetilde{Q}^3 \leq |V_0'(x)|$ for all $x \geq x_{2,2}$. Using this, we obtain

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\widetilde{Q}^3 \geq \left(\frac{C'}{20}\alpha^{-1}(x) - \frac{C}{B}\right)\widetilde{Q}^3.$$

Thus, since by (2.8) for $x \in [x_{2,2}, +\infty) \mapsto \alpha^{-1}(x)$ is increasing, we have

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\widetilde{Q}^3 \geq \left(\frac{C'}{20}\alpha^{-1}(\tilde{x}) - \frac{C}{B}\right)\widetilde{Q}^3.$$

Taking $B_2 = \frac{10}{\alpha^{-1}(x_{2,2})}\frac{C}{C'}$, it holds

$$\frac{1}{20}\alpha^{-1}(x)|V_0'(x)| - \frac{C}{B}\widetilde{Q}^3 \geq \frac{1}{2}C'\widetilde{Q}^3,$$

for all $B \geq B_2$.

Defining $B_0 = \max\{B_1, B_2\}$ and collecting the previous estimates in (2.69),

$$\begin{aligned} V_B^I(x) &\geq \frac{1}{2}\widetilde{C}\mathbf{1}_{\{1\leq x\leq\tilde{x}\}}(x) + \frac{1}{2}C'\widetilde{Q}^3\mathbf{1}_{\{x\geq\tilde{x}\}}(x) + \frac{1}{20}\alpha^{-1}(x)|V_0'(x)| \\ &\geq \frac{1}{2}\widetilde{C}\mathbf{1}_{\{1\leq x\leq\tilde{x}\}}(x) + \frac{1}{2}\left(1 + \frac{1}{10}\alpha^{-1}(x)\right)C'\widetilde{Q}^3\mathbf{1}_{\{x\geq\tilde{x}\}}(x), \end{aligned}$$

for all $B \geq B_0$. Again, using that $\alpha^{-1} : \mathbb{R} \mapsto \mathbb{R}$ is an increasing positive function, we conclude that there exists $C_1' > 0$ such that

$$V_B^I(x) \geq C_1'\widetilde{Q}^3\mathbf{1}_{x\geq 1}(x),$$

for all $x \geq 0$, $B \geq B_0$. By parity, this estimate holds for any $x \in \mathbb{R}$, obtaining (2.67). □

Now, we have to obtain some estimate for the potential V_B^{II}. For this, we prove the following result.

Lemma 5.3. *The potential V_B^{II} is strictly positive on $\mathbb{R}$. Even more, there exists $C_1'' > 0$ such that*

$$V_B^{II} \geq V_2 \quad \textit{where} \quad V_2 = C_1''\widetilde{Q}^3(x), \tag{2.70}$$

for all $x \in \mathbb{R}$.

Proof. By parity we restrict to $x \geq 0$. First, using (2.21) and the definition of $\widetilde{Q}$, we have

$$-\frac{1}{4}\widetilde{Q}'' = \frac{1}{2}\widetilde{Q}^3\left(\frac{5}{6}\widetilde{Q} - 1\right). \tag{2.71}$$

We notice that (2.71) is positive for $\widetilde{Q} > \frac{6}{5}$. If we denote $\bar{x}$ the unique positive root of (2.71), from the definition of $\widetilde{Q}$ we have

$$\overline{x} = \alpha\left(2\,\mathrm{arcosh}\left(\sqrt{\frac{4}{5}}\right)\right) \sim 0.576,$$

and we notice, recalling that $\widetilde{Q}$ is a decreasing function on $\mathbb{R}_+$, that (2.71) is positive for $|x| \leq \overline{x}$. Using this, the repulsivity of V_0 and the definition of V_B^{II}, we have that

$$V_B^{II}(x) > 0,$$

for any $x \in [0, \overline{x})$.

For $x \geq x_{2,2}$, where $x_{2,2}$ is the second positive root of V'' (see Lemma 11.4), using (2.68), the decay estimate for V_0' from Lemma 11.14, and replacing (2.21) we obtain

$$\begin{aligned}
V_B^{II}(x) \geq\ & -\frac{1}{4}\widetilde{Q}'' - \frac{1}{5}\alpha^{-1}(x)V'(x)\\
=\ & \frac{1}{2}\widetilde{Q}^3\left(\frac{5}{6}\widetilde{Q} - 1\right) + \frac{2}{5}\alpha^{-1}(x)(2 - 3\widetilde{Q})\widetilde{Q}^3\widetilde{H}\\
=\ & \left(\frac{4}{5}\alpha^{-1}(x)\widetilde{H} - \frac{1}{2}\right)\widetilde{Q}^3 + \left(\frac{5}{12} - \frac{6}{5}\alpha^{-1}(x)\widetilde{H}\right)\widetilde{Q}^4\\
=\ & k(\alpha^{-1}(x))\widetilde{Q}^3,
\end{aligned}$$

where we have defined the auxiliary function $k : \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$k(s) := \frac{4}{5}sH(s) - \frac{1}{2} + \left(\frac{5}{12} - \frac{6}{5}sH(s)\right)Q(s).$$

This is an explicit function with two positive roots $s_1 \sim 0.47$ and $s_2 \sim 2.21$. Even more, from the asymptotic behavior of $k(s)$ for $s \to \infty$ we have that

$$k(s) > 0,$$

for all $s > s_2$. Using the bijectivity of α, that $\widetilde{Q}(x_{2,2}) \sim 0.49$, $Q(s_2) \sim 0.54$, this implies that $\alpha(s_2) < x_{2,2}$, and we conclude that

$$V_B^{II}(x) \gtrsim \widetilde{Q}^3(x)$$

for all $x \geq x_{2,2}$.

For $x \in (\overline{x}, x_{2,2})$, computing we have that

$$V_B^{II}(x) > 0.$$

Considering the above cases and by parity, there exist $C, \widetilde{C} > 0$ such that

$$V_B^{II}(x) \geq C\mathbf{1}_{|x|\leq x_{2,2}}(x) + \widetilde{C}\widetilde{Q}^3\mathbf{1}_{|x|\geq x_{2,2}}(x),$$

for all $x \in \mathbb{R}$. To sum up, we have that there exists $C_1'' > 0$ where it holds

$$V_B^{II}(x) \geq C_1''\widetilde{Q}^3(x),$$

for all $x \in \mathbb{R}$. This ends the proof of (2.70). □

Using Lemmas 5.2 and 5.3, the definition of V_B in (2.65) and considering $C_1 = \min\{C_1', C_1''\}$, we obtain

$$\frac{d}{dt}\mathcal{J} \leq -\int \widetilde{Q}[(\partial_x z)^2 + C_1\widetilde{Q}^2 z^2] + \widetilde{J}_1 + J_2 + J_3. \tag{2.72}$$

To control the terms $\widetilde{J}_1$, J_2, J_3 and J_4, we need some technical estimates.

6. Technical estimates.

The following estimates are already classical, but in our context, since the decay is only algebraic, we need some particular care. We start out with estimates necessary to treat regularized functions. The proof of these are different from previous work due to the slow decay of the potential V_0. We first recall the following well-known result.

Lemma 6.1 (See [20])**.** *For any $\gamma \in (0,1)$ and $f \in L^2$,*

$$\begin{gathered}\|(1-\gamma\partial_x^2)^{-1}f\|_{L^2} \leq \|f\|_{L^2}, \quad \|(1-\gamma\partial_x^2)^{-1}\partial_x f\|_{L^2} \leq \gamma^{-\frac{1}{2}}\|f\|_{L^2}, \\ \|(1-\gamma\partial_x^2)^{-1}\partial_x^2 f\|_{L^2} \leq \gamma^{-1}\|f\|_{L^2}.\end{gathered} \tag{2.73}$$

Our second result uses the fact that, even if the decay is only polynomial, it is strong enough to perform commutator estimates.

Lemma 6.2. *Let $\alpha(\cdot)$ be the function defined in (2.8). For any $0 < K \leq 3$, $\gamma > 0$ small enough, and $f \in L^2(\mathbb{R})$ one has*

$$\|\operatorname{sech}\Big(K\alpha^{-1}(x)\Big)(1-\gamma\partial_x^2)^{-1}f\|_{L^2} \lesssim \|(1-\gamma\partial_x^2)^{-1}[\operatorname{sech}\Big(K\alpha^{-1}(x)\Big)f]\|_{L^2}, \tag{2.74}$$

and

$$\|\cosh\Big(K\alpha^{-1}(x)\Big)(1-\gamma\partial_x^2)^{-1}f\|_{L^2} \lesssim \|(1-\gamma\partial_x^2)^{-1}[\cosh\Big(K\alpha^{-1}(x)\Big)f]\|_{L^2}. \tag{2.75}$$

where the implicit constant is independent of γ and K.

Let us recall that in view of (2.23), the term $\operatorname{sech}(K\alpha^{-1}(x))$ has only polynomial decay.

Proof. We set $g = \operatorname{sech}(K\alpha^{-1})(1-\gamma\partial_x^2)^{-1} f$ and $k = (1-\gamma\partial_x^2)^{-1}[\operatorname{sech}(K\alpha^{-1})f]$. We have

$$\begin{aligned} f = &\ \cosh\Big(K\alpha^{-1}\Big)(1-\gamma\partial_x^2)k = (1-\gamma\partial_x^2)[\cosh\Big(K\alpha^{-1}\Big)g] \\ = &\ \cosh\Big(K\alpha^{-1}\Big)g - \gamma[\cosh\Big(K\alpha^{-1}\Big)'' g + 2\cosh\Big(K\alpha^{-1}\Big)'\partial_x g + \cosh\Big(K\alpha^{-1}\Big)\partial_x^2 g] \\ = &\ \cosh\Big(K\alpha^{-1}\Big)(1-\gamma\partial_x^2)g - \gamma K\cosh\Big(K\alpha^{-1}\Big)\widetilde{Q}^2\left[K - \widetilde{H}\tanh\Big(K\alpha^{-1}\Big)\right]g \\ &\ - 2\gamma K\cosh\Big(K\alpha^{-1}\Big)\widetilde{Q}\tanh\Big(K\alpha^{-1}\Big)\partial_x g. \end{aligned}$$

Thus,

$$(1-\gamma\partial_x^2)k = (1-\gamma\partial_x^2)g - \gamma K\widetilde{Q}^2\left[K - \widetilde{H}\tanh\Big(K\alpha^{-1}\Big)\right]g - 2\gamma K\widetilde{Q}\tanh\Big(K\alpha^{-1}\Big)\partial_x g.$$

Applying the operator $(1-\gamma\partial_x^2)^{-1}$ to this identity, we obtain

$$g = k + \gamma K(1-\gamma\partial_x^2)^{-1}\left\{\widetilde{Q}^2\left[K - \widetilde{H}\tanh\Big(K\alpha^{-1}\Big)\right]g\right\} + 2\gamma K(1-\gamma\partial_x^2)^{-1}\left[\widetilde{Q}\tanh\Big(K\alpha^{-1}\Big)\partial_x g\right].$$

We have from (2.73) that for $\gamma \leq \frac{1}{2}$,

$$\|(1-\gamma\partial_x^2)^{-1}\|_{\mathcal{L}(L^2,L^2)} \lesssim 1, \quad \|(1-\gamma\partial_x^2)^{-1}\partial_x\|_{\mathcal{L}(L^2,L^2)} \lesssim \gamma^{-\frac{1}{2}}. \tag{2.76}$$

Thus, for $0 < K \leq 3$

$$\left\|(1-\gamma\partial_x^2)^{-1}\left\{\widetilde{Q}^2\left[K - \widetilde{H}\tanh\Big(K\alpha^{-1}\Big)\right]g\right\}\right\|_{L^2} \lesssim \left\|\widetilde{Q}^2\left[K - \widetilde{H}\tanh\Big(K\alpha^{-1}\Big)\right]g\right\|_{L^2} \lesssim_K \|g\|_{L^2},$$

and

$$\begin{aligned} \left\|(1-\gamma\partial_x^2)^{-1}\left[\widetilde{Q}\tanh\Big(K\alpha^{-1}\Big)\partial_x g\right]\right\|_{L^2} \lesssim &\ \left\|(1-\gamma\partial_x^2)^{-1}\partial_x\left[\widetilde{Q}\tanh\Big(K\alpha^{-1}\Big)g\right]\right\|_{L^2} \\ &\ + \|(1-\gamma\partial_x^2)^{-1}[\partial_x(\widetilde{Q}\tanh\Big(K\alpha^{-1}\Big))g]\|_{L^2} \\ \lesssim &\ \gamma^{-\frac{1}{2}}K\|\widetilde{Q}\tanh\Big(\alpha^{-1}\Big)g\|_{L^2} \\ &\ + K\|\widetilde{Q}^2\left(K\operatorname{sech}^2(K\alpha^{-1}) - \widetilde{H}\tanh\Big(K\alpha^{-1}\Big)\right)g\|_{L^2} \\ \lesssim &\ \gamma^{-\frac{1}{2}}K\|g\|_{L^2} + K^2\|g\|_{L^2} \lesssim \gamma^{-\frac{1}{2}}\|g\|_{L^2}. \end{aligned}$$

We deduce that there exists a constant $C > 0$, independent of γ small and $0 < K \leq 3$, such that

$$\|g\|_{L^2} \leq \|k\|_{L^2} + C\gamma^{\frac{1}{2}}\|g\|_{L^2},$$

which implies (2.74) for γ small enough.

We prove (2.75) similarly. Setting

$$g = \cosh\Big(K\alpha^{-1}\Big)(1-\gamma\partial_x^2)^{-1}f \quad \text{and} \quad K = (1-\gamma\partial_x^2)^{-1}[\cosh\Big(K\alpha^{-1}\Big)f],$$

we compute

$$\begin{aligned} f =& \operatorname{sech}\Big(K\alpha^{-1}\Big)(1-\gamma\partial_x^2)k = (1-\gamma\partial_x^2)[\operatorname{sech}\Big(K\alpha^{-1}\Big)g] \\ =& \operatorname{sech}\Big(K\alpha^{-1}\Big)g - \gamma[\operatorname{sech}\Big(K\alpha^{-1}\Big)'' g + 2\operatorname{sech}\Big(K\alpha^{-1}\Big)'\partial_x g + \operatorname{sech}\Big(K\alpha^{-1}\Big)\partial_x^2 g] \\ =& \operatorname{sech}\Big(K\alpha^{-1}\Big)(1-\gamma\partial_x^2)g - \gamma K\widetilde{Q}\operatorname{sech}\Big(K\alpha^{-1}\Big)[K\widetilde{Q}(1-2\operatorname{sech}^2(K\alpha^{-1}))g - 2\tanh\Big(K\alpha^{-1}\Big)\partial_x g]. \end{aligned}$$

Thus, applying the operator $(1-\gamma\partial_x^2)^{-1}$ as before, we have

$$g = k + \gamma K^2(1-\gamma\partial_x^2)^{-1}[\widetilde{Q}^2(1-2\operatorname{sech}^2(K\alpha^{-1}))g] - 2\gamma K(1-\gamma\partial_x^2)^{-1}[\widetilde{Q}\tanh\Big(K\alpha^{-1}\Big)\partial_x g].$$

Using $0 < K \leq 3$ and (2.76), it follows that

$$\|(1-\gamma\partial_x^2)^{-1}[\widetilde{Q}^2(1-2\operatorname{sech}^2(K\alpha^{-1}))g]\|_{L^2} \lesssim \|\widetilde{Q}^2(1-2\operatorname{sech}^2(K\alpha^{-1}))g\|_{L^2} \lesssim \|g\|_{L^2},$$

and

$$\begin{aligned} &\|(1-\gamma\partial_x^2)^{-1}[\widetilde{Q}\tanh\Big(K\alpha^{-1}\Big)\partial_x g]\|_{L^2} \\ &\qquad \lesssim \|(1-\gamma\partial_x^2)^{-1}\partial_x[\widetilde{Q}\tanh\Big(K\alpha^{-1}\Big)g]\|_{L^2} + \|(1-\gamma\partial_x^2)^{-1}[\partial_x(\widetilde{Q}\tanh\Big(K\alpha^{-1}\Big))g]\|_{L^2} \\ &\qquad \lesssim \gamma^{-\frac{1}{2}}\|\widetilde{Q}\tanh\Big(K\alpha^{-1}\Big)g\|_{L^2} + \|\widetilde{Q}^2[K\operatorname{sech}^2(K\alpha^{-1}) - \widetilde{H}\tanh\Big(K\alpha^{-1}\Big)]g\|_{L^2} \\ &\qquad \lesssim \gamma^{-\frac{1}{2}}\|g\|_{L^2}. \end{aligned}$$

It follows that there exist $\widetilde{C} > 0$ independent of γ such that

$$\|g\|_{L^2} \leq \|k\|_{L^2} + \widetilde{C}\gamma^{\frac{1}{2}}\|g\|_{L^2}.$$

Considering γ small enough we obtain (2.75). □

Remark 6.3. *There are some interesting consequences of the previous results. Indeed, using* (2.74) *and* (2.75) *for* $K = \frac{n}{2} + \frac{2}{A}$ *with* $A \geq 2$, (2.73) *and* $n = 1, 3$ *implies the following inequalities*

$$\left\|\operatorname{sech}\left(\left(\frac{3}{2}+\frac{1}{A}\right)\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}f\right\| \lesssim \left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}\left(\left(\frac{3}{2}+\frac{1}{A}\right)\alpha^{-1}\right)f\right]\right\|, \tag{2.77}$$

and

$$\left\|\operatorname{sech}\left(\left(\frac{1}{2}+\frac{1}{A}\right)\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}f\right\| \lesssim \left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}\left(\left(\frac{1}{2}+\frac{1}{A}\right)\alpha^{-1}\right)f\right]\right\|. \tag{2.78}$$

Besides, we recall that

$$\sigma_A\widetilde{Q}^{-\frac{1}{2}} \lesssim \cosh\left(\frac{2-A}{2A}\alpha^{-1}\right) \lesssim \sigma_A\widetilde{Q}^{-\frac{n}{2}}, \tag{2.79}$$

for any $A \geq 2$, *so using* (2.75) *for* $K = \frac{2-A}{2A}$ *with* $A \geq 4$, *and* (2.79) *implies*

$$\|\sigma_A\widetilde{Q}^{-\frac{1}{2}}(1-\gamma\partial_x^2)^{-1}\partial_x f\| \lesssim \gamma^{-\frac{1}{2}}\|\sigma_A\widetilde{Q}^{-\frac{1}{2}}f\|. \tag{2.80}$$

The following result is a $\widetilde{Q}$ localized version of the radiation term.

Lemma 6.4. *For any $A \geq 1$ large, any $\gamma > 0$ small and any $u \in H^1(\mathbb{R})$, if we define v related with u by*

$$v = (1-\gamma\partial_x^2)^{-1}Uu,$$

then

$$\left\|\sigma_A\widetilde{Q}^{\frac{3}{2}}v\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\widetilde{Q}^{\frac{3}{2}}u\right\|, \tag{2.81}$$

and

$$\left\|\sigma_A\widetilde{Q}^{\frac{1}{2}}\partial_x v\right\| \lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\widetilde{Q}^{\frac{1}{2}}\partial_x u\right\| + \left\|\sigma_A\widetilde{Q}^{\frac{5}{2}}u\right\|. \tag{2.82}$$

Remark 6.5. *Estimates in* (2.80) *and Lemma 6.4 require the additional terms $\widetilde{Q}^{\frac{1}{2}}$, $\widetilde{Q}^{\frac{3}{2}}$ in order to control some nonstandard terms appearing in below estimates.*

Proof. By direct computations, we have $U = \partial_x - h_0$, where the function h_0 is bounded (see Appendix Lemma 11.6). In addition, using that

$$\sigma_A\widetilde{Q}^{\frac{n}{2}} \lesssim \operatorname{sech}\left(\left(\frac{n}{2}+\frac{1}{A}\right)\alpha^{-1}\right) \lesssim \sigma_A\widetilde{Q}^{\frac{n}{2}} \tag{2.83}$$

with $n=3$, the first estimate is a consequence of (2.77) and (2.73),

$$\begin{aligned}
\left\|\sigma_A\widetilde{Q}^{\frac{3}{2}}v\right\| &\lesssim \left\|\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)v\right\| \\
&\lesssim \left\|\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}\partial_x u\right\| + \left\|\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}[h_0u]\right\| \\
&\lesssim \left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)\partial_x u\right]\right\| + \left\|(1-\gamma\partial_x^2)^{-1}[\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)h_0u]\right\| \\
&\lesssim \left\|(1-\gamma\partial_x^2)^{-1}\partial_x\left[\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)u\right]\right\| \\
&\quad + \frac{3A+2}{2A}\left\|(1-\gamma\partial_x^2)^{-1}\left[\widetilde{Q}\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)u\right]\right\| + \left\|\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)h_0u\right\| \\
&\lesssim \gamma^{-\frac{1}{2}}\left\|\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)u\right\| + \left\|\widetilde{Q}\operatorname{sech}\left(\frac{3A+2}{2A}\alpha^{-1}\right)u\right\| + \left\|\sigma_A\widetilde{Q}^{\frac{3}{2}}h_0u\right\| \\
&\lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\widetilde{Q}^{\frac{3}{2}}u\right\| + \left\|\sigma_A\widetilde{Q}^{\frac{3}{2}}u\right\| + \left\|\sigma_A\widetilde{Q}^{\frac{3}{2}}h_0u\right\| \\
&\lesssim \gamma^{-\frac{1}{2}}\left\|\sigma_A\widetilde{Q}^{\frac{3}{2}}u\right\|.
\end{aligned}$$

This proves (2.81).

For the second estimate, we have

$$\partial_x U = \partial_x^2 - h_0\partial_x - h_0'.$$

Using (2.83) with $n=1$ and (2.78), plus the fact that h_0 is bounded and $|h_0'| \lesssim |V| \lesssim \widetilde{Q}^2$

(see Lemma 11.1, (2.128)), analogously to the previous estimate we have

$$\begin{aligned}
\|\sigma_A \widetilde{Q}^{\frac{1}{2}}\partial_x v\| \lesssim{}& \left\|\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)\partial_x v\right\| \\
\lesssim{}& \left\|\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}\partial_x^2 u\right\| + \left\|\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}[h_0\partial_x u]\right\| \\
&+ \left\|\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}[h_0' u]\right\| \\
\lesssim{}& \left\|(1-\gamma\partial_x^2)^{-1}\partial_x\left[\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)\partial_x u\right]\right\| + \left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)'\partial_x u\right]\right\| \\
&+ \left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)h_0\partial_x u\right]\right\| + \left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)h_0' u\right]\right\| \\
\lesssim{}& \gamma^{-\frac{1}{2}}\left\|\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)\partial_x u\right\| + \left\|\operatorname{sech}\left(\frac{A+2}{2A}\alpha^{-1}\right)\widetilde{Q}^2 u\right\| \\
\lesssim{}& \gamma^{-\frac{1}{2}}\left\|\sigma_A \widetilde{Q}^{\frac{1}{2}}\partial_x u\right\| + \left\|\sigma_A \widetilde{Q}^{\frac{5}{2}} u\right\|,
\end{aligned}$$

which proves (2.82). □

Lemma 6.6. *One has*

(1) Estimates on w_1.

$$\left\|\sigma_A \widetilde{Q}^{\frac{1}{2}}\partial_x u_1\right\| \lesssim \left\|\widetilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \frac{1}{A}\left\|\widetilde{Q}^{\frac{3}{2}} w_1\right\|. \tag{2.84}$$

(2) Estimates on v_1.

$$\left\|\sigma_A \widetilde{Q}^{\frac{3}{2}} v_1\right\|_{L^2} \lesssim \gamma^{-\frac{1}{2}}\left\|\widetilde{Q}^{\frac{3}{2}} w_1\right\|, \tag{2.85}$$

$$\left\|\sigma_A \widetilde{Q}^{\frac{1}{2}}\partial_x v_1\right\| \lesssim \gamma^{-\frac{1}{2}}\left(\left\|\widetilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\widetilde{Q}^{\frac{3}{2}} w_1\right\|\right). \tag{2.86}$$

Remark 6.7. *Compared with previous results in [20, 24], Lemma 6.6 contains new weighted estimates because of the variable coefficients in the model and the emergence of new weighted terms as well.*

Proof. Proof of (6.6). Using that $\sigma_A \lesssim \zeta_A$, and that from definition (2.33) $\zeta_A' \lesssim A^{-1}\widetilde{Q}\zeta_A$, we have

$$\begin{aligned}
\left\|\sigma_A \widetilde{Q}^{\frac{1}{2}}\partial_x u_1\right\| \lesssim{}& \left\|\zeta_A \widetilde{Q}^{\frac{1}{2}}\partial_x u_1\right\| \lesssim \left\|\widetilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left\|\widetilde{Q}^{\frac{1}{2}}\zeta_A' u_1\right\| \\
\lesssim{}& \left\|\widetilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \frac{1}{A}\left\|\widetilde{Q}^{\frac{3}{2}} w_1\right\|.
\end{aligned}$$

Proof of (6.6). Estimate (2.85) is direct from (2.81), using $\sigma_A \lesssim \zeta_A$ and (2.38).

Now, using (2.82) and (2.84) we have

$$\begin{aligned}
\left\|\sigma_A \widetilde{Q}^{\frac{1}{2}}\partial_x v\right\| \lesssim{}& \gamma^{-\frac{1}{2}}\left\|\sigma_A \widetilde{Q}^{\frac{1}{2}}\partial_x u\right\| + \left\|\sigma_A \widetilde{Q}^{\frac{5}{2}} u\right\| \\
\lesssim{}& \gamma^{\frac{1}{2}}\|\widetilde{Q}^{\frac{1}{2}}\partial_x w_1\| + \frac{1}{A}\gamma^{-\frac{1}{2}}\left\|\widetilde{Q}^{\frac{3}{2}} w_1\right\| + \left\|\sigma_A \widetilde{Q}^{\frac{5}{2}} u\right\| \\
\lesssim{}& \gamma^{-\frac{1}{2}}\left\|\widetilde{Q}^{\frac{1}{2}}\partial_x w_1\right\| + \left(1 + \frac{1}{A}\gamma^{-\frac{1}{2}}\right)\left\|\widetilde{Q}^{\frac{3}{2}} w_1\right\|,
\end{aligned}$$

obtaining (2.86).

□

7. Controlling error and nonlinear terms.

Now we have in a position to control the error and nonlinear terms in (2.72). By the definition of ζ_B and $\tilde{\chi}_A$ in (2.35), it holds that

$$\begin{aligned} &\zeta_B(x) \lesssim e^{-\frac{1}{B}|\alpha^{-1}(x)|}, \quad |\zeta_B'(x)| \lesssim \frac{1}{B}\widetilde{Q}e^{-\frac{1}{B}|\alpha^{-1}(x)|}, \quad |\varphi_B| \lesssim B, \\ &|\tilde{\chi}_A'| \lesssim \frac{1}{A}\widetilde{Q}, \quad |(\tilde{\chi}_A^2)'| \lesssim \frac{1}{A}\widetilde{Q}, \quad |\tilde{\chi}_A''| \lesssim \frac{1}{A}\widetilde{Q}^2, \quad |(\tilde{\chi}_A^2)'''| \lesssim \frac{1}{A}\widetilde{Q}^3. \end{aligned} \tag{2.87}$$

Even more, from the definition of χ in (2.32) we have

$$\tilde{\chi}_A'(x) = \tilde{\chi}_A''(x) = \tilde{\chi}_A'''(x) = 0, \tag{2.88}$$

if $|\alpha^{-1}(x)| < A$ or if $|\alpha^{-1}(x)| > 2A$.

7.1. Control of $\widetilde{J}_1$.

Let us now recall the definition of $\widetilde{J}_1$:

$$\begin{aligned} \widetilde{J}_1 = &\frac{1}{4}\int \widetilde{Q}(\tilde{\chi}_A^2)'(\zeta_B^2)'v_1^2 + \frac{1}{4}\int \widetilde{Q}'(\tilde{\chi}_A^2)'\zeta_B^2 v_1^2 + \frac{1}{2}\int \widetilde{Q}[3(\tilde{\chi}_A')^2 + \tilde{\chi}_A''\tilde{\chi}_A]\zeta_B^2 v_1^2 \\ &+ \frac{1}{4}\int (\tilde{\chi}_A^2)'''\varphi_B v_1^2 - \int (\tilde{\chi}_A^2)'\varphi_B(\partial_x v_1)^2 \\ = &J_{1,1} + J_{1,2} + J_{1,3} + J_{1,4} + J_{1,5}. \end{aligned} \tag{2.89}$$

For the first four terms, using that $\sigma_A \gtrsim 1$ on $[-2\alpha(A), 2\alpha(A)]$, (2.87) and (2.88), we have

$$\begin{aligned} |(\tilde{\chi}_A^2)'(\zeta_B^2)'| \lesssim \frac{1}{AB}e^{-2\frac{A}{B}}\sigma_A^2\widetilde{Q}^2, \quad |(\tilde{\chi}_A^2)'\zeta_B^2| \lesssim \frac{1}{A}e^{-2\frac{A}{B}}\sigma_A^2\widetilde{Q}, \\ (\tilde{\chi}_A')^2\zeta_B^2 + |\tilde{\chi}_A''\tilde{\chi}_A|\zeta_B^2 \lesssim \frac{1}{A}e^{-2\frac{A}{B}}\sigma_A^2\widetilde{Q}^2, \\ |(\tilde{\chi}_A^2)'''\varphi_B| \lesssim \frac{B}{A}\sigma_A^2\widetilde{Q}^3, \quad |(\tilde{\chi}_A^2)'\varphi_B| \lesssim \frac{B}{A}\sigma_A^2\widetilde{Q}. \end{aligned} \tag{2.90}$$

Thus, using the above estimates and (2.85), we have for the terms in (2.89),

$$\begin{aligned} |J_{1,1}| + |J_{1,2}| + |J_{1,3}| + |J_{1,4}| &\lesssim \frac{B}{A}\|\sigma_A\widetilde{Q}^{\frac{3}{2}}v_1\|^2 \\ &\lesssim \gamma^{-1}\frac{B}{A}\|\sigma_A\widetilde{Q}^{\frac{3}{2}}u_1\|^2 \lesssim \gamma^{-1}\frac{B}{A}\|\widetilde{Q}^{\frac{3}{2}}w_1\|^2. \end{aligned}$$

In the case of $J_{1,5}$, using (2.90) and (2.86), we obtain

$$|J_{1,5}| \lesssim \frac{B}{A}\|\sigma_A \widetilde{Q}^{\frac{1}{2}}\partial_x v_1\|^2 \lesssim \gamma^{-1}\frac{B}{A}\left(\|\widetilde{Q}^{\frac{1}{2}}\partial_x w_1\|^2 + \|\widetilde{Q}^{\frac{3}{2}}w_1\|^2\right).$$

Therefore, we conclude for this term

$$|\widetilde{J}_1| \lesssim \gamma^{-1}\frac{B}{A}\left(\int \widetilde{Q}(\partial_x w_1)^2 + \int \widetilde{Q}^3 w_1^2\right). \tag{2.91}$$

7.2. Control of J_2.

First, by the Cauchy-Schwarz inequality,

$$|J_2| \lesssim \gamma \left\|\widetilde{Q}(1-\gamma\partial_x^2)^{-1}\left(\psi_{A,B}\partial_x v_1 + \frac{1}{2}\psi'_{A,B}v_1\right)\right\| \|\widetilde{Q}^{-1}(V_0''v_1 + V_0'\partial_x v_1)\|.$$

Using the commutativity estimate (2.74), (2.73) and $\widetilde{Q} \lesssim \operatorname{sech}(\alpha^{-1}) \lesssim \widetilde{Q}$,

$$\begin{aligned}\|\widetilde{Q}(1-\gamma\partial_x^2)^{-1}(\psi_{A,B}\partial_x v_1)\| &\lesssim \|\operatorname{sech}\left(\alpha^{-1}\right)(1-\gamma\partial_x^2)^{-1}(\psi_{A,B}\partial_x v_1)\| \\ &\lesssim \|(1-\gamma\partial_x^2)^{-1}(\operatorname{sech}\left(\alpha^{-1}\right)\psi_{A,B}\partial_x v_1)\| \\ &\lesssim \|\operatorname{sech}\left(\alpha^{-1}\right)\psi_{A,B}\partial_x v_1\| \\ &\lesssim \|\widetilde{Q}\psi_{A,B}\partial_x v_1\|.\end{aligned}$$

From the definition of z in (2.62), we have

$$\partial_x z = \tilde{\chi}_A\zeta_B\partial_x v_1 + (\tilde{\chi}_A\zeta_B)'v_1 \quad \Longrightarrow \quad \tilde{\chi}_A^2\zeta_B^2(\partial_x v_1)^2 \lesssim (\partial_x z)^2 + |(\tilde{\chi}_A\zeta_B)'v_1|^2.$$

Using (2.87) and again the definition of z

$$|(\tilde{\chi}_A\zeta_B)'v_1|^2\tilde{\chi}_A^2 \lesssim \left(\frac{1}{A}+\frac{1}{B}\right)^2 \widetilde{Q}^2\tilde{\chi}_A^2\zeta_B^2 v_1^2 \lesssim \frac{1}{B^2}\widetilde{Q}^2 z^2,$$

and so

$$\tilde{\chi}_A^4\zeta_B^2(\partial_x v_1)^2 \lesssim \tilde{\chi}_A^2(\partial_x z)^2 + \frac{1}{B^2}\widetilde{Q}^2 z^2.$$

Thus, using $|\psi_{A,B}| \leq |\alpha^{-1}(x)|\tilde{\chi}_A^2$,

$$\widetilde{Q}^2|\psi_{A,B}\partial_x v_1|^2 \lesssim |\alpha^{-1}(x)|^2\widetilde{Q}^2\tilde{\chi}_A^4(\partial_x v_1)^2 \lesssim \widetilde{Q}\tilde{\chi}_A^4\zeta_B^2(\partial_x v_1)^2 \lesssim \widetilde{Q}(\partial_x z)^2 + \frac{1}{B^2}\widetilde{Q}^3 z^2.$$

So, it follows that

$$\|\widetilde{Q}\psi_{A,B}\partial_x v_1\| \lesssim \left(\widetilde{Q}(\partial_x z)^2 + \frac{1}{B^2}\widetilde{Q}^3 z^2\right)^{\frac{1}{2}}. \tag{2.92}$$

Proceeding as before and using (2.74), (2.73), for the other term we obtain

$$\|\widetilde{Q}(1-\gamma\partial_x^2)^{-1}(\psi'_{A,B}v_1)\| \lesssim \|\widetilde{Q}\psi'_{A,B}v_1\|.$$

Now, we claim

$$(\psi'_{A,B})^2 \lesssim \widetilde{Q}^2\tilde{\chi}_A^2. \tag{2.93}$$

Indeed, using (2.87), the definition of $\psi_{A,B}$ in (2.35) and that $\tilde{\chi}_A = 0$ for $|\alpha^{-1}(x)| \geq 2A$,

$$\begin{aligned}(\psi'_{A,B})^2 &= [(\tilde{\chi}_A^2)'\varphi_B + \tilde{\chi}_A^2\widetilde{Q}\zeta_B^2]^2 \lesssim (\tilde{\chi}_A\tilde{\chi}'_A\varphi_B)^2 + \widetilde{Q}^2\tilde{\chi}_A^2\zeta_B^2 \\ &\lesssim \frac{1}{A^2}\widetilde{Q}^2\tilde{\chi}_A^2|\alpha^{-1}(x)|^2 + \widetilde{Q}^2\tilde{\chi}_A^2 \lesssim \widetilde{Q}^2\tilde{\chi}_A^2.\end{aligned}$$

Using (2.93), we have that

$$|\widetilde{Q}^2(\psi'_{A,B})^2v_1^2| \lesssim \widetilde{Q}^4\tilde{\chi}_A^2v_1^2 \lesssim \widetilde{Q}^3z^2,$$

and so

$$\|\widetilde{Q}\psi_{A,B}v_1\|^2 \lesssim \int \widetilde{Q}^3z^2. \tag{2.94}$$

Collecting (2.92) and (2.94) we have

$$\left\|\widetilde{Q}(1-\gamma\partial_x^2)^{-1}\left(\psi_{A,B}\partial_x v_1 + \frac{1}{2}\psi'_{A,B}v_1\right)\right\| \lesssim \left(\int \widetilde{Q}(\partial_x z)^2 + \widetilde{Q}^3z^2\right)^{\frac{1}{2}}. \tag{2.95}$$

Now we estimate the term related with the potential V_0. By Lemma 11.14 we have $|V_0'| \lesssim \widetilde{Q}^3$, and using that

$$|h_0| \lesssim 1, \qquad h_0' = \frac{1}{4h_0}(V_0' + V'), \qquad h_0'' = V' - 2h_0h_0'$$

with

$$V_0'' = 4(h_0')^2 + 4h_0h_0'' - V'',$$

one has $|V_0''| \lesssim \widetilde{Q}^3$. Combining the above estimates,

$$|V_0''v_1| + |V_0'\partial_x v_1| \lesssim \widetilde{Q}^3|v_1| + \widetilde{Q}^3|\partial_x v_1|,$$

so

$$\|\widetilde{Q}^{-1}(V_0''v_1 + V_0'\partial_x v_1)\| \lesssim \|\widetilde{Q}^2v_1\| + \|\widetilde{Q}^2\partial_x v_1\|.$$

From the definition of z in (2.62) and the particular polynomial decay of ζ_B and $\widetilde{Q}$, we have

$$\widetilde{Q}^{\frac{1}{2}}\tilde{\chi}_A^2v_1^2 \lesssim \tilde{\chi}_A^2\zeta_B^2v_1^2 = z^2. \tag{2.96}$$

Thus, using the above and from the definition of $\tilde{\chi}_A$,

$$\widetilde{Q}^4v_1^2 = \widetilde{Q}^4v_1^2\tilde{\chi}_A^2 + \widetilde{Q}^4v_1^2(1-\tilde{\chi}_A^2) \lesssim \widetilde{Q}^{\frac{7}{2}}z^2 + e^{-A}\widetilde{Q}^{\frac{7}{2}}v_1^2.$$

From this, and using that $\widetilde{Q}^{\frac{1}{4}} \lesssim \sigma_A$ for A large enough, it follows that

$$\|\widetilde{Q}^2v_1\| \lesssim \|\widetilde{Q}^{\frac{7}{4}}z\| + e^{-\frac{A}{2}}\|\widetilde{Q}^{\frac{7}{4}}v_1\| \lesssim \|\widetilde{Q}^{\frac{3}{2}}z\| + e^{-\frac{A}{2}}\|\sigma_A\widetilde{Q}^{\frac{3}{2}}u_1\|.$$

By estimate (2.85) we obtain

$$\|\widetilde{Q}^2 v_1\| \lesssim \|\widetilde{Q}^{\frac{3}{2}} z\| + \gamma^{-\frac{1}{2}} e^{-\frac{A}{2}} \|\sigma_A \widetilde{Q}^{\frac{3}{2}} u_1\| \lesssim \|\widetilde{Q}^{\frac{3}{2}} z\| + \gamma^{-\frac{1}{2}} e^{-\frac{A}{2}} \|\widetilde{Q}^{\frac{3}{2}} w_1\|. \tag{2.97}$$

For the other term $\|\widetilde{Q}^2 \partial_x v_1\|$, differentiating $z = \tilde{\chi}_A \zeta_B v_1$ we obtain

$$\tilde{\chi}_A \zeta_B \partial_x v_1 = \partial_x z - \frac{\zeta_B'}{\zeta_B} z - \tilde{\chi}_A' \zeta_B v_1.$$

Thus, from the properties of ζ_B and $\tilde{\chi}_A$ in (2.47) and (2.87) we get

$$|\tilde{\chi}_A \zeta_B \partial_x v_1| \lesssim \partial_x z + \frac{1}{B} \widetilde{Q} z. \tag{2.98}$$

Replacing and using the polynomial decay of ζ_B, we have

$$\begin{aligned} \widetilde{Q}^4 (\partial_x v_1)^2 &= \widetilde{Q}^4 (\partial_x v_1)^2 \tilde{\chi}_A^2 + \widetilde{Q}^4 (\partial_x v_1)^2 (1 - \tilde{\chi}_A^2) \\ &\lesssim \widetilde{Q}^{\frac{7}{2}} (\partial_x z)^2 + \frac{1}{B} \widetilde{Q}^{\frac{11}{2}} z^2 + e^{-\frac{A}{2}} \widetilde{Q}^3 (\partial_x v_1)^2. \end{aligned}$$

Integrating over $\mathbb{R}$ and using (2.85), we obtain

$$\begin{aligned} \|\widetilde{Q}^2 \partial_x v_1\| &\lesssim \|\widetilde{Q}^{\frac{7}{4}} \partial_x z\| + \frac{1}{\sqrt{B}} \|\widetilde{Q}^{\frac{11}{4}} z\| + e^{-\frac{A}{4}} \|\widetilde{Q}^{\frac{7}{4}} \partial_x v_1\| \\ &\lesssim \|\widetilde{Q}^{\frac{1}{2}} \partial_x z\| + \frac{1}{\sqrt{B}} \|\widetilde{Q}^{\frac{3}{2}} z\| + e^{-\frac{A}{4}} \|\widetilde{Q}^{\frac{3}{2}} \partial_x v_1\| \\ &\lesssim \|\widetilde{Q}^{\frac{1}{2}} \partial_x z\| + \frac{1}{\sqrt{B}} \|\widetilde{Q}^{\frac{3}{2}} z\| + \gamma^{-\frac{1}{2}} e^{-\frac{A}{4}} \left(\|\widetilde{Q}^{\frac{1}{2}} \partial_x w_1\| + \|\widetilde{Q}^{\frac{3}{2}} w_1\| \right). \end{aligned} \tag{2.99}$$

It follows using (2.97) and (2.99) that

$$\|\widetilde{Q}^2 v_1\| + \|\widetilde{Q}^2 \partial_x v_1\| \lesssim \|\widetilde{Q}^{\frac{1}{2}} \partial_x z\| + \|\widetilde{Q}^{\frac{3}{2}} z\| + \gamma^{-\frac{1}{2}} e^{-\frac{A}{4}} \left(\|\widetilde{Q}^{\frac{1}{2}} \partial_x w_1\| + \|\widetilde{Q}^{\frac{3}{2}} w_1\| \right). \tag{2.100}$$

Therefore, collecting the estimates (2.95) and (2.100) we conclude

$$|J_2| \lesssim \gamma \left(\int \widetilde{Q} (\partial_x z)^2 + \widetilde{Q}^3 z^2 \right) + \gamma^{\frac{1}{2}} e^{-\frac{A}{4}} \left(\int \widetilde{Q} (\partial_x w_1)^2 + \widetilde{Q}^3 w_1^2 \right). \tag{2.101}$$

7.3. Control of J_3.

Using the Cauchy-Schwarz inequality and (2.73), we have

$$|J_3| \lesssim \left(\|\widetilde{Q}^{\frac{1}{2}} \psi_{A,B} \partial_x v_1\| + \|\widetilde{Q}^{\frac{1}{2}} \psi_{A,B}' v_1\| \right) \left\| \widetilde{Q}^{-\frac{1}{2}} \tilde{\chi}_A (1 - \gamma \partial_x^2)^{-1} U N^{\perp} \right\|. \tag{2.102}$$

For the first term, using (2.35), $|\varphi_B| \lesssim B$, estimate (2.86) and that $\zeta_A \gtrsim 1$ on $[-2\alpha(A), 2\alpha(A)]$,

$$\|\widetilde{Q}^{\frac{1}{2}} \psi_{A,B} \partial_x v_1\| \lesssim B \|\widetilde{Q}^{\frac{1}{2}} \tilde{\chi}_A^2 \partial_x v_1\| \lesssim \gamma^{-\frac{1}{2}} B (\|\widetilde{Q}^{\frac{1}{2}} \partial_x w_1\| + \|\widetilde{Q}^{\frac{3}{2}} w_1\|).$$

On the other hand, using (2.87),

$$|\psi'_{A,B}| \le |(\tilde\chi_A^2)'\varphi_B| + \widetilde{Q}^2\tilde\chi_A^2\zeta_B^2 \lesssim \frac{B}{A}\widetilde{Q}\tilde\chi_A + \widetilde{Q}^2\tilde\chi_A^2\zeta_B^2.$$

Using (2.85), it follows

$$\|\widetilde{Q}^{\frac12}\psi'_{A,B}v_1^2\| \lesssim \frac{B}{A}\|\widetilde{Q}^{\frac32}\tilde\chi_A v_1\| + \|\widetilde{Q}^2 z\| \lesssim \gamma^{-\frac12}\frac{B}{A}\|\widetilde{Q}^{\frac32}w_1\| + \|\widetilde{Q}^2 z\|.$$

Collecting these estimates, we obtain

$$\|\widetilde{Q}^{\frac12}\psi_{A,B}\partial_x v_1\| + \|\widetilde{Q}^{\frac12}\psi'_{A,B}v_1\| \lesssim \gamma^{-\frac12}B\left(\|\widetilde{Q}^{\frac12}\partial_x w_1\| + \|\widetilde{Q}^{\frac32}w_1\|\right) + \|\widetilde{Q}^2 z\|. \tag{2.103}$$

It remains to bound the second term in (2.102). Using that $N^\perp = N - N_0\phi_0$, we split it in two parts as follows

$$\left\|\widetilde{Q}^{-\frac12}\tilde\chi_A(1-\gamma\partial_x^2)^{-1}UN^\perp\right\| \le \left\|\widetilde{Q}^{-\frac12}\tilde\chi_A(1-\gamma\partial_x^2)^{-1}UN\right\| + |N_0|\left\|\widetilde{Q}^{-\frac12}\tilde\chi_A(1-\gamma\partial_x^2)^{-1}U\phi_0\right\|$$

Now, we recall the pointwise estimate for the non-linear term (2.29) and its projection over ϕ_0,

$$\begin{aligned} |N| &\lesssim \widetilde{Q}^2(a_1^2\phi_0^2 + u_1^2)\\ |N_0| &\lesssim a_1^2 + \int \widetilde{Q}^2\phi_0 u_1^2 \lesssim a_1^2 + \|u_1\|_{L^\infty}\|\widetilde{Q}^{\frac32}w_1\|. \end{aligned} \tag{2.104}$$

Thus, using $1 \lesssim \sigma_A \lesssim \zeta_A$ on $[-2\alpha(A), 2\alpha(A)]$, $U = \partial_x - h_0$ with $h_0 \lesssim 1$ and estimate (2.80),

$$\left\|\widetilde{Q}^{-\frac12}\tilde\chi_A(1-\gamma\partial_x^2)^{-1}UN\right\| \lesssim \|\sigma_A\widetilde{Q}^{-\frac12}(1-\gamma\partial_x^2)^{-1}UN\| \lesssim \gamma^{-\frac12}\|\sigma_A\widetilde{Q}^{-\frac12}N\|.$$

Inserting the pointwise estimate (2.104) into this, it follows that

$$\begin{aligned} \left\|\widetilde{Q}^{-\frac12}\tilde\chi_A(1-\gamma\partial_x^2)^{-1}UN\right\| &\lesssim \gamma^{-\frac12}\left(a_1^2\|\sigma_A\widetilde{Q}^{\frac32}\phi_0^2\| + \|\sigma_A\widetilde{Q}^{\frac32}u_1^2\|\right)\\ &\lesssim \gamma^{-\frac12}\left(a_1^2 + \|u_1\|_{L^\infty}\|\widetilde{Q}^{\frac32}w_1\|\right) \end{aligned} \tag{2.105}$$

For the remaining term, using the exponential decay of ϕ_0, (2.80) and (2.104) we have

$$\begin{aligned} \left\|\widetilde{Q}^{-\frac12}\tilde\chi_A(1-\gamma\partial_x^2)^{-1}U\phi_0\right\| &\lesssim \left\|\sigma_A\widetilde{Q}^{-\frac12}(1-\gamma\partial_x^2)^{-1}U\phi_0\right\|\\ &\lesssim \gamma^{-\frac12}\|\sigma_A\widetilde{Q}^{-\frac12}\phi_0\| \lesssim \gamma^{-\frac12}. \end{aligned} \tag{2.106}$$

Now combining the preceding estimates (2.103), (2.105), (2.106) and (2.104) with (2.102) yields

$$\begin{aligned} |J_3| \lesssim{}& \gamma^{-1}B\left(\|\widetilde{Q}^{\frac12}\partial_x w_1\| + \|\widetilde{Q}^{\frac32}w_1\|\right)\left(a_1^2 + \|u_1\|_{L^\infty}\|\widetilde{Q}^{\frac32}w_1\|\right)\\ &+ \gamma^{-\frac12}\|\widetilde{Q}^{\frac32}z\|\left(a_1^2 + \|u_1\|_{L^\infty}\|\widetilde{Q}^{\frac32}w_1\|\right). \end{aligned} \tag{2.107}$$

8. End of Proposition 5.1

Gathering (2.72), (2.91), (2.101) and (2.107), it follows that there exist $C_2 > 0$ and $C > 0$ such that

$$\begin{aligned}\frac{d}{dt}\mathcal{J} \leq & -4C_2 \int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] + \gamma^{-1}\frac{CB}{A}\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + \gamma C \int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] \\ & + \gamma^{\frac{1}{2}} C e^{-\frac{A}{4}} \int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + \gamma^{-1} CB \left(\|\widetilde{Q}^{\frac{1}{2}}\partial_x w_1\| + \|\widetilde{Q}^{\frac{3}{2}} w_1\|\right)\left(a_1^2 + \|u_1\|_{L^\infty}\|\widetilde{Q}^{\frac{3}{2}} w_1\|\right) \\ & + \gamma^{-\frac{1}{2}} C \|\widetilde{Q}^{\frac{3}{2}} z\| \left(a_1^2 + \|u_1\|_{L^\infty}\|\widetilde{Q}^{\frac{3}{2}} w_1\|\right).\end{aligned}$$

We fix $\gamma > 0$ such that $\gamma C \leq 2C_2$ and also small enough to satisfy Lemma 6.2 and Lemma 6.4.

The value of γ being now fixed, we do not mention anymore dependency of γ. Via standard inequalities and for A large enough, we obtain, for a possibly large constant $C > 0$,

$$\begin{aligned}\frac{d}{dt}\mathcal{J} \leq & -C_2 \int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] + C\left(\frac{B}{A} + e^{-\frac{A}{4}}\right)\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] \\ & + CB\left(\|\widetilde{Q}^{\frac{1}{2}}\partial_x w_1\| + \|\widetilde{Q}^{\frac{3}{2}} w_1\|\right)\left(a_1^2 + \|u_1\|_{L^\infty}\|\widetilde{Q}^{\frac{3}{2}} w_1\|\right) + C\left(a_1^2 + \|u_1\|_{L^\infty}\|\widetilde{Q}^{\frac{3}{2}} w_1\|\right)^2.\end{aligned}$$

Since $A = \delta^{-\frac{1}{4}}$ and $B = \delta^{-\frac{1}{8}}$ (see (2.39), (2.63)), using assumption (2.27) and standard inequalities, we have

$$B\left(\|\widetilde{Q}^{\frac{1}{2}}\partial_x w_1\| + \|\widetilde{Q}^{\frac{3}{2}} w_1\|\right)\left(a_1^2 + \|u_1\|_{L^\infty}\|\widetilde{Q}^{\frac{3}{2}} w_1\|\right) \leq \delta^{\frac{7}{8}}|a_1|^2 + \delta^{\frac{7}{8}}\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2].$$

Therefore, using again (2.27), for δ small enough (to absorb some constants), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{J} \leq & -C_2 \int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] + C\delta^{\frac{1}{4}}\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + C\delta^{\frac{7}{8}}|a_1|^2 \\ \leq & -C_2 \int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] + \delta^{\frac{1}{8}}\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + \delta^{\frac{3}{4}}|a_1|^2.\end{aligned}$$

This ends the proof of (2.64).

9. Proof of Theorem 2.1

sec:proof theorem 1, sec:end virial II, sec:second virial estimate, Sect:3

Before starting the proof of Theorem 2.1, we need a coercivity result to deal with the term

$$\int \widetilde{Q}^6 u_1^2$$

that appears in the virial estimate of $\mathcal{I}(t)$ (see (2.40)), being a term with enough decay to be controlled by the variables (v_1, v_2) and (z_1, z_2). In this section, the constant γ is fixed as in Proposition 5.1.

9.1. Coercivity

We prove a coercivity result adapted to the orthogonality condition $\langle u, \phi_0\rangle = 0$ in (2.25), where ϕ_0 was introduced in (2.16). The idea is to follow the strategy used in [20], where the linearized operator has an explicit unique negative single eigenvalue τ_0 associated with an explicit L^2 eigenfunction denoted Y_0. Despite our system we only have the existence of such negative eigenvalue $-\mu_0^2$ associated with ϕ_0, we still have this control given by orthogonality.

Lemma 9.1. *Let u and v be $L^2(\mathbb{R})$ functions related by*

$$v = (1-\gamma\partial_x^2)^{-1}Uu \tag{2.108}$$

and such that $\langle v, \phi_0\rangle = 0$, the following estimate holds

$$\int \widetilde{Q}^6 u^2 \lesssim \int \widetilde{Q}^{\frac{7}{2}}[(\partial_x v)^2 + v^2]. \tag{2.109}$$

Proof. Using that $U = \phi_0 \cdot \partial_x \cdot \phi_0^{-1}$, we rewrite (2.108) as

$$v - \gamma\partial_x^2 v = \phi_0 \partial_x \left(\frac{u}{\phi_0}\right).$$

and thus, after some algebra

$$\partial_x\left(\frac{u}{\phi_0} + \gamma\frac{\partial_x v}{\phi_0}\right) = \frac{1}{\phi_0}\left(v - \gamma h_0 \partial_x v\right)$$

where $h_0 = \phi_0'/\phi_0$ (see (2.125)). Integrating between 0 and $x > 0$, it follows

$$\frac{u}{\phi_0} + \gamma\frac{\partial_x v}{\phi_0} = a + \int_0^x \frac{1}{\phi_0}\left(v - \gamma h_0\partial_x v\right)$$

for some constant a. If we rewrite this last expression, multiplying by ϕ_0, it follows

$$u = a\phi_0 - \gamma\partial_x v + \tilde{u}, \tag{2.110}$$

where

$$\tilde{u} = \phi_0 \int_0^x \frac{1}{\phi_0}\left(v - \gamma h_0 \partial_x v\right).$$

Let us now estimate $\tilde{u}$. First, using the Cauchy-Schwarz inequality, a change of variables, and recalling that ϕ_0 is even and decreasing for $x > 0$, we have

$$\phi_0 \int_0^x \frac{|v|}{\phi_0} \lesssim \phi_0 \left(\int \widetilde{Q}^{\frac{7}{2}} v^2\right)^{\frac{1}{2}} \left(\int_0^x \frac{1}{\widetilde{Q}^{\frac{7}{2}}\phi_0^2}\right)^{\frac{1}{2}} \lesssim \|\widetilde{Q}^{\frac{7}{4}} v\| \left(\int_0^{\alpha^{-1}(x)} \frac{1}{\widetilde{Q}^{\frac{9}{2}}}\right)^{\frac{1}{2}} \lesssim \widetilde{Q}^{-\frac{9}{4}} \left\|\widetilde{Q}^{\frac{7}{4}} v\right\|.$$

Similarly, using that $|h_0| \lesssim 1$,

$$\phi_0 \int_0^x \frac{|h_0 \partial_x v|}{\phi_0} \lesssim \phi_0 \left(\int \widetilde{Q}^{\frac{7}{2}} (h_0\partial_x v)^2\right)^{\frac{1}{2}} \left(\int_0^x \frac{1}{\widetilde{Q}^{\frac{7}{2}}\phi_0^2}\right)^{\frac{1}{2}} \lesssim \widetilde{Q}^{-\frac{9}{4}} \left\|\widetilde{Q}^{\frac{7}{4}} \partial_x v\right\|.$$

Collecting these estimates, we obtain the uniform bound

$$\widetilde{Q}^{\frac{9}{2}}\tilde{u}^2 \lesssim \int \widetilde{Q}^{\frac{7}{2}}[(\partial_x v)^2 + v^2],$$

for all $x \geq 0$. The same result holds for $x \leq 0$. Therefore, multiplying by $\widetilde{Q}^{\frac{3}{2}}$ and integrating we obtain

$$\int \widetilde{Q}^6\tilde{u}^2 \lesssim \left(\int \widetilde{Q}^{\frac{3}{2}}\right)\left(\int \widetilde{Q}^{\frac{7}{2}}[(\partial_x v)^2 + v^2]\right) \lesssim \int \widetilde{Q}^{\frac{7}{2}}[(\partial_x v)^2 + v^2].$$

Using that $\langle u, \phi_0\rangle = 0$ and (2.16), we have

$$a = \gamma\langle \partial_x v, \phi_0\rangle - \langle \tilde{u}, \phi_0\rangle.$$

Thus, using the Cauchy-Schwarz inequality we estimate the constant a in (2.110) as follows,

$$a^2 \lesssim \left(\int \phi_0\partial_x v\right)^2 + \left(\int \phi_0\tilde{u}\right)^2 \lesssim \int \widetilde{Q}^{\frac{7}{2}}(\partial_x v)^2 + \int \widetilde{Q}^6\tilde{u}^2 \lesssim \int \widetilde{Q}^{\frac{7}{2}}[(\partial_x v)^2 + v^2].$$

We conclude (2.109) using again (2.110). □

As result of the previous lemma, we have the following transfer estimate from the variable u_1 to the transformed and localized variable z introduced in (2.62).

Lemma 9.2. *Let* (u_1, u_2) *be solution of* (2.31) *satisfying* (2.25), (w_1, w_2) *be as in* (2.38), *and* z *as in* (2.62). *Then, for any* A *large enough, it holds*

$$\int \widetilde{Q}^6 u_1^2 \lesssim \int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] + e^{-\frac{A}{4}}\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2]. \tag{2.111}$$

Proof. Since u_1 satisfies the orthogonality condition (2.25), applying (2.109)

$$\int \widetilde{Q}^6 u_1^2 \lesssim \int \widetilde{Q}^{\frac{7}{2}}[(\partial_x v_1)^2 + v_1^2].$$

Now, using that $\widetilde{Q}^{\frac{1}{4}} \lesssim \zeta_B^2$, (2.96) and(2.98), it follows

$$\begin{aligned}\widetilde{Q}^{\frac{7}{2}}[(\partial_x v_1)^2 + v_1^2] &\lesssim \int \widetilde{Q}^3\zeta_B^2(\partial_x v_1)^2 + \int \widetilde{Q}^3\zeta_B^2 v_1^2\\ &\lesssim \int \widetilde{Q}^3[(\partial_x z)^2 + \widetilde{Q}^2 z^2] + \int \widetilde{Q}^3 z^2 + e^{-2A}\int \widetilde{Q}\zeta_B^2(1-\chi_A^2)(\partial_x v_1)^2\\ &\quad + e^{-\frac{A}{4}}\int \widetilde{Q}^3\zeta_B^2(1-\chi_A^2)v^2,\end{aligned}$$

and since $\zeta_B \lesssim \zeta_A \lesssim \sigma_A$, using (2.85) and (2.86),

$$\begin{aligned}\widetilde{Q}^{\frac{7}{2}}[(\partial_x v_1)^2 + v_1^2] &\lesssim \int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] + e^{-2A}\int \sigma_A^2\widetilde{Q}(\partial_x v_1)^2 + e^{-\frac{A}{4}}\int \sigma_A^2\widetilde{Q}^3 v_1^2\\ &\lesssim \int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] + e^{-\frac{A}{4}}\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2],\end{aligned}$$

and the asserted estimate (2.111) follows.

□

9.2. Proof of Theorem 2.1.

Recall that the constants $\gamma > 0$, $\delta_1, \delta_2 > 0$ were defined and fixed in Propositions 4.1 and 5.1.

In this section we prove Theorem 2.1, in particular the conditional asymptotic stability property (2.20). In this case, the orthogonality conditions (2.25) and the dynamical equations satisfied by (a_1, a_2) in (2.28) will be of key importance. It turns out that (b_1, b_2) as in (2.26) are better suited variables to fully catch the exponential unstable behavior of the full system.

Proposition 9.3. *There exist $C_3 > 0$ and $0 < \delta_3 \leq \min(\delta_1, \delta_2)$ such that for any $0 < \delta \leq \delta_3$, the following holds. Fix $A = \delta^{-\frac{1}{4}}$ and $B = \delta^{-\frac{1}{8}}$. Assume that for all $t \geq 0$, (2.27) holds.*

Let

$$\mathcal{H} = \mathcal{J} + 8C_0^{-1}B^{-1}\mathcal{I}, \tag{2.112}$$

where $C_0 > 0$ is the constant from Proposition 4.1.

Then, for all $t \geq 0$,

$$\frac{d}{dt}\mathcal{H} \leq -B^{-1}\int \widetilde{Q}\left[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2\right] + 2\delta^{\frac{3}{4}}|a_1|^2. \tag{2.113}$$

Proof. In the context of Propositions 4.1 and 5.1, observe that fixing $A = \delta^{-\frac{1}{4}}$ and $B = \delta^{-\frac{1}{8}}$, for $\delta > 0$ small is consistent with the requirement of scales in (2.36).

First, combining (2.44) with (2.111), for $\delta_3 > 0$ small enough and $0 < \delta \leq \delta_3$, we obtain for some constants $C_0, C > 0$ fixed, and possibly choosing a smaller δ_3,

$$\begin{aligned}
\frac{d}{dt}\mathcal{I} \leq & -\frac{1}{2}C_0\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + C\int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] \\
& + Ce^{-\frac{A}{4}}\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + C\delta_3|a_1|^3 \\
\leq & -\frac{1}{4}C_0\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + C\int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] + |a_1|^3.
\end{aligned}$$

Secondly, for $\frac{d}{dt}\mathcal{J}$, using (2.64) and $0 < \delta \leq \delta_3$, we get for some constant $C_2 > 0$ fixed,

$$\frac{d}{dt}\mathcal{J} \leq -C_2\int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] + B^{-1}\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + \delta^{\frac{3}{4}}|a_1|^2.$$

Therefore, defining $\mathcal{H}$ as in (2.112) and by combining the above estimates, it follows that

$$\begin{aligned}
\frac{d}{dt}\mathcal{H} \leq & -C_2\int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] + B^{-1}\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] \\
& + \frac{8C}{C_0}B^{-1}\int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] + \left(1 + \frac{8}{C_0}\delta^{\frac{3}{8}}\right)\delta^{\frac{3}{4}}|a_1|^2.
\end{aligned}$$

Thus, possible choosing a smaller δ_3, we obtain

$$\frac{d}{dt}\mathcal{H} \leq -\frac{C_2}{2}\int \widetilde{Q}[(\partial_x z)^2 + \widetilde{Q}^2 z^2] - B^{-1}\int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] + 2\delta^{\frac{3}{4}}|a_1|^2.$$

We have that (2.113) follows directly from the above estimate. □

We define now

$$\mathcal{B} = b_+^2 - b_-^2, \tag{2.114}$$

where b_+, b_- are given in (2.26).

Lemma 9.4. *There exist $C_4 > 0$ and $0 < \delta_4 \leq \delta_3$ such that for any $0 < \delta \leq \delta_4$, the following holds. Fix $A = \delta^{-\frac{1}{4}}$. Assume that for all $t \geq 0$, (2.27) holds. Then, for all $t \geq 0$,*

$$|\dot{b}_+ - \mu_0 b_+| + |\dot{b}_- + \mu_0 b_-| \leq C_4 \left(b_+^2 + b_-^2 + \int \widetilde{Q}^3 w_1^2 \right), \tag{2.115}$$

and

$$\left| \frac{d}{dt}(b_+^2) - 2\mu_0 b_+^2 \right| + \left| \frac{d}{dt}(b_-^2) + 2\mu_0 b_-^2 \right| \leq C_4 \left(b_+^2 + b_-^2 + \int \widetilde{Q}^3 w_1^2 \right)^{\frac{3}{2}}. \tag{2.116}$$

In particular,

$$\frac{d}{dt}\mathcal{B} \geq \mu_0 (b_+^2 + b_-^2) - C_4 \int \widetilde{Q}^3 w_1^2 = \frac{\mu_0}{2}(a_1^2 + a_2^2) - C_4 \int \widetilde{Q}^3 w_1^2. \tag{2.117}$$

Proof. From (2.104) and (2.26), it holds

$$|N_0| \lesssim a_1^2 + \int \widetilde{Q}^3 w_1^2 \lesssim b_+^2 + b_-^2 + \int \widetilde{Q}^3 w_1^2.$$

From (2.28) we conclude the estimates (2.115) and (2.116). Finally, estimate (2.117) is a consequence of (2.116) taking $\delta_4 > 0$ small enough. □

Combining (2.113) and (2.117), it holds

$$\frac{d}{dt}\left(\mathcal{B} - 2C_4 B \mathcal{H} \right) \geq \frac{\mu_0}{2}(a_1^2 + a_2^2) + C_4 \int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] - 4C_4 \delta^{\frac{5}{8}} |a_1|^2,$$

and thus, for possibly smaller $\delta > 0$,

$$\frac{d}{dt}\left(\mathcal{B} - 2C_4 B \mathcal{H} \right) \geq \frac{\mu_0}{4}(a_1^2 + a_2^2) + C_4 \int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2]. \tag{2.118}$$

By the choice of $A = \delta^{-\frac{1}{4}}$, the bound $|\varphi_A| \lesssim A$, and (2.27), we have for all $t \geq 0$,

$$\mathcal{I} \lesssim A \|u_1\|_{H^1} \|u_2\|_{L^2} \lesssim \delta.$$

Similarly, using that $U = \partial_x - h_0$ and (2.73), it holds

$$\mathcal{J} \lesssim B \|v_1\|_{H^1} \|v_2\|_{L^2} \lesssim \delta.$$

Then, we have

$$|\mathcal{H}| \lesssim \delta.$$

Estimate $|\mathcal{B}| \lesssim \delta^2$ is also clear from (2.27).

Therefore, integrating estimate (2.118) on $[0,t]$ and passing to the limit as $t \to +\infty$, it follows that

$$\int_0^\infty \left\{ a_1^2 + a_2^2 + \int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2] \right\} dt \lesssim \delta.$$

Since

$$\int \widetilde{Q}^2[(\partial_x u_1)^2 + \widetilde{Q}^2 u_1^2] \lesssim \int \widetilde{Q}[(\partial_x w_1)^2 + \widetilde{Q}^2 w_1^2],$$

this implies

$$\int_0^\infty \left\{ a_1^2 + a_2^2 + \int \widetilde{Q}^2[(\partial_x u_1)^2 + \widetilde{Q}^2 u_1^2] \right\} dt \lesssim \delta. \tag{2.119}$$

Using the above equation, we will conclude the proof of Theorem 2.1. Let

$$\mathcal{K} = \int u_1 u_2 \widetilde{Q}^2 \quad \text{and} \quad \mathcal{G} = \frac{1}{2}\int [(\partial_x u_1)^2 + \widetilde{Q}^2 u_1^2 + u_2^2]\widetilde{Q}^2.$$

Using (2.31), we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{K} &= \int [\dot{u}_1 u_2 + u_1 \dot{u}_2]\widetilde{Q}^2 \\
&= \int \left[u_2^2 - u_1 \left(L u_1 + N^\perp \right) \right] \widetilde{Q}^2 \\
&= \int [u_2^2 - (\partial_x u_1)^2 - 2\widetilde{Q}^2(1-\widetilde{Q})u_1^2]\widetilde{Q}^2 + \frac{1}{2}\int (\widetilde{Q}^2)'' u_1^2 - \int N^\perp \widetilde{Q}^2 u_1.
\end{aligned}$$

From (2.104) and the exponential decay of ϕ_0 we can check that

$$\int N^\perp \widetilde{Q}^2 u_1 \lesssim a_1^2 + \int \widetilde{Q}^4 u_1^2.$$

In particular, collecting the above estimates and using that $(\widetilde{Q}^2)'' \lesssim \widetilde{Q}^4$, it follows that there exists some $C > 0$ such that

$$\int \widetilde{Q}^2 u_2^2 \leq \frac{d}{dt}\mathcal{K} + C a_1^2 + C \int \widetilde{Q}^2[(\partial_x u_1)^2 + \widetilde{Q}^2 u_1^2].$$

From this, the bound $|\mathcal{K}| \lesssim \delta^2$ and (2.119), we deduce

$$\int_0^\infty [a_1^2 + a_2^2 + \mathcal{G}] dt \lesssim \delta. \tag{2.120}$$

Analogously, we compute

$$\begin{aligned}
\frac{d}{dt}\mathcal{G} &= \int [(\partial_x \dot{u}_1)(\partial_x u_1) + \widetilde{Q}^2 \dot{u}_1 u_1 + \dot{u}_2 u_2]\widetilde{Q}^2 \\
&= \int \left[(\partial_x u_2)(\partial_x u_1) + \widetilde{Q}^2 u_2 u_1 - \left(L u_1 + N^\perp \right) u_2 \right] \widetilde{Q}^2 \\
&= -2\int \widetilde{Q}\widetilde{Q}' u_2 \partial_x u_1 + \int (\widetilde{Q}-1)\widetilde{Q}^4 u_1 u_2 - \int \widetilde{Q}^2 N^\perp u_2,
\end{aligned}$$

and so, using (2.104) as before, we obtain

$$\left|\frac{d}{dt}\mathcal{G}\right| \lesssim a_1^2 + \mathcal{G}. \tag{2.121}$$

By (2.120), there exists an increasing sequence $t_n \to +\infty$ such that

$$\lim_{n\to\infty}[a_1^2(t_n) + a_2^2(t_n) + \mathcal{G}(t_n)] = 0.$$

For $t \geq 0$, integrating (2.121) on $[t, t_n]$, and passing to the limit as $n \to \infty$, we obtain

$$\mathcal{G}(t) \lesssim \int_t^\infty [a_1^2 + \mathcal{G}]dt.$$

Using (2.120), we deduce that $\lim_{t\to\infty}\mathcal{G}(t) = 0$.

Finally, by (2.31) and (2.104), we get

$$\left|\frac{d}{dt}(a_1^2)\right| + \left|\frac{d}{dt}(a_2^2)\right| \lesssim a_1^2 + a_2^2 + \int \widetilde{Q}^2 u_1^2.$$

Similarly as before, by integration on $[t, t_n]$ and taking $n \to \infty$,

$$a_1^2(t) + a_2^2(t) \lesssim \int_t^\infty [a_1^2 + a_2^2 + \mathcal{G}]dt,$$

which proves $\lim_{t\to\infty} |a_1(t)| + |a_2(t)| = 0$. By the decomposition of solution the (2.24), this clearly implies (2.20). The proof of Theorem 2.1 is complete.

10. Linear Spectral Theory for L

In this section, we describe the spectral properties of the operator L introduced in equation (2.15). Being a variable coefficients operator with no explicit eigenfunctions, the understanding here becomes more subtle, and some interesting new features appear in the spectral analysis.

Notice that L correspond to a Schrödinger operator with potential $V(x) = 2\widetilde{Q}^2(x)(1 - \widetilde{Q}(x))$, where we have defined the function

$$\widetilde{Q}(x) = Q(\alpha^{-1}(x)) \quad \text{with} \quad \alpha(x) = \frac{1}{3}(\sinh x + x).$$

Unlike standard operators [28], L has a complicated structure with slow decay, essentially just enough to run suitable estimates.

Remark 10.1. *A direct analysis shows that the null space of $P_0 = -\partial_x^2$ is spanned by functions of the type $1, x$ as $x \to \infty$. Note that this set is linearly independent and there are no $L^2(\mathbb{R})$ integrable functions in the semi-infinite line $[0, +\infty)$. Therefore, the analysis of V becomes essential to understand the set of possible solutions in $L^2(\mathbb{R})$ for the operator L.*

Lemma 10.2. *The linear operator L defined by*

$$L[\phi] = -\partial_x^2\phi + V(x)\phi, \quad \text{with} \quad V(x) = 2\widetilde{Q}^2(x)(1-\widetilde{Q}(x)), \tag{2.122}$$

with dense domain $\mathcal{D}(L) = H^2(\mathbb{R})$, satisfies the following properties.

1. *The essential spectrum of L is $[0, +\infty)$.*
2. *$\sigma_{disc}(L) \cap \mathbb{R}_-$ is not empty.*
3. *The operator L has a first simple eigenvalue $\lambda_0 = -\mu_0^2$, with associated eigenfunction ϕ_0 that satisfies*
$$L\phi_0 = -\mu_0^2\phi_0, \quad \phi_0 \in H^2(\mathbb{R}). \tag{2.123}$$

Proof. Proof of (1). Clearly L is self-adjoint on $H^2(\mathbb{R})$, so the whole spectrum of L is contained on the real axis. Even more, since $\alpha(x)$ is strictly monotone, positive and $\alpha^{-1}(x) \to \pm\infty$ as $x \to \pm\infty$, we can see from Lemma 3.2 that the associated potential $V(x)$ goes to 0 when $x \to \pm\infty$. This imply by standard arguments (see [15], chapter XIII, section 6) that the essential spectrum of L is $[0, +\infty)$.

Proof of (2). First note that by choosing $\phi = \widetilde{Q}$ we obtain

$$\begin{aligned} L\widetilde{Q} &= -\partial_x^2\widetilde{Q} + 2\widetilde{Q}^3(1-\widetilde{Q}) = \partial_x(\widetilde{Q}^2\widetilde{H}) + 2\widetilde{Q}^3(1-\widetilde{Q}) \\ &= -2\widetilde{Q}^3\widetilde{H}^2 + \frac{1}{3}\widetilde{Q}^4 + 2\widetilde{Q}^3(1-\widetilde{Q}) = -\frac{5}{3}\widetilde{Q}^4, \end{aligned}$$

and then
$$\langle L\widetilde{Q}, \widetilde{Q}\rangle = -\frac{5}{3}\int \widetilde{Q}^5(x)dx = -\frac{5}{3}\int Q^4(y)dy < 0.$$
This conclude that $\sigma_{disc}(L) \cap \mathbb{R}_- \neq \varnothing$.

Proof of (3). First, since L is bounded from below we consider the operator $L_c = L + c$ for a large enough constant $c > 0$ such that the associated potential is strictly positive. Since for any $f \in \mathcal{C}_0^1(\mathbb{R})$ the problem

$$\begin{cases} -L_c v(y) = f(y), & y \in \mathbb{R} \\ \qquad v \in H^2(\mathbb{R}), \end{cases}$$

has a unique solution satisfying $\|v\|_{2,2} \lesssim \|f\|_{1,2}$, it follows that $L_c^{-1} : \mathcal{C}^1(\mathbb{R}) \to \mathcal{C}^1(\mathbb{R})$ is linear compact. From the strong maximum principle theorem if $f \geq 0$ then $v = L_c^{-1}f > 0$ in $\mathbb{R}$. This implies that L_c^{-1} is a strongly positive operator over the set of nonnegative functions. Now it follows from the Krein-Rutman theorem (see [10] [25]) that the radius of the operator $r(L_c^{-1})$ is a positive simple eigenvalue, and the associated eigenfunction f is nonnegative. Thus $\phi_0 = L_c^{-1}f$ satisfies
$$-L\phi_0(x) = \lambda_0\phi_0(x), \quad x \in \mathbb{R}$$
with $\phi_0 > 0$ in $\mathbb{R}$, and $\lambda_0 = r(L_c^{-1}) - c$ a simple eigenvalue. □

Eigenvalues embedded in the continuous spectrum of L depend directly on the decay and oscillation of the potential V. As emphasized in [36, chapter XIII, section 13], the existence of embedded eigenvalues in the continuous spectrum of L depends on detailed assumptions over the decay, symmetry and oscillation of the potential V.

Proposition 10.3. *The operator L has no strictly positive eigenvalues.*

Proof. By Lemma 3.2 we have a polynomial decrease of $V \sim |x|^{-2}$, and even more

$$\int_0^\infty |V(x)|dx = 2\int_0^\infty \widetilde{Q}^2(x)|1-\widetilde{Q}(x)|dx = 2\int_0^\infty Q(s)|1-Q(s)|ds \leq \int_0^\infty Q(s)ds < +\infty.$$

This, and the fact that V is a symmetric function on R, allows us apply a particular case of the Kato-Argmon-Simon Theorem (see [36, Theorem XIII.56]), where we conclude that L has no strictly positive eigenvalues. □

Lemma 10.4. *We have the following bounds for the first negative eigenvalue.*

$$0.808 \leq \mu_0 \leq 0.883.$$

Proof. Recall that

$$\lambda_0 = \inf_{\|f\|_{L^2}=1} (Lf, f).$$

We introduce now the following test function:

$$f(x) := c_0 e^{-\frac{1}{2}x^2}\left(a_4x^4 + a_2x^2 + a_0\right),$$

with

$$a_4 := -0.0574167, \quad a_2 := 0.115416, \quad a_0 := -0.761391.$$

Here, c_0 is an explicit normalizing constant, obtained from

$$1 = \int f^2 = c_0^2\int e^{-x^2}\left(a_4^2x^8 + 2a_4a_2x^6 + (a_2^2+2a_4a_0)x^4 + 2a_2a_0x^2 + a_0^2\right).$$

and the fact that from Wolfram Mathematica,

$$\int e^{-x^2} = \sqrt{\pi}, \quad \int x^2e^{-x^2} = \frac{\sqrt{\pi}}{2}, \quad \int x^4e^{-x^2} = \frac{3\sqrt{\pi}}{4},$$

and

$$\int x^6e^{-x^2} = \frac{15\sqrt{\pi}}{8}, \quad \int x^8e^{-x^2} = \frac{105\sqrt{\pi}}{16}.$$

One can easily see that $c_0 \sim 1.0000005590505727$. Then, since $\alpha(y) = \frac{1}{3}(y+\sinh y)$ is bijection,

$$\begin{aligned}(Lf,f) &= \int f'^2 + 2\int f^2\widetilde{Q}^2(1-\widetilde{Q}) \\ &= \int f'^2(x)dx + 2\int f^2(\alpha(y))Q(1-Q)(y)dy \sim -0.652,\end{aligned}$$

and therefore $\mu_0^2 \geq 0.652$ and $0.808 \leq \mu_0$. In the other sense, if

$$Q_p = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)}\right)^{1/(p-1)}, \quad p = 9/2,$$

we have $L \geq -\partial_x^2 - 0.845 Q_p^{7/2}$, with explicit first eigenfunction Q_p^m, $m = \frac{1}{40}(-35 + \sqrt{4943}) \sim 0.88$ and first eigenvalue $-m^2 \sim -0.7791$, from which $\mu_0 \leq 0.883$. □

Lemma 10.5. *For the operator L, the associated eigenfunction ϕ_0 of the first simple eigenvalue $-\mu_0^2$ satisfies, along with its derivatives, an exponential decay given by*

$$|\phi_0(x)|, |\partial_x \phi_0(x)|, |\partial_x^2 \phi_0(x)| \lesssim e^{-\frac{\sqrt{2}}{2}\mu_0 x} \tag{2.124}$$

Proof. This result follows from a standard argument of ODE (see e.g. [9]) adapted for the particular variable coefficient problem analyzed in this article. For the sake of completeness, we show it here.

By Lemma 10.2 ϕ_0 is a normalized even solution of class $H^1(\mathbb{R})$ associated with the principal eigenvalue $\lambda_0 = -\mu_0^2$ satisfying the equation

$$\partial_x^2 \phi_0 = q(x)\phi_0$$

where $q(x) = \mu_0^2 + V(x)$. In the following we restrict our analysis in the semi-infinite line $[0, +\infty)$ due to the parity of ϕ_0. Since $V \geq 0$ for $x \geq x_r$, with $x_r = \alpha(2 \operatorname{arcosh}(\sqrt{3/2}))$, one has the bound by below

$$q(x) \geq \mu_0^2,$$

for any $x \geq x_r$.

We define $v = \phi_0^2 \geq 0$, which verifies

$$\frac{1}{2}\partial_x^2 v(x) = (\partial_x u)^2(x) + q(x)u^2(x) \geq \mu_0^2 v^2(x),$$

for any $x \geq x_r$.

Now let define the auxiliary function $z = e^{-\sqrt{2}\mu_0 x}(\partial_x v + \sqrt{2}\mu_0 v)$ to compare the decreasing rate of ϕ_0 with respect to an exponential. We have

$$\partial_x z = e^{-\sqrt{2}\mu_0 x}(\partial_x^2 v - 2\mu_0^2 v) \geq 0,$$

hence z is non-decreasing on $[x_r, +\infty)$.

Next, we prove that $z \leq 0$ for $x \geq x_r$ by contradiction: If there exists a $x_0 > x_r$ such that $z(x_0) > 0$, then

$$z(x) \geq z(x_0) > 0,$$

for all $x \geq x_0$. This implies that

$$\partial_x v + \sqrt{2}\mu_0 v \geq z(x_0) e^{\sqrt{2}\mu_0},$$

then $\partial_x v + \sqrt{2}\mu_0 v$ is not integrable on $(x_0, +\infty)$. But $\phi_0 \partial_x \phi_0$ and ϕ_0^2 are integrable on $(x_0, +\infty)$, so that $\partial_x v$ and v are integrable. This is a contradiction, hence we conclude that $z(x) \leq 0$ for $x > x_r$.

In particular, we have the inequality

$$\partial_x(e^{\sqrt{2}\mu_0 x} v) = e^{2\sqrt{2}\mu_0 x} z \leq 0 \quad \text{for } x \geq x_r,$$

This implies that $v(x) \lesssim e^{-\sqrt{2}\mu_0 x}$. Replacing the definition of v, we obtain the decay estimate for the first eigenvalue given by

$$|\phi_0(x)| \lesssim e^{-\frac{\sqrt{2}}{2}x}.$$

To obtain the exponential decay of $\partial_x \phi_0$, we use the trivial bound

$$\mu_0^2 \leq q(x) \leq \mu_0^2 + 1,$$

for all $x > x_r$. Hence, integrating over (x_1, x_2)

$$\mu_0^2 \int_{x_1}^{x_2} \phi_0 \leq \partial_x\phi_0(x_2) - \partial_x\phi_0(x_1) \leq (\mu_0^2 + 1) \int_{x_1}^{x_2} \phi_0,$$

and from the exponential decay of ϕ_0, letting $x_1, x_2 \to +\infty$ proves that $\partial_x\phi_0$ has a limit at infinity. From the exponential decay of ϕ_0, this limit must be zero. Therefore

$$|\partial_x\phi_0(x)| \leq (\mu_0^2 + 1) \int_x^\infty |\phi_0| \lesssim e^{-\frac{\sqrt{2}}{2}\mu_0 x}.$$

Finally, the exponential decay for $\partial_x^2\phi_0$ follows directly from the decay of ϕ_0.

□

Corollary 10.6. *If $\phi_0 : \mathbb{R} \to \mathbb{R}$ is a positive function, then $\phi_0'(x)$ is non-positive for all $x \geq 0$, and has a unique root at 0.*

Proof. First, we denote as $x_0 > 0$ the point where $V(x_0) = -\mu_0^2$.

If $0 < x < x_0$, then integrating equation (2.16) between 0 and x, and by Corollary 11.12 we have

$$\phi'(x) = \int_0^x (\mu_0^2 + V(y))\phi_0(y)dy < 0.$$

If $x > x_0$, we integrate (2.16) and by the decay estimate over ϕ_0' we obtain that

$$\phi_0'(x) = -\int_x^\infty (\mu_0^2 + V(y))\phi_0(y)dy < 0,$$

since ϕ_0 and $\mu_0^2 + V(y)$ are positive for $y \geq x_0$.

□

11. Positivity and repulsivity of the potential V_0

Now, we focus on proving some results related to the transformed problem for the Schrödinger equation for L_0, see subsection 5.2 for details. In particular, the objective of this section is to prove the repulsivity of the potential V_0 (in the sense that $xV_0' \leq 0$ for any x), and its strict repulsivity in a particular subregion of space. Recall that this is one of the most relevant facts needed to apply a virial argument to describe the stability of the kink [36, Theorem XIII.60]. This result becomes subtle due to the lack of an explicit form for the eigenvalue, in contrast to other recent works. See also the cubic-quintic NLS case by Martel [30] and the works [32, 33] for problems in some sense similar to ours. Hence, we must establish some results with an auxiliary function that determines the transformed problem.

11.1. Key properties and positivity

We start out with a fundamental lemma. For this, let ϕ_0 be the positive, even and exponentially decaying eigenfunction satisfying (2.123), and define $h_0 : \mathbb{R}_+ \to \mathbb{R}$ as

$$h_0(x) = \frac{\phi_0'(x)}{\phi_0(x)}. \tag{2.125}$$

Finally, recall L and V from (2.122).

Lemma 11.1. *Let h_0 be as in* (2.125)*. Then one has the following:*

1. *The function h_0 is well defined over $\mathbb{R}_+$. It is non-positive and one can write the principal eigenfunction ϕ_0 of the operator L as follows*

$$\phi_0(x) = \phi_0(0) \exp\left(\int_0^x h_0(y)dy\right). \tag{2.126}$$

2. *The function h_0 is the unique solution of the initial value problem*

$$\begin{cases} h_0'(x) + h_0^2(x) = \mu_0^2 + V(x), & \text{for } x \geq 0, \\ h_0(0) = 0. \end{cases} \tag{2.127}$$

3. *We have the integral formulation*

$$h_0'(x) = -\frac{1}{\phi_0^2(x)} \int_x^\infty V'(y)\phi_0^2(y)dy \tag{2.128}$$

for all $x \geq 0$.

Proof. Proof of (1). By (2.123), the first eigenvalue $-\mu_0^2$ associated with L obey the equation

$$\phi_0''(x) = (\mu_0^2 + V(x))\phi_0. \tag{2.129}$$

From Lemma 10.2, ϕ_0 is the unique positive and even eigenfunction, and it has no roots. From Corollary 10.6 we have that $\phi_0'(x)$ is negative for $x > 0$. This proves that h_0 is well defined over $\mathbb{R}_+$, and even more, by direct integration we have that the identity

$$\phi_0(x) = \phi_0(0) \exp\left(\int_0^x h_0(y)dy\right),$$

is well defined over all $x \in [0, +\infty)$. The extension to any $x \in \mathbb{R}$ is direct.

Proof of (2). This is a direct fact from the parity of h_0 and the eigenvalue equation (2.123) that obeys ϕ_0.

Proof of (3). From (2.129) and the decay estimates (2.124) we have

$$
\begin{aligned}
(\phi_0'(x))^2 &= -\int_x^\infty (\mu_0^2 + V(y))(\phi_0^2)'(y)dy \\
&= (\mu_0^2 + V(x))\phi_0^2(x) + \int_x^\infty V'(y)\phi_0^2(y)dy.
\end{aligned}
$$

Dividing by ϕ_0^2 and by definition of h_0, we obtain

$$
h_0^2(x) = \mu_0^2 + V(x) + \frac{1}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy.
$$

Replacing in (2.127) we have (2.128).

□

Remark 11.2. *The function h_0 is primordial to understand the Darboux transformation applied in Subsection 5.1, since we can write the operators L_0, U, U^* as follows*

$$
L_0 = -\partial_x^2 + 2(h_0^2 - \mu_0^2) - V,
$$

$$
U = \partial_x - h_0, \qquad U^* = -\partial_x - h_0.
$$

Remark 11.3. *Lemma 11.1 also suggests a growing dependence of the sign of h_0' with respect to the potential V'. This fact and the convexity of h_0 will allow us to obtain useful bounds to control the derivative of the transformed potential V_0'.*

Lemma 11.4. *There exist only a unique positive root x_0 of $V(x)$, a unique positive root x_1 of $V'(x)$, and two positive roots $\{x_{2,1}, x_{2,1}\}$ of $V''(x)$. Moreover, $0 < x_{2,1} < x_0 < x_1 < x_{2,2}$ (see also Figure 2.1).*

Remark 11.5. *Explicitly, one has*

$$
\begin{cases} V(x) \le 0 & \text{for } 0 \le x \le x_0, \\ V(x) \ge 0 & \text{for } x \ge x_0. \end{cases} \qquad \begin{cases} V'(x) \ge 0 & \text{for } 0 \le x \le x_1, \\ V'(x) \le 0 & \text{for } x \ge x_1. \end{cases}
$$

$$
\begin{cases} V''(x) \ge 0 & \text{for } 0 \le x \le x_{2,1}, \\ V''(x) \le 0 & \text{for } x_{2,1} \le x \le x_{2,2}, \\ V''(x) \ge 0 & \text{for } x \ge x_{2,2}. \end{cases}
$$

Proof of Lemma 11.4. Since $Q(x)$ is positive, even, decreasing for $x > 0$, and has range $(0, \frac{3}{2})$, we easily see that for $V(x) = 2\widetilde{Q}^2(x)(1 - \widetilde{Q}(x))$, its root $x_0 > 0$ is unique. From (2.10) and (2.21), V' satisfies

$$
\begin{aligned}
V'(x) &= 4\widetilde{Q}(x)\widetilde{Q}'(x) - 6\widetilde{Q}^2(x)\widetilde{Q}'(x) \\
&= 2\widetilde{Q}^2(x)Q'(\alpha^{-1}(x))(2 - 3\widetilde{Q}(x)).
\end{aligned} \tag{2.130}
$$

By the same arguments as before, $x_1 > 0$ is unique. Moreover, $V' > 0$ in $(0, x_1)$ and negative in (x_1, ∞). Notice that $V(x_0) = 2\widetilde{Q}^2(x_0)(1 - \widetilde{Q}(x_0)) = 0$, and since $x_0 > 0$,

$$
V'(x_0) = 2\widetilde{Q}^2(x_0)Q'(\alpha^{-1}(x_0))(2 - 3\widetilde{Q}(x_0)) = -2\widetilde{Q}^3(x_0)Q'(\alpha^{-1}(x_0)) > 0.
$$

Therefore, by uniqueness $x_0 < x_1$. Since also $V'(0) = 0$, one has $0 < x_{2,1} < x_1$, where $x_{2,1} > 0$ is a root of V''. Finally,

$$V''(x) = 8\widetilde{Q}^2(x)Q'^2(\alpha^{-1}(x)) + 4\widetilde{Q}^3(x)Q''(\alpha^{-1}(x)) - 18\widetilde{Q}^3(x)Q'^2(\alpha^{-1}(x)) - 6\widetilde{Q}^4(x)Q''(\alpha^{-1}(x)).$$

Since $Q'' = Q - Q^2$ and $Q'^2 = Q^2 - \frac{2}{3}Q^3$, we obtain

$$V''(x) = 2\widetilde{Q}^4(x)\left(6 - \frac{50}{3}\widetilde{Q}(x) + 9\widetilde{Q}^2(x)\right). \tag{2.131}$$

Notice that $\widetilde{Q} \in (0, \frac{3}{2})$ in $x > 0$. The equation $9m^2 - \frac{50}{3}m + 6 = 0$ has two positive real roots: $m_\pm = \frac{1}{27}(25 \pm \sqrt{139})$, $m_- \sim 0.49$ and $m_+ \sim 1.36$, both below $\frac{3}{2}$. Since α^{-1} is a bijection this implies that V'' has only two positive roots, $x_{2,1}$ and $x_{2,2}$. Let us check that $x_{2,1} < x_0$ and $x_{2,2} > x_1$. Indeed,

$$V''(0) = 2\left(\frac{3}{2}\right)^4\left(6 - \frac{50}{3}\left(\frac{3}{2}\right) + 9\left(\frac{3}{2}\right)^2\right) \sim 12.65, \quad V''(x_0) = -\frac{5}{3} < 0,$$

therefore $x_{2,1}$ first root of V'' must satisfy $x_{2,1} < x_0$. Finally, since $\widetilde{Q}(x_1) = \frac{2}{3}$ and $V'(x_1) = 0$ as unique root, we have

$$V''(x_1) = 2\left(\frac{2}{3}\right)^4\left(6 - \frac{50}{3}\left(\frac{2}{3}\right) + 9\left(\frac{2}{3}\right)^2\right) \sim -0.44,$$

implying that $x_{2,2} > x_1$. The proof is complete. □

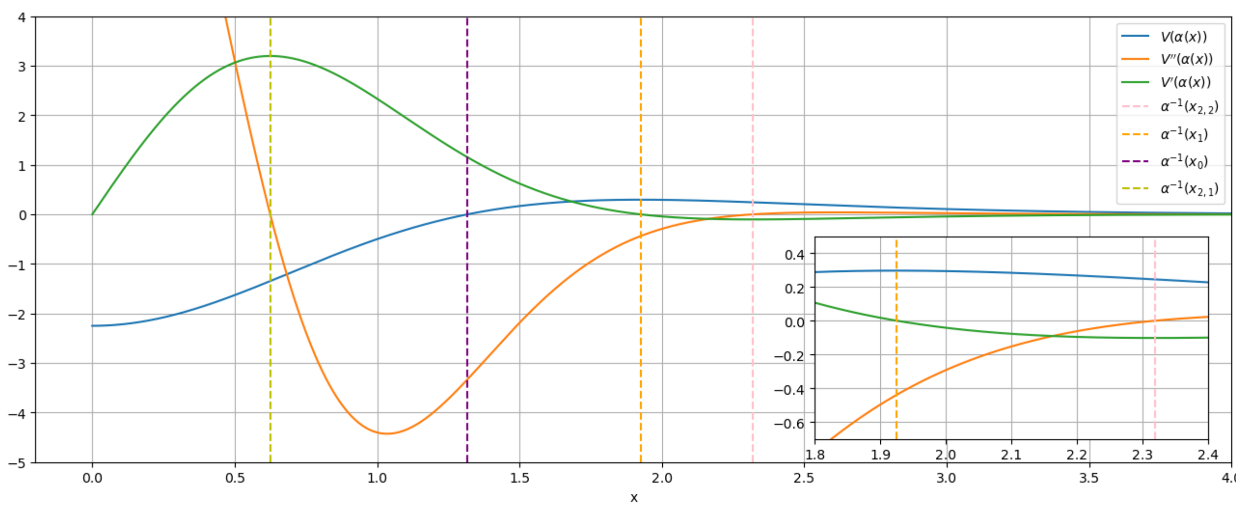

Figure 2.1: Numerical computation of $V(\alpha(x)), V'(\alpha(x)), V''(\alpha(x))$ where their roots are explicitly plotted in dashed vertical lines. In particular we observe that $0 < x_{2,1} < x_0 < x_1 < x_{2,2}$.

Recall that $h_0(x) < 0$ if $x > 0$ (Lemma 11.1).

Lemma 11.6. *If we define*

$$\widetilde{\mu}_0 := \sqrt{\mu_0^2 + \max_{y>0} V(y)}, \quad \max_{y>0} V(y) = \frac{8}{27}, \tag{2.132}$$

the following upper and lower bounds for h_0 are satisfied:

1. *For all $x \geq 0$,*
$$-\tilde{\mu}_0 \leq h_0(x). \tag{2.133}$$

2. *For all $x \geq x_0$,*
$$h_0(x) \leq -\mu_0. \tag{2.134}$$

   In addition, we have the limit

$$\lim_{x\to+\infty} h_0(x) = -\mu_0. \tag{2.135}$$

Proof. Proof of (1). By Lemma 11.4 we know that $V'(x)$ has a unique positive root x_1. Then, by (2.128) and Remark 11.5 we conclude that h_0' is positive for large x and it has at most one positive root. Now, from Lemma 10.4, (2.127), $Q(0) = \widetilde{Q}(0) = \frac{3}{2}$ and (2.122), h_0' satisfies

$$h_0'(0) = \mu_0^2 + V(0) = \mu_0^2 - \frac{9}{4} \sim -1.59.$$

Also, by Remark 11.5, and (2.128) we obtain $h_0'(x_1) > 0$. Therefore there exists a unique positive root of h_0', that we denote $\bar{x}$, with $0 < \bar{x} < x_1$. Moreover, $h_0' < 0$ in $(0, \bar{x})$ and positive in $(\bar{x}, \infty)$. Due to the sign of h_0, $\bar{x}$ must correspond to the global minimum for h_0 in the positive line. With this result, $h_0 \leq 0$ and using (2.127) and (2.132),

$$h_0^2(x) \leq h_0^2(\bar{x}) = \mu_0^2 + V(\bar{x}) \leq \mu_0^2 + \max_{y>0} V(y) = \tilde{\mu}_0^2.$$

This concludes (2.133).

Proof of (2). First, from Remark 11.5, if $x \geq x_1$ then $V(x) > 0$, $V'(x) \leq 0, \phi_0'(x) < 0$, and by (2.127) and (2.128) we have

$$\begin{aligned} \mu_0^2 - h_0^2(x) = &\ -\frac{1}{\phi_0^2(x)} \int_x^\infty V'(y)\phi_0^2(y)dy - V(x) \\ \leq &\ -\int_x^\infty V'(y)dy - V(x) = 0. \end{aligned}$$

Since $h_0(x) \leq 0$, we conclude that $h_0(x) \leq -\mu_0$.

Similarly, from Remark 11.5, if $x_0 \leq x \leq x_1$ we have that $V(x), V'(x) \geq 0$, $\phi_0'(x) < 0$.

Then

$$
\begin{aligned}
\mu_0^2 - h_0^2(x) = & -\frac{1}{\phi_0^2(x)}\int_x^{x_1} V'(y)\phi_0^2(y)dy + \frac{1}{\phi_0^2(x)}\int_{x_1}^{\infty} |V'(y)|\phi_0^2(y)dy - V(x) \\
\leq & -\frac{\phi_0^2(x_1)}{\phi_0^2(x)}\int_x^{x_1} V'(y)dy - \frac{\phi_0^2(x_1)}{\phi_0^2(x)}\int_{x_1}^{\infty} V'(y)dy - V(x) \\
= & -\left(1 - \frac{\phi_0^2(x_1)}{\phi_0^2(x)}\right)V(x) \leq 0.
\end{aligned}
$$

We conclude that $h_0(x) \leq -\mu_0$ for all $x \geq x_0$.

If we consider $x \geq x_1$ we have $V'(x) \geq 0$, and using (2.127) and (2.128) and by triangle inequality we have

$$
\left|h_0^2(x) - \mu_0^2\right| \leq \frac{1}{\phi_0^2(x)}\int_x^{\infty} V'(y)\phi_0^2(y)dy + |V(x)| \leq 2|V(x)|.
$$

Taking x to infinity in this last inequality, we obtain (2.135). □

We will need a refined version of the previous result. The next lemma will be used to obtain better bounds for h_0 in the interval $(0, x_0)$.

Lemma 11.7. *For all $x \geq 0$, one has*

$$
(\mu_0^2 - \tilde{\mu}_0^2)x - 2\widetilde{Q}(x)\widetilde{H}(x) \leq h_0(x) \leq \mu_0^2 x - 2\widetilde{Q}(x)\widetilde{H}(x), \tag{2.136}
$$

where $\tilde{\mu}_0$ is defined in (2.132)*, and $\widetilde{H}$ is the modified version by α^{-1} of the kink H satisfying* (2.6)*. Even more,*

$$
\mu_0^2 x - R(x) \leq h_0 \qquad \text{for all } x > 0, \tag{2.137}
$$

where we define the auxiliary function

$$
R(x) := 2\ln\left(\frac{3}{2}\right) - 2\ln\left(\widetilde{Q}\right) + 2\widetilde{Q}\widetilde{H} + \frac{\mu_0^2 - \tilde{\mu}_0^2}{2}x^2.
$$

Proof. First, we consider the initial value problem:

$$
\begin{cases} h_1' = \mu_0^2 + V \\ h_1(0) = 0. \end{cases} \tag{2.138}
$$

Using (2.21), and a change of variables, we have

$$
\begin{aligned}
\int_0^x V(y)dy = 2\int_0^x \widetilde{Q}^2(y)(1 - \widetilde{Q}(y))dy &= 2\int_0^{\alpha^{-1}(x)} Q(s)(1 - Q(s))ds \\
&= 2\int_0^{\alpha^{-1}(x)} Q''(s)ds = 2Q'(\alpha^{-1}(x)) \\
&= -2\widetilde{Q}(x)\widetilde{H}(x).
\end{aligned}
$$

Then, the explicit solution for (2.138) problem is given by

$$h_1(x) = \int_0^x (\mu_0^2 + V(y))dy = \mu_0^2 x + \int_0^x V(y)dy = \mu_0^2 x - 2\widetilde{Q}(x)\widetilde{H}(x).$$

Notice that $h_1(0) = h_0(0) = 0$, and from (2.138) one has $h_0'(x) \leq h_1'(x)$ for all $x \geq 0$. Thus, the inequality

$$h_0(x) \leq \mu_0^2 x - 2\widetilde{Q}(x)\widetilde{H}(x),$$

holds for all $x \geq 0$. This proves the upper bound in (2.136).

Second, we consider the initial value problem:

$$\begin{cases} h_2' = \mu_0^2 - \widetilde{\mu}_0^2 + V \\ h_2(0) = 0. \end{cases}$$

The explicit solution for this problem is given by

$$\begin{aligned} h_2(x) = & \int_0^x (\mu_0^2 - \widetilde{\mu}_0^2 + V(y))dy = (\mu_0^2 - \widetilde{\mu}_0^2)x + \int_0^x V(y)dy \\ & = (\mu_0^2 - \widetilde{\mu}_0^2)x - 2\widetilde{Q}(x)\widetilde{H}(x). \end{aligned}$$

Using (2.133), one has

$$h_2'(x) \leq \mu_0^2 + V(x) - h_0^2(x) = h_0'(x),$$

for all $x \geq 0$. Since $h_2(0) = h_0(0) = 0$, this implies that $h_2 \leq h_0$. Hence,

$$(\mu_0^2 - \widetilde{\mu}_0^2)x - 2\widetilde{Q}(x)\widetilde{H}(x) \leq h_0(x)$$

for all $x \geq 0$, obtaining the lower bound in (2.136).

We notice that we can improve this bound analogously. If we consider the initial value problem

$$\begin{cases} h_3' = \mu_0^2 - h_2^2 + V \\ h_3(0) = 0, \end{cases}$$

the explicit solution is given by

$$h_3(x) = \mu_0^2 x - 2\ln\left(\frac{3}{2}\right) + 2\ln\left(\widetilde{Q}\right) - 2\widetilde{Q}\widetilde{H} - \frac{\mu_0^2 - \widetilde{\mu}_0^2}{2}x^2.$$

Since $h_3'(x) \leq h_0'(x)$ for all $x > 0$, and $h_3(0) = h_0(0) = 0$, we conclude that $h_3 \leq h_0$, and this proves (2.137). □

Now, we are in condition to obtain estimates for h_0 in the interval $(0, x_0)$ in the next lemma, useful for the proof of repulsivity in the transformed potential.

Lemma 11.8. *One has the following properties:*

1. *For $0 \leq x \leq x_{2,1}$ we have*

$$\frac{4}{3}\left(\mu_0^2 - \frac{9}{4}\right)\widetilde{H}(x) \leq h_0(x). \tag{2.139}$$

2. For all x such that $x_{2,1} \le x \le x_0$,

$$(\mu_0^2 - \widetilde{\mu}_0^2)(x - x_0) - \widetilde{\mu}_0 \le h_0(x) \le -\frac{\mu_0}{x_0}x. \tag{2.140}$$

Proof. Proof of (1). We define the auxiliary function

$$g(x) = h_0(x) - \frac{4}{3}\left(\mu_0^2 - \frac{9}{4}\right)\widetilde{H}(x). \tag{2.141}$$

By direct calculation, we obtain $g(0) = g'(0) = 0$, and by the mean value theorem,

$$g(x) = g'(\xi)x, \tag{2.142}$$

for some $\xi \in (0, x)$. Thus, to prove the positivity of g for $0 \le x \le x_{2,1}$, it is enough to study the sign of g'. Deriving g, and using (2.127), (2.10), (2.21), one has that proving $g' \ge 0$ is equivalent to prove

$$h_0^2 \le \mu_0^2 + V - \frac{4}{9}\left(\mu_0^2 - \frac{9}{4}\right)\widetilde{Q}^2. \tag{2.143}$$

for $0 \le x \le x_{2,1}$.

Using (2.137) and Lemma 10.4 we have that

$$h_0^2 \le \mu_0^4 x^2 - 2\mu_0^2 x R(x) + R^2(x). \tag{2.144}$$

The RHS of this last equation is explicit except for μ_0, so comparing both RHSs of (2.143) and (2.144), one wants to prove the following,

$$\mu_0^4 x^2 - 2\mu_0^2 x R(x) + R^2(x) \le \mu_0^2 + V - \frac{4}{9}\left(\mu_0^2 - \frac{9}{4}\right)\widetilde{Q}^2,$$

equivalent to prove

$$\mu_0^4 x^2 + \left(\frac{4}{9}\widetilde{Q}^2 - 2xR(x) - 1\right)\mu_0^2 \le V + \widetilde{Q}^2 - R^2 \tag{2.145}$$

for all $0 \le x \le x_{2,1}$.

Now, applying Lemma 10.4, one has

$$\mu_0^4 x^2 + \left(\frac{4}{9}\widetilde{Q}^2 - 2xR(x) - 1\right)\mu_0^2 \le G(\alpha^{-1}(x)), \tag{2.146}$$

where

$$G(s) := (0.883)^4\alpha(s)^2 + \left(\frac{4}{9}\widetilde{Q}^2 - 2\alpha(s)R(\alpha(s)) - 1\right)(0.808)^2$$

is given by explicit functions. Combining these last inequalities, we obtain

$$G(s) \le V(\alpha(s)) + Q^2(s) - R^2(\alpha(s)), \tag{2.147}$$

for $0 \le s \le \alpha^{-1}(x_{2,1})$ (see Figure 2.2).

Replacing (2.147) into (2.146) we obtain (2.145), and we conclude via (2.144) that $g'(x) \ge$

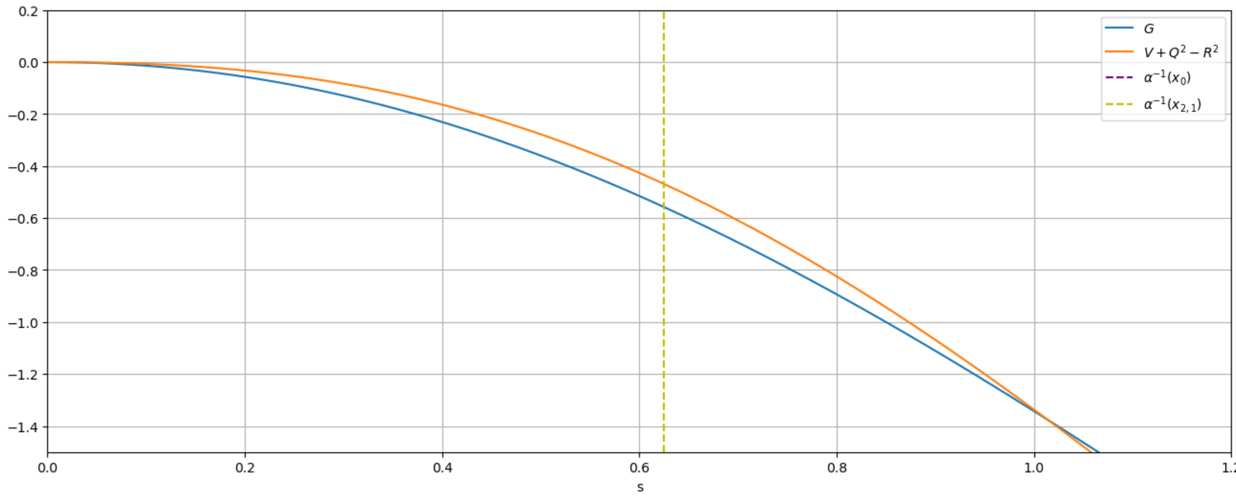

Figure 2.2: Numerical computation of auxiliary functions $G(s)$ and $V(\alpha(s)) + Q^2(s) - R^2(\alpha(s)$. In particular we observe that $G \leq V + Q^2 + R$ for $x \in (0, x_{2,1})$.

0. This proves that g is a positive function for $0 \leq x \leq x_{2,1}$. Hence, by (2.141) (2.142) we conclude (2.133).

Proof of (2). We claim that h_0 is a convex function for $x \in (0, x_0)$. First, from the proof of Lemma 11.6 we know that h_0' has a unique root denoted by $\bar{x}$, with $h_0'(x) < 0$ in $(0, \bar{x})$ and negative sign in $(\overline{x}, \infty)$. Now using that $V(x_0) = 0$, (2.134), and (2.127), we have

$$h_0'(x_0) = \mu_0^2 - h_0^2(x_0) \leq \mu_0^2 - \mu_0^2 = 0.$$

This implies that h_0' is negative in $(0, x_0)$.

In addition, if $x \in (x_{2,1}, x_0)$ we know from (2.133) that $-\widetilde{\mu}_0 \leq h_0$. Hence, replacing in (2.127), we obtain

$$\mu_0^2 - \widetilde{\mu}_0^2 + V(x) \leq \mu_0^2 - h_0^2 + V = h_0'(x) \leq 0. \tag{2.148}$$

Taking derivative in (2.127), using that $h_0', h_0 \leq 0$, the lower bounds from (2.133) (2.137) and (2.148),

$$\begin{aligned} h_0'' &= V' - 2h_0 h_0' \geq V' - 2(\mu_0^2 x - R(x))h_0' \\ &\geq V' - 2(\mu_0^2 x - R(x))(\mu_0^2 - \widetilde{\mu}_0^2 + V) \\ &\geq V' - 2(0.808^2 x - R(x)) \left(-\frac{8}{27} + V \right) \\ &=: j_1(\alpha^{-1}(x)). \end{aligned}$$

where j_1 is obtained employing Lemma 10.4. Computing this function, we have that $j_1(s) > 0$ for all $s \in (\alpha^{-1}(x_{2,1}), \alpha^{-1}(x_0))$ (see Fig. 2.3). Hence, by bijectivity of $\alpha : \mathbb{R} \to \mathbb{R}$, we conclude

$$h_0''(x) \geq j_1(\alpha^{-1}(x)) > 0,$$

for all $x \in (x_{2,1}, x_0)$. This proves the convexity of $h_0(x)$ over $(x_{2,1}, x_0)$. Using (2.127), (2.133), if $x_{2,1} \leq x \leq x_0$, by definition of convexity,

$$\begin{aligned} h_0(x) &\geq h_0'(x_0)(x - x_1) + h_0(x_0) \\ &= (\mu_0^2 - h_0^2(x_0))(x - x_0) + h_0(x_0) \geq (\mu_0^2 - \widetilde{\mu}_0^2)(x - x_0) - \widetilde{\mu}_0. \end{aligned}$$

This proves the lower bound in (2.140).

If now $0 \leq x \leq x_{2,1}$, using that $h_0', h_0 \leq 0$, $V' \geq 0$, (2.139) and (2.127) we have the following set of inequalities

$$\begin{aligned} h_0'' &= V' - 2h_0 h_0' \\ &\geq V' - 2\left(\frac{4\mu_0^2 - 9}{3}\right)\left(\mu_0^2 + V - \left(\frac{4\mu_0^2 - 9}{3}\right)^2 \widetilde{H}^2\right)\widetilde{H} \\ &\geq V' - 2\left(\frac{4(0.808)^2 - 9}{3}\right)\left((0.808)^2 + V - \left(\frac{4(0.808)^2 - 9}{3}\right)^2 \widetilde{H}^2\right)\widetilde{H} \\ &:= j_2(\alpha^{-1}(x)). \end{aligned}$$

Replacing directly V, V' and considering the variable $s = \alpha^{-1}(x)$, we obtain

$$\begin{aligned} j_2(s) &= 2Q^3 H(3Q - 2) \\ &\quad - 2\left(\frac{4(0.808)^2 - 9}{3}\right)\left((0.808)^2 - \left(\frac{4(0.808)^2 - 9}{3}\right)^2 + \frac{2}{3}\left(\frac{4(0.808)^2 - 9}{3}\right)^2 Q + 2Q^2(1 - Q)\right) \\ &= -2\left(\frac{4(0.808)^2 - 9}{3}\right)\left(\mu_0^2 - \left(\frac{4(0.808)^2 - 9}{3}\right)^2 - \frac{4}{3}\left(\frac{4(0.808)^2 - 9}{3}\right)\right)^3 Q \\ &\quad - 4\left(\frac{4(0.808)^2 - 9}{3}\right) Q^2 + 4\left(\frac{4(0.808)^2 - 9}{3} - H\right) Q^3 + 6HQ^4. \end{aligned}$$

This last expression is bounded employing Lemma 10.4. Computing (see Fig. 2.3), we have that $j_2(s) > 0$ for all $s \in (0, \alpha^{-1}(x_{2,1}))$. Hence, by bijectivity of α, we conclude $h_0''(x) > 0$ for all $x \in (0, x_{2,1})$.

This proves the convexity of h_0 over $(0, x_0)$, and it is enough to conclude (2.140). Indeed, using convexity between $(0, h_0(0))$ and $(x_0, h_0(x_0))$, and (2.134), we have

$$h_0(x) \leq \frac{h_0(x_0)}{x_0} x \leq -\frac{\mu_0}{x_0} x.$$

This proves the upper bound in (2.137), where $x_0 < x < x_1$ and we conclude the proof of Lemma 11.6. $\square$

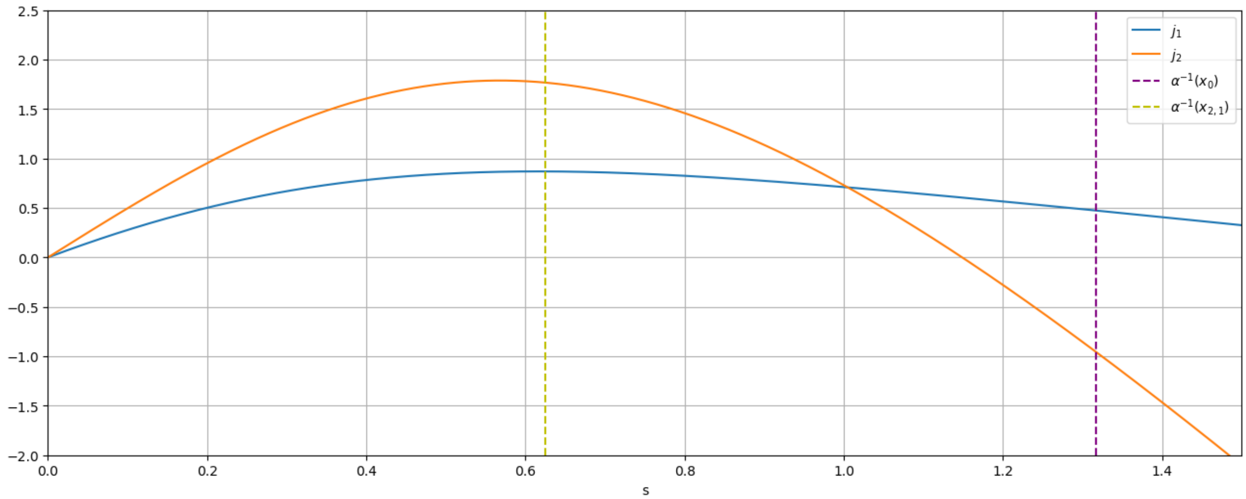

Figure 2.3: Numerical computation of $j_1(s)$, lower bound for h_0'' for s in $(x_{2,1}, x_0)$, and $j_2(s)$, lower bound for s in $(0, x_{2,1})$.

11.2. Positivity

Now, employing the estimates over h_0 in the previous subsection and the integral form of h_0', we are in position to deal with the sign of V_0.

Lemma 11.9. *The potential V_0 is non-negative over the real line. In particular L_0 has a positive first eigenvalue and positive spectrum.*

Proof. To prove the positivity of V_0, first we will obtain a convenient formulation of the potential in terms of an integral. By definition of V_0 and (2.128) we have

$$V_0(x) = V(x) + \frac{2}{\phi_0^2(x)} \int_x^\infty V'(y)\phi_0^2(y)dy.$$

Integrating by parts to eliminate the potential V on the right hand side, and using (2.126), we obtain

$$\begin{aligned} V_0(x) &= \frac{1}{\phi_0^2(x)} \int_x^\infty V'(y)\phi_0^2(y)dy - \frac{1}{\phi_0^2(x)} \int_x^\infty V(y)(\phi_0^2(y))'dy \\ &= \frac{1}{\phi_0^2(x)} \int_x^\infty [V'(y) - 2h_0(y)V(y)]\phi_0^2(y)dy. \end{aligned}$$

Thus, we have the integral formulation of V_0,

$$V_0(x) = \frac{1}{\phi_0^2(x)} \int_x^\infty K(y)\phi_0^2(y)dy,$$

where we have defined

$$K(y) := V'(y) - 2h_0(y)V(y).$$

We will prove the positivity of $K(y)$ for all $y \geq 0$.

For $y \geq x_0$ this is straightforward, since we know that $V(y), V'(y) \geq 0$ and $h_0(y) < 0$, then $K(y)$ must be non-negative.

For $x_{2,1} \le y \le x_0$, we know that $V(y), h_0(y) \le 0$. Using the bound (2.137) for $h_0(y)$, using Lemma 10.4, and replacing directly V', V, we have

$$\begin{aligned} K(y) &= V' - 2h_0 V \ge V' - 2(\mu_0^2 x - R)V \ge V' - 2(0.808^2 x - R)V \\ &= 2\widetilde{Q}^2[2(0.808^2 x - R) - 2((0.808^2 x - R) + \widetilde{H})\widetilde{Q} + 3\widetilde{H}\widetilde{Q}^2] =: 2\widetilde{Q}^2 k_1(\alpha^{-1}(y)). \end{aligned}$$

We recall that the function k_1 is explicitly known employing Lemma 10.4. Computing this, we have that $k_1(s) > 0$ for all $s \in (\alpha^{-1}(x_{2,1}), \alpha^{-1}(x_0))$ (see Fig. 2.4). Hence, by bijectivity of α, we conclude $K(y) > 0$ for all $y \in (x_{2,1}, x_0)$.

For $0 \le y \le x_{2,1}$ we just consider the bound (2.139) for h_0 instead of (2.133). Then we proceed analogously:

$$\begin{aligned} K(y) &\ge V' + \frac{8}{3}\left(\mu_0^2 - \frac{9}{4}\right)\widetilde{H}V \\ &= 2\widetilde{Q}^2\widetilde{H}\left[\frac{2}{3}(4\mu_0^2 - 9) + 2\left(1 - \frac{4}{3}\mu_0^2\right)\widetilde{Q} + 3\widetilde{Q}^2\right] \\ &\ge 4\widetilde{Q}^2\widetilde{H}\left[\frac{1}{3}(4(0.808)^2 - 9) + \left(1 - \frac{4}{3}(0.883)^2\right)\widetilde{Q} + \frac{3}{2}\widetilde{Q}^2\right] =: 4\widetilde{Q}^2\widetilde{H}k_2(\alpha^{-1}(y)), \end{aligned}$$

where k_2 is explicitly known employing Lemma 10.4. Computing, we have that $k_2(s) > 0$ for all $s \in (0, \alpha^{-1}(x_{2,1}))$ (see Fig. 2.4). Hence, by bijectivity of α, we conclude $K(y) > 0$ for all $y \in (0, x_{2,1})$. □

One of the most crucial properties about L for our analysis of the stability of the kink is that it possesses only one negative eigenvalue.

Corollary 11.10. *The operator L has a unique negative eigenvalue $-\mu_0^2 < 0$ of multiplicity one.*

Remark 11.11. *Corollary 11.10 shows the unstable character of the kink solution H, under which the asymptotic stability could only hold if one already has orbital stability.*

Proof. This is just a consequence of removing the first eigenvalue once we obtain the transformed super-symmetric partner operator L_0. We recall the following decomposition

$$L = (-\partial_x - h_0)(\partial_x + h_0) - \mu_0^2 = U^*U - \mu_0^2,$$

and changing the order of the operators U and U^*, we define

$$L_0 = (\partial_x + h_0)(-\partial_x - h_0) - \mu_0^2 = UU^* - \mu_0^2, \tag{2.149}$$

obtaining the super-symmetric relation

$$UL = L_0 U \tag{2.150}$$

which is, by construction, isospectral to L except for $\lambda = -\mu_0$. This is, we claim

$$\sigma_p(L_0) = \sigma_p(L) \setminus \{-\mu_0^2\}.$$

Let $\lambda \neq -\mu_0^2$ be an eigenvalue of L, with the corresponding eigenfunction ϕ. Then, by equation (2.150) we get $L_0(U\phi) = \lambda U\phi$. Since by Lemma 10.2 $\lambda_0 = -\mu_0^2$ is a simple eigenvalue, we have that $U\phi \not\equiv 0$. This proves that $\sigma_p(L) \setminus \{-\mu_0^2\} \subseteq \sigma_p(L_0)$. For the reversed inclusion, we only need to prove that $-\mu_0^2 \notin \sigma_p(L_0)$, since for the rest we could repeat the same procedure as above, but relative to the eigenvalues of L_0. By contradiction, we assume that there exists some $\varphi \in L^2(\mathbb{R})$ such that $L_0\varphi = -\mu_0^2\varphi$. Then, by (2.149), we obtain $UU^*\varphi = 0$, and using that $\operatorname{ran}(U^*) \perp \ker(U)$ we have that $U^*\varphi = 0$, which implies that $\varphi = \phi_0^{-1}$, which is a contradiction since $g \in L^2(\mathbb{R})$.

By Lemma 11.9 we conclude that L_0 has no negative eigenvalues, and from the above we conclude that $-\mu_0^2$ is the unique negative eigenvalue associated with the operator L. □

Corollary 11.12. *Given ϕ_0 eigenfunction associated with the unique negative eigenvalue $-\mu_0^2$, then ϕ_0 is an even function and $\partial_x\phi_0$ is odd.*

Proof. The parity follows from the fact that L is invariant over the reflection $x \to -x$, so the eigenfunctions are even or odd, and since ϕ_0 is positive in the real line we conclude it is even. Since λ_0 is the unique negative eigenvalue of multiplicity one, ϕ_0 is unique, even, and $\partial_x\phi_0$ is odd. □

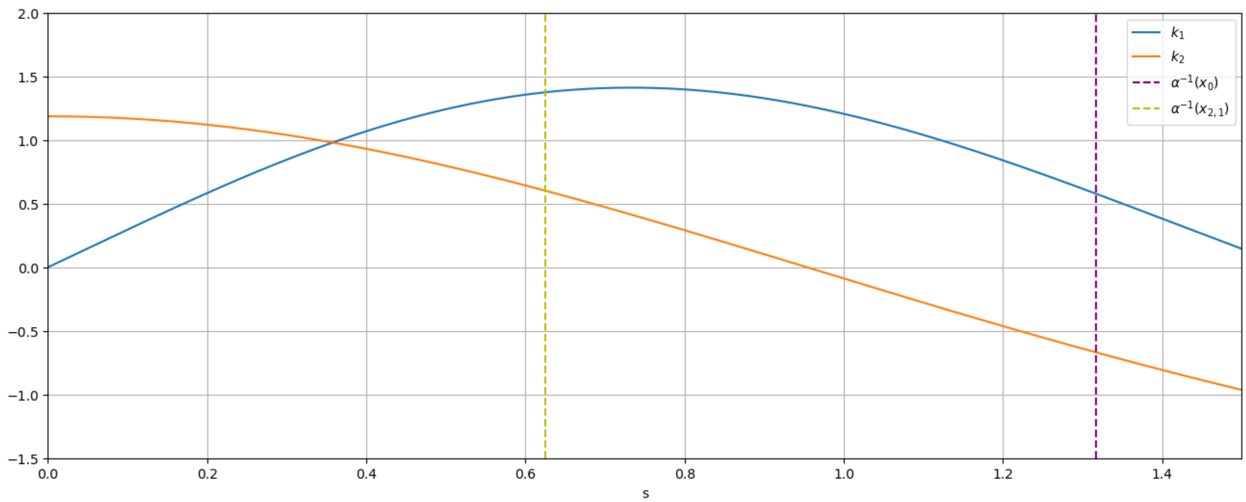

Figure 2.4: Numerical computation of $k_1(s)$, lower bound for $K(\alpha^{-1}(s))$ with s in $(x_{2,1}, x_0)$, and $k_2(s)$, lower bound for $K(\alpha^{-1}(s))$ with s in $(0, x_{2,1})$.

11.3. Repulsivity.

Lemma 11.13. *The derivative of the transformed potential $V_0'(x)$ is odd and negative for any $x \neq 0$. In particular, L_0 is repulsive.*

The rest of this section is devoted to prove Lemma 11.13.

11.3.1. An integral formula.

By (2.126) we have that $(\phi_0^2)' = 2h_0\phi_0^2$. Using this, the definition of V_0 in (2.57) and h_0, (2.128), and integration by parts, we get

$$
\begin{aligned}
V_0'(x) &= 4h_0(x)h_0'(x) - V'(x) \\
&= -\frac{2h_0(x)}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy - \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \frac{V'(y)}{h_0(y)}(\phi_0^2(y))'dy - V'(x) \\
&= -\frac{2h_0(x)}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy + \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(\frac{V'(y)}{h_0(y)}\right)'\phi_0^2(y)dy \\
&\quad - \left.\frac{h_0(x)V'(y)\phi_0^2(y)}{\phi_0^2(x)h_0(y)}\right|_x^\infty - V'(x) \\
&= \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(\frac{V'(y)}{h_0(y)} - 2V(y)\right)'\phi_0^2(y)dy \\
&= \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(\frac{V''(y)}{h_0(y)} - \frac{V'(y)h_0'(y)}{h_0^2(y)} - 2V'(y)\right)\phi_0^2(y)dy \\
&= \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(V''(y)h_0(y) - V'(y)h_0'(y) - 2V'(y)h_0^2(y)\right)\left(\frac{\phi_0}{h_0}\right)^2(y)dy,
\end{aligned}
$$

Thus, we have the equivalent formulation

$$
V_0'(x) = \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty I(y)\left(\frac{\phi_0}{h_0}\right)^2(y)dy, \tag{2.151}
$$

where, using equation (2.127), we have

$$
I(y) = V''(y)h_0(y) - V'(y)(h_0^2(y) + \mu_0^2 + V(y)). \tag{2.152}
$$

Due to the dependence of this expression on the sign of the potential and its derivatives, we will divide the proof depending on the roots $\{x_0, x_1, x_{2,1}, x_{2,2}\}$ (see Lemma 11.4).

To prove that V_0' is non positive, we restrict our analysis to the interval $(0,\infty)$ by parity. We will prove the positivity of $I(y)$ for all $y \geq 0$ by separate cases.

11.3.2. Positivity for $x_1 \leq y < \infty$.

Firstly, we consider the case $y \geq x_{2,2}$. Then Remark 11.5 ensures that $V(y), V''(y) \geq 0$, $V'(y) \leq 0$. We apply in (2.152) the bounds (2.133) and (2.134) for h_0:

$$
\begin{aligned}
I(y) = &\; - V''(y)|h_0(y)| + |V'(y)|(h_0^2(y) + \mu_0^2 + V(y)) \\
\geq &\; - \tilde{\mu}_0 V''(y) + |V'(y)|\left(2\mu_0^2 + V(y)\right) \\
\geq &\; - 1.038V''(y) + (2\cdot 0.808^2 + V(y))|V'(y)|.
\end{aligned}
$$

Replacing directly V, V', V'' and considering the variable $s = \alpha^{-1}(y)$, we obtain

$$\begin{aligned}
I(\alpha(s)) \geq & -2.075Q^4\left(6-\frac{50}{3}Q+9Q^2\right)+4(2-3Q)(0.652+Q^2-Q^3)Q^3H \\
&= 2Q^3\left[2.611-6(1.038+0.652H)Q+\left(\frac{50}{3}1.038+4H\right)Q^2-(9.342+20H)Q^3+6Q^4H\right] \\
&=: 2Q^3 i_1(s).
\end{aligned}$$

By the exponential decay of Q, we obtain explicitly via computation that $i_1(s) > 0$ for all $s \geq \alpha^{-1}(x_{2,2})$ (see Fig. 2.5). Hence, we conclude $I(y) > 0$ for all $y \geq x_{2,2}$ by the bijection of $\alpha : \mathbb{R} \to \mathbb{R}$.

If now $x_1 \leq y \leq x_{2,2}$, then $V(y) \geq 0$, $V'(y), V''(y) \leq 0$, applying (2.133), (2.134), and Lemma 10.4, replacing V, V' and V'',

$$\begin{aligned}
I(y) &= |V''(y)h_0| + |V'(y)|\left(h_0^2(y)+\mu_0^2+V(y)\right) \geq \mu_0|V''(y)| + |V'(y)|\left(2\mu_0^2+V(y)\right) \\
&\geq 0.808|V''(y)| + |V'(y)|\left(2\cdot 0.808^2+V(y)\right).
\end{aligned}$$

Again, replacing V, V', V'' and considering the variable $s = \alpha^{-1}(y)$, we obtain

$$\begin{aligned}
I(\alpha(s)) = & -2\mu_0 Q^4\left(6-\frac{50}{3}Q+9Q^2\right)+4(2-3Q)(0.808^2+Q^2-Q^3)Q^3H \\
&= 2Q^3H\left[4\cdot 0.808^2H-6\cdot 0.808(1+0.808H)Q+\left(\frac{50}{3}\cdot 0.808+4H\right)Q^2\right. \\
&\qquad \left.-(10H+9\cdot 0.808)Q^3+6HQ^4\right] \\
&=: 2Q^3H i_2(s),
\end{aligned}$$

where $\hat{k}(s)$ is explicitly known employing Lemma 10.4. Computing this function, we have that $i_2(s) > 0$ for all $s \in (\alpha^{-1}(x_1), \alpha^{-1}(x_{2,2}))$ (see Fig. 2.5). Hence, by bijectivity of α, we conclude $I(y) > 0$ for all $y \in (x_1, x_{2,2})$.

11.3.3. Positivity for $x_0 \leq y < x_1$.

In this case $V(y), V'(y) \geq 0$, and $V''(y) \leq 0$. This, combined with inequalities (2.134), (2.133) gives us that I satisfies the following inequality for all $y \in [x_0, x_1]$:

$$\begin{aligned}
I(y) &= |V''(y)h_0(y)| - V'(y)\left(h_0^2(y)+\mu_0^2+V(y)\right) \\
&\geq \mu_0|V''(y)| - V'(y)\left(\tilde{\mu}_0^2+\mu_0^2+V(y)\right) \\
&\geq 0.808|V''(y)| - V'(y)\left(1.959+V(y)\right).
\end{aligned}$$

Replacing V, V', V'' and considering the variable $s = \alpha^{-1}(y)$, we obtain

$$\begin{aligned} I(\alpha(s)) \geq & -2 \cdot 0.808Q^4 \left(6 - \frac{50}{3}Q + 9Q^2\right) + 2(2-3Q)(1.959 + 2Q^2 - 2Q^3)Q^3 H \\ &= 2Q^3 \Big[3.842H - 3(1.959 + 1.616H)Q + \left(\frac{50}{3} \cdot 0.808 + 4H\right) Q^2 \\ &\qquad - (7.272 + 10H)Q^3 + 6HQ^4\Big] \\ &=: 2Q^3 i_3(s), \end{aligned}$$

where $i_3(s)$ is explicitly known employing Lemma 10.4. Computing this function, we have that $i_3(s) > 0$ for all $s \in (\alpha^{-1}(x_0), \alpha^{-1}(x_1))$ (see Fig. 2.5). Hence, by bijectivity of α, we conclude $I(y) > 0$ for all $y \in (x_0, x_1)$.

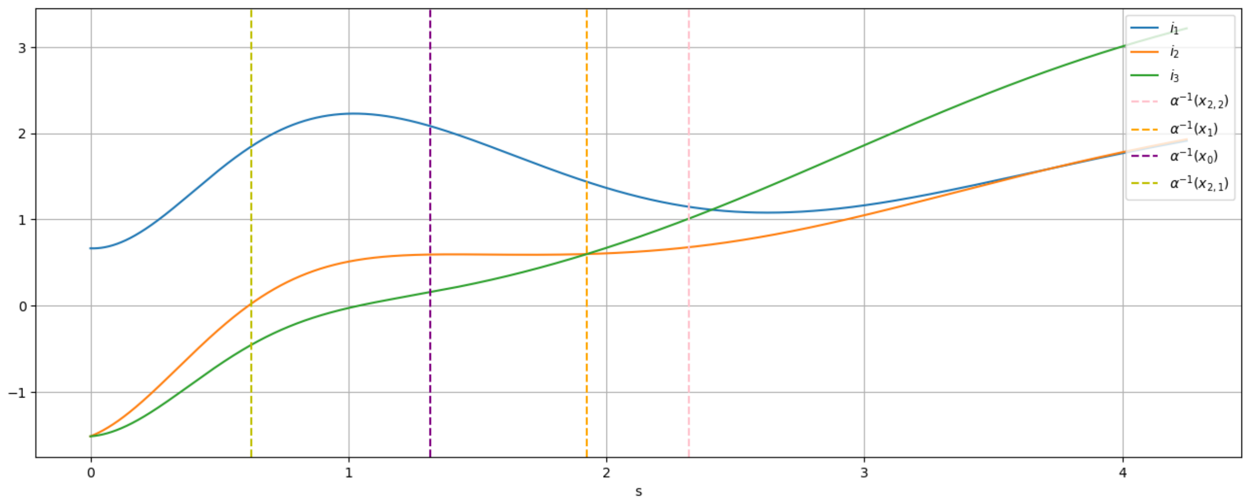

Figure 2.5: Numerical computation of the bounds for $I(\alpha(x))$ in the intervals $(\alpha^{-1}(x_0), \alpha^{-1}(x_1))$, $(\alpha^{-1}(x_1), \alpha^{-1}(x_{2,2}))$, and $(\alpha^{-1}(x_{2,2}), \infty)$.

11.3.4. Positivity for $x_{2,1} \leq y < x_0$.

If y is a positive real number such that $x_{2,1} \leq y < x_0$, then $V(y), V''(y) \leq 0$, $V'(y) \geq 0$. We separate the study in two cases.

Case 1. If $h_0^2(y) + \mu_0^2 + V(y) \leq 0$, directly by the sign of the expression in (2.152)

$$I(y) = |V''(y)h_0(y)| + |V'(y)(h_0^2(y) + \mu_0^2 + V(y))| \geq 0.$$

Case 2. On the other hand, if $h_0^2(y) + \mu_0^2 + V(y) \geq 0$, by (2.140) and Lemma 10.4 we know

$$h_0^2(y)+\mu_0^2+V(y) \geq \left(\frac{8}{27}(x-x_0)+\tilde{\mu}_0\right)^2+\mu_0^2+V(y) \geq \left(\frac{8}{27}(x-x_0)+0.974\right)^2+0.652+V(y).$$

Hence, using (2.140) and the above estimate to bound by below (2.152),

$$I(y) \geq -\frac{\mu_0}{x_0} y V''(y) - V'(y) \left(\left(\frac{8}{27}(y - x_0) + 0.974 \right)^2 + 0.652 + V(y) \right).$$

Replacing V, V', V'' and considering the variable $s = \alpha^{-1}(y)$, we obtain

$$\begin{aligned} I(\alpha(s)) \geq & -2\frac{0.808}{x_0}\alpha(s)Q^4 \left(6 - \frac{50}{3}Q + 9Q^2\right) \\ & + 2(2-3Q)Q^3 H \left(\left(\frac{8}{27}(\alpha(s) - x_0) + 0.974 \right)^2 + 0.652 + 2Q^2(1-Q) \right) \\ =:& \ m(s), \end{aligned}$$

where $m(s)$ is explicitly known employing Lemma 10.4. Computing this function, we have that $m(s) > 0$ for all $s \in (\alpha^{-1}(x_{2,1}), \alpha^{-1}(x_0))$ (see Fig. 2.6). Hence, by bijectivity of α, we conclude $I(y) > 0$ for all $y \in (x_{2,1}, x_0)$.

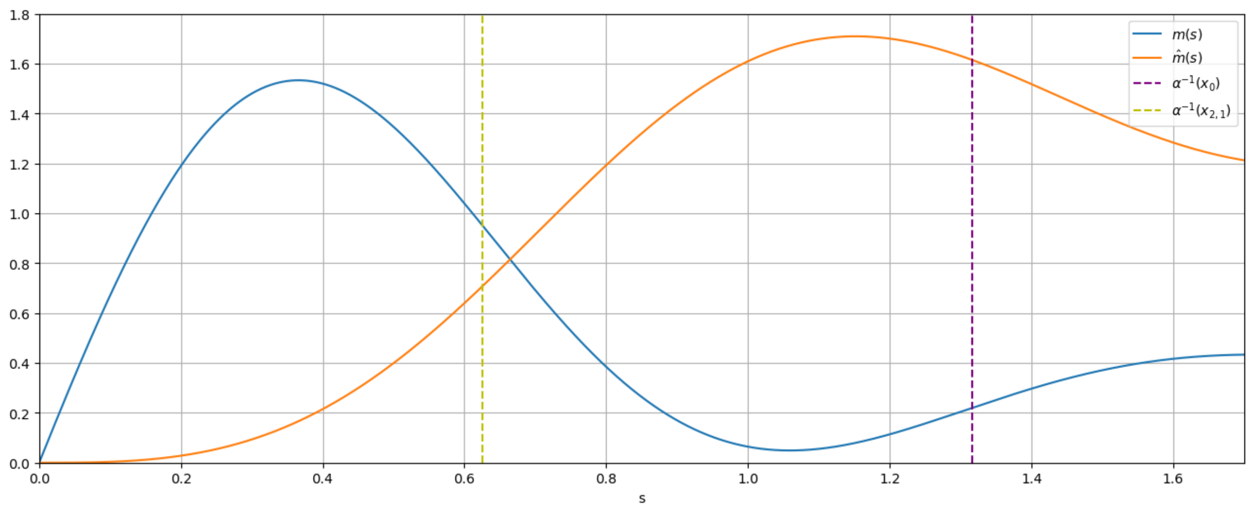

Figure 2.6: Numerical computation of the bounds for $I(\alpha(x))$ in the intervals $(0, \alpha^{-1}(x_{2,1}))$ and $(\alpha^{-1}(x_{2,1}), \alpha^{-1}(x_0))$.

11.3.5. Positivity for $0 \leq y < x_{2,1}$.

Finally, for this case $V(y) \leq 0$, $V'(y), V''(y) \geq 0$, and using (2.139) we obtain

$$h_0^2(y) + \mu_0^2 + V(y) \leq \left(\mu_0^2 - \frac{9}{4}\right)^2 y^2 + \mu_0^2 + V(y) \leq \left(0.652 - \frac{9}{4}\right)^2 y^2 + 0.78 + V(y) \leq 0,$$

where the last inequality was obtained using the bounds for μ_0 of Lemma 10.4. Hence, this combined with inequalities (2.134), (2.133) gives us that I satisfies for all $y \in (0, x_{2,1})$:

$$I(y) = V''(y)|h_0(y)| + V'(y) \left| h_0^2(y) + \mu_0^2 + V(y) \right|.$$

Bounding by below, we have

$$I(y) \geq \left(0.652 - \frac{9}{4}\right) yV''(y) - V'(y)\left(\left(0.652 - \frac{9}{4}\right)^2 y^2 + 0.652 + V(y)\right)$$

Replacing V, V', V'' and considering the variable $s = \alpha^{-1}(y)$, we obtain

$$\begin{aligned} I(\alpha(s)) \geq\ & 2\left(0.652 - \frac{9}{4}\right)\alpha(s)Q^4\left(6 - \frac{50}{3}Q + 9Q^2\right) \\ & + 2Q^3H(2-3Q)\left(\left(0.652 - \frac{9}{4}\right)^2 \alpha(s)^2 + 0.652 + 2Q^2(1-Q)\right) \\ =:\ & \hat{m}(s). \end{aligned}$$

where $\hat{m}(s)$ is explicitly known employing Lemma 10.4. Computing this function, we have that $\hat{m}(s) > 0$ for all $s \in (0, \alpha^{-1}(x_{2,1}))$ (see Fig. 2.6). Hence, by bijectivity of α, we conclude $I(y) > 0$ for all $y \in (0, x_{2,1})$.

This proves that $I(y) \geq 0$ for all $y \geq 0$.

11.3.6. Proof of Lemma 11.13.

Since $h_0(x) \leq 0$ for all $x \geq 0$, we conclude by (2.151)

$$V_0'(x) = \frac{h_0(x)}{\phi_0^2(x)} \int_x^\infty \left(V''(y)h_0(y) - V'(y)h_0'(y) - 2V'(y)h_0^2(y)\right)\left(\frac{\phi_0}{h_0}\right)^2(y)dy \leq 0$$

for $x \geq 0$.

11.4. Decay of V_0'.

In order to prove the positivity of the transformed problem, we need an upper bound for V_0'. We state the following lemma.

Lemma 11.14. *For $|x| \gg 1$ we have that V_0 is strictly negative, and decay as $V'(x)$. Even more, the following bound*

$$3V'(x) \leq V_0'(x) \leq \frac{1}{2}V'(x), \tag{2.153}$$

is satisfied for all $x \geq x_{2,2}$.

Proof. Due to the parity we restrict our analysis to the positive axis, and we can assume that $x \geq x_{2,2}$.

First, we prove the lower bound using that from Lemma 11.5 $|V'(x)|$ decrease for $x \geq x_{2,2}$, and in addition employing equations (2.126), (2.128), (2.134), we have that

$$\begin{aligned}
|V_0'(x)| &\le \left|\frac{4h_0(x)}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy\right| + |V'(x)| \\
&= \left|\frac{4h_0(x)}{\phi_0^2(x)}\int_x^\infty \frac{V'(y)}{2h_0(y)}(\phi_0^2(y))'dy\right| + |V'(x)| \\
&\le \left|\frac{2\mu_0^{-1}h_0(x)V'(x)}{\phi_0^2(x)}\int_x^\infty (\phi_0^2(y))'dy\right| + |V'(x)| \\
&\le 3|V'(x)|,
\end{aligned}$$

for all $x \ge x_{2,2}$.

Second, analogously to the proof of Lemma 11.13 we use the integral formula for h_0 and apply specific bounds. Using the definition of V_0, Lemma 11.1, equation (2.127), and integration by parts,

$$\begin{aligned}
V_0'(x) &= 4h_0(x)h_0'(x) - \frac{3}{2}V'(x) + \frac{1}{2}V'(x) \\
&= h_0(x)h_0'(x) + 3h_0(x)h_0'(x) - \frac{3}{2}V'(x) + \frac{1}{2}V'(x) \\
&= -\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy - \frac{3}{2}\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \frac{V'(y)}{h_0(y)}(\phi_0^2(y))'dy - \frac{3}{2}V'(x) + \frac{1}{2}V'(x) \\
&= -\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty V'(y)\phi_0^2(y)dy + \frac{3}{2}\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(\frac{V'(y)}{h_0(y)}\right)'\phi_0^2(y)dy \\
&\quad - \frac{3}{2}\frac{h_0(x)V'(y)\phi_0^2(y)}{\phi_0^2(x)h(y)}\bigg|_x^\infty - \frac{3}{2}V'(x) + \frac{1}{2}V'(x) \\
&= \frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(\frac{3}{2}\frac{V'(y)}{h_0(y)} - V(y)\right)'\phi_0^2(y)dy + \frac{1}{2}V'(x) \\
&= \frac{1}{2}\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(3\frac{V''(y)}{h_0(y)} - 3\frac{V'(y)h_0'(y)}{h_0^2(y)} - 2V'(y)\right)\phi_0^2(y)dy + \frac{1}{2}V'(x) \\
&= \frac{1}{2}\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty \left(3V''(y)h_0(y) - 3V'(y)h_0'(y) - 2V'(y)h_0^2(y)\right)\left(\frac{\phi_0}{h_0}\right)^2(y)dy + \frac{1}{2}V'(x).
\end{aligned}$$

Thus, we define the integral form for V_0' given by

$$V_0'(x) = \frac{1}{2}\frac{h_0(x)}{\phi_0^2(x)}\int_x^\infty J(y)\left(\frac{\phi_0}{h_0}\right)^2(y)dy + \frac{1}{2}V'(x) \tag{2.154}$$

where we have denoted $J(y)$ as the term in parenthesis in the penultimate equation. Using equation (2.128) we have

$$J(y) = 3V''(y)h_0(y) - V'(y)(3\mu_0^2 - h_0^2(y) + 3V(y)). \tag{2.155}$$

Thus, we only have to prove the positivity of $J(y)$ to obtain (2.153). Applying the bound

(2.134), and the fact that $V(y) > 0$,

$$3\mu_0^2 - h_0^2(y) + 3V(y) \geq 3\mu_0^2 - \widetilde{\mu}_0^2 + 3V(y) = 2\mu_0^2 - \frac{8}{27} > 0.$$

Bounding by below (2.155) and using Lemma 10.4, since $V'(y) < 0$,

$$J(y) \geq -3V''(y) - \left(3\mu_0^2 - \widetilde{\mu}_0^2 + 3V(y)\right) V'(y) \geq -3V''(y) - (1.3 + 3V(y))V'(y) \geq 0,$$

for all $y \geq x_{2,2}$, where we obtain the last inequality via the explicit expressions using (2.15), (2.130) and (2.131). Hence, recalling (2.154), we obtain that

$$V_0'(x) = \frac{1}{2} \underbrace{\frac{h_0(x)}{\phi_0^2(x)}}_{\leq 0} \underbrace{\int_x^\infty J(y) \left(\frac{\phi_0}{h_0}\right)^2 (y) dy}_{\geq 0} + \frac{1}{2} V'(x) \leq \frac{1}{2} V'(x) \leq 0.$$

This ends the proof of Lemma 11.14. □

Chapter 3

Conclusion

1. Conclusions

This work focused on the study of long-time asymptotic kink solutions for a $SU(2)$ Yang-Mills model over a curved spacetime background.

In the first chapter of this thesis, we made a general introduction to the problem from a mathematical and physical perspective. In Chapter 2, we have extensively analyzed the asymptotic stability dynamics for a kink $\mathbf{H}$, employing interesting variations of virial techniques substantially developed in recent years. The results obtained consist of a deep analysis of the asymptotic stability of standing waves of the $SU(2)$ Yang-Mills model on the exterior of the extremal Reissner-Nordström black hole,

$$\partial_t^2 \varphi - Q\partial_x(Q\partial_x\varphi) + Q^2(\varphi^2 - 1)\varphi = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{3.1}$$

where we proved conditional asymptotic stability for perturbations in the energy space around the kink $\mathbf{H}$. This work is the first of his type for a Yang-Mills model over black hole geometries and opens a new area of research for the next years.

2. Future Work

It is proposed in the future to work in the following directions:

1. Construction of a stable manifold, where one has the asymptotic stability of the kink for perturbations in the energy space. This construction is currently under work by Muñoz and myself. The key idea is that, even in the absence of a spectral gap for L, we can employ the above results developed in Section 10 to obtain a coercivity result related to a small perturbation operator of L. Using standard energy estimates (see Annex), we can construct a Lipschitz graph of initial data leading to stable and asymptotically stable trajectories following the spirit of [20].
2. Using the developed spectral method, study similar models from general relativity under different geometries, such as Kerr black holes, or high energy physics couplings models, such as the Yang-Mills-Higgs theory.

Bibliography

[1] A. Actor, *Classical solutions of SU (2) Yang—Mills theories*. Reviews of Modern Physics, **51**, no. 3 461 (1979).

[2] M. Alammari, and S. Snelson, *Existence and stability of near-constant solutions of variable-coefficient scalar-field equations*, to appear in Diff. Integral Eqns. (2023).

[3] M. Alammari, and S. Snelson, *Linear and orbital stability analysis for solitary-wave solutions of variable-coefficient scalar field equations*, J. Hyperbolic Diff. Eqns., 19(1) 175–201, 2022.

[4] M. F. Atiyah, and R. Bott, *The yang-mills equations over riemann surfaces*. Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, **308**, no. 1505, 523-615 (1983).

[5] H. Baum, *Eichfeldtheorie*. Berlin, Heidelberg: Springer Berlin Heidelberg (2009).

[6] R. Bartnik, and J. McKinnon, *Particlelike solutions of the Einstein-Yang-Mills equations*. Physical Review Letters, **61**, no. 2 141 (1988).

[7] P. Bizoń, T. Chmaj and A. Rostworowski, *Late-time tails of a Yang-Mills field on Minkowski and Schwarzschild backgrounds*, Classical and Quantum Gravity, **24** (2007), no. 13, F55.

[8] P. Bizoń and M. Kahl, *A Yang–Mills field on the extremal Reissner–Nordström black hole*, Classical and Quantum Gravity, **33** (2016), no. 17, 175013.

[9] H. Berestycki, P.L. Lions, *Nonlinear scalar field equations, II existence of infinitely many solutions*, Arch. Rational Mech. Anal. **82**, 347–375 (1983). https://doi.org/10.1007/BF00250556

[10] K. Deimling. *Solutions in Cones. Nonlinear Functional Analysis*, 217-255 (1985).

[11] S. M. Carroll, *Spacetime and geometry*. Cambridge University Press (2019).

[12] S. M. Chang, S. Gustafson, K. Nakanishi, and T. P. Tsai, *Spectra of linearized operators for NLS solitary waves*, SIAM J. Math. Anal., **39** (2007/8), no. 4, 1070-1111.

[13] D. Christodoulou, *Solutions globales des equations de champ de Yang-Mills*, C. R. Acad. Sci. Paris A 293, **39** (1981).

[14] P. T. Chruściel and J. Shatah, *Global existence of solutions of the Yang–Mills equations on globally hyperbolic four dimensional Lorentzian manifolds*, Asian journal of Mathematics, **1** (1997), no. 3, 530–548.

[15] N. Dunford and J. T Schwartz, *Spectral theory: self adjoint operators in Hilbert space*, Interscience publishers, 1963.

[16] D. M. Eardley and V. Moncrief, *The global existence of Yang-Mills-Higgs fields in 4-dimensional Minkowski space*, Communications in Mathematical Physics, **83** (1982), no. 2, 171–191.

[17] R. T. Glassey. *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations. Journal of Mathematical Physics*, **18** (1977), no. 9, 1794-1797.

[18] M. J. Hamilton, *Mathematical gauge theory.* New York US: Springer International Publishing AG (2017).

[19] M. Kowalczyk and Y. Martel, *Kink dynamics under odd perturbations for* $(1+1)$*-scalar field models with one internal mode*, Preprint arXiv:2203.04143.

[20] M. Kowalczyk, Y. Martel, and C. Muñoz, *Soliton dynamics for the 1D NLKG equation with symmetry and in the absence of internal modes*, J. Eur. Math. Soc. (JEMS) 24 (2022), no. 6, 2133–2167.

[21] M. Kowalczyk, Y. Martel, and C. Muñoz, *Kink dynamics in the* ϕ^4 *model: asymptotic stability for odd perturbations in the energy space*, J. Amer. Math. Soc. **30** (2017), no. 3, 769–798.

[22] M. Kowalczyk, Y. Martel, and C. Muñoz, *Nonexistence of small, odd breathers for a class of nonlinear wave equations*, Lett. Math. Phys. **107** (2017), no. 5, 921–931.

[23] M. Kowalczyk, Y. Martel, and C. Muñoz, *On asymptotic stability of nonlinear waves*, Séminaire Laurent Schwartz—Équations aux dérivées partielles et applications. Année 2016–2017, Ed. Éc. Polytech., Palaiseau, 2017, pp. Exp. No. XVIII, 27.

[24] M. Kowalczyk, Y. Martel, C. Muñoz, and H. Van Den Bosch, *A sufficient condition for asymptotic stability of kinks in general* $(1+1)$*-scalar field models*, Ann. PDE **7** (2021), no. 1, Paper No. 10, 98.

[25] M. Krein, and M. Rutman. *Linear operators leaving invariant a cone in a Banach space*, Uspekhi mat. nauk, **3**(1), 3-95 (1948).

[26] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* Macmillan (1973).

[27] Y. Li and J. Lührman, *Soliton dynamics for the 1D quadratic Klein-Gordon equation with symmetry*, Journal of Differential Equations **344** (2023), 172-202.

[28] G. Pöschl and E. Teller, *Bemerkungen zur Quantenmechanik des anharmonischen Oszillators*, Zeitschrift für Physik, **83** (1933), no. 3, 143–151.

[29] N. Manton and P. Sutcliffe, *Topological solitons*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2004.

[30] Y. Martel, *Asymptotic stability of solitary waves for the 1D cubic-quintic Schrödinger equation with no internal mode*, Probability and Mathematical Physics, Vol. 3 (2022), No. 4, 839–867 DOI: 10.2140/pmp.2022.3.839.

[31] Y. Martel, and F. Merle. *Asymptotic Stability of Solitons for Subcritical Generalized KdV Equations.* Archive for rational mechanics and analysis, 157 (2001), 219-254.

[32] C. Maulén, *Asymptotic stability manifolds for solitons in the generalized Good Boussinesq equation*, to appear in J. Math. Pures et Appliquées (2023).

[33] C. Maulén, and C. Muñoz, *Asymptotic stability of the fourth order* ϕ^4 *kink for general perturbations in the energy space*, preprint arXiv:2305.19222 (2023).

[34] Tao, T. *Nonlinear dispersive equations: local and global analysis* (No. 106). American Mathematical Soc. (2006).

[35] R. Rajaraman, *Solitons and instantons*, (1982).

[36] M. Reed, B. Simon. *Methods of modern mathematical physics: Functional analysis. IV: Analysis of Operators*, Elsevier **4** 1978.

[37] J. A. Sánchez-Monroy, and C. J. Quimbay, *Exact solutions of (n+ 1)-dimensional Yang-Mills equations in curved space-time.* Annals of Physics, **327**, no. 9, 2166-2176 (2012).

[38] S. Snelson, *Asymptotic stability for odd perturbations of the the stationary kink in the variable-speed ϕ^4 model*, Transactions of the AMS 370(10) 7437-7460, 2018.

[39] M. Srednicki. *Quantum field theory.* Cambridge University Press, 2007.

[40] W. Pauli, *Relativistic field theories of elementary particles.* Reviews of Modern Physics, **13**(3), 203 (1941).

[41] Y. S. Tyupkin, V. A. Fateev, and A. S. Shvarts, *Particle-like solutions of the equations of gauge field theories.* Teoreticheskaya i Matematicheskaya Fizika, **26** , no. 3 397-402 (1976).

[42] T. Vachaspati, *Kinks and domain walls*, Cambridge University Press, New York, 2006, An introduction to classical and quantum solitons.

[43] E. J. Weinberg, *Classical solutions in quantum field theory: Solitons and Instantons in High Energy Physics.* Cambridge University Press (2012).

[44] Yang-Mills equations. DispersiveWiki. https://dispersivewiki.org/DispersiveWiki/Yang-Mills_equations.

[45] C. N. Yang and R. L Mills, *Conservation of isotopic spin and isotopic gauge invariance*, Physical review, **96** (1954), no. 1, 191.

Annex

Conservation of energy

Performing a standard computation using (2.15) and (2.16), we expand the conservation of energy (2.13) for a solution $(\phi_1, \partial_t \phi_1) = (\phi_1, \phi_2)$ written under the preservative form (2.24) with the orthogonality conditions (2.25), to obtain for all $t \geq 0$,

$$\begin{aligned}
&2\{E(\phi_1, \phi_2) - E(\widetilde{H}, 0)\} \\
&\quad = \int w_2^2 + \int (-\partial_x^2 w_1 + V w_1) w_1 - 2\int \widetilde{Q}^2 \widetilde{H} w_1^3 + \frac{1}{4}\int \widetilde{Q}^2 w_1^4 \\
&\quad = \int (\mu_0 a_2 \phi_0 + u_2)^2 + \langle L(a_1\phi_0 + u_1), a_1\phi_0 + u_1\rangle \\
&\quad = \mu_0^2 a_2^2 + 2\mu_0 a_2 \langle \phi_0, u_2\rangle + \|u_2\|_2^2 + a_1^2\langle L\phi_0, \phi_0\rangle + a_1\langle Lu_1, \phi_0\rangle \\
&\qquad + a_1\langle L\phi_0, u_1\rangle + \langle Lu_1, u_1\rangle - \int \widetilde{Q}^2\widetilde{H}(a_1^3\phi_0^3 + u_1^3 + 3a_1^2\phi_0^2 u_1 + 3a_1\phi_0 u_1^2) \\
&\qquad + \frac{1}{4}\int \widetilde{Q}^2(a_1^4\phi_0^4 + 4a_1^3\phi_0^3 u_1 + 6a_1^2\phi_0^2 u_1^2 + 4a_1\phi_0 u_1^3 + u_1^4) \\
&\quad = \mu_0^2(a_2^2 - a_1^2) + \|u_2\|_2^2 + \langle Lu_1, u_1\rangle - \int \widetilde{Q}^2(\widetilde{H} - a_1\phi_0)u_1^3 \\
&\qquad + \frac{1}{4}\int \widetilde{Q}^2 u_1^4 + \frac{a_1^3}{4}\int \widetilde{Q}^2(a_1\phi_0 - 4\widetilde{H})\phi_0^3 + a_1^2\int \widetilde{Q}^2(a_1\phi_0 - 3\widetilde{H})\phi_0^2 u_1 \\
&\qquad + \frac{3a_1}{2}\int \widetilde{Q}^2(a_1\phi_0 - 2\widetilde{H})u_1^2.
\end{aligned}$$

If we define $b_+ = \frac{1}{2}(a_1 + a_2)$, $b_- = \frac{1}{2}(a_1 - a_2)$, we obtain:

$$\begin{aligned}
2\{E(\phi_1, \phi_2) - E(\widetilde{H}, 0)\} &= -4\mu_0^2 b_+ b_- + \|u_2\|_2^2 + \langle Lu_1, u_1\rangle \\
&\quad - \int \widetilde{Q}^2(\widetilde{H} - a_1\phi_0)u_1^3 + \frac{1}{4}\int \widetilde{Q}^2 u_1^4 + O(|a_1|^3 + \|\widetilde{Q}u_1^{3/2}\|_2^2).
\end{aligned} \tag{.1}$$

Let $\delta_0 > 0$ be defined by

$$\delta_0^2 = b_+(0) + b_-(0) + \|u_2(0)\|_2^2 + \|\partial_x u_1(0)\|_2^2 + \|\widetilde{Q}u_1\|_2^2.$$

Then, (.1) applied at $t = 0$ gives

$$|2\{E(\phi_1, \phi_2) - E(\widetilde{H}, 0)\}| \lesssim \delta_0^2.$$