UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

WANG SUBSHIFTS ON FINITELY GENERATED GROUPS

TESIS PARA OPTAR AL GRADO DE
DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

HUGO ANDRÉS MATURANA CORNEJO

PROFESOR GUÍA:
ALEJANDRO MAASS SEPÚLVEDA

MIEMBROS DE LA COMISIÓN:
MARÍA CORTEZ MUÑOZ
SEBASTIÁN DONOSO FUENTES
SERVET MARTINEZ AGUILERA
JAIME ORTEGA PALMA

Este trabajo ha sido parcialmente financiado por ANID 21170770 y CMM ANID BASAL FB10005

SANTIAGO DE CHILE
2023

RESUMEN DE LA TESIS PARA OPTAR
AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA,
MENCIÓN MODELACIÓN MATEMÁTICA
POR: HUGO ANDRÉS MATURANA CORNEJO
FECHA: 2023
PROF. GUÍA: SR. ALEJANDRO MAASS SEPÚLVEDA

Subshifts de Wang en grupos finitamente generados

En este trabajo de tesis se estudia la dinámica simbólica en grupos finitamente generados. La motivación inicial de esta investigación fueron los resultados obtenidos por S. Piantadosi [42] en dinámica simbólica de grupos libres $\mathbb{F}_k$, desarrollando la teoría de *subshifts de tipo finito*, y el trabajo realizado por E. Jeandel y M. Rao [27], donde los autores obtuvieron que la cantidad mínima de $\mathbb{Z}^2$-Wang tiles que generan un $\mathbb{Z}^2$-Wang subshift aperiódico no vacío es 11 (definición de Wang tile y Wang subshifts en 3.1). En una de las líneas de trabajo nos enfocamos en determinar la cantidad mínima de $\mathbb{F}_k$-Wang tiles necesarias para conseguir un $\mathbb{F}_k$-Wang subshift aperiódico no vacío. En otra de las líneas de trabajo el foco de la investigación es exponer la relación que existe entre condiciones necesarias para obtener un teselamiento válido en $\mathbb{Z}^2$, resultado dado por J. Chazottes y coautores en [13], y las condiciones determinadas por S. Piantadosi para $\mathbb{F}_k$, obteniendo que ambas son equivalentes, y que resultan ser condiciones necesarias para obtener un teselamiento válido en grupos promediables finitamente generados.

La teoría de Wang subshifts, desarrollada por H. Wang [46], tuvo relevancia para estudiar el *problema del dominó: dado un conjunto de restricciones (por ejemplo, un conjunto de Wang tiles), ¿existe un algoritmo que decide si es posible obtener un teselamiento de $\mathbb{Z}^2$, respetando dichas restricciones?* El problema resultó ser indecidible, obteniéndose un teselamiento aperiódico no vacío que usó 20.426 $\mathbb{Z}^2$-Wang tiles, dado por R. Berger [6]. A lo largo de los años, el número de $\mathbb{Z}^2$-Wang tiles fue disminuyendo, hasta llegar a la cantidad mínima el año 2015, que resultó ser 11. Motivados por este resultado y el desarrollo de la dinámica simbólica en grupos libres por S. Piantadosi, extendimos la teoría de Wang subshifts a grupos libres para determinar el número mínimo de $\mathbb{F}_k$-Wang tiles que determinan un $\mathbb{F}_k$-Wang subshift aperiódico no vacío, resultando en el teorema prinicipal de esta línea de trabajo, el que nos dice que dicha cantidad es 3.

En la segunda línea de trabajo, consideramos un conjunto de condiciones necesarias que son heurísticas eficientes para decidir cuándo un conjunto de Wang tiles no puede teselar un grupo. Para esto consideramos dos condiciones: la primera dada por S. Piantadosi [42] que resulta ser una condición necesaria y suficiente para decidir si un conjunto de $\mathbb{F}_k$-Wang tiles entrega un teselamiento fuertemente periódico en el grupo libre; la segunda, dada por R. Chazottes et. al [13], es una condición necesaria para decidir si un conjunto de Wang tiles consigue un teselamiento válido en $\mathbb{Z}^2$. Demostramos que ambas condiciones son equivalentes, uniendo y generalizando ambos mundos ($\mathbb{F}_k$ y $\mathbb{Z}^2$), probando que estas resultan ser condiciones necesarias para tener un teselamiento válido en cualquier grupo promediable finitamente generado.

RESUMEN DE LA TESIS PARA OPTAR
AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA,
MENCIÓN MODELACIÓN MATEMÁTICA
POR: HUGO ANDRÉS MATURANA CORNEJO
FECHA: 2023
PROF. GUÍA: SR. ALEJANDRO MAASS SEPÚLVEDA

Wang subshifts on finitely generated groups

In this thesis work we study symbolic dynamics in finitely generated groups. The initial motivation of this research comes from the results obtained by S. Piantadosi in symbolic dynamics on free groups $\mathbb{F}_k$, developing the theory of subshifts of finite type and the work developed by E. Jeandel and M. Rao. [27], where the authors obtained that the minimum number of $\mathbb{Z}^2$-Wang tiles generating an aperiodic nonempty $\mathbb{Z}^2$-Wang subshift is 11 (definition of Wang tiles and Wang subshifts in 3.1). In one of the work lines we focus on determining the minimum number of $\mathbb{F}_k$-Wang tiles needed to achieve a nonempty aperiodic $\mathbb{F}_k$-Wang subshift, in another direction the focus of the investigation is to expose the relationship between necessary conditions to obtain a valid tiling in $\mathbb{Z}^2$, result given by Chazottes et. al. [13], and the conditions determined by S. Piantadosi for $\mathbb{F}_k$, obtaining that both are equivalent and moreover that they are necessary conditions to obtain a valid tiling in finitely generated amenable groups.

Wang subshifts theory, developed by H. Wang [46], was relevant in the study of the *domino problem: given a set of restrictions (e.g., a set of Wang tiles), is there an algorithm to decide whether it is possible to obtain a tiling of* $\mathbb{Z}^2$*, respecting those restrictions?*. The problem turned out to be undecidable, obtaining a nonempty aperiodic tiling using $20,426$ $\mathbb{Z}^2$-Wang tiles, given by R. Berger. Over the years, the number of $\mathbb{Z}^2$-Wang tiles decreased, until the proof that the minimal amount of Wang tiles that can produce an aperiodic SFT is 11, which was done by E. Jeandel and M. Rao in 2015 [27]. Motivated by this result and the development of symbolic dynamics in free groups by S. Piantadosi, we developed the theory of Wang subshifts on free groups to determine the minimum number of $\mathbb{F}_k$-Wang tiles that generate a nonempty aperiodic $\mathbb{F}_k$-Wang subshift, resulting in the main theorem of this work line, where we obtain that such quantity is 3.

In a second direction, we study a set of necessary conditions which are an efficient heuristic to decide when a set of Wang tiles cannot tessellate a group. For this, we consider two conditions: the first given by S. Piantadosi [42] is a necessary and sufficient condition to decide if a set of Wang tiles gives a strongly periodic tiling of the free group; the second, given by R. Chazottes et. al. [13] is a necessary condition to decide if a set of Wang tiles gives a tiling of $\mathbb{Z}^2$. We show that both conditions are equivalent, joining and generalizing two different settings ($\mathbb{F}_k$ and $\mathbb{Z}^2$), and we prove that they are necessary for having a valid tiling of any finitely generated amenable group.

Lograr aquello que has soñado te hace feliz, pero sobre todo, te hace feliz recordar el esfuerzo empleado para lograrlo.
Rafael Nadal

Agradecimientos

Mucho antes siquiera de escribir esta página, tengo claro que me será imposible poder, en tan pocas palabras, expresar mis palabras de agradecimiento a todos/as aquellos/as personas que me acompañaron en este largo pero gratificante viaje. Un viaje que sin dudas ha tenido diversos obstáculos y muchas dificultades, pero que aún así todas forman parte a día de hoy del profesional que culmina esta etapa.

Quiero dedicar este parráfo a todos/as aquellos profesores e investigadores que fueron parte de este proceso directa e indirectamente. En primer lugar agradecer al profesor y ex coordinador del programa Marcos Kiwi por la preocupación y tiempo dedicados para que este proyecto pueda concretarse. Por supuesto agradecer a los profesores que están formando parte de esta comisión: Maria Isabel Cortez, Alejandro Maass y Sebastián Donoso, quienes han sido parte de mi formación académica. Al investigador Michael Schraudner agradezco todos los viernes de jornadas de trabajo junto Alvaro Bustos, reuniones en las que me ganaba varios retos pero terminaban siendo importantes en mi formación profesional, si bien esta finalización no es lo que esperé, siento la necesidad de escribir estas lineas para él. También hacer incapié en todo el apoyo y buana acogida que me brindó la profesora y académica Natalie Aubrun durante mi pasantía en la ENS de Lyon, agradezco las jornadas de trabajo, ideas brindadas y apoyo continuo, también los pokemones capturados y transferidos. En dicha estadía Benjamin Hellouin resultó ser un pilar fundamental, muchisimas gracias por creer en mí y querer trabajar juntos, fueron jornadas realmnente interesantes de trabajo y espero que existan muchas más. Un poco más hacia el pasado, agradezco enormemente a los profesores Ricardo S., Rosa M., Rafael L., Samuel N., Pedro U., Rodrigo C., quienes contribuyeron en mi formación de pregrado y mágister.

Quisiera expresar mis infinitas (y no numerables) gracias a mi familia, quienes me han soportado todos estos años el mal humor, a veces el no vernos y muchas otras oportunidades que, sin ganas de hablar, sólo llegaba a dormir al hogar. Mis padres y hermano han sido personas que me han acompañado implicitamente en este proceso y quisiera poder expresar mis más sinceras palabras de agradecimiento. Aún recuerdo el apañe que sentí en todo momento y sin siquiera saber qué era lo que estaba haciendo, siempre había un "Cómo va el doctorado?", también las largas llamadas desde Lyon, Francia, hablando acerca de las diferentes experiencias que estaba viviendo y asegurarse que me encontraba bien.

También no podría faltar en estos agradecimientos: mi otra familia, la familia que uno elige: mis amigos y amigas, algunos acompañando este viaje desde ya muchos años atrás y otros sumandóse desde mi llegada a Beauchef, algunas otras importantes personas apareciendo por esos azares del destino y no desapareciendo jamás. Personas muy especiales que me escucharon, apoyaron, dieron ánimos hasta en los momentos más dificiles en este proceso y sobretodo compañía. Saludar muy afectuosamente a mi amigo Pablo Dotte quien me ha apañado desde el inicio en mis estudios profesionales, siempre brindando apoyo y siendo un excelente amigo. Por estos lados agradecer a mis compañeros y amigos que conocí en el doctorado Roberto M., Javier R., Ariel P. Matias P., Alvaro B., Javiera F., Jorge O., Evelyn C., Hugo C., Jessica (2+1)Palacios, Yamit, y todos con quienes hemos compartido algún buen café, cerveza, algún juego de mesa o simplemente hablar de cosas random, sé que me quedan muchos/as en el tintero pero quiero expresar este párrafo para agradecer enormemente su apoyo. Sin olvidar también a mis amigos/as que he conocido fuera del doctorado, como la chica Natalia F. poniendo el apañe y la comida para poder seguir adelante, agradezco esas tardes de risas y caminatas al metro. Constanza S. dando muy buenos tips odontólogicos y risas en esos días más grises, Adorée A., Sandy S., Nati M., Ana. C, y al máquina Felipe Ñ.

Finalmente, agradecer a Georgina (cucky) el apoyo y cariño, Natacha A., Silvia M., Eterin A., Karen A., Oscar M., Luis M. y a Luchito (si, el caballero del cubo de Rubik fuera de 851) por esas charlas fuera de 851 y unas cuantas risas que nos sacaba junto a Jorge Olivares esas tardes de ir a comprar algo para la once.

Table of content

List of Figures

Symbols used in this book

Symbol	Description
$A \cupdot B$	disjoint union
$H \le G, H \trianglelefteq G$	subgroup, normal subgroup
$[G : H]$	Index of H over G
$H \lhd G$	H is subgroup normal of G
$X \cong Y$	isomorphism, conjugacy
$f : X \hookrightarrow Y$	injective map, monomorphism
$f : X \twoheadrightarrow Y$	surjective map, epimorphism, factor map
$\langle T \rangle, T$	the subgroup (or normal subgroup) generated by T
(G)	least cardinal of a generating set for G
$G \amalg H$	free product
$\mathbb{F}_[S]$	free group generated by S
$\ker(h)$	kernel of h: $\{x : h(x) = 0\}$
$h(G)$	image of G under h
$h^{-1}[H]$	preimage of H under h
(h)	$f[(h)]$
$\langle S \rangle$	Subgroup generated by S in G
$SYM(\Gamma)$	Symmetry group of Γ
$\mathbb{Z}^k$	group of k-adic integers, k-adic odometer (not to be confused with $\mathbb{Z}/k\mathbb{Z}$)
$P \sqsubseteq x$	the point x contains the pattern P
gP	translate of the pattern P by some $g \in G$
$[P]$	$\{x \in \mathcal{A}^G : x\|_{(P)} = P\}$
$g[P]$	$[g^{-1}P]$
CX	$\{x\|_C : x \in X\}$
X	language of X, i.e. the union of all CX
$\mathsf{X}_{\mathcal{F}}$	shift space with forbidden patterns in $\mathcal{F}$
Φ_∞	sliding block code with local function $\Phi \colon CX \to \mathcal{A}$
$\mathrm{Orb}(x)$	orbit of x
$Stab(x)$	stabilizer of x
$\mathrm{Per}_H(x), \mathrm{Per}^0_H(x)$	H-periodic points, i.e. the set of all x with $H \le (x)$ (resp., $H = (x)$)
$h_{\mathrm{top}}(X)$	entropy

Introduction

The theory of dynamical systems arises from the desire to study systems that evolve over time, many of them inspired by phenomena from physics, astronomy, computer science, etc. One line of study is topological dynamics, this area studies geometrical and topological properties of trajectories. One of the early applications is the study of periodicity in the movement of planets. The mass of the sun is considerably bigger than the mass of any planet of this planetary system, and the dominant force acting on a planet is the attraction to the sun, thus the trajectory of any planet around the sun is very close to the Keplerian elliptic. When an orbit is completed, the position and momentum of a planet is close to its initial state and momentum, if between two planets the ratio of orbit times is almost a rational number $\frac{a}{b}$, this means that one planet makes a orbits in nearly the same time as the other planet makes b orbits ([9],[8]).

In the beginning, the concept of a dynamical system consisted of a pair (X, f), where X corresponds to the phase space and $f : \mathbb{R}^+ \times X \to X$ is a function which describes the evolution of elements of X in time, i.e., if we consider $x_0 \in X$ then the set $\{f(t, x_0)\}_{t\in\mathbb{R}^+}$ shows the behavior of the point x_0 over time. This classical theory of dynamical systems was intended to respond to more qualitative problems, related to the study of the trajectory of planets in the solar system, evolution of a predator-prey ecosystem in time, etc.

An important and more recent class of dynamical systems is the collection of symbolic dynamical systems due to M. Morse and G. Hedlund ([40], [39]). The main idea is to study the behavior of a certain phenomenon by discretising the phase space. This idea was initially adopted by J. Hadamard [22], who used this technique to study geodesics on surfaces of negative curvature. A way to study a phenomenon is by getting an encoding of their behavior: if we have a partition of our phase space X given by $\mathcal{P} = \{A_1, \dots, A_n\}$, we can define the function $T : X \to \{1, \dots, n\}^{\mathbb{N}}$ as follows:

$$T(x)(n) = i \iff T^n(x) \in A_i.$$

This allows us to get a coding of the trajectory of a point, when time is considered as a discrete variable.

A special kind of symbolic dynamical systems are *subshifts*. These are systems in which the space is made up of infinite or bi-infinite sequences of symbols in a fixed alphabet $\mathcal{A}$, i.e., elements in the product space, for instance $\mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$. Using the idea explained in the last paragraph, the alphabet $\mathcal{A}$ corresponds to the set $\{1, \dots, n\}$, and the function corresponding to the system, the *shift action* σ, consists in moving every symbol of the sequence $(x_n)_{n\in\mathbb{N}}$ (if we consider one-sided sequences) one position to the left, that is:

$$\sigma((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}.$$

For example, let us consider the study of the trajectory of a ball in a Volleyball match. For this, first we label by 0 the left side and by 1 the right side. We define the sequence of numbers $x_0, x_1, x_2, \dots \in \{0, 1\}$, where $x_i = 0$ if the ball is in the left side and $x_i = 1$ in the right side, in the time i.

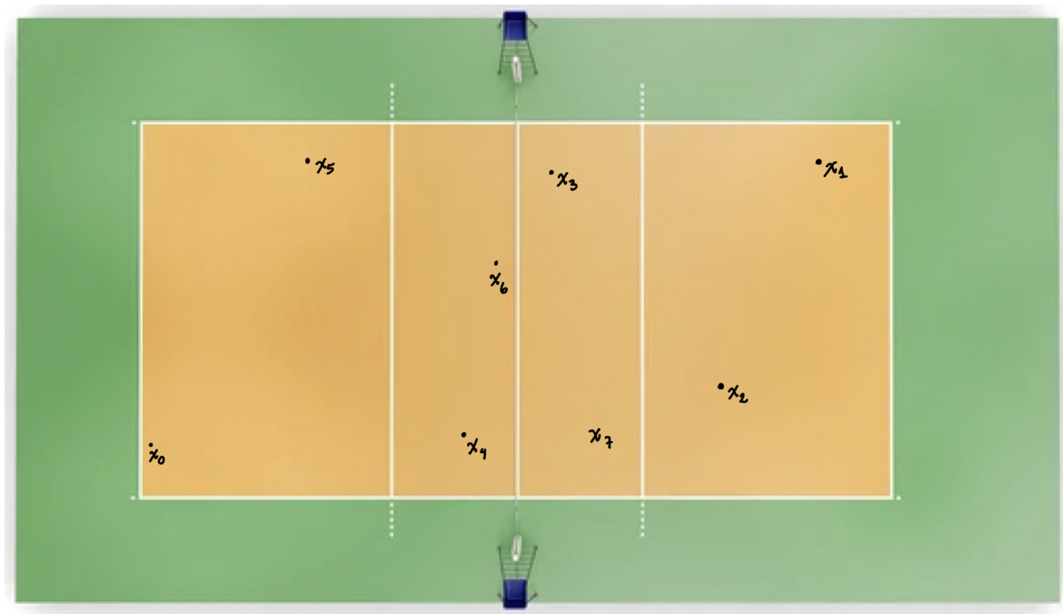


Figure 1: The first 7 seconds of the volleyball match.

As Figure 1 shows, we obtain the sequence $.01110001\dots$ and we see that the shift-action applied to a sequence corresponds precisely to the encoding of the future states of the system after moving forward in time. Nevertheless, the information we can obtain is very simplified, since it only tells us in which side of the court the ball is. This is because the space was partitioned into only two sets (both sides of the court), but if we consider a partition into more sets, we can obtain much more information and complexity.

The simplest classes of subshifts are the so called *G-subshifts of finite type* (G-SFT) and *G-sofic subshifts*, where G corresponds to a finitely generated group. Both types of subshifts have been systematically studied. In this thesis work we will be generally interested in G-SFTs. These systems can be defined by restricting their language (the set of all finite subsequences we observe when looking at any element of the shift) through forbidden words, a combinatorial viewpoint which allows to describe these systems using graph theory and the theory of finite automata, at least in the 1-dimensional case.

For $G = \mathbb{Z}$, the class of $\mathbb{Z}$-SFTs has been thoroughly studied. Corresponding subshifts can be represented by finite directed graphs such that valid $\mathbb{Z}$-colorings are in bijection with bi-infinite walks on the corresponding graph. For instance, let us consider a $\mathbb{Z}$-SFT $X_{\mathcal{F}} \subseteq \{0, 1\}^{\mathbb{Z}}$

given by the following set of forbidden patterns:

$$\mathcal{F} = \{11\}.$$

This means that the valid sequences of length 2 are $\{10, 01, 00\}$. The subshift $X_{\mathcal{F}}$ corresponds to a classic example in the one-dimensional theory called *Golden mean shift* and the corresponding adjacency graph is the follows:

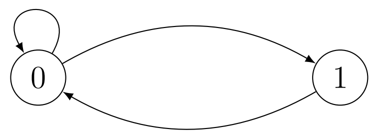


Figure 2: The Golden Mean shift as a vertex shift.

The valid sequences on $X_{\mathcal{F}}$ correspond to labels of valid infinite walks on the graph; for example, the valid sequences of length 2 correspond to $\mathcal{L} = \{00, 01, 10\}$. Moreover, we can define (in this case) an associated $\mathbb{Z}-$Wang subshift given by the $\mathbb{Z}$-Wang tiles $T = \{T_1, T_2, T_3\}$ defined in Figure 3.

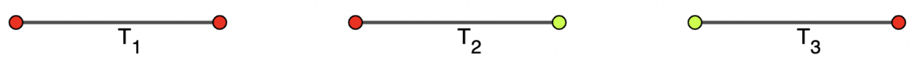


Figure 3: $\mathbb{Z}$-Wang tiles corresponding to the $\mathbb{Z}$-Wang subshift associated to the golden mean shift.

Before continuing, it is important to note that the conjugacy between the golden mean shift ($\mathbb{Z}$-SFT) and its corrresponding $\mathbb{Z}$-Wang subshift is not trivial, the increase in the amount of symbols between the original alphabet $\mathcal{A}$ and the new alphabet T of Wang tiles provides evidence of this. More details about this fact and conditions to construct this conjugacy will be seen later in this thesis.

Hence, given a collection of nearest neighbor restrictions or a finite set of Wang tiles, we can easily decide whether there is a valid $\mathbb{Z}$-coloring by verifying that the associated directed graph allows a bi-infinite path, i.e., it has a cycle. Therefore, both the Emptiness and the Domino problem on $\mathbb{Z}$ are decidable, and in fact every nonempty $\mathbb{Z}$-SFT has at least one periodic coloring (for more details see [34] and [31]).

For $G = \mathbb{Z}^2$, decidability of the Emptiness problem is much less obvious and the Domino problem in particular has a long history. In 1961, H.Wang [45] conjectured that if a finite set of $\mathbb{Z}^2$-Wang tiles can tile $\mathbb{Z}^2$ then it must also allow periodic tilings. He then observed that proving this conjecture would imply the existence of an algorithm to decide the Domino problem on the plane. In 1966, Wang's student R. Berger [6], showed that Wang's conjecture is false. He constructed an enormous alphabet of $20,426$ $\mathbb{Z}^2$-Wang tiles giving rise to a nonempty weakly aperiodic $\mathbb{Z}^2$-Wang subshift. Furthermore, using properties of that subshift, R. Berger showed that the Domino problem in $\mathbb{Z}^2$ is in fact undecidable.

After this first example, people started to work on reducing the required number of $\mathbb{Z}^2$-Wang tiles. In 1971, R. Robinson [43] constructed an aperiodic $\mathbb{Z}^2$ tiling with an alphabet comprised of only 56 tiles, followed by H. Lauchli [46] who in 1975 published an aperiodic set of 40 $\mathbb{Z}^2$-Wang tiles. After that, in 1996 J. Kari [29] invented a new method to build aperiodic tile-sets and obtained an example with 14 $\mathbb{Z}^2$-Wang tiles. In the same year, together with K. Culik [17], they reduced the set to 13 $\mathbb{Z}^2$-Wang tiles (Figure 4). Finally, in 2015 E. Jeandel and M. Rao [27] determined that 11 is the smallest cardinality of $\mathbb{Z}^2$-Wang tiles which can generate a nonempty aperiodic $\mathbb{Z}^2$-Wang subshift.

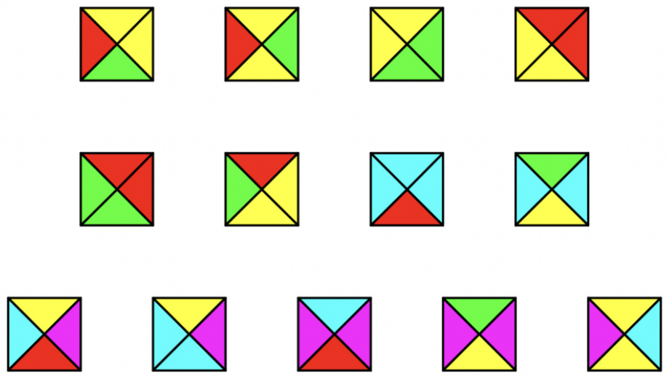


Figure 4: An aperiodic set of 13 $\mathbb{Z}^2$-Wang tiles.

On the free group $G = \mathbb{F}_k$, some basic aspects of symbolic dynamics have been developed by S. Piantadosi [42]. In particular, he studied conditions assuring non-emptiness as well as the existence of strongly periodic colorings in $\mathbb{F}_k$-*nearest neighbor subshifts of finite type* ($\mathbb{F}_k$-NNSFTs). The NNSFT are a type of SFT where the set of forbidden patterns is given in a more easy form to study; more details about this type of SFT are given in Chapter 3. In fact, the Emptiness problem is decidable for these groups, because every $\mathbb{F}_k$-NNSFT is completely determined by a family of k $\mathbb{Z}$-NNSFTs. Moreover, every non-empty $\mathbb{F}_k$-NNSFT contains a weakly periodic coloring. Nevertheless, S. Piantadosi constructs an example of a $\mathbb{F}_2$-NNSFT without any strongly periodic coloring.

It is interesting to see how these results can be different between the theory of symbolic dynamics on $\mathbb{Z}$ and when the acting group G is changed. For instance, the difference between $\mathbb{Z}$ and $\mathbb{Z}^2$, or the similar results between $\mathbb{Z}$ and $\mathbb{F}_k$. Recently, these properties have been studied in other kinds of groups. For instance, using again the existence of strongly aperiodic SFTs, the domino problem was shown to be undecidable, apart from $\mathbb{Z}^d$, on some semisimple Lie groups [41], the Baumslag-Solitar groups [2], the discrete Heisenberg group (announced, [44]), surface groups [15, 1], semidirect or direct products on $\mathbb{Z}^2$ ([5] and [4] respectively), polycyclic groups [25], some hyperbolic groups [14], etc. In contrast, with previous results, the domino problem is decidable on free groups [42] and on virtually free groups [3], and it is conjectured that these are the only groups where the domino problem is decidable.

Main contributions

The aim of this thesis is to develop the theory of Wang subshifts in finitely generated groups. There are two main directions, both of which are inspired by the results of S. Piantadosi in [42] on symbolic dynamics in free groups. In his work, S. Piantadosi studied $\mathbb{F}_k$-subshifts of finite type, focusing in particular on $\mathbb{F}_k$-nearest neighbor subshifts of finite type ($\mathbb{F}_k$-NNSFTs).

First, we show that the conditions given by J. Chazottes, J. Gambaudo and F. Gautero [13] for tiling the Euclidean plane by polygons are equivalent to the conditions given by S. Piantadosi [42] for free groups. By joining both sides of the theory (i.e., $\mathbb{Z}^2$ and free groups), we demonstrate that both conditions are necessary in finitely generated groups, but not sufficient. To support this claim, we provide a counterexample that satisfies the conditions but yields an empty tiling.

Secondly, we develop the theory of Wang subshifts on free groups and address the question of how many Wang tiles are needed to obtain a nonempty aperiodic Wang subshift. This problem was solved in $\mathbb{Z}^2$ by E. Jeandel and M. Rao [25], who showed that 11 Wang tiles were necessary. However, in our case, we generalize this result to $\mathbb{F}_k$ with $k \geq 2$ and demonstrate that the minimum number of Wang tiles required does not depend on the number of generators in the free group, which is 3.

Symbolic dynamics in one dimension has been extensively studied [34], particularly with respect to subshifts of finite type in $\mathbb{Z}$. As described before, such subshifts can be represented using graph theory, where their configurations can be viewed as bi-infinite walks in a labeled graph. This approach has proven to be useful since the adjacency matrix associated to the graph encoded important information of the system. However, subshifts of finite type in more general groups cannot always be represented using graphs. For instance, in the case of $\mathbb{Z}^2$, subshifts of finite type cannot be represented using graphs (see Example 2.6). Nevertheless, S. Piantadosi shows in [42] that subshifts of finite type in free groups can be represented using graphs, although as many graphs as the free group has generators are necessary.

The study of symbolic dynamics in free groups is particularly interesting because free groups are non-amenable, unlike previous studied cases, such as $\mathbb{Z}$ or $\mathbb{Z}^2$. Therefore, obtaining results in this area is crucial for the development of dynamical systems in increasingly general groups.

The domino problem established a connection between periodicity and the existence of a nonempty subshift of finite type. After R. Berger [43] (Wang's student) presented an example of a nonempty subshift of finite type with aperiodic tiling, which required an alphabet of $20,426$ Wang tiles, researchers became interested in obtaining subshifts of finite type with similar properties but with a smaller number of Wang tiles in their alphabet. Noteworthy, results were achieved by R. Robinson [43] and J. Kari [29] in 1996, who developed a new method to construct aperiodic tile-sets and found an example with only 13 $\mathbb{Z}^2$-Wang tiles. More recently, in 2015, E. Jeandel and M. Rao [27] showed that the minimum number of $\mathbb{Z}^2$-Wang tiles required is 11, and they utilized computers to obtain their result.

Considering the results obtained in $\mathbb{Z}^2$, some intriguing questions arise, such as: What happens in the case of free groups? Will the problem require as much time to solve as it did for $\mathbb{Z}^2$? Will computers be necessary to determine the minimum number of Wang tiles required?

This thesis aims to address these questions by determining the minimum number of $\mathbb{F}_k$-Wang tiles necessary to generate a non-empty weakly aperiodic $\mathbb{F}_k$-Wang subshift.

First, we note that not every $\mathbb{Z}$-subshift of finite type generates a $\mathbb{Z}$-Wang subshift, this is shown by Example 3.4. Our focus then turned to see which are the graphs that generates a nonempty $\mathbb{Z}$-Wang subshift, obtaining the following result:

Proposition *An essential directed graph $\Gamma = (V, E)$ determines a valid nonempty $\mathbb{Z}$-Wang subshift if and only if for every $v, w \in V$:*

1. $V^+(v) \cap V^+(w) \neq \emptyset$ *implies* $V^+(v) = V^+(w)$,
2. $V^-(v) \cap V^-(w) \neq \emptyset$ *implies* $V^-(v) = V^-(w)$.

Using the adjacency matrix of Γ, this means:

a. *Each row and column contains at least a 1 (Γ is essential).*

b. *If two rows or two columns have a 1 in the same position, then those rows or columns are identical.*

This proposition provides us with the necessary and sufficient conditions to achieve a nonempty $\mathbb{Z}$-Wang subshift. Using a strategy similar to that of S. Piantadosi [42], we show that by using k graphs with these characteristics we obtain a nonempty $\mathbb{F}_k$-Wang subshift.

Proposition *Every nonempty $\mathbb{F}_k$-Wang subshift, using n $\mathbb{F}_k$-Wang tiles, is completely determined by a family of k nonempty $\mathbb{Z}$-Wang subshifts.*

Finally, we search for the networks that satisfy the above conditions. First, the search is performed using graphs with two vertices (i.e., $\mathbb{F}_k$-Wang subshifts defined using 2 $\mathbb{F}_k$-Wang tiles), obtaining $\mathbb{F}_k$-Wang subshifts with at least a strongly periodic tiling. The main result of this work line is the following.

Theorem *Given $k \geq 2$, the minimum cardinality of a set of $\mathbb{F}_k$-Wang tiles which produces a nonempty weakly aperiodic $\mathbb{F}_k$-Wang subshift is* 3.

Following this direction, we turn our attention to characterize all possible examples of nonempty weakly aperiodic $\mathbb{F}_k$-Wang subshifts using an alphabet of 3 tiles, obtaining for every $k \geq 2$ the amount of all them. We create all 25 essential directed graphs which give rise to nonempty $\mathbb{Z}$-Wang subshifts with exactly 3 tiles and we use Proposition 3.7 to generate all possible $\mathbb{F}_k$-Wang subshifts using a three letter alphabet. Analyzing those graphs and its structure then leads to an effective method of classifying the $\mathbb{F}_k$-subshifts aperiodicity in terms of nontrivial solutions of a certain system of linear equations and allows us to determine all weakly aperiodic examples.

This result answers the question solved for $\mathbb{Z}^2$ in free groups, achieving a result that does not depend on the number of generators of the free group and also characterizing all possible examples that can be generated with such characteristics.

In another line of work, we show a set of necessary conditions which are efficient heuristics for deciding when a set of G-Wang tiles cannot tile a finitely generated amenable group G, making a link between the conditions that S. Piantadosi [42] got for free groups and conditions obtained in the work of J. Chazottes, J. Gambaudo and F. Gautero [13] in a more general context for tiling the Euclidean plane by polygons, but which is necessary for an SFT to admit a tiling of $\mathbb{Z}^2$ [28], showing that these conditions are equivalent and that they form a necessary condition for an SFT to admit a valid tiling on any finitely generated amenable group, confirming a remark of E. Jeandel ([26]).

The conditions given by S. Piantadosi, in the context of free groups, which we call $(\star)$ and $(\star\star)$ respectively are:

Definition (Condition $(\star)$) *A family of graphs $\Gamma = \{\Gamma_i\}_{1\leq i\leq d}$ whose vertices are an alphabet $\mathcal{A}$ satisfies* condition $(\star)$ *if and only if there is some nonempty $\mathcal{A}' \subset \mathcal{A}$ with a* colouring function $\Psi : \mathcal{A}' \times \mathcal{S} \to \mathcal{A}'$ *such that, for any colour $a \in \mathcal{A}'$ and any generator $g_i \in \mathcal{S}$, $a \to \Psi(a, g_i)$ is an edge in Γ_i.*

Definition (Condition $(\star\star)$) *Consider a family of graphs $\Gamma = \{\Gamma_i\}_{1\leq i\leq d}$ and $\mathcal{SC}(\Gamma_i) = \{\omega_i^j\}_{1\leq j\leq \#\mathcal{SC}(\Gamma_i)}$ the set of simple cycles for each graph Γ_i.*

We denote by $(\star\star)$ the following equation on real numbers $x_{i,j}$:

$$\forall a \in \mathcal{A}, \sum_{j=1}^{\#\mathcal{SC}(\Gamma_1)} x_{1,j}|\omega_1^j|_a = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_2)} x_{2,j}|\omega_2^j|_a = \cdots = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_d)} x_{d,j}|\omega_d^j|_a.$$

We say that the graph family satisfies condition $(\star\star)$ *if equation $(\star\star)$ is not empty (e.g., all graphs contain at least a cycle) and admits a nontrivial positive solution.*

Both were stated in [42]. The last condition was considered by J. Chazottes-J. Gambaudo-F. Gautero [13], in the context of the plane $\mathbb{Z}^2$:

Definition (Condition $(\star\star)'$) *Let T be a set of Wang tiles on colours $\mathcal{C}$ and set of generators $\mathcal{S}$. For each $g \in \mathcal{S} \cup \mathcal{S}^{-1}$ and each colour $c \in \mathcal{C}$, define c_g the subset of Wang tiles $\tau_i \in T$ such that $\tau_i(g) = c$. We call $(\star\star)'$ the following equation:*

$$\forall g \in \mathcal{S}, \forall c \in \mathcal{C}, \sum_{\tau_i \in c_g} x_i = \sum_{\tau_j \in c_{g^{-1}}} x_j.$$

We say that T satisfies condition $(\star\star)'$ *if Equation $(\star\star)'$ admits a positive nontrivial solution.*

Although conditions $(\star\star)$ and $(\star\star)'$ were introduced in very different contexts (periodic tilings of the free group and tilings of the Euclidean plane, respectively). An important result of this line of work was to show that both conditions are equivalent.

Theorem *Let T be a set of Wang tiles over the set of colours $\mathcal{C}$ and the set of generators $\mathcal{S}$.*

T satisfies condition $(\star\star)'$ if, and only if, the associated graphs satisfy condition $(\star\star)$.

Finally, we provide the necessary conditions to obtain a valid tiling of a finitely generated amenable group.

Theorem (Heuristic for tiling an amenable group) *Let G be a finitely generated amenable group, $\mathcal{S}$ a finite set of generators, and T a set of Wang tiles.*

If there is a tiling of G with the tiles T, then condition $(\star\star)$ (or equivalently $(\star\star)'$) is satisfied.

The relevance of the last result is that the heuristics can be very useful when making an exhaustive search for SFTs with desired properties; the necessary conditions in particular allow fast rejection of most empty SFTs. For example, a transducer-based heuristic was used in the search for the smallest set of $\mathbb{Z}^2$-Wang tiles that yield a strongly aperiodic $\mathbb{Z}^2$-SFT, determined by E.Jeandel and M.Rao [27]. It is also of theoretical interest to understand how the group properties impact the necessary conditions.

The text is organized as follows:

- In Chapter 1 we review prerequisites for each of the topics covered in this thesis. First, we give some basic facts about *group theory*. In particular, we talk about groups via generators and relations, because it turns out to be very important to understand the main results of the following chapters. We dedicate our attention to the understanding of *free groups* and the theory of Cayley graphs. Finally, we give the main definitions and results about *dynamical systems* and *symbolic dynamics*; in this last topic we turn our attention to symbolic dynamics on $\mathbb{Z}$ and $\mathbb{F}_k$.

- In Chapter 2, which is a collaboration with B. Hellouin de Menibus (from Université Paris-Saclay) [19], was published on *Discrete and continous dynamical system* 2020, Doi: http://dx.doi.org/10.3934/dcds.2020116 . In this work, we give a set of necessary conditions which are efficient heuristics for deciding when a set of Wang tiles cannot tile a group.

- In Chapter 3, which is a collaboration with M. Shchraudner, corresponds to an adaptation of *Weakly aperiodic Wang subshifts with minimal alphabet size on the free group* [16]. The goal of this work was to determine the minimal amount of $\mathbb{F}_k$-Wang tiles which generate a nonempty weakly aperiodic $\mathbb{F}_k$-Wang tiling. For this, we develope the theory of Wang tiles and Wang tiling on free groups, giving conditions to get a valid $\mathbb{F}_k$-Wang tiling from k nonempty $\mathbb{Z}$-Wang subshifts.

Part I

Preliminaries

Chapter 1

1.1 Elementary group theory

In this section we give the main results about elementary group theory. In general, these results are presented without the proofs and the author recommends to consult the following literature: [24], [4], [32], [37], [18], [21], [12] and [36].

Definition 1.1 (Groups) *A* group *corresponds to a pair* $(G, \cdot)$, *where* G *is a nonempty set and* $\cdot$ *is a binary operation:* $G \times G \to G$ *satisfying the following axioms:*

- *Associativity: For all* $g_1, g_2, g_3 \in G$ *we have:*
$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3.$$
- *Existence of a neutral element: There exists* $e_G \in G$ *such that for all* $g \in G$*:*
$$g \cdot e_G = e_G \cdot g = g.$$
- *Inverses: For every element* $g \in G$ *there exists an inverse element, denoted by* $g^{-1} \in G$*, such that:*
$$g \cdot g^{-1} = g^{-1} \cdot g = e_G.$$

Moreover, let us call the pair $(G, \cdot)$ *an Abelian group if it satisfies the additional axiom:*

- *Commutativity: For every* $g_1, g_2 \in G$ *we have:*
$$g_1 \cdot g_2 = g_2 \cdot g_1.$$

In order to simplify the notation, the binary operation symbol $\cdot$ is often omitted: instead of writing $g_1 \cdot g_2$ we just write $g_1 g_2$. Also, when the group operation is clear, we denote the group $(G, \cdot)$ just by the set G. For instance, we refer to $\mathbb{Z}$ as "the group of integers" when we formally mean $(\mathbb{Z}, +)$.

Definition 1.2 (Subgroups) *Let* G *be a group and* $H \subset G$*. We say* H *is a* Subgroup *of the group* G *if it satisfies the following conditions:*

- *If* $h_1, h_2 \in H$, *then* $h_1 h_2 \in H$,

- $e_G \in H$,

- $\forall h \in H$, $h^{-1} \in H$.

We use the notation $H \leq G$ *to say that* H *is a subgroup of* G. *Furthermore, if for each* $g \in G$ *and* $h \in H$ *we have* $ghg^{-1} \in H$, *we say that* H *is a* normal subgroup *of* G *and we write* $H \lhd G$.

Using subgroups we can define an equivalence relation over G as follows. Let $H \leq G$, we define the equivalence relation $\sim_H$ as $g_1 \sim g_2$ if and only if there exists $h \in H$ such that $g_1 = g_2 h$, we observe that for any $g \in G$, the corresponding equivalence class is given by:

$$[g]_{\sim_H} = \{x \in G \mid \exists h \in H: \ x = gh\} = gH.$$

The set of equivalence classes is called *left cosets* and we denote it by $G/H := G/\sim_H$. We define the *index* of H over G as:

$$[G : H] := |G/H|.$$

We observe that if $H \lhd G$ and $gH \in G/H$, as $ghg^{-1} \in H$ for each $g \in H$, we have that $gH = Hg$, therefore we can endow G/H with the binary operation defined by

$$(g_1 H)(g_2 H) := (g_1 g_2)H.$$

There is not any problem with this definition, because:

$$(g_1 H)(g_2 H) = g_1(H g_2)H = g_1(g_2 H)H = g_1 g_2 HH = g_1 g_2 H.$$

It important to note that if $g_1 H = g_1' H$ and $g_2 H = g_2' H$ then $g_1 g_2 H = g_1' g_2' H$, because $g_1 g_2 H = g_1 g_2' H = g_1 H g_2' = g_1' H g_2' = g_1' g_2' H$.

Definition 1.3 *If* H *is a normal subgroup of* G, *then the group* G/H *endowed with the operation defined by* $(g_1 H)(g_2 H) = (g_1 g_2 H)$ *is called the* quotient group *of* G *by* H.

1.1.1 Group homomorphisms

Definition 1.4 *Let* G, H *be groups. A function* $h : G \to H$ *is called a* homomorphism of groups *if it satisfies the following:*

$$h(g_1 g_2) = h(g_1)h(g_2), \forall g_1, g_2 \in G.$$

Remark A direct consequence of the definition of a homomorphism is that $h(e_G) = e_H$. Moreover, every homomorphism of groups satisfies

$$h(g^{-1}) = (h(g))^{-1}, \ \forall g \in G.$$

Definition 1.5 *Let G, H be groups and $h : G \to H$ a homomorphism of groups. Then:*

- *If h is injective it is a* monomorphism*.*
- *If h is surjective it is an* epimorphism*.*
- *If h is bijective it is an* isomorphism*.*
- *When $G = H$ the function h is called an* endomorphism*.*
- *If h is an isomorphism and an endomorphism, then h it is called an* automorphism*.*

Remark We say that the groups G and H are *isomorphic* it there exists an isomoprhism between them. Moreover, it means that both groups are "the same group" up to notation. We denote this by $G \cong H$.

An important result in the group theory is Cayley's theorem, previously we need to define the permutation group.

Definition 1.6 *A permutation corresponds to a bijection $f : A \to A$, with A a finite set. A permutation group corresponds to a finite group whose elements are permutations, group operation is the composition of functions.*

Example If $A = \{0, 1\}$, we can consider $f : A \to A$ as $f(0) = 1$ and $f(1) = 0$, clearly is a bijection and therefore corresponds to a permutation in 2 elements, other permutation over A is the identity function. Therefore, the permutation group in 2 elements is a group with 2 elements: f and the identity function 1_A.

Theorem 1.7 *Every group G is isomorphic to a permutation group of its own elements.*

Other important results are Isomorphism Theorems. For our purposes we only use the First Isomorphism Theorem, mainly to describe general groups using free groups.

Theorem 1.8 *[First Isomorphism Theorem] Let G, H be groups and $h : G \to H$ be an homomorphism. Then:*

- $\ker(h) = \{g \in G | h(g) = e_H\}$ *is a normal subgroup of G,*
- *$h(G)$ is a subgroup of H,*
- *$h(G) \cong G \diagup \ker(h)$.*

1.2 Groups via generators and relations

Definition 1.9 *If G is a group and S is a subset of elements in G, then S* generates *G if every element in G can be expressed as a finite product of elements from S and inverses of elements of S. We say that a group G is* finitely generated *if it has a finite generating set.*

Example The following examples show possible situations that we can have in finitely generated groups.

- Every finite group is a finitely generated group: it is enough to consider $S = G$.
- The group $\mathbb{Z}$ is generated by $S = \{1\}$, therefore it corresponds to a finitely generated group. Nevertheless, there are other possible generating sets with more elements, for instance $S' = \{2, 3\}$. This fact shows that the choice of S is not unique.
- The group $(\mathbb{Q}, +)$ is not a finitely generated group.

Remark We shall give an explicit description of generated subgroups. Consider G a group and $S \subset G$. Then, the subgroup generated by S in G always exists and can be described by:

$$\langle S \rangle = \bigcap \{H \ \mid \ H \subset G \text{ is a subgroup with } S \subset H\}.$$

In other words, we can write the group $\langle S \rangle$ as follows:

$$\langle S \rangle = \{s_1^{t_1} \cdots s_n^{t_n} \ \mid \ n \in \mathbb{N}, s_1 \dots s_n \in S, t_1 \dots t_n \in \{-1, 1\}\}$$

1.2.1 Cayley graphs

Definition 1.10 *A* directed graph *$\Gamma = (V, E)$ consists of a set of vertices V and a set of directed edges $E \subseteq V \times V$ given as ordered pairs of vertices. Each edge $e = (u, v) \in E$ has an initial vertex u and a terminal vertex v and we represent a graph visually as follows: each vertex corresponds to a point, while an edge is represented graphically by an arrow leading from vertex u to vertex v. We denote by* $\text{Ends}(e) = \{u, v\}$.

Definition 1.11 *A* symmetry *of a graph Γ is a bijection α taking vertices to vertices and edges to edges such that if* $\text{Ends}(e) = \{v, w\}$ *then* $\text{Ends}(\alpha(e)) = \{\alpha(v), \alpha(w)\}$. *We give the name* Symmetry group of Γ *to the collection of all symmetries and we denote it by* $\text{Sym}(\Gamma)$.

In this part, the main result is the following theorem which relates graph theory, symmetry groups and finitely generated groups. We recommend [38] to consult the proof of this theorem and more details about the theory.

Theorem 1.12 (Cayley's better theorem) *Every finitely generated group can be faithfully represented as a symmetry group of a connected, directed, and locally finite graph.*

The proof of this theorem shows how to create the *Cayley graph* associated to the group G. For simplicity, we show in the following definition this construction as well.

Definition 1.13 *Let G be a finitely generated group by S. We define the* right Cayley graph *associated to G as the graph $\Gamma_{G,S} = (V, E)$ constructed as follows:*

- *$V = G$,*

- *$(g_1, g_2) \in E$ if and only if there exists $s \in S$ such that $g_2 = g_1 s$.*

Remark It is important to note that the Cayley graph depends on the set of generators S. For instance, in the case of $\mathbb{Z}$ we obtain the following Cayley graph if $S = \{1\}$.

Figure 1.1: Cayley graph of the group $\mathbb{Z}$ using $S = \{1\}$ as generator.

Nevertheless, if we consider $S = \{2, 3\}$, the Cayley graph is different.

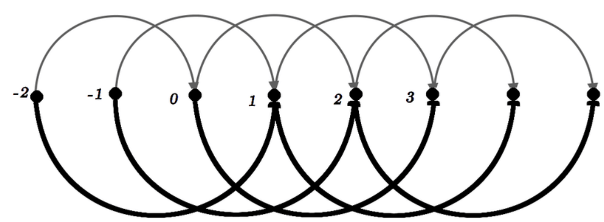


Figure 1.2: Cayley graph of the group $\mathbb{Z}$ using $S = \{2, 3\}$ as generator.

A. Cayley introduced Cayley graphs in the theory of groups in a paper published in 1878 [11]. In that article, A. Cayley states that when the group has two or more generators, we require different colours for its edges, one for each generator, as seen in Example 1.2. Regardless some authors do not distinguish edges associated to different generators by color.

1.2.2 Amenable groups

Amenable groups are an important class of groups that arise in many areas of mathematics, including analysis, geometry, topology, and algebra. They are characterized by a certain notion of "niceness" or "regularity" that makes them amenable to certain kinds of mathematical analysis.

One way to think about amenability is in terms of a certain type of averaging process. Intuitively, a group is amenable if it is possible to average functions defined on the group in a way that respects the group structure. More precisely, a group is amenable if it admits a "left-invariant mean", which is a way of assigning to each function on the group a unique "average value" that is invariant under left translations.

The study of amenable groups has led to many deep and surprising results in mathematics. For example, it is known that all finitely generated abelian groups are amenable, but there are also many examples of non-abelian amenable groups, such as the group of affine transformations of the real line. Amenability has connections to many other areas of mathematics, such as the theory of operator algebras, harmonic analysis, and geometric group theory.

Definition 1.14 (Følner sequence) *Let G be a group . A* Følner sequence *for G is a sequence of finite subsets $S_n \subset G$ such that:*

$$G = \bigcup_n S_n \quad \textit{and} \quad \forall g \in G, \frac{\#(S_n g \triangle S_n)}{\#S_n} \xrightarrow[n\to\infty]{} 0,$$

where $S_n g = \{hg \ : \ h \in S_n\}$ and $A \triangle B = (A \backslash B) \cup (B \backslash A)$ is the symmetric difference.

In the previous definition, it is easy to see that the second condition only has to be checked for g in a finite generating set. The set $S_n g \triangle S_n$ can be understood as the border of S_n, so an element of a Følner sequence must have a small border relative to its interior.

Definition 1.15 (Amenable group) *A group G is* amenable *if it admits a Følner sequence.*

This definition applies more generally for all countable groups. A few examples:

- $\mathbb{Z}^d$ is amenable and a Følner sequence is given by $S_n = [-n, n]^d$. Indeed, if $(g_i)_{1\le i\le d}$ is the canonical set of generators, then $\#S_n = (2n+1)^d$ and $\#((S_n + g_i)\triangle S_n) = 2 \cdot (2n+1)^{d-1}$.
- $\mathbb{F}_d$ for $d \geq 2$ is not amenable. In particular, the balls S_n of radius n - that is, reduced[1] words of length $\leq n$ on the set of generators $(g_i)_{1\le i\le d}$ - are not a Følner sequence. Indeed, one can easily check that $\#S_n = \Omega(d^n)$ and $\#(S_n g_i \triangle S_n) = \Omega(d^n)$.

The upcoming subsection will precisely focus on this last example, as free groups will play a crucial role in the entirety of this thesis.

1.2.3 Free groups

Definition 1.16 *Let S be a set and consider a copy $S^{-1} = \{s^{-1} \mid s \in S\}$. We say a word in $(S \cup S^{-1})^*$ is* reduced *if it does not contain ss^{-1} or $s^{-1}s$ as subwords.*

We observe that every word in $(S \cup S^{-1})^*$ can be transformed into an unique reduced word by successively eliminating every occurrence of ss^{-1} or ss^{-1}.

[1]with no $g_i^{-1} g_i$ or $g_i g_i^{-1}$ factors

Definition 1.17 *The* free group *over S*[2] *is defined as the set $\mathbb{F}[S]$ of all reduced words in $(S \cup S^{-1})^*$, endowed with word concatenation followed by reduction as its binary operation. If the cardinality of the set S is n, then we call $\mathbb{F}[S]$ the free group of rank n.*

We can describe groups in a combinatorial way. For this, free groups turn out to be very important, giving the basis for group presentations.

Theorem 1.18 *Let G be any group and let $\{g_1, \ldots, g_n\}$ be a list of elements of G, which are not necessarily distinct or non-trivial. Let $S = \{s_1, \ldots, s_n\}$ be a basis for a free group $\mathbb{F}[S]$. Then there is a group homomorphism $h : \mathbb{F}[S] \to G$ such that*

$$h(s_i) = g_i, \ \forall 1 \leq i \leq n.$$

A consequence of this theorem is that two free groups with the same amount of generators are isomorphic.

Corollary 1.19 *Any two free groups of rank n are isomorphic*

Proof. We consider G and H two free groups of rank n, with $S_G = \{g_1, \ldots, g_n\}$ and $S_H = \{h_1, \ldots, h_n\}$ the respective bases. Using Theorem 1.18 there are $h_1 : G \to H$ and $h_2 : H \to G$ homomorphisms satisfying the statement of the theorem. It follows that $h_1 \circ h_2$ is the identity automorphism of H and $h_2 \circ h_1$ the same but for G; this implies that h_1 and h_2 are bijections. □

Remark In what follows we denote by $\mathbb{F}_n$ the free group of rank n.

Corollary 1.20 *If G is generated by n elements then G is a quotient of $\mathbb{F}_n$.*

Proof. This is a consequence of Theorem 1.18 and the First Isomorphism Theorem (Theorem 1.8). It is important to note that the homomorphism $h \colon \mathbb{F}_n \to G$ given by the theorem satisfies that $h(\mathbb{F}_n)$ contains the generators of G and thus h is surjective. The result follows. □

Definition 1.21 *Let G be a group. A* relation *corresponds to an element ω in $\ker(h)$, where h is as given in the last corollary. A subset $R \subset \ker(h)$ is a set of* defining relations *if the smallest normal subgroup of $\mathbb{F}_n$ that contains R is $\ker(h)$. We say that G is* finitely presented *if there is a finite set of defining relations $R = \{\omega_1, \ldots, \omega_m\}$. In this scenario we write:*

$$G = \langle \{g_1, \cdots, g_n\} \mid \omega_1, \cdots, \omega_m \rangle$$

.

[2]The set S is called the basis of $\mathbb{F}[S]$.

Example In the case of $\mathbb{F}_n$, its presentation is given by

$$\mathbb{F}_n = \langle \{g_1, \cdots, g_n\} \mid \emptyset \rangle = \langle \{g_1, \cdots, g_n\} \mid \rangle.$$

The corresponding Cayley graph of $\mathbb{F}_2 = \langle \{a, b\} \mid \rangle$ is the following:

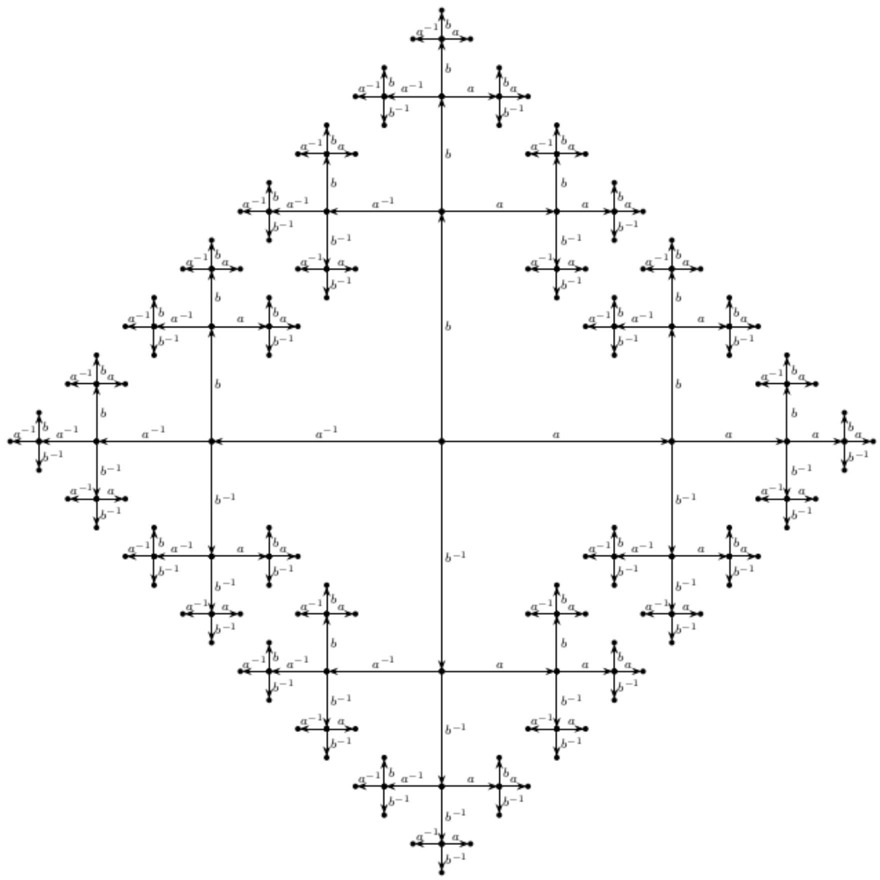


Figure 1.3: Cayley graph of the free group of rank 2.

1.3 Symbolic dynamics

The author recommends to consult the following bibliography [7], [12], [30], [31] and [35]. For more details about the results in this section these books develop symbolic dynamics on more general groups, for this thesis work we will only need to understand the theory of shift spaces defined on finitely generated groups.

1.3.1 Symbolic dynamics on finitely generated groups

Let G be a finitely generated group and $\mathcal{A}$ be a nonempty finite set. We consider the product set $\mathcal{A}^G = \{x : G \to \mathcal{A}\}$ consisting of all functions from G to $\mathcal{A}$. We refer to the set $\mathcal{A}$ as an *alphabet*, its elements $a \in \mathcal{A}$ will be called *symbols* or *colors* and the elements of $\mathcal{A}^G$ *colorings* of G. Given $x = (x_g)_{g\in G} \in \mathcal{A}^G$, each x_g corresponds to the symbol seen at position $g \in G$. The group G acts on $\mathcal{A}^G$ by left translation $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ defined coordinatewise as:

$$\sigma(g, x)_h := x_{g^{-1}h} \quad \forall h \in G.$$

We refer to σ as the shift *action* and we use the notation $\sigma_g(x)$ to denote $\sigma(g, x)$. Given a finite subset $F \subset G$, an element $P \in \mathcal{A}^F$ is called a *pattern* and $F = (P)$ its *support*. We say that a pattern P *appears* in a coloring $x \in \mathcal{A}^G$ (and we write $P \sqsubseteq x$) if there exists $g \in G$ such that $\sigma_g(x)|_F = P$. For more details see [12] and [34]. Since $\mathcal{A}$ is a finite set, $\mathcal{A}^G$, endowed with the pro-discrete topology, is a compact space and has a countable clopen basis given by the *cylinders* $[P]_g = \{x \in \mathcal{A}^G \mid \sigma_{g^{-1}}(x)|_{(P)} = P\}$.

The pair $(\mathcal{A}^G, \sigma)$ is called *G-full shift* and every closed and σ-invariant subset $X \subseteq \mathcal{A}^G$ corresponds to a *G-subshift*. An equivalent, more combinatorial way of specifying a subshift uses a set of forbidden patterns:

Proposition 1.22 *A subset $X \subseteq \mathcal{A}^G$ is a G-subshift if and only if there exists a family of patterns $\mathcal{F}$, such that X coincides with $X_{\mathcal{F}} = \{x \in \mathcal{A}^G \mid \forall P \sqsubseteq x : P \notin \mathcal{F}\}$.*

Proof. Let $\mathcal{F}$ be a set of patterns. Then $X_{\mathcal{F}} = \{x \in \mathcal{A}^G \mid \forall P \sqsubseteq x : P \notin \mathcal{F}\}$. Clearly $X_{\mathcal{F}}$ is σ-invariant and its complement $X_{\mathcal{F}}^c = \bigcup_{g\in G} \bigcup_{P\in\mathcal{F}} [P]_g$ is open as a union of open sets, thus $X_{\mathcal{F}}$ is closed.

If X is a G-subshift, X^c is open and thus a countable union of cylinders of the form $[P_i]_{g_i}$; $x \in X^c$ if and only if some pattern P_i appears in some $\sigma_g(x)$ (because X is σ-invariant). Then, taking $\mathcal{F}$ as the set of all patterns P_i, we prove that $X = X_{\mathcal{F}}$: if $x \in X_{\mathcal{F}}$, then $x \notin \bigcup_i [P_i]_{g_i}$, and thus $x \notin X^c$. Conversely, if $x \in X \setminus X_{\mathcal{F}}$, then there exists P_i such that $P_i \sqsubseteq x$, thus for some $g \in G$, $x \in [P_i]_{gF}$ (where $P_i \in \mathcal{A}^F$);then, $\sigma_{gg_i^{-1}}(x) \in [P_i]_{g_iF} \subseteq X^c$. □

If there exists a finite family of forbidden patterns $\mathcal{F}$, such that $X = X_{\mathcal{F}}$, the G-subshift X is called a *G-subshift of finite type* (G-SFT). Among G-SFTs, we distinguish two kinds that have a particulary simple description.

Definition 1.23 *A G-SFT is called a* G-nearest neighbor subshift of finite type *(G-NNSFT),* *if we can choose a set of forbidden patterns $\mathcal{F}$, such that every element in $\mathcal{F}$ has a support of the form $\{e_G, s\}$, $s \in S$.*

Remark If $X_{\mathcal{F}}$ is a G-NNSFT, we identify a pattern with support $\{e_G, s\}$, $s \in S$, with a 3-tuple $(a, s, b) \in \mathcal{A} \times S \times \mathcal{A}$ and interpret it as the following restriction: if $x \in X_{\mathcal{F}}$ and $x_g = a$, then $x_{gs} \neq b$, similarly $x_{gs} = b$ forces $x_g \neq a$.

1.3.2 Symbolic morphisms

Let $X \subseteq \mathcal{A}^G$ and $Y \subseteq \mathcal{A}^G$ be two G subshifts. A map $\phi : X \to Y$ is said to be shift commuting, if:

$$\phi \circ \sigma^g = \sigma^g \circ \phi \ , \ \forall g \in G$$

Definition 1.24 *A continuous shift commuting map between two G-subshifts is called a morphism.*

A bijetive morphism $\phi : X \to Y$ is called conjugacy, and we said that X is conjugated to Y, this fact is written $X \cong Y$. What is particular about the case of symbolic systems is that the morpshisms can be characterized combinatorially. Let $\mathcal{A}, \mathcal{B}$ be two alphabets and $F \subset G$ a finite support, we consider a map $\Phi : \mathcal{A}^F \to \mathcal{B}$, which send patterns in $\mathcal{A}^F$ to symbols in $\mathcal{B}$. It is possible to define a map $\phi : \mathcal{A}^G \to \mathcal{B}^G$ given by $\phi(x)_g = \Phi(\sigma^{g^{-1}}(x)|_F)$. Any map defined as before is called sliding-block code.

Theorem 1.25 (Curtis-Hedlund-Lyndon) *Let $X \subseteq \mathcal{A}^G$ and $Y \subseteq \mathcal{B}^G$ be G-subshifts and $\phi : X \to Y$ be a map. Then ϕ corresponds to a morphisms if and only if ϕ is a sliding-block code.*

The original theorem considered $G = \mathbb{Z}$, the lector can see the proof in [23], where M. Curtis and R.Lyndon are mentioned as co-discoverers.

The next proposition is a folklore result in symbolic dynamics. For completeness we include a formal proof, which for general groups is otherwise hard to find in the literature.

Proposition 1.26 *Every G-SFT is topologically conjugate to a G-NNSFT.*

Proof. Given a G-SFT $X_{\mathcal{F}} \subseteq \mathcal{A}^G$, let $B_r \subseteq G$ be a sufficiently large ball of radius $r \in \mathbb{N}$ with respect to the word metric given by S, such that $\bigcup_{P \in \mathcal{F}}(P) \subseteq B_r$. Let us consider the following alphabet:

$$\mathcal{A}' = \{Q \in \mathcal{A}^{B_r} | \forall P \in \mathcal{F}, P \not\sqsubseteq Q\}$$

We define $X_{\mathcal{F}'} \subseteq (\mathcal{A}')^G$ given by the set of forbidden patterns $\mathcal{F}' \subseteq \bigcup_{s \in S}(\mathcal{A}')^{\{e_G, s\}}$ with $\mathcal{F}' = \{(P, s, Q) \in \mathcal{A}' \times S \times \mathcal{A}' | \exists h \in B_r \cap sB_r : P_h \neq Q_{s^{-1}h}\}$. (This construction is similar to the definition of a higher block shift in the one-dimensional setting). By definition, the subshift $X_{\mathcal{F}'}$ corresponds to a G-NNSFT. To prove that it is topologically conjugated to $X_{\mathcal{F}}$, let $\varphi : X_{\mathcal{F}'} \to X_{\mathcal{F}}$ be the map given by $\varphi(x)_g = (x_g)_{e_G}$. Given that φ is defined locally, we can use the theorem of Curtis-Hedlund-Lyndon [12, Theorem 1.8.1] to conclude that φ is σ-invariant and corresponds to a continuous map.

Suppose there exists $x \in X_{\mathcal{F}'}$ such that $\varphi(x) \notin X_{\mathcal{F}}$, which implies that there exists $g \in G$ and $P \in \mathcal{F}$ such that $\varphi(x)_{g(P)} = P$. First, let us see that by definition of $\mathcal{F}'$ we have that for all $g \in G$, $s \in S$ and $h \in B_r$: $x_g(h) = x_{gs}(s^{-1}h)$, writing $h = s_1 \dots s_n$, with $s_1, \dots, s_n \in S \cup S^{-1}$ and $n \leq r$, we have $x_g(h) = x_{gs_1}(s_1^{-1}h)$, iterating this we obtain:

$$x_g(h) = x_{gs_1s_2}(s_2^{-1}s_1^{-1}h) = \cdots = x_{gs_1\dots s_n}(s_n^{-1}\dots s_1^{-1}h) = x_{gh}(e_G).$$

Using this last fact and given that $(P) \subseteq B_r$ we can conclude that $P \sqsubseteq x_g$, contradicting the definition of $\mathcal{A}'$. Therefore $\varphi(X_{\mathcal{F}'}) \subseteq X_{\mathcal{F}}$.

Let us consider two distinct colorings $y, z \in X_{\mathcal{F}'}$, without loss of generality, we can suppose that $y_{e_G} \neq z_{e_G}$, thus there exists $h \in B_r$ such that $y_{e_G}(h) \neq z_{e_G}(h)$. As these colorings are valid on $X_{\mathcal{F}'}$, this implies that $y_{e_G}(h) = y_h(e_G)$ (similarly for z), thus $\varphi(y)_h \neq \varphi(z)_h$ implying that φ is injective. Let $x \in X_{\mathcal{F}}$ be an arbitrary element, we can consider take $y \in X_{\mathcal{F}'}$ given by $y_g = \sigma_{g^{-1}}(x)|_{B_r}$, then $\varphi(y) = x$. Therefore, the map φ is bijective. □

A special subfamily of G-NNSFTs which will be of particular interest in this paper was introduced (in the context of $G = \mathbb{Z}^2$) by H.Wang and is called G-Wang subshifts:

Definition 1.27 *A G-*Wang tile *T corresponds to a map $T : S \cup S^{-1} \to \mathcal{A}$, with $\mathcal{A}$ a finite set. Given a finite set W of G-Wang tiles, the G-*Wang subshift *is defined as:*

$$X_W = \left\{(x_g)_{g\in G} \in W^G \mid \forall s \in S, g \in G : x_g(s) = x_{gs}(s^{-1})\right\}.$$

*The elements in X_W are called G-*Wang tilings*.*

Remark We note that every G-Wang subshift is indeed a G-NNSFT, whose set of forbidden patterns is given by:

$$\mathcal{F} = \bigcup_{s\in S} \{P : \{e_G, s\} \to W \mid P_{e_G}(s) \neq P_s(s^{-1})\}.$$

Example If we consider $\mathcal{A} = \{a, b, c, d\}$, examples of Wang tiles for each of the groups $\mathbb{Z} = \langle s \mid \rangle$, $\mathbb{Z}^2 = \langle s_1, s_2 \mid s_1s_2s_1^{-1}s_2^{-1}\rangle$ and $\mathbb{F}_2 = \langle s_1, s_2 \mid \rangle$ are given in Figure 1.4.

The $\mathbb{Z}$-Wang tile is the map $T : \{s, s^{-1}\} \to \{a, b\}$ defined by $T(s^{-1}) = a$ and $T(s) = b$. In the resuming two examples, both groups are 2-generated, therefore the map for the $\mathbb{Z}^2$- or $\mathbb{F}_2$-Wang tile, respectively, is defined as $T : \{s_1, s_2, s_1^{-1}, s_2^{-1}\} \to \{a, b, c, d\}$ given by $T(s_1) = b$, $T(s_2) = c$, $T(s_1^{-1}) = a$ and $T(s_2^{-1}) = d$. The form chosen to draw a $\mathbb{Z}^2$- or $\mathbb{F}_2$-Wang tile depends on the geometry given by the group and the dual of its Cayley graph, but the map is given abstractly and is independent of the visualization.

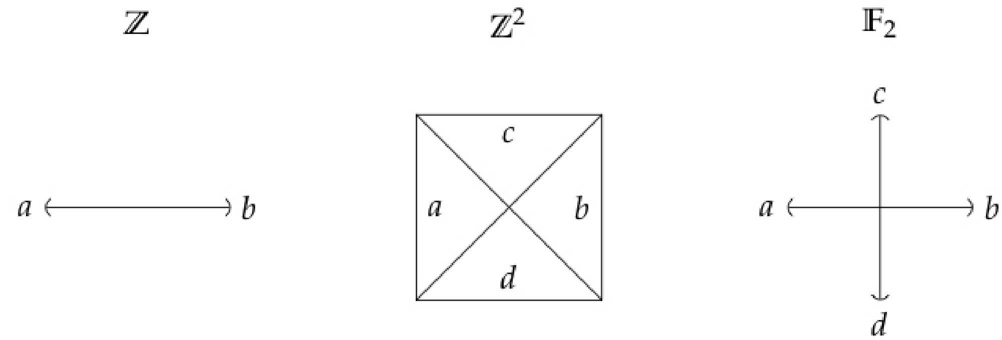


Figure 1.4: Examples of Wang-tiles

1.4 State of the art of subshifts of finite type on finitely generated groups

In this section we expose a list of results about the theory developed for subshifts of finite type defined in $\mathbb{Z}$ and $\mathbb{F}_k$. We chose to write this because the results obtained in this thesis are related to both types of groups.

In particular, we recommend to see more details about the theory in $\mathbb{Z}$ a classic book wirtten by D. Lind and B. Marcus [34] and also P. Kurka [31].

1.4.1 Shifts of finite type on $\mathbb{Z}$

The class of $\mathbb{Z}$-SFTs, also called *one-dimensional* shifts of finite type, has been thoroughly studied. Let $\mathcal{A}$ be an alphabet and $x \in A^{\mathbb{Z}}$ be a *configuration*. The configuration x may be written as

$$x = \cdots x_{-2}x_{-1}.x_0x_1x_2\cdots$$

where $x_i \in \mathcal{A}$, for each $i \in \mathbb{Z}$. The dot written between x_{-1} and x_0 is important because it indicates the "initial" position in x.

Example Some examples of $\mathbb{Z}$-SFTs are the following:

a) The $\mathbb{Z}$-full shift is a $\mathbb{Z}$-SFT. This subshift has no restrictions, therefore we may consider $\mathcal{F} = \emptyset$ as its corresponding set of forbidden patterns.

b) The Golden mean shift $X = X_{\mathcal{F}}$, where $\mathcal{F} = \{11\}$, is a classic example in the theory of $\mathbb{Z}$-SFTs. For instance: $x = \ldots 01.1001000 \cdots \in X_{\mathcal{F}}$.

c) Let us consider $\mathcal{F} = \{0000, 11\}$. The corresponding $\mathbb{Z}$-SFT $X = X_{\mathcal{F}}$ can be written using

$$\mathcal{F}' = \{0000, 1100, 0110, 0011\}$$

obtaining the same subshift but with a set of forbidden patterns whose elements have the same size. We note that, for a given subshift X, there are usually many different sets $\mathcal{F}$ such that $X = X_{\mathcal{F}}$.

Corresponding subshifts can be represented by finite directed graphs, such that valid $\mathbb{Z}$-configurations are in bijection with bi-infinite walks on said graph (see [34]).

Example Let us consider the $\mathbb{Z}$-full shift $X_\emptyset$ on 2 symbols. Its associated graph is:

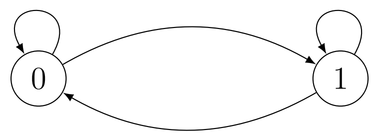


Figure 1.5: The $\mathbb{Z}$-full shift on 2 symbols as a vertex shift.

Example Considering more restrictions, take $X = X_{\mathcal{F}}$ where

$$\mathcal{F} = \{11, 00\}$$

The associated graph is:

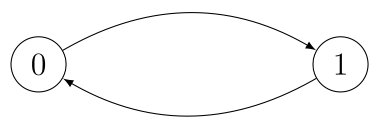


Figure 1.6: Graph associated to $X_{\{11,00\}}$.

It is not possible to get a configuration with two consecutive 1's or 0's. Every time that we see 1 must be 0 and vice versa the following symbol in the configuration.

With this in our mind, we can easily decide whether there is a valid $\mathbb{Z}$-configuration by verifying that the associated directed graph allows a bi-infinite path, i.e. it has a cycle. Therefore, both the Emptiness and the Domino problem on $\mathbb{Z}$ are decidable, and, in fact, every nonempty $\mathbb{Z}$-SFT has at least one periodic configuration.

Remark In contrast to the results in symbolic dynamics on $\mathbb{Z}$ with respect to the study of subshifts of finite type. In $\mathbb{Z}^d$, SFTs are not represented by graphs and the results obtained in $\mathbb{Z}$ do not necessarily hold. Indeed, decidability of the Emptiness problem is much less obvious and the Domino problem in particular, has a long history. In 1961, H.Wang conjectured [45], that if a finite set of $\mathbb{Z}^2$-Wang tiles can tile $\mathbb{Z}^2$ then it also allows for periodic tilings. He then observed that proving this conjecture would imply the existence of an algorithm to decide the Domino problem on the plane. In 1966, Wang's student R. Berger [6], showed that Wang's conjecture is false. He constructed an enormous alphabet of 20.426 $\mathbb{Z}^2$-Wang tiles giving rise to a nonempty weakly aperiodic $\mathbb{Z}^2$-Wang subshift. Using properties of that subshift, R. Berger furthermore showed that the Domino problem in $\mathbb{Z}^2$ is in fact undecidable.

1.5 Subshifts of finite type on Free groups $\mathbb{F}_n$

In this section we will expose a list of results about periodic properties of subshifts of finite type on free groups. Our focus is in the results obtained in [42] by S.Piantadosi, mainly in the study of nearest neighbor subshifts of finite type. Moreover, some definitions, notations and results appear in [16].

Given an alphabet $\mathcal{A}$, we consider the $\mathbb{F}_k$-full shift $\mathcal{A}^{\mathbb{F}_k}$ and study $\mathbb{F}_k$-NNSFTs $X_{\mathcal{F}} \subseteq \mathcal{A}^{\mathbb{F}_k}$. The following definition was introduced by S.Piantadosi in [42].

Definition 1.28 *Given a $\mathbb{F}_k$-NNSFT $X_{\mathcal{F}} \subseteq \mathcal{A}^{\mathbb{F}_k}$, $X_{\mathcal{F}}$ has a* coloring function *for the alphabet $\mathcal{A}$ if there exists a function $\Phi : \mathcal{A} \times \left(S \cup S^{-1}\right) \to \mathcal{A}$ such that for every $a \in \mathcal{A}$ and $s \in S$, $(a, s, \Phi(a, s)) \notin \mathcal{F}$ and $(\Phi(a, s^{-1}), s, a) \notin \mathcal{F}$.*

Using these coloring functions, S.Piantadosi gave a condition to determine whether or not a $\mathbb{F}_k$-NNSFT is empty.

Proposition 1.29 *[42, Proposition 1] A $\mathbb{F}_k$-NNSFT $X_{\mathcal{F}} \subseteq \mathcal{A}^{\mathbb{F}_k}$ is nonempty if and only if there exists a nonempty subset $\mathcal{A}' \subseteq \mathcal{A}$ and a coloring function for the alphabet $\mathcal{A}'$.*

As it turns out $\mathbb{F}_k$-NNSFTs are relatively easy to analyze. For each generator $s \in S$, we can see the set of forbidden patterns with support $\{e_{\mathbb{F}_k}, s\}$ as defining a $\mathbb{Z}$-NNSFT.

Remark Given $\{X_{\mathcal{F}_i}\}_{i=1}^{k}$ a family of k nonempty $\mathbb{Z}$-NNSFTs, we can obtain a $\mathbb{F}_k$-NNSFT $X_{\mathcal{F}}$ from these by letting $x \in X_{\mathcal{F}}$ if and only if $\forall g \in \mathbb{F}_k, 1 \leq i \leq k$, $(x_g, x_{gs_i}) \notin \mathcal{F}_i$. We will see that $X_{\mathcal{F}}$ might be empty. For this, it is enough to look at the following $\mathbb{F}_2$-example: consider $\mathcal{A} = \{0, 1, 2\}$ and $X_{\mathcal{F}_1}$, $X_{\mathcal{F}_2}$ two nonempty $\mathbb{Z}$-NNSFTs given by $\mathcal{F}_1 = \{00, 02, 11, 12, 20, 21, 22\}$ and $\mathcal{F}_2 = \{00, 01, 02, 10, 11, 20, 22\}$. Using the notation of Proposition 1.29, we have for $X_{\mathcal{F}_1}$ and $X_{\mathcal{F}_2}$, the subsets $\mathcal{A}'_1 = \{0, 1\}$ and $\mathcal{A}'_2 = \{1, 2\}$ respectively. Our claim is that nevertheless the resulting $X_{\mathcal{F}}$ is empty. Indeed, suppose the existence of $x \in X_{\mathcal{F}}$; without loss of generality suppose $x_{e_{\mathbb{F}_2}} = 0$. It is clear that $x_{s_1^{-1}} = 1 = x_{s_1}$ but it is impossible to put a valid symbol on x_{s_2} and $x_{s_2^{-1}}$, because $0 \notin \mathcal{A}'_2$, thus such x cannot exist, hence $X_{\mathcal{F}}$ is empty.

Definition 1.30 *Given a family of $\mathbb{Z}$-NNSFTs $\{X_{\mathcal{F}_i} \subseteq \mathcal{A}_i^{\mathbb{Z}}\}_{i=1}^{k}$, a nonempty set $\mathcal{A}' \subseteq \bigcup_{i=1}^{k} \mathcal{A}_i$ is called a* common alphabet *if each $X_{\mathcal{F}_i}$ admits a coloring function $\Phi_i : \mathcal{A}' \times \{s^{-1}, s\} \to \mathcal{A}'$ on $\mathcal{A}'$.*

For the example presented in previous remark, there is no such common alphabet for the resulting $\mathbb{F}_2$-NNSFT. The only possible choice for a common alphabet would be $\mathcal{A}' = \{1\} \subseteq \{0, 1, 2\}$, however every possible coloring function on $X_{\mathcal{F}_1}$ is necessarily defined over $\{0, 1\}$ (similarly for $X_{\mathcal{F}_2}$ considering $\{1, 2\}$). We will see that the existence of a common alphabet is a necessary and sufficient condition to obtain a nonempty $\mathbb{F}_k$-NNSFT from a given family of k nonempty $\mathbb{Z}$-NNSFTs.

Proposition 1.31 *If $X_{\mathcal{F}}$ is a nonempty $\mathbb{F}_k$-NNSFT. There exist $X_1, ..., X_k$ nonempty $\mathbb{Z}$-NNSFTs that completely determine X, in the sense that, for all $1 \leq i \leq k$ there is $\mathcal{F}_i$ that only considers the forbidden patterns restricted to generator $s_i \in S$.*

Proof. Let $\{X_{\mathcal{F}_i} \subseteq \mathcal{A}_i^{\mathbb{Z}}\}_{i=1}^k$ be a set of k nonempty $\mathbb{Z}$-NNSFTs. The common alphabet $\mathcal{A}' \subseteq \bigcup_{i=1}^k \mathcal{A}_i$ implies for each $X_{\mathcal{F}_i}$ the existence of a coloring function $\Phi_i : \mathcal{A}' \times \{s, s^{-1}\} \to \mathcal{A}'$. We construct the nonempty $\mathbb{F}_k$-NNSFT $X_{\mathcal{F}}$ considering $\mathcal{A}'$ and defining the coloring function $\Phi : \mathcal{A}' \times \left(S \cup S^{-1}\right) \to \mathcal{A}'$ as an extension of the coloring functions Φ_i, which means that for every $a \in \mathcal{A}'$ and $s_i \in S \cup S^{-1}$, we set $\Phi(a, s_i) = \Phi_i(a, s)$.

With this, valid colorings exist and thus $X_{\mathcal{F}}$ is nonempty. Indeed, fixing $x_{e_{\mathbb{F}_k}} \in \mathcal{A}'$ and using the coloring function Φ, we extend the coloring on $\{e_{\mathbb{F}_k}\}$ to a valid coloring on the entire ball $B_1 = \{e_{\mathbb{F}_k}\} \cup S \cup S^{-1}$ of radius 1. For every $1 \leq i \leq k$ we put $x_{s_i} = \Phi(x_{e_{\mathbb{F}_k}}, s_i)$ and $x_{s_i^{-1}} = \Phi(x_{e_{\mathbb{F}_k}}, s_i^{-1})$. This is always possible because every coloring function Φ_i uses the alphabet $\mathcal{A}'$. Inductively, we extend a valid pattern from B_n to a valid pattern on B_{n+1} as follows: let $g \in B_n \setminus B_{n-1}$ be an arbitrary element, by definition we write $g = s_1 \dots s_n$, with $s_1, \dots, s_n \in S \cup S^{-1}$, then if $s_n \in S$ we put $x_{gs} = \Phi(x_g, s)$, $\forall s \in (S \cup S^{-1}) \setminus \{s_n^{-1}\}$ and if $s_n \in S^{-1}$ then $x_{gs} = \Phi(x_g, s)$, $\forall s \in (S \cup S^{-1}) \setminus \{s_n\}$. Note that by construction none of the occurring patterns (x_g, s, x_{gs}) is forbidden. By compactness we obtain a valid coloring on $X_{\mathcal{F}}$. □

Remark Note that on general groups Proposition 1.31 does not hold. Consider two nonempty $\mathbb{Z}$-NNSFTs $X_{\mathcal{F}_1}, X_{\mathcal{F}_2} \subseteq \{0,1,2\}^{\mathbb{Z}}$ given by $\mathcal{F}_1 = \{00, 02, 10, 11, 21, 22\}$ and $\mathcal{F}_2 = \{00, 11, 12, 21, 22\}$, the alphabet $\mathcal{A}' = \{0,1,2\}$ already corresponds to a common alphabet. However, on $\mathbb{Z}^2 = \langle \{s_1, s_2\} \mid s_1 s_2 s_1^{-1} s_2^{-1} \rangle$, the corresponding $\mathbb{Z}^2$-NNSFT $X_{\mathcal{F}} \subseteq \{0,1,2\}^{\mathbb{Z}^2}$ using $\mathcal{F}_1$ and $\mathcal{F}_2$ for the generators s_1 and s_2 respectively, is empty. Indeed, let us suppose the existence of $x \in X_{\mathcal{F}}$. Without loss of generality we may suppose that $x_{e_{\mathbb{Z}^2}} = 0$ then $x_{s_1} = 1$ and $x_{s_1^2} = 2$. At the same time we have $x_{s_1 s_2} = 0 = x_{s_1^2 s_2}$. Thus no valid pattern on the subgroup $\langle s_1 \rangle$ that can be extended to all of $\mathbb{Z}^2$, since $(0, s_1, 0) \in \mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$.

1.5.1 Condition to obtain a periodic configuration on $\mathbb{F}_n$

Definition 1.32 *A* finite walk *on the directed graph Γ corresponds to a sequence of vertices $v_1 \dots v_n$ such that for every $j \in \{1, \dots, n-1\}$: $(v_j, v_{j+1}) \in E$. Similarly, a* bi-infinite walk *corresponds to a bi-infinite sequence of vertices $(v_n)_{n \in \mathbb{Z}}$ such that $(v_n, v_{n+1}) \in E$, $\forall n \in \mathbb{Z}$. A* cycle*, denoted by $\overline{v_1, \dots, v_n}$, is a closed walk on Γ, i.e., $(v_n, v_1) \in E$. A* simple cycle *is in particular a cycle but consisting only of different vertices. We denote by $SC(\Gamma)$ the set of all simple cycles on Γ. Let w be a cycle and $v \in V$. We define:*

$$\#_v(w) = |\{i \mid w_i = v, 1 \leq i \leq |w|\}|.$$

We can write $\#_v(w)$ using the simple cycles on Γ, let $SC(\Gamma) = \{c_i\}_{i=1}^{|SC(\Gamma|}$, then $\#_v(w) = \sum_{i=1}^{|SC(\Gamma)|} x_i \#_v(c_i)$, where $x_1, \dots, x_{|SC(\Gamma)|}$ are non-negative integers. If C is a set of cycles on Γ, then: $\#_v(C) = \sum_{w \in C} \#_v(w)$.

Remark Given a nonempty $\mathbb{Z}$-NNSFT $X_{\mathcal{F}} \subseteq \mathcal{A}^{\mathbb{Z}}$, it is possible to obtain a graph $\Gamma = (V, E)$ as follows: $V = \mathcal{A}$ and $(a, b) \in E$ if and only if $(a, s, b) \notin \mathcal{F}$. Moreover, we can identify each coloring of $X_{\mathcal{F}}$ with a bi-infinite walk in a essential directed graph $\Gamma' = (V', E')$, where $V' = \{v \in V | \, V^+(v) \neq \emptyset \wedge V^-(v) \neq \emptyset\}$ and $E' \subseteq V' \times V'$. The essential directed graph Γ' as before is unique and in what follows for every $\mathbb{Z}$-NNSFT we only consider the essential directed graph associated.

Corollary 1.33 *A nonempty $\mathbb{F}_k$-NNSFT is completely determined*[3] *by a family of k essential directed graphs Γ if and only if all the graphs on Γ are defined on a common set of vertices (considering the set of common vertices $V = \mathcal{A}'$).*

Studying the existence of periodic colorings, we see that every nonempty $\mathbb{F}_k$-NNSFT contains at least a weakly periodic coloring. This is a consequence of Proposition 1.31 and the fact that every nonempty $\mathbb{Z}$-NNSFT contains periodic colorings[4]. In [42] S.Piantadosi gave necessary and sufficient conditions for the existence of a strongly periodic coloring.

Theorem 1.34 *[42, Theorem 3.4] A nonempty $\mathbb{F}_k$-NNSFT X_Γ contains a strongly periodic coloring if and only if there exists a family $C = (C_i)_{i=1}^k$ of finite sets of cycles on $\Gamma = \{\Gamma_i\}_{i=1}^k$ such that for all $a \in \mathcal{A}$ and $1 \leq i < j \leq k$ we have $\#_a(C_i) = \#_a(C_j)$.*

Example Let $X_\Gamma \subseteq \{0,1\}^{\mathbb{F}_2}$ be a nonempty $\mathbb{F}_2$-NNSFT, where $\Gamma = \{\Gamma_1, \Gamma_2\}$ are given as follows:

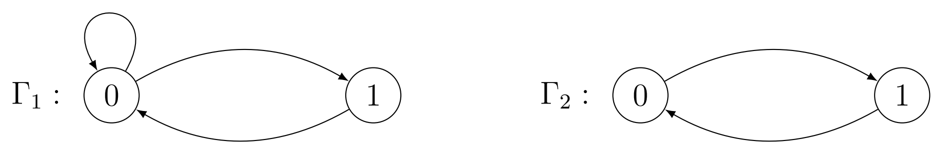


Figure 1.7: Example of a strongly periodic coloring.

There exists a strongly periodic coloring on X_Γ, because the cycle $\overline{01}$ is common to both graphs and thus in Theorem 1.34 we can use $C_1 = C_2 = \{\overline{01}\}$ to conclude.

In contrast to Example 1.5.1, S.Piantadosi also constructs a $\mathbb{F}_2$-NNSFT without any strongly periodic coloring using an alphabet with 3 symbols [42, Example 3]. We give another example of a $\mathbb{F}_2$-NNSFT without any strongly periodic coloring, also using an alphabet with 3 symbols.

Example Consider the following two directed essential graphs:

Let $X_\Gamma \subseteq \{0,1,2\}^{\mathbb{F}_2}$ with $\Gamma = \{\Gamma_1, \Gamma_2\}$. Observe that X_Γ is nonempty and X_Γ corre-

[3]In the same sense of the proposition 1.31

[4]For every nonempty $\mathbb{Z}$-NNSFT a periodic coloring stems from the existence of a cycle in its corresponding essential directed graph.

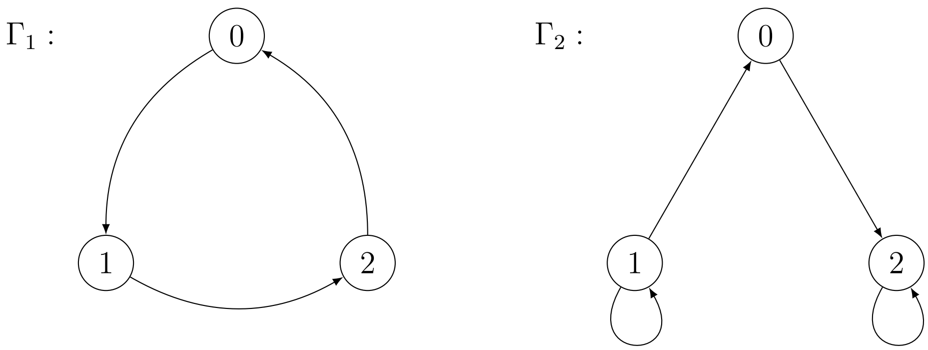


Figure 1.8: Example of a $\mathbb{F}_2$-NNSFT without any strongly periodic coloring.

sponds to a weakly aperiodic $\mathbb{F}_2$-NNSFT. Indeed, the cycles on Γ_1 necessarily hace the form $\overline{012\ldots012}$ and the possible cycles on Γ_2 are $\overline{1\ldots1}$ and $\overline{2\ldots2}$, thus for every nonempty set of cycles C_1 and C_2 on Γ_1 and Γ_2 respectively, the condition $\#_0(C_1) = \#_0(C_2)$ is always violated. Therefore, X_Γ effectively does not contain any strongly periodic coloring.

Part II

Wang subshifts on finitely generated groups

Chapter 2

Necessary conditions for tiling finitely generated amenable groups

This chapter corresponds to the manuscript developed by B. Hellouin de Menibus from Université Paris-Saclay and H.Maturana Cornejo, "*Necessary conditions for tiling finitely generated amenable groups*" [19], published on *Discrete and continous dynamical system* 2020, Doi: http://dx.doi.org/10.3934/dcds.2020116.

2.1 Introduction

$\mathbb{Z}^2$-subshifts of finite type (SFT) are a set of colourings of the 2-dimensional lattice $\mathbb{Z}^2$, or *tilings*, defined by a finite set of local restrictions. There are various equivalent ways to express the restrictions, such as the Wang tiles formalism introduced by Hao Wang [45]. This formalism was introduced to study the *domino problem*: given as input a set of restrictions (e.g. a set of Wang tiles), is there an algorithm that decides whether there is a tiling of $\mathbb{Z}^2$ that respects those restrictions?

R. Berger [6] showed that the domino problem is undecidable. The proof depends heavily on notions of periodicity and aperiodicity, more precisely on the existence of a set of Wang tiles that only tile $\mathbb{Z}^2$ in a strongly aperiodic manner. This is in stark contrast with the situation on $\mathbb{Z}$ where the domino problem is decidable thanks to a graph representation [34].

There has been a recent interest in symbolic dynamics on more general contexts, such as where the lattice $\mathbb{Z}^2$ is replaced by the Cayley graph of an infinite, finitely generated group. Using again the existence of strongly aperiodic SFTs, the domino problem was shown to be undecidable, apart from $\mathbb{Z}^d$, on some semisimple Lie groups [41], the Baumslag-Solitar groups [2], the discrete Heisenberg group (announced, [44]), surface groups [15, 1], semidirect products on $\mathbb{Z}^2$ [5] or some direct products [4], polycyclic groups [25], some hyperbolic groups [14]... It is decidable on free groups [42] and on virtually free groups [3], and it is conjectured that these are the only groups where the domino problem is decidable (Conjecture 2.8 below).

As a consequence, outside of free and virtually free groups, one can not expect to find simple necessary and sufficient conditions for admitting a valid tiling. However, heuristics can be very useful when making an exhaustive search for SFTs with desired properties; necessary conditions in particular allow fast rejection of most empty SFTs. For example, a transducer-based heuristic was used in the search for the smallest set of Wang tiles that yield a strongly aperiodic $\mathbb{Z}^2$-SFT [27]. It is also of theoretical interest to understand how the group properties impact necessary conditions.

2.1.1 Statements of results

We first consider a necessary and sufficient condition introduced by S. Piantadosi for an SFT on the free group to admit a valid tiling [42]. It is well-known that an SFT on a finitely generated group can only admit a tiling if the "corresponding" SFT on the free group does, so this becomes a necessary condition on an arbitrary f.g. group (Corollary 2.14).

The next two stronger conditions were introduced by S. Piantadosi (to decide if an SFT admits a strongly periodic tiling of the free group) and by J. Chazottes-J. Gambaudo-F. Gautero [13] in a more general context of tiling the euclidean plane by polygons, but which is necessary for an SFT to admit a tiling of $\mathbb{Z}^2$ [28]. We prove that the two conditions are equivalent (Theorem 2.12), and that they form a necessary condition for an SFT to admit a valid tiling on any finitely generated amenable group (Theorem 2.17), confirming a remark of E. Jeandel ([26], Section 3.1).

Finally, we provide for any non-free finitely generated group a counterexample that satisfies all conditions but does not provide a valid tiling.

2.2 Preliminaries

2.2.1 Symbolic dynamics on groups

Let G be an infinite, finitely generated group with unit element 1_G. We write $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ where $\mathcal{S} = \{g_1, \ldots, g_d\}$ is a finite set of generators and $\mathcal{R} = \{r_1, \ldots, r_m, \ldots\} \subset (\mathcal{S} \cup \mathcal{S}^{-1})^*$ is a (possibly infinite) set of relations. By convention $r \in \mathcal{R}$ means that $r = 1_G$.

For instance:

- the free group $\mathbb{F}_d$ is the group on d generators with no relations;
- $\mathbb{Z}^2 = \langle \{g_1, g_2\} \mid g_1 g_2 g_1^{-1} g_2^{-1} \rangle$.

Definition 2.1 (Weakly & strongly aperiodic) *For a configuration $x \in \mathcal{A}^G$, we define the orbit of the element x under the shift action as $\mathrm{Orb}_\sigma(x) = \{\sigma_g(x) | g \in G\}$ and the set of elements on G that fix the configuration x by $Stab_\sigma(x) = \{g \in G | \sigma_g(x) = x\}$. A configuration $x \in \mathcal{A}^G$ is*

strongly periodic *if $Stab_\sigma(x)$ has finite index or, equivalently, if $\mathrm{Orb}_\sigma(x)$ is finite;*

strongly aperiodic *if $Stab_\sigma(x) = \{1_G\}$.*

weakly periodic *if it is not strongly aperiodic;*

weakly aperiodic *if it is not strongly periodic.*

More generally, a subshift $X \subset \mathcal{A}^G$ is weakly/strongly aperiodic if every configuration on X is weakly/strongly aperiodic.

Example In $G = \mathbb{Z}^2$,

- the configuration x such that $x_g = 0$ for all g is strongly periodic;
- the configuration x such that $x_{g_1^n} = 0$ for all n, and $x_g = 1$ otherwise, is weakly periodic and weakly aperiodic;
- the configuration x such that $x_{(0,0)} = 0$, and $x_g = 1$ otherwise, is strongly aperiodic.

2.2.2 Wang tiles, NNSFT and graphs

Definition 2.2 (Wang tiles, Wang subshifts) *Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a finitely generated group and $\mathcal{C}$ a finite set of colours. A* Wang tile *on $\mathcal{C}$ and $\mathcal{S}$ is a map $\mathcal{S} \cup \mathcal{S}^{-1} \to \mathcal{C}$.*

*Given a set T of Wang tiles, the corresponding G-*Wang subshift *is defined as:*

$$X_T = \{(x_g) \in T^G \mid \forall g \in G, s \in \mathcal{S} \cup \mathcal{S}^{-1}, x_g(s) = x_{gs}(s^{-1})\}.$$

*We call the elements in X_T G-*Wang tilings.

Notice that the definition of a Wang tile depends only on the chosen set of generators, so that the same Wang tile can be used for $\mathbb{F}_2$ and $\mathbb{Z}^2$, for example.

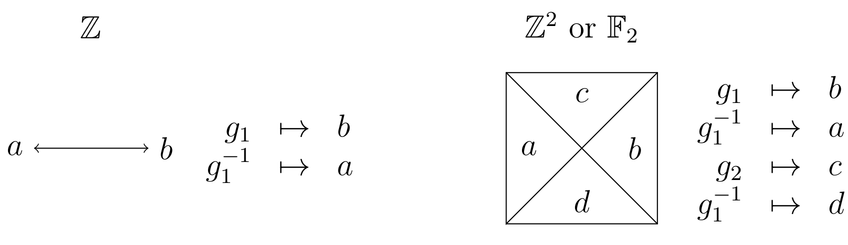


Figure 2.1: Examples of Wang tiles with colours $\mathcal{C} = \{a, b, c, d\}$ on one and two generators, respectively, with their corresponding maps.

Take any G-NNSFT X on the alphabet $\mathcal{A}$, where $G = \langle \{g_1, \ldots, g_d\} \mid \mathcal{R} \rangle$ is an arbitrary finitely generated group. Let $\mathcal{F}$ be a set of forbidden patterns with each support of the form $\{1_G, g_i\}$.

We associate to X a set of d graphs $\Gamma_1, \ldots, \Gamma_d$, where the set of vertices is $\mathcal{A}$ for all Γ_i, and

$$\forall a, b \in \mathcal{A}, \qquad a \to b \text{ in } \Gamma_i \Longleftrightarrow \left\{ \begin{array}{ccc} 1_G & \to & a \\ g_i & \to & b \end{array} \right. \notin \mathcal{F}.$$

By definition of a G-NNSFT, it follows that a configuration x belongs to X if, and only if, $x_h \to x_{hg_i}$ is an edge in Γ_i for all $h \in G$ and all $1 \le i \le d$.

Definition 2.3 (Cycles) *A* cycle *on a graph Γ is a path - with possible edge and vertex repetitions - that starts and ends on the same vertex. A cycle through the vertices $a_1 \dots a_n a_1$, with $a_i \in \mathcal{A}$, is denoted $\overline{a_1 \dots a_n}$.*

A cycle is simple *if it does not contain any vertex repetition. Denote $\mathcal{SC}(\Gamma)$ the set of simple cycles on Γ, which is a finite set.*

Remark In graph theory, cycles are sometimes called *closed walks*, in which case cycle means simple cycle. We decided to follow Piantadosi's conventions [42] for convenience.

Let w be a cycle and $a \in \mathcal{A}$. We define:

$$|w|_a = \#\{i \mid w_i = a, 1 \le i \le |w|\}.$$

In any cycle, the path between the closest repetitions is a simple cycle. By removing this simple cycle and iterating the argument, we can see that any cycle w can be decomposed into simple cycles, in the sense that there are integers λ_ω for $\omega \in \mathcal{SC}(\Gamma)$ such that:

$$\forall a \in \mathcal{A}, |w|_a = \sum_{\omega \in \mathcal{SC}(\Gamma)} \lambda_\omega |\omega|_a.$$

We say that two G-subshifts $X, Y \subset \mathcal{A}^G$ are (topologically) *conjugate* if there is a shift-commuting homeomorphism Φ (that is, $\Phi \circ \sigma_g = \sigma_g \circ \Phi$ for all $g \in G$) such that $\Phi(X) = Y$. A shift-commuting homeomorphism (or *conjugacy*) corresponds to a reversible cellular automaton: there is a finite subset $H \subset G$ and a local rule $\varphi : \mathcal{A}^H \to \mathcal{A}$ such that

$$\forall x \in X, \forall g \in G,\ \Phi(x)_g = \varphi(\sigma_{g^{-1}}(x)|_H),$$

and Φ^{-1} is itself a cellular automaton.

Proposition 2.4 *For any set of generators, each G-SFT is conjugate to a G-NNSFT and each G-NNSFT is conjugate to a G-Wang subshift.*

This is folklore. A detailed proof for the SFT - NNSFT part can be found in [33] (Propositions 1.6 and 1.7), and a proof of the NNSFT - Wang subshift part in [16].

Since the conjugacy from a G-Wang subshift to a G-NNSFT can be chosen letter-to-letter (i.e. $H = \{1_G\}$ in the definition), it is easy to see that the conjugacy does not depend on G, so we could say that a set of graphs and a set of Wang tiles are conjugate.

Proposition 2.5 *Let X and Y be two conjugate G-subshifts. X admits a valid tiling if and only if Y admits a valid tiling. The same is true for weakly/strongly (a)periodic tilings.*

2.3 S. Piantadosi's and J. Chazottes-J. Gambaudo-F. Gautero's conditions

2.3.1 State of the art on the free group and $\mathbb{Z}^2$

The first two conditions were introduced by S. Piantadosi in the context of symbolic dynamics on the free group $\mathbb{F}_d$.

Definition 2.6 (Condition $(\star)$ [42]) *A family of graphs* $\Gamma = \{\Gamma_i\}_{1\leq i\leq d}$ *whose vertices are an alphabet* $\mathcal{A}$ *satisfies* condition $(\star)$ *if and only if there is some nonempty* $\mathcal{A}' \subset \mathcal{A}$ *with a* colouring function $\Psi : \mathcal{A}' \times \mathcal{S} \to \mathcal{A}'$ *such that, for any colour* $a \in \mathcal{A}'$ *and any generator* $g_i \in \mathcal{S}$, $a \to \Psi(a, g_i)$ *is an edge in* Γ_i.

Theorem 2.7 ([42]) *Let* X *be a* $\mathbb{F}_d$*-NNSFT on the alphabet* $\mathcal{A}$. X *is nonempty if and only if the corresponding set of graphs satisfies condition* $(\star)$.

This theorem provides a decision procedure for the domino problem in free groups of any rank: find a subalphabet such that every letter admits a valid neighbour in the subalphabet for every generator.

Definition 2.8 (Condition $(\star\star)$ [42]) *Consider a family of graphs* $\Gamma = \{\Gamma_i\}_{1\leq i\leq d}$ *and* $\mathcal{SC}(\Gamma_i) = \{\omega_i^j\}_{1\leq j\leq \#\mathcal{SC}(\Gamma_i)}$ *the set of simple cycles for each graph* Γ_i.

We denote by $(\star\star)$ *the following equation on real numbers* $x_{i,j}$:

$$\forall a \in \mathcal{A}, \sum_{j=1}^{\#\mathcal{SC}(\Gamma_1)} x_{1,j}|\omega_1^j|_a = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_2)} x_{2,j}|\omega_2^j|_a = \cdots = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_d)} x_{d,j}|\omega_d^j|_a.$$

We say that the graph family satisfies condition $(\star\star)$ *if equation* $(\star\star)$ *is not empty (e.g. all graphs contain at least a cycle) and admits a nontrivial positive solution.*

Remark We formulated the previous condition in terms of simple cycles (using the formalism from Theorem 3.6 instead of Theorem 3.4 in [42]) because they form a finite set, making it easier to prove formally when the condition is not satisfied.

Theorem 2.9 ([42], Theorem 3.6) *A* $\mathbb{F}_d$*-NNSFT contains a strongly periodic configuration if and only the associated family of graphs satisfies condition* $(\star\star)$.

Example We illustrate S. Piantadosi's conditions on the following example:

The corresponding $\mathbb{F}_2$-NNSFT admits a tiling, because it satisfies condition $(\star)$ on the alphabet $\mathcal{A}' = \mathcal{A}$. However, it does not admit a periodic tiling: the simple cycles of Γ_1 are

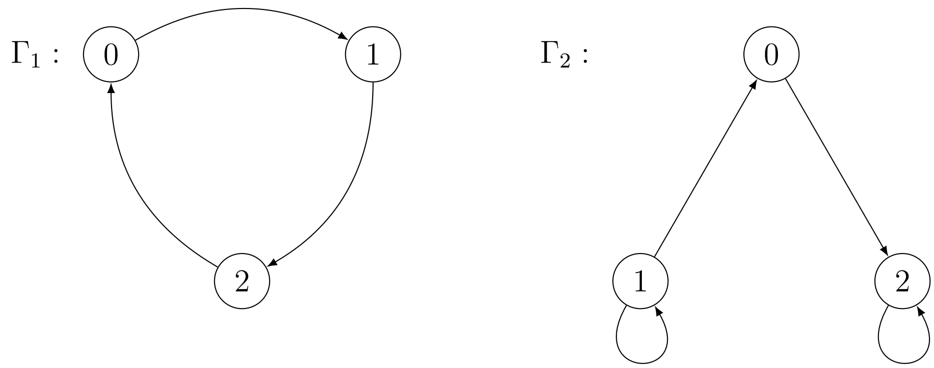


Figure 2.2: Illustrate Piantadosi's conditions

(up to shifting) $\{\overline{012}\}$ and the simple cycles of Γ_2 are $\{\overline{1}, \overline{2}\}$, so Equation $(\star\star)$ is:

$$x_{1,1} = 0 \qquad (a = 0)$$
$$x_{1,1} = x_{2,1} \qquad (a = 1)$$
$$x_{1,1} = x_{2,2} \qquad (a = 2)$$

which obviously doesn't admit a nontrivial solution. As we will see later, the corresponding $\mathbb{Z}^2$-NNSFT doesn't admit any tiling.

Remark For example, if all graphs Γ_i share a common cycle w (say $\omega_i^1 = w$ for all graphs Γ_i), then condition $(\star\star)$ admits a solution: for all i, $x_{i,1} = 1$ and $x_{i,j} = 0$ when $j \neq 1$. Therefore the corresponding $\mathbb{F}_d$-NNSFT contains a periodic configuration.

Definition 2.10 (Condition $(\star\star)'$ [13]) *Let T be a set of Wang tiles on colours $\mathcal{C}$ and set of generators $\mathcal{S}$. For each $g \in \mathcal{S} \cup \mathcal{S}^{-1}$ and each colour $c \in \mathcal{C}$, define c_g the subset of Wang tiles $\tau_i \in T$ such that $\tau_i(g) = c$. We call $(\star\star)'$ the following equation:*

$$\forall g \in \mathcal{S}, \forall c \in \mathcal{C}, \sum_{\tau_i \in c_g} x_i = \sum_{\tau_j \in c_{g-1}} x_j.$$

We say that T satisfies condition $(\star\star)'$ *if Equation $(\star\star)'$ admits a positive nontrivial solution.*

Theorem 2.11 ([13]) *If a set T of Wang tiles admits a valid tiling of $\mathbb{Z}^2$, then it satisfies condition $(\star\star)'$.*

This condition and result were introduced in [13], but a much easier presentation in our context is given in [28].

Example Example 2.3.1 is conjugate to the following set of Wang tiles. Equation $(\star\star)'$

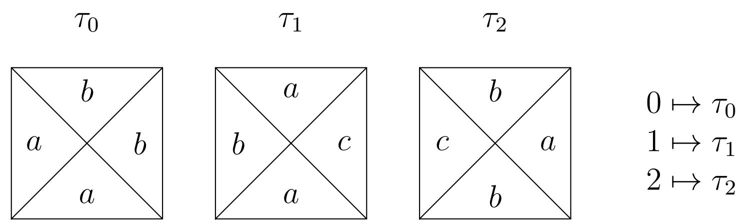


Figure 2.3: Wang tiles as example to see conditions $(\star\star)'$

becomes the following, where next to each equation is the corresponding generator and colour.

(g_1, a)	$x_2 = x_0$	(g_2, a)	$x_1 = x_0 + x_1$
(g_1, b)	$x_0 = x_1$	(g_2, b)	$x_0 + x_2 = x_2$
(g_1, c)	$x_1 = x_2$	(g_2, c)	$0 = 0$

This equation does not admit a positive nontrivial solution, so the corresponding $\mathbb{Z}^2$-Wang subshift is empty.

2.3.2 Conditions $(\star\star)$ and $(\star\star)'$ are equivalent

Although conditions $(\star\star)$ and $(\star\star)'$ were introduced in very different contexts (periodic tilings of the free group and tilings of the Euclidean plane, respectively), it turns out that they are equivalent. The fact that $(\star\star)$ is a condition on graphs (NNSFTs) and $(\star\star)'$ is a condition on sets of Wang tiles (Wang subshifts) is only cosmetic since Proposition 2.4 lets us go from one model to the other.

Theorem 2.12 *Let T be a set of Wang tiles over the set of colours $\mathcal{C}$ and the set of generators $\mathcal{S}$.*

T satisfies condition $(\star\star)'$ if, and only if, the associated graphs satisfy condition $(\star\star)$.

Proof. $(\Leftarrow)$ Let $(x_{i,j})$ be a nonnegative solution to equation $(\star\star)$. For every tile τ_i, put $x_i = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_1)} x_{1,j} |\omega_1^j|_{\tau_i}$.

Because each simple cycle of Γ_1 is a cycle, it contains as many tiles in c_{g_1} as in $c_{g_1^{-1}}$; that is, $\sum_{\tau_i \in c_{g_1}} |\omega_1^j|_{\tau_i} = \sum_{\tau_j \in c_{g_1^{-1}}} |\omega_1^j|_{\tau_i}$. Summing over all simple cycles ω_1^j, we get $\sum_{\tau_i \in c_{g_1}} x_i = \sum_{\tau_i \in c_{g_1^{-1}}} x_j$.

Since $(x_{i,j})$ is a solution to Equation $(\star\star)$, we also have $x_i = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_n)} x_{n,j} |\omega_n^j|_{\tau_i}$ for every n, so the same argument shows that (x_i) is a nonnegative solution of equation $(\star\star)'$.

$(\Rightarrow)$ Because equation $(\star\star)'$ admits a solution, it admits a rational solution, and therefore an integer solution. Let (x_i) be an integer, nonnegative solution of equation $(\star\star)'$.

For the generator g_1, consider the graph Γ_1 obtained by the letter-to-letter conjugacy of Proposition 2.4: concretely, it is the graph on vertices $\{\tau_i\}_{1\leq i\leq n}$ with $\tau_i \to \tau_j \Leftrightarrow \exists c \in \mathcal{C}, \tau_i \in c_{g_1}$ and $\tau_j \in c_{g_1^{-1}}$.

We define an auxiliary graph G_1 on vertices $\{\tau_i^k\}_{1\leq i\leq n, 1\leq k\leq x_i}$ (that is, x_i copies for each tile τ_i) as follows.

Because
$$\forall c \in \mathcal{C}, \sum_{\tau_i\in c_{g_1}} x_i = \sum_{\tau_j\in c_{g_1^{-1}}} x_j,$$
we can fix an arbitrary bijection
$$\Psi_1^c : \{\tau_i^k \ : \ \tau_i \in c_{g_1}, 1\leq k\leq x_i\} \to \{\tau_{i'}^{k'} \ : \ \tau_{i'} \in c_{g_1^{-1}}, 1\leq k'\leq x_{i'}\},$$
and put an edge $\tau_i^k \to \tau_{i'}^{k'}$ if and only if $\Psi_1^c(\tau_i^k) = \tau_{i'}^{k'}$ for some $c \in \mathcal{C}$. Because each vertex has indegree and outdegree 1, it is a (not necessarily connected) Eulerian graph and admits a finite set of cycles covering every vertex exactly once.

Notice that by construction, if G_1 has an edge $\tau_i^k \to \tau_{i'}^{k'}$, then Γ_1 has an edge $\tau_i \to \tau_{i'}$. Therefore each cycle of G_1 can be sent on a cycle in Γ_1 through the projection $\tau_i^k \mapsto \tau_i$. In this way, project the finite set of cycles obtained above and decompose them into simple cycles of Γ_1. Denote $x_{1,j}$ the total number of each simple cycle ω_1^j obtained in this way.

Because each tile τ_i was present in G_1 as a vertex in x_i copies, we have for every i: $\sum_{j=1}^{\#\mathcal{SC}(\Gamma_1)} x_{1,j}|\omega_1^j|_{\tau_i} = x_i$.

Now apply the same argument for each generator $g_2, \ldots, g_n$ and the variables $(x_{i,j})$ thus obtained are a solution to equation $(\star\star)$. □

2.4 Necessary conditions for tiling arbitrary groups

Since the above conditions apply on sets of Wang tiles or set of graphs, they actually are conditions on a family of G-SFT where G ranges over all groups with a fixed number of generators. The following proposition relates the properties of these SFT. It can be found (under a different form) in [10] (Proposition 10 and remark below)

Proposition 2.13 *Let $G_1 = \langle\{g_1, \ldots, g_d\}|\mathcal{R}\rangle$, $G_2 = \langle\{g_1, \ldots, g_d\}|\mathcal{R}'\rangle$ be finitely generated groups, with $\mathcal{R}' \subset \mathcal{R}$. Consider the canonical surjective morphism $\pi : G_2 \to G_1$ defined by $\pi(g_i) = g_i, \forall 1 \leq i \leq d$. Let $\Phi : \mathcal{A}^{G_1} \to \mathcal{A}^{G_2}$ be defined by $\Phi(x)_g = x_{\pi(g)}$. Let X_1 and X_2 be the corresponding G_1-NNSFT and G_2-NNSFT respectively, such that X_2 has the same local rules as X_1.*

We have:

1. *If x is a valid tiling for X_1 then $\Phi(x)$ is a valid tiling for X_2.*
2. *If x is weakly periodic then $\Phi(x)$ is weakly periodic. In particular, if X_1 admits a weakly periodic tiling, then X_2 admits a weakly periodic tiling.*

3. *If x is weakly aperiodic then $\Phi(x)$ is weakly aperiodic. In particular, if X_1 admits a weakly aperiodic tiling, then X_2 admits a weakly aperiodic tiling.*

The strong properties are not preserved by Φ, but of course the image of a strongly (a)periodic tiling remains weakly (a)periodic. Stronger versions with different hypotheses can be found in [10, 26].

Proof. 1. Since X_2 is an NNSFT, it is enough to check that, for all $h \in G_2$ and all $1 \leq i \leq d$, $\Phi(x)_h \to \Phi(x)_{hg_i}$ is an edge in Γ_i, that is to say, that it is not a forbidden pattern for X_2. By definition of Φ, $\Phi(x)_h = x_{\pi(h)}$ and $\Phi(x)_{hg_i} = x_{\pi(h)\pi(g_i)} = x_{\pi(h)g_i}$. Because x is a valid tiling for X_1, we have that $x_{\pi(h)} \to x_{\pi(h)g_i}$ is an edge in Γ_i, which proves the result.

2. If x is a weakly periodic tiling in X_1, then ${}_\sigma(x)$ is nontrivial by definition. We have:

$$
\begin{aligned}
{}_\sigma(\Phi(x)) &= \{g \in G_2 \,:\, \forall h \in G_2, \Phi(x)_{hg} = \Phi(x)_h\} \\
&= \{g \in G_2 \,:\, \forall h \in G_2, x_{\pi(h)\pi(g)} = x_{\pi(h)}\}.
\end{aligned}
$$

Since π is surjective, this means that $\pi({}_\sigma(\Phi(x))) =_\sigma (x)$. ${}_\sigma(x)$ is nontrivial so ${}_\sigma(\Phi(x)) = \pi^{-1}({}_\sigma(x))$ is nontrivial as well.

3. If x is a weakly aperiodic tiling in X_1, then ${}_\sigma(x)$ does not have finite index. The canonical morphism $\pi : G_2 \to G_1$ yields a morphism on the quotient:

$$
\tilde{\pi} : G_2/\pi^{-1}({}_\sigma(x)) \to G_1/{}_\sigma(x),
$$

and $\tilde{\pi}$ is surjective since π is surjective. Remember that ${}_\sigma(\Phi(x)) = \pi^{-1}({}_\sigma(x))$ by the previous point. Since ${}_\sigma(x)$ does not have finite index, $G_1/{}_\sigma(x)$ is infinite, so $G_2/\pi^{-1}({}_\sigma(x))$ is infinite as well, and ${}_\sigma(\Phi(x)) = \pi^{-1}({}_\sigma(x))$ does not have finite index.

□

Remark In the last proposition, the converse of the point (1) does not hold. For instance, consider $G = \mathbb{Z}^2 = \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} g_2^{-1} \rangle$. Example 2.3.1 provided an example of a set of graphs that satisfies condition $(\star)$ (so the corresponding $\mathbb{F}_2$-NNSFT admits a valid tiling) but does not satisfy condition $(\star\star)$ (so the corresponding $\mathbb{Z}^2$-NNSFT does not admit any valid tiling).

To understand why, notice that $ker(\pi)$ contains $g_1 g_2 g_1^{-1} g_2^{-1}$, so if a tiling $x \in \mathcal{A}^{\mathbb{F}_2}$ is such that $x_{1_{\mathbb{F}_2}} \neq x_{g_1 g_2 g_1^{-1} g_2^{-1}}$, then $\Phi^{-1}(x) = \emptyset$. If this happens for all $x \in X_2$ then X_1 is empty.

Corollary 2.14 *Let $\Gamma_1, \ldots, \Gamma_d$ be a set of graphs that does not satisfy condition $(\star)$. Then the corresponding G-NNSFT is empty for an arbitrary group G with d generators.*

Proof. If there was a valid tiling in $G = \langle g_1, \ldots, g_d \mid \mathcal{R} \rangle$ then, applying Proposition 2.13, we would obtain a tiling on $\mathbb{F}_d = \langle g_1, \ldots, g_d \mid \emptyset \rangle$, which is in contradiction with Theorem 2.7. □

2.5 Necessary conditions for tiling amenable groups

Definition 2.15 (Følner sequence) *Let G be a group. A* Følner sequence *for G is a sequence of finite subsets $S_n \subset G$ such that:*

$$G = \bigcup_n S_n \quad \textit{and} \quad \forall g \in G, \frac{\#(S_n g \triangle S_n)}{\#S_n} \xrightarrow[n\to\infty]{} 0,$$

where $S_n g = \{hg \ : \ h \in S_n\}$ and $A \triangle B = (A \backslash B) \cup (B \backslash A)$ is the symmetric difference.

In the previous definition, it is easy to see that the second condition only has to be checked for g in a finite generating set. The set $S_n g \triangle S_n$ can be understood as the border of S_n, so an element of a Følner sequence must have a small border relative to its interior.

Definition 2.16 (Amenable group) *A group G is* amenable *if it admits a Følner sequence.*

This definition applies more generally for all countable groups. A few examples:

- $\mathbb{Z}^d$ is amenable and a Følner sequence is given by $S_n = [-n, n]^d$. Indeed, if $(g_i)_{1 \le i \le d}$ is the canonical set of generators, then $\#S_n = (2n+1)^d$ and $\#((S_n + g_i) \triangle S_n) = 2 \cdot (2n+1)^{d-1}$.
- $\mathbb{F}_d$ for $d \ge 2$ is not amenable. In particular, the balls S_n of radius n - that is, reduced[1] words of length $\le n$ on the set of generators $(g_i)_{1 \le i \le d}$ - are not a Følner sequence. Indeed, one can easily check that $\#S_n = \Omega(d^n)$ and $\#(S_n g_i \triangle S_n) = \Omega(d^n)$.

Theorem 2.17 (Heuristic for tiling an amenable group) *Let G be a finitely generated amenable group, $\mathcal{S}$ a finite set of generators, and T a set of Wang tiles.*

If there is a tiling of G with the tiles T, then condition $(\star\star)$ (or equivalently $(\star\star)'$) is satisfied.

This results confirms a remark by E. Jeandel in [26], Section 3.1.

Proof. Let $x \in T^G$ be a tiling of G and S_n be a Følner sequence for G. Using notations from Definition 2.8, for a colour $c \in \mathcal{C}$ and a generator $g \in \mathcal{S}$, c_g is the set of tiles τ such that $\tau(g) = c$.

For any $h \in S_n \cap S_n g^{-1}$, we have $x_h \in c_g \Leftrightarrow x_{hg} \in c_{g^{-1}}$ (and in this case, $hg \in S_n \cap S_n g$). This means that, for all $c \in \mathcal{C}, g \in \mathcal{S}$ and $n \in \mathbb{N}$:

$$\#\{h \in S_n \cap S_n g^{-1} \ : \ x_h \in c_g\} = \#\{h \in S_n \cap S_n g \ : \ x_h \in c_{g^{-1}}\},$$

so in particular

$$|\#\{h \in S_n \ : \ x_h \in c_g\} - \#\{h \in S_n \ : \ x_h \in c_{g^{-1}}\}| \le \#(S_n g \triangle S_n) + \#(S_n g^{-1} \triangle S_n).$$

[1] with no $g_i^{-1} g_i$ or $g_i g_i^{-1}$ factors

For each tile τ_i, let $x_i^n = \frac{\#\{h \in S_n \;:\; x_h = \tau_i\}}{\# S_n}$. The previous computation implies that:

$$\forall g \in \mathcal{S},\ \forall c \in \mathcal{C},\ \left| \sum_{\tau_i \in c_g} x_i^n - \sum_{\tau_j \in c_{g^{-1}}} x_j^n \right| \leq \frac{\#(S_n g \triangle S_n)}{\# S_n} + \frac{\#(S_n g^{-1} \triangle S_n)}{\# S_n}.$$

Notice that the right-hand side tends to 0 as n tends to infinity by definition of a Følner sequence. Consider the sequence of vectors $((x_i^n)_i)_{n\in\mathbb{N}}$ and, by compacity, let (x_i) be any limit point of this sequence. Since $\sum_i x_i^n = 1$ for all n by definition, $\sum_i x_i = 1$ as well, and we have

$$\forall g \in \mathcal{S},\ \forall c \in \mathcal{C},\ \sum_{\tau_i \in c_g} x_i = \sum_{\tau_j \in c_{g^{-1}}} x_j,$$

so (x_i) is a nontrivial solution to Equation $(\star\star)$. Condition $(\star\star)'$ follows by Theorem 2.12. □

2.6 Counterexamples

It is clear that none of the $(\star)$, $(\star\star)$ or $(\star\star)'$ conditions can be a sufficient condition to admit a $\mathbb{Z}^d$-tiling, since it would be a decision procedure for the Domino problem; this argument applies to any group where the Domino problem is undecidable. For completeness, we provide explicit counterexamples for any non-free finitely generated group.

Theorem 2.18 *Let G be an arbitrary finitely generated group. If G is not free, then there exists a Wang tile set that satisfies the three conditions $(\star)$, $(\star\star)$ and $(\star\star)'$ and such that the corresponding G-Wang subshift is empty.*

Proof. Write $G = \langle g_1, \ldots, g_d \mid \mathcal{R}\rangle$, and take $r_1 : w_1 \ldots w_n \in \mathcal{R}$, with $w_1 \ldots w_n$ a reduced word on generators $g_1 \ldots g_d$ (no generator is next to its inverse).

We build a family of graphs Γ_d on vertices $\{0, \ldots, n\}$ with the following edges:

$$\forall i \leq n, \begin{cases} \text{if } w_i = g_j, \text{ then } \Gamma_j \text{ has an edge } i-1 \to i; \\ \text{if } w_i = g_j^{-1}, \text{ then } \Gamma_j \text{ has an edge } i \to i-1. \end{cases}$$

Notice that every vertex has indegree and outdegree at most 1 and we did not create any cycle in the process, so we can complete every Γ_j to be isomorphic to a n-cycle graph C_n.

Now we define a set of $n+1$ Wang tiles on $n+1$ colours $\{0, \ldots, n\}$ as follows. Tile τ_i has the following colours: for all j, $g_j^{-1} \to i$ and $g_j \to k$ if there is an edge $\tau_i \to \tau_k$ in Γ_j.

Example For $\mathbb{Z}^2$, we have $r_1 : g_1 g_2 g_1^{-1} g_2^{-1} = 1$. Therefore Γ_1 contains $0 \to 1$ and $3 \to 2$, and Γ_2 contains $1 \to 2$ and $4 \to 3$. One possible completion for Γ_1 and Γ_2 is the following:

The corresponding G-NNSFT is conjugate to the G-Wang subshift defined by the following tiles through the rewriting $i \leftrightarrow \tau_i$:

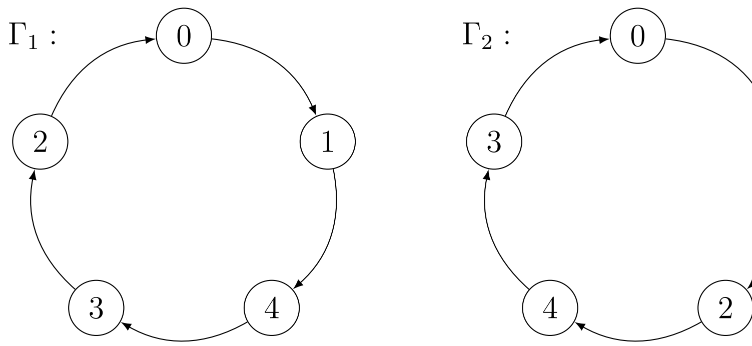


Figure 2.4: One possible completion for Γ_1 and Γ_2

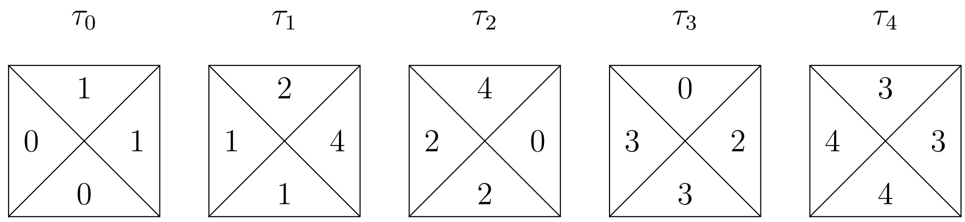


Figure 2.5: Corresponding tiles conjugates to the example

This tiling satisfies condition $(\star\star)'$ since we can assign the same weight $\frac{1}{n}$ to each tile.

It is clear that a tiling x of G using tiles $\tau_0, \ldots, \tau_n$ must contain every tile. Assume w.l.o.g that $x_1 = \tau_0$. By construction we must have $x_{w_1} = \tau_1$, $x_{w_1 w_2} = \tau_2$, and by an easy induction $x_w = \tau_n$. But since $w = 1$ in G, we have $\tau_0 = x_1 = x_w = \tau_n$, a contradiction. Therefore there is no tiling of G using tiles $\tau_0, \ldots, \tau_n$. $\square$

2.7 Extension of the result

The condition that we attributed to [13] ($(\star\star)$ in op.cit) appeared earlier in [20]. Furthermore, a combinatorial interpretation of the condition was given by T. Monteil in various talks in the context of tiling translation surfaces. We thank P. Guillon and T. Monteil for bringing this to our attention.

Lemma 2.19 *For any normal subgroup S of G, if T tiles G and there is a configuration $x \in X_T$, with $Stab(x) \subseteq S$, then T tiles G/S.*

Theorem 2.20 *Let T be a set of Wang tiles on d-generators. The following are equivalent, where all groups are finitely generated:*

1. *T tiles some group in a strongly periodic manner.*

2. *T tiles $\mathbb{F}_d$ in a strongly periodic manner.*

3. *T tiles some infinite amenable group.*

4. *T tiles some infinite amenable torsion-free group.*

5. *T tiles some finite group.*

6. *T tiles some virtually $\mathbb{Z}$ group.*

Proof. $2 \Rightarrow 1$, $4 \Rightarrow 3$ and $6 \Rightarrow 3$ are clear. $3 \Rightarrow 2$ is proved in Theorem 2.17.

$1 \Rightarrow 5$. Let G be a some group and $x \in X_T$ a strongly aperiodic configuration. Let $S = \bigcap_{y \in \mathrm{Orb}(x)} Stab(y) = \bigcap_{y \in \mathrm{Orb}(x)} g \cdot Stab(y) \cdot g^{-1}$. $S \subset Stab(x)$ and S is a normal subgroup of G, so by lemma 2.19, T tiles $G \diagup S$. We prove that $G \diagup S$ is a finite. Every σ_g for $g \in G$ is a permutation of $\mathrm{Orb}(x)$, and $\sigma : g \mapsto \sigma_g$ is a morphism from G to $\mathfrak{S}_{\mathrm{Orb}(x)}$. Bye definition, $S = \ker(\sigma)$, so $G \diagup S$ is isomorphic to some subgroup of $\mathfrak{S}_{\mathrm{Orb}(x)}$ which is finite.

$5 \Rightarrow 6$. Let G be a finite group tiled by T. G can be written as $\mathbb{F}_d \diagup S$ for some normal subgroup S. Let φ be any morphism $\mathbb{F}_d \to \mathbb{Z}$ with infinite image; for example, the morphism generated by $\varphi(g_1) = 1$ and $\varphi(g_i) = 0$, for $i > 1$. Then $\varphi(S) \neq \{0\}$, because otherwise $\mathbb{F}_d \diagup \ker(\varphi) \subset \mathbb{F}_d \diagup S$ would be infinite. So, denoting $S^\star = \ker(\varphi|_S)$, we have $\varphi(S) = k\mathbb{Z} = S \diagup S^\star$, for some $k > 0$. Then the canonical surjection $\mathbb{F}_d \diagup S^\star \to \mathbb{F}_d \diagup S$ has finite image and has kernel $S \diagup S^\star = k\mathbb{Z}$, so by Lemma 2.19, T tiles $\mathbb{F}_d \diagup S^\star$ which is virtually $\mathbb{Z}$.

$3 \Rightarrow 4$. Write G as $\mathbb{F}_d \diagup S$ for some normal subgroup S. Then $\mathbb{F}_d \diagup S'$ is torsion-free, where S' is the derived group of S. By the previous Lemma 2.19, T tiles $\mathbb{F}_d \diagup S'$. □

Corollary 2.21 *Sets of Wang tiles on d generators can be split into three types:*

1. *Those that tile no group.*

2. *Those that tile the free group but no amenable group (incl. finite groups).*

3. *Those that tile some amenable group, and there is an algorithm that, given a set of wang tiles, outputs its type.*

2.8 Conclusion

We would like to mention the two following conjectures that relate the fact of admitting a valid (periodic) tiling and the underlying group structure: [[3]] A finitely generated group has a decidable domino problem if and only if it is virtually free.

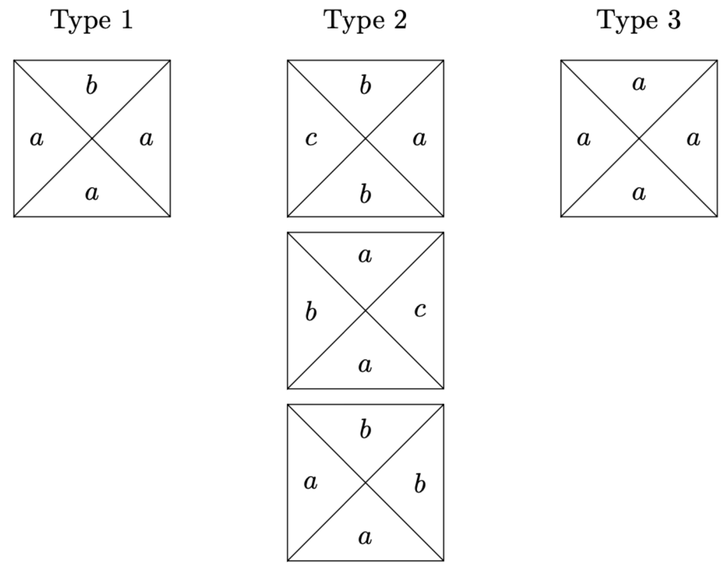


Figure 2.6: The previous theorem provide various characterisations of Type 3 sets.

[[10]] A finitely generated group has an SFT with no strongly periodic point if and only if it is not virtually cyclic.

In both cases, the "if" direction is proven and the "only if" direction is open.

If Conjecture 2.8 holds, every infinite amenable groupe has an undecidable domino problem. We ask whether the domino problem could be decidable when considering all amenable groups "at the same time", with a decision procedure given by Conditions ($\star\star$) and $(\star\star)'$.

Chapter 3

Weakly aperiodic Wang subshifts with minimal alphabet size on the Free group

This chapter corresponds to an adaptation of the original manuscript developed by M. Schraudner and H. Maturana, "Complexity and entropy in tree shifts not of finite type", in preparation.

3.1 Introduction

Given a finitely generated group G, a G-subshift corresponds to the coloring of G respecting a set of local restrictions. When there are only finitely many restrictions the G-subshift associated is a G-subshift of finite type (G-SFT). A subclass of G-SFTs considering only restrictions between adjacent elements is called nearest neighbor SFTs (G-NNSFT). A special kind of G-NNSFT, which we shall consider, are the G-Wang subshifts (Definition 3.1). In this case the restrictions are given by matching colors along edges between adjacent sites and a valid coloring of G is called a G-Wang tiling. It is a folklore result that every G-SFT is conjugated to a G-NNSFT (Proposition 1.26) and also that every G-NNSFT is conjugated to a G-Wang subshift. We give a proof for this last fact in Proposition 3.2. An important question is to know whether given a set of local restrictions, the resulting G-SFT is empty or not. This question is known as the Emptiness problem and in the case of a G-Wang subshift is also called the Domino problem.

For $G = \mathbb{Z}$, the class of $\mathbb{Z}$-SFTs has been thoroughly studied. Corresponding subshifts can be represented by finite directed graphs, such that valid $\mathbb{Z}$-colorings are in bijection with bi-infinite walks on said graph (see [34]). Hence, given a collection of nearest neighbor restrictions or a finite set of Wang tiles, we can easily decide whether there is a valid $\mathbb{Z}$-coloring by verifying that the associated directed graph allows a bi-infinite path, which is equivalent to having a cycle. Therefore, both the Emptiness and the Domino problem on $\mathbb{Z}$ are decidable and in fact, every nonempty $\mathbb{Z}$-SFT has at least one periodic coloring.

For $G = \mathbb{Z}^2$ decidability of the Emptiness problem is much less obvious and the Domino problem in particular, has a long history. In 1961, H. Wang conjectured [45], that if a finite set of $\mathbb{Z}^2$-Wang tiles can tile $\mathbb{Z}^2$ then it also allows for periodic tilings. He then observed that proving this conjecture would imply the existence of an algorithm to decide the Domino problem on the plane. In 1966, Wang's student R. Berger [6], showed that Wang's conjecture is false. He constructed an enormous alphabet of 20.426 $\mathbb{Z}^2$-Wang tiles giving rise to a nonempty weakly aperiodic $\mathbb{Z}^2$-Wang subshift. Using properties of that subshift, R. Berger furthermore showed that the Domino problem in $\mathbb{Z}^2$ is in fact undecidable.

After this first example, people started to work on reducing the required number of $\mathbb{Z}^2$-Wang tiles. In 1971, R. M. Robinson [43] constructed an aperiodic $\mathbb{Z}^2$ tiling with an alphabet comprised of only 56 tiles, followed by Lauchli [46] who in 1975 published an aperiodic set of 40 $\mathbb{Z}^2$-Wang tiles. After that, in 1996 J. Kari [29] invented a new method to build aperiodic tilesets and obtained an example with 14 $\mathbb{Z}^2$-Wang tiles. In the same year, together with K. Culik [17], they reduced the set to 13 $\mathbb{Z}^2$-Wang tiles. Finally, in 2015 E. Jeandel and M. Rao [27] determined that 11 is the smallest cardinality of $\mathbb{Z}^2$-Wang tiles which can generate a nonempty aperiodic $\mathbb{Z}^2$-Wang subshift.

On free groups $G = \mathbb{F}_k$, some basic aspects of symbolic dynamics have been developed by S. Piantadosi [42]. In particular, he studied conditions assuring non-emptiness as well as the existence of strongly periodic colorings in $\mathbb{F}_k$-NNSFTs. In fact the Emptiness problem is decidable, because every $\mathbb{F}_k$-NNSFT is completely determined by a family of k $\mathbb{Z}$-NNSFTs (Proposition 1.31). Moreover every non-empty $\mathbb{F}_k$-NNSFT contains a weakly periodic coloring, nevertheless S. Piantadosi constructs an example of a $\mathbb{F}_2$-NNSFT without any strongly periodic coloring.

The goal of this chapter is to study $\mathbb{F}_k$-Wang subshifts. To do this, it is important to note that not every $\mathbb{F}_k$-NNSFT corresponds to a $\mathbb{F}_k$-Wang subshift. We show a counterexample for $\mathbb{F}_1 = \mathbb{Z}$ in Example 3.4, by construction it is possible to extend this counterexample to $\mathbb{F}_k$, $k \geq 2$. With this in mind, we give conditions on the essential directed graphs which determine a valid nonempty $\mathbb{Z}$-Wang subshift in Proposition 3.7 and following a similar argument as before, we show that every nonempty $\mathbb{F}_k$-Wang subshift is completely determined by a family of k nonempty $\mathbb{Z}$-Wang subshifts (Proposition 3.8). In a second step we determine the minimal cardinality of $\mathbb{F}_k$-Wang tiles which generate a weakly aperiodic $\mathbb{F}_k$-Wang subshift. Clearly, a simple tile is not enough, but also the case of a two-element alphabet is quickly discarded (Proposition 3.9). Sufficiency of three distinct $\mathbb{F}_k$-Wang tiles is shown by contrasting a first example, leading to the following main result:

Theorem *Given $k \geq 2$, the minimum cardinality of a set of $\mathbb{F}_k$-Wang tiles which produces a nonempty weakly aperiodic $\mathbb{F}_k$-Wang subshift is 3.*

Creating all 25 essential directed graphs which give rise to non-empty $\mathbb{Z}$-Wang subshifts with exactly 3 tiles, we use Proposition 3.7 to generate all possible $\mathbb{F}_k$-Wang subshifts using a three alphabet. Analyzing those graphs and its structure then leads to an effective method of classifying the $\mathbb{F}_k$-subshifts aperiodicity in terms of non-trivial solutions of a certain system of linear equations and allows us to determine all weakly aperiodic examples.

3.2 Preliminaries and notations

Let G be a finitely generated group and $\mathcal{A}$ be a nonempty finite set. We consider the product set $\mathcal{A}^G = \{x : G \to \mathcal{A}\}$ consisting of all functions from G to $\mathcal{A}$. We refer to the set $\mathcal{A}$ as an *alphabet*, its elements $a \in \mathcal{A}$ will be called *symbols* and the elements of $\mathcal{A}^G$ *colorings* of G. Given $x = (x_g)_{g\in G} \in \mathcal{A}^G$, each x_g corresponds to the symbol seen at position $g \in G$. The group G acts on $\mathcal{A}^G$ by the left shift $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ defined coordinatewise as:

$$\sigma(g, x)_h := x_{g^{-1}h} \quad \forall h \in G.$$

We refer to σ as the *action* and we use the notation $\sigma_g(x)$ to denote $\sigma(g, x)$. Given a finite subset $F \subset G$, an element $P \in \mathcal{A}^F$ is called a *pattern* and $F = (P)$ its *support*. We say that a pattern P *appears* in a coloring $x \in \mathcal{A}^G$ (and we write $P \sqsubset x$) if there exists $g \in G$ such that $\sigma_g(x)|_F = P$. Since $\mathcal{A}$ is a finite set, $\mathcal{A}^G$ endowed with the pro-discrete topology, is a compact space and has a countable clopen basis given by the *cylinders* $[P]_g = \{x \in \mathcal{A}^G \mid \sigma_{g^{-1}}(x)|_{(P)} = P\}$.

The pair $(\mathcal{A}^G, \sigma)$ is called *G-full shift* and every closed and σ-invariant subset $X \subseteq \mathcal{A}^G$ corresponds to a *G-subshift*.

A special subfamily of G-NNSFTs which will be of particular interest in this chapter was introduced (in the context of $G = \mathbb{Z}^2$) by H.Wang and is called G-Wang subshifts:

Definition 3.1 *Let S be a finite generator set of G. A G-*Wang tile *T corresponds to a map $T : S \cup S^{-1} \to \mathcal{A}$, with $\mathcal{A}$ a finite set. Given a finite set W of G-Wang tiles, the G-*Wang subshift *is defined as:*

$$X_W = \left\{(x_g)_{g\in G} \in W^G \middle|\ \forall s \in S, g \in G : x_g(s) = x_{gs}(s^{-1})\right\}.$$

*The elements in X_W are called G-*Wang tilings.

Remark We note that every G-Wang subshift is indeed a G-NNSFT, whose set of forbidden patterns is given by:

$$\mathcal{F} = \bigcup_{s\in S} \{P \in W^{\{e_G,s\}} \mid P_{e_G}(s) \neq P_s(s^{-1})\}.$$

Example If we consider $\mathcal{A} = \{a, b, c, d\}$, examples of Wang tiles for each of the groups $\mathbb{Z} =< s \mid >$, $\mathbb{Z}^2 =< s_1, s_2 \mid s_1 s_2 s_1^{-1} s_2^{-1} >$ and $\mathbb{F}_2 =< s_1, s_2 \mid >$ are given in Figure 1.4.

The $\mathbb{Z}$-Wang tile is the map $T : \{s, s^{-1}\} \to \{a, b\}$ defined by $T(s^{-1}) = a$ and $T(s) = b$. In the resuming two examples, both groups are 2-generated, therefore the map for the $\mathbb{Z}^2$- or $\mathbb{F}_2$-Wang tile, respectively, is defined as $T : \{s_1, s_2, s_1^{-1}, s_2^{-1}\} \to \{a, b, c, d\}$ given by $T(s_1) = b$, $T(s_2) = c$, $T(s_1^{-1}) = a$ and $T(s_2^{-1}) = d$. The shape chosen to draw a $\mathbb{Z}^2$- or $\mathbb{F}_2$-Wang tile depends on the geometry given by the group and the dual of its Cayley graph, but the map is given abstractly and is independent of the visualization.

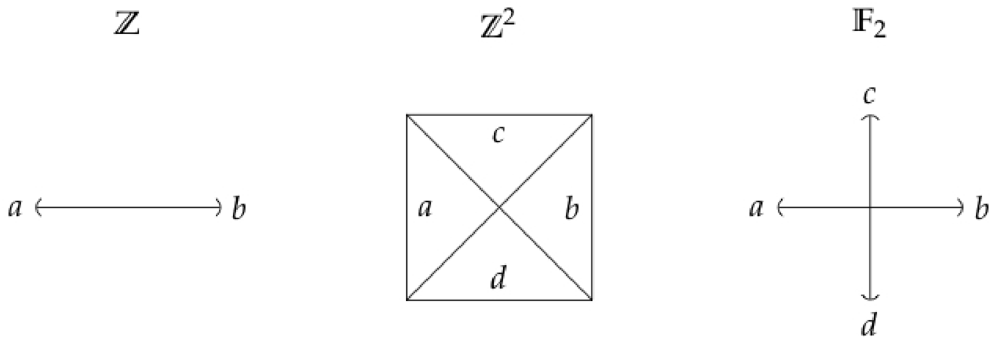


Figure 3.1: Examples of Wang-tiles

Proposition 3.2 *Every G-NNSFT is topologically conjugate to a G-Wang subshift.*

Proof. Let $X_{\mathcal{F}}$ be an arbitrary G-NNSFT, with $\mathcal{F} = \bigcup_{s\in S} \mathcal{F}_s$ where $\mathcal{F}_s \subseteq \mathcal{A}^{\{e_G,s\}}$. We denote by $\mathcal{F}_s^c = \mathcal{A}^{\{e_G,s\}} \setminus \mathcal{F}_s$.

Consider the G-Wang subshift X_W given by a set of G-Wang tiles W defined as follows: $T \in W$ if and only if $\exists a \in \mathcal{A}$, $\forall s \in S : \big(T(s^{-1}) = a \wedge \exists b \in \mathcal{A}\text{: } T(s) = b \Rightarrow (a,s,b) \in \mathcal{F}_s^c\big)$. Our claim is that $X_{\mathcal{F}}$ and X_W are topologically conjugated.

Define the map $\varphi : X_{\mathcal{F}} \to X_W$ by $\varphi((x_g)_{g\in G}) = (T_g)_{g\in G}$, such that $\forall g \in G$, $\forall s \in S$: $T_g(s) = x_{gs}$ and $T_g(s^{-1}) = x_g$. By construction,

$$\forall x \in X_{\mathcal{F}} : \forall g \in G, s \in S : (\varphi(x)_g)(s) = (\varphi(x)_{gs})(s^{-1}).$$

This implies that $\varphi(X_{\mathcal{F}}) \subseteq X_W$. Given that the map φ is defined locally, by the Theorem of Curtis-Hedlund-Lyndon we conclude that it is σ-invariant and continuous. Now consider $x, y \in X_{\mathcal{F}}$ such that $\varphi(x) = (T_g)_{g\in G} = \varphi(y)$, then for every $g \in G$ and arbitrary $s \in S$, we have $T_g(s^{-1}) = x_g = y_g$ obtaining $x = y$. Thus φ is an injective map. For every $(T_g)_{g\in G} \in X_W$, we can construct $(x_g)_{g\in G} \in X_{\mathcal{F}}$ given as $x_g = T_g(s^{-1})$, with some fixed arbitrary $s \in S$. We note that $\varphi((x_g)_{g\in G}) = (T_g)_{g\in G}$ and thus φ is a surjective map. Therefore, the map φ gives a σ-commuting homeomorphism between $X_{\mathcal{F}}$ and X_W. □

Remark In general the conjugacy between a G-NNSFT and a G-Wang subshift is not trivial. The alphabet of X_W is usually larger than the alphabet used for $X_{\mathcal{F}}$. For instance, if we consider a group G on n generators and $X_{\mathcal{F}} \subset \{0,1\}^G$ given by $\mathcal{F} = \{(1,s,1)\}_{s\in S}$, then W as constructed in the proof of Proposition 3.2 has $2^n + 1$ G-Wang tiles $\big($there are 2^n G-Wang tiles with $T(s^{-1}) = 0$, for every $s \in S$ and only one with $T(s^{-1}) = 1\big)$.

Even more, it is impossible to obtain a conjugacy map between $X_{\mathcal{F}}$ and any G-Wang subshift using only two G-Wang tiles. Indeed, let $W' = \{T_1, T_2\}$ be an arbitrary set of two distinct G-Wang tiles and suppose the existence of a conjugacy $\psi : X_{\mathcal{F}} \to X_{W'}$. Given that ψ commutes with the action σ, this map preserves fixed colorings. Thus without loss of generality we can suppose that $\psi((0)_{g\in G}) = (T_1)_{g\in G}$. This implies that $T_1(s) = T_1(s^{-1})$, $\forall s \in S$. Let $\{x^i = (x^i_g)_{g\in G}\}_{i=1}^n \subseteq X_{\mathcal{F}}$ be a set of n distinct $X_{\mathcal{F}}$-colorings, given by $x^i_g = 0$ for every $g \in G\setminus < S_i >$ and

$$x^i_{s_i^n} = \begin{cases} 1, & \text{if } n \text{ is an even integer} \\ 0, & \text{if } n \text{ is an odd integer} \end{cases}.$$

For each $1 \leq i \leq n$, we have that $\sigma_{s_i}(x^i) \neq x^i$, but $\sigma_{s_i^2}(x^i) = x^i$. Using that the map ψ commutes with σ, the Wang-tiling $\psi(x^i)$ cannot be a fixed point, thus $\psi(x^i)$ cannot be $(T_1)_{g\in G}$ or $(T_2)_{g\in G}$, however $\sigma_{s_i^2}(\psi(x^i)) = \psi(\sigma_{s_i^2}(x^i)) = \psi(x^i)$ focus the tiling $\psi(x^i)$ to have period 2 along direction s_i so necessary it must contain $\ldots T_1T_2T_1T_2\ldots$. By the definition of G-Wang tiles, we obtain: $T_1(s_i) = T_2(s_i^{-1})$ and $T_2(s_i) = T_1(s_i^{-1})$, thus $T_2(s_i) = T_2(s_i^{-1}) = T_1(s_i) = T_1(s_i^{-1})$, $\forall s_i \in S$. Therefore, we conclude that $T_1 = T_2$. Contradiction.

In the last remark, we may consider the group $\mathbb{Z} =< s \mid \ >$. If there exists an element $g \in \mathbb{Z}$ such that σ_g fixes a coloring, we say that this coloring is periodic. In particular, the coloring $x = (x_g)_{g\in\mathbb{Z}}$ defined as $x_{s^n} = 1$ for every n an even integer and $x_{s^n} = 0$ for odd n, is periodic with period 2, i.e. $\sigma_{s^2}(x) = x$. For more general groups however we have to define the concept of periodicity with more care. To see this it is enough to consider the group $\mathbb{Z}^2 =< s_1, s_2 \mid s_1s_2s_1^{-1}s_2^{-1} >$ and the coloring x^1 (as given in the last remark). Effectively, for x^1 there exists $s_1^2 \in \mathbb{Z}^2$ such that $\sigma_{s_1^2}(x^1) = x^1$, but for the generator s_2 there is no integer n, distinct of 0, such that $\sigma_{s_2^n}(x^1) = x^1$. Therefore we can say that the coloring x^1 is 2-periodic in direction s_1 but aperiodic in the direction of s_2.

3.3 $\mathbb{F}_k$-NNSFT using essential graphs

Given a nonempty $\mathbb{Z}$-NNSFT $X_{\mathcal{F}} \subseteq \mathcal{A}^{\mathbb{Z}}$, it is possible to obtain a graph $\Gamma = (V, E)$ as follows: $V = \mathcal{A}$ and $(a, b) \in E$ if and only if $(a, s, b) \notin \mathcal{F}$. Moreover, we can identify each coloring of $X_{\mathcal{F}}$ with a bi-infinite walk in a essential directed graph $\Gamma' = (V', E')$, where $V' = \{v \in V|\ V^+(v) \neq \emptyset \text{ and } V^-(v) \neq \emptyset\}$ and $E' \subseteq V' \times V'$. The essential directed graph Γ' as before is unique and in what follows for every $\mathbb{Z}$-NNSFT we only consider the essential directed graph associated.

Obtaining a valid coloring of the free group $\mathbb{F}_k$ by employing a finite set of $\mathcal{F}$ constraints corresponds to conducting these colorings individually in each of the k directions. In other words, we can consider the $X_{\mathcal{F}}$ $\mathbb{F}_k$-NNSSFT as comprising k distinct $\mathbb{Z}$-NNSFTs, with each one incorporating a specific set of forbidden patterns $\mathcal{F}_i \subseteq \mathcal{A} \times \{\sigma_i\} \times \mathcal{A}$. This significant observation is formally presented in the subsequent corollary.

Corollary 3.3 *A nonempty $\mathbb{F}_k$-NNSFT is completely determined by a family of k essential directed graphs Γ if and only if all the graphs on Γ are defined on a common set of vertices $V = \mathcal{A}'$.*

Example We consider $\mathbb{F}_2$ and $\mathcal{A} = \{0, 1\}$, we define $X_{\mathcal{F}}$ as the Fibonacci shift in the free group on two generators. Specifically, we represent $X_{\mathcal{F}}$ as $(1, s_i, 1)$, where $1 \leq i \leq 2$. By utilizing the aforementioned characterization, we observe that this $\mathbb{F}_2$-NNSFT can be effectively regarded as two distinct $\mathbb{Z}$-NNSFTs. Each $\mathbb{Z}$-NNSFT is determined by the set of forbidden patterns $\mathcal{F}_i = (1, s_i, 1)$, where $1 \leq i \leq 2$.

Definition 3.4 *As a consequence of Proposition 1.31 and Corollary 3.3, we are going to denote by X_Γ the $\mathbb{F}_k$-NNSFT defined by the set of k essential directed graphs $\Gamma = \{\Gamma_i\}_{i=1}^k$, using the common set of vertices $V = \mathcal{A}'$. Using the same notation as in the proof of Proposition 1.31, in what follows we are going to assume that $\Phi(a, s_i) = \Phi_i(a, s)$, $\forall a \in \mathcal{A}'$, $1 \leq i \leq k$. In other words the valid colorings in each coset $g < s_i >$ of a particular generator s_i and an arbitrary $g \in \mathbb{F}_k$, correspond to valid colorings of $X_{\mathcal{F}_i}$.*

Lemma 3.5 *Given a nonempty $\mathbb{F}_{k-1}$-NNSFT $X_\Gamma \subseteq V^{\mathbb{F}_{k-1}}$ and X a $\mathbb{Z}$-NNSFT defined by Γ_k, the essential directed graph associated using the same set of vertices as the family Γ. Then, it's possible to obtain a nonempty $\mathbb{F}_k$-NNSFT $X_{\Gamma'} \subseteq V^{\mathbb{F}_k}$, such that $\Gamma' = \Gamma \cup \Gamma_k$.*

Proof. In order to use Proposition 1.29, we consider as alphabet $\mathcal{A}' = V$ and we define a coloring function $\Phi_{X_{\Gamma'}}$ for $X_{\Gamma'}$ as an extension of the coloring function Φ_{X_Γ} of X_Γ obtained using Proposition 1.29 over the alphabet $\mathcal{A}'$. This means $\Phi_{X_{\Gamma'}}(a, s_i) = \Phi_{X_\Gamma}(a, s_i)$, for every $a \in \mathcal{A}'$ and $1 \leq i \leq k-1$. Given that Γ_k is an essential directed graph defined over $V = \mathcal{A}'$ implies the existence of a coloring function $\Phi_k : \mathcal{A}' \times \{s^{-1}, s\} \to \mathcal{A}'$, thus we can define $\Phi_{X_{\Gamma'}}(a, s_k) = \Phi(a, s)$ and $\Phi_{X_{\Gamma'}}(a, s_k^{-1}) = \Phi_k(a, s^{-1})$, for all $a \in \mathcal{A}'$. Therefore $X_{\Gamma'}$ is a nonempty $\mathbb{F}_k$-NNSFT. □

Next, we show how to construct a weakly aperiodic (Definition 2.1) $\mathbb{F}_k$-NNSFT, starting from a weakly aperiodic $\mathbb{F}_{k-1}$-NNSFT. With this in mind, we can extend examples defined on $\mathbb{F}_2$ to $\mathbb{F}_k$, with $k > 2$.

Lemma 3.6 *Given a weakly aperiodic $\mathbb{F}_{k-1}$-NNSFT X_Γ and X a $\mathbb{Z}$-NNSFT, with Γ_k its essential directed graph associated, using a common set of vertices as the family Γ. Then it's possible to obtain a weakly aperiodic $\mathbb{F}_k$-NNSFT $X_{\Gamma'}$, with $\Gamma' = \Gamma \cup \Gamma_k$.*

Proof. For X_Γ consider $\Gamma = \{\Gamma_1, \ldots, \Gamma_{k-1}\}$ a family of essential directed graphs defined over a common set of vertices V. Considering Γ_k a essential directed graph defined over V and using Lemma 3.5, we obtain $X_{\Gamma'}$ a nonempty $\mathbb{F}_k$-NNSFT, with $\Gamma' = \{\Gamma_1, \ldots, \Gamma_{k-1}, \Gamma_k\}$. As X_Γ is a weakly aperiodic $\mathbb{F}_{k-1}$-NNSFT, by Theorem 1.34 we obtain that for all set of cycles $C_1, \ldots, C_{k-1}$ on $\Gamma_1, \ldots, \Gamma_{k-1}$ respectively, there exists a symbol on the set of vertices $a \in V$ such that $\#_a(C_i) = \#_a(C_j)$, for some $1 \leq i < j \leq k-1$. Hence, independent of the graph Γ_k, the condition for the existence of a strongly periodic coloring given in Theorem 1.34 is always violated. Therefore, $X_{\Gamma'}$ is weakly aperiodic $\mathbb{F}_k$-NNSFT. □

Remark Let X_Γ be a nonempty $\mathbb{F}_k$-Wang subshift, with $\Gamma = \{\Gamma_1, \ldots, \Gamma_i, \ldots, \Gamma_j, \ldots \Gamma_k\}$. If consider $\Gamma' = \{\Gamma_1, \ldots, \Gamma_j, \ldots, \Gamma_i, \ldots \Gamma_k\}$, the resulting nonempty $\mathbb{F}_k$-NNSFT $Y_{\Gamma'}$ has the same characteristic of periodicity that X_Γ, this means that if X_Γ has a strongly periodic coloring then also $Y_{\Gamma'}$, equivalently if X_Γ is weakly aperiodic, also will be $Y_{\Gamma'}$.

3.4 Necessary and sufficient conditions to determine a valid nonempy $\mathbb{F}_k$-Wang subshift

We proper our attention to study $\mathbb{F}_k$-Wang subshifts. As mentioned in Section 2.1 these subshifts form a strict subclass of $\mathbb{F}_k$-NNSFTs. In contrast to Corollary 3.3, not every family of k essential directed graphs does determine a valid nonempty $\mathbb{F}_k$-Wang subshift. Counterexamples already exists for $\mathbb{F}_1 = \mathbb{Z}$.

Example Consider the nonempty $\mathbb{Z}$-NNSFT given by the following essential directed graph:

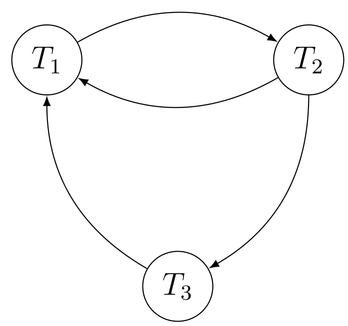


Figure 3.2: Example of a $\mathbb{Z}$-NNSFT which is not a $\mathbb{Z}$-Wang subshift

This $\mathbb{Z}$-NNSFT is not a $\mathbb{Z}$-Wang subshift. The definition of $\mathbb{Z}$-Wang tiles would imply that $T_1(s) = T_2(s^{-1})$, $T_2(s) = T_3(s^{-1})$, $T_3(s) = T_1(s^{-1})$ and $T_2(s) = T_1(s^{-1})$, forcing that $T_3(s) = T_3(s^{-1})$, which is impossible because there is no edge from T_3 to T_3.

Next we state necessary and sufficient conditions on the class of directed essential graphs which produce valid nonempty $\mathbb{Z}$-Wang subshifts.

Proposition 3.7 *An essential directed graph $\Gamma = (V, E)$ determines a valid nonempty $\mathbb{Z}$-Wang subshift if and only if for every $v, w \in V$:*

1. $V^+(v) \cap V^+(w) \neq \emptyset$ *implies* $V^+(v) = V^+(w)$,
2. $V^-(v) \cap V^-(w) \neq \emptyset$ *implies* $V^-(v) = V^-(w)$.

Using the adjacency matrix of Γ, this means:

a. *Each row and column contains at least a 1 (Γ is essential).*

b. *If two rows or two columns have a 1 in the same position, then those rows or columns are identical.*

Proof. Let $W = \{T_i\}_{i=1}^{|V|}$ be a set of $\mathbb{Z}$-Wang tiles corresponding to the set of vertices of Γ. We will identify T_i with v_i, for all $1 \leq i \leq |V|$. Thus, for every edge $(v_i, v_j) \in E$ it must be fulfilled that $T_i(s) = T_j(s^{-1})$.

($\Rightarrow$) For (1), consider $v_l \in V^+(v_i) \cap V^+(v_j)$, this implies that $(v_i, v_l), (v_j, v_l) \in E$, which in terms of $\mathbb{Z}$-Wang tiles means $T_i(s) = T_l(s^{-1})$ and $T_j(s) = T_l(s^{-1})$. If $v_p \in V^+(v_i)$ then $(v_i, v_p) \in E$, thus $T_i(s) = T_p(s^{-1})$. We obtain that $T_j(s) = T_p(s^{-1})$ or equivalently $(v_j, v_p) \in E$, implying $v_p \in V^+(v_j)$ and therefore $V^+(v_i) \subseteq V^+(v_j)$. Similarly we get $V^+(v_j) \subseteq V^-(v_i)$, obtaining $V^+(v_i) = V^+(v_j)$.

Consider $v_l \in V^-(v_i) \cap V^-(v_j)$, thus $(v_l, v_i), (v_l, v_j) \in E$, or equivalently $T_l(s) = T_i(s^{-1})$ and $T_l(s) = T_j(s^{-1})$. If $v_p \in V^-(v_i)$ then $(v_p, v_i) \in E$. This means that $T_p(s) = T_i(s^{-1})$. We obtain that $T_p(s) = T_j(s^{-1})$ and then $(v_p, v_j) \in E$, resulting that $V^-(v_i) \subseteq V^-(v_j)$. Analogously $V^-(v_j) \subseteq V^-(v_i)$ and therefore $V^-(v_i) = V^-(v_j)$ proving (2).

Finally, considering the adjacency matrix of the graph Γ, (a) is a consequence of Γ being an essential graph. and (b) is equivalent to points (1) and (2), for the rows and columns respectively.

($\Leftarrow$) Given a essential directed graph $\Gamma = (V, E)$ satisfying the conditions (1) and (2), where the set of vertices is a set of $\mathbb{Z}$-Wang tiles $V = \{T_i\}_{i=1}^{|V|}$. Considering a nonempty finite set $\mathcal{A}$, we define $T_i : S \cup S^{-1} \to \mathcal{A}$, for every $1 \leq i \leq |V|$, as follows: if $v_i \in V^+(v_j)$ then $T_j(s) = T_i(s^{-1})$ and every time that $v_i \in V^-(v_j)$ we put $T_i(s) = T_j(s^{-1})$. As Γ is an essential directed graph satisfy the conditions (1) and (2), it guarantees that we can well define the set of $\mathbb{Z}$-Wang tiles with this method and effectively this set of $\mathbb{Z}$-Wang tiles is consistent generating the graph Γ. Therefore, by definition of the elements on V, the set of every bi-infinite walks on Γ generate a valid nonempty $\mathbb{Z}$-Wang subshift. □

Proposition 3.8 *Every nonempty $\mathbb{F}_k$-Wang subshift, using n $\mathbb{F}_k$-Wang tiles, is completely determined by a family of k nonempty $\mathbb{Z}$-Wang subshifts.*

Proof. For $1 \leq i \leq k$, consider $W_i = \{T^i_j : \{s^{-1}, s\} \to \mathcal{A}_i\}_{j=1}^n$ set of $\mathbb{Z}$-Wang tiles and $X_{W_i} \subseteq W_i^{\mathbb{Z}}$ a $\mathbb{Z}$-Wang subshift. We construct a $\mathbb{F}_k$-Wang subshift from these k $\mathbb{Z}$-Wang subshifts using as alphabet the following set $W = \{T_j : S \cup S^{-1} \to \mathcal{A}\}_{j=1}^n$ a set of $\mathbb{F}_k$-Wang tiles, where $\mathcal{A} = \bigcup_{i=1}^k A_i$ and $\forall 1 \leq i \leq k$, $1 \leq j \leq n : T_j|_{\{s_i^{-1}, s_i\}} = T^i_j$.

The $\mathbb{F}_k$-Wang subshift $X_W \subseteq W^{\mathbb{F}_k}$ is nonempty by construction, because every X_{W_i} is nonempty and using Proposition 1.29 the results follows (this is a valid argument, because every $\mathbb{F}_k$-Wang subshift corresponds to a $\mathbb{F}_k$-NNSFT). Also, by construction of the alphabet W, we have that every $\mathbb{F}_k$-tiling on X_W restricted in the coset $g < s_i >$ corresponds to a $\mathbb{Z}$-tiling on X_{W_i}, i.e. the local rules on X_W in the direction of the generator s_i are the same that X_{W_i}. □

3.5 Weakly aperiodic $\mathbb{F}_k$-Wang subshifts with minimal alphabet size

Note that there is a unique nonempty $\mathbb{F}_k$-Wang subshift using a single $\mathbb{F}_k$-Wang tile, namely the one consisting of a uniform and thus strongly periodic coloring. The following proposition identifies all possibilities for an alphabet with 2 tiles.

Proposition 3.9 *Every nonempty $\mathbb{F}_k$-Wang subshift, defined on an alphabet with 2 $\mathbb{F}_k$-Wang tiles, contains strongly periodic tilings.*

Proof. Starting from a set of two $\mathbb{Z}$-Wang tiles, Proposition 3.7 gives 3 possibles essential directed graphs Γ_1, Γ_2 and Γ_3, given by the following adjacency matrices respectively:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let Γ be a nonempty family of k essential directed graphs considering Γ_1, Γ_2 and Γ_3. Using Proposition 3.8, we obtain a nonempty $\mathbb{F}_k$-Wang subshift $X_\Gamma \subseteq W^{\mathbb{F}_k}$ on 2 element alphabet $W = \{T_1, T_2\}$. By Theorem 1.34, if Γ only has k equal essential directed graphs consider the same set of cycles $C_1 = \cdots = C_k$ concluding that X_Γ has at least a strongly periodic tiling. We obtain the same result if in Γ we only consider the graphs Γ_1 and Γ_3 (or Γ_2 and Γ_3), since Γ_3 contains all the possible cycles of Γ_1 and Γ_2.

If $\Gamma_1, \Gamma_2 \in \Gamma$ we may consider the set of cycles $C_1 = \{\overline{T_1}, \overline{T_2}\}$ and $C_2 = \{\overline{T_1 T_2}\}$, for Γ_1 and Γ_2 respectively, satisfying the conditions in Theorem 1.34 and guaranteeing the existence of a strongly periodic tiling. Finally, if $\Gamma_1, \Gamma_2, \Gamma_3 \in \Gamma$ it is enough to consider for the graphs Γ_1 and Γ_2 the same set of cycles C_1 and C_2 as before, for Γ_3 any of these could be considered, concluding the existence of a strongly periodic tiling by Theorem 1.34. □

Theorem 3.10 *For every $k \geq 2$, the minimum cardinality of a set of $\mathbb{F}_k$-Wang tiles which produces a weakly aperiodic $\mathbb{F}_k$-Wang subshift is* 3.

Proof. To obtain this result it is enough to exhibit a corresponding $\mathbb{F}_2$-example and applying $k-2$ times Lemma 3.6 to conclude for $\mathbb{F}_k$, $k \geq 3$. Consider a set of three vertices $V = \{v_1, v_2, v_3\}$, and the following two essential directed graphs Γ_1 and Γ_2.

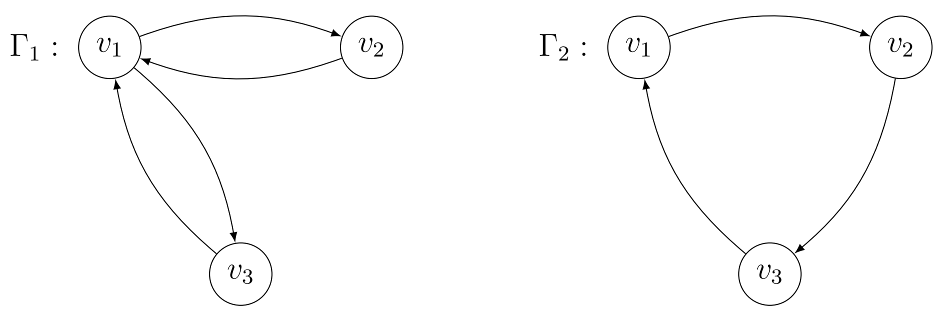


Figure 3.3: Corresponding essential graphs generates an example of weakly aperiodic $\mathbb{F}_k$-Wang subshift

Since Γ_1, Γ_2 satisfy the conditions of Proposition 3.7, this implies that Γ_1 and Γ_2 determine valid nonempty $\mathbb{Z}$-Wang subshift. Thus, they give rise to a $\mathbb{F}_2$-Wang subshift $X_\Gamma \subseteq V^{\mathbb{F}_2}$, given by $\Gamma = \{\Gamma_1, \Gamma_2\}$, considering two $\mathbb{F}_2$-Wang tiles which corresponds to extensions of the maps ($\mathbb{Z}$-Wang tiles) before obtained. Corollary 3.3 implies that X_Γ is nonempty and moreover corresponds to a weakly aperiodic $\mathbb{F}_2$-Wang subshift. First, observe that every cycle in Γ_1

has the form $\overline{v_1a_1v_1a_2v_1\ldots v_1a_n}$ with $a_i \in \{v_2, v_3\}$ and the possible cycles in Γ_2 can be $\overline{v_1v_2v_3\ldots v_1v_2v_3}$. Thus, for every nonempty set of cycles C_1 and C_2 in Γ_1 and Γ_2 respectively, we have:

$$\#_{v_1}(C_1) = \#_{v_2}(C_1) + \#_{v_3}(C_1), \tag{3.1}$$
$$\#_{v_1}(C_2) = \#_{v_2}(C_2) = \#_{v_3}(C_2). \tag{3.2}$$

Suppose the existence of a strongly periodic tiling on X_Γ. Theorem 1.34 would enforce:

$$\#_{v_1}(C_1) = \#_{v_1}(C_2), \tag{3.3}$$
$$\#_{v_2}(C_1) = \#_{v_2}(C_2), \tag{3.4}$$
$$\#_{v_3}(C_1) = \#_{v_3}(C_2). \tag{3.5}$$

Using these equations we would obtain $\#_{v_1}(C_2) = 0$, which is impossible, since Γ_2 contains no cycle avoiding v_1, and we conclude that X_Γ cannot have strongly periodic colorings. Finally, applying $k-2$ times Lemma 3.6 from X_Γ we obtain a weakly aperiodic $\mathbb{F}_k$-wang subshift, for every $k \geq 3$. □

3.6 Characterizing all the possible examples of weakly aperiodic $\mathbb{F}_k$-Wang subshifts with minimal alphabet size

We turn our attention to characterize all the possible examples of weakly aperiodic $\mathbb{F}_k$-Wang subshifts using an alphabet with 3 tiles, obtaining for every $k \geq 2$ the amount of all them. First, we enunciate an equivalent form of Theorem 1.34, which uses simple cycles. The simple cycles form a finite set, making it easier to prove formally when the condition in Theorem 1.34 is not satisfied.

In what follows, every graph considered satisfies the conditions of Proposition 3.7. We will enunciate Theorem 3.12 for $\mathbb{F}_k$-Wang subshift. Nevertheless, the result in [42] is for $\mathbb{F}_k$-NNSFTs.

Definition 3.11 *Consider a family of graphs $\Gamma = \{\Gamma_i\}_{i=1}^k$, defined over a common set of vertices V and $SC(\Gamma_i) = \{c_{i,j}\}_{j=1}^{|SC(\Gamma_i)|}$, $1 \leq i \leq k$. We denote by $(\star)$ the following equations on $x_{i,j}$:*

$$\forall v \in V, \sum_{j=1}^{|SC(\Gamma_1)|} x_{1,j}\#_v(c_{1,j}) = \cdots = \sum_{j=1}^{|SC(\Gamma_k)|} x_{k,j}\#_v(c_{k,j}).$$

We say that the family Γ satisfies the condition $(\star)$ if the equations in $(\star)$ admit a nontrivial positive solution.

Now, if we fix the attention in $\Gamma_i \in \Gamma$ and we consider an arbitrary cycle $c \in \Gamma_i$, using the *previous notation, we have the existence of constants* $x_{i,1},\ldots,x_{i,|SC(\Gamma_i)|}$, *such that:*

$$\begin{pmatrix}\#_{v_1}(c)\\ \vdots\\ \#_{v_n}(c)\end{pmatrix} = x_{i,1}\begin{pmatrix}\#_{v_1}(c_{i,1})\\ \vdots\\ \#_{v_n}(c_{i,1})\end{pmatrix} + \cdots + x_{i,|SC(\Gamma_i)|}\begin{pmatrix}\#_{v_1}(c_{i,|SC(\Gamma_i)|})\\ \vdots\\ \#_{v_n}(c_{i,|SC(\Gamma_i)|})\end{pmatrix}. \quad (\star)'$$

We denote by $W_{\Gamma_i} \subseteq \mathbb{N}_0^{|V|}$ *to the set of solutions for* Γ_i *given in* $(\star)'$.

Theorem 3.12 *[42, Theorem 3.6] A nonempty* $\mathbb{F}_k$*-Wang subshift contains a strongly periodic tiling if and only if the associated family of graphs satisfies the condition* $(\star)$.

A direct consequence from Theorem 3.12, using these sets of solutions, is the following.

Corollary 3.13 *Let* X_Γ *be a nonempty* $\mathbb{F}_k$*-Wang subshift. Then* X_Γ *contains a strongly periodic tiling if and only if* $\bigcap_{i=1}^{k} W_{\Gamma_i} \neq 0$.

Before continuing with the description of all possible examples of weakly aperiodic $\mathbb{F}_k$-Wang subshifts using an alphabet with minimal size. We are going to prove that a $\mathbb{F}_2$-Wang subshift is weakly aperiodic using Corollary 3.13.

X_Γ is a nonempty $\mathbb{F}_2$-Wang subshift by Proposition 3.8 . So, the corresponding set of simple cycles are $SC(\Gamma_1) = \{\overline{v_1 v_3}, \overline{v_2 v_3}\}$ and $SC(\Gamma_2) = \{\overline{v_1}, \overline{v_2}\}$, for Γ_1 and Γ_2 respectively. Moreover, we obtain $W_{\Gamma_1} = \left\{\begin{pmatrix}a\\ b\\ a+b\end{pmatrix} \in \mathbb{N}_0^3 \mid a,b \in \mathbb{N}_0\right\}$ and $W_{\Gamma_2} = \left\{\begin{pmatrix}a\\ b\\ 0\end{pmatrix} \in \mathbb{N}_0^3 \mid a,b \in \mathbb{N}_0\right\}$.

We note that $W_{\Gamma_1} \cap W_{\Gamma_2} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$ satisfying Corollary 3.13 and thereforeX_Γ is a weakly aperiodic $\mathbb{F}_2$-Wang subshift.

Using Proposition 3.7 we obtain all the possible graphs, defined on 3 vertices, which determine a valid nonempty $\mathbb{Z}$-Wang subshift. Satisfying these characteristics, there are a total of 25 essential directed graphs, which will be classified in 5 sets $W_1,\ldots,W_5$, whose criterion will be the set of solutions for $(\star)'$ showed in Definition 3.11. For simplicity, we expose these sets using the adjacency matrices of the corresponding graphs, as follows:

$$W_1 = \left\{ \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}, \begin{pmatrix}1&1&0\\1&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&1\\0&1&1\end{pmatrix}, \begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \right\},$$

$$W_2 = \left\{ \begin{matrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \end{matrix} \right\},$$

$$W_3 = \left\{ \begin{matrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} & & & & \end{matrix} \right\},$$

$$W_4 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\},$$

$$W_5 = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\},$$

Previously, it results important to observe that, for instance, on W_2 there are graphs $\Gamma_i, \Gamma_j \in \Gamma$ whose sets of solutions is $W_{\Gamma_i} = \left\{ \begin{pmatrix} a \\ a \\ b \end{pmatrix} \in \mathbb{N}_0^3 \mid a, b \in \mathbb{N}_0 \right\}$ and $W_{\Gamma_j} = \left\{ \begin{pmatrix} a \\ b \\ a \end{pmatrix} \in \mathbb{N}_0^3 \mid a, b \in \mathbb{N}_0 \right.$ respectively. Where both sets are practically the same set, because for our goal they behave the same, thus we will say that W_2 is isomorph to some set with these characteristics. In general, we have $W_1 \simeq \mathbb{N}_0^3$, $W_2 \simeq \left\{ \begin{pmatrix} a \\ a \\ b \end{pmatrix} \mid a, b \in \mathbb{N}_0 \right\}$, $W_3 \simeq \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \mid a, b \in \mathbb{N}_0 \right\}$, $W_4 \simeq \left\{ \begin{pmatrix} a \\ a+b \\ b \end{pmatrix} \mid a, b \in \mathbb{N}_0 \right\}$ and $W_5 \simeq \left\{ \begin{pmatrix} a \\ a \\ a \end{pmatrix} \mid a \in \mathbb{N}_0 \right\}$, this classification aims to obtain the following result.

Proposition 3.14 *The amount of examples of weakly aperiodic $\mathbb{F}_k$-Wang subshifts depends of k. For $k = 2$ this amount is 48 and when it is considered $k \geq 3$ the amount has exponential growth order $25^{o(n)}$.*

Proof. The main idea to prove this result will be to use Corollary 3.13. Then, it is necessary to obtain the possible sets which we can obtain after intersecting pairs. Doing this result the following sets appear: $W_6 \simeq \left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \mid a \in \mathbb{N}_0 \right\}$, $W_7 \simeq \left\{ \begin{pmatrix} a \\ a \\ 0 \end{pmatrix} \mid a \in \mathbb{N}_0 \right\}$, $W_8 \simeq \left\{ \begin{pmatrix} 2a \\ a \\ a \end{pmatrix} \mid a \in \mathbb{N}_0 \right\}$ and $W_9 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Next, we construct a directed graph, which will allow us to compute the examples of weakly aperiodic $\mathbb{F}_k$-Wang subshifts, for every $k \geq 2$. Let $\Gamma_W = (V_W, E_W)$ be a directed graph defined as follows: $V_W = \{W_1, \ldots, W_9\}$ and there exists an edge $(W_i, W_j) \in E_W$ every time that, for some $1 \leq t \leq 5$, some directed graph on W_t satisfies that W_t intersected with W_i results W_j. For simplicity, we going to labeling the edges of Γ_W with the amount of edges which there are between two vertices. With this in our mind, the graph Γ_W is the following:

Γ_W :

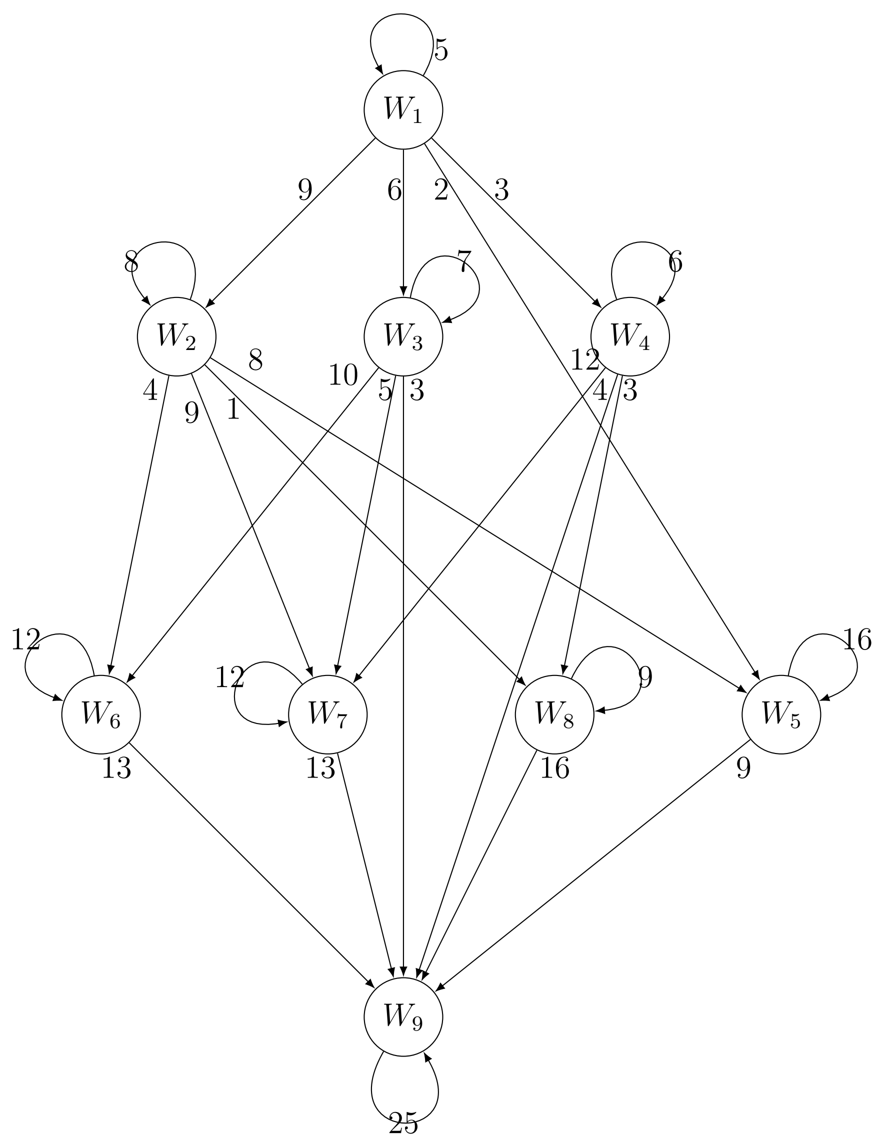
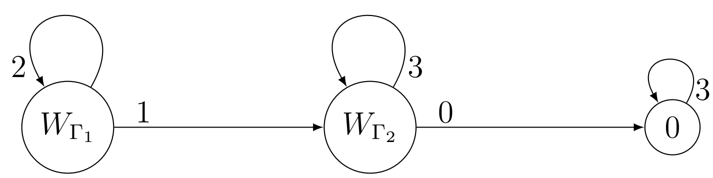


Figure 3.4: Graph that allows to compute the amount of examples of weakly aperiodic $\mathbb{F}_k$-Wang subshifts.

Using the graph Γ_W and Corollary 3.13, it is possible to compute the amount of examples of weakly aperiodic $\mathbb{F}_k$-Wang subshift. These correspond to all the paths of length k from W_1 to W_9. For instance, if we want to compute the total of examples weakly aperiodic $\mathbb{F}_2$-Wang subshifts, we must count the amount of paths of length 2 on Γ_W, resulting in 48 (because, there are 18 possibilities passing for W_3, same amount for W_5 and others 12 for W_4). Moreover, for $k \geq 3$, this amount has a exponential growth, because W_9 corresponds to the trivial set and all the possible 25 graphs satisfies the condition defined previously for the edges. Moreover, the adjacency matrix associated to Γ_W is diagonalizable, being 25 the biggest eigenvalue, the result follows.

□

Remark We may use the above to do an alternative proof of Proposition 3.9. The good reader can review that the following sets of solutions are the obtained: $W_{\Gamma_1} = W_{\Gamma_3} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{N}_0^2 \mid a, b \in \mathbb{N}_0 \right\}$ and $W_{\Gamma_2} = \left\{ \begin{pmatrix} a \\ a \end{pmatrix} \in \mathbb{N}_0^2 \mid a \in \mathbb{N}_0 \right\}$. The corresponding directed graph is:



We see that there are no paths from W_{Γ_1} to 0, then by Corollary 3.13 we can conclude that for every $k \geq 2$, always there exists a periodic tiling in the corresponding $\mathbb{F}_k$-Wang subshift.

3.7 Conclusion

The problem of determining the minimum number of $\mathbb{F}_k$-Wang tiles required to generate a nonempty weakly aperiodic $\mathbb{F}_k$-Wang subshift has been solved for free groups. In contrast to the result obtained by E. Jeandel and M. Rao in [25], for the case $G = \mathbb{Z}^2$, where there show that this quantity is 11, our result shows that this quantity is 3 in the case $G = \mathbb{F}_k$, $k \geq 2$.

Moreover, all possible examples with these characteristics have been fully characterized. Therefore, there is no further work to be done in this direction for free groups. However, a possible direction for future research could be the study of Wang subshifts in other groups, or even the resolution of the problem in $\mathbb{Z}^d$, for any d.

Part III

Bibliography

Bibliography

[1] Nathalie Aubrun, Sebastián Barbieri, and Etienne Moutot. The domino problem is undecidable on surface groups. *arXiv preprint arXiv:1811.08420*, 2018.

[2] Nathalie Aubrun and Jarkko Kari. Tiling problems on baumslag-solitar groups. *arXiv preprint arXiv:1309.1266*, 2013.

[3] Alexis Ballier and Maya Stein. The domino problem on groups of polynomial growth. *arXiv preprint arXiv:1311.4222*, 2013.

[4] Sebastián Barbieri. A geometric simulation theorem on direct products of finitely generated groups. *arXiv preprint arXiv:1706.00626*, 2017.

[5] Sebastián Barbieri and Mathieu Sablik. A generalization of the simulation theorem for semidirect products. *Ergodic Theory and Dynamical Systems*, 39(12):3185–3206, 2019.

[6] Robert Berger. *The undecidability of the domino problem*. Number 66. American Mathematical Soc., 1966.

[7] Valérie Berthé and Michel Rigo. *Combinatorics, words and symbolic dynamics*, volume 159. Cambridge University Press, 2016.

[8] George David Birkhoff. *Dynamical systems*, volume 9. American Mathematical Soc., 1927.

[9] D Birkoff. Quelques théorèmes sur le mouvement des systèmes dynamiques. *Bulletin de la Société mathématique de France*, 40:305–323, 1912.

[10] David Carroll and Andrew Penland. Periodic points on shifts of finite type and commensurability invariants of groups. *arXiv preprint arXiv:1502.03195*, 2015.

[11] Professor Cayley. Desiderata and suggestions: No. 2. the theory of groups: graphical representation. *American journal of mathematics*, 1(2):174–176, 1878.

[12] Tullio Ceccherini-Silberstein and Michel Coornaert. *Cellular automata and groups*. Springer Science & Business Media, 2010.

[13] Jean-René Chazottes, Jean-Marc Gambaudo, and François Gautero. Tilings of the plane and thurston semi-norm. *Geometriae Dedicata*, 173(1):129–142, 2014.

[14] David B Cohen, Chaim Goodman-Strauss, and YO'AV RIECK. Strongly aperiodic subshifts of finite type on hyperbolic groups. *Ergodic Theory and Dynamical Systems*, pages 1–44, 2021.

[15] David Bruce Cohen and Chaim Goodman-Strauss. Strongly aperiodic subshifts on surface groups. *Groups, Geometry, and Dynamics*, 11(3):1041–1059, 2017.

[16] H. Maturana Cornejo and M. Schraudner. Weakly aperiodic $\mathbb{F}_d$-Wang subshift with minimal alphabet size and its complexity function. *Unpublished preprint*, 2018.

[17] Karel Culik II. An aperiodic set of 13 wang tiles. *Discrete Mathematics*, 160(1-3):245–251, 1996.

[18] Pierre de La Harpe. *Topics in geometric group theory.* University of Chicago Press, 2000.

[19] Benjamin Hellouin de Menibus and Hugo Maturana Cornejo. Necessary conditions for tiling finitely generated amenable groups. *Discrete and continous dynamical systems, series A*, 2020.

[20] Jean-Marc Gambaudo. A note on tilings and translation surfaces. *Ergodic Theory and Dynamical Systems*, 26(1):179–188, 2006.

[21] Pierre Antoine Grillet. Abstract algebra, vol. 242. *Graduate Texts in Mathematics. New York: Springer-Verlag*, 2007.

[22] Jacques Hadamard. Les surfaces à courbures opposées et leurs lignes géodésique. *J. Math. pures appl.*, 4:27–73, 1898.

[23] Gustav A Hedlund. Endomorphisms and automorphisms of the shift dynamical system. *Mathematical systems theory*, 3(4):320–375, 1969.

[24] Thomas W Hungerford. *Algebra*, volume 73. Springer Science & Business Media, 2012.

[25] Emmanuel Jeandel. Aperiodic subshifts on polycyclic groups. *arXiv preprint arXiv:1510.02360*, 2015.

[26] Emmanuel Jeandel. Translation-like actions and aperiodic subshifts on groups. *arXiv preprint arXiv:1508.06419*, 2015.

[27] Emmanuel Jeandel and Michael Rao. An aperiodic set of 11 wang tiles. *arXiv preprint arXiv:1506.06492*, 2015.

[28] Emmanuel Jeandel and Pascal Vanier. The undecidability of the domino problem. In *Substitution and Tiling Dynamics: Introduction to Self-inducing Structures*, pages 293–357. Springer, 2020.

[29] Jarkko Kari. A small aperiodic set of wang tiles. *Discrete Mathematics*, 160(1-3):259–264, 1996.

[30] Bruce P Kitchens. *Symbolic dynamics: one-sided, two-sided and countable state Markov shifts*. Springer Science & Business Media, 2012.

[31] Petr Kurka. *Topological and symbolic dynamics*, volume 11. SMF, 2003.

[32] S Lang. Undergraduate algebra, utm, 1990.

[33] Sebastián Andrés Barbieri Lemp. *Shift spaces on groups: computability and dynamics*. PhD thesis, Université de Lyon, 2017.

[34] Douglas Lind and Brian Marcus. *An introduction to symbolic dynamics and coding*. Cambridge university press, 1995.

[35] M Lothaire. *Applied combinatorics on words*, volume 105. Cambridge University Press, 2005.

[36] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory: Presentations of groups in terms of generators and relations*. Courier Corporation, 2004.

[37] John Meier. *Groups, graphs and trees: an introduction to the geometry of infinite groups*. Cambridge University Press, 2008.

[38] John Meier. *Groups, graphs, and trees: an introduction to the geometry of infinite groups*. Cambridge University Press, 2008.

[39] Harold Marston Morse. A one-to-one representation of geodesics on a surface of negative curvature. *American Journal of Mathematics*, 43(1):33–51, 1921.

[40] Marston Morse and Gustav A Hedlund. Symbolic dynamics. *American Journal of Mathematics*, 60(4):815–866, 1938.

[41] Shahar Mozes. Aperiodic tilings. *Inventiones mathematicae*, 128(3):603–611, 1997.

[42] Steven T Piantadosi. Symbolic dynamics on free groups. *Discrete and Continuous Dynamical Systems*, 20(3):725, 2008.

[43] Raphael M Robinson. Undecidability and nonperiodicity for tilings of the plane. *Inventiones mathematicae*, 12(3):177–209, 1971.

[44] Ayse A Sahín, Michael Schraudner, and Ilie Ugarcovici. A strongly aperiodic shift of finite type for the discrete heisenberg group. 2014.

[45] Hao Wang. Proving theorems by pattern recognition—ii. *Bell system technical journal*, 40(1):1–41, 1961.

[46] Hao Wang. Notes on a class of tiling problems. *Fundamenta Mathematicae*, 82(4):295–305, 1975.