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A SINGULAR PERTURBATION PROBLEM MODELLING AMPLITUDE WALLS OF STRIPED PATTERNS

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Resumen

UN PROBLEMA DE PERTURBACIÓN SINGULAR QUE MODELA LA AMPLITUD SOBRE PAREDES EN PATRONES RAYADOS

El presente trabajo estudia el sistema de ecuaciones diferenciales que como modela el choque y transición de patrones rayados a través de una pared de dominio utilizando como modelo las ecuaciones de amplitud.

En el primer capitulo llamado introducción se entrega una noción general del origen de los problemas abordados en las siguientes secciones, además de las preguntas y motivaciones que generan varias de las técnicas utilizadas en este trabajo.

En el segundo capitulo son introducidos sistemas comúnmente asociados a la formación de patrones, como lo es la ecuación de Swift-Hohenberg. Se presentan además las nociones de inestabilidad lineal para el estado homogéneo, fenómeno que da origen a la generación de patrones en estos sistemas no lineales. Posteriormente se realiza una derivación de la ecuación de amplitud para la descripción espacial y evolución de estos patrones, lo que permite llegar a los problema estudiados en este texto, sistemas no lineales de ecuaciones que posee dos funciones como incógnitas a las que nos referiremos como u y v.

En el tercer capitulo son presentados aquellos resultados matemáticos que son considerados indispensables para realizar los procedimientos de los capítulos posteriores, esto con el fin de que pueda ser rápidamente consultado y permita al lector una limpia vista del texto y de una forma auto-contenida.

En el cuarto capitulo son expuestos todos aquellos procedimientos y resultados sobre los sistemas no lineales estudiados, en las secciones 4.1, 4.2 y 4.3 se demuestra la existencia de u_0, v_0 y v_0^R (este tercero para el caso perpendicular), funciones que son aproximaciones de soluciones del problema no lineal original, posteriormente, en la sección 4.4 se muestran los resultados numéricos obtenidos mediante diferentes métodos para aquellos problemas los cuales no han podido ser enfrentados de forma satisfactoria utilizando las técnicas anteriores.

Abstract

A SINGULAR PERTURBATION PROBLEM MODELLING AMPLITUDE WALLS OF STRIPED PATTERN

This work focuses on studying a system of differential equations that models the collision and transition between stripe patterns across a domain wall using amplitude equations as a model.

The first chapter, titled "Introduction", provides a general overview of the origins of the problems addressed in the following sections, along with the questions and motivations that drive the various techniques used in this work.

In the second chapter, we introduce systems commonly associated with pattern formation, including the Swift-Hohenberg equation. We also present the concept of linear instability for the homogeneous state, a phenomenon that gives rise to pattern generation in these nonlinear systems. Additionally, we derive the amplitude equation, which describes the spatial behavior and evolution of these patterns. This leads us to the problems studied in this text, specifically nonlinear systems of equations with two unknown functions denoted as u and v.

The third chapter covers essential mathematical results necessary to carry out the procedures in the subsequent chapters. This is done to ensure quick accessibility and enable the reader to have a clear understanding of the text in a self-contained manner.

In the fourth chapter, we present all the procedures and results related to the studied nonlinear systems. Sections 4.1, 4.2, and 4.3 demonstrate the existence of u_0, v_0, and v_0^R (the latter for the perpendicular case), which are functions that approximate solutions to the original nonlinear problem. Furthermore, section 4.4 provides numerical results obtained using different methods for those problems that have not been satisfactorily addressed using the previous techniques.

A mi mamá, a mi papá y a mis amigos.

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Chapter 1

Introduction

Pattern forming systems are systems where some spatial structure can be observed, this structure can be of many different shapes and forms such as labyrinths, tescelations, stripes, and others. The study of diffusion equations is the usual context where it is posible to find these patterns.

When it comes to set results over these problems it happens that there is more than one way to study how patterns are formed and more than one way to study the structure that these patterns should have. One approach to study the structure is the amplitude equation, that is the building an equation able to capture the amplitude associated to the corresponding pattern we are studying, which can be formed by stripes, squares, hexagons or another; the construction of an accurate amplitude equation is based on the different types of symmetries the physical problem obeys. When two or more different oriented structures on this pattern collide (for example stripes that collide with hexagons) it is neccesary to capture the interaction between them, this is how coupled systems of differential equations emerge as a model for this phenomena.

The problem of our interest is the coupled system of differential equations that emerges when two different oriented stripes collide

$$\begin{cases} u'' + u(1 - u^2 - \mu v^2) & = 0 \\ \delta^2 v'' + v(1 - v^2 - \mu u^2) & = 0, \end{cases}$$

where each of the functions u and v represent the amplitude of the respective set of stripes, as we can observe it is a nonlinear system and so nonlinear variational techniques may be employed to show existence and regularity of solutions for approximations of this problem, this is what we do in this text.

The limit case of the previous problem is when the angles between the stripes are perpendicular and one of them is parallel to the structure where the transition occurs (the grain boundary). The problem in this limit case is given by:

$$\begin{cases} u'' + u(1 - u^2 - \mu v^2) & = 0 \\ -\varepsilon^2 v^{(\mathrm{iv})} + v(1 - v^2 - \mu u^2) & = 0, \end{cases}$$

where ε is a real positive number, which is assumed to be small.

Chapter 2

Background and origin of the problem

2.1 Pattern-Forming nonequilibrium systems

Pattern-forming systems are commonly associated to non-equilibrium systems, this is because, to form the structures, it is necessary to inject energy in a constant or periodic way, enough to break the stability of the uniform state.

There are a lot of examples in nature where the uniform states (the medium is translationally invariant) are not stable. We will call a state unstable if it is possible to find an arbitrary small perturbation that makes the system evolve rapidly away from this initial state. In many cases the uniform state is not preserved as the system evolves, and so, another stable state may be obtained.
In our context these new stable states usually have structure and we call them patterns. We are interested in the study of these patterns and what geometrical and analytical properties they have.

2.2 Swift-Hohenberg equation as a good simplificated problem

The one-dimensional Swift-Hohenberg equation is the equation for a single field $u(x,t)$ in a one-dimensional domain $0 \leq x \leq L$ given by

$$\partial_t u = (r-1)u - 2\partial_x^2 u - \partial_x^4 u - u^3,$$

or equivallently

$$\partial_t u = ru - (\partial_x^2 + 1)^2 u - u^3, \tag{2.1}$$

this equation is commonly used to model patter-forming systems. As we see, we are dealing with an evolution equation where the right-hand side is characterized for being translationally invariant ($x \to x+s$) and rotationally invariant ($x \to -x$).

It is also clear that $u = 0$ works as a simple uniform solution of the equation, we are interested on studying if there exists any condition over the parameter r on (2.1) to make this state an unstable state.

2.2.1 Linear instability analysis

To carry out the linear stability analysis, we denote the solution $u = 0$ as the base state u_b and we refer as u_p to a perturbation such that

$$u(x,t) = u_b + u_p(x,t),$$

the right-hand side of the evolution equation is given by

$$N[u_b] = (r-1)u_b - 2\partial_x^2 u_b - \partial_x^4 u_b - u_b^3,$$

and so

$$N[u_b + u_p] = (r-1)(u_b + u_p) - 2\partial_x^2(u_b + u_p) - \partial_x^4(u_b + u_p) - (u_b + u_p)^3$$
$$N[u_b + u_p] = N[u_b] + \underbrace{(r-1)u_p - 2\partial_x^2 u_p - \partial_x^4 u_p - 3u_b^2 u_p}_{L_{u_b}[u_p]} - 3u_b u_p^2 - u_p^3$$
$$N[u_b + u_p] = N[u_b] + L_{u_b}[u_p] + o(u_p^2.$$

The linearization of the equation for u_p on u_b becomes

$$\partial_t u_p = (r - 1 - 2\partial_x^2 - \partial_x^4 - 3u_b^2)u_p,$$

and then by using the specific base state $u_b = 0$

$$\partial_t u_p = (r - 1 - 2\partial_x^2 - \partial_x^4)u_p$$

is the evolution equation for small perturbations u_p.
Given that we are facing a linear evolution equation it is natural to assume exponential type solutions

$$u_p(x,t) = Ae^{\sigma t}e^{\alpha x}.$$

By replacing in the equation we get the following condition for the perturbation

$$\sigma = r - (\alpha^2 + 1)^2.$$

If we assume an infinite boundary then α must be a purely imaginary number or the everywhere small perturbation assumption would be violated. We write $\alpha = iq$ where q is a real value. We conclude that in this case (infinite boundary) it is possible to find solutions given by

$$u_p = Ae^{\sigma_q t}e^{iqx},$$

where $\sigma_q = r - (q^2 - 1)^2$ (equation that can be interpreted as a dispersion equation) and given that we are working with a linear equation any superposition of them will be also a solution .
The key observation to make now is that if $r > 0$ then there exists infinite many values of q for which we can find perturbations that have positive σ_q and so they grow exponentially to infinity when $t \to \infty$, and so this characterizes the instability of the $u_b = 0$ state. We conclude that for $r > 0$ the uniform state $u_b = 0$ is linearly unstable.

2.3 Amplitude equation

The previous chapter leave us with exponential growing solutions, and in the unstable case the presence of unbounded solutions, which is unrealistic and so not a good model for most physical situations. The cause of the problem is clear: we linearized the equation and assume arbitrary small perturbations, which obviously gives birth to exponential type solutions. The nonlinear terms of the equations are a necessity given that they quench the exponential growth of these solutions and allows the system to reach some other stable state if possible. If the system eventually approaches a time-independent state solution (this means the linear and nonlinear terms of the equations are balanced) we call the resulting solution a saturated nonlinear steady state. We would like to study how simple Fourier modes evolves in what we call amplitude, we assume

$$u(x,t) = A(t)e^{iqx} + c.c.$$

where *c.c.* means complex conjugate such that u is a real valued function and $A(t)$ is for the amplitude that we assume is time dependent. The simplest dynamic we can find for the amplitude is the following

$$d_t A = \sigma A,$$

but as we said this dynamic is not able to capture saturation.

In order to make the saturation happen we need to obtain a higher order equation for the amplitude, there are two ways of doing this. One is to substitute the simple Fourier mode into the evolution equation and use some technique to derive the evolution equation for A. This approach is specific for the physical system or model.
A second approach is to add terms to the equation (with some unknown coefficients) that are consistent with our assumptions on smoothness and symmetry. For example, assuming a rotationally invariant system, we expand the equation of motion for A to higher order terms in a way that the transformation $A \to e^{i\theta}A$ is still a solution, this gives

$$d_t A = \sigma A - \gamma A^2 A^* + ...,$$

where γ is a constant, the dots ... denote higher-order terms negligible if the amplitude $|A|$ is small enough and A^* is for the complex conjugate of A, and so the equation may also be written as

$$d_t A = \sigma A - \gamma |A|^2 A + ...$$

In order to find a more complete description of the evolution of the system we can write

$$u_p(x,t) = A(x,t)e^{iq_c x} + c.c. + h.o.t.,$$

where the amplitude is slowly varying with respect to x and *h.o.t.* is higher order terms that are small enough in magnitude than the displayed terms.

2.3.1 Derivation of the amplitude equation

As was mentioned in the previous section, an amplitude equation for the amplitude A is usually derived by one of two possible ways, one is replace u_p into the evolution equation for

the specific problem and then by using formal expansion technique find the corresponding equation, the other one (and the one we use here) is to recognize the symmetries of the system and then deduce the form of the amplitude equation.
We argue that one-dimensional amplitude equation for a modulated stripe state near a type-I-s instability (a stationary instability details in [7]) takes the form

$$\tau_0 \partial_t A(x,t) = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A,$$

where ε is called reduced bifurcation parameter while τ_0, ξ_0 and g_0 are constants that depend on details of the physical problem which may be calculated from the known evolution equations [7].

It is not hard to notice that the previous equation is just the basic evolution equation able to capture rotational invariance ($x \to -x$ in the one-dimensional case). More complex elements may be added on the right-hand side of the equation but a more complicated equation is not necessarily a better model.

2.4 Amplitude equations for two-dimensional patterns

For two dimensional problems it is useful to introduce a notation for the coordinates on the physical system. A good way to give sense to this new coordinates is to think on the Rayleigh-Bernard convection. Given a box where energy is flowing from the bottom to the ceiling, this direction where energy flows is called parallel direction and will be denoted by $x_{\|}$, on the other hand the directions perpendicular to this one are called extended directions and denoted by $x_{\perp}$.

In the two-dimensional case rotationally invariant system become much more interesting: we may face different structures such as stripes, squares, hexagons and other more complex structures born from the interaction between Fourier modes.
Here we write an equation for the amplitude of a simple Fourier mode for a rotationally invariant two-dimensional system, and then study the problem of superimposed stripes.

2.4.1 Stripes in rotationally invariant systems

The construction of the equation for the amplitude follows the same steps as the one-dimensional case, but we now allow the amplitude to be a slowly varying function of the two coordinates in the extended dimensions, as well as of time

$$u_p(x_{\perp}, x_{\|}, t) = A(x_{\perp}, t) u_c(x_{\|}) e^{iq_c x} + c.c. + h.o.t.,$$

with $x_{\perp} = (x, y)$. Given the rotationally invariance of the system, we are forced to introduce a reference direction from which we can start building our amplitude equation, the x-direction it is going to be the normal direction to the parallel stripes.

Having this in consideration we arrive to the following lowest-order derivative evolution equation for the amplitude by adding what we call Newell-Whitehead-Segel operator to the right-hand side of the equation (details about the derivation in [7], [10])

$$\tau_0 \partial_t A(x, y, t) = \varepsilon A + \xi_0^2 \left(\partial_x - \frac{i}{2q_c} \partial_y^2 \right)^2 A - g_0 |A|^2 A.$$

By means of the change of variables

$$A = \left| \frac{g_0}{\varepsilon} \right|^{1/2} A,\ X = \frac{|\varepsilon|^{1/2} x}{\xi_0} x,\ Y = |\varepsilon|^{1/4} \left(\frac{q_c}{\xi_0} \right)^{1/2} y,\ T = \frac{\varepsilon}{\tau_0} t,$$

we get

$$\partial_T A = A + \left(\partial_X - \frac{i}{2} \partial_Y^2 \right)^2 A - |A|^2 A.$$

2.4.2 Superimposed stripes

In systems where rotational invariance is satisfied, different striped patterns with different orientations may coexists, in this cases there must be an interaction between them that we capture by adding a new term to the equation. Here we give the system for two sets of superimposed stripes [7]

$$\begin{aligned} \tau_0 \partial_t A_1 &= \varepsilon A_1 + \xi_0^2 \left(\partial_{x_1} - \frac{i}{2q_c} \partial_{y_1}^2 \right)^2 A_1 - g_0 \left(|A_1|^2 + G(\theta_{12}) |A_2|^2 \right) A_1, \\ \tau_0 \partial_t A_2 &= \varepsilon A_2 + \xi_0^2 \left(\partial_{x_2} - \frac{i}{2q_c} \partial_{y_2}^2 \right)^2 A_2 - g_0 \left(|A_2|^2 + G(\theta_{12}) |A_1|^2 \right) A_2, \end{aligned} \tag{2.2}$$

where A_i is the corresponding amplitude for the $i-th$ set of stripes and G is an even function depending on the angle between the stripes, this new addition to the equation allows the existence of corners, squares and many other defects such as the one we introduce now called grain boundary.

2.4.3 Grain boundaries

Grain boundaries are extended line defects that separate two half-spaces of different oriented patterns, for example stripes with different wave vectors, which is the case we will consider. Studying this defect we consider the coordinate system defined so that the grain boundary is along the infinite line $x = 0$, with a pattern of stripes at wave vector q_1 for $x \to \infty$ and q_2 for $x \to -\infty$, where the angles between the stripes are θ_1 and θ_2 respectively with respect to the vertical line defining boundary.

2.4.4 General Case

For the general case of θ_1 and θ_2 not close to $\pi/2$ the system (2.2) (after some scaling) becomes [7],[10]

$$A_1 + \cos^2 \theta_1 \partial_x^2 A_1 - (A_1^2 + G_{12} A_2^2) A_1 = 0,$$

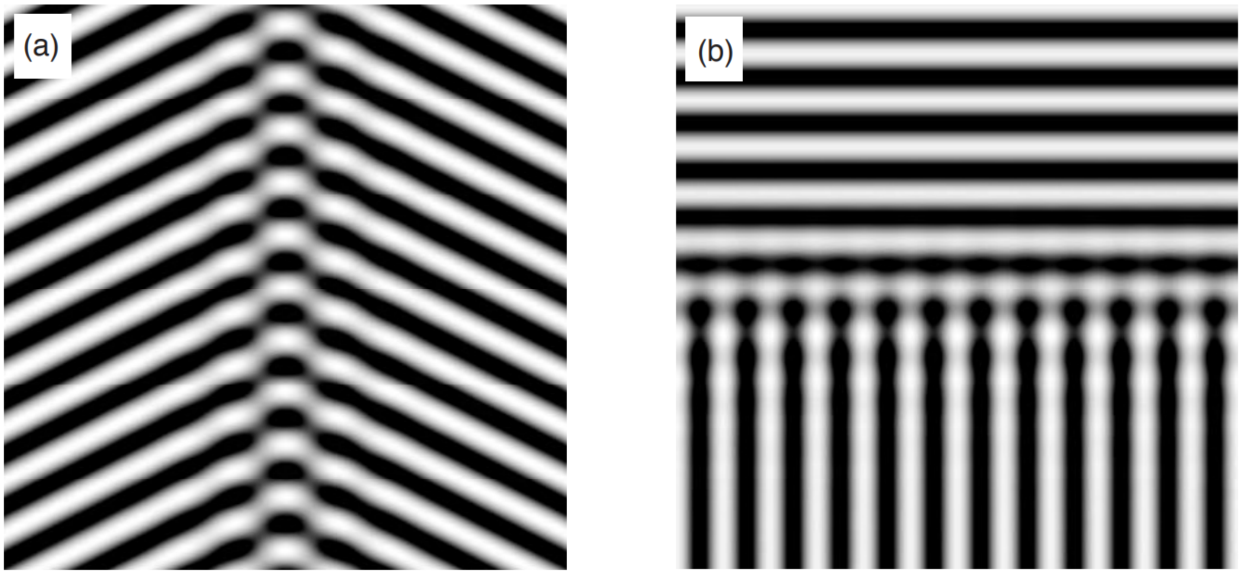

Figure 2.1: Two examples of grain boundaries, which are topological line defects. (a) Two sets of stripes with different orientations meet along a line that has a general angle relative to the stripes. (b) A perpendicular grain boundary is a special case for which the boundary is normal to one set of stripes. Figure 4.12 extracted from Cross's book [7].

$$A_2 + \cos^2\theta_2 \partial_x^2 A_2 - (A_2^2 + G_{12}A_1^2)A_2 = 0,$$

where derivatives of third and fourth order are omitted given they are small in comparison to the other terms. This can only be done when neither of the angles θ_1 or θ_2 are close to $\pi/2$, (this limit case is included in the next subsection). We consider the general case when $|\theta_i - (\pi/2 + c)| = O(\delta)$, for $i \in \{1, 2\}$ (only one of them), where c is a nonzero real constant. In this case, once again by taking a variable scaling and reordering the terms we obtain the system

$$\partial_x^2 A_1 + A_1(1 - A_1^2 + G_{12}A_2^2) = 0,$$
$$\delta^2\partial_x^2 A_2 + A_2(1 - A_2^2 + G_{12}A_1^2) = 0,$$

which is the first problem we study in the following chapter.

Perpendicular grain boundary

We present now the case where the angle between the two sets of stripes is exactly $\pi/2$, we take the setup by considering $\theta_1 = 0, \theta_2 = \pi/2$. The coupled system of amplitude equations, under a simple change of variables, becomes [10]

$$\varepsilon A_1 + \xi_0^2\partial_x^2 A_1 - g_0(A_1^2 + GA_2^2)A_1 = 0,$$

$$\varepsilon A_2 - \frac{\xi_0^2}{4q_c^2}\partial_x^4 A_2 - g_0(A_2^2 + GA_1^2)A_2 = 0.$$

The previous system emerges direcly from (2.2), where both of the operators $\partial_{y_1}^2 = \cos^2\theta_2\partial_x^2$ and $\partial_{x_2} = \cos^2\theta_2\partial_x^2$, become zero.

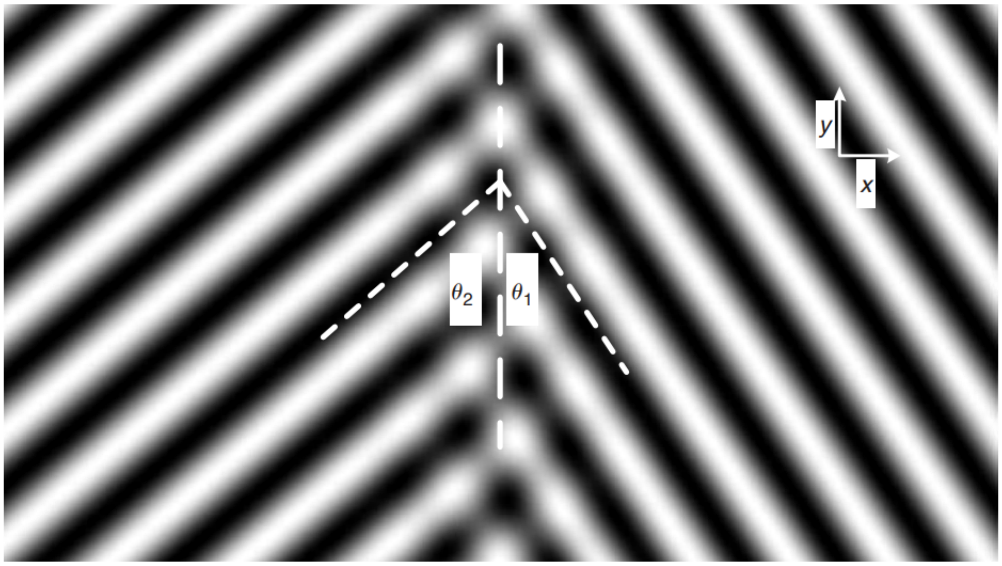

Figure 2.2: Geometry of a grain boundary. The stripes on either side make an angle θ_1 and θ_2 with the grain boundary. Figure 8.5 extracted from Cross's [7].

By means of a change of variables the previous system becomes

$$\partial_X^2 \bar{A}_1 + \bar{A}_1(1 - \bar{A}_1^{\,2} - G\bar{A}_2) = 0,$$

$$-\alpha\partial_X^4 \bar{A}_2 + \bar{A}_2(1 - \bar{A}_2^{\,2} + G\bar{A}_1^{\,2}) = 0,$$

where α is for an arbitrary small parameter. We also add the following boundary conditions to set this problem as a transition from one set of stripes to another [10]

$$\bar{A}_1(-\infty) \to 0, \bar{A}_1(\infty) \to 1 \text{ and } \bar{A}_2(-\infty) \to 1, \bar{A}_2(\infty) \to 0,$$

this is the problem we study in chapter 4.

Chapter 3

Some useful propositions

In this chapter we write useful results from real analysis, measure theory and differential equations.

3.1 Real Analysis

Theorem 3.1 (Helly's Theorem) *Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on $\mathbb{R}$ with $0 \leq f_n(x) \leq 1$ for all x and all n. Then there exists a function f and a sequence n_k such that*

$$f(x) = \lim_{k\to\infty} f_{n_k}(x).$$

Theorem 3.2 (Fatou's Lemma) *Given a measure space $(\Omega, \mathcal{F}, \mu)$ and set $A \in \mathcal{F}$, let $\{f_n\}$ be a sequence of measurable non-negative functions $f_n : A \to [0, +\infty]$. Define a function $f : A \to [0, +\infty]$ by setting $f(x) = \liminf_{n\to\infty} f_n(x)$, for every $x \in A$.*

Then f is a measurable function and also

$$\int_A f d\mu \leq \liminf_{n\to\infty} \int_A f_n d\mu.$$

Theorem 3.3 (Dominated Convergence Theorem) *Let $\{f_n\}$ be a sequence of measurable functions, that $f_n \to f$ point-wise almost everywhere as $n \to \infty$, and that $|f_n| \leq g$ for all n, where g is integrable. Then f is integrable and*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

3.2 Functional Analysis

Theorem 3.4 (Banach-Alaoglu Theorem) *If V is a neighborhood of 0 in a normed vector space X and if*

$$K = \{\Lambda \in X^* : |\Lambda x| \leq 1 \text{ for every } x \in V\}$$

then K is weak-compact.*

Theorem 3.5 (Sobolev's Inequalities) *Let U be a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary. Assume $u \in W^{k,p}(U)$.*

(i) If

$$k < \frac{n}{p},$$

then $u \in L^q(U)$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

We have in addition the estimate

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)},$$

the constant C depending only on k, p, n and U.

(ii) If

$$k > \frac{n}{p},$$

then $u \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\bar{U})$, where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, \text{ if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number} < 1, \text{ if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

We have in addition the estimate

$$\|u\|_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\bar{U})} \leq C\|u\|_{W^{k,p}(U)},$$

the constant C depending only on k, p, n, γ and U.

Theorem 3.6 (Rellich-Kondrachov) *Assume U is a bounded open subset of $\mathbb{R}^n$ and ∂U is C^1. Then we have the following compact injections:*

$$\begin{aligned} &W^{1,p} \subset L^q(U) \quad \forall q \in [1, p^*), \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}, && \text{if } p < n \\ &W^{1,p} \subset L^q(U) \quad \forall q \in [p, +\infty), && \text{if } p = n \\ &W^{1,p} \subset C(\bar{U}), && \text{if } p > n. \end{aligned}$$

In particular, $W^{1,p}(U) \subset L^q(U)$ with compact injection for all p (and all n).

3.3 Calculus of Variations

Non linear problems can be extremely hard to work on, this is because of the variety of nonlinearities that may be present on the respective problem. Let us suppose we want to solve the following non-linear equation

$$A[u] = 0,$$

where A is a nonlinear operator defined over an unspecified set of functions.

In order to find methods for solving the previous equation we think of a functional J which derivative (in a way we haven not defined yet) equals to the operator A. Then we center our attention on

$$dJ[u] = 0,$$

which can be solved using minimization of J on the respective space we are working on. We have divided the problem of solving our non-linear equation into:

1. Find a functional J such that $dJ = A$.
2. Study minimizers of J.

Specifically here we will write $dJ(u)[h]$ when we refer to the following limit:

$$\lim_{\varepsilon \to 0^+} \frac{J(u + \varepsilon h) - J(u)}{\varepsilon}.$$

A necessary condition to u be a minimum is that the previous limit is nonegative for small enough ε, this implies

$$dJ(u) = 0.$$

A much more easy way to compute $dJ(u)[h]$ is by computing

$$\frac{d}{d\varepsilon} J(u + \varepsilon h)\Big|_{\varepsilon=0},$$

and so the problem we want to solve becomes

$$\frac{d}{d\varepsilon} J(u + \varepsilon h)\Big|_{\varepsilon=0} = 0,$$

which we call the Euler-Lagrange equation associated to the functional J.

3.3.1 Direct Method in the Calculus of Variations

The direct method in the Calculus of Variations is a general method that is used to prove the existence of a minimizer for an energy functional defined over what is usually a reflexive space. It is based on the following.

Let X be a reflexive Banach space and be $J : X \longrightarrow \mathbb{R}$ an energy functional, lets consider a minimizing sequence $(u_n)_n$ over some subset of X. If J is a coercive functional ($J(u) \to \infty$ when $\|u\| \to \infty$) then the set $\{u_n\}_n$ must be a bounded set. Using that X is reflexive then this set is also compact in the weak topology. Given all this, if the functional J is at least weakly lower semi-continuous we can conclude the existence of a limit function $\bar{u}$ which is the minimum for J over X given that

$$\liminf_{n\to\infty} J(u_{n_k}) \geq J(\bar{u}),$$

where $(u_{n_k})_k$ is the sub-sequence of $(u_n)_n$ such that converges in the weak topology to $\bar{u}$, see [8] for more details.

Chapter 4

Study of the Non-Linear systems

As it was presented in the second chapter, the model for the general case of transition from one set of stripes to another over a grain boundary is given by the following system of differential equations for the amplitudes

$$\begin{cases} u'' + u(1 - u^2 - \mu v^2) & = 0 \\ \delta^2 v'' + v(1 - v^2 - \mu u^2) & = 0. \end{cases} \tag{4.1}$$

We are interested on solutions such that $u(-\infty) = v(\infty) = 0$ and $v(-\infty) = u(\infty) = 1$ [10].

4.1 The $\delta = 0$ problem

First we focus our attention on studying the system assuming $\delta = 0$, the system (4.1) becomes

$$\begin{cases} u_\varepsilon'' + u_\varepsilon(1 - u_\varepsilon^2 - \mu v_\varepsilon^2) = 0 \\ v_\varepsilon(1 - v_\varepsilon^2 - \mu u_\varepsilon^2) = 0. \end{cases} \tag{4.2}$$

We know that $\mu < 1$ may create squares in the pattern [7] and we are not interested in this case, so we just consider the case $\mu > 1$ given that we are trying to model the transition from one stripped pattern to another.
By solving the first equation of the system (4.2) for $v_\varepsilon = 0$ we can obtain $u_\varepsilon = \tanh((x - x_0)/\sqrt{2})$ where at this moment x_0 is a free constant. On the other hand if we solve now by assuming $v_\varepsilon \neq 0$ and using the second equation of the system (4.2) we get $u_\varepsilon = \sqrt{\dfrac{2}{\mu - 1}}\operatorname{sech}(\sqrt{\mu - 1}(x - x_1))$, and $v_\varepsilon = \sqrt{1 - \dfrac{2\mu}{\mu - 1}\operatorname{sech}^2(\sqrt{\mu - 1}(x - x_1))}$ where x_1 is a free constant just as x_0.
Based on what has been discussed we find the following solution for the $\delta = 0$ problem

$$u_\varepsilon(x) = \begin{cases} \sqrt{\frac{2}{\mu - 1}}\operatorname{sech}(\sqrt{\mu - 1}(x - x_1)) \text{ when } x \leq z, \\ \tanh\left(\frac{x - x_0}{\sqrt{2}}\right) \text{ when } x \geq z, \end{cases} \tag{4.3}$$

and

$$v_\varepsilon(x) = \begin{cases} \sqrt{1 - \frac{2\mu}{\mu-1}\operatorname{sech}^2(\sqrt{\mu-1}(x-x_1))} \text{ when } x \le z, \\ 0 \text{ when } x \ge z, \end{cases} \tag{4.4}$$

where z is free, which makes sense given that we are working on an autonomous system and the boundaries at infinity.
Now to make (4.3) and (4.4) continuous functions the constants x_0 and x_1 should be determined. The values that allow us to create these continuous functions always exists, this is easy to check if we notice that all we want is

$$u_\varepsilon^+(z) = u_\varepsilon^-(z)$$

$$\tanh\left(\frac{z-x_0}{\sqrt{2}}\right) = \sqrt{\frac{2}{\mu-1}}\operatorname{sech}(\sqrt{\mu-1}(z-x_1)),$$

where given that $\mu > 1$ we can obtain x_0 as a function of x_1. Now to get x_1 we consider the problem

$$v_\varepsilon^+(z) = v_\varepsilon^-(z),$$

which implies

$$\sqrt{\frac{2}{\mu-1}}\operatorname{sech}(\sqrt{\mu-1}(z-x_1)) = \frac{1}{\sqrt{\mu}}.$$

If we consider the preimage set generated by $\operatorname{sech}^{-1}(\cdot)$ we can write

$$x_1 = z - \frac{1}{\sqrt{\mu-1}}\operatorname{sech}^{-1}\left(\sqrt{\frac{\mu-1}{2\mu}}\right),$$

this gives us at least one solution for x_1, this can be observed in the graph of the function sech.

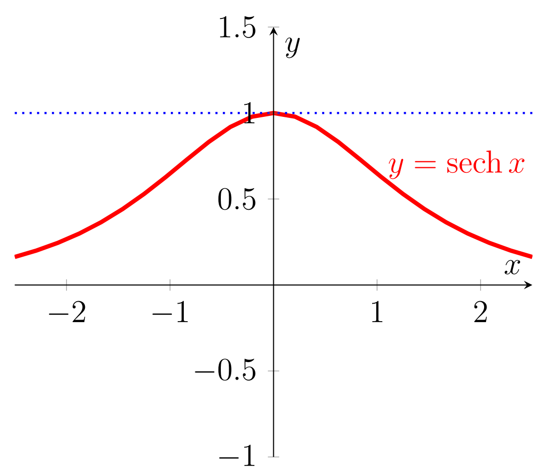

Figure 4.1: Hyperbolic Secant graph.

The existence of at least one x_1 is given by the following argument. We know the function sech takes all the values between 0 and 1 so the problem of nonexistence of x_1 shows up only if

$$\sqrt{\frac{\mu - 1}{2\mu}} > 1,$$

it is easy to check that

$$0 < \sqrt{\frac{\mu - 1}{2\mu}} \leq 1,$$

when we assume $\mu > 1$, so the problem of existence of x_1 for the continuous matching can always be solved.
Regularity of v_ε may be a useful property to determine for further results on the study of solutions for future problems and their regularity, for now can say for sure that v_ε is not a globally Lipchitz function given that

$$(v_\varepsilon^-)' = \frac{d}{dx}(1 - \mu u_\varepsilon^2)^{1/2} = \frac{1}{2}(1 - \mu u_\varepsilon^2)^{-1/2} \cdot (-2\mu u_\varepsilon u_\varepsilon'),$$

and as we know $u_\varepsilon(z) = \frac{1}{\sqrt{\mu}}$, this implies that $\lim_{x \to z^-}(v_0^-)' = -\infty$. We conclude that the function v_0 is not a Lipchitz function in $\mathbb{R}$, but it is in any interval $(-\infty, z-a) \cup (z+a, +\infty)$ for any $a > 0$.

4.1.1 v_ε regularity

As we just seen, the function v_ε is not a Lipchitz function, as the derivative grows without bound as we tend x to z. However, we can show that v_ε is a Hölder function. Let us first consider the following results

Proposition 4.1 *Let be $f : \mathbb{R} \longrightarrow \mathbb{R}$ a function defined by means of piecewise differentiable Lipchitz functions. If $|f'| \leq L$ almost everywhere for some $L > 0$, then f is also a Lipchitz function.*

Proof. If we take any two points $x, y \in \mathbb{R}$ where $x < y$, we can find $x_1 < x_2 < ... < x_n$ the points between x and y that define the functions over the corresponding intervals $[x_1, x_2], [x_2, x_3], ..., [x_{n-1}, x_n]$. Using the fundamental theorem of calculus we have

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) - f(x_1) + f(x_1) - f(x_2) + ... + f(x_n) - f(y)| \\
&\leq |f(x) - f(x_1)| + |f(x_1) - f(x_2)| + ... + |f(x_n) - f(y)| \\
&= \left|\int_x^{x_1} f'(s)ds\right| + \left|\int_{x_1}^{x_2} f'(s)ds\right| + ... + \left|\int_{x_n}^{y} f'(s)ds\right| \\
&\leq \int_x^{x_1} |f'(s)|ds + \int_{x_1}^{x_2} |f'(s)|ds + ... + \int_{x_n}^{y} |f'(s)|ds \\
&\leq L(x_1 - x) + L(x_1 - x_2) + ... + L(y - x_n) \\
&= L(y - x).
\end{aligned}$$

Then $|f(y) - f(x)| \leq L(y - x)$ when $y > x$ and so $|f(y) - f(x)| \leq L|y - x|$ for any $x, y \in \mathbb{R}$. $\square$

Lemma 4.2 *The function u_ε^2 is a Lipchitz function.*

Proof. We have that $(u_\varepsilon^2)' = 2u_\varepsilon u_\varepsilon'$ and given that u_ε is bounded and u_ε' is bounded almost everywhere, all the hypothesis of Proposition 4.1 are fulfilled to conclude that u_ε^2 is a Lipchitz function. □

Theorem 4.3 *The function v_ε defined by (4.4) is a $\frac{1}{2}$-Hölder function.*

Proof. **Considering** $x, y \le z$

$$\begin{aligned}
|v_\varepsilon(x) - v_\varepsilon(y)| &= |\sqrt{1-\mu u_\varepsilon^2(x)} - \sqrt{1-\mu u_\varepsilon^2(y)}| \\
&\le |1-\mu u_\varepsilon^2(x) - 1 + \mu u_\varepsilon^2(y)|^{1/2} \\
&\le \mu^{1/2}|u_\varepsilon^2(x) - u_\varepsilon^2(y)|^{1/2} \\
&C \cdot |x-y|^{1/2},
\end{aligned}$$

where in the first inequality we used the known fact that the square root is a $\frac{1}{2}$-Hölder function (with constant 1), and in the last one we use that u_ε^2 is a Lipchitz function.
Now the case $x \le z \le y$

$$\begin{aligned}
|v_\varepsilon(x) - v_\varepsilon(y)| &= |v_\varepsilon(x)| \\
&= \sqrt{1-\mu u_\varepsilon^2(x)} \\
&= \frac{\sqrt{1-\mu u_\varepsilon^2(x)}}{|(1-\mu u_\varepsilon^2(x)) - (1-\mu u_\varepsilon^2(y))|^\alpha} \cdot |1-\mu u_\varepsilon^2(x) - 1 + \mu u_\varepsilon^2(y)|^\alpha \\
&\le \frac{\sqrt{1-\mu u_\varepsilon^2(x)}}{|(1-\mu u_\varepsilon^2(x))|^\alpha} \cdot |\mu u_\varepsilon^2(x) - \mu u_\varepsilon^2(y)|^\alpha,
\end{aligned}$$

where in the inequality we use the fact that $1 - \mu u_\varepsilon^2(y) \le 0$ when $y \ge z$, now taking $\alpha = 1/2$ and using the fact that u_ε^2 is a Lipchitz function (for the previous lemma) we can conclude that there exists a constant $C > 0$ such that

$$|v_\varepsilon(x) - v_\varepsilon(y)| \le C|x-y|^{1/2}$$

and so v_ε is a $\frac{1}{2}$-Hölder function. □

4.2 The u_0 problem

Based on ideas presented in [14] we formulate the problem

$$u'' + u(1 - u^2 - \mu v_\varepsilon^2) = 0, \tag{4.5}$$

and we will call u_0 to the solution of this equation, where

$$v_\varepsilon = \begin{cases} \sqrt{1 - \frac{2\mu}{\mu-1}\operatorname{sech}^2(\sqrt{\mu-1}(x-x_1))} & \text{for } x \le z, \\ 0 & \text{for } x \ge z. \end{cases}$$

Lets consider the following energy functional

$$E_z(u) = \int_{-\infty}^{z} \frac{(u')^2}{2} + \frac{u^2}{2} \underbrace{\left(\mu - 1 + \frac{2\mu^2}{\mu - 1}\operatorname{sech}^2(\sqrt{\mu - 1}(x - x_1))\right)}_{K(x) = -(1 - \mu v_\varepsilon^2)} + \frac{u^4}{4} \tag{4.6}$$

$$+ \int_{z}^{+\infty} \frac{(u')^2}{2} + \frac{1}{4}(1 - u^2)^2. \tag{4.7}$$

Proposition 4.4 *The Euler-Lagrange equation associated with the functional given by* (4.6)-(4.7) *is* (4.5)*, we also have* $E_z \geq 0$.

Proof. Lets compute the Euler-Lagrange equation for the functional, to do this we define E_{z^-} and E_{z^+} as

$$E_{z^-}(u) = \int_{-\infty}^{z} \frac{(u')^2}{2} + \frac{u^2}{2} \left(\mu - 1 + \frac{2\mu^2}{\mu - 1}\operatorname{sech}^2(\sqrt{\mu - 1}(x - x_1))\right) + \frac{u^4}{4},$$

$$E_{z^+}(u) = \int_{z}^{+\infty} \frac{(u')^2}{2} + \frac{1}{4}(1 - u^2)^2,$$

where clearly $E = E_{z^-} + E_{z^+}$, then we have

$$\frac{d}{d\varepsilon}(E_z(u + \phi\varepsilon))\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}(E_{z^-}(u + \phi\varepsilon))\Big|_{\varepsilon=0} + \frac{d}{d\varepsilon}(E_{z^+}(u + \phi\varepsilon))\Big|_{\varepsilon=0},$$

by considering $K(x) = -(1 - \mu v_\varepsilon^2)$, on one hand we have

$$\begin{aligned}
\frac{d}{d\varepsilon}(E_{z^-}(u + \phi\varepsilon))\Big|_{\varepsilon=0} =& \frac{d}{d\varepsilon}\left(\int_{-\infty}^{z} \frac{(u' + \phi'\varepsilon)^2}{2} + \frac{(u + \phi\varepsilon)^2}{2}K(x) + \frac{(u + \phi\varepsilon)^4}{4}\right)\Big|_{\varepsilon=0} \\
=& \int_{-\infty}^{z} (u' + \phi'\varepsilon)\phi' + (u + \phi\varepsilon)\phi K(x) + (u + \phi\varepsilon)^3\phi\Big|_{\varepsilon=0} \\
=& \int_{-\infty}^{z} u'\phi' + u\phi K(x) + u^3\phi \\
=& u'(z)\phi(z) + \int_{-\infty}^{z} (-u'' + u(K(x) + u^2))\phi \\
=& u'(z)\phi(z) + \int_{-\infty}^{z} (-u'' - u(1 - u^2 - \mu v_\varepsilon^2))\phi.
\end{aligned}$$

On the other hand

$$\frac{d}{d\varepsilon}(E_{z^+}(u + \phi\varepsilon))\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\left(\int_{z}^{+\infty} \frac{(u' + \phi'\varepsilon)^2}{2} + \frac{1}{4}(1 - (u + \phi\varepsilon)^2)^2\right)\Big|_{\varepsilon=0},$$

and so

$$
\begin{aligned}
&= \int_z^{+\infty} (u' + \phi'\varepsilon)\phi' + \frac{1}{2}(1 - (u + \phi\varepsilon)^2) \cdot -2 \cdot (u + \phi\varepsilon)\phi\Big|_{\varepsilon=0} \\
&= \int_z^{+\infty} u'\phi' - (1 - u^2)u\phi \\
&= - u'(z)\phi(z) + \int_z^{+\infty} -u''\phi - (1 - u^2)u\phi \\
&= - u'(z)\phi(z) + \int_z^{+\infty} (-u'' - u(1 - u^2))\phi \\
&= - u'(z)\phi(z) + \int_z^{+\infty} (-u'' - u(1 - u^2 - \mu v_\varepsilon^2))\phi,
\end{aligned}
$$

the last equality is true given that $v_\varepsilon(x) = 0$ for all $x > z$. We proved

$$
\frac{d}{d\varepsilon}(E(u + \phi\varepsilon))\Big|_{\varepsilon=0} = \int_{-\infty}^{+\infty} (-u'' - u(1 - u^2 - \mu v_\varepsilon^2))\phi,
$$

and so the Euler-Lagrange equation associated with the energy functional $E(\cdot)$ is

$$
u'' + u(1 - u^2 - \mu v_\varepsilon^2) = 0.
$$

Since

$$
K(x) = \mu - 1 + \frac{2\mu^2}{\mu - 1} \operatorname{sech}^2(\sqrt{\mu - 1}(x - x_1))
$$

where $\mu > 1$ it is easy to check that E_{z^-} and E_{z^+} are nonegative functionals. $\square$

It is important to remark that the function v_ε given by (4.4) depends on the choice of z because x_1 depends on it too. We have the following proposition that relates minimums of E_z for different values of z.

Proposition 4.5 *Lets consider E_{z_1} and E_{z_2} defined by* (4.6)-(4.7)*, if u_{z_1} is a minimizer for E_{z_1}, then there exists u_{z_2} minimizer for E_{z_2} and $E_{z_1}(u_{z_1}) = E_{z_2}(u_{z_2})$, fuerthermore, if we define $u_{z_2}(x) := u_{z_1}(x + (z_1 - z_2))$ we get a minimizer for E_{z_2}.*

Proof. First lets make the following observation on the function K that appears in the definition of E_z

$$
K(x) = \mu - 1 + \frac{2\mu^2}{\mu - 1} \operatorname{sech}^2(\sqrt{\mu - 1}(x - x_1)),
$$

where $(x - x_1) = (x - z + C)$, and C is a constant. The function K depends on $(x - z)$ and so it can be thought as the displacement of $\tilde{K}$ which is defined by

$$
\tilde{K}(x - z) = K(x),
$$

we then have that

$$
E_{z_1}(u) = \int_{-\infty}^{z_1} \frac{(u')^2}{2} + \tilde{K}(x - z_1)\frac{u^2}{2} + \frac{u^4}{4} + \int_{z_1}^{+\infty} \frac{(u')^2}{2} + \frac{1}{4}(1 - u^2)^2,
$$

by applying the change of variables $y = x + (z_1 - z_2)$ and considering $\tilde{u}(y) = u(y + (z_1 - z_2))$ we get

$$E_{z_1}(u) = \int_{-\infty}^{z_2} \frac{(\tilde{u}')^2}{2} + \tilde{K}(x - z_2)\frac{\tilde{u}^2}{2} + \frac{\tilde{u}^4}{4} + \int_{z_2}^{+\infty} \frac{(\tilde{u}')^2}{2} + \frac{1}{4}(1 - \tilde{u}^2)^2 = E_{z_2}(\tilde{u}),$$

and so $E_{z_1}(u(\cdot)) = E_{z_2}(u(\cdot + (z_1 - z_2)))$. Having the previous it is easy to see that if $u_1(\cdot)$ is a minimizer of E_{z_1} then immediately $u_2(\cdot) := u_1(\cdot + (z_1 - z_2))$ is the minimum for E_{z_2}.

If we suppose that u_2 is not minimizer for E_{z_2}, that is, there exists $\bar{u}_2$ such that $E_{z_2}(\bar{u}_2) < E_{z_2}(u_2)$. Then defining $\bar{u}_1(x) := \bar{u}_2(x - (z_1 - z_2))$ we get

$$E_{z_1}(\bar{u_1}) = E_{z_2}(\bar{u}_2) \implies E_{z_1}(\bar{u}_1) = E_{z_2}(\bar{u}_2) < E_{z_2}(u_2) = E_{z_1}(u_1),$$

and so $E_{z_1}(\bar{u}_1) < E_{z_1}(u_1)$ which is a contradiction given that we are assuming u_{z_1} is a global minimum. □

Proposition 4.6 *If u_{z_1} minimizes E_{z_1} and u_{z_2} build as in the previous proposition such that minimizes E_{z_2}, we have that $u_{z_1}(z_1) = u_{z_2}(z_2)$, this means that $u_z(z)$ is independent of the choice of z.*

Proof. Based on the previous result we have $E_{z_1}(u_{z_1}) = E_{z_2}(u_{z_2})$ and $u_{z_2}(x) = u_{z_1}(x + (z_1 - z_2))$, evaluating on $x = z_2$ we obtain $u_{z_2}(z_2) = u_{z_1}(z_1)$, and so the value is independent of the choice of z. □

4.2.1 Existence of u_0

Let us consider the following set of functions

$$X = \{u : \mathbb{R} \longrightarrow \mathbb{R} | u' \in L^2(\mathbb{R}), u\mathbb{1}_{(-\infty,z]} \in L^2(\mathbb{R}), \mathbb{1}_{[z,+\infty)}(1 - u^2) \in L^2(\mathbb{R})\}$$

We have the following

Proposition 4.7 *If $u \in X$ then u is also a $C^{0,1/2}(\mathbb{R})$ function.*

Proof. Let us consider $\varepsilon > 0$ and $x, y \in (-\varepsilon, \varepsilon)$ we have

$$|u(y) - u(x)| \leq |y - x|^{1/2} \left(\int_x^y u'^2(s)ds \right)^{1/2} \leq |y - x|^{1/2} \|u'\|_{L^2(\mathbb{R})},$$

and so

$$\frac{|u(y) - u(x)|}{|y - x|^{1/2}} \leq \|u'\|_{L^2(\mathbb{R})},$$

this implies $u \in C^{0,\alpha}(-\varepsilon, \varepsilon)$ for any $0 < \alpha \leq 1/2$.
Using that ε is arbitrary, the previous bound can be extended for any $x, y \in \mathbb{R}$ to conclude that $u \in C^{0,1/2}(\mathbb{R})$. □

Lemma 4.8 *Let u be a function in X, we have that there exists another function w in X such that $0 \leq w \leq 1$, is an increasing function, and $E_z(w) \leq E_z(u)$.*

PROOF. Let's define the function w_1 as follows for $x > z$

$$w_1(x) = \begin{cases} |u(x)| & \text{if } |u(x)| \leq 1, \\ 1 & \text{if } |u(x)| > 1, \end{cases}$$

we have $(1 - w_1^2)\mathbb{1}_{[z,\infty]} \in L^2$ given

$$\begin{aligned} \int_z^\infty (1 - w_1^2)^2 &= \int_{[z,\infty)\cap|u|>1} (1 - w_1^2)^2 + \int_{[z,\infty)\cap|u(x)|\leq 1} (1 - w_1^2)^2 \\ &\leq \int_{[z,\infty)\cap|u(x)|\leq 1} (1 - u^2)^2 \\ &\leq \int_z^\infty (1 - u^2)^2 < \infty, \end{aligned}$$

and so $(1 - w_1^2)\mathbb{1}_{[z,\infty)} \in L^2$.
Now for $x \leq z$ we define

$$w_1(x) = \begin{cases} 0 & \text{if } u(x) < 0, \\ u(x) & \text{if } 0 \leq u(x) \leq 1, \\ 1 & \text{if } u(x) > 1. \end{cases}$$

We easily check that $w_1\mathbb{1}_{(-\infty,z]} \in L^2$

$$\begin{aligned} \int_{-\infty}^z w_1^2 = \int_{(-\infty,z]\cap u(x)<0} w_1^2 + \int_{(-\infty,z]\cap 0\leq u\leq 1} w_1^2 + \int_{(-\infty,z]\cap u>1} w_1^2, \\ \leq \int_{-\infty}^z u^2 < \infty. \end{aligned}$$

We have then that $|w_1(x)| \leq |u(x)|$ for all x, and $|1 - w_1^2(x)| \leq |1 - u^2(x)|$ for all $x > z$.

We now define w_2 as follows,

$$w_2(x) := \begin{cases} \sup_{y\in[z,x]} w_1(y) & \text{if } x \geq z, \\ \inf_{y\in[x,z]} w_1(y) & \text{if } x \leq z. \end{cases}$$

Given that w_1 is continuous, the function w_2 is well defined. By the definition of infimum and supremum, we obtain that w_2 is a non-decreasing function and that $0 \leq u(x) \leq w_2(x) \leq 1$ for all $x \geq z$, and $0 \leq w_2(x) \leq u(x) \leq 1$ for all $x \leq z$, this implies that

$$\int_z^{+\infty} (1 - w_2^2)^2 \leq \int_z^{+\infty} (1 - u^2)^2, \text{ and } \int_{-\infty}^z w_2^2 \leq \int_{-\infty}^z u^2,$$

respectively.

Let's study the derivative now. Given that the function w_2 is a monotone increasing function, by Lebesgue's Theorem we know that the derivative of w_2 exists almost everywhere.

We consider the set $A := \{x \in \mathbb{R} | w_2(x) = u(x)\}$. Due to the continuity of u, we have that A^c is an open set. In this set, w_2 is locally constant, which implies that w_2 is differentiable on A^c and its derivative is the zero function. Therefore, the weak derivative on this set exists and coincides with zero almost everywhere.

On the other hand, w_2 coincides with u on A, we have

$$w_2'(x) = \begin{cases} 0 & \text{for } x \in A^c, \\ u'(x) & \text{for } x \in A, \end{cases}$$

from this, its easy to conclude that

$$\int (w_2')^2 \leq \int (u')^2 < \infty.$$

Thus, we have shown that w_2 belongs to X, takes values between 0 and 1, and satisfies $E(w_2) \leq E(u)$.

Clearly, the conditions are satisfied by $w = w_2$. □

Proposition 4.9 *Given $(u_n)_n$ a minimizing sequence on X without loosing any generality we can assume that is a sequence of monotone functions taking values between 0 and 1.*

Proof. Once having the sequence $(u_n)_n$ we can build an alternative sequence $(w_n)_n$ where each function w_n is a monotone function taking values between 0 and 1, such that $E_z(w_n) \leq E_z(u_n)$. The existence of each function w_n is obtained using the two previous lemmas. Thus, we have a minimizing sequence $(w_n)_n$ consisting of monotone functions and uniformly bounded between 0 and 1. □

Theorem 4.10 *There exists a function u_0 that minimizes E_z on X, this is a monotone function, $0 \leq u \leq 1$ and also $u \in C^{2,1/2}$.*

Proof. Let us consider a minimizing sequence $(u_n)_n$ of the functional E_z on X. We have proven that we can assume $(u_n)_n$ is a sequence of uniformly bounded and monotone functions. By applying Helly's Selection theorem we can find a subsequence $(u_n)_{n_k}$ that converges pointwise to a function $\bar{u}$. We will now show that this function $\bar{u}$ belongs to X. If this is the case, we can use Fatou's Lemma to establish that $E_z(\bar{u}) \leq E_z(u_{n_k})$, indicating that $\bar{u}$ is a minimum. For clarity in notation, let us assume that $(u_n)_n$ converges to $\bar{u}$. Now, we goal is to prove that $\bar{u}' \in L^2$. However, before proceeding, we need to establish the existence of this derivative. We must find a function v such that:

$$\int v\phi = -\int \bar{u}\phi' \text{ for each } \phi \in C_0^\infty .$$

Given that $(u_n)_n$ is a minimizing sequence, we can observe that $\|u_n'\|$ forms a uniformly bounded family. Consequently, there exists a weakly convergent subsequence, which we will still denote as $(u_n)_n$, converging to a limit function denoted as v. As a result,

$$\int u_n'\phi \longrightarrow \int v\phi \text{ for each } \phi \in L^2.$$

By using the definition of the distributional derivative, we get

$$-\int u_n\phi' \longrightarrow \int v\phi \text{ for each } \phi \in C_0^\infty,$$

finally, by using dominated convergence theorem we have

$$-\int u_n\phi' \longrightarrow -\int \bar{u}\phi' \text{ for each } \phi \in C_0^\infty,$$

this implies that $-\int \bar{u}\phi' = \int v\phi$ for each $\phi \in C_0^\infty$. Therefore, the derivative exists and we have $u_n' \to \bar{u}'$ in the sense of the distributions and in the weak sense (as the first of the previous limits shows).
Now we have this weak derivative exists, we just need to prove that this function is in L^2, for this, we use that the L^2 norm is weakly lower semi-continuous. We already shown that $u_n' \rightharpoonup \bar{u}'$ and therefore

$$\liminf_{n\to\infty}\left(\int (u_n')^2\right)^{1/2} = \liminf_{n\to\infty}\|u_n'\| \geq \|\bar{u}'\| = \left(\int (\bar{u}')^2\right)^{1/2},$$

which implies $\bar{u}' \in L^2$. By means of Fatou's Lemma (here we use the known fact that u_n converges pointwise to $\bar{u}$) we are able to bound term by term the energy functional E_z to conclude that $\bar{u} \in X$ and $E_z(\bar{u}) \leq \liminf E_z(u_n)$, this means $\bar{u}$ is a minimizer. From now on we will denote u_0 to this minimum and see that it is a $C^{2,1/2}$ function.
Given that u_0 is a minimum of E_z it satisfies the corresponding Euler-Lagrange equation, this means that

$$\int_{-\infty}^{\infty}(-u'' - u(1-u^2-\mu v_\varepsilon^2))\phi = 0 \text{ for each } \phi \in C_0^\infty,$$

where u'' is the second derivative in the sense of distributions, which implies

$$u'' = -u(1-u^2-\mu v_\varepsilon^2),$$

and given that $u \in X$ is a $C^{0,1/2}$ function, this implies $u'' \in C^{0,1/2}$, and so $u \in C^{2,1/2}$. $\square$

4.3 The v_0 problem

Having the solution obtained in the previous section, we write the following problem

$$\delta^2 v'' + v(1-v^2-\mu u_0^2) = 0, \tag{4.8}$$

and call v_0 the solution of this problem. It is not hard to notice that structure of this problem is not very different from the one we already studied for u_0 and, so it is possible to procede in a very similar way.

4.3.1 Existence of v_0

Let us consider the following set of functions defined taking s such that $u_0^2(x) \geq \frac{1}{\mu}$ for any $x \geq s$, we define

$$Y = \{v : \mathbb{R} \longrightarrow \mathbb{R} | v' \in L^2, \mathbb{1}_{[s,+\infty)} v \in L^2, \mathbb{1}_{(-\infty,s]}(1 - v^2) \in L^2\},$$

we have the following results

Proposition 4.11 *The equation* (4.8) *correspond to the Euler-Lagrange equation of the following Energy functional*

$$E(v) = \int_{-\infty}^{s} \delta^2 \frac{(v')^2}{2} + \frac{\mu u_0^2 v^2}{2} + \frac{(1 - v^2)^2}{4} + \int_{s}^{\infty} \delta^2 \frac{(v')^2}{2} + \frac{v^2}{2}(\mu u_0^2 - 1) + \frac{v^4}{4}.$$

Also, the functional is nonegative over Y.

Proof. We divide the functional in the following way

$$E_{s^-} = \int_{-\infty}^{s} \delta^2 \frac{(v')^2}{2} + \frac{\mu u_0^2 v^2}{2} + \frac{(1 - v^2)^2}{4},$$

$$E_{s^+} = \int_{s}^{\infty} \delta^2 \frac{(v')^2}{2} + \frac{v^2}{2}(\mu u_0^2 - 1) + \frac{v^4}{4},$$

clearly $E_z = E_{s^-} + E_{s^+}$, then we have

$$\frac{d}{d\varepsilon}(E(v + \phi\varepsilon))\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}(E_{s^-}(v + \phi\varepsilon))\Big|_{\varepsilon=0} + \frac{d}{d\varepsilon}(E_{s^+}(v + \phi\varepsilon))\Big|_{\varepsilon=0}.$$

Calculating separately

$$\frac{d}{d\varepsilon}(E_{s^-}(v + \phi\varepsilon))\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\left(\int_{-\infty}^{s} \frac{\delta^2 (v' + \phi'\varepsilon)^2}{2} + \mu u_0^2 \frac{(v + \phi\varepsilon)^2}{2} + \frac{(1 - (v + \phi\varepsilon)^2)^2}{4}\right)\Big|_{\delta=0},$$

and so

$$\begin{aligned}
&= \int_{-\infty}^{s} \delta^2 (v' + \phi'\varepsilon)\phi' + \mu u_0^2 (v + \phi\varepsilon) - (1 - (v + \phi\varepsilon)^2)(v + \phi\varepsilon)\phi\Big|_{\varepsilon=0} \\
&= \int_{-\infty}^{s} \delta^2 v'\phi' + \mu u_0^2 v\phi - v(1 - v^2)\phi \\
&= \int_{-\infty}^{s} \delta^2 v'\phi' - v(1 - v^2 - \mu u_0^2)\phi \\
&= v'(s)\phi(s) + \int_{-\infty}^{s} (-\delta^2 v'' - v(1 - v^2 - \mu u_0^2))\phi,
\end{aligned}$$

at the same time we have

$$\frac{d}{d\varepsilon}(E_{s^+}(v + \phi\varepsilon))\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\left(\int_{s}^{\infty} \delta^2 \frac{(v' + \phi'\varepsilon)^2}{2} + \frac{(v + \phi\varepsilon)^2}{2}(\mu u_0^2 - 1) + \frac{(v + \phi\varepsilon)^4}{4}\right)\Big|_{\varepsilon=0},$$

and so

$$= \left(\int_s^\infty \delta^2(v' + \phi'\varepsilon)\phi' + (v+\phi\delta)\phi(\mu u_0^2 - 1) + (v+\phi\delta)^3\phi \right) \Big|_{\varepsilon=0}$$
$$= \int_s^\infty \delta^2 v'\phi' + v(\mu u_0^2 - 1)\phi + v^3\phi$$
$$= -v'(s)\phi(s) + \int_s^\infty (-\delta^2 v'' - v(1 - v^2 - \mu u_0^2))\phi,$$

finally we get

$$\frac{d}{d\varepsilon}(E_z(v+\phi\varepsilon))\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}(E_{s^-}(v+\phi\varepsilon))\Big|_{\varepsilon=0} + \frac{d}{d\varepsilon}(E_{s^+}(v+\phi\varepsilon))\Big|_{\varepsilon=0}$$
$$= \int_{-\infty}^s (-\delta^2 v'' - v(1 - v^2 - \mu u_0^2))\phi + \int_s^\infty (-\delta^2 v'' - v(1 - v^2 - \mu u_0^2))\phi$$
$$= \int (-\delta^2 v'' - v(1 - v^2 - \mu u_0^2))\phi,$$

and then we have

$$\frac{d}{d\varepsilon}(E_z(v+\phi\varepsilon))\Big|_{\varepsilon=0} = 0$$

which implies

$$\delta^2 v'' + v(1 - v^2 - \mu u_0^2) = 0.$$

To see that this functional is nonegative it is sufficent to note that all terms on E_{s^+} are squares of real values and so takes only positive values, for E_{s^-} we know it is positive given that we chose s such that $\mu u_0^2(x) - 1 \geq 1$ for any $x \geq s$, hence, $E(v)$ is nonegative on Y. □

Proposition 4.12 *If $v \in Y$ then $v \in C^{0,1/2}$.*

Proof. Let us consider $\varepsilon > 0$ then we have

$$|v(y) - v(x)| \leq |y - x|^{1/2} \left(\int_x^y v'^2(s)ds \right)^{1/2} \leq |y - x|^{1/2} \|v'\|_{L^2},$$

and so

$$\frac{|v(y) - v(x)|}{|y - x|^{1/2}} \leq \|v'\|_{L^2},$$

using that ε is arbitrary we conclude $v \in C^{0,1/2}$. □

Lemma 4.13 *Given a minimizing sequence (v_n) on Y we can assume without loosing generality that this sequence is uniformly bounded $0 \leq v_n \leq 1$ and each v_n is a non-increasing function.*

Proof. We just take a function v_n of the sequence of Y, it is related to another function u_n in X given by $u_n(x) = v_n(-x + (s + z))$, we can use the Lemma proved for functions in X and build a function $\tilde{u}_n$ such that its in X, is nondecreasing, bounded between 0 and 1, and $E_z(\tilde{u}_n) \leq E_z(u_n)$, now we define the function that we will use $\tilde{v}_n(x) = \tilde{u}_n(-x+(s+z))$. Clearly $\tilde{v}_n \in Y$ and using that $E_z(\tilde{u}_n) \leq E_z(u_n)$ by a change of variables we obtain $E(\tilde{v_n}) \leq E(v_n)$, by doing this for each function of the sequence we build this alternative sequence $(\tilde{v}_n)_n$ with the desired properties. □

Theorem 4.14 *The problem* (4.8) *has a weak solution on* Y*, furthermore, it is a classical solution and it lies in* $C^{2,1/2}$.

PROOF. Let us consider a minimizing sequence $(v_n)_n$ of E on Y. Without losing generality we can assume this is a uniformly bounded sequence, $0 \leq v_n \leq 1$ of non-increasing functions, and so, by Helly's selection theorem there exists a subsequence which we will denote by $(v_n)_n$ that converges pointwise to a function $\bar{v}$. We will prove that this function lies in Y. Lets see first that the derivative is a function in L^2. We have that $\|v_n'\|$ is a uniformly bounded family and so there exists a weak convergent subsequence (which we call just $(v_n')_n$) and w to the limit. We then have

$$\int v_n'\phi \longrightarrow \int w\phi, \text{ for each } \phi \in L^2,$$

this implies

$$-\int v_n\phi' \longrightarrow \int w\phi \text{ for each } \phi \in C_0^\infty,$$

finally by dominated convergence theorem we have

$$-\int v_n\phi' \longrightarrow -\int \bar{v}\phi' \text{ for each } \phi \in C_0^\infty,$$

and so $-\int \bar{v}\phi' = \int w\phi$ for each $\phi \in C_0^\infty$, that means the derivative of $\bar{v}$ exists (is the previous function w) and we have $v_n' \to \bar{v}'$ in the sense of distributions and in the weak sense.

Now that we have established the existence of the weak derivative, which is a function (obtained as the poinwise limit of the previous sequence, provided by the Banach-Alaoglu theorem), our next goal is to prove that this function belongs to L^2. To acomplish this, we utilize the fact that the L^2 norm is weakly lower semi-continuous. By applying this property, we can conclude that $v_n' \rightharpoonup \bar{v}'$, and therefore

$$\liminf_{n\to\infty}\left(\int (v_n')^2\right)^{1/2} = \liminf_{n\to\infty}\|v_n'\| \geq \|\bar{v}'\| = \left(\int (\bar{v}')^2\right)^{1/2},$$

and so $\bar{v}' \in L^2$.

By Helly's selection theorem we have (v_n) converges point-wise to $\bar{v}$, this allow us to use Fatou's Lemma to bound all terms (term by term) on $E(v_n)$ so we get $\bar{v} \in S$ and

$$E(\bar{v}) \leq \liminf E(v_n),$$

this means $\bar{v}$ is a minimizer and is contained in S.

Given that $\bar{v}$ is a minimum it satisfies the corresponding Euler-Lagrange equation and so

$$\int_\infty^\infty (-\delta^2 v'' - v(1 - v^2 - \mu u_0^2))\phi = 0, \text{ for each } \phi \in C_0^\infty,$$

where v'' is the second derivative in the sense of distributions, which implies

$$\delta^2 v'' = -v(1 - v^2 - \mu u_0^2),$$

and so v'' can be identified as a function, and given $v \in S$ we obtain $v'' \in C^{0,1/2}$, so we conclude $v \in C^{2,1/2}$. □

4.3.2 Perpendicular Case

As discussed in the second chapter, the problem we are studying has a limit case that requires us to consider a different system of differential equations. In this limit case, a fourth-order derivative term becomes significant. As a result, the equation that describes this limit case can be written as follows:

$$\begin{cases} \quad\quad u'' + u(1-u^2-\mu v^2) = 0 \\ -\varepsilon^2 \quad v^{(\mathrm{iv})} + v(1-v^2-\mu u^2) = 0. \end{cases}$$

We consider first the approximation by taking $\varepsilon = 0$ and notice that the system obtained is

$$\begin{cases} u'' + u(1-u^2-\mu v^2) & = 0 \\ v(1-v^2-\mu u^2) & = 0, \end{cases}$$

we already solved this problem in the first part of the chapter, and obtained solutions $u_\varepsilon, v_\varepsilon$. By replacing v_ε in the right hand side of the equation for u we obtain

$$u'' + u(1-u^2-\mu v_\varepsilon^2) = 0,$$

problem that has a solution u_0 on X as we proved. Now we present the fourth order equation for v considering ε not zero and the right hand side considering $u = u_0$, that is

$$-\varepsilon^2 v^{\mathrm{iv}} + v(1-v^2-\mu u_0^2) = 0. \tag{4.9}$$

This problem is quite challenging since we cannot employ the same techniques we used for the second order problem. The reason is that when working with the energy functional, we have to deal with second-order derivatives. Even if the functions on Y are continuous (which allows us to use that the derivative is a function), it is crucial to note that we cannot assume the second derivative to be a function. As a result, our techniques used to prove Lemmas 4.11 and 4.12 cannot be applied. To face this new problem we consider (4.9) on a bounded interval.

Theorem 4.15 *Let Y_R be*

$$Y_R = \{v \in H^2(-R,R) | v(-R) = 1, v(R) = 0, v'(-R) = v'(R) = 0\},$$

we have that the functional J_R defined by

$$J_R(v) = \underbrace{\int_{-R}^{s} \varepsilon^2 \frac{(v'')^2}{2} + \frac{\mu u_0^2 v^2}{2} + \frac{(1-v^2)^2}{4}}_{J_R^{s-}} + \underbrace{\int_{s}^{R} \varepsilon^2 \frac{(v'')^2}{2} + \frac{v^2}{2}(\mu u_0^2 - 1) + \frac{v^4}{4}}_{J_R^{s+}}$$

over Y_R has a minimum v_0^R, and this minimum is a $C^{4,\alpha}$ function for $0 < \alpha \leq 1/2$ and classical solution of (4.9) on $(-R,R)$.

Proof. Lets see first the connection between this functional and (4.9)

$$J_R^{s-} = \int_{-R}^{s} \varepsilon^2 \frac{(v'')^2}{2} + \frac{\mu u_0^2 v^2}{2} + \frac{(1-v^2)^2}{4},$$

$$J_R^{s^+} = \int_s^R \varepsilon^2 \frac{(v'')^2}{2} + \frac{v^2}{2}(\mu u_0^2 - 1) + \frac{v^4}{4},$$

then we have

$$\frac{d}{d\delta}(J_R(v+\phi\delta))\Big|_{\delta=0} = \frac{d}{d\delta}(J_R^{s^-}(v+\phi\delta))\Big|_{\delta=0} + \frac{d}{d\delta}(J_R^{s^+}(v+\phi\delta))\Big|_{\delta=0}.$$

Calculating separately

$$\frac{d}{d\delta}(J_R^{s^-}(v+\phi\delta))\Big|_{\delta=0} = \frac{d}{d\delta}\left(\int_{-R}^s \frac{\varepsilon^2(v''+\phi''\delta)^2}{2} + \mu u_0^2\frac{(v+\phi\delta)^2}{2} + \frac{(1-(v+\phi\delta)^2)^2}{4}\right)\Big|_{\delta=0},$$

and so

$$\begin{aligned}
&= \int_{-R}^s \varepsilon^2(v''+\phi''\delta)\phi'' + \mu u_0^2(v+\phi\delta) - (1-(v+\phi\delta)^2)(v+\phi\varepsilon)\phi\Big|_{\delta=0} \\
&= \int_{-R}^s \varepsilon^2 v''\phi'' + \mu u_0^2 v\phi - v(1-v^2)\phi \\
&= \int_{-R}^s \varepsilon^2 v''\phi'' - v(1-v^2-\mu u_0^2)\phi \\
&= v''(s)\phi'(s) - v'''(s)\phi(s) + \int_{-R}^s (\varepsilon^2 v^{\text{iv}} - v(1-v^2-\mu u_0^2))\phi,
\end{aligned}$$

at the same time we have

$$\frac{d}{d\delta}(J_R^{s^+}(v+\phi\delta))\Big|_{\delta=0} = \frac{d}{d\delta}\left(\int_s^R \varepsilon^2\frac{(v''+\phi''\delta)^2}{2} + \frac{(v+\phi\delta)^2}{2}(\mu u_0^2-1) + \frac{(v+\phi\delta)^4}{4}\right)\Big|_{\delta=0},$$

and so

$$\begin{aligned}
&= \left(\int_s^R \varepsilon^2(v''+\phi''\delta)\phi'' + (v+\phi\delta)\phi(\mu u_0^2-1) + (v+\phi\delta)^3\phi\right)\Big|_{\delta=0} \\
&= \int_s^R \varepsilon^2 v''\phi'' + v(\mu u_0^2-1)\phi + v^3\phi \\
&= -v''(s)\phi'(s) + v'''(s)\phi(s) + \int_s^R (\varepsilon^2 v^{\text{iv}} - v(1-v^2-\mu u_0^2))\phi,
\end{aligned}$$

finally we get

$$\begin{aligned}
\frac{d}{d\delta}(J_R(v+\phi\delta))\Big|_{\delta=0} &= \frac{d}{d\delta}(J_R^{s^-}(v+\phi\delta))\Big|_{\delta=0} + \frac{d}{d\delta}(J_R^{s+}(v+\phi\delta))\Big|_{\delta=0} \\
&= \int_{-R}^s (\varepsilon^2 v^{\text{iv}} - v(1-v^2-\mu u_0^2))\phi + \int_s^R (\varepsilon^2 v^{\text{iv}} - v(1-v^2-\mu u_0^2))\phi \\
&= \int_{-R}^R (\varepsilon^2 v^{\text{iv}} - v(1-v^2-\mu u_0^2))\phi,
\end{aligned}$$

and then we have

$$\frac{d}{d\delta}(J_R(v+\phi\delta))\Big|_{\delta=0} = 0$$

which implies that

$$-\varepsilon^2 v^{\text{iv}} + v(1 - v^2 - \mu u_0^2) = 0$$

is the Euler-Lagrange equation associated.

To see that this functional is nonegative it is sufficent to note that all terms on $J_R^{s^+}$ are squares of real values and so takes only nonegatives values, for $J_R^{s^-}$ we know it is positive given that we chose s such that $\mu u_0^2(x) - 1 \geq 1$ for any $x \geq s$, hence, $J_R(v)$ is nonegative on Y_R.

The rest of the proof follows the direct method described in Chapter 3, it is easy to check that the functional J_R can be written as

$$J_R(v) = \frac{\varepsilon^2}{2}\|v''\|_{L^2} + \int_{-R}^{s} \frac{\mu u_0^2 v^2}{2} + \frac{(1-v^2)^2}{4} + \int_{s}^{R} \frac{v^2}{2}(\mu u_0^2 - 1) + \frac{v^4}{4}.$$

We see that Y is a closed set over a $H^2(-R,R)$, this is useful to use the method described on Chapter 3, we only need to prove that J_R is coercive and weakly lower-semicontinuous:

Coercivity: It is easy to check that when $\|v''\|$ goes to infinity then so does $J_R(v)$. We have that

$$\begin{aligned} v(x) &= \int_{-R}^{x} v'(s)ds + 1, \\ &\leq \|v'\|_{L^2}(x+R)^{1/2} + 1, \\ &\leq \|v'\|_{L^2}(2R)^{1/2} + 1, \end{aligned}$$

and so

$$\|v\|_{L^2}^2 \leq (2R)\left(\|v'\|_{L^2}(2R)^{1/2} + 1\right)^2,$$

directly we have that $\|v\|_{L^2} \to \infty$ implies $\|v'\|_{L^2} \to \infty$.
At the same time,

$$\begin{aligned} v'(x) &= \int_{-R}^{x} v''(s)ds, \\ &\leq (2R)^{1/2}\|v''\|_{L^2}, \end{aligned}$$

and so

$$\|v'\|_{L^2}^2 \leq (2R)^2\|v''\|_{L^2}^2,$$

we conclude $\|v\|_{H^2} \to \infty \implies \|v''\| \to \infty \implies J_R(v) \to \infty$.

Wlsc: Let's divide the functional as follows

$$J_R(v) = \underbrace{\frac{\varepsilon^2}{2}\|v''\|_{L^2}^2}_{I} + \underbrace{\int_{-R}^{s} \frac{\mu u_0^2 v^2}{2}}_{II} + \underbrace{\frac{(1-v^2)^2}{4}}_{III} + \underbrace{\int_{s}^{R} \frac{v^2}{2}(\mu u_0^2 - 1)}_{IV} + \underbrace{\frac{v^4}{4}}_{V}$$

I Given that the L^2 norm is weakly lower-semicontinuous we have this term covered and so it is weakly lower-semicontinuous.

II Let $(v_n)_n$ be a sequence in Y that converges to v in the weak topology of H^2. Due to the compact embedding of H^2 into L^2 (a direct consequence of Rellich-Kondrachov's Theorem), there exists a subsequence, also denoted as v_n, that converges in the norm topology of L^2. By uniqueness of the limit, this new sequence also converges to v in L^2, we have

$$\int_{-R}^{s} \frac{\mu u_0^2}{2}(v_n^2 - v^2) \leq \int_{-R}^{s} \frac{\mu u_0^2}{2}|v_n^2 - v^2| \leq \text{Constant} \cdot \|v_n - v\|_{L^2},$$

where the last inequality is obtained using Holder's inequality. Thus, we have achieved weak lower semi-continuity for this term.

III Let $(v_n)_n$ be a sequence in Y that converges v in the weak topology of H^2. Due to the compact embedding of H^2 into L^2, there exists a subsequence of (v_n), also denoted as $(v_n)_n$, that converges in the L^2 norm to v, and so

$$\begin{aligned}\int_{-R}^{s} |(1-v_n^2)^2 - (1-v^2)^2| &\leq \int_{-R}^{s} |(1-v_n^2) + (1-v^2)| \cdot |(1-v_n^2) - (1-v^2)| \\ &\leq (\|1-v_n^2\| + \|1-v^2\|_{L^2}) \cdot \|v_n^2 - v^2\|_{L^2} \\ &\leq (\|1-v_n^2\| + \|1-v^2\|_{L^2}) \cdot \|v_n + v\| \cdot \|v_n - v\|_{L^2}\end{aligned}$$

which tends to zero due to the continuity of the norm.

IV Let $(v_n)_n$ a sequence in Y convergent to v in the H^2 weak topology. By the compact embedding of H^2 into L^2 there exists a subsequence also convergent to v in the L^2 norm (which we also call $(v_n)_n$), and so we have

$$\int_{s}^{R} \frac{v_n^2 - v^2}{2}(\mu u_0^2 - 1) \leq \int_{s}^{R} \frac{|v_n^2 - v^2|}{2}(\mu u_0^2 - 1) \leq \text{Constant} \cdot \|v_n - v\|_{L^2},$$

and so we get weakly lower semi-continuity for this term.

V Let $(v_n)_n$ a sequence of Y weakly convergent to v in H^2, by the compact embedding of H^2 into L^2 there exists a subsequence convergent to v in L^2 (which we call $(v_n)_n$) and so

$$\begin{aligned}\int_{s}^{R} \frac{1}{4}(v_n^4 - v^4) &\leq \int_{s}^{R} \frac{1}{4}|v_n^4 - v^4| \\ &= \int_{s}^{R} \frac{1}{4}|v_n^2 + v^2| \cdot |v_n + v| \cdot |v_n - v| \\ &\leq \text{Constant} \cdot \|v_n - v\|_{L^2},\end{aligned}$$

and we have the weakly lower semi-continuity.

We have all the terms on J_R are weakly lower semi-continuous and so the sum of them is weakly lower semi-continuous.

Finally by means of direct method of calculus of variations we have that there exists a minimum v_0^R on Y for J_R.

Lets check now the regularity of v_0^R, we know that $v_0^R \in Y_R$, due to Sobolev's Inequalities we also know $v_0^R \in C^{0,1/2}(-R,R)$ and that is a solution for

$$\int_{-R}^{R}(-\varepsilon^2 v^{\text{iv}} + v(1-v^2-\mu u_0^2))\phi = 0,\ \forall \phi \in C_0^{\infty}(-R,R),$$

by continuity is obtained that it also satisfies

$$-\varepsilon^2 v^{\text{iv}} + v(1-v^2-\mu u_0^2) = 0,\ \text{ for all } x \in (-R,R),$$

and so we conclude $v \in C^{4,1/2}$ and given we are working on a bounded set $(-R,R)$ is obtained that $v \in C^{4,\alpha}$ for all $0 < \alpha \leq 1/2$. □

To explore the problematic further, studying the behavior of v_0^R as $R \to \infty$ becomes neccesary. However, this particular challenge is beyond the scope of our current work. Instead, we focus on investigating this specific problem within a bounded interval and employ various numerical techniques to obtain solutions.

4.3.3 Scaled Problems

There are some natural scales to study both of the previous problems: the second order problem of the section 4.3 and the fourth order problem of the section 4.3.2. This new natural scaling will enable us to study the transition point s and the region close to this point, ureferred to as corner layer [10] [7]. The term "corner layer" is used due to the type of defect that appears in the generated patterns, similar to the ones observed in Figure 2.1.

Given that the energy functionals are divided into two parts and the functions that minimize $J_R^{s^-}$ and $J_R^{s^+}$ separately are very different (the same happens with E_{s^-} and E_{s^+} when considering the second order problem), it is of interest to study a scaled version of the problem that can capture how the function should behave close to s, transitioning from one minimizer to the other. Additionally, it is desirable for the new equation to be independent of arbitrary small values such as ε or δ. Let's choose a value s such that $u_0^2(s) = 1/\mu$ and consider the case of the second and fourth order problem:

Second order problem

By taking the expansions

$$u_0^2(s+\delta^p t) = u_0^2(s) + 2u_0(s)u_0'(s)\delta^p t + o(\delta^{2p}t^2)$$

$$v(s+\delta^p t) = \phi(t)\delta^{\alpha}.$$

The equation for ϕ obtained by replacing in (4.8) is given by

$$\delta^{2-2p}\phi'' = -\phi(1-\delta^{2\alpha}\phi^2 - \mu u_0^2(s+\delta^p t)),$$

using the previous expansion we get,

$$\delta^{2-2p}\phi'' = -\phi(1-\delta^{2\alpha}\phi^2 - \mu u_0^2(s) - 2\mu u_0(s)u_0'(s)\delta^p t + O(\delta^{2p}t^2)),$$

and so we can solve for the exponents

$$\begin{cases} 2-2p = 2\alpha \\ 2\alpha = p, \end{cases}$$

so we take $p = 2/3$, $\alpha = 1/3$, and get the following limit problem for small enough t (x close to s)

$$\phi'' - \phi(\phi^2 + Ct) = 0, \tag{4.10}$$

where $C = 2u_0(s)u_0'(s)$, a second Painlevé equation.

We are interested in solutions of (4.10) in such a way that the derivative (and the second derivative) vanishes at $\pm\infty$, we deduce

$$\begin{cases} \phi(-\infty) \approx & \sqrt{-2\mu u_0(s)u_0'(s)t} \\ \phi(\infty) = & 0, \end{cases} \tag{4.11}$$

for this problem we have the following energy functional

$$\int_{-\infty}^{0} \frac{(\phi')^2}{2} + \frac{1}{4}(\phi^2 + Ct)^2 dt + \int_{0}^{+\infty} \frac{(\phi')^2}{2} + \frac{\phi^2}{2}\left(Ct + \frac{\phi^2}{2}\right) dt. \tag{4.12}$$

Proposition 4.16 *The functional* (4.12) *has* (4.10) *as the Euler-lagrange equation associated and it is compatible with* (4.11).

Proof. Lets call $I(\phi)$ to the functional defined by (4.15), we have

$$I(\phi+\varepsilon h) = \int_{-\infty}^{0} \frac{1}{2}(\phi'+\varepsilon h')^2 + \frac{1}{4}((\phi+\varepsilon h)^2 + Ct)^2 dt$$

$$+ \int_{0}^{+\infty} \frac{1}{2}(\phi'+\varepsilon h')^2 + \frac{Ct}{2}(\phi+\varepsilon h)^2 + \frac{1}{4}(\phi+\varepsilon h)^4 dt,$$

and so

$$\frac{d}{d\varepsilon}I(\phi+\varepsilon h) = \int_{-\infty}^{0} (\phi'+\varepsilon h')h' + ((\phi+\varepsilon h)^2 + Ct)(\phi+\varepsilon h)h dt$$

$$+ \int_{0}^{+\infty} (\phi'+\varepsilon h')h' + Ct(\phi+\varepsilon h)h + (\phi+\varepsilon h)^3 h dt,$$

finally, we just evaluate $\varepsilon = 0$ and get

$$\frac{d}{d\varepsilon}I(\phi+\varepsilon h)\Big|_{\varepsilon=0} = \int_{-\infty}^{0} \phi'h' + (\phi^2+Ct)\phi h dt + \int_{0}^{+\infty} \phi'h' + Ct\phi h + \phi^3 h dt$$

$$= \int \phi'h' + \phi(\phi^2+Ct)h dt,$$

and so, if

$$\frac{d}{d\varepsilon}I(\phi+\varepsilon h)\Big|_{\varepsilon=0} = 0 \text{ for all } h \in C_c^\infty$$

then

$$-\phi'' - \phi(\phi^2 + Ct) = 0.$$

To see that the functional is compatible with the boundary conditions (4.11) we notice that to minimize the left side integral on (4.12) the function inside the integral must go to zero, this implies $(\phi^2 + Ct) \to 0$ as $t \to -\infty$ and so the first condition of (4.11) is fulfilled, the same happens when we observe the second integral in the functional which allow us to obtain the second boundary condition. □

Fourth order problem

Just as in the second order problem, we consider the expansion

$$u_0^2(s+\varepsilon^p t) = u_0^2(s) + 2u_0(s)u_0'(s)\varepsilon^p t + o(\varepsilon^{2p}t^2),$$

and the change of variables

$$v(s+\varepsilon^p t) = \phi(t)\varepsilon^\alpha,$$

in the equation (4.9), so we obtain

$$-\varepsilon^{2-4p}\phi^{iv} = -\phi(1-\varepsilon^{2\alpha}\phi^2 - \mu u_0^2(s+\varepsilon^p t)),$$

and so

$$-\varepsilon^{2-4p}\phi^{iv} = -\phi(1-\varepsilon^{2\alpha}\phi - \mu u_0^2(s) - 2\mu u_0(s)u_0'(s)\varepsilon^p x + o(\varepsilon^{2p}t^2))$$
$$-\varepsilon^{2-4p}\phi^{iv} = \phi(\varepsilon^{2\alpha}\phi^2 + 2\mu u_0(s)u_0'(s)\varepsilon^p t + o(\varepsilon^{2p}t^2)),$$

where the system for p and α is found

$$\begin{cases} 2-4p = 2\alpha \\ 2\alpha = p \end{cases}$$

and we get $p = 2/5$, $\alpha = 1/5$. The limit problem is:

$$-\phi^{iv} - \phi(\phi^2 + 2\mu u_0(s)u_0'(s)t) = 0, \text{ for all } t \in \mathbb{R}. \tag{4.13}$$

We may ask ourselves about the boundary conditions this problem should have, by means of the previous equation we deduce

$$\begin{cases} \phi(-\infty) \approx & \sqrt{-2\mu u_0(s)u_0'(s)t} \\ \phi(\infty) = & 0, \end{cases} \tag{4.14}$$

and for this problem we have the following energy functional

$$\int_{-\infty}^0 \frac{(\phi'')^2}{2} + \frac{1}{4}(\phi^2+Ct)^2 dt + \int_0^{+\infty} \frac{(\phi'')^2}{2} + \frac{\phi^2}{2}\left(Ct + \frac{\phi^2}{2}\right) dt, \tag{4.15}$$

this is proved in the following proposition:

Proposition 4.17 *The functional given by* (4.15) *has* (4.13) *as the Euler-Lagrange equations associated and is compatible with* (4.14).

Proof. Lets call $I(\phi)$ to the functional defined by (4.15), for this functional we have that

$$I(\phi+\varepsilon h) = \int_{-\infty}^{0} \frac{1}{2}(\phi''+\varepsilon h'')^2 + \frac{1}{4}((\phi+\varepsilon h)^2+Ct)^2 dt$$

$$+\int_{0}^{+\infty} \frac{1}{2}(\phi''+\varepsilon h'')^2 + \frac{Ct}{2}(\phi+\varepsilon h)^2 + \frac{1}{4}(\phi+\varepsilon h)^4 dt,$$

and so

$$\frac{d}{d\varepsilon}I(\phi+\varepsilon h) = \int_{-\infty}^{0} (\phi''+\varepsilon h'')h'' + ((\phi+\varepsilon h)^2+Ct)(\phi+\varepsilon h)hdt$$

$$+\int_{0}^{+\infty} (\phi''+\varepsilon h'')h'' + Ct(\phi+\varepsilon h)h + (\phi+\varepsilon h)^3 hdt,$$

finally, we just evaluate $\varepsilon = 0$ and get

$$\frac{d}{d\varepsilon}I(\phi+\varepsilon h)\Big|_{\varepsilon=0} = \int_{-\infty}^{0} \phi''h'' + (\phi^2+Ct)\phi hdt + \int_{0}^{+\infty} \phi''h'' + Ct\phi h + \phi^3 hdt$$

$$= \int \phi''h'' + \phi(\phi^2+Ct)hdt,$$

and so, if

$$\frac{d}{d\varepsilon}I(\phi+\varepsilon h)\Big|_{\varepsilon=0} = 0 \text{ for all } h \in C_c^\infty$$

then

$$-\phi^{\text{iv}} - \phi(\phi^2+Ct) = 0.$$

To verify that the functional is compatible with the boundary conditions (4.14), we simply need to notice that in order to minimize the left-hand side integral on (4.15), the function inside the integral must go to zero. This implies $(\phi^2+Ct) \to 0$ as $t \to -\infty$, thereby satisfying the first condition of (4.14). The same observation applies to the second integral in the functional, allowing us to obtain the second boundary condition. □

4.4 Numeric implementation and results

In this section we show the numeric results using multiple implementations of different techniques for the problem (4.5), (4.9) and (4.13).

4.4.1 Minimization of the second order problem

Numerical approximation of u_0

First we consider the problem given by

$$u'' + u(1 - u^2 - \mu v_\varepsilon^2) = 0, \quad \text{for all } x \in (-R, R),$$

$$u'(-R) = 0, u'(R) = 0.$$

The discretization of the above problem is performed by splitting the interval $[-R, R]$ using $h = 2R/n$ with $n \geq 3$ generating $n+1$ points $x_i = -R + ih$ with $0 \leq i \leq n$. We consider the following approximation for the derivative:

$$u'_j = \frac{u_{j+1} - u_j}{h},$$

and finally obtain the finite dimensional minimization problem (when considering $z = 0$)

$$\min_{u \in R^n} \sum_{i=1}^{\lfloor n/2 \rfloor} \left(\frac{(u'_i)^2}{2} + \frac{u_i^2}{2} K(x_i) + \frac{u_i^4}{4} \right) h + \sum_{i=\lfloor n/2 \rfloor + 1}^{n-1} \left(\frac{(u'_i)^2}{2} + \frac{1}{4}(1 - u_i^2)^2 \right) h,$$

subject to

$$u'(-R) = 0, u'(R) = 0.$$

The minimization was performed by the algorithm sequential least squares programming and the following is the obtained for different values of n:

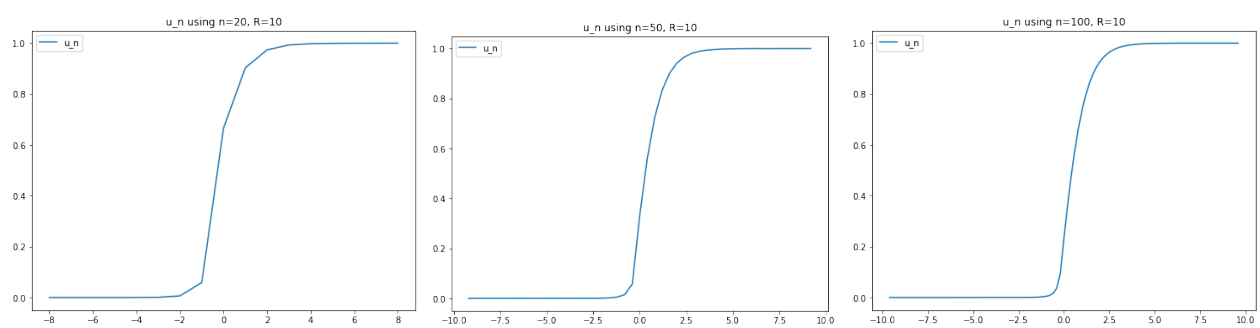

Figure 4.2: u_0 as a result of minimization of discretized energy functional.

Numerical approximation of v_0

Once having values of u_0 it is posible to confront the problem of finding v_0, that is

$$\min \int_{-R}^{s} \delta^2 \frac{(v')^2}{2} + \frac{\mu u_0^2 v^2}{2} + \frac{(1 - v^2)^2}{4} + \int_{s}^{R} \delta^2 \frac{(v')^2}{2} + \frac{v^2}{2}(\mu u_0^2 - 1) + \frac{v^4}{4},$$

subject to

$$v'(-R) = 0, v'(R) = 0.$$

To solve the previous problem we need to find s, that is easy, we just consider the first index such that $u_0[i] = 1/\mu$ and take $s = x_i$, we will call s to this index. Once knowing the value of s it is posible for us to define the finite dimensional minimization problem corresponding to v_0^R:

$$\min_{v_i \in \mathbb{R}^n} \sum_{i=1}^{s} \left(\delta^2 \frac{(v_i')^2}{2} + \frac{\mu u_{0_i}^2 v_i^2}{2} + \frac{(1 - v_i^2)^2}{4} \right) h + \sum_{i=s+1}^{n} \left(\delta^2 \frac{(v_i')^2}{2} + \frac{v_i^2}{2} (\mu u_{0_i}^2 - 1) + \frac{v_i^4}{4} \right) h,$$

subject to

$$v'(-R) = 0, v'(R) = 0.$$

Just as before we use the use sequential least squares programming algorithm to minimize this and obtain the following set of figures for different values of δ and n:

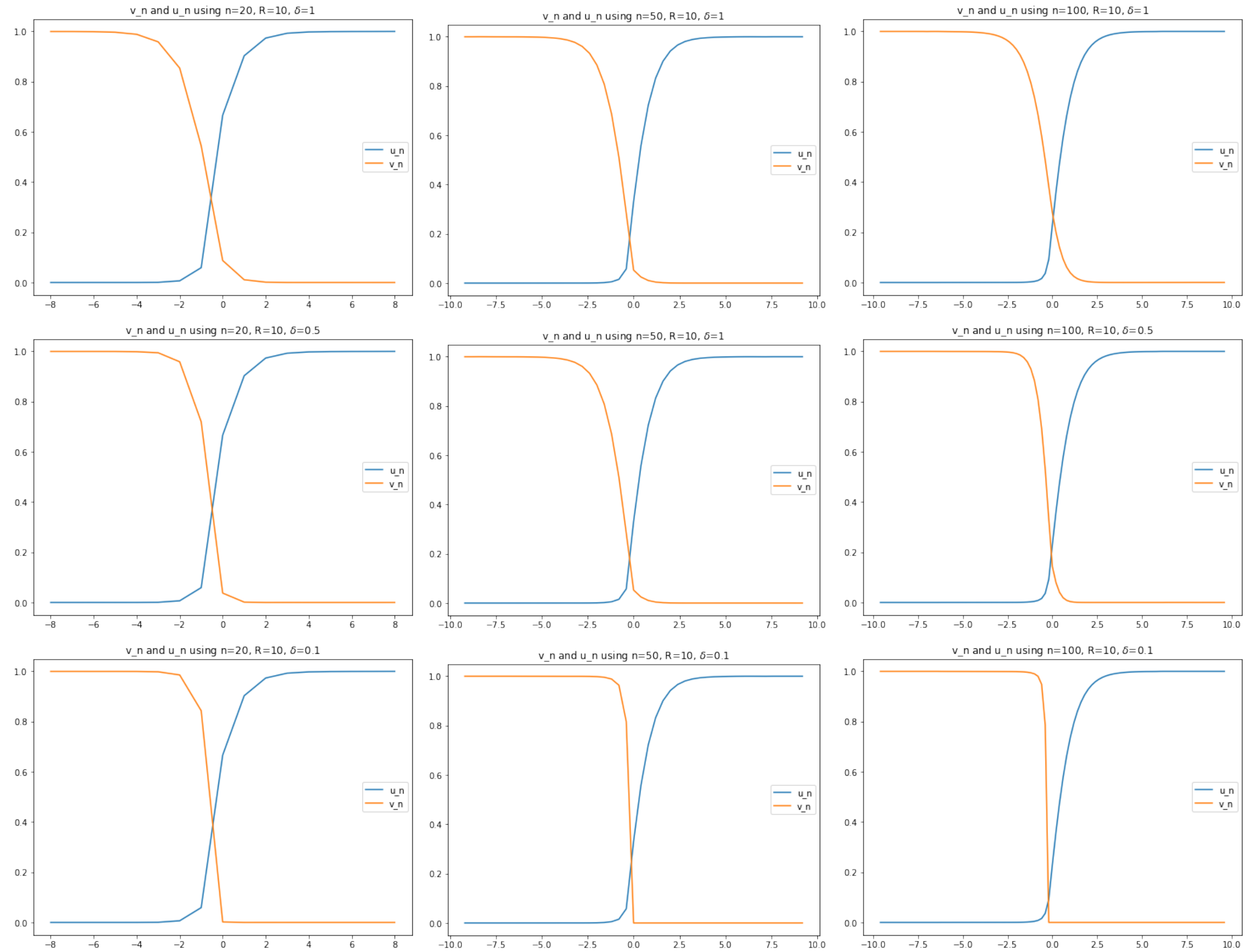

Figure 4.3: v_0 as a result of minimization of discretized energy functional.

where in each case two monotone functions are obtained. Also, we can look at an interval of lenght $O(\delta^{2/3})$ where its supposed to be localized the domain wall where v drops, we choose to graph lines on $\pm 4\delta^{2/3}$ on some of the previous graph to observe that

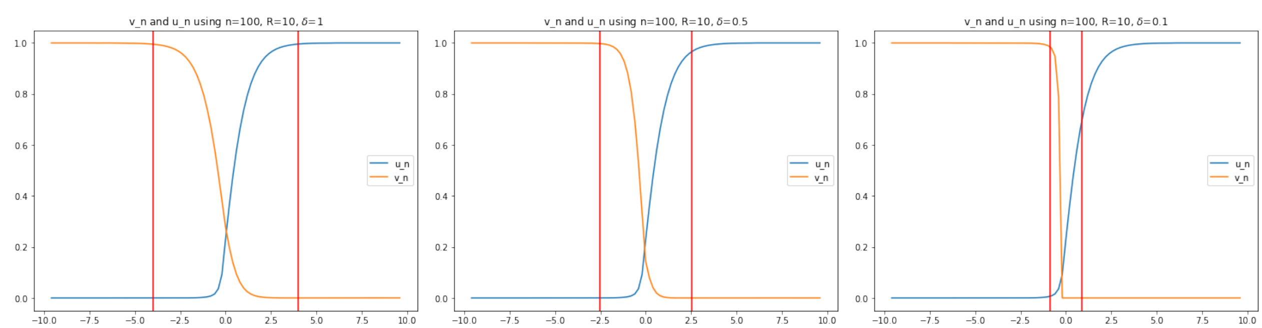

Figure 4.4: v_0 and u_0 as a result of minimization of discretized energy functional, where red lines are for $x = \pm 4\delta^{2/3}$.

4.4.2 Minimization of the fourth order problem

The fourth order has been challenging to work with due to the presence of a second derivative in its energy functional. This prevents us from constructing a uniformly bounded monotone sequence of functions to minimize the energy. Therefore, we consider the previous minimization problems as finite-dimensional minimization problems.

Numerical approximation of u_0

As we did in the previous case we consider the problem given by

$$u'' + u(1 - u^2 - \mu v_\varepsilon^2) = 0, \quad \text{for all } x \in (-R, R),$$

$$u'(-R) = 0, u'(R) = 0,$$

and compute the minimization of the finite dimensional optimization problem

$$\min_{u \in R^n} \sum_{i=1}^{\lfloor n/2 \rfloor} \left(\frac{(u_i')^2}{2} + \frac{u_i^2}{2} K(x_i) + \frac{u_i^4}{4} \right) h + \sum_{i=\lfloor n/2 \rfloor + 1}^{n-1} \left(\frac{(u_i')^2}{2} + \frac{1}{4}(1 - u_i^2)^2 \right) h,$$

subject to

$$u'(-R) = 0, u'(R) = 0,$$

where

$$u_j' = \frac{u_{j+1} - u_j}{h}.$$

Numerical approximation of v_0

Once having values of u_0 it is posible to confront the problem of finding v_0, that is

$$\min \int_{-R}^{s} \varepsilon^2 \frac{(v'')^2}{2} + \frac{\mu u_0^2 v^2}{2} + \frac{(1 - v^2)^2}{4} + \int_{s}^{R} \varepsilon^2 \frac{(v'')^2}{2} + \frac{v^2}{2}(\mu u_0^2 - 1) + \frac{v^4}{4},$$

subject to

$$v'(-R) = 0, v'(R) = 0.$$

To solve the previous problem we need to find s, that is easy, we just consider the first index such that $u_0[i] = 1/\mu$ and take $s = x_i$, we will call s to this index. Once knowing the value of s it is posible for us to define the finite dimensional minimization problem corresponding to v_0^R:

$$\min_{v_i \in \mathbb{R}^n} \sum_{i=1}^{s} \left(\varepsilon^2 \frac{(v_i'')^2}{2} + \frac{\mu u_{0_i}^2 v_i^2}{2} + \frac{(1 - v_i^2)^2}{4} \right) h + \sum_{i=s+1}^{n} \left(\varepsilon^2 \frac{(v_i'')^2}{2} + \frac{v_i^2}{2}(\mu u_{0_i}^2 - 1) + \frac{v_i^4}{4} \right) h,$$

subject to

$$v'(-R) = 0, v'(R) = 0.$$

Just as before we use the use sequential least squares programming algorithm to minimize this and obtain the following set of figures for different values of ε and n:

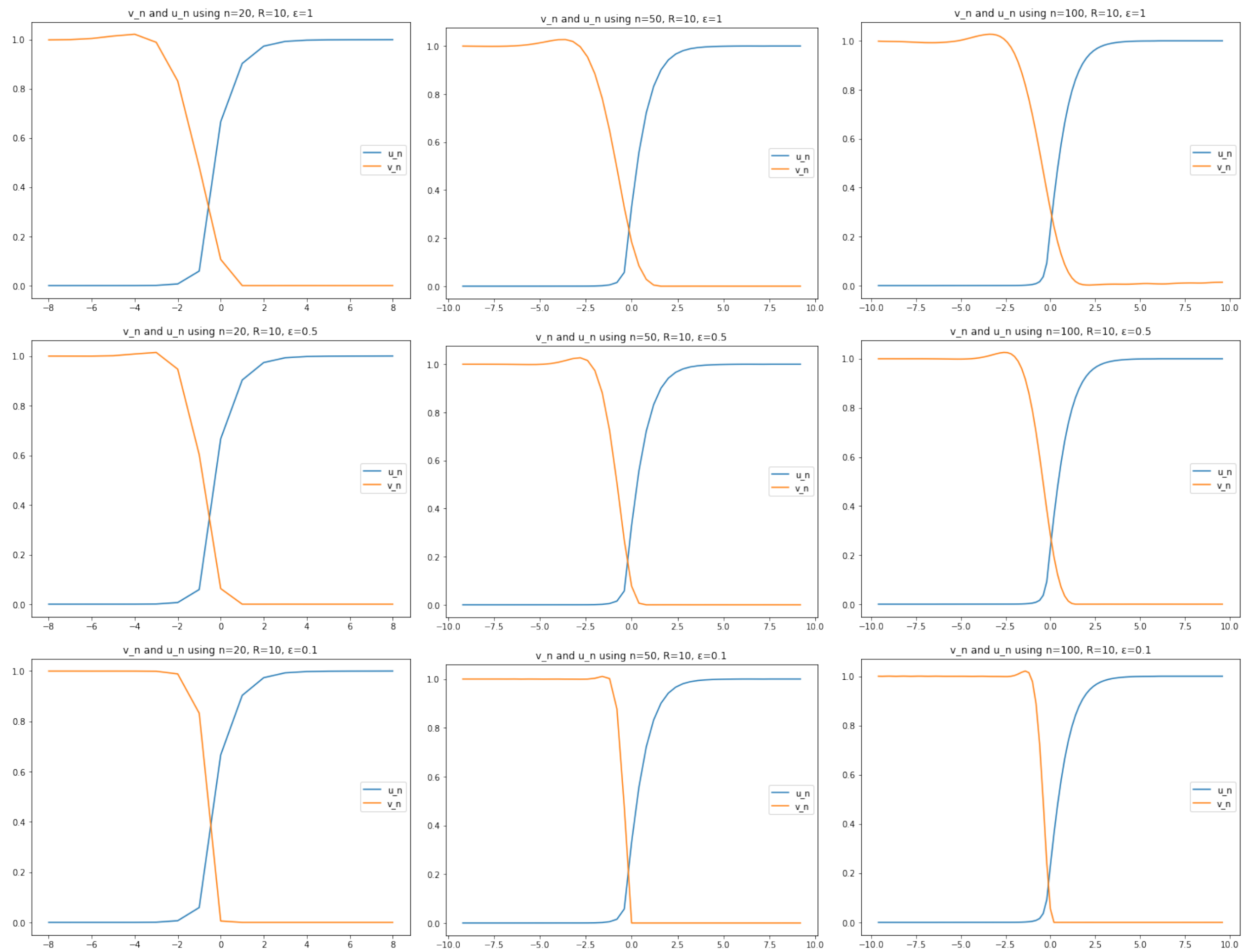

Figure 4.5: v_0 as a result of minimization of discretized energy functional.

As we observe, the solution obtained for v_0 is not a monotone function, which is consistent with the theoretical problem of constructing a non-increasing minimizing sequence. It may also be interesting to examine the size of this domain boundary in relation to ε. In the previous discussion, where the equation (4.13) was introduced, the change of scale on x was $\varepsilon^{2/5}$. This suggests that the length of this structure should be proportional to this quantity when it is small. We obtained the following figures when we considered the lines $x = \pm 4\varepsilon^{2/5}$ as reference and highlighted in red

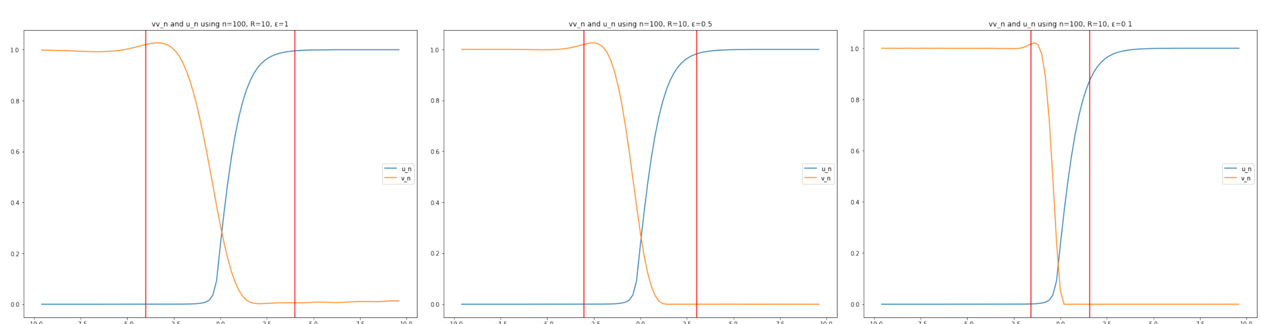

Figure 4.6: v_0 and u_0 as a result of minimization of discretized energy functional, where red lines are for $x = \pm 4\varepsilon^{2/5}$.

4.4.3 Minimization of the rescaled Calculus of Variation Problem

Second order problem

The equation (4.12) has the following minimization problem associated:

$$\min J(\phi) = \int_{-\infty}^{0} \frac{(\phi')^2}{2} + \frac{1}{4}(\phi^2 + Ct)^2 dt + \int_{0}^{+\infty} \frac{(\phi')^2}{2} + \frac{\phi^2}{2}\left(Ct + \frac{\phi^2}{2}\right) dt$$

we are interested in to capture the behavior of the solution around $t = 0$ and so we consider the numeric problem of minimizing the following integral taking big enough A:

$$\int_{-A}^{0} \frac{(\phi')^2}{2} + \frac{1}{4}(\phi^2 + Ct)^2 dt + \int_{0}^{A} \frac{(\phi')^2}{2} + \frac{\phi^2}{2}\left(Ct + \frac{\phi^2}{2}\right) dt$$

to do it we first consider a uniform discretization for n points on the interval $[-A, A]$ and the following the scheme we use for the second derivative

$$y_i' = \frac{y_{i+1} - y_i}{h},$$

where $h = 2A/n$. We set the finite-dimensional optimization problem

$$\min_{y \in R^n} \sum_{i=1}^{\lfloor n/2 \rfloor} \left(\frac{(y')_i^2}{2} + \frac{1}{4}(y_i^2 + Cx_i)^2 \right) h + \sum_{i=\lfloor n/2 \rfloor + 1}^{n-1} \left(\frac{(y')_i^2}{2} + Cx_i \frac{y_i^2}{2} + \frac{y_i^4}{4} \right) h.$$

An important observation to make here is that the value C is free at this moment given $u_0'(s)$ is unknown and yet we dont have any constraint for its value. The constant C takes place in the objective function and the boundary conditions, given that the techniques used are not able to fix it, we use the value 1 for our numerical simulations.

The following figures are the solutions for the optimization problem using scipy SLSQP algorithm from scipy library for different values of n and A.

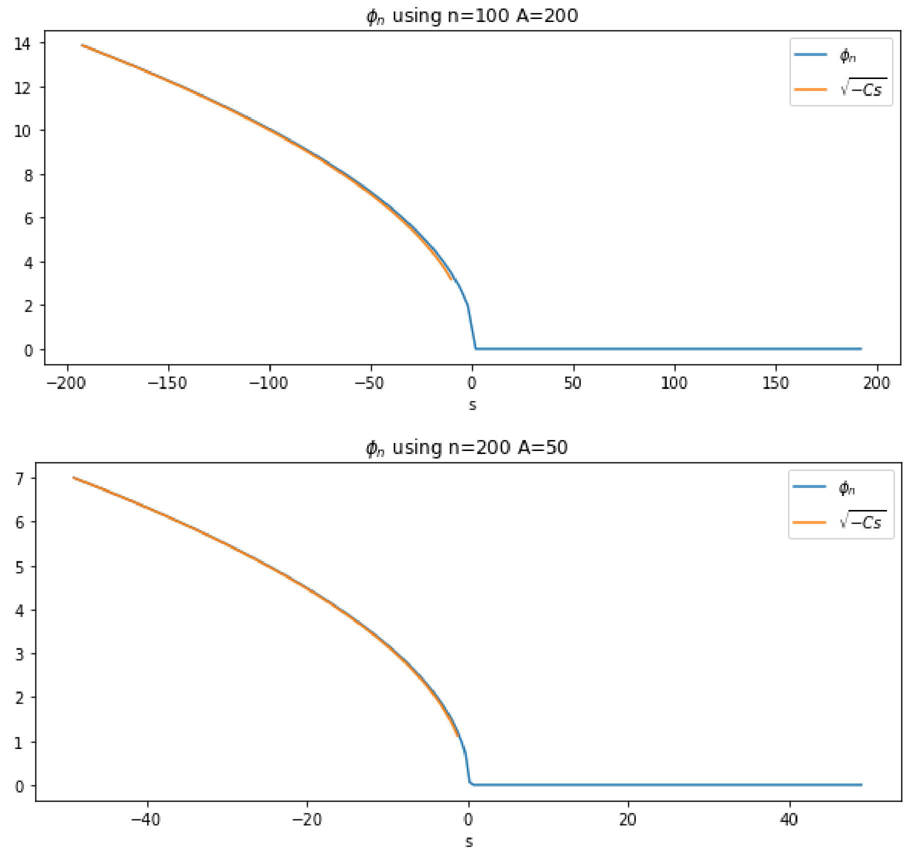

Figure 4.7: Solutions for ϕ_n for different values for n and A using numeric minimization of energy functional.

Fourth order problem

As we proved, the equation (4.13) has the following minimization problem associated:

$$\min J(\phi) = \int_{-\infty}^{0} \frac{(\phi'')^2}{2} + \frac{1}{4}(\phi^2 + Ct)^2 dt + \int_{0}^{+\infty} \frac{(\phi'')^2}{2} + \frac{\phi^2}{2}\left(Ct + \frac{\phi^2}{2}\right) dt$$

we are interested in to capture the behavior of the solution around $t = 0$ and so we consider the numeric problem of minimizing the following integral taking big enough A:

$$\int_{-A}^{0} \frac{(\phi'')^2}{2} + \frac{1}{4}(\phi^2 + Ct)^2 dt + \int_{0}^{A} \frac{(\phi'')^2}{2} + \frac{\phi^2}{2}\left(Ct + \frac{\phi^2}{2}\right) dt$$

to do it we first consider a uniform discretization for n points on the interval $[-A, A]$ and the following the scheme we use for the second derivative

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2},$$

where $h = 2A/n$. We set the finite-dimensional optimization problem

$$\min_{y \in R^n} \sum_{i=1}^{\lfloor n/2 \rfloor} \left(\frac{(y'')_i^2}{2} + \frac{1}{4}(y_i^2 + Cx_i)^2\right) h + \sum_{i=\lfloor n/2 \rfloor + 1}^{n-1} \left(\frac{(y'')_i^2}{2} + Cx_i\frac{y_i^2}{2} + \frac{y_i^4}{4}\right) h.$$

Analogous to the second order case, we have that C is free given $u_0'(s)$ is unknown and yet we dont have any constraint for its value. We use $C = 1$ for our numerical simulations.

The following figures are the solutions for the optimization problem using scipy SLSQP algorithm from scipy library for different values of n and A.

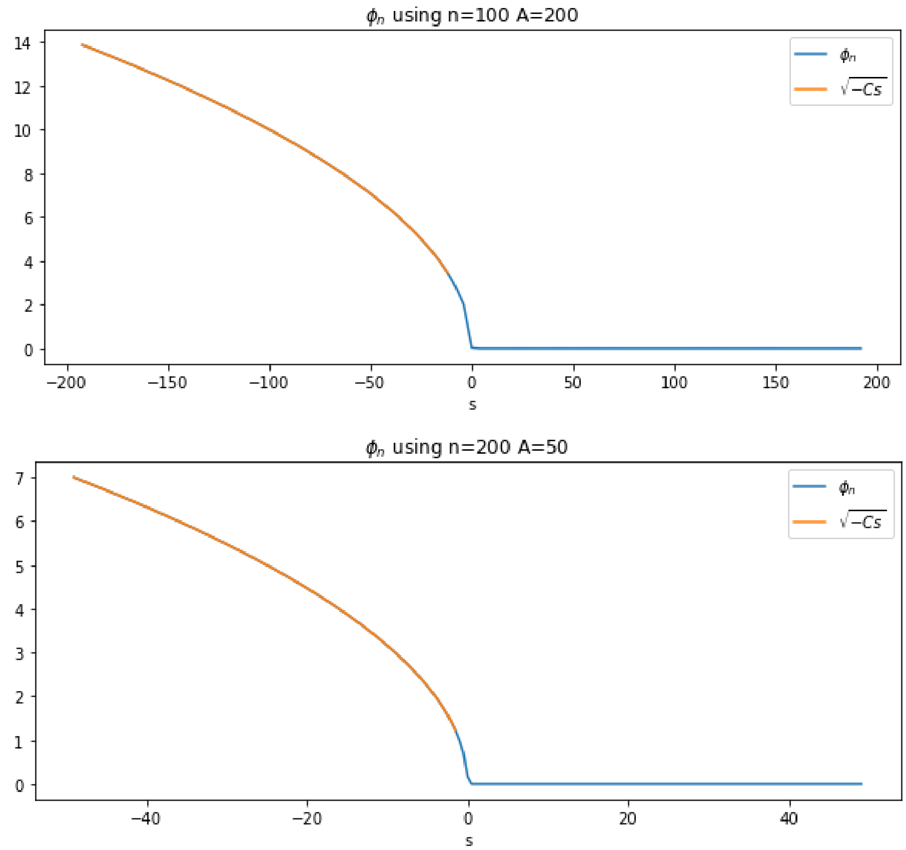

Figure 4.8: Solutions for ϕ_n for different values for n and A using numeric minimization of energy functional.

4.4.4 Finite Element Method Approach

In this section, we introduce and explain the methodology used to confront nonlinear problems using Finite Element Method and Newton's Algorithm. We also present the results obtained for the rescaled problem (4.13).

We have the following Non Linear equation we want to study

$$-\phi^{\mathrm{iv}} - \phi(\phi^2 + Ct) = 0,\, t \in \mathbb{R},$$

subject to (4.14). We are particularly interested on the behavior close to $t = 0$.

As in the previous section we work on the interval $[-A, A]$ for big enough A, and claim this will be able to capture how the solution ϕ for (4.13)-(4.14) behaves close to 0. This new problem can be written as finding $\phi \in V$ from the following variational formulation

$$\int \phi'' v'' + \phi^3 v + Cs\phi v = 0,\, \forall v \in V_0, \tag{4.16}$$

where V is given by $V = V_0 \bigoplus V_\Gamma$, where $V_0 = H_0^2(-A, A)$ (which we write H_0^2 for simplicity) and $V_\Gamma = \{u \in H^2 | u(-A) = \sqrt{AC}, u(A) = 0\}$.

The problem (4.16) is a non-linear equation which we can write using $F : V \times H_0^2 \longrightarrow \mathbb{R}$ defined by

$$F(\phi; v) = \int \phi'' v'' + \phi^3 v + Cs\phi v, \tag{4.17}$$

and so we obtain that (4.16) can be written as $F(\phi; v) = 0$, for all $v \in V_0$. The function F is non-linear on ϕ and linear on v, we then consider the Newton's Algorithm by means of displacements over V_0 to solve it, Newton's Algorithm is based on considering the following expansion (using the decomposition of the space V on V_0 and V_Γ, $u = u_0 + u_\Gamma = (u_0, u_\Gamma)$)

$$F(u_0 + \delta, u_\Gamma; v) = F(u_0, u_\Gamma; v) + \partial_{V_0} F(u_0, u_\Gamma; v)[\delta] + o(\delta),$$

and solve the equation

$$0 = F(u_0, u_\Gamma; v) + \partial_{V_0} F(u_0, u_\Gamma; v)[\delta]$$

to find δ and after each iteration we consider the update of values $u^+ = u + \delta$. The previous equation can be solved using Finite Elements Method over

$$D_V F(u; v)[\delta] = -F(u; v),$$

where δ lives in $V_0 = H_0^2$. We iterate this procedure until observe $\delta \approx 0$.

When we compute $D_V F(u; v)[\delta]$ we obtain

$$D_V F(\phi; v)[\delta] = \int \delta'' v'' + 3\phi^2 \delta + C\delta v s,$$

where $D_V F : H_0^2 \times H_0^2 \longrightarrow \mathbb{R}$.

The numeric scheme used to solve (4.16) is by considering $V_h = \mathbb{P}_2$, where $\mathbb{P}_2$ is the set of polynomials, with real coefficients, of real variable with degree less or equal to 2. We start the algorithm with $\phi_0 = \left(\frac{1}{2} - \frac{\arctan(\cdot)}{\pi}\right) \cdot \sqrt{CA}$. The figure 4.9 shows the results of iteration over this ϕ_0:

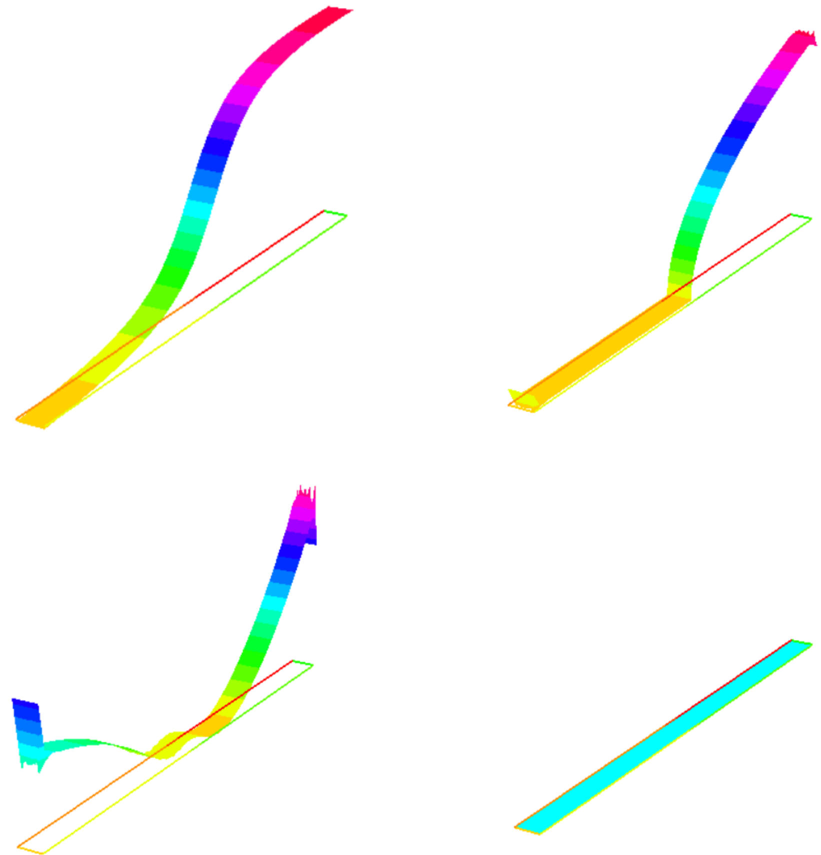

Figure 4.9: The upper-left figure is ϕ_0, the upper-right figure is the last function after iterate 14 times ϕ_{14}, the down-left figure is the first δ associated to ϕ_0 and the down-right is the last δ associated to ϕ_{13}.

Chapter 5

Conclusion

In the previous sections we have reached many results over we can set some interesting conclusions which we enumerate:

- From the results obtained, observed in Figures 4.7, 4.8 and 4.9, it is possible to conclude that the equation (4.13), which captures the transition that occurs at $x = s$ for the solutions of (4.8), cannot be satisfactorily studied using the previous methods based on weak (variational) formulations of the original problem. Neither the minimization of the calculus of variation problem nor the finite element method iteration were able to provide a smooth enough solution, especially close to $x = s$.

- The problem given by (4.8) may be a good approximation for a solution of (4.1), but a better approximation may be built if we follow the logic of how we have built problems (4.5), (4.8), and use some of the ideas in [14]. In this reference a second-order system is studied, where one of the unknowns has a perturbation parameter. They study

$$\begin{cases} x'' & = g_x(x, y) \\ \varepsilon^2 y'' & = g_y(x, y), \end{cases}$$

  where ε is a small parameter. They are interested in solutions that satisfy constraints at infinity, just as we do in our problems. The techniques used are very different, based on certain properties of the function g, but a good adaptation of the methodology may be applied to find solutions for (4.1). Our problem can be written in a similar manner as in [14] with a different function g, since our problem is indeed a variational one.

- The equations (4.10) appear as the natural scale of the second-order-problem studied. The structure of this equation is very similar to Painlevé transcendents of the second type

$$y'' = 2y^3 + xy + \alpha.$$

  Techniques used for the study of this particular Painlevé equation may be adapted. Unfortunately, the fourth order equation (4.13) does not match with any of the Painlevé equations, nevertheless, it follows a similar structure, so perhaps the techniques used to study those can also be modified to study this problem.

- The fourth-order problem introduces new difficulties when working with its variational formulation. To tackle these challenges, we have limited our analysis to a bounded interval. However, valuable insights can be considered from previous studies on fourth-order systems, as we can find in works of [13] and [4].

  In the first study, they usted sublevel sets of the respective functional and demonstrated that these sets form equicontinuous and uniformly bounded families. This allowed them to successfully apply Ascoli's theorem. We can explore and adapt some of these concepts to overcome the difficulties specific to our case.

  The second study bears similarities to our work, as they focused on specific functions within the domain of their functional and they constructed a minimizing sequence of these specific type of functions. However, instead of employing Helly's theorem, they utilized Ascoli's theorem by establishing uniform bounds on both values and derivatives of the functions. We can consider adopting some of these ideas as well for (4.9).

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Chapter 6

Annex

In this section it is included a compilation of the code and functions used on Python and Freefem++ to generate the figures above on this text.

```
import numpy as np
from scipy.optimize import minimize
import matplotlib.pyplot as plt

#Metodo que minimiza y grafica sin restricciones
def Minimizarsinrestr(A,n,C):
    def laplacianodetres(a,b,c,h):
        return (a-2*b+c)/(h**2)
    def derivada(a,b,c,h):
        return (a-b)/(h)
    def objective(y):
        return sum(laplacianodetres(y[i+1],y[i],y[i-1],h)*h/2 for i in
 range(1,n))+\
               sum(C*(-A+i*h)*(y[i]**2/2)+(y[i]**4/4) for i in range(int(n
 /2),n))*h+\
               sum((1/4)*(y[i]**2+C*(-A+i*h))**2 for i in range(1,int(n/2)
 ))*h
    h=2*A/n
    y_0=np.ones(n+1)
    y_0[0]=np.sqrt(C*A)
    y_0[-1]=0
    b= (0,None)
    bnds=()
    for j in range(n+1):
        bnds=bnds+(b,)
    sol=minimize(objective,y_0,method='SLSQP',bounds=bnds,constraints=[])
    xx=np.linspace(-A,A,n+1)
    plt.figure(figsize=(10,4))
    plt.plot(xx[2:-2],sol.x[2:-2])
    plt.plot(xx[2:int(n/2)-2],np.sqrt(-xx[2:int(n/2)-2]*C))
    plt.xlabel('s')
    plt.legend([r'$\phi_n$', r'$\sqrt{-Cs}$'])
    plt.title(r'$\phi_n$'+' using n=' +str(n)+' A='+str(A))
    plt.show()
```

```python
    return sol
```

Listing 6.1: Python example

```python
import numpy as np
from scipy.optimize import minimize
import matplotlib.pyplot as plt

def minimizarv(R,n,u_0):
    def derivada(a,b,c,h):
        return (a-b)/(h)

    def buscars(u_0):
        for k in range(len(u_0)):
            if u_0[k]**2 >= 1/mu:
                return k

    def segderivada(a,b,c,h):
        return (a-2*b+c)/(h**2)
    def objective(v):
        s=buscars(u_0)
        return sum((epsilon**2)*(segderivada(v[i+1],v[i],v[i-1],h)**2)+mu*u_0[i]*v[i]+.5*(1-v[i]**2)**2 for i in range(1,s))*h/2+\
                 sum((epsilon**2)*(segderivada(v[i+1],v[i],v[i-1],h)**2)+((v[i]**2)*(mu*u_0[i]-1)+v[i]**4/2) for i in range(s,n))*h/2

    def constraint1(v):
        return v[-1]-v[-2]
    def constraint2(v):
        return v[0]-v[1]
    con1={'type': 'eq','fun':constraint1}
    con2={'type': 'eq','fun':constraint2}
    cons=[con1,con2]
    h=2*R/n
    v_0=np.ones(n+1)
    v_0[0]=1
    v_0[-1]=0
    b=(0,None)
    bnds=()
    for j in range(n+1):
        bnds=bnds+(b,)
    sol=minimize(objective,v_0,method='SLSQP',bounds=bnds,constraints=[])
    xx=np.linspace(-R,R,n+1)

    plt.figure(figsize=(8,6))
    plt.plot(xx[2:-2],u_0[2:-2])
    plt.plot(xx[2:-2],sol.x[2:-2])
    plt.legend(['u_n','v_n'])
    plt.title('v_n and u_n using n=' +str(n)+', R='+str(R) + ', epsil='+str(epsilon))

    plt.figure(figsize=(8,6))
    plt.plot(xx[2:-2],u_0[2:-2])
    plt.plot(xx[2:-2],sol.x[2:-2])
    plt.legend(['u_n','v_n'])
    plt.axvline(x = -4*epsilon**(2/5), color = 'r', label = 'axvline - full height')
```

```python
    plt.axvline(x = 4*epsilon**(2/5), color = 'r', label = 'axvline - full height')
    plt.title('v_n and u_n using n=' +str(n)+', R='+str(R) + ', epsil='+str(epsilon))
    plt.show()
    return sol
```

Listing 6.2: Python example

```python
import numpy as np
from scipy.optimize import minimize
import matplotlib.pyplot as plt
def minimizarvord2(R,n,u_0):
    def derivada(a,b,c,h):
        return (a-b)/(h)

    def buscars(u_0):
        for k in range(len(u_0)):
            if u_0[k]**2 >= 1/mu:
                return k

    def segderivada(a,b,c,h):
        return (a-2*b+c)/(h**2)
    def objective(v):
        s=buscars(u_0)
        return sum((epsilon**2)*(derivada(v[i+1],v[i],v[i-1],h)**2)+mu*u_0[i]*v[i]+.5*(1-v[i]**2)**2 for i in range(1,s))*h/2+\
                sum((epsilon**2)*(derivada(v[i+1],v[i],v[i-1],h)**2)+((v[i]**2)*(mu*u_0[i]-1)+v[i]**4/2) for i in range(s,n))*h/2

    def constraint1(v):
        return v[-1]-v[-2]
    def constraint2(v):
        return v[0]-v[1]
    con1={'type': 'eq','fun':constraint1}
    con2={'type': 'eq','fun':constraint2}
    cons=[con1,con2]
    h=2*R/n
    v_0=np.ones(n+1)
    v_0[0]=1
    v_0[-1]=0
    b=(0,None)
    bnds=()
    for j in range(n+1):
        bnds=bnds+(b,)
    sol=minimize(objective,v_0,method='SLSQP',bounds=bnds,constraints=[])
    xx=np.linspace(-R,R,n+1)

    plt.figure(figsize=(8,6))
    plt.plot(xx[2:-2],u_0[2:-2])
    plt.plot(xx[2:-2],sol.x[2:-2])
    plt.legend(['u_n','v_n'])
    plt.title(r'v_n and u_n using n=' +str(n)+r', R='+str(R)  +', '+ r'$\delta$'+ '='+ str(epsilon))

    plt.figure(figsize=(8,6))
    plt.plot(xx[2:-2],u_0[2:-2])
```

```python
    plt.plot(xx[2:-2],sol.x[2:-2])
    plt.legend(['u_n','v_n'])
    plt.axvline(x = -4*epsilon**(2/3), color = 'r', label = 'axvline -
  full height')
    plt.axvline(x = 4*epsilon**(2/3), color = 'r', label = 'axvline - full
    height')
    plt.title(r'v_n and u_n using n=' +str(n)+r', R='+str(R)  +', '+ r'$\
  delta$'+ '='+ str(epsilon))
    plt.show()
    return sol
```

Listing 6.3: Python example

```cpp
int n=20;
real C=1;
int A=5;
real grosor=.5;
real delta=.2;
border a(t=-A,-delta){x=t ;y=0;label=1;}
border under(t=-delta,delta){x=t; y=0; label=2;}
border b(t=delta,A){x=t ;y=0;label=3;}
border c(t=0,grosor){x=A ;y=t ;label=4;}
border d(t=A,delta){x=t ;y=grosor ;label=5;}
border upper(t=delta,-delta){x=t ;y=grosor ;label=6;}
border e(t=-delta,-A){x=t ;y=grosor ;label=7;}
border f(t=grosor,0){x=-A;y=t ;label=8;}
mesh Th =buildmesh(a(n)+under(2*n)+b(n)+c(n)+d(n)+upper(2*n)+e(n)+f(n));
plot(Th,wait=1);
//Fespace
fespace Vh(Th,P2);
Vh phi,dphi,v,phicero;
phicero=(.5-atan(x)*.318)*sqrt(C*A);
//loop
phi=phicero;
int m;
real err=0;
savemesh(Th,"meshfile.msh");

plot(phi,value=false,wait=1,dim=3,fill=1,cmm="Phi_0");
for(m=1; m<15;m++){
  //Newton
  solve LinearStep(dphi,v)=int2d(Th)(dxx(dphi)*dxx(v)+3*phi^2*dphi*v+C*x*
   dphi*v)
  +int2d(Th)(dxx(phi)*dxx(v)+phi^3*v+C*x*phi*v)+on(4,dphi=0)+on(8,dphi=0);
  plot(dphi,value=false,wait=1,dim=3,fill=1,cmm="h");
  phi=phi+dphi;
  plot(phi,value=false,wait=1,dim=3,fill=1,cmm="Phi_"+(m));
}
plot(phi,value=false,wait=1,dim=3,fill=1,ps="imagen.eps");
{
ofstream fout("solution.txt");
fout << phi[];
}
```

Listing 6.4: C++ example