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ESTUDIO DE LOS SOLITONES DE LA ECUACIÓN DE KP-II.

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MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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ESTUDIO DE LOS SOLITONES DE LA ECUACIÓN DE KP-II.

Esta tesis está dedicada al estudio de las soluciones de tipo solitón y multi-solitón de la ecuación de KP-II construidas por Kodama. Estas se construyen a partir del Wronskiano de un conjunto de soluciones linealmente independientes de las primeras tres ecuaciones de la jerarquía de Burgers. A grandes rasgos, las soluciones construidas por Kodama pueden ser expresadas mediante un perfil y una fase, con fases definidas por medio de sumas de exponenciales.

Durante el desarrollo de este trabajo se comienza reesctructuando los términos de la ecuación estudiada por Kodama en función de los parámetros descritos para expresar las soluciones, perfil y fase, para luego reagruparlos de cierta forma que permita apreciar ciertas estructuras que se generan en las soluciones. Para lo anterior, se definen cuatro operadores que permiten caracterizar a las fases según los valores que se obtienen al evaluarlas en dichos operadores. Además se define un operador que permite escribir una ecuación diferencial sobre el perfil, el cuál permitirá definir las características que debe de satisfacer el perfil que genere las soluciones.

Posteriormente, se caracterizan tres tipos de soluciones. Estas son conocidas como soluciones de tipo solitón línea, multi-solitón resonante y 2-solitón. Las soluciones estudiadas en este trabajo utilizan un perfil fijo y los tipos de soluciones mencionados anteriormente se caracterizan por medio de las fases que se utilizan para construirlos. Este avance permite conocer de antemano el tipo de estructura que generarán ciertas fases al ser evaluadas bajo el perfil utilizado en el estudio, de igual forma es posible conocer la forma y características de las fases que construyen algún tipo de solución estudiado.

Para trabajos futuros, sería interesante estudiar de manera más profunda la relación que existe entre el perfil seleccionado y los tipos de soluciones que se generan. De igual manera, es de interés conocer si un par distinto de perfil-fase puede construir el mismo tipo de soluciones. Además, es de interés poder caracterizar otro tipo de soluciones más complejas, es decir que se construyen a través del Wronskiano de más soluciones linealmente independientes de las ecuaciones de la jerarquía de Burgers.

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This thesis is dedicated to the study of soliton and multi-soliton solutions of the KP-II equation constructed by Kodama. These are built from the Wronskian of a set of linearly independent solutions of the first three equations of the Burgers hierarchy. Broadly speaking, the solutions constructed by Kodama can be expressed in terms of a profile and a phase, with phases defined by sums of exponentials.

During the development of this work, the terms of the equation studied by Kodama are initially restructured in terms of the parameters described to express the solutions (profile and phase). Subsequently, they are regrouped in a certain manner to reveal certain structures that emerge in the solutions. For this purpose, four operators are defined to characterize the phases according to the values obtained by evaluating them on these operators. Furthermore, an operator is defined, which comprises a differential equation concerning the profile. This operator serves the purpose of delineating the specific characteristics that the profile must adhere to in order to engender the solutions.

Following this, three types of solutions are characterized: line-soliton solutions, resonant multi-soliton solutions, and 2-soliton solutions. The solutions examined in this study employ a fixed profile, with the mentioned solution types distinguished by the phases used in their construction. This approach facilitates the structural characteristics that certain phases will produce when evaluated under the profile employed in the study. Similarly, it enables an understanding of the form and attributes of the phases that compose the studied solution types.

For future endeavors, it would be compelling to conduct a more thorough examination of the relationship between the chosen profile and the resultant solution types. Similarly, exploring whether alternative profile-phase combinations can yield the same types of solutions is of considerable interest. Moreover, there remains an open question regarding the characterization of additional, more intricate solution types. Specifically, investigating solutions constructed through the Wronskian of a greater number of linear independent solutions from the Burgers hierarchy equations.

Para quienes crean que es imposible
y a quienes hicieron de esto posible.

Mucho éxito.

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Capítulo 1

Introduction

1.1. The KP models

The Kadomtsev-Petviashvili (KP) equation is recognized as one of the most versatile models within the realm of nonlinear wave theory. Initially, the objective was to formulate a model aimed at investigating the evolution of long, small-amplitude ion-acoustic waves propagating in plasmas. The derivation of the Kadomtsev-Petviashvili equation [13] was driven by the objective of investigating the stability of soliton solutions originating from the Korteweg-de Vries equation (KdV) when subjected to perturbations perpendicular to the direction of propagation. The KP equation falls within the category of dispersive equations. Such equations are characterized by their representation of phenomena where the propagation velocity of waves traveling within the medium varies according to their frequency.

The KP equation has been extensively studied by the mathematical community for the past approximately 40 years. For us, it will be given by

$$(-4u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0. \tag{1.1}$$

with $\sigma^2 = \pm 1$. This work primarily focuses on the study of the Kadomtsev-Petviashvili equation of type II (KP-II), mainly employed in fluid dynamics to model the behavior of fluids in shallow waters where surface tension dominates. For fluids where other forces predominate over surface tension, the KP-I equation is used. In the case where $\sigma^2 = -1$, one obtains the KP equation of type I (KP-I), whereas for $\sigma^2 = 1$, it is known as KP type II (KP-II). In the case of KP-I, one of the most well-known solutions is the lump-type equations, characterized by localized solutions in the plane. On the other hand, in the case of KP-II, one type of solution for this type of equation are the **soliton and multi-soliton solutions**, this are a particular type of structure assumed by waves as they propagate within the medium that carries them. They are characterized by maintaining a constant amplitude, provided there are no disturbances within the medium. Specifically, an N-soliton refers to a set of N line-solitons interacting with each other, thereby generating lattice patterns in the medium through which they propagate. The sign of σ^2 significantly alters the structure and algebraic properties of the equation, thereby highlighting the crucial distinction between one type of equation and the other.

KP equation serve as a fundamental model for developing and evaluating new mathematical techniques. These include variational theory for energy minimizer's existence and stability [48], addressing well-posedness in non-classical function spaces [46], and employing dynamical systems methods to study water waves [47]. It has been observed to have applications in various areas of physics, such as shallow water wave modeling [50], [49], [51], plasma physics [52], and even in fields such as string theory [57]. It has been also derived in works pertaining to dust acoustic waves in a hot dusty plasma [53]. Based on a solution of the KP equation obtained from a two-dimensional dusty plasma in the presence of two-temperature ions, the existence of compressive and rare dust acoustic solitons could be investigated [54]. Pakzad formulated the KP equation governing the dynamics of dust acoustic solitary waves in a warm dusty plasma characterized by varying dust charge, two-temperature ions, and non-thermal electrons [55]. Stability analysis revealed that solitons exhibit stability solely when the influences of dust and the dynamics of electron-ion interactions counterbalance each other. El-Shewy examined solitary wave solutions and energy characteristics within the framework of the KP equation in an unmagnetized plasma comprising both hot and cold dust species, alongside electrons and ions following Boltzmann distributions [56]. It was observed that plasma parameters significantly affect the width and amplitude of the soliton, as well as the energy of dust grains.

1.1.1. Solitons

Solitons are solitary, self-reinforcing waves that maintain their shape and velocity while propagating through a medium. The name was given by Norman Zabusky and Martin Kruskal [43, 44]. The first documented sighting of a soliton occurred in the Caledonian Canal in Scotland in 1834. Scottish engineer John Scott Russell observed this phenomenon while watching a cargo ship traveling along the canal. After the ship suddenly stopped, Scott Russell noticed a peculiar solitary wave moving up the canal without dissipating. He observed that these solitons maintained their shape and constant speed as they moved through the canal, isolated from other disturbances. His observations and experiments were detailed in an article titled “On the Wave of Translation of Bodies in Water", presented to the Royal Society of Edinburgh in 1844.

The principal characteristics associated to solitons are: stability, nonlinear nature, and interaction. Their remarkable stability enables solitons to propagate over considerable distances without dissipating or altering their shape. Within nonlinear systems, solitons arise as a result of the delicate balance between dispersion and nonlinearity, thus allowing the wave to maintain its integrity. Moreover, solitons possess the ability to interact with each other without losing their individuality. When two solitons collide, they emerge from the interaction unchanged, except for a phase shift.

The primary objective of this work is to investigate soliton type solutions constructed by Kodama [20] and to identify conditions that allow the characterization of certain structures arising within this class of solutions. In this way, it is possible to distinguish the patterns generated by each solution by verifying which conditions are satisfied.

We will provide a brief description of the types of solitons that are solutions of KP-II that will be studied throughout this work. Firstly, there is the line-soliton, which consists of a line in the $\mathbb{R}^2$ plane that moves as the time t varies. The next type of soliton consists of a line in the positive part of the y-axis that branches into different lines in the negative part of the y-axis. These different branches interact resonantly, generating a network pattern in a neighborhood of the x-axis. This neighborhood will shift along the xy-plane, outside of which the lines behave like line-solitons. We will refer to these solutions as resonant solutions.

Finally, we have the N-soliton solutions. These solutions consist of N line-solitons having non-resonant interactions between them. The patterns generated by this type of solution are much more complex, due to the fact that the number of interactions between distinct line-solitons is greater. Despite this, in regions of the plane far from the x-axis (i.e., $y \gg 0$ and $y \ll 0$), each of these branches will exhibit a behavior similar to a line-soliton.

1.1.2. Examples of KP solitons

Let us write

$$u(t,x,y) = 2\partial_x^2 F(\Theta(t,x,y)), \tag{1.2}$$

Let us recall the large family of KP solitons. The line-soliton family (see [20]) is given by

$$\Theta(t,x,y) = a_1 \exp(\theta_1) + a_2 \exp(\theta_2), \tag{1.3}$$

where $a_1, a_2 > 0$, and $\theta_j := k_j x + k_j^2 y + k_j^3 t$, $k_1, k_2 \in \mathbb{R}$. Assuming $F = \log$, the corresponding KP solution via (1.2) is given by

$$u(t,x,y) = \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2\left(\frac{1}{2}(\theta_1 - \theta_2)\right). \tag{1.4}$$

See Fig. 1.1 for details. The classical KdV soliton is recovered by setting $k_1 = -k_2 = k$, and in this case u becomes

$$Q_k(t,x,y) := 2k^2 \operatorname{sech}^2\left(kx + k^3 t\right). \tag{1.5}$$

The next case of KP solution is the resonant multi-soliton. This corresponds to the case

$$\Theta(t,x,y) = \sum_{i=1}^{M} a_i \exp(\theta_i) = \sum_{i=1}^{M} a_i \exp\left(k_i x + k_i^2 y + k_i^3 t\right), \tag{1.6}$$

where to ensure the positivity of Θ we impose that each $a_i > 0$ and $k_1 < k_2 < \cdots < k_M$.

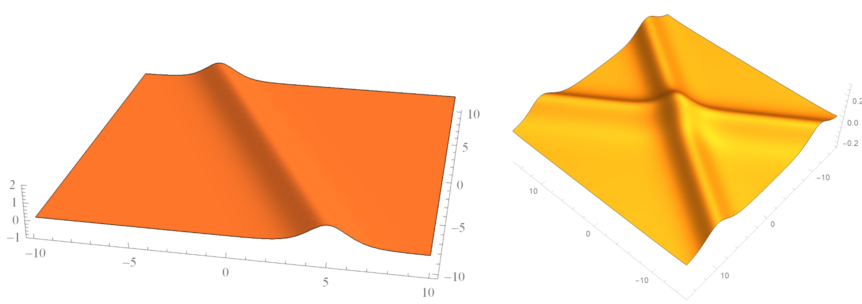


Figura 1.1: Left: One line-soliton solution (1.4) with $k_1 = -0.5$, $k_2 = 1$ and $t = 0$. This solution divides the plane into two regions, one for each exponential in the sum (1.3), and on each of these regions a different exponential dominates. Right: A 2-soliton of KP with $k_1 = -1$, $k_2 = -0.5$, $k_3 = 0.5$ and $k_4 = 1$, at time $t = 0$.

Now we recall the KP 2-soliton. In this case $\Theta = \mathrm{Wr}(\Theta_1, \Theta_2)$, where Wr is the Wronskian of two functions $\Theta_1 = \exp(\theta_1) + \exp(\theta_2)$ and $\Theta_2 = \exp(\theta_3) + \exp(\theta_4)$ being line-soliton phases. Calculating the phase Θ, one obtains the classical formula

$$\begin{aligned}\Theta = \ &(k_3-k_1)\exp(\theta_1+\theta_3)+(k_4-k_1)\exp(\theta_1+\theta_4)\\ &+(k_3-k_2)\exp(\theta_2+\theta_3)+(k_4-k_2)\exp(\theta_2+\theta_4)\,.\end{aligned} \tag{1.7}$$

In order to ensure the positivity and nondegeneracy of Θ, we require $0\le k_1<k_2<k_3<k_4$. See Fig. 1.1 for further details on the family of KP 2-solitons.

In general, a N-soliton is constructed by the profile $F=\log$ and the phase $\Theta=\mathrm{Wr}\,(\Theta_1,\ldots,\Theta_N)$, where $\Theta_i = a_i\exp\left(k_{2i-1}x+k_{2i-1}^2y+k_{2i-1}^3t\right)+b_i\exp\left(k_{2i}x+k_{2i}^2y+k_{2i}^3t\right)$, with $a_i,b_i,k_i\in\mathbb{R}$ and $i\in\{1,\ldots,N\}$. This is, N line-solitons.

1.1.3. Main new results

The main results of this work consist of the theorems that allow characterizing the different types of solutions of (1.1) that can be described in the form (1.2), with profile $F=\log$, based on conditions on the phases Θ. The phases studied correspond, broadly speaking, to sums of exponentials. We define

Definition 1.1 (Classification of phases Θ) *We shall say that Θ as in* (1.2)

(i) is of Airy type if for all $(t,x,y)\in\mathbb{R}^3$,

$$Ai\,(\Theta):=\Theta_t-\Theta_{xxx}=0.$$

(ii) Is of Heat type if for all $(t,x,y)\in\mathbb{R}^3$,

$$H\,(\Theta):=\Theta_y-\Theta_{xx}=0.$$

(iii) Is of x-Wronskian type and y-Wronskian type if $\Theta>0$, and

$$W_x\,(\Theta):=\Theta_{xxxx}-\frac{\Theta_{xx}^2}{\Theta}=0,\quad W_y\,(\Theta):=\Theta_{yy}-\frac{\Theta_y^2}{\Theta}=0, \tag{1.8}$$

respectively.

(iv) Is of $\mathcal{T}$-type if for F fixed,

$$\begin{aligned}\mathcal{T}\,(\Theta):= \ &-4F''(\Theta)\,Ai\,(\Theta)\,\Theta_x\\ &+F'(\Theta)\left(-4Ai\,(\Theta)_x+3\left(H\,(\Theta)_y+H\,(\Theta)_{xx}\right)\right)\\ &-3F'^2(\Theta)\,H\,(\Theta)\,(\Theta_y+\Theta_{xx})=0.\end{aligned} \tag{1.9}$$

The obtained results will be stated below. Assume

$$F\,(1)=0,\ F'\,(1)=1,\ F''\,(1)=-1,\text{ and } F'''\,(1)=2. \tag{1.10}$$

Our first result will be the following:

Theorem 1.1.1. *Let u be a smooth solution to KP* (1.1) *of the form* (1.2)*, with a smooth profile $F\,(\Theta)$ such that* (1.10) *hold. Then u is a KdV soliton and $F=\log$ if and only if $H\,(\Theta)=Ai\,(\Theta)=W_y\,(\Theta)=0$.*

This result allows characterizing solutions known as line-solitons. In particular, this Theorem 1.1.1 enables characterizing those line-solitons that are parallel to the y-axis, referred to as vertical line-solitons. This type of solutions is independent of the variable y and is also solutions of the KdV equation.

Theorem 1.1.2. *Let u be a smooth solution to (1.1) of the form (1.2), with a smooth profile F such that (1.10) is satisfied. Then u is an oblique line-soliton of the form (1.4)-(1.3) and $F = \log$ if and only if $H(\Theta) = Ai(\Theta) = 0$, and*

$$\Theta W_x(\Theta) = \Theta W_y(\Theta) = A(t,x)\exp(k(t,x)y), \tag{1.11}$$

for some particular functions $A > 0$, $k > 0$.

This theorem is a more general form of Theorem 1.1.1, as it allows to fully characterize all line-soliton type solutions.

Theorem 1.1.3 (Resonant multi-solitons)**.** *Let u be a solution of (1.1) of the form (1.2) with a smooth real-valued phase $\Theta > 0$ satisfying for $k = 0, 1, 2, 3$,*

$$\partial_x^k \Theta(t,0,0), \quad \partial_x^k \partial_y \Theta(t,0,0) \quad \textit{uniquely prescribed.} \tag{1.12}$$

Assume that the smooth profile F such that (1.10) holds. Then Θ corresponds to an M resonant multi-soliton (1.6) and $F = \log$ if and only if Θ satisfies $H(\Theta) = Ai(\Theta) = 0$ and $\Theta W_y(\Theta) = \Theta W_x(\Theta) \in \mathcal{W}_{\frac{1}{2}M(M-1)}$.

With this result, it is possible to characterize resonant solitons, structures that are slightly more complex than line-solitons but still share a common element with that type of solution, as both types of solutions are constructed from $\Theta = \sum_{j=1}^{M} a_i \exp\left(k_j x + k_j^2 y + k_j^3 t\right)$. This is a sum of exponentials with a linear argument.

Theorem 1.1.4 (2-solitons)**.** *Let u be a solution of (1.1) of the form (1.2) with a smooth real-valued phase $\Theta > 0$ satisfying (1.12). Assume the smooth profile $F = \log$. Then Θ corresponds to a 2-soliton (1.7) with $0 \leq k_1 < k_2 < k_3 < k_4$ if and only if $H(\Theta), Ai(\Theta)$ are contained in $\mathcal{W}_4$, $\Theta W_y(\Theta), \Theta W_x(\Theta) \in \mathcal{W}_5$, and $Ai(\Theta) = \frac{3}{2}\partial_x H(\Theta)$.*

This last result allows to characterize 2-soliton type solutions. This type of solutions consists of the interaction of two line-solitons. These are the most complex structures studied throughout this work, and the phase generating it under the profile $F = \log$ is a sum of the product of aforementioned exponentials.

1.1.4. Outline

Throughout this work, an analysis of KP solutions constructed by Kodama [20] will be done. To carry out this, the process begins by introducing and developing certain terms replaced in the form of KP, which will be studied throughout the text. Subsequently, the concepts necessary for its analysis are defined.

The study of solutions will be conducted by decomposing it into two terms, denoted as profile, F, and phase, Θ. Several symmetries exhibited by the studied solutions are illustrated, and the domains and ranges of the involved variables are defined. Subsequently, the types of phases that will be considered relevant are defined. It is worth noting that one of the definitions involves the relationship between the phase and the profile, adding a higher degree of complexity to their understanding. Next, the profile and phases of the different types of solutions that will be studied throughout this work are defined.

Capítulo 2

KP Solitons and the Grassmannian

This chapter resumes the main results in the Algebraic-Analytic Theory of KP models, and it is mainly inspired by the fundamental work by Kodama [20].

2.1. Introduction to KP Theory and KP Solitons

2.1.1. The Burgers Equation

Consider the Burgers equation for $w = w\,(t_2, x)$,

$$w_{t_2} = w_{xx} + 2ww_x = \frac{\partial}{\partial x}\left(w_x + w^2\right). \tag{2.1}$$

The Cole-Hopf Transform serves the purpose of linearization of (2.1) and corresponds to

$$w = \frac{\partial}{\partial x}\left(\ln\left(\Theta\right)\right) = \frac{\Theta_x}{\Theta}. \tag{2.2}$$

Replacing w obtained in (2.2) in the KP equation, one gets

$$\frac{\partial w}{\partial t_2} = \frac{\partial}{\partial t_2}\left(\frac{\Theta_x}{\Theta}\right) = \frac{\partial}{\partial x}\left(\frac{\Theta_{t_2}}{\Theta}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\Theta_x}{\Theta}\right) + \left(\frac{\Theta_x}{\Theta}\right)^2\right) = \frac{\partial}{\partial x}\left(\frac{\Theta_{xx}}{\Theta}\right).$$

Integrating the third and the fifth identity with respect to the variable x,

$$\Theta_{t_2} = \Theta_{xx} + c\,(t_2)\,\Theta.$$

One can choose $c\,(t_2) = 0$, so that

$$\Theta_{t_2} = \Theta_{xx}. \tag{2.3}$$

This means that Θ satisfies the linear diffusion equation.

The preceding outcome allows for the derivation of the following “Compatibility Relation",

Lemma 2.1.1 (Compatibility Relation)**.** *The following is satisfied*

$$\Theta_x = w\Theta \quad \textit{and} \quad \Theta_{t_2} = \Theta_{xx} \qquad \textit{if and only if} \qquad w_{t_2} = w_{xx} + 2ww_x.$$

This last result can be extended to consider a Burgers hierarchy.

2.1.2. Burgers Hierarchy

Let

$$\frac{\partial \Theta}{\partial t_n} := \frac{\partial^n \Theta}{\partial x^n} = \partial_x^n \Theta,$$

where $\{t_n : n = 1, 2, \ldots\}$ are "multi-time variables". It is possible to notice that, $\Theta_{t_i t_j} = \Theta_{t_j t_i}$. Thus, the **Burgers Hierarchy** corresponds to the **compatibility** between these equations and $\Theta_x = w\Theta$. Now, by defining $x = t_1$, $y = t_2$, and $t = t_3$, where x and y are spatial variables and t is a temporal variable, one gets the next result.

Theorem 2.1.1 (Construction of a KP solution)**.** *Suppose $\Theta = \Theta(t, x, y)$ is a function satisfying the initial three equations of the Burgers Hierarchy. It is possible to construct a solution to KP considering that if*

$$\Theta_x = w\Theta, \quad \Theta_y = \Theta_{xx} \quad \textit{and} \quad \Theta_t = \Theta_{xxx} \quad \textit{then} \quad u = 2w_x = 2\left(\ln\left(\Theta\right)\right)_{xx}.$$

In simpler terms, a solution for KP, denoted as u, can be constructed from a function that adheres to the three equations of the Burgers Hierarchy mentioned earlier.

Demostración. Firstly, the first two terms of the Burgers Hierarchy are computed,

$$\begin{aligned} w_y &= \left(w_x + w^2\right)_x, \\ w_t &= \left(w_{xx} + 3ww_x + w^3\right)_x. \end{aligned}$$

Compute $3w_{yy} - 4w_{xt}$,

$$3w_{yy} - 4w_{xt} = \left(-w_{xxx} - 6w_x^2\right)_x,$$

and then replace $u = 2w_x = 2\left(\ln\left(f\right)\right)_{xx}$ to obtain the KP equation,

$$\left(-4u_t + u_{xxx} + 6uu_x\right)_x + 3u_{yy} = 0.$$

The proof is complete. □

Given that the function f satisfies the linear equation in the variables y and t, it becomes feasible to express the solution in terms of the Fourier transform,

$$\Theta = \int_{C(\kappa)} \exp\left(kx + k^2 y + k^3 t\right) \mathrm{d}\mu\left(\kappa\right),$$

with $\kappa \in \mathbb{C}$ the parameter that describes the curve C and is evaluated in the measure $\mathrm{d}\mu$. Considering the particular finite-dimensional case with measure

$$\mathrm{d}\mu\left(\kappa\right) = \sum_{j=1}^{M} \rho_j \delta\left(\kappa - k_j\right). \tag{2.4}$$

When considering measure (2.4) with parameters ρ_j, $k_j \in \mathbb{R}$ one obtains

$$\Theta\left(t, x, y\right) = \sum_{j=1}^{M} \rho_j E_j\left(t, x, y\right), \tag{2.5}$$

with $E_j := \exp\left(k_j x + k_j^2 y + k_j^3 t\right)$.

This type of solution is commonly referred to as a "multi-shock" solution of the Burgers equation. Further details regarding the properties of these solutions and their relationship with KP are elucidated below.

2.1.3. Shock Solution and Line-Soliton Solution

In this subsection, the link between a specific solution of the Burgers equation and a solution (2.5) of the KP equation is demonstrated. Initially, (2.5) is selected with $M = 2$. The condition $\rho_i \rho_j > 0$ is enforced to achieve a regular solution, and without loss of generality, $\rho_i > 0$ is chosen. Consequently,

$$\begin{aligned}
\Theta &= \rho_1 E_1 + \rho_2 E_2 \\
&= \exp(\theta_1) + \exp(\theta_2) \\
&= \exp\left(\frac{1}{2}(\theta_1 + \theta_2)\right)\left(\exp\left(\frac{1}{2}(\theta_1 - \theta_2)\right) + \exp\left(-\frac{1}{2}(\theta_1 - \theta_2)\right)\right) \\
&= 2\exp\left(\frac{1}{2}(\theta_1 + \theta_2)\right)\cosh\left(\frac{1}{2}(\theta_1 - \theta_2)\right),
\end{aligned}$$

where $\theta_i = k_i x + k_i^2 y + k_i^3 t + \ln(\rho_i)$. Therefore,

$$w = (\ln(\Theta))_x = \frac{1}{2}(k_1 + k_2) + \frac{1}{2}(k_1 - k_2)\tanh\left(\frac{1}{2}(\theta_1 - \theta_2)\right).$$

This is, a shock solution of the Burgers equation. When $k_1 < k_2$ the solution behaves as follows

$$w(t,x,y) \longrightarrow \begin{cases} k_1, & \text{if } x \ll 0. \\ k_3, & \text{if } x \gg 0. \end{cases}$$

That is, the function is increasing in x for a fixed pair $(x, y) \in \mathbb{R}^2$. This monotonicity is maintained in all the regular solutions studied in this work. Constructing a solution for KP, through the equation $u = 2w_x = 2(\ln(\Theta))_{xx}$,

$$u = \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2\left(\frac{1}{2}(\theta_1 - \theta_2)\right). \tag{2.6}$$

This solution partitions the space into two regions, each corresponding to one exponential term in the sum. Within each of these regions, a single exponential dominates.

The line occurs in the region where $\theta_1 = \theta_2$. In this case, since the exponentials correspond to $E_1 = \exp(\theta_1)$ and $E_2 = \exp(\theta_2)$, this line it is called $[1, 2]$-soliton.

Note that $\theta_1 = \theta_2$ is equivalent to saying that the variables (t, x, y) satisfy

$$x + (k_1 + k_2)y + \left(k_1^2 + k_2^2 + k_1 k_2\right)t + \frac{1}{k_1 - k_2}\ln\left(\frac{\rho_1}{\rho_2}\right) = 0.$$

In general, these structures are called $[i, j]$-soliton and are the result of the intersection of the corresponding exponentials E_i, E_j. Each $[i, j]$-soliton has the same local structure as a

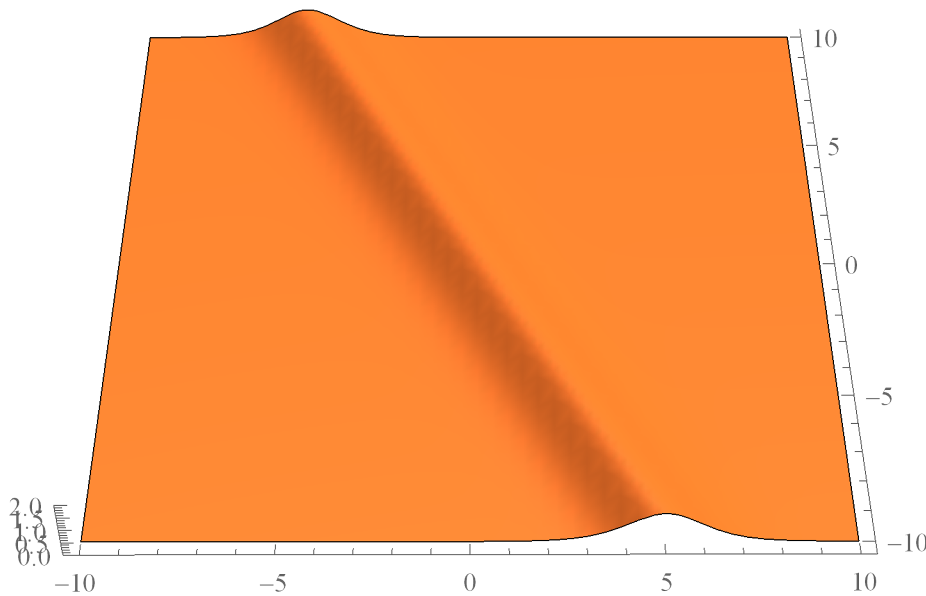


Figura 2.1: A line-soliton solution.

line-soliton, which is described by

$$u = A_{[i,j]} \operatorname{sech}^2 \left(\frac{1}{2} \left(K_{[i,j]} \cdot x - \Omega_{[i,j]} t + \Theta^0_{[i,j]} \right) \right),$$

with $\Theta^0_{i,j}$ a constant. The parameters $A_{[i,j]}$, $K_{[i,j]}$ and $\Omega_{[i,j]}$ are known by amplitude, wave-vector and frequency, respectively. These parameters are defined as

$$\begin{aligned} A_{[i,j]} &= \frac{1}{2} \left(k_j - k_i \right)^2, \\ K_{[i,j]} &= \left(k_j - k_i, k_j^2 - k_i^2 \right) = \left(k_j - k_i \right) \left(1, k_j + k_i \right), \\ \Omega_{[i,j]} &= - \left(k_j^3 - k_i^3 \right) = - \left(k_j - k_i \right) \left(k_i^2 + k_i k_j + k_j^2 \right). \end{aligned}$$

The direction of the wave vector $K_{[i,j]} = \left(K^x_{[i,j]}, K^y_{[i,j]} \right)$ is measured counter-clockwise and from the y-axis. This direction is given by

$$\frac{K^x_{[i,j]}}{K^y_{[i,j]}} = \tan \left(\Psi_{[i,j]} \right) = k_i + k_j.$$

Thus, $\Psi_{[i,j]}$ corresponds to the angle between $K \cdot x = a$ and the y-axis. Here, $a \in \mathbb{R}$ is an arbitrary constant. In this way, a line soliton can be written as a function of three parameters $A_{[i,j]}$, $\Psi_{[i,j]}$ and $x^0_{[i,j]}$, this is

$$u = A_{[i,j]} \operatorname{sech}^2 \left(\sqrt{\frac{A_{[i,j]}}{2}} \left(x + \tan \left(\Psi_{[i,j]} \right) y + C_{[i,j]} t - x^0_{[i,j]} \right) \right),$$

with, $C_{[i,j]} = k_i^2 + k_i k_j + k_j^2 = \frac{1}{2} A_{[i,j]} + \frac{3}{4} \tan^2 \left(\Psi_{[i,j]} \right)$. It is possible to note that the wave-vector,

$K_{[i,j]}$ and the frecuency $\Omega_{[i,j]}$ satisfy the soliton-dispersion relation,

$$-4\Omega_{[i,j]}K^x_{[i,j]} = \left(K^x_{[i,j]}\right)^4 + 3\left(K^y_{[i,j]}\right)^2, \tag{2.7}$$

which gives the relationship between the parameters $\left(K^x_{[i,j]}, K^y_{[i,j]}, \Omega_{[i,j]}\right)$ for the waveform $u = \exp\left(K^x_{[i,j]}x + K^y_{[i,j]}y - \Omega_{[i,j]}t\right)$ of the linear part of KP.

The velocity of the soliton $V_{[i,j]}$ travels in the direction of the wave vector, $K_{[i,j]}$ and is defined by, $K_{[i,j]} \cdot V_{[i,j]} = \Omega_{[i,j]}$, which is

$$V_{[i,j]} = \frac{\Omega_{[i,j]}}{\left|K_{[i,j]}\right|^2} = -\frac{k_i^2 + k_i k_j + k_j^2}{1 + (k_i + k_j)^2}\left(1, k_i + k_j\right).$$

Notice that $C_{[i,j]} = k_i^2 + k_i k_j + k_j^2 > 0$, and therefore the x component of the velocity is always negative, so every soliton will move towards the negative side on the x-axis.

2.1.4. Confluence of Shocks: Resonant Interaction of Line-Solitons

The well-established knowledge dictates that the collision of shocks results in the generation of a shock with an amplified amplitude. This phenomenon becomes evident in the evolution along the y-axis of a solution of the Burgers equation. Now consider the solution $w = (\ln(\Theta))_x$ with $M = 3$ in (2.5) and constants $\rho_i > 0$, with $i \in \{1, 2, 3\}$. This is,

$$\Theta = \rho_1 E_1 + \rho_2 E_2 + \rho_3 E_3.$$

Similar to the previous scenario, it is feasible to identify the dominant exponentials and analyze the structure of the solution in the xy plane.

Consider the solution f on the line $x + cy = 0$ with $c = \tan(\Psi)$. On this line the dominant exponential is $E_j = \exp\left(\eta_j(c)\, y + k_j^3 t\right)$, with $\eta_j(c) = k_j(k_j - c)$ and $E_j = \exp\left(k_j x + k_j^2 y + k_j^3 t\right)$.

Figure 2.2 illustrates that for $y \gg 0$ and for a fixed t, the exponential E_1 prevails when $c \gg 0$. As the value of c decreases (ie, rotating the line clockwise) the dominant exponential shifts to E_3. In summary,

$$w = \partial_x \ln(\Theta) \longrightarrow \begin{cases} k_1, & \text{if } x \ll 0. \\ k_2, & \text{if } x \gg 0. \end{cases}$$

The transition between dominant exponentials $E_1 \to E_3$ is marked by the condition $\eta_1 = \eta_3$, which corresponds to $c = \tan\left(\Psi_{[1,3]}\right) = k_1 + k_3$. Around this line the next approximation can be made,

$$\Theta \approx \rho_1 E_1 + \rho_3 E_3.$$

This implies the presence of a $[1,3]$-soliton on $y \gg 0$. The coefficient $\frac{\rho_3}{\rho_1}$ can be utilized to designate a particular location for the soliton.

In the scenario where $y \ll 0$, the minimum value of η_j is selected for a specific value of c. For $c \gg 0$, ($\Psi \approx \frac{\pi}{2}$, i.e. $x \gg 0$), E_3 is the dominant term. As the value of c decreases (rotating the line $x = -cy$ clockwise), the dominant term becomes E_2 when $k_2 + k_3 > c > k_1 + k_2$ and

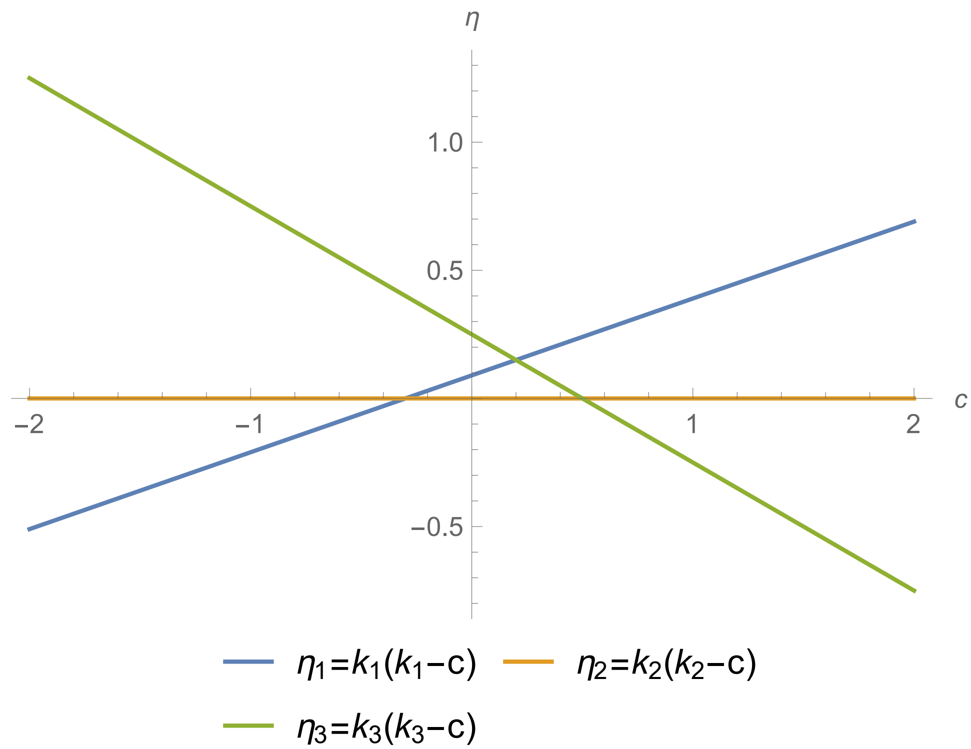


Figura 2.2: The graphs of $\eta_j(c) = k_j(k_j - c)$. The k-parameters are specified as $(k_1, k_2, k_3) = (-0.3, 0, 0.5)$.

E_1 becomes dominant when $c \ll k_1 + k_2$. Therefore, for $y \ll 0$ we have

$$w \longrightarrow \begin{cases} k_1, & \text{if } x \ll 0. \\ k_2, & \text{if } -(k_1+k_2)\,y \ll x \ll -(k_2+k_3)\,y. \\ k_3, & \text{if } x \gg 0. \end{cases}$$

The transition between dominant exponentials is given by $E_1 \longrightarrow E_2 \longrightarrow E_3$ as x increases.

In the proximity of these regions, the following approximations are valid,

$$\Theta \approx \rho_1 E_1 + \rho_2 E_2 \quad \text{and} \quad \Theta \approx \rho_2 E_2 + \rho_3 E_3,$$

which correspond to a $[1,2]$-soliton and a $[2,3]$-soliton, respectively. These solitons are in the $y \ll 0$ region.

Figure 2.3 illustrates the configuration of these solitons constructed from $\Theta = \rho_1 E_1 + \rho_2 E_2 + \rho_3 E_3$ when $t = 0$ and $\rho_i = 1$. This final condition ensures that the three line-solitons converge at the origin when $t = 0$.

It is also evident that the line-soliton at $y \gg 0$, corresponding to the $[1,3]$-soliton, aligns with the phase transition $x + c_{[1,3]}y = b$, with a direction parameter $c_{[1,3]} = k_1 + k_3$ and $b \in \mathbb{R}$ an arbitrary constant. Similarly, the line-soliton situated at $y \ll 0$, corresponding to the $[1,2]$-soliton, and $[2,3]$-soliton are positioned over their respective phase transitions with direction parameters $c_{[1,2]} = k_1 + k_2$ and $c_{[2,3]} = k_2 + k_3$, respectively. This solution represents a resonant interaction of three line-solitons. The solution to the Burgers equation in the y

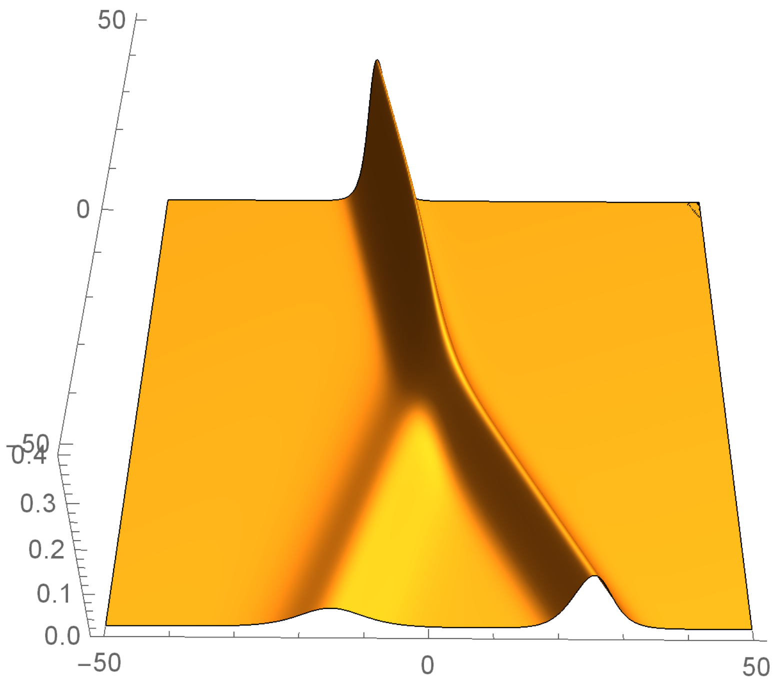


Figura 2.3: A Y-soliton. The boundaries of adjacent regions indicate the line-solitons, denoting the transition of the dominant terms . The k-parameters are the same as those in 2.2, and the line-solitons are determined from the intersection points of the $\eta_j(c)$'s in 2.2. In this case, all $\rho_i = 1$ (i.e. $\Theta = E_1 + E_2 + E_3$) ensuring that the three solitons converge at the origin at $t = 0$.

direction corresponds to the confluence of two shocks. The resonance condition for these three line-solitons is

$$K_{[1,3]} = K_{[1,2]} + K_{[2,3]}, \quad \Omega_{[1,3]} = \Omega_{[1,2]} + \Omega_{[2,3]},$$

and both satisfy with $K_{[i,j]} = \left(k_j - k_i, k_j^2 - k_i^2\right)$ and $\Omega_{[i,j]} = -\left(k_j^3 - k_i^3\right)$.

In a simple way, it is feasible to extend the earlier result for a solution constructed from an Θ with an arbitrary number of exponentials.

Theorem 2.1.2 (Kodama [20])**.** *If*

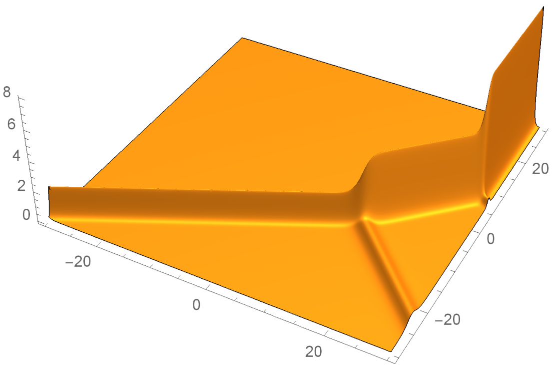
$$\Theta = \rho_1 E_1 + \rho_2 E_2 + \ldots + \rho_M E_M,$$

with $\rho_j > 0$ for $j = 1, 2, \ldots, M$. Then, the solution u exhibits the following asymptotic characteristics,

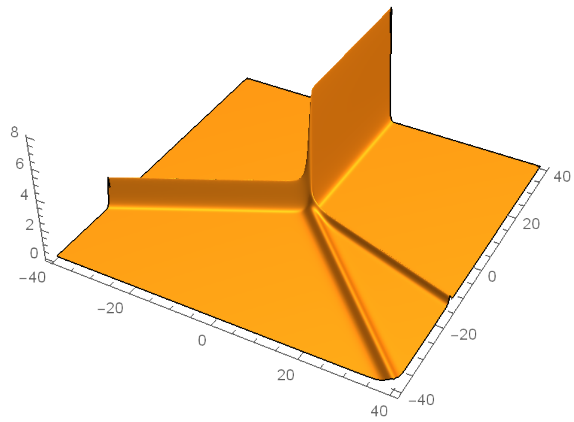
(i) for $y \gg 0$, there is only one line-soliton of the form $[1, M]$-soliton.

(ii) For $y \ll 0$, there are $M - 1$ line-solitons of the form $[k, k+1]$ with $k = 1, 2, \ldots, M - 1$, positioned in a counter-clockwise manner, spanning from the negative section to the positive part of the x-axis.

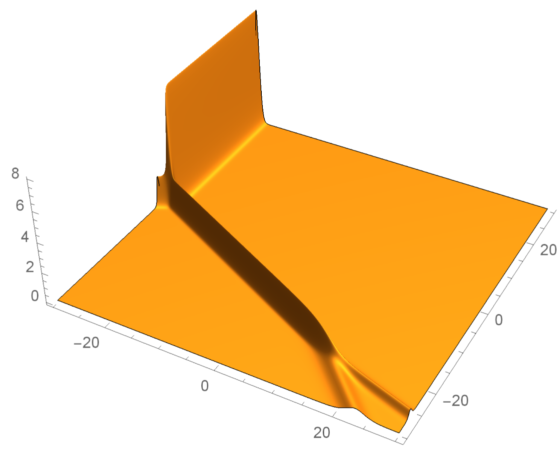
Figure 2.4 shows the evolution of a soliton solution for the case $M = 4$ and $\Theta = E_1 + E_2 + E_3 + E_4$.



(a) t=-7



(b) t=0



(c) t=7

Figura 2.4: In the left figure ($t = -7$) the finite line in the middle corresponds to a soliton of type $[1, 3]$-soliton and interacts resonantly with the solitons $[3, 4]$ and $[1, 4]$ at $y \gg 0$ and with $[1, 2]$ and $[2, 3]$ at $y \ll 0$. In the right figure ($t = 7$) the finite line in the middle corresponds to a $[2, 4]$-soliton and again there are two sets of resonant interactions with this soliton. It is noteworthy that the four exponentials divide the space into four regions, where each region is dominated by a different exponential.

2.1.5. The multi-component Burgers equation and the τ-function

We present an extension to an equation of N-components and higher order,

$$\partial_x^N \Theta = w_1 \partial_x^{N-1} \Theta + w_2 \partial_x^{N-2} \Theta + \ldots + w_N \Theta = \sum_{k=1}^{N} w_k \partial^{N-k} \Theta. \tag{2.8}$$

Expressed in matrix form,

$$\begin{pmatrix} \Theta_1^{(N)} \\ \Theta_2^{(N)} \\ \vdots \\ \Theta_N^{(N)} \end{pmatrix} = \begin{pmatrix} \Theta_1 & \Theta_1^{(1)} & \dots & \Theta_1^{(N-1)} \\ \Theta_2 & \Theta_2^{(1)} & \dots & \Theta_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \Theta_N & \Theta_N^{(1)} & \dots & \Theta_N^{(N-1)} \end{pmatrix} \begin{pmatrix} w_N \\ w_{N-1} \\ \vdots \\ w_1 \end{pmatrix}. \tag{2.9}$$

The functions Θ_i with $i \in [1, \dots, N]$ correspond to a fundamental set of solutions of the N-th order differential equation (2.8) and $\Theta_j^{(n)} := \partial_x^n \Theta_j$. Additionally, each function Θ_j is assumed to satisy the linear equations,

$$\partial_y = \partial_x^2 \Theta, \quad \partial_t \Theta = \partial_x^3 \Theta, \quad \text{for } j = 1, \dots, N.$$

The transformation between $\Theta_x = w\Theta$ and the multi-components $\{w_1, \dots, w_N\}$ correspond to the extension of the Cole-Hopf transformation. Consequently, the set of w_i satisfies a multi-component extension of the Burgers equation in the y-derivative. The additional flows with respect to "times"t_n for $n \geq 3$ characterize the symmetries of this interconnected system, forming an N-component Burgers hierarchy.

Example 2.1.1 (2-component Burger equation to (w_1, w_2))**.** *In this case,*

$$\begin{aligned} \partial_y w_1 &= 2w_1 \partial_x w_1 + \partial_x^2 w_1 + 2\partial_x w_2, \\ \partial_y w_2 &= 2w_2 \partial_x w_1 + \partial_x^2 w_2. \end{aligned}$$

It is important to note that if $w_2 = 0$, the system is reduced to the Burgers equation. By applying Cramer's rule to (2.9)*, it becomes possible to determine the values of w_j's. In particular, the function w_1 can be written in the form,*

$$w_1 = \partial_x \ln \left(\mathrm{Wr} \left(\Theta_1, \dots, \Theta_N \right) \right),$$

where $\mathrm{Wr}(\Theta_1, \dots, \Theta_N)$ *is the Wronskian with respect to the variable x. This particular Wronskian is referred to as a τ-function of KP.*

$$\tau := \mathrm{Wr} \left(\Theta_1, \dots, \Theta_N \right) = \begin{vmatrix} \Theta_1 & \Theta_1^{(1)} & \dots & \Theta_1^{(N-1)} \\ \Theta_2 & \Theta_2^{(1)} & \dots & \Theta_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \Theta_N & \Theta_N^{(1)} & \dots & \Theta_N^{(N-1)} \end{vmatrix}. \tag{2.10}$$

One has the following classical result.

Theorem 2.1.3 (τ function and KP solution)**.** *The τ-function in* (2.10) *provides a solution to the KP equation. In other words, the function*

$$u(t, x, y) = 2\partial_x^2 \ln \left(\tau (t, x, y) \right), \tag{2.11}$$

is a solution of the KP equation.

Demostración. The proof consists, roughly, of three steps: Firstly, replacing equation (2.11) in KP; secondly, expressing the terms in algebraic notation; and finally, utilizing the Plücker relation to draw conclusions.

First, inserting (2.1.3) into the KP equation, it is possible to see that if the τ-function satisfies the following bilinear form,

$$-4\left(\tau\tau_{xt}-\tau_x\tau_t\right)+\left(\tau\tau_{xxxx}-4\tau_x\tau_{xxx}+3\tau_y^2\right)+3\left(\tau\tau_{yy}-\tau_y^2\right)=0. \tag{2.12}$$

then u satisfies the KP equation. Consequently, show that (2.12) is nothing but an algebraic relation among the determinants, called a Plücker relation (see Section 2.1.6). To show this, let us express the τ-function in a symbolic form,

$$\tau=\mathrm{Wr}\left(\Theta_1,\ldots,\Theta_N\right)=:\left[0,1,\ldots,N-1\right],$$

where the numbers represent the derivatives of the functions Θ_i. In general, define $[l_0, l_1, \ldots, l_{N-1}]$ as

$$[l_0,l_1,\ldots,l_{N-1}]:=\begin{vmatrix}\Theta_1^{(l_0)} & \Theta_1^{(l_1)} & \ldots & \Theta_1^{(l_{N-1})}\\ \Theta_2^{(l_0)} & \Theta_2^{(l_1)} & \ldots & \Theta_2^{(l_{N-1})}\\ \vdots & & \ddots & \vdots\\ \Theta_N^{(l_0)} & \Theta_N^{(l_1)} & \ldots & \Theta_N^{(l_{N-1})}\end{vmatrix}, \tag{2.13}$$

where it is not assume the ordering $0\le l_0<l_1<\ldots<l_{N-1}$. The expression on the left can be viewed as an algebraic symbol representing the determinant of N vectors in $\mathbb{R}^\infty$. With this notation, the derivatives of the τ-function can be expressed as,

$$\begin{aligned}\tau_x&=[0,1,\ldots,N-2,N],\\ \tau_y&=[0,1,\ldots,N-3,N,N-1]+[0,1,\ldots,N-2,N+1]\\ &=-[0,1,\ldots,N-3,N-1,N]+[0,1,\ldots,N-2,N+1],\\ \tau_{xx}&=[0,1,\ldots,N-3,N-1,N]+[0,1,\ldots,N-2,N+1],\quad\ldots\end{aligned}$$

Note that, since $\partial_{t_n}f=\partial_x^n f$, the derivative ∂_{t_n} shifts each index l_i to l_{i+n}. Consequently, (2.12) can be expressed in the following form

$$\begin{aligned}&[0,1,\ldots,N-2,N-1]\,[0,1,\ldots,N-3,N,N+1]\\ &-[0,1,\ldots,N-2,N]\,[0,1,\ldots,N-3,N-1,N+1]\\ &\quad+[0,1,\ldots,N-2,N+1]\,[0,1,\ldots,N-3,N-1,N]=0.\end{aligned}$$

The proof concludes using Proposition 2.1.4. □

Some remarks are necessary.

Remark 2.1.1. *The derivatives above can be calculated considering the next rules,*

1. *t_n, The subscript n indicate how much to add to the terms.*
2. *The number of derivatives indicate to how many terms should be added in the rightest term.*
3. *The sum of subscripts indicate how many blocks will be at the end of the proces.*
4. *The sum start in the last term on the left.*

5. The sum in the blocks it is not on the same term, is made on the left-hand side.

The primary construction of the τ-function in a Wronskian form presented here is grounded in the Sato theory.

2.1.6. The Plücker Relations and the τ-Function.

Among the symbols $[l_0, l_1, \ldots, l_{N-1}]$ defined by (2.13), it has the following bilinear relations, known as the Plücker relations.

Theorem 2.1.4 (Plücker relations). *Let $\{\alpha_0, \alpha_1, \ldots, \alpha_{N-2}\}$ and $\{\beta_0, \beta_1, \ldots, \beta_N\}$ be a pair of two sets of nonnegative integers, α_i, $\beta_j \in \mathbb{Z}_{\geq 0}$. Then the symbols $[\alpha_0, \ldots, \alpha_{N-2}, \beta_n]$ and $\left[\beta_0, \ldots, \hat{\beta}_n, \ldots, \beta_N\right]$ for $n = 0, 1, \ldots, N$ satisfy the following Plücker relation,*

$$\sum_{n=0}^{N} (-1)^n [\alpha_0, \ldots, \alpha_{N-2}, \beta_N] [\beta_0, \ldots, \hat{\beta}_n, \ldots, \beta_N] = 0.$$

Here, $\left[\beta_0, \ldots, \hat{\beta}_n, \ldots, \beta_N\right]$ is the ordered set of $\{\beta_0, \ldots, \beta_N\}$ without β_n.

Demostración. Due to the fact that the right half of the following matrix contains a repeated block of size $N \times N$, its determinant will be equal to 0. Then, employing the Laplace expansion

$$\begin{vmatrix} \Theta_1^{(\alpha_0)} & \ldots & \Theta_1^{(\alpha_{N-2})} & \Theta_1^{(\beta_0)} & \ldots & \Theta_1^{\beta_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \Theta_N^{(\alpha_0)} & \ldots & \Theta_N^{(\alpha_{N-2})} & \Theta_N^{(\beta_0)} & \ldots & \Theta_N^{\beta_N} \\ 0 & \ldots & 0 & \Theta_1^{(\beta_0)} & \ldots & \Theta_1^{(\beta_N)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \Theta_N^{(\beta_0)} & \ldots & \Theta_N^{(\beta_N)} \end{vmatrix}.$$

□

It can be observed that by choosing

$$\alpha = \{0, 1, \ldots, N-2\} \quad \text{and} \quad \beta = \{0, 1, \ldots, N-3, N-1, N, N+1\},$$

the equation (2.12) is derived. Then, by Theorem 2.1.3, (2.10) gives a particular solution of the KP equation.

Consequently, Theorem 2.1.3 shows that the Wronskian determinant (2.10) gives a particular solution of the KP equation. From now on, it is consider finite-dimensional solutions and the functions $(\Theta_1, \ldots, \Theta_N)$ in (2.9), will be chosen in the following manner

$$\Theta_i(t, x, y) = \sum_{j=1}^{M} a_{ij} \exp(\theta_j(t, x, y)) \quad \text{with} \quad \theta_j(t, x, y) = k_j x + k_j^2 y + k_j^3 t, \tag{2.14}$$

Here, $A := (a_{ij})$ is an $N \times M$ matrix. Thus, each KP soliton expressed in the form $u = 2\partial_x^2 \ln(\mathrm{Wr}(\Theta_1, \ldots, \Theta_N))$ with (2.14) parametrized by an $N \times M$ matrix A and M parameters $(k_1, \ldots, k_M)$.

Definition 2.1 (Grassmannian) *The matrix A will be identified as a point of the real Grassmannian* $\mathrm{Gr}(N, M)$*, which is defined as the set of all N-dimensional subspaces spanned by the row vectors of A.*

In $\mathrm{Gr}(N, M)$, N is the number of lineal independents functions Θ_i and M is the maximum number of exponentials among all the Θ_i.

Example 2.1.2. *Consider the case with $N = 2$ and $M = 3$. Taking the linearly independent functions Θ_1 and Θ_2 in the form,*

$$\Theta_i = \sum_{j=1}^{3} a_{ij} E_j, \quad i = 1, 2.$$

In this case the τ-function in (2.10) *can be explicitly given by*

$$\tau = \begin{vmatrix} \Theta_1 \Theta_1^{(1)} \\ \Theta_2 \Theta_2^{(1)} \end{vmatrix} = \left| \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} E_1 & k_1 E_1 \\ E_2 & k_2 E_2 \\ E_3 & k_3 E_3 \end{pmatrix} \right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} E_{1,2} + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} E_{1,3} + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} E_{2,3},$$

where $E_{i,j} = \mathrm{Wr}(E_i, E_j) = (k_j - k_i) E_i E_j$. Consider an example with

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -b \\ 0 & 1 & a \end{pmatrix}. \tag{2.15}$$

where it is assumed a,$b > 0$. The τ-function is then given by

$$\tau = E_{1,2} + aE_{1,3} + bE_{2,3}.$$

To conduct the asymptotic analysis effectively, one must account for dominance within the set $\{\eta_{i,j} = \eta_i + \eta_j : 1 \leq i < j \leq 3\}$. This analysis remains feasible with the aid of Fig. 2.2. Specifically, for y values exceeding 0*, the transitions of the dominant exponentials adhere to the subsequent transition scheme,*

$$E_{1,2} \to E_{1,3} \to E_{2,3},$$

as c varies from large positive (i.e. x tends to negative infinity) to large negative values (i.e. x tends to positive infinity). The $[2, 3]$*-soliton solution is formed in the boundary between the regions in which dominate the exponentials $E_{1,2}$ and $E_{1,3}$. In this region the τ-function can be approximated as*

$$\begin{aligned} \tau &\approx E_{1,2} + aE_{1,3} \\ &= (k_2 - k_1) E_1 \left(E_2 + a \frac{k_3 - k_1}{k_2 - k_1} E_3 \right) \\ &= 2 (k_2 - k_1) E_1 \exp\left(\frac{1}{2} (\theta_2 + \theta_2 - \theta_{23}) \right) \cosh\left(\frac{1}{2} \left(\theta_2 - \theta_3 + \theta_{23}^0 \right) \right), \end{aligned}$$

and then

$$u = 2\partial_x^2 \ln(\tau) \approx \frac{1}{2} (k_2 - k_3)^2 \operatorname{sech}^2 \left(\frac{1}{2} \left(\theta_2 - \theta_3 + \theta_{23}^0 \right) \right),$$

where the constant θ_{23}^0 *is given by*

$$\theta_{23}^0 = \ln\left(\frac{k_2 - k_1}{k_3 - k_1}\right) - \ln(a) \quad i.e. \quad a = \frac{k_2 - k_1}{k_3 - k_1}\exp\left(-\theta_{23}^0\right).$$

The constant a *can be used to determine the location of this soliton. Similarly, at the boundary between* $E_{1,3}$ *and* $E_{2,3}$*,*

$$\tau \approx 2(k_3 - k_1)\, aE_3 \exp\left(\frac{1}{2}\left(\theta_1 + \theta_2 - \theta_{12}^0\right)\right)\cosh\left(\frac{1}{2}\left(\theta_1 - \theta_2 + \theta_{12}^0\right)\right),$$
$$u \approx \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2\left(\frac{1}{2}\left(\theta_1 - \theta_2 + \theta_{12}^0\right)\right),$$

where θ_{12}^0 *is given by*

$$\theta_{12}^0 = \ln\left(\frac{k_3 - k_1}{k_3 - k_2}\right) - \ln\left(\frac{b}{a}\right) \quad i.e. \quad b = \frac{k_2 - k_1}{k_3 - k_2}\exp\left(-\theta_{12}^0\right).$$

This gives $[1,2]$*-soliton and its location is determined by the constant* b *and the parameters* k_j*.*

For y *smaller than* 0*, there is only one transition, namely*

$$E_{2,3} \to E_{1,2},$$

as c *varies from large positive values (i.e.* x *tends to infinity) to large negative values (i.e.* x *tends to negative infinity). In this case, a* $[1,3]$*-soliton is formed for* y *smaller than 0 at the boundary of the dominant exponentials* $E_{2,3}$ *and* $E_{1,2}$*. In a similar computation as the previous cases, the soliton solution of this* $[1,3]$*-type is given by*

$$u \approx \frac{1}{2}(k_1 - k_3)^2 \operatorname{sech}^2\left(\frac{1}{2}\left(\theta_1 - \theta_3 + \theta_{13}^0\right)\right),$$

where the phase constant θ_{13}^0 *is given by*

$$\theta_{13}^0 = \ln\left(\frac{k_2 - k_1}{k_3 - k_2}\right) - \ln(b).$$

The phase constants θ_{12}^0*,* θ_{23}^0 *and* θ_{13}^0 *satisfy the resonant relation,*

$$\theta_{13}^0 = \theta_{12}^0 + \theta_{23}^0,$$

which implies that those three solitons intersect a point on the xy-plane. Note that Figure 2.5 can be obtained from Fig. 3.2 by changing (x, y) *to* $(-x, -y)$*.*

Remark 2.1.2. *Figures 3.2 and 2.5 show two types of the resonant solutions with the same set of three line-solitons of* $[1,2]$*-,* $[2,3]$*- and* $[1,3]$*-types. These solutions are generated from the different sizes of the matrix* A *with* $N = 1$ *and* $N = 2$ *with the same* $M = 3$*. In particular, note that the index set of the dominant exponentials around the intersection point is given by* $\{1,2,3\}$ *for the solution in Fig. 3.2 and* $\{12,13,23\}$ *for that in Fig. 2.5.*

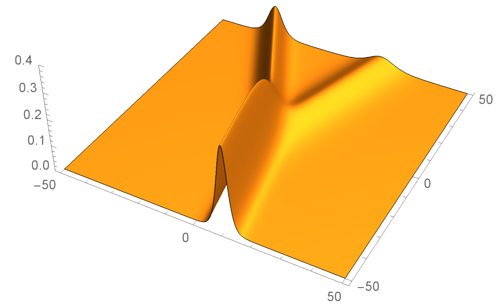


Figura 2.5: Y-soliton for the case with $N = 2$ and $M = 3$. The k-parameters are the same as those in Fig. 3.2. The parameters in the matrix A of (2.15) are chosen as the three line-solitons meet at the origin at $t = 0$.

2.1.7. The Schur Polynomials and the τ-Function

This section serves as an introduction to the Schur polynomials associated with the time variables $t = (t_1, t_2, \ldots)$ and their corresponding Young diagrams. The symbol $[l_0, l_1, \ldots, l_{N-1}]$ is interpreted as a distinct combination of derivatives of the τ-function expressed in terms of the Schur polynomial. Let us define the Schur polynomial for the index set $\{l_0, l_1, \ldots, l_{N-1}\}$,

Definition 2.2 *Let $p_l(t)$ be the polynomials generated by the identity*

$$\exp\left(\sum_{n=1}^{\infty} k^n t_n\right) = \sum_{l=0}^{\infty} k^l p_l(t), \tag{2.16}$$

where $t = (t_1, t_2, \ldots)$. These polynomials are sometimes referred to as the elementary Schur polynomials, and they are expressed as

$$p_l(t) = \sum_{n_1+2n_2+\ldots+\ell n_\ell=\ell} \frac{t_1^{n_1} \cdot \ldots \cdot t_\ell^{n_\ell}}{n_1! \cdot \ldots \cdot n_\ell!}.$$

Set $p_n(t) = 0$ when $n < 0$.

Some initial examples,

$$p_1 = t_1, \quad p_2 = t_2 + \frac{1}{2}t_1^2, \quad p_3 = t_3 + t_1 t_2 + \frac{1}{6}t_1^3.$$

The Schur polynomial corresponding to an index set $\{l_0, l_1, \ldots, l_{N-1}\}$ is defined as the Wronskian involving the set of elementary Schur polynomials, $\left\{p_{l_0}, p_{l_1}, \ldots, p_{l_{N-1}}\right\}$, this is

$$S_{l_0,l_1,\ldots,l_{N-1}}(t) = \mathrm{Wr}\left(p_{l_0}, p_{l_1}, \ldots, p_{l_{N-1}}\right).$$

The derivative in the Wronskian is taken by the variable t_1. The idea is represent τ in terms of this polynomials.

Lemma 2.1.2. *The derivatives of $p_l(t)$ are given by*

$$\frac{\partial p_l}{\partial t_n} = p_{l-n} \quad \textit{and} \quad p_m = 0 \quad \textit{if } m < 0.$$

Demostración. Taking the derivative of (2.16) with respect to t_n,

$$k^n \exp\left(\sum_{m=1}^{\infty} k^m t_m\right) = \sum_{l=0}^{\infty} k^{n+l} p_l(t) = \sum_{l=0}^{\infty} k^l \frac{\partial p_l}{\partial t_n}(t).$$

Rearranging the index in the middle term gives the formula stated in the lemma. □

The following Corollary is easy to confirm.

Corolary 2.1 *For the index set $I := \{0, 1, \ldots, N-k-1, N-k+i_1, \ldots, N-1+i_k\}$ with $0 \leq i_1 \leq i_2 \leq \cdots \leq i_k$,*

$$S_I(t) = S_{i_1, i_2, \ldots, i_k}.$$

That is, the Schur polynomial $S_{l_0, l_1 \ldots, l_N}$ depends only on the nonzero indices of the set

$$\{l_0, l_1 - 1, \ldots, l_n - n, \ldots, l_{N-1} - (N-1)\}.$$

For example, $S_{0,1,\ldots,N-1} = S_0 = p_0 = 1$.

It just matter the indexes that "jump" in I. In other words, just consider the indexes different to 0 in $[l_n - n]$.

Example 2.1.3. *Consider the case $k = 2$, this parameter represents the number of indexes. First, note that $S_{0,n} = S_{n-1} = p_{n-1}$ for $n = 1, 2, \ldots$. Some of other Schur polynomials are*

$$S_{1,2} = \begin{vmatrix} p_1 & p_2 \\ 1 & p_1 \end{vmatrix} = -t_2 + \frac{1}{2} t_1^2, \quad S_{2,3} = \begin{vmatrix} p_2 & p_3 \\ p_1 & p_2 \end{vmatrix} = -t_1 t_3 + t_2^2 + \frac{1}{12} t_1^4.$$

In $S_{0,n} = S_{n-1}$, the index can be interpreted as subtraction between the largest and the smallest indexes minus 1. Note that

$$S_{0,n} = S_{n-1} = p_{n-1} = |p_{n-1}| = \mathrm{Wr}(p_{n-1}) = \begin{vmatrix} 1 & 0 \\ p_n & p_{n-1} \end{vmatrix} = \begin{vmatrix} p_0 & p_0^{(1)} \\ p_n & p_n^{(n)} \end{vmatrix} = \mathrm{Wr}(p_0, p_n)$$

This is an example that shows that $\mathrm{Wr}(p_{n-1}) = \mathrm{Wr}(p_0, p_n)$.

Remark 2.1.3. *Write the variable t_n in the power sum of the variables $(x_1, \ldots, x_m)$,*

$$t_n = \frac{1}{n} \sum_{j=1}^{m} x_j^n,$$

with m an arbitrary positive integer. Then,

$$\begin{aligned}
\exp\left(\sum_{n=1}^{\infty} k^n t_n\right) &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{m} k^n x_j^n\right) \\
&= \exp\left(\sum_{j=1}^{m} \sum_{n=1}^{\infty} \frac{1}{n} (kx_j)^n\right) \\
&= \exp\left(-\sum_{j=1}^{m} \ln\left((1-kx_j)\right)\right) \\
&= \prod_{j=1}^{m} \frac{1}{1-kx_j} \\
&= \sum_{l=0}^{\infty} h_l(x)\, k^l.
\end{aligned}$$

The functions $h_n(x)$ are the complete homogeneous symmetric polynomials, that is

$$p_l(t) = h_l(x) = \sum_{1\leq i_1 \leq i_2 \leq \ldots \leq i_l \leq m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

Note that in the penultimate identity it has used the identity $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$*, with* $\ln(1+(-x))$ *it is obtained* $\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$*. In the last identity it has use* (2.16)*. This calculation shows the relation between the variables t and x.*

2.1.8. Young diagrams

Definition 2.3 (Young's diagram) *A Young diagram $Y = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N\}$ consists of a finite arrangement of boxes organized in left-justified rows, where the number of boxes λ_j in the jth row is weakly increasing. Consequently, the ordered set $\{l_0, l_1, \ldots, l_{N-1}\}$ with $0 \leq l_0 < l_1 < \cdots < l_{N-1}$ can be uniquely expressed by the Young diagram with $\lambda_j = l_{N-j} - (N-j)$ for $j = 1, \ldots, N$.*

For example, $\{l_0, l_1, \ldots, l_{N-1}\} = \{0, 1, \ldots, N-3, N-2, N-1\} = \emptyset$ (no boxes).

Calculations of the previous example,

$$\begin{aligned}
\lambda_1 &= l_{N-1} - (N-1) = N+1-N+1 = 2, \\
\lambda_2 &= l_{N-2} - (N-2) = N-1-N+2 = 1, \\
\lambda_3 &= l_{N-3} - (N-3) = N-3-N+3 = 0, \\
&\vdots \\
\lambda_{N-1} &= l_1 - 1 = 1-1 = 0, \\
\lambda_N &= l_0 - 0 = 0-0 = 0.
\end{aligned}$$

Then, write the symbol $[l_0, l_1, \ldots, l_{N-1}]$ as τ_Y with the Young diagram

$$Y = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N\}$$

associated to the set $\{l_0, l_1, \ldots, l_{N-1}\}$, i.e.

$$\tau_Y := [l_0, l_1, \ldots, l_{N-1}] \quad \text{with} \quad \lambda_j = l_{N-j} - (N-j)\,.$$

The Young diagram associated to $\{l_0, \ldots, l_{N-1}\}$ can be constructed as follows.

The southeast boundary of the Young diagram forms a path labeled with increasing numbers starting from 0. This path is such that the indices $\{l_0, l_1, \ldots, l_{N-1}\}$ are positioned on the vertical edges along the boundary path, as depicted below. This gives the following proposition

Proposición 2.1 *The τ-function τ_Y associated with the Young diagram Y can be expressed as*

$$\tau_Y := [l_0, l_1, \ldots, l_{N-1}] = S_Y\left(\tilde{\partial}\right)\tau,$$

where $\tilde{\partial} := \left(\partial_1, \frac{1}{2}\partial_2, \frac{1}{3}\partial_3, \ldots\right)$ and $Y = (\lambda_1, \lambda_2, \ldots, \lambda_N)$, with $\lambda_j = l_{N-j} - (N-j)$.

Demostración. The conclusion will be derived from Proposition 2.2 and the orthogonality property (2.17) of the Schur polynomials. □

Proposition 2.1 connects the algebraic symbol $[l_0, \ldots, l_{N-1}]$ with a derivative of the τ-function through the Schur polynomial. Then the Plücker relation among the symbols gives the differential equation of the τ-function.

To construct Young Diagrams, follow the next steps. Firstly, determine the number of boxes in the jth row as $l_j - j$, where l_j represents the length of the jth row, and utilize this information to establish the set of indexes, I. Write the Schur polynomial associated with I, denoted as S_I. Compute the Wronskian Wr using the respective parameters p_{i_n} where $i_n \in I$. Evaluate the operator $\tilde{\partial} = \sum_{n\geq 0} \frac{1}{n}\partial_n$, and apply the outcome from the previous step to the τ-function.

The central idea of the Sato theory is to assert that each Plücker relation corresponds to a member of the KP hierarchy. The proof of Proposition 2.1 will utilize the following identity.

Proposición 2.2 *The translation, $\tau(t+a)$ where $a = (a_1, a_2, \ldots)$ can be expanded using the Schur polynomials,*

$$\tau(t+a) = \sum_Y \tau_Y(a)\, S_Y(t)\,,$$

where the sum is taken over all the sub-Young diagram of the $(\infty)^N$ diagram (i.e. N vertical layers and an infinite number of horizontal layers).

Demostración. First note that from $\theta(t;k) = \sum_{n=1}^{\infty} k^n t_n = \sum_{l=0}^{\infty} p_l(t)\, k^n$,

$$\Theta_i(a+t) = \int_C \exp\left(\theta(a+t;k)\right) \mathrm{d}\mu(k) = \sum_{l=0}^{\infty} \Theta_i^{(l)}(a)\, p_l(t)\,.$$

Consequently,

$$\tau(a+t) = \begin{vmatrix} \Theta_1(a+t) & \Theta_2(a+t) & \dots & \Theta_N(a+t) \\ \Theta_1'(a+t) & \Theta_2'(a+t) & \dots & \Theta_N'(a+t) \\ \vdots & \vdots & \ddots & \vdots \\ \Theta_1^{(N-1)}(a+t) & \Theta_2^{(N-1)}(a+t) & \dots & \Theta_N^{(N-1)}(a+t) \end{vmatrix}$$
$$= \left| \begin{pmatrix} \Theta_1 & \Theta_1^{(1)} & \Theta_1^{(2)} & \dots & \dots \\ \Theta_2 & \Theta_2^{(1)} & \Theta_2^{(2)} & \dots & \dots \\ \dots & \dots & \dots & \ddots & \ddots \\ \Theta_N & \Theta_N^{(1)} & \Theta_N^{(2)} & \dots & \dots \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ p_1 & 1 & 0 & \dots & 0 \\ p_2 & p_1 & 1 & \dots & 0 \\ \dots & \dots & \ddots & \ddots & \dots \end{pmatrix} \right|.$$

Using the Lemma (2.1.3), the proof is finished. □

Lemma 2.1.3 (Binet-Cauchy Lemma)**.** *Let A be an $m \times n$ matrix and B an $n \times m$ matrix. Define $[n]$ as the set $\{1, \dots, n\}$, and $\binom{[n]}{m}$ as the set of combinations of m elements from. For $S \in \binom{[n]}{m}$, denoted by $A_{[m],S}$ the $m \times m$ matrix whose columns are the columns of A at indices in S, and $B_{S,[m]}$ as the $m \times m$ matrices whose rows are the rows of B at indices in S. Then, the Cauchy–Binet formula states that*

$$\det(AB) = \sum_{S \in \begin{bmatrix} [n] \\ m \end{bmatrix}} \det\left(A_{[m],S}\right) \det\left(B_{S,[m]}\right)$$

.

Then Proposition 2.1 is a consequence of the orthogonality,

$$\left\langle S_Y\left(\tilde{t}\right), S_{Y'}(t)\right\rangle := S_Y\left(\tilde{\partial}\right) S_{Y'}(t)_{t=0} = \delta_{Y,Y'}, \tag{2.17}$$

where $\tilde{t} = \left(t_1, \frac{1}{2}t_2, \frac{1}{3}t_3, \dots\right)$.

Remark 2.1.4. *Let $\mathbb{C}[[t_1, t_2, \dots]]$ be the set of formal power series ring on $\mathbb{C}^\infty$. It possesses a gradation,*

$$\mathbb{C}[[t_1, t_2, \dots]] = \bigoplus_{l \geq 0} V_l^P,$$

with the graded vector space, V_l^P, defined by

$$V_l^P := \mathrm{Span}_{\mathbb{C}} \left\{ \frac{t_1^{n_1} t_2^{n_2} \cdot \dots \cdot t_l^{n_l}}{n_1! \cdot \dots \cdot n_l!} : n_1 + 2n_2 + \dots + l n_l = l \right\}.$$

Then, each series $\Theta(t) \in \mathbb{C}[[t_1, t_2, \dots]]$ is

$$\Theta(t) = \sum_{n_k \in \mathbb{Z}_{\geq 0}} a_{n_1, n_2, \dots, n_k} \frac{t_1^{n_1} t_2^{n_2} \cdot \dots \cdot t_k^{n_k}}{n_1! n_2! \cdot \dots \cdot n_k!}.$$

The coefficients $a_{n_1,n_2,\ldots,n_k}$ can by expressed by

$$a_{n_1,n_2,\ldots,n_k} = \langle t^{|n|}, \Theta\left(t\right)\rangle \quad \textit{with} \quad t^{|n|} := t_1^{n_1} t_2^{n_2} \cdot \ldots \cdot t_k^{n_k}.$$

Here $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{C}\left[\left[t_1, t_2, \ldots\right]\right]$ defined by

$$\langle \Theta\left(t\right), g\left(t\right)\rangle := \Theta\left(\partial\right) g\left(t\right)_{t=0}.$$

In particular, the orthogonality relation is obtained,

$$\left\langle t^{|n|}, \frac{1}{m} t^{|m|} \right\rangle = \prod_j \delta_{n_j,m_j} \quad \textit{with} \quad \frac{1}{m} t^{|m|} := \frac{t_1^{m_1} t_2^{m_2} \cdot \ldots}{m_1! m_2! \cdot \ldots}.$$

The power series ring $\mathbb{C}\left[\left[t_1, t_2, \ldots\right]\right]$ can also be graded in terms of the Schur polynomials. Let V_l^S be defined by the following vector space of the Schur polynomials,

$$V_l^S := \mathrm{Span}_{\mathbb{C}} \left\{ S_Y\left(t\right) : |Y| = l \right\}.$$

Then there is an isomorphism between those vector spaces V_l^P and V_l^S. The expansion of $S_Y\left(t\right)$ in the basis of V_l^P is given by

$$S_Y\left(t\right) = \sum_{n_1+2n_2+\ldots+kn_k=|Y|} \chi_Y\left(1^{n_1} 2^{n_2} \cdot \ldots \cdot k^{n_k}\right) \frac{t_1^{n_1} t_2^{n_2} \cdot \ldots \cdot t_k^{n_k}}{n_1! n_2! \cdot \ldots \cdot n_k!},$$

where the coefficient $\chi_Y\left(1^{n_1} 2^{n_2} \cdot \ldots \cdot k^{n_k}\right)$ is the character of S_Y, which is calculated by

$$\chi_Y\left(1^{n_1} 2^{n_2} \cdot \ldots \cdot k^{n_k}\right) = \langle \partial^{|n|}, S_Y\left(t\right)\rangle.$$

Then the inversion can be expressed as

$$t_1^{n_1} t_2^{n_2} \cdot \ldots \cdot t_l^{n_l} = \frac{1}{1^{n_1} 2^{n_2} \cdot \ldots \cdot l^{n_l}} \sum_{|Y|=n_1+2n_2+\ldots+ln_l} \chi_Y\left(1^{n_1} 2^{n_2} \cdot \ldots \cdot k^{n_k}\right) S_Y\left(\tilde{t}\right).$$

where $\tilde{t} = \left(t_1, \frac{1}{2}t_2, \frac{1}{3}t_3, \ldots\right)$.

The main theorem of Sato theory of the KP hierarchy may be stated as

Theorem 2.1.5. *The set of the τ-functions $\left\{\tau_Y : Y = \text{ Young diagram } \subset (\infty)^N\right\}$ provides the Plücker coordinates of the Grassmannian $\mathrm{Gr}\left(N, \infty\right)$.*

Note that each Plücker relation gives a differential equation with the multi-time variable $t = (t_1, t_2, \ldots)$. Theorem 2.1.5 together with Theorem 2.1.3 gives the next definition of the τ-function of the KP hierarchy.

Definition 2.4 (Sato's τ-function) *A function $\phi\left(t\right)$ of multi-time variable $t = (t_1, t_2, \ldots)$ is a τ-function of the KP hierarchy if it has an expansion form,*

$$\phi\left(t\right) = \sum_Y C_Y S_Y\left(t\right),$$

where C_Y are the Plücker coordinates. Note in particular that any Schur polynomial $S_Y\left(t\right)$

provides a (rational) solution of the KP hierarchy.

2.2. Lax-Sato Formulation of the KP Hierarchy

In this section, we offer a concise overview of the Lax formulation concerning the KP hierarchy, which encompasses an infinite series of linear equations. The compatibility conditions inherent in these equations lead to the emergence of flows associated with the KP hierarchy. The principal aim of this section is to delineate the foundational framework of integrability that forms the basis of KP theory. Subsequently, we illustrate that the multi-component Burgers hierarchy, previously discussed in the preceding chapter, arises as a finite reduction within Sato theory. Particular emphasis is placed on the significance of the τ-function, elucidating its pivotal role within the context of the KP hierarchy.

2.2.1. Lax Formulation of the KP Equation

Let L the next pseudo-differential operator of first order,

$$L = \partial + u_1 + u_2\partial^{-1} + u_3\partial^{-2} + \dots,$$

with $u_i = u_i(t)$ coefficients depend on infinitely many variables $t = (t_1, t_2, t_3, \dots)$. Here, ∂ represents a differential operator while ∂^{-1} denotes formal integration. They satisfy $\partial\partial^{-1} = \partial^{-1}\partial = 1$, indicating that ∂^{-1} serves as a formal inverse of ∂. The operation ∂^{ν}, where $\nu \in \mathbb{Z}$, obeys the generalized Leibniz rule,

$$\partial^{\nu}\Theta = \sum_{n\geq 0} \binom{\nu}{j} \partial_x^j (\Theta) \partial^{\nu-j}.$$

For instance, $\partial\Theta = \Theta_x + \Theta\partial$ and $\partial^{-1} = \Theta\partial^{-1} - \Theta_x\partial^{-2} + \Theta_{xx}\partial^{-3} - \dots$ (from the typical formula of integration by parts arises the latter expression). The weights for the functions u_i and ∂^{ν} are

$$\text{wt}(u_i) = i, \quad \text{wt}(\partial^{\nu}) = \nu.$$

Thus, L possesses a homogeneous weight of one. Note that the presence of u_1 in L can be nullified through a gauge transformation involving a function g, such that $u_1 = -g^{-1}\partial_x(g) = -\frac{g_x}{g}$.

$$L \to g^{-1}Lg = \partial + \tilde{u}_2\partial^{-1} + \tilde{u}_3\partial^{-2} + \dots.$$

This principle can be extended to eliminate all u_j terms by employing an appropriate gauge operator W, as will be seen in Section 2.2.2. Consequently, we will proceed to analyze L without the u_1 term,

$$L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \dots. \tag{2.1}$$

The Lax representation of the KP hierarchy is defined by the next infinite set of nonlinear equations,

$$\partial_{t_n}(L) = [B_n, L] \quad \text{with} \quad B_n = (L^n)_{\geq 0} \quad n = 1, 2, \dots. \tag{2.2}$$

where $(L_n)_{\geq 0}$ denotes the polynomial component of L_n in ∂, and then B_n serves as a differential operator of order n and $[B_n, L] := B_nL - LB_n$ represents the standard commutator of operators.

Since

$$[B_n, L] = \left[L_n - (L_n)_{<0}, L\right] = \left[L, (L^n)_{<0}\right],$$

each side of equation (2.2) becomes a pseudo-differential operator of order ≤ -1. Here, $(L_n)_{<0}$ represents the negative component of L_n in ∂. Note that $\left[\partial, (L_n)_{<0}\right]$ has no polynomial part. Therefore, for $n > 1$, each equation in (2.2) constitutes a consistent infinite system of coupled $(1+1)$-evolution equations in t_n and x, with the variables $\{u_i : i = 2, 3, \ldots\}$. The case $n = 1$ yields the equations $\partial_{t_1} u_i = \partial_x u_i$. We identify t_1 with x. The following theorem stipulated that the infinite system (2.2) is compatible.

Theorem 2.2.1. *The differential operators $B_n = (L_n)_{\geq 0}$ have the following properties,*

$$\partial_{t_m}(B_n) - \partial_{t_n}(B_m) + [B_n, B_m] = 0. \tag{2.3}$$

Consequently, the flows defined by equation (2.2) *commute. This means that for any n, $m \geq 1$,*

$$\partial_{t_n}(\partial_{t_m}(L)) = \partial_{t_m}(\partial_{t_n}(L)).$$

Demostración. From equation (2.2), $\partial_{t_m}(L) = [B_m, L_n]$. Then, using the decomposition $L_n = B_n + (L_n)_{<0}$,

$$\begin{aligned}\partial_{t_m}(L^n) - \partial_{t_n}(L^m) &= [B_m, L^n] - [B_n, L^m] \\ &= [B_m, B_n] - \left[(L^m)_{<0}, (L^n)_{<0}\right].\end{aligned}$$

The differential part (≥ 0) of the above equation is (2.3). To demonstrate the commutability of the flows, we compute once again considering (2.2),

$$\begin{aligned}&\partial_{t_m}(\partial_{t_n}(L)) - \partial_{t_n}(\partial_{t_m}(L)) \\ &= [\partial_{t_m}(B_n), L] + [B_n, \partial_{t_m}(L)] - [\partial_{t_n}(B_m), L] - [B_m, \partial_{t_n}(L)] \\ &= [\partial_{t_m}(B_n) - \partial_{t_n}(B_m), L] + [B_n, [B_m, L]] - [B_m, [B_n, L]].\end{aligned}$$

Applying the Jacobi identity for commutators, the right-hand side of the above equations becomes $[\partial_{t_m}(B_n) - \partial_{t_n}(B_m) + [B_n, B_m], L]$, which vanishes due to (2.3). $\square$

Definition 2.5 (Jacobi identity for commutators)

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

Equation (2.3) is referred to as the Zakharov-Shabat equations for the KP hierarchy. It's noteworthy that for a given pair (n, m) with $n > m$, (2.3) yields a system of $n-1$ equations for $u_2, u_3, \ldots, u_n$, in terms of the variables t_m, t_n, and x. For instance, considering the case with $n = 3$ and $m = 2$, i.e., $B_2 = (L_2)_{\geq 0} = \partial^2 + 2u_2$ and $B_3 = (L_3)_{\geq 0} = \partial^3 + 3u_2 + 3(u_{2,x} + u_3)$, then the Zakharov-Shabat equation (2.3) for B_2 and B_3 yields the system,

$$\begin{aligned}u_{2,t_2} &= u_{2,xx} + 2u_{3,x}, \\ 2u_{2,t_3} &= 3(u_{2,x} + u_3)_{t_2} - \left(u_{2,xx} - 3u_{3,x} + 3u_2^2\right)_x.\end{aligned}$$

After setting $t_2 = y$, $t_3 = t$, and eliminating u_3 from the system, $u = 2u_2$ satisfies the KP equation. It is worth noting that the KP hierarchy (2.2) arises from the compatibility of the linear system,

$$\begin{aligned}L\phi &= k\phi,\\ \partial_{t_n}\phi &= B_n\phi, \quad n = 1, 2, \ldots,\end{aligned} \tag{2.4}$$

where, $k \in \mathbb{C}$ denotes the spectral parameter, and the eigenfunction $\phi(t;k)$ with $t = (t_1, t_2, \ldots)$ is termed the wave function of the KP hierarchy. The compatibility among the second set of equations yields the Zakharov-Shabat equations (2.3).

2.2.2. The Dressing Transformation

As previously mentioned, the Lax operator L can be gauge-transformed into the trivial operator ∂, i.e.,

$$L \to \partial = W^{-1}LW, \tag{2.5}$$

where the operator of the gauge transformation is defined as

$$W = 1 - w_1\partial^{-1} - w_2\partial^{-2} - w_3\partial^{-3} + \ldots. \tag{2.6}$$

The coefficient functions w_i are related to the coefficients u_j of L through the relation $LW = W\partial$ in equation (2.5), and we have

$$\begin{aligned}&u_2 = w_{1,x}, \quad u_3 = w_{2,x} + w_1w_{1,x}, \quad u_4 = w_{3,x} + (w_1w_2)_x - w_{1,x}^2 + w_1^2w_{1,x}, \quad \ldots\\ &u_{j+1} = w_{j,x} + F_{j+1}(w_1, w_2, \ldots, w_{j-1}), \quad \ldots\end{aligned}$$

where F_{j+1} are differential polynomials of weight $j+1$ (note $\mathrm{wt}(w_j) = j$). Consequently, w_j can be regarded as primary variables whose x-derivatives determine the KP variables. The evolutions of w_j with respect to the time variables t_n can be prescribed in a consistent manner by demanding that the gauge operator W satisfy the following system of equations,

$$\partial_{t_n}(W) = B_nW - W\partial^n \quad \text{for} \quad n = 1, 2, \ldots, \tag{2.7}$$

where B_n is now expressed as $B_n = \left(W\partial^nW^{-1}\right)_{\geq 0}$. It's worth noting that this representation of B_n as a differential operator arises from the equations $\left[\partial_{t_n}(W)W^{-1}\right]_{\geq 0} = 0$. This equation is occasionally referred to as the Sato equation.

The following theorem asserts that the KP hierarchy is generated by dressing the trivial commutation relation $[\partial_{t_n} - \partial^n, \partial] = 0$ with the operator W. Due to this result, the (inverse) gauge transformation, $\partial \to L$, is termed the dressing transformation for the KP hierarchy, and W is occasionally referred to as the dressing operator.

Theorem 2.2.2. *If the operator W satisfies the Sato equation* (2.7)*, then the operator $L = W\partial W^{-1}$ satisfies the Lax equation* (2.2) *for the KP hierarchy, and the operators $B_n = \left(W\partial^nW^{-1}\right)_{\geq 0} = (L_n)_{\geq 0}$ satisfy the Zakharov-Shabat equations* (2.3)*.*

Demostración. First, a direct calculation using $L = W\partial W^{-1}$ and $L_n = W\partial^nW^{-1} = B_n + (L_n)_{<0}$ reveals that

$$W\left(\partial_{t_n} - \partial^n\right)W^{-1} = \partial_{t_n} - \partial_{t_n}(W)W^{-1} - W\partial W^{-1} = \partial_{t_n} - \left(\partial_{t_n}(W) + W\partial^n\right)W^{-1} = \partial_{t_n} - B_n,$$

where the last equality arises from (2.7). Then, Eqs. (2.2) and (2.3) ensue from the commutator relations

$$
\begin{aligned}
0 &= W\left[\partial_{t_n} - \partial^n, \partial\right] W^{-1} = \partial_{t_n}(L) - \left[B_n, L\right], \\
0 &= W\left[\partial_{t_n} - \partial^n, \partial_{t_m} - \partial^m\right] W^{-1} = \left[\partial_{t_n} - B_n, \partial_{t_m} - B_m\right],
\end{aligned}
$$

which provide the desired formulas. □

It follows from Theorem 2.2.1 that the flows defined by the Sato equation (2.7) are commutative, i.e., they satisfy the compatibility condition $\partial_{t_m}(\partial_{t_n}(W)) = \partial_{t_n}(\partial_{t_m}(W))$.

Indeed, the compatibility condition for (2.7) is equivalent to

$$
\left[\partial_{t_n} + (L^n)_{<0}, \partial_{t_m} + (L^m)_{<0}\right] = 0.
$$

Employing $L_n = B_n + (L_n)_{<0}$, the commutator term on the left-hand side can be decomposed as $[\partial_{t_n} - B_n, \partial_{t_m} - B_m] + [\partial_{t_n} - B_n, L^m] - [\partial_{t_m} - B_m, L^n]$, all of which vanish due to Theorem 2.2.2. Finally, it's worth noting that the KP linear system (2.4) is obtained through the dressing action, i.e. $\phi = W\phi_0$, where the (vacuum) wave function ϕ_0 satisfies the bare linear system.

$$
\begin{aligned}
\partial\phi_0 &= k\phi_0, \\
\partial_{t_n}\phi_0 &= \partial^n\phi_0 = k^n\phi_0, \quad n = 1, 2, \ldots
\end{aligned}
\tag{2.8}
$$

These equations, along with Sato's equation (2.7), constitute the fundamental components of the dressing transformation. The vacuum wave function will be employed in its normalized form,

$$
\phi_0(t;k) = \exp(\theta(t;k)) \quad \text{with} \quad \theta(t;k) = \sum_{n=1}^{\infty} k^n t_n. \tag{2.9}
$$

2.2.3. Wave Function ϕ and the τ-Function

The τ-function, as introduced in Section 2.1.5, assumes a pivotal role within the Sato theory concerning the KP hierarchy. This section explicitly demonstrates the correlation between the τ-function and the dressing operator W, which satisfies the Sato equation (2.7). To maintain simplicity while capturing the essence of the broader theory, our discussion focuses on a finite truncation of the infinite order pseudo-differential operator W. Consequently, we examine the dressing operator for a finite N,

$$
W = 1 - w_1\partial^{-1} - w_2\partial^{-2} - \ldots - w_N\partial^N,
$$

and defines the differential operator,

$$
W_N := W\partial^N = \partial^N - w_1\partial^{N-1} - w_2\partial^{N-2} - \ldots - w_N.
$$

The equation $W_N f = 0$ leads to (2.8) in Section 2.1.5. Given that W satisfies the Sato equation, it follows that the operator W_N also satisfies

$$
\partial_{t_n}(W_N) = B_n W_N - W_N\partial^n \quad \text{with} \quad B_n = \left(W_N\partial^n W_N^{-1}\right)_{\geq 0}.
$$

The subsequent proposition establishes the compatibility conditions that lead to the introduction of the multi-component Burgers hierarchy in Section 2.1.5.

Proposición 2.3 *The Nth order differential equation $W_N\Theta = 0$ remains invariant under any flow of the linear heat hierarchy, $\{\partial_{t_n}\Theta = \partial_{x_n}\Theta : n = 1, 2, \ldots\}$.*

Demostración. To demonstrate this, it suffices to establish that $\partial_{t_n}(W_N f) = 0$. A straightforward computation reveals,

$$\begin{aligned}\partial_{t_n}(W_N\Theta) &= \partial_{t_n}(W_N)\Theta + W_N\partial_{t_n}\Theta \\ &= (B_nW_N - W_N\partial^n)\Theta + W_N\partial_{t_n}\Theta \\ &= B_n(W_N\Theta) + W_N(\partial_{t_n}\Theta - \partial_x^n\Theta) \\ &= 0.\end{aligned}$$

Consequently, the desired result ensues from the uniqueness of the differential equation. □

Proposition 2.3 furnishes the compatible system delineated in Section 2.1.5.

$$\begin{aligned}W_N\Theta &= 0, \\ \partial_{t_n}\Theta &= \partial_x^n\Theta, \quad n = 1, 2, \ldots.\end{aligned} \tag{2.10}$$

Moreover, a set $\{\Theta_j : j = 1, 2, \ldots, N\}$ of linearly independent solutions of $W_N\Theta = 0$ in (2.8) can be utilized to explicitly construct the coefficient functions w_i of the dressing operator W in the following manner,

$$w_i = \frac{(-1)^{i+1}}{\tau}\begin{vmatrix}\Theta_1 & \ldots & \Theta_1^{(N-i-1)} & \Theta_1^{(N-i+1)} & \ldots & \Theta_1^{(N)} \\ \Theta_2 & \ldots & \Theta_2^{(N-i-1)} & \Theta_2^{(N-i+1)} & \ldots & \Theta_2^{(N)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta_N & \ldots & \Theta_N^{(N-i-1)} & \Theta_N^{(N-i+1)} & \ldots & \Theta_N^{(N)}\end{vmatrix}. \tag{2.11}$$

Here, $\tau = \mathrm{Wr}(\Theta_1, \ldots, \Theta_N)$ (as discussed in Section 2.1.5). It is noteworthy that since the t_n-dependence of the w_i is determined by the evolution equations $\partial_{t_n}\Theta_j = \partial_x^n\Theta_j$ for $j = 1, 2, \ldots, N$, an explicit solution of the Sato equation (2.7) is also obtained.

Proposición 2.4 *The wave function ϕ of the linear system* (2.4) *can be represented as*

$$\phi(t;k) = \frac{\tau(t - [k^{-1}])}{\tau(t)}\phi_0(t;k), \tag{2.12}$$

with $\phi_0 = \exp(\theta(t;k))$ with $\theta(t;k) = \sum_{n=1}^{\infty} k^n t_n$, and

$$\left(t - \left[k^{-1}\right]\right) := \left(t_1 - \frac{1}{k}, t_2 - \frac{1}{2k^2}, t_3 - \frac{1}{3k^3}, \ldots\right).$$

Demostración. First, it is important to note that utilizing (2.11), the wave function ϕ can be expressed as

$$\phi = W_N\phi_0 = \left(1 - \frac{w_1}{k} - \frac{w_2}{k^2} - \ldots - \frac{w_N}{k^N}\right)\phi_0 = \frac{1}{\tau}\begin{vmatrix}\Theta_1 & \Theta_1^{(1)} & \ldots & \Theta_1^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ \Theta_N & \Theta_N^{(1)} & \ldots & \Theta_N^{(N)} \\ k^{-N} & k^{-N+1} & \ldots & 1\end{vmatrix}.$$

Utilizing elementary column operations, the determinant in the numerator of the aforementioned expression can be reformulated as

$$\frac{(-1)^N}{k^N}\left|\left(\Theta_i^{(j)} - k\Theta_i^{(j-1)}\right)_{1\leq i,j\leq N}\right|.$$

Given the integral representation of the functions Θ_i,

$$\Theta_i(t) = \int_C \exp\left(\theta(t;\lambda)\right)\rho_i(\lambda)\,\mathrm{d}\lambda \quad \text{for} \quad i = 1, 2, \ldots, N.$$

Here, the identity $\ln\left(1-\frac{\lambda}{k}\right) = -\sum_{n=1}^{\infty}\frac{\lambda^n}{nk^n}$ has been utilized. This concludes the proof. □

Given that the expression of ϕ in Proposition 2.4 does not explicitly rely on N, this formula remains valid in the general case of the complete untruncated version of the operator W. By expanding this formula with respect to k, we obtain an explicit formula for w_i as in (2.11),

$$w_i = -\frac{1}{\tau}p_i\left(-\tilde{\partial}\right)\tau, \tag{2.13}$$

Here, $\tilde{\partial} := \left(\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \ldots\right)$ and $p_n(z)$ are the elementary Schur polynomials as described in (2.16). It is worth noting that for an N-truncated operator W_N (i.e., $w_n = 0$ if $n > N$), the τ-function adheres to the constraints,

$$p_n\left(-\tilde{\partial}\right)\tau = 0 \quad \text{for} \quad n > N. \tag{2.14}$$

Proposition 2.4 will now be employed to derive a collection of first integrals of the KP hierarchy. The integrability of the KP hierarchy can be demonstrated by the presence of an infinite number of conservation laws in the following form,

$$\partial_{t_n}h_j = \partial_x g_{j,n},$$

for certain conserved densities h_j and the corresponding conserved fluxes $g_{j,n}$ for any j, $n \geq 1$. These functions are differential polynomials of u_i in the Lax operator L, and they can be obtained as follows. By differentiating the quantity $\phi^{-1}\partial_x\phi$ with respect to t_n and employing the evolution equation $\partial_{t_n}\phi = B_n\phi$, we initially derive the conservation law,

$$\partial_{t_n}\left(\phi^{-1}\partial_x\phi\right) = \partial_x\left(\phi^{-1}B_n\phi\right). \tag{2.15}$$

Subsequently, we invert (2.1) utilizing the generalized Leibnitz rule to express the differential operator ∂ in $\partial_x\phi$ in terms of L as

$$\partial = L - \nu_2 L^{-1} - \nu_3 L^{-2} - \ldots$$

Here, v_j are differential polynomials of u_i. Consequently, the conserved density $\phi^{-1}\partial\phi$ can be expressed as an infinite series after employing $L^n\phi = k^n\phi$, $n \in \mathbb{Z}$,

$$\phi^{-1}\partial\phi = k - \frac{\nu_2}{k} - \frac{\nu_3}{k^2} - \ldots$$

Each function v_j serves as a conserved density of the KP hierarchy. The initial ones are provided by

$$\nu_2 = u_2, \quad \nu_3 = u_3, \quad \nu_4 = u_4 + u_2^2, \quad \nu_5 = u_5 - 3u_2u_3 + u_2u_{2,x}.$$

It is noteworthy that these can be simply expressed as sums of the derivatives of $w_1 = \partial_x \ln(\tau)$. Specifically, utilizing (2.12), we obtain

$$\begin{aligned}
\phi^{-1}\partial\phi &= \partial_x \ln(\phi) \\
&= \partial_x \left[\theta(t;k) + \ln\left(\tau\left(t - \left[k^{-1}\right]\right)\right) - \ln(\tau(t))\right] \\
&= k + \partial\left[\exp\left(-\sum_{n=1}^{\infty} \frac{1}{nk^n}\partial_{t_n}\right) - 1\right] \ln(\tau(t)) \\
&= k + \sum_{n=1}^{\infty} \frac{1}{k^n}\partial_x p_n\left(-\tilde{\partial}\right)\ln(\tau).
\end{aligned}$$

This leads to

$$\nu_{n+1} = -\partial_x p_n\left(-\tilde{\partial}\right)\ln\tau = -p_n\left(-\tilde{\partial}\right)w_1. \tag{2.16}$$

Consider the next example,

$$\begin{aligned}
\nu_2 = \partial_x w_1, \quad \nu_3 &= \frac{1}{2}\left(\partial_{t_2} - \partial_x^2\right), \quad \nu_4 = \frac{1}{3}\left(\partial_{t_3} - \frac{3}{2}\partial_x\partial_{t_2} + \frac{1}{2}\partial_x^3\right)w_1, \ldots \\
\nu_{n+1} &= \frac{1}{n}\left(\partial_{t_n} + (\text{h.o.d})\right)w_1,
\end{aligned}$$

where h.o.d denotes the terms involving higher powers of the derivatives. Additionally, if the solutions u_i of the KP hierarchy decrease rapidly to zero as $|x|$ tends to infinity, the integrals C_n can be defined as

$$C_n := \int_{-\infty}^{\infty} \nu_{n+1}(x, \ldots)\,\mathrm{d}x \quad \text{for} \quad n = 1, 2, \ldots. \tag{2.17}$$

Particularly, if the τ-function yields one line-soliton of $[i,j]$-type, i.e. $\tau(t) = E_i(t) + aE_j(t)$ with $E_i(t) = \exp(\theta(t;k_i))$ for $k_i < k_j$, then according to (2.16), the integrals are

$$C_n = -p_n\left(-\tilde{\partial}\right)\ln(\tau)\Big|_{x=-\infty}^{x=\infty} = \frac{1}{n}\left(k_j^n - k_i^n\right). \tag{2.18}$$

Notice that since $k_i < k_j$, $\tau \approx E_i = \exp(\theta(t;k_i))$ for $x \ll 0$, and $\tau \approx aE_j = a\exp(\theta(t;k_j))$ for $x \gg 0$. Generally, if there are N line-solitons of $[i_l, j_l]$-type for $l = 1, \ldots, N$, then

$$C_n = \sum_{l=1}^{N} \frac{1}{n}\left(k_{j_l}^n - k_{i_l}^n\right).$$

This expression resembles the N-soliton solutions of the KdV equation. Particularly noteworthy is that for the KP equation, the C_n's are also independent of y. This aspect is consider to establish a portion of the Classification Theorem in [20].

An alternative set of conserved densities can be identified by observing the following tautological equations in the guise of conservation laws,

$$\partial_{t_m}\left(\partial_x\partial_{t_n}\ln\left(\tau\right)\right)=\partial_x\left(\partial_{t_m}\partial_{t_n}\ln\left(\tau\right)\right). \tag{2.19}$$

That is, the conserved densities obtained from these equations are given by

$$\tilde{\nu}_{n+1}:=\frac{1}{n}\partial_x\partial_{t_n}\ln\left(\tau\right),$$

it is feasible to demonstrate that the integrals $\tilde{C}_n=\int_{-\infty}^{\infty}\tilde{\nu}_{n+1}\mathrm{d}x=\int_{-\infty}^{\infty}\tilde{\nu}_{n+1}\mathrm{d}x=C_n$ hold true for all n.

2.2.4. Bilinear Identity of the τ-Function

A unified formulation for the KP hierarchy in terms of the τ-function, known as the bilinear identity of the τ-function, exists. In this section, a brief summary of this formula is provided. Initially, we revisit the Lax-Sato formulation of the KP hierarchy.

$$L=W\partial W^{-1},\quad \partial_{t_n}\left(W\right)=B_nW-W\partial^n. \tag{2.20}$$

Here, $B_n=\left(W\partial^nW^{-1}\right)_{\geq 0}$. The wave function $\phi=W\phi_0$ where $\phi_0=\exp\left(\theta\right)$ in Equation (2.9), fulfills the following condition,

$$L\phi=k\phi,$$
$$\partial_{t_n}\phi=B_n\phi.$$

The wave function ϕ can be represented in relation to the τ-function, as given in Equation (2.12) within Proposition 2.4. Now, let's define an adjoint system of the Lax pair, denoted by $\left(L^*\phi^*,B_n^*\right)$,

$$L^*\phi^*=k\phi^*,$$
$$\partial_{t_n}\phi^*=-B_n\phi.$$

Here, the (formal) adjoint operator is defined as follows,

- $\left(\partial^\nu\right)^*=\left(-1\right)^\nu\partial^\nu$ for $\nu\in\mathbb{Z}$.
- For the product of two pseudodifferential operators A and B, $\left(AB\right)^*=B^*A^*$.

Then, the adjoint wave function ϕ^* can be expressed in the form,

$$\phi^*\left(t;k\right)=\left(W^*\right)^{-1}\exp\left(-\theta\left(t;k\right)\right), \tag{2.21}$$

which is obtained by taking the adjoint of Equation (2.20). Subsequently, it is feasible to demonstrate the following main theorem.

Theorem 2.2.3. *The pair $\left\{\phi\left(t;k\right),\phi^*\left(t;k\right)\right\}$ for arbitrary t and t' satisfies the bilinear relation, commonly referred to as the bilinear identity,*

$$\oint_{C_\infty}\frac{\mathrm{d}k}{2\pi i}\phi\left(t;k\right)\phi^*\left(t';k\right)=0,$$

where C_∞ is considered to be a large circle in $\mathbb{C}$.

Then ϕ^* can be expressed in terms of the τ-function as given in Equation (2.12).

$$\phi^* (t;k) = \frac{\tau (t + [k^{-1}])}{\tau (t)} \exp (-\theta (t;k)),$$

and the bilinear identity in Theorem 2.2.3 takes the following form in terms of the τ-function,

$$\oint_{C_\infty} \frac{\mathrm{d}k}{2\pi i} \tau \left(t - \left[k^{-1}\right]\right) \tau \left(t' + \left[k^{-1}\right]\right) \exp (\theta (t - t', k)) = 0. \tag{2.22}$$

Calculating the residue of Equation (2.22), we can derive the KP hierarchy in terms of the τ-function as follows. First, we set $t \to t - y$ and $t' \to t + y$. Then

$$\begin{aligned}
0 &= \oint \frac{\mathrm{d}k}{2\pi i} \tau \left(t - y - \left[k^{-1}\right]\right) \tau \left(t + y + \left[k^{-1}\right]\right) \exp (-2\theta (y;k)) \\
&= \oint \frac{\mathrm{d}k}{2\pi i} \left(\exp\left\{\left[\sum_{n=1}^{\infty} \left(y_n + \frac{1}{nk^n}\right) D_n\right]\right\} \tau (t) \cdot \tau (t)\right) \sum_{l=0}^{\infty} p_l (-2y) k^l \\
&= \oint \frac{\mathrm{d}k}{2\pi i} \left(\left[\exp \left(\sum y_n D_n\right) \sum_{m=0}^{\infty} p_m \left(\tilde{D}\right) k^{-m}\right] \tau (t) \tau (t)\right) \sum_{l=0}^{\infty} p_l (-2y) k^l \\
&= \oint \frac{\mathrm{d}k}{2\pi i} \left(\sum_{m=0} \sum_{l=0} \frac{1}{k^{m-l}} p_l (-2y) p_m \left(\tilde{D}\right) \exp \left(\sum y_n D_n\right)\right) \tau (t) \tau (t) \\
&= \sum_{l=0}^{\infty} p_l (-2y) p_{l+1} \left(\tilde{D}\right) \exp \left(\sum y_n D_n\right) \tau (t) \tau (t).
\end{aligned}$$

From the second to the third line, we separated the exponentials. Then, from the third to the fourth line, the sums were combined. Finally, we calculate the residue.

Here, the operator D_n represents the Hirota derivative, which is defined by

$$D_n^m f \cdot g = (\partial_{t_n} - \partial_{s_n})^m f (t_n) g (s_n) |_{t_n = s_n},$$

then, we have

$$\exp ((\alpha D_n)) f (t_n) \cdot g (t_n) = f (t_n + \alpha) g (t_n - \alpha).$$

Furthermore, we have defined

$$\tilde{D} := \left(D_1, \frac{1}{2} D_2, \frac{1}{3} D_3, \dots\right).$$

Expanding the bilinear equation in terms of y, we obtain a (Hirota) bilinear equation for the τ-function, with each monomial $y_1^{i_1} \cdots y_k^{i_k}$ serving as coefficients. In particular, the equation obtained at the coefficient of y_k is as follows,

$$\left[-2p_{k+1} \left(\tilde{D}\right) + D_l D_k\right] \tau \cdot \tau = 0 \quad \text{for} \quad k = 1, 2, \dots. \tag{2.23}$$

Here, it should be noted that the equations for $k = 1$ and $k = 2$ are trivial. At $k = 3$,

$$\begin{aligned}
0 &= \left[-2p_4\left(\tilde{D} + D_1D_3\right)\right]\tau\cdot\tau \\
&= \left[-2\left(\frac{1}{4}D_4 + \frac{1}{3}D_1D_3 + \frac{1}{4}D_1^2D_2 + \frac{1}{8}D_2^2\frac{1}{24}D_1^4\right) + D_1D_3\right]\tau\cdot\tau \\
&= -\frac{1}{12}\left(-4D_1D_3 + 3D_2^2 + D_1^4\right)\tau\cdot\tau.
\end{aligned}$$

This equation corresponds to the KP equation in the τ-function form as given in Equation (2.12).

2.2.5. Hirota Perturbation Method for N-Soliton Solutions

Here is a brief explanation of how one obtains an N-soliton solution from the bilinear form using a perturbation method introduced by Hirota. The Hirota bilinear equation of the KP equation (2.12) is given by

$$P\left(D_x, D_y, D_t\right)\tau\cdot\tau := \left(-4D_xD_t + D_x^4 + 3D_y^2\right)\tau\cdot\tau = 0. \tag{2.24}$$

It is worth noting that the function $P\left(D_x, D_y, D_t\right)$ provides the soliton-dispersion relation as described in Equation (2.7),

$$P\left(K_{[i,j]}^x, K_{[i,j]}^y, \Omega_{[i,j]}\right) = -4K_{[i,j]}^x\Omega_{[i,j]} + \left(K_{[i,j]}^x\right)^4 + 3\left(K_{[i,j]}^y\right)^2 = 0.$$

The perturbation method assumes the τ-function to be expanded in the following N-term expansion with a parameter ϵ,

$$\tau = 1 + \epsilon g_1 + \epsilon^2 g_2 + \ldots + \epsilon^N g_N.$$

The coefficients of the powers of ϵ in the expansion are given by

$$\begin{aligned}
P\left(D_x, D_y, D_t\right)1\cdot 1 &= 0, \\
P\left(D_x, D_y, D_t\right)1\cdot g_1 &= 0, \\
P\left(D_x, D_y, D_t\right)g_1\cdot g_1 + 2P\left(D_x, D_y, D_t\right)1\cdot g_2 &= 0, \\
P\left(D_x, D_y, D_t\right)g_1\cdot g_2 + P\left(D_x, D_y, D_t\right)1\cdot g_3 &= 0.
\end{aligned}$$

The first equation is trivially satisfied. The second equation implies

$$P\left(\partial_x, \partial_y, \partial_t\right)g_1 = -4\partial_x\partial_t g_1 + \partial_x^4 g_1 + 3\partial_y^2 g_1 = 0,$$

which corresponds to the linearized KP equation. The solution g_1 can be expressed as

$$g_1 = \int\int \rho_1\left(p, q\right)\exp\left(px + qy + \omega t\right)\mathrm{d}p\mathrm{d}q,$$

where (p, q, ω) satisfies the dispersion relation $P\left(p, q, \omega\right) = -4p\omega + p^4 + 3q^2 = 0$.

For the 1-soliton solution ($N = 1$),

$$g_1 = \exp\left(px + qy\omega t + c\right).$$

Then one can readily demonstrate that

$$P\left(D_x, D_y, D_t\right) g_1 \cdot g_1 = 0.$$

This implies that $\tau = 1 + \epsilon g_1$ (i.e., $g_n = 0$ for $n \geq 2$) is an exact solution and the solution u of the KP equation is given by

$$u = 2\partial_x^2 \ln(\tau) = \frac{1}{2} p^2 \operatorname{sech}^2\left(\frac{1}{2}\left(px + qy + \omega t + c\right)\right).$$

Note here that the dispersion relation given by Equation (2.7) has a parametrization,

$$(p, q, \omega) = \left(k_i - k_j, k_i^2 - k_j^2, k_i^3 - k_j^3\right),$$

where k_i and k_j are arbitrary constants.

Example 2.2.1 (2-soliton)**.** *For a 2-soliton solution ($N = 2$), we have*

$$g_1 = E_1 + E_2 \quad \textit{with} \quad E_i := \exp\left(p_i x + q_i y + \omega_i t + c_i\right),$$

where (p_i, q_i, ω_i) satisfies the dispersion relation for each $i = 1, 2$. Then, the third equation at the order ϵ^2 requires

$$P\left(D_x, D_y, D_t\right) E_1 \cdot E_2 + P\left(D_x, D_y, D_t\right) 1 \cdot g_2 = 0.$$

The solution g_2 can be found in the form

$$g_2 = A_{12} E_1 E_2 \quad \textit{with} \quad A_{12} = -\frac{P\left(p_1 - p_2, q_1 - q_2, \omega_1 - \omega_2\right)}{P\left(p_1 + p_2, q_1 + q_2, \omega_1 + \omega_2\right)}.$$

One can also demonstrate that

$$\begin{aligned} P\left(D_x, D_y, D_t\right) g_1 \cdot g_2 &= A_{12} P\left(D_x, D_y, D_t\right)\left(E_1 + E_2\right) \cdot E_1 E_2 = 0, \\ P\left(D_x, D_y, D_t\right) g_2 \cdot g_2 &= A_{12}^2 P\left(D_x, D_y, D_t\right) E_1 E_2 \cdot E_1 E_2 = 0. \end{aligned}$$

The expression for τ can be represented as

$$\tau = 1 + \exp\left(\xi_1\right) + \exp\left(\xi_2\right) + A_{12} \exp\left(\xi_1 + \xi_2\right) \quad \textit{with} \quad \xi_i = p_i x + q_i y + \omega_i t + c_i',$$

where $c_i' = c_i + \ln(\epsilon)$ are arbitrary constants.

Using the dispersion relation (2.7), one can establish,

$$\begin{aligned} p_1 &= k_1 - k_2, \quad q_1 = k_1^2 - k_2^2, \quad \omega_1 = k_1^3 - k_2^3, \\ p_2 &= k_3 - k_4, \quad q_2 = k_3^2 - k_4^2, \quad \omega_2 = k_3^3 - k_4^3, \end{aligned}$$

with arbitrary parameters $(k_1, \ldots, k_4)$. Then, the τ-function can be expressed in the following form,

$$\begin{aligned} \tau &= 1 + \exp\left(\xi_1\right) + \exp\left(\xi_2\right) + A_{12} \exp\left(\xi_1 + \xi_2\right) \\ &= \frac{1}{ab E_{2,4}}\left(E_{1,3} + a E_{1,4} + b E_{2,3} + ab E_{2,4}\right), \end{aligned}$$

where $E_{i,j} = (k_j - k_i)\exp(\theta_i + \theta_j)$ *with* $\theta_i = k_i x + k_i^2 y + k_i^3 t$ *and* $A_{12} = \frac{(k_1-k_3)(k_2-k_4)}{(k_1-k_4)(k_2-k_3)}$.

The constants c_i*'s in* ξ_i*'s are determined by*

$$c_1' = -\ln\left(b\frac{k_4 - k_2}{k_4 - k_1}\right), \quad c_2' = -\ln\left(\frac{k_4 - k_2}{k_3 - k_2}\right),$$

with arbitrary constants a *and* b*, and since* $u = 2\partial_x^2 \ln(\tau)$*, the following function can be utilized for the same solution* u*,*

$$\tilde{\tau} = E_{1,3} + aE_{1,4} + bE_{2,3} + abE_{2,4},$$

which corresponds to the Wronskian form for $N = 2$ *and* $M = 4$*, with the* 2×4 *matrix*

$$A = \begin{pmatrix} 1 & b & 0 & 0 \\ 0 & 0 & 1 & a \end{pmatrix}.$$

That is to say, the $\tilde{\tau}$*-function is expressed in the Wronskian form,*

$$\tilde{\tau} = \begin{vmatrix} \Theta_1 & \partial_x \Theta_1 \\ \Theta_2 & \partial_x \Theta_2 \end{vmatrix} \quad \text{with} \quad (\Theta_1, \Theta_2) = (E_1, E_2, E_3, E_4)\, A^T,$$

where $E_i = \exp(\theta_i)$ *and* A^T *is the transpose of the matrix* A*. It is worth noting that this matrix* A *does not represent a generic form of the general* 2×4 *matrix. Thus, the solutions obtained by Hirota's perturbation method yield only a specific class of KP solitons. Additionally, it is important to observe that the* τ*-function generated by the generic* 2×4 *matrix* A *contains six exponential terms.*

Proposición 2.5 *Let* $G = \partial - \phi^{-1}\phi_x$*. Then, the operator given by*

$$\tilde{W} = GW\partial^{-1} = 1 - \tilde{w}_1\partial^{-1} - \tilde{w}_2\partial^{-2} - \dots$$

also satisfies the Sato equation (2.7)*, and the new coefficient functions* $\tilde{w}_n$ *are expressed as*

$$\tilde{w}_1 = w_1 + \phi^{-1}\phi_x, \qquad \tilde{w}_n = w_n + \phi\left(\phi^{-1}w_{n-1}\right)_x \quad \text{for } n > 1.$$

This defines the Darboux transformation for the KP hierarchy, and utilizing this transformation, it becomes possible to obtain the KP solution in the Wronskian form.

2.2.6. Introduction to the Real Grassmannian

This paragraph gives a brief introduction to the real Grassmannian $\mathrm{Gr}(N, M)$, the set of N-dimensional subspaces in $\mathbb{R}^M$, which provides a foundation of a classification of the KP solitons. A point of $\mathrm{Gr}(N, M)$ can be represented by an $N \times M$ matrix of full rank.

2.2.6.1. Soliton Solutions and the Grassmannians

In Section 2.1.2, we presented the KP solitons in the form $u(t, x, y) = 2\partial_x^2 \ln(\tau(t, x, y))$, where the τ-function is given by the Wronskian determinant,

$$\tau(t, x, y) = \mathrm{Wr}(\Theta_1, \Theta_2, \dots, \Theta_N). \tag{2.25}$$

The set of functions $\{\Theta_i(t,x,y) : i = 1,\dots,N\}$ are expressed by

$$(\Theta_1, \Theta_2, \dots, \Theta_N) = (E_1, E_2, \dots, E_N) A^T, \tag{2.26}$$

where A is a full rank $N \times M$ matrix, A^T is the transpose of A, and

$$E_j(t,x,y) = \exp(\theta_j(t,x,y)) \quad \text{with} \quad \theta_j(t,x,y) = k_j x + k_j^2 y + k_j^3 t.$$

Thus, each soliton solution is determined by the k-parameters and the matrix A. In this section, we describe matrix the A as a point of the real Grassmannian $\mathrm{Gr}(N,M)$ and a classification of those matrices in terms of several decompositions of $\mathrm{Gr}(N,M)$. In particular, we discuss $N \times M$ matrices A, whose minors are all non-negative, and how they play a key roll for the regularity of the KP solitons.

2.2.6.2. The Real Grassmannian

The real Grassmannian $\mathrm{Gr}(N,M)$ is the set of all N-dimensional subspaces of $\mathbb{R}^M$. A point ξ of $\mathrm{Gr}(N,M)$ can be expressed by an N-frame of vectors, $\xi_i \in \mathbb{R}^M$ for $i = 1,\dots,M$,

$$\xi = [\xi_1, \xi_2, \dots, \xi_N] \quad \text{with} \quad \xi_i = \sum_{j=1}^{M} a_{i,j} e_j \in \mathbb{R}^M,$$

where $\{e_i : i = 1,\dots,M\}$ is the standard basis of $\mathbb{R}^M$, and $(a_{i,j}) =: A$ is a full rank $N \times M$ matrix. Then $\mathrm{Gr}(N,M)$ can be embedded into the projectivization of the exterior product space, denoted by $\mathbb{P}\left(\wedge^N \mathbb{R}^M\right)$, which is called the Plücker embedding,

$$\begin{aligned} \mathrm{Gr}(N,M) &\hookrightarrow \mathbb{P}\left(\wedge^N \mathbb{R}^M\right) \\ [\xi_1, \dots, \xi_N] &\mapsto \xi_1 \wedge \dots \wedge \xi_N. \end{aligned}$$

Here the elements of $\wedge^N \mathbb{R}^M$ are expressed as

$$\xi_1 \wedge \dots \wedge \xi_N = \sum_{i \le i_1 < \dots < i_N \le M} \Delta_{i_1,\dots,i_N}(A)\, e_{i_1} \wedge \dots \wedge e_{i_N},$$

where the coefficients $\Delta_{i_1,\dots,i_N}(A)$ are $N \times N$ minors of the matrix A called the Plücker coordinates. Using the usual inner product on $\wedge^N \mathbb{R}^M$, $\langle \cdot, \cdot \rangle \to \mathbb{R}$ we have

$$\Delta(A) = \langle e_{i_1} \wedge \dots e_{i_N}, \xi_1 \wedge \dots \wedge \xi_N \rangle.$$

In this work, we often identify the point $\xi \in \mathrm{Gr}(N,M)$ as a full-rank $N \times N$ matrix A modulo left multiplication by nonsingular $N \times N$ matrices. In other words, two $N \times M$ matrices represent the same point in $\mathrm{Gr}(N,M)$ if and only if they can be obtained from each other by row operations. That is, for any $H \in \mathrm{GL}_N(\mathbb{R})$, the new set $\{\eta_i : i = 1,\dots,N\}$ given by

$$[\eta_1, \dots, \eta_N] = [\xi_1, \dots, \xi_N] \cdot H,$$

spans the same N-dimensional subspace. Since $H \in \mathrm{GL}_N$ gives a row operation to the matrix A, choosing appropriate H, one can put $H^T A$ in a canonical form (H^T is the transpose of

H) called a reduced row echelon form (**RREF**). We then have the isomorphism

$$\mathrm{Gr}\,(N,M) \cong \mathrm{GL}_N\,(\mathbb{R}) \setminus M_{N\times M}\,(\mathbb{R})\,,$$

where $M_{N\times M}\,(\mathbb{R})$ is the set of all $N \times M$ full-rank matrices. Then gives the dimension of $\mathrm{Gr}\,(N,M)$ as $\dim \mathrm{Gr}\,(N,M) = MN - N^2 = N\,(M-N)$.

Capítulo 3

New Uniqueness Results

In this work we consider the Kadomtsev-Petviashvilii II model placed in $\mathbb{R}_t \times \mathbb{R}_x \times \mathbb{R}_y$, in the case of smooth data, not necessarily in a localized Sobolev space. The subclass of smooth solutions that are of "soliton type" is characterized by a phase $\Theta = \Theta(t, x, y)$ and an unidimensional profile F, and every classical KP soliton and multi-soliton is in this class for suitable Θ and F. In this work, we establish concrete characterizations of KP solitons by means of a natural set of nonlinear differential equations and inclusions of functionals of Wronskian, Airy and Heat type, among others. These functional equations only depend on the new variables Θ and F. A characteristic of this method is a particular and rigid structure depending on the considered soliton. As a corollary of these equivalences, we establish uniqueness of line-solitons, multi-solitons and other degenerate solutions among a large class of KP solutions. Our results are also valid for other 2D dispersive models such as the quadratic and cubic Zakharov-Kuznetsov equations.

3.1. Introduction and Main Results

3.1.1. Setting of the problem

Let $t \in \mathbb{R}$, and $(x, y) \in \mathbb{R}^2$. In this work we consider the KP-II model

$$\left(-4u_t + u_{xxx} + 6uu_x\right)_x + 3u_{yy} = 0, \tag{3.1}$$

where $u = u(t, x, y) \in \mathbb{R}$ is the unknown. The KP equations are canonical integrable models in 2D and were first introduced by Kadomtsev and Petviashvili in 1970 [13] for modeling "long and weakly nonlinear waves" propagating essentially along the x-direction, but with a small dependence in the variable y. A rigorous derivation of both models from the symmetric *abcd* Boussinesq system was obtained by Lannes and Lannes-Saut [24, 25].

The KP-II model (3.1) (KP from now on, if there is no confusion) has an important set of symmetries. If $u = u(t, x, y)$ is solution to (3.1), then $u(t + t_0, x + x_0, y + y_0)$, with $t_0, x_0, y_0 \in \mathbb{R}$, $cu\left(c^{3/2}t, c^{1/2}x, cy\right)$, if $c > 0$, and (Galilean invariance) if $\beta \in \mathbb{R}$ is a given speed, $u\left(t, x - \frac{4\beta}{3}y + \frac{4\beta^2}{3}t, y - 2\beta t\right)$ define new solutions to KP.

Our purpose here is to establish uniqueness results for KP solitons in the class of smooth solutions that are of soliton type. More precisely, assume that u in (3.1) is sufficiently smooth and has the form

$$u(t, x, y) = 2\partial_x^2 F(\Theta(t, x, y)), \tag{3.2}$$

where, for some $s_0 > 0$, $F : [s_0, \infty) \longrightarrow \mathbb{R}$ and $\Theta = \Theta(t, x, y) \in [s_0, \infty)$ are smooth functions. Later we will justify that with no loss of generality one can choose $s_0 = 1$, consequently we shall assume this particular choice along this work. Since F can be changed by any linear affine function, we can also assume that $F(1) = 0$ and $F'(1) = 1$. We shall call Θ the *phase* of the solution u, and F will be the *profile*. Every classical KP soliton is in this class for suitable F and Θ. Indeed, one of the most important examples of solutions of the form (3.2) are given by classical solitons and multi-solitons (see e.g. Kodama [20]), namely solutions of KP of the form

$$
\begin{aligned}
u(t,x,y) &= 2\partial_x^2 \log(\Theta(t,x,y)), \\
\Theta(t,x,y) &:= \mathrm{Wr}(\Theta_1, \ldots, \Theta_n)(t,x,y), \\
\Theta_i(t,x,y) &:= \sum_{j=1}^{M_i} a_{ij} \exp\left(k_{ij}x + k_{ij}^2 y + k_{ij}^3 t\right),
\end{aligned}
\tag{3.3}
$$

where Wr represents the classical Wronskian of n functions, and k_{ij}, a_{ij} are in principle just real-valued, although particular values determine precise solutions. In this case, $F = \log$ and Θ has the additional scaling symmetry: if Θ is a valid phase, then $\lambda\Theta$ in (3.3) also does, for any $\lambda > 0$. This fact motivates the reason why requiring $\Theta > 1$ is in principle not extremely restrictive. Moreover, in the particular case where $F = \log$, given $u = u(t,x,y)$ solution of (3.1), the formula $\Theta = \exp\left(\int_0^x \int_0^t \frac{1}{2} u(s,r,y)\, \mathrm{d}r\mathrm{d}s\right)$ returns a valid phase Θ. Consequently, except by some loss of regularity, working an equation for the profile F and the phase Θ is as general as dealing with the original KP model (3.1) for u.

Taking into account the great diversity of KP solutions, our study will focus on a simple but useful characterization of soliton solutions. It turns out that this question is quite interesting and quite challenging, since Θ can assume complicated values, from simple linear functions to periodic ones. Also, the question of whether or not profile $F = \log$ is the only possibility for F is also extremely interesting.

The first step is to rewrite KP (3.1) in terms of F and Θ as in (3.2). After rearrangements, and assuming convergence to zero at infinity, we obtain the following fourth order equation for (F, Θ):

$$
\begin{aligned}
0 = &\left(F'''' + 6F''^2\right)(\Theta)\,\Theta_x^4 + 6\left(F'' + F'^2\right)'(\Theta)\,\Theta_x^2\Theta_{xx} + 3\left(F'' + F'^2\right)(\Theta)\left(\Theta_{xx}^2 + \Theta_y^2\right) \\
&- 4F''(\Theta)\,\Theta_x(\Theta_t - \Theta_{xxx}) + F'(\Theta)\left(-4(\Theta_t - \Theta_{xxx})_x\right) \\
&+ 3F'(\Theta)(\Theta_{yy} - \Theta_{xxxx}) + 3F'^2(\Theta)\left(\Theta_{xx}^2 - \Theta_y^2\right).
\end{aligned}
\tag{3.4}
$$

This is the equation that will be studied in this work. It represents a highly nonlinear equation for the two unknowns F and Θ, but its nature is certainly better than (3.1), since it has hidden structures. The most important is an splitting phenomenon between some parts concerning only F motivated terms, and others only dealing with Θ terms. Of course, this is far from being an exact "separation of variables" as in classical linear PDEs, but we mention important similarities. Indeed, it is possible to divide (3.4) into three somehow well-defined sub-equations:

(a) The first line, that can be written in terms of $\rho(s) = (F'' + F'^2)(s)$, corresponding to those terms that are equal to 0 when $\rho \equiv 0$. And ρ is equal to 0 when $F = \log$ and some particular initial conditions are met.

(b) The second line involves a modified Airy function $Ai(\Theta) := \Theta_t - \Theta_{xxx}$ and its derivative with respect to the variable x. This is a reminiscent of the 1D variable x that has a natural Airy structure for Θ (see Appendix 3.7 for further details).

(c) The third line has a complex structure represented by a Heat type term defined by $H(\Theta) := \Theta_y - \Theta_{xx}$. Notice that H is an operator in the variables y and x only.

(d) Additionally, there is a hidden structure in (3.4) represented by Wronskian type functions, see Definition 3.1.

The Heat and Airy functions are classical in the KP literature, see e.g. Kodama [20], but Wronskians and ρ functions are, as far as we understand, not so well-understood. Translated to the equation (3.4), the purpose of this work is to find suitable conditions on the Airy, Heat, Wronskian and ρ functions that characterize different soliton solutions. For this, we consider the following definitions:

Definition 3.1 (Classification of phases Θ) *We shall say that Θ as in* (3.2)

(i) is of Airy type if for all $(t,x,y) \in \mathbb{R}^3$,

$$Ai(\Theta) := \Theta_t - \Theta_{xxx} = 0.$$

(ii) Is of Heat type if for all $(t,x,y) \in \mathbb{R}^3$,

$$H(\Theta) := \Theta_y - \Theta_{xx} = 0.$$

(iii) Is of x-Wronskian type and y-Wronskian type if $\Theta > 0$, and

$$W_x(\Theta) := \Theta_{xxxx} - \frac{\Theta_{xx}^2}{\Theta} = 0, \quad W_y(\Theta) := \Theta_{yy} - \frac{\Theta_y^2}{\Theta} = 0, \tag{3.5}$$

respectively.

(iv) Is of $\mathcal{T}$-type if for F fixed,

$$\begin{aligned}\mathcal{T}(\Theta) := \; & -4F''(\Theta)\,Ai(\Theta)\,\Theta_x \\ & + F'(\Theta)\left(-4Ai(\Theta)_x + 3\left(H(\Theta)_y + H(\Theta)_{xx}\right)\right) \\ & - 3F'^2(\Theta)\,H(\Theta)\,(\Theta_y + \Theta_{xx}) = 0.\end{aligned} \tag{3.6}$$

Before stating our main results, some important comments are necessary:

Remark 3.1.1 (On the classical operators). Heat and Airy operators are naturally involved in (3.4). Indeed, it can be proved that (3.4) can be written as

$$\begin{aligned}&(\rho'' - 2F'(\Theta)\rho' + 4F''(\Theta)\rho)\,\Theta_x^4 + 6\rho'\Theta_x^2\Theta_{xx} \\ &\quad - 4F''(\Theta)\,\Theta_x Ai(\Theta) + F'(\Theta)\left(-4Ai(\Theta)_x + 3\left(H(\Theta)_y + H(\Theta)_{xx}\right)\right) \\ &\quad + 3\left(\rho\left(\Theta_{xx}^2 + \Theta_y^2\right) - F'^2(\Theta)\,H(\Theta)\,(\Theta_y + \Theta_{xx})\right) = 0,\end{aligned} \tag{3.7}$$

where $\rho(s) := F''(s) + F'^2(s)$. See Section 3.2 for further details. On the other hand, the operator $\mathcal{T}$ is the natural counterpart of the ODE type satisfied by ρ, in the sense that (3.4) reads

$$\Theta_x^4\rho'' + 2\left(6\Theta_x^2\Theta_{xx} - 2F'(\Theta)\Theta_x^4\right)\rho' + \left(3\left(\Theta_{xx}^2+\Theta_y^2\right) + 4F''(\Theta)\Theta_x^4\right)\rho + \mathcal{T}(\Theta) = 0. \tag{3.8}$$

Finally, notice that Θ being of $\mathcal{T}$-type is a condition depending on the profile F, and consequently is a more complex condition than being of Airy or Heat type, which are independent of the profile F.

Remark 3.1.2 (On the Wronskian operators)*.* The emergence of the Wronskians (3.5) in (3.4) seems obscure and nonstandard. However, it is possible to rewrite (3.4) as

$$\begin{aligned}&\left(F'''' + 6F''^2\right)(\Theta)\Theta_x^4 + 6\left(F''+F'^2\right)'(\Theta)\Theta_x^2\Theta_{xx} + 3\left(F''+F'^2\right)(\Theta)\left(\Theta_{xx}^2+\Theta_y^2\right)\\ &\quad - 4F''(\Theta)\Theta_x Ai(\Theta) - 4F'(\Theta)Ai(\Theta)_x\\ &\quad + 3F'(\Theta)\left(W_y^F(\Theta) - W_x^F(\Theta)\right) = 0,\end{aligned} \tag{3.9}$$

with W_x^F and W_y^F generalized Wronskian functionals, defined in (3.1). Later we will prove that under $H(\Theta)=0$, one has $W_x^F(\Theta) - W_y^F(\Theta) = W_x(\Theta) - W_y(\Theta) = 0$, namely one can assume that $F = \log$ in (3.1), leading to the natural definitions in (3.5). In that sense, null Wronskians are naturally related to the Heat condition $H(\Theta)=0$, however, the equivalence will not be as exact as one would prefer.

3.1.2. KP solitons

The soliton family is one of the most distinctive features present in the KP model. Distinguished by their complexity and rich character, several works have been devoted to their understanding either by integrability, algebraic and analytic techniques. We mention key works by Kodama and Williams [18, 19], where a precise description of KP-solitons with positive Grassmannian is given. See also the monograph by Kodama [20] for a complete and detailed account of this line of research.

Let us recall the large family of KP solitons. The line-soliton family (see [20]) is given by

$$\Theta(t,x,y) = a_1\exp(\theta_1) + a_2\exp(\theta_2), \tag{3.10}$$

where $a_1, a_2 > 0$, and $\theta_j := k_jx + k_j^2y + k_j^3t$, $k_1, k_2 \in \mathbb{R}$. Assuming $F = \log$, the corresponding KP solution via (3.2) is given by

$$u(t,x,y) = \frac{1}{2}(k_1-k_2)^2\operatorname{sech}^2\left(\frac{1}{2}(\theta_1-\theta_2)\right). \tag{3.11}$$

See Fig. 3.1 for details. The classical KdV soliton is recovered by setting $k_1 = -k_2 = k$, and in this case u becomes

$$Q_k(t,x,y) := 2k^2\operatorname{sech}^2\left(kx + k^3t\right). \tag{3.12}$$

The next case of KP solution is the resonant multi-soliton. This corresponds to the case

$$\Theta(t,x,y) = \sum_{i=1}^M a_i\exp(\theta_i) = \sum_{i=1}^M a_i\exp\left(k_ix + k_i^2y + k_i^3t\right), \tag{3.13}$$

where to ensure the positivity of Θ we impose that each $a_i > 0$ and $k_1 < k_2 < \cdots < k_M$.

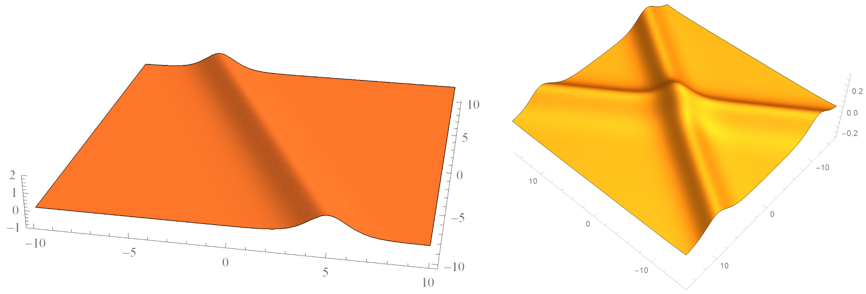


Figura 3.1: Left: One line-soliton solution (3.11) with $k_1 = -0.5$, $k_2 = 1$ and $t = 0$. This solution divides the plane into two regions, one for each exponential in the sum (3.10) and on each of these regions a different exponential dominates. Right: A 2-soliton of KP with $k_1 = -1$, $k_2 = -0.5$, $k_3 = 0.5$ and $k_4 = 1$, at time $t = 0$.

Now we recall the KP 2-soliton. In this case $\Theta = \mathrm{Wr}\left[\Theta_1, \Theta_2\right]$, where Wr is the Wronskian of two functions $\Theta_1 = \exp\left(\theta_1\right) + \exp\left(\theta_2\right)$ and $\Theta_2 = \exp\left(\theta_3\right) + \exp\left(\theta_4\right)$ being 1-soliton phases. Calculating the phase Θ, one obtains the classical formula

$$\begin{aligned} \Theta = \ & (k_3 - k_1) \exp\left(\theta_1 + \theta_3\right) + (k_4 - k_1) \exp\left(\theta_1 + \theta_4\right) \\ & + (k_3 - k_2) \exp\left(\theta_2 + \theta_3\right) + (k_4 - k_2) \exp\left(\theta_2 + \theta_4\right). \end{aligned} \tag{3.14}$$

In order to ensure the positivity and nondegeneracy of Θ, we require $0 \leq k_1 < k_2 < k_3 < k_4$. See Fig. 3.1 for further details on the family of KP 2-solitons.

3.1.3. Main Results

In this work we follow a different approach with respect to previous mentioned works. Its root is placed on the idea that each KP soliton should obey a particular "variational" characterization, expressed in the fact that critical points of a suitable nonlinear functional. Our main objective here is to provide clear and simple characterizations of the most distinctive KP solitons by means of simple "trapping" functionals that obey interesting rigidity properties, a first step in the direction previously mentioned. We believe that this idea has interesting possible applications to more complex KP solutions, but also in other related dispersive models. Our first result is a characterization of the KdV line-soliton as KP solution.

Theorem 3.1.3. *Let u be a smooth solution to KP* (3.1) *of the form* (3.2)*, with a smooth profile F such that $F(1) = 0$, $F'(1) = 1$, $F''(1) = -1$, and $F'''(1) = 2$. Then u is a KdV soliton and $F = \log$ if and only if $H(\Theta) = Ai(\Theta) = W_y(\Theta) = 0$.*

Remark 3.1.4. The four initial conditions on F may seem extremely demanding, however they are naturally explained by the fourth order equation representing (3.1). Consequently, in order to determine F, the four derivatives on x at (3.1) induce corresponding initial conditions for F and its three first derivatives. It is also easy to see that different initial conditions may lead to other solutions (F periodic, for instance), as it happens in the simpler KdV case. See [22, 23] for instance for examples of space-periodic profiles F.

Remark 3.1.5. It will be proved below (see Lemma 3.3.3) that the Heat condition $H(\Theta) = 0$ implies $W_x^F(\Theta) = W_y^F(\Theta)$ and $W_x(\Theta) = W_y(\Theta)$.

As a natural consequence of Theorem 3.1.3, we obtain the following uniqueness result of KdV solitons as extended KP solutions. As usual, we require

$$F(1) = 0, \quad F'(1) = 1, \quad F''(1) = -1, \quad \text{and} \quad F'''(1) = 2. \tag{3.15}$$

Corolary 3.1 *Assume* (3.15). *Let u be a nontrivial KP solution of the form* (3.2) *such that* $\Theta > 0$ *is a solution to* $H(\Theta) = Ai(\Theta) = W_y(\Theta) = 0$. *Then* $u = Q_k$ *in* (3.12) *for some* $k \in \mathbb{R} \setminus \{0\}$.

Theorem 3.1.3 can be extended to general KP line-solitons (3.11) as in Fig. 3.1 left, and not necessarily of KdV type. In this case, we denote them oblique line-solitons.

Theorem 3.1.6. *Let u be a smooth solution to* (3.1) *of the form* (3.2), *with a smooth profile F such that* (3.15) *is satisfied. Then u is an oblique line-soliton of the form* (3.11)-(3.10) *and* $F = \log$ *if and only if* $H(\Theta) = Ai(\Theta) = 0$, *and*

$$\Theta W_x(\Theta) = \Theta W_y(\Theta) = A(t,x)\exp(k(t,x)y), \tag{3.16}$$

for some particular functions $A > 0$, $k > 0$.

It is worth to mention that the conditions $H(\Theta) = Ai(\Theta) = 0$ do not ensure the validity of Theorem 3.1.6. Indeed, it will be proved that the class of phases satisfying these two conditions is large enough to contain many multi-soliton solutions, such as the Y structure defined below, which is not a line-soliton. Consequently, (3.16) is a necessary condition.

Remark 3.1.7. Notice that $\Theta W_x(\Theta)$ remains unchanged after a Galilean transformation (see (3.1)). This is not the case for $\Theta W_y(\Theta)$. However, one can prove that if Θ is of the form (3.10), then coincidentally for $\beta = k_1 + k_2$ one has that the Galilean transformation of Θ, denoted Θ_β (see (3.1)), satisfies $\Theta_\beta W_y(\Theta_\beta) = 0$. It is an interesting problem to fully elucidate the role of Galilean transforms in the classification of solitons as proposed in this work.

Theorem 3.1.6 puts in evidence an intriguing mathematical structure, a natural finite-dimensional cone in the variable y.

Definition 3.2 (Invariant $\mathcal{W}_n$ cones) *Given* $n \in \{1,2,3,\ldots\}$, *consider the linear, positively generated subspace (cone)*

$$\mathcal{W}_n := \left\{ \sum_{j=1}^{n} a_j(t,x)\exp(k_j(t,x)y) \ : \ \begin{array}{c} \exists k_n(t,x) > \cdots > k_2(t,x) > k_1(t,x) \geq 0, \\ \exists\, a_1(t,x), a_2(t,x), \ldots, a_n(t,x) \geq 0 \end{array} \right\}. \tag{3.17}$$

Therefore, (3.16) can be recast as $\Theta W_x(\Theta) = \Theta W_y(\Theta) \in \mathcal{W}_1$.

The space $\mathcal{W}_n$ has interesting properties, in particular its behavior under the nonlinear mapping $\Theta W_y(\Theta)$ is key to understand complex multi-soliton structures. First of all, $\Theta \in \mathcal{W}_M$ implies that $\Theta W_y(\Theta) \in \mathcal{W}_{\frac{1}{2}M(M-1)}$ (Lemma 3.4.6). Second, there is a natural "kernel" given by $\exp(ky)$: one has $\exp(ky)W_y(\exp(ky)) = 0$ for any $k = k(t,x)$. Additionally, under the gauge $\mathcal{W}_M \ni \Theta \longmapsto \Theta_k := \exp(ky)\Theta$, one has $\Theta_k W_y(\Theta_k) = \exp(2ky)\Theta W_y(\Theta) \in \mathcal{W}_{\frac{1}{2}M(M-1)}$, revealing that there is no unique nontrivial solution to the set inclusion $\Theta W_y(\Theta) \in \mathcal{W}_{\frac{1}{2}M(M-1)}$ for $\Theta \in \mathcal{W}_M$. One way to repair this gauge freedom is to ask for $\Theta(y=0)$ and $\Theta_y(y=0)$ uniquely defined, as it is done below.

On the other hand, it is easy to check that

$$\mathcal{W} := \bigcup_{n\geq 0} \mathcal{W}_n$$

is a multiplicative algebra under nonnegative coefficients. Standard topological arguments ensure that the closure of this space restricted to any compact set K of $\mathbb{R}$, under the uniform norm, is nothing but $C_+(K,\mathbb{R})$, the space of real-valued, nonnegative continuous functions defined on K.

Theorem 3.1.8 (Resonant multi-solitons)**.** *Let u be a solution of* (3.1) *of the form* (3.2) *with a smooth real-valued phase $\Theta > 0$ satisfying for $k = 0, 1, 2, 3,$*

$$\partial_x^k \Theta(t,0,0), \quad \partial_x^k \partial_y \Theta(t,0,0) \quad \textit{uniquely prescribed.} \tag{3.18}$$

Assume that F is a smooth profile such that (3.15) *holds. Then Θ corresponds to an M resonant multi-soliton* (3.13) *and $F = \log$ if and only if Θ satisfies $H(\Theta) = Ai(\Theta) = 0$ and $\Theta W_y(\Theta) = \Theta W_x(\Theta)$ has a unique value in $\mathcal{W}_{\frac{1}{2}M(M-1)}$.*

Remark 3.1.9. Notice that the condition (3.18) only requires information of Θ at $x = y = 0$. This is necessary to ensure the uniqueness of the solution Θ to $\Theta W_y(\Theta) = \Theta W_x(\Theta) \in \mathcal{W}_{\frac{1}{2}M(M-1)}$ within the class $\mathcal{W}_M$. Due to the exponential growth in x of the functions solving the equations for Θ, a less demanding sufficient condition will require to establish a Cauchy theory for the linear Airy equation $\Theta_t - \Theta_{xxx} = 0$ with initial conditions in the class $D'(\mathbb{R}_x)$, a problem that is far from trivial due to the oscillatory character of the Airy kernel.

The key in the proof of Theorem 3.1.8 is the property $\Theta \in \mathcal{W}_M$ implies $\Theta W_y(\Theta) \in \mathcal{W}_{\frac{1}{2}M(M-1)}$ (Lemma 3.4.6). This property allows us to estimate the size of the cone representing the image of $\mathcal{W}_M$ under the nonlinear mapping $\Theta W_y(\Theta)$. Then one has to establish a sort of uniqueness in the representation of Θ, which in the case $M = 1$ is easy to obtain (see Theorem 3.1.6), but in the general case it is not known to hold in generality. Under the additional prescribed data at $x = y = 0$, then uniqueness is recovered and Theorem 3.1.8 establishes the equivalence between resonant multi-solitons and Airy-Heat type phases with finite-dimensional Wronskians.

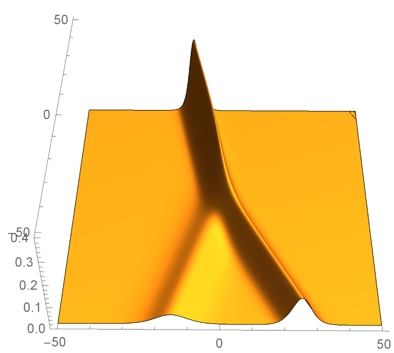


Figura 3.2: A Y-soliton characterized by $\Theta = \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3)$ in (3.13). Notice that the coefficients a_j are set equal to 1, so that the three solitons meet at the origin at $t = 0$. Here, $(k_1, k_2, k_3) = (-0.3, 0, 0.5)$.

Resonant solitons of Y-type, or Miles type (see Fig. 3.2), are essential KP solutions included in the previous result. These are usually given by [20] ($k_1 < k_2 < k_3$)

$$\Theta = a_1 \exp(\theta_1) + a_2 \exp(\theta_2) + a_3 \exp(\theta_3), \quad a_i > 0, \quad \theta_i := k_i x + k_i^2 y + k_i^3 t.$$

Sometimes referred as resonant interacting 3-solitons, Theorem 3.1.8 states that they are characterized as having zero Heat and Airy operators, but having $\Theta W_y(\Theta)$ with one more dimension than the one obtained in Theorem 3.1.6, measured in terms of the subspace $\mathcal{W}_2$. Additionally, in this case $\Theta \in \mathcal{W}_3$ and $\Theta W_y(\Theta) \in \mathcal{W}_3$, being the only phases (as far as we understand) that have this invariance under the nonlinear mapping $\Theta \longmapsto \Theta W_y(\Theta)$.

Our last result concerns the characterization of crossed 2-solitons, with Θ given in (3.14) (see also Fig. 3.1 right panel). Recall the subspace $\mathcal{W}_n$ defined in (3.17).

Theorem 3.1.10 (2-solitons)**.** *Let u be a solution of* (3.1) *of the form* (3.2) *with a smooth real-valued phase $\Theta > 0$ satisfying* (3.18) *and being at most exponentially growing in x. Assume the smooth profile $F = \log$. Then Θ corresponds to a 2-soliton* (3.14) *with $0 \leq k_1 < k_2 < k_3 < k_4$ if and only if $H(\Theta), Ai(\Theta)$ are contained in $\mathcal{W}_4$, $\Theta W_y(\Theta), \Theta W_x(\Theta)$ describe unique elements in $\mathcal{W}_5$, and $Ai(\Theta) = \frac{3}{2}\partial_x H(\Theta)$.*

Remark 3.1.11. Notice that Theorem 3.1.10 does not assume that $\Theta = \mathrm{Wr}(\Theta_1, \Theta_2)$ (standard Wronskian of Θ_1 and Θ_2), with $H(\Theta_i) = Ai(\Theta_i) = 0$ and $\Theta_1, \Theta_2 \in \mathcal{W}_2$. This is the standard and well-known definition of the 2-soliton that assumes the key wronskian substructure. Here we lift that condition and only ask conditions on Θ itself. The wronskian structure is recovered from the proof.

Remark 3.1.12. Theorem 3.1.10 can be recast as follows. It is easy to check that 2-solitons solve $\mathcal{T}(\Theta) = 0$ with $F = \log$ (Corollary 3.8). However, this equation has plenty of additional, more complicated solutions [20], and a suitable characterization of the 2-soliton subclass is desirable. In that sense, Theorem 3.1.10 and the $\mathcal{W}_n$ structure give a precise equivalence that separates 2-solitons of other more complex KP solutions.

3.1.4. Further results

Our last comments are related to possible extensions of the results presented in this work. We believe that with some work it is possible to give a suitable characterization of KP-I line-soliton and lumps in terms of particular phases. The challenge is to get a good understanding of the fact that KP-I lumps are degenerate soliton solutions, in a sense already described in [1].

Remark 3.1.13 (The KdV case). An important outcome of the proofs will be its robust character. Indeed, Theorems 3.1.3-3.1.10 have natural counterparts in the case of the 1D KdV model, where similar notions of Airy and Heat operators are introduced. Proofs are simpler and for the sake of completeness, we sketch them. See Appendix 3.7 for the complete details.

Remark 3.1.14 (The mKdV case). In concordance with the previous remark, it should be natural that the same ideas apply as well for the so-called mKdV model. It turns out that this 1D equation has a particular rich structure involving even more demanding special solutions that complicates matters. We provide for completeness a suitable short treatment of the problem in Appendix 3.7. See also [5] for a detailed account of the difficulties found when dealing with mKdV models.

3.1.5. Previous results

We mention some key results obtained for KP-II models during the past years. Bourgain [3] showed that KP is globally well-posed (GWP) in $L^2(\mathbb{R}^2)$ (see also Ukai [37] and Iório-Nunes [10] for early results). Bourgain's result was later improved by Takaoka-Tzvetkov [36], Isaza-Mejia [11], Hadac [6] and Hadac-Herr-Koch [7]. Molinet, Saut and Tzvetkov [31] proved global well-posedness of KP along the KdV line-soliton in $L^2(\mathbb{R}\times\mathbb{T})$ and $L^2(\mathbb{R}^2)$.

The long time behavior of small KP solutions has been studied by Hayashi-Kaumkin-Saut and Hayashi-Naumkin [8, 9], see also recent improvements by Niizato [32]. de Bouard-Martel [4] showed that KP has no "lump" structures, namely compact-in-space solutions. Any KdV soliton becomes an (infinite energy) line-soliton solution of KP. This structure is stable, as proved by Mizumachi and Tzvetkov [30], and asymptotically stable, see Mizumachi [28, 29]. Finally, Izasa-Linares-Ponce [12] showed propagation of regularity for this model.

Numerical studies of KP solutions have been performed in [17], see also [15, 16] for a detailed account of the KP literature via PDE methods. Multi-line-soliton structures are known to exist via Inverse Scattering (IST) methods [1, 21]. Their stability in rigorous terms has been recently considered by Wu [39, 40]. See also [33, 34] for a detailed theory of transversal stability and instability of PDE models of water waves, that applies to one line-soliton as the ones studied in Theorem 3.1.3-3.1.6. The description of small data can be found in [35, 38]. Recently, and following [26], in [27] it was shown that every solution u of KP obtained from arbitrary initial data u_0 in $L^2(\mathbb{R}^2)$ satisfies $\liminf_{t\to\infty}\int_K u^2(t,x,y)\,\mathrm{d}x\mathrm{d}y = 0$, with $K\subseteq\mathbb{R}^2$ compact. Finally, Kenig and Martel [14] showed that for any $\beta>0$ and initial data small in $L^1\cap L^2$, $\lim_{t\to\infty}\int_{x>\beta t} u^2(t,x,y)\,\mathrm{d}x\mathrm{d}y = 0$.

Organization of this work

This work is organized as follows. In Section 3.2 we introduce the basic elements needed for the proof of the main results. Section 3.2.3 is devoted to recall standard results on KP solitons. In Section 3.3 we present general results and properties about phases Θ satisfying (3.4). On the other hand, Section 3.4 presents properties satisfied by classical soliton structures. Finally, in Section 3.5 we prove the main results, Theorems 3.1.3, 3.1.6, 3.1.8, and 3.1.10. Appendix 3.6 contains some useful computations needed in this work, and finally Appendix 3.7 is devoted to the proof of similar results in the case of KdV and mKdV models.

3.2. Preliminaries

In this section we first recall some simple but important facts related to solutions of the KP model.

3.2.1. Invariances

First of all, it is clear that (3.4) is invariant under space and time shifts in the phase Θ. Additionally, (3.4) has the natural scaling invariance associated to KP:

$$\Theta(t,x,y)\longrightarrow\Theta\left(\lambda^3 t,\lambda x,\pm\lambda^2 y\right),\quad \lambda>0.$$

KP-II obeys the Galilean Transform [27]

$$u\left(t,x,y\right)\longrightarrow u\left(t,x-\frac{4\beta}{3}y+\frac{4\beta^{2}}{3}t,y-2\beta t\right)=:u_{\beta}\left(t,\tilde{x},\tilde{y}\right),\quad\beta\in\mathbb{R}.$$

This invariance naturally translates into the phase Θ

$$\Theta_{\beta}\left(t,x,y\right)=\Theta\left(t,x-\frac{4\beta}{3}y+\frac{4\beta^{2}}{3}t,y-2\beta t\right)=\Theta\left(t,\tilde{x},\tilde{y}\right), \tag{3.1}$$

that also satisfies (3.4) provided Θ does.

3.2.2. Kernel in the solitonic representation

Notice that the formulation (3.2) involves a nontrivial kernel.

Lemma 3.2.1. *One has $\partial_x^2\log\left(\Theta\left(t,x,y\right)\right)=0$ if and only if the phase Θ satisfies $\Theta\left(t,x,y\right)=\exp\left(a\left(t,y\right)x+b\left(t,y\right)\right)$, for any well-defined functions a and b.*

Demostración. Direct. □

As a consequence of the previous remark, if $F=\log$, any phase that can be expressed in the form $\Theta=\exp\left(f\left(t,x,y\right)\right)$ with $f\left(t,x,y\right)$ any smooth linear affine function in the variable x gives a trivial solution. This is the kernel of the operator which permits construct KP multi-soliton solutions based on this first seed.

3.2.3. Quick review of the simplest KP solitons

Recall the line-soliton introduced in (3.10) and (3.11). The line that separates these regions correspond to $\theta_1=\theta_2$. In this case, this line is called $[1,2]$-soliton [20]. In the general case of resonant structures such as (3.13), they are called $[i,j]$-solitons and are formed by the intersection of the corresponding exponentials i and j. Each $[i,j]$-soliton has the same local structure as a line-soliton, which is described by the form [20, p. 5]

$$u=A_{[i,j]}\operatorname{sech}^2\left(\frac{1}{2}\left(K_{[i,j]}\cdot(x,y)-\Omega_{[i,j]}t+\Theta^0_{[i,j]}\right)\right),$$

with $\Theta^0_{[i,j]}$ a constant. The parameters $A_{[i,j]}$, $K_{[i,j]}$ and $\Omega_{[i,j]}$ are known as the amplitude, wave-vector and frequency, respectively, and are defined by

$$\begin{aligned}
A_{[i,j]}&=\frac{1}{2}\left(k_j-k_i\right)^2,\\
K_{[i,j]}&=\left(k_j-k_i,k_j^2-k_i^2\right)=\left(k_j-k_i\right)\left(1,k_j+k_i\right),\\
\Omega_{[i,j]}&=-\left(k_j^3-k_i^3\right)=-\left(k_j-k_i\right)\left(k_i^2+k_ik_j+k_j^2\right).
\end{aligned}$$

If one denotes $\psi_{[i,j]}$ to the angle measured counterclockwise between the $[i,j]$-soliton and the y-axis, then $\tan\left(\psi_{[i,j]}\right)=k_i+k_j$.

Now consider the case of resonant solitons (3.13) with $M=3$. As in the previous case, it is possible to determine the dominant exponentials and analyze the structure of the solution in the xy-plane. Indeed, the line-soliton at y positive, corresponding to the $[1,3]$-soliton, is located on the phase transition $x+c_{[1,3]}y = constant$ with direction parameter $c_{[1,3]} = k_1+k_3$. In the same way, the line-soliton located at y negative, corresponding to the $[1,2]$-soliton and $[2,3]$-soliton are located over their respective phase transitions with direction parameter $c_{[1,2]} = k_1 + k_2$ and $c_{[2,3]} = k_2 + k_3$, respectively. The resonance condition of these three line-solitons is given by

$$K_{[1,3]} = K_{[1,2]} + K_{[2,3]}, \quad \Omega_{[1,3]} = \Omega_{[1,2]} + \Omega_{[2,3]},$$

and both are satisfied when $K_{[i,j]} = \left(k_j - k_i, k_j^2 - k_i^2\right)$ and $\Omega_{[i,j]} = -\left(k_j^3 - k_i^3\right)$.

In a simple way it is possible to extend the previous result for a general solution constructed from a Θ composed with an arbitrary number M of exponentials, as in (3.13).

Theorem 3.2.1 ([20], Proposition 1.2)**.** *Let Θ be an M resonant phase as in* (3.13)*. Then the solution u has the following asymptotic characteristics:*

(i) For values of $y \gg 1$, there is only one line-soliton of the form $[1, M]$-soliton.

(ii) For values of $y \ll -1$, there are $M-1$ line-solitons of the form $[k, k+1]$-soliton, with $k = 1, 2, \ldots, M-1$, located counter-clockwise from the negative part to the positive part of the x-axis.

Definition 3.3 *Let $N < M$. The Grassmannian* $\mathrm{Gr}(N, M)$ *are all the matrices which represents an N-dimensional subvectorial space contained in a M-dimensional vectorial space.*

It can be verified the following isomorphism: $\mathrm{Gr}(N, M) \cong \mathrm{GL}_N(\mathbb{R}) \setminus M_{N\times M}(\mathbb{R})$, where $\mathrm{GL}_N(\mathbb{R})$ consists of those matrices with dimension $n \times n$, real coefficients and whose determinant is non-zero.

Let $\{\Theta_i : i = 1, \ldots, N\}$ be linearly independent solutions of $Ai(\Theta) = H(\Theta) = 0$. Let $\Theta := \mathrm{Wr}(\Theta_1, \ldots, \Theta_N)$ be the Wronskian of the functions Θ_i with respect to the variable x (usually called a τ-function). It is possible to see (see [20]) that $u(t, x, y) = 2\partial_x^2 \ln(\Theta(t, x, y))$ satisfies the KP equation. A particular choice for Θ_i is given by

$$\Theta_i(t, x, y) = \sum_{j=1}^{M} a_{ij} \exp(\theta_j(t, x, y)), \quad \text{with} \quad \theta_j = k_j x + k_j^2 y + k_j^3 t,$$

where $A := (a_{ij})$ is an $N \times M$ matrix. Thus each KP soliton expressed in the previous form is parametrized by M parameters $(k_1, \ldots, k_M)$ and an $N \times M$ matrix A. The matrix A will be identified as a point of the real Grassmannain $\mathrm{Gr}(N, M)$.

3.2.4. Linear ODEs related to F

In this section we describe some ODE theory related to the equations that F must satisfy. Indeed, consider the auxiliary variable

$$\rho(s) := F''(s) + F'^2(s).$$

Lemma 3.2.2. *If $F = \log$, then $\rho = 0$. If $\rho = 0$ and $F(1) = 0$, $F'(1) = 1$, then $F = \log$.*

Demostración. Direct from solving the ODE $\rho(s) = 0$. □

Lemma 3.2.3. *Let F be a smooth profile such that $F(1) = 0$ and $F'(1) = 1$. Then the following are satisfied, for any $s \geq 1$:*

(i) If $(F'' + F'^2)(s) = 0$, then $(F'''' + 6F''^2)(s) = 0$.

(ii) If now $F''(1) = -1$ and $F'''(1) = 2$, and $(F'''' + 6F''^2)(s) = 0$, then one has $(F'' + F'^2)(s) = 0$.

(iii) If now $F(1) = 0$, $F'(1) = 1$, $F''(1) = -1$, $F'''(1) = 2$, $\Theta \in [1, \infty)$ and

$$\begin{aligned} &\left(F'''' + 6F''^2\right)(\Theta)\,\Theta_x^4 + 6\left(F'' + F'^2\right)'(\Theta)\,\Theta_x^2\Theta_{xx} \\ &\quad + 3\left(F'' + F'^2\right)(\Theta)\left(\Theta_{xx}^2 + \Theta_y^2\right) = 0, \end{aligned} \tag{3.2}$$

then $F = \log$.

Demostración. The proof of (i) is a consequence of the following identity. One has

$$\left(F'' + F'^2\right)'' - 2F'\left(F'' + F'^2\right)' + 4F''\left(F'' + F'^2\right) = F'''' + 6F''^2. \tag{3.3}$$

Therefore, $F'' + F'^2 = 0$ implies $F'''' + 6F''^2 = 0$, proving (i).

Proof of (ii). On the other hand, in the case where $F'''' + 6F''^2 = 0$ one has that (3.3) can be written in terms of ρ as

$$\rho'' - 2F'\rho' + 4F''\rho = 0. \tag{3.4}$$

This is a second order linear ODE with continuous coefficients. This solution has basis of solutions of dimension 2, say $\{\rho_1, \rho_2\}$. Consequently,

$$\rho(s) = C_1\rho_1(s) + C_2\rho_2(s), \quad C_1, C_2 \in \mathbb{R}.$$

Imposing that $F''(1) = -1$ and $F'''(1) = 2$, one has $\rho(1) = \rho'(1) = 0$, leading to $\rho(s) = 0$ for all $s \geq 1$, proving (ii).

Proof of (iii). Thanks to (3.3), equation (3.2) is equivalent to

$$\left(\rho''(s) - 2F'(\Theta)\,\rho'(s) + 4F''(\Theta)\,\rho(s)\right)\Theta_x^4 + 6\rho'(s)\,\Theta_x^2\Theta_{xx} + 3\rho(s)\left(\Theta_{xx}^2 + \Theta_y^2\right) = 0,$$

with $\rho = F'' + F'^2$. This system is analogous to (3.4), and proceeding as in step (ii) we get from Lemma 3.2.2 that $\rho(s) = 0$ for all $s \geq 1$, and $F = \log$. □

Recall from Definition 3.1 that (Θ, F) are of $\mathcal{T}$-type if (3.6) is satisfied. The following Corollary is a direct result of (iii) in Lemma 3.4.

Corolary 3.2 *Let $u = 2\partial_x^2 F(\Theta)$ solution of (3.1) and a smooth profile F that satisfies the conditions (1.10), then Θ is $\mathcal{T}$-type if and only if $F = \log$.*

Demostración. Direct from (3.8) and (iii) in Lemma 3.2.3. □

3.3. Wronskian structures

3.3.1. General Phases

The purpose of this section is to establish simple properties for the smooth phases satisfying (3.4). We start with the following

Lemma 3.3.1. *Assume that $\Theta > 0$. Then the following are satisfied:*

(i) If $F''(\Theta)\Theta_x^2 + F'(\Theta)\Theta_{xx} = 0$ for all $(t,x,y) \in \mathbb{R}^3$, then u is the trivial solution.

(ii) If $F = \log$, then a positive phase Θ that satisfies $\Theta\Theta_{xx} = \Theta_x^2$, for all $(t,x,y) \in \mathbb{R}^3$, gives the trivial solution.

(iii) If $F = \log$ then every Θ of the form $\Theta(t,x,y) = A(t,y)\exp(kx)$ with $A > 0$, gives the trivial solution.

Demostración. Proof of (i). Direct from the fact that $u = 2\partial_x^2 F(\Theta) = 2\left(F''(\Theta)\Theta_x^2 + F'(\Theta)\Theta_{xx}\right)$. Proof of (ii): also direct from the fact that $F''(\Theta)\Theta_x^2 + F'(\Theta)\Theta_{xx} = \frac{\Theta\Theta_{xx}-\Theta_x^2}{\Theta^2}$, and the previous result. Finally, (iii) is consequence of the fact that in this case $\Theta F''(\Theta) = -F'(\Theta)$, for all $(t,x,y) \in \mathbb{R}^3$. □

Remark 3.3.1. Note that in Lemma 3.3.1 we are assuming that the conditions are satisfied for all $(t,x,y) \in \mathbb{R}^3$. In the case where the conditions are satisfied just for certain points $(t,x,y) \in \mathbb{R}^3$, this points will correspond to zeros of the associated solution u.

Corolary 3.3 *If $\Theta > 0$ is a smooth phase such that $\Theta\Theta_{xx} - \Theta_x^2 = 0$, then $W_x(\Theta) = 0$.*

Demostración. Taking derivative in x, $0 = \Theta_x\Theta_{xx} + \Theta\Theta_{xxx} - 2\Theta_x\Theta_{xx} = \Theta\Theta_{xxx} - \Theta_x\Theta_{xx}$. Once again, taking derivative in x, $0 = \Theta_x\Theta_{xxx} + \Theta\Theta_{xxxx} - \Theta_{xx}^2 - \Theta_x\Theta_{xxx} = \Theta W_x(\Theta)$. □

3.3.2. General vs. simple Wronskians

It is noted that in Definition 3.1, Wronskians (3.5) do not coincide with the expected value if taken from (3.4). Indeed, the correct definition should be

$$\begin{aligned} W_y^F(\Theta) &:= \Theta_{yy} - F'(\Theta)\Theta_y^2, \\ W_x^F(\Theta) &:= \Theta_{xxxx} - F'(\Theta)\Theta_{xx}^2. \end{aligned} \tag{3.1}$$

Here F is taken general. One recovers the values stated in Definition 3.1 if $F = \log$. Notice that from (3.9) in terms of $Ai(\Theta)$, one gets

$$\begin{aligned} &\left(F'''' + 6F''^2\right)(\Theta)\Theta_x^4 + 6\left(F'' + F'^2\right)'(\Theta)\Theta_x^2\Theta_{xx} + 3\left(F'' + F'^2\right)(\Theta)\left(\Theta_{xx}^2 + \Theta_y^2\right) \\ &- 4\left(F'(\Theta)Ai(\Theta)\right)_x + 3F'(\Theta)\left(W_y^F(\Theta) - W_x^F(\Theta)\right) = 0. \end{aligned}$$

Let us study the phases related to the Wronskian conditions (3.5).

Lemma 3.3.2. *Assume that $u = 2\partial_x^2\log(\Theta)$ is solution to KP with $\Theta > 0$ and $Ai(\Theta) = 0$. Then $W_y(\Theta) = W_x(\Theta)$.*

Demostración. Since $F = \log$ one has from Lemma 3.2.3 that $\rho = F'' + F'^2 = 0$ and $F'''' + 6F''^2 = 0$. Using $Ai\,(\Theta) = 0$, from (3.9) we get

$$W_y^F\,(\Theta) - W_x^F\,(\Theta) = 0.$$

The conclusion is obtained by recalling that $F = \log$. □

Now we put our attention to the following rigidity property.

Lemma 3.3.3. *Assume F, $\Theta > 0$ general and smooth. If $H\,(\Theta) = 0$, then $W_y^F\,(\Theta) - W_x^F\,(\Theta) = W_y\,(\Theta) - W_x\,(\Theta) = 0$.*

Demostración. Fix F and $\Theta > 0$ smooth. By hypothesis $H\,(\Theta) = 0$. Then $H\,(\Theta)_y = H\,(\Theta)_{xx} = 0$. Consequently, $0 = H\,(\Theta)_y + H\,(\Theta)_{xx} = \Theta_{yxx} - \Theta_{xxxx} + \Theta_{yy} - \Theta_{xxy}$. Since $\Theta_{xxy} = \Theta_{yxx}$, one obtains $\Theta_{yy} = \Theta_{xxxx}$. Finally,

$$W_x^F\,(\Theta) := \Theta_{xxxx} - F'\,(\Theta)\,\Theta_{xx}^2 = \Theta_{yy} - F'\,(\Theta)\,\Theta_y^2 =: W_y^F\,(\Theta)\,.$$

This proves that $W_x^F\,(\Theta) = W_y^F\,(\Theta)$. Now we prove that the previous result is independent of F. First of all, one has from (3.1)

$$W_y^F\,(\Theta) - W_x^F\,(\Theta) = (\Theta_{yy} - \Theta_{xxxx}) + F'\,(\Theta)\left(\Theta_{xx}^2 - \Theta_y^2\right).$$

We compute,

$$\begin{aligned}
&W_y^F\,(\Theta) - W_x^F\,(\Theta) \\
&= W_y\,(\Theta) - W_x\,(\Theta) + \frac{1}{\Theta}\Theta_y^2 - \frac{1}{\Theta}\Theta_{xx}^2 + F'\,(\Theta)\left(\Theta_{xx}^2 - \Theta_y^2\right) \\
&= W_y\,(\Theta) - W_x\,(\Theta) + \left(\frac{1}{\Theta} - F'\,(\Theta)\right) H\,(\Theta)\,(\Theta_y + \Theta_{xx})\,.
\end{aligned}$$

Since $H\,(\Theta) = 0$, by the previous result $W_y^F\,(\Theta) - W_x^F\,(\Theta) = 0$ and $W_y\,(\Theta) - W_x\,(\Theta) = 0$. □

Remark 3.3.2. The reciproque of Lemma 3.3.3 does not hold in general. Indeed, the condition $H\,(\Theta) = 0$ is sufficient to cancel the third part of the equation (3.9). However, the conditions $W_x^F\,(\Theta) = W_y^F\,(\Theta)$ are not sufficient to ensure $H\,(\Theta) = 0$. See Corollary 3.5 for additional details.

Lemma 3.3.4. *Let $\Theta > 0$ be a smooth real-valued phase. Then Θ is of y-Wronskian type if and only if $\Theta = A\,(t,x)\exp\left(c\,(t,x)\,y\right)$, for arbitrary $A > 0$, c real-valued.*

Demostración. The sufficient condition is clear. We prove the necessary condition. By hypothesis, $W_y\,(\Theta) = 0$, i.e. $\Theta\Theta_{yy} - \Theta_y^2 = 0$. Since $\Theta > 0$, this equation can be written as $\Theta^2\partial_y\left(\frac{\Theta_y}{\Theta}\right) = 0$. Hence, $\Theta_y = c\,(t,x)\,\Theta$, with $c\,(t,x)$ a well-defined function in $\mathbb{R}^2$. Then $\Theta\,(t,x,y) = A\,(t,x)\exp\left(c\,(t,x)\,y\right)$. Note that $A\,(t,x) > 0$, since $\Theta\,(t,x,y) > 0$. □

Remark 3.3.3. It is noticed that phases satisfying the y-Wronskian condition are extremely rigid. On the contrary, the x-Wronskian condition seems less demanding.

Corolary 3.4 *Let $\Theta > 0$ be a smooth phase. Then Θ is of x-Wronskian and y-Wronskian type if and only if*

$$\Theta\,(t,x,y) = A\,(t,x)\exp\left(\left(c_1\,(t) + c_2\,(t)\,x\right)y\right),$$

with c_1, c_2 time-dependent, smooth arbitrary functions and $A > 0$ an x-Wronskian type function. In the case where $c_2(t) \neq 0$, one has $A(t,x) = c_3(t)\exp(c_4(t)x)$, with $c_3(t) > 0$ and $c_4(t) \in \mathbb{R}$ is smooth and arbitrary.

Demostración. As in the proof of Lemma 3.3.4, we only prove the necessary condition. Thanks to this last result, $\Theta(t,x,y) = A(t,x)\exp(c(t,x)y)$. Assuming now $W_x(\Theta) = 0$, i.e. $\Theta\Theta_{xxxx} - \Theta_{xx}^2 = 0$, and replacing Θ in this equation, one gets the next degree 3 polynomial in the variable y:

$$\begin{aligned} p(y) = {} & y^3 \left(4A^2c_x^2c_{xx}\right) \\ & + y^2\left(-4A_x^2c_x^2 + 4AA_{xx}c_x^2 + 8AA_xc_xc_{xx} + 2A^2c_{xx}^2 + 4A^2c_xc_{xxx}\right) \\ & + y\left(-4A_xA_{xx}c_x + 4AA_{xx}c_{xx} + 4AA_{xxx}c_x + 4AA_xc_{xxx} + A^2c_{xxxx}\right) \\ & + AA_{xxxx} - A_{xx}^2 = 0. \end{aligned}$$

By lineal independence, each of the coefficients that multiplies y^i with $i \in \{0,1,2,3\}$, must be equal to zero. Making the coefficient that multiplies y^3 equal to 0 one gets $4A^2c_x^2c_{xx} = 0$. Since $A > 0$, $(c_x^3)_x = 0$. Consequently, $c(t,x) = c_1(t) + c_2(t)x$. Now the remaining equations in powers of y return the equations

$$\begin{aligned} AA_{xxxx} - A_{xx}^2 &= 0, \\ 4c_2(t)(AA_{xxx} - A_xA_{xx}) &= 0, \\ 4c_2^2(t)\left(AA_{xx} - A_x^2\right) &= 0. \end{aligned}$$

Assume $c_2(t) \neq 0$. The third equation implies the first and second ones. We are left to solve $AA_{xx} = A_x^2$, or $A^2\left(\dfrac{A_x}{A}\right)_x = 0$. The solution is

$$A(t,x) = c_3(t)\exp(c_4(t)x).$$

Notice that A is of x-Wronskian type, that is, $AA_{xxxx} = A_{xx}^2$. Replacing A in the phase Θ,

$$\Theta(t,x,y) = c_3(t)\exp\Big(c_4(t)x + (c_1(t) + c_2(t)x)y\Big),$$

as desired.

If now $c_2(t) = 0$, then A is of x-Wronskian type. This ends the proof. □

Corolary 3.5 *Assume $\Theta > 0$. The conditions $W_y(\Theta) = W_x(\Theta) = 0$ do not necessarily implies $H(\Theta) = 0$.*

Demostración. From Corollary 3.4, and assuming $c_2(t)$ different from zero, we obtain that necessarily $\Theta(t,x,y) = c_3(t)\exp(c_4(t)x + (c_1(t) + c_2(t)x)y)$. Then

$$H(\Theta) = \Theta_y - \Theta_{xx} = \Theta\left((c_1 + c_2x) - (c_2y + c_4)^2\right).$$

Then H is identically zero only if $c_2 = 0$ and $c_1 = c_4^2$. Therefore, in general $W_x(\Theta) = W_y(\Theta) = 0$ do not implies $H(\Theta) = 0$, except if $c_2 = 0$ and $c_1 = c_4^2$. □

Lemma 3.3.5. *If $W_y(\Theta) = W_x(\Theta) = Ai(\Theta) = 0$, then $\Theta = A(t,x)\exp(cy)$, with $A(t,x)$ being of Airy type and with $c \in \mathbb{R}$.*

Demostración. From Corollary 3.4, if $W_y(\Theta) = W_x(\Theta) = 0$ one has

$$\Theta(t,x,y) = A(t,x)\exp\left(\left(c_1(t) + c_2(t)x\right)y\right).$$

If now $Ai(\Theta) = 0$,

$$\begin{aligned}
0 = \; & -y\exp\left(y\left(c_1(t) + xc_2(t)\right)\right)\left(3c_2(t)A_{xx}(t,x) + c_1'(t)\left(-A(t,x)\right) - xc_2'(t)A(t,x)\right) \\
& - 3y^2c_2^2(t)A_x(t,x)\exp\left(y\left(c_1(t) + xc_2(t)\right)\right) \\
& - \left(A_{xxx}(t,x) - A_t(t,x)\right)\exp\left(y\left(c_1(t) + xc_2(t)\right)\right) \\
& - y^3c_2^3(t)A(t,x)\left(\exp\left(y\left(c_1(t) + xc_2(t)\right)\right)\right).
\end{aligned}$$

If $A = 0$ the phase is trivial and we are done. Assuming just $c_2(t) = 0$, one gets $c_1'(t) = 0$ and then c_1 is constant, and A satisfies Airy. Then $\Theta = A(t,x)\exp(cy)$, with $A(t,x)$ being of Airy type. □

Lemma 3.3.6. *Let $u = 2\partial_x^2 F(\Theta)$ solution of (3.1). If F is a smooth profile satisfying $F(1) = 0$, $F'(1) = 1$, $F''(1) = -1$ and $F'''(1) = 2$ and if $\Theta > 0$ is a smooth phase satisfying $W_y^F(\Theta) = W_x^F(\Theta)$ and $Ai(\Theta) = 0$, then $F = \log$ and consequently $W_y^F = W_y$, $W_x^F = W_x$.*

Demostración. From (3.9) we have

$$\begin{aligned}
& \left(\rho''(s) - 2F'(\Theta)\rho'(s) + 4F''(\Theta)\rho(s)\right)\Theta_x^4 + 6\rho'(s)\Theta_x^2\Theta_{xx} + 3\rho(s)\left(\Theta_{xx}^2 + \Theta_y^2\right) \\
& \quad - 4F''(\Theta)\Theta_x Ai(\Theta) - 4F'(\Theta)Ai(\Theta)_x \\
& \quad + 3F'(\Theta)\left(W_y^F(\Theta) - W_x^F(\Theta)\right) = 0.
\end{aligned}$$

Using the hypothesis, only the first part remains

$$\left(\rho''(s) - 2F'(\Theta)\rho'(s) + 4F''(\Theta)\rho(s)\right)\Theta_x^4 + 6\rho'(s)\Theta_x^2\Theta_{xx} + 3\rho(s)\left(\Theta_{xx}^2 + \Theta_y^2\right) = 0.$$

This is an ODE for ρ with variable coefficients, exactly (3.2). Lemma 3.2.3 ensures that $\rho = 0$. Lemma 3.2.2 implies that $F = \log$ and by definition $W_y^F = W_y$, $W_x^F = W_x$. □

Corolary 3.6 *If $\Theta > 0$ is a smooth phase and F a smooth profile that satisfies* (1.10) *giving rise to a solution u of* (3.1) *such that $W_x^F(\Theta) = W_y^F(\Theta) = Ai(\Theta) = 0$, then $F = \log$.*

Demostración. Direct from Lemma 3.3.6. □

3.4. Soliton structures

3.4.1. Airy and Heat structures

Recall that the Airy type condition defined in Definition 3.1 is described by the identity $Ai(\Theta) = \Theta_t - \Theta_{xxx} = 0$ and does not coincide with linear part of the KP equation, in contrast with the KdV case where being of Airy type implies that Θ satisfies the linear part of KdV.

Some simple solutions of $Ai(\Theta) = 0$ are

$$\Theta_1(t,x,y) = A_0 \exp\left(k_1 x + k_1^2 y + k_1^3 t\right) + B_0 \exp\left(k_2 x + k_2^2 y + k_2^3 t\right) + C_0,$$

$$\Theta_2(t,x,y) = A(y) \exp\left(kx + k^3 t\right),$$

where $A_0, B_0, C_0, k_1, k_2, k \in \mathbb{R}$ and $A > 0$ is any well-defined function. Associated to the profile $F = \log$, the phase Θ_1 corresponds to a line-soliton phase while Θ_2 gives the trivial solution. Additionally, Θ_1 and Θ_2 are unique up to ponderations.

Remark 3.4.1. The phase Θ_1 serve as an example exhibiting both Airy and Heat type.

Lemma 3.4.1. *If Θ is of Heat and Airy type, then is of $\mathcal{T}$-type. The reciprocal is clearly false.*

DEMOSTRACIÓN. Direct from (3.6). □

Lemma 3.4.2. *Assume that Θ is of $\mathcal{T}$-type and $F'(\Theta)$ is different from zero for some $(t,x,y) \in \mathbb{R}^3$. If Θ is of Heat type, then exists $c_0(t,y) \in \mathbb{R}$ such that*

$$Ai(\Theta) = \frac{c_0(t,y)}{F'(\Theta)}. \tag{3.1}$$

DEMOSTRACIÓN. Notice that Θ being of $\mathcal{T}$-type as in (3.6) is equivalent to have

$$\begin{aligned} 0 = & -4F''(\Theta)\Theta_x Ai(\Theta) - 4F'(\Theta) Ai(\Theta)_x + 3F'(\Theta)\left(H(\Theta)_y + H(\Theta)_{xx}\right) \\ & - 3F'(\Theta)^2 H(\Theta)(\Theta_y + \Theta_{xx}). \end{aligned}$$

Equivalently

$$\left(F'(\Theta) Ai(\Theta)\right)_x = \frac{3F'(\Theta)}{4}\left(\left(H(\Theta)_y + H(\Theta)_{xx}\right) - F'(\Theta) H(\Theta)(\Theta_y + \Theta_{xx})\right)$$

Then (3.1) follows directly from assuming $H(\Theta) = 0$ for all (t,x,y) and solving the corresponding ODE for $Ai(\Theta)$. □

Lemma 3.4.3. *Let $\Theta > 0$ be a smooth phase satisfying $W_x(\Theta) = W_y(\Theta) = 0$. Then the following conditions are satisfied:*

(i) If $H(\Theta) = 0$, then $\Theta(t,x,y) = c_3(t) e^{\left(\sqrt{c_1(t)}x + c_1(t)y\right)} + c_4(t) e^{\left(-\sqrt{c_1(t)}x + c_1(t)y\right)}$.

(ii) If $Ai(\Theta) = 0$, then $\Theta = A(t,x)\exp(c_1 y)$ where A is of Airy type.

(iii) If both $H(\Theta) = Ai(\Theta) = 0$, then

$$\Theta = a_1 \exp\left(k_1 x + k_1^2 y + k_1^3 t\right) + a_2 \exp\left(-k_1 x + k_1^2 y - k_1^3 t\right),$$

for some $a_1, a_2 > 0$ and k_1 constants.

Demostración. Proof of (i). From $W_x(\Theta) = W_y(\Theta) = 0$, Corollary 3.4 yields $\Theta(t,x,y) = A(t,x)\exp((c_1(t) + c_2(t)x)y)$. Now, Imposing $H(\Theta) = 0$.

$$(Ac_1 - A_{xx}) + x(Ac_2) - y\left(2A_x c_2^2\right) - y^3\left(Ac_2^3\right) = 0$$

Thus, $c_2(t) = 0$ and $A = c_3(t)\exp\left(\sqrt{c_1(t)}x\right) + c_4(t)\exp\left(-\sqrt{c_1(t)}x\right)$. Remplacing in the phase Θ,

$$\Theta(t,x,y) = c_3(t)\, e^{\left(\sqrt{c_1(t)}x + c_1(t)y\right)} + c_4(t)\, e^{\left(-\sqrt{c_1(t)}x + c_1(t)y\right)}$$

This proves (i).

Proof of (ii). From Corollary 3.4 it is known that the conditions $W_y = W_x = 0$ allows to construct the phase

$$\Theta = A(t,x)\exp((c_1(t) + c_2(t)x)y).$$

Now, if $Ai(\Theta) = 0$,

$$(A_t - A_{xxx}) + y(Ac_1' - 3A_{xx}c_2) + yx(Ac_2') - y^2\left(3A_x c_2^2\right) - y^3\left(Ac_2^3\right) = 0.$$

Then $c_2(t) = 0$, $c_1(t) = c_1$ and A is Airy-type. Thus, the phase is

$$\Theta = A\exp(yc_1)$$

This prove (ii).

Proof of (iii). Since $H(\Theta) = 0$ from (i) we get

$$\Theta(t,x,y) = c_3(t)\exp\left(\left(\sqrt{c_1(t)}x + c_1(t)y\right)\right) + c_4(t)\exp\left(-\sqrt{c_1(t)}x + c_1(t)y\right). \tag{3.2}$$

Assume that $Ai(\Theta) = 0$. Then

$$\begin{aligned} Ai(\Theta) = {} & \exp(\sqrt{c_1}x + c_1 y)\left(\left(c_3' - c_1^{\frac{3}{2}}c_3\right) + \left(\frac{c_3 c_1'}{2\sqrt{c_1}}\right)x + (c_3 c_1')y\right) \\ & + \exp(-\sqrt{c_1}x + c_1 y)\left(\left(c_4' + c_1^{\frac{3}{2}}c_4\right) + \left(-\frac{c_4 c_1'}{2\sqrt{c_1}}\right)x + (c_4 c_1')y\right). \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= \left(c_3' - c_1^{\frac{3}{2}}c_3\right) + \left(\frac{c_3 c_1'}{2\sqrt{c_1}}\right)x + (c_3 c_1')y \\ 0 &= \left(c_4' + c_1^{\frac{3}{2}}c_4\right) + \left(-\frac{c_4 c_1'}{2\sqrt{c_1}}\right)x + (c_4 c_1')y \end{aligned}$$

Consequently, $c_1(t) = c_1$, $c_3(t) = a_1\exp\left(c_1^{\frac{3}{2}}\right)$, $c_4(t) = a_2\exp\left(-c_1^{\frac{3}{2}}\right)$, with c_1, a_1, a_2 arbitrary constants. Then, the phase is equal to

$$\Theta = a_1\exp\left(kx + k^2 y + k^3 t\right) + a_2\exp\left(-kx + k^2 y - k^3 t\right)$$

with $k = \sqrt{c_1}$.

On the other hand, if one starts by assuming that $Ai\left(\Theta\right)=0$, by (ii), the phase is

$$\Theta = A\left(t,x\right)\exp\left(c_1 y\right)$$

with $A\left(t,x\right)$ Airy-type.

Impossing $H\left(\Theta\right)=0$,

$$0 = \exp\left(c_1 y\right)\left(Ac_1 - A_{xx}\right)$$

Consequently, $A\left(t,x\right) = c_3\left(t\right)\exp\left(\sqrt{c_1\left(t\right)}x\right) + c_4\left(t\right)\exp\left(-\sqrt{c_1\left(t\right)}x\right)$.

Recall that $A\left(t,x\right)$ is Airy- type and then

$$\begin{aligned} Ai\left(A\right) = {} & \exp\left(\sqrt{c_1}x\right)\left(\left(c_3' - c_3 c_1^{\frac{3}{2}}\right) + \left(\frac{c_3 c_1'}{2\sqrt{c_1}}\right)x\right) \\ & + \exp\left(-\sqrt{c_1}x\right)\left(\left(c_4' + c_4 c_1^{\frac{3}{2}}\right) + \left(\frac{c_4 c_1'}{2\sqrt{c_1}}\right)x\right). \end{aligned}$$

Thus, $c_1\left(t\right) = c_1$, $c_3\left(t\right) = a_1\exp\left(c_1^{\frac{3}{2}}t\right)$, $c_4\left(t\right) = a_2\exp\left(-c_1^{\frac{3}{2}}t\right)$, with c_1, a_1, a_2 arbitrary constants. Considering this in $A\left(t,x\right)$,

$$A\left(t,x\right) = a_1\exp\left(\sqrt{c_1}x + c_1^{\frac{3}{2}}t\right) + a_2\exp\left(-\sqrt{c_1}x - c_1^{\frac{3}{2}}t\right).$$

Finally, replacing in the phase,

$$\Theta = a_1\exp\left(kx + k^2 y + k^3 t\right) + a_2\exp\left(-kx + k^2 y - k^3 t\right).$$

with $k=\sqrt{c_1}$. This proves (iii). □

Finally, we provide a quick method to construct an Airy-Heat phase Θ.

Lemma 3.4.4. *Let $\Theta_0 = \Theta_0\left(t,x\right)$ be any solution of Airy $Ai\left(\Theta_0\right)=0$ such that there are $C_1, C_2 > 0$ such that $|\Theta_0\left(t\right)| \leq C_1\exp\left(C_2|x|\right)$. Then $\Theta\left(t,x,y\right) := \exp\left(y\partial_x^2\right)\Theta_0$ solves $H\left(\Theta\right)=0$ for $y\geq 0$.*

Demostración. This result is obtained by the formula

$$\Theta\left(t,x,y\right) = \exp\left(y\partial_x^2\right)\Theta_0\left(t,x\right) = \frac{1}{\left(4\pi y\right)^{1/2}}\int_{\mathbb{R}}\exp\left(-\frac{\left(x-s\right)^2}{4\pi y}\right)\Theta_0\left(t,s\right)\mathrm{d}s$$

and the growth of Θ_0. □

Remark 3.4.2. As already mentioned in the introduction (see Remark 3.1.9), a similar result for the case of the Airy equation $Ai\left(\Theta\right)=0$ with initial condition exponentially growing in x is far from obvious.

3.4.2. Soliton structures

Recall from (3.10) that a simple KP line-soliton is obtained by the structure

$$\Theta\left(t,x,y\right) = a_1\exp\left(\theta_1\right) + a_2\exp\left(\theta_2\right), \tag{3.3}$$

where $a_1, a_2 > 0$, and $\theta_j := k_j x + k_j^2 y + k_j^3 t$, with $k_1, k_2 \in \mathbb{R}$. We shall say that Θ represents a line-soliton if Θ has the previous form. Notice that unless $F = \log$, one has that $u = 2\partial_x^2 \log(\Theta)$ is the standard KP line-soliton (3.11), but for the moment, this will not be used.

Lemma 3.4.5. *If $\Theta > 0$ smooth represents a line-soliton, then $Ai(\Theta) = H(\Theta) = 0$. Moreover,the case $k_1 = \pm k_2$, is the unique instance in where $W_x(\Theta) = W_y(\Theta) = 0$.*

Remark 3.4.3. Notice that the condition $k_1 = -k_2$ corresponds to the case of the KdV soliton (vertical line-soliton) and $k_1 = k_2$ the trivial solution.

Proof of Lemma 3.4.5. The proof comes from (3.3). Indeed

$$Ai(\Theta) = \Theta_t - \Theta_{xxx} = k_1^3 a_1 \exp(\theta_1) + k_2^3 a_2 \exp(\theta_2) - \Big(k_1^3 a_1 \exp(\theta_1) + k_2^3 a_2 \exp(\theta_2)\Big) = 0.$$

Additionally,

$$H(\Theta) = \Theta_y - \Theta_{xx} = k_1^2 a_1 \exp(\theta_1) + k_2^2 a_2 \exp(\theta_2) - \Big(k_1^2 a_1 \exp(\theta_1) + k_2^2 a_2 \exp(\theta_2)\Big) = 0.$$

Finally,

$$\begin{aligned}
\Theta W_y(\Theta) &= \Theta\Theta_{yy} - \Theta_y^2 \\
&= (a_1 \exp(\theta_1) + a_2 \exp(\theta_2)) \Big(k_1^4 a_1 \exp(\theta_1) + k_2^4 a_2 \exp(\theta_2)\Big) \\
&\quad - \Big(k_1^2 a_1 \exp(\theta_1) + k_2^2 a_2 \exp(\theta_2)\Big)^2 \\
&= k_1^4 a_1^2 \exp(2\theta_1) + k_2^4 a_1 a_2 \exp(\theta_1 + \theta_2) + k_1^4 a_1 a_2 \exp(\theta_1 + \theta_2) + k_2^4 a_2^2 \exp(2\theta_2) \\
&\quad - k_1^4 a_1^2 \exp(2\theta_1) - 2k_1^2 k_2^2 a_1 a_2 \exp(\theta_1 + \theta_2) - k_2^4 a_2^2 \exp(2\theta_2) \\
&= \Big(k_1^4 + k_2^4 - 2k_1^2 k_2^2\Big) a_1 a_2 \exp(\theta_1 + \theta_2).
\end{aligned}$$

We conclude that

$$\Theta W_y(\Theta) = \Big(k_1^2 - k_2^2\Big)^2 a_1 a_2 \exp(\theta_1 + \theta_2). \tag{3.4}$$

Notice that $W_y(\Theta) = 0$ if and only if $k_1 = \pm k_2$. Since Θ is a Heat type phase, by Lemma 3.3.3,

$$W_x(\Theta) = W_y(\Theta) = 0.$$

The proof is complete. $\square$

The previous result can be extended to smooth phases Θ of the form,

$$\Theta = \sum_{j=1}^{M} a_j \exp(\theta_j), \quad a_j > 0, \quad \theta_j := k_j x + k_j^2 y + k_j^3 t, \quad k_j \in \mathbb{R}. \tag{3.5}$$

where $k_1 < k_2 < \cdots < k_M$. In Kodama [20] this phase represents a multi estructure graphically represented by $M-1$ legs on the region y negative and 1 leg in the positive part of the y-axis. In this case, we shall say that Θ generates an M multi-line-soliton or a resonant soliton.

Lemma 3.4.6. *If $\Theta \in \mathcal{W}_M$, then*

$$\Theta W_y(\Theta) \in \mathcal{W}_{\frac{1}{2}M(M-1)}.$$

In particular, $\Theta \in \mathcal{W}_3$ *implies* $\Theta W_y(\Theta) \in \mathcal{W}_3$.

Demostración. Assume that $\Theta \in \mathcal{W}_M$, that is for $0 \leq k_1(t,x) < k_2(t,x) < \ldots < k_N(t,x)$

$$\Theta = \sum_{j=1}^{M} a_j \exp(\theta_j), \quad a_j(t,x) > 0, \; \theta_j(t,x,y) = k_j(t,x)\, y.$$

Note now that $\left(\sum_{n=1}^{M} a_n \sum_{n=1}^{M} b_n = \sum_{n=1}^{M} c_n, \text{ where } c_n = \sum_{i=0}^{n} a_i b_{n-i}\right)$,

$$\begin{aligned}
\Theta W_y(\Theta) &= \Theta\Theta_{yy} - \Theta_y^2 \\
&= \left(\sum_{i=1}^{M} a_i \exp(\theta_i)\right)\left(\sum_{i=1}^{M} k_i^4 a_i \exp(\theta_i)\right) - \left(\sum_{i=1}^{M} k_i^2 a_i \exp(\theta_i)\right)^2 \\
&= \sum_{n=1}^{M}\sum_{i=1}^{n} k_{n-i+1}^4 a_i a_{n-i+1} \exp(\theta_i + \theta_{n-i+1}) \\
&\quad - \sum_{n=1}^{M}\sum_{i=1}^{n} k_i^2 k_{n-i+1}^2 a_i a_{n-i+1} \exp(\theta_i + \theta_{n-i}) \\
&= \sum_{n=1}^{M}\sum_{i=1}^{n} a_i a_{n-i+1}\left(k_{n-i+1}^4 - k_i^2 k_{n-i+1}^2\right)\exp(\theta_i + \theta_{n-i+1}).
\end{aligned}$$

Since $\theta_i + \theta_{n-i+1} = \theta_{n-i+1} + \theta_i$,

$$\begin{aligned}
\Theta W_y(\Theta) &= \sum_{n=1}^{M}\sum_{i=1}^{n} a_i a_{n-i+1}\left(k_{n-i+1}^4 - k_i^2 k_{n-i+1}^2\right)\exp(\theta_i + \theta_{n-i+1}) \\
&= \sum_{n=1}^{M}\sum_{i=1}^{n} a_i a_{n-i+1}\left(k_{n-i+1}^4 - k_i^2 k_{n-i+1}^2 + k_i^4 - k_{n-i+1}^2 k_i^2\right)\exp(\theta_i + \theta_{n-i+1}) \\
&= \sum_{n=1}^{M}\sum_{i=1}^{n} a_i a_{n-i+1}\left(k_{n-i+1}^4 - 2k_i^2 k_{n-i+1}^2 + k_i^4\right)\exp(\theta_i + \theta_{n-i+1}) \\
&= \sum_{n=1}^{M}\sum_{i=1}^{n} a_i a_{n-i+1}\left(k_{n-i+1}^2 - k_i^2\right)^2\exp(\theta_i + \theta_{n-i+1}).
\end{aligned}$$

Under the hypotheses of the lemma, in general there are $\frac{1}{2}(M^2 - M)$ linearly independent terms in the last expression, proving the required inclusion in $\mathcal{W}_{\frac{1}{2}M(M-1)}$. The M-term corresponds to the diagonal term, and M^2 represents all the possible elements in an $M \times M$ matrix. $\square$

Corolary 3.7 *If* $\Theta > 0$ *as in* (3.5) *generates an* M *multi-line-soliton, then* $Ai(\Theta) = H(\Theta) = 0$ *but* $W_x(\Theta) = W_y(\Theta)$ *are different from zero unless* $M = 1$ *or* $M = 2$ *with* $k_1 = -k_2$.

Demostración. The proof of $Ai(\Theta) = H(\Theta) = 0$ is direct. Indeed, computing the corresponding derivatives and replacing its values in $H(\Theta)$,

$$\begin{aligned}
H(\Theta) &= \Theta_y - \Theta_{xx} \\
&= \sum_{i=1}^{M} k_i^2 a_1 \exp\left(k_i x + k_i^2 y + k_i^3 t\right) - \sum_{i=1}^{M} k_i^2 a_1 \exp\left(k_i x + k_i^2 y + k_i^3 t\right) = 0.
\end{aligned}$$

Similarly,

$$\begin{aligned} Ai(\Theta) &= \Theta_t - \Theta_{xxx} \\ &= \sum_{i=1}^{M} k_i^3 a_i \exp\left(k_i x + k_i^2 y + k_i^3 t\right) - \sum_{i=1}^{M} k_i^3 a_i \exp\left(k_i x + k_i^2 y + k_i^3 t\right) = 0. \end{aligned}$$

Therefore, if $\Theta(t,x,y) = \sum_{i=1}^{M} a_i \exp(k_i x + k_i^2 y + k_i^3 t)$ then $H(\Theta) = Ai(\Theta) = 0$.

Now, from Lemma 3.4.6, one has $\Theta W_y(\Theta) = 0$ if and only if,

$$k_i^2 = k_j^2,$$

for all $i, j \in \{1, \ldots, M\}$. If $M \geq 3$ at least one term of the sum is different to zero, since $k_1 < k_2 < \ldots < k_M$ implying W_y different from zero. In the cases $M = 2$ or $M = 1$, from Lemma 3.4.3 (iii) one has that $W_y = 0$ is equivalent to a phase associated to a KdV soliton or a trivial solution. □

3.4.3. 2-solitons

For the following result, recall the 2-soliton phase introduced in (3.14).

Lemma 3.4.7. *Let $\Theta > 0$ be a 2-soliton of scaling parameters $0 \leq k_1 < k_2 < k_3 < k_4$, and where each soliton correspond to a line-soliton. Then the following are satisfied:*

(i) In general $Ai(\Theta)$, $H(\Theta)$, $W_y(\Theta)$ and $W_x(\Theta)$ are different from zero.

(ii) In general, $Ai(\Theta)$, $H(\Theta) \in \mathcal{W}_4$ and $\Theta W_y(\Theta), \Theta W_x(\Theta) \in \mathcal{W}_5$.

(iii) If now $k_1 = -k_2$ and $k_3 = -k_4$ (this is, the case of 2 vertical line-solitons), then $W_y(\Theta) = 0$.

Demostración. After computing (see Appendix 3.6.2),

$$\begin{aligned} Ai(\Theta) = \Theta_t - \Theta_{xxx} & \\ = \; & (k_3 - k_1)\left(k_3^3 + k_1^3\right)\exp(\theta_1 + \theta_3) + (k_4 - k_1)\left(k_4^3 + k_1^3\right)\exp(\theta_1 + \theta_4) \\ & + (k_3 - k_2)\left(k_3^3 + k_2^3\right)\exp(\theta_2 + \theta_3) + (k_4 - k_2)\left(k_4^3 + k_2^3\right)\exp(\theta_2 + \theta_4) \\ & - (k_3 - k_1)(k_3 + k_1)^3 \exp(\theta_1 + \theta_3) - (k_4 - k_1)(k_4 + k_1)^3 \exp(\theta_1 + \theta_4) \\ & - (k_3 - k_2)(k_3 + k_2)^3 \exp(\theta_2 + \theta_3) - (k_4 - k_2)(k_4 + k_2)^3 \exp(\theta_2 + \theta_4). \end{aligned} \tag{3.6}$$

notice that $Ai(\Theta) = 0$ if and only if the associated exponentials coefficients are zero. Thus, for each $(j,i) \in \{(1,3),(1,4),(2,3),(2,4)\}$

$$\begin{aligned} & (k_i - k_j)\left(k_i^3 + k_j^3\right) - (k_i - k_j)(k_i + k_j)^3 \\ &= k_i^4 + k_i k_j^3 - k_i^3 k_j - k_j^4 - (k_i - k_j)\left(k_i^3 + 3k_i^2 k_j + 3k_i k_j^2 + k_j^3\right) \\ &= k_i^4 + k_i k_j^3 - k_i^3 k_j - k_i^4 - 3k_i^3 k_j - 3k_i^2 k_j^2 - k_i k_j^3 + k_i^3 k_j + 3k_i^2 k_j^2 + 3k_i k_j^3 + k_j^4 \\ &= -3k_i^3 k_j + 3k_i k_j^3 \\ &= 3k_i k_j\left(k_j^2 - k_i^2\right). \end{aligned}$$

Denoting

$$E_{ij} = (k_j - k_i) \exp(\theta_i + \theta_j),$$

then $Ai(\Theta)$ can be rewritten as

$$\begin{aligned} Ai(\Theta) = & -3\Big(k_1k_3(k_1+k_3)E_{13} + k_1k_4(k_1+k_4)E_{14} \\ & \qquad + k_2k_3(k_2+k_3)E_{23} + k_2k_4(k_2+k_4)E_{24}\Big). \end{aligned}$$

This proves that $Ai(\Theta) \in \mathcal{W}_4$.

Remark 3.4.4. Assume that all the coefficients take differents values. Let us study the condition $Ai(\Theta) = 0$. If $k_1 = 0$, then $k_2 = -k_3 = -k_4$, which is not possible. A similar argument holds if now $k_2 = 0$, or $k_3 = 0$, or $k_4 = 0$. Therefore, each k_i must be nonzero. However, in this case $k_1 = -k_3 = -k_4$, also impossible. Thus, $Ai(\Theta) \neq 0$ for a nondegenerate 2-soliton.

Let us come back to the proof. Replacing the values of the derivatives in $H(\Theta)$,

$$\begin{aligned} H(\Theta) &= \Theta_y - \Theta_{xx} \\ &= (k_3-k_1)\left(k_3^2+k_1^2\right)\exp(\theta_1+\theta_3) + (k_4-k_1)\left(k_4^2+k_1^2\right)\exp(\theta_1+\theta_4) \\ &\quad + (k_3-k_2)\left(k_3^2+k_2^2\right)\exp(\theta_2+\theta_3) + (k_4-k_2)\left(k_4^2+k_2^2\right)\exp(\theta_2+\theta_4) \\ &\quad - (k_3-k_1)(k_3+k_1)^2\exp(\theta_1+\theta_3) - (k_4-k_1)(k_4+k_1)^2\exp(\theta_1+\theta_4) \\ &\quad - (k_3-k_2)(k_3+k_2)^2\exp(\theta_2+\theta_3) - (k_4-k_2)(k_4+k_2)^2\exp(\theta_2+\theta_4). \end{aligned}$$

Repeating the procedure of the Airy condition, $H(\Theta)$ can be rewritten as

$$H(\Theta) = -2\left(k_1k_3E_{13} + k_1k_4E_{14} + k_2k_3E_{23} + k_2k_4E_{24}\right).$$

The condition $k_1 < k_2 < k_3 < k_4$ naturally forbids $H(\Theta) = 0$.

Now, Replacing the values of the derivatives in $W_y(\Theta)$,

$$\begin{aligned} &\Theta W_y(\Theta) \\ &= \Big((k_3-k_1)\exp(\theta_1+\theta_3) + (k_4-k_1)\exp(\theta_1+\theta_4) \\ &\quad + (k_3-k_2)\exp(\theta_2+\theta_3) + (k_4-k_2)\exp(\theta_2+\theta_4) \Big) \\ &\quad \cdot \Big((k_3-k_1)\left(k_3^2+k_1^2\right)^2\exp(\theta_1+\theta_3) \\ &\qquad + (k_4-k_1)\left(k_4^2+k_1^2\right)^2\exp(\theta_1+\theta_4) + (k_3-k_2)\left(k_3^2+k_2^2\right)^2\exp(\theta_2+\theta_3) \\ &\qquad + (k_4-k_2)\left(k_4^2+k_2^2\right)^2\exp(\theta_2+\theta_4) \Big) \\ &\quad - \Big((k_3-k_1)\left(k_3^2+k_1^2\right)\exp(\theta_1+\theta_3) + (k_4-k_1)\left(k_4^2+k_1^2\right)\exp(\theta_1+\theta_4) \\ &\qquad + (k_3-k_2)\left(k_3^2+k_2^2\right)\exp(\theta_2+\theta_3) + (k_4-k_2)\left(k_4^2+k_2^2\right)\exp(\theta_2+\theta_4) \Big)^2. \end{aligned}$$

Developing and rearranging,

$$\begin{aligned}
&\Theta W_y(\Theta) \\
&= \left(k_1^2 - k_2^2\right)^2 \Big(\exp(\theta_1 + \theta_2 + 2\theta_4) + \exp(\theta_1 + \theta_2 + 2\theta_3) \Big) \\
&\quad + \left(k_3^2 - k_4^2\right)^2 \Big(\exp(2\theta_1 + \theta_3 + \theta_4) + \exp(2\theta_2 + \theta_3 + \theta_4) \Big) \\
&\quad + k_{1234} \exp(\theta_1 + \theta_2 + \theta_3 + \theta_4),
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
k_{1234} :=\ & (k_3 - k_1)(k_4 - k_2)\left(\left(k_1^2 + k_3^2\right) - \left(k_2^2 + k_4^2\right)\right)^2 \\
& + (k_3 - k_2)(k_4 - k_1)\left(\left(k_1^2 + k_4^2\right) - \left(k_2^2 + k_3^2\right)\right)^2.
\end{aligned}$$

A simple observation reveals that $\Theta W_y(\Theta) = 0$ if $k_1 = \pm k_2$ and $k_3 = \pm k_4$. Since all the coefficients take differents values, one necessarily has $k_1 = -k_2$ and $k_3 = -k_4$. In this case, we also have $k_{1234} = 0$. Except by this particular case, one naturally concludes from (3.7) that $\Theta W_y(\Theta) \in \mathcal{W}_5$. Therefore, W_y can be zero in the case of a 2-soliton when $k_1 = -k_2$ and $k_3 = -k_4$. This can be true when all constants have different values.

As mentioned earlier, $H(\Theta)$ is not equal to zero and therefore it is necessary to verify both conditions, $\Theta W_y(\Theta) = 0$ and $\Theta W_x(\Theta) = 0$, separately. Then,

$$\begin{aligned}
&\Theta W_x(\Theta) \\
&= \Theta\Theta_{xxxx} - \Theta_{xx}^2 \\
&= \Big((k_3 - k_1)\exp(\theta_1 + \theta_3) + (k_4 - k_1)\exp(\theta_1 + \theta_4) + (k_3 - k_2)\exp(\theta_2 + \theta_3) \\
&\quad + (k_4 - k_2)\exp(\theta_2 + \theta_4) \Big) \cdot \Big((k_3 - k_1)(k_3 + k_1)^4 \exp(\theta_1 + \theta_3) \\
&\quad + (k_4 - k_1)(k_4 + k_1)^4 \exp(\theta_1 + \theta_4) + (k_3 - k_2)(k_3 + k_2)^4 \exp(\theta_2 + \theta_3) \\
&\quad + (k_4 - k_2)(k_4 + k_2)^4 \exp(\theta_2 + \theta_4) \Big) - \Big((k_3 - k_1)(k_3 + k_1)^2 \exp(\theta_1 + \theta_3) \\
&\quad + (k_4 - k_1)(k_4 + k_1)^2 \exp(\theta_1 + \theta_4) + (k_3 - k_2)(k_3 + k_2)^2 \exp(\theta_2 + \theta_3) \\
&\quad + (k_4 - k_2)(k_4 + k_2)^2 \exp(\theta_2 + \theta_4) \Big)^2.
\end{aligned}$$

Developing and grouping terms, we arrive to

$$\begin{aligned}
&\Theta W_x(\Theta) \\
&= \Big((k_1 + k_4)^2 - (k_1 + k_3)^2 \Big)^2 E_{13}E_{14} + \Big((k_2 + k_3)^2 - (k_1 + k_3)^2 \Big)^2 E_{13}E_{23} \\
&\quad + \left(\Big((k_2 + k_4)^2 - (k_1 + k_3)^2 \Big)^2 + \Big((k_2 + k_3)^2 - (k_1 + k_4)^2 \Big)^2 \right) E_{14}E_{23} \\
&\quad + \Big((k_1 + k_4)^2 - (k_2 + k_4)^2 \Big)^2 E_{14}E_{24} + \Big((k_2 + k_3)^2 - (k_2 + k_4)^2 \Big)^2 E_{23}E_{24}.
\end{aligned} \tag{3.8}$$

Clearly from (3.8) one concludes that in general $\Theta W_x(\Theta) \in \mathcal{W}_5$. In order to get zero value, one should have

$$|k_1 + k_4| = |k_1 + k_3| = |k_2 + k_4| = |k_2 + k_3|.$$

Under the hypothesis $k_1 < k_2 < k_3 < k_4$, this is never satisfied. Indeed,

$$k_1 + k_3 < k_2 + k_3 < k_2 + k_4, \quad k_1 + k_3 < k_1 + k_4;$$

contradicting the equality of absolute values. Then $\Theta W_x(\Theta)$ is always different from zero to a phase of a 2-soliton. $\square$

Corolary 3.8 *In general, given the profile $F = \log$, any 2-soliton as in* (3.14) *only satisfies $\mathcal{T} = 0$.*

Demostración. Direct from (3.8), Lemma 3.2.2 and Lemma 3.4.7 (i). $\square$

3.5. Proof of Main Results

3.5.1. Proof of Theorem 3.1.3

Assume $\Theta > 0$ is a smooth KdV line-soliton phase with a profile $F = \log$. Its phase is given by

$$\Theta(t, x, y) = \exp\left(kx + k^2 y + k^3 t\right) + \exp\left(-kx + k^2 y - k^3 t\right) = \exp(\theta_1) + \exp(\theta_2), \qquad (3.1)$$

with $\theta_1 = kx + k^2 y + k^3 t$ and $\theta_2 = -kx + k^2 y - k^3 t$. It will be shown that the KdV phase satisfies the following conditions

$$H(\Theta) = Ai(\Theta) = W_x(\Theta) = W_y(\Theta) = 0.$$

Computing derivatives, we obtain

$$\begin{aligned}
\Theta_x &= k\exp(\theta_1) - k\exp(\theta_2), \quad & \Theta_{xx} &= k^2\exp(\theta_1) + k^2\exp(\theta_2), \\
\Theta_{xxx} &= k^3\exp(\theta_1) - k^3\exp(\theta_2), \quad & \Theta_{xxxx} &= k^4\exp(\theta_1) + k^4\exp(\theta_2), \\
\Theta_y &= k^2\exp(\theta_1) + k^2\exp(\theta_2), \quad & \Theta_{yy} &= k^4\exp(\theta_1) + k^4\exp(\theta_2), \\
\Theta_t &= k_1^3\exp(\theta_1) - k_1^3\exp(\theta_2). & &
\end{aligned}$$

Replacing the values of Θ_y and Θ_{xx} in $H(\Theta)$,

$$\begin{aligned}
H(\Theta) &= \Theta_y - \Theta_{xx} \\
&= \left(k^2\exp(\theta_1) + k^2\exp(\theta_2)\right) - \left(k^2\exp(\theta_1) + k^2\exp(\theta_2)\right) \\
&= \left(k^2\exp(\theta_1) - k^2\exp(\theta_1)\right) + \left(k^2\exp(\theta_2) - k^2\exp(\theta_2)\right) \\
&= 0.
\end{aligned}$$

Hence $H(\Theta) = 0$, then Θ is of Heat type. Since $H(\Theta) = 0$ from Lemma 3.3.3, $W_x(\Theta) = W_y(\Theta)$. Now, replacing the values of the derivatives of Θ in $W_x(\Theta)$,

$$\begin{aligned}
W_x(\Theta) &= \Theta_{xxxx} - \frac{\Theta_{xx}^2}{\Theta} \\
&= \left(k^4 \exp(\theta_1) + k^4 \exp(\theta_2)\right) - \frac{\left(k^2 \exp(\theta_1) + k^2 \exp(\theta_2)\right)^2}{\left(\exp(\theta_1) + \exp(\theta_2)\right)} \\
&= k^4 \frac{\left(\exp(\theta_1) + \exp(\theta_2)\right)^2}{\left(\exp(\theta_1) + \exp(\theta_2)\right)} - k^4 \frac{\left(\exp(\theta_1) + \exp(\theta_2)\right)^2}{\left(\exp(\theta_1) + \exp(\theta_2)\right)} \\
&= 0.
\end{aligned}$$

Since $H(\Theta) = 0$, this also means that $W_y(\Theta) = 0$.

Finally, replacing the derivatives of Θ in $Ai(\Theta)$

$$Ai(\Theta) = \Theta_t - \Theta_{xxx} = \left(k^3 \exp(\theta_1) - k^3 \exp(\theta_2)\right) - \left(k^3 \exp(\theta_1) - k^3 \exp(\theta_2)\right) = 0.$$

Then, if $u = 2\partial_x^2 \log \Theta$ is a KdV line-soliton solution of KP (i.e. $\Theta = \exp(kx + k^2 y + k^3 t) + \exp(-kx + k^2 y - k^3 t)$) then $H(\Theta) = Ai(\Theta) = W_x(\Theta) = W_y(\Theta) = 0$.

Conversely, it will now be demonstrated that if a phase Θ satisfies $H(\Theta) = W_x(\Theta) = W_y(\Theta) = Ai(\Theta) = 0$, then the corresponding solution $u = 2\partial_x^2 F(\Theta)$ is a KdV vertical line-soliton, that is, $F = \log$ and Θ as in (3.1).

Since $H(\Theta) = Ai(\Theta) = W_x(\Theta) = W_y(\Theta) = 0$, it is sufficient to look at equation (3.7), to conclude that

$$\left(\rho(s)'' - 2F'(\Theta)\rho(s)' + 4F''(\Theta)\rho(s)\right)\Theta_x^4 + 6\rho(s)'\Theta_x^2\Theta_{xx} + 3\rho(s)\left(\Theta_{xx}^2 + \Theta_y^2\right) = 0.$$

By the hypothesis on F and the values of its derivatives at $s = 1$, if there exists a solution to the KP equation of the form $u = 2\partial_x^2 F(\Theta)$, (3.7) is satisfied, and by Lemma 3.2.3 (iii), $F = \log$.

Now, Lemma 3.3.3 implies that $W_y(\Theta) - W_x(\Theta) = 0$. Since $W_y(\Theta) = W_x(\Theta) = Ai(\Theta) = 0$, Lemma 3.3.5 ensures that $\Theta = A(t,x)\exp(cy)$, with $A(t,x)$ being of Airy type and with $c \in \mathbb{R}$.

Finally, the condition $H(\Theta) = 0$ implies

$$\Theta_y - \Theta_{xx} = (cA - A_{xx})\exp(cy) = 0.$$

We first treat the case $c = 0$. In this case

$$A(t,x) = c_{A,1}(t) + c_{A,2}(t)\,x.$$

Since A must satisfy the Airy equation for all $(t,x,y) \in \mathbb{R}^3$, one gets $c_{A,1}'(t) + c_{A,2}'(t)\,x = 0$, implying that $c_{A,1}$ and $c_{A,2}$ are constants. Θ is given in this case by

$$\Theta = c_{A,1} + c_{A,2}x,$$

corresponding to a singular soliton solution, discarded by smoothness assumptions.

Now we assume c different from zero. Here,

$$A(t,x) = c_{A,1}(t)(t)\exp(kx) + c_{A,2}(t)(t)\exp(-kx), \quad k = \sqrt{c} \in \mathbb{C} - \{0\}. \tag{3.2}$$

Solving again Airy,

$$\begin{aligned} A_t - A_{xxx} &= c'_{A,1}(t)\exp(kx) + c'_{A,2}\exp(-kx) \\ &\quad - \Big(k^3 c_{A,1}\exp(kx) - k^3 c_{A,2}\exp(-kx)\Big) = 0. \end{aligned}$$

By linear independence,

$$\begin{aligned} c'_{A,1}(t) - k^3 c_{A,1}(t) &= 0; \\ c'_{A,2}(t) + k^3 c_4(t) &= 0. \end{aligned}$$

This are independents ODE's for $c_3(t)$ and $c_4(t)$. Solving them, it is obtained

$$c_{A,1}(t) = c_{A,1,0}\exp(kt), \quad c_{A,2}(t) = c_{A,2,0}\exp(-kt), \tag{3.3}$$

with $c_{A,1,0}, c_{A,2,0} \in \mathbb{R}$ arbitrary constants. We obtain from (3.2) and (3.3)

$$\Theta(t,x,y) = c_{A,1,0}\exp\Big(kx + k^2 y + k^3 t\Big) + c_{A,2,0}\exp\Big(-kx + k^2 y - k^3 t\Big).$$

The condition $\Theta \in \mathbb{R}$ implies k real-valued. Also, $\Theta > 0$ implies $c_{A,1,0}, c_{A,2,0} > 0$. This finally shows that Θ corresponds to the phase of a KdV vertical line-soliton. The proof is complete.

3.5.2. Proof of Theorem 3.1.6

The proof of this result is based in two lemmas. Since Theorem 3.1.3 considers the case of KdV line-solitons, we concentrate ourselves in the most demanding case of oblique solitons ($A > 0$).

Lemma 3.5.1. *Let u be a smooth solution to* (3.1) *of the form* (3.2)*, with a smooth profile $F(\Theta)$ such that $F(1) = 0$, $F'(1) = 1$, $F''(1) = -1$, and $F'''(1) = 2$. Then if u is a line-soliton of the form* (3.11)-(3.10) *and $F = \log$, one has that $H(\Theta) = Ai(\Theta) = 0$, and*

$$\Theta W_x(\Theta) = \Theta W_y(\Theta) = a_1 a_2 \left(k_1^2 - k_2^2\right)^2 \exp\left((k_1 + k_2)x + \left(k_1^2 + k_2^2\right)y + \left(k_1^3 + k_2^3\right)t\right), \tag{3.4}$$

for some particular $a_1, a_2 > 0$, $k_1, k_2 \in \mathbb{R}$.

Demostración. By Lemma 3.4.5, we know that $H(\Theta) = Ai(\Theta) = 0$. Thanks to (3.4) and Lemma 3.3.3, we conclude (3.4). This proves Lemma 3.5.1. □

Lemma 3.5.2. *Let u be a smooth solution to* (3.1) *of the form* (3.2)*, with $\Theta > 0$ smooth and real-valued and F a smooth profile such that $F(1) = 0$, $F'(1) = 1$, $F''(1) = -1$, and $F'''(1) = 2$. If $H(\Theta) = Ai(\Theta) = 0$, and*

$$\Theta W_y(\Theta) = A(t,x)\exp(k(t,x)y), \tag{3.5}$$

for some particular $A > 0$, $k > 0$, then u is a line-soliton of the form (3.11)-(3.10) *and $F = \log$.*

Remark 3.5.1. Lemmas 3.5.1 and 3.5.2 conclude the proof of Theorem 3.1.6.

Proof of Lemma 3.5.2. *Step 1.* Since $H(\Theta) = Ai(\Theta) = 0$, Lemmas 3.3.3 and 3.3.6 and the hypotheses on F ensure that $F = \log$. Let us assume (3.5). Then

$$\Theta^2 \partial_y^2 (\ln(\Theta)) = A(t,x) \exp(k(t,x)y).$$

This equation can be studied like a nonlinear second order ODE on Θ and the variable y. Considering the change of variable $f = \ln(\Theta)$ one gets

$$\exp(2f) f'' = A \exp(ky), \quad \text{then} \quad \tilde{f}'' = 2A \exp\left(-\tilde{f}\right),$$

with $\tilde{f} = -ky + 2f$, and consequently $\tilde{f}'' = 2f''$. This is a classical Toda equation. The general solution $\tilde{f}$ is given by

$$\tilde{f} = 2\log\left(\frac{\sqrt{A}}{\sqrt{c_1}}\left(\exp\left(\frac{1}{2}\eta\right) + \exp\left(-\frac{1}{2}\eta\right)\right)\right),$$

with $\eta := \sqrt{c_1}(y + c_2)$, $c_1 > 0$, $c_2 \in \mathbb{R}$. Therefore,

$$\begin{aligned}\Theta = \exp(f) &= \exp\left(\frac{ky}{2}\right)\exp\left(\frac{\tilde{f}}{2}\right)\\ &= \frac{\sqrt{A}}{\sqrt{c_1}}\left(\exp\left(\left(\frac{k+\sqrt{c_1}}{2}\right)(y+c_2)\right) + \exp\left(\left(\frac{k-\sqrt{c_1}}{2}\right)(y+c_2)\right)\right).\end{aligned}$$

Defining

$$A_1(t,x) = \frac{\sqrt{A}}{\sqrt{c_1}}\exp\left(c_2\left(\frac{k+\sqrt{c_1}}{2}\right)\right), \quad A_2(t,x) = \frac{\sqrt{A}}{\sqrt{c_1}}\exp\left(c_2\left(\frac{k-\sqrt{c_1}}{2}\right)\right),$$

and

$$B_1(t,x) = \left(\frac{k+\sqrt{c_1}}{2}\right) > 0, \quad B_2(t,x) = \left(\frac{k-\sqrt{c_1}}{2}\right),$$

Θ has the form

$$\Theta = A_1(t,x)\exp(B_1(t,x)y) + A_2(t,x)\exp(B_2(t,x)y).$$

Step 2. Since $c_1 > 0$, one gets B_1 different from B_2. Replacing Θ in the condition $H(\Theta) = 0$, and using Appendix 3.6.3 in the case of two linearly independent exponentials,

$$\begin{aligned}H(\Theta) =\ & (A_1B_1\exp(B_1y) + A_2B_2\exp(B_2y)) - \Big(A_{1,xx}\exp(B_1y) + A_{2,xx}\exp(B_2y)\\ &+ y\exp(B_1y)(2A_{1,x}B_{1,x} + A_1B_{1,xx}) + y\exp(B_2y)(2A_{2,x}B_{2,x} + A_2B_{2,xx})\\ &+ y^2\exp(B_1y)A_1B_{1,x}^2 + y^2\exp(B_2y)A_2B_{2,x}^2\Big)\\ =\ & (A_1B_1 - A_{1,xx})\exp(B_1y) + (A_2B_2 - A_{2,xx})\exp(B_2y)\\ &- (2A_{1,x}B_{1,x} + A_1B_{1,xx})y\exp(B_1y) - (2A_{2,x}B_{2,x} + A_2B_{2,xx})y\exp(B_2y)\\ &- A_1B_{1,x}^2y^2\exp(B_1y) - A_2B_{2,x}^2y^2\exp(B_2y).\end{aligned} \tag{3.6}$$

In order for this term to be null, it is necessary that each of the coefficients multiplying a term $y^i \exp(B_j y)$ with $i \in \{0,1,2\}$ and $j \in \{1,2\}$ be equal to zero. Taking the above into consideration, the following system of equations is obtained

$$\begin{aligned}
&A_1 B_1 = A_{1,xx}, \quad A_2 B_2 = A_{2,xx},\\
&2A_{1,x}B_{1,x} + A_1 B_{1,xx} = 0, \quad 2A_{2,x}B_{2,x} + A_2 B_{2,xx} = 0,\\
&A_1 B_{1,x}^2 = 0, \quad A_2 B_{2,x}^2 = 0.
\end{aligned}$$

Since $A_1, A_2 > 0$, from the two bottom equations, $B_i(t,x) = B_i(t)$ for $i \in \{1,2\}$. This reduces the system to the top two equations, from which it can be concluded that $A_i(t,x) = c_i(t)\exp\left(\sqrt{B_i(t)}x\right)$. Since $\Theta > 0$ is real-valued, it is required $B_i > 0$. Therefore, the phase takes the form

$$\Theta = c_1(t)\exp\left(\sqrt{B_1(t)}x + B_1(t)\,y\right) + c_2(t)\exp\left(\sqrt{B_2(t)}x + B_2(t)\,y\right).$$

Then, using again Appendix 3.6.3 and replacing in $Ai(\Theta) = 0$,

$$\begin{aligned}
Ai(\Theta) = &\left(c_{1,t}\exp\left(\sqrt{B_1}x + B_1 y\right) + \frac{c_1 B_{1,t}}{2\sqrt{B_1}}x\exp\left(\sqrt{B_1}x + B_1 y\right)\right.\\
&+ c_1 B_{1,t}y\exp\left(\sqrt{B_1}x + B_1 y\right) + c_{2,t}\exp\left(\sqrt{B_2}x + B_2 y\right)\\
&\left.+ \frac{c_2 B_{2,t}}{2\sqrt{B_2}}x\exp\left(\sqrt{B_2}x + B_2 y\right) + c_2 B_{2,t}y\exp\left(\sqrt{B_2}x + B_2 y\right)\right)\\
&- \left(c_1\sqrt{B_1^3}\exp\left(\sqrt{B_1}x + B_1 y\right) + c_2\sqrt{B_2^3}\exp\left(\sqrt{B_2}x + B_2 y\right)\right)\\
= &\left(c_{1,t} - c_1\sqrt{B_1^3}\right)\exp\left(\sqrt{B_1}x + B_1 y\right) + \left(c_{2,t} - c_2\sqrt{B_2^3}\right)\exp\left(\sqrt{B_2}x + B_2 y\right)\\
&+ \frac{c_1 B_{1,t}}{2\sqrt{B_1}}x\exp\left(\sqrt{B_1}x + B_1 y\right) + \frac{c_2 B_{2,t}}{2\sqrt{B_2}}x\exp\left(\sqrt{B_2}x + B_2 y\right)\\
&+ c_1 B_{1,t}y\exp\left(\sqrt{B_1}x + B_1 y\right) + c_2 B_{2,t}y\exp\left(\sqrt{B_2}x + B_2 y\right).
\end{aligned}$$

To ensuring that the expression is equal to zero, it is necessary that each coefficient multiplying an exponential term be null for all values of $(x,y) \in \mathbb{R}^2$. Taking the above into consideration, the following system of equations is obtained

$$\begin{aligned}
&c_{1,t} - c_1\sqrt{B_1^3} = 0, \quad c_{2,t} - c_2\sqrt{B_2^3} = 0,\\
&c_1 B_{1,t} = 0, \quad c_2 B_{2,t} = 0.
\end{aligned}$$

Note that the last two equations are derived from the coefficients multiplying an exponential term, multiplied either by x or by y. From the last two equations, it is concluded that $B_i(t) = B_i$ for $i \in \{1,2\}$ and with $B_i \in \mathbb{R}$. Taking this into account in the first two equations is obtained $c_i = a_i\exp\left(\sqrt{B_i^3}t\right)$ for $i \in \{1,2\}$ and with $a_i > 0$. In conclusion, the phase is

$$\Theta = a_1\exp\left(\sqrt{B_1}x + B_1 y + \sqrt{B_1^3}t\right) + a_2\exp\left(\sqrt{B_2}x + B_2 y + \sqrt{B_2^3}t\right).$$

where a_1, $a_2 > 0$ are arbitrary constants. Denoting $k_i := \sqrt{B_i}$, we obtain the desired conclusion. $\square$

3.5.3. Proof of Theorem 3.1.8

From Corollary 3.7 we know that u solution of (3.1) of the form (3.2) with smooth real-valued phase $\Theta > 0$ corresponding to an M resonant multi-soliton (3.13) and $F = \log$ imply $H(\Theta) = Ai(\Theta) = 0$. Also, Lemma 3.4.6 in this particular case $\left(k_j^2 \geq 0\right)$ states that $\Theta W_y(\Theta) = \Theta W_x(\Theta) \in \mathcal{W}_{\frac{1}{2}M(M-1)}$.

Now we prove the inverse equivalence. Assume that $\Theta W_y(\Theta) \in \mathcal{W}_{\frac{1}{2}M(M-1)}$ has the form

$$\Theta W_y(\Theta) = \sum_{j=1}^{\frac{1}{2}M(M-1)} b_j \exp(m_j y). \tag{3.7}$$

After arranging terms as members of an upper triangular matrix,

$$\Theta W_y(\Theta) = \sum_{n=1}^{M} \sum_{i=1}^{n} b_{n,i} \exp(m_{n,i} y).$$

For the moment, assume

$$\Theta = \sum_{j=1}^{M} a_j(t,x) \exp(\theta_j), \tag{3.8}$$

with $\theta_j(t,x,y) = k_j(t,x)\, y$, be a phase in $\mathcal{W}_M$. From Corollary 3.4.6,

$$\Theta W_y(\Theta) = \sum_{n=1}^{M} \sum_{i=1}^{n} a_i a_{n-i+1} (k_{n-i+1} - k_i)^2 \exp(\theta_i + \theta_{n-i+1}).$$

The system

$$k_i + k_{n-i+1} = m_{n,i}, \quad i \leq n - i + 1,$$

reads

$$\begin{aligned}
k_1 + k_1 &= m_{1,1} \\
k_1 + k_2 &= m_{2,1} \\
k_1 + k_3 &= m_{3,1} \\
k_2 + k_2 &= m_{3,2} \\
\cdots \\
k_1 + k_n &= m_{n,1} \\
k_2 + k_{n-1} &= m_{n,2} \\
k_3 + k_{n-2} &= m_{n,3} \\
\cdots
\end{aligned}$$

and has a unique solution on $k = (k_1, \ldots, k_N)$, thanks to a nonsingular determinant matrix. The second system

$$\begin{aligned}
&a_i a_{n-i+1} (k_{n-i+1} - k_i)^2 = b_{n,i} \\
&\qquad \Longrightarrow \ \log a_i + \log a_{n-i+1} = \log b_{n,i} - 2 \log |k_{n-i+1} - k_i|,
\end{aligned}$$

has also a unique solution for $a = (a_1, a_2, \ldots, a_N)$ exactly following the previous argument. Consequently, $\Theta W_y(\Theta) \in \mathcal{W}_{\frac{1}{2}M(M-1)}$ has always a solution in $\mathcal{W}_M$. The fact that $\Theta W_x(\Theta) \in \mathcal{W}_{\frac{1}{2}M(M-1)}$ is direct.

Now we prove uniqueness. The key now is to use that $\Theta W_x(\Theta) \in \mathcal{W}_{\frac{1}{2}M(M-1)}$ and has a unique value. Let $F \in \mathcal{W}_{\frac{1}{2}M(M-1)}$ and let Θ_1, Θ_2 be such that $\Theta_j W_x(\Theta_j) = F$. Therefore,

$$\Theta_1\Theta_{1,xxxx} - \Theta_{1,xx}^2 = \Theta_2\Theta_{2,xxxx} - \Theta_{2,xx}^2.$$

Let $\Pi_0(x;t) := (\Theta_1 - \Theta_2)(t, x, y = 0)$. Then Π_0 satisfies the fourth order linear ODE on x:

$$\Theta_2\Pi_0'''' - (\Theta_{1,xx} + \Theta_{2,xx})\Pi_0'' + \Theta_{1,xxxx}\Pi_0 = 0,$$

and thanks to the hypothesis (3.18), $\Pi_0(x = 0; t) = \Pi_0'(x = 0; t) = \Pi_0''(x = 0; t) = \Pi_0'''(x = 0; t) = 0$, leading to $\Pi_0(x; t) \equiv 0$ and consequently $\Theta_1(t, x, 0) = \Theta_2(t, x, 0)$. Additionally, $\Pi_1(x; t) := (\Theta_{1,y} - \Theta_{2,y})(t, x, y = 0)$ satisfies

$$\Theta_2\Pi_1'''' - 2\Theta_{1,xx}\Pi_1'' + \Theta_{1,xxxx}\Pi_1 = 0.$$

(Notice that we have used that $\Pi_0(x; t) = \Pi_{0,xx}(x; t) = \Pi_{0,xxxx}(x; t) = 0$.) Again, thanks to (3.18) we conclude that $\Pi_1(x; t) \equiv 0$, leading to $\Theta_{1,y}(t, x, 0) = \Theta_{2,y}(t, x, 0)$.

Now we extend the previous uniqueness. The argument is similar to the previous case. Let $F \in \mathcal{W}_{\frac{1}{2}M(M-1)}$ unique and let Θ_1, Θ_2 be such that $\Theta_j W_y(\Theta_j) = F$. Therefore,

$$\Theta_1\Theta_{1,yy} - \Theta_{1,y}^2 = \Theta_2\Theta_{2,yy} - \Theta_{2,y}^2.$$

Let $\Pi_2(y; t, x) := (\Theta_1 - \Theta_2)(t, x, y)$. Then Π_2 satisfies the second order linear ODE on y:

$$\Theta_2\Pi_2'' - (\Theta_{1,y} + \Theta_{2,y})\Pi_2' + \Theta_{1,yy}\Pi_2 = 0,$$

and thanks to the previous step, one has $\Pi_2(0; t, x) = \Pi_{2,y}(0; t, x) = 0$. Therefore, $\Pi(y; t, x) = 0$ and the required uniqueness holds. This shows that (3.8) is the unique solution of (3.7).

Now we improve the coefficients using that Θ is Heat and Airy, showing that Θ is a line-multi-soliton. Since Heat and Airy are linear equations, the proof is similar to the proof of Lemma 3.5.2, Step 2. More precisely,

$$H(\Theta) = \sum_{j=1}^{M}\left\{(a_j k_j - a_{j,xx}) - (2a_{j,x}k_{j,x} + a_j k_{j,xx})\,y - a_j k_{j,x}^2 y^2\right\}\exp(k_j y) = 0. \tag{3.9}$$

By linear independence, we conclude $a_j(t, x) = a_{0,j}(t)\exp\left(\sqrt{k_j(t)}x\right) + b_{0,j}(t)\exp\left(-\sqrt{k_j(t)}x\right)$. Considering only exponentials with positive coefficient, that is to say $b_{0,j}(t) = 0$ for all $j \in \{1, \ldots, M\}$, the solution becomes $a_j(t, x) = a_{0,j}(t)\exp\left(\sqrt{k_j(t)}x\right)$. Computing Airy, one gets

$$Ai(\Theta) = \sum_{j=1}^{M}\left\{\left(a_{0,j,t} - a_{0,j}\sqrt{k_j^3}\right) + \frac{a_{0,j}k_{j,t}}{2\sqrt{k_j}}x + a_{0,j}k_{j,t}y\right\}\exp\left(\sqrt{k_j}x + k_j y\right) = 0. \tag{3.10}$$

From (3.9) and (3.10), and proceeding exactly as in the previous proof of Lemma 3.5.2, Step

2, it is obtained that $a_{0,j} = a_{0,0,j} \exp\left(\sqrt{k_j^3} t\right)$. This ends the proof.

3.5.4. Proof of Theorem 3.1.10

We are now ready to prove Theorem 3.1.10, the case of KP 2-solitons. Let u be a solution of (3.1) of the form (3.2) with a smooth real-valued phase $\Theta > 0$ satisfying $\Theta\,(y=0)$ and $\Theta_y\,(y=0)$ uniquely prescribed. Assume a smooth profile F such that $F\,(1) = 0$, $F'\,(1) = 1$, $F''\,(1) = -1$, and $F'''\,(1) = 2$.

Assume that Θ corresponds to a 2-soliton (3.14) with $0 \leq k_1 < k_2 < k_3 < k_4$ and $F = \log$. Let us prove that $H\,(\Theta)\,, Ai\,(\Theta) \in \mathcal{W}_4$ and $\Theta W_y\,(\Theta)\,, \Theta W_x\,(\Theta) \in \mathcal{W}_5$. Thanks to Lemma 3.4.7 (i) and (ii), this part is already proved.

Assume now that $H\,(\Theta)\,, Ai\,(\Theta) \in \mathcal{W}_4$ and $\Theta W_y\,(\Theta)\,, \Theta W_x\,(\Theta) \in \mathcal{W}_5$. We have $\Theta W_y\,(\Theta) \in \mathcal{W}_6$, and the proof of Theorem 3.1.8 and hypotheses (3.18) on Θ ensure that $\Theta \in \mathcal{W}_4$. As in (3.8), one has

$$\Theta = \sum_{j=1}^{4} a_j\,(t,x) \exp\left(\theta_j\right), \quad a_j > 0, \tag{3.11}$$

with $\theta_j\,(t,x,y) = k_j\,(t,x)\,y$. The image of (3.11) under $\Theta W_y\,(\Theta)$ is given by

$$\begin{aligned}
\Theta W_y\,(\Theta) &= \sum_{n=1}^{4}\sum_{i=1}^{n} a_i a_{n-i+1} \left(k_{n-i+1} - k_i\right)^2 \exp\left(\theta_i + \theta_{n-i+1}\right) \\
&= a_1 a_2 \left(k_1 - k_2\right)^2 E_{12} + a_1 a_3 \left(k_1 - k_3\right)^2 E_{13} + a_1 a_4 \left(k_1 - k_4\right)^2 E_{14} \\
&\quad + a_2 a_3 \left(k_2 - k_3\right)^2 E_{23} + a_2 a_4 \left(k_2 - k_4\right)^2 E_{24} + a_3 a_4 \left(k_3 - k_4\right)^2 E_{34}.
\end{aligned}$$

Since $\Theta W_y\,(\Theta) \in \mathcal{W}_5$, at least one exponential is linearly dependent with the rest of exponentials. This implies that $k_i + k_j = k_{i'} + k_{j'}$ for some i different from i', j different from j' and therefore one term above is redundant. With no loss of generality, we assume $0 \leq k_1 < k_2 < k_3 < k_4$ and $k_1 + k_4 = k_2 + k_3$. In this case we obtain $k_4 = k_2 + k_3 - k_1 > 0$, and from (3.11),

$$\Theta\,(t,x,y) = a_1 \exp\left(k_1 y\right) + a_2 \exp\left(k_2 y\right) + a_3 \exp\left(k_3 y\right) + a_4 \exp\left(\left(k_2 + k_3 - k_1\right) y\right), \tag{3.12}$$

and

$$\begin{aligned}
\Theta W_y\,(\Theta) &= a_1 a_2 \left(k_1 - k_2\right)^2 E_{12} + a_1 a_3 \left(k_1 - k_3\right)^2 E_{13} \\
&\quad + \left(a_1 a_4 \left(k_1 - k_4\right)^2 + a_2 a_3 \left(k_2 - k_3\right)^2\right) E_{14} \\
&\quad + a_2 a_4 \left(k_2 - k_4\right)^2 E_{24} + a_3 a_4 \left(k_3 - k_4\right)^2 E_{34}.
\end{aligned}$$

Θ in (3.12) can be written as follows: for $\widetilde{k}_i \geq 0$,

$$\widetilde{k}_1^2 + \widetilde{k}_3^2 := k_1, \quad \widetilde{k}_1^2 + \widetilde{k}_4^2 := k_2, \quad \widetilde{k}_2^2 + \widetilde{k}_3^2 := k_3.$$

Then $k_4 = k_2 + k_3 - k_1 = \widetilde{k}_2^2 + \widetilde{k}_4^2$, exactly as in (3.14). Therefore, our new variables will be $\widetilde{k}_i$, but in order to avoid too much notation, we drop the tildes. Replacing in (3.12), we obtain a new representation of Θ:

$$\begin{aligned}\Theta(t,x,y) = a_1 \exp\left(\left(k_1^2+k_3^2\right)y\right) + a_2 \exp\left(\left(k_1^2+k_4^2\right)y\right) \\ + a_3 \exp\left(\left(k_2^2+k_3^2\right)y\right) + a_4 \exp\left(\left(k_2^2+k_4^2\right)y\right).\end{aligned} \tag{3.13}$$

(Compare with (3.14).) Repeating again (3.9) with (3.13), we obtain e.g.

$$\begin{aligned}&H\left(a_1 \exp\left(\left(k_1^2+k_3^2\right)y\right)\right) \\ &\quad = \left(a_1\left(k_1^2+k_3^2\right) - a_{1,xx}\right)\exp\left(\left(k_1^2+k_3^2\right)y\right) \\ &\quad\quad + \left(-4a_{1,x}(k_1k_{1,x}+k_3k_{3,x}) - 2a_1(k_1k_{1,x}+k_3k_{3,x})_x\right) y \exp\left(\left(k_1^2+k_3^2\right)y\right) \\ &\quad\quad - 4a_1(k_1k_{1,x}+k_3k_{3,x})^2 y^2 \exp\left(\left(k_1^2+k_3^2\right)y\right).\end{aligned}$$

By linear independence among the exponentials, the nontrivial character of Θ and the hypothesis $H(\Theta) \in \mathcal{W}_4$, it is clear that one will obtain $k_1k_{1,x} + k_3k_{3,x} = 0$, so that if $k_j(t,0) =: k_j(t)$,

$$\left(k_1^2+k_3^2\right)(t,x) = k_1^2(t) + k_3^2(t).$$

A similar argument reveals that

$$\begin{aligned}\left(k_1^2+k_4^2\right)(t,x) &= k_1^2(t) + k_4^2(t), \\ \left(k_2^2+k_3^2\right)(t,x) &= k_2^2(t) + k_3^2(t), \\ \left(k_2^2+k_4^2\right)(t,x) &= k_2^2(t) + k_4^2(t).\end{aligned}$$

Consequently, we get in (3.13)

$$\begin{aligned}\Theta(t,x,y) = a_1(t,x) \exp\left(\left(k_1^2(t)+k_3^2(t)\right)y\right) + a_2(t,x) \exp\left(\left(k_1^2(t)+k_4^2(t)\right)y\right) \\ + a_3(t,x) \exp\left(\left(k_2^2(t)+k_3^2(t)\right)y\right) + a_4(t,x) \exp\left(\left(k_2^2(t)+k_4^2(t)\right)y\right).\end{aligned} \tag{3.14}$$

Now, repeating (3.10) with (3.14) one has

$$\begin{aligned}&Ai\left(a_1(t,x) \exp\left(\left(k_1^2(t)+k_3^2(t)\right)y\right)\right) \\ &\quad = \left(a_{1,t}(t,x) + 2\left(k_1k_{1,t}+k_2k_{2,t}\right) y a_1(t,x) - a_{1,xxx}(t,x)\right)\exp\left(\left(k_1^2(t)+k_3^2(t)\right)y\right),\end{aligned}$$

revealing that $k_1^2(t) + k_2^2(t) = k_1^2(0) + k_2^2(0) =: k_1^2 + k_2^2$, constants independent of time. Similarly,

$$\begin{aligned}\left(k_1^2+k_4^2\right)(t) &= k_1^2(0) + k_4^2(0) =: k_1^2 + k_4^2, \\ \left(k_2^2+k_3^2\right)(t) &= k_2^2(0) + k_3^2(0) =: k_2^2 + k_3^2, \\ \left(k_2^2+k_4^2\right)(t) &= k_2^2(0) + k_4^2(0) =: k_2^2 + k_4^2.\end{aligned}$$

Consequently, we get in (3.14),

$$\begin{aligned}\Theta(t,x,y) = a_1(t,x) \exp\left(\left(k_1^2+k_3^2\right)y\right) + a_2(t,x) \exp\left(\left(k_1^2+k_4^2\right)y\right) \\ + a_3(t,x) \exp\left(\left(k_2^2+k_3^2\right)y\right) + a_4(t,x) \exp\left(\left(k_2^2+k_4^2\right)y\right).\end{aligned} \tag{3.15}$$

From the hypothesis (3.18), and following $\Theta(0,x,0)$, $\Theta_y(0,x,0)$, $\Theta_{yy}(0,x,0)$ and $\Theta_{yyy}(0,x,0)$ are uniquely determined by the values from (3.14), leading to the equations

$$\begin{aligned}
&a_1(0,x)+a_2(0,x)+a_3(0,x)+a_4(0,x)=\Theta(0,x,0)\\
&\left(k_1^2+k_3^2\right)a_1(0,x)+\left(k_1^2+k_4^2\right)a_2(0,x)+\left(k_2^2+k_3^2\right)a_3(0,x)+\left(k_2^2+k_4^2\right)a_4(0,x)=\Theta_y(0,x,0)\\
&\left(k_1^2+k_3^2\right)^2a_1(0,x)+\left(k_1^2+k_4^2\right)^2a_2(0,x)+\left(k_2^2+k_3^2\right)^2a_3(0,x)+\left(k_2^2+k_4^2\right)^2a_4(0,x)=\Theta_{yy}(0,x,0)\\
&\left(k_1^2+k_3^2\right)^3a_1(0,x)+\left(k_1^2+k_4^2\right)^3a_2(0,x)+\left(k_2^2+k_3^2\right)^3a_3(0,x)+\left(k_2^2+k_4^2\right)^3a_4(0,x)=\Theta_{yyy}(0,x,0).
\end{aligned}$$

This is a classical invertible system thanks to the Vandermonde determinant and the condition $0\le k_1<k_2<k_3<k_4$. Therefore, $a_j(0,x)$, $j=1,2,3,4$ are uniquely determined:

$$a_1(0,x)=(k_3-k_1)\exp((k_1+k_3)x),\quad a_2(0,x)=(k_4-k_1)\exp((k_1+k_4)x),$$

and

$$a_3(0,x)=(k_3-k_2)\exp((k_2+k_3)x),\quad a_4(0,x)=(k_4-k_2)\exp((k_2+k_4)x).$$

Using (3.15), we compute now $Ai(\Theta)-\frac{3}{2}\partial_x H(\Theta)$:

$$\begin{aligned}
Ai(\Theta)-\frac{3}{2}\partial_x H(\Theta)= &\left(a_{1,t}+\frac{1}{2}a_{1,xxx}-\frac{3}{2}\left(k_1^2+k_3^2\right)a_{1,x}\right)\exp\left(\left(k_1^2+k_3^2\right)y\right)\\
&+\left(a_{2,t}+\frac{1}{2}a_{2,xxx}-\frac{3}{2}\left(k_1^2+k_4^2\right)a_{2,x}\right)\exp\left(\left(k_1^2+k_4^2\right)y\right)\\
&+\left(a_{3,t}+\frac{1}{2}a_{3,xxx}-\frac{3}{2}\left(k_2^2+k_3^2\right)a_{3,x}\right)\exp\left(\left(k_2^2+k_3^2\right)y\right)\\
&+\left(a_{4,t}+\frac{1}{2}a_{4,xxx}-\frac{3}{2}\left(k_2^2+k_4^2\right)a_{4,x}\right)\exp\left(\left(k_2^2+k_4^2\right)y\right)=0.
\end{aligned}$$

Therefore,

$$a_{1,t}+\frac{1}{2}\partial_x\left(a_{1,xx}-3\left(k_1^2+k_3^2\right)a_1\right)=0,\quad a_1(0,x)=(k_3-k_1)\exp((k_1+k_3)x). \tag{3.16}$$

We shall prove that the unique solution to this problem is

$$a_1(t,x)=(k_3-k_1)\exp\left((k_1+k_3)x+\left(k_1^3+k_3^3\right)t\right). \tag{3.17}$$

If we assume this equality, it is not hard to see that a similar argument reveals that

$$\begin{aligned}
a_2(t,x)= &\ (k_4-k_1)\exp\left((k_1+k_4)x+\left(k_1^3+k_4^3\right)t\right),\\
a_3(t,x)= &\ (k_3-k_2)\exp\left((k_2+k_3)x+\left(k_2^3+k_3^3\right)t\right),
\end{aligned}$$

and

$$a_4(t,x)=(k_4-k_2)\exp\left((k_2+k_4)x+\left(k_2^3+k_4^3\right)t\right).$$

This finally proves Theorem 3.1.10. Let us show (3.17). Clearly the RHS of (3.17) is a valid solution to (3.16) satisfying the initial condition $a_1(0,x)=(k_3-k_1)\exp((k_1+k_3)x)$. Let us show that it is the unique one.

First, notice that if $\widetilde{a}_1(t,y)$ is a function such that $a_1(t,x) = \widetilde{a}_1\left(t, \sqrt[3]{2}\left(x + \frac{3}{2}(k_1^2 + k_3^2)\,t\right)\right) = \widetilde{a}_1(\bar{t},\bar{x})$, then

$$Ai(\widetilde{a}_1) = \widetilde{a}_{1,t} + \widetilde{a}_{1,xxx} = 0.$$

Consequently, by the following uniqueness Lemma we get the desired result.

Lemma 3.5.3 (Uniqueness of exponentially growing Airy solutions)**.** *Let $m_1, m_2 > 0$. There is a unique solution u of*

$$\partial_t u + \partial_x^3 u = 0, \quad u(t=0,x) = m_1 \exp(m_2 x),$$

and it is given by

$$u(t,x) = m_1 \exp\left(m_2 x - m_2^3 t\right).$$

Demostración. The existence is exactly given by the explicit formula. Let us see the uniqueness, equivalent to prove that

$$\partial_t u + \partial_x^3 u = 0, \quad u(t=0,x) = 0,$$

has solution $u = 0$. This is obtained by simply taking Laplace transform, solving the corresponding obtained ODE, and using the uniqueness of the inverse Laplace transform in the exponentially growing class of solutions. □

3.6. Proof of technical results

3.6.1. Galilean actions

Evaluation Θ_β, obtained after apply the Galilean Transformation to a phase Θ, in the terms that appears in (3.1). Then the terms in Definition 3.1 satisfy,

$$\begin{aligned}
H(\Theta_\beta) &= -\frac{4\beta}{3}\partial_{\tilde{x}}\Theta + \partial_{\tilde{y}}\Theta - \partial_{\tilde{x}}^2\Theta = -\frac{4\beta}{3}\partial_{\tilde{x}}\Theta + H(\Theta),\\
Ai(\Theta_\beta) &= \frac{4\beta^2}{3}\partial_{\tilde{x}}\Theta - 2\beta\partial_{\tilde{y}}\Theta + \partial_{\tilde{t}}\Theta - \partial_{\tilde{x}}^3\Theta = \frac{4\beta^2}{3}\partial_{\tilde{x}}\Theta - 2\beta\partial_{\tilde{y}}\Theta + Ai(\Theta),\\
\Theta_\beta W_y(\Theta_\beta) &= \Theta W_{\tilde{y}}(\Theta) + \frac{16}{9}\beta^2\left(\Theta\Theta_{\tilde{x}\tilde{x}} - \Theta_{\tilde{x}}^2\right) - \frac{8}{3}\beta\left(\Theta\Theta_{\tilde{x}\tilde{y}} - \Theta_{\tilde{x}}\Theta_{\tilde{y}}\right),\\
\Theta_\beta W_x(\Theta_\beta) &= \Theta W_{\tilde{x}}(\Theta).
\end{aligned} \tag{3.1}$$

In particular H, Ai and W_y do not cancel for a nontrivial Galilean version of the vertical soliton (3.12).

3.6.2. Proof of phase computations

In this section we prove (3.6). Given Θ as in (3.14), its derivatives are

$$\begin{aligned}
\Theta_x =& \left(k_3 - k_1\right)\left(k_3 + k_1\right)\exp\left(\theta_1 + \theta_3\right) + \left(k_4 - k_1\right)\left(k_4 + k_1\right)\exp\left(\theta_1 + \theta_4\right) \\
&+ \left(k_3 - k_2\right)\left(k_3 + k_2\right)\exp\left(\theta_2 + \theta_3\right) + \left(k_4 - k_2\right)\left(k_4 + k_2\right)\exp\left(\theta_2 + \theta_4\right), \\
\Theta_{xx} =& \left(k_3 - k_1\right)\left(k_3 + k_1\right)^2\exp\left(\theta_1 + \theta_3\right) + \left(k_4 - k_1\right)\left(k_4 + k_1\right)^2\exp\left(\theta_1 + \theta_4\right) \\
&+ \left(k_3 - k_2\right)\left(k_3 + k_2\right)^2\exp\left(\theta_2 + \theta_3\right) + \left(k_4 - k_2\right)\left(k_4 + k_2\right)^2\exp\left(\theta_2 + \theta_4\right), \\
\Theta_{xxx} =& \left(k_3 - k_1\right)\left(k_3 + k_1\right)^3\exp\left(\theta_1 + \theta_3\right) + \left(k_4 - k_1\right)\left(k_4 + k_1\right)^3\exp\left(\theta_1 + \theta_4\right) \\
&+ \left(k_3 - k_2\right)\left(k_3 + k_2\right)^3\exp\left(\theta_2 + \theta_3\right) + \left(k_4 - k_2\right)\left(k_4 + k_2\right)^3\exp\left(\theta_2 + \theta_4\right), \\
\Theta_{xxxx} =& \left(k_3 - k_1\right)\left(k_3 + k_1\right)^4\exp\left(\theta_1 + \theta_3\right) + \left(k_4 - k_1\right)\left(k_4 + k_1\right)^4\exp\left(\theta_1 + \theta_4\right) \\
&+ \left(k_3 - k_2\right)\left(k_3 + k_2\right)^4\exp\left(\theta_2 + \theta_3\right) + \left(k_4 - k_2\right)\left(k_4 + k_2\right)^4\exp\left(\theta_2 + \theta_4\right), \\
\Theta_y =& \left(k_3 - k_1\right)\left(k_3^2 + k_1^2\right)\exp\left(\theta_1 + \theta_3\right) + \left(k_4 - k_1\right)\left(k_4^2 + k_1^2\right)\exp\left(\theta_1 + \theta_4\right) \\
&+ \left(k_3 - k_2\right)\left(k_3^2 + k_2^2\right)\exp\left(\theta_2 + \theta_3\right) + \left(k_4 - k_2\right)\left(k_4^2 + k_2^2\right)\exp\left(\theta_2 + \theta_4\right), \\
\Theta_{yy} =& \left(k_3 - k_1\right)\left(k_3^2 + k_1^2\right)^2\exp\left(\theta_1 + \theta_3\right) + \left(k_4 - k_1\right)\left(k_4^2 + k_1^2\right)^2\exp\left(\theta_1 + \theta_4\right) \\
&+ \left(k_3 - k_2\right)\left(k_3^2 + k_2^2\right)^2\exp\left(\theta_2 + \theta_3\right) + \left(k_4 - k_2\right)\left(k_4^2 + k_2^2\right)^2\exp\left(\theta_2 + \theta_4\right), \\
\Theta_t =& \left(k_3 - k_1\right)\left(k_3^3 + k_1^3\right)\exp\left(\theta_1 + \theta_3\right) + \left(k_4 - k_1\right)\left(k_4^3 + k_1^3\right)\exp\left(\theta_1 + \theta_4\right) \\
&+ \left(k_3 - k_2\right)\left(k_3^3 + k_2^3\right)\exp\left(\theta_2 + \theta_3\right) + \left(k_4 - k_2\right)\left(k_4^3 + k_2^3\right)\exp\left(\theta_2 + \theta_4\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
Ai\left(\Theta\right) &= \Theta_t - \Theta_{xxx} \\
&= \left(k_3 - k_1\right)\left(k_3^3 + k_1^3\right)\exp\left(\theta_1 + \theta_3\right) + \left(k_4 - k_1\right)\left(k_4^3 + k_1^3\right)\exp\left(\theta_1 + \theta_4\right) \\
&\quad + \left(k_3 - k_2\right)\left(k_3^3 + k_2^3\right)\exp\left(\theta_2 + \theta_3\right) + \left(k_4 - k_2\right)\left(k_4^3 + k_2^3\right)\exp\left(\theta_2 + \theta_4\right) \\
&\quad - \left(k_3 - k_1\right)\left(k_3 + k_1\right)^3\exp\left(\theta_1 + \theta_3\right) - \left(k_4 - k_1\right)\left(k_4 + k_1\right)^3\exp\left(\theta_1 + \theta_4\right) \\
&\quad - \left(k_3 - k_2\right)\left(k_3 + k_2\right)^3\exp\left(\theta_2 + \theta_3\right) - \left(k_4 - k_2\right)\left(k_4 + k_2\right)^3\exp\left(\theta_2 + \theta_4\right).
\end{aligned}$$

Where each of these terms has the form $\exp\left(\theta_j + \theta_i\right)$ multiplies by a term

$$\begin{aligned}
&\left(k_i - k_j\right)\left(k_i^3 + k_j^3\right) - \left(k_i - k_j\right)\left(k_i + k_j\right)^3 \\
&\quad = k_i^4 + k_i k_j^3 - k_i^3 k_j - k_j^4 - \left(k_i - k_j\right)\left(k_i^3 + 3k_i^2 k_j + 3k_i k_j^2 + k_j^3\right) \\
&\quad = k_i^4 + k_i k_j^3 - k_i^3 k_j - k_i^4 - 3k_i^3 k_j - 3k_i^2 k_j^2 - k_i k_j^3 + k_i^3 k_j + 3k_i^2 k_j^2 + 3k_i k_j^3 + k_j^4 \\
&\quad = -3k_i^3 k_j + 3k_i k_j^3 \\
&\quad = 3\left(k_i k_j^3 - k_i^3 k_j\right).
\end{aligned}$$

3.6.3. Computation of derivatives of Θ

We perform here some of the computations required in (3.6) and subsequent lines. If $\Theta = \sum_{j=1}^{M} A_j\left(t,x\right)\exp\left(B_j\left(t,x\right)y\right)$, then one has

$$
\begin{aligned}
\Theta_x &= \sum_{j=1}^{M}\left(A_{j,x}+A_jB_{j,x}\right)\exp\left(B_jy\right),\\
\Theta_{xx} &= \sum_{j=1}^{M}\left(A_{j,xx}+2A_{j,x}B_{j,x}y+A_jB_{j,xx}y+A_jB_{j,x}^2y^2\right)\exp\left(B_jy\right),\\
\Theta_{xxx} &= \sum_{j=1}^{M}\Big(A_{j,xxx}+3A_{j,xx}B_{j,x}y+3A_{j,x}B_{j,xx}y+3A_{j,x}B_{j,x}^2y^2\\
&\qquad +A_jB_{j,xxx}y+3A_jB_{j,x}B_{j,xx}y^2+A_jB_{j,x}^3y^3\Big)\exp\left(B_jy\right),\\
\Theta_{xxxx} &= \sum_{j=1}^{M}\Big(A_{j,xxxx}+4A_{j,xxx}B_{j,x}y+6A_{j,xx}B_{j,xx}y\\
&\qquad +4A_{j,x}B_{j,xxx}y+A_jB_{j,xxxx}y+6A_{j,xx}B_{j,x}^2y^2\\
&\qquad +j2A_{j,x}B_{j,x}B_{j,xx}y^2+3A_jB_{j,xx}^2y^2+A_jB_{j,x}B_{j,xxx}y^2\\
&\qquad +4A_{j,x}B_{j,x}^3y^3+6A_jB_{j,x}^2B_{j,xx}y^3+A_jB_{j,x}^4y^4\Big)\exp\left(B_jy\right),\\
\Theta_y &= \sum_{j=1}^{M}A_jB_j\exp\left(B_jy\right),\qquad \Theta_{yy}=\sum_{j=1}^{M}A_jB_j^2\exp\left(B_jy\right),\\
\Theta_t &= \sum_{j=1}^{M}\left(A_{j,t}+A_jB_{j,t}y\right)\exp\left(B_jy\right).
\end{aligned}
$$

3.7. The KdV and mKdV cases

For the sake of completeness, we provide sketch of proofs for Remarks 3.1.13 and 3.1.14.

3.7.1. The KdV case

Consider the KdV model

$$-4u_t+u_{xxx}+6uu_x=0, \tag{3.1}$$

where $u=u\left(t,x\right)\in\mathbb{R}$ and $t,x\in\mathbb{R}$. By considering

$$u\left(t,x\right)=2\partial_x^2F\left(\Theta\left(t,x\right)\right), \tag{3.2}$$

where, with no loss of generality, we consider $F:[1,\infty)\longrightarrow\mathbb{R}$ is smooth and $\Theta=\Theta\left(t,x\right)\in[1,\infty)$ is also smooth. Since F can be changed by any linear affine function, we can assume that $F\left(1\right)=0$, $F'\left(1\right)=1$. The simplest case, the KdV soliton, is found as

$$F\left(s\right):=\log\left(s\right),\quad \Theta\left(t,x\right)=1+\exp\left(ax-ta^3/4\right),\quad a\in\mathbb{R}. \tag{3.3}$$

Exactly as in the KP case, we consider the following definitions:

Definition 3.4 (Classification of phases Θ) *We shall say that Θ as in* (3.2) *is*

(i) Is of Airy type if for all $(t,x) \in \mathbb{R}^2$,

$$Ai\left(\Theta\right) := -4\Theta_t + \Theta_{xxx} = 0.$$

(ii) Is of Wronskian type if $\{\Theta_x, \Theta_{xx}\}(t,x)$ are linearly dependent, for any (t,x);

(iii) Is of $\mathcal{T}$-type if for F fixed,

$$\mathcal{T}\left(\Theta\right) := \left(Ai\left(\Theta\right)_x + F'\left(\Theta\right)\left(3W\left(\Theta\right) - \Theta_x Ai\left(\Theta\right)\right)\right) = 0, \tag{3.4}$$

where $W\left(\Theta\right) := \Theta_{xx}^2 - \Theta_x\Theta_{xxx}$.

Some comments are necessary.

Remark 3.7.1. The Airy type condition naturally describes that the phase of the nonlinear KdV solution u solves the classical Airy linear equation

$$4\Theta_t - \Theta_{xxx} = 0.$$

This is an interesting coincidence that confirms the complex integrable structure of the KdV model. Standard solutions to the Airy equation have a complex oscillatory behavior for $x < 0$, but some simple solutions are

$$\Theta\left(t,x\right) = \exp\left(ax - \frac{1}{4}a^3 t + C\right), \quad \Theta\left(t,x\right) = t + \frac{2}{3}x^3 + C, \tag{3.5}$$

where $C \in \mathbb{R}$ is any constant. The first phase corresponds to the 1-soliton phase, and the second will represent and interesting counterexample to our main results.

Remark 3.7.2. Unlike KP, here in KdV the Wronskian type phase is extremely restrictive. Multi-soliton solutions will not be of this type.

Remark 3.7.3. Exactly as in KP, notice that Θ of $\mathcal{T}$-type is a condition depending on the profile F, and consequently is a more complex condition than being of Wronskian or Airy type, which are independent of the profile F. Additionally, the 1-soliton phase (3.3) is of Airy type as well.

Replacing (3.2) in (3.1), and arranging similar terms, one easily arrives to

$$\begin{aligned} &\left(F'''' + 6F''^2\right)\left(\Theta\right)\Theta_x^4 + 6\left(F'' + F'^2\right)'\left(\Theta\right)\Theta_x^2\Theta_{xx} \\ &\quad + \left(F'' + F'^2\right)\left(\Theta\right)\left(3\Theta_{xx}^2 - 4\Theta_x\left(\Theta_t - \Theta_{xxx}\right)\right) + F'\left(\Theta\right)\left(-4\Theta_t + \Theta_{xxx}\right)_x \\ &\quad + F'^2\left(\Theta\right)\left(3\left(\Theta_{xx}^2 - \Theta_x\Theta_{xxx}\right) - \Theta_x\left(-4\Theta_t + \Theta_{xxx}\right)\right) = 0. \end{aligned} \tag{3.6}$$

(Compare with (3.4).) As in KP, consider

$$\rho\left(s\right) := F''\left(s\right) + F'^2\left(s\right).$$

If $\rho = 0$ and $F\left(1\right) = 0$, $F'\left(1\right) = -1$, then $F = \log$. This follows directly from solving the ODE $\rho\left(s\right) = 0$. Notice that from (3.3) and Definition 3.4, one has that (3.6) can be written as

$$
\begin{aligned}
&\left(\rho''-2F'\rho'+4F''\rho\right)\Theta_x^4+6\rho'\Theta_x^2\Theta_{xx}\\
&\quad+\rho\left(3\Theta_{xx}^2-4\Theta_x\left(\Theta_t-\Theta_{xxx}\right)\right)+F'\left(\Theta\right)\left(Ai\left(\Theta\right)\right)_x\\
&\quad+F'^2\left(\Theta\right)\left(3W\left(\Theta\right)-\Theta_x Ai\left(\Theta\right)\right)=0,
\end{aligned}
$$

and arranging terms,

$$
\begin{aligned}
&\Theta_x^4\rho''+2\left(3\Theta_x^2\Theta_{xx}-F'\left(\Theta\right)\Theta_x^4\right)\rho'\\
&\quad+\left(3\Theta_{xx}^2-4\Theta_x(\Theta_t-\Theta_{xxx})+4F''\left(\Theta\right)\Theta_x^4\right)\rho+\mathcal{T}\left(\Theta\right)=0.
\end{aligned}
\tag{3.7}
$$

Our main result is

Theorem 3.7.4. *The following are satisfied:*

(i) *Assume that Θ is of $\mathcal{T}$-type, Θ_x different from 0, and F satisfies $F''(1)=1$ and $F'''(1)=-2$. Then $F=\log$.*

(ii) *Assume that Θ is of Airy and Wronskian type, $\Theta_x\neq 0$, and F satisfies $F''(1)=1$ and $F'''(1)=-2$. Then $F=\log$ and $\Theta=1+\exp\left(ax-a^3/4t\right)$ or $\Theta=ax+1$, for any $a\in\mathbb{R}$.*

(iii) *Assume that u as in (3.2) solves KdV with $F(s)=\log(s)$, and Θ is of Wronskian type. Then u is a soliton.*

(iv) *Assume that u is a nontrivial multisoliton with $F=\log$. Then Θ is of $\mathcal{T}$-type, but it cannot be of Wronskian nor Airy type.*

The simplest phase to characterize is that of Wronskian type.

Lemma 3.7.1. *Assume that Θ is of Wronskian type. Then*

(i) *One has $W(\Theta)=\Theta_{xx}^2-\Theta_x\Theta_{xx}=0$.*

(ii) *Additionally, there are $a=a(t)$ and $b=b(t)$ such that either*

$$
\Theta(t,x)=1+\exp\left(a(t)x+b(t)\right),
$$

or

$$
\Theta(t,x)=a(t)x+b(t),
$$

and with $b(0)=1$.

The previous result establishes that phases Θ of Wronskian type are directly related to soliton solutions. This fact is independent of the value of F, that will be determined independently.

Demostración. The proof follows directly from Definition 3.4 *(ii)*. Indeed, the proof of *(i)* is direct from the definition of linear dependence and the the fact that

$$
W(\Theta)=\Theta_{xx}^2-\Theta_x\Theta_{xxx}=\mathrm{Wr}\left(\Theta_x,\Theta_{xx}\right),
$$

where Wr denotes the Wronskian. The proof of *(ii)* follows from the fact that $\eta:=\Theta_x$ satisfies the ODE $\eta_x^2=\eta\eta_{xx}$, which has solutions $\eta(s)=\exp\left(a(t)s+b(t)\right)$ and $\eta=a(t)$. The final result is obtained by integrating in space and imposing the condition $\Theta(0,0)=1$. □

Remark 3.7.5. 1. Notice that multisolitons cannot have a phase of Wronskian type.

2. An important example of phase of Airy type but not being of Wronskian type is the one given in (3.5): $\Theta(t,x) = t + \frac{2}{3}x^3 + 1$. In this case, $W(\Theta) = 8x^2$.

3. If Θ is of Wronskian and Airy type, then is of $\mathcal{T}$-type. The reciprocal is clearly false. This follows directly from (3.4).

Lemma 3.7.2. *Assume that Θ is of $\mathcal{T}$-type. Then*

(*i*) *There exists $c_0 \in \mathbb{R}$ such that*

$$Ai(\Theta) = c_0 \exp(F(\Theta)) - 3W(\Theta) + 3F(\Theta)\int_0^x \exp(-F(\Theta))(W(\Theta))_x(t,s)\,ds. \quad (3.8)$$

(*ii*) *If Θ is of Airy and $\mathcal{T}$-type, and $F'(\Theta)$ different from 0, then it is of Wronskian type.*

(*iii*) *If Θ is of Wronskian and $\mathcal{T}$-type, then there exists $c \in \mathbb{R}$ such that $Ai(\Theta) = c_0 \exp(\Theta)$.*

Demostración. Let us prove (*i*): notice that Θ being of $\mathcal{T}$-type as in (3.4) is equivalent to have

$$(Ai(\Theta))_x - F'(\Theta)\Theta_x Ai(\Theta) = 3F'(\Theta)W(\Theta).$$

Then (3.8) follows directly from solving the corresponding ODE for $Ai(\Theta)$. In order to prove (*ii*), notice that from (3.4) one has $0 = F'(\Theta)W(\Theta)$, proving (*ii*). Finally, to prove (*iii*), from (3.8) one has $Ai(\Theta) = c_0 \exp(F(\Theta))$, as required. □

Now we are ready to prove the main result in the KdV case.

Proof of Theorem 3.7.4. Now we are ready to prove Theorem 3.7.4.

Proof of (*i*). Under $\mathcal{T}(\Theta) = 0$, we have from (3.7) an ODE of the form

$$\Theta_x^4\rho'' + 2\left(3\Theta_x^2\Theta_{xx} - F'(\Theta)\Theta_x^4\right)\rho' + \left(3\Theta_{xx}^2 - 4\Theta_x(\Theta_t - \Theta_{xxx}) + 4F''(\Theta)\Theta_x^4\right)\rho = 0.$$

This homogeneous ODE for ρ has zero as unique solution provided $\rho(1) = \rho'(1) = 0$, which is indeed the case. Consequently, from Lemma 3.2.2, we get $F = \log$.

Proof of (*ii*). From Θ being of Airy and Wronskian type we have Θ of $\mathcal{T}$-type. The previous result ensures $F = \log$. Finally, Lemma 3.7.1 proves the final result, after checking that Θ is of Airy type.

Proof of (*iii*). If $F = \log$ then $\rho = 0$, and from (3.8), $\Theta^{-1}\mathcal{T}(\Theta) = 0$. Consequently, Θ is of $\mathcal{T}$-type. Since it is additionally of Wronskian type, it is of Airy type and (*ii*) applies.

Proof of (*iv*). The multisoliton Θ is of $\mathcal{T}$-type, but it does not satisfy being of Airy nor Wronskian type. This ends the proof of Theorem 3.7.4. □

3.7.2. The mKdV case

Let us consider the mKdV model

$$-4u_t + u_{xxx} + 6u^2u_x = 0, \quad (3.9)$$

where $u = u(t,x) \in \mathbb{R}$ and $(t,x) \in \mathbb{R}^2$. Let us extend our previous results to the mKdV case. Consider

$$u(t,x) = \partial_x F(\Theta(t,x)), \tag{3.10}$$

(notice that we only consider one derivative in space). We consider $F : \mathbb{R} \longrightarrow \mathbb{R}$ smooth and the phase $\Theta = \Theta(t,x) \in \mathbb{R}$ also smooth. Since F can be changed by any constant, we can assume that $F(0) = 0$.

Classical mKdV solitons (see e.g. [2]) are given by

$$\begin{aligned} Q_k(t,x) &= 2\partial_x \arctan(\Theta_k(t,x)) \\ &= k \,\mathrm{sech}\left(kx + \frac{1}{4}k^3 t + b_0\right), \quad \Theta_k := \exp\left(kx + \frac{1}{4}k^3 t + b_0\right), \quad b_0 \in \mathbb{R}. \end{aligned} \tag{3.11}$$

Replacing (3.10) in (3.9), we obtain

$$(\Theta_{xxx} - 4\Theta_t) F'(\Theta) + 3\Theta_{xx}\Theta_x F''(\Theta) + \Theta_x^3 \left(F''' + 2F'^3\right)(\Theta) = 0. \tag{3.12}$$

Trivial phases $\Theta = const.$ will be discarded now, so that we assume $\Theta_x \neq 0$. Note that Θ_k in (3.11) satisfies $Ai(\Theta) = \Theta_{xxx} - 4\Theta_t = 0$. Consequently, one must have

$$3\Theta_{xx}F''(\Theta) + \Theta_x^2 \left(F''' + 2F'^3\right)(\Theta) = 0.$$

Let $h(s) := \frac{1}{2}F'(s)$. This term can be written as

$$3\left(\Theta_{xx} - \frac{\Theta_x^2}{\Theta}\right) h'(\Theta) + \Theta_x^2 \left(h'' + \frac{3}{s}h' + 8h^3\right)(\Theta) = 0. \tag{3.13}$$

The first term contains nothing but $W_x(\Theta) := \Theta_{xx} - \frac{\Theta_x^2}{\Theta}$, which will be required to be zero. Notice that Θ_k in (3.11) does satisfy this condition. In the opposite direction, $W_x(\Theta) = 0$ implies $\Theta\left(\frac{\Theta_x}{\Theta}\right)_x = 0$, leading to $\Theta(t,x) = \exp(k(t)x + b(t))$. From the Airy condition $\Theta_{xxx} - 4\Theta_t = 0$ one gets

$$k^3 - 4k'x - 4b' = 0,$$

leading to $k(t) = k$ constant and $b = \frac{1}{4}k^3 t + b_0$, b_0 constant. Therefore, $\Theta = \exp\left(kx + \frac{1}{4}k^3 t + b_0\right)$. Notice that the case $k = 0$ returns the trivial solution, therefore it will be left out of the subsequent analysis. In particular, the image of Θ is $(0, \infty)$.

Finally, the second term in (3.13) (being zero now) can be written as the radial solution $h = h(s)$, $s = |x| > 0$ of $\Delta h + 8h^3 = 0$ in dimension 4.

Requiring positive solutions (by hypothesis, $h(s) := \frac{1}{2}F'(s) > 0$), these are in $H^1(\mathbb{R}^4)$ and by classical Talenti-Aubin arguments, one has $h(s) = \frac{1}{1+s^2}$, giving finally $F = 2\arctan$ if we assume $F(0) = 0$. We conclude the following result:

Theorem 3.7.6. *Let u be a smooth solution to mKdV* (3.9) *of the form* (3.10)*, with smooth profile $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(0) = 0$ and F strictly increasing in $\mathbb{R}$. Then a nontrivial Θ is a soliton* (3.11) *and $F = 2\arctan$ if and only if $W_x(\Theta) = Ai(\Theta) = 0$.*

Notice that the condition F strictly increasing in $\mathbb{R}$ can be replaced by F strictly monotone in $\mathbb{R}$, since by symmetries of the equation (3.9), if u is a solution, then $-u$ also does. However, it is noticed that "cn" periodic solutions may appear if this condition is lifted.

3.7.3. General pure-power KdV models

We finish with some words about the gKdV case, where the situation is considerably less interesting. One has

$$-4u_t + u_{xxx} + pu^{p-1}u_x = 0, \quad p = 4, 5, \ldots$$

In this case the soliton is given by

$$Q_k(t,x) := \left(\frac{k^2(p+1)}{2\cosh^2\left(\frac{(p-1)}{2}\left(kx + \frac{1}{4}k^3 t\right)\right)} \right)^{\frac{1}{p-1}}, \quad k \in \mathbb{R}.$$

Assuming now $u(t,x) = F(\Theta(t,x))$, one arrives to (compare with (3.12))

$$(\Theta_{xxx} - 4\Theta_t) F'(\Theta) + 3\Theta_x\Theta_{xx}F''(\Theta) + \Theta_x^3 F^{(3)}(\Theta) + p\Theta_x F^{p-1}(\Theta) F'(\Theta) = 0.$$

As usual, one imposes the Airy condition $Ai(\Theta) = \Theta_{xxx} - 4\Theta_t = 0$. Now, unlike mKdV, we will search for Θ possibly becoming zero at some point, but $\Theta_x \neq 0$. One has

$$3\Theta_{xx}F''(\Theta) + \Theta_x^2 F^{(3)}(\Theta) + (F^p)'(\Theta) = 0.$$

If now $\Theta_{xx} = 0$, then $\Theta_x = k \in \mathbb{R} - \{0\}$, so that $k^2F''(s) + F^p(s) = 0$.

Capítulo 4

Conclusions

In conclusion, throughout the work, various properties satisfied by the phases, Θ, of the solutions of (3.1) of type (3.2) were studied. The original equation was rewritten in a way that made certain structures within the equation clearer, and through this, it was possible to characterize the different solutions studied throughout the work.

We shall concentrate our conclusions in the chapter containing new results. In Section 3.2, several definitions regarding the phases were introduced, and various conditions satisfied by the profile when constructing a solution of (3.1) of type (3.2) were demonstrated.

In Section 3.3, various conditions that allow constructing trivial or zero solutions of the function were studied. The definitions of the Wronskians, W_x and W_y, were also examined in more detail. It is noteworthy that solutions of type W_y are more stringent than those of type W_x, with the phases of type W_y having an explicit form given by $\Theta = A(t,x)\exp(c(t,x)\,y)$. This implies that solutions that cannot be written in this form do not satisfy the condition $W_y = 0$. Additionally, it was demonstrated that in equation (3.4), the term written in terms of $H(\Theta)$ can be nullified with a weaker condition, $W_y^F = W_x^F$.

In Section 3.4, the relationship between the variables Ai and H was studied in greater depth. An explicit form for the term Ai in terms of the profile derivative was found when the phase is of Heat type, $H(\Theta) = 0$. Then, different phases were constructed assuming the conditions $Ai(\Theta) = 0$, $H(\Theta) = 0$, $W_x(\Theta) = 0$, and $W_y(\Theta) = 0$, separately and then together to determine the type of phases that satisfy these conditions. This is useful for proving one of the main results, Theorem 3.1.3.

Later, the values of $Ai(\Theta)$, $H(\Theta)$, $W_x(\Theta)$, and $W_y(\Theta)$ for phases corresponding to the sum of different numbers of exponentials were studied. From this, it was possible, using the spaces $\mathcal{W}_n$, to find a relationship between the number of exponentials present in a phase and the range of the aforementioned operators.

Finally, the phase of the 2-soliton was studied, and it was verified that this solution is only of type $\mathcal{T}$, i.e., $\mathcal{T}(\Theta) = 0$. Recalling that this is a condition relating the phase to the profile of the solution, it is noted that this result is obtained for a solution constructed using the profile $F = \log$.

Finally, in Section 3.5, the necessary and sufficient conditions that the phases must satisfy to generate the different types of solutions studied were demonstrated. These are the solutions generated from the Wronskian of one and two functions Θ_i, corresponding to the sum of exponentials.

The main results of the work can be summarized in Theorems 3.1.6, 3.1.8, and 3.1.10, where the solutions of line-soliton, resonant soliton, and 2-soliton types are characterized by their phases. Of particular interest is the fact that in all three theorems, only the definitions of Airy, Heat, W_x, and W_y are used.

4.1. Results and Future Work

4.1.1. Range Space of the Operators Ai, H, W_x, and W_y for the General Case

For the case of the line-soliton, it was possible to verify that the four definitions used to describe the phases, excluding $\mathcal{T}$ which is defined through the relationship between the profile and the phase, have the same value. This is

$$Ai\,(\Theta) = H\,(\Theta) = W_x\,(\Theta) = W_y\,(\Theta) = 0.$$

For the case of resonant solutions, constructed from only one function Θ_i, it was possible to maintain the previous equality for the definitions Ai and H. On the other hand, the value of the definitions W_x and W_y changes and lies within the space $\mathcal{W}$, defined previously. Note that, so far, the phases of both types of solutions are described using only one function $\Theta_i = \sum_{i=1}^{M} a_i \exp\left(k_i x + k_i^2 y + k_i^3 t\right)$.

Finally, the 2-soliton type solutions were characterized. In this case, it is noteworthy that each of the definitions used so far has non-zero values, except for the case of two vertical line solitons, in this case $W_y\,(\Theta) = 0$. There seems to be a tendency to group them in pairs, as the definitions Ai and H end up in the same set, $\mathcal{W}_4$, while the definitions W_x, W_y belong to the set $\mathcal{W}_5$.

It is possible that these phases are constructed via the Wronskian of two functions Θ_i is the reason each of these terms has a value belonging to one of the spaces $\mathcal{W}_n$ with $n \geq 0$. There might be an exact way to determine the space to which the terms Ai, H, W_x, and W_y will belong for a phase that, under a particular profile, especially $F = \log$, constructs an N-soliton type solution, with $N \geq 3$.

4.1.2. Uniqueness of the pairs (Θ, F)

It is important to recall that the solutions constructed throughout this work are of the form $u = 2\partial_x^2 F\,(\Theta)$, where F is the profile and Θ is the phase. The phases studied generally correspond to the Wronskian of exponential sums of the form $\sum_{j=1}^{M} a_j \exp\left(k_j x + k^2 j y + k^3 j\right)$, which is

$$\Theta = \begin{vmatrix} \Theta_1 & \Theta_2 & \cdots & \Theta_N \\ \Theta_1^{(1)} & \Theta_2^{(2)} & \cdots & \Theta_N^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \Theta_1^{(N-1)} & \Theta_2^{(N-1)} & \cdots & \Theta_N^{(N-1)} \end{vmatrix},$$

with $\Theta_i = \sum_{j=1}^{M_i} a_{ij} \exp\left(k_{ij} x + k_{ij}^2 y + k_{ij}^3\right)$ and $\Theta^{(k)} = \partial_x^k \Theta$.

An interesting topic for future research is to explore alternative structures that relate the profile and the phase. It is worth noting that the phases of the three types of solutions are of type $\mathcal{T}$ with $F = \log$, and these phases construct their respective solutions under this profile. It is possible that there are other types of solutions constructed using a different profile, in which case, the phases studied in this work may either enable the construction of other types of solutions or cease to be solutions. In other words, there is not a unique relationship associating a type of solution with a pair (Θ, F). It is possible that another profile for a phase, another phase for a profile, or another pair of phase and profile could allow the construction of the same type of solution.

4.2. Perspectives

Based on the results obtained in this study, it is expected that in the future it will be possible to construct line-soliton, resonant soliton, or 2-soliton solutions more easily, knowing in advance the conditions that satisfy the phases of each of these solutions..

Similarly, it is interesting to consider the possibility of taking the reverse approach, namely, deriving the phases that constructed known solutions from equations they must satisfy.

It is also important to consider that the vast majority of results obtained in this study rely on specific conditions regarding the profile F, which serves the role of being the function that modifies the phase to construct the solution. In this case, it can be observed that when $F = \ln$ and Θ is a sum of exponentials, there exists a certain relationship between the terms composing the phase and the profile, as the terms in the phase sum are the inverse of the profile. Generally,

$$\Theta = \sum_{j=1}^{M} F^{-1}\left(\theta_j\right), \tag{4.1}$$

It would be interesting to investigate whether equivalent results exist when maintaining the relationship (4.1) as in the case where $F = \ln$ and $F^{-1} = \exp$. This would allow for the construction of a larger group of solutions from other invertible profiles in the evolution of θ_j, which may satisfy the conditions studied in Section 3.2.

In summary, it is expected that the results obtained will allow for the creation of phases that construct the desired types of solutions, while also making it possible to recognize, based on the structure of a known solution, the form that the generating phase must have.

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