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SOLITON DYNAMICS FOR EINSTEIN FIELD EQUATIONS

TESIS PARA OPTAR AL GRADO DE
DOCTORA EN CIENCIAS DE LA INGENIERÍA,
MENCIÓN MODELACIÓN MATEMÁTICA

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RESUMEN DE LA TESIS PARA OPTAR AL GRADO DE
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DINÁMICA DE SOLITONES PARA ECUACIONES DE CAMPO DE EINSTEIN

Esta tesis está dedicada al estudio de Ecuaciones en Derivadas Parciales hiperbólicas que surgen de la Teoría de Relatividad General. Dada la complejidad de las ecuaciones de Einstein, frecuentemente es una buena opción estudiar una cuestión de interés en el marco de una clase restringida de soluciones. Una forma de imponer tales restricciones es considerar soluciones que satisfagan una condición de simetría dada. En este trabajo se considera la clase particular de soluciones con dos vectores de campo de Killing tipo-espacio. Más concretamente, nos enfocaremos en el modelo de Einstein en el vacío $R_{\mu\nu}(\tilde{g}) = 0$, donde $\tilde{g}$ es el tensor métrico y $R_{\mu\nu}$ es el tensor de Ricci, en la formulación de Belinski-Zakharov.

El objetivo principal de esta tesis es describir con rigor las condiciones para la existencia global de soluciones pequeñas, y su decaimiento en el cono de luz, así como la estabilidad de un primer conjunto de soluciones solitónicas, para la denominada *ecuación de Einstein reducida*, vista como una identificación de la ecuación de Campo Quiral Principal (PCF). El manuscrito se divide en 4 capítulos centrales, que pueden ser leídos de manera independiente.

En primer lugar, en el Capítulo 2 se describe rigurosamente la teoría de existencia local y global de soluciones con dato inicial pequeño para el modelo PCF. Además, se aborda el estudio de la dinámica a largo plazo de las soluciones con energía finita, proponiendo estimaciones viriales adecuadas para el modelo. Finalmente se propone una solución explícita tipo solitón, que pertenece a la familia de soluciones descrita en la teoría de existencia global.

En el Capítulo 3 se abordan las ecuaciones de campo de Einstein, por lo que naturalmente tenemos un problema más complejo de analizar. Bajo los supuestos apropiados de regularidad y datos iniciales pequeños, se obtiene la teoría global para el problema, así como una descripción apropiada de la energía y el momento en el caso de soluciones de tipo cosmológico en Relatividad General. Además, se proponen estimaciones viriales para esta clase de soluciones, lo que nos permite dar cuenta del decaimiento de soluciones con energía finita.

En el Capítulo 4 regresamos al modelo PCF, interesados en estudiar la estabilidad de las soluciones explícitas descritas en el Capítulo 2. A diferencia del enfoque clásico, combinaremos técnicas de estabilidad asintótica y preservación de la energía local, para proporcionar una caracterización de las perturbaciones de las soluciones de solitones regulares de PCF.

En lo que respecta al Capítulo 5, se abordará un problema diferente. Se propone estudiar la tasa de blow-up para el modelo modificado Zakharov-Kuznetsov. Este problema es particularmente interesante, dado que aún no se tiene un resultado concreto respecto a la existencia de soluciones blow-up. Sin embargo, estudios numéricos sugieren que, de tener una solución singular, la tasa de blow-up podría acotarse en cierto rango. El propósito de este estudio es contribuir al entendimiento y desarrollo de este desafiante problema.

Finalmente, en el Capítulo 6, presentamos las conclusiones de los distintos tópicos abordados, así como la descripción de problemas abiertos para considerarse en el futuro.

SUMMARY OF THE THESIS TO OBTAIN THE DEGREE OF
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SOLITONS DYNAMICS FOR EINSTEIN EQUATIONS

This thesis is devoted to the study of hyperbolic Partial Differential Equations arising from General Relativity Theory. Given the complexity of the Einstein equations, it is often a good choice to study a question of interest in the framework of a restricted class of solutions. One way to impose such restrictions is to consider solutions that satisfy a given symmetry condition. This work is concerned with the particular class of spacetimes that admit two space-like Killing vector fields. More precisely, we will focus on the Einstein vacuum model $R_{\mu\nu}(\tilde{g}) = 0$, where $\tilde{g}$ is the metric tensor and $R_{\mu\nu}$ is the Ricci tensor, in the Belinski-Zakharov setting. This ansatz is compatible with the well-known Gowdy symmetry.

The main goal of this thesis is to describe rigorously the conditions for the global existence of small solutions, and their decay in the light cone, as well as the stability of a first set of solitonic solutions (gravisolitons), for the so-called *reduced Einstein equation*, viewed as an identification of the Principal Chiral Field (PCF) model. In the following, the results of this thesis will be described. The manuscript is divided into four central chapters, which can be read independently of each other.

Chapter 2 firstly describes rigorously the theory of local and global existence of solutions with small initial data for the PCF model. Secondly, the study of the long-term dynamics of finite energy solutions is addressed, proposing suitable virial estimates for the model. Finally, an explicit soliton-type solution belonging to the family of solutions described in the global existence theory is proposed.

Chapter 3 addresses the Einstein field equations, where, under appropriate assumptions of regularity and small initial data, the local and global theory for the problem is obtained, as well as an adequate description of the energy and momentum in the case of cosmological type solutions in General Relativity. In addition, virial estimates are proposed for this class of solutions, which allows to account for the decay of solutions with finite energy.

In Chapter 4, we return to the PCF model, this time, interested in studying the orbital stability of the explicit solutions described in Chapter 2. Unlike the classical approach, we will combine asymptotic stability techniques and preservation of local energy to provide a near complete characterization of perturbations of regular soliton solutions of PCF model.

Regarding Chapter 5, a problem with a completely different focus will be addressed. This chapter proposes to study the blow-up rate for the modified Zakharov-Kuznetsov model. This problem is particularly interesting since there are still no concrete results on the existence of blow-up solutions for the model. However, numerical studies suggest that if there is a singular solution, the blow-up rate could be restricted to a certain range. The purpose of this study is to contribute to the state of the art on this challenging problem.

Finally, in Chapter 6, we present the conclusions of the different topics addressed and some open problems to be considered in the future.

A Michael & Darko.
A mis Padres & Hermanos.

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Part I

Introduction

Chapter 1

Introduction

The Einstein vacuum equation determines a 4-dimensional manifold $\mathcal{M}$ with a Lorentzian metric $\tilde{g}$ with vanishing Ricci curvature

$$R_{\mu\nu}(\tilde{g}) = 0. \tag{1.0.1}$$

This dissertation is devoted especially to the study of global existence and long time dynamics of vacuum Einstein's equations in General Relativity within the Belinski-Zakharov formalism. This model, together with an appropriate ansatz, can be identified with a **quasi-linear wave system**. In this setting, using suitable change of coordinates, we can describe $\tilde{g} = \tilde{g}(\Lambda, \phi, \alpha, f)$, where the fields Λ, ϕ, α, f are smooth functions of t and x.

Based on this, we will be able to identify the equation (1.0.1), with a special case for a general system of nonlinear wave equations in $\mathbb{R}^{1+1}_{t,x}$:

$$\partial_\mu(m^{\mu\nu}\alpha\partial_\nu\Psi) = F(\Psi, \partial\Psi), \tag{1.0.2}$$

here, $m^{\mu\nu}$ corresponds to the components of the Minkowski metric with $\mu, \nu \in \{0, 1\}$, $\Psi := (\Lambda, \phi)$ and the nonlinear term $F = (F_1, F_2)$ can be partially described in terms of the well-known fundamental null form[1] as follows

$$F_1 := Q_0(\ln\alpha, \Lambda) - 2\sinh(2\lambda + 2\Lambda)Q_0(\phi, \phi) \quad \text{and} \quad F_2 := Q_0(\ln\alpha, \phi) + \frac{\sinh(2\Lambda)}{\sinh^2(\Lambda)}Q_0(\phi, \Lambda).$$

This identification allows to use part of the machinery known for hyperbolic models, such as energy estimates and null vector fields, as well as key tools in the theory of dispersive nonlinear equations, such as virial estimates.

The first goal of this thesis is to study the global existence theory and the long-term dynamics of solutions for the Principal Chiral Field (PCF) equation. This equation arises under similar hypotheses, and corresponds to the particular case in (1.0.2) (taking α a constant). Despite the substantial contrast between the properties and dynamics of these models, this analysis is a first major step toward understanding the dynamics of the (1.0.2) model.

[1]For the forthcoming analysis it is convenient to introduce a fundamental null form, which is defined as the following bilinear form:

$$Q_0(\phi, \Lambda) = m^{\alpha\beta}\partial_\alpha\phi\partial_\beta\Lambda,$$

where $m_{\alpha\beta}$ to denote the standard Minkowski metric on $\mathbb{R}^{1+1}$.

We can summarize the second goal as follows: under the appropriate assumptions of regularity and small initial data, we rigorously describe a detailed local and global theory for the model (1.0.2), as well as an appropriate description of the energy and momentum for cosmological-type solutions in General Relativity. These types of solutions are particularly interesting since they include specific models such as Kasner's metric and other Bianchi-type models, which have a significant impact on physical applications. In addition, we propose virial estimates for these types of solutions, which allow us to account for the decay of solutions with finite energy.

The first result described earlier, concerning the PCF equation, leads us to a natural continuation of the research: the stability of the explicit solutions described for the Chiral model. Specifically, for this solution, the orbital stability is studied. Recall that stability theory is important because it provides robustness to the results previously obtained for the model equation.

The last problem addressed focuses on the *blow-up rate* associated with the Zakharov-Kusnetzov model. Using all the tools of harmonic analysis and the local existence theory already established for the model, we seek to establish a lower bound for the explosion rate of the solutions of the modified Zakharov-Kuznetsov model. In this case, the study of oscillatory integrals allows us to explicitly rewrite the estimates for both the linear model and the nonlinear problem, which enables us to establish a blow-up rate, under the assumption that there is a singular solution to the problem.

This work was carried out with important collaborations and research stays. A large part of it was developed during different long visits that I made in 2022-2024. My deepest thanks to Professors Frédéric Rousset, Jacques Smulevici, Jérémie Szeftel, Miguel A. Alejo, Magdalena Caballero, Alma Albujer and Gong Chen for their support, interesting discussions and very helpful comments during these visits.

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In what follows in this chapter, we will briefly describe interesting facts associated with the Theory of General Relativity, as well as the preliminary elements that will allow us to understand how to describe the dynamics of the model (1.0.1) from the study of the system of wave equations (1.0.2).

1.1 Brief Introduction to General Relativity Theory

General Relativity is one of the most beautiful physical theories ever invented. It describes one of the most pervasive features of the world we experience – gravitation – in terms of an elegant mathematical structure (the differential geometry of curved spacetime) leading to unambiguous predictions that have received spectacular experimental confirmation. The essential idea is perfectly straightforward: while most forces of nature are represented by fields defined on spacetime (such as the electromagnetic field, or the short-range fields characteristic of subnuclear forces), gravity is inherent in spacetime itself. In particular, what we experience as "gravity" is a manifestation of the curvature of spacetime. Carroll in [18] provides a clear introduction to General Relativity.

The background of the Theory of General Relativity was built on fascinating physical theories developed hundreds of years ago, let us briefly look at some of the history (see [35]):

We start in 1600, when ***Galileo Galilei*** stated the principle that the laws of motion and mechanics (those of electromagnetism were not yet discovered) were the same in all reference frames of constant velocity and found that the acceleration of bodies in free fall is universal[2].

Some years later, in 1666 ***Isaac Newton*** formulates the universal law of gravity and the equations of motion of classical mechanics. It states that the force that attracts two matter particles in the universe is directly proportional to the product of both their masses and inversely proportional to the the square of the distance between them. At the end of the XVII century, trying to explain whether light would behave in the same way as sound waves, two views emerged: ***Newton*** thought that light was a beam of emitted particles and ***Christiaan Huygens*** said that light should be considered as a wave motion.

Almost two centuries later, in 1854, ***Georg Friedrich Bernhard Riemann*** interprets space as a medium and introduces the notion of distance through a metric[3]. Riemann showed that the basic properties (the structure) of a curved space are determined by the differential cuadratic form ds^2 (distance notion), this would lead to the formulation of Differential Geometry.

For his part, at the end of the XIX century, ***James Clerk Maxwell*** confirmed that Huygens was right. Maxwell formulates the complete equations of electro-dynamics. In addition, Maxwell's theory provides a model of light as an electromagnetic effect and correctly predicts the speed of light. It became clear that light was the visible manifestation of a whole range of electromagnetic waves.

This conclusion raised the question of what was the medium in which light propagated and if its motion was indeed relative, to what did its speed turn out to be relative? The answer

[2]In his *Dialogue on the two maximal systems of the world*, Galileo tried to defend the Copernican idea that the Earth did not remain stationary at the center of the universe while everything else revolved around it by giving an excellent description of relativity as applied to systems moving with constant velocity relative to each other.

[3]Euclidean geometry describes plane surfaces but on curved surfaces it loses its validity. Gauss and others had developed different types of geometry that could describe the surface of spheres and other curved surfaces but it was Riemann who took things even further; he developed a way to describe a curved space with any number of dimensions.

seemed to be that the light waves constituted a perturbation of an invisible medium, called the *ether*[4], and that their velocity was relative to it. All this would lead to a desperate search for the *ether* at the end of the XIX century. Finally, in 1887 ***Albert Abraham Michelson*** and ***Edward Morley*** show through experiments that the existence of the *ether* is ruled out[5].

In 1905, ***Albert Einstein*** formulates the theory of special relativity, which leads to a true revolution in the structure of space and time. His first postulate was the principle of relativity, which stated that all the laws of physics, including Maxwell's equations, are the same for all observers moving at constant speed relative to each other, and he also postulated that, the speed of a ray of light was constant no matter how fast its source was moving.

Unfortunately, the postulate of light seemed incompatible with the principle of relativity until he found the key to the problem: analyzing the concept of time. Thus he concluded that time cannot be defined absolutely and that there is an inseparable relation between time and the speed of the signal.

In 1915 ***Albert Einstein*** formulates the theory of General Relativity. The result he obtained after great efforts and many years is known as the field equations, which are authentically covariant and, consequently, he succeeded in making his theory incorporate all forms of motion: inertial, accelerated, rotating or arbitrary. With his theory of special relativity, Einstein had shown that space and time did not have an independent existence but instead shaped the structure of space-time together. The curved and wavy structure of space-time explained gravity, its equivalence with acceleration and also - according to Einstein - the General Relativity of all forms of motion. With the equations of the gravitational field of his general theory of relativity, Einstein laid the foundations for the study of the nature and evolution of the universe, thus becoming the father of modern cosmology.

A few months later, early 1916, ***Karl Schwarzschild*** discovers the first non-trivial vacuum solution of the Einstein equations, describing the external gravitational field to a spherically symmetric mass configuration. As we will come to understand several decades later, this solution also describes an **unrotating black hole**. In the same year Einstein predicts the existence of **gravitational waves**.

The General Relativity Theory has continued to result in significant advances. The high point of success of the theory came in 1919, with the experiments of ***Arthur Eddington*** and ***Frank Dyson***, who measure the deflection of light during a solar eclipse and find that it agrees with the prediction of General Relativity. After that, in 1927, ***Georges Lemaitre*** predicts the expansion of the universe, based on Einstein's equations. A couple of years later, ***Edwin Hubble's*** work confirms this prediction. In 1963 ***Roy Kerr*** discovers the

[4]The *ether* must permeate the entire known universe, be thin and so ethereal that it would have no effect on the planets or on a feather passing through it; but at the same time it must be sufficiently rigid to allow a wave to vibrate through it at an enormous speed (the speed of light, the measure of which was already known).

[5]Einstein read an article by Wilhelm Wien describing the null results of thirteen experiments performed to detect ether, including the Michelson-Morley and Fizeau experiments. For Einstein, the significance of those experimental results was that they reinforced what he already believed: that Galileo's principle of relativity was applicable to light waves; that is, that from the mechanical point of view, any experiment performed will give the same results regardless of the frame of reference.

generalization of the Schwarzschild metric to the rotated case. ***LIGO*** experiment: in 2016 was obtained the first direct detection of gravitational waves produced by a binary black hole system. A new window into our universe is opened.

1.2 Preliminaries

1.2.1 Mathematical General Relativity

As a consequence of the beauty of the physical theory that gives us Einstein's theory of gravitation and its inherent mathematical background problem, appears the Mathematical General Relativity, which it ties fundamental problems of gravitational physics with beautiful questions in mathematics. Let us briefly recall the most important developments related to the Cauchy theory in General Relativity. Einstein field equations can be recast as quasilinear wave equations (or as a nonlinear elliptic model) under a suitable choice of gauge, therefore, it shares with the Partial Differential Equations (PDEs) theory, issues of existence, well-posedness and stability, as well as, existence of solitonic solutions and their dynamics, this issue is specific to the theory of dispersive PDEs. Choquet-Bruhat [21] gave a foundational mathematical description of the evolution of initial data, subsequently globalized by Choquet-Bruhat and Geroch [22]. After her, Newman- Penrose [92] analyzed gravitational radiation and the proper definition of gravitational waves. The stability of the Kerr BH was recently obtained in [64, 65, 66, 67, 68]. In the case of the Schwarzschild BH, the authors in [28, 29, 30] showed co-dimensional stability[6], and in [54] the authors proved stability under de Sitter gravity. When matter is included, compact objects with a larger range of allowed densities and redshifts are allowed [14]. Despite the impressive advances these years, the rigorous understanding of multiple symmetric compact objects is still open.

In the following, we will briefly present some definitions to understand the context and the elements that allow us to describe the equations of relativity in a PDEs context.

Elements of Lorentzian geometry

The equations of the Theory of General Relativity, describe how the gravitational field produces a curvature in spacetime, that is, replacing the action at a distance that produces a gravitational field created by a body, according to the classical theory of gravitation, by the idea that such a body, what it produces is a distortion of the space around it which is measured by the curvature.

In General Relativity, and related theories, the space of physical events is represented by a Lorentzian manifold. A *Lorentzian manifold* is a smooth (Hausdorff, paracompact) manifold $\mathcal{M} = \mathcal{M}^{n+1}$ of dimension $n+1$, equipped with a *Lorentzian metric* $\tilde{g}$ such that, $\tilde{g}$ assigns to each point $p \in \mathcal{M}$ a non-degenerate symmetric bilinear form on the tangent space $T_p\mathcal{M}$ of *signature* $(-++\cdots+)$.

Hence, if $\{e_0, e_1, ..., e_n\}$ is an orthonormal basis for $T_p\mathcal{M}$ with respect to $\tilde{g}$, then, perhaps after reordering the basis, the matrix $[\tilde{g}(e_i, e_j)] = \mathrm{diag}(-1, +1, \cdots, +1)$. A vector $v = \sum v^\mu e_\mu$

[6] For details of this result see specifically [30, Theorem I.3.1, pp. 5]

then has "square norm",

$$\tilde{g}(u,v) = -(v^0)^2 + \sum (v^i)^2,$$

which can be positive, negative or zero. This leads to the causal character of vectors, and indeed to the causal theory of Lorentzian manifolds. On a coordinate neighborhood $(U, x^\mu) = (U, x^0, x^1, ..., x^n)$ the metric $\tilde{g}$ is completely determined by its metric component functions on U,

$$\tilde{g}_{\mu\nu} := \tilde{g}\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right), \quad 0 \leq \mu, \nu \leq n.$$

That is, for $v = v^\mu \frac{\partial}{\partial x^\mu}, w = w^\nu \frac{\partial}{\partial x^\nu} \in T_p\mathcal{M}$, $p \in U$, $\tilde{g}(v,w) = \tilde{g}_{\mu\nu} v^\mu w^\nu$. Classically the metric in coordinates is displayed via the "line element"

$$ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu.$$

The prototype Lorentzian manifold is Minkowski space $\mathbb{R}^{n+1}$, the space-time of special relativity. This is $\mathbb{R}^{n+1}$, equipped with the Minkowski metric, which, with respect to Cartesian coordinates $(x^0, x^1, ..., x^n)$, is given by

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^n)^2.$$

Each tangent space of a Lorentzian manifold is isometric to the Minkowski space, and in this way the local accuracy of special relativity is built into General Relativity. For full details see [25].

In this order of ideas, the main object of study of the Theory of Mathematical General Relativity is a Lorentzian manifold that satisfies the Einstein field equations given by

$$\underbrace{R_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}R}_{curvature-expression} = \underbrace{8\pi T_{\mu\nu}}_{energy-momentum-tensor}. \tag{1.2.1}$$

Here R is the scalar curvature, $R_{\mu\nu}$ is the Ricci tensor which is defined by

$$R_{\mu\nu} := \partial_\sigma \Gamma^\sigma_{\mu\nu} - \partial_\nu \Gamma^\sigma_{\mu\sigma} + \Gamma^\sigma_{\gamma\sigma}\Gamma^\gamma_{\mu\nu} - \Gamma^\sigma_{\mu\gamma}\Gamma^\gamma_{\nu\sigma}, \tag{1.2.2}$$

the Christoffel symbols $\Gamma^\sigma_{\mu\nu}$ depend on the metric $\tilde{g}_{\mu\nu}$ and its inverse $\tilde{g}^{\mu\nu}$

$$\Gamma^\sigma_{\mu\nu} := \frac{1}{2}\tilde{g}^{\sigma\gamma}(\partial_\mu \tilde{g}_{\gamma\nu} + \partial_\nu \tilde{g}_{\mu\gamma} - \partial_\gamma \tilde{g}_{\mu\nu}), \tag{1.2.3}$$

this reveals the highly nonlinear nature of the set of ten equations in (1.2.1) in the components of the metric $\tilde{g}_{\mu\nu}$.

In particular, if it is considered that in the exterior of the body there are no physical phenomena of a non-gravitational nature and in this case, the metric representing the gravitational field must have null Ricci tensor, i.e., the Vacuum Einstein equation

$$R_{\mu\nu}(\tilde{g}) = 0. \tag{1.2.4}$$

In this thesis work we will focus precisely on the study of vacuum Einstein equations, which include many of the most important solutions in physics, such as **black holes** and **cosmological solutions**. In addition, the dynamics of the vacuum solutions can tell us what happens in the case where matter is present. This is a remarkable aspect of the General Relativity Theory, in contrast to Newton gravitation theory: the equation (1.2.4) is non-trivial even in the absence of matter. The focus of this thesis is the understanding of outstanding solutions of (1.2.4) in the setting of Belinski-Zakharov spacetimes.

1.2.2 The Belinski-Zakharov ansatz: an overview

Belinski and Zakharov in 1978 recalled the particular case in which the metric tensor $\widetilde{g}_{\mu\nu}$ depends on two variables only, which correspond to spacetimes that admit two commuting Killing vector fields, i.e. an Abelian two-parameter group of isometries. In this work we focus in the case when both Killing vector fields are space-like. This assumption allowed them to propose the so-called Belinski-Zakharov (BZ) transform to obtain solitonic solutions, so-called gravisolitons[7]. Gravisolitons have an unusual number of features, however, it is known that spacetimes highly important in physics and cosmology applications, such as, Schwarzschild and Kasner spacetimes, can be identified as gravisolitons [8, 9]. In this section, we introduce the ansatz used by BZ as well as the simplified version of the Einstein equation,

$$R_{\mu\nu}(\tilde{g}) = 0. \tag{1.2.5}$$

The reduced equation to be obtained is still extremely relevant in mathematics and physics, as will be explained below.

Consider a spacetime interval of a block diagonal form

$$ds^2 = g_{ij}dx^i dx^j + g_{ab}dx^a dx^b, \tag{1.2.6}$$

where we use the summation convention on $i, j, k, ... \in \{0, 3\}$ and $a, b, c, ... \in \{1, 2\}$, and assume that g_{ij} and g_{ab} depend only on the coordinates x^k. Due to the axioms of general relativity the tensor $g = (g_{ab})$ must be real and symmetric. By using a proper change of coordinates, one can always cast the matrix g_{ij} to the conformally flat form $g_{ij} = f\eta_{ij}$ where $f > 0$ and

$$\eta_{ij} = \begin{pmatrix} -e & 0 \\ 0 & 1 \end{pmatrix},$$

is a constant matrix with $e = \pm 1$. Note that $\eta_{\mu\nu}$ is the metric of a flat Minkowski 4-dimensional spacetime, while η_{ij} is the metric of its 2-dimensional subspace. η^{ij} denotes the inverse of η_{ij}. We will denote the determinants of each of the 2×2 blocks by

$$\det(g_{ij}) := -ef^2, \qquad \det(g_{ab}) := e\alpha^2,$$

their product is the determinant of the full metric tensor

$$\det \tilde{g}_{\mu\nu} = -f^2\alpha^2.$$

[7]The term gravisoliton refers to the explicit solutions generated by the Belinski-Zakharov transform [9].

From now on, we denote $g := (g_{ab})$, thus, the matrix representation of the metric is

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -ef & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & 0 \\ 0 & g_{21} & g_{22} & 0 \\ 0 & 0 & 0 & ef \end{pmatrix}. \tag{1.2.7}$$

It is important to recall that the structure of the metric (1.2.6) is not restrictive, since, from the physical point of view, we find many applications that can be described according to (1.2.6). Such spacetimes describe cosmological solutions of General Relativity, gravitational waves and their interactions [8]. In fact, we could highlight the following two scenarios:

Definition 1.2.1 (Non-stationary spacetimes)**.** *We consider as non-stationary spacetimes, a Lorentzian manifold with metrics that depend on time and one spatial coordinates, and are typically expressed using Cartesian coordinates in which* $x^i \in \{t, x\}$ *and* $x^a \in \{z, y\}$. *Here* $e = 1$ *in* (1.2.7)*, and the spacetime interval is*

$$ds^2 = f(t,x)(dx^2 - dt^2) + g_{ab}(t,x)dx^a dx^b.$$

Definition 1.2.2 (Stationary and axially-symmetric spacetimes)**.** *We consider as stationary spacetime, a Lorentzian manifold with metrics that depend on two spatial coordinates. This metrics can be conveniently expressed in cylindrical coordinates* $x^i \in \{\rho, z\}$, $x^a \in \{t, \phi\}$ *with* $e = -1$. *Then the spacetime interval is*

$$ds^2 = f(\rho,z)(d\rho^2 - dz^2) + g_{ab}(\rho,z)dx^a dx^b.$$

Such spacetimes describe fields of stationary compact objects (e.g. black holes). More precisely, among them one can find

- Classical solutions of the Robinson-Bondi plane waves [15].
- The Einstein-Rosen cylindrical wave solutions and their two polarization generalizations [17, 34].
- The homogeneous cosmological models of Bianchi types I–VII including the Kasner model [60].
- (In the "static" setting) the Schwarzschild and Kerr solutions, and Weyl axisymmetric solutions [111].
- 2-solitons, corresponding in a particular case to the Kerr-NUT (Newman-Unti-Tamburino) black-hole solution of three parameters including Kerr, Schwartzschild and Taub-NUT metrics [108].

To visualize how the mentioned spacetimes can be identified with the matrix representation (1.2.7), let us look at the following examples:

Example 1.2.1. *Line element for some special cases:*

- *Kasner metric:*

$$ds^2 = t^{(d^2-1)/2}(dz^2 - dt^2) + t^{1+d}dx^2 + t^{1-d}dy^2,$$

  where $x^a = (y, z)$, $x^i = (t, x)$.

- *Einstein Rosen metric*

$$ds^2 = f(t,r)(-dt^2 + dr^2) + e^{\Lambda(t,r)}(rd\phi)^2 + e^{-\Lambda(t,r)}dz^2,$$

where $x^a = (\phi, z)$, $x^i = (t, r)$.

- *Schwarzschild metric:* ***Kruskal-Szekeres coordinates***

$$ds^2 = -\frac{4r_s^2}{r}e^{-r/r_s}(dT^2 - dR^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

with $T^2 - R^2 = (1 - r/r_s)e^{r/r_s}$, *and* $x^a = (\phi, T)$, $x^i = (\theta, r)$.

For additional bibliography the reader may consult [74, 78, 79] and references therein. All this shows that, despite its relative simplicity, a metric of the type (1.2.6) encompasses a wide variety of physically relevant compact objects.

Reduced Einstein Equation

In order to simplify the Einstein equation (1.2.5), it is necessary to obtain the components of the Ricci tensor. Let us start by computing each of Christoffel symbol through Eq. (1.2.3):

$$\begin{aligned}\Gamma^k_{ij} =& \frac{1}{2}\left(\tilde{g}^{k0}(\partial_i\tilde{g}_{0j} + \partial_j\tilde{g}_{i0} + \partial_0\tilde{g}_{ij}) + \tilde{g}^{k3}(\partial_i\tilde{g}_{3j} + \partial_j\tilde{g}_{i3} + \partial_3\tilde{g}_{ij})\right)\\ =& \frac{1}{2}\tilde{g}^{kl}(\partial_i\tilde{g}_{lj} + \partial_j\tilde{g}_{il} - \partial_l\tilde{g}_{ij}),\end{aligned}$$

here, the notation ∂_μ indicates the derivative with respect to x^μ. Next

$$\Gamma^k_{ab} = -\frac{1}{2}\tilde{g}^{k\gamma}(\partial_\gamma\tilde{g}_{ab}) = -\frac{1}{2}\tilde{g}^{kl}(\partial_l\tilde{g}_{ab}).$$

Finally is directly that

$$\Gamma^c_{ib} = \frac{1}{2}\tilde{g}^{cd}(\partial_i\tilde{g}_{db}),$$

and

$$\begin{aligned}\Gamma^c_{ab} &= -\frac{1}{2}\tilde{g}^{al}\partial_l\tilde{g}_{bc} = 0.\\ \Gamma^k_{ib} &= \frac{1}{2}\tilde{g}^{k\gamma}\left(\partial_i\tilde{g}_{\gamma b} + \partial_b\tilde{g}_{i\gamma} - \partial_\gamma\tilde{g}_{ib}\right) = 0.\\ \Gamma^c_{ij} &= \frac{1}{2}\tilde{g}^{c\gamma}\left(\partial_i\tilde{g}_{\gamma j} + \partial_j\tilde{g}_{i\gamma} - \partial_\gamma\tilde{g}_{ij}\right)\\ &= \frac{1}{2}\tilde{g}^{cb}\left(\partial_i\tilde{g}_{bj} + \partial_j\tilde{g}_{ib} - \partial_b\tilde{g}_{ij}\right) = 0.\end{aligned}$$

The rest of the of the non-zero Christoffel symbols can be obtained through the symmetry $\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$. The trace of the Christoffel symbols is

$$\Gamma^\sigma_{i\sigma} = \partial_i(\ln(\alpha f)) \quad \text{and} \quad \Gamma^\sigma_{a\sigma} = 0.$$

The Christoffel symbols and their trace can be used in Eq. (1.2.2) to obtain the components of the Ricci tensor:

$$
\begin{aligned}
R_{ij} =& \frac{1}{2}\partial_k\left[\tilde{g}^{kl}(\partial_i\tilde{g}_{lj} + \partial_j\tilde{g}_{il} - \partial_l\tilde{g}_{ij})\right] - \partial_{ji}(\ln(\alpha f)) \\
&+ \frac{1}{2}\tilde{g}^{kl}(\partial_i\tilde{g}_{lj} + \partial_j\tilde{g}_{il} - \partial_l\tilde{g}_{ij})\partial_k(\ln(\alpha f)) \\
&- \frac{1}{4}g^{lm}g^{kn}(\partial_i\tilde{g}_{mk} + \partial_k\tilde{g}_{im} - \partial_m\tilde{g}_{ik})(\partial_j\tilde{g}_{ln} + \partial_l\tilde{g}_{jn} - \partial_n\tilde{g}_{jl}) \\
&- \frac{1}{4}g^{bd}g^{ac}(\partial_i g_{ad})(\partial_j g_{bc}),
\end{aligned} \tag{1.2.8}
$$

$$
\begin{aligned}
R_{ib} =& \partial_k\Gamma^k_{ib} - \partial_b\Gamma^\sigma_{i\sigma} + \Gamma^\sigma_{a\sigma}\Gamma^a_{ib} + \Gamma^\sigma_{k\sigma}\Gamma^k_{ib} - \Gamma^\sigma_{ik}\Gamma^k_{b\sigma} - \Gamma^\sigma_{ia}\Gamma^a_{b\sigma} \\
=& -\Gamma^c_{ik}\Gamma^k_{bc} - \Gamma^l_{ik}\Gamma^k_{bl} - \Gamma^c_{ia}\Gamma^a_{bc} - \Gamma^l_{ia}\Gamma^a_{bl} = 0,
\end{aligned}
$$

$$
R_{ab} = -\frac{1}{2\alpha f}\partial_k\left(\alpha f\tilde{g}^{kl}\tilde{g}^{cd}\partial_l\tilde{g}_{ac}\right)g_{db}.
$$

The complete system of Einstein's vacuum in Eq. (1.2.5) decomposes into two sets of equations. Simplifying the third equation in (1.2.8), the equation $R_{ab} = 0$ can be written as a single tensorial (matrix) equation

$$
\eta^{ij}\partial_j\left(\alpha\partial_i(g_{ac})g^{cb}\right) = 0. \tag{1.2.9}
$$

In this work we focus in the Eq. (1.2.9), and we shall refer to it as the ***reduced Einstein equation***. In particular, we will address the non-stationary case, for which the Eq. (1.2.9) is given by

$$
\partial_t\left(\alpha\partial_t g g^{-1}\right) - \partial_x\left(\alpha\partial_x g g^{-1}\right) = 0. \tag{1.2.10}
$$

The trace of the reduced Einstein equation is the trace equation:

$$
\eta^{ij}\partial_{ij}\alpha = 0,
$$

which implies in this case that the function α satisfies the one-dimensional wave equation, i.e.,

$$
\partial_{tt}\alpha - \partial_{xx}\alpha = 0.
$$

The second set follows from the equations $R_{00} + R_{33} = 0$ and $R_{03} = 0$, and gives the metric coefficient $f(t, x)$ in terms of the matrix g, solution of (1.2.10). These will be discussed later, specifically in the Section 1.3.1.

1.3 Introduction to the Einstein Models

1.3.1 1+1 vacuum Einstein equation in geometrical coordinates

Recall the vacuum Einstein equations

$$
R_{\mu\nu}(\tilde{g}) = 0, \tag{1.3.1}
$$

in this work, in addition to considering the Belinski-Zakharov ansatz presented previously, we will use a change of coordinates that will allow us to identify the reduced Einstein equation (1.2.10) as a quasilinear system of wave equations. This new representation of the Einstein equation (1.3.1) will be the central focus of study in this thesis.

Let us keep in mind that for the vacuum Einstein equation (1.3.1), in the context of non-stationary spacetimes, we will choose a metric tensor depending on a time-like coordinate x^0, and one space-like coordinate x^1 (possibly nonnegative). This choice, as mentioned before in Def. 1.2.1, corresponds to considering non-stationary gravitational fields, often referred to as Gowdy models [49], even when no compact spatial sections are considered. They are also often mentioned as generalized Einstein-Rosen spacetimes [17]. In the particular case where one has diagonal metrics these are called Einsten-Rosen spacetimes, first considered in 1937 by Einstein and Rosen [34].

Let us return to the description of the line element. We take the time-like coordinate $x^0 = t$ and the space-like coordinate $x^1 = x$. In this case the coordinates are typically expressed using Cartesian coordinates in which $x^i \in \{t, x\}$ with $i \in \{0, 1\}$, and $x^a, x^b \in \{y, z\}$, where the Latin indexes $a, b \in \{2, 3\}$. Then the spacetime interval is a simplified block diagonal form:

$$ds^2 = f(t,x)(dx^2 - dt^2) + g_{ab}(t,x)dx^a dx^b. \tag{1.3.2}$$

Recall that repeated indexes mean sum, following the classical Einstein convention. Recall that denote $g = g_{ab}$. Due to the axioms of general relativity the tensor g must be real and symmetric. The *reduced Einstein equation* for the non-stationary case is given by

$$\partial_t \left(\alpha \partial_t g g^{-1}\right) - \partial_x \left(\alpha \partial_x g g^{-1}\right) = 0, \quad \det g = \alpha^2. \tag{1.3.3}$$

The fact that the 2×2 tensor g is symmetric allows us to diagonalize it for fixed t and x. We write $g = RDR^T$, where D is a diagonal tensor and R is a rotation tensor, of the form

$$D = \begin{pmatrix} \alpha e^{\Lambda} & 0 \\ 0 & \alpha e^{-\Lambda} \end{pmatrix}, \qquad R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \tag{1.3.4}$$

clearly

$$\det g = \alpha^2. \tag{1.3.5}$$

Here Λ is the scalar field that determines the eingenvalues of g, and the scalar field ϕ determines the deviation of g from being a diagonal tensor. Since ϕ is considered as an angle, we assume $\phi \in [0, 2\pi]$. Therefore Λ, ϕ and α in (1.3.4) can be considered as the three degrees of freedom in the symmetric tensor g, [52]. Written explicitly, the tensor g is given now by

$$g = \alpha \begin{pmatrix} \cosh\Lambda + \cos 2\phi \sinh\Lambda & \sin 2\phi \sinh\Lambda \\ \sin 2\phi \sinh\Lambda & \cosh\Lambda - \cos 2\phi \sinh\Lambda \end{pmatrix}. \tag{1.3.6}$$

Some analog representations have been used in various associated results, for example in the Einstein-Rosen metric [17]. Note that Minkowski $\widetilde{g}_{\mu\nu} = (-1, 1, 1, 1)$ can be recovered by taking $\Lambda = 0$, $\alpha = 1$ and ϕ free. The equation (1.3.1) reads now

$$\begin{cases} \partial_t(\alpha\partial_t\Lambda) - \partial_x(\alpha\partial_x\Lambda) = 2\alpha \sinh 2\Lambda((\partial_t\phi)^2 - (\partial_x\phi)^2), \\ \partial_t(\alpha \sinh^2\Lambda\partial_t\phi) - \partial_x(\alpha \sinh^2\Lambda\partial_x\phi) = 0, \\ \partial_t^2\alpha - \partial_x^2\alpha = 0, \\ \partial_t^2(\ln f) - \partial_x^2(\ln f) = G, \end{cases} \tag{1.3.7}$$

where $G = G[\Lambda, \phi, \alpha]$ is given by

$$\begin{aligned} G := &-\big(\partial_t^2(\ln\alpha) - \partial_x^2(\ln\alpha)\big) - \frac{1}{2\alpha^2}((\partial_t\alpha)^2 - (\partial_x\alpha)^2) \\ &- \frac{1}{2}((\partial_t\Lambda)^2 - (\partial_x\Lambda)^2) - 2\sinh^2\Lambda((\partial_t\phi)^2 - (\partial_x\phi)^2). \end{aligned} \tag{1.3.8}$$

Note that the equation for $\alpha(t,x)$ is the standard one dimensional wave equation, and can be solved independently of the other variables. Also, given $\alpha(t,x)$, $\Lambda(t,x)$ and $\phi(t,x)$, solving for $\ln f(t,x)$ reduces to use D'Alembert formula for linear one dimensional wave with nonzero source term. Consequently, the only nontrivial equations in (1.3.7) are given by the equations for $\Lambda(t,x)$ and $\phi(t,x)$, for α solution to linear 1D wave.

The local behavior of the spacetime described before is defined by the function α. In our setting, α will be an always positive and bounded function. These characteristics will be provided by the initial conditions that will be imposed on the problem. The gradient of the function $\alpha(t,x)$ can be *timelike, spacelike* or *null*, see [8, 17, 34],

- The case where α is *spacelike* everywhere in spacetime ($(\partial_x\alpha)^2 - (\partial_t\alpha)^2 > 0$) corresponds to spacetimes said "with cylindrical symmetry", which corresponds to the Einstein Rosen spacetime, for example. They give an approach to the description of gravitational waves.
- When the gradient of α is globally *null*, ($(\partial_x\alpha)^2 - (\partial_t\alpha)^2 = 0$), it corresponds to the plane-symmetric waves.
- When the gradient of α is globally *timelike* ($(\partial_x\alpha)^2 - (\partial_t\alpha)^2 < 0$) is used to describe cosmological models and colliding gravitational waves.

The focus of this work will be precisely the *timelike* case. This classification for the gradient of the function α is necessary in order to propose an appropriate definition of energy and to be capable of providing a description of the decay of the solution associated with the system.

On another hand, as one can see from (1.3.7), solutions are not unique. These fields satisfy the gauge invariance

$$\begin{aligned} &(\Lambda, \phi, \alpha, f) \quad \text{solution, then} \\ &(\Lambda, \phi + k\pi, C_1\alpha, C_2 f) \quad \text{is also solution,} \qquad k \in \mathbb{Z}, \quad C_1, C_2 > 0. \end{aligned} \tag{1.3.9}$$

Since $\alpha \mapsto C_1\alpha$ is just a conformal transformation in (1.3.6), with no loss of regularity we can always assume $C_1 = C_2 = 1$ in (1.3.9). It should be noted that although (1.3.7) are strictly non-linear in the fields $\Lambda(t,x)$, $\phi(t,x)$, $\alpha(t,x)$ and $f(t,x)$, it shares many similarities with the classical linear wave and Born-Infeld equations [1]: given any $\mathcal{C}^2$ real-valued profiles $h(s), k(s), \ell(s), m(s)$, then the following functions are solutions for (1.3.7):

$$\begin{cases} \Lambda(t,x) = h(x \pm t), & \phi(t,x) = k(x \pm t), \\ \alpha(t,x) = \ell(x \pm t), & f(t,x) = m(x \pm t). \end{cases}$$

This property will be key when establishing the connection between the local theory that will be presented in the following section and the analysis of explicit solutions to the equation in the Section 3.6.

1.3.2 Principal Chiral Field equation

Consider the Principal Chiral Field Models given by

$$\partial_t \left(\partial_t g g^{-1}\right) - \partial_x \left(\partial_x g g^{-1}\right) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1.3.10}$$

valid for a 2×2 Riemannian metric g . The Principal Chiral Field is a nonlinear σ-model which is related to various classical spinor fields and received huge attention in the 1980s and 1990s. In this work, we study a particular case of the reduction problem on "symmetric spaces" such as the work of [7, 9, 52]. The symmetric space considered is the invariant manifold of symmetric matrices sitting in the Lie group $\mathrm{SL}(2;\mathbb{R})$. This space is not a Lie group, but it can be identified with a Hyperboloid in Minkowski spacetime, see [88].

The equation (1.3.10) is compatible with a certain class of constraints on the metric g that effectively "identified" this equation with a system of semilinear wave equations, as follows: if one settles $\alpha \equiv 1$ constant in (1.3.3), in this case the metric (1.3.2) is diffeomorphic to Minkowski [8, 52]. In this case, we avoid this oversimplification by only taking the first three equations in (1.3.7) with $\alpha \equiv 1$, not considering the function f, namely

$$\begin{cases} \partial_t^2 \Lambda - \partial_x^2 \Lambda = -2\sinh(2\Lambda)((\partial_x \phi)^2 - (\partial_t \phi)^2), \\ \partial_t^2 \phi - \partial_x^2 \phi = -\dfrac{\sinh(2\Lambda)}{\sinh^2(\Lambda)}(\partial_t \phi \partial_t \Lambda - \partial_x \phi \partial_x \Lambda). \end{cases} \tag{1.3.11}$$

System (1.3.11) is a Hamiltonian system, having the conserved energy (see [52, pp. 66]):

$$E[\Lambda,\phi](t) := \int \left(\frac{1}{2}((\partial_t \Lambda)^2 + (\partial_x \Lambda)^2) + 2\sinh^2 \Lambda((\partial_t \phi)^2 + (\partial_x \phi)^2) \right)(t,x)dx. \tag{1.3.12}$$

1.4 Introduction of the Zakharov-Kuznetsov Model

In this section, we introduce the generalized Zakharov-Kuznetsov (ZK) model. Although, this equation is not directly associated to the PDEs models in general relativity, it was part of my thesis work, giving me the opportunity to gain experience and to learn new tools, which contribute to improve my ability to tackle relevant research problems. More precisely, it was motivated by my research visit to Prof. Chen Gong at Georgia Institute of Thecnology.

1.4.1 Setting of the model

The generalized Zakharov-Kuznetsov equation is given by

$$u_t + (u_{xx} + u_{yy} + u^p)_x = 0. \tag{1.4.1}$$

This equation can be considered as a multi dimensional generalisation of the Korteweg-de Vries (KdV) equation. During their existence, solutions to ZK have several conserved quantities, relevant to this work are the L^2 norm (or mass), and the energy (or Hamiltonian):

$$M[u(t)] = \int_{\mathbb{R}^2} u^2(t) = M[u(0)],$$
$$E[u(t)] = \frac{1}{2}\int_{\mathbb{R}^2} (u_x^2(t) + u_y^2(t)) - \frac{1}{p+1}\int_{\mathbb{R}^2} u^{p+1}(t) = E[u(0)].$$

An important symmetry in the evolution equations is the scaling invariance, which states that an appropriately rescaled version of the original solution is also a solution of the equation. For the equation (1.4.1) it is

$$u(t,x,y) = \lambda^{\frac{d}{p-1}} u(\lambda^3 t, \lambda x, \lambda y).$$

This symmetry makes invariant the Sobolev norm $\dot{H}^s$ with $s = 1 - \frac{d}{p-1}$, since $\|u_\lambda\|_{\dot{H}^s} = \lambda^{\frac{d}{p-1}+s-1} \|u\|_{\dot{H}^s}$. Moreover, the index s gives rise to the critical-type classification of (1.4.1):

- when $s < 0$, or $p < d+1$, the equation (1.4.1) is called the L^2-subcritical equation (in this case is $p = 2$);
- if $p > d+1$, or $s > 0$, the equation is L^2-supercritical (we use $p = 4$), and
- with $p = d+1$, or $s = 0$, it is L^2**-critical**. This classification is important in the study of long time behaviour of solutions.

In Chapter 5, we will consider the case of the equation with $p = 3$, which corresponds to the modified Zakharov-Kuznetsov equation, which is given by

$$\begin{cases} u_t + u_{xxx} + u_{xyy} + u^2 u_x = 0, \quad (x,y) \in \mathbb{R}^2, \quad t > 0, \\ u(0,x,y) = u_0(x,y), \end{cases}$$

where $u = u(t,x,y)$ is a real valued function.

1.5 The Virial Method

In this section, we will describe one of the methods used in the development of this work. In particular, to understand the asymptotic behavior of the solutions. Next, let us introduce the mindset of virial identities.

Given an equation, it is always a good idea to read as much as possible out of it. Thus, we must always ask ourselves which are the rigid restrictions that an equation imposes a priori to its solutions. For example, the conservation laws. These conservation laws are particularly useful for controlling the long-time dispersive behavior or short-time smoothing behavior of nonlinear PDEs. A very useful variant of a conservation law is that of a *monotonicity formula*, a quantity $G(u(t),t)$ depending on the solution $u(t)$, and perhaps on the time t, which is always monotone increasing in time t, or perhaps monotone decreasing in time t, (see [107]).

A virial identity, in the most simplistic way, is a variation of a conservation law, often rewritten as a monotony law, that a solution has to satisfy, and are related to some symmetries and invariances that a solution to a certain equation can exhibit. These monotone quantities can be used to obtain long-time control of a solution in several ways.

If a quantity $G(u(t),t)$ is large at some initial time and is monotonically increasing, then it will clearly remain large for all times (later $t > t_0$). Conversely, if $G(u(t),t)$ is bounded at time t_0, is monotonically decreasing and is nonnegative, then it remains bounded for all later times $t > t_0$. If $G(u(t), t)$ is monotonically increasing and itself the time derivative of another quantity $K(t)$, then we also learn that $K(t)$ is convex in time, which can be useful in a number of ways. Finally, if we know that $G(u(t), t)$ is uniformly bounded in time, (through uses of conservation laws) and is monotonic, then we conclude from the fundamental theorem of calculus that $\partial_t G(u(t), t)$ is absolutely integrable in time and therefore decays to zero when $t \longrightarrow \infty$, at least in some average sense. This type of long-time decay is especially useful for understanding the asymptotic behaviour of the solution.

Monotonic quantities are used systematically in the context of elliptic equations, in this case, it is known as Pohozhaev's identity, which is applicable to localized solutions to the stationary nonlinear Schrödinger equation. In physics, the Virial Theorem gives a relation between the average total kinetic energy and the total potencial energy of the system. Although the virial estimates go back to the 70's only recently they have been used, together with their variations, in a surprisingly powerful way in the context of dispersive equations. More precisely, the virial identities in its modern form were introduced by Glassy [46] to show blow up for certain focusing nonlinear Schröndiger equation.

We describe in simple words how our virial works. As mentioned before, the base of the argument is the election of a conserved quantity, for example, for the vacuum Einstein model we constructed new energies and virial functionals to provide a description of the energy decay of smooth global cosmological metrics inside the light cone. It is possible to introduce the following adapted virial functional

$$\begin{aligned} \mathcal{I}(t) := & -\int \rho\left(\frac{x-vt}{\omega(t)}\right)\kappa\partial_x\alpha\left(4\sinh^2(\Lambda)((\partial_t\phi)^2+(\partial_x\phi)^2)+(\partial_t\Lambda)^2+(\partial_x\Lambda)^2\right)dx \\ & +\int \rho\left(\frac{x-vt}{\omega(t)}\right)\kappa\partial_t\alpha\left(2\partial_x\Lambda\partial_t\Lambda+8\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right)dx, \end{aligned} \tag{1.5.1}$$

with $v \in (-1, 1)$, and κ, ω and ρ are chosen and defined from the proper characterization of the problem. On the other hand, (Λ, ϕ, α) correspond to the solution of the quasilinear wave system obtained from the Einstein equations under the B-Z symmetry ansatz, and a change of appropriate coordinates. From the virial functional (1.5.1), one can obtain a characterization of the decay dynamics of the solutions with finite energy, which is described in the following results:

Lemma 1.5.1 (Virial identity). *One has $\mathcal{I}(t)$ well-defined and bounded in time, and*

$$\begin{aligned} \frac{d}{dt}\mathcal{I}(t) = & -\frac{\omega'(t)}{\omega(t)}\int \frac{x-vt}{\omega(t)}\rho'\left(\frac{x-vt}{\omega(t)}\right)\hat{p}(t,x) \\ & +\frac{v}{\omega(t)}\int \rho'\left(\frac{x-vt}{\omega(t)}\right)\hat{p}(t,x) \\ & +\frac{1}{\omega(t)}\int \rho'\left(\frac{x-vt}{\omega(t)}\right)\hat{e}(t,x). \end{aligned}$$

where $\hat{e} = \hat{e}(\kappa, \Lambda, \phi, \alpha)$ and $\hat{p} = \hat{p}(\kappa, \Lambda, \phi, \alpha)$. As a consequence, we can prove the following result, which accounts for the decay of the solution, for the particular case of the cosmological spacetime:

Theorem 1.5.1 (Decay of cosmological finite-energy spacetimes Theorem 3.1.3, Chapter 3)**.** *The cosmological 1-soliton* (Λ, ϕ, α) *obtained from a nonsingular generalized Kasner*[8] *metric of parameter* $d \geq 1$ *is globally defined under suitable small perturbations in the case where* α *satisfies*

$$\lim_{t\to+\infty} \int_{|x-vt|\leq\omega(t)} \Big[(\partial_t\Lambda)^2 + (\partial_x\Lambda)^2 + \sinh^2(\Lambda)\Big((\partial_t\phi)^2 + (\partial_x\phi)^2\Big)\Big](t,x)dx = 0.$$

1.6 Main results

The results proved in this thesis are included in the following articles and preprints:

1. J. Trespalacios. *Global Existence and Long Time Behavior in the 1+1 dimensional Principal Chiral Model with Applications to Solitons*, Annales Henri Poincaré, (2024). (Chapter 2).
2. C. Muñoz and J. Trespalacios. *Global Existence and Long Time Behavior in Einstein-Belinski-Zakharov Soliton Spacetimes*, arXiv:2305.01414. Submitted 2024. (Chapter 3).
3. M. Alejo, C. Muñoz, J. Trespalacios. *Nonlinear Stability of nonsingular solitons of the Principal Chiral Field equation.* arXiv:2408.09969. Submitted 2024. (Chapter 4).
4. J. Trespalacios. *Blow-up rate for modified Zakharov-Kuznetsov Equation.* Preprint. (Chapter 5).

Now we briefly describe a slightly shorter version of the results:

1.6.1 Principal Chiral Field Model

As already mentioned, the simplest case for the reduced Einstein equation corresponds to the Principal Chiral Field model. Hadad in [52] studied this identification using geometrical coordinates (or Gowdy coordinates) obtaining a system of quasilinear wave equations, then, using the Belinski-Zakharov transform, he presented some explicit solutions of this model.

In the Chapter 2, we focus on formally describing both, the local and global existence theory of the associated semilineal wave system. The main problem lies in the fact that waves in one dimension do not decay as they do in higher dimensions. However, inspired by the results for the one-dimensional wave equation with null condition [85], we were able to show that, with appropriate assumptions on the initial data, it is possible to obtain the global existence of solutions in the energy space associated with the system. The chapter considers a special case of the Principal Chiral Field model in (1 + 1)- dimensions as a simplified version of the Einstein vacuum field equations under Belinksi-Zakharov symmetry.

There are four main results in this chapter: a local well-posedness statement, a global well-posedness statement, a global-in-time decay result of some solutions, and an application to

[8]This metric correspond to a particular cosmological spacetime studied in Chapter 3.

solitons. All of the statements together make for a relatively self-contained, new, interesting, and comprehensive introduction to this special case of the Principal Chiral Field model. Moreover, this work has led to some impressive follow up work that can be see in the Chapter 3.

Local and Global Existence.

Consider the following $(1+1)-$dimensional system of the semilinear wave equations, which corresponds to the Principal Chiral Field model (PCFE) using the so called Gowdy coordinates (see [109]):

$$\begin{cases} \partial_t^2\Lambda - \partial_x^2\Lambda = -2\sinh(2\Lambda)((\partial_x\phi)^2 - (\partial_t\phi)^2), \\ \partial_t^2\phi - \partial_x^2\phi = -\dfrac{\sinh(2\Lambda)}{\sinh^2(\Lambda)}(\partial_t\phi\partial_t\Lambda - \partial_x\phi\partial_x\Lambda). \end{cases} \tag{1.6.1}$$

If we take function $\Lambda(t,x)$ in the form

$$\Lambda(t,x) := \lambda + \tilde{\Lambda}(t,x), \quad \lambda \neq 0.$$

Without loss of generality, we assume $\lambda > 0$. The basic idea is to establish the conditions that are required on λ and $\tilde{\Lambda}$ in order to obtain the desired regularity results. Let us consider the following notation:

$$\begin{cases} \Psi = \left(\tilde{\Lambda}, \phi\right), \quad \partial\Psi = \left(\partial_t\tilde{\Lambda}, \partial_x\tilde{\Lambda}, \partial_t\phi, \partial_x\phi\right), \\ |\partial\Psi|^2 = \left|\partial_t\tilde{\Lambda}\right|^2 + \left|\partial_x\tilde{\Lambda}\right|^2 + |\partial_t\phi|^2 + |\partial_x\phi|^2, \end{cases}$$

where

$$(\Psi, \partial_t\Psi) \in \mathcal{H} := H^3(\mathbb{R}) \times H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^2(\mathbb{R}).$$

Recall that, an evolution equation is said to be well-posed in the sense of Hadamard, if existence, uniqueness of solutions and continuous dependence on initial data hold. A short version of our first result is the following.

Proposition 1.6.1 (Proposition 2.1.1, Chapter 2)**.** *If* $(\Psi, \partial_t\Psi)|_{\{t=0\}} = (\Psi_0, \Psi_1) \in \mathcal{H}$, *satisfies certain smallness conditions, then we have a Local Well Posedness for* (1.6.1)

Having established the existence of solutions, our second result involves whether or not local solutions can be extended globally in time. This is not an easy problem, mainly because $\Lambda(t,x)$ may achieve the zero value in finite time. Therefore, an important aspect of the proof will be to ensure uniform distance from zero of the function $\Lambda(t,x)$.

Theorem 1.6.1 (Theorem 2.1.1, Chapter 2)**.** *Consider the semilinear wave system* (1.6.1) *posed in* $\mathbb{R}^{1+1}$, *with the following initial conditions:*

$$\begin{cases} (\phi, \tilde{\Lambda})|_{\{t=0\}} = \varepsilon(\phi_0, \tilde{\Lambda}_0), \quad (\phi_0, \tilde{\Lambda}_0) \in C_c^\infty(\mathbb{R}\times\mathbb{R}), \\ (\partial_t\phi, \partial_t\tilde{\Lambda})|_{\{t=0\}} = \varepsilon(\phi_1, \tilde{\Lambda}_1), \quad (\phi_1, \tilde{\Lambda}_1) \in C_c^\infty(\mathbb{R}\times\mathbb{R}). \end{cases} \tag{1.6.2}$$

Then, there exists ε_0 *sufficiently small such that if* $\varepsilon < \varepsilon_0$, *the unique solution remains smooth for all time and have finite conserved energy.*

Long time behavior and Applications to solitons.

Finally, we discuss the long time behavior of globally defined solutions. Here, virial identities will be key to the long-time description.

Theorem 1.6.2 (Theorem 2.1.2, Chapter 2). *Let* (Λ, ϕ) *be a global solution to* (1.6.1) *such that its energy* $E[\Lambda, \phi](t)$ *(see* (1.3.12)*) is conserved and finite. Then, for any* $v \in (-1, 1)$ *and* $\omega(t) = t/\log^2 t$*, one has*

$$\lim_{t\to+\infty} \int_{vt-\omega(t)}^{vt+\omega(t)} \left((\partial_t\Lambda)^2 + (\partial_x\Lambda)^2 + \sinh^2\Lambda((\partial_t\phi)^2 + (\partial_x\phi)^2)\right)(t,x)dx = 0.$$

This result establishes that inside the light cone, all finite-energy solutions must converge to zero as time tends to infinity.

An important outcome of our previous results is a clear background for the study of soliton solutions of (1.6.1). In this work, we propose a finite energy soliton solution. Although it is not so clear that they are physically meaningful, these new solutions have finite energy and local well-posedness properties in a vicinity.

Indeed, consider a smooth function $\theta \in C_c^2(\mathbb{R})$. Additionally, consider the constraint $0 < \mu < 1$. For any $\lambda > 0$, and $\varepsilon > 0$ small, let

$$\Lambda_\varepsilon^{(0)} := \lambda + \varepsilon\theta(t + x), \quad \phi^{(0)} := 0.$$

Clearly $\Lambda_\varepsilon^{(0)}$ solves the wave equation in $1D$ and has finite energy $E[\Lambda_\varepsilon^{(0)}, \phi_\varepsilon^{(0)}] < +\infty$. This will be for us the background seed. The corresponding 1-soliton is now

$$g^{(1)} = \begin{bmatrix} \dfrac{e^{\lambda+\varepsilon\theta}\operatorname{sech}(\beta(\lambda+\varepsilon\theta))}{\operatorname{sech}(\beta(\lambda+\varepsilon\theta)-x_0)} & -\dfrac{1}{\sqrt{c}}\operatorname{sech}(\beta(\lambda+\varepsilon\theta)) \\ -\dfrac{1}{\sqrt{c}}\operatorname{sech}(\beta(\lambda+\varepsilon\theta)) & \dfrac{e^{-(\lambda+\varepsilon\theta)}\operatorname{sech}(\beta(\lambda+\varepsilon\theta))}{\operatorname{sech}(\beta(\lambda+\varepsilon\theta)+x_0)} \end{bmatrix}, \qquad \beta = \frac{\mu+1}{\mu-1}, \tag{1.6.3}$$

which also has finite energy. Perturbations of the fields Λ and ϕ associated with this soliton will be globally defined according to the Theorem 1.6.1:

Corollary 1.6.1 (Corollary 2.1.1, Chapter 2). *Suitable perturbations of any soliton as in* (1.6.3) *are globally well-defined.*

This result has an important outcome: it allows us to try to study the stability of these solutions, which will be done in Chapter 4.

1.6.2 Einstein-Belinski-Zakharov Spacetimes

In the Chapter 3 we consider the Vacuum Einstein equation in the setting of Belinski-Zakharov ansatz. As seeing before, using Gowdy coordinates (geometrical coordinates), the model can be rewrite as the Eq. (1.3.7). For the forthcoming analysis it is convenient to introduce a fundamental null form, which is defined as the following bilinear form:

$$Q_0(\phi, \tilde{\Lambda}) = m^{\alpha\beta}\partial_\alpha\phi\partial_\beta\tilde{\Lambda},$$

where $m_{\alpha\beta}$ to denote the standard Minkowski metric on $\mathbb{R}^{1+1}$. Then, using this definition, one can rewrite the first two equations of the system (1.3.7) in terms of null forms as follows:

$$\begin{cases} \Box\tilde{\Lambda} = Q_0(\ln\alpha, \tilde{\Lambda}) - 2\sinh(2\lambda + 2\tilde{\Lambda})Q_0(\phi, \phi), \\ \Box\phi = Q_0(\ln\alpha, \phi) + \dfrac{\sinh(2\lambda + 2\tilde{\Lambda})}{\sinh^2(\lambda + \tilde{\Lambda})}Q_0(\phi, \tilde{\Lambda}), \end{cases} \tag{1.6.4}$$

and the remaining equations are

$$\begin{cases} \partial_t^2\alpha - \partial_x^2\alpha = 0, \\ \partial_t^2(\ln f) - \partial_x^2(\ln f) = G, \end{cases} \tag{1.6.5}$$

where $G = G[\Lambda, \phi, \alpha]$ is given by

$$\begin{aligned} G := &- \big(\partial_t^2(\ln\alpha) - \partial_x^2(\ln\alpha)\big) - \frac{1}{2\alpha^2}((\partial_t\alpha)^2 - (\partial_x\alpha)^2) \\ &- \frac{1}{2}((\partial_t\Lambda)^2 - (\partial_x\Lambda)^2) - 2\sinh^2\Lambda((\partial_t\phi)^2 - (\partial_x\phi)^2). \end{aligned} \tag{1.6.6}$$

In addition, we introduce the following modified energy of the system, which in the case of cosmological type solutions will be highly relevant (see Chapter 3 Section 3.4):

$$E[\Lambda, \phi; \alpha](t) := -\int [\kappa\partial_t\alpha(h_1 - 2h_2)](t, x)dx, \tag{1.6.7}$$

where $\kappa(t, x) = \dfrac{\alpha}{(\partial_x\alpha)^2 - (\partial_t\alpha)^2}$,

$$h_1(t, x) = (\partial_t\Lambda)^2 + (\partial_x\Lambda)^2 + 4\sinh^2(\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2), \tag{1.6.8}$$

and

$$h_2(t, x) = \partial_t\Lambda\partial_x\Lambda + 4\sinh^2(\Lambda)\partial_t\phi\partial_x\phi.$$

This (nonconserved) energy is a modified version of the one introduced by Hadad [52], which was not sufficiently useful in our purposes. Here (1.6.7) has important modifications to ensure the positivity of the energy functional. Compared with our previous results [109] in the case of the Principal Chiral equation, here the energy and momentum terms require deeper understanding and much more work than before. Summarizing, in this chapter we described in the Belinski-Zakharov-Einstein setting the following: **The local and global existence** of solutions of the model, the definition of a **modified energy and momentum**, the **global-in-time decay** result of solutions of finite energy, and finally **The application to gravisolitons** of cosmological type. The following is a brief description of these results:

Global existence of small solution.

Our first result in this work is the global existence of solutions. For the system, we consider constraints on the initial conditions for $\alpha(t, x)$. Using the D'Alembert formula we have an explicit expression for α that allows us to obtain tight control over appropriate terms by also using the central structure related to null forms. Although the nonlinearity is not purely defined in terms of null forms, we can follow and adapt properly in the case of variable coefficients the certain weighted energy estimates proposed. Keeping this in mind, we have the following result:

Theorem 1.6.3 (Small data global existence, Theorem 3.1.1, Chapter 3)**.** *Let $\lambda > 0, c_1 > 0$ be fixed, and set*

$$\Lambda =: \lambda + \tilde{\Lambda}, \quad \text{and} \quad \alpha := 1 + \tilde{\alpha}. \tag{1.6.9}$$

Consider the wave system (1.6.4) *posed in* $\mathbb{R}^{1+1}$*, with the following initial conditions:*

$$\text{(IC)} \quad \begin{cases} (\phi, \tilde{\Lambda}, \alpha, f)|_{\{t=0\}} = (\varepsilon\phi_0, \varepsilon\tilde{\Lambda}_0, 1 + \tilde{\alpha}_0, c_1 + f_0), \\ (\partial_t\phi, \partial_t\tilde{\Lambda}, \partial_t\alpha, \partial_t f)|_{\{t=0\}} = (\varepsilon\phi_1, \varepsilon\tilde{\Lambda}_1, \alpha_1, f_1), \\ (\phi_0, \tilde{\Lambda}_0, \tilde{\alpha}_0, f_0) \in \left(C_c^\infty(\mathbb{R})\right)^4, \\ (\phi_1, \tilde{\Lambda}_1, \alpha_1, f_1) \in C_c^\infty(\mathbb{R}) \times C_c^\infty(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}). \end{cases}$$

Assume the following bounds on the initial conditions:

1. $\alpha_1(\cdot) > 0$,
2. $\max_{n=0,1,2} \left(\|\partial_x^{(n)} \tilde{\alpha}_0\|_\infty + \|\partial_x^{(n)} \alpha_1\|_\infty \right) < \frac{1}{2}\gamma$, *where γ is a fixed sufficiently small constant, but independent on ε.*
3. $||f_0||_\infty \leq \frac{c_1}{2}$,
4. *the initial data satisfy the compatibility conditions required by Einstein's field equations.*

Then, there exists ε_0 sufficiently small such that if $\varepsilon < \varepsilon_0$, the unique solution remains smooth for all time.

A direct consequence of Theorem 1.6.3 is the global existence of the Belinski-Zakharov metric (1.3.2):

Corollary 1.6.2. *Under the assumptions in Theorem 1.6.3, g and f in* (1.3.2) *are globally well-defined.*

Decay of cosmological finite-energy spacetimes.

The second result in this chapter concerns the decay of a specific type of the solutions of the Einstein equations in the vacuum. Specifically of cosmological type solutions, which are of special interest in physics and cosmology. This type of solutions include the Kasner type spacetimes, as well as some Bianchi type models, see [8]. We will prove, using well-chosen virial estimates that for solutions to system (1.6.4) with finite energy (in particular, globally defined small solutions from Theorem 1.6.3), they must decay to zero locally in space, provided that the gradient of the function $\alpha(t,x)$ is globally timelike.

The following theorem constitutes the most important result in this chapter:

Theorem 1.6.4 (Decay of cosmological finite-energy spacetimes, Theorem 3.1.3, Chapter 3)**.** *Under the hypotheses in Theorem 1.6.3, assume in addition that one has*

(a) bounded energy condition:

$$\sup_{t \geq 0} E[\Lambda, \phi; \alpha](t) < +\infty;$$

(b) for some $c_0 > 0$ *one has*

$$\alpha(t,x) > c_0 \quad \text{and} \quad \partial_t \alpha \text{ is in the Schwartz class uniformly in time.}$$

Then, for any $v \in \mathbb{R}$, $|v| < 1$, *and* $\omega(t) = t(\log t)^{-2}$, *one has*

$$\lim_{t \longrightarrow +\infty} \int_{|x-vt| \lesssim \omega(t)} \left[(\partial_t \Lambda)^2 + (\partial_x \Lambda)^2 + \sinh^2(\Lambda) \Big((\partial_t \phi)^2 + (\partial_x \phi)^2 \Big) \right] dx = 0.$$

A simple corollary in terms of the spacetime tensor g can be obtained:

Corollary 1.6.3. *Under the hypotheses in Theorem 1.6.3, one has that* g *in* (1.3.2) *satisfies*

$$\lim_{t \longrightarrow +\infty} \int_{|x-vt| \leq \omega(t)} \left((\partial_t \det g)^2 + (\partial_x \det g)^2 \right) (t,x) dx = 0. \tag{1.6.10}$$

Vanishing property (1.6.10) can be understood as the manifestation that the spacetime is of cosmological type, and information propagates with the speed of light, supported on the light cone.

Finally, we apply Theorem 1.6.3 and 1.6.4 to the cosmological 1-soliton obtained from a *nonsingular generalized Kasner metric*, In particular, we shall prove:

Theorem 1.6.5 (Theorem 3.1.4, Chapter 3)**.** *The cosmological 1-soliton* (Λ, ϕ, α) *obtained from a nonsingular generalized Kasner metric of parameter* $d \geq 1$ *is globally defined under suitable small perturbations in the case where* α *satisfies the hypotheses of Theorem 1.6.3, and satisfies*

$$\lim_{t \to +\infty} \int_{|x-vt| \leq \omega(t)} \left[(\partial_t \Lambda)^2 + (\partial_x \Lambda)^2 + \sinh^2(\Lambda) \Big((\partial_t \phi)^2 + (\partial_x \phi)^2 \Big) \right] (t,x) dx = 0.$$

in the case where α *is of cosmological type and satisfies the hypotheses of Theorem 1.6.4. Moreover, it propagates with the speed of light.*

1.6.3 Stability for the Principal Chiral Soliton

In the Chapter 4 we consider the orbital stability study for the 1-soliton of the Principal Chrial Field equation. Stability theory is an important line of research in nonlinear partial differential equations, in the sense that it gives solidity to the results related to global existence of the solutions, asymptotic behavior, the dynamics of solutions. In addition to the motivation arising from Principal Chiral Model, the study of the stability for hyperbolic equations is of independent interest because of the connections with other branches of physics, for example, the study of the irrotational compressible Euler equations, also, the Einstein-Maxwell equation, as well as, the wave map equation.

In this work we want to study hyperbolic PDEs arising from PCF Model

$$\partial_t \left(\partial_t g g^{-1} \right) - \partial_x \left(\partial_x g g^{-1} \right) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R},$$

valid for a 2×2 Riemannian metric g, into the Lie group $\mathrm{SL}(2;\mathbb{R})$. In particular we are interested in the orbital stability of special solutions of these model with small initial data perturbations. The main idea is to use viral techniques in a new way, this time, to obtain orbital stability of the explicit solutions that can be previously constructed for the problem.

One of the important results of the global existence problem for the Principal Chiral Field equation presented in Section 1.6.1, was the proposal of the 1-soliton. Recall that the equation can be identified with the semilinear wave system

$$\begin{cases} \partial_t^2\Lambda - \partial_x^2\Lambda = -2\sinh(2\Lambda)((\partial_x\phi)^2 - (\partial_t\phi)^2), \\ \partial_t^2\phi - \partial_x^2\phi = -\dfrac{\sinh(2\Lambda)}{\sinh^2(\Lambda)}(\partial_t\phi\partial_t\Lambda - \partial_x\phi\partial_x\Lambda). \end{cases} \tag{1.6.11}$$

The Finite energy solitons for this model are given by:

$$g^{(1)} = \begin{bmatrix} \dfrac{e^{\lambda+\varepsilon\theta}\operatorname{sech}(\beta(\lambda+\varepsilon\theta))}{\operatorname{sech}(\beta(\lambda+\varepsilon\theta)-x_0)} & -\dfrac{1}{\sqrt{c}}\operatorname{sech}(\beta(\lambda+\varepsilon\theta)) \\ -\dfrac{1}{\sqrt{c}}\operatorname{sech}(\beta(\lambda+\varepsilon\theta)) & \dfrac{e^{-(\lambda+\varepsilon\theta)}\operatorname{sech}(\beta(\lambda+\varepsilon\theta))}{\operatorname{sech}(\beta(\lambda+\varepsilon\theta)+x_0)} \end{bmatrix}, \qquad \beta = \frac{\mu+1}{\mu-1}, \tag{1.6.12}$$

where $\theta \in C_c^2(\mathbb{R})$, $0<\mu<1$, and for any $\lambda>0$, $x_1 \in \mathbb{R}$ and $\varepsilon>0$ small, we defined

$$\Lambda_\varepsilon^{(0)} = \Lambda_\varepsilon^{(0)}(t,x) := \lambda + \varepsilon\theta(t+x), \quad \phi^{(0)} := 0.$$

For its part, the parameters are given by

$$c = \left(\frac{2\mu}{\mu^2-1}\right)^2, \quad v = -\frac{\mu^2+1}{2\mu} < -1, \quad \text{and} \quad x_0 = \frac{\ln|\mu|}{\sqrt{c}}.$$

For the 1-soliton (1.6.12), the corresponding fields $\hat{\Lambda}_\varepsilon$ and $\hat{\phi}_\varepsilon$, are given by:

$$\begin{aligned} \hat{\Lambda}_\varepsilon &= B := \operatorname{arcosh}\left(|v|\cosh\left(\Lambda_\varepsilon^{(0)}\right) - \frac{1}{\sqrt{c}}\tanh\left(\beta\left(\Lambda_\varepsilon^{(0)}\right)\right)\sinh\left(\Lambda_\varepsilon^{(0)}\right)\right), \\ \hat{\phi}_\varepsilon &= D := \frac{\pi}{4} - \frac{1}{2}\arctan\left(\cosh\left(\beta\left(\Lambda_\varepsilon^{(0)}\right)\right)\cosh\left(\Lambda_\varepsilon^{(0)}\right)\left(\tanh\left(\beta\left(\Lambda_\varepsilon^{(0)}\right)\right) + v\sqrt{c}\tanh\left(\Lambda_\varepsilon^{(0)}\right)\right)\right). \end{aligned} \tag{1.6.13}$$

The purpose of this work is to give a first proof of the fact that the 1-soliton (1.6.12) of the PCF model is orbital stable under small perturbations well-defined in the natural energy space associated to the problem. The stability study will be done by addressing equation (1.6.11) and using the description of the 1-solution (1.6.12) in terms of the fields Λ and ϕ given by (1.6.13). In this order of ideas, we consider 1-soliton perturbed initial data of the form

$$\begin{cases} (\Lambda,\phi)|_{\{t=0\}} = (B+\varepsilon z_0, D+\varepsilon s_0), \quad (z_0,s_0) \in C_c^\infty(\mathbb{R})^2, \\ (\partial_t\Lambda,\partial_t\phi)|_{\{t=0\}} = (B_t+\varepsilon w_0, D_t+\varepsilon m_0), \quad (w_0,m_0) \in C_c^\infty(\mathbb{R})^2. \end{cases} \tag{1.6.14}$$

Then there exists ε_0 such that if $\varepsilon<\varepsilon_0$, the unique solution remains smooth for all time and have finite conserved energy. More precisely we can prove what follows result:

Theorem 1.6.6 (Theorem 4.1.2, Chapter 4). *There exists* $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$, *the following holds. There exist* $C, \delta_0 > 0$ *such that, if* $0 < \delta < \delta_0$, *and* $[z_0, w_0, s_0, m_0]$ *is given as in* (1.6.14), *then the following are satisfied:*

1. *External energetic control. Assume that*

$$\int \left(\frac{1}{2}(w_0^2 + z_{0,x}^2) + 2\sinh^2(B + z_0)(s_{0,x}^2 + m_0^2) \right)(x)dx < \delta.$$

Then

- *For all time, one has a crossed global control:*

$$\int_{\mathbb{R}} \left(\frac{1}{2}(z_x - w)^2 + 2\sinh^2(B + z)(s_x - m)^2 \right)(t, x)dx < 3\delta. \tag{1.6.15}$$

- *Inside light-cone convergence. For any* $v \in (-1, 1)$ *and* $\omega(t) = t/\log^2 t$, *one has*

$$\lim_{t\to+\infty} \int_{vt-\omega(t)}^{vt+\omega(t)} \left(w^2 + z_x^2 + \sinh^2(B + z)(m^2 + s_x^2)\right)(t, x)dx = 0. \tag{1.6.16}$$

- *Exterior stability: for all time* $t \geq 0$,

$$\int_{|x+t|\geq R} \left(\frac{1}{2}(w^2 + z_x^2) + 2\sinh^2(B + z)(s_x^2 + m^2) \right)(t, x)dx < \delta. \tag{1.6.17}$$

2. *Full orbital stability. Assume now* $[z_0, w_0, s_0, m_0] \in C_c^\infty(\mathbb{R})^4$ *be initial data as in* (4.1.11) *such that*

$$\sum_{k=0,1} \int (1 + |x|^2)^{1+\gamma} \left((\partial_x^k w_0)^2 + (\partial_x^{k+1} z_0)^2 + (\partial_x^k m_0)^2 + (\partial_x^{k+1} s_0)^2\right) dx < \delta^2, \tag{1.6.18}$$

for $0 < \gamma < \frac{1}{3}$. *Then the corresponding global solution to* (4.1.6) *given as*

$$(B + z, \partial_t B + w, D + s, \partial_t D + m) \tag{1.6.19}$$

satisfies the same bounds for all times:

$$\sup_{t\geq 0} \left(\mathcal{E}(t) + \overline{\mathcal{E}}(t)\right) \leq C\delta^2. \tag{1.6.20}$$

(See (4.4.7) *for the definition of these norms.)*

Here $\mathcal{E}(t)$ and $\overline{\mathcal{E}}(t)$ describe the energy norms associated with the problem, which are introduced in the Chapter 4.

1.6.4 Blow-up rate for the modified Zakharov-Kuztnesov equation

Chapter 5 is devoted to the study of a lower bound for the blow up rate of that solution for the modfied Zakharov-Kuznetsov

$$\begin{cases} u_t + u_{xxx} + u_{xyy} + u^2 u_x = 0, \quad (x,y) \in \mathbb{R}^2, \quad t > 0, \\ u(0,x,y) = u_0(x,y), \end{cases}$$

where $u = u(t,x,y)$ is a real valued function.

Our analysis relies on the local well-posedness results of Linares and Pastor [81] in H^s, $s > 3/4$. The approach is to start with important linear estimates given by Faminskii [37], and then move on to non-linear estimates given by Linares and Pastor in [81]. In particular, we carefully keep track of the power of time involved in the estimates as it is central for the analysis of the lower bound for the blow-up rate. In a second stage, we will try to adapt the idea of Colliander et. al. [27]. The original idea comes from an argument used for the heat equation made by Weissler [113] and later extended to nonlinear Schrödinger equations by Cazenave and Weissler [19] to obtain a lower bound of blow-up for Sobolev norms of the solution. More precisely, we can prove what follows result

Theorem 1.6.7 (Theorem 5.1.1, Chapter 5)**.** *Consider the IVP* (5.1.2) *with initial conditions* $u_0 \in H^s_{xy}(\mathbb{R})$ *with* $s > 3/4$*. Assume that the solution* $u(t,x,y)$ *blows up in a finite time* T^* *in* $H^s_{xy}(\mathbb{R})$*. Then, we have the following lower bound for the blow-up rate:*

$$C(s) \left\| u(t) \right\|_{H^s} > (T^* - t)^{-7/48}, \quad t \uparrow T^*. \tag{1.6.21}$$

Part II

The Einstein Models

Chapter 2

Global existence and long time behavior in the 1+1 dimensional Principal Chiral model with applications to solitons

Abstract: In this paper, we consider the 1+1 dimensional vector valued Principal Chiral Field model (PCF) obtained as a simplification of the Vacuum Einstein Field equations under the Belinski-Zakharov symmetry. PCF is an integrable model, but a rigorous description of its evolution is far from complete. Here we provide the existence of local solutions in a suitable chosen energy space, as well as small global smooth solutions under a certain non degeneracy condition. We also construct virial functionals which provide a clear description of decay of smooth global solutions inside the light cone. Finally, some applications are presented in the case of PCF solitons, a first step towards the study of its nonlinear stability.

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Contents

2.1 Introduction and main results

2.1.1 Setting

The *Einstein field vacuum* equations and its consequences are key in the Physics of the past century. For a gravitational Lorentzian field $\widetilde{g} = \widetilde{g}_{\mu\nu}$ of local signature $(-1, 1, 1, 1)$, one seeks for solving the vanishing of the Ricci tensor

$$\mathrm{R}_{\mu\nu}(\widetilde{g}) = 0. \tag{2.1.1}$$

This equation can be seen as a system of nonlinear quasilinear wave equations. Its importance lies in the fact that many of the characteristic features of the dynamics of the Einstein field equations, are already present in the study of the vacuum equations. See Wald [111] for a detailed description of the Physics behind these equations.

Under certain symmetries and assumptions, the Einstein field equation can be identified and reduced to the integrable *Symmetric Principal Chiral Field Equation*,

$$\partial_t \left(\partial_t g g^{-1}\right) - \partial_x \left(\partial_x g g^{-1}\right) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{2.1.2}$$

valid for a 2×2 Riemannian metric g. This last equation will be the main subject of this work. This equation is compatible with a certain class of constraints on the metric g that effectively "reduce" the equation (2.1.2) to a system of quasilinear wave equations. We will prove existence of local solutions, global small solutions, and describe in part the asymptotic behavior of globally defined solutions. The Principal Chiral Field is a nonlinear σ-model the which is related to various classical spinor fields and received huge attention in the 1980s and 1990s. The first description of the integrability of this model in the language of the commutative representation (2.1.2) was given in [119], subsequently, different results associated with integrability, conserved quantities and soliton solutions were obtained [10, 41, 93], as well as different analyses of this equation using Backlund transformation, Darboux transformation [31, 53]. In the literature there are several results associated with the study of the reduction of the PCF equation in homogeneous spaces of Lie groups. In particular, Zakharov and Mikhailov in [117] studied the model of the Principal Chiral Field for the special unitary group $\mathrm{SU}(N)$, as well as in [118], they studied the connection of this equation with the Nambu-Jona-Lasinian model. In this work, we study a particular case of the reduction problem on "symmetric spaces" such as the work of [7, 9, 52]. The symmetric space considered is the invariant manifold of symmetric matrices sitting in the Lie group $\mathrm{SL}(2;\mathbb{R})$. This space is not a Lie group, but it can be identified with a Hyperboloid in Minkowski spacetime, see [88].

In order to explain the emergence of (2.1.2) starting from (2.1.1), one needs to consider the so-called Belinski-Zakharov symmetry ansatz [9]. Symmetry has been a successful method for understanding complicated dynamics in a series of works related to dispersive models, see e.g. [34, 45, 104]. On the other hand, this assumption is not restrictive, in the sense that several important cases of physical Einstein vacuum metrics are contained under this restriction.

2.1.2 Belinski-Zakharov spacetimes

Belinski and Zakharov recalled the particular case in which the metric tensor $\widetilde{g}_{\mu\nu}$ depends on two variables only, which correspond to spacetimes that admit two commuting Killing vector fields, i.e. an Abelian two-parameter group of isometries, [8, 9]. The metric depends on a time-like coordinate x^0, and one space-like coordinate x^1 (possibly nonnegative). This choice, as will stay clear below, corresponds to considering non-stationary gravitational fields and was first consider by Kompaneets [71]. In the particular case that one has a diagonal metric this type of spacetime is often referred to as Einsten-Rosen spacetimes and was first considered in 1937 by Einstein and Rosen [34].

In this work we take these variables to be the time-like and the space-like coordinates $x^0 = t$ and $x^1 = x$ respectively. In this case the coordinates are typically expressed using Cartesian coordinates in which $x^i \in \{t, x\}$ with $i \in \{0, 1\}$, and $x^a, x^b \in \{y, z\}$, where the Latin indexes $a, b \in \{2, 3\}$. Then the spacetime interval is a simplified block diagonal form:

$$ds^2 = f(t,x)(dx^2 - dt^2) + g_{ab}(t,x)dx^a dx^b. \tag{2.1.3}$$

Recall that repeated indexes mean sum, following the classical Einstein convention. Here with abuse of notation we denote $g = g_{ab}$. Due to the axioms of general relativity the matrix g must be real and symmetric. As mentioned above, the structure of this metric is not restrictive, since, from the physical point of view, we find many applications that can be described according to (2.1.3). Such spacetimes describe cosmological solutions of general relativity, gravitational waves and their interactions. Also they have many applications in gravitational theory, [8], we can emphasize that these types of spacetimes belong to the classical solutions of the Robinson–Bondi plane waves [15], the Einstein–Rosen cylindrical wave solutions and their two polarization generalizations, the homogeneous cosmological models of Bianchi types I–VII including the Kasner model [59], the Schwarzschild and Kerr solutions, Weyl axisymmetric solutions, etc. For many more contemporary results the reader can refer to [74]. All this shows that in spite of its relative simplicity a metric of the type (2.1.3) encompasses a wide variety of physically relevant cases.

In order to reduce Einstein vacuum equations (2.1.1), one needs to compute the Ricci curvature tensor in terms of the components of the metric $g = g_{ab}$. The consideration of the metric in the form (3.1.2) results in that the components R_{0a} and R_{3a} of the Ricci tensor are identically zero. Therefore, one can see that system of the Einstein vacuum equations (2.1.1) decomposes into two sets of equations. The first one follows from equations $\mathrm{R}_{ab} = 0$, this equation can be written as single matrix equation

$$\partial_t \left(\alpha \partial_t g g^{-1}\right) - \partial_x \left(\alpha \partial_x g g^{-1}\right) = 0, \quad \det g = \alpha^2. \tag{2.1.4}$$

We shall refer to this equation as the *reduced Einstein equation.* The trace of the equation (2.1.4) reads

$$\partial_t^2 \alpha - \partial_x^2 \alpha = 0. \tag{2.1.5}$$

This is the so-called *trace* equation; the function $\alpha(t,x)$ satisfies the 1D wave equation, for details of the derivation of equations (2.1.4) and (2.1.5) see [8, p. 11] and [52, pp. 27 and 147]. The second set of equations expresses the metric coefficient $f(t,x)$ in terms of explicit terms of α and g, where $\det \widetilde{g}_{\mu\nu} := -f^2\alpha^2$. For the moment, this expression is not relevant in this introduction, for more details see [8].

2.1.3 New coordinates

The fact that the 2×2 matrix g is symmetric allows one to diagonalize it for fixed t and x. One writes $g = RDR^T$, where D is a diagonal matrix and R is a rotation matrix, of the form

$$D = \begin{pmatrix} \alpha e^{\Lambda} & 0 \\ 0 & \alpha e^{-\Lambda} \end{pmatrix}, \qquad R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

Clearly $\det g = \alpha^2$. Here Λ is the scalar field that determines the eigenvalues of g, and the scalar field ϕ determines the deviation of g from being a diagonal matrix. Since ϕ is considered as an angle, we assume $\phi \in [0, 2\pi]$. Therefore Λ, ϕ and α can be considered as the three degrees of freedom in the symmetric matrix g, [52]. Written explicitly, the matrix g is given now by

$$g = \alpha \begin{pmatrix} \cosh\Lambda + \cos 2\phi \sinh\Lambda & \sin 2\phi \sinh\Lambda \\ \sin 2\phi\sinh\Lambda & \cosh\Lambda - \cos 2\phi\sinh\Lambda \end{pmatrix}. \tag{2.1.6}$$

Some analog representations have been used in various results associated, for example to the Einstein-Rosen metric [17]. Note that Minkowski $g_{\mu\nu} = (-1,1,1,1)$ can be recovered by taking $\Lambda = 0$, $\alpha = 1$ and ϕ free. Now, with this representation, the equation (2.1.4) reads

$$\begin{cases} \partial_t(\alpha\partial_t\Lambda) - \partial_x(\alpha\partial_x\Lambda) = 2\alpha\sinh 2\Lambda((\partial_t\phi)^2 - (\partial_x\phi)^2), \\ \partial_t(\alpha\sinh^2\Lambda\partial_t\phi) - \partial_x(\alpha\sinh^2\Lambda\partial_x\phi) = 0, \\ \partial_t^2\alpha - \partial_x^2\alpha = 0, \end{cases} \tag{2.1.7}$$

and

$$\partial_t^2(\ln f) - \partial_x^2(\ln f) = G, \tag{2.1.8}$$

where $G = G[\Lambda,\phi,\alpha]$ is given by

$$\begin{aligned} G :=& -\big(\partial_t^2(\ln\alpha) - \partial_x^2(\ln\alpha)\big) - \frac{1}{2\alpha^2}((\partial_t\alpha)^2 - (\partial_x\alpha)^2) \\ &- \frac{1}{2}((\partial_t\Lambda)^2 - (\partial_x\Lambda)^2) - 2\sinh^2\Lambda((\partial_t\phi)^2 - (\partial_x\phi)^2). \end{aligned} \tag{2.1.9}$$

Note that the equation for α is the standard one dimensional wave equation, and can be solved independently of the other variables. Also, given α, Λ and ϕ, solving for $\ln f$ reduces to use D'Alembert formula for linear one dimensional wave with nonzero source term. Consequently, the only nontrivial equations in (2.1.7) are given by

$$\begin{cases} \partial_t(\alpha\partial_t\Lambda) - \partial_x(\alpha\partial_x\Lambda) = 2\alpha\sinh 2\Lambda((\partial_t\phi)^2 - (\partial_x\phi)^2) \\ \partial_t(\alpha\sinh^2\Lambda\partial_t\phi) - \partial_x(\alpha\sinh^2\Lambda\partial_x\phi) = 0, \end{cases} \tag{2.1.10}$$

for α solution to linear 1D wave. Because of the difficulties found dealing with this system, we shall concentrate efforts in a more modest case. If one settles $\alpha \equiv 1$ constant, in this case the metric (2.1.3) is diffeomorphic to Minkowski [8, 52]. In this paper, we avoid this oversimplification by only taking (2.1.10) with $\alpha \equiv 1$, not considering the function f, namely

$$\begin{cases} \partial_t^2\Lambda - \partial_x^2\Lambda = -2\sinh(2\Lambda)((\partial_x\phi)^2 - (\partial_t\phi)^2), \\ \partial_t^2\phi - \partial_x^2\phi = -\dfrac{\sinh(2\Lambda)}{\sinh^2(\Lambda)}(\partial_t\phi\partial_t\Lambda - \partial_x\phi\partial_x\Lambda). \end{cases} \tag{2.1.11}$$

(2.1.11) is a set of coupled quasilinear wave equations, with a rich analytical and algebraic structure, as we shall see below. Also, it coincides with (2.1.2) under g as in (2.1.6) and $\alpha \equiv 1$. Understanding this particular case will be essential to fully understand the general case of non constant α. It is very notable that the basic set of equations of the Einstein equation for the metric (2.1.3) coincides with the Principal Chiral Field equation (2.1.2) when α is constant. If we only consider this set of equations, the PCF equation formally admits nontrivial solutions which would correspond to a special subclass of Chiral Field theory solutions, [8]. It should be noted that in the particle case where α is constant, the *reduced Einstein equation* (2.1.4) corresponds to the chiral field equation (2.1.2), as mentioned above, however, as we will see later, from the definitions of energy and momentum densities of the Chiral Field equation, we cannot deduce relevant results from the Einstein field equation when α is an arbitrary function, the non-constant α case requires a different treatment. As a consequence of the above observation, in the constant α case, the equation (2.1.8) is no longer coupled to the system to be worked.

***Remark* 2.1.1.** *Notice that the choice $\alpha \equiv 1$ is aso made because the equations* (2.1.7) *may have a different behavior depending on the properties of the function α. Even in this case ($\alpha \equiv 1$), the PCF model is sufficiently rich to produce a complex dynamics. In our recent result [91], posted online very recently, we consider the more demanding case α non constant, but still under some particular conditions that are natural generalizations of the hypotheses presented here. Finally, the current work has been essential to obtain the general results presented in [91].*

As we can see from the matrix form (2.1.6) the solutions in terms of the fields Λ and ϕ are not unique, since these fields satisfy a gauge invariance, that is,

$$(\Lambda, \phi) \quad \text{solution}, \qquad (\Lambda, \phi + k\pi) \quad \text{solution}, k \in \mathbb{Z}. \tag{2.1.12}$$

It should be noted that although (2.1.11) is strictly non-linear in the fields $\Lambda(t, x)$ and $\phi(t, x)$, it has many similarities with the classical linear wave equation and with Born-Infeld equation [1]: given any $\mathcal{C}^2$ real-valued profiles $h(s), k(s)$, then the following functions are solutions for Eqns. (2.1.11)

$$\Lambda(t, x) = h(x \pm t) = h(s), \qquad \phi(t, x) = k(x \pm t) = k(s). \tag{2.1.13}$$

This property will be key when establishing the connection between the local theory that will be presented in the following section and the analysis of explicit solutions to the equation in the Section 2.5. System (2.1.11) is a Hamiltonian system, having the conserved energy

$$E[\Lambda, \phi](t) := \int \left(\frac{1}{2}((\partial_t\Lambda)^2 + (\partial_x\Lambda)^2) + 2\sinh^2\Lambda((\partial_t\phi)^2 + (\partial_x\phi)^2)\right)(t, x)dx. \tag{2.1.14}$$

Note that the energy is well-defined if $(\Lambda, \partial_t \Lambda) \in \dot{H}^1 \times L^2$, but a suitable space for $(\phi, \partial_t \phi)$ strongly depends on the weight $\sinh^2 \Lambda$, which can easily grow exponentially in space, since $\dot{H}^1$ can easily contain unbounded functions. In this sense, understanding $E\Lambda, \phi$ (even for classical solutions such as solitons) is subtle and requires a deep and careful analysis, which will be done later.

The notion of the energy and the law of conservation of energy play a key role in all mathematics-physical theories. However, the definition of energy in relativity is a complex matter, and this problem has been given a lot of attention in the literature [111, 112]. The most likely candidate for the energy density for the gravitational field in general relativity would be a quadratic expression in the first derivatives of the components of the metric [111]. In this case we have a particular structure of spacetime and the equation (2.1.6) gives us a decomposition of the metric in terms of the fields Λ and ϕ.

Coming back to our problem, and using inverse scattering techniques, Belinski and Zakharov [9] considered (2.1.4) giving a first approach to this problem. They introduce a Lax-pair for (2.1.4)-(3.1.4), together with a general method for solving it. Localized structures and multi-coherent were found, but they are not solitons in the standard sense, unless α is constant, a more in-depth study on the subject, is also made in [8, 9]. More recently Hadad [52] explored the Belinski-Zakharov transformation for the 1+1 Einstein equation. It is used to derive explicit formula for solutions on arbitrary diagonal background, in particular on the Einstein-Rosen background.

2.1.4 Main results

One of the main purposes of this paper is to give a rigorous description of the dynamics for (2.1.2) in the so-called energy space associated to the problem, and close to important exact solutions. We will present three different results: local, global existence, and long time behavior of solutions, in particular solitons.

Our first result is a classical local existence result for solutions in the energy space. As mentioned above, the system (2.1.11) is a set of coupled quasilinear wave equations, with a rich analytical and algebraic structure.

Clearly in the analysis of the initial value problem for this system, we have a component of difficulty related to the regularity of the term $\frac{\sinh(2\Lambda)}{\sinh^2(\Lambda)}$ when the function $\Lambda(t,x)$ is zero, which must be carefully analyzed in order to be able to construct a result of local well-posedness associated to (2.1.11). In order to develop the results related to the local theory for the nonlinear wave equation, let us write the function $\Lambda(t,x)$ in the form

$$\Lambda(t,x) := \lambda + \tilde{\Lambda}(t,x), \quad \lambda \neq 0. \tag{2.1.15}$$

Notice that this choice makes sense with the energy in (2.1.14), in the sense that $\Lambda \in \dot{H}^1$ and $\partial_t \Lambda \in L^2$. Without loss of generality, we assume $\lambda > 0$. The basic idea is to establish the conditions that are required on λ and $\tilde{\Lambda}$ in order to obtain the desired regularity results. With this choice, the system (2.1.11) can be written in terms of the function $\tilde{\Lambda}(t,x)$ as follows:

$$\begin{cases} \partial_t^2\tilde{\Lambda} - \partial_x^2\tilde{\Lambda} = -2\sinh(2\lambda+2\tilde{\Lambda})((\partial_x\phi)^2 - (\partial_t\phi)^2), \\ \partial_t^2\phi - \partial_x^2\phi = -\dfrac{\sinh(2\lambda+2\tilde{\Lambda})}{\sinh^2(\lambda+\tilde{\Lambda})}(\partial_t\phi\partial_t\tilde{\Lambda} - \partial_x\phi\partial_x\tilde{\Lambda}). \end{cases} \tag{2.1.16}$$

This is the system we are going to analyze along this paper.

Let us consider the following notation :

$$\begin{cases} \Psi = \left(\tilde{\Lambda},\phi\right), \quad \partial\Psi = \left(\partial_t\tilde{\Lambda},\partial_x\tilde{\Lambda},\partial_t\phi,\partial_x\phi\right), \\ |\partial\Psi|^2 = \left|\partial_t\tilde{\Lambda}\right|^2 + \left|\partial_x\tilde{\Lambda}\right|^2 + |\partial_t\phi|^2 + |\partial_x\phi|^2, \\ F(\Psi,\partial\Psi) = (F_1,F_2), \\ F_1(\Psi,\partial\Psi) := 2\sinh(2\lambda+2\tilde{\Lambda})\left((\partial_x\phi)^2 - (\partial_t\phi)^2\right), \\ F_2(\Psi,\partial\Psi) := \dfrac{\sinh(2\lambda+2\tilde{\Lambda})}{\sinh^2(\lambda+\tilde{\Lambda})}\left(\partial_t\phi\partial_t\tilde{\Lambda} - \partial_x\phi\partial_x\tilde{\Lambda}\right). \end{cases} \tag{2.1.17}$$

With this notation, the initial value problem for (2.1.16) can be written in vector form as follows

$$\begin{cases} \partial_\alpha(m^{\alpha\beta}\partial_\beta\Psi) = F(\Psi,\partial\Psi) \\ (\Psi,\partial_t\Psi)|_{\{t=0\}} = (\Psi_0,\Psi_1) \in \mathcal{H}. \end{cases} \tag{2.1.18}$$

Where $m^{\alpha\beta}$ are the components of the Minkowski metric with $\alpha,\beta \in \{0,1\}$, and

$$(\Psi,\partial_t\Psi) \in \mathcal{H} := H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}).$$

Notice that from (2.1.15), $\Lambda \in \dot{H}^1$. We are also going to impose the following condition on the initial data

$$\|(\Psi_0,\Psi_1)\|_{\mathcal{H}} \le \frac{\lambda}{2D}, \tag{2.1.19}$$

where the assumptions on the constant $D \ge 1$ will be indicated below. An evolution equation is said to be well-posed in the sense of Hadamard, if existence, uniqueness of solutions and continuous dependence on initial data hold.

The following proposition shows that the equation (2.1.18), in terms of the function $\tilde{\Lambda}$ introduced in (2.1.15), is locally well-posed in the space $L^\infty([0,T];\mathcal{H})$ with the norm in this space defined by

$$\|(\Psi,\partial_t\Psi)\|_{L^\infty([0,T];\mathcal{H})} = \sup_{t\in[0,T]}\left(\|\Psi\|_{H^1(\mathbb{R})\times H^1(\mathbb{R})} + \|\partial_t\Psi\|_{L^2(\mathbb{R})\times L^2(\mathbb{R})}\right),$$

with $(\Psi,\partial_t\Psi)$ introduced in (2.1.17). Our first result is the following.

Proposition 2.1.1. *If (Ψ_0,Ψ_1) satisfies the condition (2.1.19) with an appropriate constant $D \ge 1$, then:*

(1) *(Existence and uniqueness of local-in-time solutions). There exists*

$$T = T\left(\left\|\left(\tilde{\Lambda}_0, \phi_0\right)\right\|_{H^1(\mathbb{R})\times H^1(\mathbb{R})}, \left\|\left(\tilde{\Lambda}_1, \phi_1\right)\right\|_{L^2(\mathbb{R})\times L^2(\mathbb{R})}, \lambda\right) > 0,$$

such that there exists a solution Ψ *to* (2.1.18) *with*

$$(\Psi, \partial_t \Psi) \in L^\infty([0,T]; \mathcal{H}).$$

Moreover, the solution is unique in this function space. If the data has more regularity, the solution is classical, see Appendix 2.6.3.

(2) *(Continuous dependence on the initial data). Let* $\Psi_0^{(i)}, \Psi_1^{(i)}$ *be sequence such that* $\Psi_0^{(i)} \longrightarrow \Psi_0$ *in* $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ *and* $\Psi_1^{(i)} \longrightarrow \Psi_1$ *in* $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ *as* $i \longrightarrow \infty$. *Then taking* $T > 0$ *sufficiently small, we have*

$$\left\|\left(\Psi^{(i)} - \Psi, \partial_t(\Psi^{(i)} - \Psi)\right)\right\|_{L^\infty([0,T];\mathcal{H})} \longrightarrow 0.$$

Here Ψ *is the solution arising from data* (Ψ_0, Ψ_1) *and* $\Psi^{(i)}$ *is the solution arising from data* $\left(\Psi_0^{(i)}, \Psi_1^{(i)}\right)$.

Note that the above proposition does not directly give us a classical solution to the problem, however, if it is assumed that the initial data is sufficiently regular, in fact the solution can be understood as classical, see Appendix 2.6.3.

Having established the existence of solutions, our second result involves whether or not local solutions can be extended globally in time. This is not an easy problem, mainly because $\Lambda(t,x)$ may achieve the zero value in finite time. Therefore, an important aspect of the proof will be to ensure uniform distance from zero of the function $\Lambda(t,x)$.

Theorem 2.1.1. *Consider the semilinear wave system* (2.1.18) *posed in* $\mathbb{R}^{1+1}$, *with the following initial conditions:*

$$\begin{cases} (\phi, \tilde{\Lambda})|_{\{t=0\}} = \varepsilon(\phi_0, \tilde{\Lambda}_0), \quad (\phi_0, \tilde{\Lambda}_0) \in C_c^\infty(\mathbb{R}\times\mathbb{R}), \\ (\partial_t\phi, \partial_t\tilde{\Lambda})|_{\{t=0\}} = \varepsilon(\phi_1, \tilde{\Lambda}_1), \quad (\phi_1, \tilde{\Lambda}_1) \in C_c^\infty(\mathbb{R}\times\mathbb{R}). \end{cases} \tag{2.1.20}$$

Then, there exists ε_0 *sufficiently small such that if* $\varepsilon < \varepsilon_0$, *the unique solution remains smooth for all time and have finite conserved energy* (2.1.14).

The condition that the initial data is compactly supported can be relaxed, but it is essential to have enough decay. For simplicity of exposition, we shall assume that the data is compactly supported, as it is usually done in the literature, see for example [106].

The global existence problem stated above is a key part of the analysis comes from the fact that (2.1.18) can be written as

$$\begin{cases} \Box\tilde{\Lambda} = -2\sinh(2\lambda + 2\tilde{\Lambda})Q_0(\phi,\phi), \\ \Box\phi = \dfrac{\sinh(2\lambda + 2\tilde{\Lambda})}{\sinh^2(\lambda+\tilde{\Lambda})}Q_0(\phi,\tilde{\Lambda}), \end{cases} \tag{2.1.21}$$

where Q_0 represents the well-known fundamental null form

$$Q_0(\phi, \tilde{\Lambda}) = m^{\alpha\beta}\partial_\alpha\phi\partial_\beta\tilde{\Lambda}, \tag{2.1.22}$$

where $m_{\alpha\beta}$ to denote the standard Minkowski metric on $\mathbb{R}^{1+1}$. The smallness in the initial data implies that the nonlinear equation can be solved over a long period of time and the global solution can be constructed once the non-linearity decays enough. Moreover, the slower decay rate in low dimensions can be compensated by the special structure of the nonlinearity.

Global existence of small solutions to nonlinear wave equations with null conditions has been a subject under active investigation for the past four decades. The approach to understand the small data problem with null condition was introduced by Klainerman in the pioneering works [63] and by Christodoulou [23], for the global existence of classical solutions for nonlinear wave equations with null conditions in three space dimensions. Alinhac in [3] studied the problem for the case of two space dimensions. We remark here that in $\mathbb{R}^{3+1}$ the null condition is a sufficient but not necessary condition to obtain a small-data-global-existence result, see e.g. [83, 84]. More recently Huneau and Stingo [58] studied the global existence for a toy model for the Einstein equations with additional compact dimensions, where the nonlinearity is linear combinations of the classical quadratic null forms. In one space dimension waves do not decay, and nonlinear resonance can lead to finite time blow up. Nevertheless, Luli, Yang and Yu in [85] proved, for Cauchy problems of semilinear wave equations with null conditions in one space dimension, the global existence of classical solutions with small initial data. The authors proposed a weighted energy and use the bootstrap method for obtain the result. The system in the Theorem (2.1.1) does not obey the classical null condition. However, the factors $Q_0(\phi, \phi)$ and $Q_0(\phi, \tilde{\Lambda})$ provide decay and with the appropriate condition on λ, global regularity can be obtained. Inspired by Luli, Yang and Yu's result [85] in the semilinear case, it is natural to conjecture that the Cauchy problem for one-dimension system of quasilinear wave equations (2.3.1) admits a global classical solution for small initial data. The main aim of this theorem is to verify this conjecture.

Now we discuss the long time behavior of globally defined solutions. Here, virial identities will be key to the long-time description.

Theorem 2.1.2. *Let (Λ, ϕ) be a global solution to (2.1.11) such that its energy $E[\Lambda, \phi](t)$ is conserved and finite. Then, for any $v \in (-1, 1)$ and $\omega(t) = t/\log^2 t$, one has*

$$\lim_{t\to+\infty} \int_{vt-\omega(t)}^{vt+\omega(t)} \left((\partial_t\Lambda)^2 + (\partial_x\Lambda)^2 + \sinh^2\Lambda((\partial_t\phi)^2 + (\partial_x\phi)^2)\right)(t, x)dx = 0.$$

This result establishes that inside the light cone, all finite-energy solutions must converge to zero as time tends to infinity. It is also in concordance with the solutions found in (2.1.13), which are a natural counterexample in the case $v = \pm 1$. A similar outcome has been recently found in [1], where the less involved Born-Infeld model is considered. Note that Theorem 2.1.2 is valid under general data, and compared with the obtained asymptotic result in Theorem 2.1.1, reveals that the decay property may hold under very general initial data, and unlike [1], our model is in some sense semilinear.

As a final comment on this part of our results, we should mention the work by Yan [115] dealing with the blow-up description in the Born-Infeld theory. We strongly believe that the blow-up mechanism in the PCF model is triggered by the threshold $\Lambda = 0$.

2.1.5 Application to soliton solutions

An important outcome of our previous results is a clear background for the study of soliton solutions of (2.1.11). Belinski and Zakharov in [9] proposed that the Eq. (2.1.4) has $N-$soliton solutions, see also [8] for further details. Hadad [52] also showed explicit examples of soliton solutions for the equation (2.1.11) using diagonal backgrounds, also called "seed metric". Basically, one starts with a background solution of the form

$$g^{(0)} = \begin{bmatrix} e^{\Lambda^{(0)}} & 0 \\ 0 & e^{-\Lambda^{(0)}} \end{bmatrix}. \tag{2.1.23}$$

The function $\Lambda^{(0)}(t,x)$ satisfies the wave equation $\partial_t^2\Lambda^{(0)} - \partial_x^2\Lambda^{(0)} = 0$. In this case, if we want to identify the solution in terms of the equation (2.1.6), we have that $\Lambda = \Lambda^{(0)}$, $\phi = n\pi$, with $n \in \mathbb{Z}$, and $\alpha = 1$. The gauge choice for us will be $n = 0$.

As expressed in [52], an important case is the one-soliton solution, which is obtained by taking $\Lambda^{(0)}$ time-like and equals to t and $\phi^{(0)} = 0$. Note that with this choice the energy is not well-defined, but a suitable modification will make this metric regular again. Indeed, the energy proposed in (2.1.14) is not finite, but one can consider the following modified energy

$$E_{\text{mod}}[\Lambda,\phi](t) := \int \left(\frac{1}{2}\left((\partial_t\Lambda)^2 - 1 + (\partial_x\Lambda)^2\right) + 2\sinh^2(\Lambda)((\partial_t\phi)^2 + (\partial_x\phi)^2) \right), \tag{2.1.24}$$

which is also conserved and identically zero. Hadad computed the corresponding 1-soliton solution using Belinski and Zakharov techniques, obtaining

$$g^{(1)} = \begin{bmatrix} \dfrac{e^t Q_c(x-vt)}{Q_c(x-vt-x_0)} & -\dfrac{1}{c}Q_c(x-vt) \\ -\dfrac{1}{c}Q_c(x-vt) & \dfrac{e^{-t}Q_c(x-vt)}{Q_c(x-vt+x_0)} \end{bmatrix}, \tag{2.1.25}$$

where, for a fixed parameter $\mu > 1$, one has

$$Q_c(\cdot) = \sqrt{c}\,\text{sech}(\sqrt{c}(\cdot)), \quad c = \left(\frac{2\mu}{\mu^2-1}\right)^2, \quad v = -\frac{\mu^2+1}{2\mu} < -1, \quad \text{and} \quad x_0 = \frac{\ln|\mu|}{\sqrt{c}}.$$

Notice that the first component of $g^{(1)}$ grows in time. The parameter μ represents a pole in terms of scattering techniques, however this point of view will be considered in another work. Therefore, we have a traveling superluminal soliton which travels to the left (if $\mu > 0$). Also, representing $g^{(1)}$ in terms of corresponding functions $\Lambda^{(1)}, \phi^{(1)}$ is complicated, and done in Section 2.5.

In this paper, we propose a modification of this "degenerate" soliton solution by cutting off the infinite energy part profiting of the wave-like character of solutions $\Lambda^{(0)}$. Although it is not so clear that they are physically meaningful, these new solutions have finite energy and local well-posedness properties in a vicinity.

Indeed, consider a smooth function $\theta \in C_c^2(\mathbb{R})$. Additionally, consider the constraint $0 < \mu < 1$. For any $\lambda > 0$, and $\varepsilon > 0$ small, let

$$\Lambda_\varepsilon^{(0)} := \lambda + \varepsilon\theta(t+x), \quad \phi^{(0)} := 0.$$

Clearly $\Lambda_\varepsilon^{(0)}$ solves the wave equation in $1D$ and has finite energy $E[\Lambda_\varepsilon^{(0)}, \phi_\varepsilon^{(0)}] < +\infty$. This will be for us the background seed. The corresponding 1-soliton is now

$$g^{(1)} = \begin{bmatrix} \dfrac{e^{\lambda+\varepsilon\theta}\operatorname{sech}(\beta(\lambda+\varepsilon\theta))}{\operatorname{sech}(\beta(\lambda+\varepsilon\theta)-x_0)} & -\dfrac{1}{\sqrt{c}}\operatorname{sech}(\beta(\lambda+\varepsilon\theta)) \\ -\dfrac{1}{\sqrt{c}}\operatorname{sech}(\beta(\lambda+\varepsilon\theta)) & \dfrac{e^{-(\lambda+\varepsilon\theta)}\operatorname{sech}(\beta(\lambda+\varepsilon\theta))}{\operatorname{sech}(\beta(\lambda+\varepsilon\theta)+x_0)} \end{bmatrix}, \qquad \beta = \frac{\mu+1}{\mu-1}, \tag{2.1.26}$$

which also has finite energy. Perturbations of the fields Λ and ϕ associated with this soliton will be globally defined according to the Theorem 2.1.1:

Corollary 2.1.1. *Suitable perturbations of any soliton as in* (2.1.26) *are globally well-defined.*

This result has an important outcome: it allows us to try to study the stability of these solutions, which will be done in a forthcoming work. Additionally, there are other possible choices of metrics in Einstein's field equations that lead to the KdV model, see e.g. [102].

Organization of this chapter

This paper is organized as follows: Section 2.2 is devoted to the proof of local existence of solutions. In Section 2.3 we prove global existence of small solutions close to a nonzero value. Section 2.4 is devoted to the long time behavior of solutions, and finally Section 2.5 consider the particular case of solitons.

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2.2 The initial value problem: local existence

This section is devoted to the proof of Proposition 2.1.1. First, recall the following result [106], that we will use to prove Proposition 2.1.1.

Lemma 2.2.1. *Let $\psi : I \times \mathbb{R} \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$, be the solution of the initial value problem*

$$\begin{cases} \partial_t^2\psi - \Delta\psi = f(t,x), & (t,x) \in I \times \mathbb{R}, \\ (\psi, \partial_t\psi)|_{\{t=0\}} = (\psi_0, \psi_1) \in H^k(\mathbb{R}) \times H^{k-1}(\mathbb{R}), \end{cases} \tag{2.2.1}$$

where k be a positive integer. Then for some positive constant $C = C(k)$, the following energy estimate holds

$$\sup_{t\in[0,T]} \|(\psi, \partial_t\psi)\|_{H^k(\mathbb{R})\times H^{k-1}(\mathbb{R})} \leq C(1+T)\left(\|(\psi_0,\psi_1)\|_{H^k(\mathbb{R})\times H^{k-1}(\mathbb{R})} + \int_0^T \|f\|_{H^{k-1}(\mathbb{R})}(t)dt \right). \tag{2.2.2}$$

Proof of Proposition 2.1.1. The proof is standard in the literature, but for the sake of completeness, we include it here.

(1). This part of the Proposition is proved by Picard's iteration. Using a density argument it is sufficient to assume the initial data $(\Psi_0, \Psi_1) \in \mathcal{S}^4$ ($\mathcal{S}$ being the Schwartz class), along with condition (2.1.19). Define a sequence of smooth functions $\Psi^{(i)}$, with $i \geq 1$ such that

$$\Psi^{(1)} = (0, 0),$$

and for $i \geq 2$, $\Psi^{(i)}$ is iteratively defined as the unique solution to the system

$$\begin{cases} \partial_\alpha (m^{\alpha\beta} \partial_\beta \Psi^{(i)}) = F(\Psi^{(i-1)}, \partial \Psi^{(i-1)}) \\ (\Psi^{(i)}, \partial_t \Psi^{(i)})|_{\{t=0\}} = (\Psi_0, \Psi_1) \in \mathcal{H}. \end{cases} \tag{2.2.3}$$

It is important to note that from (2.1.17) and (2.1.19) we can assure that for $j = 1, 2$,

$$\sum_{\gamma=0,1} \sup_{|x|,|p| \leq \frac{\lambda}{2}} |\partial^\gamma_{x,p} F_j|(x, p) \leq C_{j,\frac{1}{2}\lambda}. \tag{2.2.4}$$

Indeed, this can be seen from the fact that for $(x, p) = (x_1, x_2, p_1, p_2, p_3, p_4)$ and $|x| \leq \frac{\lambda}{2}$,

$$F_1(x, p) = 2 \sinh(2\lambda + 2x_1) \left(p_4^2 - p_3^2\right), \quad F_2(x, p) = \frac{\sinh(2(\lambda + x_1))}{\sinh^2(\lambda + x_1)} \left(p_3 p_1 - p_2 p_4\right).$$

Define bounded functions in the class C^1.

It is important to note that condition (2.2.4) allows this iterative definition of the functions $\Psi^{(i)}$ to be possible, since it maintains each component of F with the required regularity, see [106]. First, it will be shown that for a sufficiently small $T > 0$, the sequence $(\Psi, \partial_t \Psi)$ is uniformly (in i) bounded in $L^\infty([0, T]; \mathcal{H})$, then it will be shown that it is also a Cauchy sequence. For the first part, the idea is to use the energy estimates (2.3.8), we want to prove that there is a constant $0 < A \leq \frac{\lambda}{2}$ such that

$$\left\| \left(\Psi^{(i-1)}, \partial_t \Psi^{(i-1)} \right) \right\|_{L^\infty([0,T];\mathcal{H})} \leq A, \tag{2.2.5}$$

implies that

$$\left\| \left(\Psi^{(i)}, \partial_t \Psi^{(i)} \right) \right\|_{L^\infty([0,T];\mathcal{H})} \leq A.$$

The energy estimation (2.3.8) allows us to write for (2.2.3) the following estimate:

$$\begin{aligned} \sup_{t \in [0,T]} \left\| \left(\Psi^{(i)}, \partial_t \Psi^{(i)} \right) \right\|_{\mathcal{H}} \leq & \; C(1 + T)(\|(\Psi_0, \Psi_1)\|_{\mathcal{H}}) \\ & + C(1 + T) \int_0^T \left(\left\| F_1 \left(\Psi^{(i-1)}, \partial \Psi^{(i-1)} \right) \right\|_{L^2(\mathbb{R})} + \left\| F_2 \left(\Psi^{(i-1)}, \partial \Psi^{(i-1)} \right) \right\|_{L^2(\mathbb{R})} \right) (t) dt. \end{aligned} \tag{2.2.6}$$

With this estimate, our goal is to bound the integral on the right hand side of the inequality above. For this, we will use the conditions (2.1.19) for each F_j which is satisfied by the hypothesis in (2.2.5), which results in the following, if $B = \max\{C_{1,\frac{\lambda}{2}}, C_{2,\frac{\lambda}{2}}\}$, then

$$\sup_{t \in [0,T]} \left\| \left(\Psi^{(i)}, \partial_t \Psi^{(i)} \right) \right\|_{\mathcal{H}} \leq C(1 + T) \left(\|(\Psi_0, \Psi_1)\|_{\mathcal{H}} + 2BT \right), \tag{2.2.7}$$

we can choose $T > 0$ sufficiently small such that

$$2BT \leq \|(\Psi_0, \Psi_1)\|_{\mathcal{H}},$$

so

$$\left\|\left(\Psi^{(i)}, \partial_t \Psi^{(i)}\right)\right\|_{L^\infty([0,T];\mathcal{H})} \leq 2C \|(\Psi_0, \Psi_1)\|_{\mathcal{H}}.$$

If we choose $D > 4C$ in (2.1.19) and $A := 2C||(\Psi_0, \Psi_1)||_{\mathcal{H}} \leq \frac{2C\lambda}{D} \leq \frac{\lambda}{2}$. We have thus shown the desired implication.

We now move to the second part in which we show that the sequence is Cauchy in a larger space $L^\infty([0,T];\mathcal{H})$. For every $i \geq 3$ we consider the equation for $\Psi^{(i)} - \Psi^{(i-1)}$:

$$\partial_\alpha (m^{\alpha\beta} \partial_\beta (\Psi^{(i)} - \Psi^{(i-1)})) = F(\Psi^{(i-1)}, \partial\Psi^{(i-1)}) - F(\Psi^{(i-2)}, \partial\Psi^{(i-2)}).$$

Using the condition (2.2.5) and the mean value theorem to show that there exists some $C > 0$ (depending on initial data but independent of i and T) such that

$$\left\|F_j\left(\Psi^{(i-1)}, \partial\Psi^{(i-1)}\right) - F_j\left(\Psi^{(i-2)}, \partial\Psi^{(i-2)}\right)\right\|_{L^2(\mathbb{R})} \leq C\left\|\partial(\Psi^{(i-1)} - \Psi^{(i-2)})\right\|_{L^2(\mathbb{R})\times L^2(\mathbb{R})}.$$

Now, applying again the energy estimation (2.3.8)

$$\sup_{t\in[0,T]} \left\|(\Psi^{(i)} - \Psi^{(i-1)}, \partial_t(\Psi^{(i)} - \Psi^{(i-1)}))\right\|_{\mathcal{H}} \leq CT\left\|\left(\Psi^{(i-1)} - \Psi^{(i-2)}, \partial_t(\Psi^{(i-1)} - \Psi^{(i-2)})\right)\right\|_{\mathcal{H}}.$$

Using (2.2.5) we have

$$\sup_{t\in[0,T]} \left\|\left(\Psi^{(2)} - \Psi^{(1)}, \partial_t(\Psi^{(2)} - \Psi^{(1)})\right)\right\|_{\mathcal{H}} \leq C_1,$$

therefore, choosing T sufficiently small, for $i \geq 3$ we have

$$\sup_{t\in[0,T]} \left\|\Psi^{(i)} - \Psi^{(i-1)}\right\|_{H^1\times H^1} \leq \frac{1}{2} \sup_{t\in[0,T]} \left\|\Psi^{(i-1)} - \Psi^{(i-2)}\right\|_{H^1\times H^1},$$

which implies that

$$\sup_{t\in[0,T]} \left\|\Psi^{(i)} - \Psi^{(i-1)}\right\|_{H^1\times H^1} \leq \frac{C_1}{2^{i-2}}.$$

Therefore we have that the sequence is a Cauchy sequence on $L^\infty([0,T];\mathcal{H})$, hence convergent. That is, there exists $(\Psi, \partial_t\Psi)$ in $L^\infty([0,T];\mathcal{H})$. The uniqueness proof, is the result of considering again the energy estimation.

Finally, for the continuous dependence on initial data taking $i \in \mathbb{N}$ sufficiently large, let us bound the difference $\Psi^{(i)} - \Psi$ and use again the energy estimate for the equation:

$$\partial_\alpha (m^{\alpha\beta} \partial_\beta (\Psi^{(i)} - \Psi)) = F(\Psi^{(i)}, \partial\Psi^{(i)}) - F(\Psi, \partial\Psi)$$

Applying the same reasoning as above and the energy estimation we can again write:

$$\sup_{s\in[0,t]} \left\|\left(\Psi^{(i)} - \Psi, \partial_t\Psi^{(i)} - \partial_t\Psi\right)\right\|_{\mathcal{H}} \leq C\left\|\left(\Psi_0^{(i)} - \Psi_0, \Psi_1^{(i)} - \Psi_1\right)\right\|_{\mathcal{H}}$$

$$+ C \int_0^t \left\| \left(\Psi^{(i)} - \Psi, \partial_t \Psi^{(i)} - \partial_t \Psi \right) \right\|_{\mathcal{H}}.$$

Using Gronwall's inequality (see Appendix 2.6.1) and (2.3.8), we have for a constant $C = C(T) > 0$,

$$\sup_{t\in[0,T]} \left\| \left(\Psi^{(i)} - \Psi, \partial_t \Psi^{(i)} - \partial_t \Psi \right) \right\|_{\mathcal{H}} \leq C \left\| \left(\Psi_0^{(i)} - \Psi_0, \Psi_1^{(i)} - \Psi_1 \right) \right\|_{\mathcal{H}}.$$

Taking $i \longrightarrow \infty$ the right-hand side of the inequality tends to zero, then

$$\sup_{s\in[0,t]} \left\| \left(\Psi^{(i)} - \Psi, \partial_t \Psi^{(i)} - \partial_t \Psi \right) \right\|_{\mathcal{H}} \longrightarrow 0.$$

This last property ends the proof of Proposition 2.1.1. □

2.3 Global Solutions for Small Initial Data

In this Section we prove Theorem 2.1.1. As in the previous section, let us consider the field $\Lambda(t,x)$ described as $\Lambda(t,x) := \lambda + \tilde{\Lambda}(t,x)$, then, let us establish the conditions on λ that guarantee the regularity conditions necessary to study the system (2.1.16)

$$\begin{cases} \partial_t^2 \tilde{\Lambda} - \partial_x^2 \tilde{\Lambda} = -2\sinh(2\lambda + 2\tilde{\Lambda}) \left((\partial_x \phi)^2 - (\partial_t \phi)^2 \right) = -2\sinh(2\lambda + 2\tilde{\Lambda}) Q_0(\phi, \phi), \\ \partial_t^2 \phi - \partial_x^2 \phi = \dfrac{\sinh(2\lambda + 2\tilde{\Lambda})}{\sinh^2(\lambda + \tilde{\Lambda})} \left(\partial_x \phi \partial_x \tilde{\Lambda} - \partial_t \phi \partial_t \tilde{\Lambda} \right) = \dfrac{\sinh(2\lambda + 2\tilde{\Lambda})}{\sinh^2(\lambda + \tilde{\Lambda})} Q_0(\phi, \tilde{\Lambda}), \end{cases} \tag{2.3.1}$$

with Q_0 given in (2.1.22). The constant $\lambda > 0$ will play an important role in the overall analysis of the problem and the conditions assumed on it will be verified using a continuity method. We will use two coordinate systems: the standard Cartesian coordinates (t,x) and the null coordinates $(u, \underline{u})$:

$$u := \frac{t-x}{2}, \quad \underline{u} := \frac{t+x}{2}.$$

***Remark* 2.3.1.** *Consider the two null vector fields defined globally as*

$$L = \partial_t + \partial_x, \quad \underline{L} = \partial_t - \partial_x.$$

*Then, one can rewrite the right-hand side of (*2.3.1*) as*

$$(\partial_x \phi)^2 - (\partial_t \phi)^2 = Q_0(\phi, \phi) = 2L\phi \underline{L}\phi, \tag{2.3.2}$$

$$\partial_x \phi \partial_x \tilde{\Lambda} - \partial_t \phi \partial_t \tilde{\Lambda} = Q_0(\phi, \tilde{\Lambda}) = \frac{1}{2} L\phi \underline{L}\tilde{\Lambda} + \frac{1}{2} L\tilde{\Lambda}\underline{L}\phi. \tag{2.3.3}$$

It can be noticed that the null structure commutes with derivatives:

$$\partial_x Q_0(\phi, \tilde{\Lambda}) = Q_0(\partial_x \phi, \tilde{\Lambda}) + Q_0(\phi, \partial_x \tilde{\Lambda}). \tag{2.3.4}$$

Also, based on this, we have the following inequality

$$Q_0(\partial_x^p \phi, \partial_x^q \phi) \lesssim |L\partial_x^p \phi||\underline{L}\partial_x^q \phi| + |\underline{L}\partial_x^p \phi||L\partial_x^q \phi|. \tag{2.3.5}$$

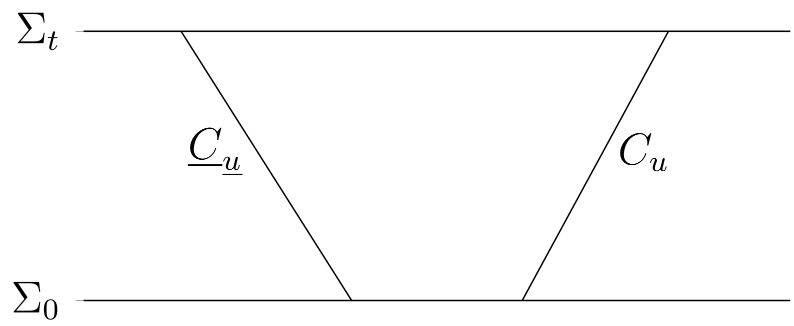

Figure 2.1: The entire region enclosed by Σ_0 and Σ_t is D_t.

Before presenting the proof, there are certain results and definitions to be mentioned before, for details and proofs see [4, 85]. From now on, we will consider the conformal killing vector field on $\mathbb{R}^{1+1}$ given by

$$(1+|\underline{u}|^2)^{1+\delta}L, \quad (1+|u|^2)^{1+\delta}\underline{L},$$

with $0<\delta<1$, and the following integration regions: Σ_{t_0} denotes the following time slice in $\mathbb{R}^{1+1}$:

$$\Sigma_{t_0} := \{(t,x):\ t=t_0\}. \tag{2.3.6}$$

D_{t_0} denotes the following region of spacetime

$$D_{t_o} := \{(t,x):\ 0\le t\le t_0\}, \quad D_{t_0} = \bigcup_{0\le t\le t_0} \Sigma_{t_0}. \tag{2.3.7}$$

The level sets of the functions u and $\underline{u}$ define two global null foliations of D_{t_0}. More precisely, given $t_0>0$, u_0 and $\underline{u}_0$, we define the rightward null curve segment C_{u_0} as :

$$C_{u_0} := \left\{(t,x):\ u=\frac{t-x}{2}=u_0,\ 0\le t\le t_0\right\},$$

and the segment of the null curve to the left $\underline{C}_{\underline{u}_0}$ as:

$$\underline{C}_{\underline{u}_0} := \left\{(t,x):\ \underline{u}=\frac{t+x}{2}=\underline{u}_0,\ 0\le t\le t_0\right\}.$$

The space time region D_{t_0} is foliated by $\underline{C}_{\underline{u}_0}$ for $\underline{u}\in\mathbb{R}$, and by C_{u_0} for $u\in\mathbb{R}$. Let us also consider the following energy estimate proposed in [4, 85] for the scalar linear equations $\Box\psi=\rho$ given by:

$$\begin{aligned}
&\int_{\Sigma_t}\left[(1+|u|^2)^{1+\delta}|\underline{L}\psi|^2+(1+|\underline{u}|^2)^{1+\delta}|L\psi|^2\right]dx\\
&\quad+\sup_{\underline{u}\in\mathbb{R}}\int_{\underline{C}_{\underline{u}}}(1+|u|^2)^{1+\delta}|\underline{L}\psi|^2d\tau+\sup_{u\in\mathbb{R}}\int_{C_u}(1+|\underline{u}|^2)^{1+\delta}|L\psi|^2d\tau\\
&\quad\le C_0\int_{\Sigma_0}\left[(1+|u|^2)^{1+\delta}|\underline{L}\psi|^2+(1+|\underline{u}|^2)^{1+\delta}|L\psi|^2\right]dx\\
&\qquad+C_0\iint_{D_t}\left[(1+|u|^2)^{1+\delta}|\underline{L}\psi|+(1+|\underline{u}|^2)^{1+\delta}|L\psi|\right]|\rho|.
\end{aligned} \tag{2.3.8}$$

Motivated by the above estimation (2.3.8) and [85], we define the space-time weighted energy norms valid for $k = 0, 1$:

$$
\begin{aligned}
\mathcal{E}_k(t) &= \int_{\Sigma_t} \left[(1+|u|^2)^{1+\delta}|\underline{L}\partial_x^k\tilde{\Lambda}|^2 + (1+|\underline{u}|^2)^{1+\delta}|L\partial_x^k\tilde{\Lambda}|^2\right] dx,\\
\overline{\mathcal{E}}_k(t) &= \int_{\Sigma_t} \left[(1+|u|^2)^{1+\delta}|\underline{L}\partial_x^k\phi|^2 + (1+|\underline{u}|^2)^{1+\delta}|L\partial_x^k\phi|^2\right] dx,\\
\mathcal{F}_k(t) &= \sup_{\underline{u}\in\mathbb{R}} \int_{\underline{C}_{\underline{u}}} (1+|u|^2)^{1+\delta}\left|\underline{L}\partial_x^k\tilde{\Lambda}\right|^2 ds + \sup_{u\in\mathbb{R}} \int_{C_u} (1+|\underline{u}|^2)^{1+\delta}|L\partial_x^k\tilde{\Lambda}|^2 ds,\\
\overline{\mathcal{F}}_k(t) &= \sup_{\underline{u}\in\mathbb{R}} \int_{C_{\underline{u}}} (1+|u|^2)^{1+\delta}\left|\underline{L}\partial_x^k\phi\right|^2 ds + \sup_{u\in\mathbb{R}} \int_{C_u} (1+|\underline{u}|^2)^{1+\delta}|L\partial_x^k\phi|^2 ds.
\end{aligned}
\tag{2.3.9}
$$

Finally, we define the total energy norms as follows:

$$\mathcal{E}(t) = \mathcal{E}_0(t) + \mathcal{E}_1(t).$$

Analogously one defines $\mathcal{F}(t)$, $\overline{\mathcal{E}}(t)$, and $\overline{\mathcal{F}}(t)$.

***Remark* 2.3.2.** *We note that if $t = 0$ then $\mathcal{F}(0) = \overline{\mathcal{F}}(0) = 0$ and for $\mathcal{E}(t)$ the initial data determines a constant C_1 so that*

$$\mathcal{E}(0) = C_1\varepsilon^2. \tag{2.3.10}$$

We will use the method of continuity as follows: we assume that the solution $\tilde{\Lambda}$ exists for $t \in [0, T^*]$ so that it has the following bound

$$\mathcal{E}(t) + \mathcal{F}(t) \le 6C_0C_1\varepsilon^2, \tag{2.3.11}$$

$$\overline{\mathcal{E}}(t) + \overline{\mathcal{F}}(t) \le 6C_0\overline{C}_1\varepsilon^2, \tag{2.3.12}$$

and

$$\sup_{t\in[0,T^*]} \left\|\tilde{\Lambda}\right\|_{L^\infty(\mathbb{R})} \le \frac{\lambda}{2}. \tag{2.3.13}$$

We want to show that for all $t \in [0, T^*]$ there exists a universal constant ε_0 (independent of T^*) such that the estimates are improved for all $\varepsilon \le \varepsilon_0$. It is important recall that, the terms related to the functions $\sinh(\cdot)$, $\cosh(\cdot)$, $\coth(\cdot)$ and $\operatorname{csch}(\cdot)$ can be written using the Taylor expansion as:

$$
\begin{cases}
\sinh(2\lambda + 2\tilde{\Lambda}) = \sinh(2\lambda) + 2\cosh(2\lambda)\tilde{\Lambda} + 4\sinh(2\lambda + 2\xi_1)\tilde{\Lambda}^2,\\
\cosh(2\lambda + 2\tilde{\Lambda}) = \cosh(2\lambda) + 2\sinh(2\lambda)\tilde{\Lambda} + 4\cosh(2\lambda + 2\xi_2)\tilde{\Lambda}^2,
\end{cases}
\tag{2.3.14}
$$

and under (2.3.13),

$$
\begin{cases}
\coth(\lambda + \tilde{\Lambda}) = \coth(\lambda) - \operatorname{csch}(\lambda)\tilde{\Lambda} - \operatorname{csch}(\lambda + \xi_3)\coth(\lambda + \xi_3)\tilde{\Lambda}^2,\\
\operatorname{csch}^2(\lambda + \tilde{\Lambda}) = \operatorname{csch}^2(\lambda) - 2\operatorname{csch}^2(\lambda)\coth(\lambda)\tilde{\Lambda}\\
\qquad\qquad + \left\{2\operatorname{csch}^2(\lambda + \xi_4)\coth^2(\lambda + \xi_4) + \operatorname{csch}^4(\lambda + \xi_4)\right\}\tilde{\Lambda}^2,
\end{cases}
\tag{2.3.15}
$$

with $\xi_1, \xi_2, \xi_3, \xi_4$ between 0 and $\tilde{\Lambda}$, which satisfies (2.3.13). Then, from this condition (2.3.13) and (2.3.14) one has

$$
\begin{aligned}
&|\sinh(2\lambda + 2\tilde{\Lambda})| \le \lambda_0(\lambda), \quad |\cosh(2\lambda + 2\tilde{\Lambda})| \le \lambda_1(\lambda).\\
&|\coth(\lambda + \tilde{\Lambda})| \le \lambda_3(\lambda), \quad \text{and} \quad |\operatorname{csch}(\lambda + \tilde{\Lambda})| \le \lambda_4(\lambda).
\end{aligned}
\tag{2.3.16}
$$

Using the assumptions (2.3.11) and (2.3.12) the following pointwise bounds were established in [85].

Lemma 2.3.1 ([85], Lemma 3.2). *Under assumptions* (2.3.11)-(2.3.13), *there exists a universal constant* $C_2 > 0$ *such that:*

$$|L\tilde{\Lambda}(t,x)| \leq \frac{C_2\varepsilon}{(1+|\underline{u}|^2)^{1/2+\delta/2}}, \qquad |L\phi(t,x)| \leq \frac{C_2\varepsilon}{(1+|\underline{u}|^2)^{1/2+\delta/2}},$$
$$|\underline{L}\tilde{\Lambda}(t,x)| \leq \frac{C_2\varepsilon}{(1+|u|^2)^{1/2+\delta/2}}, \qquad |\underline{L}\phi(t,x)| \leq \frac{C_2\varepsilon}{(1+|u|^2)^{1/2+\delta/2}}.$$

Proof. It is sufficient to prove one of the four inequalities, since the other inequalities are completely analogous. The proof is based on the boostrap assumptions (2.3.11)-(2.3.13). Indeed, according to the Sobolev inequality on $\mathbb{R}$, since

$$|\partial_x(1+|\underline{u}|^2)^{1/2+\delta/2}| \leq (1+|\underline{u}|^2)^{1/2+\delta/2},$$

one has

$$\begin{aligned}
|(1+|\underline{u}|^2)^{1/2+\delta/2}L\phi(t,x)|^2 &\lesssim ||(1+|\underline{u}|^2)^{1/2+\delta/2}L\phi||^2_{L^2(\mathbb{R}_x)} + ||\partial_x((1+|\underline{u}|^2)^{1/2+\delta/2}L\phi)||^2_{L^2(\mathbb{R}_x)} \\
&\lesssim ||(1+|\underline{u}|^2)^{1/2+\delta/2}L\phi||^2_{L^2(\mathbb{R}_x)} + ||\partial_x((1+|\underline{u}|^2)^{1/2+\delta/2})L\phi||^2_{L^2(\mathbb{R}_x)} \\
&\quad + ||(1+|\underline{u}|^2)^{1/2+\delta/2}L\partial_x(\phi)||^2_{L^2(\mathbb{R}_x)} \\
&\lesssim ||(1+|\underline{u}|^2)^{1/2+\delta/2}L\phi||^2_{L^2(\mathbb{R}_x)} + ||(1+|\underline{u}|^2)^{1/2+\delta/2}L\partial_x(\phi)||^2_{L^2(\mathbb{R}_x)} \\
&\lesssim \overline{\mathcal{E}}(t) \\
&\lesssim 6C_0\overline{C}_1\varepsilon^2.
\end{aligned}$$

Consequently, we have the desired inequality. □

Now we have all the ingredients to prove Theorem 2.1.1.

2.3.1 Proof of the Theorem 2.1.1

For simplicity, we work with the first equation of the system (2.1.21). On the other hand, we can substitute in the following proof $\tilde{\Lambda}$ by ϕ and then sum the estimates to complete the test for the original system, see Appendix 2.6.2 for details of the estimates for the second equation in (2.3.1), which complete the proof. We prove this using the bootstrap method; i.e., we will assume that this weighted energy is bounded by some constant. Then, we can show that the solution decays. Since the initial data are small, this allows us to show that the weighted energy is bounded by some better constant. Thus, by continuity, we conclude that the weighted energy cannot grow to infinity in any finite time interval and therefore, using the local existence theorem, the solution exists for all time.

Proof. Using (2.3.2) and (2.3.4) in the first equation of the (2.3.1) we obtain:

$$\Box\partial_x\tilde{\Lambda} = -2\Big[\sinh(2\lambda+2\tilde{\Lambda})\left(Q_0(\partial_x\phi,\phi)+Q_0(\phi,\partial_x\phi)\right) + 2\partial_x\tilde{\Lambda}\cosh(2\lambda+2\tilde{\Lambda})Q_0(\phi,\phi)\Big]. \tag{2.3.17}$$

We can see that the null structure is "quasi-preserved" after differentiating with respect to x. We will use a bootstrap argument as in the (3+1)-dimensional case [63]. Fix $\delta \in (0,1)$. Under the assumptions (2.3.11)-(2.3.12)-(2.3.13) for all $t \in [0, T^*]$, we assume that the solution remains regular, to later show that these bounds are maintained, with a better constant.

Consider $k = 0, 1$. Using (2.3.8) on (2.3.17), with $\psi = \partial_x^k \tilde{\Lambda}$, and taking the sum over $k = 0, 1$, we obtain

$$\begin{aligned}
&\mathcal{E}(t) + \mathcal{F}(t) \leq 2C_0\mathcal{E}(0) \\
&\quad + 2C_0 \iint_{D_t} \left((1+|u|^2)^{1+\delta}|\underline{L}\tilde{\Lambda}| + (1+|\underline{u}|^2)^{1+\delta}|L\tilde{\Lambda}| \right) |\sinh(2\lambda + 2\tilde{\Lambda})||Q_0(\phi,\phi)| \\
&\quad + 4C_0 \iint_{D_t} \left((1+|u|^2)^{1+\delta}|\underline{L}\partial_x\tilde{\Lambda}| + (1+|\underline{u}|^2)^{1+\delta}|L\partial_x\tilde{\Lambda}| \right) |\sinh(2\lambda + 2\tilde{\Lambda})||Q_0(\phi,\partial_x\phi)| \\
&\quad + 4C_0 \iint_{D_t} \left((1+|u|^2)^{1+\delta}|\underline{L}\partial_x\tilde{\Lambda}| + (1+|\underline{u}|^2)^{1+\delta}|L\partial_x\tilde{\Lambda}| \right) |\partial_x\tilde{\Lambda}\cosh(2\lambda + 2\tilde{\Lambda})||Q_0(\phi,\phi)| \\
&=: 2C_0\mathcal{E}(0) + 2C_0 \sum_{j=1}^{6} I_j,
\end{aligned} \tag{2.3.18}$$

where the integrals $I_i, i \in \{1, 2, 3..., 6\}$ are defined as follows:

$$\begin{aligned}
I_1 &:= \iint_{D_t} \left((1+|u|^2)^{1+\delta}|\underline{L}\tilde{\Lambda}| \right) |\sinh(2\lambda + 2\tilde{\Lambda})||Q_0(\phi,\phi)|, \\
I_2 &:= \iint_{D_t} \left((1+|\underline{u}|^2)^{1+\delta}|L\tilde{\Lambda}| \right) |\sinh(2\lambda + 2\tilde{\Lambda})||Q_0(\phi,\phi)|, \\
I_3 &:= 2\iint_{D_t} \left((1+|u|^2)^{1+\delta}|\underline{L}\partial_x\tilde{\Lambda}| \right) |\sinh(2\lambda + 2\tilde{\Lambda})||Q_0(\phi,\partial_x\phi)|, \\
I_4 &:= 2\iint_{D_t} \left((1+|\underline{u}|^2)^{1+\delta}|L\partial_x\tilde{\Lambda}| \right) |\sinh(2\lambda + 2\tilde{\Lambda})||Q_0(\phi,\partial_x\phi)|, \\
I_5 &:= 2\iint_{D_t} \left((1+|u|^2)^{1+\delta}|\underline{L}\partial_x\tilde{\Lambda}| \right) |\partial_x\tilde{\Lambda}\cosh(2\lambda + 2\tilde{\Lambda})||Q_0(\phi,\phi)|, \\
I_6 &:= 2\int_{D_t} \left((1+|\underline{u}|^2)^{1+\delta}|L\partial_x\tilde{\Lambda}| \right) |\partial_x\tilde{\Lambda}\cosh(2\lambda + 2\tilde{\Lambda})||Q_0(\phi,\phi)|.
\end{aligned} \tag{2.3.19}$$

The goal is to control the right-hand side of the above estimate. Essentially we have six terms to control, but several are equivalent and essentially we only need to consider two cases. Indeed, it will be sufficient to bound the terms corresponding to $\underline{L}\tilde{\Lambda}$ and $\underline{L}\partial_x\tilde{\Lambda}$, since by symmetry, the procedure for the other terms will be analogous. First, we start to bound the term:

$$\begin{aligned}
I_{35} := I_3 + I_5 = &\iint_{D_t} \left((1+|u|^2)^{1+\delta}|\underline{L}\partial_x\tilde{\Lambda}| \right) \\
&\left(2|\partial_x\tilde{\Lambda}\cosh(2\lambda + \tilde{\Lambda})||Q_0(\phi,\phi)| + 2|\sinh(2\lambda + 2\tilde{\Lambda})||Q_0(\phi,\partial_x\phi)| \right).
\end{aligned} \tag{2.3.20}$$

Taking into account (2.3.5), (2.3.13) and (2.3.14), we can write for (2.3.20):

$$\begin{aligned}
I_{35} &\lesssim \iint_{D_t} \left((1+|u|^2)^{1+\delta}|\underline{L}\partial_x\tilde{\Lambda}| \right) \left(|\partial_x\tilde{\Lambda}||L\phi||\underline{L}\phi| + |L\partial_x\phi||\underline{L}\phi| + |L\phi||\underline{L}\partial_x\phi| \right) \\
&=: I_{35,1} + I_{35,2} + I_{35,3}.
\end{aligned} \tag{2.3.21}$$

Since $\partial_x\tilde{\Lambda} = \frac{1}{2}(L-\underline{L})\tilde{\Lambda}$, we have

$$\begin{aligned}
I_{35,1} &= \iint_{D_t} (1+|u|^2)^{1+\delta}|\underline{L}\partial_x\tilde{\Lambda}||\partial_x\tilde{\Lambda}||L\phi||\underline{L}\phi| \\
&\leq \frac{1}{2}\iint_{D_t} (1+|u|^2)^{1+\delta}|\underline{L}\partial_x\tilde{\Lambda}||L\tilde{\Lambda}||L\phi||\underline{L}\phi| + (1+|\underline{u}|^2)^{1+\delta}|\underline{L}\partial_x\tilde{\Lambda}||\underline{L}\tilde{\Lambda}||L\phi||\underline{L}\phi| \\
&=: I_{35,1,1} + I_{35,1,2}.
\end{aligned} \tag{2.3.22}$$

Recall that by Fubini's Theorem the spacetime D_t in (2.3.7) is foliated by $\underline{C}_{\underline{u}}$ for $u \in \mathbb{R}$, and also by $\{t\}\times\Sigma_t$, $t\in\mathbb{R}$. Using Lemma 2.3.1 and defining $\varphi(x) = (1+|x|^2)^{1+\delta}$ (to simplify the notation), we have the following:

$$\begin{aligned}
I_{35,1,1} &\lesssim \iint_{D_t} \varepsilon\, \underbrace{\left(\varphi(\underline{u})^{-3/4}\varphi(u)^{1/2}|\underline{L}\partial_x\tilde{\Lambda}|\right)}_{L^2_tL^2_x} \underbrace{\left(\varphi^{1/2}(\underline{u})|L\phi|\right)}_{L^\infty_tL^2_x} \underbrace{\left(\varphi(\underline{u})^{-1/4}\varphi(u)^{1/2}|\underline{L}\phi|\right)}_{L^2_tL^\infty_x} \\
&\lesssim \varepsilon \underbrace{\left(\iint_{D_t} \frac{\varphi(u)|\underline{L}\partial_x\tilde{\Lambda}|^2}{\varphi(\underline{u})^{3/2}}\right)^{1/2}}_{T_1} \underbrace{\sup_{t\in[0,T^*]}\left(\int_{\Sigma_t}\varphi(\underline{u})|L\phi|^2\right)^{1/2}}_{T_2} \underbrace{\left(\int_0^t \left\|\frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/4}}|\underline{L}\phi|\right\|^2_{L^\infty(\Sigma_\tau)} d\tau\right)^{1/2}}_{T_3}.
\end{aligned}$$

Let us study each of the factors T_j. For T_1, one has:

$$T_1^2 \leq \int_{\mathbb{R}}\left[\int_{\underline{C}_{\underline{u}}} \frac{\varphi(u)|\underline{L}\partial_x\tilde{\Lambda}|^2}{\varphi(\underline{u})^{3/2}}ds\right]d\underline{u} = \int_{\mathbb{R}}\frac{1}{\varphi(\underline{u})^{3/2}}\underbrace{\left[\int_{\underline{C}_{\underline{u}}}\varphi(u)|\underline{L}\partial_x\tilde{\Lambda}|^2 ds\right]}_{\lesssim \mathcal{F}_1(t)}d\underline{u} \lesssim \int_{\mathbb{R}}\frac{\varepsilon^2}{\varphi(\underline{u})^{3/2}}d\underline{u},$$

since the integral is finite, we have $T_1 \lesssim \varepsilon$. The integral T_2 is part of the energy norm $\overline{\mathcal{E}}_0(t)$ in (2.3.9) then $T_2 \lesssim \varepsilon$. For the term T_3 one can use the same argument as in [85]: using Lemma 2.6.2 one gets

$$\begin{aligned}
T_3 &\lesssim \left(\int_0^t \left\|\frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/4}}\underline{L}\phi(t,x)\right\|^2_{L^2(\Sigma_\tau)} + \int_0^t \left\|\frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/4}}\underline{L}\partial_x\phi(t,x)\right\|^2_{L^2(\Sigma_\tau)}\right)^{1/2} \\
&\lesssim \left(\iint_{D_t}\frac{\varphi(u)}{\varphi(\underline{u})^{1/2}}|\underline{L}\phi|^2 + \iint_{D_t}\frac{\varphi(u)}{\varphi(\underline{u})^{1/2}}|\underline{L}\partial_x\phi|^2\right)^{1/2}.
\end{aligned}$$

Both terms above are of the same form as T_1 and then we have that $T_3 \lesssim \varepsilon$. We conclude that $I_{35,1,1} \lesssim \varepsilon^4$.

Now we control the integral $I_{35,1,2}$ in (2.3.22). Using again Lemma 2.3.1 we have:

$$\begin{aligned}
I_{35,1,2} &= \iint_{D_t}(1+|u|^2)^{1+\delta}|\underline{L}\partial_x\tilde{\Lambda}||\underline{L}\tilde{\Lambda}||L\phi||\underline{L}\phi| \lesssim \iint_{D_t}\varepsilon^2|\underline{L}\partial_x\tilde{\Lambda}||L\phi| \\
&= \iint_{D_t}\varepsilon^2\frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/2}}|\underline{L}\partial_x\tilde{\Lambda}|\frac{\varphi(\underline{u})^{1/2}}{\varphi(u)^{1/2}}|L\phi| \\
&\lesssim \iint_{D_t}\varepsilon^2\left(\frac{\varphi(u)}{\varphi(\underline{u})}|\underline{L}\partial_x\tilde{\Lambda}|^2 + \frac{\varphi(\underline{u})}{\varphi(u)}|L\phi|^2\right)
\end{aligned}$$

$$\lesssim \int_{\mathbb{R}} \frac{\varepsilon^2}{\varphi(\underline{u})} \underbrace{\left[\int_{\underline{C}_{\underline{u}}} \varphi(u)|\underline{L}\partial_x\tilde{\Lambda}|^2 ds\right]}_{\lesssim \mathcal{F}_1(t)} d\underline{u} + \int_{\mathbb{R}} \frac{\varepsilon^2}{\varphi(u)} \underbrace{\left[\int_{C_u} \varphi(\underline{u})|L\phi|^2 ds\right]}_{\lesssim \overline{\mathcal{F}}_0(t)} du \lesssim \varepsilon^4.$$

Putting all estimates together for $I_{35,1}$, we can conclude that $I_{35,1} \lesssim \varepsilon^4$. A similar result is obtained for $I_{46} := I_4 + I_6$.

Now we treat the term $I_1 + I_2 + I_{35,2} + I_{35,3}$ from (4.4.14) and (2.3.21). We have from (2.3.5) and (4.2.10),

$$\iint_{D_t} \varphi(u)|\underline{L}\partial_x\tilde{\Lambda}| \left(|L\partial_x\phi||\underline{L}\phi| + |L\phi||\underline{L}\partial_x\phi|\right) + \iint_{D_t} \left(\varphi(u)|\underline{L}\tilde{\Lambda}| + \varphi(\underline{u})|L\tilde{\Lambda}|\right) \left(|L\phi||\underline{L}\phi| + |L\phi||\underline{L}\phi|\right).$$

Using the condition (2.3.13), the situation matches Case 1 developed in [85]. All these integrals can be written as

$$\sim \iint_{D_t} \left(\varphi(u)|\underline{L}\partial_x\tilde{\Lambda}||L\phi||\underline{L}\partial_x\phi| + \varphi(u)|\underline{L}\partial_x\tilde{\Lambda}||L\partial_x\phi||\underline{L}\phi|\right).$$

We bound this term in the following form: take $j, k \in \{0, 1\}$, $\psi = \tilde{\Lambda}, \phi$, so that

$$\begin{aligned}
\iint_{D_t} \varphi(u) \left|\underline{L}\partial_x^k\psi\right| |L\psi| \left|\underline{L}\partial_x^j\psi\right| &\lesssim \iint_{D_t} \frac{\varepsilon}{\varphi(\underline{u})^{1/2}} \varphi(u) \left|\underline{L}\partial_x^k\psi\right| \left|\underline{L}\partial_x^j\psi\right| \\
&\lesssim \iint_{D_t} \frac{\varepsilon}{\varphi(\underline{u})^{1/2}} \left(\varphi(u) \left|\underline{L}\partial_x^k\psi\right|^2 + \varphi(u) \left|\underline{L}\partial_x^j\psi\right|^2\right) \\
&\lesssim \int_{\mathbb{R}} \left[\int_{\underline{C}_{\underline{u}}} \frac{\varepsilon}{\varphi(\underline{u})^{1/2}} \left(\varphi(u) \left|\underline{L}\partial_x^k\psi\right|^2 + \varphi(u) \left|\underline{L}\partial_x^j\psi\right|^2\right) ds\right] d\underline{u} \\
&= \int_{\mathbb{R}} \frac{\varepsilon}{\varphi(\underline{u})^{1/2}} \underbrace{\left[\int_{\underline{C}_{\underline{u}}} \left(\varphi(u) \left|\underline{L}\partial_x^k\psi\right|^2 + \varphi(u) \left|\underline{L}\partial_x^j\psi\right|^2\right) ds\right]}_{\lesssim \mathcal{E}+\mathcal{F}+\overline{\mathcal{E}}+\overline{\mathcal{F}}} d\underline{u} \\
&\lesssim \int_{\mathbb{R}} \frac{\varepsilon^3}{\varphi(\underline{u})^{1/2}} d\underline{u} \lesssim \varepsilon^3.
\end{aligned} \tag{2.3.23}$$

See also Luli, Yan and Yu [85] for detailed computations. So we can conclude that in this case we can bound them by ε^3. Finally, from the energy estimate (2.3.8), we can take all the estimates together for some universal constants C_4, C_5 we have that for all $t \in [0, T^*]$:

$$\mathcal{E}(t) + \mathcal{F}(t) \leq 2C_0C_1\varepsilon^2 + C_4\varepsilon^3 + C_5\varepsilon^4. \tag{2.3.24}$$

Now, we take ε_0 such that

$$\varepsilon_0 \leq \frac{C_0C_1}{C_4}, \quad \varepsilon_0^2 \leq \frac{C_0C_1}{C_5}, \tag{2.3.25}$$

we can see that for all $0 < \varepsilon \leq \varepsilon_0$ and for all $t \in [0, T]$, we have

$$\mathcal{E}(t) + \mathcal{F}(t) \leq 4C_0C_1\varepsilon^2. \tag{2.3.26}$$

This improves the constant in (2.3.11) In the same way, an analogous reasoning is used for the analysis of the equation in terms of ϕ, using in this case the equation (2.3.3), which results in an improvement of the constant involved in the estimate (2.3.12).

To improve condition (2.3.13), using the Fundamental Theorem of Calculus and Lemma 2.3.1, one can write $\tilde{\Lambda}(t,x)$, $t \geq 0$, in the following form:

$$\begin{aligned}
|\tilde{\Lambda}(t,x)| &\leq \varepsilon|\tilde{\Lambda}_0(x)| + \int_0^t |\partial_t\tilde{\Lambda}(\tau,x)|d\tau \\
&\leq \varepsilon K_1 + \frac{1}{2}\int_0^t |L\tilde{\Lambda} + \underline{L}\tilde{\Lambda}|d\tau \\
&\leq \varepsilon K_1 + \frac{1}{2}\int_0^t \left(\frac{C_2\varepsilon}{\varphi(\underline{u})^{1/2}} + \frac{C_2\varepsilon}{\varphi(u)^{1/2}}\right)d\tau \\
&\leq \varepsilon K_1 + \varepsilon C_2 K_2 \leq K\varepsilon,
\end{aligned}$$

for some universal constant K. Next, we take $\varepsilon_0 > 0$ that satisfies the condition (2.3.25) and such that

$$K\varepsilon_0 < \frac{\lambda}{4}, \tag{2.3.27}$$

taking sup over $t \in [0, T^*]$, we conclude that for all $0 < \varepsilon \leq \varepsilon_0$ we improved estimate (2.3.13). As mentioned before, the proof is completed by doing an analogous study in terms of the ϕ field and then taking the sum over the estimates for the final conclusion.

□

2.4 Long time behavior

Recall the energy introduced in (2.1.14):

$$E[\Lambda,\phi](t) = \int \left(\frac{1}{2}((\partial_x\Lambda)^2 + (\partial_t\Lambda)^2) + 2\sinh^2(\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2)\right)(t,x)dx.$$

We first start with a simple computation, already present in [52].

Lemma 2.4.1. *If $\Lambda(t,x), \phi(t,x)$ are the solutions of (2.1.2) with $\Lambda(t,x) \in C_0^\infty(\mathbb{R})$ and $\phi(x) \in C_0^\infty(\mathbb{R})$ then the energy of the system is conserved, that is*

$$\frac{d}{dt}E[\Lambda,\phi](t) = 0.$$

2.4.1 Energy and momentum densities

In terms of the fields Λ and ϕ, let us introduce the energy and momentum densities

$$\begin{aligned}
p(t,x) &:= \partial_x\Lambda\partial_t\Lambda + 4\sinh^2(\Lambda)\partial_x\phi\partial_t\phi, \\
e(t,x) &:= \frac{1}{2}((\partial_x\Lambda)^2 + (\partial_t\Lambda)^2) + 2\sinh^2(\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2).
\end{aligned} \tag{2.4.1}$$

Lemma 2.4.2. *Using the definition above in Eq. (2.4.1), one has the following continuity equations*

$$\begin{aligned}
\partial_t p(t,x) &= \partial_x e(t,x), \\
\partial_t e(t,x) &= \partial_x p(t,x),
\end{aligned} \tag{2.4.2}$$

and the inequality

$$|p(t,x)| \leq e(t,x). \tag{2.4.3}$$

Proof. First we prove $\partial_t p(t,x) = \partial_x e(t,x)$. Using (2.1.11) we can prove the continuity equation (2.4.2). Let us start with the first derivatives

$$\begin{aligned}&\partial_x \left(-\frac{1}{2}((\partial_x\Lambda)^2 + (\partial_t\Lambda)^2) - 2\sinh^2(\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2)\right)\\ &= -\partial_x\Lambda\partial_x^2\Lambda + \partial_t\Lambda\partial_{tx}\Lambda - 2\sinh(2\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2)\partial_x\Lambda - 4\sinh^2(\Lambda)(\partial_x\phi\partial_x^2\phi + \partial_t\phi\partial_{tx}\phi),\end{aligned}$$

and

$$\begin{aligned}&\partial_t \left(\partial_x\Lambda\partial_t\Lambda + 4\sinh^2(\Lambda)\partial_x\phi\partial_t\phi\right)\\ &= \partial_{xt}\Lambda\partial_t\Lambda + \partial_x\Lambda\partial_t^2\Lambda + 4\sinh(2\Lambda)\partial_t\Lambda\partial_x\phi\partial_t\phi + 4\sinh^2(\Lambda)\partial_{xt}\phi\partial_t\phi + 4\sinh^2(\Lambda)\partial_z\phi\partial_t^2\phi.\end{aligned}$$

Subtracting these two last equations gives:

$$\begin{aligned}&2\sinh 2\Lambda((\partial_x\phi)^2 + (\partial_t\phi)^2)\partial_x\Lambda + \partial_x\Lambda(\partial_x^2\Lambda - \partial_t^2\Lambda) + 4\sinh^2(\partial_x^2\phi - \partial_t^2\phi)\partial_x\phi\\ &\quad - 4\sinh(2\Lambda)\partial_t\Lambda\partial_x\phi\partial_t\phi = 4\sinh(2\Lambda)(\partial_x\phi)^2\partial_x\Lambda - 4(\partial_x\phi)^2\partial_x\Lambda\sinh(2\Lambda)\\ &\quad + 4\sinh(2\Lambda)\partial_x\phi\partial_t\phi\partial_t\Lambda - 4\sinh(2\Lambda)\partial_t\Lambda\partial_x\phi\partial_t\phi = 0.\end{aligned}$$

Second, we prove $\partial_t e(t,x) = \partial_x p(t,x)$; in effect, using (2.1.11) we have

$$\begin{aligned}\partial_t e(t,x) =& \partial_x\Lambda\partial_{xt}\Lambda + \partial_t\Lambda\partial_{tt}\Lambda + 2\partial_t\Lambda\sinh(2\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2)\\ &+ 4\sinh^2(\Lambda)\partial_x\phi\partial_{xt}\phi + 4\sinh^2(\Lambda)\partial_t\phi\partial_{tt}\phi\\ =& -2\partial_t\Lambda\sinh(2\Lambda)((\partial_x\phi)^2 - (\partial_t\phi)^2) + 2\partial_t\Lambda\sinh(2\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2)\\ &+ 4\sinh^2(\Lambda)\partial_x\phi\partial_{xt}\phi + 4\partial_t\phi\partial_{xx}\phi\sinh^2(\Lambda) + 4\sinh(2\Lambda)\partial_t\phi\partial_x\phi\partial_t\Lambda\\ &+ \partial_x\Lambda\partial_{xt}\Lambda + \partial_t\Lambda\partial_{xx}\Lambda - 4(\partial_t\phi)^2\partial_t\Lambda\sinh(2\Lambda)\\ =& \partial_x(\partial_x\Lambda\partial_t\Lambda + 4\sinh^2(\Lambda)\partial_t\phi\partial_x\phi).\end{aligned}$$

Then, the equation (2.4.2) is satisfied. As we can see, the continuity equation can be written explicitly as

$$\begin{aligned}&\partial_t \left(\partial_x\Lambda\partial_t\Lambda + 4\sinh^2\Lambda\partial_x\phi\partial_t\phi\right)\\ &\quad - \partial_x\left(\frac{1}{2}((\partial_x\Lambda)^2 + (\partial_t\Lambda)^2) + 2\sinh^2(\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2)\right) = 0.\end{aligned} \tag{2.4.4}$$

To prove the inequality, let us take into account Cauchy's inequality, then

$$|\partial_x\Lambda\partial_t\Lambda| \leq \frac{1}{2}\left((\partial_x\Lambda)^2 + (\partial_t\Lambda)^2\right), \qquad |\partial_x\phi\partial_t\phi| \leq \frac{1}{2}\left((\partial_x\phi)^2 + (\partial_t\phi)^2\right),$$

so that

$$|p(t,x)| \leq \frac{1}{2}((\partial_x\Lambda)^2 + (\partial_t\Lambda)^2) + 2\sinh^2(\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2). \tag{2.4.5}$$

That is, the energy density exerts a control on the momentum density, which will be of key importance, since all the analysis and results will attempt to establish the energy space of the coupled system. □

2.4.2 Virial estimate

The purpose of this section is to present a Virial identity which is related to the energy presented above. Let us take into account certain considerations following a proposal similar to the one used in [1]. However, in our case the semilinear character of the model enters and no smallness in a smaller space is needed. In what follows, we consider $t \geq 2$ only, and

$$\omega(t) := \frac{t}{\log^2 t}, \qquad \frac{\omega'(t)}{\omega(t)} = \frac{1}{t}\left(1 - \frac{2}{\log t}\right). \tag{2.4.6}$$

Furthermore, let us consider (Λ, ϕ) continuous in time such that $E[\Lambda, \phi](t) < +\infty$ is conserved. We introduce a Virial identity for the chiral field equation (2.1.11). Indeed, let $\rho := \tanh(\cdot)$, and let $\mathcal{I}(t)$ be defined as

$$\mathcal{I}(t) := -\int_{\mathbb{R}} \rho\left(\frac{x - vt}{\omega(t)}\right)\left(\partial_x\Lambda\partial_t\Lambda + 4\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right)dx, \quad v \in (-1, 1). \tag{2.4.7}$$

A time-dependent weight was also considered in [1], with the same goals. The choice of $\mathcal{I}(t)$ is motivated by the momentum and energy densities. Recall that $\int = \int_{\mathbb{R}}$.

Lemma 2.4.3 (Virial identity). *We have*

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}(t) = {} & \frac{\omega'(t)}{\omega(t)}\int \frac{x - vt}{\omega(t)}\rho'\left(\frac{x - vt}{\omega(t)}\right)(\partial_x\Lambda\partial_t\Lambda + 4\partial_x\phi\partial_t\phi\sinh^2(\Lambda)) \\
& + \frac{1}{\omega(t)}\int \rho'\left(\frac{x - vt}{\omega(t)}\right)\left(\frac{1}{2}(\partial_x\Lambda)^2 + 2(\partial_t\phi)^2\sinh^2(\Lambda)\right) \\
& + \frac{1}{\omega(t)}\int \rho'\left(\frac{x - vt}{\omega(t)}\right)\left(\frac{1}{2}(\partial_t\Lambda)^2 + 2(\partial_x\phi)^2\sinh^2(\Lambda)\right) \\
& + \frac{v}{\omega(t)}\int \rho'\left(\frac{x - vt}{\omega(t)}\right)\left(\partial_x\Lambda\partial_t\Lambda + 4\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right).
\end{aligned} \tag{2.4.8}$$

Proof. From (2.4.2) we readily have

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}(t) = {} & \frac{\omega'(t)}{\omega(t)}\int \rho'\left(\frac{x - vt}{\omega(t)}\right)\frac{x - vt}{\omega(t)}p(t, x) + \frac{v}{\omega(t)}\int \rho'\left(\frac{x - vt}{\omega(t)}\right)p(t, x) \\
& - \int \rho\left(\frac{x - vt}{\omega(t)}\right)\partial_x e(t, x),
\end{aligned}$$

using integration by parts and the Lemma 2.4.2

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}(t) = {} & \frac{\omega'(t)}{\omega(t)}\int \rho'\left(\frac{x - vt}{\omega(t)}\right)\frac{x - vt}{\omega(t)}p(t, x) + \frac{v}{\omega(t)}\int \rho'\left(\frac{x - vt}{\omega(t)}\right)p(t, x) \\
& + \frac{1}{\omega(t)}\int \rho'\left(\frac{x - vt}{\omega(t)}\right)e(t, x).
\end{aligned}$$

This proves (2.4.8) after replacing (2.4.1). □

2.4.3 Integration of the dynamics

The goal of this subsection is prove the Theorem 2.1.2; let us start with the following integral estimate

Lemma 2.4.4. *Let $\omega(t)$ given as in* (2.4.6). *Assume that the solution $(\Lambda, \phi)(t)$ of the system* (2.1.11) *satisfies*

$$E[\Lambda, \phi](t) < +\infty. \tag{2.4.9}$$

Then we have the averaged estimate

$$\int_2^{\infty} \frac{1}{\omega(t)} \int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x - vt}{\omega(t)}\right) e(t, x) dx dt \lesssim 1, \tag{2.4.10}$$

Moreover, there exists an increasing sequence $t_n \to +\infty$ such that

$$\lim_{n \longrightarrow +\infty} \int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x - vt_n}{\omega(t_n)}\right) e(t_n, x) dx = 0. \tag{2.4.11}$$

In order to show Lemma (2.4.4), we use the new Virial identity for (2.4.7) presented for the Chiral Field Equation (2.1.2).

Proof. First note that, from the condition (2.4.9) we have that clearly $\mathcal{I}(t)$ in (2.4.7) is well defined. Here we use the fact that both $\partial_x \Lambda$ and $\partial_t \Lambda$ are small in L^∞ thanks to the Sobolev embedding and in of the same form $\partial_x \phi$ and $\partial_t \phi$. Moreover

$$\sup_{t \in \mathbb{R}} |\mathcal{I}(t)| \lesssim E[\Lambda, \phi](t) \lesssim 1. \tag{2.4.12}$$

On the other hand, from Lemma 2.4.3, we have the identity

$$\frac{d}{dt} \mathcal{I}(t) = \mathcal{J}_1 + \mathcal{J}_2, \tag{2.4.13}$$

where

$$\mathcal{J}_1 = \frac{\omega'(t)}{\omega(t)} \int \rho'\left(\frac{x - vt}{\omega(t)}\right) \frac{x - vt}{\omega(t)} \left(\partial_x \Lambda \partial_t \Lambda + 4 \partial_x \phi \partial_t \phi \sinh^2(\Lambda)\right),$$

and $\mathcal{J}_2$ is the remaining term of (2.4.8). From the definition of $\omega(t)$, (2.4.9) and using Cauchy's inequality for $\delta > 0$ small, we have:

$$\begin{aligned}
|\mathcal{J}_1| &\leq \frac{2}{t} \int \frac{|x - vt|}{\omega(t)} \rho'\left(\frac{x - vt}{\omega(t)}\right) \left(|\partial_x \Lambda||\partial_t \Lambda| + 4|\partial_x \phi||\partial_t \phi| \sinh^2(\Lambda)\right) \\
&\leq \frac{8C_\delta}{t^2} \int \frac{(x - vt)^2}{\omega(t)} \rho'\left(\frac{x - vt}{\omega(t)}\right) \left(\frac{1}{2}(\partial_t \Lambda)^2 + 2(\partial_t \phi)^2 \sinh^2(\Lambda)\right) \\
&\quad + \frac{\delta}{\omega(t)} \int \rho'\left(\frac{x - vt}{\omega(t)}\right) \left(\frac{1}{2}(\partial_x \Lambda)^2 + 2(\partial_x \phi)^2 \sinh^2(\Lambda)\right) \\
&\leq \frac{8C_\delta \omega(t)}{t^2} \sup_{x \in \mathbb{R}} \left(\frac{(x - vt)^2}{\omega^2(t)} \rho'\left(\frac{x - vt}{\omega(t)}\right)\right) \int \left(\frac{1}{2}(\partial_t \Lambda)^2 + 2(\partial_t \phi)^2 \sinh^2(\Lambda)\right) \\
&\quad + \frac{\delta}{\omega(t)} \int \rho'\left(\frac{x - vt}{\omega(t)}\right) \left(\frac{1}{2}(\partial_x \Lambda)^2 + 2(\partial_t \phi)^2 \sinh^2(\Lambda)\right) \\
&\leq \frac{C}{t \log^2 t} + \frac{\delta}{\omega(t)} \int \rho'\left(\frac{x - vt}{\omega(t)}\right) \left(\frac{1}{2}(\partial_x \Lambda)^2 + 2(\partial_t \phi)^2 \sinh^2(\Lambda)\right).
\end{aligned}$$

Furthermore, for $\mathcal{J}_2(t)$ we have

$$\begin{aligned}\frac{|v|}{\omega(t)}\int \rho'\left(\frac{x-vt}{\omega(t)}\right)\left|\partial_x\Lambda\partial_t\Lambda+4\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right| &= \frac{|v|}{\omega(t)}\int \rho'\left(\frac{x-vt}{\omega(t)}\right)|p(t,x)|\\ &\leq \frac{|v|}{\omega(t)}\int \rho'\left(\frac{x-vt}{\omega(t)}\right)e(t,x).\end{aligned}$$

With this estimate on $\mathcal{J}_1$ to obtain $1-|v|-\delta>0$ for $\delta>0$ sufficiently small, and

$$\frac{d}{dt}\mathcal{I}(t)\geq \frac{1-|v|-\delta}{\omega(t)}\int \rho'\left(\frac{x-vt}{\omega(t)}\right)e(t,x)-\frac{C}{t\log^2 t}. \tag{2.4.14}$$

After integration in time we get (2.4.10). Finally, (2.4.11) is obtained from (2.4.10) and the fact that $\omega^{-1}(t)$ is not integrable in $[2,\infty)$. ☐

Proof of Theorem 2.1.2. Let us consider $\psi(\cdot)=(\rho')^2=\operatorname{sech}^4(\cdot)$, then

$$\begin{aligned}\frac{d}{dt}\int \psi\left(\frac{x-vt}{\omega(t)}\right)e(t,x) = & -\frac{\omega'(t)}{\omega(t)}\int \frac{x-vt}{\omega(t)}\psi'\left(\frac{x-vt}{\omega(t)}\right)e(t,x)-\frac{v}{\omega(t)}\int \psi'\left(\frac{x-vt}{\omega(t)}\right)e(t,x)\\ &+\frac{1}{\omega(t)}\int \psi'\left(\frac{x-vt}{\omega(t)}\right)p(t,x).\end{aligned}$$

Since $|\frac{x-vt}{\omega(t)}\psi'\left(\frac{x-vt}{\omega(t)}\right)|\lesssim \operatorname{sech}^2\left(\frac{x-vt}{\omega(t)}\right)$ and $|p(t,x)|\leq e(t,x)$ we have:

$$|\frac{d}{dt}\int \psi\left(\frac{x-vt}{\omega(t)}\right)e(t,x)|\leq \frac{C}{\omega(t)}\int \operatorname{sech}^2\left(\frac{x-vt}{\omega(t)}\right)e(t,x), \tag{2.4.15}$$

furthermore

$$\lim_{n\longrightarrow\infty}\int \operatorname{sech}^4\left(\frac{x-vt_n}{\omega(t_n)}\right)e(t_n,x)=0. \tag{2.4.16}$$

Finally using (2.4.15) for $t<t_n$

$$|\int \psi\left(\frac{x-vt_n}{\omega(t_n)}\right)e(t_n,x)-\int \psi\left(\frac{x-vt}{\omega(t)}\right)e(t,x)|\leq \int_t^{t_n}\frac{2}{\omega(s)}\int \operatorname{sech}^2\left(\frac{x-vs}{\omega(s)}\right)e(s,x)dxds,$$

sending n to infinity, and using (2.4.16) we have

$$|\int \psi\left(\frac{x-vt}{\omega(t)}\right)e(t,x)|\leq \int_t^{\infty}\frac{2}{\omega(s)}\int \operatorname{sech}^2\left(\frac{x-vs}{\omega(s)}\right)e(s,x)dxds, \tag{2.4.17}$$

which implies, thanks to Lemma 2.4.4,

$$\lim_{t\longrightarrow\infty}\int \operatorname{sech}^4\left(\frac{x-vt}{\omega(t)}\right)e(t,x)=0,$$

which finally shows the validity of Theorem 2.1.2. ☐

2.5 Application to soliton solutions

In this section, we apply our previous results to prove existence of global solutions around a new class of soliton solutions of finite energy. First, we consider the case treated by Hadad in [52]. See also [33, 78, 79] for other cases of soliton-like solutions not treated here.

2.5.1 Singular solitons

Consider the soliton introduced in (2.1.25). We claim that this solution is singular in the narrow sense that the energy of the system for this soliton is not finite. Our first result is the following straightforward computation:

Lemma 2.5.1. *One has,*

$$\begin{aligned}\Lambda(t,x) =& \ln(|v|\cosh(t))\\ &+\ln\left(1-\frac{\tanh(t)\tanh(\sqrt{c}(x-vt))}{|v|\sqrt{c}}+\sqrt{\left(1-\frac{\tanh(t)\tanh(\sqrt{c}(x-vt))}{|v|\sqrt{c}}\right)^2-\frac{\operatorname{sech}^2(t)}{|v|^2}}\right),\\ \phi(t,x) =&\frac{\pi}{4}-\frac{1}{2}\arctan\left[\cosh(t)\cosh(\sqrt{c}(x-vt))(\tanh(\sqrt{c}(x-vt))+v\sqrt{c}\tanh(t))\right].\end{aligned} \tag{2.5.1}$$

Moreover, for E_{mod} given in (2.1.24)*, the previous solution gives*

$$E_{mod}[\Lambda,\phi](t)=0. \tag{2.5.2}$$

***Remark* 2.5.1.** *Notice that $g^{(0)}$ in* (2.1.23) *has also zero modified energy. This is in concordance with the fact that $g^{(1)}$ is obtained from $g^{(0)}$ as seed.*

Proof. We use the notation in (2.1.25) and $\gamma:=\sqrt{c}(x-vt)$. Comparing the soliton (2.1.25) with (2.1.6) we have the following equations:

$$e^{t}[\cosh(\ln\mu)-\sinh(\ln\mu)\tanh(\gamma)] = \cosh(\Lambda)+\cos(2\phi)\sinh(\Lambda), \tag{2.5.3}$$

$$e^{-t}[\cosh(\ln\mu)+\sinh(\ln\mu)\tanh(\gamma)] = \cosh(\Lambda)-\cos(2\phi)\sinh(\Lambda), \tag{2.5.4}$$

$$-\frac{1}{\sqrt{c}\cosh(\gamma)} = \sin(2\phi)\sinh(\Lambda), \tag{2.5.5}$$

where

$$\cosh(\ln(\mu))=\frac{\mu^2+1}{2\mu}=-v,\quad \sinh(\ln(\mu))=\frac{\mu^2-1}{2\mu}=\frac{1}{\sqrt{c}},$$

adding the first two equations we obtain:

$$-v\cosh(t)-\frac{1}{\sqrt{c}}\sinh(t)\tanh(\gamma)=\cosh\Lambda.$$

Then, since we have the constraint $\mu>1$ we can write the expression for Λ as

$$\begin{aligned}\Lambda(t,x) =& \ln(|v|\cosh(t))\\ &+\ln\left(1-\frac{\tanh(t)\tanh(\gamma)}{|v|\sqrt{c}}+\sqrt{\left(1-\frac{\tanh(t)\tanh(\gamma)}{|v|\sqrt{c}}\right)^2-\frac{1}{|v|^2}\operatorname{sech}^2(t)}\right).\end{aligned}$$

Next, subtracting the same equations and using (2.5.5)

$$-v\sinh(t)-\tfrac{1}{\sqrt{c}}\cosh(t)\tanh(\gamma)=\cos(2\phi)\sinh(\Lambda),$$

$$\begin{gathered}-v\sinh(t)-\tfrac{1}{\sqrt{c}}\cosh(t)\tanh(\gamma)=-\tfrac{1}{\sqrt{c}}\cot(2\phi)\operatorname{sech}(\gamma)\\ \sinh(\gamma)\cosh(t)+\sqrt{c}v\sinh(t)\cosh(\gamma)=\cot(2\phi),\end{gathered}$$

In order to make sense, one needs $\sin(2\phi)\neq 0$, i.e., $\phi\neq\dfrac{n\pi}{2}$. Therefore, we can write:

$$\begin{aligned}\phi(t,x)=&\frac{\pi}{4}-\frac{1}{2}\arctan\left(\sinh(\gamma)\cosh(t)+\sqrt{c}v\sinh(t)\cosh(\sqrt{c}(x-vt))\right)\\ =&\frac{\pi}{4}-\frac{1}{2}\arctan\left(\cosh(t)\cosh(\sqrt{c}(x-vt))(\tanh(\sqrt{c}(x-vt))+v\sqrt{c}\tanh(t))\right).\end{aligned}$$

Now, let us study the derivatives of the Λ and ϕ fields. Assuming the constraints for the parameter μ we have that

$$\begin{aligned}\partial_x\Lambda&=-\frac{\sinh(t)\operatorname{sech}^2(\gamma)}{\sqrt{(-v\cosh(t)-\frac{1}{\sqrt{c}}\sinh(t)\tanh(\gamma))^2-1}},\\ \partial_t\Lambda&=\frac{\tanh(\gamma)\left(-v\sinh(t)\tanh(\gamma)-\frac{1}{\sqrt{c}}\cosh(t)\right)}{\sqrt{(-v\cosh(t)-\frac{1}{\sqrt{c}}\sinh(t)\tanh(\gamma))^2-1}}.\end{aligned}$$

Additionally,

$$\begin{aligned}\partial_x\phi&=-\frac{1}{2}\left(\frac{\sqrt{c}\cosh(t)\cosh(\gamma)(1+v\sqrt{c}\tanh(t)\tanh(\gamma))}{1+(\sinh(\gamma)\cosh(t)+v\sqrt{c}\sinh(t)\cosh(\gamma))^2}\right),\\ \partial_t\phi&=-\frac{1}{2}\left(\frac{(1-cv^2)\sinh(t)\sinh(\gamma)}{1+(\sinh(\gamma)\cosh(t)+v\sqrt{c}\sinh(t)\cosh(\gamma))^2}\right).\end{aligned}$$

Simplifying, the energy density is:

$$\begin{aligned}&(\partial_x\Lambda)^2+(\partial_t\Lambda)^2-1=\\ &\frac{\sinh^2(t)\operatorname{sech}^4(\gamma)-v^2\sinh^2(t)\operatorname{sech}^2(\gamma)(\tanh^2(\gamma)+1)-\frac{v}{\sqrt{c}}\sinh(2t)\tanh(\gamma)\operatorname{sech}^2(\gamma)-v^2+1}{(-v\cosh(t)-\frac{1}{\sqrt{c}}\sinh(t)\tanh(\gamma))^2-1},\end{aligned}$$

and

$$\begin{aligned}&\sinh^2(\Lambda)((\partial_x\phi)^2+(\partial_t\phi)^2)\\ &=\frac{c\cosh^2(t)\cosh^2(\gamma)-\frac{vc}{2}\sinh(2t)\sinh(2\gamma)+(c^2v^4-cv^2+1)\sinh^2(t)\sinh^2(\gamma)}{(1+(\sinh(\gamma)\cosh(t)+v\sqrt{c}\sinh(t)\cosh(\gamma))^2)^2}.\end{aligned}$$

Then, the integrals can be calculated with the help of the computer algebra system Mathematica, obtaining that the soliton has finite modified energy, in fact, we have that

$$E_{\text{mod}}[\Lambda,\phi](t)=0.$$

With the results obtained we can see that the Λ and ϕ fields associated to the soliton (2.1.25) do not belong to the energy space proposed in the previous sections. □

2.5.2 Finite energy solitons

In this final section we consider the case of finite energy solitons, their perturbations, and a corresponding global well-posedness result.

Proof of Corollary 2.1.1: Identifying the 1-soliton in (2.1.26)

$$g^{(1)} = \begin{bmatrix} \dfrac{e^{\lambda+\varepsilon\theta}\operatorname{sech}(\beta(\lambda+\varepsilon\theta))}{\operatorname{sech}(\beta(\lambda+\varepsilon\theta)-x_0)} & -\dfrac{1}{\sqrt{c}}\operatorname{sech}(\beta(\lambda+\varepsilon\theta)) \\ -\dfrac{1}{\sqrt{c}}\operatorname{sech}(\beta(\lambda+\varepsilon\theta)) & \dfrac{e^{-(\lambda+\varepsilon\theta)}\operatorname{sech}(\beta(\lambda+\varepsilon\theta))}{\operatorname{sech}(\beta(\lambda+\varepsilon\theta)+x_0)} \end{bmatrix}, \qquad \beta = \frac{\mu+1}{\mu-1}, \tag{2.5.6}$$

with the geometrical representation (2.1.6), one gets the corresponding fields $\hat{\Lambda}_\varepsilon$ and $\hat{\phi}_\varepsilon$, which have the following form:

$$\hat{\Lambda}_\varepsilon(t,x) := \cosh^{-1}\left(|v|\cosh(\lambda+\varepsilon\theta) - \frac{1}{\sqrt{c}}\tanh(\beta(\lambda+\varepsilon\theta))\sinh(\lambda+\varepsilon\theta)\right),$$

$$\hat{\phi}_\varepsilon(t,x) := \frac{\pi}{4} - \frac{1}{2}\arctan\left(\cosh(\beta(\lambda+\varepsilon\theta))\cosh(\lambda+\varepsilon\theta)(\tanh(\beta(\lambda+\varepsilon\theta)) + v\sqrt{c}\tanh(\lambda+\varepsilon\theta))\right),$$

which are solutions for (2.1.21). From now on we drop ε to make the notation less cumbersome.

We claim that $\hat{\Lambda}$ have the desired local and global well-posedness properties. Indeed, note that since $0<\mu<1$, then $|v|>1$ and $\beta<0$, so, for all $t,x\in\mathbb{R}$

$$\begin{aligned} &|v|\cosh(\lambda+\varepsilon\theta) - \frac{1}{\sqrt{c}}\tanh(\beta(\lambda+\varepsilon\theta))\sinh(\lambda+\varepsilon\theta) \\ &\geq |v| + \frac{1}{\sqrt{c}}\tanh(|\beta|(\lambda+\varepsilon\theta))\sinh(\lambda+\varepsilon\theta) > 1, \end{aligned}$$

therefore, $\hat{\Lambda}$ is well-defined and $\hat{\Lambda}(t,x)>0$ for all $t,x\in\mathbb{R}$. Also, since $\theta\in L^\infty(\mathbb{R})$, for each t, we have to that $\hat{\Lambda}$ is a bounded function. Since $\theta\in C_0^2$, we have that

$$\hat{\Lambda}(t=0,x) = C(\lambda), \quad \forall x\in\mathbb{R}\setminus\operatorname{supp}\theta,$$

then, define $\tilde{\lambda} := C(\lambda)$, which allows us to write $\hat{\Lambda} := \tilde{\Lambda} + \tilde{\lambda}$. For the function $\tilde{\Lambda}$ one has

$$\tilde{\Lambda}|_{\{t=0\}} = \varepsilon\tilde{\Lambda}_0, \quad \text{with} \quad \tilde{\Lambda}_0 \in C_0^2(\mathbb{R}).$$

where $\tilde{\Lambda}_0$ is defined as:

$$\begin{aligned} \tilde{\Lambda}_0(x) := &\frac{1}{\varepsilon}\left(\cosh^{-1}\left(|v|\cosh(\lambda+\varepsilon\theta(x)) - \frac{1}{\sqrt{c}}\tanh(\beta(\lambda+\varepsilon\theta(x)))\sinh(\lambda+\varepsilon\theta(x))\right)\right) \\ &- \frac{1}{\varepsilon}\left(\cosh^{-1}\left(|v|\cosh(\lambda) - \frac{1}{\sqrt{c}}\tanh(\beta\lambda)\sinh(\lambda)\right)\right). \end{aligned}$$

The dependence associated with ε for this function, is suitable in the sense that we can demonstrate straightforwardly that $\tilde{\Lambda}_0$ is a bounded function when ε tends to zero, indeed, we have that the $\lim\limits_{\varepsilon\longrightarrow 0}\tilde{\Lambda}_0$ can be calculated using L'Hôpital's rule:

$$\lim_{\varepsilon\longrightarrow 0}\tilde{\Lambda}_0 = \lim_{\varepsilon\longrightarrow 0}\frac{\theta(x)\left(|v|\sinh(\lambda+\varepsilon\theta) - \frac{\beta}{\sqrt{c}}\operatorname{sech}^2(\beta(\lambda+\varepsilon\theta)) - \frac{1}{\sqrt{c}}\tanh(\beta(\lambda+\varepsilon\theta))\cosh(\lambda+\varepsilon\theta)\right)}{\sqrt{(|v|\cosh(\lambda+\varepsilon\theta) - \frac{1}{\sqrt{c}}\tanh(\beta(\lambda+\varepsilon\theta))\sinh(\lambda+\varepsilon\theta))^2 - 1}}$$

$$= \frac{\theta(x)\left(|v|\sinh(\lambda) - \frac{\beta}{\sqrt{c}}\operatorname{sech}^2(\beta\lambda) - \frac{1}{\sqrt{c}}\tanh(\beta\lambda)\cosh(\lambda)\right)}{\sqrt{(|v|\cosh(\lambda) - \frac{1}{\sqrt{c}}\tanh(\beta\lambda)\sinh(\lambda))^2 - 1}} = C\theta(x)$$

On the other hand, the derivative of $\tilde{\Lambda}$ is given by

$$\partial_t\tilde{\Lambda} = \frac{\varepsilon\theta'\left(|v|\tanh(\gamma) - \frac{1}{\sqrt{c}}\beta\operatorname{sech}^2(\beta\gamma)\tanh(\gamma) - \frac{1}{\sqrt{c}}\tanh(\beta\gamma)\right)}{\operatorname{sech}(\gamma)\sqrt{\left(|v|\cosh(\gamma) - \frac{1}{\sqrt{c}}\sinh(\gamma)\tanh(\beta\gamma)\right)^2 - 1}},$$

in this case $\gamma := \lambda + \varepsilon\theta$, then, is clearly that $\partial_t\tilde{\Lambda}|_{\{t=0\}} \in C_0^2(\mathbb{R})$.

Next, for the field $\hat{\phi}$, we have a bounded function and $\hat{\phi}(t,x) > 0$ for all $t, x \in \mathbb{R}$. Again, since $\theta \in C_0^2(\mathbb{R})$ we have,

$$\hat{\phi}(t=0,x) = C_1(\lambda), \quad \forall x \in \mathbb{R} \setminus \operatorname{supp}\theta,$$

and we can define

$$\phi(t,x) = \hat{\phi} - \epsilon \quad \text{with} \quad \epsilon = C_1(\lambda).$$

With this definition one has:

$$\phi(t=0,x) = \hat{\phi}(t=0,x) - \epsilon,$$

then

$$\phi(t=0,x) = \hat{\phi}(t=0,x) - \epsilon = 0 \quad \forall x \in \mathbb{R} \setminus \operatorname{supp}\theta,$$

if we choose

$$\varepsilon\phi_0(x) = \phi(t=0,x) = \hat{\phi}(t=0,x) - \epsilon,$$

where ϕ_0 is given as:

$$\phi_0(x) = \frac{1}{2\varepsilon}\arctan\left(\cosh(\beta\lambda)\cosh(\lambda)(\tanh(\beta\lambda) + v\sqrt{c}\tanh(\lambda))\right)$$
$$- \frac{1}{2\varepsilon}\arctan\left(\cosh(\beta(\lambda+\varepsilon\theta(x)))\cosh(\lambda+\varepsilon\theta(x))(\tanh(\beta(\lambda+\varepsilon\theta(x))) + v\sqrt{c}\tanh(\lambda+\varepsilon\theta(x)))\right),$$

note that this definition is suitable, we can compute the $\lim_{\varepsilon\longrightarrow 0}\phi_0$ using L'Hôpital's rule:

$$\begin{aligned}\lim_{\varepsilon\longrightarrow 0}\phi_0 &= \lim_{\varepsilon\longrightarrow 0}\frac{\theta(x)((1+\beta v\sqrt{c})\sinh(\gamma)\sinh(\beta\gamma) + (\beta + v\sqrt{c})\cosh(\gamma)\cosh(\beta\gamma)}{2(1 + (\cosh(\gamma)\sinh(\beta\gamma) + \beta\sqrt{c}\sinh(\gamma)\cosh(\beta\gamma))^2)} \\ &= C_1\theta(x).\end{aligned}$$

The function ϕ has the desired local and global well-posedness properties. Indeed, the derivative of this function is:

$$\partial_t\phi = \frac{-\varepsilon\theta'(\beta + v\sqrt{c} + (1+\beta v\sqrt{c})\tanh(\beta\gamma)\tanh(\gamma))}{2\operatorname{sech}(\beta\gamma)\operatorname{sech}(\gamma)((\cosh(\gamma)\sinh(\beta\gamma) + v\sqrt{c}\sinh(\gamma)\cosh(\beta\gamma))^2 + 1)},$$

which is also a localized function. Finally from the previous analysis, we can conclude that for $\partial_t\Lambda, \partial_t\phi \in L^2(\mathbb{R})$, with $\Lambda(t,x) = \hat{\Lambda}(t,x)$, then

$$E[\Lambda, \phi] < \infty.$$

In the end, $\hat{\Lambda}$ reads as

$$\hat{\Lambda} = \ln(\cosh(\gamma)) + \ln\left(|v| - \frac{\tanh(\gamma)\tanh(\beta\gamma)}{\sqrt{c}} + \sqrt{\left(|v| - \frac{\tanh(\gamma)\tanh(\beta\gamma)}{\sqrt{c}}\right)^2 - 1}\right).$$

This finishes the proof. □

2.6 Appendix

2.6.1 Some useful inequalities

This section start by presenting the well-known Gronwall's lemma:

Lemma 2.6.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a positive continuous function and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a positive integrable function such that*

$$f(t) \leq A + \int_0^t f(s)g(s)ds,$$

for some $A \geq 0$ for every $t \in [0, T]$. Then

$$f(t) \leq A \exp\left(\int_0^t g(s)ds\right),$$

for every $t \in [0, T]$.

The second result to be presented is related to another pointwise bounds that were presented for Luli et. al. in [85] for the study of the global problem in the Section 2.3:

Lemma 2.6.2. *Under the assumption (2.3.11) and (2.3.12) exits a universal constant C_3 so that*

$$\begin{aligned}&\left\|\frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/4}}\underline{L}\tilde{\Lambda}(t,x)\right\|_{L^\infty(\Sigma_t)}\\ &\qquad\leq C_3\left(\left\|\frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/4}}\underline{L}\tilde{\Lambda}(t,x)\right\|_{L^2(\Sigma_t)} + \left\|\frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/4}}\underline{L}\partial_x\tilde{\Lambda}(t,x)\right\|_{L^2(\Sigma_t)}\right),\end{aligned}$$

$$\begin{aligned}&\left\|\frac{\varphi(\underline{u})^{1/2}}{\varphi(u)^{1/4}}L\tilde{\Lambda}(t,x)\right\|_{L^\infty(\Sigma_t)}\\ &\qquad\leq C_3\left(\left\|\frac{\varphi(\underline{u})^{1/2}}{\varphi(u)^{1/4}}L\tilde{\Lambda}(t,x)\right\|_{L^2(\Sigma_t)} + \left\|\frac{\varphi(\underline{u})^{1/2}}{\varphi(u)^{1/4}}L\partial_x\tilde{\Lambda}(t,x)\right\|_{L^2(\Sigma_t)}\right),\end{aligned}$$

and

$$\begin{aligned}&\left\|\frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/4}}\underline{L}\phi(t,x)\right\|_{L^\infty(\Sigma_t)}\\ &\qquad\leq C_3\left(\left\|\frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/4}}\underline{L}\phi(t,x)\right\|_{L^2(\Sigma_t)} + \left\|\frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/4}}\underline{L}\partial_x\phi(t,x)\right\|_{L^2(\Sigma_t)}\right),\end{aligned}$$

$$\left\| \frac{\varphi(\underline{u})^{1/2}}{\varphi(u)^{1/4}} L\phi(t,x) \right\|_{L^\infty(\Sigma_t)} \leq C_3 \left(\left\| \frac{\varphi(\underline{u})^{1/2}}{\varphi(u)^{1/4}} L\phi(t,x) \right\|_{L^2(\Sigma_t)} + \left\| \frac{\varphi(\underline{u})^{1/2}}{\varphi(u)^{1/4}} L\partial_x\phi(t,x) \right\|_{L^2(\Sigma_t)} \right).$$

2.6.2 Ending of proof of Theorem 2.1.1

In this section, we describe the details of the estimates for the second equation in (2.3.1) that complete the proof of Theorem 2.1.1.

For simplicity, in Section 2.3 we worked with the first equation of system (2.3.1). Now we prove the estimates for the second equation.

Proof. The first step is the following: Using (2.3.3) and (2.3.4) in the second equation of (2.3.1), we obtain:

$$\square\,(\partial_x\phi) = 2\Big[\coth(\lambda+\tilde{\Lambda})\left(Q_0(\partial_x\phi,\tilde{\Lambda}) + Q_0(\phi,\partial_x\tilde{\Lambda})\right) - 2\partial_x\tilde{\Lambda}\,\mathrm{csch}^2(\lambda+\tilde{\Lambda})Q_0(\phi,\tilde{\Lambda})\Big]. \tag{2.6.1}$$

As in Section 2.3, fix $\delta \in (0,1)$, under the assumptions (2.3.11)-(2.3.12)-(2.3.13) for all $t \in [0,T^*]$, we assume that the solution remains regular, to later show that these bounds are maintained, with a better constant.

Consider $k = 0,1$. Using (2.3.8) on (2.3.17), with $\psi = \partial_x^k\phi$, and taking the sum over $k = 0,1$, we obtain

$$\begin{aligned}
&\overline{\mathcal{E}}(t) + \overline{\mathcal{F}}(t) \leq 2C_0\overline{\mathcal{E}}(0) \\
&\quad + 2C_0 \iint_{D_t} \left((1+|u|^2)^{1+\delta}|\underline{L}\phi| + (1+|\underline{u}|^2)^{1+\delta}|L\phi|\right) 2|\coth(\lambda+\tilde{\Lambda})||Q_0(\phi,\tilde{\Lambda})| \\
&\quad + 4C_0 \iint_{D_t} \left((1+|u|^2)^{1+\delta}|\underline{L}\partial_x\phi| + (1+|\underline{u}|^2)^{1+\delta}|L\partial_x\phi|\right) |\coth(\lambda+\tilde{\Lambda})||(Q_0(\partial_x\phi,\tilde{\lambda}) + Q_0(\phi,\partial_x\tilde{\Lambda}))| \\
&\quad + 4C_0 \iint_{D_t} \left((1+|u|^2)^{1+\delta}|\underline{L}\partial_x\phi| + (1+|\underline{u}|^2)^{1+\delta}|L\partial_x\phi|\right) |\partial_x\tilde{\Lambda}\,\mathrm{csch}^2(\lambda+\tilde{\Lambda})||Q_0(\phi,\tilde{\Lambda})| \\
&=: 2C_0\overline{\mathcal{E}(0)} + 2C_0\sum_{j=1}^{8} I_j.
\end{aligned} \tag{2.6.2}$$

In this case, the integrals $I_j, i \in \{1,2,..,8\}$ are defined as follows:

$$
\begin{aligned}
I_1 &:= 2C_0 \iint_{D_t} \big((1+|u|^2)^{1+\delta}|\underline{L}\phi|\big)\,|\coth(\lambda+\tilde\Lambda)||Q_0(\phi,\tilde\Lambda)|\\
I_2 &:= 2C_0 \iint_{D_t} \big((1+|\underline{u}|^2)^{1+\delta}|L\phi|\big)\,|\coth(\lambda+\tilde\Lambda)||Q_0(\phi,\tilde\Lambda)|\\
I_3 &:= 2C_0 \iint_{D_t} \big((1+|u|^2)^{1+\delta}|\underline{L}\partial_x\phi|\big)\,|\coth(\lambda+\tilde\Lambda)||(Q_0(\partial_x\phi,\tilde\Lambda)|\\
I_4 &:= 2C_0 \iint_{D_t} \big((1+|u|^2)^{1+\delta}|\underline{L}\partial_x\phi|\big)\,|\coth(\lambda+\tilde\Lambda)||Q_0(\phi,\partial_x\tilde\Lambda))|\\
I_5 &:= 2C_0 \iint_{D_t} \big((1+|\underline{u}|^2)^{1+\delta}L\partial_x\phi|\big)\,|\coth(\lambda+\tilde\Lambda)||(Q_0(\partial_x\phi,\tilde\Lambda)|\\
I_6 &:= 2C_0 \iint_{D_t} \big((1+|\underline{u}|^2)^{1+\delta}|L\partial_x\phi|\big)\,|\coth(\lambda+\tilde\Lambda)||Q_0(\phi,\partial_x\tilde\Lambda))|\\
I_7 &:= 2C_0 \iint_{D_t} \big((1+|u|^2)^{1+\delta}|\underline{L}\partial_x\phi|\big)\,|\partial_x\tilde\Lambda\,\mathrm{csch}^2(\lambda+\tilde\Lambda)||Q_0(\phi,\tilde\Lambda)|\\
I_8 &:= 2C_0 \iint_{D_t} \big((1+|\underline{u}|^2)^{1+\delta}|L\partial_x\phi|\big)\,|\partial_x\tilde\Lambda\,\mathrm{csch}^2(\lambda+\tilde\Lambda)||Q_0(\phi,\tilde\Lambda)|.
\end{aligned}
\tag{2.6.3}
$$

The goal is to control the right-hand side of the above estimate. Essentially we have eight terms to control, but several are equivalent and essentially we only need to consider two cases. Indeed, it will be sufficient to bound the terms corresponding to $\underline{L}\partial_x\phi$ and $\underline{L}\phi$, since by symmetry, the procedure for the other terms will be analogous. First, we start to bound the term I_7, that represents the most attention, given that it has different sub-terms to estimate, recalling that we define $\varphi(x) = (1+|x|^2)^{1+\delta}$, with $0<\delta\ll 1$.

Taking into account (2.3.5), (2.3.13) and (2.3.15)-(4.2.10), and writing $\partial_x\tilde\Lambda = \frac{1}{2}(L-\underline{L})\tilde\Lambda$, we get

$$
\begin{aligned}
I_7 \lesssim{}& C_0\iint_{D_t}\varphi(u)|\underline{L}\partial_x\phi||L\tilde\Lambda||L\phi||\underline{L}\tilde\Lambda| + C_0\iint_{D_t}\varphi(u)|\underline{L}\partial_x\phi||L\tilde\Lambda|^2|\underline{L}\phi|\\
&+C_0\iint_{D_t}\varphi(u)|\underline{L}\partial_x\phi||L\tilde\Lambda|^2|L\phi| + C_0\iint_{D_t}\varphi(u)|\underline{L}\partial_x\phi||\underline{L}\tilde\Lambda||\underline{L}\phi||L\tilde\Lambda|\\
:={}& I_{7,1}+I_{7,2}+I_{7,3}+I_{7,4}.
\end{aligned}
\tag{2.6.4}
$$

Recall that by Fubini's Theorem the spacetime D_t in (2.3.7) is foliated by $\underline{C}_{\underline{u}}$ for $u\in\mathbb{R}$, and also by $\{t\}\times\Sigma_t, t\in\mathbb{R}$. Using again the Lemma 2.3.1, we obtain

$$
\begin{aligned}
I_{7,1} \lesssim{}& \iint_{D_t}\varepsilon\,\underbrace{\big(\varphi(\underline{u})^{-3/4}\varphi(u)^{1/2}|\underline{L}\partial_x\phi|\big)}_{L^2_tL^2_x}\underbrace{\big(\varphi^{1/2}(\underline{u})|L\tilde\Lambda|\big)}_{L^\infty_tL^2_x}\underbrace{\big(\varphi(\underline{u})^{-1/4}\varphi(u)^{1/2}|\underline{L}\tilde\Lambda|\big)}_{L^2_tL^\infty_x}\\
\lesssim{}&\varepsilon\underbrace{\left(\iint_{D_t}\frac{\varphi(u)|\underline{L}\partial_x\phi|^2}{\varphi(\underline{u})^{3/2}}\right)^{1/2}}_{T_1}\underbrace{\sup_{t\in[0,T^*]}\left(\int_{\Sigma_t}\varphi(\underline{u})|L\tilde\Lambda|^2\right)^{1/2}}_{T_2}\underbrace{\left(\int_0^t\left\|\frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/4}}|\underline{L}\tilde\Lambda|\right\|^2_{L^\infty(\Sigma_\tau)}d\tau\right)^{1/2}}_{T_3}.
\end{aligned}
$$

Let us study each of the factors T_j. For T_1, one has:

$$T_1^2 \leq \int_{\mathbb{R}} \left[\int_{\underline{C}_{\underline{u}}} \frac{\varphi(u)|\underline{L}\partial_x\phi|^2}{\varphi(\underline{u})^{3/2}} ds \right] d\underline{u} = \int_{\mathbb{R}} \frac{1}{\varphi(\underline{u})^{3/2}} \underbrace{\left[\int_{\underline{C}_{\underline{u}}} \varphi(u)|\underline{L}\partial_x\phi|^2 ds \right]}_{\lesssim \overline{\mathcal{F}}_1(t)} d\underline{u} \lesssim \int_{\mathbb{R}} \frac{\varepsilon^2}{\varphi(\underline{u})^{3/2}} d\underline{u},$$

since the integral is finite, we have $T_1 \lesssim \varepsilon$. The integral T_2 is part of the energy norm $\mathcal{E}_0(t)$ in (2.3.9) then $T_2 \lesssim \varepsilon$. For the term T_3 one can use the same argument as in [85]: using Lemma 2.6.2 one gets

$$\begin{aligned} T_3 \lesssim & \left(\int_0^t \left\| \frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/4}} \underline{L}\tilde{\Lambda}(t,x) \right\|^2_{L^2(\Sigma_\tau)} + \int_0^t \left\| \frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/4}} \underline{L}\partial_x\tilde{\Lambda}(t,x) \right\|^2_{L^2(\Sigma_\tau)} \right)^{1/2} \\ \lesssim & \left(\iint_{D_t} \frac{\varphi(u)}{\varphi(\underline{u})^{1/2}} |\underline{L}\tilde{\Lambda}|^2 + \iint_{D_t} \frac{\varphi(u)}{\varphi(\underline{u})^{1/2}} |\underline{L}\partial_x\tilde{\Lambda}|^2 \right)^{1/2}, \end{aligned}$$

both terms above are of the same form as T_1 and then we have that $T_3 \lesssim \varepsilon$. We conclude that $I_{7,1} \lesssim \varepsilon^4$. Now we control the integral $I_{7,2}$ in (2.6.4), using again the Lemma 2.3.1, the assumption (2.3.12) and Cauchy–Schwarz inequality. We have:

$$\begin{aligned} I_{7,2} = & C_0 \iint_{D_t} \varphi(u)|\underline{L}\partial_x\phi||L\tilde{\Lambda}|^2|\underline{L}\phi| \leq \iint_{D_t} C_2\varepsilon^2 \frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/2}} |\underline{L}\partial_x\phi| \frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/2}} |\underline{L}\phi| \\ \leq & C_2\varepsilon^2 \left(\iint_{D_t} \frac{\varphi(u)}{\varphi(\underline{u})} |\underline{L}\partial_x\phi|^2 \right)^{1/2} \left(\iint_{D_t} \frac{\varphi(u)}{\varphi(\underline{u})} |\underline{L}\phi|^2 \right)^{1/2} \lesssim \varepsilon^4. \end{aligned}$$

To finish with the term I_7 we need to estimate the terms $I_{7,3}$ and $I_{7,4}$ in (2.6.4), which are similar in structure, for this case we get:

$$\begin{aligned} I_{7,34} = & \; I_{7,3} + I_{7,4} \lesssim \iint_{D_t} \varepsilon^2 |\underline{L}\partial_x\phi||L\phi| + \iint_{D_t} \varepsilon^2 |\underline{L}\partial_x\phi||L\tilde{\Lambda}| \\ = & \iint_{D_t} \varepsilon^2 \frac{\varphi(u)^{1/2}}{\varphi(\underline{u})^{1/2}} |\underline{L}\partial_x\phi| \left(\frac{\varphi(\underline{u})^{1/2}}{\varphi(u)^{1/2}} |L\phi| + \frac{\varphi(\underline{u})^{1/2}}{\varphi(u)^{1/2}} |L\tilde{\Lambda}| \right) \\ \lesssim & \iint_{D_t} \varepsilon^2 \left(\frac{\varphi(u)}{\varphi(\underline{u})} |\underline{L}\partial_x\phi|^2 \right) + \iint_{D_t} \varepsilon^2 \left(\frac{\varphi(\underline{u})}{\varphi(u)} |L\phi|^2 \right) + \iint_{D_t} \varepsilon^2 \left(\frac{\varphi(\underline{u})}{\varphi(u)} |L\tilde{\Lambda}|^2 \right) \\ \lesssim & \int_{\mathbb{R}} \frac{\varepsilon^2}{\varphi(\underline{u})} \underbrace{\left[\int_{\underline{C}_{\underline{u}}} \varphi(u)|\underline{L}\partial_x\phi|^2 ds \right]}_{\lesssim \overline{\mathcal{F}}_1(t)} d\underline{u} + \int_{\mathbb{R}} \frac{\varepsilon^2}{\varphi(u)} \underbrace{\left[\int_{C_u} \varphi(\underline{u})|L\phi|^2 ds \right]}_{\lesssim \overline{\mathcal{F}}_0(t)} du \\ & + \int_{\mathbb{R}} \frac{\varepsilon^2}{\varphi(u)} \underbrace{\left[\int_{C_u} \varphi(\underline{u})|L\tilde{\Lambda}|^2 ds \right]}_{\lesssim \mathcal{F}_0(t)} du \lesssim \varepsilon^4. \end{aligned}$$

Putting all estimates together for I_7, we conclude that $I_7 \lesssim \varepsilon^4$. A similar result is obatained for I_8.

Now we treat the term $I_1+I_3+I_4$ from (2.6.2). We have from (2.3.5) and (2.3.15)-(4.2.10),

$$\iint_{D_t} \varphi(u)|\underline{L}\partial_x\phi| \left(|L\partial_x\phi||\underline{L}\tilde{\Lambda}| + |\underline{L}\partial_x\phi||L\tilde{\Lambda}| + |L\phi||\underline{L}\partial_x\tilde{\Lambda}| + |\underline{L}\phi||L\partial_x\tilde{\Lambda}|\right) + \iint_{D_t} (\varphi(u)|\underline{L}\phi|) \left(|L\phi||\underline{L}\tilde{\Lambda}| + |L\tilde{\Lambda}||\underline{L}\phi|\right).$$

Using the condition (2.3.13), the situation matches Case 1 developed in [85]. All these integrals can be written as

$$\sim \iint_{D_t} \left(\varphi(u)|\underline{L}\partial_x\tilde{\Lambda}||L\phi||\underline{L}\partial_x\phi| + \varphi(u)|\underline{L}\partial_x\tilde{\Lambda}||L\partial_x\phi||\underline{L}\phi|\right).$$

Then, we can use the estimate (2.3.23) in Section 2.3 to conclude the bounds on these terms, which again are of order ε^3.

□

2.6.3 Classical Solution: Local Theory

As we can see, Proposition 2.1.1 does not directly provide us with a classical solution for the initial value problem (2.1.16). In order to obtain such a classical solution, we need an initial data with sufficient regularity, which allows us to control the terms associated with the nonlinearity. The idea of the proof still has the same structure.

Recall that the initial value problem for (2.1.16) can be written in vector form as follows

$$\begin{cases} \partial_\alpha(m^{\alpha\beta}\partial_\beta\Psi) = F(\Psi,\partial\Psi) \\ (\Psi,\partial_t\Psi)|_{\{t=0\}} = (\Psi_0,\Psi_1) \in \hat{\mathcal{H}}. \end{cases} \tag{2.6.5}$$

Where $m^{\alpha\beta}$ are the components of the Minkowski metric with $\alpha,\beta \in \{0,1\}$, and

$$(\Psi,\partial_t\Psi) \in \hat{\mathcal{H}} := H^3(\mathbb{R}) \times H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^2(\mathbb{R}). \tag{2.6.6}$$

We are also going to impose the following condition on the initial data

$$\|(\Psi_0,\Psi_1)\|_{\hat{\mathcal{H}}} \leq \frac{\lambda}{2D}, \tag{2.6.7}$$

where the assumptions on the constant $D \geq 1$ will be indicated below.

The following proposition shows that the equation (2.6.5), in terms of the function $\tilde{\Lambda}$ introduced in (2.1.15), is locally well-posed in the space $L^\infty([0,T];\hat{\mathcal{H}})$ with the norm in this space defined by

$$\|(\Psi,\partial_t\Psi)\|_{L^\infty([0,T];\mathcal{H})} = \sup_{t\in[0,T]} \left(\|\Psi\|_{H^3(\mathbb{R})\times H^3(\mathbb{R})} + \|\partial_t\Psi\|_{H^2(\mathbb{R})\times H^2(\mathbb{R})}\right),$$

with $(\Psi,\partial_t\Psi)$ introduced in (2.1.17). The result is the following.

Proposition 2.6.1. *If (Ψ_0, Ψ_1) satisfies the condition (2.6.7) with an appropriate constant $D \geq 1$, then:*

(1) (Existence and uniqueness of local-in-time solutions). There exists

$$T = T\left(\left\|\left(\tilde{\Lambda}_0, \phi_0\right)\right\|_{H^3(\mathbb{R})\times H^3(\mathbb{R})}, \left\|\left(\tilde{\Lambda}_1, \phi_1\right)\right\|_{H^2(\mathbb{R})\times H^2(\mathbb{R})}, \lambda\right) > 0,$$

such that there exists a (classical) solution Ψ to (2.6.5) with

$$(\Psi, \partial_t\Psi) \in L^\infty([0,T]; \hat{\mathcal{H}}).$$

Moreover, the solution is unique in this function space.

(2) (Continuous dependence on the initial data). Let $\Psi_0^{(i)}, \Psi_1^{(i)}$ be sequence such that $\Psi_0^{(i)} \longrightarrow \Psi_0$ in $H^3(\mathbb{R}) \times H^3(\mathbb{R})$ and $\Psi_1^{(i)} \longrightarrow \Psi_1$ in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ as $i \longrightarrow \infty$. Then taking $T > 0$ sufficiently small, we have

$$\left\|\left(\Psi^{(i)} - \Psi, \partial_t(\Psi^{(i)} - \Psi)\right)\right\|_{L^\infty([0,T];H^s(\mathbb{R})\times H^s(\mathbb{R}))\times L^\infty([0,T];H^{s-1}(\mathbb{R})\times H^{s-1}(\mathbb{R}))} \longrightarrow 0.$$

as $i \longrightarrow \infty$ for every $1 \leq s < 3$. Here Ψ is the solution arising from data (Ψ_0, Ψ_1) and $\Psi^{(i)}$ is the solution arising from data $\left(\Psi_0^{(i)}, \Psi_1^{(i)}\right)$.

Proof of Proposition 2.6.1. (1). This part of the Proposition is proved by Picard's iteration. Using a density argument it is sufficient to assume the initial data $(\Psi_0, \Psi_1) \in \mathcal{S}^4$ ($\mathcal{S}$ being the Schwartz class), along with condition (2.6.7). Define a sequence of smooth functions $\Psi^{(i)}$, with $i \geq 1$ such that

$$\Psi^{(1)} = (0,0),$$

and for $i \geq 2$, $\Psi^{(i)}$ is iteratively defined as the unique solution to the system

$$\begin{cases} \partial_\alpha(m^{\alpha\beta}\partial_\beta\Psi^{(i)}) = F(\Psi^{(i-1)}, \partial\Psi^{(i-1)}) \\ (\Psi^{(i)}, \partial_t\Psi^{(i)})|_{\{t=0\}} = (\Psi_0, \Psi_1) \in \mathcal{H}. \end{cases} \tag{2.6.8}$$

It is important to note that from (2.1.17) and (2.6.7) we can assure that for $j = 1, 2$,

$$\sum_{\gamma=0}^{2} \sup_{|x|,|p|\leq\frac{\lambda}{2}} |\partial_{x,p}^\gamma F_j|(x,p) \leq C_{j,\frac{1}{2}\lambda}. \tag{2.6.9}$$

Indeed, this can be seen from the fact that for $(x,p) = (x_1, x_2, p_1, p_2, p_3, p_4)$ and $|x| \leq \frac{\lambda}{2}$,

$$F_1(x,p) = 2\sinh(2\lambda + 2x_1)\left(p_4^2 - p_3^2\right), \quad F_2(x,p) = \frac{\sinh(2(\lambda + x_1))}{\sinh^2(\lambda + x_1)}\left(p_3p_1 - p_2p_4\right).$$

Define bounded functions in the class C^1.

It is important to note that condition (2.6.9) allows this iterative definition of the functions $\Psi^{(i)}$ to be possible, since it maintains each component of F with the required regularity, see [106]. First, it will be shown that for a sufficiently small $T > 0$, the sequence $(\Psi, \partial_t\Psi)$ is uniformly (in i) bounded in $L^\infty([0,T];\hat{\mathcal{H}})$, then it will be shown that it is also a Cauchy sequence. For the first part, the idea is to use the energy estimates (2.3.8), we want to prove that there is a constant $0 < A \leq \frac{\lambda}{2}$ such that

$$\left\|\left(\Psi^{(i-1)}, \partial_t \Psi^{(i-1)}\right)\right\|_{L^\infty([0,T];\hat{\mathcal{H}})} \leq A, \tag{2.6.10}$$

implies that

$$\left\|\left(\Psi^{(i)}, \partial_t \Psi^{(i)}\right)\right\|_{L^\infty([0,T];\hat{\mathcal{H}})} \leq A.$$

The energy estimation (2.3.8) allows us to write for (2.6.5) the following estimate:

$$\begin{aligned}\sup_{t\in[0,T]} \left\|\left(\Psi^{(i)}, \partial_t \Psi^{(i)}\right)\right\|_{\hat{\mathcal{H}}} &\leq C(1+T)(\|(\Psi_0,\Psi_1)\|_{\hat{\mathcal{H}}}) \\ &+C(1+T)\int_0^T \left(\left\|F_1\left(\Psi^{(i-1)}, \partial\Psi^{(i-1)}\right)\right\|_{H^2(\mathbb{R})} + \left\|F_2\left(\Psi^{(i-1)}, \partial\Psi^{(i-1)}\right)\right\|_{H^2(\mathbb{R})}\right)(t)dt.\end{aligned} \tag{2.6.11}$$

With this estimate, our goal is to bound the integral on the right hand side of the inequality above. That is, we want to prove that there exists $B = B(A,F) > 0$ such that for $t \in [0,T]$, we have

$$\sum_{n=0}^{2} ||\partial_x^n F(\Psi^{(i-1)}, \partial\Psi^{(i-1)})||_{L^2}(t) \leq B. \tag{2.6.12}$$

For this, we will use the conditions (2.6.7) for each F_j which is satisfied by the hypothesis in (2.6.10), if $B_1 = \max\{C_{1,\frac{\lambda}{2}}, C_{2,\frac{\lambda}{2}}\}$, and using chain rule we get

$$\begin{aligned}\sum_{n=0}^{2} ||\partial_x^n F(\Psi^{(i-1)}, \partial\Psi^{(i-1)})||_{L^2} &\leq B_1 + B_1||\partial_x\Psi^{i-1}||_{L^2} + B_1||\partial\partial_x\Psi^{i-1}||_{L^2} + B_1||\partial\Psi^{i-1}||^2_{H^2} \\ &+ B_1||\partial_x^2\Psi^{i-1}||_{L^2} + B_1||\partial\partial_x\Psi^{i-1}\cdot\partial\partial_x\Psi^{i-1}||_{L^2} + ||\partial\partial_x^2\Psi^{i-1}||_{L^2} \\ &\leq B,\end{aligned}$$

where $B = B(B_1, A, \lambda)$, which results in the following estimate

$$\sup_{t\in[0,T]} \left\|\left(\Psi^{(i)}, \partial_t \Psi^{(i)}\right)\right\|_{\hat{\mathcal{H}}} \leq C(1+T)\left(\|(\Psi_0,\Psi_1)\|_{\hat{\mathcal{H}}} + 2BT\right), \tag{2.6.13}$$

we can choose $T > 0$ sufficiently small such that

$$2BT \leq \|(\Psi_0,\Psi_1)\|_{\hat{\mathcal{H}}},$$

so

$$\left\|\left(\Psi^{(i)}, \partial_t \Psi^{(i)}\right)\right\|_{L^\infty([0,T];\hat{\mathcal{H}})} \leq 2C\,\|(\Psi_0,\Psi_1)\|_{\hat{\mathcal{H}}}.$$

If we choose $D > 4C$ in (2.6.7) and $A := 2C||(\Psi_0,\Psi_1)||_{\hat{\mathcal{H}}} \leq \frac{2C\lambda}{D} \leq \frac{\lambda}{2}$. We have thus shown the desired implication.

In the Section 2.2 we showed that the last sequence is of Cauchy type in the larger space $L^\infty([0,T];\mathcal{H})$. Therefore, the sequence is Cauchy on $L^\infty([0,T];\mathcal{H})$, and hence convergent. That is, there exists $(\Psi, \partial_t\Psi)$ in $L^\infty([0,T];\mathcal{H})$. The uniform bounds (on i) in $L^\infty([0,T],\hat{\mathcal{H}})$ guarantees that the limit in fact lies in the smaller space $L^\infty([0,T],\hat{\mathcal{H}})$, that is, for almost

$t \in [0,T]$, $(\Psi^{(i)}, \partial_t \Psi^{(i)})(t) \in \hat{\mathcal{H}}$, uniform in i, and therefore by Banach-Alaoglu's Theorem, there is a weak limit in $\hat{\mathcal{H}}$ (up to a subsequence). But the uniqueness of the limit ensures that this limit must agree with $(\Psi, \partial_t \Psi)(t)$. This concludes the proof of existence.

Finally, for the continuous dependence on initial data, we prove in the Section 2.2 that taking $i \longrightarrow \infty$, we get

$$\sup_{s\in[0,t]} \left\| \left(\Psi^{(i)} - \Psi, \partial_t \Psi^{(i)} - \partial_t \Psi \right) \right\|_{\mathcal{H}} \longrightarrow 0.$$

To obtain the result in general for $1 \leq s < 3$, simply observe that

$$\begin{aligned} &\sup_{t\in[0,T]} ||(\Psi^{(i)} - \Psi, \partial_t \Psi^{(i)} - \partial_t \Psi)||_{H^s \times H^s \times H^{s-1} \times H^{s-1}}(t) \\ &\quad \leq C \sup_{t\in[0,T]} \left(||(\Psi^{(i)} - \Psi, \partial_t \Psi^{(i)} - \partial_t \Psi)||_{H^1 \times H^1 \times L^2 \times L^2}(t) \right)^{\frac{3-s}{2}} \\ &\qquad \times \left(||(\Psi^{(i)} - \Psi, \partial_t \Psi^{(i)} - \partial_t \Psi)||_{H^3 \times H^3 \times H^1 \times H^1}(t) \right)^{\frac{s-1}{2}} \longrightarrow 0. \end{aligned}$$

This last property ends the proof of Proposition 2.1.1. $\square$

Chapter 3

Global existence and long time behavior in Einstein-Belinski-Zakharov Soliton spacetimes

Abstract: We consider the vacuum Einstein field equations under the Belinski-Zakharov symmetries. Depending on the chosen signature of the metric, these spacetimes contain most of the well-known special solutions in General Relativity, including well-known black holes. In this paper, we prove global existence of small Belinski-Zakharov spacetimes under a natural nondegeneracy condition. We also construct new energies and virial functionals to provide a description of the energy decay of smooth global cosmological metrics inside the light cone. Finally, some applications are presented in the case of generalized Kasner solitons.

This chapter is contained in: C. Muñoz and J. Trespalacios, *Global Existence and Long Time Behavior in Einstein-Belinski-Zakharov Soliton Spacetimes*, arXiv:2305.01414. Submitted 2024.

Contents

3.1 Introduction and main result

The *Einstein vacuum* equation determines a 4−dimensional manifold $\mathcal{M}$ with a Lorentzian metric $\tilde{g}$ with vanishing *Ricci* curvature

$$\mathrm{R}_{\mu\nu}(\widetilde{g}) = 0. \tag{3.1.1}$$

These equations can be written under certain gauge choices as a difficult system of quasilinear equations. This is a remarkable aspect of the general relativity theory, in contrast to Newton gravitation theory: the equation (3.1.1) is non-trivial even in the absence of matter. The focus of this paper is the understanding of outstanding solutions of (3.1.1) in the setting of Belinski-Zakharov spacetimes.

Salam and Strathdee [101] discussed black holes as possible solitons. Belinski and Zakharov [7, 9] (see also Kompaneets [71] and [117, 118]) proposed an application for the Inverse Scattering Transform for spacetimes that admit two commuting Killing vector fields. Using this ansatz Einstein's vacuum field equations can be recast as a 1+1 system of four *quasilinear wave equations*. In this paper we will follow their ansatz and describe rigorously symmetric spacetimes and their long time dynamics. Symmetry has been a successful method for understanding complicated dynamics in a series of works related to dispersive models, see e.g. [34, 45, 104] and references therein.

3.1.1 The Belinski-Zakharov Integrability ansatz

Belinski and Zakharov recalled the particular case in which the metric tensor $\widetilde{g}_{\mu\nu}$ depends on two variables only, which correspond to spacetimes that admit two commuting Killing vector fields, i.e. an Abelian two-parameter group of isometries. This assumption allowed them to propose the so-called Belinski-Zakharov transform to obtain solitonic solutions. Gravisolitons have an unusual number of features, however, it is known that spacetimes highly important in physics and cosmology applications, such as Kasner spacetimes, can be identified as gravisolitons [8, 9].

We will choose here a metric tensor depending on a time-like coordinate x^0, and one space-like coordinate x^1 (possibly nonnegative). This choice, as will stay clear below, corresponds to considering non-stationary gravitational fields, often referred to as Gowdy models [49], even when no compact spatial sections are considered. They are also often mentioned as generalized Einstein-Rosen spacetimes [17]. In the particular case where one has diagonal metrics these are called Einsten-Rosen spacetimes, first considered in 1937 by Einstein and Rosen [34].

Set $x^0 = t$ and $x^1 = x$. In this case the coordinates are typically expressed using Cartesian coordinates in which $x^a, x^b \in \{y, z\}$, where the Latin indexes $a, b \in \{2, 3\}$. Then the spacetime interval is a simplified block diagonal form: $x^a, x^b \in \{y, z\}$, where the Latin indexes $a, b \in \{2, 3\}$. Then the spacetime interval is a simplified block diagonal form:

$$ds^2 = f(t,x)(dx^2 - dt^2) + g_{ab}(t,x)dx^a dx^b. \tag{3.1.2}$$

Recall that repeated indexes mean sum, following the classical Einstein convention. Here with a slight abuse of notation we shall denote $g = g_{ab}$. Due to the axioms of general relativity the tensor g must be real and symmetric.

It is important to recall that the structure of the metric (3.1.2) is not restrictive, since, from the physical point of view, we find many applications that can be described according to (3.1.2). Such spacetimes describe cosmological solutions of general relativity, gravitational waves and their interactions [8]. Among them one can find

- classical solutions of the Robinson-Bondi plane waves [15],
- the Einstein-Rosen cylindrical wave solutions and their two polarization generalizations [17, 34],
- the homogeneous cosmological models of Bianchi types I–VII including the Kasner model [60],
- (in the "static" setting) the Schwarzschild and Kerr solutions, and Weyl axisymmetric solutions, see Section 8.3 in [8],
- 2-solitons, corresponding in a particular case to the Kerr-NUT (Newman-Unti-Tamburino) black-hole solution of three parameters including Kerr, Schwartzschild and Taub-NUT metrics [108].

For additional bibliography the reader may consult [74, 78, 79] and references therein. All this shows that, despite its relative simplicity, a metric of the type (3.1.2) encompasses a wide variety of physically relevant compact objects. Additionally, Belinski-Zakharov metrics contain the so-called Gowdy spacetimes [49, 89], where the initial topology differs from our setting. See also Section 4.1 in [8] for a deeper discussion.

In order to reduce Einstein vacuum equations (3.1.1), one needs to compute the Ricci curvature tensor in terms of the components of the metric $g = g_{ab}$. The consideration of the metric in the form (3.1.2) leads to components R_{0a} and R_{3a} of the Ricci tensor that are identically zero. Therefore, one can see that Einstein vacuum equations (3.1.1) decompose into two sets of equations. The first one follows from $\mathrm{R}_{ab} = 0$; this equation can be written as the single tensor equation

$$\partial_t\left(\alpha\partial_t g g^{-1}\right) - \partial_x\left(\alpha\partial_x g g^{-1}\right) = 0, \quad \det g = \alpha^2. \tag{3.1.3}$$

We shall refer to this equation as the *reduced Einstein equation.* The trace of the equation (3.1.3) reads

$$\partial_t^2\alpha - \partial_x^2\alpha = 0. \tag{3.1.4}$$

Therefore, the function $\alpha(t,x)$ satisfies the 1D wave equation. These equations may be recast as equivalent to the "dynamical part" of the Einstein equations. The second set of equations expresses the metric coefficient $f(t,x)$ in terms of explicit terms of α and g, where $\det \widetilde{g}_{\mu\nu} := -f^2\alpha^2$.

Geometrical coordinates

The fact that the 2×2 tensor g is symmetric allows one to diagonalize it for fixed t and x. One writes $g = RDR^T$, where D is a diagonal tensor and R is a rotation tensor, of the form

$$D = \begin{pmatrix} \alpha e^{\Lambda} & 0 \\ 0 & \alpha e^{-\Lambda} \end{pmatrix}, \qquad R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{3.1.5}$$

Clearly

$$\det g = \alpha^2. \tag{3.1.6}$$

Here Λ is the scalar field that determines the eingenvalues of g, and the scalar field ϕ determines the deviation of g from being a diagonal tensor. Since ϕ is considered as an angle, we assume $\phi \in [0, 2\pi]$. Therefore Λ, ϕ and α in (3.1.5) can be considered as the three degrees of freedom in the symmetric tensor g, [52]. Written explicitly, the tensor g is given now by

$$g = \alpha \begin{pmatrix} \cosh\Lambda + \cos 2\phi \sinh\Lambda & \sin 2\phi \sinh\Lambda \\ \sin 2\phi \sinh\Lambda & \cosh\Lambda - \cos 2\phi \sinh\Lambda \end{pmatrix}. \tag{3.1.7}$$

Some analog representations have been used in various associated results, for example in the Einstein-Rosen metric [17]. Note that Minkowski $\widetilde{g}_{\mu\nu} = (-1, 1, 1, 1)$ can be recovered by taking $\Lambda = 0$, $\alpha = 1$ and ϕ free. The equation (3.1.3) reads now

$$\begin{cases} \partial_t(\alpha\partial_t\Lambda) - \partial_x(\alpha\partial_x\Lambda) = 2\alpha\sinh 2\Lambda((\partial_t\phi)^2 - (\partial_x\phi)^2), \\ \partial_t(\alpha\sinh^2\Lambda\partial_t\phi) - \partial_x(\alpha\sinh^2\Lambda\partial_x\phi) = 0, \\ \partial_t^2\alpha - \partial_x^2\alpha = 0, \\ \partial_t^2(\ln f) - \partial_x^2(\ln f) = G, \end{cases} \tag{3.1.8}$$

where $G = G[\Lambda, \phi, \alpha]$ is given by

$$\begin{aligned} G := & -\big(\partial_t^2(\ln\alpha) - \partial_x^2(\ln\alpha)\big) - \frac{1}{2\alpha^2}((\partial_t\alpha)^2 - (\partial_x\alpha)^2) \\ & - \frac{1}{2}((\partial_t\Lambda)^2 - (\partial_x\Lambda)^2) - 2\sinh^2\Lambda((\partial_t\phi)^2 - (\partial_x\phi)^2). \end{aligned} \tag{3.1.9}$$

Note that the equation for $\alpha(t,x)$ is the standard one dimensional wave equation, and can be solved independently of the other variables. Also, given $\alpha(t,x)$, $\Lambda(t,x)$ and $\phi(t,x)$, solving for $\ln f(t,x)$ reduces to use D'Alembert formula for linear one dimensional wave with nonzero source term. Consequently, the only nontrivial equations in (3.1.8) are given by the equations for $\Lambda(t,x)$ and $\phi(t,x)$, for α solution to linear 1D wave.

As one can see from (3.1.8), solutions are not unique. These fields satisfy the gauge invariance

$$\begin{aligned} &(\Lambda, \phi, \alpha, f) \quad \text{solution, then} \\ &(\Lambda, \phi + k\pi, C_1\alpha, C_2 f) \quad \text{is also solution,} \qquad k \in \mathbb{Z}, \quad C_1, C_2 > 0. \end{aligned} \tag{3.1.10}$$

Since $\alpha \mapsto C_1\alpha$ is just a conformal transformation in (3.1.7), with no loss of regularity we can always assume $C_1 = C_2 = 1$ in (3.1.10). It should be noted that although (3.1.8) are strictly non-linear in the fields $\Lambda(t,x)$, $\phi(t,x)$, $\alpha(t,x)$ and $f(t,x)$, it shares many similarities with the classical linear wave and Born-Infeld equations [1]: given any $\mathcal{C}^2$ real-valued profiles $h(s), k(s), \ell(s), m(s)$, then the following functions are solutions for (3.1.8):

$$\begin{aligned} \Lambda(t,x) &= h(x \pm t), \qquad & \phi(t,x) &= k(x \pm t), \\ \alpha(t,x) &= \ell(x \pm t), \qquad & f(t,x) &= m(x \pm t). \end{aligned}$$

This property will be key when establishing the connection between the local theory that will be presented in the following section and the analysis of explicit solutions to the equation in the Section 3.6.

Coming back to our problem, and using Inverse Scattering techniques, Belinski and Zakharov [9] considered (3.1.3) giving first foundational results, see also [119]. They proposed the application of the Inverse Scattering method to the equations of general relativity and the procedure of calculating exact solitonic solutions of the equation. They introduce a Lax-pair for (3.1.3)-(3.1.4), together with a general method for solving it. Localized and multi-coherent structures were found, but they are not solitons in the standard sense, unless α is constant. A more in-depth study on the subject is also made in [8, 9]. More recently, Hadad [52] explored the Belinski-Zakharov transformation for the 1+1 dimensional setting, obtaining explicit formulae for solutions constructed on arbitrary diagonal backgrounds, in particular on the Einstein-Rosen background. With the detection of gravitational waves obtained by the twin LIGO interferometers and their description as a merger of two black holes, the study of gravitational soliton dynamics has gained huge importance. It should be noted that the class of gravitational soliton solutions, as mentioned above, includes cosmological solutions which describe non-homogeneous cosmological models, i.e. waves propagating with subluminal velocity.

The local behavior of the spacetime described before is defined by the function α. In our setting, α will be an always positive and bounded function. These characteristics will be provided by the initial conditions that will be imposed on the problem. The gradient of the function $\alpha(t,x)$ can be *timelike, spacelike* or *null*. The case where α is spacelike everywhere in spacetime ($(\partial_x\alpha)^2 - (\partial_t\alpha)^2 > 0$) corresponds to spacetimes said "with cylindrical symmetry", which corresponds to the Einstein Rosen spacetime, for example. They give an approach to the description of gravitational waves. When the gradient of α is globally null, ($(\partial_x\alpha)^2 - (\partial_t\alpha)^2 = 0$), it corresponds to the plane-symmetric waves. Finally, the last case, when the gradient of α is globally timelike ($(\partial_x\alpha)^2 - (\partial_t\alpha)^2 < 0$) is used to describe cosmological models and colliding gravitational waves, see [6, 8, 17, 34]. It will be precisely the *timelike* case the focus in this work. This classification for the gradient of the function α is necessary in order to propose an appropriate definition of energy and to be capable of giving a description of the decay of the solution associated with the system.

In a previous work [109] one of us considered the case when α is a constant function. Such consideration simplified the system (3.1.8) and identified it with the Principal Chiral Field model (PCF). This approach allowed us to give a first global existence result and local decay in space. It should be noted that, in the case of constant α, the results obtained cannot be extrapolated to the case of the Einstein equation in vacuum since essentially PCF is not exactly the case $\alpha = const.$ in (3.1.8), but instead one has to completely eliminate the equation for f. A different situation is obtained when considering the case in which $\alpha(t,x)$ is a more general function; in this case, the results are completely identifiable with the Einstein equation, so it automatically becomes a more interesting and complex problem to analyze. Unfortunately we are forced to consider only half of the α axis because, in general, the points $\alpha = 0$ correspond to the physical singularity through which the metric cannot be extended [6, 8].

The study of hyperbolic nonlinear differential equations has been developed enormously since the early 1980s, following the pioneering work of F. John, D. Christodoulou, L. Hormander, S. Klainerman, and others. Much of the effort was focused on understanding the global existence and blow-up for quasilinear wave equations or systems. An overview of the main results can be found in [55]. Furthermore, a description from the geometrical analysis is presented in [5, 106], where the stability results of the Minkowski space, demonstrated by Christodoulou and Klainerman [24], are explained. It is also described how these results meant the starting point for the mathematical development of the framework of general relativity.

In the particular setting of $\mathbb{R}^{1+3}$, the nonlinear wave equation with *null condition*[1] has been intensively studied, and many deep applications in physics and geometry have been found. Klainerman, in his seminal work [63], introduces the celebrated null condition. Using an approach subject to suitable small initial data, he constructs global solutions for the problem, setting a trend and line of work in that direction. Christodoulou [23] also showed independently the existence of smooth solutions to the nonlinear wave equation with small initial data. It should be noted that the null condition is a sufficient, but not necessary condition for global existence, see for example [83, 84]. Alinhac in [3] showed global existence for small initial data in two dimensions in space, conditioning with a more restrictive null form on nonlinearity.

Global small solutions in 1+1 dimensions may not exist in general [43, 44]. In particular, in $\mathbb{R}^{1+1}$ we have an added difficulty, since waves do not decay in the same way in higher dimensions. However, the special structure in the nonlinearity can give rise to important results related to the asymptotic behavior of solutions, as in the case of the wave map [20]. In a recent paper, Luli et. al. [85] used weighted estimates for linear waves in $\mathbb{R}^{1+1}$, and the null condition, to construct global solutions for the associated nonlinear equation. These energy estimates allowed them to improve the decay on the null form.

3.1.2 Main results

Our first result in this paper is the global existence of solutions. For (3.1.8) we consider constraints on the initial conditions for $\alpha(t,x)$. Using the D'Alembert formula we have an explicit expression for α that allows us to obtain tight control over appropriate terms by also using the central structure related to null forms. Although the nonlinearity is not purely defined in terms of null forms, we can follow and adapt properly in the case of variable coefficients the weighted energy estimates proposed in [85] to approach the problem and finally obtain a small data global existence result for (3.1.8).

[1]For the forthcoming analysis it is convenient to introduce a fundamental null form, which is defined as the following bilinear form:

$$Q_0(\phi, \tilde{\Lambda}) = m^{\alpha\beta}\partial_\alpha\phi\partial_\beta\tilde{\Lambda},$$

where $m_{\alpha\beta}$ to denote the standard Minkowski metric on $\mathbb{R}^{1+1}$.

Theorem 3.1.1 (Small data global existence)**.** *Let $\lambda > 0, c_1 > 0$ be fixed, and set*

$$\Lambda =: \lambda + \tilde{\Lambda}, \quad \textit{and} \quad \alpha := 1 + \tilde{\alpha}. \tag{3.1.11}$$

Consider the wave system (3.1.8) *posed in $\mathbb{R}^{1+1}$, with the following initial conditions:*

$$\textit{(IC)} \quad \begin{cases} (\phi, \tilde{\Lambda}, \alpha, f)|_{\{t=0\}} = (\varepsilon\phi_0, \varepsilon\tilde{\Lambda}_0, 1+\tilde{\alpha}_0, c_1 + f_0), \\ (\partial_t\phi, \partial_t\tilde{\Lambda}, \partial_t\alpha, \partial_t f)|_{\{t=0\}} = (\varepsilon\phi_1, \varepsilon\tilde{\Lambda}_1, \alpha_1, f_1), \\ (\phi_0, \tilde{\Lambda}_0, \tilde{\alpha}_0, f_0) \in (C_c^\infty(\mathbb{R}))^4, \\ (\phi_1, \tilde{\Lambda}_1, \alpha_1, f_1) \in C_c^\infty(\mathbb{R}) \times C_c^\infty(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}). \end{cases} \tag{3.1.12}$$

Assume the following bounds on the initial conditions:

1. $\alpha_1(\cdot) > 0$,
2. $\max_{n=0,1,2}\left(\|\partial_x^{(n)}\tilde{\alpha}_0\|_\infty + \|\partial_x^{(n)}\alpha_1\|_\infty\right) < \frac{1}{2}\gamma$, *where γ is a fixed sufficiently small constant, but independent on ε.*
3. $||f_0||_\infty \leq \frac{c_1}{2}$,
4. *the initial data satisfy the compatibility conditions required by Einstein's field equations.*

Then, there exists ε_0 sufficiently small such that if $\varepsilon < \varepsilon_0$, the unique solution remains smooth for all time.

***Remark* 3.1.1.** *Note that the conditions on α_1 and f are less demanding than the ones required for α_0. Indeed, one only needs data in the Schwartz class $S(\mathbb{R})$ and compact support is not necessary; this will be useful in some applications.*

Recall that α is solution to the linear wave equation in 1D but *far from zero*. Along the paper we will see that this condition is necessary and natural in view of (3.1.6). Consequently, one only expects decay in the $\dot{H}^1 \times L^2$ norm, precisely as in [1]. A direct consequence of Theorem 3.1.1 is the global existence of the Belinski-Zakharov metric (3.1.2):

Corollary 3.1.1. *Under the assumptions in Theorem 3.1.1, g and f in* (3.1.2) *are globally well-defined.*

***Remark* 3.1.2.** *In view of the fact that the Einstein equations are a geometrical problem, it is important to note that, in order to formulate the equations in general relativity as a initial value problem, the initial data must satisfy compatibility conditions known as Einstein constraint equations. Yvonne Choquet-Bruhat in her pioneering work [21] proved the local existence and uniqueness for the Einstein equations in vacuum when given a set of initial data $(\Sigma_t, \mathring{g}, \mathring{k})$ where Σ_t is a spacelike hypersurface of $\mathcal{M}$, $\mathring{g}$ a Riemannian metric on Σ_t and $\mathring{k}$ the associated second fundamental form. The result is valid when $\mathring{g}$ and $\mathring{k}$ satisfy the so-called constraint equations, which are geometric conditions on the problem, see also [22] to help understand the importance and complexity of the problem.*

The Einstein constraint equations constitute a problem of great interest since from them emerge a nontrivial system of elliptic equations, which has been studied from different interesting fronts. Huneau in [57] obtained the existence of solutions for these compatibility *conditions, assuming the existence of a translational spacelike Killing field in the asymptotically flat case, this hypothesis allowed her to move from a 3+1 dimensional problem to a 2+1 dimensional one and from that, rewrite the Einstein constraint equations in a suitable form. Also, Premoselli [97], using conformal method, obtains an admissible initial data for the conformal Einstein-scalar constraint system. Recently Fournoduvlos et. al. [42], assumes a constant main curvature condition on the hypersurface, to study the development of singularities for a generalized Kasner metric. In this work we assume that the initial conditions associated with the fields Λ, α, ϕ and f allow us to describe a spatial metric $\mathring{g}$ on the hypersurface associated with $t = 0$, such that the Einstein constraint equations are satisfied. It is of independent interest to make a description and in-depth study of the constraint equations for the specific cases of metrics that are identified with the formalism proposed by Belinski and Zakharov.*

***Remark* 3.1.3.** *The problem of global existence under general data size is delicate. Indeed,* (3.1.8) *is clearly singular in the case where Λ reaches the zero value. Also, the global structure of the solution in this case seems not clear unless one has further assumptions on the initial data. These geometric problems will be considered in a forthcoming publication.*

The second result in this work concerns the decay of a specific type of solutions of the Einstein equations in the vacuum. Specifically of cosmological type solutions, which are of special interest in physics and cosmology. This type of solutions include the Kasner type spacetimes, as well as some Bianchi type models, see [8]. We will prove, using well-chosen virial estimates that for solutions to (3.1.8) with finite energy (in particular, globally defined small solutions from Theorem 3.1.1), they must decay to zero locally in space, provided that the gradient of the function $\alpha(t,x)$ is globally timelike.

Indeed, virial functionals can describe in great generality the decay mechanism for models where standard scattering is not available, either because the dimension is too small, or the nonlinearity is long range, see e.g. [72, 73]. We will prove this result inspired by the results obtained for the Born-Infeld equation in 1+1 dimensions [1].

Before proving this result, we introduce the following modified energy of the system, which in the case of cosmological type solutions will be highly relevant (see Section 3.4):

$$E[\Lambda, \phi; \alpha](t) := -\int [\kappa \partial_t \alpha (h_1 - 2h_2)](t,x)dx, \tag{3.1.13}$$

where $\kappa(t,x) = \dfrac{\alpha}{(\partial_x \alpha)^2 - (\partial_t \alpha)^2}$,

$$h_1(t,x) = (\partial_t \Lambda)^2 + (\partial_x \Lambda)^2 + 4\sinh^2(\Lambda)((\partial_x \phi)^2 + (\partial_t \phi)^2), \tag{3.1.14}$$

and

$$h_2(t,x) = \partial_t \Lambda \partial_x \Lambda + 4\sinh^2(\Lambda)\partial_t \phi \partial_x \phi.$$

This (nonconserved) energy is a modified version of the one introduced by Hadad [52], which was not sufficiently useful in our purposes. Here (3.1.13) has important modifications to ensure the positivity of the energy functional. Compared with our previous results [109] in the case of the Principal Chiral Equation, here the energy and momentum terms require deeper understanding and much more work than before.

For this theorem we shall assume the cosmological condition

$$\begin{aligned}&\alpha(t,x)>0, \quad \partial_t\alpha(t,x)>0,\\ &(\partial_t\alpha)^2(t,x)-(\partial_x\alpha)^2(t,x)<0, \quad \forall (t,x)\in[0,\infty)\times\mathbb{R}.\end{aligned} \tag{3.1.15}$$

Theorem 3.1.2 (Existence of a modified energy)**.** *Let* $(\Lambda,\phi,\alpha)(t)$ *be a smooth solution of the system* (3.1.8) *such that* α *satisfies* (3.1.15). *Then the modified energy* $E[\Lambda,\phi;\alpha](t)$ *is well-defined and nonnegative.*

Recall that the existence of a suitable energy is one of the key elements needed to study long time behavior in Hamiltonian-type systems. In our setting, the energy E will not be preserved in time, but under suitable conditions, already satisfied by solutions in Theorem 3.1.1, it will be bounded in time. The following remark clarifies this point:

***Remark* 3.1.4** (On the cosmological type condition)**.** *Condition* (3.1.15) *is not empty. Indeed, in the case of small data as in Theorem 3.1.1, a sufficient condition to ensure* (3.1.15) *is that*

$$|\alpha_0'(x)|<\alpha_1(x), \quad \forall x\in\mathbb{R}.$$

This condition is in concordance with (3.1.12), *where* α_1 *has been chosen to belong to a not compactly supported space.*

Now we are ready to state the result that we consider the most important in this work.

Theorem 3.1.3 (Decay of cosmological finite-energy spacetimes)**.** *Under the hypotheses in Theorem 3.1.2, assume in addition that one has*

(a) bounded energy condition:

$$\sup_{t\geq 0} E[\Lambda,\phi;\alpha](t)<+\infty; \tag{3.1.16}$$

(b) for some $c_0>0$ *one has*

$$\alpha(t,x)>c_0 \quad \text{and} \quad \partial_t\alpha \text{ is in the Schwartz class uniformly in time.} \tag{3.1.17}$$

Then, for any $v\in\mathbb{R}$, $|v|<1$, *and* $\omega(t)=t(\log t)^{-2}$, *one has*

$$\lim_{t\longrightarrow+\infty}\int_{|x-vt|\lesssim\omega(t)}\Big[(\partial_t\Lambda)^2+(\partial_x\Lambda)^2+\sinh^2(\Lambda)\Big((\partial_t\phi)^2+(\partial_x\phi)^2\Big)\Big]\,dx=0. \tag{3.1.18}$$

***Remark* 3.1.5** (On the finite energy condition)**.** *Globally defined solutions obtained from Theorem 3.1.1 satisfy the finite energy condition* (3.1.16) *thanks to suitable weighted estimates. Moreover, they also satisfy* (3.1.17) *in the case where the first line in* (3.1.15) *is satisfied. In that sense, Theorem 3.1.3 is more general and might be satisfied by large solutions, as explained in Section 3.6 where applications to Kasner spacetimes are presented.*

A simple corollary in terms of the spacetime tensor g can be obtained:

Corollary 3.1.2. *Under the hypotheses in Theorem 3.1.3, one has that* g *in* (3.1.2) *satisfies*

$$\lim_{t\longrightarrow+\infty}\int_{|x-vt|\leq\omega(t)}\left((\partial_t\det g)^2+(\partial_x\det g)^2\right)(t,x)dx=0. \tag{3.1.19}$$

Vanishing property (3.1.19) can be understood as the manifestation that the spacetime is of cosmological type, and information propagates with the speed of light, supported on the light cone.

Aplications to gravisolitons. One of the motivations of Belinski and Zakharov was to show the existence of gravitational solitons (gravisolitons). From the mathematical point of view, these are of solitonic type, although they exhibit a number of features unusual in this type of solutions [8]. In this paper, we apply Theorem 3.1.1 and 3.1.3 to the cosmological 1-soliton obtained from a *nonsingular generalized Kasner metric*, see (3.6.2) and (3.6.9)-(3.6.10) for the explicit formulae. In particular, we shall prove (Corollaries 3.6.1 and 3.6.2):

Theorem 3.1.4. *The cosmological 1-soliton* (Λ, ϕ, α) *obtained from a nonsingular generalized Kasner metric of parameter* $d \geq 1$ *is globally defined under suitable small perturbations in the case where* α *satisfies the hypotheses of Theorem 3.1.1, and satisfies*

$$\lim_{t\to+\infty} \int_{|x-vt|\leq\omega(t)} \left[(\partial_t\Lambda)^2 + (\partial_x\Lambda)^2 + \sinh^2(\Lambda)\Big((\partial_t\phi)^2 + (\partial_x\phi)^2\Big)\right](t,x)dx = 0.$$

in the case where α *is of cosmological type and satisfies the hypotheses of Theorem 3.1.3. Moreover, it propagates with the speed of light.*

Notice that conditions in Theorem 3.1.4 are essentially only depending on α, and in some sense this function determines the final behavior of solutions. The generalized Kasner metric discussed in Theorem 3.1.4 avoids some undesirable bad behavior at the time origin, although we believe that standard Kasner metrics should satisfy a result similar to Theorem 3.1.4.

3.1.3 More results and future research

The study of Einstein's field equations has a long history of important developments. Choquet-Bruhat [21, 22] gave a foundational mathematical description of the evolution of initial data. A complete mathematical understanding of well-known black holes has taken many years. The stability of the Kerr black hole was recently obtained in a series of works by Klainerman, Szeftel and Giorgi [47, 64, 65, 66, 67]. In the case of the Schwarzschild black hole, Dafermos, Holzegel, Rodnianski and Taylor [28, 29, 30] showed codimensional stability and the asymptotic stability. Finally, Hintz and Vasy [54] proved nonlinear stability of Kerr under de Sitter gravity.

In the case of the Einstein equations, symmetries are very important (see e.g. the binary black hole merging LIGO simulations). Given the complexity of the Einstein equations, this is a natural form to approach otherwise untreatable problems. A particular result is the *strong cosmic censorship conjecture*, which states that for a generic initial data, the MGHD[2] is inextensible. In vaccuum, Ringström provided important results in the framework of the so-called Gowdy symmetry, see [99, 100]. Smulevici studied the same issue for T2-symmetric[3]

[2]Yvonne Choquet-Bruhat showed that it is possible to formulate the Einstein vacuum equations can be viewed as an initial value problem [21], and given the initial data there is a part of spacetime, the so-called maximum global hyperbolic development (MGHD), which is uniquely determined up to isometry.

[3]A spacetime (M, g) is said to be T^2-symmetric if the metric is invariant under the action of the Lie group T^2 and the group orbits are spatial. These solutions constitute a class of spacetimes admitting a torus action.

spacetimes with positive cosmological constant [105]. Gowdy spacetimes have also been considered as a model to study gravitational waves and mathematical cosmology [49]. The compatibility of the initial data with the conditions known as constraint equations is another important issue. Huneau et. al. considered spacetimes with a translational Killing vector, i.e. a symmetry with respect to one of the spatial coordinates [57].

Organization of this chapter

This chapter is organized as follows. Section 3.2 presents a summary of the local existence result for system (3.1.8), which relies, as in [109], on a particular energy estimate. In the Section 3.3 we prove the small initial data global existence result, namely Theorem 3.1.1. Section 3.4 is focused on presenting a formalism suitable for the energy and momentum densities for (3.1.8), in the particular case of cosmological type solutions. Then in Section 3.5 we present and prove the long term behavior result, Theorem 3.1.3. Finally, Section 3.6 is devoted to an application in the case of Kasner metrics.

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3.2 Local existence

Before presenting the proof of global existence for the system, it is important to make some remarks to convince us that we first have a theory of local existence for the system (3.1.8). The first thing that we need is to set the initial conditions for the one-dimensional wave equation for α, which allow us to obtain a bounded and positive solution of this equation. These conditions are not only needed to establish the local existence, but also to obtain the global existence and to be subsequently able to make an analysis of the long-term behavior of the corresponding finite energy solution, as we will see in the further sections. In order to develop the results related to the local theory for the nonlinear wave equation, let us write the function $\Lambda(t,x)$ in the form

$$\Lambda(t,x) := \lambda + \tilde{\Lambda}(t,x), \quad \lambda \neq 0. \tag{3.2.1}$$

Notice that this choice makes sense with the energy in (3.1.13), in the sense that $\Lambda \in \dot{H}^1$ and $\partial_t \Lambda \in L^2$. Without loss of generality, we assume $\lambda > 0$. We consider the following vector notation

$$\begin{cases} \Psi = \left(\tilde{\Lambda}, \phi\right), \quad \partial\Psi = \left(\partial_t\tilde{\Lambda}, \partial_x\tilde{\Lambda}, \partial_t\phi, \partial_x\phi\right), \\ |\partial\Psi|^2 = \left|\partial_t\tilde{\Lambda}\right|^2 + \left|\partial_x\tilde{\Lambda}\right|^2 + |\partial_t\phi|^2 + |\partial_x\phi|^2, \\ F(\Psi, \partial\Psi) = (F_1, F_2), \\ F_1(\Psi, \partial\Psi) := 2\sinh(2\lambda + 2\tilde{\Lambda})\left((\partial_x\phi)^2 - (\partial_t\phi)^2\right), \\ F_2(\Psi, \partial\Psi) := \dfrac{\sinh(2\lambda + 2\tilde{\Lambda})}{\sinh^2(\lambda + \tilde{\Lambda})}\left(\partial_t\phi\partial_t\tilde{\Lambda} - \partial_x\phi\partial_x\tilde{\Lambda}\right). \end{cases}$$

With this notation, the initial value problem for (3.1.8) can be studied by first focusing on the following initial-value problem for $(\Psi, \partial_t\Psi)$:

$$\begin{cases} \partial_\mu(m^{\mu\nu}\alpha\partial_\nu\Psi) = F(\Psi, \partial\Psi) \\ (\Psi, \partial_t\Psi)|_{\{t=0\}} = (\Psi_0, \Psi_1) \in \mathcal{H}. \end{cases} \tag{3.2.2}$$

Here $m^{\mu\nu}$ are the components of the Minkowski metric with $\mu, \nu \in \{0, 1\}$, and the function $\alpha := 1 + \tilde{\alpha}$, satisfies the following initial valued problem

$$\begin{cases} \partial_t^2\alpha - \partial_x^2\alpha = 0 \\ (\alpha, \partial_t\alpha)|_{t=0} = (1 + \tilde{\alpha}_0, \alpha_1) \\ \text{with} \quad (\tilde{\alpha}_0, \alpha_1) \in C_c^\infty(\mathbb{R}) \times \mathcal{S}(\mathbb{R}). \end{cases} \tag{3.2.3}$$

Assume the following bounds on the initial conditions in (3.2.3):

1. $\alpha_1(\cdot) > 0$,
2. $\max_{n=0,1,2}\left(\|\partial_x^{(n)}\tilde{\alpha}_0\|_\infty\right) + \max_{n=0,1}\left(\|\partial_x^{(n)}\alpha_1\|_\infty\right) < \frac{1}{2}\gamma$, where γ is a fixed sufficiently small constant, but independent on ε.

Notice that these conditions are already stated in Theorem 3.1.1. In addition, we will seek for solutions in the space

$$(\Psi, \partial_t\Psi) \in \mathcal{H} := H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}).$$

Notice that from (3.2.1), $\Lambda \in \dot{H}^1$. We are also going to impose the following condition on the initial data

$$\|(\Psi_0, \Psi_1)\|_{\mathcal{H}} \leq \frac{\lambda}{2D}. \tag{3.2.4}$$

where D is a suitable constant. In order to state a local existence result for the initial value problems (3.2.2), is important to recall the following result [106]:

Lemma 3.2.1. *Let $\psi : I \times \mathbb{R} \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$, be the solution of the initial value problem*

$$\begin{cases} \partial_\mu(a^{\mu\nu}\partial_\nu\psi) = f(t, x), \quad (t, x) \in I \times \mathbb{R}, \\ (\psi, \partial_t\psi)|_{\{t=0\}} = (\psi_0, \psi_1) \in H^k(\mathbb{R}) \times H^{k-1}(\mathbb{R}), \end{cases}$$

where k be a positive integer and a and all its derivatives (of all orders) are bounded in $[0,T]\times\mathbb{R}$. Then for some positive constant $C = C(k)$, the following energy estimate holds

$$\sup_{t\in[0,T]} \|(\psi,\partial_t\psi)\|_{\mathcal{H}} \leq C\left(\|(\psi_0,\psi_1)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} + \int_0^T \|f\|_{H^{k-1}(\mathbb{R})}(t)dt\right)\exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R})}(t)\right). \tag{3.2.5}$$

Now, we can propose the following result for the initial-value problem (3.2.2):

Proposition 3.2.1. *If (Ψ_0,Ψ_1) satisfies the condition (3.2.4) with an appropriate constant $D\geq 1$, then:*

(1) (Existence and uniqueness of local-in-time solutions). There exists

$$T = T\left(\left\|\left(\tilde{\Lambda}_0,\phi_0\right)\right\|_{H^1(\mathbb{R})\times H^1(\mathbb{R})}, \left\|\left(\tilde{\Lambda}_1,\phi_1\right)\right\|_{L^2(\mathbb{R})\times L^2(\mathbb{R})},\lambda\right) > 0,$$

such that there exists a (classical) solution Ψ to (3.2.2) with

$$(\Psi,\partial_t\Psi)\in L^\infty([0,T];\mathcal{H}).$$

Moreover, the solution is unique in this function space.

(2) (Continuous dependence on the initial data). Let $\Psi_0^{(i)},\Psi_1^{(i)}$ be sequence such that $\Psi_0^{(i)}\longrightarrow\Psi_0$ in $H^1(\mathbb{R})\times H^1(\mathbb{R})$ and $\Psi_1^{(i)}\longrightarrow\Psi_1$ in $L^2(\mathbb{R})\times L^2(\mathbb{R})$ as $i\longrightarrow\infty$. Then taking $T>0$ sufficiently small, we have

$$\left\|\left(\Psi^{(i)}-\Psi,\partial_t(\Psi^{(i)}-\Psi)\right)\right\|_{L^\infty([0,T];\mathcal{H})}\longrightarrow 0.$$

Here Ψ is the solution arising from data (Ψ_0,Ψ_1) and $\Psi^{(i)}$ is the solution arising from data $\left(\Psi_0^{(i)},\Psi_1^{(i)}\right)$.

Sketch of proof. The idea of the proof is standard in the literature (see e.g. Luk [86]), in this case we must identify the component of a in (3.2.5) con $a^{00} = a^{11} := \alpha(t,x)$, and $a^{01} = a^{10} = 0$. Then we can use the energy estimate (3.2.5). The rest of the proof, for this particular system, can be seen into detail in [109, Proposition 1], only an adaptation in the estimation of energy to be used is required. This ends the proof of Proposition 3.2.1. □

3.3 Proof of Global Existence – Theorem 3.1.1

3.3.1 Preliminaries

We begin this section by introducing some basic definitions, and certain important results that will be useful for describe our result. For full details on the notation and considered norms, see [5, 85] and our previous work [109]. We will use two coordinate systems: the standard Cartesian coordinates (t, x) and the null coordinates $(u, \underline{u})$:

$$u := \frac{t+x}{2}, \quad \underline{u} := \frac{t-x}{2}, \tag{3.3.1}$$

and consider the two null vector fields defined globally as

$$L = \partial_t + \partial_x, \quad \underline{L} = \partial_t - \partial_x. \tag{3.3.2}$$

In the same way as in [5, 85, 109] we consider the weight function φ defined as

$$\varphi(u) := (1 + |u|^2)^{1+\delta} \quad \text{with} \quad 0 < \delta < 1/3. \tag{3.3.3}$$

Recall that from initial conditions (3.1.12) we have $\alpha_0 := 1+\tilde{\alpha}_0$ and also have the following facts, which are easy to check:

(i) Since $\tilde{\alpha}_0 \in C_c^\infty(\mathbb{R})$ and $\alpha_1 \in \mathcal{S}(\mathbb{R})$, with $\alpha_1 > 0$, one has for some fixed constant $K_1, K_2 > 0$ such that,

$$||\tilde{\alpha}_0||_\infty < \frac{\gamma}{2}, \qquad |\alpha_0^{(n)}(2u)| \leq \frac{K_1\gamma}{\varphi^{3/4}(u)}, \quad n = 1, 2, \tag{3.3.4}$$

and

$$|\alpha_1^{(n)}(2u)| \leq \frac{K_2\gamma}{\varphi^{3/4}(u)}, \quad n = 0, 1. \tag{3.3.5}$$

(ii) Using the classical D'Alambert formula in the third equation in (3.1.8) which correspond to one-dimensional wave equation for α, we obtain :

$$\alpha(t,x) = \frac{1}{2}\left(2 + \tilde{\alpha}_0(2u) + \tilde{\alpha}_0(-2\underline{u}) + \int_0^{2u} \alpha_1(s)ds - \int_0^{-2\underline{u}} \alpha_1(s)ds\right), \tag{3.3.6}$$

(iii) Moreover, the derivatives of the function α will be described as:

$$\begin{cases} \partial_x\alpha = \frac{1}{2}\left(\tilde{\alpha}_0'(2u) + \tilde{\alpha}_0'(-2\underline{u}) + \alpha_1(2u) - \alpha_1(-2\underline{u})\right) \\ \partial_t\alpha = \frac{1}{2}\left(\tilde{\alpha}_0'(2u) - \tilde{\alpha}_0'(-2\underline{u}) + \alpha_1(2u) + \alpha_1(-2\underline{u})\right) \\ \partial_x^2\alpha = \frac{1}{2}\left(\tilde{\alpha}_0''(2u) + \tilde{\alpha}_0''(-2\underline{u}) + \alpha_1'(2u) - \alpha_1'(-2\underline{u})\right) \\ \partial_{t,x}\alpha = \frac{1}{2}\left(\tilde{\alpha}_0''(2u) - \tilde{\alpha}_0''(-2\underline{u}) + \alpha_1'(2u) + \alpha_1'(-2\underline{u})\right). \end{cases} \tag{3.3.7}$$

(iv) The following relations for the null vector field L and $\underline{L}$ hold:

$$\begin{aligned} |L(\ln\alpha)| &\leq |\partial_x\alpha + \partial_t\alpha| = |\tilde{\alpha}_0'(2u) + \alpha_1(2u)| \lesssim \frac{K_1\gamma}{\varphi^{3/4}(u)}. \\ |L(\partial_x(\ln\alpha))| &\leq \frac{1}{2}\left|\left(\tilde{\alpha}_0'(2u) + \tilde{\alpha}_0'(-2\underline{u}) + \alpha_1'(2u) - \alpha_1'(-2\underline{u})\right)\left(\tilde{\alpha}_0'(2u) + \alpha_1(2u)\right)\right| \\ &\quad + |\tilde{\alpha}_0''(2u) + \alpha_1'(2u)| \lesssim \frac{K_1\gamma}{\varphi^{3/4}(u)}. \\ |\underline{L}(\partial_x(\ln\alpha))| &\leq \frac{1}{2}\left|\left(\tilde{\alpha}_0'(2u) + \tilde{\alpha}_0'(-2\underline{u}) + \alpha_1'(2u) - \alpha_1'(-2\underline{u})\right)\left(\tilde{\alpha}_0'(2u) + \alpha_1(2u)\right)\right| \\ &\quad + |\tilde{\alpha}_0''(2u) + \alpha_1'(2u)| \lesssim \frac{K_1\gamma}{\varphi^{3/4}(u)}. \end{aligned} \tag{3.3.8}$$

From now on, we will consider the conformal Killing vector fields on $\mathbb{R}^{1+1}$ given by

$$(1+|u|^2)^{1+\delta}L, \quad (1+|\underline{u}^2|)^{1+\delta}\underline{L},$$

with $0<\delta 1/3$. We also consider the following integration regions:

- S_{t_0} denotes the following time slice in $\mathbb{R}^{1+1}$:

$$S_{t_0} := \{(t,x):\ t=t_0\}\,.$$

- D_{t_0} denotes the following region of spacetime

$$D_{t_o} := \{(t,x):\ 0\le t\le t_0\}\,, \quad D_{t_0} = \bigcup_{0\le t\le t_0} S_{t_0}.$$

The level sets of the functions u and $\underline{u}$ define two global null foliations of D_{t_0}. More precisely, given $t_0>0$, u_0 and $\underline{u}_0$, we define the rightward null curve segment $C_{t_0,\underline{u}_0}$ as:

$$C_{t_0,\underline{u}_0} := \left\{(t,x):\ u=\frac{t-x}{2}=\underline{u}_0,\ 0\le t\le t_0\right\}, \tag{3.3.9}$$

and the segment of the null curve to the left C_{t_0,u_0} as:

$$C_{t_0,u_0} := \left\{(t,x):\ u=\frac{t+x}{2}=u_0,\ 0\le t\le t_0\right\}. \tag{3.3.10}$$

The space time region D_{t_0} is foliated by $C_{t_0,\underline{u}_0}$ for $\underline{u}\in\mathbb{R}$, and by C_{t_0,u_0} for $u\in\mathbb{R}$.

Finally, we will consider the following energy estimate proposed in [4, 85] for the scalar linear wave equation $\Box\psi=\rho$ ($\tau\in[0,t]$ in $C_{t,\underline{u}}$ and $C_{t,u}$). There exists $C_0>0$ such that

$$\begin{aligned}
&\int_{S_t}\left[(1+|\underline{u}|^2)^{1+\delta}|\underline{L}\psi|^2+(1+|u|^2)^{1+\delta}|L\psi|^2\right]dx\\
&\quad+\sup_{u\in\mathbb{R}}\int_{C_{t,u}}(1+|\underline{u}|^2)^{1+\delta}|\underline{L}\psi|^2d\tau+\sup_{\underline{u}\in\mathbb{R}}\int_{C_{t,\underline{u}}}(1+|u|^2)^{1+\delta}|L\psi|^2d\tau\\
&\quad\le C_0\int_{S_0}\left[(1+|\underline{u}|^2)^{1+\delta}|\underline{L}\psi|^2+(1+|u|^2)^{1+\delta}|L\psi|^2\right]dx\\
&\qquad+C_0\iint_{D_t}\left[(1+|\underline{u}|^2)^{1+\delta}|\underline{L}\psi|+(1+|u|^2)^{1+\delta}|L\psi|\right]|\rho|d\tau dx.
\end{aligned} \tag{3.3.11}$$

3.3.2 Global existence for $(\tilde{\Lambda},\phi)$

Recall that α was already solved in (3.3.6) and from (3.1.8) $\ln f$ is completely determined if we know (Λ,ϕ). Now we state a modified version of the main theorem, written in the variables $(\tilde{\Lambda},\phi)$, introduced in (3.1.11).

For the forthcoming analysis it is convenient to introduce a fundamental null form, which is defined as the following bilinear form:

$$Q_0(\phi,\tilde{\Lambda}) = m^{\alpha\beta}\partial_\alpha\phi\partial_\beta\tilde{\Lambda},$$

where $m_{\alpha\beta}$ to denote the standard Minkowski metric on $\mathbb{R}^{1+1}$. Then, using this definition, one can rewrite the first two equations of the system (3.1.8) in terms of null forms as follows:

$$\begin{cases} \Box\tilde{\Lambda} = Q_0(\ln\alpha, \tilde{\Lambda}) - 2\sinh(2\lambda + 2\tilde{\Lambda})Q_0(\phi,\phi), \\ \Box\phi = Q_0(\ln\alpha, \phi) + \dfrac{\sinh(2\lambda + 2\tilde{\Lambda})}{\sinh^2(\lambda+\tilde{\Lambda})} Q_0(\phi, \tilde{\Lambda}). \end{cases} \tag{3.3.12}$$

It can be also noticed that the null structure is "quasi-preserved" after differentiating with respect to x, in the sense that

$$\partial_x Q_0(\phi,\tilde{\Lambda}) = Q_0(\partial_x\phi, \tilde{\Lambda}) + Q_0(\phi, \partial_x\tilde{\Lambda}). \tag{3.3.13}$$

Additionally, we have the following relation between the null form and the Killing vector fields L and $\underline{L}$

$$Q_0(\partial_x^p\phi, \partial_x^q\tilde{\Lambda}) \lesssim |L\partial_x^p\phi||\underline{L}\partial_x^q\tilde{\Lambda}| + |\underline{L}\partial_x^p\phi||L\partial_x^q\tilde{\Lambda}|, \tag{3.3.14}$$

where the implicit constant is independent of $(\tilde{\Lambda}, \phi)$.

Motivated by estimation (4.2.6) and [5, 85, 109], we define the space-time weighted energy norms, valid for $k = 0, 1$:

$$\begin{aligned} \mathcal{E}_k(t) &= \int_{S_t} \left[(1+|\underline{u}|^2)^{1+\delta}|\underline{L}\partial_x^k\tilde{\Lambda}|^2 + (1+|u|^2)^{1+\delta}|L\partial_x^k\tilde{\Lambda}|^2\right] dx, \\ \overline{\mathcal{E}}_k(t) &= \int_{S_t} \left[(1+|\underline{u}|^2)^{1+\delta}|\underline{L}\partial_x^k\phi|^2 + (1+|u|^2)^{1+\delta}|L\partial_x^k\phi|^2\right] dx, \\ \mathcal{F}_k(t) &= \sup_{u\in\mathbb{R}} \int_{C_{t,u}} (1+|\underline{u}|^2)^{1+\delta}\left|\underline{L}\partial_x^k\tilde{\Lambda}\right|^2 ds + \sup_{\underline{u}\in\mathbb{R}} \int_{C_{t,\underline{u}}} (1+|u|^2)^{1+\delta}|L\partial_x^k\tilde{\Lambda}|^2 ds, \\ \overline{\mathcal{F}}_k(t) &= \sup_{u\in\mathbb{R}} \int_{C_{t,u}} (1+|\underline{u}|^2)^{1+\delta}\left|\underline{L}\partial_x^k\phi\right|^2 ds + \sup_{\underline{u}\in\mathbb{R}} \int_{C_{t,\underline{u}}} (1+|u|^2)^{1+\delta}|L\partial_x^k\phi|^2 ds. \end{aligned} \tag{3.3.15}$$

Then, using (4.4.7) we define the total energy norms as follows:

$$\mathcal{E}(t) = \mathcal{E}_0(t) + \mathcal{E}_1(t).$$

Analogously one defines $\mathcal{F}(t)$, $\overline{\mathcal{E}}(t)$, and $\overline{\mathcal{F}}(t)$.

***Remark* 3.3.1.** *We note that if $t = 0$ then from* (3.3.9) *and* (3.3.10) *one has $\mathcal{F}(0) = \overline{\mathcal{F}}(0) = 0$. Also, for $\mathcal{E}(t)$ the initial data determines a constant C_1 so that*

$$\mathcal{E}(0) = C_1\varepsilon^2. \tag{3.3.16}$$

This exact bound will be used by the end of the proof of global existence, more specifically in (3.3.26).

We are now ready to state and to prove the main result of this section:

Theorem 3.3.1. *Under the assumptions in Theorem 3.1.1, the following are satisfied. Assume that the solution $(\tilde{\Lambda}, \phi)$ of the system* (3.3.12) *exists for $t \in [0, T^*]$ satisfying the bounds*

$$\mathcal{E}(t) + \mathcal{F}(t) \leq 6C_0C_1\varepsilon^2, \tag{3.3.17}$$

$$\overline{\mathcal{E}}(t) + \overline{\mathcal{F}}(t) \leq 6C_0\overline{C}_1\varepsilon^2, \tag{3.3.18}$$

and

$$\sup_{t\in[0,T^*]} \left\|\tilde{\Lambda}\right\|_{L^\infty(\mathbb{R})} \leq \frac{\lambda}{2}. \tag{3.3.19}$$

Then for all $t \in [0, T^*]$ there exists a universal constant ε_0 (independent of T^*) such that the previous estimates are improved for all $\varepsilon \leq \varepsilon_0$.

The previous result ensures that the solution $(\tilde{\Lambda}, \phi)$ constructed via an iterative method is global in time and satisfies the bounds (4.4.8)-(4.4.10), consequently we can finally conclude the proof of the Theorem 3.1.1.

3.3.3 Proof of Theorem 4.4.1

For simplicity, we work with the first equation of the system (3.3.12). An analogous study of the equation for the field ϕ shows the same outcome, proving that ϕ is also globally defined.

The proof is based on the bootstrap method; i.e., we will assume that the weighted energies $\mathcal{E}(t)$, $\mathcal{F}(t)$ are bounded by some particular constant. Then, we will show that the corresponding solution defined in $[0, T^*]$ decays. Since the initial data are small, this allows us to show that the weighted energies are bounded by some better constant. Thus, by continuity, we conclude that the weighted energy cannot grow to infinity in any finite time interval and therefore, using the local existence theorem, the solution exists for all time.

This procedure has been done before in several works, see e.g. [85, 109]. however, in this work we have several complications coming from the new wave field α, which has to be correctly estimated in order to preserve the wave-like character of the system (3.3.12).

Deriving the first equation of (3.3.12) and using (4.4.2) we obtain:

$$\Box\partial_x\tilde{\Lambda} = \rho_1 + \rho_2, \tag{3.3.20}$$

where

$$\begin{cases} \rho_1 := Q_0(\partial_x(\ln\alpha), \tilde{\Lambda}) + Q_0(\ln\alpha, \partial_x\tilde{\Lambda}), \\ \rho_2 := -2\Big[\sinh(2\lambda + 2\tilde{\Lambda})\left(Q_0(\partial_x\phi, \phi) + Q_0(\phi, \partial_x\phi)\right) + 2\partial_x\tilde{\Lambda}\cosh(2\lambda + 2\tilde{\Lambda})Q_0(\phi, \phi)\Big]. \end{cases} \tag{3.3.21}$$

We can see that the null structure is "quasi-preserved" after differentiating with respect to x. We will use a bootstrap argument as in the (3+1)-dimensional case [63]. Fix $\delta \in (0, 1)$. Under the assumptions (4.4.8)-(4.4.9)-(4.4.10) for all $t \in [0, T^*]$, we assume that the solution remains regular, to later show that these bounds are maintained, with a better constant.

Consider $k = 0, 1$. Using (4.2.6) on (3.3.12), with $\psi = \partial_x^k \tilde{\Lambda}$ and (4.4.11)-(4.4.12). Taking the sum over $k = 0, 1$, we obtain

$$\begin{aligned}
&\mathcal{E}(t) + \mathcal{F}(t) \leq 2C_0\mathcal{E}(0) \\
&\quad + 2C_0 \iint_{D_t} \Big((1+|\underline{u}|^2)^{1+\delta}|\underline{L}\tilde{\Lambda}| + (1+|u|^2)^{1+\delta}|L\tilde{\Lambda}|\Big) \Big|Q_0(\ln\alpha, \tilde{\Lambda})\Big|\, |Q_0(\phi,\phi)| \\
&\quad + 2C_0 \iint_{D_t} \Big((1+|\underline{u}|^2)^{1+\delta}|\underline{L}\tilde{\Lambda}| + (1+|u|^2)^{1+\delta}|L\tilde{\Lambda}|\Big) \Big|\sinh(2\lambda + 2\tilde{\Lambda})\Big|\, |Q_0(\phi,\phi)| \\
&\quad + 2C_0 \iint_{D_t} \Big((1+|\underline{u}|^2)^{1+\delta}|\underline{L}\partial_x\tilde{\Lambda}| + (1+|u|^2)^{1+\delta}|L\partial_x\tilde{\Lambda}|\Big) |\rho_1| \\
&\quad + 2C_0 \iint_{D_t} \Big((1+|\underline{u}|^2)^{1+\delta}|\underline{L}\partial_x\tilde{\Lambda}| + (1+|u|^2)^{1+\delta}|L\partial_x\tilde{\Lambda}|\Big) |\rho_2| =: \sum_{j=0}^{5} A_j.
\end{aligned} \tag{3.3.22}$$

In the framework of the energy integrals already established, and given the symmetry of the terms, it is sufficient to establish the control of the terms $A_1 + A_3$ in (4.4.14), as follows:

$$\begin{aligned}
A_1 + A_3 = &\overbrace{\iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}||Q_0(\ln\alpha, \tilde{\Lambda})|}^{I_1} + \overbrace{\iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}||Q_0(\partial_x(\ln\alpha), \tilde{\Lambda}) + Q_0(\ln\alpha, \partial_x\tilde{\Lambda})|}^{I_2} \\
&+ \underbrace{\iint_{D_t} \varphi(u)|L\tilde{\Lambda}||Q_0(\ln\alpha, \tilde{\Lambda})|}_{I_3} + \underbrace{\iint_{D_t} \varphi(u)|L\tilde{\Lambda}||Q_0(\partial_x(\ln\alpha), \tilde{\Lambda}) + Q_0(\ln\alpha, \partial_x\tilde{\Lambda})|}_{I_4}.
\end{aligned} \tag{3.3.23}$$

Let us start with the integral I_1 in the term below, using (4.4.3) we get:

$$\begin{aligned}
I_1 &:= \iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}||Q_0(\ln\alpha, \tilde{\Lambda})| \\
&\lesssim \iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}|[|L(\ln\alpha)||\underline{L}\tilde{\Lambda}| + |\underline{L}(\ln\alpha)||L\tilde{\Lambda}|] =: I_{1,1} + I_{1,2}.
\end{aligned} \tag{3.3.24}$$

We will analyze into detail each part in this integral. For this, we recall the following result due to Luli et. al. in [85]:

Lemma 3.3.1 ([85], Lemma 3.2)**.** *Under assumptions* (4.4.8) *and* (4.4.9)*, there exists a universal constant* $C_2 > 0$ *such that:*

$$|L\tilde{\Lambda}(t,x)| \leq \frac{C_2\varepsilon}{(1+|u|^2)^{1/2+\delta/2}}, \qquad |L\phi(t,x)| \leq \frac{C_2\varepsilon}{(1+|u|^2)^{1/2+\delta/2}},$$
$$|\underline{L}\tilde{\Lambda}(t,x)| \leq \frac{C_2\varepsilon}{(1+|\underline{u}|^2)^{1/2+\delta/2}}, \qquad |\underline{L}\phi(t,x)| \leq \frac{C_2\varepsilon}{(1+|\underline{u}|^2)^{1/2+\delta/2}}.$$

Consider now Lemma 4.4.1 and the definition of φ in (4.2.5). Also, consider the inequalities for α (4.4.5), and (4.4.6). We obtain

$$I_{1,1} := \iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}|^2|L(\ln\alpha)| \lesssim \int_{\mathbb{R}} \frac{K_1\gamma}{\varphi^{3/4}(u)} \underbrace{\left[\int_{C_{t,u}} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}|^2 ds\right]}_{\lesssim \mathcal{F}(t)} du \lesssim K_1\gamma\varepsilon^2.$$

For the second integral in (3.3.24) consider

$$K := \max\{K_1, K_2\}. \tag{3.3.25}$$

Using again (3.3.7), (4.4.4)-(4.4.5) and (4.4.6) and (4.4.1) we have

$$\begin{aligned}
I_{1,2} :=& \iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}||L\tilde{\Lambda}||\underline{L}(\ln\alpha)| \lesssim \left(\iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}|^2|L\tilde{\Lambda}|\right)^{1/2} \left(\iint_{D_t} \varphi(\underline{u})\frac{K^2\gamma^2}{\varphi^{3/2}(\underline{u})}|L\tilde{\Lambda}|\right)^{1/2}\\
\lesssim& \left(\int_{\mathbb{R}} \frac{C_2\varepsilon}{\varphi^{1/2}(u)}\int_{C_{t,u}} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}|^2 du\right)^{1/2} \left(\int_{\mathbb{R}} \frac{C_2\varepsilon}{\varphi^{1/2}(u)}\int_{C_{t,u}} \frac{4K^2\gamma^2}{\varphi^{1/2}(\underline{u})} du\right)^{1/2}\\
\lesssim& K(C_2\varepsilon^3)^{1/2}(C_2\gamma^2\varepsilon)^{1/2} = KC_2\gamma\varepsilon^2.
\end{aligned}$$

For the integral I_2 in (3.3.23), we have from (4.4.3) that

$$\begin{aligned}
I_2 =& \iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}||Q_0(\partial_x(\ln\alpha), \tilde{\Lambda}) + Q_0(\ln\alpha, \partial_x\tilde{\Lambda})|\\
\leq& \iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}|^2|L(\partial_x(\ln\alpha)| + \iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}||L\tilde{\Lambda}||\underline{L}\partial_x(\ln\alpha)|\\
&+ \iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}||\underline{L}\partial_x\tilde{\Lambda}||L(\ln\alpha)| + \iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}||\underline{L}(\ln\alpha)||L\partial_x(\tilde{\Lambda})|\\
=:& I_{2,11} + I_{2,12} + I_{2,21} + I_{2,22}.
\end{aligned}$$

Using (4.4.6) and similar computations to the previous ones, we get

$$I_{2,11} \lesssim K_1\gamma\varepsilon^2.$$

Next, using Cauchy-Schwarz,

$$\begin{aligned}
I_{2,12} \lesssim& \left(\iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}|^2|L\tilde{\Lambda}|\right)^{1/2} \left(\iint_{D_t} \varphi(\underline{u})|L\tilde{\Lambda}||\underline{L}\partial_x(\ln\alpha)|^2\right)^{1/2}\\
\lesssim& \left(\int_{\mathbb{R}} \frac{C_2\varepsilon}{\varphi^{1/2}(u)}\underbrace{\left[\int_{C_{t,u}} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}|^2 ds\right]}_{\lesssim\mathcal{F}(t)} du\right)^{1/2} \left(\int_{\mathbb{R}} \frac{C_2\varepsilon}{\varphi^{1/2}(u)}\left[\int_{C_{t,u}} \frac{K^2\gamma^2}{\varphi^{1/2}(\underline{u})} ds\right] du\right)^{1/2}\\
\lesssim& C_2K\gamma\varepsilon^2.
\end{aligned}$$

Using the same analysis as before,

$$\begin{aligned}
I_{2,21} \lesssim& \left(\iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}|^2|L(\ln\alpha)|\right)^{1/2} \left(\iint_{D_t} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}|^2|L(\ln\alpha)|\right)^{1/2}\\
\lesssim& \left(\int_{\mathbb{R}} \frac{K\gamma}{\varphi^{3/4}(u)}\left[\int_{C_{t,u}} \varphi(\underline{u})|\underline{L}\tilde{\Lambda}|^2 ds\right] du\right)^{1/2} \left(\int_{\mathbb{R}} \frac{K\gamma}{\varphi^{3/4}(u)}\left[\int_{C_{t,u}} \varphi(\underline{u})|\underline{L}\partial_x\tilde{\Lambda}|^2 ds\right] du\right)^{1/2}\\
\lesssim& C_2K\gamma\varepsilon^2.
\end{aligned}$$

The last estimate involves Cauchy-Schwarz to obtain

$$
\begin{aligned}
I_{2,22} &\lesssim \left(\iint_{Dt} \frac{\varphi(\underline{u})}{\varphi(u)}|\underline{L}\tilde{\Lambda}|^2\right)^{1/2} \left(\iint_{D_t} \varphi(\underline{u})|\underline{L}(\ln\alpha)|^2|L\partial_x\tilde{\Lambda}|^2\varphi(u)\right)^{1/2} \\
&\lesssim \varepsilon \left(\int_{\mathbb{R}} \frac{K^2\gamma^2}{\varphi^{3/2}(\underline{u})}\left[\int_{C_{t,\underline{u}}} \varphi(u)|L\partial_x\tilde{\Lambda}|\right] d\underline{u}\right)^{1/2} \lesssim K\gamma\varepsilon^2.
\end{aligned}
$$

The remaining integrals are analogous and we have for both expressions that:

$$
\begin{aligned}
I_3 &:= \iint_{D_t} \varphi(u)|L\tilde{\Lambda}||Q_0(\ln\alpha, \tilde{\Lambda})| \lesssim \gamma\varepsilon^2, \\
I_4 &:= \iint_{D_t} \varphi(u)|L\tilde{\Lambda}||Q_0(\partial_x(\ln\alpha), \tilde{\Lambda}) + Q_0(\ln\alpha, \partial_x\tilde{\Lambda})| \lesssim \gamma\varepsilon^2.
\end{aligned}
$$

For the other term, which correspond to ρ_2 in (4.4.11), the analysis is the same as described in our recently completed work [109]. See this reference for full details.

Finally, from the energy estimate (4.2.6), we can arrange all the previous estimates together, and for universal constants C_4, C_5, K with $K_1, K_2 \leq K$, (see (3.3.25)), one has for all $t \in [0, T^*]$:

$$
\mathcal{E}(t) + \mathcal{F}(t) \leq (2C_0C_1 + K\gamma)\varepsilon^2 + C_4\varepsilon^3 + C_5\varepsilon^4, \tag{3.3.26}
$$

where C_1 is given in (3.3.16). Now, if we take ε_0 such that

$$
\varepsilon_0 \leq \frac{C_0C_1}{C_4}, \quad \varepsilon_0^2 \leq \frac{C_0C_1}{C_5}, \tag{3.3.27}
$$

and γ such that

$$
K\gamma < \frac{C_0C_1}{2},
$$

we can see that for all $0 < \varepsilon \leq \varepsilon_0$ and for all $t \in [0, T]$, we have

$$
\mathcal{E}(t) + \mathcal{F}(t) \leq \frac{9}{2}C_0C_1\varepsilon^2.
$$

By taking a suitable γ and ε_0, we have the desired control. This improves the constant in (4.4.8).

To improve condition (4.4.10), using the Fundamental Theorem of Calculus, (3.3.2) and Lemma 4.4.1, one can write $\tilde{\Lambda}(t,x)$, $t \geq 0$, in the following form:

$$
\begin{aligned}
|\tilde{\Lambda}(t,x)| &\leq \varepsilon|\tilde{\Lambda}_0(x)| + \int_0^t |\partial_\tau\tilde{\Lambda}(\tau,x)|d\tau \\
&\leq \varepsilon M_1 + \frac{1}{2}\int_0^t |L\tilde{\Lambda} + \underline{L}\tilde{\Lambda}|d\tau \\
&\leq \varepsilon M_1 + \frac{1}{2}\int_0^t \left(\frac{C_2\varepsilon}{\varphi(u)^{1/2}} + \frac{C_2\varepsilon}{\varphi(\underline{u})^{1/2}}\right) d\tau \\
&\leq \varepsilon M_1 + \varepsilon C_2M_2 \leq M\varepsilon,
\end{aligned} \tag{3.3.28}
$$

for some universal constant M. Next, we take $\varepsilon_0 > 0$ that satisfies the condition (3.3.27) and such that

$$M\varepsilon_0 < \frac{\lambda}{4}, \tag{3.3.29}$$

taking sup over $t \in [0, T^*]$, we conclude that for all $0 < \varepsilon \leq \varepsilon_0$ we have improved via (3.3.29) the key estimate (4.4.10).

The above estimates, prove that the solution $\tilde{\Lambda}$ is global. A similar argument, as established before, shows that ϕ is also globally defined. This ends the existence proof in Theorem 4.4.1.

3.3.4 End of proof of Theorem 3.1.1

Since α, $\tilde{\Lambda}$ and ϕ have been completely determined in previous steps, we only need to determine the behavior of the function f in (3.1.8). Note that α satisfies (3.3.6) and in the system (3.1.8) we have that $\ln f$ satisfies the nonhomogeneous wave equation with initial conditions (3.1.12), then, we can use D'Alembert's solution to describe the function f, but previously, let us analyze the following result:

Lemma 3.3.2. *Let G be defined as in* (3.1.9)*. Under the hypotheses of Theorem 3.1.1, and under the consequences of Theorem 4.4.1, the following is satisfied:*

- *For each $t \in \mathbb{R}$, $G(t, \cdot) \in (L^1 \cap L^\infty)(\mathbb{R})$;*
- *There exists $C > 0$ such that $\sup_{t \geq 0} \|G(t)\|_{L^1 \cap L^\infty} \leq C$.*

Proof. Since G is given by (3.1.9), one has

$$\begin{aligned} G := &-\big(\partial_t^2(\ln\alpha) - \partial_x^2(\ln\alpha)\big) - \frac{1}{2\alpha^2}((\partial_t\alpha)^2 - (\partial_x\alpha)^2) \\ &- \frac{1}{2}((\partial_t\Lambda)^2 - (\partial_x\Lambda)^2) - 2\sinh^2\Lambda((\partial_t\phi)^2 - (\partial_x\phi)^2). \end{aligned}$$

From (3.3.7) and (4.4.4), we can simplify

$$\begin{aligned} G = &\frac{1}{2\alpha^2}(\tilde{\alpha}_0'(2u) + \alpha_1(2u))(\alpha_1(-2\underline{u}) - \tilde{\alpha}_0'(-2\underline{u})) \\ &- \frac{1}{2}((\partial_t\Lambda)^2 - (\partial_x\Lambda)^2) - 2\sinh^2\Lambda((\partial_t\phi)^2 - (\partial_x\phi)^2) =: G_1 + G_2. \end{aligned}$$

It can be seen that the regularity of the term G depends on the initial conditions for the function α, and on the functions Λ, ϕ. The hypotheses in Theorem 3.1.1 ensure that, for all $t \in \mathbb{R}$,

$$G_1 = \frac{1}{2\alpha^2}(\tilde{\alpha}_0'(2u) + \alpha_1(2u))(\alpha_1(-2\underline{u}) - \tilde{\alpha}_0'(-2\underline{u})) \in S(\mathbb{R}).$$

Moreover, $\sup_{t\in\mathbb{R}} \|G_1(t)\|_{L^1\cap L^\infty} \leq C$. On the other hand, G_2 satisfies from (3.1.14)

$$\big|(\partial_t\Lambda)^2 - (\partial_x\Lambda)^2 - 2\sinh^2\Lambda((\partial_t\phi)^2 - (\partial_x\phi)^2)\big|\,(t,x) \leq h_1(t,x).$$

Now we use the following result to conclude:

Lemma 3.3.3.

$$|h_1(t,x)| \lesssim \frac{\varepsilon^2}{\varphi(u)} + \frac{\varepsilon^2}{\varphi(\underline{u})}.$$

Assuming this result, we finally get $G(t,\cdot) \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ with uniform bounds in time. □

Proof of Lemma 3.3.3. First of all, we have (see (3.3.2), Lemma (4.4.1) and (4.2.5))

$$|\partial_t \Lambda| \lesssim |L\tilde{\Lambda}| + |\underline{L}\tilde{\Lambda}| \le \frac{\varepsilon}{\varphi(u)^{1/2}} + \frac{\varepsilon}{\varphi(\underline{u})^{1/2}}.$$

Similarly,

$$|\partial_x \Lambda| + |\partial_x \phi| + |\partial_x \phi| \lesssim \frac{\varepsilon}{\varphi(u)^{1/2}} + \frac{\varepsilon}{\varphi(\underline{u})^{1/2}}.$$

Finally, thanks to (3.3.28),

$$\sinh^2(\Lambda) \lesssim \sinh^2(\lambda).$$

Gathering these results we conclude. □

The previous result allows us to describe the function f using d'Alembert formula for the nonhomogeneous linear wave and (3.1.12). Consider the initial data problem for $v(t,x) := \ln f(t,x)$ given by

$$\begin{cases} \partial_t^2 v - \partial_x^2 v = G(t,x) \\ v(0,x) = \ln(f(0,x)) = \ln(f_0 + c_1) \\ \partial_t v(t,x)|_{\{t=0\}} = \frac{f_1}{c_1+f_0}. \end{cases} \tag{3.3.30}$$

We get

$$\begin{aligned} v(t,x) :=& \frac{1}{2}\left[\ln(c_1 + f_0(x+t)) + \ln(c_1 + f_0(x-t))\right] \\ &+ \frac{1}{2}\int_{x-t}^{x+t} \frac{f_1(s)ds}{c_1+f_0(s)} + \frac{1}{2}\int_0^t \left[\int_{x+s-t}^{x+t-s} G(s,y)dy\right] ds =: v_1 + v_2 + v_3. \end{aligned} \tag{3.3.31}$$

It is clear that v_1 and v_2 are globally defined, bounded in time and space members. On the other hand, thanks to (3.3.3),

$$\begin{aligned} \left|\int_{x+s-t}^{x+t-s} G(s,y)dy\right| &\lesssim \varepsilon^2 \int_{x+s-t}^{x+t-s} \left(\frac{1}{\varphi(s+y)} + \frac{1}{\varphi(s-y)}\right) dy \\ &\lesssim \varepsilon^2 \int_{x+2s-t}^{x+t} \frac{dy}{\varphi(y)} + \varepsilon^2 \int_{x-t}^{x+t-2s} \frac{dy}{\varphi(y)} \lesssim \frac{\varepsilon^2(t-s)}{\varphi(x+t)} + \frac{\varepsilon^2(t-s)}{\varphi(x-t)}. \end{aligned}$$

Consequently,

$$|v_3| \lesssim \int_0^t \left(\frac{\varepsilon^2(t-s)}{\varphi(x+t)} + \frac{\varepsilon^2(t-s)}{\varphi(x-t)}\right) ds \lesssim t^2 \left(\frac{\varepsilon^2}{\varphi(x+t)} + \frac{\varepsilon^2}{\varphi(x-t)}\right).$$

Now we conclude the proof of the theorem. From (3.3.31) the function f is given by

$$f(t,x) = \rho(t,x) \exp\left(\frac{1}{2}\int_{x-t}^{x+t} \frac{f_1(s)ds}{c_1+f_0(s)} + \frac{1}{2}\int_0^t \left[\int_{x+s-t}^{x+t-s} G(s,y)dy\right] ds\right),$$

with

$$\rho(t,x) = \sqrt{(c_1 + f_0(x+t))(c_1 + f_0(x-t))}.$$

Notice that f is strictly positive everywhere in time and space. Given the initial conditions imposed on the function f, the integrals are well-defined. Additionally the function f is positive consistent with Belinski-Zakharov proposal.

3.4 Energy-momentum formulation

The aim of this section is first to introduce a correct definition of energy and momentum densities for one type of solutions of the Einstein equations in vacuum, and then to give a proper description of the decay of these solutions in the framework of the global existence theory presented in the previous section.

The notion of the energy and the law of conservation of energy play a key role in all mathematics-physical theories. The definition of energy in relativity is a complex matter, and this problem has been given a lot of attention in the literature [111, 112]. For this and other reasons it is very interesting to study and to define what could be considered a good definition of "energy". However, the most likely candidate for the energy density of the gravitational field in general relativity would be a quadratic expression in the first derivatives of the components of the metric [111], or as in this case, in terms of the fields defining the components of the metric. For the particular case of spacetimes admitting two commutative Killing vectors, the energy formulation is constrained by the function $\alpha(t,x)$, which we recall, in this setting, is a positive solution of the one-dimensional wave equation.

What we must keep in mind is that these spacetimes can be used to describe both cylindrical gravitational waves and inhomogeneous cosmological models in vacuum, but the former are less suited to study decay properties, for the reasons exposed below. Roughly speaking, for gravitational cylindrical wave solutions, the gradient of $\alpha(t,x)$ must be spacelike, while for the description of cosmological models, it must be timelike.

In this section, we propose an adequate description of the energy and momentum densities, according to the type of spacetime being analyzed, i.e., subject to the sign of the gradient of the function α.

3.4.1 Energy-Momentum formalism

We begin by proposing an initial definition for energy and momentum densities of the system (3.1.8). In the spirit of the definition proposed by Hadad in [52, p.73], we will expose this new description for these densities in the suitable terms of the field Λ, ϕ, α, and study whether or not it is a conserved quantity and to find local conservation laws.

Recall (3.1.13). In terms of the fields Λ, ϕ and the function $\alpha(t,x)$, let us introduce the following densities:

$$\begin{aligned} e(t,x) := \kappa\partial_t\alpha &\left[\frac{(\partial_x\alpha)^2+(\partial_t\alpha)^2}{\alpha^2} + 4\sinh^2(\Lambda)\Big((\partial_t\phi)^2+(\partial_x\phi)^2\Big) + (\partial_t\Lambda)^2+(\partial_x\Lambda)^2\right] \\ &- 2\kappa\partial_x\alpha\left(\frac{\partial_x\alpha\partial_t\alpha}{\alpha^2} + \partial_x\Lambda\partial_t\Lambda + 4\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right), \end{aligned} \tag{3.4.1}$$

where

$$\kappa(t,x) = \frac{\alpha}{(\partial_x\alpha)^2-(\partial_t\alpha)^2}, \tag{3.4.2}$$

and

$$\begin{aligned} p(t,x) := \kappa\partial_x\alpha &\left[\frac{(\partial_x\alpha)^2+(\partial_t\alpha)^2}{\alpha^2} + 4\sinh^2(\Lambda)\Big((\partial_t\phi)^2+(\partial_x\phi)^2\Big) + (\partial_t\Lambda)^2+(\partial_x\Lambda)^2\right] \\ &- 2\kappa\partial_t\alpha\left(\frac{\partial_x\alpha\partial_t\alpha}{\alpha^2} + \partial_x\Lambda\partial_t\Lambda + 4\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right). \end{aligned} \tag{3.4.3}$$

It should be noted that, in providing these densities, certain constraints, regarding the region in which $(\partial_x\alpha)^2-(\partial_t\alpha)^2$ is null, must be considered. These considerations will be studied in more detail in the following section. Now, in order to have an suitable definition of these densities, we propose the following redefinition:

$$\begin{aligned} \tilde{e} = \tilde{e}[\Lambda,\phi,\alpha] := \kappa\partial_t\alpha &\left[(\partial_t\Lambda)^2+(\partial_x\Lambda)^2 + 4\sinh^2(\Lambda)\Big((\partial_t\phi)^2+(\partial_x\phi)^2\Big)\right] \\ &- 2\kappa\partial_x\alpha\left(\partial_x\Lambda\partial_t\Lambda + 4\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right), \end{aligned} \tag{3.4.4}$$

and

$$\begin{aligned} \tilde{p} = \tilde{p}[\Lambda,\phi,\alpha] := \kappa\partial_x\alpha &\left[(\partial_t\Lambda)^2+(\partial_x\Lambda)^2 + 4\sinh^2(\Lambda)\Big((\partial_t\phi)^2+(\partial_x\phi)^2\Big)\right] \\ &- 2\kappa\partial_t\alpha\left(\partial_x\Lambda\partial_t\Lambda + 4\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right). \end{aligned} \tag{3.4.5}$$

For the densities e, p, we can state the following identities

Lemma 3.4.1. *Let (Λ,ϕ,α) be a solution to* (3.1.8). *Let $e(t,x)$ and $p(t,x)$ be as introduced in* (3.4.1)-(3.4.3). *Assume that (t,x) lies in an open region of spacetime such that $(\partial_x\alpha)^2-(\partial_t\alpha)^2\neq 0$. Then one has*

$$\begin{aligned} &\partial_t p(t,x) + \partial_x e(t,x) = 0, \\ &\partial_t e(t,x) + \partial_x p(t,x) = 4\sinh^2(\Lambda)\left(\phi_t^2-\phi_x^2\right) + \Lambda_t^2 - \Lambda_x^2 + \partial_x\left(\frac{\alpha_x}{\alpha}\right) - \partial_t\left(\frac{\alpha_t}{\alpha}\right). \end{aligned} \tag{3.4.6}$$

Equations (3.4.6) are a modified version of the continuity equations for the energy and momentum densities. A perfectly behaved relation was found in [109] in the case of the integrable Principal Chiral model. The situation here is more subtle, and there is no sign of a perfectly behaved continuity equation, manly because of the functions α and f.

Part of the proof of the first equation is essentially contained in Hadad [52], but the technical details, as well as the proof of the second equation are included in Appendix 3.7.1. As a corollary we also have the following identities for the redefined densities $\tilde{e}(t,x)$, and $\tilde{p}(t,x)$:

Corollary 3.4.1. *Let $\tilde{e}(t,x)$ and $\tilde{p}(t,x)$ be as introduced in* (3.4.4)-(3.4.5)*. Under the assumptions of Lemma 3.4.1, one has*

$$\begin{aligned} &\partial_t \tilde{p}(t,x) + \partial_x \tilde{e}(t,x) = 0, \\ &\partial_t \tilde{e}(t,x) + \partial_x \tilde{p}(t,x) = 4\sinh^2(\Lambda)\left(\phi_t^2 - \phi_x^2\right) + \Lambda_t^2 - \Lambda_x^2. \end{aligned}$$

Proof. The proof follows immediately from the definition and description of the densities obtained in the Lemma 3.4.6. □

Note the symmetry in the terms defining the densities, however, the derivatives of the function $\alpha(t,x)$ make a significant change in the nature of these densities, (as compared to the PCF equation case, where $\alpha(t,x)$ was considered as a constant, see [109]). This implies a deeper analysis regarding the correct formulation of energy densities. As mentioned in the introduction, the local behavior of the spacetime is defined by the nature of the function $\alpha(t,x)$.

This function may have a gradient spacelike in all the spacetime (corresponds to spacetimes with cylindrical symmetry), globally null (corresponds to the plane-symmetric waves) or timelike (cosmological type-solutions), see [6, 8, 17, 34, 51] for more details. The following sections propose appropriate definitions of the energy and momentum densities associated with each type of solution, i.e., depending on the nature of the gradient of the alpha function.

3.4.2 Cosmological-type solutions

As mentioned before, spacetimes in the Belinski-Zakharov setting can be used to represent inhomogeneous vacuum cosmological models. In these, the universe is assumed to contain gravitational waves propagating in opposite spatial directions, see [6, 8]. To describe this class of models, it is appropriate to take the function $\alpha(t,x)$ timelike, i.e., with negative gradient norm. Let us start with some preliminary definitions and results.

Definition 3.4.1 (Timelike condition)**.** *Given the function $\alpha(t,x)$, we will say that $\alpha(t,x)$ is timelike, if its gradient satisfies*

$$(\partial_x \alpha)^2 - (\partial_t \alpha)^2 < 0, \quad \forall (t,x) \in \mathbb{R}^2. \tag{3.4.7}$$

In this case, we will say that our model is of cosmological type.

Definition 3.4.1 is taken from [17, p. 965]. It is relevant to remark that, as expressed in [17], other cosmological type models are of interest, such as Gowdy models. For more details, the reader can consult the aforementioned work and references therein.

Using the same notation as in (3.3.6) for the initial conditions for α, as a solution to the wave equation, when $\alpha(t,x)$ is timelike everywhere, we have the following result:

Lemma 3.4.2. *Assume that $\alpha(t,x)$ is timelike in the whole spacetime. Then*

(i) one has

$$|\partial_x \alpha| < |\partial_t \alpha|. \tag{3.4.8}$$

(ii) if additionally, (t, x) *is such that*

$$\partial_t \alpha(t, x) > 0, \tag{3.4.9}$$

then, $|\partial_x \alpha| < \partial_t \alpha$ *if and only if the initial data of the function* α *satisfy*

$$|\tilde{\alpha}_0'(\cdot)| < \alpha_1(\cdot). \tag{3.4.10}$$

(iii) if additionally

$$\alpha(t, x) > 0 \quad \forall (t, x) \in \mathbb{R}^2, \tag{3.4.11}$$

then the parameter κ *defined in* (3.4.2) *is well-defined and it is negative, and*

$$-\kappa \partial_t \alpha > 0. \tag{3.4.12}$$

Proof. The proof is obtained from a straightforward calculation and the use of (3.4.9). □

***Remark* 3.4.1.** *Notice that condition* (3.4.7) *is ensured if the initial data for* α *satisfies* (3.4.10)*, in addition, this condition allows us to propose an* α *function that is in consistency with the hypotheses of the Theorem* 3.1.1*.*

Now, in order to define a positive energy density and being possible to set a control of this density over the momentum density, we propose:

Definition 3.4.2. *For cosmological-type solutions, the energy and momentum densities will be defined as*

$$\hat{e}(t, x) = -\tilde{e}(t, x), \tag{3.4.13}$$

and

$$\hat{p}(t, x) = \tilde{p}(t, x). \tag{3.4.14}$$

3.4.3 Proof of Theorem 3.1.2

Theorem 3.1.2 will be a consequence of the following lemma.

Lemma 3.4.3. *Under* (3.4.8)*,* (3.4.9) *and* (3.4.11)*, the energy density defined in* (3.4.13) *is nonnegative. Moreover, one has the improved estimate*

$$\hat{e} \geq |\kappa|(|\partial_t \alpha| - |\partial_x \alpha|) \left[(\partial_t \Lambda)^2 + (\partial_x \Lambda)^2 + 4 \sinh^2(\Lambda) \Big((\partial_t \phi)^2 + (\partial_x \phi)^2 \Big) \right].$$

Proof of Lemma 3.4.3. We compute: from (3.4.4) and Lemma 3.4.2 (ii),

$$\begin{aligned}
\hat{e} = -\tilde{e} = & -\kappa\partial_t\alpha\left[(\partial_t\Lambda)^2 + (\partial_x\Lambda)^2 + 4\sinh^2(\Lambda)\Big((\partial_t\phi)^2 + (\partial_x\phi)^2\Big)\right] \\
& \quad + 2\kappa\partial_x\alpha\left(\partial_x\Lambda\partial_t\Lambda + 4\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right) \\
& \geq |\kappa\partial_t\alpha|\left[(\partial_t\Lambda)^2 + (\partial_x\Lambda)^2 + 4\sinh^2(\Lambda)\Big((\partial_t\phi)^2 + (\partial_x\phi)^2\Big)\right] \\
& \quad - 2|\kappa\partial_x\alpha|\left(|\partial_x\Lambda||\partial_t\Lambda| + 4|\partial_x\phi||\partial_t\phi|\sinh^2(\Lambda)\right) \\
& = |\kappa|(|\partial_t\alpha| - |\partial_x\alpha|)\left[(\partial_t\Lambda)^2 + (\partial_x\Lambda)^2 + 4\sinh^2(\Lambda)\Big((\partial_t\phi)^2 + (\partial_x\phi)^2\Big)\right] \\
& \quad + |\kappa\partial_x\alpha|\Big((\partial_t\Lambda)^2 + (\partial_x\Lambda)^2 - 2|\partial_x\Lambda||\partial_t\Lambda|\Big) \\
& \quad + 4|\kappa||\partial_x\alpha|\sinh^2(\Lambda)\Big((\partial_t\phi)^2 + (\partial_x\phi)^2 - 2|\partial_x\phi||\partial_t\phi|\Big) \\
& \geq |\kappa|(|\partial_t\alpha| - |\partial_x\alpha|)\left[(\partial_t\Lambda)^2 + (\partial_x\Lambda)^2 + 4\sinh^2(\Lambda)\Big((\partial_t\phi)^2 + (\partial_x\phi)^2\Big)\right] \geq 0.
\end{aligned} \tag{3.4.15}$$

The proof is complete. □

Furthermore, under this same hypothesis, we can establish an appropriate control of the energy density on the momentum density. Recall that the condition that $\partial_t\alpha > 0$, necessarily implies, that the function $\alpha_1(s) > 0, \forall s \in \mathbb{R}$ which is in correspondence with the setting proposed for function $\alpha(t,x)$ in the previous global existence theory. We can obtain the following result:

Lemma 3.4.4. *Under* (3.4.8), (3.4.9) *and* (3.4.11),

$$|\hat{p}(t,x)| \leq \hat{e}(t,x). \tag{3.4.16}$$

Proof. To simplify the notation, let us define:

$$h_1(t,x) = (\partial_t\Lambda)^2 + (\partial_x\Lambda)^2 + 4\sinh^2(\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2) \geq 0, \tag{3.4.17}$$

and

$$h_2(t,x) = \partial_t\Lambda\partial_x\Lambda + 4\sinh^2(\Lambda)\partial_t\phi\partial_x\phi, \tag{3.4.18}$$

then, the energy density and the momentum density can be written as

$$\hat{e} = -\kappa(\partial_t\alpha h_1 - 2\partial_x\alpha h_2), \qquad \hat{p} = \kappa(\partial_x\alpha h_1 - 2\partial_t\alpha h_2).$$

Recall that, $\hat{e} \geq |\kappa|(|\partial_t\alpha| - |\partial_x\alpha|)h_1 \geq 0$ thanks to (3.4.15). Now, let us prove (3.4.26). Using the Cauchy inequality and the condition (3.4.7), one has $2|h_2| \leq h_1$. Therefore, using that $\kappa < 0$, $\kappa h_1 \leq 2\kappa h_2$. Since $\partial_t\alpha + \partial_x\alpha > 0$, one has

$$\kappa h_1(\partial_t\alpha + \partial_x\alpha) \leq 2\kappa h_2(\partial_t\alpha + \partial_x\alpha).$$

Consequently,

$$\kappa\partial_x\alpha h_1 - 2\kappa\partial_t\alpha h_2 < -\kappa\partial_t\alpha h_1 + 2\kappa\partial_x\alpha h_2,$$

which proves that $\hat{p} \leq \hat{e}$.

For the other direction we have $\partial_t\alpha - \partial_x\alpha > 0$, $2\kappa h_2 \leq -\kappa h_1$, so that

$$\begin{aligned} -\kappa h_1(\partial_t\alpha - \partial_x\alpha) &\geq 2\kappa h_2(\partial_t\alpha - \partial_x\alpha), \\ \kappa\partial_x\alpha h_1 - 2\kappa\partial_t\alpha h_2 &\geq \kappa\partial_t\alpha h_1 - 2\kappa\partial_x\alpha h_2, \\ \hat{p} &\geq -\hat{e}. \end{aligned}$$

Therefore, we obtain a control of energy density over momentum density

$$|\hat{p}(t,x)| \leq \hat{e}(t,x).$$

The proof is complete. □

With the previous definitions of hat-densities (3.4.13) and (3.4.14) and the identities obtained in Corollary 3.4.1, one has the following consequences (modified continuity equations):

Corollary 3.4.2. *Let* (Λ, ϕ, α) *solutions of the system* (3.1.8). *Under the assumptions of Lemma 3.4.1, one has*

$$\begin{aligned} &\partial_t\hat{p}(t,x) - \partial_x\hat{e}(t,x) = 0, \\ &\partial_t\hat{e}(t,x) - \partial_x\hat{p}(t,x) = 4\sinh^2(\Lambda)\left(\phi_x^2 - \phi_t^2\right) + \Lambda_x^2 - \Lambda_t^2. \end{aligned} \tag{3.4.19}$$

Lemma 3.4.4 and Corollary 3.4.2 will become very important, in the sense that, the control that new energy density has over the momentum density, allow us to propose virial estimate, and analyzed the long time behavior of the cosmological type solution, as we will be discussing in the subsequent sections. We now discuss the energy formulation for the case where the gradient of the function α is spacelike.

3.4.4 Cylindrical Gravitational waves

Let $u^{(0)} = u^{(0)}(t,r)$ be a solution to the cylindrical wave equation in 2D:

$$\partial_t^2 u^{(0)} = \frac{1}{r}\partial_r\left(r\partial_r u^{(0)}\right), \quad (t,r) \in \mathbb{R}_t \times (0,\infty).$$

As usual, α satisfies the 1D wave equation in (t,r). Let us introduce the following line element of a cylindrically symmetric spacetime as follows:

$$ds^2 = f^{(0)}(-dt^2 + dr^2) + e^{-u^{(0)}}(\alpha d\phi)^2 + e^{u^{(0)}}dz^2, \tag{3.4.20}$$

with $x^a = \{\phi, z\}$ and $x^i = \{t, r\}$ and $r > 0$. This line element belongs to the class of solutions considered in the Belinski-Zakharov spacetime setting where

$$g = \begin{bmatrix} \alpha^2 e^{-u^{(0)}} & 0 \\ 0 & e^{u^{(0)}} \end{bmatrix}.$$

A particular case of the metric (3.4.20) is the one given by the Einstein-Rosen model, where $\alpha \equiv r$. See (3.6.12) and (3.6.13) for more details.

As mentioned before, the local behavior of the considered spacetime is defined by the gradient of the function α. In the case that this gradient is spacelike, it actually corresponds to cylindrical spacetimes. Notice that metric (3.4.20) is a particular case of (3.1.2) in cylindrical coordinates, where the fields described by the geometric representation (3.1.7), are given as follows: the field ϕ is a constant, and the field $\Lambda = u^{(0)}$. In this section, we will consider precisely this general setting. Consider the system (3.1.8) with α as a positive solution of the one-dimensional wave equation, and satisfying the so-called space-like condition, which will be described below. Thus we capture the essential condition describing the Einstein-Rosen gravitational wave metric [17].

As in the previous subsection, we introduce some preliminary definitions and results.

Definition 3.4.3 (Spacelike Condition)**.** *We say that $\alpha(t,x)$ is spacelike if its gradient satisfies*

$$(\partial_r\alpha)^2 - (\partial_t\alpha)^2 > 0, \quad \forall (t,r). \tag{3.4.21}$$

The spacelike condition (3.4.21) contrasts with the timelike one in (3.4.7) not only by the obvious reason (opposite signs), but also because it will allow not decaying solutions to the problem. In this sense, one can guess that no general virial theorem is present in this situation, unless we assume additional hypothesis on $u^{(0)}$ and α.

Coming back to (3.4.20), and using the same notation for the initial conditions for α as in (3.3.6), with $\alpha(t,x)$ spacelike everywhere, one has the following result:

Lemma 3.4.5. *If the function α is spacelike,*

$$|\partial_t\alpha| < |\partial_r\alpha|,$$

and the following are satisfied:

(i) if (t,r) is such that

$$\partial_r\alpha(t,r) > 0, \tag{3.4.22}$$

then, $|\partial_t\alpha| < \partial_r\alpha$ if and only if the initial data of the function α satisfy

$$|\alpha_1(\cdot)| < \tilde{\alpha}_0'(\cdot)$$

(ii) the parameter κ defined in (3.4.2) *(with x replaced by the variable r) is well-defined and positive, and*

$$\kappa\partial_r\alpha > 0. \tag{3.4.23}$$

Proof of Lemma 3.4.5. The proof is obtained from a straightforward calculation as in Lemma 3.4.2. □

Comparing with (3.4.9) and (3.4.12), one can see that (3.4.22) and (3.4.23) are "dual" to the former ones. Although one can think that these properties are not harmful, it turns the case that this is exactly the case: these signs are bad for decay purposes by natural reasons: spacelike dynamics tends to be unphysical in reality.

To ensure that we have an appropriate energy and to be possible to set a control of this density over the momentum one, we define

Definition 3.4.4. *For cylindrical-type solutions, the energy and momentum densities are defined as*

$$\hat{e}(t,r) = \tilde{p}(t,r), \tag{3.4.24}$$

and

$$\hat{p}(t,r) = \tilde{e}(t,r). \tag{3.4.25}$$

Notice that in this case, when the gradient of the function $\alpha(t,r)$ is spacelike, the parameter κ defined in (3.4.2), is positive. Now, with these redefinitions, we provide analogous estimates to those obtained in the case of cosmological type solutions.

Lemma 3.4.6. *If $\partial_r\alpha(t,r) > 0$ globally in spacetime, then the energy density $\hat{e}(t,r)$ is always nonnegative. Moreover, we have*

$$|\hat{p}(t,r)| \leq \hat{e}(t,r). \tag{3.4.26}$$

The proof of (3.4.16), considering the constraints, is obtained in a similar way as in the previous section, see Lemma 3.4.4 for more details. Lemma 3.4.6 is always useful to understand the right notion of energy.

Finally, similar to the previous section, with the formulation of energy and momentum densities given in (3.4.24)-(3.4.25), the identity equations obtained in the Corollary 3.4.1, provide the following modified continuity equations:

Lemma 3.4.7. *Let (Λ, ϕ, α) solutions of the system* (3.1.8)*, and $\alpha(t,r)$ spacelike, then, we have the following continuity equations*

$$\begin{aligned} &\partial_t\hat{e}(t,r) + \partial_r\hat{p}(t,r) = 0, \\ &\partial_t\hat{p}(t,r) + \partial_r\hat{e}(t,r) = 4\sinh^2(\Lambda)\left(\phi_r^2 - \phi_t^2\right) + \Lambda_r^2 - \Lambda_t^2. \end{aligned} \tag{3.4.27}$$

The proof of this result is obtained in a similar way as in the previous subsection. An important remark obtained from (3.4.27) is the following: in this set of identities the role of energy is played by the momentum, and vice versa. This somehow harmless condition destroys possible computations of decay by showing that the quantity that decays has no particular positivity. However, we expect to consider Lemma 3.4.7 in forthcoming works.

3.5 Virial Estimates for Cosmological-type Solutions

Let us come back to the setting already worked in Subsection 3.4.2. In what follows, let us consider (Λ, ϕ, α) globally defined in time and continuous such that

$$\text{(3.1.15) and (3.1.17) are satisfied.} \tag{3.5.1}$$

Note that (3.4.8) is a consequence of assuming (3.1.15) in Theorem 3.1.3. Finally,

$$E[\Lambda, \phi; \alpha] := \int_{\mathbb{R}} \hat{e}(t,x)dx$$

is well-defined for all time and bounded:

$$0 \leq E[\Lambda, \phi; \alpha] \leq \sup_{t\in\mathbb{R}} E[\Lambda, \phi; \alpha] < +\infty. \tag{3.5.2}$$

Notice that this time $E[\Lambda, \phi; \alpha]$ is not conserved (see (3.4.19)).

***Remark* 3.5.1.** *Condition* (3.5.2) *is not empty, for instance if the data is given as in Theorem 3.1.1 and α satisfies the time-like condition* (3.4.7) *and* (3.4.9)*, then* (4.4.8)-(4.4.9) *ensures that the energy is bounded in time as in* (3.5.2)*. See Lemma 3.3.3 for a proof.*

We introduce a Virial identity for the Einstein field equation (3.1.8). Indeed, let ρ be a smooth bounded function with $L^1 \cap L^\infty$ integrable derivative. Let $\omega(t)$ be a smooth positive function to be chosen later, not necessarily varying in time. Finally, for $v \in (-1, 1)$ let

$$\begin{aligned}\mathcal{I}(t) := & -\int \rho\left(\frac{x-vt}{\omega(t)}\right)\kappa\partial_x\alpha\left(4\sinh^2(\Lambda)((\partial_t\phi)^2+(\partial_x\phi)^2)+(\partial_t\Lambda)^2+(\partial_x\Lambda)^2\right)dx \\ & +\int \rho\left(\frac{x-vt}{\omega(t)}\right)\kappa\partial_t\alpha\left(2\partial_x\Lambda\partial_t\Lambda+8\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right)dx \\ =: & -\int \rho\left(\frac{x-vt}{\omega(t)}\right)\hat{p}(t,x)dx.\end{aligned} \tag{3.5.3}$$

A time-dependent weight $\omega(t)$ was already considered in [1, 109], but $\omega(t) = const.$ is also perfectly possible. The choice of $\mathcal{I}(t)$ is motivated by the momentum and energy densities.

Lemma 3.5.1 (Virial identity)**.** *One has $\mathcal{I}(t)$ well-defined and bounded in time, and*

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(t) = & -\frac{\omega'(t)}{\omega(t)}\int \frac{x-vt}{\omega(t)}\rho'\left(\frac{x-vt}{\omega(t)}\right)\hat{p}(t,x) \\ & +\frac{v}{\omega(t)}\int \rho'\left(\frac{x-vt}{\omega(t)}\right)\hat{p}(t,x) \\ & +\frac{1}{\omega(t)}\int \rho'\left(\frac{x-vt}{\omega(t)}\right)\hat{e}(t,x).\end{aligned} \tag{3.5.4}$$

Proof. The proof of (3.5.4) follows immediately from the Lemma 3.4.2. The proof of boundedness of $\mathcal{I}(t)$ goes as follows: from (3.5.3), the boundedness of ρ and (3.5.1),

$$|\mathcal{I}(t)| \le \int |\rho|\left(\frac{x-vt}{\omega(t)}\right)|\hat{p}|(t,x)dx \lesssim \int |\hat{p}|(t,x)dx \le \int \hat{e}(t,x)dx,$$

therefore from (3.5.2) we obtain $\sup_{t\ge 0}|\mathcal{I}(t)| < +\infty$. □

3.5.1 Virial estimates

Now we are ready to use previous identities.

We choose ω and ρ. Let $\omega(t) = const.$ or

$$\omega(t) := \frac{t}{\log^2 t}, \qquad \frac{\omega'(t)}{\omega(t)} = \frac{1}{t}\left(1-\frac{2}{\log t}\right). \tag{3.5.5}$$

and

$$\rho := \tanh, \quad \rho' = \operatorname{sech}^2. \tag{3.5.6}$$

Theorem 3.5.1. *Let ω and ρ be given as in* (3.5.5)-(3.5.6). *Assume that the solution* $(\Lambda, \phi, \alpha)(t)$ *of the system* (3.1.8) *is such that α satisfies* (3.5.1) *and the finite energy condition* (3.5.2) *is satisfied. Then we have the averaged estimate*

$$\int_2^\infty \frac{1}{\omega(t)} \int \operatorname{sech}^2\left(\frac{x-vt}{\omega(t)}\right) \hat{e}(t,x)dxdt \lesssim 1, \tag{3.5.7}$$

Moreover, there exists an increasing sequence $t_n \to +\infty$ such that

$$\lim_{n \longrightarrow +\infty} \int \operatorname{sech}^2\left(\frac{x-vt_n}{\omega(t_n)}\right) \hat{e}(t_n,x)dx = 0. \tag{3.5.8}$$

In order to show Theorem 3.5.1, we use the new Virial identity for (3.5.3) presented for the Einstein field equation (3.1.8).

Proof. On the other hand, recall that, we are considering that α is a positive solution of the one-dimensional wave equation, with time-like gradient and with positive time derivative in all spacetime, therefore, we can use the Lemma 3.4.4 and the Lemma 3.5.1, then, we get

$$\frac{d}{dt}\mathcal{I}(t) =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.$$

First of all, we consider $\mathcal{J}_1$. If $\omega(t)$ is constant, there is nothing to prove. Assume now $\omega(t)$ given as in (3.5.5). We have

$$|\mathcal{J}_1| \leq \frac{|\omega'(t)|}{\omega(t)} \int \frac{|x-vt|}{\omega(t)} \rho'\left(\frac{x-vt}{\omega(t)}\right) |\hat{p}(t,x)|dx \leq \frac{|\omega'(t)|}{\omega(t)} \int \frac{|x-vt|}{\omega(t)} \rho'\left(\frac{x-vt}{\omega(t)}\right) \hat{e}(t,x)dx.$$

From the definition of $\omega(t)$ and using Cauchy's inequality for $\delta > 0$ small, we have:

$$\begin{aligned} |\mathcal{J}_1| \leq & \frac{C_\delta \omega(t)}{t^2} \sup_{x\in\mathbb{R}} \left(\frac{(x-vt)^2}{\omega^2(t)} |\rho'| \left(\frac{x-vt}{\omega(t)}\right)\right) \int \hat{e}(t,x)dx \\ & + \frac{\delta}{\omega(t)} \int \rho'\left(\frac{x-vt}{\omega(t)}\right) \hat{e}(t,x)dx \\ \leq & \frac{C}{t\log^2 t} + \frac{\delta}{\omega(t)} \int \rho'\left(\frac{x-vt}{\omega(t)}\right) \hat{e}(t,x)dx. \end{aligned}$$

Now,

$$|\mathcal{J}_2(t)| \leq \frac{|v|}{\omega(t)} \int \rho'\left(\frac{x-vt}{\omega(t)}\right) |\hat{p}(t,x)| \leq \frac{|v|}{\omega(t)} \int \rho'\left(\frac{x-vt}{\omega(t)}\right) \hat{e}(t,x).$$

Finally, $\mathcal{J}_3(t)$ does not need any bound at all. In any case, $\omega(t) = \mathit{const.}$ or $\omega(t)$ as in (3.5.5), one has the following: if $\delta > 0$ is small:

$$\frac{d}{dt}\mathcal{I}(t) \geq \frac{1-|v|-\delta}{\omega(t)} \int \rho'\left(\frac{x-vt}{\omega(t)}\right) \hat{e}(t,x) - \frac{C_\delta}{t\log^2 t}. \tag{3.5.9}$$

After integration in time in (3.5.9) and since the term $\frac{C}{t\log^2 t}$ integrates finite, we get (3.5.7) Finally, (3.5.8) is obtain from (3.5.7) and the fact that $\omega^{-1}(t)$ is not integrable in $[2, \infty)$. ☐

3.5.2 Proof of the Theorem 3.1.3

First of all, notice that the RHS in (3.4.19) satisfies (with h_1 given in (3.4.17))

$$\left|4\sinh^2(\Lambda)\left(\phi_x^2-\phi_t^2\right)+\Lambda_x^2-\Lambda_t^2\right|\leq h_1.$$

Using the the Lemma 3.4.4, Lemma 3.4.2, (3.5.11) and integration by part we have

$$\begin{aligned}
&\left|\frac{d}{dt}\int \operatorname{sech}^4\left(\frac{x-vt}{\omega(t)}\right)\hat{e}(t,x)dx\right|\\
&\leq \frac{|\omega'(t)|}{\omega(t)}\int \frac{x-vt}{\omega(t)}|(\operatorname{sech}^4)'|\left(\frac{x-vt}{\omega(t)}\right)\hat{e}(t,x)dx \;+\frac{4|v|}{\omega(t)}\int \operatorname{sech}^4\left(\frac{x-vt}{\omega(t)}\right)\hat{e}(t,x)dx\\
&\quad+\int\frac{1}{\omega(t)}\operatorname{sech}^4\left(\frac{x-vt}{\omega(t)}\right)|\hat{p}|dx+\int \operatorname{sech}^4\left(\frac{x-vt}{\omega(t)}\right)h_1dx\\
&\leq \frac{2|v|+1+|\omega'(t)|}{\omega(t)}\int \operatorname{sech}^2\left(\frac{x-vt}{\omega(t)}\right)\hat{e}(t,x)dx+\frac{2}{\omega(t)}\int \operatorname{sech}^4\left(\frac{x-vt}{\omega(t)}\right)\omega(t)\frac{\partial_t\alpha}{\alpha}\hat{e}(t,x)dx.
\end{aligned}$$

Finally, notice that from (3.3.7), (3.3.1) and (3.5.5),

$$\omega(t)\operatorname{sech}\left(\frac{x-vt}{\omega(t)}\right)\frac{\partial_t\alpha}{\alpha}\lesssim 1.$$

This estimation is possible since α_0' and α_1 are compactly supported and Schwartz, respectively. Therefore, for every $n\geq 0$,

$$\alpha\geq\frac{1}{2},\quad \partial_t\alpha\lesssim_n\frac{1}{\varphi^n(u)}+\frac{1}{\varphi^n(\underline{u})},\quad u,\ \underline{u}\text{ as in (3.3.1)}.$$

Consequently for n sufficiently large but fixed,

$$\begin{aligned}
\omega(t)\operatorname{sech}\left(\frac{x-vt}{\omega(t)}\right)\frac{\partial_t\alpha}{\alpha}&\lesssim\omega(t)\operatorname{sech}\left(\frac{x-vt}{\omega(t)}\right)\left(\frac{1}{\varphi^n(u)}+\frac{1}{\varphi^n(\underline{u})}\right)\\
&\lesssim\frac{\omega(t)}{\varphi^n((1-|v|)|t|)}+\omega(t)\operatorname{sech}\left((1-|v|)\frac{|t|}{\omega(t)}\right)\\
&\lesssim 1+\omega(t)\operatorname{sech}\left((1-|v|)\log^2 t\right)\lesssim\omega(t)t^{-(1-|v|)\log t}\lesssim 1.
\end{aligned}$$

We conclude that

$$\left|\frac{d}{dt}\int \operatorname{sech}^4\left(\frac{x-vt}{\omega(t)}\right)\hat{e}(t,x)dx\right|\lesssim\frac{1}{\omega(t)}\int \operatorname{sech}^2\left(\frac{x-vt}{\omega(t)}\right)\hat{e}(t,x)dx.$$

Then integrating in time for $t<t_n$, we have

$$\begin{aligned}
&\left|\int \operatorname{sech}^4\left(\frac{x-vt_n}{\omega(t)}\right)\hat{e}(t_n,x)dx-\int \operatorname{sech}^4\left(\frac{x-vt}{\omega(t)}\right)\hat{e}(t,x)dx\right|\\
&\qquad\leq\int_t^{t_n}\frac{1}{\omega(t)}\int \operatorname{sech}^2\left(\frac{x-vs}{\omega(t)}\right)\hat{e}(s,x)dxds.
\end{aligned}$$

sending $n\longrightarrow\infty$ and using (3.5.8), we get

$$\left|\int \operatorname{sech}^4\left(\frac{x-vt}{\omega(t)}\right)\hat{e}(t,x)dx\right|\leq\int_t^{\infty}\frac{1}{\omega(t)}\int \operatorname{sech}^2\left(\frac{x-vs}{\omega(t)}\right)\hat{e}(s,x)dxds.$$

Now, sending $t \longrightarrow \infty$, we get

$$\lim_{t \longrightarrow \infty} \int \operatorname{sech}^4\left(\frac{x - vt}{\omega(t)}\right) \hat{e}(t,x)dx = 0.$$

Using again the definition of the h_1 and h_2 given in (3.4.17)-(3.4.18), and the condition (3.4.7) that ensures $2|h_2| \leq h_1$, one has

$$\hat{e} = -\kappa\partial_t\alpha h_1 + 2\kappa\partial_x\alpha h_2 \geq (\partial_t\alpha - |\partial_x\alpha|)|\kappa|h_1. \tag{3.5.10}$$

In addition, using that $\alpha > 0$, $\partial_t\alpha > 0$ and Lemma 3.4.2, we estimate $|\kappa|(\partial_t\alpha - |\partial_x\alpha|)$ as follows

$$\begin{aligned} |\kappa|(\partial_t\alpha - |\partial_x\alpha|) &= \frac{\alpha}{|(\partial_x\alpha)^2 - (\partial_t\alpha)^2|}(\partial_t\alpha - |\partial_x\alpha|) \\ &= \frac{\alpha}{|\partial_x\alpha| + \partial_t\alpha} \geq \frac{1}{2}\frac{\alpha}{\partial_t\alpha}. \end{aligned}$$

Therefore,

$$\frac{1}{2}\frac{\alpha}{\partial_t\alpha}h_1 \leq \hat{e}. \tag{3.5.11}$$

Now, for δ sufficiently small and $|v| < 1$, the first term in the right side of the equation (3.5.9) in the Theorem 3.5.1, can be estimated as follows

$$\frac{1 - |v| - \delta}{\omega(t)} \int \rho'\left(\frac{x - vt}{\omega(t)}\right) \hat{e}(t,x) \gtrsim \frac{1}{\omega(t)} \int \rho'\left(\frac{x - vt}{\omega(t)}\right) \frac{\alpha}{2\partial_t\alpha}h_1(t,x).$$

Finally, from the hypothesis (3.1.17) we have $(\partial_t\alpha)^{-1} > c_0 > 0$, and from the inequalities (3.5.10) and (3.5.11) we get the lower bound

$$\begin{aligned} &\int \operatorname{sech}^2\left(\frac{x - vt}{\omega(t)}\right) \left((\partial_x\Lambda)^2 + (\partial_t\Lambda)^2 + \sinh^2(\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2)\right)(t,x)dx \\ &\qquad \lesssim \int \operatorname{sech}^2\left(\frac{x - vt}{\omega(t)}\right) \hat{e}(t,x)dx, \end{aligned}$$

which finally shows the validity of Theorem 3.1.3 and the proof of (3.1.18).

***Remark* 3.5.2.** *Notice that from* (3.3.7) *we have* $\partial_t\alpha$ *uniformly in the Schwartz class, and one has the lower bound* $(\partial_t\alpha)^{-1} > c_0 > 0$. *Indeed,*

$$\frac{\alpha}{\partial_t\alpha} \gtrsim \frac{1}{\alpha_1 - |\alpha_0'|} \gtrsim 1.$$

consequently, the solutions from Theorem 3.1.1 satisfy Theorem 3.1.3 as well.

3.6 Applications to gravitational solitons

The purpose of this section is to analyze the dynamics of certain exact solutions to the Einstein field equations that can be derived from the Belinski-Zakaharov transform.

3.6.1 Generalized Kasner metric background

We begin our analysis by considering vacuum cosmologies described by the Kasner-type model. The Kasner metric, being one of the first known exact solutions in relativistic cosmology, remains one of the most important exact solutions in General Relativity. The generalized Kasner metric can be written in the diagonal form as follows:

$$ds^2 = f_0(t,x)(dx^2 - dt^2) + \alpha e^{u_0} dy^2 + \alpha e^{-u_0} dz^2, \tag{3.6.1}$$

where the function u_0 is given by

$$u_0(t,x) = d \ln \alpha, \tag{3.6.2}$$

and d is an arbitrary parameter, the *Kasner parameter*. It can be chosen either positive or negative, for instance $d = \pm 1$ corresponds to a region of Minkowski, $d = 0$ is an LRS space with Petrov type metric D. The x axis expands as time evolves if $|d| > 1$ and contracts if $|d| < 1$ [8]. The original Kasner metric [59] is obtained by taking $\alpha = t$ (timelike) and describes an anisotropic universe without matter. The original Kasner's choice does not fit into the assumptions of Theorem 3.1.1, and will be studied elsewhere.

In this work we will assume that $d \geq 1$ to ensure the correct finite energy condition. Naturally one has

$$\det g = \alpha^2, \quad g = \alpha \operatorname{diag}\left(e^{u_0}, e^{-u_0}\right).$$

As mentioned in the previous section, in order to identify the spacetime (3.6.1) with a cosmological model, the function $\alpha(t,x)$ must be globally timelike. If one compares (3.6.1) with (3.1.7), we have that Λ and ϕ should be given by

$$\Lambda^{(0)}(t,x) = u_0, \quad \text{and} \quad \phi^{(0)} = n\pi, \quad n \in \mathbb{Z}. \tag{3.6.3}$$

Lemma 3.6.1. *If the function $\alpha(t,x)$ satisfies the hypotheses of the Theorem 3.1.1 with $|\tilde{\alpha}_0| < \alpha_1$ and $\partial_t \alpha > 0$, then, the Kasner-type seed solution $(\Lambda^{(0)}, \phi^{(0)})$ of the* (3.1.8) *has finite nonnegative energy.*

Proof. The energy density proposed in (3.4.13), in this case, has the following structure:

$$\hat{e}_0 = -\frac{\alpha \partial_t \alpha}{(\partial_x \alpha)^2 - (\partial_t \alpha)^2}\left[(\partial_x \Lambda^{(0)})^2 + (\partial_t \Lambda^{(0)})^2\right] + \frac{2\alpha \partial_x \alpha}{(\partial_x \alpha)^2 - (\partial_t \alpha)^2} \partial_x \Lambda^{(0)} \partial_t \Lambda^{(0)}.$$

Using (3.6.3) we get:

$$\begin{cases} \partial_x \Lambda^{(0)} = \dfrac{d \partial_x \alpha}{\alpha}, \\ \partial_t \Lambda^{(0)} = \dfrac{d \partial_t \alpha}{\alpha}. \end{cases}$$

Then, since $(\partial_x \alpha)^2 - (\partial_t \alpha)^2 < 0$ we can simplify the expression (3.6.1) as:

$$\hat{e}_0 = \frac{d^2 \alpha \partial_t \alpha}{(\partial_x \alpha)^2 - (\partial_t \alpha)^2}\left(\frac{(\partial_x \alpha)^2 - (\partial_t \alpha)^2}{\alpha^2}\right) = d^2 \partial_t (\ln \alpha). \tag{3.6.4}$$

Notice that $\hat{e}_0$ is nonnegative and well-defined thanks to the timelike condition on α. And a similar way, we have the following momentum density:

$$\hat{p}_0 = d^2 \partial_x (\ln \alpha).$$

Given the definition of $\partial_t \alpha$ in terms of initial conditions $\tilde{\alpha}_0, \alpha_1$, and using (4.4.6), we can conclude that the energy (3.6.4) corresponding to this background metric is finite, i.e.

$$E[\Lambda^{(0)}, \phi^{(0)}; \alpha](t) = \int \hat{e}_0 dx = \int \frac{1}{2}[L(\ln \alpha) + \underline{L}(\ln \alpha)]dx < \infty.$$

The proof is complete. □

Theorems 3.1.1 and 3.1.3 imply in this case that for any $|v| < 1$ and $\omega(t) = t(\log t)^{-2}$,

$$\lim_{t \to +\infty} \int_{|x-vt| \leq \omega(t)} \frac{1}{\alpha^2} \left(\alpha_t^2 + \alpha_x^2\right)(t, x)dx = 0,$$

as naturally expected for solutions of the 1D wave equation. What is more interesting is the case of 1-soliton solutions.

***Remark* 3.6.1.** *As we can notice, until now it has been enough to impose certain constraints on the function α, to understand how we must define the energy in each spacetime and to understand how the solution of the system behaves in long time. At this point, it is important to emphasize that, in the framework of the Inverse Scattering theory for the Einstein equation, proposed by Belinski and Zakharov, in addition to the function $\alpha(t, x)$, it is necessary to introduce its conjugate derivative $\beta(t, x)$, related to $\alpha(t, x)$ by the identities*

$$\partial_t \beta = \partial_x \alpha, \quad \text{and} \quad \partial_x \beta = \partial_t \alpha,$$

β is introduced with the aim of describing the 1-soliton solution. In the setting of Theorem 3.1.1, from (3.3.6) *one sees that this function is given by*

$$\beta(t, x) := C + \tilde{\alpha}_0(2u) - \tilde{\alpha}_0(-2\underline{u}) + \int_0^{2u} \alpha_1(s)ds + \int_0^{-2\underline{u}} \alpha_1(s)ds, \quad C \in \mathbb{R}. \tag{3.6.5}$$

$\beta(t, x)$ is a second independent solution of the one-dimensional wave equation, and will be automatically spacelike in our setting. Indeed, from (3.3.1),

$$\beta_t = \tilde{\alpha}_0'(2u) + \tilde{\alpha}_0'(-2\underline{u}) + \alpha_1(2u) - \alpha_1(-2\underline{u}),$$

and

$$\beta_x = \tilde{\alpha}_0'(2u) - \tilde{\alpha}_0'(-2\underline{u}) + \alpha_1(2u) + \alpha_1(-2\underline{u}) > 0.$$

Consequently,

$$\beta_x - \beta_t = -2\tilde{\alpha}_0'(-2\underline{u}) + 2\alpha_1(-2\underline{u}) > 0,$$

and

$$\beta_x + \beta_t = 2\tilde{\alpha}_0'(2u) + 2\alpha_1(2u) > 0.$$

Belinski and Zakharov postulate that there is a smooth, one-to-one, surjective mapping between t, x and α, β, see [6].

In the setting of Theorem 3.1.1, it is clear that β defines a bounded function in spacetime. Additionally,

$$\lim_{t \to +\infty} \int_{|x-vt| \leq \omega(t)} \frac{1}{\beta^2} \left(\beta_t^2 + \beta_x^2\right)(t, x)dx = 0.$$

One Soliton Solution

Belinski and Verdaguer [8, p. 47] introduced the one soliton solution with *Kasner background*. Let $\omega \in \mathbb{R}$ be a fixed parameter. Let μ be

$$\mu := w - \beta - \sqrt{(w-\beta)^2 - \alpha^2}, \tag{3.6.6}$$

where β solves (3.6.5), namely $\partial_t \beta = \partial_x \alpha$. Then the 1-soliton with Kasner background is given by

$$g^{(1)} = \frac{1}{\mu \cosh(\rho)} \begin{bmatrix} e^{u_0}(\mu^2 e^{\rho} + \alpha^2 e^{-\rho}) & \alpha^2 - \mu^2 \\ \alpha^2 - \mu^2 & e^{-u_0}(\alpha^2 e^{\rho} + \mu^2 e^{-\rho}) \end{bmatrix}, \tag{3.6.7}$$

where

$$\rho = d \ln\left(\frac{\mu}{\alpha}\right) + C, \quad C \in \mathbb{R},$$
$$f = f^{(0)} \alpha^{1/2} \mu \cosh(\rho)(\alpha^2 - \mu^2)^{-1}.$$

Some important remarks are in order. First, from (3.6.6) one can see that if ω is sufficiently large, μ is real-valued. Assume $\omega > 0$ sufficiently large such that μ is real valued and positive. For the purposes of this work, we need a further simplification of (3.6.7). Assuming for simplicity $C = 0$ in ρ, after some computations we get

$$e^{u_0} = \alpha^d, \quad e^{\rho} = \left(\frac{\mu}{\alpha}\right)^d, \quad e^{-\rho} = \left(\frac{\mu}{\alpha}\right)^{-d}, \quad \cosh\rho = \frac{1}{2}\left(\left(\frac{\mu}{\alpha}\right)^d + \left(\frac{\mu}{\alpha}\right)^{-d}\right),$$

and for

$$m := \frac{\mu}{\alpha} = \frac{1}{\alpha}(w - \beta - \sqrt{(w-\beta)^2 - \alpha^2}),$$

one obtains

$$\begin{aligned} g^{(1)} &= \frac{2}{\mu\left(\left(\frac{\mu}{\alpha}\right)^d + \left(\frac{\mu}{\alpha}\right)^{-d}\right)} \begin{bmatrix} \alpha^d\left(\mu^2\left(\frac{\mu}{\alpha}\right)^d + \alpha^2\left(\frac{\mu}{\alpha}\right)^{-d}\right) & \alpha^2 - \mu^2 \\ \alpha^2 - \mu^2 & \alpha^{-d}\left(\alpha^2\left(\frac{\mu}{\alpha}\right)^d + \mu^2\left(\frac{\mu}{\alpha}\right)^{-d}\right) \end{bmatrix} \\ &= \frac{2\alpha}{m^d + m^{-d}} \begin{bmatrix} \alpha^d\left(m^{d+1} + m^{-d-1}\right) & \frac{1}{m} - m \\ \frac{1}{m} - m & \alpha^{-d}\left(m^{d-1} + m^{-d+1}\right) \end{bmatrix}. \end{aligned} \tag{3.6.8}$$

A first glimpse of the $g^{(1)}$ reveals that it will behave closely to the functions α and β. In this sense, we can say that the associated propagation speed must coincide with a support on the light cone. Comparing (3.6.8) with (3.1.7), we find that

$$\cosh\Lambda + \cos 2\phi \sinh\Lambda = \frac{2\alpha^d\left(m^{d+1} + m^{-d-1}\right)}{m^d + m^{-d}}$$
$$\cosh\Lambda - \cos 2\phi \sinh\Lambda = \frac{2\alpha^{-d}\left(m^{d-1} + m^{-d+1}\right)}{m^d + m^{-d}}$$
$$\sin 2\phi \sinh\Lambda = \frac{2\left(m^{-1} - m\right)}{m^d + m^{-d}}.$$

and therefore

$$\cosh\Lambda = \frac{\alpha^d\left(m^{d+1} + m^{-d-1}\right) + \alpha^{-d}\left(m^{d-1} + m^{-d+1}\right)}{m^d + m^{-d}}, \tag{3.6.9}$$

$$\tanh 2\phi = \frac{2\left(m^{-1}-m\right)}{\alpha^{d}\left(m^{d+1}+m^{-d-1}\right)-\alpha^{-d}\left(m^{d-1}+m^{-d+1}\right)}. \quad (3.6.10)$$

Since $\alpha, m > 0$ by hypothesis and $a + \frac{1}{a} \geq 1$ for $a > 0$, we get

$$\begin{aligned}&\frac{\alpha^{d}\left(m^{d+1}+m^{-d-1}\right)+\alpha^{-d}\left(m^{d-1}+m^{-d+1}\right)}{m^{d}+m^{-d}}\\&\quad=\frac{(\alpha^{d}m+\alpha^{-d}m^{-1})m^{d}+(\alpha^{d}m^{-1}+\alpha^{-d}m)m^{-d}}{m^{d}+m^{-d}}\geq 1.\end{aligned}$$

Using that $\sinh(\text{arccosh}x) = \sqrt{x^2-1}$ for $|x| \geq 1$, we get

$$\sinh^2\Lambda = \left(\frac{(\alpha^{d}m+\alpha^{-d}m^{-1})m^{d}+(\alpha^{d}m^{-1}+\alpha^{-d}m)m^{-d}}{m^{d}+m^{-d}}\right)^2 - 1. \quad (3.6.11)$$

As a first application, we use Theorem 3.1.1 to provide the following global existence result:

Corollary 3.6.1. *Under the smallness hypotheses on α from Theorem 3.1.1, suitable perturbations of the 1-soliton with Kasner metric background* (3.6.8) *are globally defined.*

Additionally, it is not difficult to realize that (Λ, ϕ, α) define globally defined finite energy solutions. Consequently, Theorem 3.1.3 allows us to conclude

Corollary 3.6.2. *Under the hypothesis on α obtained from Theorem 3.1.1, $g^{(1)}$ in the form (Λ, ϕ, α) satisfies the assumptions in Theorem 3.1.3, and consequently*

$$\lim_{t\to+\infty}\int_{|x-vt|\leq\omega(t)}\left(\Lambda_t^2+\Lambda_x^2+\sinh^2(\Lambda)(\phi_t^2+\phi_x^2)\right)(t,x)dx = 0.$$

Both corollaries prove Theorem 3.1.4.

Proof of Corollary 3.6.1. We have to verify the hypotheses in Theorem 3.1.1. Indeed, notice that from (3.6.5) (by choosing $C = 1$)

$$\alpha = 1 + \tilde{\alpha}_0, \quad \beta = 1 + \tilde{\beta}_0.$$

Similarly, m has the same asymptotic behavior, converging to a constant as time tends to infinity. It is then revealed that $\Lambda = \lambda + \widetilde{\Lambda}$ and ϕ in (3.6.9)-(3.6.10) follow an analogous structure, where perturbations can be made arbitrarily small, depending on a parameter ε. The rest of the hypotheses are standard and satisfied in a standard fashion. □

Proof of Corollary 3.6.2. Assume (3.1.15) and (3.1.17). In order to apply Theorems 3.1.2 and 3.1.3 we only need to check the finite energy condition for all time. This is clear from the form of $\Lambda = \lambda + \widetilde{\Lambda}$ and ϕ in (3.6.9)-(3.6.10): every squared time and space derivative will involve squared derivatives on α, μ and β, which have bounded in time finite energy. Finally, (3.6.11) ensures the last part of the energy condition. □

3.6.2 The Einstein-Rosen Metric

We study now a metric with cylindrical symmetry where our decay results do not apply. We will choose $\alpha = r > 0$ as solution to 1D waves, such that

$$\alpha_t = 0, \quad \alpha_r^2 - \alpha_t^2 = 1 > 0, \quad \frac{\alpha_t}{\alpha} = 0.$$

The cylindrical coordinates are $x^\mu = t, r$ and $x^a = \varphi, z$. The metric will be also diagonal ($\varphi = 0$). We have then the following spacetime interval [34]:

$$ds^2 := f^{(0)}(-dt^2 + dr^2) + e^{u_0}(rd\phi)^2 + e^{-u_0}dz^2, \tag{3.6.12}$$

where $f^{(0)} > 0$ and u_0 are functions of t, r and $u_0(t, r)$ satisfies the “cylindrical” wave equation

$$\partial_t^2 u_0 = \frac{1}{r}\partial_r(r\partial_r u_0). \tag{3.6.13}$$

This is the *Einstein-Rosen diagonal form.* As in the previous case, the Belinski-Zakharov setting is

$$g = \alpha \operatorname{diag}(e^{\Lambda^{(0)}}, e^{-\Lambda^{(0)}}), \quad \alpha = r, \quad u_0(t, r) = \Lambda^{(0)} - \ln r.$$

Then $\Lambda^{(0)}$ is as (3.1.7) if $\phi^{(0)} = n\pi$. It satisfies the equation

$$\partial_t^2 \Lambda^{(0)} = \frac{1}{r}\partial_r(r\partial_r \Lambda^{(0)}).$$

A particular choice for $\Lambda^{(0)}$ is given by

$$\Lambda^{(0)} = J_0(r)\sin(t), \tag{3.6.14}$$

where J_0 denotes the 0-th order Bessel function. From (3.6.14) clearly $\Lambda^{(0)}$ does not decay in time. For this case, the densities are given as follows:

$$\begin{aligned} e_0 &= r((\partial_t \Lambda^{(0)})^2 + (\partial_r \Lambda^{(0)})^2) \\ p_0 &= -2r\partial_t \Lambda^{(0)}\partial_r \Lambda^{(0)}. \end{aligned}$$

For completeness, the one soliton solution in this case was studied by Hadad in [52], and it is given as

$$g^{(1)} = \frac{1}{\mu\cosh(\gamma)}\begin{bmatrix} r^2 e^{u_0}\cosh(\gamma + \tilde{\gamma}) & \frac{r^2-\mu^2}{2\mu} \\ \frac{r^2-\mu^2}{2\mu} & e^{-u_0}\cosh(\gamma - \tilde{\gamma}) \end{bmatrix},$$

where $\omega \in \mathbb{R}$,

$$\begin{cases} \mu = \omega - t \pm \sqrt{(\omega - t)^2 - r^2} \\ \tilde{\gamma} = \ln\left(\frac{r}{|\mu|}\right) \mp \cosh^{-1}\left(\frac{\omega - t}{r}\right) \\ \gamma = K + u_0 + 2\rho + \tilde{\gamma} \qquad K = \ln(C), \quad C > 0 \\ \partial_t \rho = \frac{r}{\mu^2 - r^2}(r\partial_t u_0 + \mu\partial_r u_0) \\ \partial_r \rho = \frac{r}{\mu^2 - r^2}(r\partial_r u_0 + \mu\partial_t u_0) \\ f = C_0\sqrt{r}\frac{\mu}{\mu^2 - r^2}\cosh(\gamma) f^0. \end{cases}$$

Then, the fields Λ and ϕ are given as:

$$\Lambda = \cosh^{-1}\left(\frac{r}{2}e^{u_0}\cosh(\gamma+\tilde{\gamma}) + \frac{e^{-u_0}}{2r}\cosh(\gamma-\tilde{\gamma})\right)$$

$$\phi = \frac{1}{2}\tan^{-1}\left(\frac{r^2-\mu^2}{4\mu\cosh(\gamma)h}\right),$$

where

$$h = \frac{r}{2}e^{u_0}\cosh(\gamma+\tilde{\gamma}) - \frac{e^{-u_0}}{2r}\cosh(\gamma-\tilde{\gamma}).$$

A further study of this metric with other techniques will be done elsewhere.

3.7 Appendix

3.7.1 Proof of the Lemma 3.4.1

In this section we prove, for completeness, the modified continuity equations (3.4.6). First, let us start by writing the derivatives of the energy and momentum densities (respectively). Recall that, for this case, we have a full form for h_1 and h_2 introduced in (3.4.17) and (3.4.18), as follow:

$$h_1 = \frac{(\partial_x\alpha)^2 + (\partial_t\alpha)^2}{\alpha^2} + 4\sinh^2(\Lambda)\Big((\partial_t\phi)^2 + (\partial_x\phi)^2\Big) + (\partial_t\Lambda)^2 + (\partial_x\Lambda)^2$$

and

$$h_2 = \frac{\partial_x\alpha\partial_t\alpha}{\alpha^2} + \partial_x\Lambda\partial_t\Lambda + 4\partial_x\phi\partial_t\phi\sinh^2(\Lambda).$$

Then, we can write the energy and momentum density as:

$$e(t,x) = \kappa\partial_t\alpha h_1 - 2\kappa\partial_x\alpha h_2,$$
$$p(t,x) = \kappa\partial_x\alpha h_1 - 2\kappa\partial_t\alpha h_2,$$

where

$$\kappa := \frac{\alpha}{\alpha_x^2-\alpha_t^2}, \quad \text{and}$$

$$\partial_t\kappa = \frac{\alpha_t\alpha_x^2 - \alpha_t\alpha_t^2 - 2\alpha\alpha_x\alpha_{xt} + 2\alpha\alpha_t\alpha_{tt}}{(\alpha_x^2-\alpha_t^2)^2},$$

$$\partial_x\kappa = \frac{\alpha_x\alpha_x^2 - \alpha_x\alpha_t^2 - 2\alpha\alpha_x\alpha_{xx} + 2\alpha\alpha_t\alpha_{tx}}{(\alpha_x^2-\alpha_t^2)^2}.$$

For the derivatives of h_1 and h_2 we have:

$$
\begin{aligned}
\partial_t h_1 =& 2\frac{(\alpha_x\alpha_{xt}+\alpha_t\alpha_{tt})\alpha^2-\alpha\alpha_t(\alpha_x^2+\alpha_t^2)}{\alpha^4}+4\sinh(2\Lambda)\Lambda_t(\phi_x^2+\phi_t^2)\\
&+8\sinh^2(\Lambda)(\phi-x\phi_{xt}+\phi_t\phi_{tt})+2\Lambda_x\Lambda_{xt}+2\Lambda_t\Lambda_{tt}\\
\partial_x h_1 =& 2\frac{(\alpha_x\alpha_{xx}+\alpha_t\alpha_{tx})\alpha^2-\alpha\alpha_x(\alpha_x^2+\alpha_t^2)}{\alpha^4}+4\sinh(2\Lambda)\Lambda_x(\phi_x^2+\phi_t^2)\\
&+8\sinh^2(\Lambda)(\phi_x\phi_{xx}+\phi_t\phi_{tx})+2\Lambda_x\Lambda_{xx}+2\Lambda_t\Lambda_{tx}\\
\partial_t h_2 =& \frac{(\alpha_{tt}\alpha_x+\alpha_t\alpha_{xt})\alpha^2-2\alpha\alpha_t^2\alpha_x}{\alpha^2}+\Lambda_{xt}\Lambda_t+\Lambda_x\Lambda_{tt}\\
&+4\sinh(2\Lambda)\Lambda_t\phi_t\phi_x+4\sinh^2(\Lambda)(\phi_{xt}\phi_t+\phi_x\phi_{tt})\\
\partial_x h_2 =& \frac{(\alpha_{xx}\alpha_t+\alpha_x\alpha_{xt})\alpha^2-2\alpha\alpha_x^2\alpha_t}{\alpha^2}+\Lambda_{xx}\Lambda_t+\Lambda_x\Lambda_{tx}\\
&+4\sinh(2\Lambda)\Lambda_x\phi_t\phi_x+4\sinh^2(\Lambda)(\phi_{xx}\phi_t+\phi_x\phi_{tx})
\end{aligned}
$$

The first step will be to proof the first equation in (3.4.6), taking derivative in x for energy density and derivative in t for the momentum density, we have

$$
\begin{aligned}
\partial_x e(t,x) =& \kappa_x\alpha_t h_1-2\kappa_x\alpha_x h_2+\kappa\left(\alpha_{tx}h_1+\alpha_t\partial_x h_1-2\alpha_{xx}h_2-2\alpha_x\partial_x h_2\right)\\
=& Te_\alpha+Te_\Lambda+Te_\phi,\\
\partial_t p(t,x) =& \kappa_t\alpha_x h_1-2\kappa_t\alpha_t h_2+\kappa\left(\alpha_{tx}h_1+\alpha_x\partial_t h_1-2\alpha_{tt}h_2-2\alpha_t\partial_t h_2\right)\\
=& Tp_\alpha+Tp_\Lambda+Tp_\phi.
\end{aligned}
$$

Where the terms, for example, $Te_\alpha, Te_\Lambda, Te_\phi$ represent the terms in $\partial_x e$ that are related with α, Λ, and ϕ respectively, as follows:

$$
\begin{aligned}
Te_\alpha &:= \partial_x\left(\kappa\partial_t\alpha\frac{(\partial_x\alpha)^2+(\partial_t\alpha)^2}{\alpha^2}-2\kappa\partial_x\alpha\frac{\partial_x\alpha\partial_t\alpha}{\alpha^2}\right)\\
Te_\phi &:= \partial_x\left(4\kappa\partial_t\alpha\sinh^2(\Lambda)\Big((\partial_t\phi)^2+(\partial_x\phi)^2\Big)-8\kappa\partial_x\alpha\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right)\\
Te_\Lambda &:= \partial_x\left(\kappa\partial_t\alpha((\partial_t\Lambda)^2+(\partial_x\Lambda)^2)-2\kappa\partial_x\alpha\partial_x\Lambda\partial_t\Lambda\right).
\end{aligned}
$$

In the same way for $Tp_\alpha, Tp_\phi, Tp_\Lambda$, this time respect to $\partial_t p$:

$$
\begin{aligned}
Tp_\alpha &:= \partial_t\left(\kappa\partial_x\alpha\frac{(\partial_x\alpha)^2+(\partial_t\alpha)^2}{\alpha^2}-2\kappa\partial_t\alpha\frac{\partial_x\alpha\partial_t\alpha}{\alpha^2}\right)\\
Tp_\phi &:= \partial_t\left(4\kappa\partial_x\alpha\sinh^2(\Lambda)\Big((\partial_t\phi)^2+(\partial_x\phi)^2\Big)-8\kappa\partial_t\alpha\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right)\\
Tp_\Lambda &:= \partial_t\left(\kappa\partial_x\alpha((\partial_t\Lambda)^2+(\partial_x\Lambda)^2)-2\kappa\partial_x\alpha\partial_t\Lambda\partial_t\Lambda\right).
\end{aligned}
$$

Now, we are going to compute the sum of these two expressions, term by term, taking into account the structure of each term, starting by Te_α, Tp_α:

$$
\begin{aligned}
Tp_\alpha =& \frac{\alpha_x\alpha_x^2\alpha_t}{(\alpha_x^2-\alpha_t^2)^2\alpha^2}+\frac{\alpha_x\alpha_t\alpha_t^2}{(\alpha_x^2-\alpha_t^2)^2\alpha^2}-\frac{2\alpha\alpha_x^2\alpha_x^2\alpha_{xt}}{(\alpha_x^2-\alpha_t^2)^2\alpha^2}-\frac{2\alpha\alpha_x^2\alpha_{xt}\alpha_t^2}{(\alpha_x^2-\alpha_t^2)^2\alpha^2}+\frac{2\alpha\alpha_t\alpha_x\alpha_x^2\alpha_{tt}}{(\alpha_x^2-\alpha_t^2)^2\alpha^2}\\
&+\frac{2\alpha\alpha_t\alpha_x\alpha_t^2\alpha_{tt}}{(\alpha_x^2-\alpha_t^2)^2\alpha^2}-\frac{2\alpha_t^2\alpha_x\alpha_t}{(\alpha_x^2-\alpha_t^2)\alpha^2}+\frac{4\alpha\alpha_x^2\alpha_t^2\alpha_{xt}}{(\alpha_x^2-\alpha_t^2)^2\alpha^2}-\frac{4\alpha\alpha_t\alpha_x\alpha_t^2\alpha_{tt}}{(\alpha_x^2-\alpha_t^2)^2\alpha^2}+\frac{2\alpha\alpha_x^2\alpha_{xt}}{(\alpha_x^2-\alpha_t^2)\alpha^2}\\
&+\frac{2\alpha\alpha_x\alpha_t\alpha_{tt}}{(\alpha_x^2-\alpha_t^2)\alpha^2}-\frac{2\alpha_x\alpha_t\alpha_x^2}{(\alpha_x^2-\alpha_t^2)\alpha^2}-\frac{2\alpha_t^2\alpha_x\alpha_t}{(\alpha_x^2-\alpha_t^2)\alpha^2}-\frac{2\alpha_{tt}\alpha\alpha_x\alpha_t}{(\alpha_x^2-\alpha_t^2)\alpha^2}-\frac{2\alpha_t\alpha\alpha_{tt}\alpha_x}{(\alpha_x^2-\alpha_t^2)\alpha^2}\\
&-\frac{2\alpha_t^2\alpha\alpha_{xt}}{(\alpha_x^2-\alpha_t^2)\alpha^2}+\frac{4\alpha_t\alpha_t^2\alpha_x}{(\alpha_x^2-\alpha_t^2)\alpha^2}+\frac{\alpha\alpha_{tt}\alpha_x^2}{(\alpha_x^2-\alpha_t^2)\alpha^2}+\frac{\alpha\alpha_{xt}\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2}.
\end{aligned}
$$

Arranging terms,

$$
\begin{aligned}
Tp_\alpha =& \frac{\alpha_x^2}{\alpha_x^2-\alpha_t^2}\left(\frac{\alpha\alpha_{xt}-\alpha_x\alpha_t}{\alpha^2}\right) - \frac{\alpha_t^2}{\alpha_x^2-\alpha_t^2}\left(\frac{\alpha\alpha_{xt}-\alpha_t\alpha_x}{\alpha^2}\right) + \frac{2\alpha\alpha_x^2\alpha_t^2\alpha_{xt}}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} - \frac{2\alpha\alpha_t\alpha_x\alpha_{tt}\alpha_t^2}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} \\
&- \frac{2\alpha\alpha_x^2\alpha_x^2\alpha_{xt}}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} + \frac{2\alpha\alpha_t\alpha_x\alpha_x^2\alpha_{tt}}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} + \frac{2\alpha\alpha_x^2\alpha_{xt}}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha_t\alpha\alpha_{tt}\alpha_x}{(\alpha_x^2-\alpha_t^2)\alpha^2} \\
=& \frac{\alpha_x^2}{\alpha_x^2-\alpha_t^2}\partial_x\left(\frac{\alpha_t}{\alpha}\right) - \frac{\alpha_t^2}{\alpha_x^2-\alpha_t^2}\partial_x\left(\frac{\alpha_t}{\alpha}\right) - \frac{2\alpha_x^2\alpha\alpha_{xt}}{(\alpha_x^2-\alpha_t^2)\alpha^2} + \frac{2\alpha\alpha_t\alpha_x\alpha_{tt}}{(\alpha_x^2-\alpha_t^2)\alpha^2} \\
&+ \frac{2\alpha_x^2\alpha\alpha_{xt}}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha\alpha_t\alpha_x\alpha_{tt}}{(\alpha_x^2-\alpha_t^2)\alpha^2} = \partial_x\left(\frac{\alpha_t}{\alpha}\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
Te_\alpha =& \frac{\alpha_x^2}{\alpha_x^2-\alpha_t^2}\left(\frac{\alpha_x\alpha_t-\alpha\alpha_{tx}}{\alpha^2}\right) - \frac{\alpha_x\alpha_t\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} + \frac{3\alpha_{tx}\alpha\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha\alpha_x^2\alpha_t^2\alpha_{tx}}{(\alpha_x^2-\alpha_t^2)\alpha^2} \\
&+ \frac{2\alpha\alpha_x^2\alpha_x\alpha_t\alpha_{xx}}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} - \frac{2\alpha\alpha_x\alpha_{xx}\alpha_t\alpha_t^2}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} + \frac{2\alpha\alpha_t^2\alpha_{xt}\alpha_t^2}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} - \frac{2\alpha\alpha_{xx}\alpha_x\alpha_t}{s\alpha^2} \\
=& -\frac{\alpha_x^2}{\alpha_x^2-\alpha_t^2}\partial_x\left(\frac{\alpha_t}{\alpha}\right) - \frac{\alpha_x\alpha_t\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} + \frac{3\alpha_{tx}\alpha\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha_t^2\alpha_{tx}\alpha}{(\alpha_x^2-\alpha_t^2)\alpha^2} \\
&+ \frac{2\alpha\alpha_{xx}\alpha_t\alpha_x}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha\alpha_{xx}\alpha_x\alpha_t}{(\alpha_x^2-\alpha_t^2)\alpha^2} \\
=& -\frac{\alpha_x^2}{\alpha_x^2-\alpha_t^2}\partial_x\left(\frac{\alpha_t}{\alpha}\right) + \frac{\alpha_t^2}{\alpha_x^2-\alpha_t^2}\left(\frac{\alpha_{tx}\alpha-\alpha_x\alpha_t}{\alpha^2}\right) \\
=& -\frac{\alpha_x^2}{\alpha_x^2-\alpha_t^2}\partial_x\left(\frac{\alpha_t}{\alpha}\right) + \frac{\alpha_t^2}{\alpha_x^2-\alpha_t^2}\partial_x\left(\frac{\alpha_t}{\alpha}\right) \\
=& -\partial_x\left(\frac{\alpha_t}{\alpha}\right)\frac{\alpha_x^2-\alpha_t^2}{\alpha_x^2-\alpha_t^2} = -\partial_x\left(\frac{\alpha_t}{\alpha}\right),
\end{aligned}
$$

therefore $Te_\alpha + Tp_\alpha = 0$. We continue with the terms that depend mainly on Λ and its derivatives:

$$
\begin{aligned}
Te_\Lambda =& \frac{\alpha_x\alpha_t\Lambda_t^2}{\alpha_x^2-\alpha_t^2} + \frac{\alpha_x\alpha_t\Lambda_x^2}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha_x^2\Lambda_x\Lambda_t}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha\alpha_x\alpha_{xx}\alpha_t\Lambda_t^2}{(\alpha_x^2-\alpha_t^2)^2} - \frac{2\alpha\alpha_x\alpha_{xx}\alpha_t\Lambda_x^2}{(\alpha_x^2-\alpha_t^2)^2} + \frac{4\alpha\alpha_x^2\alpha_{xx}\Lambda_x\Lambda_t}{(\alpha_x^2-\alpha_t^2)^2} \\
&+ \frac{\alpha\alpha_{xt}\Lambda_t^2}{\alpha_x^2-\alpha_t^2} + \frac{\alpha\alpha_{xt}\Lambda_x^2}{\alpha_x^2-\alpha_t^2} + \frac{2\alpha\alpha_t\Lambda_x\Lambda_{xx}}{\alpha_x^2-\alpha_t^2} + \frac{2\alpha\alpha_t\Lambda_t\Lambda_{tx}}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha\alpha_{xx}\Lambda_x\Lambda_t}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha\alpha_x\Lambda_{xx}\Lambda_t}{\alpha_x^2-\alpha_t^2} \\
&- \frac{2\alpha\alpha_x\Lambda_x\Lambda_{tx}}{\alpha_x^2-\alpha_t^2} + \frac{2\alpha\alpha_t^2\alpha_{tx}\Lambda_t^2}{(\alpha_x^2-\alpha_t^2)^2} + \frac{2\alpha\alpha_t^2\alpha_{tx}\Lambda_x^2}{(\alpha_x^2-\alpha_t^2)^2} - \frac{4\alpha\alpha_t\alpha_{tx}\alpha_x\Lambda_x\Lambda_t}{(\alpha_x^2-\alpha_t^2)^2},
\end{aligned}
$$

and for p we have:

$$
\begin{aligned}
Tp_\Lambda =& \frac{\alpha_x\alpha_t\Lambda_t^2}{\alpha_x^2-\alpha_t^2} + \frac{\alpha_x\alpha_t\Lambda_x^2}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha_x^2\Lambda_x\Lambda_t}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha\alpha_x^2\alpha_{xt}\alpha_t\Lambda_t^2}{(\alpha_x^2-\alpha_t^2)^2} - \frac{2\alpha\alpha_x^2\alpha_{xt}\alpha_t\Lambda_x^2}{(\alpha_x^2-\alpha_t^2)^2} + \frac{4\alpha\alpha_x\alpha_t\alpha_{xt}\Lambda_x\Lambda_t}{(\alpha_x^2-\alpha_t^2)^2} \\
&+ \frac{\alpha\alpha_{xt}\Lambda_t^2}{\alpha_x^2-\alpha_t^2} + \frac{\alpha\alpha_{xt}\Lambda_x^2}{\alpha_x^2-\alpha_t^2} + \frac{2\alpha\alpha_x\Lambda_t\Lambda_{tt}}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha\alpha_t\Lambda_t\Lambda_{tx}}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha\alpha_{xx}\Lambda_x\Lambda_t}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha\alpha_t\Lambda_{tt}\Lambda_t}{\alpha_x^2-\alpha_t^2} \\
&+ \frac{2\alpha\alpha_x\Lambda_x\Lambda_{tx}}{\alpha_x^2-\alpha_t^2} + \frac{2\alpha\alpha_t\alpha_x\alpha_{tt}\Lambda_t^2}{(\alpha_x^2-\alpha_t^2)^2} + \frac{2\alpha\alpha_t\alpha_x\alpha_{tt}\Lambda_x^2}{(\alpha_x^2-\alpha_t^2)^2} - \frac{4\alpha\alpha_t^2\alpha_{tt}\Lambda_x\Lambda_t}{(\alpha_x^2-\alpha_t^2)^2}.
\end{aligned}
$$

If we sum these two terms and using the first equation in the (3.1.8), we get

$$
\begin{aligned}
&Te_\Lambda + Tp_\Lambda = \\
&= \frac{2(\alpha_x\Lambda_t - \alpha_t\Lambda_x)[\alpha\Lambda_{tt} - \alpha\Lambda_{xx}]}{\alpha_x^2 - \alpha_t^2} + \frac{2\alpha_t\Lambda_t(\alpha_x\Lambda_t - \alpha_t\Lambda_x)}{\alpha_x^2 - \alpha_t^2} - \frac{2\alpha_x\Lambda_x(\alpha_x\Lambda_t - \alpha_t\Lambda_x)}{\alpha_x^2 - \alpha_t^2} + R \\
&= \frac{2}{\alpha_x^2 - \alpha_t^2}(\alpha_x\Lambda_t - \alpha_t\Lambda_x)\left[\partial_t(\alpha\Lambda_t) - \partial_x(\alpha\Lambda_x)\right] + R \\
&= \frac{2}{\alpha_x^2 - \alpha_t^2}(\alpha_x\Lambda_t - \alpha_t\Lambda_x)\left[2\alpha\phi_t^2\sinh(2\Lambda) - 2\alpha\phi_x^2\sinh(2\Lambda)\right] + R,
\end{aligned}
$$

where R represents the remainder of the terms in the sum above. After simplification, we have that R is actually equal to zero, indeed

$$
\begin{aligned}
R = &\frac{2\alpha_{xt}\alpha\Lambda_t^2}{(\alpha_x^2 - \alpha_t^2)^2}(\alpha_t^2 - \alpha_x^2) + \frac{2\alpha_{xt}\alpha\Lambda_x^2}{(\alpha_x^2 - \alpha_t^2)^2}(\alpha_t^2 - \alpha_x^2) + \frac{4\alpha_{tt}\alpha\Lambda_t\Lambda_x}{(\alpha_x^2 - \alpha_t^2)^2}(\alpha_x^2 - \alpha_t^2) + \frac{2\alpha_{xt}\alpha\Lambda_t^2}{\alpha_x^2 - \alpha_t^2} \\
&+ \frac{2\alpha_{xt}\alpha\Lambda_x^2}{(\alpha_x^2 - \alpha_t^2)^2} - \frac{4\alpha_{tt}\alpha\Lambda_t\Lambda_x}{\alpha_x^2 - \alpha_t^2} = 0.
\end{aligned}
$$

The last terms to simplify are the terms that depend mainly on ϕ, first, let us start with the terms related to ϕ in momentum density derivatives:

$$
\begin{aligned}
Tp_\phi = &\frac{4\alpha_t\alpha_x\sinh^2(\Lambda)\phi_t^2}{\alpha_x^2 - \alpha_t^2} + \frac{4\alpha_t\alpha_x\sinh^2(\Lambda)\phi_x^2}{\alpha_x^2 - \alpha_t^2} - \frac{8\alpha_t^2\sinh^2(\Lambda)\phi_x\phi_t}{\alpha_x^2 - \alpha_t^2} - \frac{8\alpha\alpha_x^2\alpha_{tx}\sinh^2(\Lambda)\phi_x^2}{(\alpha_x^2 - \alpha_t^2)^2} \\
&- \frac{8\alpha\alpha_x^2\alpha_{tx}\sinh^2(\Lambda)\phi_t^2}{(\alpha_x^2 - \alpha_t^2)^2} + \frac{16\alpha\alpha_x\alpha_{xt}\alpha_t\sinh^2(\Lambda)\phi_x\phi_t}{(\alpha_x^2 - \alpha_t^2)^2} + \frac{8\alpha\alpha_t\alpha_{tt}\alpha_x\sinh^2(\Lambda)\phi_t^2}{(\alpha_x^2 - \alpha_t^2)^2} \\
&+ \frac{8\alpha\alpha_t\alpha_{tt}\alpha_x\sinh^2(\Lambda)\phi_x^2}{(\alpha_x^2 - \alpha_t^2)^2} - \frac{16\alpha\alpha_t^2\alpha_{tt}\sinh^2(\Lambda)\phi_t\phi_x}{(\alpha_x^2 - \alpha_t^2)^2} + \frac{4\alpha\alpha_{xt}\sinh^2(\Lambda)\phi_t^2}{\alpha_x^2 - \alpha_t^2} + \frac{4\alpha\alpha_{xt}\sinh^2(\Lambda)\phi_x^2}{\alpha_x^2 - \alpha_t^2} \\
&+ \frac{4\alpha\alpha_x\sinh(2\Lambda)\Lambda_t\phi_x^2}{\alpha_x^2 - \alpha_t^2} + \frac{4\alpha\alpha_x\sinh(2\Lambda)\Lambda_t\phi_t^2}{\alpha_x^2 - \alpha_t^2} + \frac{8\alpha\alpha_x\sinh^2(\Lambda)\phi_x\phi_{xt}}{\alpha_x^2 - \alpha_t^2} + \frac{8\alpha\alpha_x\sinh^2(\Lambda)\phi_x\phi_{tt}}{\alpha_x^2 - \alpha_t^2} \\
&- \frac{8\alpha\alpha_{tt}\sinh^2(\Lambda)\phi_x\phi_t}{\alpha_x^2 - \alpha_t^2} - \frac{8\alpha\alpha_t\sinh(2\Lambda)\Lambda_t\phi_x\phi_t}{\alpha_x^2 - \alpha_t^2} - \frac{8\alpha\alpha_t\sinh^2(\Lambda)\phi_{xt}\phi_t}{\alpha_x^2 - \alpha_t^2} - \frac{8\alpha\alpha_t\sinh^2(\Lambda)\phi_{tt}\phi_x}{\alpha_x^2 - \alpha_t^2}
\end{aligned}
$$

Now, let us pass to the terms in the derivative of the energy density

$$
\begin{aligned}
Te_\phi = &\frac{4\alpha_t\alpha_x\sinh^2(\Lambda)\phi_t^2}{\alpha_x^2 - \alpha_t^2} + \frac{4\alpha_t\alpha_x\sinh^2(\Lambda)\phi_x^2}{\alpha_x^2 - \alpha_t^2} - \frac{8\alpha_x^2\sinh^2(\Lambda)\phi_x\phi_t}{\alpha_x^2 - \alpha_t^2} - \frac{8\alpha\alpha_x\alpha_t\alpha_{xx}\sinh^2(\Lambda)\phi_t^2}{(\alpha_x^2 - \alpha_t^2)^2} \\
&- \frac{8\alpha\alpha_x\alpha_t\alpha_{xx}\sinh^2(\Lambda)\phi_x^2}{(\alpha_x^2 - \alpha_t^2)^2} + \frac{16\alpha\alpha_x^2\alpha_{xx}\sinh^2(\Lambda)\phi_x\phi_t}{(\alpha_x^2 - \alpha_t^2)^2} + \frac{8\alpha\alpha_t^2\alpha_{tx}\sinh^2(\Lambda)\phi_t^2}{(\alpha_x^2 - \alpha_t^2)^2} \\
&+ \frac{8\alpha\alpha_t^2\alpha_{tx}\sinh^2(\Lambda)\phi_x^2}{(\alpha_x^2 - \alpha_t^2)^2} - \frac{16\alpha\alpha_t\alpha_{tx}\alpha_x\sinh^2(\Lambda)\phi_t\phi_x}{(\alpha_x^2 - \alpha_t^2)^2} + \frac{4\alpha\alpha_{xt}\sinh^2(\Lambda)\phi_t^2}{\alpha_x^2 - \alpha_t^2} + \frac{4\alpha\alpha_{xt}\sinh^2(\Lambda)\phi_x^2}{\alpha_x^2 - \alpha_t^2} \\
&+ \frac{4\alpha\alpha_t\sinh(2\Lambda)\Lambda_x\phi_x^2}{\alpha_x^2 - \alpha_t^2} + \frac{4\alpha\alpha_t\sinh(2\Lambda)\Lambda_x\phi_t^2}{\alpha_x^2 - \alpha_t^2} + \frac{8\alpha\alpha_t\sinh^2(\Lambda)\phi_x\phi_{xx}}{\alpha_x^2 - \alpha_t^2} + \frac{8\alpha\alpha_t\sinh^2(\Lambda)\phi_t\phi_{tx}}{\alpha_x^2 - \alpha_t^2} \\
&- \frac{8\alpha\alpha_{tt}\sinh^2(\Lambda)\phi_x\phi_t}{\alpha_x^2 - \alpha_t^2} - \frac{8\alpha\alpha_x\sinh(2\Lambda)\Lambda_x\phi_x\phi_t}{\alpha_x^2 - \alpha_t^2} - \frac{8\alpha\alpha_x\sinh^2(\Lambda)\phi_{xx}\phi_t}{\alpha_x^2 - \alpha_t^2} - \frac{8\alpha\alpha_x\sinh^2(\Lambda)\phi_x\phi_{tx}}{\alpha_x^2 - \alpha_t^2}.
\end{aligned}
$$

In the next step we will sum $Te_\phi + Tp_\phi + Te_\Lambda + Tp_\Lambda$, then, simplify the similar terms and cancel the corresponding terms, then we obtain the following expression:

$$
\begin{aligned}
&Te_\phi + Tp_\phi + Te_\Lambda + Tp_\Lambda \\
&= \frac{8\alpha_x\alpha_t \sinh^2(\Lambda)\phi_t^2}{\alpha_x^2-\alpha_t^2} + \frac{8\alpha_x\alpha_t \sinh^2(\Lambda)\phi_x^2}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha_x^2 \sinh^2(\Lambda)\phi_t\phi_x}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha_t^2 \sinh^2(\Lambda)\phi_t\phi_x}{\alpha_x^2-\alpha_t^2} \\
&+ \frac{8\alpha\alpha_{tx} \sinh^2(\Lambda)\phi_t^2}{(\alpha_x^2-\alpha_t^2)^2}(\alpha_t^2-\alpha_x^2) + \frac{8\alpha\alpha_{tx} \sinh^2(\Lambda)\phi_x^2}{(\alpha_x^2-\alpha_t^2)^2}(\alpha_t^2-\alpha_x^2) \\
&+ \frac{16\alpha\alpha_{xx} \sinh^2(\Lambda)\phi_t\phi_x}{(\alpha_x^2-\alpha_t^2)^2}(\alpha_x^2-\alpha_t^2) + \frac{8\alpha\alpha_{tx} \sinh^2(\Lambda)\phi_t^2}{\alpha_x^2-\alpha_t^2} + \frac{8\alpha\alpha_{tx} \sinh^2(\Lambda)\phi_x^2}{\alpha_x^2-\alpha_t^2} \\
&+ \frac{4\alpha\alpha_t \sinh(2\Lambda)\Lambda_x\phi_x^2}{\alpha_x^2-\alpha_t^2} + \frac{4\alpha\alpha_t \sinh(2\Lambda)\Lambda_x\phi_t^2}{\alpha_x^2-\alpha_t^2} + \frac{4\alpha\alpha_x \sinh(2\Lambda)\Lambda_t\phi_x^2}{\alpha_x^2-\alpha_t^2} \\
&+ \frac{4\alpha\alpha_x \sinh(2\Lambda)\Lambda_t\phi_t^2}{\alpha_x^2-\alpha_t^2} - \frac{16\alpha\alpha_{tt} \sinh^2(\Lambda)\phi_x\phi_t}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha\alpha_x \sinh(2\Lambda)\Lambda_x\phi_x\phi_t}{\alpha_x^2-\alpha_t^2} \\
&- \frac{8\alpha\alpha_x \sinh^2(\Lambda)\phi_{xx}\phi_t}{\alpha_x^2-\alpha_t^2} + \frac{8\alpha\alpha_x \sinh^2(\Lambda)\phi_t\phi_{tt}}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha\alpha_t \sinh(2\Lambda)\Lambda_t\phi_x\phi_t}{\alpha_x^2-\alpha_t^2} \\
&- \frac{8\alpha\alpha_t \sinh^2(\Lambda)\phi_x\phi_{tt}}{\alpha_x^2-\alpha_t^2} + \frac{4\alpha\alpha_x \sinh(2\Lambda)\Lambda_t\phi_t^2}{\alpha_x^2-\alpha_t^2} - \frac{4\alpha\alpha_x \sinh(2\Lambda)\Lambda_t\phi_x^2}{\alpha_x^2-\alpha_t^2} \\
&- \frac{4\alpha\alpha_t \sinh(2\Lambda)\Lambda_x\phi_t^2}{\alpha_x^2-\alpha_t^2} + \frac{4\alpha\alpha_t \sinh(2\Lambda)\Lambda_x\phi_x^2}{\alpha_x^2-\alpha_t^2} + \frac{8\alpha\alpha_t \sinh^2(\Lambda)\phi_x\phi_{xx}}{\alpha_x^2-\alpha_t^2}.
\end{aligned}
$$

In this expression we have several terms that will cancel out. They can be gathered in such a way that we can use the second equation in the system (3.1.8). We have

$$
\begin{aligned}
Te_\phi + Tp_\phi + Te_\Lambda + Tp_\Lambda =& \sinh^2(\Lambda)(\alpha_t\phi_t - \alpha_x\phi_x + \alpha\phi_{tt} - \alpha\phi_{tt})\left(\frac{8\alpha_x\phi_t - 8\alpha_t\phi_x}{\alpha_x^2-\alpha_t^2}\right) \\
&+ \sinh(2\Lambda)(\alpha\phi_t\Lambda_t - \alpha\phi_x\Lambda_x)\left(\frac{8\alpha_x\phi_t - 8\alpha_t\phi_x}{\alpha_x^2-\alpha_t^2}\right) \\
=&(\partial_t(\alpha\sinh^2\Lambda\partial_t\phi) - \partial_x(\alpha\sinh^2\Lambda\partial_x\phi))\left(\frac{8\alpha_x\phi_t - 8\alpha_t\phi_x}{\alpha_x^2-\alpha_t^2}\right) = 0.
\end{aligned}
$$

We conclude that:

$$\partial_t p(t,x) + \partial_x e(t,x) = 0.$$

In the second part of the proof, we are going to show the second equation in (3.4.6). For this, we will again use the notation for grouping terms in the following way

$$
\begin{aligned}
\partial_t e(t,x) &= \gamma_t\alpha_t h_1 - 2\gamma_t\alpha_x h_2 + \gamma\left(\alpha_{tt}h_1 + \alpha_t\partial_t h_1 - 2\alpha_{xt}h_2 - 2\alpha_x\partial_t h_2\right) \\
&= Te_\alpha + Te_\Lambda + Te_\phi, \\
\partial_x p(t,x) &= \partial_x\gamma\left(\alpha_x h_1 - 2\alpha_t h_2\right) + \gamma\left(\alpha_{xx}h_1 + \alpha_x\partial_x h_1 - 2\alpha_{tx}h_2 - 2\alpha_t\partial_x h_2\right) \\
&= Tp_\alpha + Tp_\Lambda + Tp_\phi,
\end{aligned}
$$

where the terms $Te_\alpha, Te_\phi, Te_\Lambda$ has the same form than in the before, but, this time respect to the terms for $\partial_t e$ (or $\partial_x p$ respectively) Let us simplify each term, starting with the terms in α:

$$
\begin{aligned}
Te_\alpha =& -\frac{\alpha\alpha_x^2\alpha_{tt}}{(\alpha_x^2-\alpha_t^2)\alpha^2} + \frac{2\alpha_x^2\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} + \frac{3\alpha\alpha_t^2\alpha_{tt}}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha\alpha_{tx}\alpha_x\alpha_t}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha_t^2\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} \\
& - \frac{\alpha_t^2\alpha_x^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} + \frac{\alpha_t^2\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha\alpha_x\alpha_t\alpha_{xt}\alpha_x^2}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} - \frac{2\alpha\alpha_x\alpha_{xt}\alpha_t^2\alpha_t}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} + \frac{2\alpha\alpha_t^2\alpha_{tt}\alpha_x^2}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} \\
& + \frac{2\alpha\alpha_t^2\alpha_{tt}\alpha_t^2}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} + \frac{4\alpha\alpha_x\alpha_x^2\alpha_{xt}\alpha_t}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} - \frac{4\alpha\alpha_t^2\alpha_{tt}\alpha_x^2}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} \\
=& -\frac{\alpha\alpha_x^2\alpha_{tt}}{(\alpha_x^2-\alpha_t^2)\alpha^2} + \frac{2\alpha_x^2\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} + \frac{\alpha_t^2}{(\alpha_x^2-\alpha_t^2)}\partial_x\left(\frac{\alpha_x}{\alpha}\right) + \frac{2\alpha\alpha_t^2\alpha_{tt}}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{\alpha_t^2\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} \\
& - \frac{2\alpha\alpha_{tx}\alpha_x\alpha_t}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha\alpha_t^2\alpha_{tt}}{(\alpha_x^2-\alpha_t^2)\alpha^2} + \frac{2\alpha\alpha_x\alpha_{xt}\alpha_t}{(\alpha_x^2-\alpha_t^2)\alpha^2} \\
=& -\frac{\alpha_t^2}{(\alpha_x^2-\alpha_t^2)}\partial_x\left(\frac{\alpha_x}{\alpha}\right) - \frac{\alpha_{tt}}{\alpha} + \frac{\alpha_t^2}{(\alpha_x^2-\alpha_t^2)}\partial_t\left(\frac{\alpha_t}{\alpha}\right) = -\partial_t\left(\frac{\alpha_t}{\alpha}\right),
\end{aligned}
$$

for another hand, using a similar simplification as above, we obtain for Tp_α the following expression:

$$
\begin{aligned}
Tp_\alpha =& \frac{\alpha_x^2\alpha_x^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} + \frac{\alpha_x^2\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha_x^2\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha\alpha_x^2\alpha_x^2\alpha_{xx}}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} - \frac{2\alpha\alpha_x^2\alpha_t^2\alpha_{xx}}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} \\
& + \frac{4\alpha\alpha_x^2\alpha_t^2\alpha_{xx}}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} + \frac{2\alpha\alpha_t\alpha_x\alpha_{xt}\alpha_x^2}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} + \frac{2\alpha\alpha_t\alpha_x\alpha_{xt}\alpha_t^2}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} - \frac{4\alpha\alpha_t\alpha_x\alpha_{xt}\alpha_t^2}{(\alpha_x^2-\alpha_t^2)^2\alpha^2} + \frac{\alpha\alpha_{xx}\alpha_x^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} \\
& + \frac{\alpha\alpha_{xx}\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} + \frac{2\alpha\alpha_{xx}\alpha_x^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} + \frac{2\alpha\alpha_{tx}\alpha_x\alpha_t}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha_x^2\alpha_x^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha_x^2\alpha_t^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} \\
& - \frac{2\alpha\alpha_x\alpha_t\alpha_{tx}}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha\alpha_t^2\alpha_{xx}}{(\alpha_x^2-\alpha_t^2)\alpha^2} - \frac{2\alpha\alpha_t\alpha_x\alpha_{tx}}{(\alpha_x^2-\alpha_t^2)\alpha^2} + \frac{4\alpha_t^2\alpha_x^2}{(\alpha_x^2-\alpha_t^2)\alpha^2} \\
=& \,\partial_x\left(\frac{\alpha_x}{\alpha}\right).
\end{aligned}
$$

Now, for the terms in Te_Λ and Tp_Λ we have

$$
\begin{aligned}
Te_\Lambda =& \frac{\alpha_t^2\Lambda_t^2}{\alpha_x^2-\alpha_t^2} + \frac{\alpha_t^2\Lambda_x^2}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha_x\alpha_t\Lambda_x\Lambda_t}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha\alpha_x\alpha_{tx}\alpha_t\Lambda_t^2}{(\alpha_x^2-\alpha_t^2)^2} - \frac{2\alpha\alpha_x\alpha_{tx}\alpha_t\Lambda_x^2}{(\alpha_x^2-\alpha_t^2)^2} + \frac{4\alpha\alpha_x^2\alpha_{xt}\Lambda_x\Lambda_t}{(\alpha_x^2-\alpha_t^2)^2} \\
& + \frac{2\alpha\alpha_{tt}\alpha_t^2\Lambda_t^2}{\alpha_x^2-\alpha_t^2} + \frac{2\alpha\alpha_t^2\alpha_{tt}\Lambda_x^2}{\alpha_x^2-\alpha_t^2} - \frac{4\alpha\alpha_t\alpha_x\alpha_{tt}\Lambda_x\Lambda_t}{\alpha_x^2-\alpha_t^2} + \frac{\alpha\alpha_{tt}\Lambda_t^2}{\alpha_x^2-\alpha_t^2} + \frac{\alpha\alpha_{xx}\Lambda_x^2}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha\alpha_x\Lambda_x\Lambda_{tt}}{\alpha_x^2-\alpha_t^2} \\
& + \frac{2\alpha\alpha_x\alpha_t\Lambda_x\Lambda_{tx}}{\alpha_x^2-\alpha_t^2} + \frac{2\alpha\alpha_t\Lambda_t\Lambda_{tt}}{(\alpha_x^2-\alpha_t^2)^2} - \frac{2\alpha\alpha_{tx}\Lambda_x\Lambda_t}{(\alpha_x^2-\alpha_t^2)^2} - \frac{2\alpha\alpha_x\Lambda_{xt}\Lambda_t}{(\alpha_x^2-\alpha_t^2)^2},
\end{aligned}
$$

and, for p we have:

$$
\begin{aligned}
Tp_\Lambda =& \frac{\alpha_x^2\Lambda_t^2}{\alpha_x^2-\alpha_t^2} + \frac{\alpha_x^2\Lambda_x^2}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha_x\alpha_t\Lambda_x\Lambda_t}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha\alpha_x^2\alpha_{xx}\Lambda_t^2}{(\alpha_x^2-\alpha_t^2)^2} - \frac{2\alpha\alpha_x^2\alpha_{xx}\Lambda_x^2}{(\alpha_x^2-\alpha_t^2)^2} + \frac{4\alpha\alpha_x\alpha_t\alpha_{xx}\Lambda_x\Lambda_t}{(\alpha_x^2-\alpha_t^2)^2} \\
& + \frac{2\alpha\alpha_{xt}\alpha_x\alpha_t\Lambda_t^2}{\alpha_x^2-\alpha_t^2} + \frac{2\alpha\alpha_{xt}\alpha_x\alpha_t\Lambda_x^2}{\alpha_x^2-\alpha_t^2} - \frac{4\alpha\alpha_t^2\alpha_{xt}\Lambda_t\Lambda_x}{\alpha_x^2-\alpha_t^2} - \frac{\alpha\alpha_{xx}\Lambda_t^2}{\alpha_x^2-\alpha_t^2} + \frac{\alpha\alpha_{xx}\Lambda_x^2}{\alpha_x^2-\alpha_t^2} + \frac{2\alpha\alpha_x\Lambda_{xx}\Lambda_x}{\alpha_x^2-\alpha_t^2} \\
& + \frac{2\alpha\alpha_x\Lambda_t\Lambda_{tx}}{\alpha_x^2-\alpha_t^2} - \frac{2\alpha\alpha_{xt}\Lambda_t\Lambda_x}{(\alpha_x^2-\alpha_t^2)^2} - \frac{2\alpha\alpha_t\alpha_x\Lambda_t\Lambda_{xx}}{(\alpha_x^2-\alpha_t^2)^2} - \frac{2\alpha\alpha_t\Lambda_x\Lambda_{tx}}{(\alpha_x^2-\alpha_t^2)^2}.
\end{aligned}
$$

If we sum up the terms and using the first equation in the system (3.1.8) and after simplification we obtain

$$
\begin{aligned}
Te_\Lambda + Tp_\Lambda &= \left(\frac{2\alpha_t\Lambda_t - 2\alpha_x\Lambda_x}{\alpha_x^2-\alpha_t^2}\right)(\alpha_t\Lambda_t - \alpha_x\Lambda - \alpha\Lambda_{xx} + \alpha\Lambda_{tt}) \\
&\quad + \frac{(\alpha_t^2-\alpha_x^2)\Lambda_x^2}{\alpha_x^2-\alpha_t^2} + \frac{(\alpha_x^2-\alpha_t^2)\Lambda_t^2}{\alpha_x^2-\alpha_t^2} \\
&= \left(\frac{2\alpha_t\Lambda_t - 2\alpha_x\Lambda_x}{\alpha_x^2-\alpha_t^2}\right)(\alpha_t\Lambda_t + \alpha\Lambda_{tt} - \alpha_x\Lambda_x - \alpha\Lambda_{xx}) + \Lambda_t^2 - \Lambda_x^2 \\
&= \left(\frac{2\alpha_t\Lambda_t - 2\alpha_x\Lambda_x}{\alpha_x^2-\alpha_t^2}\right)\big(2\alpha\phi_t^2\sinh(2\Lambda) - 2\alpha\phi_x^2\sinh(2\Lambda)\big) + \Lambda_t^2 - \Lambda_x^2.
\end{aligned}
$$

To conclude the result, let us simplify the terms in ϕ:

$$
\begin{aligned}
Tp_\phi = {} & \frac{4\alpha_x^2\sinh^2(\Lambda)\phi_t^2}{\alpha_x^2-\alpha_t^2} + \frac{4\alpha_x^2\sinh^2(\Lambda)\phi_x^2}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha_t\alpha_x\sinh^2(\Lambda)\phi_x\phi_t}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha\alpha_x^2\alpha_{xx}\sinh^2(\Lambda)\phi_t^2}{(\alpha_x^2-\alpha_t^2)^2} \\
& - \frac{8\alpha\alpha_x^2\alpha_{xx}\sinh^2(\Lambda)\phi_x^2}{(\alpha_x^2-\alpha_t^2)^2} + \frac{16\alpha\alpha_x\alpha_{xx}\alpha_t\sinh^2(\Lambda)\phi_x\phi_t}{(\alpha_x^2-\alpha_t^2)^2} + \frac{8\alpha\alpha_t\alpha_{tx}\alpha_x\sinh^2(\Lambda)\phi_t^2}{(\alpha_x^2-\alpha_t^2)^2} \\
& + \frac{8\alpha\alpha_t\alpha_{tx}\alpha_x\sinh^2(\Lambda)\phi_x^2}{(\alpha_x^2-\alpha_t^2)^2} - \frac{16\alpha\alpha_t^2\alpha_{tx}\sinh^2(\Lambda)\phi_t\phi_x}{(\alpha_x^2-\alpha_t^2)^2} + \frac{4\alpha\alpha_{xx}\sinh^2(\Lambda)\phi_t^2}{\alpha_x^2-\alpha_t^2} + \frac{4\alpha\alpha_{xx}\sinh^2(\Lambda)\phi_x^2}{\alpha_x^2-\alpha_t^2} \\
& + \frac{4\alpha\alpha_x\sinh(2\Lambda)\Lambda_x\phi_x^2}{\alpha_x^2-\alpha_t^2} + \frac{4\alpha\alpha_x\sinh(2\Lambda)\Lambda_x\phi_t^2}{\alpha_x^2-\alpha_t^2} + \frac{8\alpha\alpha_x\sinh^2(\Lambda)\phi_x\phi_{xx}}{\alpha_x^2-\alpha_t^2} + \frac{8\alpha\alpha_x\sinh^2(\Lambda)\phi_t\phi_{tt}}{\alpha_x^2-\alpha_t^2} \\
& - \frac{8\alpha\alpha_{tx}\sinh^2(\Lambda)\phi_x\phi_t}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha\alpha_t\sinh(2\Lambda)\Lambda_x\phi_x\phi_t}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha\alpha_t\sinh^2(\Lambda)\phi_{xx}\phi_t}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha\alpha_t\sinh^2(\Lambda)\phi_{tx}\phi_x}{\alpha_x^2-\alpha_t^2}.
\end{aligned}
$$

Now, let us pass to the terms in the derivative of the energy density

$$
\begin{aligned}
Te_\phi = {} & \frac{4\alpha_t^2\sinh^2(\Lambda)\phi_t^2}{\alpha_x^2-\alpha_t^2} + \frac{4\alpha_t^2\sinh^2(\Lambda)\phi_x^2}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha_x\alpha_t\sinh^2(\Lambda)\phi_x\phi_t}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha\alpha_x\alpha_t\alpha_{xt}\sinh^2(\Lambda)\phi_t^2}{(\alpha_x^2-\alpha_t^2)^2} \\
& - \frac{8\alpha\alpha_x\alpha_t\alpha_{xt}\sinh^2(\Lambda)\phi_x^2}{(\alpha_x^2-\alpha_t^2)^2} + \frac{16\alpha\alpha_x^2\alpha_{xt}\sinh^2(\Lambda)\phi_x\phi_t}{(\alpha_x^2-\alpha_t^2)^2} + \frac{8\alpha\alpha_t^2\alpha_{tt}\sinh^2(\Lambda)\phi_t^2}{(\alpha_x^2-\alpha_t^2)^2} \\
& + \frac{8\alpha\alpha_t^2\alpha_{xx}\sinh^2(\Lambda)\phi_x^2}{(\alpha_x^2-\alpha_t^2)^2} - \frac{16\alpha\alpha_t\alpha_{tt}\alpha_x\sinh^2(\Lambda)\phi_t\phi_x}{(\alpha_x^2-\alpha_t^2)^2} + \frac{4\alpha\alpha_{tt}\sinh^2(\Lambda)\phi_t^2}{\alpha_x^2-\alpha_t^2} + \frac{4\alpha\alpha_{tt}\sinh^2(\Lambda)\phi_x^2}{\alpha_x^2-\alpha_t^2} \\
& + \frac{4\alpha\alpha_t\sinh(2\Lambda)\Lambda_t\phi_x^2}{\alpha_x^2-\alpha_t^2} + \frac{4\alpha\alpha_t\sinh(2\Lambda)\Lambda_t\phi_t^2}{\alpha_x^2-\alpha_t^2} + \frac{8\alpha\alpha_t\sinh^2(\Lambda)\phi_x\phi_{xt}}{\alpha_x^2-\alpha_t^2} + \frac{8\alpha\alpha_t\sinh^2(\Lambda)\phi_t\phi_{tt}}{\alpha_x^2-\alpha_t^2} \\
& - \frac{8\alpha\alpha_{xt}\sinh^2(\Lambda)\phi_x\phi_t}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha\alpha_x\sinh(2\Lambda)\Lambda_t\phi_x\phi_t}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha\alpha_x\sinh^2(\Lambda)\phi_{xt}\phi_t}{\alpha_x^2-\alpha_t^2} - \frac{8\alpha\alpha_x\sinh^2(\Lambda)\phi_x\phi_{tt}}{\alpha_x^2-\alpha_t^2}.
\end{aligned}
$$

The last step is to perform the sum of the all term in Λ, ϕ, and simplify similar terms. We will use the second equation in the system (3.1.8), then, we can write

$$
\begin{aligned}
&Te_\phi + Tp_\phi + Te_\Lambda + Tp_\Lambda \\
&= \frac{4\alpha_t^2 \sinh^2(\Lambda)\phi_t^2}{\alpha_x^2 - \alpha_t^2} + \frac{4\alpha_t^2 \sinh^2(\Lambda)\phi_x^2}{\alpha_x^2 - \alpha_t^2} + \frac{4\alpha\alpha_t \sinh(2\Lambda)\Lambda_t\phi_x^2}{\alpha_x^2 - \alpha_t^2} + \Lambda_t^2 - \Lambda_x^2 \\
&\quad + \frac{4\alpha\alpha_t \sinh(2\Lambda)\Lambda_t\phi_t^2}{\alpha_x^2 - \alpha_t^2} - \frac{4\alpha\alpha_t \sinh(2\Lambda)\Lambda_t\phi_x^2}{\alpha_x^2 - \alpha_t^2} - \frac{16\alpha_t\alpha_x \sinh^2(\Lambda)\phi_x\phi_t}{\alpha_x^2 - \alpha_t^2} \\
&\quad + \frac{8\alpha\alpha_t \sinh^2(\Lambda)\phi_t\phi_{tt}}{\alpha_x^2 - \alpha_t^2} - \frac{8\alpha\alpha_x \sinh(2\Lambda)\Lambda_t\phi_x\phi_t}{\alpha_x^2 - \alpha_t^2} + \frac{16\alpha\alpha_{xt} \sinh^2(\Lambda)\phi_x\phi_t}{\alpha_x^2 - \alpha_t^2} \\
&\quad - \frac{8\alpha\alpha_x \sinh^2(\Lambda)\phi_x\phi_{tt}}{\alpha_x^2 - \alpha_t^2} + \frac{4\alpha_x^2 \sinh^2(\Lambda)\phi_t^2}{\alpha_x^2 - \alpha_t^2} + \frac{4\alpha_x^2 \sinh^2(\Lambda)\phi_x^2}{\alpha_x^2 - \alpha_t^2} \\
&\quad + \frac{4\alpha\alpha_x \sinh(2\Lambda)\Lambda_x\phi_x^2}{\alpha_x^2 - \alpha_t^2} + \frac{4\alpha\alpha_x \sinh(2\Lambda)\Lambda_x\phi_t^2}{\alpha_x^2 - \alpha_t^2} + \frac{8\alpha\alpha_x \sinh^2(\Lambda)\phi_x\phi_{xx}}{\alpha_x^2 - \alpha_t^2} \\
&\quad - \frac{8\alpha\alpha_t \sinh(2\Lambda)\Lambda_x\phi_x\phi_t}{\alpha_x^2 - \alpha_t^2} - \frac{8\alpha\alpha_t \sinh^2(\Lambda)\phi_t\phi_{xx}}{\alpha_x^2 - \alpha_t^2} + \frac{4\alpha\alpha_t \sinh(2\Lambda)\Lambda_t\phi_t^2}{\alpha_x^2 - \alpha_t^2} \\
&\quad - \frac{4\alpha\alpha_x \sinh(2\Lambda)\Lambda_x\phi_t^2}{\alpha_x^2 - \alpha_t^2} + \frac{4\alpha\alpha_x \sinh(2\Lambda)\Lambda_x\phi_x^2}{\alpha_x^2 - \alpha_t^2} - \frac{16\alpha\alpha_{tx} \sinh^2(\Lambda)\phi_t\phi_x}{\alpha_x^2 - \alpha_t^2} \\
&= 4\sinh^2(\Lambda)\left(\phi_t^2 - \phi_x^2\right) + \left(\frac{8\alpha_t\phi_t - 8\alpha_x\phi_x}{\alpha_x^2 - \alpha_t^2}\right)(\alpha\phi_t\Lambda_t - \alpha\phi_t\Lambda_t)\sinh(2\Lambda) \\
&\quad + \left(\frac{8\alpha_t\phi_t - 8\alpha_x\phi_x}{\alpha_x^2 - \alpha_t^2}\right)\sinh^2(\Lambda)(\alpha_t\phi_t - \alpha_x\phi_x + \alpha\phi_{tt} - \alpha\phi_{xx}) + \Lambda_t^2 - \Lambda_x^2 \\
&= 4\sinh^2(\Lambda)\left(\phi_t^2 - \phi_x^2\right) + \Lambda_t^2 - \Lambda_x^2.
\end{aligned}
$$

We can then conclude that:

$$
\partial_t e(t,x) + \partial_x p(t,x) = 4\sinh^2(\Lambda)\left(\phi_t^2 - \phi_x^2\right) + \Lambda_t^2 - \Lambda_x^2 + \partial_x\left(\frac{\alpha_x}{\alpha}\right) - \partial_t\left(\frac{\alpha_t}{\alpha}\right),
$$

as desired.

Chapter 4

Nonlinear Stability of nonsingular solitons of the Principal Chiral Field equation

Abstract: We consider the Principal Chiral Field model posed in 1+1 dimensions into the Lie group $\mathrm{SL}(2,\mathbb{R})$. This model is characterized for being integrable with infinitely many conserved quantities. The Belinski-Zakharov formalism has been applied to this model by Hadad, showing the existence of 1-solitons of singular type obtained by a dressing method from background seeds. In a previous work, it was proved the existence of finite-energy, nonsingular solitons, extending the class of physically meaningful solutions. In this work we show the nonlinear stability of sufficiently small non singular solitons. The method of proof involves the use of vector field methods as in a previous work by the second and third authors dealing with the Einstein's field equations under the Belinski-Zakharov formalism, extending for all times the size of suitable null weighted norms of the perturbations at time zero.

This chapter is contained in: M.A. Alejo, C. Muñoz and J. Trespalacios. *Nonlinear Stability of nonsingular solitons of the Principal Chiral Field equation*, arXiv:2408.09969. Submitted 2024.

Contents

4.1 Introduction

Consider the Principal Chiral Field model (PCF) given by

$$\partial_t \left(\partial_t g g^{-1}\right) - \partial_x \left(\partial_x g g^{-1}\right) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{4.1.1}$$

valid for a 2×2 Riemannian metric g, into the Lie group $\mathrm{SL}(2; \mathbb{R})$. Chiral fields on Lie groups represent equivalence classes of the integrable relativistic two-dimensional systems, consequently, (4.1.1) represents an integrable system. Zakharov and Mikhailov [117] showed that classical spinor fields are connected with each such system. The Principal Chiral Field is a nonlinear σ-model. The first description of the integrability of this model in the language of the commutative representation (4.1.1) was given in [119]. Subsequently, different results associated with integrability, conserved quantities and soliton solutions were obtained [10, 41, 93], as well as different descriptions of this equation using Bäcklund and Darboux transformations [31, 53]. There are several results associated with the study of the reduction of the Principal Chiral Field equation in homogeneous spaces of Lie groups. In particular, Zakharov and Mikhailov [117] studied PCF (4.1.1) for the special unitary group $\mathrm{SU}(N)$. In [118], they studied the connection of this equation with the Nambu-Jona-Lasinian model. In this work, we follow a different approach; we shall study PCF in the particular case of the reduction problem on "symmetric spaces", following the works [7, 9, 52]. The symmetric space considered is the invariant manifold of symmetric matrices sitting in the Lie group $\mathrm{SL}(2; \mathbb{R})$. This space is not a Lie group, but it can be identified with an hyperboloid in Minkowski spacetime, see [88].

In this work we shall study stability of regular soliton solutions of the PCF model. In particular, we are interested in the notion of *orbital and asymptotic stability* of special solutions of these model with small initial data perturbations. As far as we know, this seems the first rigorous results in this direction for these kind of solutions. The stability theory is an important line of research in nonlinear PDEs, in the sense that it gives solidity to the results related to global existence of the solutions, asymptotic behavior, the dynamics of solutions. In addition to the motivation arising from the PCF model, the study of the stability for hyperbolic equations is of independent interest because of the connections with other branches of physics. Indeed, the study of stability elucidates our understanding of whether such models can provide mathematically reasonable models for physical phenomena.

Unlike many previous results related to orbital stability, in this paper we do not follow the classical approach since it is nearly useless. PCF is a model where standard techniques fail and one needs a new approach. We will combine asymptotic stability techniques and preservation of local energy to provide a complete characterization of perturbations of regular soliton solutions of PCF.

4.1.1 Setting of the model

Following the same approach that in a previous work [109], in this paper we consider the identification of the PCF Equation (4.1.1) with the following $(1+1)$-dimensional system of the semilinear wave equations

$$\begin{cases} \partial_t^2\Lambda - \partial_x^2\Lambda = -2\sinh(2\Lambda)((\partial_x\phi)^2 - (\partial_t\phi)^2), \\ \partial_t^2\phi - \partial_x^2\phi = -\dfrac{\sinh(2\Lambda)}{\sinh^2(\Lambda)}(\partial_t\phi\partial_t\Lambda - \partial_x\phi\partial_x\Lambda). \end{cases} \tag{4.1.2}$$

This system corresponds to the PCF equation using the so-called Gowdy coordinates (see [91] for full details). The fact that the 2×2 matrix g is symmetric allows one to diagonalize it for fixed t and x. One writes $g = RDR^T$, where D is a diagonal matrix and R is a rotation matrix, of the form

$$D = \begin{pmatrix} e^{\Lambda} & 0 \\ 0 & e^{-\Lambda} \end{pmatrix}, \qquad R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

Clearly $\det g = 1$. Here Λ is the scalar field that determines the eigenvalues of g, and the scalar field ϕ determines the deviation of g from being a diagonal matrix. Since ϕ is considered as an angle, we can assume without loss of generality that $\phi \in [0, 2\pi]$. Therefore Λ, ϕ can be considered as the two degrees of freedom in the symmetric matrix g, [52]. Written explicitly, the matrix g is given now by

$$g = \begin{pmatrix} \cosh\Lambda + \cos 2\phi \sinh\Lambda & \sin 2\phi \sinh\Lambda \\ \sin 2\phi \sinh\Lambda & \cosh\Lambda - \cos 2\phi \sinh\Lambda \end{pmatrix}. \tag{4.1.3}$$

Therefore, the components of the matrix g are completely determined by the fields Λ and ϕ. For more details see [52, 109]. The system (4.1.2) is a set of coupled quasilinear wave equations, with a rich analytical and algebraic structure. Solutions of the (4.1.2) are invariant under space and time translations. Indeed, for any $t_0, x_0 \in \mathbb{R}$, $\Lambda(t-t_0, x-x_0), \phi(t-t_0, x-x_0)$ is also a solution. The conservation laws for the PCF equation (4.1.2) are as follow: in first place, we have the energy

$$E[\Lambda, \Lambda_t, \phi, \phi_t] = \int \left(\frac{1}{2}((\partial_x\Lambda)^2 + (\partial_t\Lambda)^2) + 2\sinh^2(\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2)\right)(t, x)dx, \tag{4.1.4}$$

and in second place we have the momentum

$$P[\Lambda, \Lambda_t, \phi, \phi_t] = \int \left(\partial_x\Lambda\partial_t\Lambda + 4\sinh^2(\Lambda)\partial_x\phi\partial_t\phi\right)(t, x)dx. \tag{4.1.5}$$

Note that the energy is well-defined if $(\Lambda, \partial_t\Lambda) \in \dot{H}^1 \times L^2$, but a suitable space for $(\phi, \partial_t\phi)$ strongly depends on the weight $\sinh^2\Lambda$, which can easily grow exponentially in space, since $\dot{H}^1$ can easily contain unbounded functions. In this regard, making sense of the energy (even for classical solution such as solitons) is subtle and requires a deep and careful analysis.

Local and global solutions

Among the results presented in [109, Proposition 1.1], one has the local existence result for solutions in the energy space. In the analysis of the initial value problem for this system, the regularity of the term $\frac{\sinh(2\Lambda)}{\sinh^2(\Lambda)}$ is delicate, specially when the function $\Lambda(t, x)$ is zero. This case must be carefully analyzed in order to be able to construct a local existence result associated to PCF (4.1.2). In [109], it was proposed to write the function $\Lambda(t,x)$ in the form

$$\Lambda(t,x) := \tilde{\lambda} + \tilde{\Lambda}(t,x), \quad \tilde{\lambda} \neq 0.$$

Notice that this choice makes sense with the energy in (4.1.4), in the sense that $\Lambda \in \dot{H}^1$ and $\partial_t \Lambda \in L^2$. Without loss of generality, we assume $\tilde{\lambda} > 0$. The basic idea for LWP was to establish the conditions that are required on $\tilde{\lambda}$ and $\tilde{\Lambda}$ in order to obtain the desired regularity results. With this choice, the system (4.1.2) can be written in terms of the function $\tilde{\Lambda}(t,x)$ as follows:

$$\begin{cases} \partial_t^2 \tilde{\Lambda} - \partial_x^2 \tilde{\Lambda} = -2\sinh(2\tilde{\lambda} + 2\tilde{\Lambda})((\partial_x \phi)^2 - (\partial_t \phi)^2), \\ \partial_t^2 \phi - \partial_x^2 \phi = -\dfrac{\sinh(2\tilde{\lambda} + 2\tilde{\Lambda})}{\sinh^2(\lambda + \tilde{\Lambda})}(\partial_t \phi \partial_t \tilde{\Lambda} - \partial_x \phi \partial_x \tilde{\Lambda}). \end{cases} \tag{4.1.6}$$

Having established the existence of solutions, the second result presented in [109] involves whether or not local solutions can be extended globally in time. One has [109, Theorem 1.1]:

Theorem 4.1.1. *Consider the semilinear wave system* (4.1.2) *posed in* $\mathbb{R}^{1+1}$*, with the following initial conditions:*

$$\begin{cases} (\phi, \tilde{\Lambda})|_{\{t=0\}} = \varepsilon(\phi_0, \tilde{\Lambda}_0), \quad (\phi_0, \tilde{\Lambda}_0) \in C_c^\infty(\mathbb{R})^2, \\ (\partial_t \phi, \partial_t \tilde{\Lambda})|_{\{t=0\}} = \varepsilon(\phi_1, \tilde{\Lambda}_1), \quad (\phi_1, \tilde{\Lambda}_1) \in C_c^\infty(\mathbb{R})^2. \end{cases}$$

Then, there exists ε_0 *such that if* $\varepsilon < \varepsilon_0$*, the unique solution remains smooth for all time and have finite conserved energy* (4.1.4).

In the same work [109], it was also established that the family of solutions satisfying the hypotheses of Theorem 4.1.1 is non-empty. Using the ideas proposed by Belinski-Zakharov, it was proved that there exists a family of solitons-like solutions for the PCF equation (4.1.1). The construction and characterization of these solutions is briefly shown in the following section.

4.1.2 Soliton solutions

Belinski and Zakharov in [9] (see also [8]) showed that (4.1.1) has N-soliton solutions. More precisely, they proposed a transformation that systematically constructs the so-called gravisolitons in the context of the vacuum Einstein equations in General Relativity, under certain conditions of symmetry in their coordinates. Hadad [52] explicitly showed, using this transformation, the structure of the N-soliton for the PCF model (4.1.1). The application of this transformation can be done since this model can be identified with the so-called *reduced Einstein equation* given by

$$\partial_t \left(\alpha \partial_t g g^{-1}\right) - \partial_x \left(\alpha \partial_x g g^{-1}\right) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{4.1.7}$$

with $\det g = \alpha$. Note that (4.1.1) corresponds to the case $\alpha = 1$ in (4.1.7). It is important to note that although the identification of the equation (4.1.1) with (4.1.7) can be made, this special class of solutions has no relevance for the gravitational field. However, even in the case ($\alpha \equiv 1$), the PCF model is sufficiently rich to produce a complex dynamics and would formally have nontrivial solutions even when α is constant. In our recent result [91], we considered the more demanding case when α non constant. We believe that the current work will be essential to further study the stability of the cosmological type solutions as the ones studied in [91].

We will now briefly discuss the general structure of the 1-soliton of the (4.1.1), as well as the particular elements that characterize it. One starts using diagonal backgrounds, also called "seed metric", of the form

$$g^{(0)} = \begin{bmatrix} e^{\Lambda^{(0)}} & 0 \\ 0 & e^{-\Lambda^{(0)}} \end{bmatrix}.$$

The function $\Lambda^{(0)}(t,x)$ satisfies the 1+1 wave equation $\partial_t^2\Lambda^{(0)} - \partial_x^2\Lambda^{(0)} = 0$. In this case, if we want to identify the solution in terms of the fields Λ and ϕ in the equation (4.1.2), we must set (4.1.3) $\Lambda = \Lambda^{(0)}$, $\phi = n\pi$, with $n \in \mathbb{Z}$, and $\alpha = 1$. The gauge choice for us will be $n = 0$.

As can be seen into detail in [52], when applying the algorithm proposed by Belinski-Zakharov, the following 1-soliton structure is obtained for the equation (4.1.1):

$$g^{(1)} = \frac{1}{\cosh\gamma_1}\begin{bmatrix} e^{\Lambda^{(0)}}\cosh(\gamma_1+\tilde{\gamma}_1) & \dfrac{1-\mu_1^2}{2\mu_1} \\ \dfrac{1-\mu_1^2}{2\mu_1} & e^{-\Lambda^{(0)}}\cosh(\gamma_1+\tilde{\gamma}_1) \end{bmatrix}, \qquad (4.1.8)$$

where the parameters $\tilde{\gamma}_1, \gamma_1$, are given in terms of the parameter μ_1[1], as follow

$$\tilde{\gamma}_1 := -\ln|\mu_1|, \qquad \gamma_1 := K_1 + \Lambda^{(0)} + 2B_1, \quad K_1 := \ln|C_1|, \quad C_1 \in \mathbb{R},$$
$$B_1 := \frac{1}{\mu_1^2 - 1}\left(\Lambda^{(0)} + \mu_1\tilde{\Lambda}^{(0)}\right), \quad \text{where} \quad \partial_x\tilde{\Lambda}^{(0)} = \partial_t\Lambda^{(0)}.$$

The parameter μ_1 represents a pole in terms of scattering techniques, however for this particular cases μ_1 is a constant. It is easy to verify that $\det g^{(1)} = 1$, as required.

Singular solitons

Hadad [52] described the 1-soliton solution, which is obtained by taking $\Lambda^{(0)} = t$ (time-like) and $\phi^{(0)} = 0$. Note that with this choice the energy in (4.1.4) is not well-defined. Indeed, the energy proposed in (4.1.4) is not finite, but one can consider the following modified energy

$$E_{\text{mod}}[\Lambda,\phi](t) := \int \left(\frac{1}{2}\left((\partial_t\Lambda)^2 - 1 + (\partial_x\Lambda)^2\right) + 2\sinh^2(\Lambda)((\partial_t\phi)^2 + (\partial_x\phi)^2)\right),$$

which is also conserved and identically zero. Hadad computed the corresponding 1-soliton solution using Belinski and Zakharov techniques, obtaining the corresponding form (4.1.8),

[1] This parameter appears naturally in the application of the Belinski-Zakharov transform to the equations of General Relativity, for full details see [8, 117, 118, 119].

as follows

$$g^{(1)} = \begin{bmatrix} \dfrac{e^{t}Q_c(x-vt)}{Q_c(x-vt-x_0)} & -\dfrac{1}{c}Q_c(x-vt) \\ -\dfrac{1}{c}Q_c(x-vt) & \dfrac{e^{-t}Q_c(x-vt)}{Q_c(x-vt+x_0)} \end{bmatrix},$$

where, for a fixed parameter $\mu > 1$, one has

$$Q_c(\cdot) = \sqrt{c}\,\mathrm{sech}(\sqrt{c}(\cdot)), \quad c = \left(\frac{2\mu}{\mu^2-1}\right)^2, \quad v = -\frac{\mu^2+1}{2\mu} < -1, \quad \text{and} \quad x_0 = \frac{\ln|\mu|}{\sqrt{c}}.$$

Notice that the first component of $g^{(1)}$ grows in time. Therefore, we have a traveling superluminal soliton which travels to the left (if $\mu > 0$).

In [109], we propose a modification of this "degenerate" soliton solution by cutting off the infinite energy part profiting of the wave-like character of solutions $\Lambda^{(0)}$. Although it is not so clear that they are physically meaningful, these new solutions have finite energy and local well-posedness properties in a vicinity. However, the stability of this solution was not clear at the moment.

Finite energy, nonsingular solitons

Consider a smooth function $\theta \in C_c^2(\mathbb{R})$, and $0 < \mu < 1$. For any $\lambda > 0$ and $\varepsilon > 0$ small, let

$$\Lambda_\varepsilon^{(0)}(t,x) := \lambda + \varepsilon\theta(t+x), \quad \phi^{(0)} := 0,$$

(it is also possible to take shifts, but this freedom is nearly unimportant for our results.) Clearly $\Lambda_\varepsilon^{(0)}$ solves the wave equation in $1+1$ dimensions and has finite energy $E[\Lambda_\varepsilon^{(0)}, \phi_\varepsilon^{(0)}] < +\infty$. This will be for us the background seed. The corresponding 1-soliton, in correspondence with the form (4.1.8), is now

$$g^{(1)} = \begin{bmatrix} \dfrac{e^{\lambda+\varepsilon\theta}\,\mathrm{sech}(\beta(\lambda+\varepsilon\theta))}{\mathrm{sech}(\beta(\lambda+\varepsilon\theta)-x_0)} & -\dfrac{1}{\sqrt{c}}\mathrm{sech}(\beta(\lambda+\varepsilon\theta)) \\ -\dfrac{1}{\sqrt{c}}\mathrm{sech}(\beta(\lambda+\varepsilon\theta)) & \dfrac{e^{-(\lambda+\varepsilon\theta)}\,\mathrm{sech}(\beta(\lambda+\varepsilon\theta))}{\mathrm{sech}(\beta(\lambda+\varepsilon\theta)+x_0)} \end{bmatrix}, \qquad \beta = \frac{\mu+1}{\mu-1}, \tag{4.1.9}$$

with

$$c = \left(\frac{2\mu}{\mu^2-1}\right)^2, \quad v = -\frac{\mu^2+1}{2\mu} < -1, \quad \text{and} \quad x_0 = \frac{\ln|\mu|}{\sqrt{c}}.$$

For this, the corresponding fields $\hat{\Lambda}_\varepsilon$ and $\hat{\phi}_\varepsilon$ in (4.1.2), are given by $\gamma := \lambda + \varepsilon\theta(t+x)$,

$$\begin{aligned} \hat{\Lambda}_\varepsilon &= B = \operatorname{arcosh}\left(|v|\cosh\gamma - \frac{1}{\sqrt{c}}\tanh(\beta\gamma)\sinh\gamma\right), \\ \hat{\phi}_\varepsilon &= D = \frac{\pi}{4} - \frac{1}{2}\arctan\left(\cosh(\beta\gamma)\cosh\gamma(\tanh(\beta\gamma) + v\sqrt{c}\tanh\gamma)\right). \end{aligned} \tag{4.1.10}$$

Finally, we define the time derivative of B and D as follows

$$\partial_t B = \frac{\varepsilon\theta'\left(|v|\tanh(\gamma) - \frac{1}{\sqrt{c}}\beta\,\mathrm{sech}^2(\beta\gamma)\tanh(\gamma) - \frac{1}{\sqrt{c}}\tanh(\beta\gamma)\right)}{\mathrm{sech}(\gamma)\sqrt{\left(|v|\cosh(\gamma) - \frac{1}{\sqrt{c}}\sinh(\gamma)\tanh(\beta\gamma)\right)^2 - 1}}.$$

Clearly $\partial_t B|_{\{t=0\}} \in C_0^1(\mathbb{R})$, and

$$\partial_t D = \frac{-\varepsilon\theta'(\beta + v\sqrt{c} + (1+\beta v\sqrt{c})\tanh(\beta\gamma)\tanh(\gamma))}{2\,\mathrm{sech}(\beta\gamma)\,\mathrm{sech}(\gamma)((\cosh(\gamma)\sinh(\beta\gamma) + v\sqrt{c}\sinh(\gamma)\cosh(\beta\gamma))^2 + 1)},$$

which is also a localized function. Notice that $\partial_t B, \partial_t D \in L^2(\mathbb{R})$, and the total energy is finite.

From [109] one has that B is well-defined, $B(t,x) > 0$ for all $t, x \in \mathbb{R}$, and for each t, B is a bounded function. Again following [109], one can write $B = \tilde{\lambda} + \tilde{B}$, $\tilde{\lambda} := \lim_{x\to\infty} B(t = 0, x) > 0$. Moreover,

$$\tilde{B}|_{\{t=0\}} = \varepsilon\tilde{B}_0, \quad \text{with} \quad \tilde{B}_0 \in C_0^2(\mathbb{R}).$$

where $\tilde{B}_0$ is bounded in ε. A similar computation can be done for D and its time derivatives, see [109]. Therefore, the setting of Theorem 4.1.1 is satisfied.

In this paper, in order to get global solutions, we shall assume 1-soliton perturbed initial data of the form

$$\begin{cases} (\Lambda, \phi)|_{\{t=0\}} = (B + \varepsilon z_0, D + \varepsilon s_0), \quad (z_0, s_0) \in C_c^\infty(\mathbb{R})^2, \\ (\partial_t\Lambda, \partial_t\phi)|_{\{t=0\}} = (B_t + \varepsilon w_0, D_t + \varepsilon m_0), \quad (w_0, m_0) \in C_c^\infty(\mathbb{R})^2. \end{cases} \tag{4.1.11}$$

Then, by the previous non degeneracy analysis and Theorem 4.1.1, there exists ε_0 such that if $\varepsilon < \varepsilon_0$, the unique solution remains smooth for all time and it has finite conserved energy (4.1.4). This does not ensure that the perturbations $[z, w, s, m](t)$ will remain small compared with $\tilde{B}$ and $\tilde{D}$ after a large time, but our goal will be to guarantee that if they are small at time zero, they will remain small at large time.

4.1.3 Main results

Having described in detail in Subsection 4.1.2 the PCF 1-soliton solutions, the purpose of this paper is to give a first proof of the fact that the 1-soliton (4.1.9) of the PCF model is orbital stable under small perturbations well-defined in the natural energy space associated to the problem. The stability study will be done by addressing equation (4.1.2) and using the description of the 1-soliton (4.1.9) in terms of the fields B and D given by (4.1.10). Our main theorem is the following:

Theorem 4.1.2. *There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, the following holds. There exist $C, \delta_0 > 0$ such that, if $0 < \delta < \delta_0$, and $[z_0, w_0, s_0, m_0]$ is given as in (4.1.11), then the following are satisfied:*

1. *External energetic control. Assume that*

$$\int \left(\frac{1}{2}(w_0^2 + z_{0,x}^2) + 2\sinh^2(B + z_0)(s_{0,x}^2 + m_0^2)\right)(x)dx < \delta.$$

Then

- *For all time, one has a crossed global control:*

$$\int_{\mathbb{R}} \left(\frac{1}{2}(z_x - w)^2 + 2\sinh^2(B+z)(s_x - m)^2 \right)(t,x)dx < 3\delta. \qquad (4.1.12)$$

- *Inside light-cone convergence. For any* $v \in (-1,1)$ *and* $\omega(t) = t/\log^2 t$*, one has*

$$\lim_{t\to+\infty} \int_{vt-\omega(t)}^{vt+\omega(t)} \left(w^2 + z_x^2 + \sinh^2(B+z)(m^2+s_x^2)\right)(t,x)dx = 0. \qquad (4.1.13)$$

- *Exterior stability: for all time* $t \geq 0$*,*

$$\int_{|x+t|\geq R} \left(\frac{1}{2}(w^2 + z_x^2) + 2\sinh^2(B+z)(s_x^2 + m^2) \right)(t,x)dx < \delta. \qquad (4.1.14)$$

2. *Full orbital stability. Assume now* $[z_0, w_0, s_0, m_0] \in C_c^\infty(\mathbb{R})^4$ *be initial data as in* (4.1.11) *such that*

$$\sum_{k=0,1} \int (1+|x|^2)^{1+\gamma} \left((\partial_x^k w_0)^2 + (\partial_x^{k+1} z_0)^2 + (\partial_x^k m_0)^2 + (\partial_x^{k+1} s_0)^2 \right) dx < \delta^2, \quad (4.1.15)$$

for $0 < \gamma < \frac{1}{3}$*. Then the corresponding global solution to* (4.1.6) *given as*

$$(B+z, \partial_t B + w, D+s, \partial_t D + m) \qquad (4.1.16)$$

satisfies the same bounds for all times:

$$\sup_{t\geq 0} \left(\mathcal{E}(t) + \overline{\mathcal{E}}(t) \right) \leq C\delta^2. \qquad (4.1.17)$$

(See (4.4.7) *for the definition of these norms.)*

Notice that (4.1.12) is a global in time and space property satisfied by perturbations of solitons. However, having direct control on each component of the differences is an important open question, that probably requires the introduction of better decay properties of the initial data, as it is done in (4.1.15).

Additionally, condition (4.1.15) is part of a more general energy norm condition, described as

$$\mathcal{E}[z_0, w_0] + \overline{\mathcal{E}}[s_0, m_0] < \delta^2,$$

(see (4.4.7) for the definition of these norms). It turns out that these norms are key to translate the smallness information of the problem, in addition to the smallness of the 1-soliton solution, represented by the parameter ε, which enters when computing space and time derivatives of the solution.

One may think that standard energetic Lyapunov control in terms of energy and momentum is the key to prove orbital stability. Unfortunately, in the case of PCF solitons, this is not the case. Indeed, energy and momentum are useless since they do not control the central region around the soliton. The best example of this fact is the estimate (4.1.12). The proof of Theorem 4.1.2 needs new ideas. In [109], the author showed that perturbations of *small* 1-solitons lead to a global solution, but the smallness of the perturbation could not be preserved. In this paper we avoid this problem by proposing a new idea obtained from the general case of the Einstein's field equations under the Belinski-Zakharov formalism [91]. Solitons are perturbations of the nonlinear primordial equations, and their size will be controlled using well-defined weighted norms. It will be particularly important to notice that solitons are assumed sufficiently small, but perturbations are considered smaller.

Organization of this work

This work is organized as follows. In Section 4.2 some important definitions, notation and previous work are stated. In Section 4.3 we will introduce energy and momentum for perturbations of the soliton solution, and compute useful virial identities valid for data only in the energy space. Finally, in Section 4.4 we prove the nonlinear stability of perturbations of nonsingular PCF solitons.

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4.2 Preliminaries

4.2.1 Soliton profiles

Let $\xi := t+x$, and $\varepsilon > 0$. From now on, we will use the following notation: let $\gamma := \lambda+\varepsilon\theta(\xi)$. Consider the soliton profiles

$$\begin{aligned} B(\xi) &= \cosh^{-1}\left(|v|\cosh\gamma - \frac{1}{\sqrt{c}}\tanh(\beta\gamma)\sinh\gamma\right), \\ D(\xi) &= \frac{\pi}{4} - \frac{1}{2}\arctan\left(\cosh(\beta\gamma)\cosh\gamma\left(\tanh(\beta\gamma) + v\sqrt{c}\tanh\gamma\right)\right). \end{aligned}$$

If $\xi = t+x$, then B and D are solutions to (4.1.6). We shall also need the derivatives of B and D:

$$\begin{aligned} B'(\xi) &= \frac{\varepsilon\theta'\left(|v|\tanh\gamma - \frac{1}{\sqrt{c}}\beta\operatorname{sech}^2(\beta\gamma)\tanh(\gamma) - \frac{1}{\sqrt{c}}\tanh(\beta\gamma)\right)}{\operatorname{sech}(\gamma)\sqrt{\left(|v|\cosh(\gamma) - \frac{1}{\sqrt{c}}\sinh(\gamma)\tanh(\beta\gamma)\right)^2 - 1}}, \\ D'(\xi) &= \frac{-\varepsilon\theta'\left(\beta + v\sqrt{c} + (1+\beta v\sqrt{c})\tanh(\beta\gamma)\tanh\gamma\right)}{2\operatorname{sech}(\beta\gamma)\operatorname{sech}\gamma\left(\left(\cosh(\gamma)\sinh(\beta\gamma) + v\sqrt{c}\sinh\gamma\cosh(\beta\gamma)\right)^2 + 1\right)}. \end{aligned} \tag{4.2.1}$$

Notice that both functions have compact support since θ' has compact support.

The vector soliton is denoted as $\mathbf{B} = [B, B', D, D']$, and the perturbation will be denoted as $\mathbf{u} = [z, w, s, m] \in C_0^\infty(\mathbb{R})^4$.

4.2.2 Previous bounds on global solutions

Let us recall some previous information [109] about the global solution in our problem. Consider

$$u := \frac{t-x}{2}, \quad \underline{u} := \frac{t+x}{2}.$$

Consider the two null vector fields defined globally as

$$L = \partial_t + \partial_x, \quad \underline{L} = \partial_t - \partial_x.$$

Let $0 < \delta < 1$. Σ_{t_0} denotes the region

$$\Sigma_{t_0} := \{(t,x) : \ t = t_0\}. \tag{4.2.2}$$

D_{t_0} denotes the following region of spacetime

$$D_{t_o} := \{(t,x) : \ 0 \le t \le t_0\}, \quad D_{t_0} = \bigcup_{0\le t \le t_0} \Sigma_{t_0}.$$

The level sets of the functions u and $\underline{u}$ define two global null foliations of D_{t_0}. More precisely, given $t_0 > 0$, u_0 and $\underline{u}_0$, we define the rightward null curve segment C_{u_0} as :

$$C_{u_0} := \left\{(t,x) : \ u = \frac{t-x}{2} = u_0, \, 0 \le t \le t_0\right\}, \tag{4.2.3}$$

and the segment of the null curve to the left $\underline{C}_{\underline{u}_0}$ as:

$$\underline{C}_{\underline{u}_0} := \left\{(t,x) : \ \underline{u} = \frac{t+x}{2} = \underline{u}_0, \, 0 \le t \le t_0\right\}. \tag{4.2.4}$$

The space time region D_{t_0} is foliated by $\underline{C}_{\underline{u}_0}$ for $\underline{u} \in \mathbb{R}$, and by C_{u_0} for $u \in \mathbb{R}$.

In the same way as in [85, 109] we consider the weight function φ defined as

$$\varphi(u) := (1 + |u|^2)^{1+\gamma} \quad \text{with} \quad 0 < \gamma < 1/3. \tag{4.2.5}$$

Finally, we will consider the following energy estimate proposed in [4, 85] for the scalar linear wave equation $\Box\psi = \rho$ ($\tau \in [0,t]$ in $\underline{C}_{\underline{u}}$ and C_u). There exists $C_0 > 0$ such that

$$\begin{aligned} &\int_{\Sigma_t} \left[\varphi(u)|\underline{L}\psi|^2 + \varphi(\underline{u})|L\psi|^2\right] dx \\ &\quad + \sup_{\underline{u}\in\mathbb{R}} \int_{\underline{C}_{\underline{u}}} \varphi(u)|\underline{L}\psi|^2 d\tau + \sup_{u\in\mathbb{R}} \int_{C_u} \varphi(\underline{u})|L\psi|^2 d\tau \\ &\quad \le C_0 \int_{\Sigma_0} \left[\varphi(u)|\underline{L}\psi|^2 + \varphi(\underline{u})|L\psi|^2\right] dx + C_0 \iint_{D_t} \left[\varphi(u)|\underline{L}\psi| + \varphi(\underline{u})|L\psi|\right] |\rho| d\tau dx. \end{aligned} \tag{4.2.6}$$

Based on this estimate, and in the setting of equation (4.1.6), we define the space-time weighted energy norms valid for $k = 0, 1$:

$$\begin{aligned}
\mathcal{E}_k[\tilde{\Lambda}](t) &= \int_{\Sigma_t} \left[\varphi(u)|\underline{L}\partial_x^k\tilde{\Lambda}|^2 + \varphi(\underline{u})|L\partial_x^k\tilde{\Lambda}|^2\right] dx, \\
\overline{\mathcal{E}}_k[\tilde{\phi}](t) &= \int_{\Sigma_t} \left[\varphi(u)|\underline{L}\partial_x^k\phi|^2 + \varphi(\underline{u})|L\partial_x^k\phi|^2\right] dx, \\
\mathcal{F}_k[\tilde{\Lambda}](t) &= \sup_{\underline{u}\in\mathbb{R}} \int_{\underline{C}_{\underline{u}}} \varphi(u) \left|\underline{L}\partial_x^k\tilde{\Lambda}\right|^2 ds + \sup_{u\in\mathbb{R}} \int_{C_u} \varphi(\underline{u})|L\partial_x^k\tilde{\Lambda}|^2 ds, \\
\overline{\mathcal{F}}_k[\tilde{\phi}](t) &= \sup_{\underline{u}\in\mathbb{R}} \int_{C_{\underline{u}}} \varphi(u) \left|\underline{L}\partial_x^k\phi\right|^2 ds + \sup_{u\in\mathbb{R}} \int_{C_u} \varphi(\underline{u})|L\partial_x^k\phi|^2 ds.
\end{aligned} \tag{4.2.7}$$

Finally, we define the total energy norms as follows:

$$\mathcal{E}(t) = \mathcal{E}_0(t) + \mathcal{E}_1(t).$$

Analogously one defines $\mathcal{F}(t)$, $\overline{\mathcal{E}}(t)$, and $\overline{\mathcal{F}}(t)$. Then from [109] we know that there exists $C > 0$ such that for all $t \geq 0$,

$$\mathcal{E}(t) + \mathcal{F}(t) \leq C\varepsilon^2, \qquad \overline{\mathcal{E}}(t) + \overline{\mathcal{F}}(t) \leq C\varepsilon^2, \tag{4.2.8}$$

and ($\tilde{\lambda} := \lim_{x\to\infty} B(t = 0, x) > 0$)

$$\sup_{t\geq 0} \left\|\tilde{\Lambda}\right\|_{L^\infty(\mathbb{R})} \leq \frac{\tilde{\lambda}}{2}. \tag{4.2.9}$$

From this last fact, we have the following corollary:

Corollary 4.2.1. *Let $z(t)$ be defined in* (4.1.16)*. One has for all time $t \geq 0$,*

$$\begin{aligned}
0 < c_0(\tilde{\lambda}) \leq \sinh(B + z) \leq c_1(\tilde{\lambda}), \\
0 < d_0(\tilde{\lambda}) \leq \cosh(B + z) \leq d_1(\tilde{\lambda}).
\end{aligned} \tag{4.2.10}$$

Proof. Recall B in (4.1.10). Since $B + z = \tilde{\lambda} + \tilde{\Lambda}$, we first have $\frac{\lambda}{4} \leq \gamma = \lambda + \varepsilon\theta(\xi) \leq \frac{5}{4}\lambda$, and from (4.2.9),

$$\frac{1}{2}\tilde{\lambda} \leq |B + z| \leq \frac{3}{2}\tilde{\lambda}.$$

The remaining bounds in (4.2.10) are direct consequences of the previous inequality. □

4.2.3 Local decay

Recall the following result from [109]:

Theorem 4.2.1. *Under a finite energy assumption, every globally defined solution to the PCF model satisfies the following convergence property: for any $v \in (-1, 1)$ and $\omega(t) = t/\log^2 t$, one has*

$$\lim_{t\to+\infty} \int_{vt-\omega(t)}^{vt+\omega(t)} \left((\partial_t\Lambda)^2 + (\partial_x\Lambda)^2 + \sinh^2\Lambda((\partial_t\phi)^2 + (\partial_x\phi)^2)\right)(t, x)dx = 0.$$

This results shows that finite-energy global PCF solutions must decay inside the light cone. Notice that this is a property satisfied by any globally-defined finite-energy solution. An immediate consequence of the previous result is the following decay property, which proves (4.1.13).

Corollary 4.2.2. *For any* $v \in (-1,1)$ *and* $\omega(t) = t/\log^2 t$, *one has*

$$\lim_{t\to+\infty} \int_{vt-\omega(t)}^{vt+\omega(t)} \left(w^2 + z_x^2 + \sinh^2(B+z)(m^2+s_x^2)\right)(t,x)dx = 0.$$

Proof. An immediate consequence of (4.1.16), (4.2.1) and the fact that B' and D' are supported in $|x+t| \leq R$. □

4.3 Exterior stability bounds

Recall from (4.1.4) and (4.1.5) the energy and momentum

$$E[\Lambda, \Lambda_t, \phi, \phi_t] = \int e(t,x)dx, \quad P[\Lambda, \Lambda_t, \phi, \phi_t] = \int p(t,x)dx,$$

where e and p are the energy and momentum densities:

$$e := \frac{1}{2}((\partial_x\Lambda)^2 + (\partial_t\Lambda)^2) + 2\sinh^2(\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2), \tag{4.3.1}$$

$$p := \partial_x\Lambda\partial_t\Lambda + 4\sinh^2(\Lambda)\partial_x\phi\partial_t\phi. \tag{4.3.2}$$

The system (4.1.2) can be expanded around the perturbation of the soliton $\mathbf{B} + \mathbf{u}$, as follows: for any $\mathbf{u} = [z, w, s, m]$, we have that $\mathbf{u}$ formally satisfies the following system

$$\begin{cases} z_{tt} - z_{xx} = 2\sinh(2(B+z))\left(2D'(s_t - s_x) + s_t^2 - s_x^2\right), \\ s_{tt} - s_{xx} = -\dfrac{\sinh(2(B+z))}{\sinh^2(B+z)}\left(D'(z_t - z_x) + B'(s_t - s_x) + s_t z_t - s_x z_x\right). \end{cases} \tag{4.3.3}$$

This system can be written in matrix form as

$$\begin{cases} z_t = w, \quad s_t = m, \\ w_t - z_{xx} = 2\sinh(2(B+z))\left(2D'(m - s_x) + m^2 - s_x^2\right), \\ m_t - s_{xx} = -\dfrac{\sinh(2(B+z))}{\sinh^2(B+z)}\left(D'(w - z_x) + B'(m - s_x) + mw - s_x z_x\right). \end{cases} \tag{4.3.4}$$

Recall that the conservation laws of the model are given by (4.1.4) and (4.1.5), if you consider the expansion of the energy and momentum density around the 1-soliton we get

Lemma 4.3.1. *Let* (t,x) *be such that* $\mathbf{u}(t,x)$ *is well-defined. Then a.e.*

$$2(e-p)[\mathbf{B}+\mathbf{u}] = (z_x - w)^2 + 4\sinh^2(B+z)(s_x - m)^2 \geq 0, \tag{4.3.5}$$

where e *and* p *are given in* (4.3.1) *and* (4.3.2).

Proof. First of all, notice that

$$\begin{aligned} e[\mathbf{B}+\mathbf{u}] = {} & B'^2 + B'z_x + B'w + \frac{1}{2}z_x^2 + \frac{1}{2}w^2 \\ & + 4\sinh^2(B+z)\left(D'^2 + D'(s_x+m) + \frac{1}{2}(s_x^2+m^2)\right). \end{aligned} \tag{4.3.6}$$

On the other hand,

$$p[\mathbf{B}+\mathbf{u}] = B'^2 + B'z_x + B'w + z_x w + 4\sinh^2(B+z)\left(D'^2 + D'(s_x+m) + s_x m\right). \tag{4.3.7}$$

Subtracting both identities we get (4.3.5):

$$2(e-p)[\mathbf{B}+\mathbf{u}] = (z_x - w)^2 + 4\sinh^2(B+z)(s_x-m)^2.$$

The proof is complete. □

Now we are ready to prove (4.1.12).

Corollary 4.3.1. *For all times $t \geq 0$,*

$$\begin{aligned} & \int_{\mathbb{R}} \left((z_x - w)^2 + 4\sinh^2(B+z)(s_x-m)^2\right)(t,x)dx \\ & = \int_{\mathbb{R}} \left((z_{0,x} - w_0)^2 + 4\sinh^2(B+z_0)(s_{0,x}-m_0)^2\right)(x)dx < 3\delta. \end{aligned}$$

The previous result establishes that the differences $z_x - w$ and $s_x - m$ have good behavior in time, and remain bounded. An important part of the proof of Theorem 4.1.2 will be to get better control on each part of the perturbation $\mathbf{u}$ by separate.

Let us calculate the variation with respect to t of the quantity (4.3.5), using the system (4.3.4). Let

$$\begin{aligned} \hat{e}[\mathbf{u}](t) &:= \frac{1}{2}(w^2 + z_x^2) + 2\sinh^2(B+z)(s_x^2+m^2), \\ \hat{p}[\mathbf{u}](t) &:= z_x w + 4\sinh^2(B+z)s_x m, \\ F_p[\mathbf{u}](t) &:= 2\sinh(2(B+z))\left(B'(s_x^2 - m^2) + 2D'(z_x m - w s_x)\right), \\ F_e[\mathbf{u}](t) &:= 2\sinh(2(B+z))\left(B'\left(s_x^2 - m^2\right) + 2D'\left(D'(w - z_x) + mw - s_x z_x\right)\right). \end{aligned} \tag{4.3.8}$$

In this case, $\hat{e}$ and $\hat{p}$ are localized versions of the energy and momentum densities. Notice that $|\hat{p}[\mathbf{u}](t)| \leq |\hat{e}[\mathbf{u}](t)|$. Then, we claim the following

Lemma 4.3.2. *For all $t \geq 0$, it holds that*

$$\partial_t \hat{p}[\mathbf{u}](t) = \partial_x \hat{e}[\mathbf{u}](t) + F_p[\mathbf{u}](t), \quad \partial_t \hat{e}[\mathbf{u}](t) = \partial_x \hat{p}[\mathbf{u}](t) + F_e[\mathbf{u}](t). \tag{4.3.9}$$

Proof. From [109, Lemma 4.2], we know that $\partial_t e = \partial_x p$ and $\partial_t p = \partial_x e$, with e and p given in (4.3.1)-(4.3.2). We have from (4.3.6)-(4.3.7), and $(B'^2)_t = (B'^2)_x$,

$$\begin{aligned} \frac{d}{dt}\hat{e}[\mathbf{u}](t) = {} & -\frac{d}{dt}\left(B'z_x + B'w + 4\sinh^2(B+z)D'\left(D' + s_x + m\right)\right) \\ & + \partial_x\left(B'z_x + B'w + 4\sinh^2(B+z)D'\left(D' + s_x + m\right) + \hat{p}[\mathbf{u}]\right). \end{aligned}$$

Using (4.3.4),

$$\begin{aligned}
&\frac{d}{dt}\left(B'z_x + B'w + 4\sinh^2(B+z)D'\left(D'+s_x+m\right)\right)\\
&= B''z_x + B'w_x + B''w + B'\left(z_{xx} - 2\sinh(2(B+z))\left(2D'(s_x-m)+s_x^2-m^2\right)\right)\\
&\quad + 4\sinh(2(B+z))(B'+w)D'\left(D'+s_x+m\right) + 4\sinh^2(B+z)D''\left(D'+s_x+m\right)\\
&\quad + 4\sinh^2(B+z)D'\left(D''+m_x+s_{xx}\right)\\
&\quad - 4D'\sinh(2(B+z))\left(D'(w-z_x)+B'(m-s_x)+mw-s_xz_x\right).
\end{aligned}$$

Simplifying,

$$\begin{aligned}
&\frac{d}{dt}\left(B'z_x + B'w + 4\sinh^2(B+z)D'\left(D'+s_x+m\right)\right)\\
&= \partial_x\left(B'z_x + B'w + 4\sinh^2(B+z)D'\left(D'+s_x+m\right)\right) - 2\sinh(2(B+z))B'\left(s_x^2-m^2\right)\\
&\quad - 4\sinh(2(B+z))D'\left(D'(w-z_x)+mw-s_xz_x\right).
\end{aligned}$$

We conclude now that

$$\frac{d}{dt}\hat{e}[\mathbf{u}](t) = \partial_x\hat{p}[\mathbf{u}](t) + F_e[\mathbf{u}](t).$$

This is the first identity in (4.3.9). On the other hand, using (4.3.4) again,

$$\begin{aligned}
&\frac{d}{dt}\left(z_xw + 4\sinh^2(B+z)s_xm\right)\\
&= ww_t + z_xz_{xt} + 2\sinh(2(B+z))(B'+z_t)(s_x^2+m^2) + 4\sinh^2(B+z)(s_xs_{xt}+mm_t)\\
&= w\left(z_{xx} - 2\sinh(2(B+z))\left(2D'(s_x-m)+s_x^2-m^2\right)\right)\\
&\quad + z_xw_x + 2\sinh(2(B+z))(B'+w)(s_x^2+m^2) + 4\sinh^2(B+z)s_xm_x\\
&\quad + 4\sinh^2(B+z)ms_{xx} - 4m\sinh(2(B+z))\left(D'(w-z_x)+B'(m-s_x)+mw-s_xz_x\right).
\end{aligned}$$

Rearranging appropriately and simplifying,

$$\begin{aligned}
&\frac{d}{dt}\hat{p}[\mathbf{u}](t)\\
&= \frac{1}{2}\partial_x(w^2+z_x^2) + \sinh(2(B+z))\left(-4D'z_xs_x + 4D'z_xm\right)\\
&\quad - 2\sinh(2(B+z))s_x^2z_x + 2\sinh(2(B+z))z_xm^2 + 4\sinh(2(B+z))(B's_xm + s_xwm)\\
&\quad + 4\sinh^2(B+z)m_xm + 2\sinh^2(B+z)\partial_x(s_x^2) - 4\sinh(2(B+z))D'(w-z_x)s_x\\
&\quad - 4\sinh(2(B+z))(B's_xm + s_xwm) + 4\sinh(2(B+z))(B'+z_x)s_x^2\\
&= \frac{1}{2}\partial_x\left(z_x^2 + 4\sinh^2(B+z)s_x^2\right) + 2\sinh(2B+2z)B's_x^2 + 2\sinh(2B+2z)z_xs_x^2\\
&\quad + \sinh(2(B+z))[-4D'z_xs_x + 4D'z_xm - 2z_xs_x^2 + 2z_xm^2]\\
&\quad + 2\sinh^2(B+z)\partial_x(m^2) - 4\sinh(2(B+z))D'ws_x + 4\sinh(2(B+z))D'z_xs_x\\
&\quad + 2\sinh(2(B+z))B'm^2 - 2\sinh^2(2(B+z))B'm^2.
\end{aligned}$$

Finally,

$$\frac{d}{dt}\hat{p}[\mathbf{u}](t) = \partial_x\hat{e}[\mathbf{u}](t) + 2\sinh(2(B+z))\left(B'(s_x^2-m^2) + 2D'(z_xm - ws_x)\right).$$

Finally, this proves the second identity in (4.3.9). □

The Lemma 4.3.2, allows us to propose the following integral estimation, which will be the focus in the subsequent section

Lemma 4.3.3. *Let* ψ *a smooth bounded weight function and* $r \in \mathbb{R}$. *Then*

$$\begin{aligned}\frac{d}{dt}\int \hat{e}[\mathbf{u}](t)\psi(x+rt)dx &= r\int \hat{e}[\mathbf{u}](t)\psi'(x+rt)dx - \int \hat{p}[\mathbf{u}](t)\psi'(x+rt)dx \\ &\quad + \int F_e[\mathbf{u}](t)\psi(x+rt)dx, \\ \frac{d}{dt}\int \hat{p}[\mathbf{u}](t)\psi(x+rt)dx &= r\int \hat{p}[\mathbf{u}](t)\psi'(x+rt)dx - \int \hat{e}[\mathbf{u}](t)\psi'(x+rt)dx \\ &\quad + \int F_p[\mathbf{u}](t)\psi(x+rt)dx.\end{aligned} \tag{4.3.10}$$

Proof. The proof of the Lemma 4.3.3 is direct from the Lemma 4.3.2 and from using integration by parts. □

Note that from (4.2.1) B', D' are compactly supported functions. Assume that at time $t = 0$ one has $\text{supp}(B') \cup \text{supp}(D') \subseteq \{\xi \in \mathbb{R} : |\xi| < R\}$, for some $R > 0$.

Lemma 4.3.4 (Exterior stability)**.** *Assume that* $\boldsymbol{B} + \boldsymbol{u}$ *is globally defined. Then for any* $0 < \delta < R$ *one has*

$$\int_{|x+t|\geq R} \hat{e}[\mathbf{u}](t)dx \leq \int_{|x|\geq R} \hat{e}[\mathbf{u}](0)dx.$$

This bound proves (4.1.14).

Proof. Let ψ be a smooth cut-off function such that

$$0 \leq \psi \leq 1, \quad \psi' \leq 0, \quad \psi(s) = 0,\ s \geq -R, \quad \psi(s) = 1,\ s \leq -R-1.$$

Now, if $r = 1$, notice that $\psi(x+t)$ has support in $x + t \leq -R$. Since $|\hat{p}| \leq \hat{e}$, we have from (4.3.10),

$$\frac{d}{dt}\int \hat{e}[\mathbf{u}](t)\psi(x+t)dx \leq \int F_e[\mathbf{u}](t)\psi(x+t)dx = 0,$$

proving the stability estimate for the left side. The other side is proved similarly.

□

4.4 Interior control

In this section we prove the last estimate in (4.1.17). It is important to notice that one can write (4.3.3) as follows

$$\begin{cases} z_{tt} - z_{xx} = -2\sinh(2(B+z))\left(2Q_0(D,s) + Q_0(s,s)\right), \\ s_{tt} - s_{xx} = \dfrac{\sinh(2(B+z))}{\sinh^2(B+z)}\left(Q_0(D,z) + Q_0(B,s) + Q_0(s,s)\right). \end{cases} \tag{4.4.1}$$

where Q_0 is the fundamental null form

$$Q_0(\phi_1, \phi_2) = m^{\alpha\beta}\partial_\alpha\phi_1\partial_\beta\phi_2,$$

and where $m_{\alpha\beta}$ to denote the standard Minkowski metric on $\mathbb{R}^{1+1}$ with signature $(-1, 1)$. It can be also noticed that the null structure is "quasi-preserved" after differentiating with respect to x, in the sense that

$$\partial_x Q_0(\phi, \tilde{\Lambda}) = Q_0(\partial_x\phi, \tilde{\Lambda}) + Q_0(\phi, \partial_x\tilde{\Lambda}). \tag{4.4.2}$$

Additionally, we have the following relation between the null form and the Killing vector fields L and $\underline{L}$

$$Q_0(\partial_x^p\phi, \partial_x^q\tilde{\Lambda}) \lesssim |L\partial_x^p\phi||\underline{L}\partial_x^q\tilde{\Lambda}| + |\underline{L}\partial_x^p\phi||L\partial_x^q\tilde{\Lambda}|, \tag{4.4.3}$$

where the implicit constant is independent of $(\tilde{\Lambda}, \phi)$.

It is easy to check that (see (4.2.5)):

(i) Since $0 < \gamma < 1/3$, using (4.2.5) and the Bernoulli's inequality, we get

$$\varphi^{3/4}(\cdot) \leq (1 + |2(\cdot)|)^2.$$

(ii) Thus, since $B', D' \in C_c^\infty(\mathbb{R})$, one has that for some fixed constant $K_1, K_2 > 0$,

$$|B^{(n+1)}(2\underline{u})| \leq \frac{K_1\varepsilon}{\varphi^{3/4}(\underline{u})}, \quad n = 0, 1, \tag{4.4.4}$$

and

$$|D^{(n+1)}(2\underline{u})| \leq \frac{K_2\varepsilon}{\varphi^{3/4}(\underline{u})}, \quad n = 0, 1. \tag{4.4.5}$$

(iii) The following relations for the null vector field L and $\underline{L}$ hold:

$$\begin{aligned} |L(D(2\underline{u}))| &= 2|D'(2\underline{u})| \leq \frac{K_1\varepsilon}{\varphi^{3/4}(\underline{u})}, \\ |L(D'(2\underline{u}))| &= 2|D''(2\underline{u})| \leq \frac{K_1\varepsilon}{\varphi^{3/4}(\underline{u})}, \quad |\underline{L}D)| = 0. \end{aligned} \tag{4.4.6}$$

and similar estimates for B'.

Recall Σ_t, C_u and $\underline{C}_u$ as introduced in (4.2.2), (4.2.3) and (4.2.4), respectively. Recall the energies introduced in (4.2.7). We will adapt the proof of Theorem 3.1 in [91] to the case of solitons, to obtain that the following energies remain small: for $k = 0, 1$:

$$\begin{aligned} \mathcal{E}_k[z, w](t) &= \int_{\Sigma_t} \left[\varphi(u)|\underline{L}\partial_x^k z|^2 + \varphi(\underline{u})|L\partial_x^k z|^2\right] dx, \\ \overline{\mathcal{E}}_k[s, m](t) &= \int_{\Sigma_t} \left[\varphi(u)|\underline{L}\partial_x^k s|^2 + \varphi(\underline{u})|L\partial_x^k s|^2\right] dx, \\ \mathcal{F}_k[z, w](t) &= \sup_{\underline{u}\in\mathbb{R}} \int_{\underline{C}_{\underline{u}}} \varphi(u)\left|\underline{L}\partial_x^k z\right|^2 + \sup_{u\in\mathbb{R}} \int_{C_u} \varphi(\underline{u})|L\partial_x^k z|^2, \\ \overline{\mathcal{F}}_k[s, m](t) &= \sup_{\underline{u}\in\mathbb{R}} \int_{C_{\underline{u}}} \varphi(u)\left|\underline{L}\partial_x^k s\right|^2 + \sup_{u\in\mathbb{R}} \int_{C_u} \varphi(\underline{u})|L\partial_x^k s|^2. \end{aligned} \tag{4.4.7}$$

Then, using (4.4.7) we define the total energy norms as follows:

$$\mathcal{E}(t) := \mathcal{E}_0[z,w](t) + \mathcal{E}_1[z,w](t).$$

Analogously one defines $\mathcal{F}(t)$, $\overline{\mathcal{E}}(t)$, and $\overline{\mathcal{F}}(t)$. From (4.1.15) we know that $\mathcal{E}(0) + \overline{\mathcal{E}}(0) < \delta^2$. The stability of the soliton perturbation (4.1.17) will be a consequence of the following more complete result:

Theorem 4.4.1. *Assume that there exists $T^* > 0$ such that, for all $t \in [0,T^*]$, the estimates*

$$\mathcal{E}(t) + \mathcal{F}(t) \leq 6CC_0\delta^2, \tag{4.4.8}$$
$$\overline{\mathcal{E}}(t) + \overline{\mathcal{F}}(t) \leq 6CC_0\delta^2, \tag{4.4.9}$$

hold for some $C_0 > 0$ and

$$\sup_{t\in[0,T^*]} \|z\|_{L^\infty(\mathbb{R})} \leq \frac{\lambda}{2}. \tag{4.4.10}$$

Then for all $t \in [0,T^*]$ there exists a universal constant δ_0 (independent of T^*) such that the previous estimates are improved for all $\delta \leq \delta_0$.

4.4.1 Proof of Theorem 4.4.1

As usual, we work with the system for z in (4.4.1), the one for s being very similar in both nature and estimates.

Deriving (4.4.1) and using (4.4.2) we obtain:

$$\Box \partial_x z = \rho_1 + \rho_2, \tag{4.4.11}$$

where

$$\begin{cases} \rho_1 := -4\sinh(2(B+z))\left(Q_0(\partial_x D, s) + Q_0(D,\partial_x s) + Q_0(s,\partial_x s)\right), \\ \rho_2 := -4\cosh(2(B+z))(B' + z_x)\left(2Q_0(D,s) + Q_0(s,s)\right). \end{cases} \tag{4.4.12}$$

Under the assumptions (4.4.8)-(4.4.9)-(4.4.10) for all $t \in [0,T^*]$, we assume that the solution remains regular, to later show that these bounds are maintained, with a better constant.

Consider $k = 0,1$. Using (4.2.6), with $\psi = \partial_x^k z$ and (4.4.11)-(4.4.12). Taking the sum over $k = 0,1$, and using (4.4.10), we obtain

$$\begin{aligned} \mathcal{E}(t) + \mathcal{F}(t) &\leq 2C_0\mathcal{E}(0) \\ &+ C_\lambda C_0 \iint_{D_t} \left(\varphi(u)|\underline{L}z| + \varphi(\underline{u})|Lz|\right)|Q_0(D,s)| + C_\lambda C_0 \iint_{D_t} \left(\varphi(u)|\underline{L}z| + \varphi(\underline{u})|Lz|\right)|Q_0(s,s)| \\ &+ 2C_\lambda C_0 \iint_{D_t} \left(\varphi(u)|\underline{L}\partial_x z| + \varphi(\underline{u})|L\partial_x z|\right)|\rho_1| + 2C_\lambda C_0 \iint_{D_t} \left(\varphi(u)|\underline{L}\partial_x z| + \varphi(\underline{u})|L\partial_x z|\right)|\rho_2|, \end{aligned} \tag{4.4.13}$$

where $C_\lambda := C(c_0(\lambda), c_1(\lambda))$, same as in (4.2.1). Recall the following result due to Luli et. al. in [85]:

Lemma 4.4.1 ([85], Lemma 3.2). *Under assumptions* (4.4.8) *and* (4.4.9), *there exists a universal constant* $C_2 > 0$ *such that:*

$$|Lz(t,x)| \leq \frac{C_2\delta}{(1+|u|^2)^{1/2+\gamma/2}}, \qquad |Ls(t,x)| \leq \frac{C_2\delta}{(1+|u|^2)^{1/2+\gamma/2}},$$
$$|\underline{L}z(t,x)| \leq \frac{C_2\delta}{(1+|\underline{u}|^2)^{1/2+\gamma/2}}, \qquad |\underline{L}s(t,x)| \leq \frac{C_2\delta}{(1+|\underline{u}|^2)^{1/2+\gamma/2}}.$$

Note that we can conveniently write

$$\partial_x z = \frac{1}{2}\left(Lz - \underline{L}z\right) \quad \text{and} \quad B' = \frac{1}{2}LB,$$

then, using (4.4.3) we can estimate (4.4.14) as follows

$$\begin{aligned}
&\mathcal{E}(t) + \mathcal{F}(t) \lesssim 2C_0\mathcal{E}(0)\\
&+ C_0 \iint_{D_t} \left(\varphi(u)|\underline{L}z| + \varphi(\underline{u})|Lz|\right)|\underline{L}s|\,|LD| + 2C_0 \iint_{D_t} \left(\varphi(u)|\underline{L}z| + \varphi(\underline{u})|Lz|\right)|Ls||\underline{L}s|\\
&+ 2C_0 \iint_{D_t} \left(\varphi(u)|\underline{L}\partial_x z| + \varphi(\underline{u})|L\partial_x z|\right)|\underline{L}\partial_x s||LD| + 2C_0 \iint_{D_t} \left(\varphi(u)|\underline{L}\partial_x z| + \varphi(\underline{u})|L\partial_x z|\right)|\underline{L}s||LD'|\\
&+ 2C_0 \iint_{D_t} \left(\varphi(u)|\underline{L}\partial_x z| + \varphi(\underline{u})|L\partial_x z|\right)|\underline{L}s||L\partial_x s| + 2C_0 \iint_{D_t} \left(\varphi(u)|\underline{L}\partial_x z| + \varphi(\underline{u})|L\partial_x z|\right)|\underline{L}\partial_x s||Ls|\\
&+ 2C_0 \iint_{D_t} \left(\varphi(u)|\underline{L}\partial_x z| + \varphi(\underline{u})|L\partial_x z|\right)|LB||\underline{L}s|||LD| + 2C_0 \iint_{D_t} \left(\varphi(u)|\underline{L}z| + \varphi(\underline{u})|Lz|\right)|\underline{L}z||Ls||\underline{L}s|\\
&=: 2C_0\mathcal{E}(0) + 2C_0\sum_{n=1}^{8} T_n.
\end{aligned} \tag{4.4.14}$$

Taking into account the estimates already established in [109], we will focus on the key terms in this framework, that is, the terms T_i, for $i \in \{1,3,4,7\}$. Let us start with T_1, using Hölder inequality we get:

$$\begin{aligned}
T_1 = T_{1,1} + T_{1,2} &:= \iint_{D_t} \varphi(u)|\underline{L}z|\,|\underline{L}s|\,|LD| + \iint_{D_t} \varphi(\underline{u})|Lz||\underline{L}s||LD|\\
&\lesssim \left(\iint_{D_t} \varphi(u)|\underline{L}z|^2|LD|\right)^{1/2}\left(\iint_{D_t} \varphi(u)|\underline{L}s|^2|LD|\right)^{1/2}\\
&\quad + \left(\iint_{D_t} \varphi(\underline{u})|Lz|^2|\underline{L}s|\right)^{1/2}\left(\iint_{D_t} \varphi(\underline{u})|\underline{L}s||LD|^2\right)^{1/2}\\
&\leq \left(\int_{\mathbb{R}} \frac{\varepsilon K_2}{\varphi^{3/4}(\underline{u})}\left[\int_{\underline{C}_{\underline{u}}} \varphi(u)|\underline{L}z|^2 ds\right] d\underline{u}\right)^{1/2}\left(\int_{\mathbb{R}} \frac{\varepsilon K_2}{\varphi^{3/4}(\underline{u})}\left[\int_{\underline{C}_{\underline{u}}} \varphi(u)|\underline{L}s|^2 ds\right] d\underline{u}\right)^{1/2}\\
&\quad + \left(\int_{\mathbb{R}} \frac{C_2\delta}{\varphi^{1/2}(u)}\left[\int_{C_u} \varphi(\underline{u})|Lz|^2 ds\right] du\right)^{1/2}\left(\int_{\mathbb{R}} \frac{C_2\delta}{\varphi^{1/2}(u)}\left[\int_{C_u} \frac{K_2^2\varepsilon^2}{\varphi^{1/2}(\underline{u})} ds\right] du\right)^{1/2}\\
&\lesssim K_2\varepsilon\delta^2.
\end{aligned}$$

Now, for T_3 we have

$$T_3 = T_{3,1} + T_{3,2} := \iint_{D_t} \varphi(u)|\underline{L}\partial_x z||\underline{L}\partial_x s||LD| + \iint_{D_t} \varphi(\underline{u})|L\partial_x z||\underline{L}\partial_x s||LD|,$$

note that $T_{3,1} \sim T_{1,1}$, thus $T_{3,1} \lesssim \varepsilon\delta^2$, for the integral $T_{3,2}$ we obtain

$$\begin{aligned} T_{3,2} &= \iint_{D_t} \frac{\varphi^{1/2}(\underline{u})}{\varphi^{1/2}(u)} |L\partial_x z| \varphi^{1/2}(u)\varphi^{1/2}(\underline{u}) |\underline{L}\partial_x s||LD| \\ &\leq \left(\iint_{D_t} \frac{\varphi(\underline{u})|L\partial_x z|^2}{\varphi(u)}\right)^{1/2} \left(\iint_{D_t} \frac{K_2^2\varepsilon^2\varphi(u)|\underline{L}\partial_x s|^2}{\varphi^{1/2}(\underline{u})}\right)^{1/2} \lesssim K_2\varepsilon\delta^2. \end{aligned}$$

The next term we would like to estimate is T_4, note that if we write

$$T_4 = T_{4,1} + T_{4,2} = \iint_{D_t} \varphi(u)|\underline{L}\partial_x z||\underline{L}s||LD'| + \iint_{D_t} \varphi(\underline{u})|L\partial_x z||\underline{L}s||LD'|,$$

then $T_{4,1} \sim T_{1,1}$ and $T_{4,2} \sim T_{3,2}$, thus $T_4 \lesssim K_2\varepsilon\delta^2$. Finally for T_7 we get

$$\begin{aligned} T_7 &= \iint_{D_t} \varphi(u)|\underline{L}\partial_x z||LB||\underline{L}s||LD| + \iint_{D_t} +\varphi(\underline{u})|L\partial_x z||LB||\underline{L}s||LD| \\ &\leq \left(\iint_{D_t} \varphi(u)|\underline{L}\partial_x z|^2|LB|^2\right)^{1/2} \left(\iint_{D_t} \varphi(u)|\underline{L}s|^2|LD|^2\right)^{1/2} \\ &\quad + \left(\iint_{D_t} \varphi(\underline{u})|L\partial_x z|^2|\underline{L}s|\right)^{1/2} \left(\iint_{D_t} \varphi(\underline{u})|\underline{L}s||LB|^2|LD|^2\right)^{1/2} \\ &\leq \left(\iint_{D_t} \frac{K_1^2\varepsilon^2}{\varphi^{3/2}(\underline{u})}\varphi(u)|\underline{L}\partial_x z|^2\right)^{1/2} \left(\iint_{D_t} \frac{K_2^2\varepsilon^2}{\varphi^{3/2}(\underline{u})}\varphi(u)|\underline{L}s|^2\right)^{1/2} \\ &\quad + \left(\iint_{D_t} \frac{C_2\delta}{\varphi^{1/2}(u)}\varphi(\underline{u})|L\partial z|^2\right)^{1/2} \left(\iint_{D_t} \frac{K_1^2K_2^2\varepsilon^2\delta}{\varphi^2(\underline{u})\varphi^{1/2}(u)}\right)^{1/2} \\ &\lesssim K_1K_2\varepsilon^2\delta^2. \end{aligned}$$

For the other term, which correspond to T_i, for $i \in \{2,5,6\}$ in (4.4.14), the analysis is the same as described in our recently completed work [109]. See this reference for full details. More precisely

$$T_j \lesssim \delta^3, \quad \text{for} \quad j \in \{2,5,6\} \qquad \text{and} \qquad T_8 \lesssim \delta^4.$$

Finally, from the energy estimate , we can arrange all the previous estimates together, and for universal constants C_3, C_4, C_5, C_6, K, with $K = \max\{K_1, K_2\}$, we have that for all $t \in [0, T^*]$:

$$\mathcal{E}(t) + \mathcal{F}(t) \leq 2C_1C_0\delta^2 + C_3\varepsilon K\delta^2 + C_4K^2\varepsilon^2\delta^2 + C_5\delta^3 + C_6\delta^4.$$

Now, we take δ_0 such that

$$\delta_0 \leq \frac{C_1C_0}{C_5} \quad \text{and} \quad \delta_0^2 \leq \frac{C_1C_0}{C_6},$$

and $0 < \varepsilon < \varepsilon_0$ such that

$$\max\left\{C_3\varepsilon K, C_4\varepsilon^2K^2\right\} \leq \frac{C_1C_0}{2}.$$

Then, we can see that for all $0 < \delta < \delta_0$ and for all $t \in [0, T^*]$, we have

$$\mathcal{E}(t) + \mathcal{F}(t) \leq 5C_1C_0\delta^2.$$

This improves the constant in (4.4.8).

Part III

Modified Zakharov-Kuznetsov Model

Chapter 5

Blow-up rate for modified Zakharov-Kuznetsov Equation

Abstract: In this short note we consider the modified Zakharov-Kuznetsov equation in $\mathbb{R}^2$, for initial conditions in the Sobolev space H^s with $s > 3/4$. This equation is L^2 or mass critical. Assuming that there is a blow up solution at finite time T^*, we set a lower bound for the blow up rate of that solution, expressed in terms of a lower bound for the H^s norm of the solution. The analysis is based on properly examining the linear estimates given by Faminskii [37], as well as, the local well-posedness theory of Linares and Pastor [81], combined with an argument developed by Weissler [113] and Colliander-Czuback-Sulem [27] in the context of the semilinear heat equation.

This chapter is contained in: J. Trespalacios, *Rate blow-up for modified Zakharov-Kuznetsov Equation.* Preprint 2024.

Contents

5.1 Introduction and main results

5.1.1 Setting

Consider the two-dimensional (2D) generalized Zakharov-Kuznetsov (ZK) equation

$$u_t + (u_{xx} + u_{yy} + u^p)_x = 0. \tag{5.1.1}$$

This equation is an extension of the well-known generalized Korteweg-de Vries (KdV) equation to two spatial dimensions. The ZK equation in 3D with a quadratic nonlinearity ($p = 2$), was originally proposed by Zakharov and Kuznetsov [116], to model waves in magnetized plasmas, see also [75], and a rigorously justified derivation in [76] from the Euler-Poisson system for uniformly magnetized plasma. From the local theory it follows that solutions to the ZK equation has a maximal forward lifespan $[0, T)$ with either $T = +\infty$ or $T < \infty$. Also, in the case $T < +\infty$, one has $||\nabla u(t)||_{L^2(\mathbb{R}^2)} \nearrow \infty$ as $t \longrightarrow T$, although the unbounded growth of the gradient might also happen in infinite time. In [37] Faminskii considered the case $p = 2$ in 2D. He showed local and global well-posedness for initial data in $H^m(\mathbb{R}^2)$, $m \geq 1$, integer. His method of proof was inspired by the one given by Kenig, Ponce and Vega in [61] to show local well-posedness for the IVP associated to the KdV equation. To prove global results, he made use of the L^2 and H^1 conserved quantities for solutions of (5.1.1).

In this note we consider the particular case when $p = 3.$, which is so-called the modified ZK equation. Notice that aso in this case mZK has a physical meaning. Indeed, it appears as an asymptotic model in the context of weakly nonlinear ion-acoustic waves in a plasma of cold ions and hot isothermal electrons with a uniform magnetic field [90]. On the other hand, the case $p = 4$ does not seem to appear as a physically relevant model but (as the generalized KdV equation) can be used as a mathematical toy model to investigate the competition between nonlinearity and dispersion.

5.1.2 Modified ZK

Consider the two-dimensional initial value problem (IVP) associated to the modified Zakharov-Kuznetsov (mZK) equation (which correspond to (5.1.1) with $p = 3$ and with a rescaling)

$$\begin{cases} u_t + u_{xxx} + u_{xyy} + u^2 u_x = 0, \quad (x, y) \in \mathbb{R}^2, \quad t > 0, \\ u(0, x, y) = u_0(x, y), \end{cases} \tag{5.1.2}$$

where $u = u(t, x, y)$ is a real valued function. During their existence, solutions to ZK have several conserved quantities, relevant to this work is the L^2 norm (or mass), and the energy (or Hamiltonian):

$$M[u(t)] = \int_{\mathbb{R}^2} u^2(t) = M[u(0)],$$

$$E[u(t)] = \frac{1}{2}\int_{\mathbb{R}^2} \left(u_x^2(t) + u_y^2(t)\right) - \frac{1}{4}\int_{\mathbb{R}^2} u^4(t) = E[u(0)],$$

under appropriate conditions on the smoothness of the solution $u(t, x, y)$ and its decay at the infinity. An important symmetry in the evolution equation (5.1.1) is the *the scaling invariance*, which states that an appropriately rescaled version of the original solution is also a solution of the equation. For the equation (5.1.2) it is

$$u_\lambda(t, x, y) = \lambda u(\lambda^3 t, \lambda x, \lambda y).$$

This symmetry makes invariant the Sobolev norm $\dot{H}^s$ with $s = 0$, where $\dot{H}^s(\mathbb{R}^2)$ denotes the homogeneous Sobolev space of order s, since $\|u_\lambda\|_{L^2} = \|u\|_{L^2}$. Therefore, the index s gives rise to the critical-type classification of (5.1.1). Since the mass of the solution (the L^2 norm) of the equation (5.1.2) is scaling-invariant, the 2D mZK equation is said to be mass-critical. The mZK equation has a family of localized traveling waves, which travels only in x direction

$$u(t, x, y) = Q_c(x - ct, y),$$

where Q_c satisfies

$$\Delta Q_c - cQ_c + Q_c^3 = 0,$$

and defining the ground state solution. We note that $Q_c \in C^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$. The solitons $Q_c(x,y)$ are related to the soliton $Q_1(x,y) =: Q(x,y)$ for $c > 0$ as follows:

$$Q_c(x,y) = \sqrt{c}Q(\sqrt{c}x, \sqrt{c}y);$$

thus, it suffices to consider $c = 1$. One can prove that Q is radial, and $\partial_r Q(r) < 0$ for any $r = |(x,y)| > 0$. Moreover, Q has exponential decay: $\partial^\alpha Q(x,y) \leq c_\alpha e^{-c_0 r}$ for any multi-index α and any $(x,y) \in \mathbb{R}^2$. The asymptotic stability of this solution was studied in [26, 39, 96], while the existence of breathers is discussed in [40].

The 2D mZK equation (5.1.2) has been extensively studied in recent years. Biagini-Linares [13] studied the Local Well-Posedness (LWP) in $H^1(\mathbb{R}^2)$. Linares and Pastor in [81], proved that the mZK equation is locally well-posed for data in $H^s(\mathbb{R}^2)$, $s > 3/4$ and they also showed the ill-posedness, in the sense that the data-to-solution map fails to be uniformly continuous (non-uniform data to solution map) for $s \leq 0$, so one cannot expect semilinear well-posedness in the critical space $L^2(\mathbb{R}^2)$. This LWP result was improved by Ribaud and Vento [98], who proved LWP for data in $H^s(\mathbb{R}^2)$, $s > 1/4$. More recently, Kinoshita in [62] established LWP at regularity $s = 1/4$, which is in fact optimal for the Picard iteration approach. Regarding to Global Well Posedness (GWP), Linares and Pastor [82] proved the GWP in $H^s(\mathbb{R}^2)$ for $s > 53/63$, when the mass of the initial data is smaller than the mass of the ground state in the focusing case. Bhattacharya et. al. in [11] used the I-method to obtain global well-posedness in $H^s(\mathbb{R}^2)$ space for $s > 3/4$, thus, improving the result of Linares and Pastor [82].

Using the energy and mass conservation together with the Gagliardo-Nirenberg inequality an its sharp constant expressed in terms of the solution mass, one has

$$\|\nabla u\|_{L^2}^2 \leq \left(1 - \frac{\|u\|_{L^2}^2}{\|Q\|_{L^2}^2}\right)^{-1} E[u].$$

Thus, if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then solutions with the initial condition u_0 exists also globally in time, while the blow-up might be possible if the initial mass $\|u_0\|_{L^2}$ is greater or equal to that of the soliton Q, see [69]. Recently, in [12] it was proved mass-concentration of low-regularity blow-up solutions, hinting an important step towards the proof of finite time blow-up. Recall the definition for blow up solution:

Definition 5.1.1 (Blow-up solution)**.** *We say that the solution $u(t,x,y)$ to the IVP* (5.1.2) *with $u_0 \in H^s(\mathbb{R}^2)$ blows up in finite time if there exists $0 < T^* < \infty$ such that*

$$\lim_{t \uparrow T^*} \|u(t,x,y)\|_{H^s(\mathbb{R}^2)} = \infty.$$

Concerning the study of the existence of blow up solutions, significant efforts have been made with respect to the case of the 2D mZK equation. Recall that this equation is mass critical, and therefore, can be compared to the critical generalized KdV equation in 1D. For this equation, Martel and Merle [87] proved the existence of solutions that blow up in the

$H^1(\mathbb{R}^2)$ norm in finite time. However, no exhaustive analogous results are yet available for the 2D mZK equation. When the initial data $u_0 \in H^1(\mathbb{R}^2)$ Farah et. al. [38] proved that there exists $\alpha > 0$ such that the solution to 2D mZK blows up in finite or infinite time, if the energy is negative and if the mass of the initial data satisfies $\|Q\|_{L^2(\mathbb{R}^2)} < \|u_0\|_{L^2(\mathbb{R}^2)} < \|Q\|_{L^2(\mathbb{R}^2)} + \alpha$. This is referred to as near-threshold blow-up phenomenon for the negative energy solutions. Klein-Roudenko-Stoilov in [69] investigated the $H^1(\mathbb{R}^2)$ blow-up phenomenon for the 2D mZK equation, numerically. In particular, they propose the following conjecture:

Conjecture 1 (L^2- critical case, [69])**.** *Consider the 2D critical mZK equation* (5.1.2)*. Then*

- *If u_0 is such that $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the solution is dispersed.*
- *If u_0 is sufficiently localized and such that $\|u_0\|_{L^2} > \|Q\|_{L^2}$, then the solution blows up in finite time $t = T^*$ and such that as $t \longrightarrow T^*$*

$$u(t,x,y) - \frac{1}{L(t)} Q\left(\frac{x - x_m(t)}{L(t)}, \frac{y - y_m(t)}{L(t)}\right) \longrightarrow \tilde{u} \in L^2,$$

with $L(t) \sim \sqrt{T^* - t}$,

$$\|u_x(t)\|_{L^2} \sim \frac{1}{L(t)}, \quad \textit{and} \quad x_m(t) \sim \frac{1}{T^* - t}, \quad y_m(t) \longrightarrow y^* < \infty.$$

This statement conjectured that blow-up happens in finite time and that blow-up solutions have some resemblance of being self-similar, i.e., the blow-up core forms a rightward moving self-similar type rescaled profile with the blow-up happening at infinity. For another hand, in a very recent result, Bozgan et. al. in [16], proposes a proof for existence of a blow up solution for the mZK equation, if the initial data satisfies $\|u_0\|_{L^2} > \|Q\|_{L^2(\mathbb{R}^2)} + \varepsilon_0$, with an additional condition on the $H^1(\mathbb{R}^2)$ norm of ε_0. In addition the authors propose a rate for the blow up of the singular solution, however, there are some issues in the proposed proof that are under discussion. The results in [16] propose that the blow-up rate exponent is 3/4, in contrast to the one presented in Conjecture 1, This discrepancy highlights a fundamental issue about the difficulty of understanding the dynamics of blow up solutions for this model.

5.1.3 Main Result

In this work we provide a lower bound on the blow-up rate for solutions to mZK in 2D. Our analysis relies on the LWP results of Linares and Pastor [81] in H^s, $s > 3/4$. The approach is to start with important linear estimates given by Faminskii [37], and then move on to non-linear estimates given by Linares and Pastor in [81]. In particular, we carefully keep track of the power of time involved in the estimates as it is central for the analysis of the lower bound for the blow-up rate. In a second stage, we will try to adapt the idea of Colliander et. al. [27]. The original idea comes from an argument used for the heat equation made by Weissler [113] and later extended to nonlinear Schrödinger equations by Cazenave and Weissler [19] to obtain a lower bound of blow-up for Sobolev norms of the solution. More precisely,

Theorem 5.1.1. *Consider the IVP* (5.1.2) *with initial conditions $u_0 \in H^s_{xy}(\mathbb{R})$ with $s > 3/4$. Assume that the solution $u(t,x,y)$ blows up in a finite time T^* in $H^s_{xy}(\mathbb{R})$. Then, we have the following lower bound for the blow-up rate:*

$$C(s)\,\|u(t)\|_{H^s} > (T^* - t)^{-7/48}, \quad t \uparrow T^*. \tag{5.1.3}$$

Compared with Conjecture 1, we are somehow far from the proposed rate of decay. However, it was noticed in [69] that no particular rate of decay was extremely favored from numerical experiments, leading to an unclear picture of the possible rates of decay for critical ZK.

Here is a brief description of the guidelines that we will follow in this work. First of all, the proof is based in a previous result obtained for the case of the Zakharov system by Colliander-Czuback-Sulem [27] and previous well-posedness results by Faminskii [37] and Linares-Pastor [81]. The goal is to obtain an explicit quantitative version of the LWP obtained in [37], key to apply the arguments from [27]. In Section 5.2 we present in detail the linear estimates associated with the equation (5.1.2). The goal is to show that we can handle the linear part in the right side of (5.2.1), and produce *additional* powers of the existence time T in the process. More precisely, we shall study in detail and improve the following bound from [37]:

Lemma 5.1.1. *Let $u_0 \in H^s(\mathbb{R}^2)$, $s > 3/4$. Then,*

$$\|U(t)u_0\|_{L^2_x L^\infty_{yT}} \leq c(s)T^{1/8} \|u_0\|_{H^1_{xy}},$$

with $c(s)$ is a constant depending on s.

Next, in Section 5.3 we will study nonlinear estimates for the full problem, in the spirit of Linares and Pastor [81]. Our goal is to carry out all the powers in the variable T. In specific, we will improve with quantitative bounds in time the following result:

Theorem 5.1.2 (Theorem 1.1 in [81])**.** *For any $u_0 \in H^s(\mathbb{R}^2)$, $s > 3/4$, there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution of the IVP (5.1.2), say $u(\cdot)$, defined in the interval $[0,T]$ such that*

$$\begin{aligned} &u \in C\left([0,T]; H^s(\mathbb{R}^2)\right), \\ &\|D^s_x u_x\|_{L^\infty_x L^2_{yT}} + \left\|D^s_y u_x\right\|_{L^\infty_x L^2_{yT}} < \infty, \\ &\|u\|_{L^3_T L^\infty_{xy}} + \|u_x\|_{L^{9/4}_T L^\infty_{xy}} < \infty, \end{aligned}$$

and

$$\|u\|_{L^2_x L^\infty_{yT}} < \infty.$$

Finally in the Section 5.4 we will assume that there exists a blow-up solution in finite time. Then, collecting all the previously obtained estimates, and following [27] with minor differences, we will obtain (5.1.3).

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5.2 Linear estimates

5.2.1 Notation

For $\alpha \in \mathbb{C}$, the operators D_x^α and D_y^α are defined via Fourier transform by $\widehat{D_x^\alpha f}(\xi\eta) = |\xi|^\alpha \hat{f}(\xi.\eta)$ and $\widehat{D_y^\alpha f}(\xi\eta) = |\eta|^\alpha \hat{f}(\xi,\eta)$. The mixed space-time norm is defined by (for $1 \leq p,q,r < \infty$)

$$\|f\|_{L_x^pL_y^qL_T^r} = \left(\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\left(\int_0^T |f(t,x,y)|^r dt\right)^{q/r} dy\right)^{p/q} dx\right)^{1/p},$$

with natural modification is either $p=\infty$, $q=\infty$, or $r=\infty$. For another hand, consider the Duhamel representation of the solution of the mZK equation (5.1.2), that takes the form:

$$u(t,\cdot,\cdot) = U(t)u_0 + \int_0^t U(t-t')(u^2u_x)(t')dt'. \tag{5.2.1}$$

Define the metric spaces

$$\mathcal{Y}_T = \{u \in C([0,T];H^s(\mathbb{R}^2)); \|\|u\|\| < \infty\},$$

and

$$\mathcal{Y}_T^a = \{u \in \mathcal{Y}_T; \|\|u\|\| < a\},$$

with

$$\|\|u\|\| := \|u\|_{L_T^\infty H_{xy}^s} + \|u\|_{L_T^3L_{xy}^\infty} + \|u\|_{L_T^{9/4}L_{xy}^\infty} + \|D_x^s u_x\|_{L_x^\infty L_{yT}^2} + \left\|D_y^s u_x\right\|_{L_x^\infty L_{yT}^2} + \|u\|_{L_x^2L_{yT}^\infty}.$$

In this section consider the IVP

$$\begin{cases} u_t + \partial_x \Delta u = 0, & (x,y) \in \mathbb{R}^2 \quad t \in \mathbb{R}, \\ u(0,x,y) = u_0(x,y). \end{cases} \tag{5.2.2}$$

The solution of the (5.2.2) is given by the unitary group $\{U(t)\}_{t=-\infty}^{\infty}$:

$$u(t) = U(t)u_0(x,y) = \int_{\mathbb{R}^2} e^{i(t(\xi^3+\xi\eta^2)+x\xi+y\eta)}\hat{u}_0(\xi,\eta)d\xi d\eta. \tag{5.2.3}$$

The estimates associated with the solution (5.2.3) are well-known and can be seen in detail in [37, 81]. In particular, as we are interested in tracking the time power involved in the estimation, we will focus in the important linear estimate in Lemma 5.1.1, from [37, Theorem 2.4]. First, it is necessary to follow the details of the following auxiliary result, see [37, Lemma 2.2]:

Lemma 5.2.1. *For any $T>0$ and $k \geq 0$ there exists a constant $c>0$ and a function $H_{k,T}(x) > 0$ such that*

$$\int_0^{+\infty} H_{k,T}(x)dx \leq c \cdot T^{1/4}2^{3k/2}(k+1)^2, \tag{5.2.4}$$

and

$$\left| \iint \exp(i(t\xi^3 + t\xi\eta^2 + x\xi + y\eta))\psi_1(\xi)\psi_2(\eta)d\xi d\eta \right| \leq H_{k,T}(|x|), \tag{5.2.5}$$

for $|t| \leq T$ *and* $(x, y) \in \mathbb{R}^2$, *where* $\psi_1(\xi) = \mu(a - |\xi|)$ *and* $\psi_2(\eta) = \mu(b - |\eta|)$ *for any* $a, b \leq 2^{k+1}$.

Here, the function $\mu(x)$ denotes a nondecreasing infinitely differentiable function on $\mathbb{R}$ such that $\mu(x) = 0$ for $x \leq 0$ and $\mu(x) = 1$ for $x \geq 1$. Note that, with this definition, the function $\mu(a - |\xi|)$ is a cut-off function defined as,

$$\mu(a - |\xi|) = \begin{cases} 1, & |\xi| \leq a - 1 \\ 0, & |\xi| \geq a, \end{cases}$$

in addition, $\mu(|\xi| - b)$ is given by

$$\mu(|\xi| - b) = \begin{cases} 1, & |\xi| \geq b + 1 \\ 0, & |\xi| \leq b. \end{cases}$$

Proof of the Lemma 5.2.1. We closely follow Faminskii [37]. Without loss of generality one can assume that $T \geq 1$ and $t \geq 0$. By $J(t, x, y)$ we denote the analog of the left-hand side in (5.2.5) in which the function $\psi_1 = \mu(a - |\xi|)$ is replaced by $\psi_1 = \mu(a - |\xi|)\mu(|\xi| - 1)$ (then $\psi_1(\xi) = 0$ for $|\xi| \leq 1$), i.e.,

$$J = \iint \exp(i(t\xi^3 + t\xi\eta^2 + x\xi + y\eta))\psi_1(\xi)\psi_2(\eta)d\xi d\eta.$$

For $|x| \leq 2^{-k/2}$ we use the inequality

$$|J| \leq c \cdot 2^{2k}, \tag{5.2.6}$$

which corresponds to the size of the domain in this region. Next, suppose that either $x \geq 2^{-k/2}$, or $x < -\max\{2^{-k/2}, 32t \cdot 2^{2k}\}$. Set $\varphi_1(\xi, \eta) := t\xi^3 + t\xi\eta^2 + x\xi$; then $(\varphi_1)_\xi = t(3\xi^2 + \eta^2) + x$ and $|(\varphi_1)_\xi| \geq \max\{t(3\xi^2 + \eta^2), |x|/2\}$ for $(\xi, \eta) \in \operatorname{supp} \psi_1(\xi)\psi_2(\eta)$. Integrating by parts, we obtain

$$\begin{aligned} J &= \int \psi_2(\eta)e^{iy\eta} \left(\int e^{i\varphi_1}\psi_1(\xi)d\xi \right) d\eta \\ &= \int \psi_2(\eta)e^{iy\eta} \left(- \int \left(\frac{\psi_1(\xi)}{i(\varphi_1)'} \right)' e^{i\varphi_1} d\xi \right) d\eta. \end{aligned} \tag{5.2.7}$$

Then

$$\begin{aligned} |J| &\leq \int \psi_2(\eta) \int \left| \left(\frac{\psi_1(\xi)}{i(\varphi_1(\xi))_\xi} \right)' \right| d\xi d\eta \\ &\leq \int \psi_2(\eta) \int \left(\left| \frac{\psi_1'}{\varphi_1'} \right| + \left| \frac{\psi_1 \varphi_1''}{(\varphi_1')^2} \right| \right) d\xi d\eta \\ &\lesssim \int \psi_2(\eta) \int (\psi_1'|x|^{-1} + \psi_1 t|\xi||x|^{-2}) d\xi d\eta \leq c \cdot 2^k |x|^{-1} \quad \text{for} \quad 2^{-k/2} \leq |x| \leq 1. \end{aligned}$$

For the remaining case, integrating by parts again in (5.2.7), and by simplifying, we get

$$|J| \leq \int \psi_2(\xi) \int \left| \left(\frac{1}{\varphi_1'} \left(\frac{\psi_1}{\varphi_1'} \right)' \right)' \right| d\xi d\eta \leq c \cdot 2^k |x|^{-2} \quad \text{for} \quad |x| \geq 1. \tag{5.2.8}$$

The main difficulty is in the consideration of the remaining case $-32t \cdot 2^{2k} < x < -2^{-k/2}$. Let us reduce J as follws:

$$J = \int \Phi(y - z) \underbrace{\left(\int \left(\frac{\pi}{t|\xi|} \right)^{1/2} \exp\left(i \left(t\xi^3 + x\xi - \frac{z^2}{4\xi t} + \frac{\pi}{4} \operatorname{sgn} \xi \right) \right) \psi_1(\xi) d\xi \right)}_{J_1} dz, \tag{5.2.9}$$

where $\Phi \equiv \mathcal{F}_y^{-1}(\psi_2)$. Let us estimate the inner integral $J_1(t, x, y)$ with respect to ξ on the right-hand side in (5.2.9).

Case $z^2 \geq x^2/4$. Let us split the real axis into two parts

$$\Omega_1 = \{\xi : \xi^2 > |x|/(32t)\}, \qquad \text{and} \qquad \Omega_2 = \{\xi : \xi^2 < |x|/(32t)\}.$$

From now on, the integrals J_{1i} will correspond to the integral J_1 on each of the regions Ω_i. If $\varphi(\xi) = t\xi^3 + x\xi - z^2/(4\xi t)$, after the analysis for this region [37] one gets

$$\begin{cases} |J_{11}| \leq ct^{-7/12}|x|^{-1/4} \leq c_1 \cdot 2^{7k/6}|x|^{-5/6}, \\ |J_{12}| \leq c \dfrac{(t|\xi|^{-1/2})}{|\varphi'|} \Big|_{\partial\Omega_2} + c \displaystyle\int_{\Omega_2} \left| \left(\frac{|t\xi|^{-1/2}\psi_1}{\varphi'} \right)' \right| d\xi \leq c_1 2^{k/2}|x|^{-3/2}. \end{cases} \tag{5.2.10}$$

Case $z^2 \leq x^2/4$. Let $z^2 = px^2$, $0 \leq p \leq 1/4$. Consider the following domains:

$$\Omega_3 = \left\{ \xi : \xi^2 > \frac{|x|}{6t} \right\}, \quad \Omega_4 = \left\{ \xi : \frac{p|x|}{2t} < \xi^2 \leq \frac{|x|}{6t} \right\}, \quad \text{and} \quad \Omega_5 = \left\{ \xi : \xi^2 < \frac{p|x|}{2t} \right\}.$$

For these regions we have to

$$\begin{cases} |J_{13}| \leq c 2^{7k/6}|x|^{-5/6}, \\ |J_{14}| \leq c_1 2^k |x|^{-3/2}, \\ |J_{15}| \leq c_1 2^{3k/2}|x|^{-1}. \end{cases} \tag{5.2.11}$$

Using the inequality $\int |\Phi(y)| dy \leq c(k+1)$ and the estimates (5.2.10)-(5.2.11), from the equation (5.2.9) we obtain

$$|J| \leq c(k+1)(2^{7k/6}|x|^{-5/6} + 2^k|x|^{-3/2} + 2^{3k/2}|x|^{-1}), \tag{5.2.12}$$

for $-32t \cdot 2^{2k} < x < -2^{-k/2}$ and for any y. Combining inequalities (5.2.6)-(5.2.8) and (5.2.12) yields the estimate (5.2.5) for J.

Principal estimate: This is the main contribution of this work. Let us return to the structure for J in (5.2.9), but now consider the case of

$$J_0 = \int \Phi(y - z) \left(\int \left(\frac{\pi}{t|\xi|} \right)^{1/2} \exp\left(i \left(t\xi^3 + x\xi - \frac{z^2}{4\xi t} + \frac{\pi}{4} \operatorname{sgn} \xi \right) \right) \psi_0(\xi) d\xi \right) dz, \tag{5.2.13}$$

where $\Phi \equiv \mathcal{F}_y^{-1}(\psi_2)$, and the function $\psi_0(\xi) \equiv \mu(2-|\xi|)$, instead of $\psi_1(\xi)$ ocurring with J in (5.2.9), then we have

$$|J_0| \leq c \cdot 2^k, \qquad \text{for} \quad |x| \leq 128T, \tag{5.2.14}$$

and

$$|J_0| \leq c \cdot 2^k x^{-2}, \qquad \text{for} \quad x \geq 128T \quad \text{or} \quad x \leq -\max(128T, 32t \cdot 2^{2k}). \tag{5.2.15}$$

But if $-32t \cdot 2^{2k} < x < -128T$, then we must estimate the inner integral $J_{0i}(t,x,z)$ with respect to ξ on the right-hand side in (5.2.13), that contains the function $\psi_0(\xi) \equiv \mu(2-|\xi|)$ instead of $\psi_1(\xi)$ ocurring in J, and will be analyzed in different regions Ω_{0i}, $i \in \{1,2,3,4\}$, which will be described below. That is

$$J_{0i} := \int_{\Omega_{0i}} \left(\frac{\pi}{t|\xi|}\right)^{1/2} \exp\left(i\left(t\xi^3 + x\xi - \frac{z^2}{4\xi t} + \frac{\pi}{4}\operatorname{sgn}\xi\right)\right)\psi_0(\xi)d\xi.$$

Case 1: Consider $z^2 \geq \frac{x^2}{4}$. For this case, let us split the real axis into two

$$\Omega_{01} = \{\xi : \xi^2 > |x|/(32t)\}, \qquad \Omega_{02} = \{\xi : \xi^2 < |x|/(32t)\}. \tag{5.2.16}$$

Note that if $\xi \in \operatorname{supp}\psi_0$, then $|\xi| < |x|/32t$, i.e. $\Omega_{01} = \emptyset$ and, since $|\xi|^{-1/2}\exp(i\pi\operatorname{sgn}(\xi/4)) = (-i\xi)^{-1/2}$, it follows that, similar to (5.2.10), integration by parts yields

$$|J_0| = \left|\int \Phi(y-z)J_{02}dz\right| \leq cx^{-2}.$$

Case 2: Consider $z^2 \leq \frac{x^2}{4}$. Set $\varphi(\xi) = t\xi^3 + x\xi - z^2/(4\xi t)$ and $z^2 = px^2$, $0 \leq p \leq 1/4$. We divide the real axis into three parts (here $\Omega_{03} = \{\xi : \xi^2 > |x|/(6t)\} = \emptyset$):

$$\begin{cases} \Omega_{04} := \Omega_4 \cap \{\xi : |\xi| \geq |x|^{-1/2}\}; \quad \Omega_4 = \left\{\xi : \frac{p|x|}{2t} < \xi^2 \leq \frac{|x|}{6t}\right\}, \\ \Omega_{05} := \Omega_5 \cap \{\xi : |\xi| \geq |x|^{-1/2}\}; \quad \Omega_5 = \left\{\xi : \xi^2 < \frac{p|x|}{2t}\right\}, \\ \Omega_{06} := \{\xi : |\xi| \leq |x|^{-1/2}\}. \end{cases} \tag{5.2.17}$$

Let us start estimating J_{04}, that it is, the integral in the region Ω_{04}. Note that $\varphi'(\xi) = 3t\xi^2 + x + \frac{z^2}{4\xi^2 t} = 3t\xi^2 + x + \frac{px^2}{4\xi^2 t}$ and

$$\varphi'\left(\left(\frac{p|x|}{(2t)}\right)^{1/2}\right) = \varphi'\left(\left(\frac{|x|}{(6t)}\right)^{1/2}\right) = |x|\frac{3p-1}{2} \leq -\frac{|x|}{8}. \tag{5.2.18}$$

Now, $\varphi''' = 6t + \frac{3z^2}{2t}\xi^{-4} > 0$. It follows that $|\varphi'(\xi)| \geq \frac{|x|}{8}$ for $\xi \in \Omega_4$, thus also in Ω_{04} and

$$|\varphi''| = \left|6t\xi - \frac{1}{2}\frac{p}{t}\frac{|x|^2}{\xi^3}\right| \leq |6t\xi| + 3\frac{|x|}{|\xi|} \leq c\left(t|\xi| + \left|\frac{x}{\xi}\right|\right). \tag{5.2.19}$$

Also, we can write Ω_{04} as follows:

$$\Omega_{04} = [-b,-a] \cup [a,b], \quad \text{with} \quad a := \max\left\{\left(\frac{p|x|}{2t}\right)^{1/2}, |x|^{-1/2}\right\} \quad \text{and} \quad b := \left(\frac{|x|}{6t}\right)^{1/2}.$$

Since $|\xi|^{-1/2}\exp(i\pi\operatorname{sgn}\xi/4) = (-i\xi)^{-1/2}$, then the integral to be estimated can be written as

$$J_{04} = \int_{\Omega_{04}} \frac{\pi^{1/2}}{t^{1/2}}(|\xi|)^{-1/2}\exp(i\varphi(\xi))\psi_0 d\xi.$$

By integration by parts

$$|J_{04}| \lesssim \pi^{1/2}\frac{|(t|\xi|)^{-1/2}\psi_0(\xi)|}{|\varphi'(\xi)|}\bigg|_{\partial\Omega_{04}} + \int_{\Omega_{04}}\left|\left(\frac{|t\xi|^{-1/2}\psi_0(\xi)}{\varphi'(\xi)}\right)'\exp(i\varphi\xi)\right|d\xi.$$

Let us estimate the integral of the right-hand side of the inequality. First we have for the derivative

$$\begin{aligned}&\left|\frac{d}{d\xi}\left(\frac{(t|\xi|)^{-1/2}\psi_0(\xi)}{\varphi'(\xi)}\right)\right| \\ &= \frac{1}{\varphi'(\xi)^2}\left(\left(-\frac{1}{2}t^{-1/2}\frac{\xi}{|\xi|}\xi^{-3/2}\psi_0 + (t|\xi|)^{-1/2}\psi_0'\right)\varphi'(\xi) - (t|\xi|)^{-1/2}\psi_0\varphi''(\xi)\right).\end{aligned}$$

Using (5.2.18)-(5.2.19) we get

$$\begin{aligned}\int_{\Omega_{04}}\left|\left(\frac{|t\xi|^{-1/2}\psi_0(\xi)}{\varphi'(\xi)}\right)'\exp(i\varphi\xi)\right|d\xi &\lesssim |x|^{-1}t^{-1/2}\int_{\Omega_{04}}|\xi|^{-3/2}\psi_0 d\xi + |x|^{-1}t^{-1/2}\int_{\Omega_{04}}|\xi|^{-1/2}\psi_0' d\xi \\ &\quad + |x|^{-2}t^{-1/2}\int_{\Omega_{04}}|\xi|^{-1/2}\left(t|\xi| + \frac{|x|}{|\xi|}\right)\psi_0 d\xi \\ &= I_1 + I_2 + I_3.\end{aligned}$$

We are going to estimate each integral I_i, $i \in \{1,2,3\}$, conveniently using the definition of the domain as follow

$$I_1 \leq |x|^{-1}t^{-1/2}\int_{|x|^{-1/2}}^{2}|\xi|^{-3/2}d\xi \lesssim |x|^{-1}t^{-1/2}|x|^{1/4} = t^{-1/2}|x|^{-3/4}. \tag{5.2.20}$$

Next for I_2 we have

$$I_2 \leq |x|^{-1}t^{-1/2}\int_1^2|\xi|^{-1/2}d\xi \lesssim |x|^{-1}t^{-1/2}|x|^{1/4} = t^{-1/2}|x|^{-3/4}. \tag{5.2.21}$$

We have used that $T \geq 1$ and $x < -128T$. For I_3 we get

$$\begin{aligned}I_3 &\leq t^{1/2}|x|^{-2}\int_{|x|^{-1/2}}^{\left(\frac{|x|}{6t}\right)^{1/2}}|\xi|^{1/2}d\xi + t^{-1/2}|x|^{-1}\int_{|x|^{-1/2}}^{2}|\xi|^{-3/2}d\xi \\ &\lesssim t^{1/2}|x|^{-2}\left(\frac{|x|}{6t}\right)^{3/4} + t^{-1/2}|x|^{-3/4} \leq (T^{1/4}+1)t^{-1/2}|x|^{-3/4}.\end{aligned} \tag{5.2.22}$$

Again, we have used that $T \geq 1$ and $x < -128T$. Finally, collecting estimates (5.2.20)-(5.2.22) we can conclude

$$\int_{\Omega_{04}}\left|\left(\frac{|t\xi|^{-1/2}\psi_0(\xi)}{\varphi'(\xi)}\right)'\exp(i\varphi\xi)\right|d\xi \lesssim (T^{1/4}+1)t^{-1/2}|x|^{-3/4}. \tag{5.2.23}$$

Now, $\partial\Omega_{04} = \{-a, -b, a, b\}$, with $a := \max\left\{\left(\frac{p|x|}{2t}\right)^{1/2}, |x|^{-1/2}\right\}$ and $b := \left(\frac{|x|}{6t}\right)^{1/2}$. Then, using (5.2.18), for the boundary term we have

$$\pi^{1/2}\frac{|(t|\xi|)^{-1/2}\psi_0(\xi)|}{|\varphi'(\xi)|}\Big|_{\partial\Omega_{04}} \lesssim |x|^{-1}t^{-1/2}|x|^{1/4} + |x|^{-1}t^{-1/2}\left(\frac{|x|}{6t}\right)^{-1/4}$$
$$\lesssim |x|^{-3/4}t^{-1/2} + |x|^{-1}\left(\frac{|x|}{6t}\right)^{-1/4}t^{-1/2} \lesssim T^{1/4}t^{-1/2}|x|^{-3/4}. \tag{5.2.24}$$

So then, by collecting (5.2.23)-(5.2.24) we get

$$|J_{04}| \lesssim T^{1/4}t^{-1/2}|x|^{-3/4}.$$

The remaining integrals can be estimated as in [37] to get:

$$\begin{aligned} |J_{05}| &\leq ct^{-1/2}|x|^{-1/4}, \\ |J_{06}| &\leq ct^{-1/2}|x|^{-1/4}, \end{aligned} \tag{5.2.25}$$

Combining the estimates (5.2.1)-(5.2.25), we obtain, similarly to (5.2.12):

$$|J_0| \leq c\cdot T^{1/4}(k+1)t^{-1/2}|x|^{-1/4} \lesssim c\cdot T^{1/4}(k+1)2^k|x|^{-3/4}.$$

□

Lemma 5.2.2 (Theorem 2.4 [37])**.** *Let $v_0 \in H^s$ for some $s > 3/4$. Then*

$$\|U(t)u_0\|_{L^2_xL^\infty_{yT}} \leq c(s)T^{1/8}\|u_0\|_{H^s_{xy}},$$

where $c(s)$ is a constant depending on s.

Proof. Let us introduce the sequence of functions $\psi_k(\xi,\eta)$ as follows:

1. $\psi_0(\xi,\eta) = \mu(2-|\xi|)\mu(2-|\eta|)$, and
2. $\psi_k(\xi,\eta) = \mu(2^{k+1}-|\xi|)\mu(2^{k+1}-|\eta|)\mu(|\eta|-2^k+1)+\mu(2^{k+1}-|\xi|)\mu(|\xi|-2^k+1)\mu(2^k-|\eta|)$, for $k \geq 1$.

Note that, using induction, we can show that

$$\sum_{k=0}^{m}\psi_k(\xi,\eta) = \mu(2^{m+1}-|\xi|)\mu(2^{m+1}-|\eta|), \quad \text{with} \quad \operatorname{supp}\left(\sum_{k=0}^{m}\psi_k(\xi,\eta)\right) = [-2^{m+1},2^{m+1}],$$

thus we can conclude that $\sum_{k=0}^{\infty}\psi_k(\xi,\eta) \equiv 1$.

For any function $f \in L^2$, we set $B_kf = \mathcal{F}^{-1}(\psi_k^{1/2}\hat{f})$. Note that $\|B_ku_0\|_{L^2} \leq c\cdot 2^{-ks}\|u_0\|_{H^s}$. If $g \in C_0^\infty(\mathbb{R})^3$, then, by (5.2.5), we have

$$\left|\int_{-T}^{T}(U(t-\tau))B_k^2 g(\tau,\cdot,\cdot)(x,y)d\tau\right| \leq \left(H_{k,2T}(|\cdot|) * \int_{-T}^{T}\int |g(\tau,\cdot,y)|d\tau dy\right)(x), \tag{5.2.26}$$

for $|t| \leq T$. Let us introduce operators A_k acting from $L_1([-T,T];L^2)$ as follows:

$$A_k = \int \chi_T(\tau)U(-\tau) \times B_k g(\tau,\cdot,\cdot)d\tau.$$

Then, by virtue of the unitary of $U(t)$, the self-adjointness of B_k, and the permutability of $U(t)$ and B_k, the operator A_k^* (acting from L^2 into $L^\infty([-T,T];L^2)$) is specified by the formula $A_k^* v = U(t)B_k v$.

Set $X = L_x^2(\mathbb{R}; L_{ty}^1([-T,T]\times\mathbb{R}))$; then $X^* = L_x^2(\mathbb{R}; L_{ty}^\infty([-T,T]\times\mathbb{R}))$ and, by virtue of (5.2.4) and (5.2.26), one has

$$\begin{aligned}\|A_k^* A_k g\|_{X^*} &= \left\| \int_{-T}^{T} U(t-\tau)B^2 g(\tau,\cdot,\cdot))(x,y)d\tau \right\|_{X^*} \\ &\leq \int H_{k,2T}(|x|)dx \, \|g\|_X \leq cT^{1/4}2^{3k/2}\|g\|_X\,,\end{aligned}$$

for $g \in C_0^\infty(\mathbb{R}^3)$. Therefore, by Lemma 2.1 in [48], we have

$$\left\|U(t)B_k^2 v_0\right\|_{X^*} \leq c^{1/2}T^{1/8}2^{3k/4}(k+1)\|B_k u_0\|_{L^2} \leq c^{1/2}T^{1/8}2^{-k(s-3/4)}(k+1)\|u_0\|_{H^s}\,.$$

Thus, we obtain

$$\begin{aligned}\|U(t)u_0\|_{X^*} &\leq c\left(\sum_{k=0}^{\infty} 2^{\varepsilon k}\left\|U(t)B_k^2 v_0\right\|_{X^*}^2\right)^{1/2} \\ &\leq c\left(\sum_{k=0}^{\infty} 2^{\varepsilon k}(c^{1/2}T^{1/8}2^{-k(s-3/4)}(k+1)\|u_0\|_{H^s})^2\right)^{1/2} \\ &\leq c\cdot T^{1/8}\left(\sum_{k=0}^{\infty} 2^{\varepsilon k}2^{-k(2s-3/2)}(k+1)^2\right)^{1/2}\|u_0\|_{H^s} \leq c_1(s)T^{1/8}\|u_0\|_{H^s}\,,\end{aligned}$$

if $0 < \varepsilon < 2s - 3/2$. □

5.3 Nonlinear estimates

As mentioned before, the intention of this section is to trace the powers of T that appear in the non-linear estimation associated to the equation (5.1.2), for this, we reanalyze the LWP result of Linares and Pastor [81].

Proof of the Theorem 5.1.2. We closely follow Theorem 1.1 in [81]. Consider the integral operator:

$$\Psi(u) = \Psi(u)(t) = U(t)u_0 + \int_0^t U(t-t')(u^2 u_x)(t')dt'.$$

We assume that $s > 3/4$ and, without loss of generality $T \leq 1$. We start by estimating the H^s-norm of $\Psi(u)$. Let $u \in \mathcal{Y}_T$. By using Minkowski's inequality, group properties, and

Hölder inequality, we have

$$\begin{aligned}\|\Psi(u)(t)\|_{L^2_{x,y}} &\leq c\|u_0\|_{H^s} + c\int_0^T \|u\|_{L^2_{xy}}\|uu_x\|_{L^\infty_{xy}}\,dt' \\ &\leq c\|u_0\|_{H^s} + cT^{2/9}\|u\|_{L^\infty_T L^2_{xy}}\|u\|_{L^3_T L^\infty_{xy}}\|u_x\|_{L^{9/4}_T L^\infty_{xy}}.\end{aligned} \tag{5.3.1}$$

Recall the Leibniz rule for fractional derivatives: Let $0<\alpha<1$ and $1<p<\infty$. Then:

$$\|D^\alpha(fg) - fD^\alpha g - gD^\alpha f\|_{L^p(\mathbb{R})} \leq c\|g\|_{L^\infty(\mathbb{R})}\|D^\alpha f\|_{L^p(\mathbb{R})}, \tag{5.3.2}$$

where D^α denote D^α_x or D^α_y. Now, using group properties, Minkowski and Hölder inequalities, and twice the Leibniz rule (5.3.2) we have:

$$\begin{aligned}\|D^s_x\Psi(u)(t)\|_{L^2_{xy}} &\leq \|D^s_x u_0\|_{L^2_{xy}} + \int_0^T \left\|D^s_x(u^2u_x)(t')\right\|_{L^2_{xy}}dt' \\ &\leq c\|u_0\|_{H^s} + c\int_0^T \|u_x\|_{L^\infty_{xy}}\left\|D^s_x(u^2)\right\|_{L^2_{xy}}dt' + c\int_0^T \left\|u^2D^s_x u_x\right\|_{L^2_{xy}}dt' \\ &\leq c\|u_0\|_{H^s} + c\int_0^T \|u_x\|_{L^\infty_{xy}}\|u_x\|_{L^\infty_{xy}}\|D^s_x u\|_{L^2_{xy}}dt' + c\int_0^T \left\|u^2D^s_x u_x\right\|_{L^2_{xy}}dt' \\ &\leq c\|u_0\|_{H^s} + c\|u\|_{L^\infty_T H^s_{xy}}\int_0^T \|u_x\|_{L^\infty_{xy}}\|u_x\|_{L^\infty_{xy}}dt' + c\int_0^T \left\|u^2D^s_x u_x\right\|_{L^2_{xy}}dt'.\end{aligned} \tag{5.3.3}$$

From Hölder inequality and the argument in (5.3.1), we get

$$\int_0^T \|u_x\|_{L^\infty_{xy}}\|u_x\|_{L^\infty_{xy}}dt' \leq T^{2/9}\|u\|_{L^3_T L^\infty_{xy}}\|u_x\|_{L^{9/4}_T L^\infty_{xy}}, \tag{5.3.4}$$

and

$$\begin{aligned}\int_0^T \left\|u^2D^s_x u_x\right\|_{L^2_{xy}}dt' &\leq \int_0^T \|u_x\|_{L^\infty_{xy}}\|uD^s_x u_x\|_{L^2_{xy}}dt' \\ &\leq \left(\int_0^T \|u\|^2_{L^\infty_{xy}}dt'\right)^{1/2}\|uD^s_x u_x\|_{L^2_{xyT}} \\ &\leq cT^{1/6}\|u\|_{L^3_T L^\infty_{xy}}\|u\|_{L^2_x L^\infty_{yT}}\|D^s_x u_x\|_{L^\infty_x L^2_{yT}}.\end{aligned} \tag{5.3.5}$$

Thus, combining (5.3.3), (5.3.4) and (5.3.5), we obtain

$$\begin{aligned}\|D^s_x\Psi(u)(t)\|_{L^2_{xy}} &\leq c\|u_0\|_{H^s} + cT^{2/9}\|u\|_{L^\infty_T H^s_{xy}}\|u\|_{L^3_T L^\infty_{xy}}\|u\|_{L^{9/4}_T L^\infty_{xy}} \\ &\quad + cT^{1/6}\|u\|_{L^3_T L^\infty_{xy}}\|u\|_{L^2_T L^\infty_{yT}}\left\|D^s_y u_x\right\|_{L^\infty_x L^2_{yT}}.\end{aligned} \tag{5.3.6}$$

Similarly for $\left\|D^s_y\Psi(u)(t)\right\|_{L^2_{xy}}$,

$$\begin{aligned}\left\|D^s_y\Psi(u)(t)\right\|_{L^2_{xy}} &\leq c\|u_0\|_{H^s} + cT^{2/9}\|u\|_{L^\infty_T H^s_{xy}}\|u\|_{L^3_T L^\infty_{xy}}\|u\|_{L^{9/4}_T L^\infty_{xy}} \\ &\quad + cT^{1/6}||u||_{L^3_T L^\infty_{xy}}\|u\|_{L^2_T L^\infty_{yT}}\left\|D^s_y u_x\right\|_{L^\infty_x L^2_{yT}}.\end{aligned} \tag{5.3.7}$$

Therefore, from (5.3.1), (5.3.6) and (5.3.7), we deduce

$$\|\Psi(u)\|_{L^\infty_T H^s} \leq c\|u_0\|_{H^s} + cT^{1/6}|||u|||^3. \tag{5.3.8}$$

Next, from the oscillatory inequality, group properties, and the argument in (5.3.1), we get

$$\begin{aligned}\|\Psi(u)\|_{L^3_T L^\infty_{xy}} &\leq \|U(t)u_0\|_{L^3_T L^\infty_{xy}} + \left\| U(t)\left(\int_0^t U(-t')(u^2u_x)(t')dt'\right)\right\|_{L^3_T L^\infty_{xy}} \\ &\leq c\,\|u_0\|_{L^2_{xy}} + c\int_0^T \left\|(u^2u_x)(t')\right\|_{L^2_{xy}} dt' \\ &\leq c\,\|u_0\|_{H^s} + cT^{2/9}|||u|||^3.\end{aligned} \tag{5.3.9}$$

By choosing $\varepsilon \sim 1/2$ such that $1-\varepsilon/2 \leq s$, an application of Lemma 2.6 in [81] together with arguments similar to (5.3.6) yield

$$\begin{aligned}\|\partial_x\Psi(u)\|_{L^{4/9}_T L^\infty_{xy}} \leq\ & \|U(t)\partial_x u_0\|_{L^{4/9}_T L^\infty_{xy}} \\ & + \left\| U(t)\left(\int_0^t U(-t')\partial_x(u^2u_x)(t')dt'\right)\right\|_{L^{4/9}_T L^\infty_{xy}} \\ &\leq c\left\|D_x^{-\varepsilon/2}\partial_x u_0\right\|_{L^2_{xy}} + c\int_0^T \left\|D_x^{-\epsilon/2}\partial_x(u^2u_x)(t')\right\|_{L^2_{xy}} dt' \\ &\leq c\,\|u_0\|_{H^s} + c\int_0^T \left\|(u^2u_x)(t')\right\|_{L^2_{xy}} dt' + c\int_0^T \left\|D_x^s(u^2u_x)(t')\right\|_{L^2_{xy}} dt' \\ &\leq c\,\|u_0\|_{H^s} + cT^{1/6}|||u|||^3.\end{aligned} \tag{5.3.10}$$

Applying Lemma (2.7.i) in [81], group properties, and Minkowski and Hölder inequalitiies, we obtain

$$\begin{aligned}\|D_x^s\partial_x\Psi(u)\|_{L^\infty_x L^2_{yT}} &\leq \|\partial_x U(t)D_x^s u_0\|_{L^\infty_x L^2_{yT}} \\ &\leq \left\|\partial_x U(t)\left(\int_0^t U(-t')D_x^s(u^2u_x)(t')dt'\right)\right\|_{L^\infty_x L^2_{yT}} \\ &\leq c\,\|D_x^s u_0\|_{L^2_{xy}} + c\int_0^T \left\|D_x^s(u^2u_x)(t')\right\|_{L^2_{xy}} dt' \\ &\leq c\,\|u_0\|_{H^s} + cT^{1/6}|||u|||^3\end{aligned}$$

and

$$\begin{aligned}\left\|D_y^s\partial_x\Psi(u)\right\|_{L^\infty_x L^2_{yT}} &\leq \|\partial_x U(t)D_x^s u_0\|_{L^\infty_x L^2_{yT}} \\ &\leq \left\|\partial_x U(t)\left(\int_0^t U(-t')D_x^s(u^2u_x)(t')dt'\right)\right\|_{L^\infty_x L^2_{yT}} \\ &\leq c\,\|D_x^s u_0\|_{L^2_{xy}} + c\int_0^T \left\|D_x^s(u^2u_x)(t')\right\|_{L^2_{xy}} dt' \\ &\leq c\,\|u_0\|_{H^s} + cT^{1/6}|||u|||^3.\end{aligned}$$

At last, and application of Lemma 5.2.2, Minkowski's inequality, group properties, and arguments previously used yield

$$\begin{aligned}\|\Psi(u)\|_{L^2_xL^\infty_{yT}} &\leq \|U(t)u_0\|_{L^2_xL^\infty_{yT}} + \left\|U(t)\left(\int_0^t U(-t')(u^2u_x)(t')dt'\right)\right\|_{L^2_xL^\infty_{yT}} \\ &\leq c(s)T^{1/8}\left\|u_0\right\|_{H^s} + c(s)T^{1/8}\int_0^T \left\|(u^2u_x)(t')\right\|_{H^s} dt' \\ &\leq c(s)T^{1/8}\left\|u_0\right\|_{H^s} + c(s)T^{1/8}T^{1/6}|||u|||^3.\end{aligned} \tag{5.3.11}$$

Therefore, from (5.3.8)-(5.3.11), we deduce

$$|||\Psi(u)||| \leq c(s)T^{1/8}\left\|u_0\right\|_{H^s} + c(s)T^{1/8}T^{1/6}|||u|||^3. \tag{5.3.12}$$

Choose $a = 2c(s)T^{1/8}\left\|u_0\right\|_{H^s}$ and $T > 0$ such that

$$a^2c(s)T^{1/8}T^{1/6} \leq \frac{1}{4}. \tag{5.3.13}$$

Then we get

$$|||\Psi(u)||| \leq \left(\frac{1}{2} + 4c(s)^3T^{3/8+1/6}\left\|u_0\right\|^2_{H^s}\right)2c(s)T^{1/8}\left\|u_0\right\|_{H^s} \leq \frac{3}{4}a.$$

Thus, to see that $\Psi : \mathcal{Y}^a_T \longrightarrow \mathcal{Y}^a_T$ is well defined. Moreover, similar arguments show that Ψ is a contraction. □

5.4 Lower bound for the rate of blow-up of singular solutions

Taking into account the estimate (5.3.12) and denoting

$$\mathcal{Y}(M,T) = \{u : u(t=0,x) = u_0, |||u||| \leq M\},$$

the LWP theory is obtained by a contraction argument in the space $\mathcal{Y}(M,T)$, provided the smallness condition $c(s)T^{\frac{1}{8}+\frac{1}{6}}M^2 < 1/4$ is satisfied. We adapt arguments develop by Colliander et. al [27] for the Zakharov system, which were originally developed by Weissler [113] for the heat equation, to prove a lower bound on the rate of blow up. Denote by T^* the supremum of all $T > 0$ for which there exists a solution u of the mZK satisfying

$$\|u\|_{L^\infty_TH^s_{xy}} + \|u\|_{L^3_TL^\infty_{xy}} + \|u\|_{L^{9/4}_TL^\infty_{xy}} + \|D^s_xu_x\|_{L^\infty_xL^2_{yT}} + \|D^s_yu_x\|_{L^\infty_xL^2_{yT}} + \|u\|_{L^2_xL^\infty_{yT}} < \infty.$$

The LWP theory shows $T^* > 0$ and for all $t \in [0, T^*)$

$$\|u(t)\|_{H^s_{xy}} < \infty.$$

By maximality of T^*, it is imposible that

$$\|u(t)\|_{L^\infty_{[0,T^*]}H^s_{xy}} < \infty,$$

since, otherwise, it would be possible to demonstrate LWP for $T_1 > T^*$ by taking as initial condition $u_0 = u(T^*, x)$. Therefore, if $T^* < \infty$, blow-up occurs:

$$\|u(t)\|_{H^s_{xy}} \longrightarrow \infty, \quad \text{as} \quad t \longrightarrow T^*.$$

Consider $u(t)$ posed at some time $t_0 \in [0, T^*]$. If for some M

$$c(s, (T - t_0))\|u(t_0)\|_{H^s} + c(s, (T - t_0))(T - t_0)^{1/6} M^3 < M,$$

then, we have LWP in $[t_0, T]$, thus $T < T^*$. Therefore, $\forall M > 0$

$$c(s)(T^* - t_0)^{1/8}\|u(t_0)\|_{H^s} + c(s)(T^* - t_0))^{1/8}(T^* - t_0)^{1/6} M^3 > M.$$

Choosing $M = 2c(s)\|u(t_0)\|_{H^s}$, we have

$$2c(s)M^2 > (2 + (t^* - t_0)^{1/8})(T^* - t_0)^{-(1/8+1/6)} > (T^* - t_0)^{-(1/8+1/6)}.$$

Equivalently,

$$2c(s)^2\|u(t_0)\|^2_{H^s} > (T^* - t_0)^{-(1/8+1/6)}, \quad \text{or} \quad \sqrt{2}c(s)\|u(t_0)\|_{H^s} > (T^* - t_0)^{-(7/48)}.$$

Thus we get (5.1.3).

Part IV

Conclusions

Chapter 6

Conclusions and Future Work

6.1 Conclusions

This thesis work is concerned with the study of global existence for small solutions to the Einstein equations in vacuum in the Belinski Zakharov setting, as well as the study of the long-term behavior of the solutions, and their stability. The first result corresponds to a particular case identified with the Principal Chiral Field equation, then, the Einstein equations in vacuum are properly addressed. Finally, an additional result related to the Zakharov Kuznetsov model is presented.

The results obtained in Part II are obtained from the identification of the vacuum Einstein equation with a system of quasilinear wave equations, for the global existence, and the use of the virial technique to obtain the asymptotic behavior of the solution, as well as, certain stability result.

Global existence for Pincipal Chiral Field Equations

In a first work, it was considered a special case of the Principal Chiral Field model in (1 + 1)- dimensions as a simplified version of the Einstein vacuum field equations under Belinksi-Zakharov symmetry [52, 119].

$$\partial_t \left(\partial_t g g^{-1}\right) - \partial_x \left(\partial_x g g^{-1}\right) = 0, \quad \det g = 1.$$

There are four main results in this paper:

- **Local and global well-posedness**,
- **Global-in-time decay** result of some solutions, and finally
- **The application to solitons**.

The statements together make for a relatively self-contained, new, interesting, and comprehensive introduction to this special case of the Principal Chiral Field model. Energy estimates for the wave equation were key to prove the local wellposedness result. Having established the existence of solutions, our second result involves whether or not local solutions can be extended globally in time, for this purpose, a condition of smallness is stated for the initial data. The smallness in the initial data implies that the nonlinear equation can be solved over a long period of time and the global solution can be constructed once the non-linearity decays enough. Moreover, the slower decay rate in low dimensions can be compensated by the special structure of the nonlinearity.

Belisnki-Zakharov spacetimes

The *Einstein vacuum* equation determine a 4−dimensional manifold $\mathcal{M}$ with a Lorentzian metric $\tilde{g}$ with vanishing *Ricci* curvature

$$\mathrm{R}_{\mu\nu}(\widetilde{g}) = 0. \tag{6.1.1}$$

The focus is understanding of outstanding solutions of (6.1.1) in the setting of Belinski-Zakharov spacetimes. Belinski and Zakharov recalled the particular case in which the metric tensor $\widetilde{g}_{\mu\nu}$ depends on two variables only. The following statements have been provided:

- A rigorous description of the **global existence theory** for the Einstein equations in vacuum, approached, via Gowdy coordinates (geometrical coordinates), as a system of quasilinear wave equations. Although the nonlinearity is not purely defined in terms of null forms, we can follow and adapt properly in the case of variable coefficients the weighted energy estimates proposed in [85] to approach the problem and finally obtain a small data global existence result.
- Formulation of conserved quantities for cosmological setting: we were able to propose a suitable formulation of energy and momentum, which allowed us to study **decay of a cosmological-type solutions** of the Einstein equations in the vacuum.
- The description of the asymptotic behavior for soliton-like solutions that can be identified as cosmological-like solutions of the Einstein equations: we proved, using **well-chosen virial estimates** that for solutions with finite energy, they must decay to zero locally in space.

Stability Results.

In this work we studied stability of particular soliton solutions to PCF. In particular, *orbital stability* of special solutions of these model with small initial data perturbations. It seems the first rigorous results in this direction for these kind of solutions. The stability theory is an important line of research in nonlinear PDEs, in the sense that it gives solidity to the results related to global existence of the solutions, asymptotic behavior, the dynamics of solutions.

Unlike many previous results related to orbital stability, in this work we do not follow the classical approach since it is nearly useless. PCF is a model where standard techniques fail and one needs a new approach. We combined asymptotic stability techniques and preservation of local energy to provide a near complete characterization of perturbations of regular soliton solutions of PCF. The main idea is to use viral techniques in a new way, this time, to obtain orbital stability of the explicit solutions that can be previously constructed for the problem. In this work was given a first proof of the fact that the 1-soliton (4.1.9) of the PCF model are orbital stable under small perturbations well-defined in the natural energy space associated to the problem.

The results obtained in Part III concerned to the ***modified Zakharov-Kuznetsov equation***, more precisely:

Rate Blow-up for modified Zakharov-Kuznetsov equation

By considering the modified Zakharov.Kuznetsov equation, given by

$$\begin{cases} u_t + u_{xxx} + u_{xyy} + u^2 u_x = 0, \quad (x,y) \in \mathbb{R}^2, \quad t > 0, \\ u(0,x,y) = u_0(x,y), \end{cases}$$

for initial conditions in the Sobolev space H^s with $s > 3/4$. Assuming that there is a blow up solution at finite time t^*, we set a lower bound for the blow up rate of that solution.

It is important to emphasize that the existence of blow-up solutions for this model has not been demonstrated, however, there are some numerical works that suggest that they exist and attempt to describe the rate for this blow-up. The results presented in this work are intended to contribute to the advances in the state of the art of the problem.

6.2 Future Works

In the following I would like to raise certain problems that are of great interest to me and in which I find a lot of potential, based on the results we have obtained for Belinski Zakharov spacetimes. These questions are quite varied and very robust in each case. Let us see a brief discussion of each of these problems below.

6.2.1 Integrability and Solitons in General Relativity

In this project I will be interested in the study of stationary axisymmetric spacetimes, for which one of the two **Killing fields is timelike**. Indeed, I will focus on the **Kerr and Kerr-Nut solutions** and its generalizations, in the framework of the Belinski-Zakharov transfom, as solutions of 2-solitons on the **Minkowski background**. These spacetimes have been studied from the physical point of view in the last decades, since it is possible to rely on these models to describe the gravitational fields of astrophysical sources, such as black holes.

Integrability in Axisymmetric spacetimes

For the **stationary axisymmetric spacetimes**, the more general case in which the sources are rotating but their fields remain stationary, the line element can be extended to what is known as the Weyl–Lewis–Papapetrou form see [80, 94, 114],

$$ds^2 = -f\left(dt + A(z,\rho)d\varphi\right)^2 + f^{-1}\left(e^{2\gamma(z,\rho)}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right), \quad \begin{cases} f(z,\rho) > 0, \quad \rho > 0, \\ z \in \mathbb{R}, \quad \varphi \in [0, 2\pi). \end{cases}$$

This block diagonal form is guaranteed in vacuum by Papapetrou's theorem [94], assuming a regular symmetry axis and the presence of the two commuting Killing vector fields ∂_t (timelike) and ∂_φ (spacelike).

Now, the equation $R_{ab} = 0$ can be expressed in terms of two sets of nonlinear equations for f and $\phi = \phi(\rho, f, A)$. After some work, one gets that the complex function $\mathcal{E}(\rho, z) := f + i\phi$ satisfies the so-called **Ernst equation**:

$$(\mathcal{E} + \overline{\mathcal{E}})\Delta\mathcal{E} = 2g^{\mu\nu}\mathcal{E}_\mu\mathcal{E}_\nu, \quad \text{and} \quad \Delta\mathcal{E} = \frac{1}{\sqrt{-\det g}}\partial_\mu\left(\sqrt{-\det g}g^{\mu\nu}\mathcal{E}_\nu\right). \tag{6.2.1}$$

The function $\mathcal{E}$ is called the Ernst potential. Then, solutions to the stationary axisymmetric Einstein equations in vacuum can be obtained by solving the Ernst equation. In fact, this equation can be obtained in any space-time having two commutative Killing vectors [50, 70, 77]. Ernst's original motivation in finding the Ernst equation [36, 70] was to provide a simple scheme to construct the Kerr metric as a solution to the stationary axisymmetric Einstein equations in vacuum. This model is considered integrable in the sense of the existence of a *zero curvature representation* (Lax representation in the case of the KdV equation). Interesting problems have been addressed using the integrability structure of the Ernst equation [50]. Among them, we mention the collision of two plane gravitational waves [77]. This shows the relevance and potential of this form of integrability, and the need to be fully understood and rigorously described.

Proposed Problem: Description of the axisymmetric 2-gravisoliton spacetimes

In a first stage, I will propose new classification results for (6.2.1) in the case of 2-gravisoliton spacetimes. Following the road map introduced in [2], I expect to uniquely **describe Black-Holes in terms of suitable profile-phase functions** [110].The purpose is to identify, in this context, the operators describing the dynamics of the so-called phase function, as well as to describe the solitonic solutions belonging to this class. Although some solutions have been well studied in the literature and show interesting features, there are still open questions about their meaning, singularities, topology, and conserved charges. In this objective, I aim to contribute to a new classification of gravitational solitons.

In a second stage, I will describe **dynamical properties of 2-gravisoliton** spacetimes in the case of **perturbed symmetric data** by using generalized virial identities in the spirit of [91]. In the case of axially symmetric spacetimes, and from the identification of the Ernst equation in the setting of the Belinski-Zakharov ansatz, I will understand and take advantage of the integrability structure to extend and describe the long time dynamics as I previously did in [91].

6.2.2 Stiff fluids in FLRW spacetimes

A classical assumption is that our large-scale universe is governed by the cosmological principle, with a Universe homogeneous and isotropic. In this setting, the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$(FLRW) \qquad ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2\right)$$

is commonly used to describe the Universe at great scales and it is one of the basis of the current Big-Bang theory, usually referred as the ΛCDM cosmological model. The other key component of the model is the Cold Dark Matter (CDM) theory. A key component of the FLRW metric (FLRW) is the scale factor, usually denoted as $a(t)$, which measures either the expansion or contraction of the Universe with respect to a time scale variable. The parameter K measures the curvature of the universe, being $K = 0$ a flat spacetime. Assuming this as a perfect fluid leads to the Friedmann's equations, one has an energy momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \quad p = \text{pressure}, \quad \rho = \text{density},$$

that through **Einstein's field equations** couples the scalar factor with the content of the universe, in the sense that

$$\dot{\rho} + 3H(\rho + p) = 0, \quad H(t) := \frac{\dot{a}(t)}{a(t)}.$$

Usually one assumes that H is constant. The precise value of today's Hubble parameter (and the consequent equipartition of mass-energy of the universe) is matter of a hot controversy [32, 103], between essentially two methods of measuring H (among other important observables) that differ in their outputs: measures using the local distance ladder and those inferred using the CMB[1] and galaxy surveys. Density ρ in the Universe is a tough question. Precisely, this last ingredient is another issue in Cosmology.

However, the Big-Bang theory as it is does not explain several puzzling observations of our current universe. These are the **horizon, the flatness, and the initial conditions** problems. The horizon problem refers to the impressive homogeneity of our universe, given the lack of causal connection among extreme sections of it. The flatness problem corresponds to the extreme flatness of the Universe today, given the fact that expansion and time evolution should make our universe even flatter. Finally, the Big-Bang model does not explain the initial conditions required to fulfill today's universe. Precisely, the **theory of inflation** was introduced to repair these unsolved issues of the model, and solves with great success the two first problems. For the third one must consider perturbations of inflation.

The large structure of the present Universe seems to be isotropic and spatially homogeneous. Physical cosmology is based on the relativistic Friedmann-Lemaître-Robertson-Walker (FLRW) models which describe the Universe as completely homogeneous and isotropic in all its evolution. Under the appropriate symmetry restrictions, Einstein equations coupled to a massless scalar field ϕ (a massless Klein-Gordon field) reads

$$R_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi, \quad \partial^\mu \partial_\mu \phi = 0$$

It is well known that a solution of this system have an interpretation as a stiff perfect fluid in the FLRW setting, as follows: if ϕ is timelike, i.e. $\partial_\mu \phi \partial_\nu \phi < 0$, p is the pressure, ρ the

[1]Hubble [56], based on the observation of local galaxies, observed that the universe expands and proposed the famous formula $v = H_0 d$, where v is the radial velocity of nearby galaxies and d its distance. The parameter H_0, today known as the Hubble's constant (or parameter) describes the expansion of our universe. A posterior analysis, with the discovery of the Cosmic Microwave Background (CMB) [95], a vestige of the decoupling between matter and radiation, confirmed the idea of a primordial universe more compact than the actual one

density, and u_μ the 4-velocity, defined as

$$\rho = p = -\frac{1}{2}\partial_\mu \phi \partial \nu \phi, \quad u_\mu = \frac{\partial_\mu \phi}{\sqrt{-\partial_\mu \phi \partial^\nu \phi}},$$

then the energy tensor $T_{\mu\nu}$ represents a perfect fluid: $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$. Stiffness is defined by $\rho = p$, and the speed of sound equals the speed of light. Under a **2-Killing vector field isometries**, $g_{\mu\nu}$ can be written using the Belinski-Zakharov ansatz, and the recently developed theory [91, 109] applies.

Proposed problem. To explore stiff-fluids using the long time behavior techniques developed previously mentioned, such as virial estimates, to obtain rigorous insights about the cosmological dynamics. This proposal is part of the **Regional Program MATH-AmSud** macro project in collaboration with **Diego Chamorro** (U. Paris-Saclay) and **Claudio Muñoz** (University of Chile).

6.2.3 Stability in the Energy Space for Einstein Equation

The stability theory is an important line of research in nonlinear partial differential equations, in the sense that it gives solidity to the results related to global existence of the solutions, asymptotic behavior, the dynamics of solitonic solutions. Given the description of the Einstein equations as a nonlinear model of wave equations, either from the Belinski-Zakharov formalism or using the so-called wave coordinates, we can study the different stability results developed for certain dispersive models, see for example [72], in order to establish sufficient conditions for stability in energy spaces for the equations associated with General Relativity (GR). In addition to the motivation arising from GR, the study of the stability for hyperbolic equations is of independent interest because of the connections with other branches of physics, for example, the study of the irrotational compressible Euler equations, which describe the dynamics of a compressible gas, also, the Einstein-Maxwell equation, as well as, the wave map equation.

In this problem I want study hyperbolic PDEs arising from General Relativity. In particular I am interested in the orbital and asymptotic stability of special solutions of these model against small initial data perturbations. The study of stability elucidates our understanding of whether such PDEs can provide mathematically reasonable models for physical phenomena in our universe. This problem is a naturally continuation of my PhD. research, In a previous work [91], we achieved the description of a type of cosmological solution, under the setting of the Belinski-Zakharov symmetries described below, the main goal is to explore the stability of these objects under small perturbations in the appropriate energy space, by addressing the problem using a variational approach.

Proposed problem. To understand the stability and asymptotic stability theory of Belinksi-Zakharov gravisolitons following our recent works and the previously developed techniques in [91].

6.2.4 The Einstein Constraint Equations with two Killing Vectors Fields

In view of the fact that the Einstein equations are a geometrical problem, it is important to note that, in order to formulate the equations in General Relativity as a initial value problem, the initial data must satisfy compatibility conditions known as Einstein constraint equations. Yvonne Choquet-Bruhat in her pioneering work [21] proved the local existence and uniqueness for the Einstein equations in vacuum when given a set of initial data $(\Sigma_t, \mathring{g}, \mathring{k})$ where Σ_t is a spacelike hypersurface of $\mathcal{M}$, $\mathring{g}$ a Riemannian metric on Σ_t and $\mathring{k}$ the associated second fundamental form. The result is valid when $\mathring{g}$ and $\mathring{k}$ satisfy the so-called constraint equations, which are geometric conditions on the problem, see also [22] to help understand the importance and complexity of the problem. These constraint equations are described as: the Hamiltonian and momentum constraint equations, which are respectively

$$\begin{cases} \mathring{R} - |\mathring{k}|^2 + (tr\mathring{k})^2 = 0, \\ d\mathring{i}v\mathring{k} - \mathring{\nabla} tr\mathring{k} = 0, \end{cases}$$

where $\mathring{\nabla}$ is the Levi-Civita connection of $\mathring{g}$ and $\mathring{R}$ is the scalar curvature of $\mathring{g}$. The Einstein constraint equations constitute a problem of great interest since from them emerge a non-trivial system of elliptic equations, which has been studied from different interesting fronts. Huneau in [57] obtained the existence of solutions for these compatibility conditions, assuming the existence of a translational spacelike Killing field in the asymptotically flat case, this hypothesis allowed her to move from a 3+1 dimensional problem to a 2+1 dimensional one and from that, rewrite the Einstein constraint equations in a suitable form. Also, Premoselli [97], using conformal method, obtains an admissible initial data for the conformal Einstein-scalar constraint system. Recently Fournoduvlos et. al. [42], assumes a constant main curvature condition on the hypersurface, to study the development of singularities for a generalized Kasner metric.

Proposed problem. To make a description and in-depth study of the constraint equations for the specific cases of metrics that are identified with the formalism proposed by Belinski and Zakharov. This study includes the existence of solutions to the proposed "elliptic" model.

Chapter 7

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