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Representations of groups of automorphisms on compact Riemann surfaces

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Abstract

A group action on a compact Riemann surface induces an action on the vector space of holomorphic differentials; the analytic representation of the action. This thesis deals with dihedral actions. First, a bijective correspondence between geometric signatures and analytic representations is obtained. Second, a refinement of a result of Bujalance, Cirre, Gamboa, and Gromadzki about signature realization is provided. Finally, we relate our results to decomposition of Jacobians by Prym varieties and elliptic curves, extending results of Carocca, Recillas, and Rodríguez.

Resumen

Una acción de grupo en una superficie de Riemann compacta induce una acción en el espacio vectorial de diferenciales holomorfos; la representación analítica de la acción. Esta tesis trata sobre acciones dihedrales. Primero, se obtiene una correspondencia biyectiva entre signaturas geométricas y representaciones analíticas. Segundo, se entrega un refinamiento de un resultado de Bujalance, Cirre, Gamboa y Gromadzki sobre realización de signaturas. Finalmente, relacionamos nuestros resultados con descomposiciones de Jacobianas por variedades de Prym y curvas elípticas, extendiendo resultados de Carocca, Recillas y Rodríguez.

Contents

Introduction

With the advent of significant advancements in computer algebra systems and finite group theory at the turn of the twenty first century, the investigation of automorphism groups of compact Riemann surfaces has witnessed a renewed surge of interest.

While the field rests upon foundational results established by Riemann, Klein, Wiman, and Hurwitz, a vast majority of the literature has been published within the past three decades. This renewed interest is based in part in two key aspects of the area. First, it boasts a rich landscape of open problems. Second, its capacity to leverage concepts and techniques from various mathematical disciplines—including geometry/topology, algebra, combinatorics, number theory, and Galois theory—provides a rich toolkit for tackling these problems.

The following examples should provide some insight into this interplay and the various points of view that have emerged in the past few decades.

(1) Construction of Riemann surfaces with actions can be achieved through various methods. Some methods draw from algebraic topology (monodromy epimorphisms), the uniformization theorem (surface kernel epimorphisms), algebraic geometry (defining equations), or hyperbolic geometry (tilings by fundamental domains).

(2) The action of a group of automorphisms can be classified up to signature. This classification in turn is equivalent to finding subgroups of Fuchsian groups. Thus, many open problems involve studying Fuchsian groups, their subgroups, and their over groups.

(3) Riemann's existence theorem provides two conditions under which a group acts on a Riemann surface with a given signature. While the first condition is a simple arithmetic check, the second one (the existence of a surface kernel epimorphism) is computationally nontrivial and the focus of much current research.

(4) The study of the variation in the automorphism group among all possible Riemann surfaces of a fixed genus has been carried out with the help of moduli and Teichmüller spaces.

(5) As compact Riemann surfaces are in natural bijection with smooth, irreducible projective curves, studying Riemann surfaces provides insight into questions about curves.

(6) Torelli's theorem establishes a strong correspondence between curves and Jacobian varieties. Furthermore, the automorphism group of a curve can be used to decompose its Jacobian variety. Most importantly for this thesis, the theory of abelian varieties provides a natural way to lift a group action on a curve into its *analytic representation*.

For an excellent overview of the field, we refer to [4] for a historical perspective and to [5] for an expository article on open problems and future directions.

Broughton's seminal 1990 paper [2] established the modern perspective on classifying finite group actions on compact Riemann surfaces, with Fuchsian groups playing a central role. We refer to [3] for an up-to-date treatment of this topic. Databases of actions on low genus can be consulted in [12, 20].

The theory of group algebra decomposition of Jacobian varieties is a relatively recent development, pioneered by the works of Lange and Recillas [22], and Carocca and Rodríguez [9] in the early 2000s. Shortly thereafter, Rojas in [37] introduced the notion of geometric signature, a generalization of the usual signature of an action. Rojas' article proved that the geometric signature captures much information: the geometric structure of the lattice of intermediate covers, the isotypical decomposition of the rational representation of the group action, and the dimension of the subvarieties of the group algebra decomposition. For a survey of the broader area of group actions on abelian varieties, we refer to [35]. For more recent developments, see [8, 34].

Dihedral actions on compact Riemann surfaces serve as a rich study case. Notably, they are foundational examples for the theory of group algebra decompositions of Jacobians, providing valuable insights into the general structure of these decompositions. Concretely, Recillas and Rodríguez in [31] worked out the case $S_3 \cong D_3$ (1998), and later, Carocca, Recillas, and Rodríguez in [7] gave a more general treatment of dihedral actions (2002). In a different line of research, Bujalance, Cirre, Gamboa, and Gromadzki in [6] provided necessary and sufficient conditions for a signature to admit a dihedral action (2003). For related works on Riemann surfaces or abelian varieties with dihedral actions, see [17, 24, 32].

In this thesis we deal with dihedral actions on compact Riemann surfaces of genus $g \geq 2$. Our aim is to understand in detail the interplay between different notions of geometric data of an action. The main results are:

(1) We prove that there is a bijective correspondence between geometric signatures and analytic representations of dihedral actions on compact Riemann surfaces (Theorem 2.3). Explicit formulas are provided.

(2) We state necessary and sufficient conditions for a *geometric* signature to admit a dihedral action (Theorem 3.4 and Theorem 3.5); this result is a refinement of [6].

(3) We solve the problem of deciding when a $\mathbb{C}$-representation is the analytic representation of a dihedral action (Theorem 3.6 and Theorem 3.7).

(4) We prove that the dihedral group D_n is Prym-affordable if and only if n is the power of a prime number (Theorem 4.3), extending results of [7].

(5) We characterize the group algebra components of a Jacobian with D_n-action (n odd) that are isogenous to the Prym variety of an intermediate cover (Theorem 4.4).

(6) We provide an exhaustive list of (geometric) signatures of D_n for which the group algebra decomposition provides a complete decomposition of the Jacobian (Theorem 4.5).

(7) We generalize (6) and give an exhaustive list of (geometric) signatures of D_n for which the group algebra decomposition provides a decomposition of the Jacobian into factors of equal dimension (Theorem 4.6).

This thesis is organized as follows. In Chapter §1 we shall briefly review the basic background: group actions on Riemann surfaces, $\mathbb{Q}$-representations, abelian varieties, and representations of automorphism groups. The main results of this thesis will be stated and proved in Chapter §2 (interplay between geometric data) and Chapter §3 (geometric signature realization). Finally, Chapter §4 will be concerned with applications to decomposition of Jacobians.

Chapter 1

Preliminaries

In this chapter we briefly review the basic background needed to state and prove the results of this thesis.

1.1 Group actions on Riemann surfaces

A *Riemann surface* is a connected complex analytic manifold of complex dimension 1. A map between Riemann surfaces is *holomorphic* if its expression in local charts is holomorphic. An *action* of a group G on a Riemann surface S is a monomorphism $G \to \mathrm{Aut}(S)$ into the automorphism group of S. Henceforth, S denotes a compact Riemann surface of genus $g \geq 2$ with a G-action. A classical result due to Hurwitz states that G is finite and $|G| \leq 84(g-1)$.

Each G-action on S induces a *Galois covering* $\pi_G : S \to S_G$, where S_G denotes the quotient Riemann surface given by the action of G on S. The *signature* of the action is the tuple $(\gamma; m_1, \ldots, m_v)$, where γ is the genus of S_G and $m_1, \ldots, m_v$ are the ramification indices of the branch values of π_G. If the action has signature as above, then it satisfies the *Riemann-Hurwitz formula*

$$2g - 2 = |G| \left[2\gamma - 2 + \sum_{j=1}^{v} \left(1 - \frac{1}{m_j} \right) \right].$$

Definition 1.1. Following [37], the *geometric signature* of the Galois covering $\pi_G : S \to S_G$ is the $(v+1)$-tuple

$$(\gamma; G_1, \ldots, G_v),$$

where γ is the genus of S_G, and $G_1, \ldots, G_v$ are the stabilizer subgroups of G associated to the (G-orbits of the) ramification points of π_G. Such subgroups are determined up to conjugation and reenumeration. (Strictly speaking, the geometric signature is given by the conjugacy classes of the stabilizers, but we denote them by a representative.)

It is worth mentioning that two actions of the same group with the same signature can have different geometric signatures, as we shall see later.

Remark 1.1. If $g \in G$ and $a \in \mathbb{Z}_+$, then the symbol $\langle g \rangle^a$ in a geometric signature abbreviates $\langle g \rangle, \overset{a}{\ldots}, \langle g \rangle$. Similarly, if $m \geq 2$ is an integer, then the symbol m^a in a signature abbreviates $m, \overset{a}{\ldots}, m$.

For any pair of subgroups $H \leq K$ of G, the induced maps $\pi_K^H : S_H \to S_K$ are called *intermediate coverings* of $\pi_G : S \to S_G$. The genus and the ramification data of the intermediate coverings of π_G are determined by the geometric signature of the G-action. We refer to [37, §3] for more details.

Let $\mathbb{H}$ denote the upper half-plane. We recall that $\mathrm{Aut}(\mathbb{H}) \cong \mathbb{P}\mathrm{SL}(2, \mathbb{R})$.

Definition 1.2. A *Fuchsian group* Δ is a discrete subgroup of $\mathrm{Aut}(\mathbb{H})$. A *surface Fuchsian group* is a torsion free Fuchsian group.

Let Δ be a co-compact Fuchsian group ($\mathbb{H}_\Delta$ is compact). The universal covering map $\mathbb{H} \to \mathbb{H}_\Delta$ is unramified if and only if Δ is torsion free. If γ denotes the genus of $\mathbb{H}_\Delta$ and $m_1, \ldots, m_v$ are the ramification indices of the branch values of the covering map $\Delta \to \mathbb{H}_\Delta$, then the tuple $(\gamma; m_1, \ldots, m_v)$ is the *signature* of Δ, and Δ has a canonical presentation

$$\left\langle \alpha_1, \ldots, \alpha_\gamma, \beta_1, \ldots, \beta_\gamma, x_1, \ldots, x_v : x_1^{m_1}, \ldots, x_v^{m_v}, \prod_{i=1}^{\gamma} [\alpha_i, \beta_i] \prod_{j=1}^{v} x_j \right\rangle,$$

where the square bracket denotes the commutator. The elements $\alpha_1, \ldots, \alpha_\gamma$ and $\beta_1, \ldots, \beta_\gamma$ are called the *hyperbolic generators* of Δ, whereas $x_1, \ldots, x_v$ are the *elliptic generators* of Δ.

Theorem 1.1. *Let S be a compact Riemann surface of genus $g \geq 2$. Then*

(1) there exists a (co-compact) surface Fuchsian group Γ such that $S \cong \mathbb{H}_\Gamma$;

(2) a group G acts on $S \cong \mathbb{H}_\Gamma$ with signature σ if and only if there exists a Fuchsian group Δ of signature σ and an exact sequence of groups

$$1 \to \Gamma \to \Delta \xrightarrow{\theta} G \to 1.$$

In this case $S_G \cong \mathbb{H}_\Delta$ and g satisfies the Riemann-Hurwitz formula.

The first statement of the previous theorem is called the *uniformization theorem*, and the second one is called *Riemann's existence theorem.* Note that in the theorem above the signature of Δ and the signature of the action of G agree. We say that the action is represented by the *surface kernel epimorphism* (ske) $\theta : \Delta \to G$. It is common to identify θ with the $(2\gamma + v)$-tuple or *generating vector*

$$(\theta(\alpha_1), \dots, \theta(\alpha_\gamma), \theta(\beta_1), \dots, \theta(\beta_\gamma), \theta(x_1), \dots, \theta(x_v)) \in G^{2\gamma+v}.$$

Definition 1.3. Let S_j be a Riemann surface with an action $\varepsilon_j : G \to \mathrm{Aut}(S_j)$, for $j = 1, 2$. The actions ε_1 and ε_2 are *topologically equivalent* if there exist a group automorphism $\omega \in \mathrm{Aut}(G)$ and an orientation-preserving homeomorphism $T : S_1 \to S_2$ such that

$$T\varepsilon_1(\omega(g)) = \varepsilon_2(g)T$$

for all $g \in G$. If T is holomorphic then we speak of *analytic equivalence.*

Topological equivalence of actions can be detected in a Fuchsian groups formulation. More precisely, following [2], two actions of G represented by the skes $\theta_j : \Delta \to G$ $(j = 1, 2)$ are topologically equivalent if and only if there exist $\omega \in \mathrm{Aut}(G)$ and $\phi^* \in \mathcal{B} < \mathrm{Aut}(\Delta)$ such that

$$\theta_2 = \omega \circ \theta_1 \circ \phi^*,$$

where $\mathcal{B}$ is the group of automorphisms of Δ induced by the homeomorphisms T as in Definition 1.3.

Proposition 1.1. *If G is abelian and S_G has genus zero, then $\mathcal{B}$ acts on a ske by permuting the images $\theta(x_j)$ of the elliptic generators so that the orders of the elements are preserved.*

Proof. See [2, Proposition 2.6]. □

An introductory treatment on Riemann surfaces can be found in [26]. For more advanced accounts, see [15] or [16].

1.2 $\mathbb{Q}$-representations

Let G be a finite group and let $\mathbb{F}$ be a field of characteristic zero. An $\mathbb{F}$-*representation* of G is a homomorphism $\rho : G \to \mathrm{GL}(V)$ into the general linear group of a (finite-dimensional) $\mathbb{F}$-vector space V. We will usually abuse notation and simply write V instead of ρ. The *degree* d_V of V is the dimension of V as an $\mathbb{F}$-vector space, and the *character* χ_V of V is the map that sends each element $g \in G$ into the trace of $V(g)$. Two representations are *equivalent* if their characters agree; we write $V_1 \cong V_2$.

The *character field* K_V of V is the extension of $\mathbb{F}$ by the values of the character of V. The *Schur index* s_V is the smallest positive integer such that there exists a degree s_V extension of fields $L_V > K_V$ over which V can be defined; L_V is called a *field of definition*.

A representation is *irreducible* if its invariant subspaces are the trivial ones. There are finitely many pairwise nonequivalent irreducible $\mathbb{F}$-representations of G; we denote by $\mathrm{Irr}_{\mathbb{F}}(G)$ the set formed by them. If V is an $\mathbb{F}$-representation and $\mathrm{Irr}_{\mathbb{F}}(G) = \{U_1, \ldots, U_v\}$, then for each j there exist a unique nonnegative integer a_j, the *multiplicity* of U_j in V, such that

$$V = a_1 U_1 \oplus \cdots \oplus a_v U_v,$$

where $a_j U_j = U_j \oplus \overset{a_j}{\cdots} \oplus U_j$. The integer a_j agrees with $\langle V, U_j \rangle / \langle U_j, U_j \rangle$, where

$$\langle V, U \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_U}(g).$$

It is known that for $W \in \mathrm{Irr}_{\mathbb{Q}}(G)$ there exists $V \in \mathrm{Irr}_{\mathbb{C}}(G)$ such that

$$W \otimes_{\mathbb{Q}} \mathbb{C} \cong s_V(\oplus_\sigma V^\sigma) = (\oplus_\sigma V^\sigma) \oplus \overset{s_V}{\cdots} \oplus (\oplus_\sigma V^\sigma),$$

where the direct sum is taken over the Galois group associated to the field extension $K_V > \mathbb{Q}$. We say that V and W are *Galois associated*.

Let H be a subgroup of G. The *fixed subspace* of V under H is

$$V^H = \{v \in V \mid \rho(h)(v) = v \text{ for all } h \in H\};$$

we denote its dimension by d_V^H. We refer to [38] and [39] for more details.

1.3 Abelian varieties

A g-dimensional *complex torus* $X = V_\Lambda$ is the quotient of a g-dimensional $\mathbb{C}$-vector space V by a discrete subgroup Λ of V of maximal rank. Each complex torus is an abelian group and a g-dimensional compact connected complex analytic manifold. Homomorphisms between complex tori are holomorphic maps which are also group homomorphisms. Every tori homomorphism $f : X_1 \to X_2$ is induced by a unique $\mathbb{C}$-linear map $\rho_a(f) : V_1 \to V_2$ that sends Λ_1

into Λ_2. The restriction of $\rho_a(f)$ to Λ_1 is a $\mathbb{Z}$-linear map $\rho_r(f) : \Lambda_1 \to \Lambda_2$. The corresponding monomorphisms

$$\rho_a : \mathrm{Hom}(X_1, X_2) \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_2),$$
$$\rho_r : \mathrm{Hom}(X_1, X_2) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda_1, \Lambda_2),$$

are called the *analytic representation* and *rational representation* of $\mathrm{Hom}(X_1, X_2)$, respectively. Both representations can be extended to

$$\mathrm{Hom}_{\mathbb{Q}}(X_1, X_2) := \mathrm{Hom}(X_1, X_2) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The (extension to $\mathbb{C}$ of the) rational representation is equivalent to the direct sum of the analytic representation and its complex conjugate:

$$\rho_r \otimes 1 \cong \rho_a \oplus \overline{\rho_a}. \tag{1.1}$$

An *isogeny* of tori is a surjective homomorphism with finite kernel. Two isogenous tori are denoted by $X_1 \sim X_2$.

An *abelian variety* is a complex torus which is also a complex projective algebraic variety. The *Jacobian variety* JS of a compact Riemann surface S is an (irreducible principally polarized) abelian variety: it is the quotient

$$JS = \Omega^1(S)^*/\Lambda$$

of linear functionals on the space of holomorphic differentials modulo periods. (Λ is the canonical injection of the first homology group $H_1(S, \mathbb{Z})$ on $\Omega^1(S)^*$.) The dimension of JS is the genus of S. Torelli's theorem states that

$$S \cong S' \iff JS \cong JS',$$

where $JS \cong JS'$ is an isomorphism of principally polarized abelian varieties.

Given a (ramified) covering map $f : S_1 \to S_2$ between compact Riemann surfaces, the pullback $f^* : JS_2 \to JS_1$ is an isogeny onto its image $f^*(JS_2)$. By Poincare's theorem there exists a complementary abelian subvariety of $f^*(JS_2)$, the *Prym variety* $P(f)$ of f, and then

$$JS_1 \sim JS_2 \times P(f).$$

We refer to [1] for more details on abelian varieties.

1.4 Representations of automorphism groups

Let G be a finite group. Every action of G on a compact Riemann surface (of genus $g \geq 2$) induces a G-action on its Jacobian variety.

Definition 1.4. Let S be a compact Riemann surface of genus $g \geq 2$ with a G-action. The homomorphisms

$$G \to \mathrm{Aut}(JS) \xrightarrow{\rho_a} \mathrm{GL}(\Omega^1(S)^*)$$
$$G \to \mathrm{Aut}(JS) \xrightarrow{\rho_r} \mathrm{GL}(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q})$$

are called, respectively, the *analytic representation* and the *rational representation* of the action. Abusing notation, we denote them by ρ_a and ρ_r.

Set $\zeta_n = e^{2\pi i/n}$.

Definition 1.5. Let G be a finite group. We define the function

$$\mathcal{N} : \mathrm{Irr}_{\mathbb{C}}(G) \times G \to \mathbb{Q} \quad \text{as} \quad \mathcal{N}(V, g) = \sum_{\alpha=1}^{|g|} N_{g,\alpha} \frac{|g| - \alpha}{|g|},$$

where $N_{g,\alpha}$ is the number of eigenvalues of $V(g)$ that are equal to $\zeta_{|g|}^{\alpha}$.

The following theorem is a classical result due to Chevalley and Weil [11]. A modern proof can be found in [28].

Theorem 1.2 (Chevalley-Weil formula)**.** *Let S be a compact Riemann surface of genus $g \geq 2$ with a G-action, and let ρ_a be its analytic representation. Assume that the action has signature $(\gamma; m_1, \ldots, m_v)$ and is represented by the surface kernel epimorphism $\theta : \Delta \to G$. If $V \in \mathrm{Irr}_{\mathbb{C}}(G)$ is nontrivial, then*

$$\langle \rho_a, V \rangle = d_V(\gamma - 1) + \sum_{j=1}^{v} \mathcal{N}(V, c_j),$$

where $c_1, \ldots, c_v$ are the images of the v elliptic generators of Δ. If V is the trivial representation then $\langle \rho_a, V \rangle = \gamma$.

Theorem 1.3. *Let S be a compact Riemann surface of genus $g \geq 2$ with a G-action, and let ρ_r be its rational representation. Assume that the action has geometric signature $(\gamma; G_1, \ldots, G_v)$. If $V \in \mathrm{Irr}_{\mathbb{C}}(G)$ is nontrivial, then*

$$\langle \rho_r, V \rangle = 2d_V(\gamma - 1) + \sum_{k=1}^{v} (d_V - d_V^{G_k}).$$

If V is the trivial representation then $\langle \rho_r, V \rangle = 2\gamma$.

Proof. See [37, Theorem 5.10]. □

Observe that the theorem above says that the rational representation of an action is determined by its geometric signature. The same holds for the analytic representation when $\rho_a \cong \overline{\rho_a}$.

Corollary 1.1. *If $\rho_a \cong \overline{\rho_a}$ then $\rho_r \cong 2\rho_a$. In particular,*

$$\langle \rho_r, V\rangle = 2\langle \rho_a, V\rangle \text{ for } V \in \mathrm{Irr}_{\mathbb{C}}(G).$$

In this case, ρ_a is determined by the geometric signature of the action.

Proof. This is a direct consequence of Theorem 1.3, which states that $\langle \rho_r, V\rangle$ is determined by the geometric signature, coupled with Equation (1.1). □

Chapter 2

Geometric signature and analytic representation

We begin this chapter with some general remarks about the interplay between the analytic representation of a group action, its geometric signature and other significant geometric data. As an example, we briefly consider the cyclic actions. The rest of the chapter deals with dihedral actions.

2.1 General considerations

Proposition 2.1. *Let S_j be a compact Riemann surface with a G-action, for $j = 1, 2$. If the actions are topologically equivalent, then their geometric signatures either agree or differ by an outer automorphism of G.*

Proof. As the geometric signature is preserved under inner group automorphisms, we only need to verify that conjugation by (orientation-preserving) homeomorphisms does not affect the stabilizer groups, and leaves invariant the geometric signature.

Let $T : S_1 \to S_2$ be an (orientation preserving) homeomorphism, and let ψ_j denote a G-action on S_j $(j = 1, 2)$ such that $\psi_2(g) = T \circ \psi_1(g) \circ T^{-1}$ for all $g \in G$. We observe that, for $p \in S_1$, the automorphism $\psi_1(g)$ fixes p if and only if $\psi_2(g) = T \circ \psi_1(g) \circ T^{-1}$ fixes $T(p)$, as desired. □

Corollary 2.1. *If G does not have outer automorphisms, then the analytic representation is constant over classes of topological equivalence of actions. That is, if two actions are topologically equivalent then they share the geometric signature.*

Example 2.1. Let $D_4 = \langle s, r : r^4, s^2, (sr)^2 \rangle$ be the dihedral group of order 8, and consider the outer automorphism of D_4 given by

$$\omega : s \mapsto sr, \; r \mapsto r.$$

It is known that D_4 acts in genus 3 with signature $(0; 2, 2, 4, 4)$. The generating vectors

$$\mathcal{V}_1 = (s, s, r, r^3) \text{ and } \mathcal{V}_2 = (sr, sr, r, r^3)$$

of signature $(0; 2, 2, 4, 4)$ are related by the automorphism ω. Hence they represent actions that are topologically equivalent. Since $\langle s \rangle$ and $\langle sr \rangle$ are not conjugate, the geometric signatures of $\mathcal{V}_1$ and $\mathcal{V}_2$ are distinct.

Now, let us study the analytic representation of cyclic actions. As complex conjugation permutes the irreducible $\mathbb{C}$-representations of a given cyclic group, the relation $\rho_a \cong \overline{\rho_a}$ does not hold for most cyclic actions.

Example 2.2. Consider the cyclic group $\mathbb{Z}_p = \langle r \rangle$ of prime order p. Let us compute the values of the function $\mathcal{N} : \mathrm{Irr}_{\mathbb{C}}(\mathbb{Z}_p) \times \mathbb{Z}_p \to \mathbb{Q}$, introduced in Definition 1.5. The group $\mathbb{Z}_p$ has p irreducible $\mathbb{C}$-representations, of degree one, given by

$$\rho_h : r \mapsto \zeta_p^h \text{ for } 0 \leq h \leq p-1,$$

where $\zeta_p = e^{2\pi i/p}$. For $0 \leq h, k \leq p-1$, the eigenvalue of $\rho_h(r^k)$ is ζ_p^{hk}. Then,

$$\mathcal{N}(\rho_h, r^k) = \sum_{\alpha=1}^{p} N_{r^k,\alpha} \frac{p-\alpha}{p} = \frac{p - \mathrm{mod}_p(hk)}{p},$$

where $\mathrm{mod}_p(a)$ is the remainder $0 < b \leq p$ of a after integer division by p.

For cyclic actions, the analytic representations are not in bijective correspondence with neither the actions (modulo hyperbolic generators), nor the (geometric) signatures, nor the classes of topological equivalence of actions, as we will see in the next two examples.

Example 2.3. Consider the group $\mathbb{Z}_5 = \langle r \rangle$. The generating vector $\mathcal{V}_1 = (r, r, r^3)$ guarantees that $\mathbb{Z}_5$ acts on a Riemann surface of genus $g = 2$ with signature $(0; 5, 5, 5)$. The outer automorphism $r \mapsto r^3$ shows that such an action is also represented by $\mathcal{V}_2 = (r^3, r^3, r^4)$. However, $\langle \rho_a, \rho_1 \rangle = 1$ for $\mathcal{V}_1$ whereas $\langle \rho_a, \rho_1 \rangle = 0$ for $\mathcal{V}_2$.

Example 2.4. The group $\mathbb{Z}_5 = \langle r \rangle$ acts in genus $g = 6$ with signature $(0; 5^5)$. The generating vectors

$$\mathcal{V}_1 = (r, r^2, r^2, r^2, r^3) \text{ and } \mathcal{V}_2 = (r, r, r^2, r^2, r^4)$$

have the same signature $(0; 5^5)$ and the same analytic representation

$$\rho_a \cong 2\rho_1 \oplus \rho_2 \oplus 2\rho_3 \oplus \rho_4,$$

but they are topologically distinct. The last claim is a direct consequence of the fact that there is no automorphism of $\mathbb{Z}_5$ sending $\mathcal{V}_1$ to $\mathcal{V}_2$ together with Proposition 1.1. Finally, we observe that ρ_a is not equivalent to its complex conjugate

$$\overline{\rho_a} \cong \rho_1 \oplus 2\rho_2 \oplus \rho_3 \oplus 2\rho_4.$$

2.2 Interlude: divisor transform

Consider the poset of positive integers $\mathbb{Z}_+$ ordered by divisibility. Let

$$\mathbb{Z}^{|n} = \{q \in \mathbb{Z}_+ : q \text{ divides } n\}$$

be the subset of all positive divisors of $n \in \mathbb{Z}_+$.

Definition 2.1. Set $n, q \in \mathbb{Z}_+$. We say that q is a *k-divisor* of n if q divides n and n/q is a product of exactly k distinct prime numbers. We denote the set of all k-divisors of n by

$$\mathbb{Z}^{|n}_k = \{q \in \mathbb{Z}^{|n} : q \text{ is a } k\text{-divisor of } n\}.$$

By definition $\mathbb{Z}^{|n}_0 = \{n\}$.

Definition 2.2. Let $\Psi, \Phi : \mathbb{Z}_+ \to \mathbb{Z}$ be two functions. The *divisor transform* of Ψ is the function $\widehat{\Psi} : \mathbb{Z}_+ \to \mathbb{Z}$ given by

$$\widehat{\Psi}(n) = \sum_{q \in \mathbb{Z}^{|n}} \Psi(q).$$

The *inverse divisor transform* of Φ is the function $\widetilde{\Phi} : \mathbb{Z}_+ \to \mathbb{Z}$ given by

$$\widetilde{\Phi}(n) = \sum_{k \geq 0} (-1)^k \sum_{q \in \mathbb{Z}^{|n}_k} \Phi(q).$$

Remark 2.1. For each $n \in \mathbb{Z}_+$ there exists $k \in \mathbb{Z}_+$ such that $\mathbb{Z}^{|n}_{k'}$ is empty for all $k' \geq k$, hence $\widetilde{\Phi}$ is well-defined. If the prime decomposition of n is $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then for each $1 \leq k \leq r$ one has that

$$\sum_{q \in \mathbb{Z}^{|n}_k} \Phi(q) = \sum_{1 \leq j_1 < \cdots < j_k \leq r} \Phi\left(\frac{n}{p_{j_1} \cdots p_{j_k}}\right).$$

In particular, $\sum_{q \in \mathbb{Z}^{|n}_0} \Phi(q) = \Phi(n)$ and $\sum_{q \in \mathbb{Z}^{|n}_r} \Phi(q) = \Phi(\frac{n}{p_1 \cdots p_r})$.

Example 2.5. Let $n = p^2qr$ where p, q, r are distinct prime numbers. We have that $\mathbb{Z}_0^{|n} = \{n\}$, $\mathbb{Z}_1^{|n} = \{pqr, p^2q, p^2r\}$, $\mathbb{Z}_2^{|n} = \{pr, pq, p^2\}$ and $\mathbb{Z}_3^{|n} = \{p\}$. If $\Phi : \mathbb{Z}_+ \to \mathbb{Z}$, then

$$\widetilde{\Phi}(n) = \Phi(n) - \Phi(pqr) - \Phi(p^2q) - \Phi(p^2r) + \Phi(pr) + \Phi(pq) + \Phi(p^2) - \Phi(p).$$

Lemma 2.1. *Let $\Psi, \Phi : \mathbb{Z}^+ \to \mathbb{Z}$ be two functions. For a given positive integer n the following conditions are equivalent:*

(1) $\widehat{\Phi}(q) = \Psi(q)$ for all $q \in \mathbb{Z}^{|n}$;

(2) $\widetilde{\Psi}(q) = \Phi(q)$ for all $q \in \mathbb{Z}^{|n}$.

Proof. $(1 \to 2)$ Let $q \in \mathbb{Z}^{|n}$ with prime decomposition $q = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ ($\alpha_j \geq 1$ for $1 \leq j \leq r$). For $k \geq 0$, Condition (1) implies that

$$\sum_{s \in \mathbb{Z}_k^{|q}} \Psi(s) = \sum_{s \in \mathbb{Z}_k^{|q}} \widehat{\Phi}(s) = \sum_{s \in \mathbb{Z}_k^{|q}} \sum_{t' \in \mathbb{Z}^{|s}} \Phi(t'). \tag{2.1}$$

Note that if t is not a divisor of q then $\Phi(t)$ does not appear in (2.1).

For $t \in \mathbb{Z}^{|q}$, let us write $q/t = p_1^{\beta_1} \cdots p_r^{\beta_r}$, where $0 \leq \beta_j \leq \alpha_j$ for $1 \leq j \leq r$. Observe that if q/t has $r' = \#\{\beta_j \geq 1\}$ distinct prime factors then the number of times that $\Phi(t)$ appears in (2.1) is $\binom{r'}{k}$, where $\binom{a}{b} = \frac{a!}{(a-b)!b!}$ is the binomial coefficient. It follows that if t is different from q then $\Phi(t)$ appears

$$\sum_{k=0}^{r'} (-1)^k \binom{r'}{k} = (1-1)^{r'} = 0$$

times in the total sum $\widetilde{\Psi}(q) = \sum_{k=0}^{r} (-1)^k \sum_{s \in \mathbb{Z}_k^{|q}} \Psi(s)$. The term $\Phi(q)$ appears exactly once, which proves the desired result.

$(2 \to 1)$ The proof is by strong induction on the number N of prime factors (counting multiplicities) of $q \in \mathbb{Z}^{|n}$. The base case $N = 0$ is trivial:

$$\Psi(1) = \widetilde{\Psi}(1) = \Phi(1) = \widehat{\Phi}(1).$$

Let us consider the case $N = 1$, and let $p \in \mathbb{Z}^{|n}$ be a prime number. By Condition (2) we have that $\widetilde{\Psi}(p) = \Phi(p)$ and $\widetilde{\Psi}(1) = \Phi(1)$. Then

$$\widehat{\Phi}(p) = \Phi(p) + \Phi(1) = \widetilde{\Psi}(p) + \widetilde{\Psi}(1) = (\Psi(p) - \Psi(1)) + \Psi(1) = \Psi(p),$$

as expected. Let us assume that we have proved Condition (1) for all divisors of n with less than $N \geq 2$ prime factors, and pick $q \in \mathbb{Z}^{|n}$ with N prime factors. By Condition (2) and the strong inductive hypothesis, we have that

$$\begin{aligned}\Phi(q) &= \widetilde{\Psi}(q)\\ &= \Psi(q) + \sum_{k=1}^{r}(-1)^k \sum_{s\in\mathbb{Z}_k^{|q}} \Psi(s)\\ &= \Psi(q) + \sum_{k=1}^{r}(-1)^k \sum_{s\in\mathbb{Z}_k^{|q}} \widehat{\Phi}(s)\\ &= \Psi(q) + \sum_{k=1}^{r}(-1)^k \sum_{s\in\mathbb{Z}_k^{|q}} \sum_{t'\in\mathbb{Z}^{|s}} \Phi(t'),\end{aligned} \tag{2.2}$$

where r is the number of distinct prime factors of q. Using the same counting argument of the previous part of the proof, we conclude that for each proper divisor t of q, $\Phi(t)$ appears

$$\sum_{k=1}^{r}(-1)^k\binom{r}{k} = (1-1)^r - 1 = -1$$

times in the sum of Equation (2.2). It follows that

$$\widehat{\Phi}(q) = \sum_{s\in\mathbb{Z}^{|q}} \Phi(s) = \Psi(q),$$

as desired. □

Proposition 2.2. $\Phi \mapsto \widehat{\Phi}$ *and* $\Phi \mapsto \widetilde{\Phi}$ *are inverse operations.*

Proof. To check that $\widehat{\widetilde{\Phi}} = \Phi$ and $\widetilde{\widehat{\Phi}} = \Phi$ it suffices to apply the above lemma to the pairs of functions $\widetilde{\Phi}, \Phi$ and $\Phi, \widehat{\Phi}$ respectively. □

2.3 Dihedral actions: Chevalley-Weil formula

Let $D_n = \langle s, r \mid r^n, s^2, (sr)^2\rangle$ be the dihedral group of order $2n$. The irreducible $\mathbb{C}$-representations of D_n are well-known (see for example [38, §5.3]). Namely, if n is even then D_n has four $\mathbb{C}$-representations of degree one:

$$\psi_1 : r \mapsto 1,\ s \mapsto 1, \qquad \psi_2 : r \mapsto 1,\ s \mapsto -1,$$
$$\psi_3 : r \mapsto -1,\ s \mapsto 1, \qquad \psi_4 : r \mapsto -1,\ s \mapsto -1,$$

and $(n-2)/2$ irreducible $\mathbb{C}$-representations of degree two:

$$\rho^h : r \mapsto \mathrm{diag}(\omega^h, \overline{\omega}^h),\ s \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right),$$

where $\omega = \zeta_n = e^{2\pi i/n}$ and $1 \le h \le (n-2)/2$. If n is odd then D_n has two $\mathbb{C}$-representations of degree one, ψ_1 and ψ_2, and $(n-1)/2$ irreducible $\mathbb{C}$-representations of degree two, given by ρ^h for $1 \le h \le (n-1)/2$.

Remark 2.2. As the character field of each irreducible $\mathbb{C}$-representation V of D_n is real, one has that V is equivalent to its complex conjugate $\overline{V}$. Thus, if ρ_a is the analytic representation of a dihedral action, then $\rho_a \cong \overline{\rho_a}$. Moreover, ρ_a is determined by the geometric signature of the action and

$$\langle \rho_r, V \rangle = 2\langle \rho_a, V \rangle,\ \text{for } V \in \mathrm{Irr}_{\mathbb{C}}(G).$$

In what follows we compute the values of the function $\mathcal{N} : \mathrm{Irr}_{\mathbb{C}}(D_n) \times D_n \to \mathbb{Q}$, introduced in Definition 1.5. Note that $\mathcal{N}(\psi_1, g) = 0$ for any $g \in D_n$. (This is true regardless of the group under consideration.)

Lemma 2.2. *Let $n \ge 3$ be an odd integer and let $q \in \mathbb{Z}^{|n} \setminus \{1\}$. Then*

$\mathcal{N}$	s	$r^{n/q}$
ψ_1	0	0
ψ_2	$1/2$	0
ρ^h	$1/2$	ε

where $\varepsilon = 0$ if q divides h, and $\varepsilon = 1$ otherwise.

Proof. The eigenvalue of $\psi_2(s)$ is $\zeta_2^1 = -1$ and the eigenvalue of $\psi_2(r^{n/q})$ is $\zeta_2^2 = 1$. It follows that

$$\mathcal{N}(\psi_2, s) = \sum_{\alpha=1}^{2} N_{s,\alpha} \frac{2-\alpha}{2} = \frac{2-1}{2} = \frac{1}{2},$$

$$\mathcal{N}(\psi_2, r^{n/q}) = \sum_{\alpha=1}^{2} N_{r^{n/q},\alpha} \frac{2-\alpha}{2} = \frac{2-2}{2} = 0.$$

The eigenvalues of $\rho^h(s)$ are $\zeta_2^1 = -1$ and $\zeta_2^2 = 1$. Thus

$$\mathcal{N}(\rho^h, s) = \sum_{\alpha=1}^{2} N_{s,\alpha} \frac{2-\alpha}{2} = \frac{2-1}{2} + \frac{2-2}{2} = \frac{1}{2}.$$

The eigenvalues of $\rho^h(r^{n/q})$ are $\omega^{hn/q}$ and $\omega^{-hn/q}$ with $\omega = \zeta_n$. Observe that $|r^{n/q}| = q$ and $\omega^{hn/q} = \zeta_q^h = \zeta_q^{\mathrm{mod}_q(h)}$, where $\mathrm{mod}_q(a)$ is the remainder $0 <$

$b \le q$ of a after integer division by q. If $\mathrm{mod}_q(h) = q$, then both eigenvalues are equal to $\zeta_q^q = 1$ and

$$\mathcal{N}(\rho^h, r^{n/q}) = \sum_{\alpha=1}^{q} N_{r^{n/q},\alpha} \frac{q-\alpha}{q} = 2\frac{q-q}{q} = 0.$$

If $\mathrm{mod}_q(h) \neq q$, then the eigenvalues are $\zeta_q^{\mathrm{mod}_q(h)}$ and $\zeta_q^{q-\mathrm{mod}_q(h)}$, hence

$$\mathcal{N}(\rho^h, r^{n/q}) = \sum_{\alpha=1}^{q} N_{r^{n/q},\alpha} \frac{q-\alpha}{q} = \frac{q-\mathrm{mod}_q(h)}{q} + \frac{q-(q-\mathrm{mod}_q(h))}{q} = 1.$$

The proof of the lemma follows after noticing that $\mathrm{mod}_q(h) = q$ if and only if q divides h. □

Lemma 2.3. *Let $n \ge 2$ be an even integer and let $q \in \mathbb{Z}^{|n} \setminus \{1\}$. Then*

$\mathcal{N}$	s	sr	$r^{n/q}$
ψ_1	0	0	0
ψ_2	1/2	1/2	0
ψ_3	0	1/2	$\delta/2$
ψ_4	1/2	0	$\delta/2$
ρ^h	1/2	1/2	ε

where $\delta = 0$ if $2q$ divides n, and $\delta = 1$ otherwise; and $\varepsilon = 0$ if q divides h and $\varepsilon = 1$ otherwise. In particular, for $r^{n/2}$ we have that $\delta = 0$ if $n \in 4\mathbb{Z}$, and $\delta = 1$ otherwise; and $\varepsilon = 0$ if h is even, and $\varepsilon = 1$ otherwise.

Proof. We only give a proof for ψ_3 and ψ_4, because the remaining cases are covered by the proof of Lemma 2.2. The eigenvalue of $\psi_3(s)$ and $\psi_3(r^{\mathrm{even}})$ is $\zeta_2^2 = 1$, whereas the eigenvalue of $\psi_3(sr)$ and $\psi_3(r^{\mathrm{odd}})$ is $\zeta_2^1 = -1$. Thus

$$\mathcal{N}(\psi_3, r^{\mathrm{even}}) = \mathcal{N}(\psi_3, s) = \sum_{\alpha=1}^{2} N_{s,\alpha} \frac{2-\alpha}{2} = \frac{2-2}{2} = 0,$$
$$\mathcal{N}(\psi_3, r^{\mathrm{odd}}) = \mathcal{N}(\psi_3, sr) = \sum_{\alpha=1}^{2} N_{sr,\alpha} \frac{2-\alpha}{2} = \frac{2-1}{2} = \frac{1}{2}.$$

On the other hand, the eigenvalue of $\psi_4(s)$ and $\psi_4(r^{\text{odd}})$ is $\zeta_2^1 = -1$, whereas the eigenvalue of $\psi_4(sr)$ and $\psi_4(r^{\text{even}})$ is $\zeta_2^2 = 1$. Thus

$$\mathcal{N}(\psi_4, r^{\text{odd}}) = \mathcal{N}(\psi_4, s) = \sum_{\alpha=1}^{2} N_{s,\alpha}\frac{2-\alpha}{2} = \frac{2-1}{2} = \frac{1}{2},$$
$$\mathcal{N}(\psi_4, r^{\text{even}}) = \mathcal{N}(\psi_4, sr) = \sum_{\alpha=1}^{2} N_{sr,\alpha}\frac{2-\alpha}{2} = \frac{2-2}{2} = 0.$$

To finish the proof, note that n/q is even if and only if $2q$ divides n. □

2.4 Analytic representation formulas

Let S be a compact Riemann surface of genus $g \geq 2$ with a dihedral action represented by the ske $\theta : \Delta \to D_n$. The geometric signature of the action has the form

$$(\gamma; \langle s\rangle^a, \langle sr\rangle^b, C_1, \ldots, C_v),$$

where $a, b \geq 0$ are nonnegative integers, and $C_j = \langle r^{n/m_j}\rangle$ is a cyclic subgroup of D_n of order $m_j \geq 2$. We recall that we employ the notations introduced in Remark 1.1. Observe that if n is odd then the signature encodes the same information as the geometric signature. (In such a case $\langle s\rangle \sim \langle sr\rangle$ and the relevant number is $t := a + b$.)

Definition 2.3. Let $\theta : \Delta \to D_n$ be a ske. The *signature function* $\Psi_\theta : \mathbb{Z}_+ \to \mathbb{Z}$ of the action θ is given by

$$\Psi_\theta(q) = \#\{1 \leq j \leq v : m_j = q\}.$$

Note that $\widehat{\Psi}_\theta(n) = v$ and $\widehat{\Psi}_\theta(1) = 0$. For instance, when $n \geq 4$ is even $\Psi_\theta(2)$ is the number of times that the center $\langle r^{n/2}\rangle$ appears in the geometric signature, and consequently the number of fixed points of the automorphism represented by $r^{n/2}$ not fixed by other non-trivial powers of r is $n\Psi_\theta(2)$.

Lemma 2.4. *Let $\theta : \Delta \to D_n$ be a ske, and let q be a positive integer. Then*

(1) $\widehat{\Psi}_\theta(q) = \widehat{\Psi}_\theta((n, q))$;

(2) $\widehat{\Psi}_\theta(n) = \widehat{\Psi}_\theta(\frac{n}{q})$ if and only if $\operatorname{lcm}(m_1, \ldots, m_v)$ divides $\frac{n}{q}$, for $q \in \mathbb{Z}^{|n}$;

(3) $\widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(q)$ is the number of cyclic subgroups C_j appearing in the geometric signature of the action such that m_j does not divide q.

Proof. Let us show the previous statements.

(1) If $t \in \mathbb{Z}_+$ does not divide n, then $\Psi_\theta(t) = 0$. It follows that $\widehat{\Psi}_\theta(q)$ is equal to the sum of Ψ_θ over the common divisors of q and n. That is,

$$\widehat{\Psi}_\theta(q) = \sum_{t \in \mathbb{Z}^{|q}} \Psi_\theta(t) = \sum_{t \in \mathbb{Z}^{|q} \cap \mathbb{Z}^{|n}} \Psi_\theta(t) = \sum_{t \in \mathbb{Z}^{|(n,q)}} \Psi_\theta(t) = \widehat{\Psi}_\theta((n,q)).$$

(2) Observe that $\widehat{\Psi}_\theta(n) = \widehat{\Psi}_\theta(\frac{n}{q})$ if and only if $\{m_1, \ldots, m_v\} \subset \mathbb{Z}^{|n/q}$. Which in turn is true if and only if $\mathrm{lcm}(m_1, \ldots, m_v)$ divides n/q.

(3) It follows directly from the definition of $\widehat{\Psi}_\theta$.

□

Proposition 2.3. *Let $n \geq 3$ be an odd integer. Consider the collection of all D_n-actions on compact Riemann surfaces with a given signature.*

(1) The analytic representation is constant over classes of topological equivalence.

(2) Actions that are topologically (hence, analytically) distinct share the same analytic representation.

Proof. We know that the analytic representation of dihedral actions is determined by its geometric signature, and that when n is odd the geometric signature is determined by the signature. □

Now, we determine ρ_a explicitly in terms of the geometric signature. As usual, there is a difference in flavor when considering dihedral groups of order $2\times$odd and $2\times$even, so they must be treated separately. Whereas the odd case is rather simple, the existence of a nontrivial central element and three classes of involution makes the treatment of the even case slightly more involved.

Theorem 2.1. *Let $n \geq 3$ be an odd integer, and let S be a compact Riemann surface of genus $g \geq 2$ with dihedral action represented by the surface kernel epimorphism $\theta : \Delta \to D_n$. If the action has signature $(\gamma; 2^t, m_1, \ldots, m_v)$, then its analytic representation is given by*

$$\rho_a \cong \gamma\psi_1 \oplus \mu_2\psi_2 \oplus \bigoplus_{h=1}^{(n-1)/2} \nu_h \rho^h,$$

where

$$\mu_2 = \langle \rho_a, \psi_2 \rangle = \gamma - 1 + \tfrac{1}{2}t,$$
$$\nu_h = \langle \rho_a, \rho^h \rangle = 2(\gamma - 1) + \tfrac{1}{2}t + \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(h),$$

for $1 \leq h \leq (n-1)/2$.

Proof. The Chevalley-Weil formula states that

$$\begin{aligned} \langle \rho_a, \psi_2 \rangle &= 2(\gamma - 1) + t\mathcal{N}(\psi_2, s) + \sum_{q \in \mathbb{Z}^{|n}} \Psi_\theta(q)\mathcal{N}(\psi_2, r^{n/q}), \\ \langle \rho_a, \rho^h \rangle &= 2(\gamma - 1) + t\mathcal{N}(\rho^h, s) + \sum_{q \in \mathbb{Z}^{|n}} \Psi_\theta(q)\mathcal{N}(\rho^h, r^{n/q}), \end{aligned}$$

for $1 \leq h \leq (n-1)/2$. By Lemma 2.2 one has that

$$\mathcal{N}(\psi_2, s) = \tfrac{1}{2},\ \mathcal{N}(\psi_2, r^{n/q}) = 0,\ \text{and } \mathcal{N}(\rho^h, s) = \tfrac{1}{2}.$$

Moreover,

$$\sum_{q \in \mathbb{Z}^{|n}} \Psi_\theta(q)\mathcal{N}(\rho^h, r^{n/q}) = \sum_{q \in \mathbb{Z}^{|n} \setminus \mathbb{Z}^{|h}} \Psi_\theta(q) \cdot 1$$

is the number of cyclic groups C_j appearing in the geometric signature such that m_j does not divide h. Then we conclude by Lemma 2.4. □

Theorem 2.2. *Let $n \geq 2$ be an even integer, and let S be a compact Riemann surface of genus $g \geq 2$ with a dihedral action represented by a surface kernel epimorphism $\theta : \Delta \to D_n$. If the action has geometric signature $(\gamma; \langle s \rangle^a, \langle sr \rangle^b, C_1, \ldots, C_v)$, then its analytic representation is given by*

$$\rho_a \cong \gamma\psi_1 \oplus \mu_2\psi_2 \oplus \mu_3\psi_3 \oplus \mu_4\psi_4 \oplus \bigoplus_{h=1}^{(n-2)/2} \nu_h \rho^h,$$

where

$$\begin{aligned} \mu_2 &= \langle \rho_a, \psi_2 \rangle = \gamma - 1 + \tfrac{1}{2}(a+b), \\ \mu_3 &= \langle \rho_a, \psi_3 \rangle = \gamma - 1 + \tfrac{1}{2}[b + \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(\tfrac{n}{2})], \\ \mu_4 &= \langle \rho_a, \psi_4 \rangle = \gamma - 1 + \tfrac{1}{2}[a + \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(\tfrac{n}{2})], \\ \nu_h &= \langle \rho_a, \rho^h \rangle = 2(\gamma - 1) + \tfrac{1}{2}(a+b) + \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(h), \end{aligned}$$

for $1 \leq h \leq (n-2)/2$.

Proof. The proof is analogous to the proof of the previous theorem, except that we require Lemma 2.3 instead of Lemma 2.2. □

Remark 2.3. $\widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(h)$ is the number of cyclic subgroups C_j appearing in the geometric signature such that m_j does not divide h (Lemma 2.4). In particular, if n is even then $\widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(\frac{n}{2})$ is the number of cyclic subgroups C_j appearing in the geometric signature generated by an odd power of r.

Remark 2.4. Theorem 2.2 remains valid for the Klein group D_2.

As a final observation, there is a bijection between $\mathbb{Z}^{|n} \setminus \{1,2\}$ and the set of irreducible $\mathbb{Q}$-representations of D_n of degree ≥ 2, given by

$$\mathbb{Z}^{|n} \setminus \{1,2\} \to \mathrm{Irr}_{\mathbb{Q}}(D_n), \quad q \mapsto W(q) = \oplus_\sigma (\rho^{n/q})^\sigma.$$

Assuming the notation of Theorem 2.2, if n is even then the analytic representation of a D_n-action can be written as

$$\rho_a \cong \mu_1\psi_1 \oplus \cdots \oplus \mu_4\psi_4 \oplus \bigoplus_{q \in \mathbb{Z}^{|n} \setminus \{1,2\}} \nu_{n/q} W(q).$$

Indeed, this is a direct consequence of using Statement (1) of Lemma 2.4 on the formulas given in Theorem 2.2. An analogous result is obtained for n odd.

2.5 Applications

The case D_p with p prime

Let $p \geq 3$ be a prime number. If the dihedral group D_p acts on a compact Riemann surface S with signature $(\gamma; 2^t, p^l)$, then the genus of S is

$$g = 1 + (2\gamma - 2)p + \tfrac{1}{2}tp + l(p-1).$$

(For necessary and sufficient conditions under which such an action exists, see Theorem 3.2 and Theorem 3.3.) As a particular case of Theorem 2.1, the analytic representation of the action is

$$\rho_a \cong \gamma\psi_1 \oplus (\gamma - 1 + \tfrac{1}{2}t)\psi_2 \oplus [2(\gamma-1) + \tfrac{1}{2}t + l]W(p),$$

where $W(p) \cong \bigoplus_{h=1}^{(p-1)/2} \rho^h$.

Remark 2.5. Two D_p-actions $\Delta \to D_p$ and $\Delta' \to D_p$ have equivalent analytic representations if and only if they have the same signature:

$$\rho_a \cong \rho_a' \iff s(\Delta) = s(\Delta').$$

Of course, this is valid for a very special family of groups. (There is a unique conjugation class of involutions, for example.) As we will see in the next section, this is true for any D_n with n odd.

Example 2.6. The dihedral group D_7 acts in genus $g = 12$ with signature $(0; 2, 2, 7, 7, 7)$. There are four topological equivalence classes of such actions, represented by the generating vectors

$$(s, s, r^5, r, r), \quad (s, s, r^4, r^2, r), \quad (s, sr, r^3, r^2, r), \quad (s, sr^2, r^3, r, r).$$

By Proposition 2.3, these four actions share the same analytic representation; given by

$$\rho_a \cong 2(\rho^1 \oplus \rho^2 \oplus \rho^3).$$

Nonzero multiplicities

Latter on, in §4.3, we will want to know which irreducible $\mathbb{C}$-representations of D_n of degree two have non-vanishing multiplicities in the analytic representation ρ_a of a given dihedral action. The following set quantifies this notion:

$$Q_\theta = \{q \in \mathbb{Z}^{|n} \setminus \{1, 2\} : \langle \rho_a, \rho^{n/q} \rangle \geq 1\}.$$

Proposition 2.4. *Let S be a compact Riemann surface of genus $g \geq 2$ with a dihedral action represented by a ske $\theta : \Delta \to D_n$. Assume that the action has geometric signature $(\gamma; \langle s \rangle^a, \langle sr \rangle^b, C_1, \ldots, C_v)$. Then*

(1) if $\gamma \geq 1$, or $\gamma = 0$ and $a + b \geq 6$, then $Q_\theta = \mathbb{Z}^{|n} \setminus \{1, 2\}$;

(2) if $\gamma = 0$ and $a + b = 4$ then $Q_\theta = (\mathbb{Z}^{|n} \setminus \mathbb{Z}^{|n/\alpha}) \setminus \{1, 2\}$ for $\alpha = \mathrm{lcm}(m_1, \ldots, m_v)$. Furthermore, $Q_\theta \neq \emptyset$.

Proof. In order to show the two previous statements, we apply the formulas of Theorem 2.1 (if n is odd) and Theorem 2.2 (if n is even). Set $q \in \mathbb{Z}^{|n} \setminus \{1, 2\}$.

(1) It is easy to see that if $\gamma \geq 2$, $\gamma = 1$ and $a + b \geq 2$, or $\gamma = 0$ and $a + b \geq 6$, then $\langle \rho_a, \rho^{n/q} \rangle \geq 1$. Assume that $\gamma = 1$ and $a + b \leq 1$. As will be proven in the next chapter (Theorem 3.2 and Theorem 3.5) the existence of an action implies that $a + b$ is even, hence $a + b = 0$. Furthermore, $\mathrm{lcm}(m_1, \ldots, m_v) = n$ or $n/2$. Then, by Lemma 2.4 one has that $\langle \rho_a, \rho^{n/q} \rangle = \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(\frac{n}{q}) = 0$ if and only if $\mathrm{lcm}(m_1, \ldots, m_v)$ divides n/q, a contradiction.

(2) Assume that $\gamma = 0$ and $a + b = 4$. Lemma 2.4 states that $\langle \rho_a, \rho^{n/q} \rangle = \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(\frac{n}{q}) = 0$ if and only if $q \in \mathbb{Z}^{|n/\alpha}$ for $\alpha = \mathrm{lcm}(m_1, \ldots, m_v)$. Since $g \geq 2$ it follows that $\{m_1, \ldots, m_v\}$ is non-empty (otherwise $g = 1$, a contradiction). Thus, $n/\alpha < n$ and $Q_\theta \neq \emptyset$.

This concludes the proof. □

2.6 Geometric signature formulas

In §2.4 we determined the analytic representation of a dihedral action from a given geometric signature. Here we deal with the converse problem.

Definition 2.4. Let V be a $\mathbb{C}$-representation of D_n. The *pre-signature function* $\Phi_V : \mathbb{Z}_+ \to \mathbb{Z}$ is given by

$$\Phi_V(q) = \begin{cases} \langle V, \rho^1\rangle - \langle V, \psi_1 \oplus \psi_2\rangle + 1, & \text{if } (n,q) = n, \\ \langle V, \rho^1\rangle - \langle V, \psi_3 \oplus \psi_4\rangle, & \text{if } n \text{ is even and } (n,q) = \frac{n}{2}, \\ \langle V, \rho^1\rangle - \langle V, \rho^{(n,q)}\rangle, & \text{if } (n,q) < \frac{n}{2}. \end{cases}$$

For the sake of clearness, we recall here that the divisor transform of a function $\Psi : \mathbb{Z}_+ \to \mathbb{Z}$ is the function

$$\widehat{\Psi} : \mathbb{Z}_+ \to \mathbb{Z}, \quad \widehat{\Psi}(n) = \sum_{q \in \mathbb{Z}^{|n}} \Psi(q).$$

Also, the inverse divisor transform of a function $\Phi : \mathbb{Z}_+ \to \mathbb{Z}$ is the function

$$\widetilde{\Phi} : \mathbb{Z}_+ \to \mathbb{Z}, \quad \sum_{k \geq 0} (-1)^k \sum_{q \in \mathbb{Z}_k^{|n}} \Phi(q),$$

where $\mathbb{Z}_k^{|n}$ is the set of k-divisors of n. The definition of the pre-signature function Φ_V is dual to that of the signature function Ψ_θ. More precisely:

Proposition 2.5. *If ρ_a is the analytic representation of a dihedral action represented by a ske $\theta : \Delta \to D_n$, then Ψ_θ is the inverse divisor transform of Φ_{ρ_a}. In other words,*

$$\Psi_\theta = \widetilde{\Phi}_{\rho_a} \text{ and } \widehat{\Psi}_\theta = \Phi_{\rho_a}.$$

Proof. Assume that n is even. A quick look at Theorem 2.2 gives

$$\begin{aligned} \widehat{\Psi}_\theta(n) &= \langle \rho_a, \rho^1\rangle - \langle \rho_a, \psi_1 \oplus \psi_2\rangle + 1, \\ \widehat{\Psi}_\theta(\tfrac{n}{2}) &= \langle \rho_a, \rho^1\rangle - \langle \rho_a, \psi_3 \oplus \psi_4\rangle, \\ \widehat{\Psi}_\theta(h) &= \langle \rho_a, \rho^1\rangle - \langle \rho_a, \rho^h\rangle, \end{aligned}$$

for $1 \leq h \leq (n-2)/2$. By Lemma 2.4 one has that

$$\widehat{\Psi}_\theta(q) = \widehat{\Psi}_\theta((n,q)) = \Phi_{\rho_a}(q) \quad \text{for } q \in \mathbb{Z}_+.$$

We conclude by Proposition 2.2. The proof is analogous for n odd. □

An immediate consequence is that we obtain an explicit formula for the geometric signature of the action in terms of its analytic representation.

Proposition 2.6. *Let $n \geq 3$ be an odd integer, and let S be a Riemann surface of genus $g \geq 2$ with a dihedral action represented by a ske $\theta : \Delta \to D_n$. Set $\mathbb{Z}^{|n} = \{1 < m_1 < \ldots < m_v\}$. If ρ_a is the analytic representation of the action, then the signature*

$$(\gamma; 2^t, m_1^{l_1}, \ldots, m_v^{l_v}),$$

is given by

$$\begin{aligned} \gamma &= \langle \rho_a, \psi_1 \rangle, \\ t &= 2\langle \rho_a, \psi_2 \rangle - 2\langle \rho_a, \psi_1 \rangle + 2, \\ l_j &= \widetilde{\Phi}_{\rho_a}(m_j) \text{ for } 1 \leq j \leq v. \end{aligned}$$

Proof. The formulas for γ and t are clear as a direct consequence of Theorem 2.1. Besides, by definition, $l_j = \Psi_\theta(m_j)$ for $1 \leq j \leq v$, and the proof follows from Proposition 2.5. □

Example 2.7. Let V be a $\mathbb{C}$-representation of D_{15} of the form

$$V = \nu_1(\rho^1 \oplus \rho^2 \oplus \rho^4 \oplus \rho^7) \oplus \nu_3(\rho^3 \oplus \rho^6) \oplus \nu_5 \rho^5,$$

for ν_1, ν_3, ν_5 nonnegative integers. The pre-signature function Φ_V is determined by the values $\Phi_V(15) = \nu_1 + 1$, $\Phi_V(5) = \nu_1 - \nu_5$ and $\Phi_V(3) = \nu_1 - \nu_3$. It follows that

$$\widetilde{\Phi}_V(15) = \nu_3 + \nu_5 - \nu_1 + 1, \quad \widetilde{\Phi}_V(5) = \nu_1 - \nu_5, \quad \widetilde{\Phi}_V(3) = \nu_1 - \nu_3.$$

If V is the analytic representation of a D_{15}-action, then its signature is

$$(0; 2, 2, 3^{\nu_1 - \nu_3}, 5^{\nu_1 - \nu_5}, 15^{\nu_3 + \nu_5 - \nu_1 + 1}).$$

Proposition 2.7. *Let $n \geq 2$ be an even integer, and let S be a Riemann surface of genus $g \geq 2$ with a dihedral action represented by a ske $\theta : \Delta \to D_n$. Set $\mathbb{Z}^{|n} = \{1 < m_1 < \ldots < m_v\}$. If ρ_a is the analytic representation of the action, then the geometric signature*

$$(\gamma; \langle s \rangle^a, \langle sr \rangle^b, \langle r^{n/m_1} \rangle^{l_1}, \ldots, \langle r^{n/m_v} \rangle^{l_v}),$$

is given by

$$\begin{aligned} \gamma &= \langle \rho_a, \psi_1 \rangle, \\ a &= \langle \rho_a, \psi_2 \oplus \psi_4 \rangle - \langle \rho_a, \psi_1 \oplus \psi_3 \rangle + 1, \\ b &= \langle \rho_a, \psi_2 \oplus \psi_3 \rangle - \langle \rho_a, \psi_1 \oplus \psi_4 \rangle + 1, \\ l_j &= \widetilde{\Phi}_{\rho_a}(m_j) \text{ for } 1 \leq j \leq v. \end{aligned}$$

Proof. This is a direct consequence of Theorem 2.2. The formula for γ is clear. In order to see the formulas for a and b, note that

$$\begin{aligned} \tfrac{1}{2}(a+b) &= \langle \rho_a, \psi_2 \rangle - \langle \rho_a, \psi_1 \rangle + 1, \\ \tfrac{1}{2}(a-b) &= \langle \rho_a, \psi_4 \rangle - \langle \rho_a, \psi_3 \rangle. \end{aligned}$$

Finally, by Proposition 2.5, $l_j = \Psi_\theta(m_j) = \widetilde{\Phi}_{\rho_a}(m_j)$ for $1 \leq j \leq v$. □

Theorem 2.3. *There is a bijective correspondence between geometric signatures and analytic representations of dihedral actions on compact Riemann surfaces of genus $g \geq 2$.*

Proof. Set $\mathbb{Z}^{|n} = \{1 < m_1 < \cdots < m_v\}$. Assume that n is even and that the action has geometric signature

$$(\gamma; \langle s \rangle^a, \langle sr \rangle^b, \langle r^{n/m_1} \rangle^{l_1}, \ldots, \langle r^{n/m_v} \rangle^{l_v}).$$

We know that the analytic representation ρ_a is determined by the geometric signature. By Proposition 2.7, the geometric signature induced by ρ_a is

$$(\gamma'; \langle s \rangle^{a'}, \langle sr \rangle^{b'}, \langle r^{n/m_1} \rangle^{l'_1}, \ldots, \langle r^{n/m_v} \rangle^{l'_v}),$$

where

$$\begin{aligned} \gamma' &= \langle \rho_a, \psi_1 \rangle, \\ a' &= \langle \rho_a, \psi_2 \oplus \psi_4 \rangle - \langle \rho_a, \psi_1 \oplus \psi_3 \rangle + 1, \\ b' &= \langle \rho_a, \psi_2 \oplus \psi_3 \rangle - \langle \rho_a, \psi_1 \oplus \psi_4 \rangle + 1, \\ l'_j &= \widetilde{\Phi}_{\rho_a}(m_j) \text{ for } 1 \leq j \leq v. \end{aligned}$$

By Theorem 2.2 and Proposition 2.5 one has that $\gamma' = \gamma$, $a' = a$, $b' = b$, and $l'_j = l_j$. We conclude that both geometric signatures are equal.

Conversely, assume that the action has analytic representation ρ_a. Proposition 2.7 states that the geometric signature induced by ρ_a is

$$(\gamma; \langle s \rangle^a, \langle sr \rangle^b, \langle r^{n/m_1} \rangle^{l_1}, \ldots, \langle r^{n/m_v} \rangle^{l_v}),$$

where

$$\begin{aligned} \gamma &= \langle \rho_a, \psi_1 \rangle, \\ a &= \langle \rho_a, \psi_2 \oplus \psi_4 \rangle - \langle \rho_a, \psi_1 \oplus \psi_3 \rangle + 1, \\ b &= \langle \rho_a, \psi_2 \oplus \psi_3 \rangle - \langle \rho_a, \psi_1 \oplus \psi_4 \rangle + 1, \\ l_j &= \widetilde{\Phi}_{\rho_a}(m_j) \text{ for } 1 \leq j \leq v. \end{aligned}$$

Now, by Theorem 2.2 and Proposition 2.5, the analytic representation ρ'_a induced by the previous geometric signature is equivalent to ρ_a. Indeed,

$$\begin{aligned}
\langle \rho'_a, \psi_1 \rangle &= \gamma = \langle \rho_a, \psi_1 \rangle, \\
\langle \rho'_a, \psi_2 \rangle &= \gamma - 1 + \tfrac{1}{2}(a+b) = \langle \rho_a, \psi_2 \rangle, \\
\langle \rho'_a, \psi_3 \rangle &= \gamma - 1 + \tfrac{1}{2}[b + \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(\tfrac{n}{2})] = \langle \rho_a, \psi_3 \rangle, \\
\langle \rho'_a, \psi_4 \rangle &= \gamma - 1 + \tfrac{1}{2}[a + \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(\tfrac{n}{2})] = \langle \rho_a, \psi_4 \rangle, \\
\langle \rho'_a, \rho^h \rangle &= 2(\gamma - 1) + \tfrac{1}{2}(a+b) + \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(h) = \langle \rho_a, \rho^h \rangle,
\end{aligned}$$

for $1 \leq h \leq (n-2)/2$, as desired. The proof of the case n odd is analogous. $\square$

Chapter 3

Existence of dihedral actions

Bujalance, Cirre, Gamboa, and Gromadzki in [6] provided necessary and sufficient conditions for a signature to admit a surface kernel epimorphism onto a dihedral group. In this chapter we provide a refinement of their results from signatures to geometric signatures. Then we apply our results to address the problem of deciding when a $\mathbb{C}$-representation is the analytic representation of a dihedral action.

3.1 Signature realization

For the sake of completeness, in this section we briefly review the main results of [6]. Along this section, Δ is a Fuchsian group of signature

$$(\gamma; 2^t, m_1, \ldots, m_v),$$

with $m_j \geq 3$ for $j = 1, \ldots, v$. Also, set $m_{j+v} = 2$ for $1 \leq j \leq t$.

Theorem 3.1. *Let $n \geq 2$ be an even integer and let $\gamma > 0$. Then necessary and sufficient conditions for the existence of a surface kernel epimorphism $\theta : \Delta \to D_n$ are:*

(1) m_j divides n for $j = 1, \ldots, v$;

(2) if $t \leq 2$ then there is an even number of integers $1 \leq j \leq v + t$ such n/m_j is an odd integer;

(3) if $\gamma = 1$ and $t \leq 1$ then $\operatorname{lcm}(2^t, m_1, \ldots, m_v) = n$ or $n/2$. In the latter case, if $n \in 4\mathbb{Z}$ then there is an odd number of integers $1 \leq j \leq v + t$ such that $n/2m_j$ is an odd integer.

Proof. See [6, Theorem 2.1]. $\square$

Theorem 3.2. *Let $n \geq 3$ be an odd integer and let $\gamma > 0$. Then necessary and sufficient conditions for the existence of a surface kernel epimorphism $\theta : \Delta \to D_n$ are:*

(1) m_j divides n for $j = 1, \ldots, v$;

(2) t is even;

(3) if $\gamma = 1$ and $t = 0$ then $\operatorname{lcm}(m_1, \ldots, m_v) = n$.

Proof. See [6, Theorem 2.2]. □

Theorem 3.3. *Set $\gamma = 0$. Then necessary and sufficient conditions for the existence of a surface kernel epimorphism $\theta : \Delta \to D_n$ are:*

(1) m_j divides n for $j = 1, \ldots, v$;

(2) $t \geq 2$, and it is even if n is odd;

(3) if $t = 2$ then $\operatorname{lcm}(m_1, \ldots, m_v) = n$;

(4) if $t = 3$ then $\operatorname{lcm}(2, m_1, \ldots, m_v) = n$.

Proof. See [6, Theorem 2.3]. □

We emphasize that the previous theorems provide sufficient, but not necessary, conditions under which there exist actions with a given geometric signature. As the following example illustrates, there are signatures for which only some (of its associated) geometric signatures are realized.

Example 3.1. In accordance with the theorem above, the group D_6 acts in genus 5 with signature $(0; 2^4, 3)$. The generating vector (s, s, sr^3, sr, r^2) represents an action with geometric signature $(0; \langle s \rangle^2, \langle sr \rangle^2, \langle r^2 \rangle)$. In contrast, there are no D_6-actions in genus 5 with geometric signature $(0; \langle s \rangle, \langle sr \rangle, \langle r^3 \rangle^2, \langle r^2 \rangle)$. If this were the case, then there would exist a generating vector of the form

$$(sr^{2\xi_1}, sr^{2\xi_2+1}, r^3, r^3, r^{\pm 2}),$$

where $\xi_1, \xi_2 \in \mathbb{Z}$ satisfy

$$2(\xi_2 - \xi_1) \pm 2 + 7 \in 6\mathbb{Z}.$$

However, the number above is odd, a contradiction.

3.2 Geometric signature realization

Here we extend the results of the previous section from signatures to geometric signatures for actions of the dihedral group D_n. As for n odd the signature is equivalent to the geometric signature, we only consider the case $n \geq 2$ even.

Let Δ be a (co-compact) Fuchsian group of signature $(\gamma; 2^{a+b}, m_1, \ldots, m_v)$, with $m_j \in \mathbb{Z}^{|n} \backslash \{1\}$, canonically presented by hyperbolic generators $\alpha_1, \ldots, \alpha_\gamma$, $\beta_1, \ldots, \beta_\gamma$, elliptic generators $x_1, \ldots, x_{a+b}$, $y_1, \ldots, y_v$, and relations

$$x_i^2 = y_j^{m_j} = \prod_{t=1}^{\gamma} [\alpha_t, \beta_t] \prod_{k=1}^{a+b} x_k \prod_{l=1}^{v} y_l = 1, \tag{3.1}$$

for $i = 1, \ldots, a+b$ and $j = 1, \ldots, v$. Assume that there exists a ske $\theta : \Delta \to D_n$ that represents an action with geometric signature

$$(\gamma; \langle s \rangle^a, \langle sr \rangle^b, C_1, \ldots, C_v),$$

where $C_j = \langle r^{n/m_j} \rangle$ is a cyclic subgroup of order $m_j \geq 2$. Without loss of generality, it follows that

$$\theta(x_k) = \begin{cases} sr^{\text{even}}, & 1 \leq k \leq a, \\ sr^{\text{odd}}, & a+1 \leq k \leq a+b, \end{cases}$$

and

$$\theta(y_j) = (r^{n/m_j})^{q_j} \text{ with } (q_j, m_j) = 1, \tag{3.2}$$

for $1 \leq j \leq v$. Observe that

$$\begin{aligned} \langle \theta(y_1), \ldots, \theta(y_v) \rangle &= \langle (r^{n/m_1})^{q_1}, \ldots, (r^{n/m_v})^{q_v} \rangle \\ &= \langle r^{n/m_1}, \ldots, r^{n/m_v} \rangle \\ &= \langle r^{n/\operatorname{lcm}(m_1, \ldots, m_v)} \rangle. \end{aligned} \tag{3.3}$$

For the rest of this section, we set

$$\begin{aligned} A &= \#\{1 \leq j \leq v : n/m_j \text{ is an odd integer}\}, \\ B &= \#\{1 \leq j \leq v : n/2m_j \text{ is an odd integer}\}. \end{aligned}$$

Lemma 3.1.

(1) *There are integers* ξ_1*,* ξ_2*, and* ξ_3 *such that*

$$\prod_{t=1}^{\gamma} \theta([\alpha_t, \beta_t]) = r^{2\xi_1}, \ \prod_{j=1}^{a+b} \theta(x_j) = r^{\xi_2}, \ \prod_{j=1}^{v} \theta(y_j) = r^{\xi_3},$$

and

$$2\xi_1 + \xi_2 + \xi_3 \in n\mathbb{Z}.$$

Concretely, $\xi_3 = \sum_{j=1}^{v} nq_j/m_j$*.*

(2) *The integers* a*,* b*,* ξ_2*,* ξ_3*, and* A *have the same parity;* $a+b$ *is even.*

(3) *If* $A = 0$ *then every term* nq_j/m_j *is even. In this case, if* $n \in 4\mathbb{Z}$ *then the integers* $\xi_3/2$ *and* B *have the same parity.*

Proof. Let us prove the statements above.

(1) From (3.2) it follows that $\prod_{j=1}^{v} \theta(y_j) = r^{\xi_3}$ with $\xi_3 = \sum_{j=1}^{v} nq_j/m_j$. Since the derived subgroup of D_n is $\langle r^2 \rangle$, it follows that $\prod_{t=1}^{\gamma} \theta([\alpha_t, \beta_t]) = r^{2\xi_1}$ for some $\xi_1 \in \mathbb{Z}$. Thus, the long relation (3.1) of Δ implies that $\prod_{j=1}^{a+b} \theta(x_j) = r^{\xi_2}$ for $\xi_2 \in \mathbb{Z}$ and

$$2\xi_1 + \xi_2 + \xi_3 \in n\mathbb{Z}.$$

(2) As $\prod_{j=1}^{a+b} \theta(x_j) = r^{\xi_2}$, a and b must have the same parity. The parity of ξ_2 is determined by the number of terms of the form sr^{odd} in the product $\prod_{j=1}^{a+b} \theta(x_j)$. Concretely, ξ_2 is odd if and only if b is odd. By the relation above ξ_2 and ξ_3 must have the same parity. Now, observe that $\xi_3 = \sum_{j=1}^{v} nq_j/m_j$ is odd if and only if there is an odd number of integers $1 \leq j \leq v$ such that nq_j/m_j is odd. Since $(q_j, m_j) = 1$, nq_j/m_j is odd if and only if n/m_j is odd. We conclude that ξ_3 and A have the same parity.

(3) Assume that $A = 0$. Thus, every term n/m_j is even and the same holds for nq_j/m_j. The integer $\xi_3/2 = \sum_{j=1}^{v} nq_j/2m_j$ is odd if and only if there is an odd number of integers $1 \leq j \leq v$ such that $nq_j/2m_j$ is odd. Now, assume that $n \in 4\mathbb{Z}$. To conclude it suffices to show that $nq_j/2m_j$ is odd if and only if $n/2m_j$ is odd. Indeed, if $n/2m_j$ is odd then m_j is even, hence q_j is odd and so is $nq_j/2m_j$. The converse is direct.

This finishes the proof. □

Theorem 3.4. *Let $n \geq 2$ be an even integer and let $\gamma = 0$. Then necessary and sufficient conditions for the existence of a surface kernel epimorphism $\theta : \Delta \to D_n$ of geometric signature $(0; \langle s\rangle^a, \langle sr\rangle^b, C_1, \dots, C_v)$ are:*

(1) a, b, and A have the same parity;

(2) $a + b \geq 2$ is even;

(3) if $a + b = 2$ then $\operatorname{lcm}(m_1, \dots, m_v) = n$;

(4) if $a + b > 2$, and $a = 0$ or $b = 0$ then $A > 0$.

Proof. Let us prove that the conditions are **necessary**.

(1) This has already been proved in Lemma 3.1.

(2) Assume that $a + b < 2$. Then $a = b = 0$ and $\theta(\Delta) \leq \langle r\rangle$, a contradiction with the surjectivity of θ.

(3) Assume that $a + b = 2$. By Lemma 3.1 we have that $\langle \theta(x_1), \theta(x_2)\rangle = \langle sr^k, r^{\xi_3}\rangle$ for some $k \in \mathbb{Z}$ and $\xi_3 = \sum_{j=1}^{v} nq_j/m_j$. Observe that

$$\begin{aligned}\theta(\Delta) &= \langle \theta(x_1), \theta(x_2), \theta(y_1), \dots, \theta(y_v)\rangle \\ &= \langle sr^k, r^{\xi_3}, r^{n/m_1}, \dots, r^{n/m_v}\rangle \\ &= \langle sr^k, r^{n/\operatorname{lcm}(m_1,\dots,m_v)}\rangle.\end{aligned}$$

Since θ is onto it follows that $\operatorname{lcm}(m_1, \dots, m_v) = n$.

(4) Assume that $a + b > 2$, and $a = 0$ or $b = 0$. Since a and b have the same parity, either $a \geq 4$ and $b = 0$, or $b \geq 4$ and $a = 0$. If $a \geq 4$ and $b = 0$ then $\langle \theta(x_1), \dots, \theta(x_a)\rangle \leq \langle s, r^2\rangle$. Since θ is onto, there must exist some $\theta(y_j) = r^{nq_j/m_j}$ with n/m_j odd, and therefore $A \neq 0$. The same argument holds for $a = 0$ and $b \geq 4$.

Let us now show that the conditions are **sufficient**. Let a and b be two nonnegative integers as in Condition (2). Then, one of the following cases occurs:

(i) $a, b \geq 2$ are even,

(ii) $a \geq 3$ and $b \geq 1$ are odd,

(iii) $a \geq 1$ and $b \geq 3$ are odd,

(iv) $a \geq 4$ is even and $b = 0$,

(v) $a = 0$ and $b \geq 4$ is even,

(vi) $a = 2$ and $b = 0$,

(vii) $a = 0$ and $b = 2$,

(viii) $a = b = 1$.

For Cases (i), (ii), (iv), (vi) and (viii), we explicitly construct a ske $\theta : \Delta \to D_n$ (in the form of a generating vector) of geometric signature

$$(0; \langle s\rangle^a, \langle sr\rangle^b, C_1, \dots, C_v).$$

The remaining cases will follow after considering the outer automorphism $r \mapsto r$, $s \mapsto sr$.

Set $\xi_3 = \sum_{j=1}^{v} n/m_j$. By definition, A is odd if and only ξ_3 is odd. Condition (1) guarantees that the integers a, b, ξ_3 and A have the same parity.

(i) Assume that $a, b \geq 2$ are even. The tuple

$$(s, \overset{a}{\dots}, s, sr, \overset{b-1}{\dots}, sr, sr^{1-\xi_3}, r^{n/m_1}, \dots, r^{n/m_v})$$

is a generating vector. Note that ξ_3 is even, hence $\langle sr^{1-\xi_3}\rangle \sim \langle sr\rangle$.

(ii) Assume that $a \geq 3$ and $b \geq 1$ are odd. The tuple

$$(s, \overset{a-1}{\dots}, s, sr^{1+\xi_3}, sr, \overset{b}{\dots}, sr, r^{n/m_1}, \dots, r^{n/m_v})$$

is a generating vector. Note that ξ_3 is odd, hence $\langle sr^{1+\xi_3}\rangle \sim \langle s\rangle$.

(iv) Assume that $a \geq 4$ is even and $b = 0$. The tuple

$$(s, \overset{a-2}{\dots}, s, sr^2, sr^{2-\xi_3}, r^{n/m_1}, \dots, r^{n/m_v})$$

is a generating vector. In fact, Condition (4) implies that $A > 0$, and consequently some n/m_j is odd. It follows that $\langle r, r^2, r^{n/m_j}\rangle = D_n$. Also, note that ξ_3 is even, hence $\langle sr^{2-\xi_3}\rangle \sim \langle s\rangle$.

(vi) Assume that $a = 2$ and $b = 0$. The tuple

$$(s, sr^{-\xi_3}, r^{n/m_1}, \dots, r^{n/m_v})$$

is a generating vector. By Condition (3), $\operatorname{lcm}(m_1, \dots, m_v) = n$ and

$$\theta(\Delta) = \langle s, r^{n/m_1}, \dots, r^{n/m_v}\rangle = \langle s, r^{n/\operatorname{lcm}(m_1,\dots,m_v)}\rangle = \langle s, r\rangle = D_n.$$

In particular, $v > 0$. Also, note that ξ_3 is even, hence $\langle sr^{-\xi_3}\rangle \sim \langle s\rangle$.

(viii) Assume that $a = b = 1$. The previous generating vector remains a suitable choice. Observe that, in this case, ξ_3 is odd and $\langle sr^{-\xi_3}\rangle \sim \langle sr\rangle$.

This concludes the proof. □

Remark 3.1. Observe that $\operatorname{lcm}(m_1, \dots, m_v) > 1$ implies that $v > 0$.

Theorem 3.5. *Let $n \geq 2$ be an even integer and let $\gamma > 0$. Then necessary and sufficient conditions for the existence of a surface kernel epimorphism $\theta : \Delta \to D_n$ of geometric signature $(\gamma; \langle s\rangle^a, \langle sr\rangle^b, C_1, \ldots, C_v)$ are:*

(1) a, b, and A have the same parity ($a + b$ is even);

(2) if $\gamma = 1$ and $a = b = 0$ then $\operatorname{lcm}(m_1, \ldots, m_v) = n$ or $n/2$. In the latter case, if $n \in 4\mathbb{Z}$ then B is odd.

Proof. Let us prove that the conditions are **necessary**. Condition (1) has already been proved in Lemma 3.1.

In order to show Condition (2), assume that $\gamma = 1$ and $a = b = 0$. As θ is onto at least one of the hyperbolic generators of Δ must be sent to $D_n \setminus \langle r\rangle$. After considering the outer automorphism $r \mapsto r$, $s \mapsto sr^l$, we can assume that $\theta(\alpha_1) = s$. Note that $\theta(\beta_1) = sr^{\xi_1}$ or $r^{-\xi_1}$ for some $\xi_1 \in \mathbb{Z}$. In each case, $\theta([\alpha_1, \beta_1]) = r^{2\xi_1}$. By Lemma 3.1, the integers ξ_1 and $\xi_3 = \sum_{j=1}^{v} nq_j/m_j$ satisfy the relation

$$2\xi_1 + \xi_3 \in n\mathbb{Z}. \tag{3.4}$$

Thus, Equation (3.3) implies that $r^{2\xi_1} \in \langle r^{n/\operatorname{lcm}(m_1,\ldots,m_v)}\rangle$. Surjectivity requires that $\langle s, r^{\xi_1}, r^{n/\operatorname{lcm}(m_1,\ldots,m_v)}\rangle = D_n$, and hence $\langle s, r^{n/\operatorname{lcm}(m_1,\ldots,m_v)}\rangle$ has index 1 or 2 in D_n. Or equivalently,

$$\operatorname{lcm}(m_1, \ldots, m_v) = n \text{ or } n/2.$$

Assume that $\operatorname{lcm}(m_1, \ldots, m_v) = n/2$. It follows that each n/m_j is even ($A = 0$) and ξ_1 is odd. (If ξ_1 is even then $\theta(\Delta) = \langle s, r^{\xi_1}, r^2\rangle \neq D_n$.) Now, assume that $n \in 4\mathbb{Z}$. Relation (3.4) turns into

$$\xi_1 + \xi_3/2 \in \tfrac{n}{2}\mathbb{Z}.$$

Since ξ_1 is odd and $n/2$ is even, $\xi_3/2$ must be odd. We conclude by Lemma 3.1.

Let us now show that the conditions are **sufficient**. Let a and b be non-negative integers as in Condition (1). Then, one of the following cases occurs:

(i) $\gamma \geq 2$,

(ii) $\gamma = 1$ and $a \neq 0$,

(iii) $\gamma = 1$ and $b \neq 0$,

(iv) $\gamma = 1$ and $a = b = 0$.

For cases (i), (ii) and (iv) we explicitly construct a ske $\theta : \Delta \to D_n$ (in the form of a generating vector) of geometric signature

$$(\gamma; \langle s\rangle^a, \langle sr\rangle^b, C_1, \ldots, C_v).$$

The remaining case will follow after considering the outer automorphism $r \mapsto r,\ s \mapsto sr$. Set $\xi_2 = 0$ if b is even and $\xi_2 = 1$ if b is odd. We also set $\xi_3 = \sum_{j=1}^{v} n/m_j$. By definition, A is odd if and only if ξ_3 is odd. Condition (1) guarantees that the integers a, b, ξ_2, ξ_3 and A have the same parity.

(i) Assume that $\gamma \geq 2$. The tuple

$$(s, r, \overset{\gamma-1}{\ldots}, r, r^{(\xi_2+\xi_3)/2}, r, \overset{\gamma-1}{\ldots}, r; s, \overset{a}{\ldots}, s, sr, \overset{b}{\ldots}, sr, r^{n/m_1}, \ldots, r^{n/m_v})$$

is a generating vector. Note that $\xi_2 + \xi_3$ is even.

(ii) Assume that $\gamma = 1$ and $a \neq 0$. The tuple

$$(sr, r^{(\xi_2+\xi_3)/2}; s, \overset{a}{\ldots}, s, sr, \overset{b}{\ldots}, sr, r^{n/m_1}, \ldots, r^{n/m_v})$$

is a generating vector. Note that $\xi_2 + \xi_3$ is even.

(iv) Assume that $\gamma = 1$ and $a = b = 0$. Let $\xi_1 = (\delta n + \xi_3)/2$ for $\delta \in \{0, 1\}$, and consider the tuple

$$(s, r^{\xi_1}; r^{n/m_1}, \ldots, r^{n/m_v}).$$

Observe that the product $[s, r^{\xi_1}]r^{\xi_3} = 1$ for $\delta = 0$ and $\delta = 1$, and that

$$\theta(\Delta) = \langle s, r^{\xi_1}, r^{n/m_1}, \ldots, r^{n/m_v}\rangle = \langle s, r^{\xi_1}, r^{n/\operatorname{lcm}(m_1,\ldots,m_v)}\rangle.$$

Condition (2) implies that $\operatorname{lcm}(m_1, \ldots, m_v) = n$ or $n/2$. In the first case it is clear that the tuple is a generating vector. Now, assume that $\operatorname{lcm}(m_1, \ldots, m_v) = n/2$. Then, every term n/m_j is even ($A = 0$) and $\theta(\Delta) = \langle s, r^{\xi_1}, r^2\rangle$. (a) If $n \in 4\mathbb{Z}$ then, by Condition (2), B is odd, that is, there is an odd number of integers $1 \leq j \leq v$ such that $n/2m_j$ is odd. It follows that $\xi_3/2$ is odd, and we choose $\delta = 0$. Thus, ξ_1 is odd and $\theta(\Delta) = \langle s, r^{\xi_1}, r^2\rangle = D_n$, satisfying that θ is a ske. (b) If $n \notin 4\mathbb{Z}$ ($n/2$ is odd) and $\xi_3/2$ is odd then we choose $\delta = 0$. It follows that ξ_1 is odd, and we conclude as in (a). (c) If $n \notin 4\mathbb{Z}$ and $\xi_3/2$ is even then we choose $\delta = 1$. It follows that ξ_1 is odd, and we conclude in the same way.

This concludes the proof. □

3.3 Analytic representation criteria

We briefly recall some previous notions and introduce a pair of useful lemmas. The divisor transform of a function $\Psi : \mathbb{Z}_+ \to \mathbb{Z}$ is the function

$$\widehat{\Psi} : \mathbb{Z}_+ \to \mathbb{Z}, \quad \widehat{\Psi}(n) = \sum_{q \in \mathbb{Z}^{|n}} \Psi(q).$$

The inverse divisor transform of a function $\Phi : \mathbb{Z}_+ \to \mathbb{Z}$ is the function

$$\widetilde{\Phi} : \mathbb{Z}_+ \to \mathbb{Z}, \quad \widetilde{\Phi}(n) = \sum_{k \geq 0} (-1)^k \sum_{q \in \mathbb{Z}_k^{|n}} \Phi(q),$$

where $\mathbb{Z}_k^{|n}$ is the set of k-divisors of n. Assume that $\theta : \Delta \to D_n$ is a ske that represents an action with geometric signature $(\gamma; \langle s \rangle^a, \langle sr \rangle^b, C_1, \ldots, C_v)$, where $C_j = \langle r^{n/m_j} \rangle$ with $m_j \in \mathbb{Z}^{|n} \setminus \{1\}$. The signature function of θ is

$$\Psi_\theta : \mathbb{Z}_+ \to \mathbb{Z}, \quad \Psi_\theta(q) = \#\{1 \leq j \leq v : m_j = q\}.$$

If V is a $\mathbb{C}$-representation of D_n then its pre-signature function $\Phi_V : \mathbb{Z}_+ \to \mathbb{Z}$ is given by

$$\Phi_V(q) = \begin{cases} \langle V, \rho^1 \rangle - \langle V, \psi_1 \oplus \psi_2 \rangle + 1, & \text{if } (q, n) = n, \\ \langle V, \rho^1 \rangle - \langle V, \psi_3 \oplus \psi_4 \rangle, & \text{if } n \text{ is even and } (q, n) = \frac{n}{2}, \\ \langle V, \rho^1 \rangle - \langle V, \rho^{(q,n)} \rangle, & \text{if } (q, n) < \frac{n}{2}. \end{cases}$$

If ρ_a is the analytic representation of the dihedral action represented by the ske θ then, as proved in Proposition 2.5,

$$\widetilde{\Phi}_{\rho_a} = \Psi_\theta \text{ and } \Phi_{\rho_a} = \widehat{\Psi}_\theta.$$

We also recall that

$$A = \#\{1 \leq j \leq v : n/m_j \text{ is an odd integer}\},$$
$$B = \#\{1 \leq j \leq v : n/2m_j \text{ is an odd integer}\}.$$

Analytic representation criteria

Not every $\mathbb{C}$-representation of a dihedral group comes from an action, as shows the following example.

Example 3.2. Let V be a $\mathbb{C}$-representation of D_7 given by

$$V = 2\rho^1 \oplus \rho^2 \oplus \rho^3.$$

We observe that V cannot be the analytic representation of an action. Indeed, we have that $\widetilde{\Phi}_V(7) = 3$. Thus, by Proposition 2.6, if V were the analytic representation of a D_7-action then the signature would be $(0; 2, 2, 7, 7, 7)$. However, the analytic representation ρ_a associated to (an action with) this signature is

$$\rho_a = 2(\rho^1 \oplus \rho^2 \oplus \rho^3).$$

The following results answer the question of when a given $\mathbb{C}$-representation is the analytic representation of a dihedral action. Let us denote by $\operatorname{Supp}\Psi = \{q \in \mathbb{Z}_+ : \Psi(q) \neq 0\}$ the *support* of the function $\Psi : \mathbb{Z}_+ \to \mathbb{Z}$.

Theorem 3.6. *Let V be a $\mathbb{C}$-representation of D_n with $n \geq 3$ odd. Then necessary and sufficient conditions for V to be the analytic representation of a D_n-action are:*

(1) $\langle V, \psi_2\rangle + 1 \geq \langle V, \psi_1\rangle$;

(2) $\widetilde{\Phi}_V(q) \geq 0$ *for each* $q \in \mathbb{Z}^{|n} \setminus \{1\}$;

(3) $\langle V, \rho^h\rangle = \langle V, \rho^{(n,h)}\rangle$ *for* $1 \leq h \leq (n-1)/2$;

(4) if $\langle V, \psi_1\rangle \leq 1$ *and* $\langle V, \psi_2\rangle = 0$ *then* $\operatorname{lcm}(\operatorname{Supp}\widetilde{\Phi}_V) = n$.

Set $\mathbb{Z}^{|n} = \{1 < m_1 < \ldots < m_v\}$. *In this case, the action has signature*

$$(\langle V, \psi_1\rangle; 2^t, m_1^{l_1}, \ldots, m_v^{l_v}),$$

where $t = 2(\langle V, \psi_2\rangle - \langle V, \psi_1\rangle + 1)$ *and* $l_j = \widetilde{\Phi}_V(m_j)$ *for* $j = 1, \ldots, v$.

Proof. Let us prove that the conditions are **necessary**. Assume that V is the analytic representation of the action represented by the ske $\theta : \Delta \to D_n$ of signature $(\gamma; 2^t, m_1^{l_1}, \ldots, m_v^{l_v})$, where $l_j \geq 0$.

(1) By Proposition 2.6, $0 \leq \frac{1}{2}t = \langle V, \psi_2\rangle - \langle V, \psi_1\rangle + 1$.

(2) By Proposition 2.6, $0 \leq l_j = \widetilde{\Phi}_V(m_j)$ for each $m_j \in \mathbb{Z}^{|n} \setminus \{1\}$.

(3) By definition, $\Phi_V(q) = \Phi_V((n, q))$ for $q \in \mathbb{Z}_+$. Since $\Phi_V = \widehat{\Psi}_\theta$, Theorem 2.1 implies that $\langle V, \rho^h\rangle = \langle V, \rho^{(n,h)}\rangle$ for $1 \leq h \leq (n-1)/2$.

(4) Observe that $\operatorname{lcm}(m_1^{l_1}, \ldots, m_v^{l_v}) = \operatorname{lcm}(\operatorname{Supp}\Psi_\theta) = \operatorname{lcm}(\operatorname{Supp}\widetilde{\Phi}_V)$. Then, Theorem 2.1 and Theorem 3.3 give the following implications

$$\langle V, \psi_1\rangle = \langle V, \psi_2\rangle = 0 \implies \gamma = 0 \text{ and } t = 2 \implies \operatorname{lcm}(\operatorname{Supp}\widetilde{\Phi}_V) = n.$$

Similarly, Theorem 2.1 and Theorem 3.2 give

$$\langle V, \psi_1\rangle = 1 \text{ and } \langle V, \psi_2\rangle = 0 \implies \gamma = 1 \text{ and } t = 0 \implies \operatorname{lcm}(\operatorname{Supp}\widetilde{\Phi}_V) = n.$$

Let us now prove that the conditions are **sufficient**. Let V be a $\mathbb{C}$-representation of D_n that satisfies Conditions $(1), \ldots, (4)$. As V satisfies (1) and (2), the formulas given in Proposition 2.6 induce a well-defined signature. More precisely, V induces the signature $\sigma = (\gamma; 2^t, m_1^{l_1}, \ldots, m_v^{l_v})$, where

$$\begin{aligned}\gamma &= \langle V, \psi_1 \rangle,\\ t &= 2\langle V, \psi_2 \rangle - 2\langle V, \psi_1 \rangle + 2,\\ l_j &= \widetilde{\Phi}_V(m_j) \text{ for } 1 \leq j \leq v.\end{aligned}$$

We verify that the signature σ satisfies the sufficient conditions to admit a dihedral action given in Theorem 3.3 and Theorem 3.2. (a) Assume that $\gamma = 0$, that is, $\langle V, \psi_1 \rangle = 0$. It is clear that $t \geq 2$ is even. Also, if $t = 2$ then $\langle V, \psi_2 \rangle = 0$, and Condition (4) implies that $\operatorname{lcm}(\operatorname{Supp} \widetilde{\Phi}_V) = \operatorname{lcm}(m_1^{l_1}, \ldots, m_v^{l_v}) = n$. Then, Theorem 3.3 guarantees that there exists a ske $\theta : \Delta \to D_n$ of signature σ. (b) Now, assume that $\gamma \geq 1$. It is clear that t is even. And, if $\gamma = 1$ and $t = 0$, then $\langle V, \psi_1 \rangle = 1$ and $\langle V, \psi_2 \rangle = 0$, and Condition (4) implies that $\operatorname{lcm}(\operatorname{Supp} \widetilde{\Phi}_V) = \operatorname{lcm}(m_1^{l_1}, \ldots, m_v^{l_v}) = n$. Then, Theorem 3.2 guarantees that there exists a ske $\theta : \Delta \to D_n$ of signature σ.

In both cases (a) and (b), let ρ_a be the analytic representation associated to the corresponding ske $\theta : \Delta \to D_n$ of signature σ. By Theorem 2.1,

$$\begin{aligned}\langle \rho_a, \psi_1 \rangle &= \gamma,\\ \langle \rho_a, \psi_2 \rangle &= \gamma - 1 + \tfrac{1}{2},\\ \langle \rho_a, \rho^h \rangle &= 2(\gamma - 1) + \tfrac{1}{2}t + \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(h),\end{aligned}$$

for $1 \leq h \leq (n-1)/2$. It is clear that

$$\langle \rho_a, \psi_j \rangle = \langle V, \psi_j \rangle \text{ for } j = 1, 2.$$

By Proposition 2.6, $\widetilde{\Phi}_{\rho_a}(m_j) = \Psi_\theta(m_j) = l_j$ for $1 \leq j \leq v$, and hence

$$\widetilde{\Phi}_{\rho_a}(q) = \widetilde{\Phi}_V(q) \text{ for each } q \in \mathbb{Z}^{|n}.$$

By Lemma 2.1 and Proposition 2.2, it follows that

$$\Phi_V(q) = \Phi_{\rho_a}(q) \text{ for each } q \in \mathbb{Z}^{|n}.$$

By definition, $\langle V, \rho^{(n,h)} \rangle = \langle \rho_a, \rho^{(n,h)} \rangle$ for all $1 \leq h \leq (n-1)/2$. To conclude, Condition (3) implies that $\langle V, \rho^h \rangle = \langle \rho_a, \rho^h \rangle$ for all $1 \leq h \leq (n-1)/2$. Hence $V \cong \rho_a$ is the analytic representation of a dihedral action. □

Remark 3.2. It follows from the proof of the theorem above that Conditions (1) and (2) guarantee that the induced signature is well-defined. Condition

(3) ensures that V has the "structure" of an analytic representation, that is, that the Galois associated $\mathbb{C}$-representations have the same multiplicity in the decomposition of V. Finally, Condition (4) has to do with the existence of a dihedral action.

Lemma 3.2. *Let $n \geq 2$ be an even integer.*

(1) $A = \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(\frac{n}{2}) = \Phi_{\rho_a}(n) - \Phi_{\rho_a}(\frac{n}{2})$.

(2) $B = \widehat{\Psi}_\theta(\frac{n}{2}) - \widehat{\Psi}_\theta(\frac{n}{4}) = \Phi_{\rho_a}(\frac{n}{2}) - \Phi_{\rho_a}(\frac{n}{4})$ *for* $n \in 4\mathbb{Z}$.

Proof. Recall Lemma 2.4 and the fact that $\widehat{\Psi}_\theta = \Phi_{\rho_a}$. □

Theorem 3.7. *Let V be a $\mathbb{C}$-representation of D_n with $n \geq 2$ even. Then necessary and sufficient conditions for V to be equivalent to the analytic representation of a D_n-action are:*

(1) $\langle V, \psi_2\rangle + 1 \geq \langle V, \psi_1\rangle + |\langle V, \psi_3\rangle - \langle V, \psi_4\rangle|$;

(2) $\widetilde{\Phi}_V(q) \geq 0$ *for each* $q \in \mathbb{Z}^{|n} \setminus \{1\}$;

(3) $\langle V, \rho^h\rangle = \langle V, \rho^{(n,h)}\rangle$ *for* $1 \leq h \leq (n-2)/2$;

(4) if $\langle V, \psi_1\rangle = \langle V, \psi_2\rangle = 0$ *then* $\mathrm{lcm}(\mathrm{Supp}\,\widetilde{\Phi}_V) = n$;

(5) if $\langle V, \psi_1\rangle = 0$, $\langle V, \psi_2\rangle \geq 1$ *and* $|\langle V, \psi_3\rangle - \langle V, \psi_4\rangle| = \langle V, \psi_2\rangle + 1$ *then* $\Phi_V(n) > \Phi_V(\frac{n}{2})$;

(6) if $\langle V, \psi_1\rangle = 1$ *and* $\langle V, \psi_2\rangle = 0$ *then* $\mathrm{lcm}(\mathrm{Supp}\,\widetilde{\Phi}_V) = n$ *or* $n/2$. *In the latter case, if* $n \in 4\mathbb{Z}$ *then* $\Phi_V(\frac{n}{2}) - \Phi_V(\frac{n}{4})$ *is odd.*

Set $\mathbb{Z}^{|n} = \{1 < m_1 < \ldots < m_v\}$. *In this case, the action has geometric signature*

$$(\langle V, \psi_1\rangle; \langle s\rangle^a, \langle sr\rangle^b, \langle r^{n/m_1}\rangle^{l_1}, \ldots, \langle r^{n/m_v}\rangle^{l_v}),$$

where $l_j = \widetilde{\Phi}_V(m_j)$ *for* $j = 1, \ldots, v$, *and*

$$a = \langle V, \psi_2 \oplus \psi_4\rangle - \langle V, \psi_1 \oplus \psi_3\rangle + 1,$$
$$b = \langle V, \psi_2 \oplus \psi_3\rangle - \langle V, \psi_1 \oplus \psi_4\rangle + 1.$$

Proof. Let us prove that the conditions are **necessary**. Assume that V is the analytic representation of the action represented by the ske $\theta : \Delta \to D_n$ of geometric signature

$$(\gamma; \langle s\rangle^a, \langle sr\rangle^b, \langle r^{n/m_1}\rangle^{l_1}, \ldots, \langle r^{n/m_v}\rangle^{l_v}).$$

(1) By Proposition 2.7,

$$a \geq 0 \implies \langle V, \psi_2 \rangle + 1 \geq \langle V, \psi_1 \rangle + (\langle V, \psi_3 \rangle - \langle V, \psi_4 \rangle),$$
$$b \geq 0 \implies \langle V, \psi_2 \rangle + 1 \geq \langle V, \psi_1 \rangle + (\langle V, \psi_4 \rangle - \langle V, \psi_3 \rangle).$$

(2) By Proposition 2.7, $0 \leq l_j = \widetilde{\Phi}_V(m_j)$ for $m_j \in \mathbb{Z}^{|n} \setminus \{1\}$.

(3) By definition, $\Phi_V(q) = \Phi_V((n, q))$ for $q \in \mathbb{Z}_+$. Since $\Phi_V = \widehat{\Psi}_\theta$, Theorem 2.2 implies that $\langle V, \rho^h \rangle = \langle V, \rho^{(n,h)} \rangle$ for $1 \leq h \leq (n-2)/2$.

(4) Observe that $\operatorname{lcm}(m_1^{l_1}, \ldots, m_v^{l_v}) = \operatorname{lcm}(\operatorname{Supp} \Psi_\theta) = \operatorname{lcm}(\operatorname{Supp} \widetilde{\Phi}_V)$. Then, Theorem 2.2 coupled with Theorem 3.4 give

$$\langle V, \psi_1 \rangle = \langle V, \psi_2 \rangle = 0 \implies \gamma = 0 \text{ and } a+b = 2 \implies \operatorname{lcm}(\operatorname{Supp} \widetilde{\Phi}_V) = n.$$

(5) By Theorem 2.2 one has that

$$\begin{array}{c} \langle V, \psi_1 \rangle = 0,\ \langle V, \psi_2 \rangle \geq 1 \text{ and} \\ |\langle V, \psi_3 \rangle - \langle V, \psi_4 \rangle| = \langle V, \psi_2 \rangle + 1 \end{array} \implies \begin{array}{c} \gamma = 0,\ a+b > 2, \text{ and} \\ a = 0 \text{ or } b = 0. \end{array}$$

Then, Theorem 3.4 implies that $A > 0$. To conclude, Lemma 3.2 states that $A = \Phi_V(n) - \Phi_V(\frac{n}{2})$.

(6) By Theorem 2.2,

$$\langle V, \psi_1 \rangle = 1 \text{ and } \langle V, \psi_2 \rangle = 0 \implies \gamma = 1 \text{ and } a = b = 0.$$

Then, Theorem 3.5 implies that

$$\operatorname{lcm}(m_1^{l_1}, \ldots, m_v^{l_v}) = \operatorname{lcm}(\operatorname{Supp} \widetilde{\Phi}_V) = n \text{ or } n/2,$$

and, in the latter case, if $n \in 4\mathbb{Z}$ then B is odd. To conclude, Lemma 3.2 states that $B = \Phi_V(\frac{n}{2}) - \Phi_V(\frac{n}{4})$.

Let us now prove that the conditions are **sufficient**. Let V be a $\mathbb{C}$-representation of D_n that satisfies conditions (1), ..., (6). The formulas given in Proposition 2.7 show that, as V satisfies (1) and (2), V induces a well-defined geometric signature. Concretely, V induces the geometric signature

$$\sigma = (\gamma; \langle s \rangle^a, \langle sr \rangle^b, \langle r^{n/m_1} \rangle^{l_1}, \ldots, \langle r^{n/m_v} \rangle^{l_v}),$$

where

$$\begin{aligned}\gamma &= \langle V, \psi_1 \rangle,\\ a &= \langle V, \psi_2 \oplus \psi_4 \rangle - \langle V, \psi_1 \oplus \psi_3 \rangle + 1,\\ b &= \langle V, \psi_2 \oplus \psi_3 \rangle - \langle V, \psi_1 \oplus \psi_4 \rangle + 1,\\ l_j &= \widetilde{\Phi}_V(m_j) \text{ for } 1 \le j \le v.\end{aligned}$$

Lemma 3.2, which easily extends to geometric signatures that do not admit actions, implies that $B = \Phi_V(\frac{n}{2}) - \Phi_V(\frac{n}{4})$ and

$$A = \Phi_V(n) - \Phi_V(\tfrac{n}{2}) = \langle V, \psi_3 \oplus \psi_4 \rangle - \langle V, \psi_1 \oplus \psi_2 \rangle + 1.$$

Assume that $\gamma = 0$. We verify that the geometric signature σ satisfies the existence conditions $(i), \ldots, (iv)$ of Theorem 3.4.

(i) It is clear that $a+b$ and $a+A$ are even, hence a, b, and A have the same parity.

(ii) Since $\langle V, \psi_1 \rangle = \gamma = 0$, it follows that $a + b \ge 2$.

(iii) Assume that $a+b = 2$. Then $\langle V, \psi_2 \rangle = 0$ and Condition (4) implies that $\text{lcm}(m_1^{l_1}, \ldots, m_v^{l_v}) = \text{lcm}(\text{Supp}\, \widetilde{\Phi}_V) = n$.

(iv) Assume that $a + b > 2$, and $a = 0$ or $b = 0$. Then $\langle V, \psi_2 \rangle \ge 1$ and $|\langle V, \psi_3 \rangle - \langle V, \psi_4 \rangle| = \langle V, \psi_2 \rangle + 1$. Moreover, Condition (5) implies that $A = \Phi_V(n) - \Phi_V(\frac{n}{2}) > 0$.

Now, assume that $\gamma \ge 1$. We verify that the geometric signature σ satisfies the existence conditions (i) and (ii) of Theorem 3.5.

(i) As before, it is clear that $a + b$ and $a + A$ are even. Hence a, b, and A have the same parity.

(ii) Assume that $\gamma = 1$ and $a = b = 0$. Then $\langle V, \psi_1 \rangle = 1$ and $\langle V, \psi_2 \rangle = 0$. Moreover, Condition (6) implies that

$$\text{lcm}(m_1^{l_1}, \ldots, m_v^{l_v}) = \text{lcm}(\text{Supp}\, \widetilde{\Phi}_V) = n \text{ or } n/2.$$

Also, Condition (6) states that, in the latter case, if $n \in 4\mathbb{Z}$ then $B = \Phi_V(\frac{n}{2}) - \Phi_V(\frac{n}{4})$ is odd.

We conclude that there always exists a ske $\theta : \Delta \to D_n$ of geometric signature σ. The rest of the proof is analogous to that of Theorem 3.6. (Up to we must employ Theorem 2.2 instead of Theorem 2.1.) □

Example 3.3. Let V be a $\mathbb{C}$-representation of D_p with $p \geq 3$ prime. There exist unique nonnegative integers $\mu_1, \mu_2, \nu_1, \ldots, \nu_{(p-1)/2}$ such that

$$V = \mu_1\psi_1 \oplus \mu_2\psi_2 \oplus \nu_1\rho^1 \oplus \cdots \oplus \nu_{(p-1)/2}\rho^{(p-1)/2}.$$

We apply Conditions $(1), \ldots, (4)$ of Theorem 3.6 to find necessary and sufficient conditions on $\mu_1, \mu_2, \nu_1, \ldots, \nu_{(p-1)/2}$ such that V is the analytic representation of a D_p-action. First, we observe that $\widetilde{\Phi}_V(p) = \nu_1 - (\mu_1 + \mu_2) + 1$ and that Condition (3) is equivalent to

$$\nu_1 = \cdots = \nu_{(p-1)/2}.$$

Now, we consider the cases (i) $\mu_1 = 0$, (ii) $\mu_1 = 1$, and (iii) $\mu_1 \geq 2$.

(i) Assume that $\mu_1 = 0$. Conditions (1) and (4) hold. Then, by Condition (2), one has that
$$\nu_1 + 1 \geq \mu_2.$$

(ii) Assume that $\mu_1 = 1$. Condition (1) holds, Condition (2) is equivalent to $\nu_1 \geq \mu_2$, and Condition (3) states that if $\mu_2 = 0$ then $\widetilde{\Phi}_V(p) = \nu_1 \geq 1$. Thus,
$$\nu_1 \geq \mu_2 \text{ and } \nu_1 \geq 1.$$

(iii) Assume that $\mu_1 \geq 2$. Condition (1) states that $\mu_2 \geq \mu_1 - 1 \geq 1$, and Condition (4) trivially holds. Condition (2) is equivalent to $\widetilde{\Phi}_V(p) = \nu_1 - (\mu_1 + \mu_2) + 1 \geq 0$. Thus,
$$\nu_1 + 1 \geq \mu_1 + \mu_2 \text{ and } \mu_2 \geq \mu_1 - 1.$$

Example 3.4. Let V be a $\mathbb{C}$-representation of D_4. There exist nonnegative integers $\mu_1, \ldots, \mu_4$ and ν_1 such that

$$V = \mu_1\psi_1 \oplus \cdots \oplus \mu_4\psi_4 \oplus \nu_1\rho^1.$$

We apply Conditions $(1), \ldots, (6)$ of Theorem 3.6 to find conditions on $\mu_1, \ldots, \mu_4$ and ν_1 such that V is the analytic representation of some D_4-action. First, we observe that Condition (3) trivially holds and that

$$\widetilde{\Phi}_V(4) = \Phi_V(4) - \Phi_V(2) = \mu_3 + \mu_4 - (\mu_1 + \mu_2) + 1,$$
$$\widetilde{\Phi}_V(2) = \Phi_V(2) = \nu_1 - (\mu_3 + \mu_4).$$

By Condition (1), one of the following cases occurs: (i) $\mu_1 = 0$ and $\mu_2 = 0$, (ii) $\mu_1 = 1$ and $\mu_2 = 0$, and (iii) $\mu_1 \geq 1$.

(i) Assume that $\mu_1 = \mu_2 = 0$. Conditions (5) and (6) trivially hold. Since $\widetilde{\Phi}_V(4) = \mu_3 + \mu_4 + 1 \geq 1$ it follows that Condition (4) holds. Condition (1) is equivalent to $|\mu_3 - \mu_4| \leq 1$. Finally, Condition (2) states that $\widetilde{\Phi}_V(4), \widetilde{\Phi}_V(2) \geq 0$. Thus,

$$|\mu_3 - \mu_4| \leq 1 \text{ and } \nu_1 \geq \mu_3 + \mu_4.$$

(ii) Assume that $\mu_1 = 1$ and $\mu_2 = 0$. Conditions (4) and (5) trivially hold. Condition (1) states that $\mu_3 = \mu_4$, and Condition (2) is equivalent to $\widetilde{\Phi}_V(4), \widetilde{\Phi}_V(2) \geq 0$, that is, $\nu_1 \geq \mu_3 + \mu_4$. Finally, Condition (6) states that $\mathrm{lcm}(\mathrm{Supp}\,\widetilde{\Phi}_V) = 4$ or 2, and, in the latter case, $\Phi_V(2)$ is odd. If $\widetilde{\Phi}_V(4) \geq 1$ then we are done. Otherwise, $\widetilde{\Phi}_V(4) = 0$, $\widetilde{\Phi}_V(2) \geq 1$ and $\Phi_V(2)$ is odd. That is, $\mu_3 = \mu_4 = 0$ and $\nu_1 \geq 1$ is odd. Thus,

$$\mu_3 = \mu_4, \ \nu_1 \geq 2\mu_3, \text{ and}$$
$$\mu_3 = 0 \implies \nu_1 \geq 1 \text{ is odd.}$$

(iii) Assume that $\mu_2 \geq 1$. Condition (4) and (6) trivially hold. Conditions (1) is equivalent to $\mu_2 + 1 \geq \mu_1 + |\mu_3 - \mu_4|$, and (2) states that $\widetilde{\Phi}_V(4), \widetilde{\Phi}_V(2) \geq 0$. Finally, we observe that Condition (5) holds. Indeed, it states that if $\mu_1 = 0$ and $|\mu_3 - \mu_4| = \psi_2 + 1$ then $\Phi_V(4) > \Phi_V(2)$. The previous inequality is equivalent to $\widetilde{\Phi}_V(4) \geq 0$, which is already guaranteed by Condition (2). Thus,

$$\nu_1 \geq \mu_3 + \mu_4 \geq \mu_1 + \mu_2 + 1 \geq 2\mu_1 + |\mu_3 - \mu_4|.$$

For instance, whereas the $\mathbb{C}$-representations of D_4,

$$3\psi_3 \oplus 4\psi_4 \oplus 7\rho^1, \ \psi_1 \oplus \rho^1, \text{ and } 2\psi_2 \oplus 3\psi_3 \oplus 3\rho^1,$$

are analytic representations of a D_4-action, the $\mathbb{C}$-representations of D_4,

$$3\psi_3 \oplus 5\psi_4 \oplus 7\rho^1, \ \psi_1 \oplus 2\rho^1, \text{ and } \psi_2 \oplus 3\psi_3 \oplus 3\rho^1,$$

are not analytic representations of any D_4-action.

Lemma 3.3. *For $n \geq 3$, if ρ_a is the analytic representation of a D_n-action in genus $g \geq 2$, then $\langle \rho_a, \rho^1 \rangle \geq 1$.*

Proof. Assume that $n \geq 4$ is even. By Theorem 2.2 we know that

$$\langle \rho_a, \rho^1 \rangle = 2(\gamma - 1) + \tfrac{1}{2}(a + b) + \widehat{\Psi}_\theta(n).$$

If $\langle \rho_a, \rho^1 \rangle = 0$ then either:

(1) $\gamma = 1$, $a + b = 0$ and $\widehat{\Psi}_\theta(n) = 0$;

(2) $\gamma = 0$, $a + b = 2$ and $\widehat{\Psi}_\theta(n) = 1$;

(3) $\gamma = 0$, $a + b = 4$ and $\widehat{\Psi}_\theta(n) = 0$.

Note that Theorem 3.5 tells us that no action satisfies (1). Besides, by Theorem 3.4, the only geometric signature compatible with (2) is $(0; \langle s\rangle, \langle sr\rangle, \langle r\rangle)$. Also, (3) has geometric signature $(0; \langle s\rangle^2, \langle sr\rangle^2)$. In each case the genus g is less than 2, a contradiction. If $n \geq 3$ is odd then we set $t := a + b$ and obtain the same conclusions. □

Proposition 3.1. *There is a D_n-action in genus $g \geq 2$ with irreducible analytic representation ρ_a if and only if $n \in \{3, 4, 6\}$. In each case, $\rho_a \cong \rho^1$ and the action is in genus 2.*

Proof. ($\rightarrow$) Assume that ρ_a is irreducible. By Lemma 3.3 one has that $\langle \rho_a, \rho^1\rangle$ is always positive, hence $\rho_a \cong \rho^1$. However, by Theorem 3.6 (if n is odd) and Theorem 3.7 (if n is even) one has that if $\langle \rho_a, \rho^1\rangle \neq 0$ then $\langle \rho_a, (\rho^1)^\sigma\rangle \neq 0$ for all σ in the Galois group of ρ^1. It follows that the Galois group of ρ^1 must be trivial, and this only happens for D_n with $n \in \{3, 4, 6\}$. (The size of the Galois group of ρ^1 is $\frac{1}{2}\phi(n)$, where ϕ is Euler's totient function.)

($\leftarrow$) Consider the following (geometric) signatures:

$$\begin{aligned} D_3: &\qquad \sigma_3 = (0; 2, 2, 3, 3), \\ D_4: &\qquad \sigma_4 = (0; \langle s\rangle, \langle sr\rangle, \langle r^2\rangle, \langle r\rangle), \\ D_6: &\qquad \sigma_6 = (0; \langle s\rangle, \langle sr\rangle, \langle r^3\rangle, \langle r^2\rangle). \end{aligned}$$

In each case, the (geometric) signature realizes as an action and its associated analytic representation satisfies $\rho_a \cong \rho^1$. (The realization of the signature σ_3 follows from Theorem 3.3, and the realization of the geometric signatures σ_4 and σ_6 is guaranteed by Theorem 3.4. Then, using the analytic representation formulas in Theorem 2.1 and Theorem 2.2, it is easy to check that $\rho_a \cong \rho^1$.)

For the sake of clearness, we compute the geometric signature σ associated to $\rho_a \cong \rho^1$ for D_6, and verify that $\sigma = \sigma_6$. By Proposition 2.7,

$$\sigma = (\gamma; \langle s\rangle^a, \langle sr\rangle^b, \langle r^3\rangle^{l_1}, \langle r^2\rangle^{l_2}, \langle r\rangle^{l_3}),$$

where $\gamma = \langle \rho_a, \psi_1 \rangle = 0$,

$$\begin{aligned}
a &= \langle \rho_a, \psi_2 \oplus \psi_4 \rangle - \langle \rho_a, \psi_1 \oplus \psi_3 \rangle + 1 = 1, \\
b &= \langle \rho_a, \psi_2 \oplus \psi_3 \rangle - \langle \rho_a, \psi_1 \oplus \psi_4 \rangle + 1 = 1, \\
l_1 &= \widetilde{\Phi}_{\rho_a}(2) = \langle \rho_a, \rho^1 \rangle - \langle \rho_a, \rho^2 \rangle = 1, \\
l_2 &= \widetilde{\Phi}_{\rho_a}(3) = \langle \rho_a, \rho^1 \rangle - \langle \rho_a, \psi_3 \oplus \psi_4 \rangle = 1, \\
l_3 &= \widetilde{\Phi}_{\rho_a}(6) = (\langle \rho_a, \rho^1 \rangle - \langle \rho_a, \psi_3 \oplus \psi_4 \rangle + 1) - 2 = 0.
\end{aligned}$$

(Note that $\widetilde{\Phi}_{\rho_a}(6) = \Phi_{\rho_a}(6) - \Phi_{\rho_a}(3) - \Phi_{\rho_a}(2)$.) This ends the proof. □

Chapter 4

Group algebra decomposition

Carocca, Recillas, and Rodríguez in [7] studied compact Riemann surfaces with dihedral actions and provided the associated group algebra decomposition of their Jacobians. In this chapter, equipped with tools not available at that time, we deal with the problem of determining when such a decomposition is affordable by Prym varieties. Along the way, we recover some of the results of [7]. Finally, we relate our results with the classical Ekedahl-Serre problem of completely decomposable Jacobians.

4.1 Preliminaries

Let G be a finite group and set $\mathrm{Irr}_{\mathbb{Q}}(G) = \{W_1, \dots, W_v\}$.

Theorem 4.1. *Let A be an abelian variety with a G-action. There are abelian subvarieties $B_1, \dots, B_v$ of A and a G-equivariant isogeny*

$$A \sim B_1^{n_1} \times \cdots \times B_v^{n_v},$$

where G acts on $B_i^{n_j}$ via the representation W_j.

Proof. See [21, Theorem 2.2]. □

The isogeny above is the *group algebra decomposition* of A respect to G. The subvariety B_j, which is defined up to isogeny, is called the *group algebra component* of A associated to W_j. If $A = JS$ and W_1 denotes the trivial representation, then (as we shall do in this chapter) $B_1 \sim JS_G$ and $n_1 = 1$.

Proposition 4.1. *For $j = 2, \dots, v$, the dimension of the group algebra component B_j of JS is*

$$\dim B_j = \tfrac{1}{2} k_{V_j} \langle \rho_r, V_j \rangle,$$

where $V_j \in \mathrm{Irr}_{\mathbb{C}}(G)$ *is Galois associated to* W_j*, and* $k_{V_j} = s_{V_j}|\,\mathrm{Gal}(K_{V_j}/\mathbb{Q})|$.

Proof. See [37, Theorem 5.12] and Theorem 1.3. □

It is worth observing that whereas the dimension of B_j depends on the action, the integer v only depends on the algebraic structure of G.

Theorem 4.2. *Let* S *be a compact Riemann surface of genus* $g \geq 2$ *with a* G*-action. Let* $H \leq K$ *be subgroups of* G*. If* $B_1, \dots, B_v$ *are the group algebra components of* JS*, then*

$$JS_H \sim JS_G \times B_2^{u_2} \times \cdots B_v^{u_v},$$

where $u_j = d_{V_j}^H / s_{V_j}$ *for* $2 \leq j \leq v$*. In addition,*

$$P(\pi_K^H) \sim B_2^{t_2} \times \cdots \times B_v^{t_v},$$

where $t_j = (d_{V_j}^H - d_{V_j}^K)/s_{V_j}$ *for* $2 \leq j \leq v$.

Proof. See [9, Theorem 5.2 and Corollary 5.4]. □

A detailed account on decomposition of Jacobians by Prym varieties can be found in [23].

4.2 Jacobians with dihedral actions

We recall that the Schur index of the dihedral representations is equal to 1. Also, there is a bijection between $\mathbb{Z}^{|n} \setminus \{1,2\}$ and the set of irreducible $\mathbb{Q}$-representations of D_n of degree ≥ 2, given by

$$\mathbb{Z}^{|n} \setminus \{1,2\} \to \mathrm{Irr}_{\mathbb{Q}}(D_n), \quad q \mapsto W(q) = \oplus_\sigma (\rho^{n/q})^\sigma.$$

For the sake of clearness, we begin this section with a simple example.

Example 4.1. The group algebra decomposition of JS with respect to D_3 is

$$JS \sim JS_{D_3} \times B_2 \times B(3)^2,$$

where B_2 and $B(3)$ are the group algebra components associated to ψ_2 and $W(3)$, respectively. The dimensions of the fixed subgroups are

d_V^H	ψ_2	ρ^1
$\langle r \rangle$	1	0
$\langle s \rangle$	0	1

It follows from Theorem 4.2 that $B_2 \sim P(\pi_{D_3}^{\langle r \rangle})$ and $B(3) \sim P(\pi_{D_3}^{\langle s \rangle})$. Thus,

$$JS \sim JS_{D_3} \times P\big(\pi_{D_3}^{\langle r \rangle}\big) \times P\big(\pi_{D_3}^{\langle s \rangle}\big)^2.$$

The general case for dihedral groups is as follows. For n even, the group algebra decomposition of JS with respect to D_n is

$$JS \sim JS_{D_n} \times B_2 \times B_3 \times B_4 \times \prod_{q \in \mathbb{Z}^{|n} \setminus \{1,2\}} B(q)^2,$$

where B_j is the group algebra component associated to the nontrivial degree one representation ψ_j, and $B(q)$ is the group algebra component associated to $W(q)$. If n is odd then we just omit B_3 and B_4, and hence

$$JS \sim JS_{D_n} \times B_2 \times \prod_{q \in \mathbb{Z}^{|n} \setminus \{1,2\}} B(q)^2.$$

Hereafter, ϕ denotes the *Euler totient function*.

Proposition 4.2. *The dimension of the group algebra components are:*

(1) $\dim B_j = \langle \rho_a, \psi_j \rangle$ *for* $j = 1, \dots, 4$*;*

(2) $\dim B(q) = \frac{1}{2}\phi(q)\langle \rho_a, \rho^{n/q} \rangle$ *for* $q \in \mathbb{Z}^{|n} \setminus \{1, 2\}$.

Proof. Since $\rho_a \cong \overline{\rho_a}$, by Corollary 1.1 and Proposition 4.1 one has that $\dim B = |\operatorname{Gal}(K_V/\mathbb{Q})| \langle \rho_a, V \rangle$. We recall the well-known fact that

$$|\operatorname{Gal}(K_V/\mathbb{Q})| = \begin{cases} 1, & \text{if } V = \psi_j, \\ \frac{1}{2}\phi(q), & \text{if } V = \rho^{n/q}. \end{cases}$$

This completes the proof. □

The dimensions d_V^H are known for the dihedral groups. We include them here for latter use.

Lemma 4.1. *Let* $H \leq D_n$ *and* $V \in \operatorname{Irr}_{\mathbb{C}}(D_n)$*. For* $\alpha, q \in \mathbb{Z}^{|n}$*,*

d_V^H	ψ_1	ψ_2	ψ_3	ψ_4	$\rho^{n/q}$
$\langle r^{n/\alpha} \rangle$	1	1	δ	δ	2ε
$\langle s, r^{n/\alpha} \rangle$	1	0	δ	0	ε
$\langle sr, r^{n/\alpha} \rangle$	1	0	0	δ	ε

where $\delta = 1$ *if* n/α *is even, and* $\delta = 0$ *otherwise; and* $\varepsilon = 1$ *if* α *divides* n/q*, and* $\varepsilon = 0$ *otherwise. If* n *is odd then we must omit columns* ψ_3 *and* ψ_4*.*

Now, we derive expressions for the decompositions of JS_H and $P(\pi_K^H)$ induced by the group algebra decomposition of JS. For $\alpha \in \mathbb{Z}^{|n}$, let

$$H_\alpha = \langle s, r^{n/\alpha} \rangle,\ K_\alpha = \langle sr, r^{n/\alpha} \rangle,\ \text{and } C_\alpha = \langle r^{n/\alpha} \rangle,$$

be subgroups of D_n. In fact, this covers all of the subgroups of D_n modulo conjugation. Also, $H_n = K_n = D_n$. If α is a proper divisor of $\beta \in \mathbb{Z}^{|n}$, then

$$C_\alpha < H_\alpha < H_\beta,\ C_\alpha < K_\alpha < K_\beta,\ \text{and } C_\alpha < C_\beta,$$

with associated intermediate coverings

$$S_{C_\alpha} \to S_{H_\alpha} \to S_{H_\beta},\ S_{C_\alpha} \to S_{K_\alpha} \to S_{K_\beta},\ \text{and } S_{C_\alpha} \to S_{C_\beta}.$$

We observe that, modulo conjugation, all possible group inclusions and intermediate coverings are depicted above.

Proposition 4.3. *For $\alpha \in \mathbb{Z}^{|n}$, set $Q_\alpha = \mathbb{Z}^{|n/\alpha} \setminus \{1,2\}$. Then*

(1) $JS_{H_\alpha} \sim \begin{cases} JS_{D_n} \times \prod_{q \in Q_\alpha} B(q), & \text{if } \frac{n}{\alpha} \text{ is odd} \\ JS_{D_n} \times B_3 \times \prod_{q \in Q_\alpha} B(q), & \text{if } \frac{n}{\alpha} \text{ is even} \end{cases}$

(2) $JS_{K_\alpha} \sim \begin{cases} JS_{D_n} \times \prod_{q \in Q_\alpha} B(q), & \text{if } \frac{n}{\alpha} \text{ is odd} \\ JS_{D_n} \times B_4 \times \prod_{q \in Q_\alpha} B(q), & \text{if } \frac{n}{\alpha} \text{ is even} \end{cases}$

(3) $JS_{C_\alpha} \sim \begin{cases} JS_{D_n} \times B_2 \times \prod_{q \in Q_\alpha} B(q)^2, & \text{if } \frac{n}{\alpha} \text{ is odd} \\ JS_{D_n} \times B_2 \times B_3 \times B_4 \times \prod_{q \in Q_\alpha} B(q)^2, & \text{if } \frac{n}{\alpha} \text{ is even} \end{cases}$

If n is odd then we just discard the components B_3 and B_4 ($\frac{n}{\alpha}$ is odd).

Proof. Assume that n is even. Following Theorem 4.2,

$$JS_{H_\alpha} \sim JS_{D_n} \times B_2^{u_2} \times B_3^{u_3} \times B_4^{u_4} \times \prod_{q \in \mathbb{Z}^{|n} \setminus \{1,2\}} B(q)^{u(q)},$$

where $u_j = d_{\psi_j}^{H_\alpha}$ for $j \in \{2,3,4\}$, and $u(q) = d_{\rho^{n/q}}^{H_\alpha}$ for $q \in \mathbb{Z}^{|n} \setminus \{1,2\}$. By Lemma 4.1, one has that $u_2 = 0$, $u_3 = \delta$, $u_4 = 0$ and $u(q) = \varepsilon(q)$, where

$$\delta = \begin{cases} 1, & \text{if } \frac{n}{\alpha} \text{ is even} \\ 0, & \text{if } \frac{n}{\alpha} \text{ is odd} \end{cases} \quad \text{and} \quad \varepsilon(q) = \begin{cases} 1, & \text{if } q \in \mathbb{Z}^{|n/\alpha} \\ 0, & \text{if } q \notin \mathbb{Z}^{|n/\alpha} \end{cases}$$

Therefore,

$$JS_{H_\alpha} \sim JS_{D_n} \times B_2^0 \times B_3^\delta \times B_4^0 \times \prod_{q \in \mathbb{Z}^{|n} \setminus \{1,2\}} B(q)^{\varepsilon(q)}.$$

Similarly,

$$JS_{K_\alpha} \sim JS_{D_n} \times B_2^0 \times B_3^0 \times B_4^\delta \times \prod_{q\in\mathbb{Z}^{|n}\setminus\{1,2\}} B(q)^{\varepsilon(q)},$$
$$JS_{C_\alpha} \sim JS_{D_n} \times B_2^1 \times B_3^\delta \times B_4^\delta \times \prod_{q\in\mathbb{Z}^{|n}\setminus\{1,2\}} B(q)^{2\varepsilon(q)}.$$

The proof for the case n odd is analogous. □

Proposition 4.4. *For $\alpha, \beta \in \mathbb{Z}^{|n}$, set $Q_{\alpha,\beta} = (\mathbb{Z}^{|n/\alpha} \setminus \mathbb{Z}^{|n/\beta}) \setminus \{2\}$. If α is a proper divisor of β, then*

(1) $P(\pi^{H_\alpha}_{H_\beta}) \sim \begin{cases} B_3 \times \prod_{q\in Q_{\alpha,\beta}} B(q), & \text{if } \frac{n}{\alpha}+\frac{n}{\beta} \text{ is odd} \\ \prod_{q\in Q_{\alpha,\beta}} B(q), & \text{if } \frac{n}{\alpha}+\frac{n}{\beta} \text{ is even} \end{cases}$

(2) $P(\pi^{K_\alpha}_{K_\beta}) \sim \begin{cases} B_4 \times \prod_{q\in Q_{\alpha,\beta}} B(q), & \text{if } \frac{n}{\alpha}+\frac{n}{\beta} \text{ is odd} \\ \prod_{q\in Q_{\alpha,\beta}} B(q), & \text{if } \frac{n}{\alpha}+\frac{n}{\beta} \text{ is even} \end{cases}$

(3) $P(\pi^{C_\alpha}_{C_\beta}) \sim P(\pi^{H_\alpha}_{H_\beta}) \times P(\pi^{K_\alpha}_{K_\beta})$*;*

(4) $P(\pi^{C_\alpha}_{H_\beta}) \sim B_2 \times P(\pi^{H_\alpha}_{H_\beta}) \times P(\pi^{K_\alpha}_{D_n})$.

(5) $P(\pi^{C_\alpha}_{K_\beta}) \sim B_2 \times P(\pi^{K_\alpha}_{K_\beta}) \times P(\pi^{H_\alpha}_{D_n})$.

If n is odd then we just discard the components B_3 and B_4 ($\frac{n}{\alpha}+\frac{n}{\beta}$ is even).

Proof. Assume that n is even. Following Theorem 4.2,

$$P(\pi^{H_\alpha}_{H_\beta}) \sim B_2^{t_2} \times B_3^{t_3} \times B_4^{t_4} \times \prod_{q\in\mathbb{Z}^{|n}\setminus\{1,2\}} B(q)^{t(q)},$$

where $t_j = d^{H_\alpha}_{\psi_j} - d^{H_\beta}_{\psi_j}$ for $j \in \{2,3,4\}$, and $t(q) = d^{H_\alpha}_{\rho^{n/q}} - d^{H_\beta}_{\rho^{n/q}}$ for $q \in \mathbb{Z}^{|n} \setminus \{1,2\}$. By Lemma 4.1, one has that $t_2 = 0$, $t_3 = \delta - \delta'$, $t_4 = 0$ and $t(q) = \varepsilon(q) - \varepsilon'(q)$, where

$$\delta = \begin{cases} 1, & \text{if } \frac{n}{\alpha} \text{ even} \\ 0, & \text{if } \frac{n}{\alpha} \text{ odd} \end{cases} \qquad \varepsilon(q) = \begin{cases} 1, & \text{if } q \in \mathbb{Z}^{|n/\alpha} \\ 0, & \text{if } q \notin \mathbb{Z}^{|n/\alpha} \end{cases}$$
$$\delta' = \begin{cases} 1, & \text{if } \frac{n}{\beta} \text{ even} \\ 0, & \text{if } \frac{n}{\beta} \text{ odd} \end{cases} \qquad \varepsilon'(q) = \begin{cases} 1, & \text{if } q \in \mathbb{Z}^{|n/\beta} \\ 0, & \text{if } q \notin \mathbb{Z}^{|n/\beta} \end{cases}$$

Therefore,

$$P(\pi^{H_\alpha}_{H_\beta}) \sim B_2^0 \times B_3^{\delta-\delta'} \times B_4^0 \times \prod_{q\in\mathbb{Z}^{|n}\setminus\{1,2\}} B(q)^{\varepsilon(q)-\varepsilon'(q)}.$$

Similarly,

$$P(\pi^{K_\alpha}_{K_\beta}) \sim B_2^0 \times B_3^0 \times B_4^{\delta-\delta'} \times \prod_{q\in\mathbb{Z}^{|n}\setminus\{1,2\}} B(q)^{\varepsilon(q)-\varepsilon'(q)},$$
$$P(\pi^{C_\alpha}_{C_\beta}) \sim B_2^0 \times B_3^{\delta-\delta'} \times B_4^{\delta-\delta'} \times \prod_{q\in\mathbb{Z}^{|n}\setminus\{1,2\}} B(q)^{2(\varepsilon(q)-\varepsilon'(q))},$$
$$P(\pi^{C_\alpha}_{H_\beta}) \sim B_2^1 \times B_3^{\delta-\delta'} \times B_4^{\delta} \times \prod_{q\in\mathbb{Z}^{|n}\setminus\{1,2\}} B(q)^{2\varepsilon(q)-\varepsilon'(q)},$$
$$P(\pi^{C_\alpha}_{K_\beta}) \sim B_2^1 \times B_3^{\delta} \times B_4^{\delta-\delta'} \times \prod_{q\in\mathbb{Z}^{|n}\setminus\{1,2\}} B(q)^{2\varepsilon(q)-\varepsilon'(q)}.$$

The proof for the case n odd is analogous. □

The following corollary, which was proven in [7, Theorems 6.1, 7.1, and 8.1], follows directly from the previous proposition.

Corollary 4.1. *Let S be a compact Riemann surface of genus $g \geq 2$ with a D_n-action. Then for each n, $P(\pi^{\langle r\rangle}_{D_n}) \sim B_2$. Moreover, if n is even, then $P(\pi^{\langle s,r^2\rangle}_{D_n}) \sim B_3$ and $P(\pi^{\langle sr,r^2\rangle}_{D_n}) \sim B_4$.*

Proof. Observe that

$$P(\pi^{\langle r\rangle}_{D_n}) = P(\pi^{C_n}_{H_n}) \sim B_2 \times P(\pi^{H_n}_{H_n}) \times P(\pi^{K_n}_{K_n}) = B_2 \times P(\pi^{D_n}_{D_n})^2 \sim B_2.$$

Similarly for the other cases. □

4.3 Prym affordable Jacobians

Let S be a compact Riemann surface of genus $g \geq 2$ with a G-action. The group algebra decomposition of JS is a powerful tool to study the following two questions:

Question 1. When does the Jacobian JS decomposes as a product of Jacobians of quotient surfaces of S?

Question 2. When does the Jacobian JS decomposes as a product of JS_G and Prym varieties of intermediate coverings of $\pi : S \to S_G$?

In 1989, preceding the group algebra decomposition theorems, Kani and Rosen in [19] provided partial results regarding the first question. More recently, Reyes-Carocca and Rodríguez in [33] generalized their results. The second question was considered by Moraga in [27] for actions of affine groups over finite fields.

Definition 4.1. Let S be a compact Riemann surface of genus $g \geq 2$ with a G-action. Following [27], the group algebra decomposition of JS (with respect to G) is called *affordable by Prym varieties* if each group algebra component (with respect to G) of JS is isogenous to the Prym variety of an intermediate covering.

Definition 4.2. A group G is *Prym-affordable* if every group algebra decomposition of a Jacobian with respect to G is affordable by Prym varieties.

Example 4.2. The cyclic group $\mathbb{Z}_p$ of prime order p is Prym-affordable. Indeed, $\mathbb{Z}_p$ has two irreducible $\mathbb{Q}$-representations and every irreducible $\mathbb{C}$-representation has degree and Schur index equal to one. Thus, if $G \cong \mathbb{Z}_p$ acts on a Riemann surface S of genus $g \geq 2$ then

$$JS \sim JS_G \times B_2 \sim JS_G \times P(\pi_G^{\{1\}}).$$

Partial results for the problem of determining when D_n is Prym-affordable have been obtained. Concretely, Carocca, Recillas, and Rodríguez proved in [7, Theorems 6.4 and 7.1] that D_p ($p \geq 3$ prime) and D_{2^e} ($e \geq 2$) are Prym-affordable, respectively. Besides, Lange and Recillas in [21, §4.4] pointed out that D_{2p} is not Prym-affordable.

The following result gives a complete answer to the problem.

Theorem 4.3. *The dihedral group D_n is Prym-affordable if and only if $n = p^e$ for p prime and $e \geq 1$.*

Proof. For $\alpha, \beta \in \mathbb{Z}^{|n}$, let $H_\alpha = \langle s, r^{n/\alpha} \rangle$ be a subgroup of D_n, and let

$$Q_{\alpha,\beta} = (\mathbb{Z}^{|n/\alpha} \setminus \mathbb{Z}^{|n/\beta}) \setminus \{2\}.$$

($\leftarrow$) Let $n = p^e$ for p prime and $e \geq 1$, and assume that D_n acts on a compact Riemann surface S of genus $g \geq 2$. After considering Corollary 4.1, it suffices to prove that for each $q \in \mathbb{Z}^{|n} \setminus \{1, 2\}$, the subvariety $B(q)$ of JS is isogenous to the Prym variety of an intermediate covering. Set $\alpha_j = n/p^j$ and $\beta_j = n/p^{j-1}$ for $j \in \{1, \ldots, e\}$. Observe that

$$Q_{\alpha_j,\beta_j} = \mathbb{Z}^{|p^j} \setminus \mathbb{Z}^{|p^{j-1}} = \{p^j\}.$$

Then, by Proposition 4.4(1), one has that

$$P(\pi_{H_{\beta_j}}^{H_{\alpha_j}}) \sim \prod_{q \in Q_{\alpha_j,\beta_j}} B(q) \sim B(p^j),$$

for $1 \leq j \leq e$ if $p \geq 3$, and for $2 \leq j \leq e$ if $p = 2$, as desired. (If $p \geq 3$ then $\mathbb{Z}^{|n} \setminus \{1, 2\} = \{p, \ldots, p^e\}$, otherwise $\mathbb{Z}^{|n} \setminus \{1, 2\} = \{2^2, \ldots, 2^e\}$. Moreover, $\frac{n}{\alpha_j} + \frac{n}{\beta_j} = p^j + p^{j-1}$ is even in each of the previous cases.)

($\rightarrow$) Let us prove the contrapositive. Assume that n is not the power of a prime. Then, one of the following cases occurs:

(i) $n \in pq\mathbb{Z}$ for $p, q \geq 3$ primes, or

(ii) $n \in 2p\mathbb{Z}$ for $p \geq 3$ prime.

We now proceed to prove that for each of the cases above, given a suitable choice of action, there is a group algebra component that is not isogenous to the Prym variety of an intermediate covering. Let $\theta : \Delta \rightarrow D_n$ be a ske such that the quotient surface has genus $\gamma \geq 2$. (The existence of θ is guaranteed by Theorems 3.1 and 3.2.) Then, by the Chevalley-Weil formula and Proposition 4.2, one has that all the group algebra components (associated to the action of θ) have positive dimension.

(i) Assume that $n \in pq\mathbb{Z}$. By Proposition 4.4, if $B(pq) \sim P(\pi^H_K)$ for $H < K$ subgroups of D_n then, as every group algebra component has positive dimension, there are integers $\alpha, \beta \in \mathbb{Z}^{|n}$, with α a proper divisor of β, such that

$$Q_{\alpha,\beta} = (\mathbb{Z}^{|n/\alpha} \setminus \mathbb{Z}^{|n/\beta}) \setminus \{2\} = \{pq\}.$$

It follows that pq divides n/α but does not divide n/β. Since p and q are prime, at least one of them does not divide n/β and $Q_{\alpha,\beta}$ is not a singleton, a contradiction.

(ii) Assume that $n \in 2p\mathbb{Z}$. By Proposition 4.4, if $B(pq) \sim P(\pi^H_K)$ for $H < K$ subgroups of D_n (and every group algebra component has positive dimension), then there are integers $\alpha, \beta \in \mathbb{Z}^{|n}$, with α a proper divisor of β, such that

$$Q_{\alpha,\beta} = (\mathbb{Z}^{|n/\alpha} \setminus \mathbb{Z}^{|n/\beta}) \setminus \{2\} = \{2q\}.$$

In particular, $2p$ is a divisor of $\frac{n}{\alpha}$ (even) but not of $\frac{n}{\beta}$. Since $p \notin Q_{\alpha,\beta}$ it follows that p divides $\frac{n}{\beta}$ and therefore 2 does not divide $\frac{n}{\beta}$ (odd). We conclude that $\frac{n}{\alpha} + \frac{n}{\beta}$ is odd and thus

$$P(\pi^K_H) \sim B_3 \times B(2p) \text{ or } P(\pi^K_H) \sim B_4 \times B(2p).$$

a contradiction.

This concludes the proof. $\square$

Moving forward, we refine our analysis to consider the possibility that some group algebra components may have dimension zero.

Definition 4.3. For each subset $Q \subset \mathbb{Z}_+$, we define the function

$$L_Q : \mathbb{Z}_+ \to \mathbb{Z}_+ \text{ given by } L_Q(q) = \operatorname{lcm}(Q \cap \mathbb{Z}^{|q} \setminus \{q\}).$$

Lemma 4.2. *Set $Q \subset \mathbb{Z}_+$ and $q \in \mathbb{Z}_+$. Then*

(1) $L_Q(q)$ is a divisor of q;

(2) $L_Q(q) \neq q$ if and only if $L_Q(q)$ is a proper divisor of q;

(3) if q is the power of a prime number then $L_Q(q) \neq q$;

(4) if $q \in Q$ then $L_Q(q) \neq q$ if and only if $Q \cap \mathbb{Z}^{|q} \setminus \mathbb{Z}^{|L_Q(q)} = \{q\}$;

(5) if $q \in Q$ and $L_Q(q) \neq q$, then $L_Q(q) = \gcd\{t \in \mathbb{Z}^{|q} : Q \cap \mathbb{Z}^{|q} \setminus \mathbb{Z}^{|t} = \{q\}\}$.

Proof. The least common multiple of a subset of $\mathbb{Z}^{|q}$ is a divisor of q. This proves **(1)**, which in turn implies **(2)**. To see that **(3)** holds, note that

$$Q \cap \mathbb{Z}^{|p^e} \setminus \{p^e\} \subset \{p, \ldots, p^{e-1}\} \text{ for } p \text{ prime.}$$

Set $q \in Q$. If $L_Q(q) = q$ then $Q \cap \mathbb{Z}^{|q} \setminus \mathbb{Z}^{|L_Q(q)}$ is empty. By definition, if $L_Q(q) \neq q$ then each element of $Q \cap \mathbb{Z}^{|q} \setminus \{q\}$ divides $L_Q(q)$. This shows **(4)**.

Finally, let us prove **(5)**. Set $q \in Q$ and assume that $L_Q(q) \neq q$. Choose $t \in \mathbb{Z}^{|q}$ such that $Q \cap \mathbb{Z}^{|q} \setminus \mathbb{Z}^{|t} = \{q\}$. Since every element of $Q \cap \mathbb{Z}^{|q} \setminus \{q\}$ divides t it follows that $L_Q(q)$ divides t, as desired. □

Definition 4.4. Let ρ_a be the analytic representation of a dihedral action represented by a ske $\theta : \Delta \to D_n$. Set

$$Q_\theta = \{t \in \mathbb{Z}^{|n} \setminus \{1, 2\} : \langle \rho_a, \rho^{n/t} \rangle \geq 1\}.$$

Theorem 4.4. *Let $n \geq 3$ be an odd integer, and let S be a compact Riemann surface of genus $g \geq 2$ with a dihedral action represented by a surface kernel epimorphism $\theta : \Delta \to D_n$. For $q \in Q_\theta$, the subvariety $B(q)$ of JS is isogenous to the Prym variety of an intermediate covering if and only if $L_Q(q) \neq q$. In this case,*

$$B(q) \sim P(\pi^H_K) \text{ for } H = \langle s, r^q \rangle \text{ and } K = \langle s, r^{L_{Q_\theta}(q)} \rangle.$$

Proof. Assume that $L_{Q_\theta}(q) \neq q$. Set $\alpha = n/q$ and $\beta = n/L_{Q_\theta}(q)$, and recall that $H_\alpha = \langle s, r^{n/\alpha}\rangle$ and $Q_{\alpha,\beta} = (\mathbb{Z}^{|n/\alpha} \setminus \mathbb{Z}^{|n/\beta})$. By Lemma 4.2(2), α is a proper divisor of β and hence $H_\alpha < H_\beta$. Moreover, by Lemma 4.2(4),

$$Q_\theta \cap Q_{\alpha,\beta} = Q_\theta \cap (\mathbb{Z}^{|q} \setminus \mathbb{Z}^{|L_{Q_\theta}(q)}) = \{q\}.$$

Then Proposition 4.4(1) implies that

$$P\big(\pi^{H_\alpha}_{H_\beta}\big) \sim \prod_{t \in Q_{\alpha,\beta}} B(t) \sim \prod_{t \in Q_\theta \cap Q_{\alpha,\beta}} B(t) \sim B(q).$$

For the converse, assume that $B(q) \sim P(\pi^K_H)$ for $H < K$ subgroups of D_n. By Proposition 4.4, we can assume that $H = H_{n/q}$ and $K = H_{n/t}$ with t a proper divisor of q such that

$$Q_\theta \cap (\mathbb{Z}^{|q} \setminus \mathbb{Z}^{|t}) = \{q\}.$$

By Lemma 4.2(5), we conclude that $L_{Q_\theta}(q)$ divides t and $L_{Q_\theta}(q) \neq q$. □

Consider a dihedral action in genus $g \geq 2$ represented by a ske $\theta : \Delta \to D_n$ with n odd. Let $(\gamma; 2^t, m_1, \ldots, m_v)$ be its signature. By Lemma 3.3, one has that $n \in Q_\theta$, that is, Q_θ is not empty. Moreover, Proposition 2.4(1) states that if $\gamma \geq 1$, or $\gamma = 0$ and $t \geq 6$, then $Q_\theta = \mathbb{Z}^{|n} \setminus \{1, 2\}$. In this case, all the subvarieties $B(q)$ of JS have positive dimension (Proposition 4.2). Therefore, in our search for examples of group algebra decomposition's with dimension zero subvarieties, we will examine the following cases:

(1) $\gamma = 0$ and $t = 2$;

(2) $\gamma = 0$ and $t = 4$.

Also, in Case (2) we know, by Proposition 2.4(2), that

$$Q_\theta = \mathbb{Z}^{|n} \setminus \mathbb{Z}^{|n/\operatorname{lcm}(m_1,\ldots,m_v)}.$$

As we will see in the following examples, group algebra decompositions of Jacobian varieties, respect to non Prym-affordable dihedral groups, can still be affordable by Prym varieties. Moreoever, we provide an infinite family of such decompositions.

Example 4.3. Recall that $H_\alpha = \langle s, r^{n/\alpha}\rangle$, $C_\alpha = \langle r^{n/\alpha}\rangle$ and $W(q) = \oplus_\sigma (\rho^{n/q})^\sigma$.

(1) Let $p, q \geq 3$ be distinct prime numbers, and let $e \geq 1$. By Theorem 3.3, one has that the group $D_{p^e q}$ acts in genus $g = 1 + p^e(q-1)$ with signature $(0; 2, 2, 2, 2, q)$. Since $\operatorname{lcm}(m_1, \ldots, m_v) = q$, it follows that

$$Q_\theta = \mathbb{Z}^{|p^e q} \setminus \mathbb{Z}^{|p^e} = \{q, pq, p^2 q, \ldots, p^e q\}.$$

Therefore, $L_{Q_\theta}(q) = 1$ and

$$L_{Q_\theta}(p^j q) = \operatorname{lcm}\{q, pq, \dots, p^{j-1}q\} = p^{j-1}q \text{ for } 1 \le j \le e.$$

By Theorem 4.4, one has that

$$\begin{aligned} JS &\sim B_2 \times B(q)^2 \times B(pq)^2 \times \cdots \times B(p^e q)^2, \\ &\sim P\big(\pi^{\langle r\rangle}_{D_{p^e q}}\big) \times P\big(\pi^{\langle s,r^q\rangle}_{D_{p^e q}}\big)^2 \times P\big(\pi^{\langle s,r^{pq}\rangle}_{\langle s,r^q\rangle}\big)^2 \times \cdots \times P\big(\pi^{\langle s\rangle}_{\langle s,r^{p^{e-1}q}\rangle}\big)^2, \\ &= P\big(\pi^{C_{p^e q}}_{D_{p^e q}}\big) \times P\big(\pi^{H_{p^e}}_{D_{p^e q}}\big)^2 \times P\big(\pi^{H_{p^{e-1}}}_{H_{p^e}}\big)^2 \times \cdots \times P\big(\pi^{H_1}_{H_p}\big)^2. \end{aligned}$$

(2) Let $p, q \ge 3$ be distinct prime numbers. By Theorem 3.3, one has that the group $D_{p^2 q}$ acts in genus $g = (p^2 - 1)(q - 1)$ with signature $(0; 2, 2, q, p^2)$. Then, by Theorem 2.1, the analytic representation of the action satisfies $\rho_a \cong W(p^2 q) \oplus W(pq)$. Since $Q_\theta = \{pq, p^2 q\}$, it follows that $L_{Q_\theta}(pq) = 1$ and $L_{Q_\theta}(p^2 q) = pq$. By Theorem 4.4, we conclude that

$$JS \sim B(pq)^2 \times B(p^2 q)^2 \sim P\big(\pi^{\langle s,r^{pq}\rangle}_{D_{p^2 q}}\big)^2 \times P\big(\pi^{\langle s\rangle}_{\langle s,r^{pq}\rangle}\big)^2 = P\big(\pi^{H_p}_{D_{p^2 q}}\big)^2 \times P\big(\pi^{H_1}_{H_p}\big)^2.$$

(3) Let $p, q, r \ge 3$ be distinct prime numbers. By Theorem 3.3, one has that the group D_{pqr} acts in genus $g = 1 + 2pqr - (qr + pr + p)$ with signature $(0; 2, 2, p, q, qr)$. Then, by Theorem 2.1, the analytic representation of the action is given by

$$\rho_a \cong 2W(pqr) \oplus 2W(pq) \oplus W(qr) \oplus W(pr) \oplus W(q),$$

and hence $Q_\theta = \{pqr, pq, qr, pr, q\}$. Then,

$$\begin{gathered} L_{Q_\theta}(pqr) = \operatorname{lcm}(pq, qr, pr, q) = pqr, \\ L_{Q_\theta}(pq) = L_{Q_\theta}(qr) = q \text{ and } L_{Q_\theta}(pr) = L_{Q_\theta}(q) = 1. \end{gathered}$$

It follows from Theorem 4.4 that $B(pqr)$ is not isogenous to the Prym variety of any intermediate cover.

4.4 Completely decomposable Jacobians

An abelian variety is *completely decomposable* if it is isogenous to a product of elliptic curves. Ekedahl and Serre in [14] posed the following two questions:

Question 1. Is it true that, for all positive integers g, there exists a curve of genus g whose Jacobian is completely decomposable?

Question 2. Is the set of genera for which a curve with completely decomposable Jacobian exists infinite?

Since the publication of [14], the study of completely decomposable Jacobians has attracted considerable interest. See for example [10, 13, 18, 25, 29]. Despite recent advancements, the previous questions still remain open.

Notably, Ekedahl and Serre provided a list of genera (up to 1297) with curves exhibiting completely decomposable Jacobians, but significant gaps remained. To address this, Paulhus and Rojas in [30] used computational tools to systematically construct examples in new genera, relying on the group algebra decomposition of Jacobian varieties. See also the recent article [36].

Here we determine all Riemann surfaces with dihedral action such that the group algebra decomposition yields a complete decomposition of the Jacobian.

Theorem 4.5. *Let $n \geq 3$ be a positive integer, and let S be a compact Riemann surface of genus $g \geq 2$ with a D_n-action. The group algebra decomposition (with respect to D_n) provides a complete decomposition of JS if and only if one of the following cases occurs.*

n	*genus*	*signature*	*geometric signature*	*complete decomposition of JS*
3	2	$(0;2,2,3,3)$		$B(3)^2$
	3	$(0;2,2,2,2,3)$		$B_2 \times B(3)^2$
		$(1;3)$		$JS_{D_3} \times B(3)^2$
	4	$(1;2,2)$		$JS_{D_3} \times B_2 \times B(3)^2$
4	2	$(0;2,2,2,4)$	$(0;\langle s\rangle,\langle sr\rangle,\langle r^2\rangle,\langle r\rangle)$	$B(4)^2$
	3	$(1;2)$	$(1;\langle r^2\rangle)$	$JS_{D_4} \times B(4)^2$
		$(0;2,2,2,2,2)$	$(0;\langle s\rangle^2,\langle sr\rangle^2,\langle r^2\rangle)$	$B_2 \times B(4)^2$
		$(0;2,2,4,4)$	$(0;\langle sr\rangle^2,\langle r\rangle^2)$	$B_3 \times B(4)^2$
			$(0;\langle s\rangle^2,\langle r\rangle^2)$	$B_4 \times B(4)^2$
	4	$(0;2,2,4)$	$(0;\langle s\rangle,\langle sr\rangle^3,\langle r\rangle)$	$B_2 \times B_3 \times B(4)^2$
			$(0;\langle s\rangle^3,\langle sr\rangle,\langle r\rangle)$	$B_2 \times B_4 \times B(4)^2$
	5	$(1;2)$	$(1;\langle sr\rangle^2)$	$JS_{D_4} \times B_2 \times B_3 \times B(4)^2$
			$(1;\langle s\rangle^2)$	$JS_{D_4} \times B_2 \times B_4 \times B(4)^2$
6	2	$(0;2,2,2,3)$	$(0;\langle s\rangle,\langle sr\rangle,\langle r^3\rangle,\langle r^2\rangle)$	$B(6)^2$
	3	$(0;2,2,2,6)$	$(0;\langle sr\rangle^2,\langle r^3\rangle,\langle r\rangle)$	$B_3 \times B(6)^2$
			$(0;\langle s\rangle^2,\langle r^3\rangle,\langle r\rangle)$	$B_4 \times B(6)^2$
	4	$(0;2^5)$	$(0;\langle s\rangle,\langle sr\rangle^3,\langle r^3\rangle)$	$B_2 \times B_3 \times B(6)^2$
			$(0;\langle s\rangle^3,\langle sr\rangle,\langle r^3\rangle)$	$B_2 \times B_4 \times B(6)^2$
		$(0;2,2,3,6)$	$(0;\langle s\rangle,\langle sr\rangle,\langle r^2\rangle,\langle r\rangle)$	$B(3)^2 \times B(6)^2$
	5	$(1;3)$	$(1;\langle r^2\rangle)$	$JS_{D_6} \times B(3)^2 \times B(6)^2$
		$(0;2,2,2,2,3)$	$(0;\langle s\rangle^2,\langle sr\rangle^2,\langle r^2\rangle)$	$B_2 \times B(3)^2 \times B(6)^2$
		$(0;2,2,6,6)$	$(0;\langle sr\rangle^2,\langle r\rangle^2)$	$B_3 \times B(3)^2 \times B(6)^2$
			$(0;\langle s\rangle^2,\langle r\rangle^2)$	$B_4 \times B(3)^2 \times B(6)^2$
	6	$(0;2,2,2,2,6)$	$(0;\langle s\rangle,\langle sr\rangle^3,\langle r\rangle)$	$B_2 \times B_3 \times B(3)^2 \times B(6)^2$
			$(0;\langle s\rangle^3,\langle sr\rangle,\langle r\rangle)$	$B_2 \times B_4 \times B(3)^2 \times B(6)^2$
	7	$(1;2,2)$	$(1;\langle sr\rangle^2)$	$JS_{D_6} \times B_2 \times B_3 \times B(3)^2 \times B(6)^2$
			$(1;\langle s\rangle^2)$	$JS_{D_6} \times B_2 \times B_4 \times B(3)^2 \times B(6)^2$

Proof. According to Proposition 4.2, if ρ_a is the analytic representation of an action of D_n then $\dim B(n) = \frac{1}{2}\phi(n)\langle \rho_a, \rho^1 \rangle$. We observe that $\phi(n) = 2$ if and only if $n \in \{3,4,6\}$. This fact coupled with Lemma 3.3, which states that $\langle \rho_a, \rho^1 \rangle \geq 1$ in genus $g \geq 2$, imply that $\dim B(n) \geq 2$ for $n \neq 3,4,6$.

Now, we will determine all the (geometric) signatures of D_3, D_4, and D_6 whose associated group algebra decomposition of JS provides a complete decomposition of JS. Our approach will be to list all the "potential analytic representations" and use Theorem 3.6 and Theorem 3.7 to determine which among them actually are the analytic representation of a dihedral action. Then, obtaining the (geometric) signature is a straightforward application of Proposition 2.6 and Proposition 2.7.

(1) Consider the group D_3. Assume that the group algebra decomposition,

$$JS \sim JS_{D_3} \times B_2 \times B(3)^2,$$

with respect to D_3 provides a complete decomposition of JS. Then, by Proposition 4.1, the multiplicities of the irreducible $\mathbb{C}$-representations in ρ_a are zero or one. That is, ρ_a is equivalent to one of the following cases:

(i) ρ^1

(ii) $\psi_1 \oplus \rho^1$

(iii) $\psi_2 \oplus \rho^1$

(iv) $\psi_1 \oplus \psi_2 \oplus \rho^1$

Observe that

$$\widetilde{\Phi}_{\rho_a}(3) = \Phi_{\rho_a}(3) = \langle \rho_a, \rho^1 \rangle - \langle \rho_a, \psi_1 \oplus \psi_2 \rangle + 1.$$

By Theorem 3.6, each of the representations above is the analytic representation of some D_3-action. For the sake of example, we include the computation of the signature $\sigma = (1;3)$ associated to $\rho_a \cong \psi_1 \oplus \rho^1$. By Proposition 2.6 we have that $\sigma = (\gamma; 2^t, 3^l)$, where

$$\begin{aligned}
\gamma &= \langle \rho_a, \psi_1 \rangle = 1, \\
t &= 2\langle \rho_a, \psi_2 \rangle - 2\langle \rho_a, \psi_1 \rangle + 2 = 0, \\
l &= \widetilde{\Phi}_{\rho_a}(3) = 1.
\end{aligned}$$

(2) Consider the group D_4. Assume that the group algebra decomposition,

$$JS \sim JS_{D_4} \times B_2 \times B_3 \times B_4 \times B(4)^2,$$

with respect to D_4 provides a complete decomposition of JS. As before, the multiplicities of the irreducible $\mathbb{C}$-representations in ρ_a are zero or one. That is, ρ_a is equivalent to one of the following cases:

(i) ρ^1
(ii) $\psi_1 \oplus \rho^1$
(iii) $\psi_2 \oplus \rho^1$
(iv) $\psi_3 \oplus \rho^1$
(v) $\psi_4 \oplus \rho^1$
(vi) $\psi_1 \oplus \psi_2 \oplus \rho^1$
(vii) $\psi_1 \oplus \psi_3 \oplus \rho^1$
(viii) $\psi_1 \oplus \psi_4 \oplus \rho^1$
(ix) $\psi_2 \oplus \psi_3 \oplus \rho^1$
(x) $\psi_2 \oplus \psi_4 \oplus \rho^1$
(xi) $\psi_3 \oplus \psi_4 \oplus \rho^1$
(xii) $\psi_1 \oplus \psi_2 \oplus \psi_3 \oplus \rho^1$
(xiii) $\psi_1 \oplus \psi_2 \oplus \psi_4 \oplus \rho^1$
(xiv) $\psi_1 \oplus \psi_3 \oplus \psi_4 \oplus \rho^1$
(xv) $\psi_2 \oplus \psi_3 \oplus \psi_4 \oplus \rho^1$
(xvi) $\psi_1 \oplus \psi_2 \oplus \psi_3 \oplus \psi_4 \oplus \rho^1$

Observe that

$$\widetilde{\Phi}_{\rho_a}(4) = \Phi_{\rho_a}(4) - \Phi_{\rho_a}(2) = \langle \rho_a, \psi_3 \oplus \psi_4 \rangle - \langle \rho_a, \psi_1 \oplus \psi_2 \rangle + 1,$$
$$\widetilde{\Phi}_{\rho_a}(2) = \Phi_{\rho_a}(2) = \langle \rho_a, \rho^1 \rangle - \langle \rho_a, \psi_3 \oplus \psi_4 \rangle.$$

By Theorem 3.7, the representations (vi), (vii), (viii), (xi), (xiv), (xv) and (xvi) are not the analytic representation of any D_4-action. More precisely: the cases (vii) and (viii) do not satisfy statement (1) of Theorem 3.7; and cases (vi), (xi), (xiv), (xv) and (xvi) do not satisfy statement (2) of Theorem 3.7. The remaining statements of Theorem 3.7 hold in each case. We conclude that the remaining representations are the analytic representation of some D_4-action.

(3) Consider the group D_6. Assume that the group algebra decomposition,

$$JS \sim JS_{D_6} \times B_2 \times B_3 \times B_4 \times B(3)^2 \times B(6)^2,$$

with respect to D_6 provides a complete decomposition of JS. As before, the multiplicities of the irreducible $\mathbb{C}$-representations in ρ_a are zero or one. That is, ρ_a is equivalent to one of the following cases:

(i) ρ^1
(ii) $\psi_1 \oplus \rho^1$
(iii) $\psi_2 \oplus \rho^1$
(iv) $\psi_3 \oplus \rho^1$
(v) $\psi_4 \oplus \rho^1$
(vi) $\psi_1 \oplus \psi_2 \oplus \rho^1$
(vii) $\psi_1 \oplus \psi_3 \oplus \rho^1$
(viii) $\psi_1 \oplus \psi_4 \oplus \rho^1$
(ix) $\psi_2 \oplus \psi_3 \oplus \rho^1$
(x) $\psi_2 \oplus \psi_4 \oplus \rho^1$
(xi) $\psi_3 \oplus \psi_4 \oplus \rho^1$
(xii) $\psi_1 \oplus \psi_2 \oplus \psi_3 \oplus \rho^1$

(xiii) $\psi_1 \oplus \psi_2 \oplus \psi_4 \oplus \rho^1$

(xiv) $\psi_1 \oplus \psi_3 \oplus \psi_4 \oplus \rho^1$

(xv) $\psi_2 \oplus \psi_3 \oplus \psi_4 \oplus \rho^1$

(xvi) $\psi_1 \oplus \psi_2 \oplus \psi_3 \oplus \psi_4 \oplus \rho^1$

(xvii) $\rho^2 \oplus \rho^1$

(xviii) $\psi_1 \oplus \rho^2 \oplus \rho^1$

(xix) $\psi_2 \oplus \rho^2 \oplus \rho^1$

(xx) $\psi_3 \oplus \rho^2 \oplus \rho^1$

(xxi) $\psi_4 \oplus \rho^2 \oplus \rho^1$

(xxii) $\psi_1 \oplus \psi_2 \oplus \rho^2 \oplus \rho^1$

(xxiii) $\psi_1 \oplus \psi_3 \oplus \rho^2 \oplus \rho^1$

(xxiv) $\psi_1 \oplus \psi_4 \oplus \rho^2 \oplus \rho^1$

(xxv) $\psi_2 \oplus \psi_3 \oplus \rho^2 \oplus \rho^1$

(xxvi) $\psi_2 \oplus \psi_4 \oplus \rho^2 \oplus \rho^1$

(xxvii) $\psi_3 \oplus \psi_4 \oplus \rho^2 \oplus \rho^1$

(xxviii) $\psi_1 \oplus \psi_2 \oplus \psi_3 \oplus \rho^2 \oplus \rho^1$

(xxix) $\psi_1 \oplus \psi_2 \oplus \psi_4 \oplus \rho^2 \oplus \rho^1$

(xxx) $\psi_1 \oplus \psi_3 \oplus \psi_4 \oplus \rho^2 \oplus \rho^1$

(xxxi) $\psi_2 \oplus \psi_3 \oplus \psi_4 \oplus \rho^2 \oplus \rho^1$

(xxxii) $\psi_1 \oplus \psi_2 \oplus \psi_3 \oplus \psi_4 \oplus \rho^2 \oplus \rho^1$

Observe that

$$\begin{aligned}
\widetilde{\Phi}_{\rho_a}(6) &= \Phi_{\rho_a}(6) - \Phi_{\rho_a}(3) - \Phi_{\rho_a}(2) \\
&= \langle \rho_a, \rho^2 \oplus \psi_3 \oplus \psi_4 \rangle - \langle \rho_a, \rho^1 \oplus \psi_1 \oplus \psi_2 \rangle + 1 \\
\widetilde{\Phi}_{\rho_a}(3) &= \Phi_{\rho_a}(3) = \langle \rho_a, \rho^1 \rangle - \langle \rho_a, \psi_3 \oplus \psi_4 \rangle \\
\widetilde{\Phi}_{\rho_a}(2) &= \Phi_{\rho_a}(2) = \langle \rho_a, \rho^1 \rangle - \langle \rho_a, \rho^2 \rangle
\end{aligned}$$

By Theorem 3.7, the representations (ii), (iii), (vi), (vii), (viii), (xi), (xii), (xiii), (xiv), (xv), (xvi), (xxii), (xxiii), (xxiv), (xxvii), (xxx), (xxxi), and (xxxii) are not the analytic representation of any D_6-action. More precisely: the cases (vii), (viii), (xxiii) and (xxiv) do not satisfy Statement (1) of Theorem 3.7; the cases (ii), (iii), (vi), (xi), (xii), (xiii), (xiv), (xv), (xvi), (xxii), (xxvii), (xxx), (xxxi) and (xxxii) do not satisfy Statement (2) of Theorem 3.7; and the cases (vii) and (viii) do not satisfy Statement (6) of Theorem 3.7. The remaining statements of Theorem 3.7 hold in each case. We conclude that the remaining representations are the analytic representation of some D_6-action.

This ends the proof. □

The following corollary is a quick consequence of the theorem above.

Corollary 4.2. *If $n \geq 3$ is different from 3, 4, and 6 then the group algebra decomposition with respect to each action of D_n does not provide a complete decomposition of the Jacobian.*

A generalization of the Ekedahl-Serre problem

An interesting generalization of the Ekedahl-Serre problem involves seeking decompositions of Jacobians where each subvariety in the decomposition has the same dimension.

Definition 4.5. Let k be a nonnegative integer. An abelian variety is k-*decomposable* if it is isogenous to a product of abelian varieties of dimension k. We say that the product is a k-*decomposition* of JS.

Lemma 4.3. *Let D_n act in genus $g \geq 2$, and let ρ_a be the analytic representation of the action. For $q \in \mathbb{Z}^{|n} \setminus \{1, 2\}$:*

(1) $\langle \rho_a, \rho^1 \rangle \geq \langle \rho_a, \rho^{n/q} \rangle$*;*

(2) $\dim B(n) \geq 1$*;*

(3) $\dim B(n) \geq \dim B(q)$*;*

(4) if $\dim B(n) = \dim B(q)$ *then* $q \in \{n, \frac{n}{2}\}$ *for* $n \in 2\mathbb{Z} \setminus 4\mathbb{Z}$*, and* $q = n$ *otherwise.*

Proof. Recall that $\dim B(q) = \frac{1}{2}\phi(q)\langle \rho_a, \rho^{n/q} \rangle$. To prove Statement (1) note that $\widehat{\Psi}_\theta(n) \geq \widehat{\Psi}_\theta(n) - \widehat{\Psi}_\theta(\frac{n}{q})$ and apply Theorem 2.1 and Theorem 2.2. Statement (2) follows from (1) and the fact that $\phi(n) \geq \phi(q)$. Statement (3) is a quick consequence of Lemma 3.3.

Let us prove Statement (4). By (1) and (2) one has that $\dim B(n) = \dim B(q)$ if and only if $\phi(n) = \phi(q)$ and $\langle \rho_a, \rho^1 \rangle = \langle \rho_a, \rho^{n/q} \rangle$. To conclude, we observe that $\phi(n) = \phi(q)$ implies that $n = q$ or $n = 2q$ (for q odd). □

Proposition 4.5. *Let S be a compact Riemann surface of genus $g \geq 2$ with a D_n-action. If the group algebra decomposition of JS with respect to D_n yields a k-decomposition of JS, then $k = \dim B(n)$ is a multiple of $\frac{1}{2}\phi(n)$.*

Proof. Assume that the group algebra decomposition yields a k-decomposition of JS. Then, $\dim B(n)$ equals zero or k. By Lemma 4.3 one has that $\dim B(n) \geq 1$ and hence $k = \dim B(n)$. □

Theorem 4.6. *Let $n \geq 3$ be a positive integer, and let S be a compact Riemann surface of genus $g \geq 2$ with a D_n-action. For $k \geq 2$, the group algebra decomposition (with respect to D_n) provides a k-decomposition of JS if and only if one of the following cases occurs.*

n	*genus g*	*g/k*	*signature*	*geometric signature*	*k-decomposition of JS*
3	$2m$	2	$(0; 2, 2, 3^{m+1})$		$B(3)^2$
	$3m$	3	$(0; 2^{2m+2}, 3)$		$B_2 \times B(3)^2$
4	$2m$	2	$(0; 2^{m+2}, 4)$	$(0; \langle s\rangle, \langle sr\rangle, \langle r^2\rangle^m, \langle r\rangle)$	$B(4)^2$
	$4m$	4	$(0; 2^{2m+2}, 4)$	$(0; \langle s\rangle, \langle sr\rangle^{2m+1}, \langle r\rangle)$	$B_2 \times B_3 \times B(4)^2$
	$4m$	4		$(0; \langle s\rangle^{2m+1}, \langle sr\rangle, \langle r\rangle)$	$B_2 \times B_4 \times B(4)^2$
5	$4m$	2	$(0; 2, 2, 5^{m+1})$		$B(5)^2$
	6	3	$(0; 2^6)$		$B_2 \times B(5)^2$
6	$4m$	4	$(0; 2, 2, 3^m, 6)$	$(0; \langle s\rangle, \langle sr\rangle, \langle r^2\rangle^m, \langle r\rangle)$	$B(3)^2 \times B(6)^2$
	$6m$	6	$(0; 2^{2m+2}, 6)$	$(0; \langle s\rangle, \langle sr\rangle^{2m+1}, \langle r\rangle)$	$B_2 \times B_3 \times B(3)^2 \times B(6)^2$
	$6m$	6		$(0; \langle s\rangle^{2m+1}, \langle sr\rangle, \langle r\rangle)$	$B_2 \times B_4 \times B(3)^2 \times B(6)^2$
p	$(p-1)m$	2	$(0; 2, 2, p^{m+1})$		$B(p)^2$
p^e	$p^{e-1}(p-1)m$	2	$(0; 2, 2, p^m, (p^e))$		$B(p^e)^2$
pq	$(p-1)(q-1)$	2	$(0; 2, 2, p, q)$		$B(pq)^2$
2^e	$2^{e-1}m$	2	$(0; 2^{m+2}, (2^e))$	$(0; \langle s\rangle, \langle sr\rangle, \langle r^{2^{e-1}}\rangle^m, \langle r\rangle)$	$B(2^e)^2$
$2p$	$p-1$	2	$(0; 2, 2, 2, p)$	$(0; \langle s\rangle, \langle sr\rangle, \langle r^p\rangle, \langle r^2\rangle)$	$B(2p)^2$
	$2(p-1)m$	4	$(0; 2, 2, p^m, 2p)$,	$(0; \langle s\rangle, \langle sr\rangle, \langle r^2\rangle^m, \langle r\rangle)$	$B(p)^2 \times B(2p)^2$

Here m is a positive integer; if $n \in \{3, 4, 5\}$ then $m \geq 2$. For the last five rows of the table above we make, in descending order, the following assumptions: (i) $p \geq 7$ is prime; (ii) $p \geq 3$ is prime and $e \geq 2$; (iii) $p, q \geq 3$ are distinct primes; (iv) $e \geq 3$; and (v) $p \geq 5$ is prime. Also, for $n = p^e$, we introduce the notation (p^e) to emphasize that the integer p^e appears one time in the signature. Similarly for $n = 2^e$.

Proof. Let us first prove that if for some $k \in \mathbb{Z}_+$ the group algebra decomposition (with respect to D_n) provides a k-decomposition of JS, then n is either a prime power or the product of two primes.

Let ρ_a be the analytic representation of a D_n-action. Fix $m = \langle \rho_a, \rho^1 \rangle$, which is a positive integer by Lemma 3.3. We proceed by contradiction. Assume that n is not a prime power nor the product of two primes. There are two cases:

(1) Set $n \notin 2\mathbb{Z} \setminus 4\mathbb{Z}$. By Lemma 4.3 one has that $\langle \rho_a, \rho^{n/q} \rangle = 0$ for all $q \in \mathbb{Z}^{|n} \setminus \{1, 2, n\}$. It follows that

$$\Phi_{\rho_a}(q) = \begin{cases} m - \langle \rho_a, \psi_1 \oplus \psi_2 \rangle + 1, & \text{if } (n, q) = n \\ m - \langle \rho_a, \psi_3 \oplus \psi_4 \rangle, & \text{if } n \text{ is even and } (n, q) = \frac{n}{2} \\ m, & 1 < (n, q) < \frac{n}{2} \\ 0, & (n, q) = 1 \end{cases}$$

If n is odd then there are distinct odd primes $p_1, p_2 \in \mathbb{Z}^{|n}$ such that $n \neq p_1 p_2$, hence $\widetilde{\Phi}_{\rho_a}(p_1 p_2) = -m < 0$. Thus, Statement (2) of Theorem 3.6 is not satisfied and therefore ρ_a is not the analytic representation of a D_n-action, a contradiction. If n is even then there is an odd prime

$p \in \mathbb{Z}^{|n}$ such that $n \neq 2p$, hence $\widetilde{\Phi}_{\rho_a}(2p) \leq -m < 0$. Similarly, we reach a contradiction by Statement (2) of Theorem 3.7.

(2) Set $n \in 2\mathbb{Z} \setminus 4\mathbb{Z}$. Note that $\frac{n}{2}$ is odd. By Lemma 4.3 one has that $\langle \rho_a, \rho^{n/q} \rangle = 0$ for all $q \in \mathbb{Z}^{|n} \setminus \{1, 2, \frac{n}{2}, n\}$, and $\langle \rho_a, \rho^2 \rangle = 0$ or m. It follows that

$$\Phi_{\rho_a}(q) = \begin{cases} m - \langle \rho_a, \psi_1 \oplus \psi_2 \rangle + 1, & \text{if } (n,q) = n \\ m - \langle \rho_a, \psi_3 \oplus \psi_4 \rangle, & \text{if } n \text{ is even and } (n,q) = \frac{n}{2} \\ m, & 2 < (n,q) < \frac{n}{2} \\ m - \langle \rho_a, \rho^2 \rangle, & (n,q) = 2 \\ 0, & (n,q) = 1 \end{cases}$$

If there are distinct odd prime numbers $p_1, p_2 \in \mathbb{Z}^{|n}$, then $\widetilde{\Phi}_{\rho_a}(p_1 p_2) < 0$ and we conclude as in (1). Now, assume that $n = 2p^e$ for some odd prime p and $e \geq 2$. If $e \geq 3$ then $\widetilde{\Phi}_{\rho_a}(2p^{e-1}) = -m < 0$, a contradiction. If $e = 2$ then $n = 2p^2$, and

$$\begin{aligned} \widetilde{\Phi}_{\rho_a}(2p^2) &= \langle \rho_a, \psi_3 \oplus \psi_4 \rangle - \langle \rho_a, \psi_1 \oplus \psi_2 \rangle + 1 - m, \\ \widetilde{\Phi}_{\rho_a}(p^2) &= -\langle \rho_a, \psi_3 \oplus \psi_4 \rangle, \\ \widetilde{\Phi}_{\rho_a}(2p) &= \langle \rho_a, \rho^2 \rangle - m, \\ \widetilde{\Phi}_{\rho_a}(p) &= m, \\ \widetilde{\Phi}_{\rho_a}(2) &= m - \langle \rho_a, \rho^2 \rangle. \end{aligned}$$

We observe that Statement (2) of Theorem 3.7 tells us that $\widetilde{\Phi}_{\rho_a}(q) \geq 0$ for $q \in \{2p^2, p^2, 2p, p, 2\}$. Then, $\langle \rho_a, \psi_j \rangle = 0$ for $j = 1, \ldots, 4$, $\langle \rho_a, \rho^2 \rangle = m$ and $m = 1$. However, $\operatorname{lcm}(\operatorname{Supp} \widetilde{\Phi}_{\rho_a}) = p \neq 2p^2$, a contradiction by Statement (4) of Theorem 3.7.

Now, if $n \geq 3$ is a prime power or a product of two primes, then one of the following cases occurs:

(1) $n = 3$,

(2) $n = 4$,

(3) $n = 5$,

(4) $n = 6$,

(5) $n = p$ for $p \geq 7$ prime,

(6) $n = p^e$ for $p \geq 3$ prime and $e \geq 2$,

(7) $n = pq$ for $p, q \geq 3$ distinct primes,

(8) $n = 2^e$ for $e \geq 3$,

(9) $n = 2p$ for $p \geq 5$ prime.

For each of the cases above, we will determine all the (geometric) signatures of D_n whose associated group algebra decomposition provides a k-decomposition of the Jacobian for some $k \geq 2$.

Our approach will be to apply Theorem 3.6 and Theorem 3.7 to find all the compatible analytic representations. Then, as in the proof of Theorem 4.5, obtaining the (geometric) signature is a straightforward application of Proposition 2.6 and Proposition 2.7.

Let ρ_a be the analytic representation of a D_n-action as above. Proposition 4.5 states that $k = \frac{1}{2}\phi(n)m$ with $m = \langle \rho_a, \rho^1 \rangle$ a positive integer. Proposition 4.2 together with Lemma 4.3 imply that $\langle \rho_a, \rho^{n/q} \rangle = 0$ for $q \in \mathbb{Z}^{|n} \setminus \{1, 2, n\}$; unless $n \in 2\mathbb{Z} \setminus 4\mathbb{Z}$, in which case we can also have $\langle \rho_a, \rho^2 \rangle = m$. Moreover, by Theorem 3.6(3) and Theorem 3.7(3), $\langle \rho_a, (\rho^{n/q})^\sigma \rangle = \langle \rho_a, \rho^{(n,h)} \rangle$ for each element σ of the Galois group of $\rho^{n/q}$. In summary:

$$\langle \rho_a, \rho^h \rangle = \begin{cases} m, & \text{if } (n,h) = 1, \\ 0 \text{ or } m, & \text{if } n \in 2\mathbb{Z} \setminus 4\mathbb{Z} \text{ and } (n,h) = \frac{n}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Also, by Proposition 4.2 one has that $\langle \rho_a, \psi_j \rangle = 0$ or k. Finally, we recall that $W(q) = \oplus_\sigma (\rho^{n/q})^\sigma$.

(1) Assume that $n = 3$. In this case, $k = m \geq 2$ and

$$\widetilde{\Phi}_{\rho_a}(3) = m - \langle \rho_a, \psi_1 \oplus \psi_2 \rangle + 1.$$

Statement (2) of Theorem 3.6 says that $\widetilde{\Phi}_{\rho_a}(3) \geq 0$. That is, $\langle \rho_a, \psi_1 \oplus \psi_2 \rangle \leq m + 1$. Thus, either $\langle \rho_a, \psi_1 \oplus \psi_2 \rangle = 0$ or m. By Statement (1) of Theorem 3.6, which tells us that $\langle \rho_a, \psi_2 \rangle + 1 \geq \langle \rho_a, \psi_1 \rangle$, if $\langle \rho_a, \psi_1 \oplus \psi_2 \rangle = m$ then $\langle \rho_a, \psi_1 \rangle = 0$ and $\langle \rho_a, \psi_2 \rangle = m$. We conclude, by Theorem 3.6, that the compatible analytic representations are:

(i) $mW(3)$,

(ii) $m\psi_2 \oplus mW(3)$.

By Proposition 3.6, the signature $(\gamma; 2^t, 3^l)$ associated to ρ_a is given by

$$\gamma = \langle \rho_a, \psi_1 \rangle, \quad t = 2\langle \rho_a, \psi_2 \rangle - 2\langle \rho_a, \psi_1 \rangle + 2, \quad \text{and} \quad l = \widetilde{\Phi}_{\rho_a}(3).$$

For instance, if $\rho_a \cong mW(3)$ then $\gamma = 0$, $t = 2$, and $l = m+1$. We conclude that the representations above have signatures $(0; 2, 2, 3^{m+1})$ and $(0; 2^{2m+2}, 3)$, respectively.

(2) Assume that $n = 4$. Then, $k = m \geq 2$ and

$$\widetilde{\Phi}_{\rho_a}(4) = \langle \rho_a, \psi_3 \oplus \psi_4 \rangle - \langle \rho_a, \psi_1 \oplus \psi_2 \rangle + 1,$$
$$\widetilde{\Phi}_{\rho_a}(2) = m - \langle \rho_a, \psi_3 \oplus \psi_4 \rangle.$$

Statement (2) of Theorem 3.7 requires that $\widetilde{\Phi}_{\rho_a}(4), \widetilde{\Phi}_{\rho_a}(2) \geq 0$. In particular, $\langle \rho_a, \psi_3 \oplus \psi_4 \rangle \leq m$. If $\langle \rho_a, \psi_3 \oplus \psi_4 \rangle = 0$ then $\langle \rho_a, \psi_1 \oplus \psi_2 \rangle = 0$, because $\widetilde{\Phi}_{\rho_a}(4) \geq 0$. By Statement (1) of Theorem 3.7, which says that $\langle \rho_a, \psi_2 \rangle + 1 \geq \langle \rho_a, \psi_1 \rangle + m$, if $\langle \rho_a, \psi_3 \oplus \psi_4 \rangle = m$ then $\langle \rho_a, \psi_1 \rangle = 0$ and $\langle \rho_a, \psi_2 \rangle = m$. We conclude, by Theorem 3.7, that the compatible analytic representations are:

(i) $mW(4)$,

(ii) $m\psi_2 \oplus m\psi_3 \oplus mW(4)$,

(iii) $m\psi_2 \oplus m\psi_4 \oplus mW(4)$.

By Proposition 2.3, the geometric signature $(\gamma; \langle s \rangle^a, \langle sr \rangle^b, \langle r^2 \rangle^{l_1}, \langle r \rangle^{l_2})$ associated to ρ_a is given by

$$\gamma = \langle \rho_a, \psi_1 \rangle,$$
$$a = \langle \rho_a, \psi_2 \oplus \psi_4 \rangle - \langle \rho_a, \psi_1 \oplus \psi_3 \rangle + 1,$$
$$b = \langle \rho_a, \psi_2 \oplus \psi_3 \rangle - \langle \rho_a, \psi_2 \oplus \psi_4 \rangle + 1,$$
$$l_1 = \widetilde{\Phi}_{\rho_a}(2),$$
$$l_2 = \widetilde{\Phi}_{\rho_a}(4).$$

For instance, if $\rho_a \cong m\psi_2 \oplus m\psi_3 \oplus mW(4)$ then $\gamma = 0$, $a = 1$, $b = 2m+1$, $l_1 = 0$, and $l_2 = 1$. We conclude that the representations above have geometric signatures $(0; \langle s \rangle, \langle sr \rangle, \langle r^2 \rangle^m, \langle r \rangle)$, $(0; \langle s \rangle, \langle sr \rangle^{2m+1}, \langle r \rangle)$, and $(0; \langle s \rangle^{2m+1}, \langle sr \rangle, \langle r \rangle)$, respectively. Henceforth, we omit the (geometric) signature computations.

(3) Assume that $n = 5$. Then, $k = 2m$ and

$$\widetilde{\Phi}_{\rho_a}(5) = m - \langle \rho_a, \psi_1 \oplus \psi_2 \rangle + 1.$$

Statement (2) of Theorem 3.6 says that $\widetilde{\Phi}_{\rho_a}(5) \geq 0$. If $m \geq 2$ then $\langle \rho_a, \psi_1 \oplus \psi_2 \rangle = 0$. On the other hand, if $m = 1$ then $\langle \rho_a, \psi_1 \oplus \psi_2 \rangle \leq$

$k = 2$. Thus, by Statement (1) of Theorem 3.6, which tells us that $\langle \rho_a, \psi_2 \rangle + 1 \geq \langle \rho_a, \psi_1 \rangle$, one has that $\langle \rho_a, \psi_1 \oplus \psi_2 \rangle = 0$, or $\langle \rho_a, \psi_1 \rangle = 0$ and $\langle \rho_a, \psi_2 \rangle = 2$. We conclude, by Theorem 3.6, that the compatible analytic representations are:

(i) $mW(5)$,

(ii) $2\psi_2 \oplus W(5)$.

(4) Assume that $n = 6$. Then, $k = m \geq 2$ and

$$\begin{aligned}
\widetilde{\Phi}_{\rho_a}(6) &= \langle \rho_a, \psi_3 \oplus \psi_4 \oplus \rho^2 \rangle - \langle \rho_a, \psi_1 \oplus \psi_2 \rangle - m + 1, \\
\widetilde{\Phi}_{\rho_a}(3) &= m - \langle \rho_a, \psi_3 \oplus \psi_4 \rangle, \\
\widetilde{\Phi}_{\rho_a}(2) &= m - \langle \rho_a, \rho^2 \rangle.
\end{aligned}$$

Statement (2) of Theorem 3.7 requires that $\widetilde{\Phi}_{\rho_a}(q) \geq 0$ for $q = 2, 3, 6$. In particular, as $\widetilde{\Phi}_{\rho_a}(3) \geq 0$, $\langle \rho_a, \psi_3 \oplus \psi_4 \rangle \leq m$. If $\langle \rho_a, \psi_3 \oplus \psi_4 \rangle = 0$ then $\langle \rho_a, \psi_1 \oplus \psi_2 \rangle = 0$, and $\langle \rho_a, \rho^2 \rangle = m$, because $\widetilde{\Phi}_{\rho_a}(6) \geq 0$. Now, if $\langle \rho_a, \psi_3 \oplus \psi_4 \rangle = m$ then $\langle \rho_a, \psi_1 \rangle = 0$, $\langle \rho_a, \psi_2 \rangle = k$ and $\langle \rho_a, \rho^2 \rangle = k$. Indeed, this is a consequence of $\widetilde{\Phi}_{\rho_a}(6) \geq 0$ and Statement (1) of Theorem 3.7, which says that $\langle \rho_a, \psi_2 \rangle + 1 \geq \langle \rho_a, \psi_1 \rangle + m$. We conclude, by Theorem 3.7, that the compatible analytic representations are:

(i) $mW(3) \oplus mW(6)$,

(ii) $m\psi_2 \oplus m\psi_3 \oplus mW(3) \oplus mW(6)$,

(iii) $m\psi_2 \oplus m\psi_4 \oplus mW(3) \oplus mW(6)$.

(5) Assume that $n = p$ for $p \geq 7$ prime. Then, $k = \frac{1}{2}(p-1)m \geq 3m$ and

$$\widetilde{\Phi}_{\rho_a}(p) = m - \langle \rho_a, \psi_1 \oplus \psi_2 \rangle + 1.$$

Statement (2) of Theorem 3.6 says that $\widetilde{\Phi}_{\rho_a}(p) \geq 0$. In other words, $\langle \rho_a, \psi_1 \oplus \psi_2 \rangle \leq m + 1$. Since $\langle \rho_a, \psi_j \rangle = 0$ or $k \geq 3m$, it follows that $\langle \rho_a, \psi_1 \oplus \psi_2 \rangle = 0$. We conclude, by Theorem 3.6, that the compatible analytic representations are:

(i) $mW(p)$.

(6) Assume that $n = p^e$ for $p \geq 3$ prime and $e \geq 2$. Then, $k = \frac{1}{2}\phi(p^e)m \geq 3m$ and

$$\begin{aligned} \widetilde{\Phi}_{\rho_a}(p^e) &= 1 - \langle \rho_a, \psi_1 \oplus \psi_2 \rangle, \\ \widetilde{\Phi}_{\rho_a}(p^j) &= 0 \text{ for } 1 < j < e, \\ \widetilde{\Phi}_{\rho_a}(p) &= m. \end{aligned}$$

As before, Statement (2) of Theorem 3.6 implies that $\langle \rho_a, \psi_1 \oplus \psi_2 \rangle = 0$. We conclude, by Theorem 3.6, that the compatible analytic representations are:

(i) $mW(p^e)$.

(7) Assume that $n = pq$ for $p, q \geq 3$ distinct primes. Then, we have that $k = \frac{1}{2}(p-1)(q-1)m \geq 4m$ and

$$\begin{aligned} \widetilde{\Phi}_{\rho_a}(pq) &= 1 - m - \langle \rho_a, \psi_1 \oplus \psi_2 \rangle, \\ \widetilde{\Phi}_{\rho_a}(p) &= m, \\ \widetilde{\Phi}_{\rho_a}(q) &= m. \end{aligned}$$

By Statement (2) of Theorem 3.6, which requires that $\widetilde{\Phi}_{\rho_a}(pq) \geq 0$, one has that $m = 1$ and $\langle \rho_a, \rho_1 \oplus \psi_2 \rangle = 0$. We conclude, by Theorem 3.6, that the compatible analytic representations are:

(i) $W(pq)$.

(8) Assume that $n = 2^e$ for $e \geq 3$. Then, $k = 2^{e-2}m \geq 2m$ and

$$\begin{aligned} \widetilde{\Phi}_{\rho_a}(2^e) &= \langle \rho_a, \psi_3 \oplus \psi_4 \rangle - \langle \rho_a, \psi_1 \oplus \psi_2 \rangle + 1, \\ \widetilde{\Phi}_{\rho_a}(2^{e-1}) &= -\langle \rho_a, \psi_3 \oplus \psi_4 \rangle, \\ \widetilde{\Phi}_{\rho_a}(2^j) &= 0 \text{ for } 1 < j < e-1, \\ \widetilde{\Phi}_{\rho_a}(2) &= m. \end{aligned}$$

Statement (2) of Theorem3.7 says that $\widetilde{\Phi}_{\rho_a}(q) \geq 0$ for $q \in \mathbb{Z}^{|2^e} \setminus \{1\}$. Since $\widetilde{\Phi}_{\rho_a}(2^{e-1}) \geq 0$ it follows that $\langle \rho_a, \psi_3 \oplus \psi_4 \rangle = 0$. Also, $\langle \rho_a, \psi_1 \oplus \psi_2 \rangle = 0$ because $\widetilde{\Phi}_{\rho_a}(2^e) \geq 0$. We conclude, by Theorem 3.7, that the compatible analytic representations are:

(i) $mW(2^e)$.

(9) Assume that $n = 2p$ for $p \geq 5$ prime. Then, $k = \frac{1}{2}(p-1)m \geq 2m$ and

$$\begin{aligned}\widetilde{\Phi}_{\rho_a}(2p) &= \langle \rho_a, \psi_3 \oplus \psi_4 \oplus \rho^2 \rangle - \langle \rho_a, \psi_1 \oplus \psi_2 \oplus \rho^1 \rangle + 1 \\ \widetilde{\Phi}_{\rho_a}(p) &= \langle \rho_a, \rho^1 \rangle - \langle \rho_a, \psi_3 \oplus \psi_4 \rangle \\ \widetilde{\Phi}_{\rho_a}(2) &= \langle \rho_a, \rho^1 \rangle - \langle \rho_a, \rho^2 \rangle\end{aligned}$$

Statement (2) of Theorem 3.7 says that $\widetilde{\Phi}_{\rho_a}(q) \geq 0$ for $q = 2, p, 2p$. Since $\widetilde{\Phi}_{\rho_a}(p) \geq 0$ and $k \geq 2m$, it follows that $\langle \rho_a, \psi_3 \oplus \psi_4 \rangle = 0$. Also, $\langle \rho_a, \psi_1 \oplus \psi_2 \rangle = 0$ because $\widetilde{\Phi}_{\rho_a}(2p) \geq 0$. Moreover, if $\langle \rho_a, \rho^2 \rangle = 0$ then $m = 1$. We conclude, by Theorem 3.7, that the compatible analytic representations are:

(i) $W(2p)$,

(ii) $mW(p) \oplus mW(2p)$ for $m \geq 1$.

This concludes the proof. □

The following is a quick corollary of the previous theorem.

Corollary 4.3. *For $k \geq 2$, if n is neither a prime power nor the product of two primes, then the group algebra decomposition (with respect to any action of D_n) does not provide a k-decomposition of the Jacobian.*

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