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DYNAMICS ON THE THICK POINTS OF THE GAUSSIAN FREE FIELD

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RESUMEN DE LA TESIS PARA OPTAR AL GRADO DE
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Y MEMORIA PARA OPTAR AL TÍTULO DE
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POR: FELIPE ALEJANDRO ESPINOSA VERGARA
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DINAMICAS EN LOS PUNTOS ALTOS DEL CAMPO LIBRE GAUSSIANO

El campo libre gaussiano (GFF por sus siglas en ingles) bidimensional es la generalización del movimiento browniano cuando el tiempo se sustituye por un dominio espacial. En las últimas décadas, el GFF ha sido un objeto de gran interés en el estudio de la teoría de probabilidades debido a sus conexiones con la geometría conforme aleatoria. Aunque el GFF en sí no es una función sino más bien una distribución de Schwartz, hay una buena comprensión de sus propiedades geométricas. A pesar de esto, se sabe relativamente poco sobre las dinámicas para las cuales el GFF es la medida invariante.

En esta tesis, investigamos las características geométricas de dos dinámicas naturales distintas que conservan la GFF como su distribución estacionaria. En concreto, nos centramos en los denominados puntos altos. A grandes rasgos, tales puntos se pueden entender como aquellos donde el GFF es del orden de su varianza en vez de su desviación estandar. Estos puntos son de gran interés ya que codifican la medida de Liouville del campo. Sorprendentemente, el comportamiento de estos puntos altos varía significativamente en función de la dinámica impuesta al GFF.

La primera dinámica que exploramos es el que denominamos como Ornstein-Uhlenbeck GFF (OU-GFF). Este proceso se puede entender como un proceso Ornstein-Uhlenbeck a valores GFF. Nuestro principal objeto de estudio en este contexto en este contexto es la función de altura. Esta cuantifica como varia la altura de un punto en el espacio a través del tiempo. Demostramos que, casi seguramente, para todo punto en el espacio, su función de altura es continua en el tiempo. Más aun, tal familia de funciones forman una familia equicontinua.

El principal resultado para el OU-GFF se refiere a la existencia de puntos en el espacio cuya función de altura coincide con una función determinista dada f. Establecemos que la existencia de tales puntos viene determinada por una energía explícita $\mathcal{E}$. En concreto, si $\mathcal{E}(f) > 4$, tal punto no existe, mientras que si $\mathcal{E}(f) < 4$, casi seguramente existen infinitos puntos en el espacio con f como función de altura.

La segunda dinámica que examinamos es la solución de la ecuación del calor estocástica aditiva. En este caso, la función de altura no es continua. Sin embargo, demostramos que, a pesar de que un GFF típicamente no posee puntos más altos que 2, es posible encontrar puntos con cualquier altura en el rango $[0, 2\sqrt{2})$ bajo esta dinámica.

El resultado central para este segundo modelo implica el estudio de los momentos excepcionales en los que el campo exhibe puntos que tienen una altura estrictamente mayor a 2. Demostramos que, casi con seguridad, no hay momentos en los que el campo tenga infinitos puntos que sean estrictamente más que 2-altos. De hecho, identificamos infinitas transiciones de fase en $\gamma > 2$ convergentes a 2, correspondientes a momentos en los que hay a lo más N puntos en el espacio que son γ-altos.

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DYNAMICS ON THE THICK POINTS OF THE GAUSSIAN FREE FIELD

The 2-dimensional Gaussian Free Field (GFF) is the generalization of Brownian motion when time is replaced by a spatial domain. Over the past few decades, the GFF has become a central object of study in probability due to its profound connections with conformal geometry. Although the GFF itself is not a function and is best viewed as a Schwartz distribution, most of its geometric properties are well-understood. However, relatively little is known about the dynamics for which the GFF is an invariant measure.

In this thesis, we investigate the geometric features of two distinct natural dynamics that preserve the GFF as their stationary distribution. Specifically, we focus on the so-called thick points —locations where the local average of the field behaves like the variance rather than the standard deviation. These points are of interest because they encode the Liouville measure of the field. Surprisingly, the behavior of these thick points varies significantly depending on the dynamics imposed on the GFF.

The first dynamic we explore is what we call the Ornstein-Uhlenbeck GFF (OU-GFF). This process can be viewed as an Ornstein-Uhlenbeck with GFF values. Our primary object of study in this context is the thickness function, which quantifies how "thick" a point is at any given time and location in space. We prove that, almost surely, the thickness function is continuous in time for all points in space. Moreover, the family of thickness functions forms an equicontinuous family.

The main result for the OU-GFF concerns the existence of points in space whose thickness function matches a given deterministic function f. We establish that the existence of such points is determined by an explicit energy $\mathcal{E}$. Specifically, if $\mathcal{E}(f) > 4$, no such point exists, while if $\mathcal{E}(f) < 4$, there almost surely exists infinitely many points in space with f as its thickness function.

The second dynamic we examine is the solution to the additive stochastic heat equation. In this case, the thickness function is not continuous. Nevertheless, we demonstrate that, even though a GFF typically does not possess points thicker than 2, it is possible to find points with any thickness in the range $[0, 2\sqrt{2})$ under this dynamic.

The central result for this second model involves the study of exceptional times when the field exhibits points that are strictly more than 2-thick. We prove that, almost surely, there are no times at which the field has infinitely many points that are strictly more than 2-thick. In fact, we identify infinitely many phase transitions in $\gamma > 2$ converging to 2, corresponding to times when there are more than N points in space that are γ-thick.

There are no answers, only cross-references

Norbert Wiener-Law of libraries

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Table of Content

Introduction

The present thesis can be understood as a story that began with the following question

“***What happens to the geometry of a planar Gaussian free field when it evolves in time under a natural dynamic ?***”

To be more precise we work with Markovian processes whose stationary law is given by the Gaussian free field (GFF). As a naive first approach, the GFF can viewed as a generalization of the Brownian motion when the time domain is replaced by a multidimensional one.

The Gaussian free field is a universal object that appears as the scaling limit of many models, in particular, it is the scaling limit of the height function of the dimer model [Ken01], the fluctuation of the log of the characteristic polynomial of certain random matrices models [RV07], and as the scaling limit of the Integer value GFF [BPR22].

The Gaussian free field is ubiquitous in the study of two-dimensional planar geometry. However, it cannot be realized as a proper function. In fact, it is only a generalized function living in a Solobev space of negative exponent, which has made the study of its geometry quite technical.

The geometry of the Gaussian free field can be studied from many different perspectives. One can study its extreme values[1] as in [HMP10], and their relation with the Liouville measure as in [BL19], [DS11] and [DMS14]. One can also study its level sets as in [ALS20], [ASW19], or its flow lines as in [MS12].

As a geometric feature of study, we decided to focus on the set of thick points. Such choice is done since it is the simplest geometric aspect to study. At an intuitive level, a thick point can be though as a point where the size of the GFF around him is proportional to its variance instead of his standard deviation. This makes them highly unlikely. It is known that such points are also highly informative as one can see in [BSS14]. Therefore, the set thick point is not only a geometric aspect of interest, they are also of interest from a probabilistic point of view.

The results of this thesis is a proper study of the thick points under two given evolution. Such processes have the common property that their stationary law is a GFF. But the dynamic behavior on the thick points differs drastically as we will see.

From now on, we consider D as a subset of $\mathbb{R}^2$ open and bounded. We mention (and specify later) that we need a condition on ∂D in order to have certain results and properties related to the Poisson equation.

Orstein-Uhlenbeck evolution

As a first approach, we could first consider a process that generalizes the Brownian motion to infinite dimensions. In other words, we define a process $(X_t)_{t\geq 0}$, that at the beginning is

[1] In the literature such values are also known as the thick points

0, has independent increments, and for every $t > s$, $(X_t - X_s) \sim (t-s) \cdot GFF$. Then, we take the process defined as

$$\Phi(t) = e^{-\frac{1}{2}t} X_{e^t}.$$

This could be understood as an infinite dimensional Orstein-Uhlenbeck. Therefore, we now have a continuous process whose stationary law is a GFF. In order to understand from another perspective this process, one can easily check that is the solution of the following stochastic differential equation

$$(OU) \begin{cases} \partial_t \Phi = -\frac{1}{2}\Phi + W, \\ \Phi(0) = \Gamma. \end{cases}$$

Here we take as the initial condition a given GFF Γ. The random force to consider W is a noise that we will introduce properly later. But could be considered as a white noise of some Hilbert space of the form $L^2(\mathbb{R}, \mathcal{H})$. To our knowledge this process has not been studied before in the literature, from now on we call it OU-GFF.

To study the thick points through time on this dynamic, we define for each point in space $x \in D$ its thickness function γ_x as

$$\gamma_x(t) = \limsup_{\varepsilon \to 0} \frac{\Phi_\varepsilon(x,t)}{\ln\left(\frac{1}{\varepsilon}\right)},$$

where Φ_ε indicates an approximation of the field via certain convolution. One can understand this function as the fluctuation in time of the thickness of a point. We first show in Proposition 2.2 that such fluctuations are always continuous.

As a first direct corollary of the continuity, is that the thickness function of a given point in space $x \in D$ is always 0. From this, in order to find an interesting dynamic behavior, one could look to the thickness function of a point sampled from the Liouville measure related to the initial condition. In Lemma 2.3, we prove that such points have an exponential decay in time.

Motivated by the last result mentioned above. We ask ourselves for which functions, there exist a point in space whose thickness function is the desire one. In order to answer such question we have to introduce the following energy for a given f

$$\mathcal{E}(f) = \int_{\mathbb{R}} |f' + \frac{1}{2}f|^2 ds.$$

The above energy it is related to the deterministic version of (OU) when the space domain is a point.

We prove that when $\mathcal{E}(f) < 4$ almost surely there exists a point in space whose thickness function is given by f and the set $T^f = \{x \in D;\ \gamma_x = f\}$, has Hausdorff dimension given by

$$dim_H(T^f) = 2 - \frac{\mathcal{E}(f)}{2}.$$

To prove this, we introduce what we call the integrated field

$$h = \int_{\mathbb{R}} \Phi(s)\varphi(s) ds.$$

For some φ given. One can easily prove that h is a GFF up to a constant $c(\varphi)$. From here, we use the theory of GMC of the GFF to conclude the existence of those points. One can see that a typical point for this measure has the desired thickness function, and under the correct choice of φ one has that $c(\varphi)^2 = \mathcal{E}(f)$.

One can also prove that when $\mathcal{E}(f) > 4$, no such point in space exists. This is $T^f = \emptyset$. Broadly speaking, to prove this one check that the probability of finding a point in a space-box whose thickness function is given by f is smaller than the number of boxes used to cover the space.

The heat equation model

The other process we study may be a more natural one for the GFF. This evolution is given by the additive stochastic heat equation

$$(SHE) \begin{cases} \partial_t \Phi = \frac{1}{2}\Delta \Phi + \xi, \\ \Phi(0) = \Gamma. \end{cases}$$

Here the initial condition Γ has the law of a GFF and ξ is a white noise independent of Γ. This process is the simplest example of a non-trivial SPDE, however there has not been much study on the evolution of its geometric properties. This, to our knowledge, has only been partially done in [Gar20] for a similar SPDE used to construct the Liouville field.

We first study the thickness function of a typical point chosen according to the GMC measure of Γ. We show in Lemma 3.2, we show that for any given time $t > 0$ a.s. the thickness of that point is 0. And as a consequence of this, we can conclude directly that not every thickness function is continuous.

The above result motivates a change of perspective. In order to see an interesting dynamic behavior, we consider our process $(\Phi(t))_{0<t<T}$ as a field in $D_T = D \times [0, T]$. In this context we are interested now in the set of thick points defined as

$$T^\gamma := \{(x, t) \in D_T;\ \gamma_x(t) = \gamma\}.$$

The first result regarding thick point is that the set T^γ is not empty for every γ in $(0, 2\sqrt{2})$. Furthermore, we compute in Proposition 3.10 its Hausdorff dimension

$$\dim_H(T^\gamma) = \begin{cases} 3 - \frac{\gamma^2}{4} & \text{For } \gamma \in [0, 2] \\ 4 - \frac{\gamma^2}{2} & \text{For } \gamma \in (2, 2\sqrt{2}). \end{cases} \tag{0.1}$$

The lower bound of this is done by constructing the space-time GMC measure of our field. This is not straightforward from the classical work of Berestycki [Ber17] as the field is not log-correlated. However, a variation of that construction as in [Gar20] can show existence. In such work the focus was related to the SPDE natural for Liouville quantum gravity and not in the behaviour of thick points.

To prove the upper bound of (0.1), one proceeds using a box-counting argument, were one have to separate the cases of boxes of length-side "$\varepsilon \times \varepsilon \times \sqrt{\varepsilon}$", and length-side "$\varepsilon^2 \times \varepsilon^2 \times \varepsilon$".

In [HMP10] it was proven that for a given GFF, the set γ-thick points is empty for $\gamma > 2$. However, we have already seen that for $\gamma \in (2, 2\sqrt{2})$ the set T^γ it is not. This implies that there are certain times τ where $\Phi(\tau)$ behaves quite differently from a GFF. This motivates us to study the exceptional times as τ where $\Phi(\cdot, \tau)$ has γ-thick points for $\gamma > 2$. As a geometric aspect of this set, in Proposition 3.11, we compute its Hausdorff dimension as a function of the thickness parameter γ.

Furthermore, for a given exceptional time τ, we study how many γ-thick points $\Phi(\tau)$ has. We show, in Proposition 3.18, that the number of thick points related to an exceptional time are almost surely finite. Furthermore, for each given γ in $(2, 2\sqrt{2})$, we have that almost surely if

$$N < \frac{4}{\gamma^2 - 4},$$

there are exceptional times τ, where $\phi(\tau)$ has at least N γ-thick points. Furthermore, are no exceptional times with N thick points if $N(\gamma^2 - 4) > 4$.

As in the above case, the proof of this result is done constructing an appropriate Liouville measure which we call an N-Liouville measure on $D^N \times [0, T]$. For the non-existence, we use a box-counting argument on the same space (Proposition 3.26).

Remarkably, a consequence of the result mentioned above is that this field exhibits a countable number of phase transitions accumulating to the right of $\gamma = 2$ given by

$$\gamma_N = \sqrt{\frac{4}{N} + 4}.$$

A naive interpretation of this phenomenon could be from an energetic point of view. Thick points that are more than 2-thick are a highly energetic however there is only finite energy available at any given point in time. Furthermore, the closer to 2 one is, the less energy is necessary to have that thickness and this is what produces this different phase transitions.

Chapter 1

Preliminaries

In the present chapter we present the basic tools related to stochastic processes that we need in order to construct the GFF and other stochastic processes we need. Then we define the set of thick point and show his fractal behavior. Afterwards we do a small review on the theory of linear stochastic partial differential equations (or SPDE). In particular, we intend to give a sense of what does it mean to be a solution, and show the basics results related to existence and uniqueness. We then do a small review of some basic aspects of the Orstein-Uhlenbeck process. To finalize with some basic properties about the heat equations that will be useful to study his stochastic counterpart.

1.1. Gaussian processes

First we fix $(\Omega, \mathcal{F}, \mathbb{P})$ a probability measure space. Let us first start for the context of Gaussian processes (See Section 1.6 of [HH80] related to Gaussian systems, which is an analogue). To do this let us consider a set Λ, and then consider a family of random variables indexed by this set $(X_\lambda)_{\lambda\in\Lambda}$. A or which we know their law and correlations. A First important result is

Proposition 1.1 *For Λ some index set, $M : \Lambda \to \mathbb{R}$ and $C : \Lambda \times \Lambda \to \mathbb{R}$ such that for every finite set $J \subset \Lambda$, $(C(a, b))_{a,b\in J}$ defines a positive matrix. There exists a Gaussian process $(X_\lambda)_{\lambda\in\Lambda}$ with mean given by M and covariance given by C. And if X' is another system such that have the same properties, then X and X' has the same law.*

The proof of this proposition can be done using the Kolmogorov consistency theorem (See Theorem 9.2.5 from [SM18]). From Proposition 1.1 we can see that the Gaussian process are characterized by their mean and covariance. Therefore, a natural way to define the Gaussian random variable of $\mathcal{H}$ would be the following one.

Definition 1.1 *The process $(\mathcal{X}_h)_{h\in\mathcal{H}}$ is the standard Gaussian variable of $\mathcal{H}$, if*

- *For every $h \in \mathcal{H}$, $\mathcal{X}_h \sim \mathcal{N}(0, |h|)$.*
- *For every $h, h' \in \mathcal{H}$, $\mathbb{E}[\mathcal{X}_h\mathcal{X}_{h'}] = \langle h, h'\rangle_{\mathcal{H}}$.*

One can easily check that the definitions follows from Proposition 1.1. As a first consequence of this, is that the σ-algebra considered is the product one. Hence, some important events are not measurable. For example

- For $h, h' \in \mathcal{H}$ given, take $\{\mathcal{X}_{h+h'} = \mathcal{X}_h + \mathcal{X}_{h'}\}$.

- For $h \in \mathcal{H}, c \in \mathbb{R}$ given, take $\{\mathcal{X}_{ch} = c\mathcal{X}_h\}$.

To avoid this problem, we add the hypothesis that $\mathcal{H}$ is a separable space. Therefore, we know that this space have an orthonormal basis $(e_k)_{k\in\mathbb{N}}$. From this we can consider the following process

Proposition 1.2 *For $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ a separable Hilbert space and $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis of this space. Taking $(\alpha_k)_{k\in\mathbb{N}}$ an i.i.d. sequence of standard random variables, the process $(X_h)_{h\in\mathcal{H}}$ defined as*

$$X_h = \lim_{N\to\infty} \sum_{k=0}^{N} \alpha_k \langle h, e_k\rangle_{\mathcal{H}},$$

where the limit is in L^2 fulfills Definition 1.1.

Proof. We first check that such limit exists. To do this we take an arbitrary element $h \in \mathcal{H}$ and consider the sequence

$$X_h^n = \sum_{k=0}^{n} \alpha_k \langle h, e_k\rangle_{\mathcal{H}}. \tag{1.1}$$

And let us check we have a Cauchy sequence in L^2 Taking without loss of generality $n > m \in \mathbb{N}$, we can see that

$$\mathbb{E}[|X_h^n - X_h^m|^2] = \sum_{k=m+1}^{n} \langle h, e_k\rangle_{\mathcal{H}}^2 \tag{1.2}$$

here we used that $(\alpha_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence with variance 1. And since the series on the right-hand side of 1.2 converges. We can conclude that for every $h \in \mathcal{H}$, $(X_h^n)_{n\in\mathbb{N}}$ is a Cauchy sequence. And therefore the limit X_h exists. To check it has the same law of the process from Definition 1.1. We check the mean and covariance. Since we are in a probability space. The L^2 convergence implies the L^1. Therefore,

$$\mathbb{E}[X_h] = \lim_{n\to\infty} \mathbb{E}[X_h^n] = 0.$$

And we have the correct mean. On the other hand, for $h, h' \in \mathcal{H}$

$$\begin{aligned}\mathbb{E}[X_h X_{h'}] &= \lim_{n\to\infty} \mathbb{E}[X_h^n X_{h'}^n] \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \langle e_k, h\rangle_{\mathcal{H}} \langle e_k, h'\rangle_{\mathcal{H}} \\ &= \langle h, h'\rangle_{\mathcal{H}}.\end{aligned} \tag{1.3}$$

Here we again used that $(\alpha_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of normal distributions. From (1.1) and (1.3) we can conclude the desired. □

From the above, we can see for example that for any given $h, h' \in \mathcal{H}$ and $\lambda \in R$ we have almost surely that

$$X_{\lambda h + h'} = \lambda X_h + X_{h'}. \tag{1.4}$$

Making it a measurable event. From this, we can give an expression that implies (1.1), and is considering

$$X = \sum_{k\in\mathbb{N}} \alpha_k e_k \tag{1.5}$$

for $(\alpha_k)_{k\in\mathbb{N}}$ an i.i.d. sequence of normal distributions. From here it is straightforward that $X_h = \langle X, h\rangle_{\mathcal{H}}$. Hence, we could think of X as an element of the dual space $\mathcal{H}^*$. From the Riesz representation theorem. This implies that $X \in \mathcal{H}$, and using the Parsvebal-Bessel identity, we have then

$$|X|^2 = \sum_{k\in\mathbb{N}} \alpha_k^2. \tag{1.6}$$

But since $(\alpha_k)_{k\in\mathbb{N}}$ is i.i.d. and $\mathbb{P}(\alpha_k > 1) = c$ for some constant $c \in (0,1)$ independent of k. By Borel-Cantelli II we can conclude that almost surely, there exists an increasing sequence $(k_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $\alpha_{k_n} > 1$. Therefore, we have that

$$\mathbb{P}(|X|^2 = \infty) = 1. \tag{1.7}$$

Which contradicts the fact of $X \in \mathcal{H}$. The answer on where such random variable lives, is related to what in the literature sometimes is called *measurable norms*. Such concept is mentioned in [She07], Section 2.2 and gives a rough idea on the concept. It can be found in more detail in [Gro67].

As a last corollary we show that a way to define the Gaussian distribution X, is via a dense set.

Corollary 1.1 *For $\mathcal{H}$ a separable Hilbert space and $V \subset \mathcal{H}$ a dense linear space. If X is the Gaussian variable of $\mathcal{H}$ and $(\hat{X}_v)_{v\in V}$ has the same mean and correlations. Then we have that $\hat{X}$ ca be extended to $\mathcal{H}$, and such extension have the same law of X.*

Proof. It is direct that for every $h \in \mathcal{H}$ and $(v_n)_{n\in\mathbb{N}} \subset V$ such that $v_n \to h$. Then $(\hat{X}_{v_n})_{n\in\mathbb{N}}$ is a Cauchy sequence and in the limit we have the wright correlations and mean to conclude, since in L^2 the Gaussian variables are a closed set. □

Let us consider now the construction as a Gaussian process. We could consider $(Y_h)_{h\in\mathcal{H}}$ a centered Gaussian process indexed by $\mathcal{H}$ with a correlation function given by C_X. And assume C_X is a bilinear function that generates the same topology on $\mathcal{H}$ as $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Then by Lax-Milgram one can proof there exists a bijective isometry T between $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ and $(\mathcal{H}, C_x(\cdot,\cdot))$. Therefore, it is easily proven that if X is the standard Gaussian Hilbert process from Proposition 1.2, we have for every $h \in \mathcal{H}$, $Y_h \stackrel{\mathcal{L}}{=} X_{Th}$.

1.1.1. The Cameron-Martin space

An important notion for Gaussian processes on separable Hilbert spaces, is to understand under which translations my process has some invariance or symmetry that becomes useful to study it. A helpful notion of such invariance is the preservation of null sets. In other word we are interested in the directions where the original process and the shifted one are absolutely continuous one another. The set of directions that fulfils this form of symmetry is called Cameron-Martin space. One can find a formal definition of this space can be found in [Hai09] Section 4.2. However, in order to keep the preliminaries as simple as possible, we use the main property of this space as a characterization.

Definition 1.2 *For $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ a separable Hilbert space, and $(X_h)_{h\in\mathcal{H}}$ a centered Gaussian process indexed by $\mathcal{H}$ with correlations given by a bilinear function C_X. We define its Cameron-Martin space $\mathcal{H}_X$ of X, as follows*

$$\mathcal{H}_X := \left\{ \hat{h} \in \mathcal{H};\ X \textit{ and } X + \hat{h} \textit{ are absolutely continuous with respect each other} \right\}.$$

The above definition is actually known as the Cameron-Martin theorem in the literature. The geometric aspects of this subspace are well studied in [Hai09] Section 4.2 for the more general case of Gaussian measures over Banach spaces. Another important result related to the Cameron-Martin space is the following one

Theorem 1.1 *Let X be a Gaussian process on a separable Hilbert space $\mathcal{H}$ with Cameron-Martin space $\mathcal{H}_X$ and for $(\mathcal{H}')$ another separable Hilbert space with a respective Gaussian process Y. Let $A : \mathcal{H}_X \to \mathcal{H}'$ a bounded linear operator such that for $h, h' \in \mathcal{H}'$, $C_Y(h, h') = C_X(A^\star h, A^\star h')$. Then, there exists a unique linear measurable extension $\hat{A}$ of A to $\mathcal{H}$, up to sets of measure 0.*

The proof can be found in [Hai09], Theorem 4.51. Such theorem allows us later to define certain type of processes that we need in order to present the theory of stochastic partial differential equations.

1.1.2. Cylindrical Wiener processes and stochastic integration

Let us consider $\mathcal{H}$ a separable Hilbert, and $\mathcal{H}'$ a larger Hilbert space containing $\mathcal{H}$ as a dense subset and such the inclusion map $\iota : \mathcal{H} \hookrightarrow \mathcal{H}'$ is Hilbert-Schmidt. For every $\mathcal{H}$ separable Hilbert space we can find such $\mathcal{H}'$, since we take $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis of $\mathcal{H}$ and take the closure of it under the norm

$$\|x\|_2^2 = \sum_{k\in\mathbb{N}} r_n^2 \langle x, e_k\rangle_{\mathcal{H}}$$

for some sequence $r \in \ell^2(\mathbb{N})$ such that for every $n \in \mathbb{N}$ one has $r_n > 0$ (As an example consider $r_n = (n+1)^{-1}$) since $\iota\iota^\star e_k = r_k^2 e_k$ we have the desired. Under this setting we have the following proposition

Proposition 1.3 *For $\mathcal{H}$ and $\mathcal{H}'$ as above. The Gaussian process W on $\mathcal{H}'$ with covariance $\iota\iota^\star$ has as Cameron-Martin space $\mathcal{H}$.*

The proof is in Proposition 4.61 from [Hai09]. From this we define

Definition 1.3 *Let $\mathcal{H}$ and $\mathcal{H}'$ as above. We say $(W(s))_{s>0}$ is a cylindrical Weiner process on $\mathcal{H}$, if for every $s > 0$ is a centered Gaussian process in $\mathcal{H}'$. And for every $s, t > 0$ and $h, h' \in \mathcal{H}'$ we have*

$$\mathbb{E}[\langle h, W(s)\rangle_{\mathcal{H}'} \langle h', W(t)\rangle_{\mathcal{H}'}] = (s \wedge t)\langle \iota^\star h, \iota^\star h'\rangle_{\mathcal{H}}. \tag{1.8}$$

The idea behind define such process is to extend the notion of stochastic integral to the context of separable Hilbert space processes.

Remark 1.1 *In [Hai09], it is defined the Banach space* $C_\rho(\mathbb{R},\mathcal{H})$ *as*

$$C_\rho(\mathbb{R},\mathcal{H}') := \{f \in C(\mathbb{R},\mathcal{H}');\ \lim_{t\to\infty} \frac{f(t)}{\rho(t)} \text{ exists } \}. \tag{1.9}$$

And it is endowed with the norm $\|f\|_\rho = \|f/\rho\|_\infty$. *When one take* $w = 1+t^2$. *The cylindrical process* W *from the above definition can be understood as the Gaussian process of* $C_w(\mathbb{R},\mathcal{H}')$ *with the respective desired covariance.*

The idea to consider this type of process is to define a Hilbert space-valued stochastic integral. To achieve this let us now consider 2 separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and consider the space $\mathcal{L}_{HS}(\mathcal{H},\mathcal{K})$ of linear Hilbert-Schmidt operators (For a definition of Hilbert-Schmidt operator see [Bré11] Chapter 6, comment 2), with the inner product given by

$$\langle A, B\rangle_{\mathcal{L}_{HS}(\mathcal{H},\mathcal{K})} = \sum_{k\in\mathbb{N}} \langle Ae_k, Be_k\rangle_{\mathcal{K}}, \tag{1.10}$$

for $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis of $\mathcal{H}$. Then consider the filtration $\mathcal{F}_t$ as the σ-algebra generated by $(W_t)_{t\leq s}$. We start by defining the elementary process as follows

Definition 1.4 (Elementary process) *For* $((s_k,t_k])_{k=1}^N$ *a family of disjoints intervals of* $\mathbb{R}^+$, *consider for each* $k = 1,...,N$, *the* $\mathcal{F}_{s_k}$*-measurable random variable* Φ_k *that takes values at* $\mathcal{L}_{HS}(\mathcal{H},\mathcal{K})$. *Then* Φ *defined as*

$$\Phi(t) = \sum_{k=1}^N \Phi_k 1_{(s_k,t_k]}(t) \tag{1.11}$$

is an elementary process on $\mathcal{K}$.

Remark 1.2 *For* $\mathcal{H}$, $\mathcal{H}'$ *as in Definition 1.3. And for some fixed* H *separable Hilbert space such that there exists* $A : \mathcal{H} \to H$ *that is Hilbert-Schmidt. One can do a measurable extension of* A *to* $\mathcal{H}'$ *such that* AW *makes sense.*

The above definition allows us to first define the stochastic integral as follows

Definition 1.5 *For* Φ *an elementary process and* W *a cylindrical Weiner process on* $\mathcal{H}$. *We define the* H*-valued stochastic integral as*

$$\int_0^\infty \Phi dW(s) = \sum_{k=1}^N \Phi_k(W(t_k) - W(s_k)). \tag{1.12}$$

As a first observation, from Theorem 1.1 and proposition 1.3. The above is well-defined and even more, it is not depending on the choice of $\mathcal{H}'$. On the other hand, one can notice that

$$\mathbb{E}\left[\| \int_0^\infty \Phi(s)dW(s)\|_{\mathcal{K}}^2\right] = \sum_{i=1}^N \mathbb{E}\left[\langle \Phi_k(W(t_k)-W(s_k)), \Phi_k(W(t_k)-W(s_k))\rangle_{\mathcal{K}}\right]$$
$$= \sum_{i=1}^N \mathbb{E}\left[\langle \Phi_k, \Phi_k\rangle_{\mathcal{L}_{HS}(\mathcal{H},\mathcal{K})}(t_k - s_k)\right]$$
$$= \mathbb{E}\left[\int_0^\infty \langle \Phi(s), \Phi(s)\rangle_{\mathcal{L}_{HS}(\mathcal{H},\mathcal{K})} ds\right].$$

This implies that the stochastic integral is an isometry between $L^2(\mathbb{R}^+ \times \Omega; \mathcal{L}_{HS}(\mathcal{H},\mathcal{K}))$ and $L^2(\Omega, \mathcal{K})$. From this we first need the following definition

Definition 1.6 *We define the predictable σ-algebra $\mathcal{F}_{pr}$ over $\mathbb{R}^+ \times \Omega$ and generated by the sets of the form $(s,t] \times A$ for $A \in \mathcal{F}_s$. A measurable function $X : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is call predictable, if it is $\mathcal{F}_{pr}$ measurable*

From the above, we have the following proposition

Proposition 1.4 *The set of elementary processes is dense in $L^2_{pr}(\mathbb{R}^+ \times \Omega; \mathcal{L}_{HS}(\mathcal{H},\mathcal{K}))$ the set of predictable $\mathcal{L}_{HS}(\mathcal{H},\mathcal{K})$-valued processes*

The proof it is in [Hai09], Proposition 4.68. As a direct corollary we have the following

Corollary 1.2 *The stochastic integral is well-defined for $\Phi \in L^2_{pr}(\mathbb{R}^+ \times \Omega; \mathcal{L}_{HS}(\mathcal{H},\mathcal{K}))$.*

As a last result related to the above. We construct the Cylindrical Weiner process for a given $\mathcal{H}$.

Proposition 1.5 *For $\mathcal{H}$, $\mathcal{H}'$ and $\iota : \mathcal{H} \to \mathcal{H}'$ as in Definition 1.3. IF $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$ then, for $(B_k)_{k\in\mathbb{N}}$ an i.i.d. sequence of Brownian motions*

$$W = \sum_{k\in\mathbb{N}} B_k \iota(e_k), \tag{1.13}$$

is a Cylindrical Wiener process in $\mathcal{H}$

Proof. To prove the proposition we only need to check for the correlations to be the wright ones since we are in the context of Gaussian processes. Therefore, take $h, g \in \mathcal{H}'$, and notice that

$$\mathbb{E}[\langle \sum_{k\in\mathbb{N}} B_k(t)\iota e_k, h\rangle_{\mathcal{H}'} \langle \sum_{k\in\mathbb{N}} B_k(s)\iota e_k, g\rangle_{\mathcal{H}'}] = \mathbb{E}[\sum_{k,j\in\mathbb{N}} B_k(s)B_l(s)\langle \iota e_k, g\rangle_{\mathcal{H}} \langle \iota e_j, g\rangle_{\mathcal{H}'}]$$
$$= \sum_{k\in\mathbb{N}} (s\wedge t)\langle e_k, \iota^\star h\rangle_{\mathcal{H}} \langle e_k, \iota^\star g\rangle_{\mathcal{H}}$$
$$= (s\wedge t)\langle \iota^\star h, \iota^\star g\rangle_{\mathcal{H}}.$$

Since we have the correct correlations, we conclude the proposition □

As a direct corollary from the above proposition we have that

Corollary 1.3 *For $\Phi \in L^2_{pr}(\mathbb{R}^+ \times \Omega; \mathcal{L}_{HS}(\mathcal{H},\mathcal{K}))$, and $h \in \mathcal{K}$. If W is the cylindrical Weiner* *process on $\mathcal{H}$. Then we have that*

$$\langle \int_0^\infty \Phi(s) dW_s, h\rangle_{\mathcal{K}} = \sum_{k\in\mathbb{N}} \int_0^\infty \langle \Phi(s)e_k, h\rangle_{\mathcal{K}} dB_s^k. \tag{1.14}$$

Where $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$.

1.1.3. White noise

The first important example of a Normal in a Hilbert space is the White noise, which is defined as the Gaussian random variable of the space $L^2(\mathbb{R}^d)$ for some $d \geq 1$, or a subset of it. Therefore, for $D \subseteq \mathbb{R}^d$ we will say that W is a white noise in D if it is the normal distribution of $L^2(D)$. Therefore, taking an orthonormal basis $(b_l)_{l\in\mathbb{N}}$ of $L^2(D)$, we can see that

$$\xi = \sum_{l\in\mathbb{N}} \alpha_l b_l.$$

With, as before, $(\alpha_l)_{l\in\mathbb{N}}$ an i.i.d. sequence of normal random variables. In the context of SPDEs, this type of random variable makes the roll of our random force. From now on $\langle \cdot, \cdot \rangle$ indicates the inner product of $L^2(D)$ for a given D. The first case of interest in the white noise is the following one.

Lemma 1.1 *The derivative of the Brownian motion $(B_t)_{t\geq 0}$ in the Schwartz sense, is the white noise in $\mathbb{R}^+$*

Proof. To prove this we use corollary 1.1 and consider $f, g \in C_c^\infty(\mathbb{R})$. And proceed as follows, when applied Itô formula for $F(s, x) = f(s)x$ we can conclude that

$$\langle f, B' \rangle = -\langle f', B\rangle = -\int_{\mathbb{R}^+} f'(s) B(s) ds$$

From here we can see that $\langle f, B'\rangle \sim \mathcal{N}(0, \langle f, f\rangle)$, and also it is straightforward to check the correlations. □

Now, we focus on how to use the above definition to construct other noises. In particular, we focus on the case of domains of the form $A \times B$. For this case we have the following proposition

Proposition 1.6 (Product case) *For, $A \subset \mathbb{R}^{d_1}$, $B \subset \mathbb{R}^{d_2}$ measurable sets, and ξ_A, ξ_B the respective white noises. If $(a_n)_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2(A)$ and $(b_l)_{l\in\mathbb{N}}$ the same for $L^2(B)$. We have then that the White noise ξ of $L^2(A \times B)$ can be written as*

$$\xi = \sum_{l\in\mathbb{N}} \xi_A^l b_l \tag{1.15}$$

Whit $(\xi_A^l)_{l\in\mathbb{N}}$ an i.i.d. sequence of white noise from A. We can do the same for ξ_B and obtain.

$$\xi = \sum_{n\in\mathbb{N}} \xi_B^n a_n \tag{1.16}$$

Proof. We know that $(a_n b_l)_{n,l\in\mathbb{N}}$ is an orthonormal basis for $L^2(A \times B)$. Therefore, we have that

$$\xi = \sum_{n,l\in\mathbb{N}} \alpha_{n,l} a_n b_l \tag{1.17}$$

for $(\alpha_{n,l})_{n,l\in\mathbb{N}}$ an i.i.d. sequence of normal random variables. From expression (1.17), one can easily check the desired when we check the mean and correlations in the dense $\{fg;\ f \in L^2(A), g \in L^2(B)\}$. □

Using the above we could ask ourselves about more complexes noises. The following proposition is an extension of the above result

Proposition 1.7 *For $\mathcal{H}$ a separable Hilbert space, and W the Gaussian standard for $L^2([0,T];\mathcal{H})$ under the inner product given as the integration of the inner product form $\mathcal{H}$. If $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$, then, $w = \langle W, e_k\rangle_{\mathcal{H}}$ is the Gaussian distribution of $L^2([0,T];\mathbb{R})$.*

1.2. Gaussian free field

We now consider $D \subset \mathbb{R}^2$ open and bounded, such that ∂D satisfies a Poincaré cone condition (see [MP10], Section 3.1). Consider denote the usual inner product of $L^2(D)$ as $\langle \cdot,\cdot\rangle$. On the other hand, for the Solobev space $H_0^1(D)$, we consider the following inner product

$$\langle f, g\rangle_\nabla = \frac{1}{2\pi}\int_D \nabla f \cdot \nabla g dx,\ \ \forall f, g \in H_0^1(D).$$

Where ∇f indicates the distributional derivatives. From this, we have the Hilbert space $(H_0^1, \langle\cdot,\cdot\rangle_\nabla)$. The Gaussian free field then is defined as follows.

Definition 1.7 *The Gaussian free field Γ (from now on GFF) is defined as the standard Gaussian variable of $(H_0^1(D), \langle\cdot,\cdot\rangle_\nabla)$.*

In [Bré11] Section 9.8. It is proven that the eigenfunctions $(f_k)_{k\in\mathbb{N}}$ for the Dirichlet Laplace operator form an orthonormal basis of H_0^1 with the usual inner product. From this, it is straightforward then when one takes $e_k = \sqrt{2\pi} f_k$, we have then an orthonormal basis for $(H_0^1(D), \langle\cdot,\cdot\rangle_\nabla)$. From now on, we name $(e_k)_{k\in\mathbb{N}}$ as the spectral basis of $H_0^1(D)$ or spectral basis when there is no possible confusion. Using then (1.5) we have that the GFF Γ can be written as

$$\Gamma = \sum_{k\in\mathbb{N}} \alpha_k e_k. \tag{1.18}$$

At prior, we have that we know how to test the weak derivative of the GFF with respect to some function, even though, from (1.7), we know that almost surely the GFF is not n element of $H_0^1(D)$. In order to understand a bit more where the GFF lives. We could try to understand its correlations in space. In order to do this, one could do the following heuristic computation. Take $f, g \in C_c^\infty(D)$, by integration by parts we have

$$\langle g, f\rangle_\nabla = \langle g, (-\frac{1}{2\pi}\Delta) f\rangle.$$

Therefore, if we want to understand how to integrate with respect to the GFF. We could try to give some sense to the inverse of $-\Delta$, to then have, for $f \in L^2(D)$

$$\langle \Gamma, f\rangle = \langle \Gamma, (-\frac{1}{2\pi}\Delta)(-2\pi\Delta^{-1})f\rangle = \langle \Gamma, (-2\pi\Delta^{-1})f\rangle_{\nabla}. \tag{1.19}$$

Therefore, in order to understand more about this process. We need first to recall some basic properties related to $(-\Delta)^{-1}$. This function can be understood as the function that gives the solution to the Poisson equation, which is the following PDE

$$(P)\begin{cases} -\Delta u = f & \text{in } D, \\ u = 0 & \text{in } \partial D. \end{cases} \tag{1.20}$$

For $f \in C(D)$, in [Eva22], Section 2.2.4. Theorem 12, it is proven that the solution of (1.20) is given in by

$$u(x) = \int_D G^D(x,y)f(y)dy. \tag{1.21}$$

Where $G^D(x,y)$ is called Green function. In relation with this function, we have the following

Proposition 1.8 *For $D \subset \mathbb{R}^2$, an open bounded domain such that his boundary satisfies the Poincaré cone condition. We have that the respective Green function G^D. Satisfies*

(i) For every $x, y \in D$ different, $G^D(x,y) = G^D(y,x)$.

(ii) There exists a function $g \in C^2(D\times D)\cap C(\bar{D}\times\bar{D})$ such that $G^D(x,y) = -(2\pi)^{-1}\ln(|x-y|)+ g(x,y)$.

(iii) For every $y \in D$ the function $G^D(\cdot,y)$ is in $L^1(D)$. Even more $\sup_{y\in D}\|G^D(\cdot,y)\|_{L^1} < \infty$.

The proofs of (i) and (ii) can be found in [Eva22], Section 2.2.4. For (iii), using $f = 1$ in (1.21) one can conclude directly. As an observation, from (ii) it is direct that for every $x \in D$, we have that $G^D(x,x) = \infty$. On the other hand, in the following corollary we extend (1.21).

Corollary 1.4 *The function*

$$(-\Delta)^{-1}f = \int_D G^D(x,y)f(y)dy,$$

can be extended to a continuous bijective operator $(-\Delta)^{-1} : L^2 \to H^1_0(D)$. Even more $u = (-\Delta)^{-1}f$ solves in the weak sense equation (1.20)

For a definition of weak solution we recommend [Eva22], Section 6.2.

Proof. The extension of $(-\Delta)^{-1}$ to L^2 is a direct consequence of Proposition 1.8, statement (iii). Using that same statement plus the Sobolev inequality (see [Bré11], Section 9.3), one can conclude the continuity. To prove is bijective we can prove that $u = (-\Delta)^{-1}f$ is a weak solution of (1.20) and conclude due to uniqueness theorem (see[Bré11], Section 9.5). To prove that it gives a weak solution we recall that in the Schwartz sense we have that

$$(-\Delta)G^D(\cdot,y) = \delta_y.$$

Where δ_y indicates the Dirac distribution at y. To prove this, from Proposition 1.8, statement (iii) we have that for every $\phi \in C_c^\infty(D)$

$$\int_D (-\Delta)^{-1} f(x)(-\Delta)\phi(x)dx = \int_D f(y)(-\Delta)^{-1}(-\Delta)\phi(y)dy = \int_D f(y)\phi(y)dy,$$

and in the last equality we used (1.21). This allows us to conclude. □

Remark 1.3 *As a curiosity, we mention that in [MP10], Section 3.1, there is proven that under the Poincaré cone condition one can also express a strong solution of 1.20 in terms of the Brownian motion. In particular, it is shown the relation between the Green function and the Brownian motion.*

From now on, G^D indicates the Kernel of $(2\pi)(-\Delta)^{-1}$. And this is nothing more but the Green functions times 2π. Using (ii) from Proposition 1.8, we have then

$$G^D(x, y) = -\ln(|x - y|) + g^D(x, y),$$

where $g^D \in C^2(D \times D) \cap C(\bar{D} \times \bar{D})$. At the same time we now use $(-\Delta)^{-1}$ to indicate the linear function $(2\pi - \Delta^{-1})$.

The relation between the Green function and the GFF it is explained in the Following proposition.

Proposition 1.9 *The GFF Γ can be understood as a centered Gaussian process indexed $L^2(D)$, where the correlations are given by the Green function. In other words, for f, g in $L^2(D)$ we have that*

$$\mathbb{E}[\langle \Gamma, f\rangle\langle \Gamma, g\rangle] = \langle f, (-\Delta)^{-1} g\rangle = \int_D \int_D f(x)g(y)G^D(x, y)dxdy.$$

The proof is similar to the heuristic computations done in (1.19). At an intuitive level, one could say that the Green function gives the correlations between 2 points for the GFF. Since such function explodes in the diagonal, we cannot expect that the GFF is even pointwise defined. To answer the question related to where this variable lives, we first need to recall some properties related to the Dirichlet Laplace operator and its eigenfunctions and eigenvalues. An important result we need to understand the regularity of the GFF, is the following proposition, also known as Weil formula.

Proposition 1.10 *Let $(\lambda_k)_{k\in\mathbb{N}}$ the eigenvalues of the Dirichlet Laplace operator in a bounded domain $D \subset \mathbb{R}^d$. Then, there exists a constant $C_W = C_W(D, d)$ such that*

$$\lim_{n\to\infty} \frac{\lambda_n}{n^{\frac{2}{d}}} = C_W. \tag{1.22}$$

And in particular this implies there exists two positive constants $c < C$ (that depends on D, d), such that for every $n \in \mathbb{N}$,

$$cn^{\frac{2}{d}} < \lambda_n < Cn^{\frac{2}{d}} \tag{1.23}$$

This result was first proven in [Wey12]. In [Gei14] one can find a stronger result that implies 1.10. Considering this estimation and considering $(\lambda_k, f_k)_{k\in\mathbb{N}}$ the eigenfunction basis of the Dirichlet Laplace operator with is respective eigenvalue, we define the spectral fractional

Laplace operator

Definition 1.8 *Let $f \in C_c^\infty(D)$. We define the fractional Laplacian for $\beta \in \mathbb{R}$ as*

$$(-\Delta)^\beta f = \sum_{k\in\mathbb{N}} \lambda_k^\beta \langle f_k, f\rangle_\nabla f_k. \tag{1.24}$$

One can easily probe that for $f \in C_c^\infty(D)$ that for every $\beta \in \mathbb{R}$ we have that $\langle f, (-\Delta)^\beta f\rangle$ is positive. And even more, one can define the inner product given by $\langle f, g\rangle_\beta = \langle f, (-\Delta)^\beta f\rangle$ and check that is a bilinear form. From this we can define the fractional Sobolev spaces as follows

Definition 1.9 (Fractional Sobolev space) *For $\beta \in \mathbb{R}$, we define $H_0^\beta(D)$ as*

$$H_0^\beta(D) := \{f = \sum_{k\in\mathbb{N}} C_k f_k;\ \|f\|_\beta < \infty\}.$$

Such space can be understood as the completion of $C_c^\infty(D)$ under the norm

$$\|f\|_\beta^2 := \langle f, (-\Delta)^\beta f\rangle. \tag{1.25}$$

Under this definition we have the following proposition related to the regularity of the GFF

Proposition 1.11 *If Γ indicates the GFF in $D \subset \mathbb{R}^2$ bounded and open. We have that almost surely $\Gamma \in H_0^{-\varepsilon}$ for every $\varepsilon > 0$.*

Proof. For a given $\beta \in \mathbb{R}$. For $(\alpha_k)_{k\in\mathbb{N}}$ an i.i.d sequence of normal Gaussian variables, we take

$$\Gamma_N = \sum_{k=0}^N \alpha_k e_k.$$

We know that for each $h \in H_0^1(D)$, in L^2 we have $\langle \Gamma_N, h\rangle_\nabla$ tends to $\langle \Gamma, h\rangle_\nabla$. Therefore, we can first study Γ_N

$$\begin{aligned}\mathbb{E}[\|\Gamma_N\|_\beta^2] &= \mathbb{E}[\sum_{k=0}^N \alpha_k^2 \lambda_k^{\beta-1}]\\ &= \sum_{k=0}^N \lambda_k^{\beta-1}.\end{aligned}$$

Using 1.22. We have that there exists some constants $c < C$ such that for every $N \in \mathbb{N}$,

$$c\sum_{k=1}^N (k+1)^{\beta-1} < \sum_{k=0}^N \lambda_k^{\beta-1} < C\sum_{k=0}^N (k+1)^{\beta-1}.$$

This implies that when $\beta < 0$ the limit as N tends to infinity exists. Using that Γ_N is a Gaussian process whose covariance tends to ones for the GFF. We have then that Γ_N tends to Γ in Law. This allows us to conclude the proposition since we have

$$\lim_{N\to\infty} \mathbb{E}[\|\Gamma_N\|_\beta^2] = \mathbb{E}[\|\Gamma\|_\beta^2].$$

□

From this proposition we can see that the GFF is not a random function, it is a random Schwartz distribution.

1.2.1. Circle average approximation

As we mentioned in the Introduction. Part of the interest on the GFF, it is related to some of its geometric properties, but as we just saw, it is not defined point-wise. Therefore, a way to study the geometry of the field could be approximate it via a convolution with some regularization kernel. In order to define such kernel, we first present the harmonic measure

Definition 1.10 *For $D \subset \mathbb{R}^2$ an open bounded set that satisfies the Poincaré cone condition, and $x \in D$. We define its harmonic measure at x, $\mu_{D,x}$ as*

$$\int f(z)\mu_D(dz) = \mathbb{E}[f(B^x_{\tau_D}], \ \forall f \in C(\overline{D}),$$

where, $\tau_D = \inf(t > 0; B^x_t \notin D)$ is the exit time of D.

Notice that the measure $\mu_{D,x}$ is supported in ∂D for every $x \in D$. An important property related to this measure is the following proposition

Proposition 1.12 *For $D \subset \mathbb{R}^2$ an open bounded set that fulfils the Poincaré cone condition. Then, for a given $\phi \in C(\partial D)$ we have that the solution of the Dirichlet problem*

$$\begin{cases} -\Delta u = 0, & \text{in } D, \\ u = \phi, & \text{in } \partial D, \end{cases}$$

is given by

$$u(x) = \int_{\partial D} \phi(z)\mu_{D,x}(dz)$$

Proof. The proof of the proposition can be founded in [LG16], Proposition 7.7. □

In the above proposition we can see in some sense that the harmonic measure has some regularization effect. An important property we need to recall related to Harmonic functions is the maximum principle

Theorem 1.2 (Maximum principle) *For $u \in C(D) \cap C^2(D)$ a harmonic function, we have that the maximum is attached in the boundary of D.*

Proof. A proof can be found in [MP10], Theorem 3.5. □

From the above, we know consider the kernel given by an specific harmonic measure

Definition 1.11 *For every $x \in D$ and $\varepsilon >$, we define the circle average measure $\mu_{\varepsilon,x}$ in D as*

$$\int_D f(z)\mu_{\varepsilon,x}(dz) = \mathbb{E}[f(B^x_{\tau_{\varepsilon,x}})],\ \forall f \in C(D)$$

Where $\mu_{\varepsilon,x}$ indicates the harmonic measure of $D \cap B(\varepsilon, x)$.

From this, we now want to understand $\langle \Gamma, \mu_{\varepsilon,x}\rangle$. This could be interpreted as a dual product, more formally

Proposition 1.13 *The function*

$$h(y) = \int_D G^D(y,z)\mu_{\varepsilon,x}(dz)$$

belongs to $H^1_0(D)$.

The proof is using Proposition 1.8 and Fubini's theorem. From this proposition, we can do the following definition

Definition 1.12 *We define the circle average of a GFF Γ as*

$$\Gamma_\varepsilon(x) = \langle \Gamma, \mu_{\varepsilon,x}\rangle = \langle \Gamma, (-\Delta)^{-1}\mu_{\varepsilon,x}\rangle_\nabla.$$

Thanks to Proposition 1.13. We know the above definition has no problems. It is direct that for $x, y \in D$ and $\varepsilon, \delta > 0$ we have

$$\mathbb{E}[\Gamma_\varepsilon(x)\Gamma_\delta(y)] = \int_D\int_D G^D(z,z')\mu_{\varepsilon,x}(dz)\mu_{\delta,y}(dz').$$

From now on, we denote as $G^D_{\varepsilon,\delta}(x,y)$ to the right-hand side of the above equation. One could check that the field $\Gamma_\varepsilon(x)$ tends in probability to the GFF in the space of distributions. As a first result related to the correlations we do the following prior estimation.

Proposition 1.14 *There exists a constant $K > 0$, such that uniformly on $x, y \in D$ and $\varepsilon, \delta > 0$ we have that*

$$\mathbb{E}[\Gamma_\varepsilon(x)\Gamma_\delta(y)] = G^D_{\varepsilon,\delta}(x,y) \le \ln\left(\frac{1}{|x-y| \vee \varepsilon \vee \delta}\right) + K.$$

Proof. Let us assume for simplicity that $B(\varepsilon, x) \subset D$ For $z \in B(\varepsilon, x)$, define $F(z) = \mathbb{E}[G^D(B^z_{\tau_{\varepsilon,x}}, B'_{\tau_{\delta,0}+y})$. We know that this implies that F solves the equation

$$\begin{cases} -\Delta F = 0 & B(\varepsilon, x) \\ F = \mathbb{E}[G^D(\cdot, B'_{\tau_{\delta,0}+y})] & \partial B(\varepsilon, x). \end{cases}$$

By the maximum principle we have that $\max\left\{F(z), z \in \overline{B(\varepsilon,x)}\right\} = \max\left\{F(z), z \in \partial B(\varepsilon, x)\right\}$, and therefore, $F(x)$ is upper bounded by

$$\max_{z\in\partial B(\varepsilon,x)} \mathbb{E}[G^D(B^z_{\tau_{\varepsilon,x}}, B'_{\tau_{\delta,0}} + y) \le \max_{z\in\partial B(\varepsilon,0)} \mathbb{E}\left[\ln\left(\frac{1}{|(z - B'_{\tau_{\delta,0}}) + (x-y)|}\right)\right] + K$$

and from here we can conclude the desired since $|(z-B'_{\tau_{\delta,0}})+(x-y)|\geq |x-y|\vee\varepsilon\vee\delta$. $\square$

To understand more about this function $\Gamma_\varepsilon(x)$, we first need to extend Lemma C1 from [HMP10] to the following one

Theorem 1.3 (Extended Kolmogorov continuity theorem) *For $D\subset\mathbb{R}^d$, a bounded open set, and $X: D\times(0,1]\to\mathbb{R}$ a time-varying random field satisfying*

$$\mathbb{E}|X(z,r)-X(w,s)|^\alpha]\leq C\left(\frac{d((z,r),(w,s))}{r\wedge s}\right)^{d+1+\beta}.$$

For some $\alpha,\beta>0$. For each $\zeta>\alpha^{-1}$ and $\gamma\in(0,\beta/\alpha)$, X has a modification (that we identify also with X) satisfying

$$|X(z,r)-X(w,r)|\leq M(\ln\Big(\frac{1}{r\wedge s}\Big))^\zeta\frac{d((z,r),(w,s))^\gamma}{(r\wedge s)^{\hat{\gamma}}}.$$

Where $r\wedge s=\min\{s,r\}$ and $r\vee s=\max\{r,s\}$. And

$$\hat{\gamma}=\frac{d+\beta}{\alpha},$$

for $z,w\in D$ and $r,s\in(0,1]$, with $1/2\leq(r\wedge s)/(r\vee s)\leq 2$, and d a metric on $\mathbb{R}^{d+1}$ that generates the usual topology.

The proof is the same as in [HMP10] Appendix C. We do the proof under the necessary adjustments.

Proof. As a simplification, since D is bounded, we can assume without loss of generality that $D\subset[0,1]^d$, since otherwise we could rescale. For every $n,T\in\mathbb{N}$ we define

$$\mathcal{Q}_{n,T}:=\left\{(z,t)\in 2^{-n}\mathbb{Z}^{d+1};(z,t)\in D\times[2^{-T},2^{-T+1}]\right\}.$$

Consider $\Delta_{k,T}$ the set of pairs (a,b), with $a,b\in\mathcal{Q}_{k,T}$ such that $d(a,b)\leq 2^{-k}$. Notice that $|\Delta_{k,T}|$ is of order $O(2^{(k+1)(d+1)-T})$. And for every $a,b\in\Delta_{k,T}$, we have that $d(a,b)\leq 2^{-k+1}$

$$K_i=\sup_{T\geq 1}\left(\frac{2^{-\hat{\gamma}T}}{T^\zeta}\sup_{(a,b)\in\Delta_{i,T}}|X(a)-X(b)|\right).$$

Therefore, we have that

$$\begin{aligned}\mathbb{E}[K_i^\alpha]&\leq\sum_{T\geq 1}\frac{2^{-\alpha\hat{\gamma}T}}{T^{\alpha\zeta}}\sum_{(a,b)\in\Delta_{i,T}}\mathbb{E}[|X(a)-X(b)|^\alpha]\\&\leq C\sum_{T\geq 1}\frac{2^{-(d+\beta)T}}{T^{\alpha\zeta}}2^{(i+1)(d+1)-T}2^{(T-i)(d+1+\beta)}\\&\leq O\left(2^{-i\beta}\right)\end{aligned}$$

Where the constant C might depend on d and β. We used that $\alpha\zeta>1$. Now, we consider the following dense set in (D, d)

$$\mathcal{Q}_T = \bigcup_{n\in\mathbb{N}} \mathcal{Q}_{n,T}.$$

And then consider for $a, a' \in \mathcal{Q}_T$ the order $a \le a'$ as the component-wise order. Hence, we can see that for each $a \in \mathcal{Q}_T$ there exists an increasing sequence that fulfills $a_n \in \mathcal{Q}_{n,T}$. And that there exists some $n_0 \in \mathbb{N}$ where for every $n \ge n_0$, $a_n = a$. If for $a, b \in \mathcal{Q}_T$ with respective sequences $(a_n)_{n\in\mathbb{N}}$, and $(b_n)_{n\in\mathbb{N}}$, and assume that $d(a,b) \le 2^{-m}$. We have that

$$X(a) - X(b) = X(a_m) - X(b_m) + \sum_{i=m}^{\infty} (X(a_{i+1}) - X(a_i)) + \sum_{i=m}^{\infty} (X(b_{i+1}) - X(b_i)).$$

This implies

$$\frac{2^{-\hat{\gamma}T}}{T^{\zeta}}|X(a) - X(b)| \le K_m + 2\sum_{i=m+1}^{\infty} K_i \le 2\sum_{i=m}^{\infty} K_i. \tag{1.26}$$

From this, we can see that

$$\begin{aligned} A &:= \sup_{n,T\in\mathbb{N}} \left(\sup\left\{ \frac{2^{-\hat{\gamma}T}|X(a) - X(b)|}{T^{\zeta} d(a,b)^{\gamma}};\ a, b \in \mathcal{Q}_T, 2^{-(n+1)} < d(a,b) < 2^{-n} \right\} \right) \\ &\le \sup_{n\in\mathbb{N}} \left(2^{\gamma(n+1)+1} \sum_{i\ge n} K_i \right) \le 2\sum_{i\ge 1} K_i \end{aligned}$$

Hence, we have that $\mathbb{E}[A^{\alpha}]$ is finite. This implies almost surely there exists M such that for every $(x,t), (y,s) \in \mathcal{Q}_T$ we have that, since $s, t \in (2^{-T}, 2^{-T+1})$

$$|X(x,t) - X(y,s)| \le M\frac{T^{\zeta}}{2^{-\hat{\gamma}T}} d((x,t),(y,s))^{\gamma} \le M\left(\ln\left(\frac{1}{s\wedge t}\right)\right)^{\zeta} \frac{d((x,t),(y,s))^{\gamma}}{(s\wedge t)^{\hat{\gamma}}}$$

Therefore, the above inequality also holds for $\mathcal{Q} = \cup_{T\in N} \mathcal{Q}_T$. Then, taking

$$\hat{X}(a) = \lim_{b\in\mathcal{Q}; b\to a} X(b),$$

gives us the desired modification. □

Using the above theorem we can prove

Proposition 1.15 *The circle average approximation $\Gamma_{\varepsilon}(x)$ possesses a modification (that we identify also as $\Gamma_{\varepsilon}(x)$) such that for every $0 < \gamma < 1/2$ and $\varepsilon, \zeta > 0$, for $0 < r < s < 1$ such that $1/2 < r/s < 2$. Almost surely, there exists a constant $M = M(\gamma, \varepsilon\zeta)$ such that*

$$|\Gamma_r(x) - \Gamma_s(y)| \le M\left(\ln\left(\frac{1}{r}\right)\right)^{\zeta} \frac{|(x,s) - (y,r)|^{\gamma}}{r^{\gamma+\varepsilon}}$$

The proof can be found in [HMP10], Proposition 2.1. From now on we call $\Gamma_{\varepsilon}(x)$ the circle average approximation of the GFF, we use Γ_{ε} to denote it.

1.3. Thick points of the GFF

We now present the geometric object of interest. Such aspect can be understood as its extreme values. Since the GFF is a random Schwartz distribution, in order to define an extreme value we used the circle average approximation. More formally, we defined the set of γ-thick points as follows

Definition 1.13 *For a given $\gamma > 0$. We define the set of γ-thick points T^γ of Γ as*

$$T^\gamma := \left\{ x \in D;\ \limsup_{\epsilon \to 0} \frac{\Gamma_\epsilon(x)}{\ln\left(\frac{1}{\epsilon}\right)} = \gamma \right\}.$$

Using that the field Γ_ε is continuous, one can easily see that the set T^γ is always a measurable set for every $\gamma > 0$. Nonetheless, find such points in space in not a trivial task, for example, if we take some $x \in D$, we have that for every $\varepsilon > 0$ that

$$\mathbb{P}(\Gamma_\varepsilon \geq \gamma \ln\left(\frac{1}{\varepsilon}\right)) \leq Ce^{-\frac{\gamma^2}{2}\ln\left(\frac{1}{\varepsilon}\right)}.$$

Therefore, we have that

$$\mathbb{P}(x \in T^\gamma) \leq \mathbb{P}(\Gamma_\varepsilon \geq \gamma \ln\left(\frac{1}{\varepsilon}\right)) \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

In other words, if we take a point in space, almost surely such point is not γ-thick point for every $\gamma > 0$. One could ask if then this set is always empty, and the answer is not. Or at least not always. More specific, we have the following result

Lemma 1.2 *For $\gamma \in [0, 2]$, the set T^γ is not empty and for $\gamma > 2$ the set is almost surely empty*

The proof of this lemma can be found in [HMP10], Section 3. In the same paper it is studied the Hausdorff dimension of the set. In particular, in the same section it is proved the following

Lemma 1.3 *For $\gamma \in (0, 2)$ the Hausdorff dimension of the set of γ-thick points has the following upper bound*

$$\dim_H(T^\gamma) \leq 2 - \frac{\gamma^2}{2}.$$

The proof of this upper bound is done using a box-counting argument, that can be done thanks to Proposition 1.15.

1.3.1. Gaussian multiplicative chaos

The theory of Gaussian multiplicative chaos (from now on GMC) was initiated by Kahane in 1985 [Kah85], and its goal in general terms is to study the measures of the form

$$: e^{\gamma h} : dx := e^{\gamma h(x) - \frac{\gamma^2}{2}\mathbb{E}[h^2(x)]} dx. \tag{1.27}$$

Where h is a centered Gaussian field over an open and bounded domain $D \subset \mathbb{R}^d$, and its correlations are given by a kernel of the form.

$$K(x,y) = \ln\left(\frac{1}{|x-y|}\right) + g(x,y),$$

here $|\cdot|$ indicates the usual norm of $\mathbb{R}^d$ and $g \in C(D \times D)$. Such fields are usually called log-correlated fields. One can see that usually such fields are not point-wise defined. Therefore, in order to understand (1.27), one first consider the measure

$$\mu_\varepsilon^\gamma(dx) = e^{\gamma h_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon^2(x)]} dx, \tag{1.28}$$

with h_ε the field h convoluted with some regularization kernel $\rho_{\varepsilon,x}$. Then, one study the convergence of μ_ε^γ in the weak topology. The existence of the limit measure, it is proven in [Ber17], in order to prove the convergence for $\gamma \in (0, \sqrt{2d})$, one need to use the following approximation

$$\mu_\varepsilon^\gamma(dx) = e^{\gamma h_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon^2(x)]} 1_{G_{\alpha,\varepsilon}(x)} dx. \tag{1.29}$$

Where $G_{\alpha,\varepsilon}(x)$ indicates that x is a good point. More formally, this is defined fixing first some ε_0, and then take $\varepsilon < \varepsilon_0$ to define

$$G_{\alpha,\varepsilon}(x) := \{h_r \leq \alpha \ln\left(\frac{1}{r}\right);\ \forall r \in (\varepsilon, \varepsilon_0)\}.$$

Then, in [Ber17], Theorem 1.1. It is proving in particular the following

Theorem 1.4 *The approximation $\mu_\varepsilon^\gamma(dx)$ converges in probability to a measure μ^γ on the topology of weak convergence of measures on D when $\gamma < \sqrt{2d}$.*

It is also proven that the limit measure does not depend on the regularization kernel used for h_ε. And one can also prove that when one take ε_0 to 0, and consider μ^γ as the limit measure, we have that for every $f \in C(D)$

$$\mathbb{E}[\int_D f(x)\mu^\gamma(dx)] = \mathbb{E}[\int_D f(x) : e^{\gamma h} : (dx)].$$

When one consider the case $h = \Gamma$ the GFF in $D \subset \mathbb{R}^2$, the limit measure μ it is usually known as Liouville measure. As we mention before, the interest on study the Liouville measure its due to its relation with the Liouville quantum gravity (as one can see in [DS11]).

From now on, and along this work, we use the following notations for the Liouville measure $M^\gamma(\Gamma, dx)$ from [Aru20], μ^γ, in order to be more coherent with the notation μ_ε^γ, and also we consider the Wick exponent notation $: e^{\gamma\Gamma} : dx$ (for details related to the Wick exponent one can see [Sim74] Section 1.1).

The proofs on the existence and convergence for this case are really alike the measure we will construct in Section 3.3.1. Therefore, we do not do it here.

Remark 1.4 *Up to this point, we do the observation that the Liouville measure it is constructed for the GFF in dimension 2 in bounded domains. For the GFF in dimensions equal or greater than 3 the Liouville measure is trivial. Nonetheless, we also mention that the Liouville measure it is also constructed for the GFF in the Riemann sphere (see [DKRV16]),* *and in the complex Tori (see [DRV16]). As a last curiosity, there is also an infinite analogue that has been studied lately in [BLM24].*

The theory of GMC and Liouville measure it is strongly related also with the thick points. Such connection can be seen in Claim 2.2 from [Aru20]. Here, since we will do a sketch of the proof, we present such claim as a proposition.

Proposition 1.16 *For Γ a given GFF and $\gamma \in (0,2)$. If we sample $X \sim M^\gamma(\Gamma, dx)/M^\gamma(\Gamma, D)$ we have that almost surely X is a γ-thick point of Γ.*

This result gives us the connection between the Liouville measure and the thick points. The notion here is that a "typical point" for the Liouville measure looks like a thick point for the respective GFF. In [Aru20], to understand this notion of typical point, they introduce what they called a rooted measure. This is a measure in $\mathcal{D}'(D) \times D$, where $\mathcal{D}'(D)$ indicates the space of Schwartz distributions on D, and it is defined as

$$\mathcal{Q}_\varepsilon^\gamma(d\Gamma dx) = e^{\gamma\Gamma_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[\Gamma_\varepsilon(x)^2]} \frac{dx}{\lambda(D)} \mathbb{P}(d\Gamma).$$

Where $\lambda(D)$ indicates the Lebesgue measure of D and $\mathbb{P}$ the law of a GFF. This measure give us a connection between the Liouville measure of a given Γ GFF and the field itself. Such relation it is given in the following lemma

Lemma 1.4 *The conditional law of x on D conditioned to Γ is*

$$\mu_\varepsilon(dx) = \frac{\exp\Big(\gamma\Gamma_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[\Gamma_\varepsilon(x)]\Big)}{M^\gamma(\Gamma_\varepsilon, D)} dx. \tag{1.30}$$

And on the other hand, if $\Gamma \sim \tilde{\mathbb{P}}$, where $\tilde{\mathbb{P}}$ is the conditional law of Γ on $\mathcal{D}(D)'$ conditioned to x, we have that

$$\Gamma = \hat{\Gamma} + \gamma Cov(\cdot, \Gamma_\varepsilon(x)) \tag{1.31}$$

where $\hat{\Gamma}$ under $Q_\varepsilon^\gamma(\cdot|x)$ has the law of a GFF.

A way to understand this lemma, is that for every $F : \mathcal{D}'(D) \times D \to \mathbb{R}$ that is bounded and continuous, we have that

$$\mathbb{E}_{\mathbb{P}}[\int_D F(\Gamma, x)\mu_\varepsilon^\gamma(dx)] = \mathbb{E}_{\mathbb{P}}[\int_D F(\Gamma + \gamma G(\cdot, x)_\varepsilon, x)dx].$$

Where $G(\cdot, x)_\varepsilon = \mathbb{E}[\Gamma_\varepsilon(\cdot)\Gamma_\varepsilon(x)]$ is the correlation function of the circle average approximation between x and another point. The equality can also be though as a Cameron-Martin shift. The last expression gives us a stronger result, that makes more clear the one seen in Lemma 1.4.

Proposition 1.17 *The measure $\mathcal{Q}_\varepsilon^\gamma$ converges to a measure $\mathcal{Q}^\gamma$ in the weak topology. And such measure fulfills*

$$\mathbb{E}_{\mathcal{Q}^\gamma}[F(\Gamma, x)] = \mathbb{E}_{\mathbb{P}}[\int_D F(\Gamma, x)M^\gamma(\Gamma, dx)] = \mathbb{E}_{\mathbb{P}}[\int_D F(\Gamma + \gamma G(\cdot, x), x)dx]$$

Where $G(\cdot, x)$ is the limit in the Schwartz sense of $G_\varepsilon(\cdot, x)$.

The above proposition, together with 1.4, allows us to conclude Proposition 1.16. Notice that now we have a measure that almost surely it is supported in the set of thick points of a given GFF.

1.3.2. Fractal dimension of the thick points

Since we just saw that the Liouville measure is supported in the thick points on the respective GFF. In order to study the fractal behavior of T^γ we could use potential theoretic methods. As it is mentioned in [MP10] Section 4.3. This type of techniques replace the Mass distribution principle (For a reference on this last technique, see [MP10] Section 4.2). These techniques are in some sense related to study certain energy in order to obtain lower bounds of the Hausdorff dimensions of certain fractals. From [Fal04], Section 4.3, it is defined, the α-energy, for some $\alpha > 0$, as follows

Definition 1.14 *For $\alpha > 0$ and F a given subset of $\mathbb{R}^d$. We define the α-energy of a given measure μ on F as*

$$I_\alpha(\mu) = \int_F \int_F \frac{\mu(dx)\mu(dy)}{|x-y|^\alpha}.$$

The following theorem gives us a first relation between this energy and the Hausdorff dimension

Theorem 1.5 *Let $F \subset \mathbb{R}^d$, If there exists a mass distribution μ supported on F, such that*

$$I_s(\mu) < \infty,$$

then, in particular $\dim_H(F) \geq s$.

The proof of this theorem can be found in [Fal04], Theorem 4.13. This theorem allows us to conclude the following proposition

Proposition 1.18 *For every $\gamma \in (0,2)$, the set of γ-thick points of the GFF is almost surely*

$$\dim_H(T^\gamma) = 2 - \frac{\gamma^2}{2}.$$

We already saw in 1.3 that the right-hand side is an upper bound. In order to prove is also a lower bound, thanks to Theorem 1.5, we only need to prove the following theorem.

Proposition 1.19 *For every $\alpha < 2 - \frac{\gamma^2}{2}$ we have, almost surely $I_\alpha(M^\gamma(\Gamma)) < \infty$.*

We do a sketch of the proof of this last proposition since the techniques are similar to the work done later in Chapter III

Proof. To prove the desired, we study $\mathbb{E}[I_\alpha(M^\gamma)]$. Since we know that μ_ε^γ from 1.29 converges almost surely to $M^\gamma(\Gamma)$ we have that $E[I_\alpha(M^\gamma)]$ is bounded by above by

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \mathbb{E}[I_\alpha(\mu_\varepsilon^\gamma)] \\
&= \liminf_{\varepsilon\to 0} \int_D\int_D f(x)f(y)\mathbb{E}[e^{\gamma h_\varepsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(x)^2]}e^{\gamma h_\varepsilon(y)-\frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(y)^2]}1_{G_{\alpha,\varepsilon}(x)}1_{G_{\alpha,\varepsilon}(y)}]dxdy \\
&= \liminf_{\varepsilon\to 0} \int_D\int_D f(x)f(y)e^{\gamma^2 G_{\varepsilon,\varepsilon}(x,y)}\mathbb{E}_{\tilde{\mathbb{P}}}[1_{G_{\alpha,\varepsilon}(x)}1_{G_{\alpha,\varepsilon}(y)}]dxdy.
\end{aligned}$$

Where we consider the measure

$$\tilde{\mathbb{P}}(d\Gamma) = e^{\gamma h_\varepsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(x)^2]}e^{\gamma h_\varepsilon(y)-\frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(y)^2]}\mathbb{P}(d\Gamma)$$

From here, one have to prove the following claim

Claim 1.1 *There exists a constant $C >$, such that for every $x, y \in D$, we have that*

$$\tilde{\mathbb{P}}(G_{\alpha,\varepsilon}(x), G_{\alpha,\varepsilon}(y)) \leq Cr^{\frac{(2\alpha-\gamma)^2}{2}}.$$

Where $r = |x-y| \vee \varepsilon$.

We do not prove this claim here, later on we prove Claim 3.1 and the techniques used are the same as it is done in [Ber17]. Using this claim one hand we used the estimation (1.14), the conclusion is direct. We do not do the details since the techniques are quite similar to the things done in Chapter III. □

Up to this point, one can see that in order to study the set of thick points of the GFF, and obtain his Hausdorff dimension. One can do as follows

(*i*) Obtain a modulus of continuity of the field using an extension of the Kolmogorov continuity theorem.

(*ii*) From the modulus of continuity, do a count-box argument to obtain the upper bound for the Hausdorff dimension.

(*iii*) Construct a GMC measure and obtain the Lower bound using potential theoretic methods.

This 3-step list is what we plan to use in order to study our processes of interest.

1.4. Orstein-Uhlenbeck process

As we mentioned in the introduction. One of our process of interest could be though as an infinite dimensional analogue of the Orstein-Uhlenbeck process. Since such process will also appear in the context of the additive stochastic heat equation. We see some of its properties.

The Orstein-Uhlenbeck process (from now on OU or OU process). Can be defined as the solution of the following SDE

$$\begin{cases} dX_t = -\theta X_t dt + \lambda dB_t, \\ X(0) = x_0 \end{cases} \tag{OU}$$

Where B indicates a Brownian motion, usually one calls θ the speed or rate of the process, and λ its volatility. The OU process it is already a process of interest due to its applications in physics, financial mathematics, and evolutionary biology. One have that the solution of the SDE is the following

Lemma 1.5 *For a given initial condition X_0 of* (OU). *We have that the solution is given by*

$$X_t = X_0 e^{-\theta t} + \lambda \int_0^t e^{-\theta(t-s)} dB_s.$$

As it is mentioned in [LG16] Section 8.4. To prove that the above is the solution, one could apply Itô's formula with the process $Y_s = (s, X_s)$ in the function $F(t,x) = e^{\theta t} x$. As a small observation, notice that if $X_0 \sim \mathcal{N}(0, \sigma^2)$. We have then that the OU process is a Gaussian process.

Since we saw that the distributional derivative of the Brownian motion is the time white noise η, the solution of (OU) can be re-written as

$$X_t = X_0 e^{-\theta t} + \lambda \langle f_t(s), \eta \rangle.$$

Where $f_t(s) = e^{-\theta(t-s)} 1_{[0,t]}$. From this expression one can see the following lemma

Lemma 1.6 *For a centered OU process X_t with rate θ, volatility λ, and initial condition a centered random variable X_0 independent of η and with second moment. We have that the correlations at two given times $s, t > 0$ are*

$$\mathbb{E}[X_s X_t] = e^{-\theta(t+s)} \mathbb{E}[X_0^2] + \frac{\lambda^2}{2\theta} e^{-\theta(t+s)} \left(e^{2\theta(t \wedge s)} - 1 \right).$$

The proof can be done using the independence of the initial condition and η plus the already known covariance structure of the white noise from Section 1.1.3. As a direct corollary one have the following

Corollary 1.5 *For X_t an OU process with initial condition independent $X_0 \sim \mathcal{N}(0,1)$ of η. If $\lambda^2 = 2\theta$, we have that for any two given times $s, t > 0$, the correlations are*

$$\mathbb{E}[X_s X_t] = e^{-\theta|t-s|}.$$

In particular, we have then that the stationary law of X_t is a standard normal distribution.

From now on, when we speak about an OU process we assume that its initial condition is a standard Gaussian variable, unless it is otherwise said. Since we know that the Gaussian processes are characterized by its covariance structure, Corollary 1.5 allows us to prove the following Lemma

Lemma 1.7 *If X is an OU process with rate θ. Then we have that*

$$X_t \overset{\mathcal{L}}{=} e^{-\theta t} B_{e^{2\theta t}}.$$

Where B indicates a Brownian motion.

Proof. Since we are dealing with Gaussian processes, it is enough to check that both processes have the same mean and covariance structure in time. □

As a direct corollary of this, we have the following result

Corollary 1.6 *the OU process X with rate θ is time-reversible, and therefore it can be indexed by $\mathbb{R}$.*

The above properties will be in used later in order to study the evolution's of the GFF.

1.5. Stochastic partial differential equations

Since in the introduction we mentioned that the dynamics of interest are given by stochastic partial differential equations (from now on SPDEs). In the present chapter we introduce the basics of the theory of linear SPDEs presented in [Hai09] Chapter 6. We only focus and present the case of Hilbert spaces. Therefore, from now on, we are interested in describe certain evolution of Gaussian variables at values in $\mathcal{H}$ some given Hilbert space

We are interested in equations that be seen as evolution equations with a random force in the given Hilbert space

$$\partial_t u = Lu + QdW. \tag{1.32}$$

Where L is a linear operator with domain given by $D(L)$ a subset of $\mathcal{H}$ such that generates some Feller semigroup $(P_t)_{t>0}$ (for a definition we suggest [LG16] Section 6.2). And W indicates a cylindrical Gaussian process. The term dW can be understood as noise. Such noise can be understood as a random force or random fluctuations that gives our dynamic its random behavior.

Given that in this concept we have a random force term given by a Gaussian process in an infinite dimensional Hilbert space, one could not expect that the solution lives in $D(L)$. Therefore, one might not be able to obtain a strong solution of the desired equation. In order to give a sense to the equation, we first consider the weak approach, and hence, we define a weak solution as follows

Definition 1.15 *Let us consider $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ a Hilbert space, and $L : D(L) \subset \mathcal{H} \to \mathcal{H}$ a linear operator. For W a cylindrical Gaussian process over another Hilbert space $(\mathcal{H}', \langle\cdot,\cdot\rangle_{\mathcal{H}'})$ and $Q : \mathcal{H} \to \mathcal{H}'$ a bounded linear operator. We say u is a weak solution of*

$$\begin{cases} d_t u = Ludt + QdW(t) \\ u(0) = u_0 \end{cases} \tag{1.33}$$

if and only if,

$$\int_0^t \|u(s)\|_{\mathcal{H}} < \infty$$

and for every $\ell \in D(L^\star)$ we have

$$\langle \ell, u(t)\rangle_{\mathcal{H}^\star,\mathcal{H}} - \langle \ell, u_0\rangle_{\mathcal{H}^\star,\mathcal{H}} = \int_0^t \langle L^\star \ell, u(s)\rangle_{\mathcal{H}^\star,\mathcal{H}} ds + \int_0^t Q^\star \ell dW(s).$$

Here the second term of the right-hand side is in the sense of 1.2 since $Q^\star\ell \in \mathcal{L}_{HS}(\mathcal{H}', \mathbb{R})$.

In other words, we are in a strong-weak sense. Strong in the sense of SDEs, but weak under the perspective of PDE theory. In order to give a solution of the above equation. We first define what is call a mild solution

Definition 1.16 *We said u is a mild solution. If it is a Gaussian process that takes values in $\mathcal{H}$ and almost surely for every $t > 0$ satisfies*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)QdW_s.$$

Where $(S(t))_{t>0}$ indicates the Feller semi-group generated by L.

We present this result, since one have the following equivalence

Proposition 1.20 *If the mild solution is integrable. Then, it is an also weak solution. Conversely, every weak solution is a mild solution.*

the proof is rather technical but can be founded in [Hai09] proof of Proposition 6.7. From the above proposition it is direct that the existence and uniqueness of solution for (1.33)

Proposition 1.21 *If there exists a solution sense of 1.15. Then it is unique*

Proof. Is a direct consequence of Proposition 1.20. Since every weak solution is equal to the mild solution. □

Since we are in the Hilbert case, one can prove the following proposition in order to characterize the solution

Proposition 1.22 (Solution of SPDE in Hilbert case) *For $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis for $D(L^\star)$. We have that u is a solution in the sense of 1.15 if and only if, for every $k \in \mathbb{N}$, we have that*

$$\langle e_k, u(t)\rangle_{D(L^\star),D(L)} - \langle e_k, u(0)\rangle_{D(L^\star),D(L)} = \int_0^t \langle L^\star e_k, u(s)\rangle_{\mathcal{H}^\star,\mathcal{H}} ds + \int_0^t Q^\star e_k dW(s)$$

The proof is use the decomposition of every $\ell \in D(L^\star)$. If one considers corollary 1.3, we have then, for $(\hat{e}_l)_{l\in\mathbb{N}}$ a basis for $\mathcal{H}'$ that

$$\int_0^t Q^\star e_k dW(s) = \sum_{l\in\mathbb{N}} \int_0^t \langle Q^\star e_k, \hat{e}_l\rangle_{(\mathcal{H}')^\star,\mathcal{H}'} dB_s^l,$$

where $(B^l)_{l\in\mathbb{N}}$ is an i.i.d sequence of Brownian motions.

1.6. The heat equation

In this section we present some basics results regarding the heat equations. In particular, we are interested in solve this equation using the Brownian motion since this will be useful for the stochastic counterpart. Since we are now considering a space-time domain. For a given $T > 0$, we introduce the notation $D_T = D \times (0, T)$.

As it usually done in PDE theory. In order to obtain a solution of an equation and sometimes have uniqueness. One usually needs to consider information of the problem. Such information can be of the form of an initial condition or a boundary condition or others. In the context of the heat, a sufficient condition to have uniqueness of solution is to consider information on part of the space-time boundary, the respective part of the boundary is call parabolic boundary, and it is defined as follows

Definition 1.17 *For $D \subset \mathbb{R}^d$ open and bounded, and $T > 0$ we define the parabolic boundary as*

$$\partial_p D_T = \overline{D} \times \{0\} \cup \partial D \times (0, T)$$

In [Eva22], Section 2.3. It is proven how a condition on the parabolic boundary implies uniqueness in the solution via an argument of maximum principle. Assuming this uniqueness, we now give an expression for the solution given a parabolic boundary condition.

Lemma 1.8 *For a given $f \in C(\partial_p D_T)$, we have that a strong solution of the heat equation*

$$(H) \begin{cases} \partial_t u = \frac{1}{2}\Delta u, & \text{in } D_T \\ u = f & \text{in } \partial_p D_T. \end{cases}$$

is given by

$$u(x, t) = \mathbb{E}[f(t - (t \wedge \tau_D), B^x_{t\wedge\tau_D})].$$

Where, B indicates a Brownian motion, and τ_D indicates the exit time of D.

Proof. Using the regularization property of the heat equation (see for example Theorem 8 in [Eva22]) we know that $u \in C^\infty(D_T)$. Hence, if we consider the process $X^x_s = (t - s, B^x_s)$, and apply Itô's formula on u at time $t \wedge \tau_D$, doing some elementary computations we have that

$$u(B^x_{t\wedge T_D}, t - t \wedge \tau_D) - u(x, t) = \int_0^{t\wedge T_D} \nabla_x u(B^x_s, t - s) \cdot dB^x_s.$$

Since B is a martingale, we have that the stochastic integral is also a martingale. Therefore, when we take expectation we have that

$$u(t, x) = \mathbb{E}[f(B^x_s, t - t \wedge \tau_D)],$$

since, almost surely $(B^x_{t\wedge\tau_D}, t - t \wedge \tau_D) \in \partial_p D_T$. □

The above result, has a useful and interesting corollary, when one consider a initial condition in $C^\alpha(D)$ the family of α-Hölder function. This is

Corollary 1.7 *For $f \in C^\alpha(\bar{D})$ for $\alpha \in (0, 1)$, we have that if u is the strong solution of the equation*

$$(H) \begin{cases} \partial_t u = \frac{1}{2}\Delta u, & \text{in } D_T \\ u(\cdot, 0) = f & \text{in } \bar{D}, \\ u = 0 & \text{in } \partial D \times [0, T] \end{cases}$$

then, we will have that u is also α Hölder when in D_T we consider the next metric

$$d^p((x, t), (y, s)) = |x - y| + \sqrt{|t - s|}.$$

PROOF. If u indicates the solution, and f the initial condition. We have that

$$\begin{aligned}|u(x,t)-u(y,s)| &= |\mathbb{E}[f(B^x_{t\wedge\tau_D})]-\mathbb{E}[f(B^y_{s\wedge\tau_D})]| \\ &\le K\left(\mathbb{E}[|B^x_t-B^x_s|^2]\right)^{\alpha/2} = Kd^p((x,t),(y,s))^\alpha.\end{aligned}$$

□

The above result is an important one since it shows us that for the heat equation. The metric d^p defined above. Usually, this metric can be found in the literature as the *parametric metric*. As a comment, most of the results for the field related to the stochastic heat equations, is due to this metric in one or other sense.

Chapter 2

The Orstein-Uhlenbeck GFF

In the present chapter the main object of interest is what we call thickness function (See Definition 2.2) for a specific dynamic on the GFF. The notion of thickness function can be understood as the fluctuations of the thickness of a point through time. We first prove that such fluctuations are always continuous. Then, we focus on the problem related to find points in space such that the thickness function follows some specific given f trajectory. We characterize when such points exist under an energy constraint, that it is some sense a continuous analogue to the maximum thickness for the case of the GFF.

Remark 2.1 *In this chapter, C represents a finite positive constant that may vary between lines. Any dependencies on variables will be explicitly noted.*

In Section 1.2, we saw that a way to construct the GFF was to write it in the spectral basis of $H_0^1(D)$. To do this we needed $(\beta_k)_{k\in\mathbb{N}}$ an i.i.d. sequence of normal random variables. In order to construct a dynamic that has the GFF as a stationary distribution, we can restrict ourselves to generate a dynamic on $(\beta_k)_{k\in\mathbb{N}}$ that keeps the standard normal variable as a stationary distribution. An elementary way to do this is, for each k, consider $\alpha_k(t)$ the solution of an SDE where the initial condition is β_k, and have the desired stationary law. As a first choice for such process we considered the Orstein-Uhlenbeck. Under this choice, we present the first dynamic of interest that we call the Orstein-Uhlenbeck GFF as follows

Definition 2.1 (Orstein-Uhlenbeck GFF) *Take $(\alpha_k(t))_{k\in\mathbb{N}}$ an i.i.d. sequence of Orstein-Uhlenbeck processes with rate $1/2$ and initial condition a Gaussian distribution. The Orstein-Uhlenbeck GFF (from now on OU-GFF) is defined as*

$$\Phi(t) = \sum_{k\in\mathbb{N}} \alpha_k(t) e_k, \tag{2.1}$$

where $(e_k)_{k\in\mathbb{N}}$ is the spectral basis for $H_0^1(D)$.

Remark 2.2 *Since we know that the Orstein-Uhlenbeck processes are reversible in time. It is direct that the same holds for the OU-GFF. Therefore, our process Φ is indexed in time by $\mathbb{R}$.*

In order to study a bit more our process, we first proof the following lemma

Lemma 2.1 *The Orstein-Uhlenbeck GFF Φ fulfills the following equation*

$$\begin{cases} \partial_t \Phi = -\frac{1}{2}\Phi + W, \\ \Phi(0) = \Gamma. \end{cases} \tag{2.2}$$

Here W indicate the standard Gaussian variable of $L^2(\mathbb{R}; H_0^1(D))$ and Γ a GFF independent of W.

Proof. Taking $(e_k)_{k\in\mathbb{N}}$ as the spectral basis of $H_0^1(D)$ and using Proposition 1.7 we know that $\langle W, e_k\rangle_\nabla$ is nothing more but the Gaussian distribution of $L^2(\mathbb{R})$, and that is nothing more but the distributional derivative of B^k a Brownian motion. On the other hand, it is straightforward that $\langle \Phi(t), e_k\rangle_\nabla = \alpha_k(t)$. Therefore, since $\alpha_k(t)$ is an Orstein-Uhlenbeck with rate $1/2$ we have that

$$\begin{cases} d_t\alpha_k(t) = -\frac{1}{2}\alpha_k dt + dB_t^k \\ \alpha_k(0) = n_k. \end{cases} \tag{OU}$$

Here $(n_k)_{k\in\mathbb{N}}$ is an i.i.d sequence of normal random variables. From (OU) the lemma is direct plus Proposition 1.22. We can conclude the lemma. □

Notice that, from here, due to Proposition 1.22 we have that Definition 2.1 does not depend on the basis considered. We now start to look up some basics results for this process.

2.1. Basic results

In this section we start the study of the OU-GFF. In particular, we compute expression for the correlations. Present the circle average approximation as in Section 1.2.1, and prove the continuity of this approximation.

Lemma 2.2 *Let us consider $s, t \in \mathbb{R}$, and $f, g \in H_0^1(D)$, then, the covariance between $\langle \Phi(t), f\rangle_\nabla$ and $\langle \Phi(s), g\rangle_\nabla$ is given by*

$$\mathbb{E}[\langle \Phi(t), f\rangle_\nabla \langle \Phi(s), g\rangle_\nabla] = e^{-\frac{1}{2}|s-t|}\langle f, g\rangle_\nabla.$$

As a direct consequence. We have that for $f, g \in L^2(D)$ the respective correlations are

$$\mathbb{E}[\langle \Phi_{r_n}(t), f\rangle \langle \Phi(s), g\rangle] = e^{-\frac{1}{2}|t-s|}\langle f, (-\Delta)^{-1} g\rangle.$$

Proof. To check this we can simply compute using Expression (2.1). Since for s, t fixed we know that such series converges in L^2. We have then used the independence of the sequence $(\alpha_k(\cdot))_{k\in\mathbb{N}}$ that

$$\mathbb{E}[\langle \Phi(t), f\rangle_\nabla \langle \Phi(s), g\rangle_\nabla] = \sum_{k\in\mathbb{N}} \mathbb{E}[\alpha_k(s)\alpha_k(t)]\langle f, e_k\rangle_\nabla \langle g, e_l\rangle_\nabla = \sum_{k\in\mathbb{N}} e^{-\frac{1}{2}|s-t|}\langle f, e_k\rangle_\nabla \langle g, e_l\rangle_\nabla.$$

Therefore, we can conclude the desire equality. □

Now we do as we did for the case of the GFF and consider circle average approximation given in Definition 1.12 and for $t \in \mathbb{R}$ take

$$\Phi_\varepsilon(x,t) = \langle \Phi(t), \mu_{\varepsilon,x} \rangle.$$

Notice that the integration is only done in space since we are interested in the behavior of the thick points through time. We now show the continuity of this process (up to a modification) in (x,t,ε) in the following proposition

Proposition 2.1 *The process $\Phi_\varepsilon(x,t)$ possesses a modification (that we identify with Φ as well) such that for all $\alpha \in (0,1/2)$ and $\rho, \zeta > 0$, almost surely there exists a constant $M = M(\alpha,\rho,\zeta)$ such that*

$$|\Phi_\varepsilon(x,t) - \Phi_\delta(y,s)| \leq M(\ln\Big(\frac{1}{\varepsilon \wedge \delta}\Big))^\zeta \frac{(|(x,t,\varepsilon) - (y,s,\delta)|)^\alpha}{(\varepsilon \wedge \delta)^{\alpha+\rho}}$$

for all $(x,t),(y,s) \in D \times [0,T]$ and $\varepsilon, \delta \in [0,1]$ such that $1/2 \leq (\varepsilon \wedge \delta)/(\varepsilon \vee \delta) \leq 2$.

Proof. As we did for the GFF, to prove this we will the modified Kolmogorov's theorem we saw in [HMP10] Appendix C. Take (x,t,ε) and $((y,s,\delta))$ and consider

$$\mathbb{E}[|\Phi_\varepsilon(x,t) - \Phi_\delta(y,s)|^2] = \mathbb{E}[\Phi^2_\varepsilon(x,t) - \Phi_\varepsilon(x,t)\Phi_\delta(y,s)] + \mathbb{E}[\Phi^2_\varepsilon(y,s) - \Phi_\varepsilon(x,t)\Phi_\delta(y,s)].$$

Without loss of generality, we focus on

$$\mathbb{E}[\Phi^2_\varepsilon(x,t) - \Phi_\varepsilon(x,t)\Phi_\delta(y,s)] = G^D_\varepsilon(x) - e^{-\frac{1}{2}|t-s|} G^D_{\varepsilon,\delta}(x,y)$$

where we recall that $G^D_{\varepsilon,\delta}(x,y)$ indicates the Green function integrated with respect to the product measure $\mu_{\varepsilon,x} \otimes \mu_{\delta,y}$. Using that for all $x > 0$ we have $-e^{-x} \leq x - 1$. Therefore, we have

$$\mathbb{E}[\Phi^2_\varepsilon(x,t) - \Phi_\varepsilon(x,t)\Phi_\delta(y,s)] \leq \frac{1}{2}|t-s| G^D_{\varepsilon,\delta}(x,y) + C\frac{|(x,\varepsilon) - (y,\delta)|}{\varepsilon \wedge \delta}. \tag{2.3}$$

Where in the last inequality we applied the one saw in the proof of Proposition 2.1 from [HMP10]. On the other hand, Proposition 1.14 implies that

$$\frac{1}{2}|t-s| G_{\varepsilon,\delta}(x,y) \leq \frac{1}{2}|t-s|(\ln\Big(\frac{1}{\varepsilon \vee \delta \vee |x-y|}\Big) + K).$$

Using $\ln(x) \leq x$ for all $x > 0$ we have then

$$\frac{1}{2}|t-s| G_{\varepsilon,\delta}(x,y) \leq \frac{C}{2}|t-s| \frac{1}{\varepsilon \wedge \delta}. \tag{2.4}$$

Therefore, replacing in (2.3) with (2.4), we have

$$\mathbb{E}[\Phi_\varepsilon(x,t)^2 - 2\Phi_\varepsilon(x,t)\Phi_\delta(y,s) + \Phi_\delta(y,s)^2] \leq \frac{C}{\varepsilon \wedge \delta}(|(x,\varepsilon) - (y,\delta)| + |t-s|)$$

therefore, we can conclude the proposition applying Proposition 1.3 since in finite dimensional vector spaces all the norms are equivalent. □

2.2. The thickness functions

In Section 1.3, we introduce the notion of the thickness of a point. The dynamic analogue of this definition is what we call *thickness function* defined as follows

Definition 2.2 *Let Φ the OU-GFF. For $x \in D$ we define his* ***thickness function*** *as*

$$t \in \mathbb{R} \mapsto \gamma_x(t) = \limsup_{\varepsilon \to 0} \frac{\Phi_\varepsilon(x,t)}{\ln\left(\frac{1}{\varepsilon}\right)}. \tag{2.5}$$

The main result we prove related to this random function is the following.

Proposition 2.2 *Almost surely, for every $x \in D$ the function $\gamma_x(\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous.*

As a first corollary from this proposition is that we cannot have points in space that his thickness jumps from 0 to a positive value in an instant.

Corollary 2.1 *For X sampled uniformly on D independent of Φ. We have that*

$$\gamma_X = 0, \; \forall t \in \mathbb{R}.$$

As a second consequence. When we look up the set of thick points in space and time. We can obtain for which values of γ such set is not empty.

Corollary 2.2 *The set $T^\gamma := \{(x,t) \in D_T;\; \gamma_x(t) = \gamma\}$ of thick points in D_T is not empty if and only if $\gamma \in [0,2]$.*

As a third corollary. We look up the thickness function related to the points sampled from the Liouville measure of the initial condition.

Lemma 2.3 *Take Φ a OU-GFF and $\Gamma = \Phi(0)$ a GFF. Take $\gamma \in (0,2)$, and $M^\gamma(\Gamma, dx)$ the respective normalized Liouville measure of Γ. If X is sampled from this measure, we have then*

$$\gamma_X(t) = e^{\frac{-1}{2}|t|}\gamma \; \forall t \in \mathbb{R}.$$

Proof. Since the process is reversible in time we only need to check for $t > 0$. Let us now take $X \sim M^\gamma(\Gamma, dz)$, the chaos measure related to the initial condition, and take $t > 0$. We first claim the following.

Claim 2.1 *There exist Φ^* an independent GFF of Γ, such that*

$$\Phi_\epsilon(x,t) = e^{-\frac{1}{2}t}\Gamma_\epsilon(x) + \sqrt{1-e^{-t}}\Phi^*_\epsilon(x). \tag{2.6}$$

From the claim, it follows that, when we evaluate X in $\Phi_\varepsilon(t)$, and divide by $\ln\left(\frac{1}{\varepsilon}\right)$ we have

$$\frac{\Phi_\epsilon(X,t)}{\ln\left(\frac{1}{\varepsilon}\right)} \overset{(\mathcal{L})}{=} e^{-\frac{1}{2}t}\frac{\Gamma_\epsilon(X)}{\ln\left(\frac{1}{\varepsilon}\right)} + \sqrt{1-e^{-t}}\frac{\Phi^*_\epsilon(X)}{\ln\left(\frac{1}{\varepsilon}\right)},$$

since Φ^* is independent of the initial condition and X is sampled from the Liouville measure related to it, when taking the limit of $\varepsilon \to 0$ we have

$$\lim_{\varepsilon\to 0} \frac{\Phi_\epsilon(X,t)}{\ln\left(\frac{1}{\varepsilon}\right)} = e^{-\frac{1}{2}t}\gamma.$$

Using that γ_x is always continuous we can conclude the desired lemma by proving the claim.

Proof of Claim 2.1. Consider

$$\Phi^\star = (1-e^{-t})^{-\frac{1}{2}}\left(\Phi(t) - e^{-\frac{1}{2}t}\Gamma\right).$$

It is direct is a Gaussian process since it is a linear combination of them. Let us check now that is independent of Φ. To do this since there are Gaussian processes we can conclude the desired by calculating the correlations. Taking $f, g \in L^2(D)$ we have that

$$\begin{aligned}\mathbb{E}[\langle\Phi^\star, f\rangle\langle\Phi(s), g\rangle] &= (1-e^{-t})^{-\frac{1}{2}}\left(\mathbb{E}[\langle\Phi(t), f\rangle\langle\Phi(0), g\rangle] - e^{-\frac{1}{2}t}\mathbb{E}[\langle\Phi(s), g\rangle\langle\Gamma, f\rangle]\right)\\ &= (1-e^{-t})^{-\frac{1}{2}}\left(e^{-\frac{1}{2}t}\langle f, (-\Delta)^{-1}g\rangle - e^{-\frac{1}{2}t}\langle f, (-\Delta)^{-1}g\rangle\right) = 0\end{aligned}$$

Hence it is an independent process. To check that it is a GFF. We only need to corroborate the correlations are the right ones since it is direct that the mean will always be 0. We take then $f, g \in L^2(D)$

$$\mathbb{E}[\langle\Phi^\star, f\rangle\langle\Phi^\star, g\rangle] = ((1-e^{-t}))^{-1}\left(\langle f, (-\Delta)^{-1}g\rangle - e^{-t}\langle f, (-\Delta)^{-1}g\rangle\right) = \langle f, (-\Delta)^{-1}g\rangle.$$

Therefore, the claim is proven. □

□

We show these lemmas to justify the study of the thickness function. To prove the main proposition of this section we first need the following lemma.

Lemma 2.4 *There exists a deterministic sequence $(r_n)_{n\in\mathbb{N}}$ that goes to 0 as n goes to infinity such that a.s. for every $(x,t) \in D_T$*

$$\limsup_{\varepsilon\to 0} \frac{\Phi_\varepsilon(x,t)}{\ln\left(\frac{1}{\varepsilon}\right)} = \limsup_{n\to\infty} \frac{\Phi_{r_n}(x,t)}{\ln\left(\frac{1}{r_n}\right)}.$$

Proof. Notice that since r_n is a sequence that converges to 0 when $n \to \infty$, it is direct that

$$\limsup_{\varepsilon\to 0} \frac{\Phi_\varepsilon(x,t)}{\ln\left(\frac{1}{\varepsilon}\right)} \geq \limsup_{n\to\infty} \frac{\Phi_{r_n}(x,t)}{\ln\left(\frac{1}{r_n}\right)}.$$

Therefore, we only need to check the other inequality. To do this we follow [HMP10] Section 3. From Proposition 2.1 we know that for every given $\alpha, \zeta, \delta \in (0, 1/2)$ almost surely there exists a constant $M > 0$ such that

$$|\Phi_\varepsilon(x,t) - \Phi_{r_n}(x,t)| \leq M(\ln\left(\frac{1}{\varepsilon\wedge r_n}\right))^\zeta \frac{|\varepsilon - r_n|^\alpha}{(\varepsilon\wedge r_n)^{\alpha+\delta}} \leq M(\ln\left(\frac{1}{r_{n+1}}\right))^\zeta \frac{|r_n - r_{n+1}|^\alpha}{(r_{n+1})^{\alpha+\delta}},$$

for $r_{n+1} \le \varepsilon \le r_n$. Now we need to control the right-hand side. Take $r_n = n^{-K}$ for some $K \in \mathbb{N}$ such that $0 < K < \alpha/\delta$, with this choice we have that there exists a constant $C = C(K)$ such that

$$\frac{|r_n - r_{n+1}|^\alpha}{(r_{n+1})^{\alpha+\delta}} = (n+1)^{K(\alpha+\delta)} \frac{|n^K - (n+1)^K|^\alpha}{((n+1)n)^{K\alpha}} \le C \frac{(n+1)^{K\delta}}{n^\alpha}, \tag{2.7}$$

where this last inequality was achieved using the mean value theorem. Therefore, we can conclude that (2.7) is uniformly upper bounded by some constant C. Hence, replacing in (2.2) we have

$$|\Phi_\varepsilon(x,t) - \Phi_{r_n}(x,t)| \le C(\ln\left(\frac{1}{r_{n+1}}\right))^\zeta.$$

Then dividing by $\ln(1/r_n)$ we have

$$\frac{\Phi_\varepsilon(x,t)}{\ln(1/r_n)} - \frac{\Phi_{r_n}}{\ln(1/r_n)} \le \frac{C(\ln\left(\frac{1}{r_{n+1}}\right))^\zeta}{\ln(1/r_n)} \le C \ln\left(\frac{1}{r_{n+1}}\right)^{\zeta-1}.$$

This implies that

$$\frac{\Phi_\varepsilon(x,t)}{\ln(1/r_n)} \le \frac{\Phi_{r_n}}{\ln(1/r_n)} + \ln\left(\frac{1}{r_n}\right)^{\zeta-1}, \tag{2.8}$$

and is valid for $r_{n+1} \le \varepsilon \le r_n$, and $\zeta < 1$. Then, since $\ln\left(\frac{1}{r_n}\right)/\ln\left(\frac{1}{\varepsilon}\right) < 1$, and using (2.8), we can conclude that

$$\sup_{\varepsilon \in (r_{n+1}, r_n)} \frac{\Phi_\varepsilon(x,t)}{\ln\left(\frac{1}{\varepsilon}\right)} \le \frac{\Phi_{r_n}}{\ln(1/r_n)} + \ln\left(\frac{1}{r_n}\right)^{\zeta-1}. \tag{2.9}$$

When we take the limsup for n tending to infinity on (2.9) we can conclude the lemma. □

This result simplifies the limit from Definition 2.2. From this simplification we now prove the main result.

Proof of Proposition 2.2. Fix $q, \delta \in \mathbb{Q}$. We find a continuous function ρ

$$A_n(\delta, q) := \left\{ \sup_{x \in D} \sup_{s \in [q, q+\delta]} |\Phi_{r_n}(x,q) - \Phi_{r_n}(x,s)| \le \ln\left(\frac{1}{r_n}\right)\rho(\delta) \right\}. \tag{2.10}$$

For some function $\rho(\cdot)$ continuous and such that $\rho(0) = 0$. Let us first assume the following claim and see how implies the proposition.

Claim 2.2 *There exists $\rho : \mathbb{R} \to \mathbb{R}$ a continuous function that $\rho(0) = 0$, such that for every $\delta, q \in \mathbb{Q}$, we have that*

$$\mathbb{P}(\liminf_{n\to\infty} A_n(\delta, q)) = 1.$$

From claim 2.2 it follows that a.s. there exists some $n_0 \in \mathbb{N}$ such that for all $n > n_0$

$$|\Phi_{r_n}(x,q) - \Phi_{r_n}(x,s)| \le \ln\left(\frac{1}{r_n}\right)\rho(\delta),\ \forall (x,s) \in D \times [q-\delta, q+\delta], 3 \tag{2.11}$$

since it is also valid for$\hat{q} = q - \delta$. Then dividing by $\ln(1/r_n)$ and taking the limit of $n \to \infty$.

We can conclude that a.s.

$$|\gamma_x(s) - \gamma_x(q)| \le \rho(\delta)\ \forall (x,s) \in D \times [q-\delta, q+\delta]$$

Since this is a.s. for every $\delta, q \in \mathbb{Q}$. Taking the intersection we can conclude that for every $s > 0$ and s_n tending to s. Taking $|s_n - s| \le \delta/4$. We can find a rational at distance at most $\delta/2$ of both of them and conclude using the triangular inequality. To conclude we just need to prove the claim

PROOF OF CLAIM 2.2. Since the event $A_n(\delta, q)$ is too complicated to study directly. We start by approximating the domain using squares. Taking r_n as in Lemma 2.4 and then $\hat{r}_n = r_n^{1+\varepsilon}$ for some $\varepsilon > 0$. We consider the collection of squares given by

$$\mathcal{Q}_n := \{(z + [k\hat{r}_n, (k+1)\hat{r}_n] \times [l\hat{r}_n, (l+1)\hat{r}_n])\,;\ z \in \mathbb{Z}^2,\ k, l \in \{0, ..., \lceil (r_n)^{-1} \rceil - 1\}\}.$$

For $\Box \in \mathcal{Q}_n$ let $z_\Box$ denote the center of it. From Proposition 2.1, we know that for $\alpha, \varepsilon, \zeta$ in $(0, 1/2)$ a.s. there exist a constant C such that

$$|\Phi_{r_n}(x,t) - \Phi_{r_n}(z_\Box, t)| \le C(\ln\Big(\frac{1}{r_n}\Big)^\zeta)\frac{1}{r_n^{\hat{\alpha}}}|z_\Box - x|^\alpha \le C(\ln\Big(\frac{1}{r_n}\Big)^\zeta), \tag{2.12}$$

where $\hat{\alpha} = \alpha(1+\varepsilon)$, and for the last inequality we use the side length of the square. From this inequality we can conclude that for $x \in \Box$ and $s \in [q, q+\delta]$ we have that $|\Phi_{r_n}(x,q) - \Phi_{r_n}(x,s)|$ is upper bounded by

$$|\Phi_{r_n}(x,q) - \Phi_{r_n}(z_\Box, q)| + |\Phi_{r_n}(z_\Box, s) - \Phi_{r_n}(z_\Box, q)| + |\Phi_{r_n}(x,s) - \Phi_{r_n}(z_\Box, s)|\,.$$

This inequality plus (2.12) implies

$$\begin{aligned} &\sup_{x\in D}\sup_{s\in[q,q+\delta]} |\Phi_{r_n}(x,q) - \Phi_{r_n}(x,s)| \\ &= \max_{\Box\in\mathcal{Q}_n; \Box\cap D\neq\emptyset}\sup_{x\in\Box}\sup_{s\in I(q,\delta)} |\Phi_{r_n}(x,q) - \Phi_{r_n}(x,s)| \\ &\le \max_{\Box\in\mathcal{Q}_n; \Box\cap D\neq\emptyset}\sup_{s\in I(q,\delta)} |\Phi_{r_n}(z_\Box,q) - \Phi_{r_n}(z_\Box,s)| + C\ln\Big(\frac{1}{r_n}\Big)^\zeta. \end{aligned} \tag{2.13}$$

And This expression implies that $\mathbb{P}(A_n^c(\delta,q)) \le$ is upper bounded by

$$\mathbb{P}\left(\max_{\Box\in\mathcal{Q}_n; \Box\cap D\neq\emptyset}\sup_{s\in I(q,\delta)} |\Phi_{r_n}(z_\Box,q) - \Phi_{r_n}(z_\Box,s)| \ge \ln\Big(\frac{1}{r_n}\Big)\left(\rho(\delta) - C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}\right)\right). \tag{2.14}$$

From here we focus on the right-hand side of (2.13). We focus first on the event

$$\left\{\sup_{s\in[q,q+\delta]} |\Phi_{r_n}(z_\Box,q) - \Phi_{r_n}(z_\Box,s)| \ge \ln\Big(\frac{1}{r_n}\Big)\left(\rho(\delta) - C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}\right)\right\}.$$

To continue, we consider the process given by $X_s = \Phi_{r_n}(z_\Box, s)/\sqrt{\ln\big(\frac{1}{r_n}\big)}$. From the previous work we know that this has the law of an Orstein-Uhlenbeck process. Therefore, the desired event can be written as

$$\sup_{s\in[0,\delta]} |X_{s+q} - X_q| \geq \sqrt{\ln\Big(\frac{1}{r_n}\Big)}\left(\rho(\delta) - C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}\right).$$

Since X_s is an O.U. we know that is equal in law to a Brownian motion changed in time and re-escalated, more specifically we have the equality $X_s = e^{-s/2}B_{e^s}$ in law. Hence, the event (2.2) is equal to the event

$$\sup_{s\in[0,\delta]} |e^{-(s+q)/2}B_{e^{s+q}} - e^{-q/2}B_{e^q}| \geq \sqrt{\ln\Big(\frac{1}{r_n}\Big)}\left(\rho(\delta) - C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}\right).$$

Since we know that for a Brownian motion B we have that $B_{C^2 s} \stackrel{\mathcal{L}}{=} CB_s$. The event (2.2) is equal to

$$\sup_{s\in[0,\delta]} |e^{-s/2}B_{e^s} - B_1| \geq \sqrt{\ln\Big(\frac{1}{r_n}\Big)}\left(\rho(\delta) - C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}\right).$$

We need to do a small before continue observation. For $x, y \in \mathbb{R}^2$, it is direct that

$$|x - y| \leq 2|x_1 - y_1| \vee |x_2 - y_2|. \tag{2.15}$$

From this inequality, we do an abuse of notation and for now B indicate a 1 dimensional Brownian motion, and using the symmetry of this around the $\hat{x}$ axis. The event (2.2) is equivalent to

$$\sup_{s\in[0,\delta]} e^{-s/2}B_{e^s} - B_1 \geq \sqrt{\ln\Big(\frac{1}{r_n}\Big)}\left(\rho(\delta) - C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}\right). \tag{2.16}$$

From the symmetry in law for the Brownian motion we know that

$$\sup_{s\in[0,\delta]} e^{-s/2}B_{e^s} \leq \sup_{s\in[0,\delta]} B_{e^s}.$$

And here from [MP10], Theorem 2.34. We know that $\sup_{s\in[0,T]} B_s \stackrel{\mathcal{L}}{=} |B_T|$. From this equality, we can then upper bound the probability of event (2.16), up to a constant, by the event

$$B_{e^\delta} - B_1 \geq \sqrt{\ln\Big(\frac{1}{r_n}\Big)}\left(\rho(\delta) - C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}\right). \tag{2.17}$$

This last one can be easily bounded using the classic bound for tales of normal distributions. Since $B_{e^\delta} - B_1 \sim \mathcal{N}(0, (e^\delta - 1))$, and assume that $\rho(\delta) = G\sqrt{e^\delta - 1}$, whit $G > 4K + 2$ and dividing by it. We have nothing more than, when $\mathbb{P}$ taken, that

$$\mathbb{P}\left(\mathcal{N}(0,1) \geq G\sqrt{\ln\Big(\frac{1}{r_n}\Big)}\right) \leq Ce^{-\frac{G^2}{2}\ln\left(\frac{1}{r_n}\right)},$$

where for this upper bound we have used that for $a, b > 0$ we have that $-(a-b)^2 \leq -(a^2+b^2)$.

Therefore, we have that

$$\mathbb{P}\left(\sup_{s\in I(q,\delta)}|\Phi_{r_n}(z_\square,q)-\Phi_{r_n}(z_\square,s)|\geq \ln\left(\frac{1}{r_n}\right)\rho(\delta)\right)\leq Ce^{-\frac{G^2}{2}\ln\left(\frac{1}{r_n}\right)}.$$

From this prior bounds we have that

$$\begin{aligned}\sum_{\square\in\mathcal{Q}_n;\square\cap D\neq\emptyset}&\mathbb{P}\left(\sup_{s\in I(q,\delta)}|\Phi_{r_n}(z_\square,q)-\Phi_{r_n}(z_\square,s)|\geq \ln\left(\frac{1}{r_n}\right)\rho(\delta)\right)\\ \leq& C\#\{\square\in\mathcal{Q}_n;\square\cap D\neq\emptyset\}e^{-\frac{G^2}{2}\ln\left(\frac{1}{r_n}\right)}.\end{aligned}$$

Since $\#\{\square\in\mathcal{Q}_n;\square\cap D\neq\emptyset\}=O(1/r_n^2)$, we have then

$$\mathbb{P}\left(\max_{\square\in\mathcal{Q}_n;\square\cap D\neq\emptyset}\sup_{s\in I(q,\delta)}|\Phi_{r_n}(z_\square,q)-\Phi_{r_n}(z_\square,s)|\geq \ln\left(\frac{1}{r_n}\right)\rho(\delta)\right)\leq \hat{C}n^{K(2-\frac{G^2}{2})}.$$

And by the choice of G we know that $K(2-\frac{G^2}{2})<-1$. Therefore, the claim follows from Borel-Cantelli. □

□

2.3. Points with a given thickness function via GMC

In the present section we are concerned on, for a given function f, how to find a point $x\in D$ such that $\gamma_x=f$. We know that a way to actually find a thick point for a related field is to use the associated chaos measure. Which can be understood as the measure given by take the Wick exponent of the field. And we saw in Section 1.3.1 that the Liouville measure can be constructed up to a restriction in the thickness parameter γ. The respective constraint for our problem of interest will be given by an energy functional $\mathcal{E}$. The above discussion lead us to the following proposition that we will prove in this section.

Proposition 2.3 *If $f\in H_0^1(\mathbb{R})$ such that $\mathcal{E}(f)<4$, then, there exists a point $x\in D$ Such that $f=\gamma_x(\cdot)$.*

Here the energy functional to consider is

$$\mathcal{E}(f)=\int_{\mathbb{R}}|f'+\frac{1}{2}f|^2ds. \tag{2.18}$$

To prove such proposition. We first introduce the following field

$$h=\int_{\mathbb{R}}\Phi(s)\varphi(s)ds,$$

and here we consider $\varphi\in C_c^\infty(\mathbb{R})$. And we have the following lemma related to this field.

Lemma 2.5 *If h is given as in Equation (2.3) for $\varphi\in C_c^\infty(\mathbb{R})$, and Φ is the OU-GFF process. Then, there exists a constant $c=c(\varphi)$ dependent on φ such that*

$$h\overset{\mathcal{L}}{=}c(\varphi)\Gamma^o,$$

where $\Gamma^o \sim GFF$.

Proof. First, notice that

$$\langle h, g\rangle_\nabla = \int_{\mathbb{R}} \langle \Phi(s), g\rangle_\nabla \varphi(s) ds$$

since the integral can be approximated via sums. This also implies that we have then that $\langle h, g\rangle_\nabla$ is a normal distribution. It follows then that the field h is a Gaussian process. Hence, to prove the lemma we only need to check for the correlations. Taking $f, g \in H_0^1(D)$ we have

$$\begin{aligned}\mathbb{E}[\langle h, f\rangle_\nabla \langle h, g\rangle_\nabla] &= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(s)\varphi(t)\mathbb{E}[\langle h(s), f\rangle_\nabla \langle h(t), g\rangle_\nabla] dsdt \\ &= \langle f, g\rangle_\nabla \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\frac{1}{2}|t-s|}\varphi(s)\varphi(t) dsdt.\end{aligned}$$

Therefore, taking

$$c^2(\varphi) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\frac{1}{2}|t-s|}\varphi(s)\varphi(t) dsdt$$

We have then that the correlations are the desired ones up to a constant and the lemma is proven. ☐

For now, we use h to indicates the field defined in 2.3, Γ^o for the related GFF, and $c(\varphi)$ for the respective constant.

Now we can introduce an intermediate step to prove Proposition 2.3.

Proposition 2.4 *Taking* $\varphi \in C_c^\infty(\mathbb{R})$*, such that* $c(\varphi) < 2$*. If we sample* $X \sim M^{c(\varphi)}(\Gamma^o, dz)$ *and look up the thickness function. We have that*

$$\gamma_X(t) = \int_{\mathbb{R}} e^{-\frac{1}{2}|t-s|}\varphi(s) ds.$$

Proof. Notice first that we can do the same proof as in Proposition 1.16. And prove that the pair (x, Φ), whit $\Phi \sim OU - GFF$ and x under the chaos measure of Γ^o. Is the same of first take x sampled uniformly on D and Φ under

$$\tilde{\mathbb{P}}_\varepsilon(d\Phi) = e^{c(\varphi)\Gamma_\varepsilon(X) - \frac{c(\varphi)^2}{2}\mathbb{E}[\Gamma_\varepsilon(X)^2]}\mathbb{P}(d\Phi). \tag{2.19}$$

Let us call (x, Φ) to this pair. From the Cameron-Martin theorem from [Hai09] that Φ under (2.19) Φ fulfills

$$\Phi = \hat{\Phi} + \mathrm{Cov}(h_\varepsilon(X), \cdot).$$

Here $\hat{\Phi}$ is a standard OU-GFF under $\tilde{\mathbb{P}}_\varepsilon$ independent of Φ. Since from Proposition 2.2 we know that

$$\mathrm{Cov}(h_\varepsilon(X), \Phi_\delta(y, t)) = G^D_{\varepsilon,\delta}(X, y)\int_{\mathbb{R}} \varphi(s) e^{-\frac{1}{2}|t-s|} ds.$$

Therefore, we can conclude than when take ε going to 0. The measure $\tilde{\mathbb{P}}_\varepsilon$ tends to a measure $\tilde{\mathbb{P}}$. Such that, Φ fulfills for all $\varepsilon > 0$

$$\Phi_\varepsilon(X, t) = \hat{\Phi}_\varepsilon(X, t) + G^D_\varepsilon(X, X)\int_{\mathbb{R}} \varphi(s) e^{-\frac{1}{2}|t-s|} ds.$$

Dividing by $\ln(1/\varepsilon)$ and taking the limsup of $\varepsilon \to 0$, we have

$$\limsup_{\varepsilon\to 0} \frac{\Phi_\varepsilon(X,t)}{\ln\left(\frac{1}{\varepsilon}\right)} = \limsup_{\varepsilon\to 0} \frac{\hat{\Phi}_\varepsilon(X,t)}{\ln\left(\frac{1}{\varepsilon}\right)} + \int_{\mathbb{R}} e^{-\frac{1}{2}|t-s|}\varphi(s)ds.$$

Since $\hat{\Phi}$ is a standard OU-GFF independent of Φ, the respective limsup has to be 0. Therefore, we can conclude that at the given time t

$$\gamma_X(t) = \int_{\mathbb{R}} e^{-\frac{1}{2}|t-s|}\varphi(s)ds.$$

And since we saw that γ_X has to be continuous. The equality holds for all $t \in \mathbb{R}$. $\square$

This last proposition also motivates to define the application

$$T\varphi = \int_{\mathbb{R}} e^{-\frac{1}{2}|t-s|}\varphi(s)ds.$$

At prior this function is defined from $C_c^\infty(\mathbb{R})$ to $C_0(\mathbb{R})$ the functions that are continuous and vanishing at infinity. From Proposition 2.4 can be understood as follows: Taking X from the Liouville measure of h defined in 2.3 for φ. The associated thickness function fulfills that

$$\gamma_X = T\varphi,$$

whenever $c(\varphi) < 2$. This constant is nothing else but

$$c(\varphi)^2 = a(\varphi,\varphi) = \langle \varphi, T\varphi\rangle. \tag{2.20}$$

Let us assume for now that a is an inner product. From this, we can take the spaces given by $(C_c^\infty(\mathbb{R}), a)$ and $(C_0(\mathbb{R}), \mathcal{E})$. Then T is nothing more but a linear functional between these spaces. Proposition 2.4 could be understood as a first step to prove Proposition 2.3. Since it reduces the problem to study the operator T. As a first result that we will use is the continuity of this operator. This is done in the following lemma.

Lemma 2.6 *The operator T is an isometry between $(C_c^\infty(\mathbb{R}), a)$, with $a(\cdot,\cdot)$ from (2.20), and $(C_0(\mathbb{R}), \mathcal{E})$.*

Proof. To prove this, We first check for $\varphi \in C_c^\infty(\mathbb{R})$. To compute $\mathcal{E}(T\varphi)$ we need to know the derivative of $T\varphi$. We first separate as follows

$$T\varphi(t) = \int_t^\infty e^{-\frac{1}{2}(s-t)}\varphi(s)ds + \int_{-\infty}^t e^{-\frac{1}{2}(t-s)}\varphi(s)ds.$$

Using that $e^{-|s|}$ and φ are in $L^1(\mathbb{R})$. We can differentiate both functions easily using the Leibniz rule and the fundamental theorem of calculus to compute the derivative. On one hand we have

$$(\int_t^\infty e^{-\frac{1}{2}(s-t)}\varphi(s)ds)' = \frac{1}{2}e^{\frac{t}{2}}\int_t^\infty e^{-\frac{1}{2}s}\varphi(s)ds - \varphi(t).$$

On the other hand

$$(\int_{-\infty}^{t} e^{-\frac{1}{2}(t-s)}\varphi(s)ds)' = (-\frac{1}{2})e^{-\frac{1}{2}t}\int_{-\infty}^{t} e^{\frac{1}{2}s}\varphi(s)ds + \varphi(t).$$

By summing these 2 expressions we have

$$T\varphi(t)' = \frac{1}{2}\int_{t}^{\infty} e^{-\frac{1}{2}|t-s|}\varphi(s)ds + (-\frac{1}{2})\int_{-\infty}^{t} e^{-\frac{1}{2}|t-s|}\varphi(s)ds. \tag{2.21}$$

Using that $1/2 = 1 - 1/2$, replacing in the equation we have that

$$\begin{aligned}(T\varphi)' &= \int_{\mathbb{R}} (1_{\{(t-s)<0\}} - \frac{1}{2})e^{-\frac{1}{2}|t-s|}\varphi(s)ds \\ &= \int_{\mathbb{R}} 1_{\{(t-s)<0\}}e^{-\frac{1}{2}|t-s|}\varphi(s)ds - \frac{1}{2}T\varphi.\end{aligned}$$

Now we evaluate $T\varphi$ in $\mathcal{E}$ to have

$$\begin{aligned}\mathcal{E}(T\varphi) &= \int_{\mathbb{R}} |\left(\int_{\mathbb{R}} 1_{\{(t-s)<0\}}e^{-\frac{1}{2}|t-s|}\varphi(s)ds - \frac{1}{2}T\varphi(t)\right) + \frac{1}{2}T\varphi(t)|^2 dt \\ &= \int_{\mathbb{R}} |\int_{\mathbb{R}} 1_{\mathbb{R}} e^{-\frac{1}{2}|t-s|}\varphi(s)ds|^2 dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} 1_{\{(t-s_1)<0\}}e^{-\frac{1}{2}|t-s_1|}\varphi(s_1)1_{\{(t-s_2)<0\}}e^{-\frac{1}{2}|t-s_2|}\varphi(s_2)ds_2ds_1dt.\end{aligned}$$

Applying Fubini's theorem we have

$$\int_{\mathbb{R}} \varphi(s_1)\int_{\mathbb{R}} \varphi(s_2)\int_{\mathbb{R}} 1_{\{(t-s_1)<0\}}e^{-\frac{1}{2}|t-s_1|}1_{\{(t-s_2)<0\}}e^{-\frac{1}{2}|t-s_2|}dtds_2ds_1. \tag{2.22}$$

Now we compute the integral from the middle

$$\begin{aligned}\int_{-\infty}^{s_1\wedge s_2} e^{-\frac{1}{2}((s_1-t)+(s_2-t))}dt &= e^{-\frac{1}{2}((s_1+s_2))}\int_{-\infty}^{s_1\wedge s_2} e^{t}dt \\ &= e^{-\frac{1}{2}((s_1+s_2))+s_1\wedge s_2}\end{aligned}$$

Using that $-\frac{1}{2}((s_1+s_2)) + s_1 \wedge s_2 = -\frac{1}{2}|s_1 - s_2|$. Replacing in Equation (2.22) we have that

$$\begin{aligned}\mathcal{E}(T\varphi) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(s_1)\varphi(s_2)e^{-\frac{1}{2}|s_1-s_2|}ds_1ds_2 \\ &= a(\varphi,\varphi).\end{aligned}$$

From this last equality we can conclude the desired lemma. □

As a direct corollary we have that

Corollary 2.3 *The bilinear form a is an inner product for $C_0^1(\mathbb{R})$.*

From the above corollary, we call $\mathcal{H}$ the completion of $C_0^\infty(\mathbb{R})$ under a. We won't talk too much about this space, only mention his existence. On the other hand, if in $C_0^1(\mathbb{R})$ we put the inner product associated with energy $\mathcal{E}$, it is direct that

$$\begin{aligned}\mathcal{E}(f) &= \int_{\mathbb{R}}((f')^2 + (ff') + \frac{1}{4}f^2)ds \\ &= \int_{\mathbb{R}}(f')^2 ds + \frac{1}{4}\int_{\mathbb{R}}(f)^2 + \frac{1}{2}(f^2)|_{-\infty}^{\infty} \\ &= \|f'\|_{L^2} + \frac{1}{4}\|f\|_{L^2}.\end{aligned}$$

Therefore,

$$\frac{1}{4}\|f\|_{H_0^1} \leq \mathcal{E}(f) \leq \|f\|_{H_0^1}.$$

This inequality implies that when we complete $C_0^1(\mathbb{R})$ under $\mathcal{E}$ we end with the Sobolev space $H_0^1(\mathbb{R})$. And from lemma 2.6 we have then the following Corollary.

Corollary 2.4 *T can be extender to a linear isometry between $\mathcal{H}$ and $H_0^1(\mathbb{R})$.*

This corollary plus Proposition 2.4 implies the following corollary

Corollary 2.5 *For $\varphi \in \mathcal{H}$ such that $c(\varphi) < 4$. There exists a point $x \in D$ such that $\gamma_x = T\varphi$.*

This last corollary plus Proposition 2.4 and a small detail implies Proposition 2.3. The small detail is given in the following lemma

Lemma 2.7 *The function $T : \mathcal{H} \to H_0^1(\mathbb{R})$ is surjective*

Proof. From [SS09] we know that for H a Hilbert space, if $S \subset H$ is a linear subspace, we know the orthogonal complement $S^\perp$ exists and has trivial intersection with S. And H is the direct sum between S and $S^\perp$. We know also that

$$S^\perp := \{f \in H;\ \langle f, g\rangle_H = 0,\ \forall g \in S\}.$$

With this idea in mind, we can conclude that T is surjective if $T(H)^\perp = \{0\}$. To achieve this we will check that the only orthogonal element in $H_0^1(\mathbb{R})$ of $T(\mathcal{H})$ is the 0. Let us consider $f \in H_0^1(\mathbb{R})$ orthogonal to $T(\mathcal{H})$. We have then in particular that $\varphi \in C_0^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}}(f' + \frac{1}{2}f)(T\varphi' + \frac{1}{2}T\varphi)dt = 0.$$

Using the value of $T\varphi' + \frac{1}{2}T\varphi$ from the proof of Lemma 2.6 and replacing we have

$$\int_{\mathbb{R}}(f' + \frac{1}{2}f)(\int_{\mathbb{R}} 1_{\{(t-s<0)\}}\varphi(s)e^{-\frac{1}{2}|t-s|}ds)dt = 0-$$

Applying Fubini we have

$$\int_{\mathbb{R}}\varphi(s)\left(\int_{-\infty}^{s}(f' + \frac{1}{2}f)e^{-\frac{1}{2}|t-s|}dt\right)ds = 0.$$

Since this equation holds for every $\varphi \in C_c^\infty(\mathbb{R})$. Taking

$$F(s) = \int_{-\infty}^{s}(f' + \frac{1}{2}f)e^{-\frac{1}{2}|t-s|}dt.$$

We have that

$$\int_{\mathbb{R}} \varphi(s)F(s)ds = 0.$$

It is direct that the above equality holds for every $\varphi \in C_b(\mathbb{R})$ the functions continuous and bounded. This implies that for every $s \in \mathbb{R}$

$$F(s) = 0.$$

Equivalently

$$\int_{-\infty}^{s} (f' + \frac{1}{2}f)e^{-\frac{1}{2}|t-s|}dt = 0.$$

This implies

$$\int_{-\infty}^{s} (f' + \frac{1}{2}f)e^{\frac{1}{2}t}dt = 0.$$

This implies that for every $s, s' \in \mathbb{R}$ $s' < s$, that

$$\int_{s'}^{s} (f' + \frac{1}{2}f)e^{\frac{1}{2}t}dt = 0.$$

Therefore, we have

$$(f' + \frac{1}{2}f) = 0.$$

From here we can see, taking the square and integrating, since $f(\pm\infty) = 0$ we will have that the $H_0^1(\mathbb{R})$ norm is 0 and consequently $f = 0$, and therefore we can conclude that T is surjective. □

This last lemma allows us to prove Proposition 2.3. Let us check this.

PROOF OF PROPOSITION 2.3. Let us take $f \in H_0^1(\mathbb{R})$ that fulfills $\mathcal{E}(f) < 4$. From Lemma 2.7 and 2.4 we know this implies the existence of some $\varphi \in H$ that $T\varphi = f$ and $\mathcal{E}(f) = a(\varphi, \varphi) = c(\varphi)$. Therefore, we can apply Corollary 2.5 to conclude the Proposition. □

Now that we know under which conditions for f there exists the desired point. We proceed to see that when the energy restrain is broken no such point exists.

2.4. Functions that are not thickness functions

Proposition 2.3 could be interpreted as an existence result. Since it tells us when, for a given function f, there exists a point $x \in D$ with a desired dynamical property. In the present section on the other hand. We deal with the non-existence. In particular, we will prove the following.

Proposition 2.5 *If for a given $f \in H_0^1(\mathbb{R})$, we have that $\mathcal{E}(f) > 4$. Then we have that there are no points $x \in D$ such that $\gamma_x = f$.*

Before we gave the proof. Let us show some interesting corollaries that follow from this result.

Corollary 2.6 *If we consider a constant and positive time $C, T > 0$, such that $TC^2 > 16$ we have that there are no points $x \in D$ such that $\gamma_x = C$ in $[0, T]$.*

And we also have a regularity constraint. This is more clear in the following corollary

Corollary 2.7 *For B the realization of a Brownian bridge, independent of Φ, and that is always in $(-2,2)$. There are no points $x \in D$ such that $\gamma_x = B$.*

We now proceed with the proof.

PROOF OF PROPOSITION 2.5. Let us first start in the case $f \in C^1(\mathbb{R})$, and $\mathcal{E}(f) > 4$, by the definition and after doing some translations, we can assume that exists $T > 0$ such that

$$\int_0^T |f' + \frac{1}{2}f|(s)ds > 4.$$

Now we proceed in taking a discretization of $D \times [0,T]$. To do this we discretize D and $[0,T]$ separately. In space, we take $r_n = n^{-K}$ whit K big enough (and maybe dependent on f). From this we focus on squares of sides $\hat{r}_n = r_n^{1+\varepsilon}$ for some ε positive and small enough. We use the above to define

$$\hat{\mathcal{Q}}_n := \{z + [kr_n, (k+1)r_n] \times [lr_n, (l+1)r_n];\ z \in \mathbb{Z}^2,\ k, l \in \{0, ..., \lceil (r_n)^{-1} \rceil - 1\}\}.$$

Then we take $\mathcal{Q}_n$ the set of squares in $\hat{\mathcal{Q}}_n$ such that intersects the domain D, or at least that the center of the square is in D. On the other hand, we discretize the time interval $[0,T]$ using the following set

$$T_n = \{T\frac{k}{2^n},\ k \in \{0, ..., 2^n\}\}.$$

Now we consider the set T^f, the set of points in space such that the thickness function is given by f. We are interested in "count boxes", in particular we study the following

$$\mathbb{E}[\#\{\square \in \mathcal{Q};\ \square \cap T^f \neq \emptyset\}] = \sum_{\square \in \mathcal{Q}} \mathbb{P}\left(\square \cap T^f \neq \emptyset\right).$$

To study this expectation we start by looking up the event $\{\square \cap T^f \neq \emptyset\}$. A first observation is that such event is contained in the event given by

$$A_n(\square) = \{\exists x \in \square;\ \forall t \in T_n;\ \limsup_{m \to \infty} \frac{\Phi_{r_m}(x,t)}{\ln\left(\frac{1}{r_m}\right)} = f(t)\}.$$

We now take $z_\square$ to denotes the center of $\square$. From Proposition 2.1, we know that for $\zeta, \varepsilon \in (0,1)$ and $\alpha \in (0, 1/2)$ almost surely exists $C > 0$ such that

$$|\Phi_{r_n}(x,t) - \Phi_{r_n}(z_\square, t)| \le C(\ln\left(\frac{1}{r_n}\right))^\zeta \frac{1}{r_n^{\alpha(1+\varepsilon)}}.|x - z_\square|^\alpha$$

Using that the length side of $\square$ is $r_n^{1+\varepsilon}$, we can conclude that

$$\Phi_{r_n}(x,t) - C(\ln\left(\frac{1}{r_n}\right))^\zeta \le \Phi_{r_n}(z_\square, t) \le \Phi_{r_n}(x,t) + C(\ln\left(\frac{1}{r_n}\right))^\zeta.$$

And since we are under the assumption of $x \in T^f$. The above implies for n big enough, we have that for every $t \in T_n$

$$(f(t) - C(\ln\Big(\frac{1}{r_n}\Big))^{\zeta-1}) \ln\Big(\frac{1}{r_n}\Big) \leq \Phi_{r_n}(z_\square, t) \leq (f(t) + C(\ln\Big(\frac{1}{r_n}\Big))^{\zeta-1}) \ln\Big(\frac{1}{r_n}\Big). \tag{2.23}$$

Now we take $t_k = T(k/2^n) \in T_n$ for $k \in \{0, ..., 2^n - 1\}$. We know that we have

$$\Phi_{r_n}(z_\square, t_{k+1}) \overset{\mathcal{L}}{=} e^{-\frac{1}{2}|t_{k+1}-t_k|}\Phi_{r_n}(z_\square, t_k) + \sqrt{1 - e^{-|t_{k+1}-t_k|}}\Phi_{r_n}(z_\square)^k, \tag{2.24}$$

where Φ^k indicates an independent GFF of Φ. This equality plus the upper inequality from (2.23) implies

$$\Phi_{r_n}(z_\square, t_{k+1}) \leq \left(e^{-\frac{1}{2}|t_{k+1}-t_k|}f(t_k) + C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}\right)\ln\Big(\frac{1}{r_n}\Big) + \sqrt{1 - e^{-|t_{k+1}-t_k|}}\Phi^k_{r_n}(z_\square). \tag{2.25}$$

Using the lower inequality in (2.23) in the above. We can conclude that

$$\left((|f(t_{k+1}) - e^{-\frac{1}{2}|t_{k+1}-t_k|}f(t_k))| - C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}\right)\ln\Big(\frac{1}{r_n}\Big) \leq \sqrt{1 - e^{-|t_{k+1}-t_k|}}\Phi^k_{r_n}(z_\square). \tag{2.26}$$

This holds for every $k \in \{0, ..., 2^n - 1\}$. Taking $\Delta_k = |t_{k+1} - t_k|$, we have that is a constant independent of k. Inequality (2.26), implies

$$\ln\Big(\frac{1}{r_n}\Big)\sum_{k=0}^{2^n-1}\left(|f(t_{k+1}) - e^{\frac{-\Delta_k}{2}}f(t_k)| - 2C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}\right) \leq \sqrt{1 - e^{-\Delta_k}}\sum_{k=0}^{2^n-1}\Phi^k_{r_n}(z_\square). \tag{2.27}$$

From the work done, we know that the probability of this event is an upper bound of $\mathbb{P}(\square \cap T^f)$, And dividing by the variance of $\sqrt{1 - e^{-\Delta_k}}\sum_{k=0}^{2^n-1}\Phi^k_{r_n}(z_\square)$ in 2.27. We have that the probability then is equal to

$$2^{-n/2}\ln\Big(\frac{1}{r_n}\Big)^{\frac{1}{2}}(\sum_{k=0}^{2^n-1}\left(|f(t_{k+1}) - e^{\frac{-\Delta_k}{2}}f(t_k)| - 2C\ln\Big(\frac{1}{r_n}\Big)^{\zeta}\right))(1 - e^{-\Delta_k})^{-1/2} \leq \mathcal{N}(0, 1)$$

Now we use the exponential tails of the normal distribution and bound the above, up to a constant, by

$$\exp\left(\frac{-2^{-n}}{2}\ln\Big(\frac{1}{r_n}\Big)\left(\sum_{k=0}^{2^n-1}(|f(t_{k+1}) - e^{\frac{-\Delta_k}{2}}f(t_k))|(1 - e^{-\Delta_k})^{-1/2}\right)^2 + (C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1})^2\right). \tag{2.28}$$

We now focus on the sum expression since it looks like an integral. Recall that if $\Delta_k << 1$ we have that

$$e^{-\Delta_k} = 1 - \Delta_k + O(\Delta_k)^2 \tag{2.29}$$

and, at the same time

$$(1 - e^{-\Delta_k})^{-1} = O(\Delta_k^{-2}). \tag{2.30}$$

Estimations 2.29 and 2.30 implies

$$\left(\sum_{k=0}^{2^n-1}(|f(t_{k+1})-e^{\frac{-\Delta_k}{2}}f(t_k))|(1-e^{-\Delta_k})^{-1/2}\right)^2=\sum_{k=0}^{2^n-1}|(f(t_{k+1})-f(t_k)+\frac{1}{2}f(t_k)\Delta_k)|^2(\Delta_k)^{-2}+O(1).$$

Since $f\in C^1$, when factorizing $\sqrt{\Delta_k}$ we can use the mean value theorem to obtain the above is equal to

$$\sum_{k=0}^{2^n-1}|f'(t_k+\rho_n)+f(t_k)|^2+O(1),$$

where ρ_n tends to 0 when n goes to infinity. Therefore, we can upper bound the probability of 2.28 by

$$\exp\left(-\frac{1}{2}\ln\Big(\frac{1}{r_n}\Big)2^{-n}\sum_{k=0}^{2^n-1}|f'(t_k+)+f(t_k)|^2+O(\Delta_k)+C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}\right)$$
$$\leq\hat{M}\left(\frac{1}{r_n}\right)^{-\frac{1}{2}\sum_{k=0}^{2^n-1}2^{-n}|f'(t_k+)+f(t_k)|^2}$$

for some constant $\hat{M}$. This, plus the fact that $|\mathcal{Q}_n|=O((r_n)^{-2})$, allows us to conclude that

$$\mathbb{E}[\#\{\square\in\mathcal{Q};\ \square\cap T^f\neq\emptyset\}]\leq\hat{M}r_n^{-2}\left(\frac{1}{r_n}\right)^{-\frac{1}{2}\sum_{k=0}^{2^n-1}2^{-n}|f'(t_k+\rho_n)+f(t_k)|^2}$$

and here we will use that

$$\sum_{k=0}^{2^n-1}2^{-n}|f'(t_k+\rho_n)+f(t_k)|^2\underset{n\to\infty}{\longrightarrow}\int_0^T|f'(s)+\frac{1}{2}f(s)|^2ds. \tag{2.31}$$

This implies that for n big enough we have

$$(2-\frac{1}{2}\sum_{k=0}^{2^n-1}2^{-n}|f'(t_k+\rho_n)+f(t_k)|^2)<0,$$

and therefore

$$\mathbb{E}[\#\{\square\in\mathcal{Q};\ \square\cap T^f\neq\emptyset\}]\underset{n\to\infty}{\longrightarrow}0.$$

Since $\#\{\square\in\mathcal{Q};\ \square\cap T^f\neq\emptyset\}$ is a positive random value, we have then that almost sure

$$\#\{\square\in\mathcal{Q};\ \square\cap T^f\neq\emptyset\}\underset{n\to\infty}{\longrightarrow}0.$$

This implies that no such point exists. At prior this result is valid for $f\in C_0^1(\mathbb{R})$. But to extend the result we only need to check the limit from (2.31). Therefore, we only need to check that for $f\in H^1([0,T])$ we have that

$$2^{-n}\sum_{k=0}^{2^n-1}|\frac{(f(t_{k+1})-f(t_k))}{\Delta_n}+\frac{1}{2}f(t_k)|^2\underset{n\to\infty}{\longrightarrow}\int_0^T|f'(s)+\frac{1}{2}f(s)|^2ds \tag{2.32}$$

we can conclude the nonexistence of the point using the same argument, so next part we will show this convergence.

First notice that

$$\frac{f(s+h)-f(s)}{h} \xrightarrow{L^2} f'(s) \tag{2.33}$$

when $h \to 0$. To prove this, we first need to recall that since $f \in H_0^1(\mathbb{R})$, we know exists $\hat{f} \in C(\mathbb{R})$ such that almost everywhere $f = \hat{f}$, and for all $x, y \in \mathbb{R}$

$$\hat{f}(y) - \hat{f}(x) = \int_x^y f'(s)ds, \tag{2.34}$$

since $\hat{f} = f$ almost everywhere, we can replace $\hat{f}$ with f, in the equality. Now we have

$$\frac{f(s+h)-f(s)}{h} = \frac{1}{h}\int_h^{s+h} f's(ds)$$

almost everywhere. Since $f' \in L^2(\mathbb{R})$, we can conclude (2.33) from the Lebesgue differentiation theorem (a reference for this result is in the appendix E of [Eva22]). This theorem also tells us than the convergence is also almost everywhere. From the above, we know come back to check Limit (2.32). To do this, we notice we are interested in study the sequence given by

$$S_n = 2^{-n}\sum_{k=0}^{2^n-1} \left|\frac{(f(t_{k+1})-f(t_k))}{\Delta_n} + \frac{1}{2}f(t_k)\right|^2 \tag{2.35}$$

$$= 2^{-n}\sum_{k=0}^{2^n-1} \left(\frac{(f(t_{k+1})-f(t_k))}{\Delta_n}\right)^2 + \left(f(t_k)\left(\frac{(f(t_{k+1})-f(t_k))}{\Delta_n}\right)\right) + \left(\frac{1}{2}f(t_k)\right)^2 \tag{2.36}$$

We call S_1, S_2 and S_3 the sums form (2.36) respectively. And now we check each one of them.

Step 1 To check the 1st sum. We use 2.34 to notice that

$$\begin{aligned} S_1 &= \sum_{k=0}^{2^n-1} 2^{-n}\left|\int_0^T 1_{[t_k,t_{k+1}]}\frac{1}{\Delta_k}f'(s)ds\right|^2 \\ &= \int_0^T f'(s)\sum_{k=0}^{2^n-1} 1_{[t_k,t_{k+1}]}(s)\int_0^T 1_{[t_k,t_{k+1}]}(t)\frac{1}{\Delta_k}f'(t)dtds. \end{aligned} \tag{2.37}$$

Here we also used that $\Delta_k = 2^{-n}$. Define

$$F_n(s) = \sum_{k=0}^{2^n-1} 1_{[t_k,t_{k+1}]}(s)\int_0^T 1_{[t_k,t_{k+1}]}(t)\frac{1}{\Delta_k}f'(t)dt. \tag{2.38}$$

It is direct that almost everywhere in $[0,T]$, we have the punctual limit

$$F_n(s) \underset{n\to\infty}{\longrightarrow} f'(s).$$

And it is also direct that the limits holds in L^2 strongly. This implies the limit also holds in the weak topology. Therefore, (2.37) is the same as

$$S_3 = \langle f', F_n\rangle \underset{n\to\infty}{\longrightarrow} \langle f', f'\rangle$$

and, we can conclude here.

Step 2 Now we check the 2^{nd} sum. Notice using again (2.34). We have

$$\begin{aligned} S_2 &= \sum_{k=0}^{2^n-1} 2^{-n} f(t_k) \frac{1}{\Delta_k} \int_0^T 1_{[t_k,t_{k+1}]} f'(s) ds \\ &= \int_0^T f'(s) \sum_{k=0}^{2^n-1} 1_{[t_k,t_{k+1}]}(s) f(t_k) ds. \end{aligned} \tag{2.39}$$

Taking now

$$F_n(s) = \sum_{k=0}^{2^n-1} 1_{[t_k,t_{k+1}]}(s) f(t_k). \tag{2.40}$$

We can easily check that $F_n \to \hat{f}$, almost surely. And from here it is direct that the convergence is also strongly and therefore weakly in L^2. Therefore, in 2.39 we can do

$$S_2 = \langle f', F_n \rangle \underset{n\to\infty}{\longrightarrow} \langle f', \hat{f} \rangle = 2\langle f', \frac{1}{2} f \rangle \tag{2.41}$$

Step 3 Finally the 3^{rd} sum it is direct that

$$S_3 = \sum_{k=0}^{2^n-1} 2^{-n} \left(\frac{1}{2} f(t_k) \right)^2 \underset{n\to\infty}{\longrightarrow} \int_0^T f^2 ds. \tag{2.42}$$

Here can do a Riemann sums argument since f is continuous.

Therefore, we can conclude the limit (2.32) and hence the proposition. □

One can see that many of the results related to the thickness function can be generalized to the case of

$$\Phi = \sum_{k\in\mathbb{N}} \alpha_k e_k,$$

where $(\alpha_k)_{k\in\mathbb{N}}$ is now an i.i.d. sequence of some given Itô process whit stationary law a standard normal distribution. At an intuitive level, the reason for this is the correlation of this process in space and time, since it will be nothing more but the correlation given in time by the Itô process times the correlations of the GFF.

Chapter 3

Thick points of the stochastic heat equation

We now change the dynamic of interest. Although its still an evolution of GFF, we have to do a change of perspective in relation to the object of study. In the previous chapter, we focus on the thickness function, and part of the motivation for this was due to the continuity of these functions. However, as a consequence of Lemma 3.2, we cannot expect the same here. Therefore, we adjust our attention now to the set of thick points in the space-time domain (See Definition 3.1). When one look up this new domain, we have that the set is not empty for values up to $2\sqrt{2}$ of the thickness parameter, this tell us that in particular, we have space-time points with thickness greater than 2 and motivates us to talk about "thick points" (thickness in $(0, 2]$), and "super-thick" points (thickness between $(2, 2\sqrt{2})$. As a geometric result for this set is the Hausdorff dimension as a function of the thickness parameter ((3.22)). Afterwards, we focus on the exceptional times given by the times where there is a space point with thickness greater than 2. We first study its Hausdorff dimension (see (3.11)). Afterwards we focus on the fibers in space related to the exceptional times, and saw that the number of super-thick points in that fiber is always finite (see Proposition 3.13), even more, we find the Hausdorff dimension of exceptional times with a given number of space-thick points (see 3.18).

Remark 3.1 *In this chapter, C represents a finite positive constant that may vary between lines. Any dependencies on variables will be explicitly noted.*

We now focus on the process given by the solution of the additive stochastic heat equation, this is the following SPDE

$$(SHE) \begin{cases} \partial_t \Phi = \frac{1}{2}\Delta \Phi + \xi, \\ \Phi(0) = \Gamma. \end{cases} \tag{3.1}$$

Here, ξ a white noise of $L^2(D \times \mathbb{R}_{>0})$ and Γ a GFF independent of ξ.

Remark 3.2 *From now on, we will talk for the solution of* (3.1) *as SHEF, and we will use Φ to denote this field unless is it said otherwise.*

To solve this equation we will use the results from Proposition 1.22 using that $\langle \xi, e_k \rangle_\nabla = \sqrt{\lambda_k} \langle \xi, \sqrt{\lambda_k} e_k \rangle$. From proposition 1.6, we can conclude that, $X^k = \langle \Phi(t), e_k \rangle_\nabla$ follows the equation of an Orstein-Uhlenbeck with velocity given by λ_k. And from Proposition 1.5, we know that

$$X^k(t) = \alpha_0 e^{\frac{-\lambda_k}{2}t} + \sqrt{\lambda_k}\int_0^t e^{\frac{-\lambda_k}{2}(t-s)} dB_s^k, \tag{3.2}$$

where $\alpha_0 = \langle \Gamma, e_k\rangle_\nabla \sim \mathcal{N}(0,1)$ and $(B^k)_{k\in\mathbb{N}}$ a sequence of i.i.d. Brownian motions.

The above discussion allows us to obtain an expression for Φ in the orthonormal basis of $H_0^1(D)$ given by the eigenfunctions of the Laplace operator. That is to say

$$\Phi(t) = \sum_{k\in\mathbb{N}} X^k(t) e_k,$$

for X^k the i.i.d. sequence of Orstein-Uhlenbeck process obtained.

3.1. Basics results

In this section we present some elementary results for the SHEF. We believe that most of these results might have been done elsewhere, but since we could not find a good reference, we will state them and give the respective proofs when necessary. From expression (3) it is direct that for a given $t \in [0,T]$, $\Phi(t) \sim GFF$. Furthermore, taking $s,t \in [0,T]$ and $f,g \in H_0^1(D)$, it follows that

$$\begin{aligned}\mathbb{E}[\langle \Phi(s), f\rangle_\nabla \langle \Phi(t), g\rangle_\nabla] &= \sum_{k\in\mathbb{N}} \langle f, e_k\rangle_\nabla \langle g, e_k\rangle_\nabla \mathbb{E}[X^k(t)X^k(s)] \\ &= \sum_{k\in\mathbb{N}} e^{\frac{-\lambda_k}{2}|t-s|} \langle f, e_k\rangle_\nabla . \langle g, e_k\rangle_\nabla .\end{aligned}$$

As a consequence, when we take $f,g \in L^2(D)$, the respective covariance is

$$\mathbb{E}[\langle \Phi(s), f\rangle \langle \Phi(t), g\rangle] = \sum_{k\in\mathbb{N}} e^{\frac{-\lambda_k}{2}|t-s|} \langle \hat{e}_k, f\rangle \langle (-\Delta)^{-1}\hat{e}_k, g\rangle, \tag{3.3}$$

where $\hat{e}_k = \sqrt{\lambda_k} e_k$, the normalizes basis of L^2.

We now take $\Phi_\epsilon(x,t) = \langle \Phi(t), \mu_{\varepsilon,x}\rangle$, where we do as in Definition 1.12. Using Formula (3.3) it follows that

$$\mathbb{E}[\Phi_\varepsilon(x,t)\Phi_\delta(y,0)] = \sum_{k\in\mathbb{N}} e^{\frac{-\lambda_k}{2}t} \langle \mu_{\varepsilon,x}, e_k\rangle \langle e_k, \mu_{\delta,y}\rangle, \ \forall t \in [0,T],\ x\ y \in D, \tag{3.4}$$

this equality actually motivates the following lemma.

Lemma 3.1 *Fixing $\varepsilon, \delta > 0$ and $y \in D$. Take $u(x,t) = \mathbb{E}[\Phi_\varepsilon(x,t)\Phi_\delta(y,0)]$, we have that this function fulfill the heat equation given by*

$$(H)\begin{cases}\partial_t u = \frac{1}{2}\Delta u, & \text{in } D \times (0,T) \\ u(\cdot, 0) = G^D_{\varepsilon,\delta}(\cdot, y), & \text{in } \partial_p D \times (0,T).\end{cases}$$

recall that $\partial_p D_\varepsilon \times (0,T)$ indicates the parabolic boundary from Definition 1.17

Proof. First, notice that from definition at $t = 0$ and $x \in D$ we will have that

$$u(x,0) = \mathbb{E}[\Phi_\varepsilon(x,0)\Phi_\delta(y,0)]$$
$$= G^D_{\varepsilon,\delta}(x,y),$$

and the last equality is from the case of the GFF. On the other hand, we know from Section 1.2.1, that when $x \in \partial D$ we will have that $G^D_{\varepsilon,\delta}(x,y) = 0$, and considering Expression (3.4), $x \in \partial D$ implies $\langle \mu_{\varepsilon,x}, e_k\rangle = 0$, and hence $u(x,t)$ is also 0.

Now let us check that the equations holds. And to do this, notice first that

$$\Delta\langle e_k, \mu_{\varepsilon,x}\rangle = \langle \Delta e_k, \mu_{\varepsilon,x}\rangle, \tag{3.5}$$

to check this equation, taking $\tau_{\varepsilon,x} = \tau_{B(\varepsilon,x)}$, we notice that

$$\langle e_k, \mu_{\varepsilon,x}\rangle = \mathbb{E}[e_k(B^x_{\tau_{\varepsilon,x}})] \tag{3.6}$$
$$= \mathbb{E}[e_k(B_{\tau_{\varepsilon,0}} + x)] \tag{3.7}$$

and therefore due to the dominated convergence theorem we can conclude (3.5).

On the other hand, since $-\Delta e_k = \lambda_k e_k$, and they are orthonormal in $H^1_0(D)$, we have that

$$\int_D |e_k(x)|^2 dx = \frac{1}{\lambda_k},$$

and we know from PDE theory that the eigenvalues of the Laplace operator tends to infinity, and since it is also known that and since we know that the eigenfunctions are in $C^\infty_c(D)$, we will have that they converge to 0 for the supreme norm, and therefore are uniformly bounded and therefore there are also $\partial_i e_k$ and $\partial_i^2 e_k$. Recall the series from (3.4) is given as the sum of the sequence

$$a_k(x,t) = e^{\frac{-\lambda_k}{2}t}\langle \mu_{\varepsilon,x}, e_k\rangle\langle e_k, \mu_{\delta,y}\rangle$$

And thanks to (3.7) we can see that $\partial_t a_k = 1/2\Delta a_k$. From this last observation, to conclude we only need to confirm that the series and the series for the derivatives are bounded. To do this, since e_k and their derivatives are uniformly bounded, from Expression (3.4) we can see that there exists a constant C such that

$$\sum_{k\in\mathbb{N}} |\partial_t a_k(x,t)| \leq C \sum_{k\in\mathbb{N}} \lambda_k e^{\frac{-\lambda_k}{2}t} < \infty$$

and the same for the first and second order of the spatial derivatives. From this uniform bound we can conclude the desired.

Since we have the uniform convergence of the series from Equation (3.4) we have that $u(x,t)$ goes to the initial condition continuously. And hence, it is a strong solution of Equation (H). □

A direct corollary of this result is the following one

Corollary 3.1 *For $u(x,t) = Cov(\Phi_\varepsilon(x,t), \Phi_\delta(y,s))$, we have the following prior bounds*

$$|\ln\left(\frac{1}{\varepsilon \vee \delta \vee |x-y| \vee \sqrt{|t-s|}}\right) - u(x,t)| \leq O(1).$$

Proof. The proof can be found in [HS16], Lemma 3.9. □

Before continuing the study of the solution of (SHE). Let us present the *parabolic metric*, defined as

$$d_p((x,t),(y,s)) = \max\{|x-y|, \sqrt{|t-s|}\},$$

for $(x,t),(y,s) \in \mathbb{R}^2 \times \mathbb{R}$ this metric is actually quite natural in the context of the heat equation as we will see later on. A first result where appears this metric is related to the continuity of our field, and is given in the following proposition.

Proposition 3.1 *If $\Phi_\epsilon(x,t)$ is the harmonic measure approximation for the solution of* (3.1) *then, this process accepts a modification (that we also denote byΦ_ε such that for every $0 < \alpha < 0.5$, and $\delta, \zeta > 0$, almost surely, for $r < \varepsilon$, there exists $C = C(\alpha,\delta,\zeta)$ such that*

$$|\hat{\Phi}_r(x,t) - \hat{\Phi}_r(x,t)| \leq M\left(\ln\left(\frac{1}{r}\right)^\zeta\right)\frac{1}{(r)^{\alpha+\delta}}\left(|\varepsilon - r| + d_p((x,t),(y,s))\right)^\alpha,$$

where $(x,t),(y,s) \in D \times [0,T]$, and $1/2 < r/\varepsilon < 2$.

Proof. This result is an application of the modified Kolmogorov extension theorem given in Proposition 1.3 and more precisely is an extension of the result given in [HMP10], Appendix C. To apply this proposition we need to estimate $\mathbb{E}\left[|\Phi_\varepsilon(x,t) - \Phi_\delta(y,s)|^2\right]$ for $x,y \in D$, since this is equal to

$$\Big(\mathbb{E}[\Phi_\varepsilon(x,t)^2] - \mathbb{E}[\Phi_\varepsilon(x,t)\Phi_\delta(y,s)]\Big) + \Big(\mathbb{E}[\Phi_\delta(y,s)^2] - \mathbb{E}[\Phi_\varepsilon(x,t)\Phi_\delta(y,s)]\Big),$$

without loss of generality we can focus on one of the two terms and then conclude. For now let us assume without loss of generality that $t > s$. From Lemma 1.8, taking

$$u(x,t) = \mathbb{E}[G^D_{\varepsilon,\delta}(B^x_{|t-s|\wedge\tau_D}, y)]$$

we will have that

$$\begin{aligned}\Big|\mathbb{E}[\Phi_\varepsilon(x,t)^2] - \mathbb{E}[\Phi_\varepsilon(x,t)\Phi_\delta(y,s)]\Big| &= |\mathbb{E}[G^D_{\varepsilon,\varepsilon}(x,x)] - \mathbb{E}[G^D_{\varepsilon,\delta}(B^x_{|t-s|\wedge\tau_D}, y)]| \\ &\leq \mathbb{E}[|G^D_{\varepsilon,\varepsilon}(x,x) - G^D_{\varepsilon,\delta}(B^x_{|t-s|\wedge\tau_D}, y)|].\end{aligned} \tag{3.8}$$

Here, we recall that in [HMP10], Proposition 2.1, they prove that

$$|G_{\varepsilon,\varepsilon}(x,x) - G_{\varepsilon,\delta}(x,y)| \leq C\frac{(|\varepsilon - \delta)| + |x-y|}{\varepsilon \wedge \delta}$$

and therefore replacing in 3.8, we have that

$$\begin{aligned}\left|\mathbb{E}[\varPhi_\varepsilon(x,t)^2]-\mathbb{E}[\varPhi_\varepsilon(x,t)\varPhi_\delta(y,s)]\right| &\le C\mathbb{E}[\frac{(|\varepsilon-\delta)|+|B^x_{|t-s|\wedge\tau_D}-y|}{\varepsilon\wedge\delta}] \\ &= \frac{C}{\varepsilon\wedge\delta}\left(|\varepsilon-\delta|+\mathbb{E}[|B^x_{|t-s|\wedge\tau_D}-y|].\right).\end{aligned} \tag{3.9}$$

From an elementary computation we can notice that

$$\mathbb{E}[|B^x_{|t-s|\wedge\tau_D}-y|]\le\sqrt{|x-y|^2+|t-s|}$$

and since we know that for $a,b>0$ we know that $a+b\le 2a\vee b$, replacing in (3.9) we conclude that

$$\left|\mathbb{E}[\varPhi_\varepsilon(x,t)^2]-\mathbb{E}[\varPhi_\varepsilon(x,t)\varPhi_\delta(y,s)]\right|\le\frac{C}{\varepsilon\wedge\delta}\left(|\varepsilon-\delta|+d_p((x,t),(y,s))\right).$$

From we can conclude as in [HMP10] using Proposition 1.15. □

This first section related to the continuity and the correlations of this field, will be the basics for study the geometric aspects of the thick points, this will be the focus on the next section.

Remark 3.3 *Before we continue, we do some comments related on the notation throughout the chapter. First, from now on, z and w indicates space-time variables in D_T unless is said otherwise. On the other hand, we use x and y for space variables in D, and s,t for time variables.*

3.2. Thick points

In Section 1.3 we presented the set of γ-thick points of the GFF, studied their fractal dimension and saw the relation between this set and the GMC measure. In particular, we saw that a "typical" point taken from this measure, is a thick point for the related GFF.

We now concentrate on the dynamical behavior on the thick points of the GFF. The first result we present is the behavior for the thick points given by the Liouville measure of the initial condition at a certain given time.

Lemma 3.2 *For $\varPhi$ the SHEF with initial condition Γ, and $\gamma\in(0,2)$, if we sample χ from $M^\gamma(\Gamma,dx)/M^\gamma(\Gamma,D)$, the normalized Liouville measure of the initial condition. Take $t>0$ a given deterministic time, then*

$$\limsup_{\varepsilon\to0}\frac{\varPhi_\varepsilon(\chi,t)}{\ln\left(\frac{1}{\varepsilon}\right)}=0.$$

In other words, a typical γ-thick point for the initial condition is 0-thick for every given deterministic time.

Proof. As seen in [Aru20], Proposition 3.1, a way to sample for a GFF given a thick point, is first sample a uniform point in space and then sample the respective field by the measure given by (3.10), from this idea we proceed as follows. We first sample $\chi \sim Unif(D)$ and then sample Γ from

$$\tilde{\mathbb{P}}_\chi(d\Gamma) =: e^{\gamma\Gamma_\varepsilon(\chi)} : \mathbb{P}(d\Gamma) \tag{3.10}$$

then we have that,

$$\Gamma \stackrel{\mathcal{L}}{=} \hat{\Gamma} + \gamma G^D(\cdot, \chi)$$

with $\hat{\Gamma}$ an independent GFF according to $\tilde{\mathbb{P}}_\chi$. Since we know that $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are absolutely continuous between each other, we could first check the result is almost surely for $\tilde{\mathbb{P}}$, which will imply it is for $\mathbb{P}$. To do this, if we sample Γ from $\tilde{\mathbb{P}}$. Using the uniqueness from Proposition 1.21, we can see that (SHE) can be separated as

$$(HE)\begin{cases}\partial_t \Phi^h(t) = \frac{1}{2}\Delta \Phi^h(t) + \xi, \\ \Phi^h(0) = \hat{\Gamma},\end{cases} \quad ,\text{(Shift)}\begin{cases}\partial_t \Phi^s(t) = \frac{1}{2}\Delta \Phi^s(t), \\ \Phi^s(0) = \gamma G^D(\cdot, \chi),\end{cases} \tag{3.11}$$

and since (Shift) is a determinist equation, we will have that bot respective fields are independent. This separation implies that if Φ is a solution of (3.1) with initial condition given by Γ as before, then

$$\frac{\Phi_\varepsilon(\chi,t)}{\ln\left(\frac{1}{\varepsilon}\right)} = \frac{\Phi^h_\varepsilon(\chi,t)}{\ln\left(\frac{1}{\varepsilon}\right)} + \frac{\Phi^s_\varepsilon(\chi,t)}{\ln\left(\frac{1}{\varepsilon}\right)} -$$

Since $\hat{\Gamma}$ is independent of Γ, when take the limsup at $\varepsilon \to 0$, we will have that

$$\limsup_{\varepsilon\to 0} \frac{\Phi^h_\varepsilon(\chi,t)}{\ln\left(\frac{1}{\varepsilon}\right)} = 0.$$

On the other hand, from Lemma 1.8, we know that

$$\Phi^s_\varepsilon(\chi,t) = \mathbb{E}[\gamma G^D_\varepsilon(B^\chi_{s\wedge\tau_D}, \chi)]$$

and this implies that

$$\limsup_{\varepsilon\to 0} \frac{\Phi^s_\varepsilon(\chi,t)}{\ln\left(\frac{1}{\varepsilon}\right)} = 0.$$

And therefore, we conclude the desired lemma. □

This first result motivates us to study the behavior of thick points in times as a subset of the space-time domain D_T. This idea motivates the definition

Definition 3.1 (**Thick points in space-time domain**) *For a $\gamma > 0$ given, and taking $\Phi_\varepsilon(x,t)$, the circle average*

$$T^\gamma := \left\{(x,t) \in D_T;\ \limsup_{\varepsilon\to 0} \frac{\Phi_\varepsilon(x,t)}{\ln\left(\frac{1}{\varepsilon}\right)} = \gamma\right\},$$

the set of thick points in D_T.

We will now focus on understanding this space-time set have an interesting behavior thanks to the dynamic in consideration.

3.3. Fractal dimension in space-time domain

In this section, we want to study the basic geometric aspects of T^γ. In particular, we are interested in knowing up to which value of γ the set is not empty. And, as a more geometric property, what is its Hausdorff dimension as a function of γ.

As in the case of the GFF, to study this type of problems we use mainly to ideas. For the upper bound we can do a box-counting argument using the modulus of continuity. And for the lower bound we construct a GMC measure supported on T^γ.

3.3.1. Gaussian multiplicative chaos on the SHEF

The main goal of this section is to construct the GMC measure for $\varPhi$ the SHEF. It is well known from [Ber17], that this measure can be constructed for log-correlated fields. But in our case we have done not have a log-correlated field as we saw in lemma 3.1 and more clear by the prior estimation from Corollary 3.1. Therefore, we will have to do some small extra work to confirm the existence of this measure. This means that in this section our goal is to prove the following proposition

Proposition 3.2 (GMC measure) *The measure*

$$\mu_\varepsilon^\gamma(dz) = \exp^{\gamma\varPhi_\varepsilon(z)-\frac{\gamma^2}{2}\mathbb{E}[\varPhi_\varepsilon^2(z)]}dz$$

almost surely converges in probability to a random finite positive measure $M^\gamma(\varPhi, dxdt)$ in D_T.

The main idea now is related to extend the work from [Ber17]. This could be done in two ways. The first way is generalized the same construction for the context of $\mathbb{R}^d$ with a generic metric given. To then use this result for the parabolic metric d_p. On the other hand is to extend the result for the limit considering the Euclidean metric in $\mathbb{R}^3$. In this section we will proceed with the last idea. We take $f \in C(D_T)$ and consider

$$J_\varepsilon = \int_{D_T} f(z)e^{\gamma\varPhi_\varepsilon(x,t)-\frac{\gamma^2}{2}\mathrm{Var}(\varPhi_\varepsilon(x,t))}dz.$$

For the moment, we fix an $\epsilon_0 > 0$, and take $\epsilon < \epsilon_0$. We define the event that (x,t) is a "good point" of order α as

$$G_{\varepsilon,\alpha}(z) := \{\varPhi_r(z) \le \alpha \ln\left(\frac{1}{r}\right),\ \forall r \in (\varepsilon, \varepsilon_0)\}, \tag{3.12}$$

and therefore we separate J_ε as the sum

$$J_\varepsilon = I_\varepsilon + I_\varepsilon^c, \tag{3.13}$$

where

$$I_\varepsilon = \int_{D_T} f(z)1_{G_{\varepsilon,\alpha}(z)}\mu_\varepsilon(dxdt), \quad I_\varepsilon^c = \int_{D_T} f(z)1_{G^c_{\varepsilon,\alpha}(z)}\mu_\varepsilon(dxdt).$$

The idea now is to see that the term with $G^c_{\varepsilon,\alpha}$ goes to 0 in L^1 as ε_0 tends to 0. To prove this, we will need the following lemma

Lemma 3.3 *For $\alpha > \gamma$, we have that exists p such that*

$$\mathbb{E}[1_{G_{\varepsilon,\alpha}(z)}e^{\gamma\Phi_\varepsilon(z)-\frac{\gamma^2}{2}\mathbb{E}[\Phi_\varepsilon^2(z)]}] \geq 1-p(\varepsilon_0),$$

for p a function that goes to 0 when $\varepsilon_0 \to 0$. This implies that

$$\mathbb{E}[1_{G^c_{\varepsilon,\alpha}(z)}e^{\gamma\Phi_\varepsilon(z)-\frac{\gamma^2}{2}\mathbb{E}[\Phi_\varepsilon^2(z)]}] \underset{\varepsilon_0\to 0}{\longrightarrow} 0.$$

Proof. Since in our definition of Φ_ε we are only approximating in space. The proof of the lemma is the same computation realized in [Ber17], Section 3, and therefore, we do not do it. □

The above result, told us that for the L^1 convergence of J_ε, wee only need to focus on I_ε. As in [Ber17] we will start by studying this variable first in L^2, and from that conclude the convergence in L^1. To do this, first we will prove the following proposition

Proposition 3.3 *The sequence $(I_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in L^2 for $\gamma < 2\sqrt{2}$ and α sufficiently close to γ.*

Proof. First, notice that, a direct computation gives us that

$$\mathbb{E}[I_\varepsilon^2] = \int_{D_T}\int_{D_T} f(z)f(w)e^{\gamma^2\mathrm{Cov}(\Phi_\varepsilon(z),\Phi_\delta(w))}\tilde{\mathbb{P}}(G_{\varepsilon,\alpha}(w), G_{\delta\alpha}(w))dzdw$$

in the last line we have done the change of measure from $\mathbb{P}$ to $\tilde{\mathbb{P}}$, and the new measure is given by the next Radon-Nikodyn derivative

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \frac{e^{\gamma(\Phi_\varepsilon(z)+\Phi_\varepsilon(w))}}{\mathbb{E}[e^{\gamma(\Phi_\varepsilon(z)+\Phi_\varepsilon(w))}]}.$$

We claim that there exists a constant C such that

$$\begin{aligned}\widetilde{\mathbb{P}}(\tilde{\Phi}_{\hat{r}}(x,t) \leq (2\alpha-\gamma)\ln(1/\hat{r})+C) &\leq Ce^{-\frac{1}{2}(2(\alpha-\gamma))^2\ln\left(\frac{1}{\hat{r}}\right)+C(\ln(1/r))^{-1}} \\ &\leq C\hat{r}^{\frac{1}{2}(2\alpha-\gamma)^2},\end{aligned}$$

let us assume for now this proposition, we will prove if afterwards.

On the other hand, from Corollary 3.1 we can see that there exist some constants $0 < c < C$ such that, for $\hat{r} = \varepsilon \vee \delta \vee |x-y| \vee \sqrt{|t-s|}$ we have

$$\ln\Big(\frac{1}{\hat{r}}\Big) + c \leq \mathrm{Cov}(\Phi_\varepsilon(x,t), \Phi_\delta(y,s)) \leq \ln\Big(\frac{1}{\hat{r}}\Big) + C.$$

Thanks to these priors estimations when α is near γ we have that

$$\mathbb{E}[I_\varepsilon^2] \leq C\int_{D_T}\int_{D_T} f(z)f(w)(\frac{1}{\varepsilon \vee |x-y| \vee \sqrt{|t-s|}})^{\frac{\gamma^2}{2}}dzdw,$$

and we want that the function $(\varepsilon \vee |x-y| \vee \sqrt{|t-s|})^{-\frac{\gamma^2}{2}}$ be an integrable function, and since the problems are when $|x-y| \vee \sqrt{|t-s|} \leq \varepsilon$ we could study the next integral to see the behavior of the desired function, in particular we want to check that for $\gamma < \sqrt{8}$, this integral is finite. And to do this, since we are roughly speaking seen a term with a certain cylindrical symmetry, when we see this integral locally around a fixed $(y, s) \in D_T$, we can estimate his value (or at least study it) via the next integral

$$\int_0^1 \int_0^1 (\frac{1}{r \vee \sqrt{s}})^{\frac{\gamma^2}{2}} r dr ds, \tag{3.14}$$

and therefore we want to see for which values of γ this integral is finite, and computing the integral by separating the domain in $\{r \leq \sqrt{s}\}$ and $\{r \geq \sqrt{s}\}$. We can see that for $\gamma < 2\sqrt{2}$, the integral in (3.14) is finite. Hence, the sequence I_ε is uniformly bounded in L^2.

We are only left to prove the claim given as follows.

Claim 3.1 *If we consider the measure,*

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \frac{e^{\gamma(\Phi_\varepsilon(x,t)+\Phi_\varepsilon(y,s))}}{\mathbb{E}[e^{\gamma(\Phi_\varepsilon(x,t)+\Phi_\varepsilon(y,s))}]}.$$

We have that exists some positive constant C, such that

$$\begin{aligned}\widetilde{\mathbb{P}}(\tilde{\Phi}_{\hat{r}}(x,t) \leq (2\alpha - \gamma)\ln(1/\hat{r}) + C) &\leq Ce^{-\frac{1}{2}(2(\alpha-\gamma))^2 \ln\left(\frac{1}{\hat{r}}\right) + C(\ln(1/r))^{-1}} \\ &\leq C\hat{r}^{\frac{1}{2}(2\alpha-\gamma)^2},\end{aligned}$$

Proof. From the Cameron-Martin theorem, we can see that

$$\Phi_\varepsilon(x,t) \overset{\mathcal{L}}{=} \tilde{\Phi}_\varepsilon(x,t) + \gamma \mathrm{Var}(\Phi_\varepsilon(x,t)),$$

where $\tilde{\Phi}$ is the SHEF for $\tilde{\mathbb{P}}$. Therefore, we have that $\widetilde{\mathbb{P}}(G_{\varepsilon,\alpha}(x,t)G_{\delta,\alpha}(y,s))$ is upper bounded by

$$\begin{aligned}&\widetilde{\mathbb{P}}\left(\Phi_r(x,t) \leq \alpha \ln(1/r) + C, \Phi_r(y,s) \leq \gamma \ln(1/r) + C \ , \ \forall r \in (\varepsilon, \varepsilon_0)\right) \\ &\leq \widetilde{\mathbb{P}}(\Phi_{\hat{r}}(x,t) \leq 2\alpha \ln(1/\hat{r}) + C)\end{aligned}$$

here we took $\hat{r} = \varepsilon \vee |x-y| \vee \sqrt{|t-s|}$ and assumed that $|x-y| \vee \sqrt{|t-s|} < \varepsilon_0$, to achieve this last bound and here we know that $\Phi_r(x,t)$ is a Gaussian variable with mean $\gamma \mathrm{Var}(\Phi_{\hat{r}}(x,t))$ and variance $\mathrm{Var}(\Phi_r(x,t))$, dividing by $\ln(1/\hat{r})$, and assuming that $|x-y| \vee \sqrt{|t-s|} \leq \varepsilon_0$, we can upper bound using the classic Gaussian inequality

$$\begin{aligned}\widetilde{\mathbb{P}}(\tilde{\Phi}_{\hat{r}}(x,t) \leq (2\alpha - \gamma)\ln(1/\hat{r}) + C) &\leq Ce^{-\frac{1}{2}(2(\alpha-\gamma))^2 \ln\left(\frac{1}{\hat{r}}\right) + C(\ln(1/r))^{-1}} \\ &\leq C\hat{r}^{\frac{1}{2}(2\alpha-\gamma)^2}\end{aligned}$$

and hence, the lemma is proven. □

□

Now let us focus on the convergence in L^2 of $(I_\varepsilon)_{\varepsilon>0}$. The main goal here is to see that the sequence is Cauchy in this space, and to do this, notice that for $\varepsilon, \delta > 0$, a direct computation tell us that

$$\begin{aligned}\mathbb{E}[|I_\varepsilon - I_\delta|^2] &= \mathbb{E}[I_\varepsilon^2] - 2\mathbb{E}[I_\varepsilon I_\delta] + \mathbb{E}[I_\delta^2] \\ &= \Big(\mathbb{E}[I_\varepsilon^2] - \mathbb{E}[I_\varepsilon I_\delta]\Big) + \Big(\mathbb{E}[I_\delta^2] - \mathbb{E}[I_\varepsilon I_\delta]\Big)\end{aligned}$$

and this last equality motivates to study the terms of the form $\mathbb{E}[I_\varepsilon^2]$ and $\mathbb{E}[I_\varepsilon I_\delta]$. For this we will have the next two lemmas.

Lemma 3.4 *For $\mathbb{E}[I_\varepsilon I_\delta]$ we have that the liminf its lower bounded by*

$$\int_{D_T}\int_{D_T} f(z)f(w)g_\alpha(z,w)\mathbb{E}\left[\frac{\exp\Big(\gamma^2 G^D(B^x_{|t-s|\wedge\tau_D}, y)\Big)}{|B^x_{|t-s|\wedge\tau_D} - y|^{\gamma^2}}\right] dzdw$$

Proof. To prove this we first notice that, from the discussion of above, we know that

$$\mathbb{E}[I_\varepsilon I_\delta] = \int_{D_T}\int_{D_T} ff\tilde{\mathbb{P}}(G_{\varepsilon,\alpha}(x,t), G_{\delta,\alpha}(y,s))\exp\Big(\gamma^2\mathrm{Cov}(\Phi_\varepsilon(x,t), \Phi_\delta(y,s))\Big)$$

replacing in the covariance with the expression given by the solution of the heat equation, we will have

$$\mathbb{E}[I_\varepsilon I_\delta] = \int_{D_T}\int_{D_T} ff\tilde{\mathbb{P}}(G_{\varepsilon,\alpha}(x,t), G_{\delta,\alpha}(y,s))\exp\Big(\gamma^2\mathbb{E}[G^D_{\varepsilon,\delta}(B^x_{|t-s|\wedge\tau_D}, y)\Big)]$$

and here we can do as in [Ber17], and for η small enough, we separate this integral in the events $|x-y| \le \eta$ and $|x-y| > \eta$. If we first see under $|x-y| > \eta$, we will have that since the law of the pair sequences $(\Phi_r(x,t))_{\varepsilon<r<\varepsilon_0}, (\Phi_{r'}(y,s))_{\delta<r'<\varepsilon_0}$ under $\widetilde{\mathbb{P}}$ has the same covariance that under $\mathbb{P}$ but with a shifted mean given by the Cameron-Martin theorem, and the respective shifted means converges to another one when ε and δ goes to 0, the same happens with the covariance structure. Therefore, when we take the limit for ε, δ to 0, we will have than they converge in law to $(\Phi_r(x,t))_{r<\varepsilon_0}, (\Phi_{r'}(y,s))_{r<\varepsilon_0}$, and even more, we can see that the punctual limit is given by

$$\widetilde{\mathbb{P}}(G_{\varepsilon,\alpha}(z), G_{\delta,\alpha}(z)) \to \widetilde{\mathbb{P}}(G_{0,\alpha}(z), G_{0,\alpha}(z))g_\alpha(z,w)$$

and since therefore we can see that this function is bounded, one can check too than this limits holds uniformly in $|x-y| > \eta$. Thanks to this first convergence, if we take the liminf, thanks to Fatou's lemma we have that the liminf of $\mathbb{E}[I_\varepsilon I_\delta]$ as $\varepsilon, \delta \to 0$, when $|x-y| \ge \eta$ is lower bounded by

$$\iint_{[0,T]^2}\iint_{|x-y|>\eta} f(z)f(w)\liminf_{\varepsilon,\delta\to 0}\tilde{\mathbb{P}}(G_{\varepsilon,\alpha}(z), G_{\delta,\alpha}(w))\exp\Big(\gamma^2\mathbb{E}[G^D_{\varepsilon,\delta}(B^x_{|t-s|\wedge\tau_D}, y)\Big)]dzdw$$

but using the convergence we saw and that the exponential is an increasing and continuous function and again Fatou's lemma, we have that the above is lower bounded by

$$\iint_{[0,T]^2}\iint_{|x-y|>\eta} f(z)f(w)g_\alpha(z,w)\exp\left(\gamma^2\mathbb{E}\left[\liminf_{\varepsilon,\delta\to 0} G^D_{\varepsilon,\delta}(B^x_{|t-s|\wedge\tau_D}, y)\right]\right) dzdw$$

and from here we can conclude the desired due to the work done by [Ber17]. On the other hand, since we have that for $\gamma < 2\sqrt{2}$, the process I_ε is uniformly integrable, we will have that the term with $|x-y| \leq \eta$, will go to 0 when η goes to 0 as well, and for this same reason, the term with the complement will be finite in the limit, since g_α will have the same bound we saw before for $\tilde{\mathbb{P}}(G_{\varepsilon,\alpha}, G_{\delta,\alpha})$, hence we can conclude the desired. □

Notice that with this we control the term given by $\mathbb{E}[I_\delta I_\varepsilon]$, now we need the next lemma such that we can control the ones of the form $\mathbb{E}[I_\varepsilon^2]$.

Lemma 3.5 *For $\mathbb{E}[I_\varepsilon^2]$ we have that its limsup a is upper bounded by*

$$\int_{D_T}\int_{D_T} f(z)f(w)g_\alpha(z,w)\mathbb{E}\left[\frac{\exp\Big(\gamma^2 G^D(B^x_{|t-s|\wedge\tau_D}, y)\Big)}{|B^x_{|t-s|\wedge\tau_D} - y|^{\gamma^2}}\right]$$

Proof. The proof for this other inequality is directly the same has before under the difference that we will have the process $(\Phi_r(x,t), \Phi_r(y,s))_{\varepsilon<r<\varepsilon_0}$, instead of $(\Phi_r(x,t))_{\varepsilon<r<\varepsilon_0}, (\Phi_{r'}(y,s))_{\delta<r'<\varepsilon_0}$ but the argument of this is exactly still the same. □

Notice that these 2 lemmas implies the next proposition

Proposition 3.4 *The sequence I_ε is Cauchy in L^2, and therefore converges to a random variable I. And this limit holds in L^1.*

Proof. The proof is direct from Lemmas 3.5 and 3.4, and noticing that

$$\mathbb{E}[|I_\varepsilon - I_\delta|^2] \leq \Big|\mathbb{E}[I_\varepsilon^2] - \mathbb{E}[I_\varepsilon I_\delta]\Big| + \Big|\mathbb{E}[I_\delta^2] - \mathbb{E}[I_\varepsilon I_\delta]\Big|.$$

And therefore we conclude. □

With this result, it is direct to check that almost surely, the measure μ_ε converges weakly to a measure μ, so we have an almost surely convergence.

Proposition 3.5 *The measure $\mu_\varepsilon(dxdt)$ converges to a measure $\mu(dxdt)$ in probability in the weak topology*

Proof. The proof is the same as in [Ber17], Section 6. □

From now on, this limit measure will be called GMC measure for the SHEF. The next section will be related to the connection between this measure and the thick points.

3.3.2. The SHEF from a GMC typical point

Now we do a small change on the notation since we know that for almost surely μ_ε converges in the weak topology, and this measures depends on Φ and γ, we will say that $\mu_\varepsilon(dxdt) = M^\gamma(\Phi_\varepsilon, dxdt)$. To understand the link between this measure and the thick points of the field we introduce the so-called rooted measure (also known as a "Peyriére measure", since it first appears in [Pey74]). In [Aru20] it is explained that the motivation behind the idea of rooted measure could be a way to understand how a measure looks around a "typical point" of it. From this same paper we can see that for our context the rooted measure is the probability measure in $Ps(D,[0,T])\times D_T$, where $Ps(D,[0,T])$ is the space of functions from $[0,T]$ to the space of Schwartz distributions $\mathcal{D}'(D)$. We can define the root measure on $Ps(D,[0,T])\times D_T$ as

$$Q_\varepsilon^\gamma(d\Phi, dxdt) = \frac{M^\gamma(\Phi_\varepsilon, dxdt)}{\lambda(D_T)}\mathbb{P}(d\Phi). \tag{3.15}$$

Here λ denotes the Lebesgue measure in D_T. This expression in more detail will be

$$Q_\varepsilon^\gamma(d\Phi, dxdt) = \exp\left(\gamma\Phi_\varepsilon(x,t) - \frac{\gamma^2}{2}\mathbb{E}[\Phi_\varepsilon^2(x,t)]\right)\frac{\lambda(dxdt)}{\lambda(D_T)}\mathbb{P}(d\Phi). \tag{3.16}$$

From this idea given by the rooted measure, the main result in this section is the following one.

Proposition 3.6 *For Φ given and $M^\gamma(\Phi, dxdt)$ the chaos measure associated, we will have that a typical point of this measure is a γ-thick point for the field Φ associated.*

To prove this result we first need to understand better the measure Q_ε^γ. In particular, we do as in [Aru20] Section 2. And first check some basic properties of 3.16. To then study the limit to conclude.

Since we want to study the limit of 3.16, we first need the following lemma.

Lemma 3.6 *Take (x,t,Φ) a tuple chose according to $\mathcal{Q}_\varepsilon^\gamma$. The law of (x,t) given Φ is*

$$\nu_\varepsilon^{\gamma,N}(dxdt) := \frac{\exp\left(\gamma\Phi_\varepsilon(x,t) - \frac{\gamma^2}{2}\mathbb{E}[(\Phi_\varepsilon(x,t))^2]\right)}{\mu_\varepsilon^\gamma(D_T)}dxdt. \tag{3.17}$$

And on the other hand, the law of Φ conditionally on (x,t), is equal to

$$\hat{\Phi} + \gamma Cov(\cdot, \Phi_\varepsilon(x,t)), \tag{3.18}$$

where $\hat{\Phi}$ has the law of the SHEF on $Ps(D,[0,T])$.

Proof. We first start noticing if U is the measure given by the uniform random variable on D_T. Then Q_ε^γ under $\mathbb{P}\otimes U$, has a Radon-Nykodin derivative given by

$$f(\Phi,(x,t)) = \exp\left(\gamma\Phi_\varepsilon(x,t) - \frac{\gamma^2}{2}\mathbb{E}[\Phi_\varepsilon^2(x,t)]\right). \tag{3.19}$$

Taking g bounded and measurable on D_T and F bounded and measurable in $Ps(D,[0,T])$, we have that

$$\begin{aligned}\mathbb{E}_{Q_\varepsilon^\gamma}[F(\Phi)g(x,t)] &= \mathbb{E}_{\mathbb{P}\otimes U}\left[F(\Phi)g(x,t)f(\Phi,(x,t))\right]\\ &= \mathbb{E}_{\mathbb{P}}\left[F(\Phi)\mathbb{E}_U(g(x,t)f(\Phi,(x,t)))\right]\\ &= \mathbb{E}_{\mathbb{P}}\left[F(\Phi)\mathbb{E}_U[g(x,t)f(\Phi,(x,t))]\frac{f(\Phi)}{f(\Phi)}\right],\end{aligned}$$

where $f(\Phi)$ indicates the marginal law of Φ under Q_ε^γ. From this it follows that

$$\mathbb{E}_{Q_\varepsilon^\gamma}(F(\Phi)g(x,t)) = E_{\mathbb{P}}\left(F(\Phi)G(\Phi)f(\Phi)\right), \tag{3.20}$$

where

$$G(\Phi) = \mathbb{E}_U(g(x,t)\frac{f(\Phi,(x,t))}{f(\Phi)})$$

since $f(\Phi) = M^\gamma(\Phi_\varepsilon, D_T)$, and $G(\Phi)$ is nothing more than g integrated with respect the desired measure we can conclude equality (3.17). An analog computation will tell us that the conditional law of Φ under (x,t), it will be given by

$$\tilde{\mathbb{P}}(d\Phi) = \frac{f(\Phi,(x,t))(\Phi,(x,t))}{f(x,t)},$$

where now $f(x,t)$ is the marginal distribution of $f(\Phi,(x,t))$. It is direct in this case $f(x,t) = 1$. Therefore, we have that

$$\tilde{\mathbb{P}}(d\Phi) = \exp\left(\gamma\Phi_\varepsilon(x,t) - \frac{\gamma^2}{2}\mathbb{E}[\Phi_\varepsilon(x,t)^2]\right)\mathbb{P}(d\Phi)$$

and here to conclude the desired. From [Hai09], Proposition 4.11. We know that in order to conclude the desired we can check the Fourier transform. Therefore, we can consider $t \in [0,T]$, $f \in H_0^1$, and notice that

$$\mathbb{E}_{\tilde{\mathbb{P}}}(e^{\rho\langle\Phi(t),f\rangle_\nabla}) = \mathbb{E}_{\mathbb{P}}(e^{\rho\langle\Phi(t),f\rangle_\nabla}e^{\gamma\Phi_\varepsilon(x,t)-\frac{\gamma^2}{2}\mathbb{E}[\Phi_\varepsilon(x,t)^2)}]$$

since we are dealing with Gaussian random variables we have that

$$\mathbb{E}_{\tilde{\mathbb{P}}}(e^{\rho\langle\Phi(t),f\rangle_\nabla}) = e^{\frac{\rho^2}{2}\mathbb{E}[\langle\Phi(t),f\rangle_\nabla]+\rho\gamma\mathbb{E}[\langle\Phi(t),f\rangle_\nabla\Phi_\varepsilon(x,t)]}.$$

Using Definition 4.1 from [Hai09], we can conclude that under $\tilde{\mathbb{P}}$ we have that

$$\Phi = \hat{\Phi} + \gamma\mathrm{Cov}(\cdot,\Phi_\varepsilon(x,t)).$$

Which is nothing more but equality (3.18). Therefore, we can conclude the lemma. □

From this first result, we can actually conclude the desired convergence. As is done in the following proposition.

Proposition 3.7 *The measure Q_ε^γ converges weakly to a measure Q^γ when $\varepsilon \to 0$.*

Proof. To prove this, let us consider $F : Ps(D,[0,T]) \times D_T \to \mathbb{R}$ bounded and measurable, since we already know that for all $\varepsilon > 0$ we have that

$$\mathbb{E}_{Q_\varepsilon^\gamma}(F(\Phi,(x,t))) = \mathbb{E}_{\mathbb{P}\times U}(F(\Phi + \mathrm{Cov}(\cdot,\Phi_\varepsilon(x,t)),(x,t)))$$

and using the continuity of F and the theorem of dominated convergence we can conclude the desired convergence for $\varepsilon \to 0$. Even more we can see that in the limit, the law of Φ is shifted by the Schwartz distribution given by the integral functional with kernel given by the limit of $\gamma\mathrm{Cov}(\cdot,\Phi_\varepsilon(x,t))$. □

From the above proposition, we will call $\mathrm{Cov}(\cdot,(x,t))$ as the limit distribution of $\mathrm{Cov}(\cdot,\Phi_\varepsilon(x,t))$. Now we have all the tools to prove Proposition 3.6.

Proof of Proposition 3.6. To prove this proposition notice that from Proposition 3.7, we know that if we first sample $(x,t) \sim U(D_T)$ and then Φ sampled by $\tilde{\mathbb{P}}$ defined in Lemma 3.6, we have

$$\Phi = \hat{\Phi} + \gamma \mathrm{Cov}(\cdot,(x,t)).$$

And therefore when we approximate now Φ via the circle average, we have

$$\frac{\Phi_\varepsilon(x,t)}{\ln\left(\frac{1}{\varepsilon}\right)} = \frac{\hat{\Phi}_\varepsilon(x,t)}{\ln\left(\frac{1}{\varepsilon}\right)} + \gamma \frac{\mathrm{Var}(\Phi_\varepsilon(x,t))}{\ln\left(\frac{1}{\varepsilon}\right)}$$

for all $\varepsilon > 0$. Since we know that $\mathrm{Var}(\Phi_\varepsilon(x,t)) = \ln\left(\frac{1}{\varepsilon}\right) + O(1)$. We can conclude that almost surely (x,t) is γ-thick point. And since $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are absolutely continuous with each other. The proposition follows. □

From the connection saw in this Section, we can obtain geometric information related to the geometry of T^γ.

3.3.3. Lower bound for the Hausdorff dimension

The work done above in relation to the GMC measure for the SHEF and thick points will allow us to prove the following proposition.

Proposition 3.8 *For $\gamma \in (0, 2\sqrt{2})$, we have that, almost surely,*

$$\min\{3 - \frac{\gamma^2}{4}, 4 - \frac{\gamma^2}{2}\} \leq dim_H(T^\gamma).$$

To achieve this, we will use the connections we saw between the GMC measure and the thick points. This connection motivates us to look up for potential theoretic methods to calculate Hausdorff dimensions. In particular, we recall from Section 1 we recall Theorem 1.5. Using this result we now prove our proposition.

Proof of Proposition 3.8. For apply Theorem 1.5, we take M^γ the GMC measure. From Section 3.3.2 we know that almost surely M^γ is supported in the set of thick points. We want to know, for which values of α, the energy $I_\alpha(M^\gamma)$ is finite. To do this, first we take a small notation simplification given by $L = 1_{G_\gamma(x,t), G_\gamma(y,s)}$, the event of been no more than γ-thick. To estimate $I_\alpha(M^\gamma)$ we see his first moment.

$$\begin{aligned}
&\mathbb{E}[\int\int_{D_T \times D_T} \frac{L M^\gamma(\Phi, dxdt) M^\gamma(\Phi, dyds)}{|(x,t)-(y,s)|^\alpha}] \\
&\qquad = \mathbb{E}[\liminf_{\varepsilon \to 0} \int\int_{D_T \times D_T} \frac{L M^\gamma(\Phi_\varepsilon, dxdt) M^\gamma(\Phi_\varepsilon, dyds)}{|(x,t)-(y,s)|^\alpha}] \\
&\qquad \leq \liminf_{\varepsilon \to 0} \mathbb{E}[\int\int_{D_T \times D_T} \frac{L M^\gamma(\Phi_\varepsilon, dxdt) M^\gamma(\Phi_\varepsilon, dyds)}{|(x,t)-(y,s)|^\alpha}]
\end{aligned}$$

and using the definition of $M^\gamma(\varPhi_\varepsilon, dxdt)$ we have

$$\mathbb{E}[\int\int_{D_T\times D_T} \frac{LM^\gamma(\varPhi_\varepsilon, dxdt)M^\gamma(\varPhi_\varepsilon, dyds)}{|(x,t)-(y,s)|^\alpha}]$$

$$= \int_{D_T}\int_{D_T} \frac{\mathbb{E}[Le^{\gamma\varPhi_\varepsilon(x,t)+\gamma\varPhi_\varepsilon(y,s)}]e^{-(\frac{\gamma^2}{2}\mathbb{E}[\varPhi_\varepsilon^2(x,t)]\frac{\gamma^2}{2}\mathbb{E}[\varPhi_\varepsilon]^2(y,s))}}{|(x,t)-(y,s)|^\alpha}$$

$$= \int_{D_T}\int_{D_T} \frac{e^{\gamma^2\mathrm{Cov}(\varPhi_\varepsilon(x,t),\varPhi_\varepsilon(y,s))}\tilde{\mathbb{P}}(G_\gamma(x,t),G_\gamma(y,s))}{|(x,t)-(y,s)|^\alpha}-$$

and as we saw earlier, $\tilde{\mathbb{P}}$ is the measure from Lemma 3.6. From Lemma 3.1, we know in particular that

$$\mathrm{Cov}(\varPhi_\varepsilon(x,t)) \le \ln\left(\frac{1}{|x-y|\vee\sqrt{|t-s|}\vee\varepsilon}\right) + C.$$

Also, from 3.1 we know that

$$\tilde{\mathbb{P}}(G_\gamma(x,t), G_\gamma(y,s)) \le C(|x-y|\vee\sqrt{|t-s|}\vee\varepsilon)^{\frac{\gamma^2}{2}}.$$

Using these 2 inequalities we have then, that $\mathbb{E}[I_\alpha(M^\gamma)$ is upper bounded by

$$\int\int_{D_T\times D_T} \frac{1}{|(x,t)-(y,s)|^\alpha}\frac{1}{(|x-y|\vee\sqrt{|t-s|}\vee\varepsilon)^{-\frac{\gamma^2}{2}}}$$

and if we take $\varepsilon = 0$, to understand when this integral is finite, we see locally around a point (x,t). Using a cylindrical symmetry, the above problem is reduced to check for which values of α the following integral is finite

$$\int_0^1\int_0^1 \frac{1}{(r\vee s)^\alpha}\frac{1}{(r\vee\sqrt{s})^{\frac{\gamma^2}{2}}}rdrds.$$

To check this integral, we need to separate in the cases $r \le s$, $s \le r \le \sqrt{s}$ and $\sqrt{s} \le r$. Let us call this 3 separations 1,2 and 3 respectively. Now we check each one.

1^{rst} *Case:* We start by checking the case $r \le s$, here, the integral will be

$$\int_0^1\int_r^1 \frac{1}{(s)^\alpha}\frac{1}{(\sqrt{s})^{\frac{\gamma^2}{2}}}dsrdr$$

which is equal to

$$\int_0^1\int_r^1 \frac{1}{(s)^{\alpha+\frac{\gamma^2}{4}}}dsrdr$$

and when we calculate the first integral. We see we only need to check

$$\int_0^1 \frac{r}{r^{\alpha+\frac{\gamma^2}{4}-1}}dr.$$

Here we can see that

$$\alpha < 3 - \frac{\gamma^2}{4}.$$

This first step gave us a first value, let us check now the second interval.

2^{nd} *Case:* Now we see the case $s \leq r \leq \sqrt{s}$. Here we have that the integral is equal to

$$\int_0^1 \frac{1}{(\sqrt{s})^{\frac{\gamma^2}{2}}} \int_s^{\sqrt{s}} \frac{r}{(r)^\alpha} drds$$

and performing the respective calculations. For the first integral, we have

$$\int_0^1 \frac{1}{(\sqrt{s})^{\frac{\gamma^2}{2}}} (\frac{1}{(\sqrt{s})^{\alpha-2}} - \frac{1}{(s)^{\alpha-2}}) = \int_0^1 \frac{1}{s^{\frac{1}{2}(\alpha-2+\frac{\gamma^2}{2})}} - \frac{1}{s^{(\alpha-2+\frac{\gamma^2}{4})}}$$

and here we will need the next 2 inequalities

$$\alpha - 2 + \frac{\gamma^2}{4} < 1, \quad \frac{1}{2}(\alpha - 2 + \frac{\gamma^2}{2}) < 1.$$

This implies that

$$\alpha < \min\{3 - \frac{\gamma^2}{4}, 4 - \frac{\gamma^2}{2}\}.$$

And again, we have the respective values.

3^{rd} *Case:* And now we finalize checking the case $\sqrt{s} < r$, in this case we have that the integral can be written as

$$\int_0^1 \int_{\sqrt{s}}^1 \frac{1}{r^{\alpha+\frac{\gamma^2}{2}}} rdrds.$$

When we resolve the integral, as before, we can see that the only therm that is matter is

$$\int_0^1 \frac{1}{s^{\frac{1}{2}(\alpha+\frac{\gamma^2}{2}-2)}} ds.$$

And therefore we need that

$$\alpha < 4 - \frac{\gamma^2}{2}.$$

And since we need that the 3 inequalities hold, we have that

$$\alpha < \min\{3 - \frac{\gamma^2}{4}, 4 - \frac{\gamma^2}{2}\}.$$

Therefore, by Theorem 1.5 we conclude the proof. □

And at least but not less important. If in D_T we put the metric given by d_p, and perform the same calculations as before. We have that, since we will only use steps 1 and 3. Under a small change in the first one. We can conclude that

$$\dim_{H,d_p}(T^\gamma) > 4 - \frac{\gamma^2}{2}, \tag{Ob}$$

and with this, we can do the interpretation that the change of behavior is due to the fact that the natural metric for this field is this last one and not the Euclidean one.

Now that we have an interesting lower bound, we only have to check than it is also an upper bound, the next section will be checking this.

3.3.4. Upper bound for the Hausdorff dimension

In the context of random fractals, a simple way to obtain upper bounds for the Hausdorff dimension is to estimate a Box-counting dimension, such as the Minkowski dimension. This section it will be devoted to prove the following proposition.

Proposition 3.9 *For $\gamma \in (0, 2\sqrt{2})$ we will have that*

$$\dim_H(T^\gamma) \leq \min\{4 - \frac{\gamma 2}{2}, 3 - \frac{\gamma^2}{4}\}$$

and if $\gamma > 2\sqrt{2}$, we will have that T^γ is empty.

So we can conclude the equality. To achieve this, we first need to do a small work to simplify the limit in the definition of thickness. To achieve this, we first need the following lemma.

Lemma 3.7 *There exists a deterministic discrete sequence r_n such that $r_n \to 0$ when n goes to infinity. And almost surely for all $(x,t) \in D_T$, we have that*

$$\limsup_{\varepsilon \to 0} \frac{\Phi_\varepsilon(x,t)}{\ln\left(\frac{1}{\varepsilon}\right)} = \limsup_{n \to \infty} \frac{\Phi_{r_n}(x,t)}{\ln\left(\frac{1}{r_n}\right)}$$

Proof. Taking $r_n = n^{-K}$ for some $K \in \mathbb{N}$ big enough. On one hand it is direct that

$$\limsup_{\varepsilon \to 0} \frac{\Phi_\varepsilon(x,t)}{\ln\left(\frac{1}{\varepsilon}\right)} \geq \limsup_{n \to \infty} \frac{\Phi_{r_n}(x,t)}{\ln\left(\frac{1}{r_n}\right)}.$$

And therefore we only need to check the other inequality. To do this we will proceed the same way that is done in [HMP10]. From Proposition 3.1 we know that a.s. for every $\alpha \in (0, 1/2)$ and given $\zeta, \delta \in (0,1)$ there exists a constant $C > 0$ such that

$$|\Phi_\varepsilon(x,t) - \Phi_{r_n}(x,t)| \leq C(\ln\left(\frac{1}{\varepsilon \wedge r_n}\right))^\zeta$$

To prove this we can do as in the proof of Proposition 2.4 to obtain that

$$|\Phi_\varepsilon(x,t) - \Phi_{r_n}(x,t)| \leq C(\ln\left(\frac{1}{r_{n+1}}\right))^\zeta.$$

Dividing by $\ln(1/r_n)$ we have

$$\frac{\Phi_\varepsilon(x,t)}{\ln(1/r_n)} - \frac{\Phi_{r_n}}{\ln(1/r_n)} \leq \frac{C(\ln\left(\frac{1}{r_{n+1}}\right))^\zeta}{\ln(1/r_n)}.$$

And since $\ln(1/r_n) \le \ln(1/r_{n+1})$ we have that

$$\frac{\Phi_\varepsilon(x,t)}{\ln(1/r_n)} - \frac{\Phi_{r_n}}{\ln(1/r_n)} \le \ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}.$$

From where we can see that

$$\frac{\Phi_\varepsilon(x,t)}{\ln(1/r_n)} \le \frac{\Phi_{r_n}}{\ln(1/r_n)} + \ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}$$

since this is valid for $r_{n+1} \le \varepsilon \le r_n$, and $\zeta - 1 < 0$, we can conclude the desire lemma. $\square$

From this lemma we can prove the desired proposition.

PROOF OF PROPOSITION 3.9. Taking r_n as in Lemma 3.7. And first take $\hat{r}_n = r_n^{1+\delta}$, for some $\delta > 0$, for then define

$$\hat{\mathcal{Q}}^1_n := \{[kr_n, (k+1)\hat{r}_n] \times [lr_n, (l+1)\hat{r}_n) \times [i\hat{r}_n^2, (i+1)\hat{r}_n^2);\ k, l \in \{0, ..., \lceil(\hat{r}_n)^{-1}\rceil - 1\},\\ i \in \{0, ..., \lceil(\hat{r}_n)^{-2}\rceil - 1\}\}$$

a set of cubes that covers $[0,1]^3$. From this set, we define

$$\mathcal{Q}^1_n := \{\square \in \hat{\mathcal{Q}}^1_n + \mathbb{Z}^3;\ \square \cap D_T \ne \emptyset\}$$

the family of cubes that covers D_T. We start estimating

$$\mathbb{E}[\#\{\square \in \mathcal{Q}^1_n;\ \square \cap T^\gamma \ne \emptyset\}] = \sum_{\square \in \mathcal{Q}^1_n} \mathbb{P}(\square \cap T^\gamma \ne \emptyset).$$

A first observation is that

$$\square \cap T^\gamma \ne \emptyset \iff \exists (x,t) \in \square, \limsup_{r_n \to 0} \frac{\Phi_{r_n}(x,t)}{\ln\big(\frac{1}{r_n}\big)} = \gamma$$

and if we assume that (x,t) is a thick point. From the modulus of continuity, taking $(x_\square, t_\square)$ as the center of the square $\square$. We have that for $\alpha \in (0, 1/2)$, almost surely

$$|\Phi_{r_n}(x,t) - \Phi_{r_n}(x_\square, t_\square)| \le C \ln\Big(\frac{1}{r_n}\Big)^\zeta \frac{1}{r_n^{\hat{\alpha}}} d_p((x,t),(x_\square,t_\square))^\alpha$$

since $\hat{\alpha} = \alpha(1+\delta)$ for some $\delta > 0$. From the fact that $(x,t), (x_\square, t_\square)$ are in $\square$. It is direct that $d_p((x,t),(x_\square,t_\square)) < C\hat{r}_n$. Therefore, we have

$$|\Phi_{r_n}(x,t) - \Phi_{r_n}(x_\square, t_\square)| \le C \ln\Big(\frac{1}{r_n}\Big)^\zeta.$$

And from this we can conclude then

$$\mathbb{P}(\square \cap T^\gamma \ne \emptyset) \le \mathbb{P}\left(\Phi_{r_n}(x_\square, t_\square) \ge (\gamma - C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1}) \ln\Big(\frac{1}{r_n}\Big)\right)$$

or equivalently

$$\mathbb{P}(\Box \cap T^\gamma \neq \emptyset) \leq \mathbb{P}\left(\frac{\Phi_{r_n}(x_\Box, t_\Box)}{\sqrt{\ln\left(\frac{1}{r_n}\right)}} \geq (\gamma - C\ln\left(\frac{1}{r_n}\right)^{\zeta-1})\sqrt{\ln\left(\frac{1}{r_n}\right)} \right).$$

Since here we have a normal distribution we can apply the exponential tails of it to see that

$$\begin{aligned}\mathbb{P}(\Box \cap T^\gamma \neq \emptyset) &\leq C\exp\left(-\frac{1}{2}\ln\left(\frac{1}{r_n}\right)(\gamma - C\ln\left(\frac{1}{r_n}\right)^{\zeta-1})^2\right) \\ &\leq C\exp\left(-\frac{1}{2}\ln\left(\frac{1}{r_n}\right)(\gamma^2 + C\ln\left(\frac{1}{r_n}\right)^{2(\zeta-1)})\right).\end{aligned}$$

And considering $\zeta < 0.5$, we can see that

$$\mathbb{P}(\Box \cap T^\gamma \neq \emptyset) \leq C\exp\left(-\frac{1}{2}\ln\left(\frac{1}{r_n}\right)\gamma^2\right).$$

On the other hand, $|\mathcal{Q}_n^1| = O((1/r_n)^4)$, since we cover 2 times in space by r_n, and in time by r_n^2. We have then

$$\mathbb{E}[\#\{\Box \in \mathcal{Q}_n^1;\ \Box \cap T^\gamma \neq \emptyset\}] \leq C\left(\frac{1}{r_n}\right)^{4-\frac{\gamma^2}{2}}.$$

This first inequality told us that when $\gamma > \sqrt{8}$ the set has to be empty, since almost surely there is no box hitting T^γ. On the other hand, we will be able to bound the Hausdorff dimension as follows. Take $\delta \in (\hat{r}_{n+1}, \hat{r}_n)$, since

$$H_\delta^\alpha(T^\gamma) = \inf\left(\sum_{n\in\mathbb{N}} diam^\alpha(X_n);\ T^\gamma \subset \bigcup_{n\in\mathbb{N}} X_n,\ \forall n \in \mathbb{N},\ diam(X_n) \leq \delta \right).$$

We have then,

$$\begin{aligned}H_\delta^\alpha(T^\gamma) &\leq \sum_{\Box\in\mathcal{Q}_n^1} diam^{4-\frac{\gamma^2}{2}}(\Box)1_{\Box\cap T^\gamma\neq\emptyset} \\ &\leq Cr_n^{4-\frac{\gamma^2}{2}}\#\{\Box \in \mathcal{Q}_n^1;\ \Box \cap T^\gamma \neq \emptyset\}.\end{aligned}$$

And when taking expectation, we have, due to the bound obtain, that

$$\mathbb{E}[H_\delta^{4-\frac{\gamma^2}{2}}(T^\gamma)] \leq C$$

for a constant C independent of n. Taking the limit of n to infinity we have then that $\delta \to 0$. And therefore the expectation of $H^{4-\frac{\gamma^2}{2}}(T^\gamma)$ is finite. And that implies is 0. Hence, we have

$$\dim_H(T^\gamma) \leq 4 - \frac{\gamma^2}{2}.$$

Notice than from this first inequality is come from the fact that while in space we cover by squares of side ε in time we covered it by "squares" of side ε^2. From this rather naive observation. We can expect that the second inequality comes from cover in time by ε^2, but by $\sqrt{\varepsilon}$ in space. To check this, we know consider the next cubes,

$$\hat{\mathcal{Q}}_n^2 := \{[kr_n, (k+1)\sqrt{\hat{r}}_n] \times [lr_n, (l+1)\sqrt{\hat{r}}_n) \times [i\hat{r}_n^2, (i+1)\hat{r}_n^2);\ k, l \in \{0, ..., \lceil(\sqrt{\hat{r}^{-1}}_n)\rceil - 1\},$$
$$i \in \{0, ..., \lceil(\hat{r}_n)^{-2}\rceil - 1\}\}.$$

We do as before and define $\mathcal{Q}_n^2$ as the set of cubes in $\hat{\mathcal{Q}}_n^2 + \mathbb{Z}^3$ that intersects D_T. Notice that the idea is to do the same as before, and therefore, we want to estimate

$$\mathbb{E}[\#\{\square \in \mathcal{Q}_n^2;\ \square \cap T^\gamma \neq \emptyset\}] = \sum_{\square \in \mathcal{Q}_n^2} \mathbb{P}(\square \cap T^\gamma \neq \emptyset).$$

Doing as before but for $\Phi_{\sqrt{r}_n}$, here. The modulus of continuity tells us that for $(x,t) \in \square$, we have

$$|\Phi_{\sqrt{r}_n}(x,t) - \Phi_{\sqrt{r}_n}(x_\square, t_\square)| \leq C \ln\left(\frac{1}{\sqrt{r}_n}\right)^\zeta \frac{1}{\sqrt{r}_n^{\alpha(1+\delta)}} d^\alpha((x,t),(x_\square, t_\square))$$

and therefore, we have that

$$|\Phi_{\sqrt{r}_n}(x,t) - \Phi_{\sqrt{r}_n}(x_\square, t_\square)| \leq C \ln\left(\frac{1}{\sqrt{r}_n}\right)^\zeta.$$

From this we have

$$\mathbb{P}(\square \cap T^\gamma \neq \emptyset) \leq \mathbb{P}(\Phi_{\sqrt{r}_n} \geq (\gamma - \ln\left(\frac{1}{\sqrt{r}_n}\right)^{\zeta-1})) \ln\left(\frac{1}{\sqrt{r}_n}\right).$$

And doing as before we now obtain the bound

$$\mathbb{P}(\square \cap T^\gamma \neq \emptyset) \leq C \exp\left(-\frac{\gamma^2}{4} \ln\left(\frac{1}{r_n}\right)\right).$$

And, since for $\mathcal{Q}_n^2$ the sides of the squares are $\sqrt{r}_n$, $\sqrt{r}_n$ and r_n, we have that $|\mathcal{Q}_n^2| = O(\frac{1}{r_n^3})$. Therefore, we see that

$$\mathbb{E}[\#\{\square \in \mathcal{Q}_n^2;\ \square \cap T^\gamma \neq \emptyset\}] \leq C(\frac{1}{r_n})^{3-\frac{\gamma^2}{4}}.$$

And we can use the same argument as before to conclude that

$$\dim_H(T^\gamma) \leq 3 - \frac{\gamma^2}{4},$$

And hence, the proposition is proved. □

From this section, we can conclude the following proposition

Proposition 3.10 *We can conclude that for $\gamma \in (0, 2\sqrt{2})$,*

$$\dim_H(T^\gamma) = \min\{4 - \frac{\gamma^2}{2}, 3 - \frac{\gamma^2}{4}\}. \tag{3.21}$$

And the proof is a direct consequence of Propositions 3.8 and 3.9.

Let us re-write (3.21) as

$$\dim_H(T^\gamma) = \begin{cases} 3 - \frac{\gamma^2}{4} & \text{For } \gamma \in [0,2] \\ 4 - \frac{\gamma^2}{2} & \text{For } \gamma \in (2, 2\sqrt{2}). \end{cases} \tag{3.22}$$

This way to write the fractal dimension makes more clear than we could talk about 2 regimes. The first one, the natural one for the GFF, it would be $[0, 2]$. We call this regime as "thick-regime" or "thick points". But we have also another regime. The one given by $(2, 2\sqrt{2})$. We refer to this regime as "super-thick". This due to the fact that there are no points in the GFF with this "thickness".

The separation of the regimes "thick" and "super-thick" is motivated by 2 observations. The first interesting thing to notice, is that since $T^\gamma \subset D_T$, and it has non-trivial Hausdorff dimension. There have to exist times when the behavior of the model is not the one from GFF. And at prior we do not know which are those times. But they have to exist, and it cannot be just a few ones. On the other hand, if we see the Hausdorff dimension of T^γ as a function of γ, from the expression 3.22, we can see that at $\gamma = 2$ the function it is not analytic, even more, it is not differentiable. Hence, we could speak about some form of "phase transition". Therefore, we can expect that something has to change in the behavior of the thick points between the 2 mentioned regimes. These 2 observations motivates the last 2 sections of this chapter.

A small observation is that, the above result is valid when in D_T we consider the Euclidean distance. If we use the parabolic one, the above calculations plus the observation given by Ob help us realize that if we change the metric in D_T to the parabolic one. And consider $T^\gamma_{d_p}$ as the set of thick points on D_T with this different metric. We have that

$$\dim_H(T^\gamma_{d_p}) = 4 - \frac{\gamma^2}{2}.$$

This observation will be useful in the next section related to the existence of the existence of the just mentioned points.

3.4. Exceptional times

Since T^γ has positive Hausdorff dimension in the super-thick regime. We can conclude that, in particular, exists certain times τ, where $\Phi(\tau)$ has super-thick points. This simple observation motivate us to talk about this set of "exceptional times", and study a little about the geometry of it.

To put the above idea in a formality. Taking $\pi_{\hat{t}}$ as the projection from D_T to $[0, T]$. The set of exceptional times is defined as follows

Definition 3.2 *For $\gamma \in (2, 2\sqrt{2})$. The set ST^γ of exceptional times is defined as*

$$ST^\gamma = \pi_{\hat{t}}(T^\gamma).$$

Since T^γ is not empty in the super-thick regime, we know that ST^γ is also not empty in the same regime. A first natural question is what happens with the intersection given by $\mathbb{Q} \cap ST^\gamma$. Since that if we take a $q \in \mathbb{Q}$, we will have that

$$\mathbb{P}(q \in ST^\gamma) = \mathbb{P}(\exists x \in D, \text{ s.t. } \limsup_{n\to\infty} \frac{\Phi_{r_n}(x,q)}{\ln\left(\frac{1}{r_n}\right)} = \gamma)$$

but since at time q we know that $\Phi(q)$ has the law of a GFF, this last probability is equal to 0. Therefore, we can conclude that almost surely $\mathbb{Q} \cap ST^\gamma = \emptyset$. From this observation, a first simple lemma about this set would be the next one

Lemma 3.8 *Almost surely, the set of exceptional time is totally disconnect.*

This is the first geometric aspect we can study as we just saw how. We know want to see a more complex geometric notion as is the Hausdorff dimension.

We now proceed to continue our calculations of dimensions, and we put as a goal to obtain the fractal dimension of the set ST^γ. In particular, we show the following proposition

Proposition 3.11 *For $\gamma \in (2, 2\sqrt{2})$. Almost surely the Hausdorff dimension of the exceptional times is*

$$\dim_H(ST^\gamma) = 2 - \frac{\gamma^2}{2}.$$

Thanks to we work we have done, this result is rather simple to obtain. To start, let us check the upper bound. To do this we recall the next lemma from [MP10] chapter 4. On the context of metric spaces.

Lemma 3.9 *Let (X_1, d_1) and (X_2, d_2), two complete and locally compact metric spaces, and consider $f : (X_1, d_1) \to (X_2, d_2)$ a surjective map for which exists $C > 0$ and $\alpha > 0$ such that for all $x, y \in X_1$, we have that*

$$d_2(f(x), f(y)) \le C d_1(x, y)^\alpha.$$

Then, we have that

$$\dim_H(X_2) \le \frac{1}{\alpha} \dim_H(X_1).$$

The proof of this lemma is usually and exercise. We show the proof in an annex.

Assuming this lemma, we can prove the next proposition rather easily

Lemma 3.10 *For $\gamma \in (2, 2\sqrt{2})$, we will have that*

$$\dim_H(ST^\gamma) \le 2 - \frac{\gamma^2}{2}.$$

PROOF. To prove this lemma, let us consider the following surjective map,

$$\begin{aligned}\pi_{\hat{t}} : (D_T, d_p) &\longrightarrow ([0,T], |\cdot|) \\ (x,t) &\longmapsto t.\end{aligned}$$

Here notice that it is direct that this map is surjective. On the other hand, since in the domain we considered the parabolic metric. We can see that,

$$\begin{aligned}|\pi_{\hat{t}}(x,t) - \pi_{\hat{t}}(y,s)| &= |t-s| \\ &\leq |t-s| \vee |x-y|^2 \\ &= d((x,t),(y,s))^2.\end{aligned}$$

Therefore, the map $\pi_{\hat{t}}$ is a function that fulfills the hypothesis of the lemma 3.9 for $\alpha = 2$, and hence, if we consider the map $\pi_{\hat{t}}|_{T^{\gamma}_{d_p}}$, it also fulfills requirements. Since $\pi_{\hat{t}}(T^{\gamma}_{d_p}) = ST^{\gamma}$. We have then

$$\begin{aligned}\dim_H(ST^{\gamma}) &\leq \frac{1}{2}\dim_H(T^{\gamma}) \\ &\leq \frac{1}{2}(4 - \frac{\gamma^2}{2})\end{aligned}$$

and hence, the lemma is true. □

Now that we have the upper bound for the Hausdorff dimension, we only need to check the lower one. To do this, we take the push-foward measures of the chaos measure associated to the field. This is

$$\sigma^{\gamma}(dt) = (\pi_{\hat{t}})_{\#}(M^{\gamma}(\Phi, dxdt)). \tag{3.23}$$

And from the previous section we know than this measure is well-defined, and even more, a "typical time" for this measure is an exceptional time. Hence, we can prove the following lemma as we did in Proposition 3.8

Proposition 3.12 *For γ in the super-thick regime, we have that*

$$\dim_H(ST^{\gamma}) \geq 2 - \frac{\gamma^2}{4}.$$

PROOF. We can use Theorem 1.5 with the measure given in (3.23). And check the respective energy functional. And again, this can be reduced to study the following liminf.

$$\liminf_{\varepsilon\to 0} \mathbb{E}\left[\int_0^T \int_0^T \frac{\sigma^{\gamma}_{\varepsilon}(dt)\sigma^{\gamma}_{\varepsilon}(ds)}{|t-s|^{\alpha}}\right]. \tag{3.24}$$

First notice that it is direct that, up to a normalized constant

$$\sigma^{\gamma}_{\varepsilon}(dt) \propto \int_D 1_{G_{\gamma}(x,t)} e^{\gamma\Phi_{\varepsilon}(x,t) - \frac{\gamma^2}{2}\mathbb{E}[\Phi^2_{\varepsilon}(x,t)]} dxdt.$$

Therefore, we have that $\mathbb{E}[I_\alpha(\sigma_\varepsilon^\gamma$ is equal to

$$\int_0^T\int_0^T \frac{1}{|t-s|^\alpha}\int_D\int_D \mathbb{E}[e^{\gamma\Phi_\varepsilon(x,t)-\frac{\gamma^2}{2}\mathbb{E}[\Phi_\varepsilon^2(x,t)]\gamma\Phi_\varepsilon(y,s)-\frac{\gamma^2}{2}\mathbb{E}[\Phi_\varepsilon^2(y,s)]}]dzdw. \tag{3.25}$$

And here we can do as we did in Proposition 3.8 to notice that 3.25 is upper bounded by

$$\int_0^T\int_0^T \frac{1}{|t-s|^\alpha}\int_D\int_D \frac{dxdydtds}{(\sqrt{|t-s|}\vee|x-y|\vee\varepsilon)^{\gamma^2/2}}.$$

And again as in proposition 3.8 to estimate this integral we only need to check for

$$\int_0^1 \frac{1}{s^\alpha}\int_0^1 \frac{r}{(r\vee\sqrt{s})^{\gamma^2/2}}drds.$$

From here, we need to separate the case where $r<\sqrt{s}$ and $\sqrt{s}<r$. After the respective computations we conclude that

$$\alpha < 2-\frac{\gamma^2}{4}.$$

From Theorem 1.5we can conclude the lemma. □

Notice that with Lemmas 3.10 and 3.12, Proposition 3.11 follows directly.

We saw a first geometric aspect related to this set of exceptional time. The following section is about a change of behavior between the thick regime and the super-thick regime. And it is also related to this set of exceptional times.

3.4.1. Exceptional times with N thick points

We saw from (3.22) that at $\gamma=2$, the dimension seen has a function of the thickness is not differentiable. This change of the behavior motivate us to ask ourselves if something was happening after this value, since the Hausdorff dimension obtained is constant almost surely, we could think it as an intrinsic property of the field, and this "change of behavior" of the dimension at 2 make us ask ourselves if something else is changing, and since the change is at 2, something is happening when we move from the thick regime to the super-thick one.

And the expected change of behavior can be seen in the fibers in space related to the exceptional times. In particular, in the thick regimen, the number of thick points in the fiber would be infinite almost-surely. But in the super regime, the fiber of the exceptional time will actually be a finite set. This is what we will prove in the following proposition.

Proposition 3.13 *For γ in the super-thick regime. If we defined the set $Fib(\tau)=\pi_{\hat{t}}^{-1}(\tau)$. We will have that $|Fib(\tau)|$ is finite. Even more, if we call N_γ the maximum number of super-thick points in the fiber. We have that*

$$N_\gamma < \frac{4}{\gamma^2-4}. \tag{3.26}$$

Proof. To prove this proposition, we first study the set

$$T_\delta^{\gamma,N} := \left\{(x_1,...,x_N,t)\in D^N\times[0,T];\ \forall i\in\{1,...,N\},\ (x_i,t)\in T^\gamma,\ d(x_i,x_j)>\delta \text{ when } i\neq j\right\}.$$

The idea here is to find for which values of N this set is not-empty. A way to check this is to see when we can "hit" the set with boxes. To do this we take as before $r_n = n^{-K}$, for the space approximation. And $\hat{r}_n = r_n^{1+\varepsilon}$ for $K \in \mathbb{N}$ with $\varepsilon > 0$ fixed and K big enough (and maybe dependent on γ) for the box length side. We then define the family of boxes $\hat{\mathcal{Q}}_n$, of the form

$$\left(\prod_{j=1}^{N}[(l_j-1)\hat{r}_n, l_j\hat{r}_n] \times [(l'_j-1)\hat{r}_n, l'_j\hat{r}_n]\right) \times [(i-1)\hat{r}_n^2, i\hat{r}_n^2],$$

for $l, l' \in \{1, ..., \lceil \hat{r}_n^{-1} \rceil\}^P$, and $i \in \{1, ..., \lceil \hat{r}_n^{-2} \rceil\}$. From this first set that covers $[0,1]^3$. We take from $\hat{\mathcal{Q}}_n + \mathbb{Z}^{2N+1}$ the subset $\mathcal{Q}_n$, defined as the family of boxes that covers D_T. A first observation is to notice that

$$|\mathcal{Q}_n| = O((\frac{1}{r_n})^{2N+2}). \tag{3.27}$$

The idea now is to estimate the following

$$\mathbb{E}[\#\{\square \in \mathcal{Q}_n;\ \square \cap T^{\gamma,N} \neq \emptyset\}] = \sum_{\square \in \mathcal{Q}_n} \mathbb{P}(\square \cap T^{\gamma,N} \neq \emptyset). \tag{3.28}$$

For this, we first look up the event

$$\square \cap T^{\gamma,N} \neq \emptyset. \tag{3.29}$$

And since $\square \subset D^N \times [0,T]$. We take $\square_i$ as a subset of D_T defined by the box given by the projection in the i^{th} copy of D in D^N times the projection in time of the original box. Therefore, we have that if $(\vec{x}, t) \in \square \cap T^{\gamma,N}$. We have the condition on the distance for the components of $\vec{x}$, and also

$$\forall i = 1, ..., P,\ (x_i, t) \in \square_i\ , \limsup_{n\to\infty} \frac{\Phi_{r_n}(x_i, t)}{\ln\left(\frac{1}{r_n}\right)} = \gamma.$$

Now, taking $(z_i, t_\square)$ as the center of the square $\square_i$. Using the modulus of continuity of SHEF we know that the event from (3.29) is contained in

$$d(z_i, z_j) \geq \delta \text{ when } i \neq j,\ \forall i = 1, ..., N,\ \Phi_{r_n}(z_i, t_\square) \geq (\gamma - C\ln\left(\frac{1}{r_n}\right)^{\zeta-1})\ln\left(\frac{1}{r_n}\right).$$

Therefore, we have that $\mathbb{P}(\square \cap T^{\gamma,N} \neq \emptyset)$ is upper bounded by

$$\mathbb{P}\left(d(z_i, z_j) \geq \delta \text{ for } i \neq j,\ \forall i = 1, ..., N,\ \Phi_{r_n}(z_i, t_\square) \geq (\gamma - C\ln\left(\frac{1}{r_n}\right)^{\zeta-1})\ln\left(\frac{1}{r_n}\right)\right). \tag{3.30}$$

If we look

$$\hat{\Phi}_{r_n}(x,t) = \frac{\Phi_{r_n}(x,t)}{\sqrt{\mathrm{Var}(\Phi_{r_n}(x,t))}} \tag{3.31}$$

We have then that the right-hand side of Inequality (3.30) is equivalent to

$$\hat{p} = \mathbb{P}\left(d(z_i, z_j) \geq \delta \text{ when } i \neq j,\ \forall i = 1, ..., N,\ \hat{\Phi}_{r_n}(z_i, t_{\square}) \geq (\gamma - C \ln\Big(\frac{1}{r_n}\Big)^{\zeta - 1}) \ln^{1/2}(\frac{1}{r_n})\right). \tag{3.32}$$

We can easily check that for $\hat{\Phi}_{r_n}$, the covariance between 2 different points tends to 0. Even more, if C^n Denotes the covariance matrix of 3.31 we have strongly that $C^n \to Id$, where Id indicates the identity matrix of $\mathbb{R}^n$. This implies that the above also happens for $(C^n)^1$. Using the formula for the distribution of a Gaussian vector in finite dimension. We can conclude that for n large enough. There exists a constant $A > 0$ such that

$$\hat{p} \leq A\mathbb{P}\left(d(z_i, z_j) \geq \delta \text{ when } i \neq j,\ \forall i = 1, ..., N,\ \chi_i \geq \ln^{1/2}(\frac{1}{r_n})\right). \tag{3.33}$$

Here now χ is a standard Gaussian vector. Using the independence and the tail decay for the normal distribution, we can conclude from Equation (3.30)

$$\begin{aligned} \mathbb{P}(\square \cap T^{\gamma,N} \neq \emptyset) &\leq A \exp\left(-\frac{1}{2}(\gamma - C\ln\Big(\frac{1}{r_n}\Big)^{\zeta-1})^2 N \ln\Big(\frac{1}{r_n}\Big)\right) \\ &\leq \hat{A}\exp\left(-\frac{\gamma^2 N}{2}\ln\Big(\frac{1}{r_n}\Big)\right). \end{aligned} \tag{3.34}$$

We now use this upper bound in Equation (3.28) and Estimation from (3.27) to conclude that

$$\mathbb{E}[\#\{\square \in \mathcal{Q}_n;\ \square \cap T^{\gamma,N} \neq \emptyset\}] \leq \mathcal{O}\left(\Big(\frac{1}{r_n}\Big)^{2N+2-\frac{\gamma^2 N}{2}}\right). \tag{3.35}$$

Now notice that when

$$N > \frac{4}{\gamma^2 - 4} \tag{3.36}$$

we have that

$$\mathbb{E}[\#\{\square \in \mathcal{Q}_n;\ \square \cap T^{\gamma,N} \neq \emptyset\}] \leq \mathcal{O}\left(r_n^{\rho}\right) \tag{3.37}$$

for some $\rho > 0$. Hence, when we take $n \to \infty$. We have that the expectation from (3.28) tends to 0. Therefore, we can conclude also that almost surely

$$\#\{\square \in \mathcal{Q}_n;\ \square \cap T^{\gamma,N} \neq \emptyset\} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.38}$$

And then the proposition is straightforward for $T^{\gamma,N}_{\delta}$ and since the above is independent of δ. We can conclude the same for

$$T^{\gamma,N} := \Big\{(x_1, ..., x_N, t) \in D^N \times [0, T];\ \forall i \in \{1, ..., N\},\ (x_i, t) \in T^{\gamma},\ d(x_i, x_j) > 0 \text{ when } i \neq j\Big\}.$$

And from here, the proposition is direct. □

As a direct corollary of this proposition we have you know that

Corollary 3.2 *If ST^{γ}_N denotes the set of times of N thick points, we have that for N that* *fulfills* (3.26)

$$\dim_H(ST_N^\gamma) \leq (N+1) - \frac{N\gamma^2}{4}. \tag{3.39}$$

We would like to check for the opposite inequality of this dimension. We will do this in the following section

3.4.2. N-Liouville measure

In Section 3.3.1, we construct the Chaos measure related to the SHEF and how this measure is related to the thick points. In the present section we do a small generalization in the expectation of find exceptional times with N thick points related. To do this, we find the respective Chaos measure. And as we saw in the mentioned section, we start our study considering the following measure

$$\mu_\varepsilon^{\gamma,N}(d\vec{x}dt) = \exp\left(\gamma\sum_{i=1}^N \Phi_\varepsilon(x_i,t) - \frac{\gamma^2}{2}\mathbb{E}[(\sum_{i=1}^N \Phi_\varepsilon(x_i,t))^2]\right) 1_{A_N} d\vec{x}dt. \tag{3.40}$$

Here we define

$$A_N := \{\vec{x} \in D^N;\ \text{ for every } i,j = 1,...,N,\ i \neq j,\ d(x_i,x_j) > 0\},$$

since we are interested in the case of find N different points. Let us now fix ε_0, from this we define the event $G_{\alpha,\varepsilon}(x,t)$ of been a good point as in (3.12). From this we consider the generalization of have N good points as

$$G^N_{\alpha,\varepsilon}(\vec{x},t) = \bigcap_{i=1}^N G_{\alpha,\varepsilon}(x_i,t).$$

Then, when we consider for some $f \in C(D^N \times [0,T])$, the random variable J_ε as the integration of f with respect to the measure $\mu_\varepsilon^{\gamma,N}$. To simplify notation consider $D_T^N = D^N \times [0,T]$. From this, notice that J_ε is equal to the sum of

$$I_\varepsilon = \int_{D_T^N} 1_{G^N_{\alpha,\varepsilon}(\vec{x},t)} f \mu_\varepsilon^{\gamma,N}(d\vec{x}dt)\ I_\varepsilon^c = \int_{D_T^N} 1_{(G^{N_{\alpha,\varepsilon}}(\vec{x},t))^c} f \mu_\varepsilon^{\gamma,N}(d\vec{x}dt).$$

From this decomposition we have the following lemma

Lemma 3.11 *We have that* $\mathbb{E}[I_\varepsilon^c] \to 0$ *as* $\epsilon \to 0$.

PROOF. The proof of the lemma is the same as Lemma 3.3. □

As a consequence of this lemma, we focus on I_ε. We start by checking that the sequence is uniformly bounded in L^2.

Proposition 3.14 *For* $\alpha > \gamma$ *and near* γ, *we have that the sequence* $(I_\varepsilon)_{\varepsilon>0}$ *is uniformly bounded in* L^2 *if*

$$N\left(\frac{\gamma^2}{4} - 1\right) < 1.$$

PROOF. Let $\varepsilon > 0$ arbitrary. After some elementary computations we can see that

$$\mathbb{E}[I_\varepsilon^2] = \int_{D_T^N}\int_{D_T^N} f(z_1)f(z_2)e^{\gamma^2\sum_{i=1}^N \mathrm{Cov}(\Phi_\varepsilon(x_i,t),\Phi_\varepsilon(y_i,s))}\mathbb{E}_{\tilde{\mathbb{P}}}[1_{G^N_{\alpha,\varepsilon}(\vec{x},t)}1_{G^N_{\alpha,\varepsilon}(\vec{y},s)}]1_{A_N}(\vec{x})1_{A_N}(\vec{y})dz_1dz_1, \tag{3.41}$$

where

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{\exp\left(\sum_{i=1}^N \gamma(\Phi_\varepsilon(x,t)+\Phi_\varepsilon(y,s))\right)}{\mathbb{E}\left[\exp\left(\sum_{i=1}^N \gamma(\Phi_\varepsilon(x,t)+\Phi_\varepsilon(y,s))\right)\right]}.$$

Since $u(x_i,t)_{y_i,s,\varepsilon} = \mathrm{Cov}(\Phi_\varepsilon(x_i,t),\Phi_\varepsilon(y_i,s))$ solves a heat equation with initial condition given by $G^D_{\varepsilon,ee}(x_i,y_i)$. Using (3.1) we can conclude that there exists some constant $C > 0$ such that

$$e^{\gamma^2\sum_{i=1}^N \mathrm{Cov}(\Phi_\varepsilon(x_i,t),\Phi_\varepsilon(y_i,s))} \leq C\prod_{i=1}^N\left(\frac{1}{|x_i-y_i|\vee\sqrt{|s-t|}\vee\varepsilon}\right)^{\gamma^2}. \tag{3.42}$$

On the other hand we need the following claim

Claim 3.2 *There exists a constant C, such that*

$$\tilde{\mathbb{P}}(G^N_{\alpha,\varepsilon}(\vec{x},t), G^N_{\alpha,\varepsilon}(\vec{y},s)) \leq C\prod_{i=1}^N (r_i)^{\frac{1}{2}(2\alpha-\gamma)^2}, \tag{3.43}$$

where $r_i = |x_i-y_i|\vee\sqrt{|t-s|}\vee\varepsilon$ and we consider $r_i \leq \varepsilon_0$ for every $i = 1,...,N$.

From (3.42) and Claim 3.43, in (3.41) we have then when consider $\alpha \approx \gamma$

$$\mathbb{E}[I_\varepsilon^2] \leq C\int_{D_T^N}\int_{D_T^N} 1_{A_N}(\vec{x})1_{A_N}(\vec{y})f(z)f(w)\prod_{i=1}^N\left(\frac{1}{|x_i-y_i|\vee\sqrt{|s-t|}\vee\varepsilon}\right)^{\frac{\gamma^2}{2}} dzdw \tag{3.44}$$

From here we know that the problem can be reduced to study when the following integral is finite

$$I_N = \int_0^1\int_{[0,1]^N}\prod_{i=1}^N\left(\frac{1}{r_i\vee\sqrt{s}}\right)^{\frac{\gamma^2}{2}} r_idr_ids = \int_0^1\prod_{i=1}^N\int_0^1\left(\frac{1}{r_i\vee\sqrt{s}}\right)^{\frac{\gamma^2}{2}} r_idr_ids. \tag{3.45}$$

To know for which values of γ the above integral is finite, we first notice that

$$\begin{aligned}\int_0^1\left(\frac{1}{r_i\vee\sqrt{s}}\right)^{\frac{\gamma^2}{2}} r_idr &= \int_{\sqrt{s}}^1 \frac{1}{r_i^{\frac{\gamma^2}{2}-1}}dr_i + \int_0^{\sqrt{s}}\left(\frac{1}{\sqrt{s}}\right)^{\frac{\gamma^2}{2}} r_idr_i \\ &= c + (\frac{1}{2}-c)\frac{1}{s^{\frac{\gamma^2}{4}-1}}\end{aligned}$$

for $c = (\frac{\gamma^2}{2}-2)^{-1}$. When $\gamma \in (2, 2\sqrt{2})$ the above constant is always positive and finite. From this computation we have then

$$I_N = \int_0^1 \left(c + (\frac{1}{2} - c) \frac{1}{s^{\frac{\gamma^2}{4} - 1}} \right)^N ds.$$

And for this integral to be finite, we need that

$$N(\frac{\gamma^2}{4} - 1) < 1.$$

Therefore, to conclude the proposition we only need to prove the claim

Proof of Claim 3.2. It is straightforward that for $r_i = |x_i - y_i| \vee \sqrt{|t-s} \vee \varepsilon$

$$\tilde{\mathbb{P}}(G^N_{\alpha,\varepsilon}(\vec{x}, t), G^N_{\alpha,\varepsilon}(\vec{y}, s)) \leq \tilde{\mathbb{P}}(\forall i = 1, ..., N,\ \Phi_{r_i}(x_i, t) \leq 2\alpha \ln(1/r_i)).$$

Since under $\tilde{\mathbb{P}}$, we have that

$$\Phi_{r_i}(x_i, t) \overset{\mathcal{L}}{=} \tilde{\Phi}_{r_i}(x_i, t) + \gamma \sum_{j=1}^{N} \mathrm{Cov}(\Phi_{r_i}(x_i, t), (\Phi_{r_i}(x_j, t) + \Phi_{r_i}(y_j, s))),$$

where $\tilde{\Phi}$ is a SHEF under $\tilde{\mathbb{P}}$. From here we can do as in the proof of Claim 3.1 to obtain that the above is bounded by

$$\tilde{\mathbb{P}}(\forall i = 1, ..., N,\ \frac{\tilde{\Phi}_{r_i}(x_i, t)}{\ln(1/r_i)} \leq (2\alpha - \gamma) \ln^{\frac{1}{2}}(1/r_i) + C)$$

here we also divided by $\ln(1/r_i)$. Taking $(X_i)_{i=1}^N$ as $\frac{\tilde{\Phi}_{r_i}(x_i,t)}{\sqrt{\mathrm{Var}(\tilde{\Phi}_{r_i}(x_i,t))}}$. We have that the correlations of such process tends to 0 as n goes to infinity. This implies that there exist some constant A (that might change between lines) that

$$\tilde{\mathbb{P}}(\forall i = 1, ..., N,\ X_i \leq (2\alpha - \gamma) \ln^{\frac{1}{2}}(1/r_i) + C) \leq A \prod_{i=1}^{N} \tilde{\mathbb{P}}(\frac{\tilde{\Phi}_{r_i}(x_i, t)}{\ln(1/r_i)} \leq (2\alpha - \gamma) \ln^{\frac{1}{2}}(1/r_i) + C)$$

now we apply the classic Gaussian bound to obtain

$$\tilde{\mathbb{P}}(\forall i = 1, ..., N,\ X_i \leq (2\alpha - \gamma) \ln^{\frac{1}{2}}(1/r_i) + C) \leq A \prod_{i=1}^{N} (r_i)^{\frac{1}{2}(2\alpha-\gamma)^2}.$$

Hence, the claim is proven. And therefore the proposition. □

□

From this result, we have that

Proposition 3.15 *The sequence $(I_\varepsilon)_{\varepsilon>0}$ is Cauchy in L^2 when $N(\gamma^2 - 4) < 4$.*

The proof of this proposition is a direct consequence of the following lemmas

Lemma 3.12 *For $(I_\varepsilon)_{\varepsilon>0}$ we have that*

$$\limsup_{\varepsilon\to 0}\mathbb{E}[I_\varepsilon^2] \le \int_{D_T^N}\int_{D_T^N} f(z_1)f(z_2)g_\alpha(\vec{x},\vec{y},t,s)\prod_{i=1}^N \mathbb{E}[\frac{\exp\Big(\gamma^2 g(B^{x_i}_{|t-s|\wedge\tau_D}, y_i)\Big)}{|B^{x_i}_{|t-s|\wedge\tau_D} - y|^{\gamma^2}}]dz_1dz_2$$

Proof. The proof is analogue computation as the one done for Lemma 3.5, up to the change that now we have to sum over N different covariances. □

Lemma 3.13 *For $(I_\varepsilon)_{\varepsilon>0}$ we have that*

$$\liminf_{\varepsilon,\delta\to 0}\mathbb{E}[I_\varepsilon I_\delta] \ge \int_{D_T^N}\int_{D_T^N} f(z_1)f(z_2)g_\alpha(\vec{x},\vec{y},t,s)\prod_{i=1}^N \mathbb{E}[\frac{\exp\Big(\gamma^2 g(B^{x_i}_{|t-s|\wedge\tau_D}, y_i)\Big)}{|B^{x_i}_{|t-s|\wedge\tau_D} - y|^{\gamma^2}}]dz_1dz_2$$

Proof. The same as before, but now with Lemma 3.4. □

We can see the convergence in L^2 and therefore in L^1 of $(I_\varepsilon)_{\varepsilon>0}$. From here we can see that

Proposition 3.16 *The measure $\mu_\varepsilon^{\gamma,N}(d\vec{x}dt)$ converges to a measure $\mu^{\gamma,N}$ in probability in the weak topology when*

$$N(\frac{\gamma^2}{4} - 1) < 1.$$

Proof. The proof is the same as in [Ber17] Section 6. □

The above implies the following corollary

Corollary 3.3 *There exists a subsequence $(\mu_{\varepsilon_k}^{\gamma,N})_{k\in\mathbb{N}}$ that converges almost surely to $\mu^{\gamma,N}$.*

3.4.3. The SHEF from an N-Liouville typical point

Now that we have constructed $\mu^{\gamma,N}$. In the present section we want to prove the following proposition

Proposition 3.17 *Let Φ be a SHEF and $\mu^{\gamma,N}$ be its N-Liouville measure of parameter γ. The support of $\mu^{\gamma,N}$ are pairs $((x_i)_{i=1}^N, t)$ such that all x_i are γ-thick points of $\Phi(\cdot,t)$.*

To prove the above proposition, we do as in Section 3.3.2, and consider first the rooted measure given by

$$\mathcal{Q}_\varepsilon^{\gamma,N} := \frac{\exp\Big(\gamma\sum_{i=1}^N \Phi_\varepsilon(x_i,t) - \frac{\gamma^2}{2}\mathbb{E}[(\sum_{i=1}^N \Phi_\varepsilon(x_i,t))^2]\Big)}{\lambda(D_T^N)}\mathbb{P}(d\Phi)d\vec{x}dt.$$

From this measure we have the following property.

Lemma 3.14 *Take $(\vec{x},t,\Phi)$ a tuple chose according to $\mathcal{Q}_\varepsilon^{\gamma,N}$. The law of $(\vec{x},t)$ given Φ is*

$$\nu_\varepsilon^{\gamma,N}(d\vec{x}dt) := \frac{\exp\Big(\gamma\sum_{i=1}^N \Phi_\varepsilon(x_i,t) - \frac{\gamma^2}{2}\mathbb{E}[(\sum_{i=1}^N \Phi_\varepsilon(x_i,t))^2]\Big)}{\mu_\varepsilon^{\gamma,N}(D_T^N)}d\vec{x}dt.$$

And on the other hand, the law of Φ conditionally on $(\vec{x}, t)$, is equal to

$$\hat{\Phi} + \gamma \sum_{i=1}^{N} Cov(\cdot, \Phi_\varepsilon(x_i, t)),$$

where $\hat{\Phi}$ has the law of the SHEF on $Ps(D, [0, T])$.

Proof. The proof is the same, analogous to that of Lemma 3.6, in particular, since we still have a Gaussian process, we are able to apply the Cameron-Martin theorem. □

A direct corollary of the lemma concerns the computation of expected values under the measure $\mathcal{Q}_\varepsilon^{\gamma,N}$.

Corollary 3.4 *For every $F : Ps(D, [0, T]) \times D_T^N \to \mathbb{R}$, continuous and bounded, we have that*

$$\mathbb{E}_{\mathbb{Q}_\varepsilon^{\gamma,N}}[F(\Phi, (\vec{x}, t))] = \mathbb{E}_{\mathbb{P}}\left[\int_{D_T^N} F\left(\Phi + \gamma \sum_{i=1}^{N} Cov(\cdot, \Phi_\varepsilon(x_i, t), (\vec{x}, t))\right)\right]$$

From this corollary we can easily prove to the following lemma

Lemma 3.15 *The measure $\mathcal{Q}_\varepsilon^{\gamma,N}$ converges in the weak topology to a measure $\mathcal{Q}^{\gamma,N}$ that fulfills*

$$\mathbb{E}_{\mathbb{Q}^{\gamma,N}}[F(\Phi, (\vec{x}, t))] = \mathbb{E}_{\mathbb{P}}\left[\int_{D_T^N} F\left(\Phi + \gamma \sum_{i=1}^{N} Cov(\cdot, (x_i, t)), (\vec{x}, t)\right)\right],$$

where $Cov(\cdot, (x_i, t))$ is the limit in the Schwartz sense of $Cov(\cdot, \Phi_\varepsilon(x_i, t))$.

Proof. The proof is analogue to Proposition 3.7. □

After this work we can easily prove Proposition 3.17 the same way we prove Proposition 3.6.

3.4.4. Hausdorff dimension of exceptional times with N points

The above work related to the N-Liouville measure $\mu^{\gamma,N}$ allows us to prove a last result related to the set of exceptional times ST^γ for γ in the super-thick regime. This would be

Proposition 3.18 *For $\gamma \in (2, 2\sqrt{2})$, if we define $ST^\gamma(N)$ as the set of exceptional times with N super-thick points. We have that*

$$\dim_H(ST_N^\gamma) = (N + 1) - \frac{N\gamma^2}{4}$$

In Corollary 3.2 we saw the upper bound. To obtain the lower bound, we use Theorem 1.5. To do this, we consider the following measure

$$\sigma^{\gamma,N}(dt) = (\pi_{\hat{t}}^N)_\# \mu^{\gamma,N}(dt). \tag{3.46}$$

Here $\pi_{\hat{t}}^N$ is the projection from D_T^N to $[0, T]$. We know that a typical time for $\sigma^{\gamma,N}$ is an exceptional time with N thick points, we proceed to use this measure to prove the desired.

Proof of Proposition 3.18. For $\sigma^{\gamma,N}$ from (3.46), since we want that for some α that $I_\alpha(\sigma\gamma, N)$ since it is a positive random variable. We can conclude the desired by checking the expectation of it,

$$\mathbb{E}[I_s(\sigma^{\gamma,N})] \leq \int_0^T \int_0^T \int_{D^N} \int_{D^N} \liminf_{\varepsilon\to 0} \mathbb{E}[\frac{\mu_\varepsilon^{\gamma,N}(d\vec{x}dt)\mu_\varepsilon^{\gamma,N}(d\vec{y}ds)}{|t-s|^\alpha}]$$
$$\leq C \liminf_{\varepsilon\to 0} \int_0^T \int_0^T \int_{D^N} \int_{D^N} \frac{1}{|t-s|^\alpha} \prod_{i=1}^N \frac{1}{(|x_i - y_i| \vee \sqrt{|t-s|} \vee \varepsilon)^{\frac{\gamma^2}{2}}}.$$

And we know that to know when the above integral is finite. We need to check for which values of α the following integral is finite

$$I_N = \int_0^1 \int_{[0,1]^N} \frac{1}{s^\alpha} \prod_{i=1}^N \frac{1}{r_i \vee \sqrt{s}} dr_i dr ds.$$

Using (3.4.2), we can see that

$$I_N = \int_0^1 \frac{1}{s^\alpha}(c_1 + c_2 \frac{1}{s^{\frac{\gamma^2}{4}-1}})^N ds. \tag{3.47}$$

From here it is direct that to conclude the desired, we need for

$$\alpha + N(\frac{\gamma^2}{4} - 1) < 1,$$

or, equivalently

$$\alpha < N + 1 - \frac{N\gamma^2}{4}. \tag{3.48}$$

Therefore, we can conclude the desired. □

From here, we can see that, the sequence defined by

$$\gamma_N = \sqrt{\frac{4}{N} + 4}, \tag{3.49}$$

gives us a sequence of phase transitions of this model. Since we have that $\dim_H(ST_N^{\gamma_N} = 0$, it is not clear if for the critical value there exists the respective thick points in space and time. Such problem unfortunately will not be solved in this thesis. However, as we will mention in the conclusion, one can expect that $ST_N^{\gamma_N}$ is not empty.

Conclusions and future work

We studied two specific dynamics on the GFF. In some sense, both are a natural evolution for the GFF. However, at a geometric level, these processes differs. In the present thesis we saw a specific geometric aspect related to the thick points, where those difference are quite drastic. Nonetheless, even for this two dynamics, there are still many question one could pose. Many of them are related to understand other features of the geometry of the GFF, as are its flow lines, or level lines. However, there are also question we did not answer in this thesis related to the thick points of both, the OU-GFF process and the SHEF.

For the OU-GFF, a first problem is related to the critical case of $\mathcal{E}(f) = 4$. On the other hand, we are still pending to confirm that the energy $\mathcal{E}$ characterizes every thickness function. In other words, we want to know that almost surely, for $f \in H_0^1(\mathbb{R})$ such that $\mathcal{E}(f) < 4$ then, there exists $x \in D$ such that $f = \gamma_x$.

For the SHEF, there are still many questions of interest related to a critical parameter. The first one is the existence of thick points in space and time for $\gamma = \gamma_{\text{critic}} = 2\sqrt{2}$. We conjecture that in this case the set $T^{\gamma_{\text{critic}}}$ is not empty and has Hausdorff dimension 0. The other critical value is the maximum value of thick points in a exceptional time space-fiber. In this case, we still do not know if for γ_N from (3.49) the set $T^{\gamma_N, N}$ is empty or not.

On the problem related to the critical values mentioned for both processes. We think that the techniques to address these problems are the same ones (up to some small modifications) as for the GFF. In particular, we conjecture that for these critical values one could do a work similar (if not as a consequence) to the case of the critical Liouville measure. Unfortunately, we did not achieve a proof or sketch of this.

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