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THICK POINTS STUDY OF LOG-CORRELATED FIELDS

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MATEMÁTICAS APLICADAS

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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PUNTOS ALTOS DE CAMPOS LOG-CORRELACIONADOS.

Los campos log-correlacionados son campos aleatorios cuyas correlaciones dependen logarítmicamente de la distancia entre los puntos. Se cree ampliamente que, bajo condiciones suaves, los valores extremos de cualquier campo log-correlacionado caen dentro de la misma clase de universalidad. En otras palabras, su comportamiento será similar al de un ejemplo representativo de esta clase.

El campo log-correlacionado más estudiado es el Gaussian Free Field (GFF) en dos dimensiones, un campo Gaussiano centrado cuya función de correlación es la función de Green del Laplaciano. Aunque el GFF en sí mismo no es una función en el sentido tradicional, sus valores extremos pueden estudiarse de manera significativa regularizando el campo. Específicamente, el punto γ-alto del campo se puede definir como

$$T(\gamma) \doteq \left\{ x \in D : \limsup_{\varepsilon \to 0} \frac{\Phi_\varepsilon(x)}{\log(1/\varepsilon)} = \gamma \right\},$$

donde Φ_ε denota la aproximación de circle-average del campo.

Con el objetivo de explorar las propiedades de universalidad de los thick points en los campos log-correlacionados, presentamos un nuevo objeto que captura la esencia de todas las posibles variantes de estos campos: el pseudo Gaussian Free Field (pGFF). Este es un campo cuyas correlaciones están regidas por la función de Green. En la primera parte de esta tesis, describimos las características fundamentales que hacen del pGFF un objeto adecuado para abordar nuestro estudio. Posteriormente, analizamos los valores extremos del pGFF en el caso unidimensional.

En la parte final de esta tesis, estudiamos el comportamiento de los thick points para una clase específica de pGFF. Para esta clase restringida, demostramos que la dimensión de los thick points exhibe un comportamiento universal, coincidiendo con el del GFF. La idea central es utilizar la convergencia mod-Gaussiana para mostrar que las probabilidades de cola de ciertas variables aleatorias se comportan de manera similar a las de las variables Gaussianas.

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THICK POINTS STUDY OF LOG-CORRELATED FIELDS

Log-correlated fields are random fields whose correlations depend logarithmically on the distance between points. It is widely believed that, under mild conditions, the extreme values of any log-correlated field fall within the same universality class. In other words, their behavior will resemble that of a representative example from this class.

The best-understood log-correlated field is the Gaussian Free Field (GFF) in two dimensions, a centered Gaussian field whose correlation function is the Green's function of the Laplacian. Although the GFF itself is not a function in the traditional sense, its extreme values can be meaningfully studied by regularizing the field. Specifically, the γ-thick point of the field can be defined as

$$T(a) \doteq \left\{ x \in D : \limsup_{\varepsilon \to 0} \frac{\Phi_\epsilon(x)}{\log(1/\varepsilon)} = \gamma \right\},$$

where Φ_ϵ denotes the circle-average approximation of the field.

With the aim of exploring the universality properties of thick points in log-correlated fields, we introduce a new object that captures the essence of all possible variants of these fields: the pseudo Gaussian Free Field (pGFF). This is a field whose correlations are governed by the Green's function. In the first part of this thesis, we describe the fundamental characteristics that make the pGFF a suitable object for our study. Subsequently, we analyze the extreme values of the pGFF in the one-dimensional case.

In the final part of this thesis, we study the behavior of thick points for a specific class of pGFF. For this restricted class, we demonstrate that the dimension of thick points exhibits universal behavior, matching that of the GFF. The core idea is to use mod-Gaussian convergence to show that the tail probabilities of certain random variables behave similarly to those of Gaussian variables.

Arranca por la derecha
el genio del fútbol mundial

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Introduction

Log-correlated fields constitute a fascinating topic within theoretical physics and applied mathematics, particularly in field theory and stochastic processes. They are distinguished by having correlations between their values that decay logarithmically with distance, allowing them to model phenomena involving long-range interactions. These fields are essential for describing extreme behavior situations, such as the appearance of exceptional points in physical and mathematical systems. These types of fields have been the subject of extensive study in Gaussian Multiplicative Chaos due to their connection with Liouville Quantum Gravity ([Aru20a], [BSS14]).

One of the most notable examples of a log-correlated field is the two-dimensional Gaussian Free Field (GFF), which models complex physical and mathematical phenomena, such as random interfaces and quantum field theories. More rigorously, the GFF is defined as the standard Gaussian random variable in the space H_0^1, and its correlation structure is determined by the Green's function. As in dimension bigger than or equal to 2 the Green's function explodes in the diagonal, the field cannot be interpreted as a function in the classical sense and, instead, must be understood as a distribution in the sense of Schwartz. See Figure 1 for a two dimensional simulation of the GFF. As observed in Figure 1, the pronounced peaks in this random surface

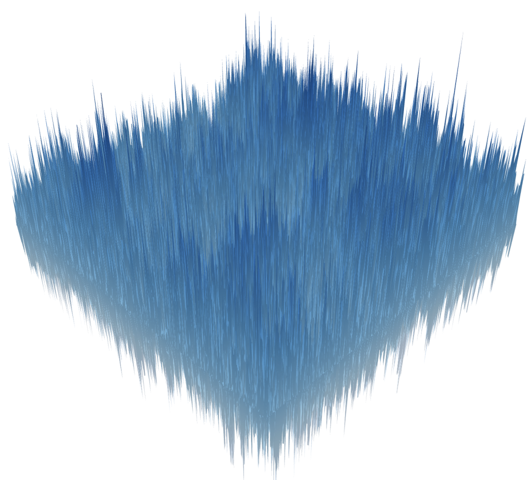

Figure 1: Simulation of a two dimensional Gaussian Free Field

stand out. These exceptional points, known as thick points, are a primary focus of the study of the geometric landscape induced by the GFF. Since it is not possible to evaluate the GFF pointwise, the value of a GFF Φ at a point $x \in D$ is approximated using a regularizing function known as the circle average $\mu_{x,\varepsilon}$. With this approach, the set of a-high points is defined as

$$T(a) \doteq \left\{ x \in D : \limsup_{\varepsilon \to 0} \frac{\langle \Phi, \mu_{x,\varepsilon} \rangle}{\log(1/\varepsilon)} = \gamma \right\}.$$

The study of these exceptional points has been addressed in various works, such as in [APP22], where it was shown that they form a totally disconnected set. Furthermore, their significance has been highlighted in [Kah85], emphasizing them as the support of limiting measures in Multiplicative Chaos theory. This underscores the importance of analyzing these points, not only because they are the most prominent in simulations but also because they represent a peculiar set with unique characteristics and great theoretical value.

This raises a fundamental question: how general is the behavior of thick points in log-correlated fields? Although the GFF is a good starting point for exploring this question, its structure is not general enough to encompass the full range of log-correlated fields. This is why this work introduces the pseudo Gaussian Free Field (pGFF), an object that retains the correlation structure of the GFF, maintaining it as a log-correlated field, but allowing for some flexibility by relaxing the Gaussianity property. With the pGFF, we aim to explore the limits of universality of thick points in fields with logarithmic correlations.

The interesting aspect of working with the pseudo GFF is that, given a domain $D \subseteq \mathbb{R}$, these fields can be characterized by a basis $(e_n)_{n \in \mathbb{N}}$ of the Sobolev space $H_0^1(D)$ along with a sequence of random variables $(\alpha_n)_{n \in \mathbb{N}}$. In the case of the GFF, the random variables used are Gaussian; however, in the pseudo GFF, we use variables whose correlation structure is determined by

$$\mathbb{E}\left[\alpha_n\right] = 0, \qquad\qquad \mathbb{E}\left[\alpha_n \alpha_m\right] = \mathbb{1}_{n=m}.$$

Then, Υ is a pseudo GFFs, if that can be written as

$$\Upsilon = \sum_n \alpha_n e_n.$$

In this work, we used two specific bases: the Levy basis in the one-dimensional case and the Fourier basis in the two-dimensional case. However, this choice does not compromise the generality of our study, as demonstrated in Section 3, where it is shown that being a pGFF does not depend on the basis. This property gives the pGFF considerable flexibility, justifying its usefulness for the objectives of this work.

To poke at the universality question, we began by working with the pseudo Brownian motion (pBM), which corresponds to the pGFF in the one-dimensional case. The strategy is to compare this object with its closest counterpart, the Brownian motion (BM), to identify and understand the differences and similarities that emerge in this scenario. Once these distinctions are established, the next step is to generalize the ideas and tools developed to the two-dimensional case. See Figure 2 for a simulation of a pBM

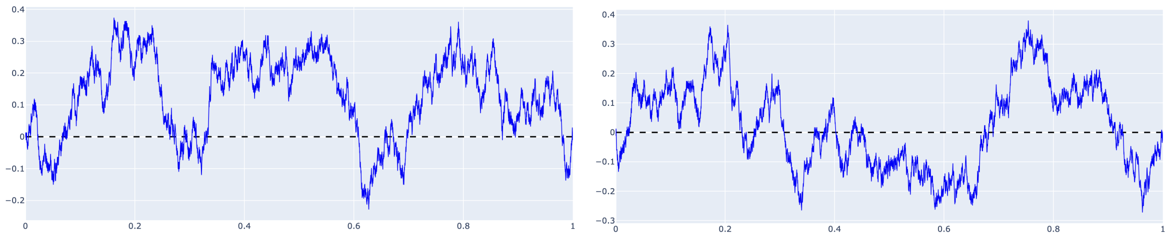

Figure 2: The figures show a Brownian bridge and a pseudo-Brownia bridge. Can you guess which is which?, Propositions 3.3 and 3.4 give us pointers on where to look.

Figure 2 shows that at first glance, it is extremely difficult to distinguish between the two processes. However, at a microscopic level, they exhibit significant differences in their behavior. In this work, these differences were analyzed in terms of regularity, specifically focusing on the modulus of continuity, which projects the idea of exceptional points in the one-dimensional context. In Propositions 3.3 and 3.4, we show that the pBM is a continuous object, but its modulus of continuity is not universal, as it depends significantly on the variables with which it is defined.

After working on and understanding the one-dimensional case, we turned our attention to the two-dimensional case, using a basis of the space $H_0^1(D)$ where $D \subseteq \mathbb{R}^2$. At this stage, our goal is to calculate the the Hausdorff dimension of the thick points and compare these results with those of the GFF, to assess whether there is any form of universality. Figure 3 shows a simulation

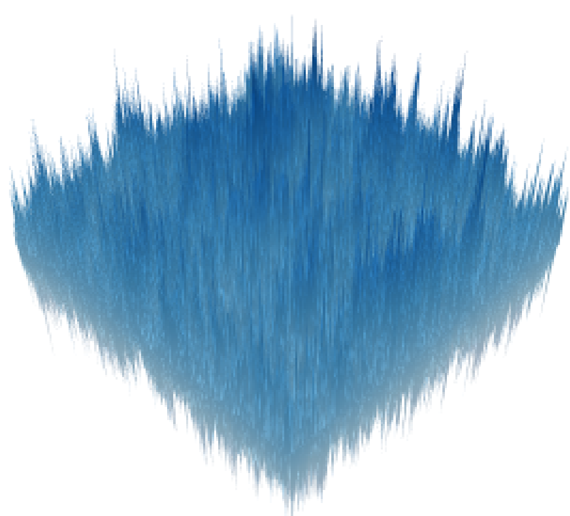

Figure 3: Simulation of the pseudo Gaussian Free Field. The main question of this thesis is whether this picture is close to Figure 1 or not.

As observed the Figure 1, the two-dimensional pGFF exhibits behavior that is notably similar to its Gaussian counterpart. Moreover, the extreme peaks and valleys stand out visibly, intensifying the need to address the question: How does the behavior of these extreme points differ compared to their Gaussian equivalent? Furthermore, how general is the behavior of these points in log-correlated fields?

To address these issues, we restrict our work to a smaller subset of the pGFF, where we demonstrate behavior similar to that of the GFF (4.1). Within this subset, we are able to calculate the Hausdorff dimension of the set of thick points. First, we obtain the upper bound using an approach similar to that employed in the Gaussian context. That is, we apply a regularity estimate (4.1) combined with a box-counting argument (4.5).

To establish the lower bound, discussed in Section 4.6, we employed tools from both mod-ϕ convergence theory and Liouville Quantum Gravity. The latter was particularly challenging because the approach outlined by Berestycki in [Ber17] relies heavily on the Cameron-Martin theorem, which essentially depends on the Gaussian nature of the variables involved. Since we lack this property in our case, we turned to the mod-ϕ convergence theory, addressed in Section 4.4. This theory focuses on the normalization of the moment generating function rather than normalizing each random variable individually, allowing us to obtain precise bounds for the generating function, thus facilitating its handling and application.

The thesis is composed of four chapters that we describe below.

Chapter I: We introduce the basic concepts of this thesis. We start by defining the Gaussian Free Field as a standard Gaussian variable in Hilbert spaces, using this approach to explain its fundamental properties, particularly its correlation structure given by the Green's function. Next, we focus on this function, presenting its detailed calculation for the two-dimensional case. We then explore the Hausdorff dimension and its application in the analysis of thick points both in the one-dimensional and two-dimensional GFF. Subsequently, we introduce the concept of mod-ϕ convergence, with an emphasis on mod-Gaussian convergence. We also define strongly sub-Gaussian variables, highlighting their main properties. Finally, we present the basis of the space that we will use in this work, emphasizing their definition and useful properties for the development of this research.

Chapter II: We present the main object of study of this work, starting with its formal definition and continuing with an analysis of the properties that underpin its utility for achieving the objectives outlined in this research.

Chapter III: We address the one-dimensional case of the pGFF. In this chapter, we demonstrate that this object is continuous and calculate the upper and lower bounds of its modulus of continuity.

Chapter IV: We focus on the two-dimensional case of the pGFF. We begin by defining the concept of circle average and thick points within this new non-Gaussian context. We then show that the pBM naturally embeds into the circle average of a planar pGFF and demonstrate that this object satisfies the mod-Gaussian convergence property. Subsequently, we calculate the upper bound of the Hausdorff dimension of the thick points of the pGFF. Next, we develop the theory of Multiplicative Chaos in this new non-Gaussian context, and finally, calculate the lower bound of the Hausdorff dimension of the pGFF.

Chapter 1

Preliminares

1.1 Gaussian Free Field

In probability theory and statistical mechanics, the Gaussian Free Field (GFF) is a random field that serves as a fundamental model for representing random surfaces. Its continuous version can be defined both in $\mathbb{R}^d$ and in bounded subdomains of $\mathbb{R}^d$, and it is considered a natural extension of one-dimensional Brownian motion to the context of d temporal dimensions.

A key property of this field is that its correlation structure is governed by the Green's function. In particular, in two dimensions, the GFF belongs to the family of log-correlated fields, which provides a solid theoretical foundation for addressing the central goal of this thesis.

In this chapter, we will present the main features of the GFF. In Section 1.1.1, we will describe its formal construction step by step. Then, in Section 1.1.2, we will analyze its regularity to determine which space it belongs to. Finally, in Section 1.1.3, we will delve into its correlation structure, providing a detailed understanding of this fundamental aspect.

1.1.1 Construction

The main focus of this thesis is deeply inspired by the GFF, making it useful to present its construction, as many of the properties of the pGFF stem from it. In this section, we will address the fundamental construction of the Gaussian Free Field. We will begin by reviewing the concept of a Gaussian variable in Euclidean spaces and then extend this notion to Hilbert spaces. Through a step-by-step approach, we will explore the construction of the GFF, assessing the advantages and limitations that arise at each stage, until we reach an optimized definition of its properties.

Definition 1.1 (Standard Gaussian in a d-dimensional Space)**.** We say that the random variable $X \in \mathbb{R}^d$ is a standard if and only if

1. The law of X is given by

$$\mathbb{P}(X \in dx) = \frac{1}{(2\pi)^{d/2}} e^{\frac{-\|x\|^2}{2}} dx.$$

2. For any deterministic vector $v \in \mathbb{R}^d$, we have that

$$\langle X, v\rangle \sim \mathcal{N}(0, \langle v, v\rangle).$$

3. For any deterministic vector $v \in \mathbb{R}^d$,

$$\mathbb{E}[e^{\langle X,v\rangle}] = e^{\frac{\|v\|^2}{2}}$$

With the notion of normal variable in mind, it is more coherent to direct our efforts towards the generalization of some of the notions present in the previous definition. This is why we define the Gaussian variable on Hilbert spaces as follows.

Definition 1.2 (Standard Gaussian in a Hilbert Space)**.** Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space. We say that a collection of Gaussian random variables $(X_h)_{h\in\mathcal{H}}$ is the standard Gaussian if and only if the following are true

- For any $h \in \mathcal{H}$, $X_h \sim \mathcal{N}(0, \langle h, h\rangle)$.
- For any $h_1, h_2 \in \mathcal{H}$ and $\lambda \in \mathbb{R}$ we have that a.s

$$X_{\lambda h_1 + h_2} = \lambda\langle X, h_1\rangle + \langle X, h_2\rangle$$

The first property is the generalization of the second in Definition 1.1, while the second requires that the object satisfies the linearity of the inner product in the space.

It is logical to ask now whether there exists a family that satisfies the conditions established in the previous definition. This result is a direct consequence of Kolmogorov's Consistency Theorem.

Theorem 1.1 (Kolmogorov's Consistency for Gaussian Variables)**.** *Let E be a set and $A : E \times E \to \mathbb{R}$ be a function such that for any finite set $\tilde{E} \subseteq E$ the matrix $(A(e, e'))_{e,e'\in\tilde{E}}$ is symmetric positive definite. Then, there exists a unique probability measure μ in $\mathbb{R}^E$ on the product of σ-algebras that satisfies the following. If X is a random variable distributed as μ, then for any finite set $\tilde{E} \subseteq E$ the law of $(X_e)_{e\in\tilde{E}}$ is a centered Gaussian random variable with covariance $(A(e, e'))_{e,e'\in\tilde{E}}$.*

Given the above theorem, it is not difficult to prove the existence of the Gaussian variable is Hilbert spaces by means of the following corollary
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Corollary 1.2 (Existence of Standard Gaussian in a Hilbert Space)**.** *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Then there exists a probability measure μ on $\mathbb{R}^{\mathcal{H}}$ for the product σ-algebra such that if X is a random variable distributed as μ, then X es a standard Gaussian in $\mathcal{H}$.*

Proof: Let $A : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be the function $A(h, h') = \langle h, h'\rangle_{\mathcal{H}}$. The commutative property is obvious by the definition of inner product. Note that if $E \subseteq \mathcal{H}$ is a finite set, we have that the matrix $M \doteq (A(h, h'))_{h,h'\in E}$ is a positive definite matrix because for any $\lambda \in \mathbb{R}^{|E|}$,

$$\lambda^T M\lambda = \sum_{h,h'\in E} \lambda_h \langle h, h'\rangle_{\mathcal{H}} \lambda_{h'} = \langle \sum_{h\in E} \lambda_h h, \sum_{h\in E} \lambda_{h'} h'\rangle_{\mathcal{H}} \geq 0.$$

We can use Theorem 1.1 to obtain a measure μ, such that if X distributed as μ, for any finite $E \subseteq \mathcal{H}$, $(X_h)_{h\in E}$ is distributed as a centred Gaussian random variable with covariance matrix $(A(h,h')_{h,h'\in E})$. In fact, for all $h \in \mathcal{H}$, X_h is a centered normal random variable with variance $\mathbb{E}[X^2] = \langle h, h\rangle_{\mathcal{H}}$. On the other hand, for fixed $h_1, h_1 \in \mathcal{H}$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}[(X_{\lambda h_1+h_2} - \lambda X_{h_1} + X_{h_2})^2] \\
&= \mathbb{E}[X^2_{\lambda h_1+h_2} + \lambda^2 X^2_{h_1} + X^2_{h_2} - 2\lambda X_{\lambda h_1+h_2} X_{h_1} - 2X_{\lambda h_1+h_2} X_{h_2} + 2\lambda X_{h_1} X_{h_2}] \\
&= 0,
\end{aligned}$$

which proves that a.s $X_{\lambda h_1+h_2} = \lambda X_{h_1} + X_{h_2}$ and X is a Gaussian variable of $\mathcal{H}$. □

The problem with the above definition is the σ-algebra on which it is defined, since it is quite small. We will now turn to a discussion of this.

Remark. In a Hilbert space $\mathcal{H}$, the σ-algebra induced by the cylinders of $\mathbb{R}^{\mathcal{H}}$ is reduced in the sense that it only allows measuring "countable questions". That is, events indexed by an uncountable set are not measurable. For example, the following events are not measurable

1. For $h \in \mathcal{H}$ fix. The event $\{X_{\lambda h} = \lambda X_h\}$.
2. For $h \in \mathcal{H}$ fix. The event $\{X_{h+h'} = X_h + X_{h'}\}$

Informally, the main idea of this problem is that the only measurable events in the product σ-algebra are those to which a "countable number" of questions can be asked.

In general, it is useful to construct the *GFF* in a Hilbert space that makes certain events indexed by uncountable sets to be measurable. However, the cost of this is the need to add the separability hypothesis to the Hilbert space in which we are working. Within this context, we can state the following proposition.

Proposition 1.3. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a separable Hilbert space with $(e_n)_{n\in\mathbb{N}}$ an orthonormal base and $(\alpha_n)_{n\in\mathbb{N}}$ an i.i.d sequence of standard Gaussian variables. If we define X_h as the following limit in L^2,*

$$X_h = \langle X, h\rangle_{\mathcal{H}} = \sum_{n\in\mathbb{N}} \alpha_n \langle h, e_n\rangle_{\mathcal{H}} = \lim_{N\to\infty} \sum_{n=0}^{N} \alpha_n \langle h, e_n\rangle_{\mathcal{H}}, \quad \forall h \in \mathcal{H},$$

then X is the Gaussian variable of $\mathcal{H}$.

Proof: Let us first verify that the variable is well-defined as a limit in L^2 and then that it is a Gaussian variable in the Hilbert space $\mathcal{H}$. Let $h \in \mathcal{H}$ be fixed and for $N \in \mathbb{N}$, denote the partial sum as:

$$X_N = \sum_{k=0}^{N} \alpha_k \langle h, e_k\rangle_{\mathcal{H}}.$$

Then, for $N, M \in \mathbb{N}$, with $N \geq M$, it holds that:

$$\mathbb{E}[(X_h^N - X_h^M)^2] = \mathbb{E}\left[\left(\sum_{k=M+1}^{N} \alpha_k \langle h, e_k\rangle_{\mathcal{H}}\right)^2\right] = \sum_{k=M+1}^{N} \langle h, e_k\rangle_{\mathcal{H}}.$$

Now, taking $N, M \to \infty$, we obtain that $\mathbb{E}[(X_h^N - X_h^M)^2] \to 0$ since the above series corresponds to the tail of $\langle h, h\rangle_{\mathcal{H}}$. With this, we conclude that $(X_h^N)_{h\in\mathcal{H}}$ is Cauchy in L^2, and therefore, the limit is well-defined.

To verify that it is a Gaussian variable in $\mathcal{H}$, we will rely on the result stated in [Gal18, Prop 1.1]. With this property,

$$\lim_{N\to\infty} \mathbb{E}[(X_h^N)^2] = \lim \sum_{k=0}^{N} \langle h, e_k\rangle_{\mathcal{H}}^2 = \langle h, h\rangle_{\mathcal{H}}.$$

Finally, to prove that it satisfies the second condition for Gaussians in Hilbert spaces, we take $h_1, h_2 \in \mathcal{H}$ and $\lambda \in \mathbb{R}$. Then,

$$X_{\lambda h_1 + h_2} = \lim_{N\to\infty} \sum_{k=0}^{N} \alpha_k \langle \lambda h_1 + h_2, e_k\rangle_{\mathcal{H}} = \lambda X_{h_1} + X_{h_2},$$

and we conclude. □

Note from the previous proposition that the equality $X_{\lambda h_1+h_2} = \lambda X_{h_1} + X_{h_2}$ holds with full probability, not just almost surely. Furthermore, this allows us to define $X_{\lambda h}$ simultaneously for all $\lambda \in \mathbb{R}$, thus understanding X as a linear operator from $\mathcal{H}$ to $\mathbb{R}$. However, a natural question arises: does X belong to $\mathcal{H}$? The answer is negative, and we will discuss this in the following remark.

Remark. Let $\mathcal{H}$ denote a separable Hilbert space and define the random variable

$$X = \lim_{N\to\infty} \sum_{n=0}^{N} \alpha_n e_n.$$

We remark that X a.s. does not belong to the space $\mathcal{H}$. The reason for this is that

$$\|X\|_{\mathcal{H}}^2 = \langle X, X\rangle_{\mathcal{H}} = \sum_{n=0}^{\infty} \alpha_n^2.$$

We know that, for every $n \in \mathbb{N}$, $\mathbb{P}(\alpha_n > 1) > 0$, which implies that $\sum_{n\in\mathbb{N}} \mathbb{P}(\alpha_n > 1)$ diverges. The Borel-Cantelli Lemma ensures that there are infinitely many terms in the above sum that are greater than 0, thereby guaranteeing that the norm of X in $\mathcal{H}$ is infinite.

This leaves open the question of whether the random variable X can be defined in a bigger Hilbert space. The answer to that question is given in Section 1.1.2.

To close this chapter, let us establish the formal definition of the GFF as a Gaussian variable of the space $H_0^1(D)$. Let $D \subseteq \mathbb{R}^d$ be an open set and consider the inner product in $\mathcal{C}_0^\infty(D)$.

$$\langle f, g\rangle_\nabla = \frac{1}{2\pi} \int_D \nabla f(x) \nabla g(x) dx, \tag{1.1}$$

where the term $\frac{1}{2\pi}$ is used to normalize the Green's function that we will see later.

We define the Sobolev space H_0^1 by completing $\mathcal{C}_0^\infty$ with the inner product defined before, that is,

$$H_0^1(D) = \overline{\mathcal{C}_0^\infty(D)}^{\|\cdot\|_\nabla}.$$

When $f \in H_0^1$ it is possible to define $\nabla f \in L^2(D)$. With this, the inner product $H_0^1(D)$ is given by

$$\langle f, g \rangle_\nabla = \frac{1}{2\pi} \int_D \nabla f(x) \nabla g(x) dx, \quad \forall f, g \in H_0^1(D).$$

Definition 1.3 (Gaussian Free Field)**.** The GFF Φ in D is defined as the standard Gaussian variable of $(H_0^1(D), \langle \cdot, \cdot \rangle_\nabla)$.

1.1.2 Regularity of the GFF

En esta sección, continuaremos en el estudio del GFF, enfocándonos en su regularidad y en la identificación del espacio funcional adecuado al que pertenece. Para ello, Comenzaremos introduciendo la generalización del espacio de Sobolev $H_0^1(D)$ hacia el espacio $H_0^\beta(D)$, con $\beta > 0$. A continuación, determinaremos la regularidad precisa, caracterizada por el parámetro β, que define el espacio al que pertenece el GFF.

Let us consider a domain $D \subseteq \mathbb{R}^d$ and the space of functions $(L^2(D), \langle \cdot, \cdot \rangle)$. Let $(e_k)_{k\in\mathbb{N}}$ denote the orthonormal basis of this space, formed by the eigenfunctions of the operator $(-\Delta)$ with zero boundary conditions, and let $(\lambda_k)_{k\in\mathbb{N}}$ denote the corresponding eigenvalues. With this, we can obtain an orthonormal family of the space H_0^1 as follows

$$\langle e_n, e_m \rangle_\nabla = \langle e_n, (-\Delta) e_m \rangle = \langle e_n, \lambda_m e_m \rangle = \lambda_m \langle e_n, e_m \rangle = \lambda_n \mathbb{1}_{n=m}.$$

This implies that $((\sqrt{\lambda_k})^{-1} e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of the space $H_0^1(D)$ and let us denote by $(\hat{e}_k)_{k\in\mathbb{N}}$ this family. We then consider the generalization of this space given by

$$H_0^\beta(D) \doteq \{f \in L^2(D) : \sum_{k\geq 0} \lambda_k^\beta \langle f, \hat{e}_k \rangle^2 < \infty\}.$$

In $H_0^\beta(D)$, we consider the natural norm $\|\cdot\|_\beta$ defined by

$$\|f\|_\beta^2 = \sum_{k\geq 0} \lambda_k^\beta \, \langle f, \hat{e}_k \rangle^2. \tag{1.2}$$

The goal is to find the appropriate value of β for which the GFF belongs to $H_0^\beta(D)$. Before proceeding, we will present a small lemma, whose proof can be found in [Wey12], that provides the growth rate of the eigenvalues of $(-\Delta)$. This lemma will be essential for analyzing the finiteness of the norm defined in (1.2) for the GFF.

Lemma 1.4 (Weyl's Law)**.** *There exist constants $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$,*

$$C_1 \, n^{2/d} \leq \lambda_n \leq C_2 \, n^{2/d}.$$

We now present the central result of this section:

Proposition 1.5. *For $\beta < d/2 - 1$, we have that $\Phi \in H_0^\beta(D)$.*

Proof: The proof of this proposition is relatively simple by following the following calculation. For Φ a GFF and $\beta \in \mathbb{R}$, we have that

$$\|\Phi\|_\beta^2 = \sum_{k\geq 0} \lambda_k^\beta \langle \Phi, \hat{e}_k\rangle^2 = \sum_{k\geq 0} \lambda_k^{\beta-1} \langle \Phi, \lambda_k \hat{e}_k\rangle^2 = \sum_{k\geq 0} \lambda_k^{\beta-1} \langle \Phi, \hat{e}_k\rangle_\nabla^2$$

Then, using 1.4, we can compute

$$\mathbb{E}[\|\Phi\|_\beta^2] = \sum_{k\geq 0} \lambda_k^{\beta-1} \leq \sum_{k\geq 0} k^{2/d(\beta-1)} \tag{1.3}$$

which is finite if $b < d/2 - 1$. □

1.1.3 Correlations

In this section, we will delve into the correlation structure of the GFF Φ, defined in a domain $D \subseteq \mathbb{R}^d$. As we previously observed, for a function $f \in H_0^1(D)$, the corresponding Gaussian random variable is given by $\Phi_f \sim \mathcal{N}(0, \langle f, f\rangle_\nabla)$. This result raises a fundamental question: what more can we deduce about the term $\langle f, f\rangle_\nabla$? Our goal here is to address this question and provide a deeper insight into that expression. To do so, we will first derive an equality that links the term of interest to the Laplace operator. Subsequently, we will introduce the Poisson problem, which plays a key role in understanding the Green's function. Finally, we will discuss a crucial property that relates any stochastic process with Green's correlations with the GFF.

Using the property of integration by parts, we obtain the following equality:

$$\langle f, f\rangle_\nabla = \langle \nabla f, \nabla f\rangle = \langle f, -\Delta f\rangle. \tag{1.4}$$

This equality motivates us to explore the term $-\Delta f$. To do this, we need the following definition:

Definition 1.4 (Poisson Problem)**.** Let $D \subseteq \mathbb{R}^d$ and $f \in \mathcal{C}^2(D)$. We define the Poisson problem as

$$(P_f) \begin{cases} -\Delta u(x) = f(x), & x \in D, \\ u(x) = 0, & x \in \partial D. \end{cases}$$

We know that the problem defined above has a solution given by

$$u(x) = \int_D G^D(x,y) f(y)\, dy,$$

where G^D is the Green's function defined over D, of which we will study in more detail in Section 1.2. Using this solution, we can reformulate the result obtained in 1.4 as follows:

$$\begin{aligned}\langle u, u\rangle_\nabla = \langle u, -\Delta u\rangle = \langle \int_D G^D(\cdot, y) f(y)\, dy, -\Delta u\rangle &= \langle \int_D G^D(\cdot, y) f(y)\, dy, f\rangle \\ &= \int_{D\times D} f(x) G^D(x,y) f(y)\, dx\, dy.\end{aligned}$$

This final equality leads us to formulate the following property:

Proposition 1.6. *Every Gaussian process with Green's correlations is a GFF. In other words, if ψ is a Gaussian process with*

$$\mathbb{E}[\langle \Psi, f\rangle \langle \Psi, g\rangle] = \int_{D\times D} f(x) G^D(x,y) g(y)\, dx\, dy,$$

then Ψ is a GFF.

1.2 Estimates on the Green's Function

The Green's function is a widely studied concept in mathematics, particularly in the field of partial differential equations (PDEs). We have already mentioned its intrinsic connection to the correlations of the GFF, and notably, when working in two-dimensional spaces, this function takes on a logarithmic form. In other words, the GFF becomes a log-correlated field when $D \subseteq \mathbb{R}^2$. Given its importance, in this section, we will thoroughly examine the structure and specific form of the Green's function. We will begin by presenting it as the solution to the Poisson problem, then define it in terms of the transition density in general domains, and finally, demonstrate its logarithmic form in the two-dimensional context.

As shown in [MP10, Remark 8.7], the solution to the Poisson problem defined in Definition 1.4 can be expressed as

$$f(x) = \mathbb{E}^x \left[\int_0^{\tau_D} F(B(t)) dt \right], \tag{1.5}$$

where $\mathbb{E}^x$ denotes expectation starting from a point $x \in D$, B is a d-dimensional Brownian motion, and $\tau_D = \inf\{t > 0 : B(t) \notin D\}$. Moreover, it follows from [MP10, Theorem 3.30] that there exists a transition density $p^D : [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ such that

$$\mathbb{E}^x[F(B_t)\mathbb{1}_{t\leq\tau}] = \int_D p^D(t,x,y) dy.$$

Using this transition density, we can provide a formal definition of the Green's function.

Definition 1.5 (Green's Function)**.** Let $D \subseteq \mathbb{R}^d$ be a bounded domain. We define the Green's function of D as

$$G^D(x,y) \doteq \int_0^\infty p^D(t,x,y) dt$$

Computing the Green's function using the transition density p gives us explicit expressions that vary significantly depending on the dimension of the space. In this work, we focus exclusively on the two-dimensional case, where the specific properties of the Green's function play a crucial role in analyzing log-correlated fields and the associated singularities.

Proposition 1.7. *Let $D \subseteq \mathbb{R}^2$ be a subset. The Green's function on D is given by*

$$G^D(x,y) = \log\left(\frac{1}{\|x-y\|}\right)(1 + o(1)).$$

Proof: From the explicit form of the transition density p,

$$G^D(x,y) = \int_0^\infty \frac{e^{-t}}{t} \cdot e^{\frac{-\|x-y\|}{2t}} dt.$$

Rewriting the exponential in integral form, we obtain

$$G(x,y) = \int_0^\infty \frac{e^{-t}}{t} \cdot e^{\frac{-\|x-y\|^2}{2t}} dt = \int_0^\infty \frac{e^{-t}}{t} \int_{\frac{\|x-y\|}{2t}} e^{-s} ds\, dt = \int_0^\infty e^{-s} \int_{\frac{\|x-y\|}{2s}}^\infty \frac{e^{-t}}{t} dt\, ds.$$

The idea now is to find upper and lower bounds for the inner integral. For the upper bound, we have

$$\int_{\frac{\|x-y\|}{2s}}^\infty \frac{e^{-t}}{t} dt \leq \begin{cases} \log(\frac{2s}{\|x-y\|^2}+1), & \text{if } \|x-y\| \leq 2s, \\ 1, & \text{if } \|x-y\| > 2s. \end{cases}$$

For $\|x-y\| \leq 1$, we have on one hand,

$$\int_0^1 \frac{e^{-t}}{t} \log(\frac{2s}{\|x-y\|^2}) + 1 = (1-e^{-1})(\log(2) + 1 - 2\log\|x-y\|).$$

And on the other hand,

$$\int_1^\infty \frac{e^{-t}}{t} \log(\frac{2s}{\|x-y\|^2} + 1) = e^{-1}(1 + \log(2) - 2\log\|x-y\|) + \tilde{\xi},$$

where $\tilde{\xi} = \int_1^\infty e^{-s}\log(s)ds < \infty$.

The previous computations give us the upper bound

$$G(x,y) \leq 1 + \log(2) + \tilde{\xi} - 2\log\|x-y\|.$$

To calculate the lower bound, we use the fact that

$$\int_{\frac{\|x-y\|}{2s}}^\infty \frac{e^{-t}}{t} dt \geq \log(\frac{2s}{\|x-y\|^2}) - 1.$$

Denoting by $\xi = -\int_0^\infty e^{-s}\log(s)ds$ denoting Euler's constant, we obtain

$$G(x,y) \geq 1 + \log(2) - \xi - 2\log\|x-y\|.$$

Thus, we conclude that

$$G(x,y) = \log\left(\frac{1}{\|x-y\|}\right)(1+o(1)).$$

□

1.3 Fractal Dimensions

The concept of fractal dimension is used in geometry to quantify, through a real-valued index, the roughness of a set. Fractal dimensions are applied to describe a wide range of objects, from abstract structures to practical phenomena. In probability theory, the Hausdorff and Minkowski dimensions have been employed to measure the fractality or irregularity of various stochastic processes. In this case, we will use the Hausdorff dimension to quantify the rarity of the set of thick points of the GFF and, subsequently, of the pGFF.

In Section 1.3.1, we introduce the fundamental concepts necessary to understand the definition of the Hausdorff dimension, followed by its formal definition. Next, in Section 1.3.2, we explore the fast points of Brownian motion and calculate the dimension of the set they form. Finally, in Section 1.3.3, we present the calculation of the dimension of the thick points of the GFF.

1.3.1 Hausdorff Dimension

The Hausdorff dimension measures the local size of a set by considering the distance between its points. More specifically, it seeks to determine the minimum exponent needed so that the sum of the diameters of the balls covering the set goes to zero as the radius of these balls tends to zero. Throughout this thesis, we will use this concept to analyze the behavior of the Thick Points of the pGFF. In this section, we will introduce the fundamental concepts underlying the definition of the Hausdorff dimension and then formalize this definition.

Definition 1.6. Let (X, d) be a metric space. Let $A \subseteq X$ and $\delta > 0$ a real number. Define

$$H^b_\delta(A) \doteq \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^b : A \subseteq \bigcup_{i=1}^{\infty} U_i,\ \operatorname{diam}(U_i) < \delta\right\}$$

Let us note that $H^b_\delta(A)$ is non-increasing in δ, since as δ increases, more collections of sets are allowed. Therefore, it makes sense to take $\lim_{\delta\to 0} H^b_\delta(A)$, although it could be infinite. We then define the Hausdorff outer measure as

Definition 1.7 (Hausdorff Mesure)**.** Let (X, d) be a metric space. Let $A \subseteq X$ and $\delta > 0$ a real number. Define

$$H^b(A) \doteq \lim_{\delta\to 0} H^b_\delta(A) = \sup_{\delta>0} H^b_\delta(A)$$

It turns out that $H^b(A)$ can have a finite, non-zero value for at least one $\bar{b} > 0$. In other words, the Hausdorff measure is zero for any value above $\bar{b}$ and infinite below $\bar{b}$, analogous to the idea that the area of a line is zero and the length of a 2D figure is, in a certain sense, infinite. This leads us to define the Hausdorff dimension as follows

Definition 1.8 (Hausdorff Dimension)**.** Let (X, d) be a metric space. Define the Hausdorff Measure as

$$\dim_H(A) = \inf\{b > 0 : H^b(A) = 0\} = \sup\{b > 0 : H^b(A) = \infty\}$$

With this definition it can be seen that the idea of this dimension is to determine the minimum exponent necessary so that the sum of the diameters of the balls that cover the set is zero.

1.3.2 Extremes Values in Dimension 1

In this section, we will explore the fast times of Brownian motion, exceptional moments when the velocity of the process exceeds the expectations set by the law of the iterated logarithm. Our goal is to demonstrate how the Hausdorff dimension can be used to quantify the rarity and structure of these exceptional points in the one-dimensional context.

We will begin by reviewing the law of the iterated logarithm, which describes the expected behavior of points in a Brownian motion. Next, we will present the definition and properties of fast times and, finally, perform the calculation of the dimension of the set of these points, thus illustrating a concrete application of the Hausdorff dimension in this case.

The law of the iterated logarithm describes the magnitude of the fluctuations of a random walk. Specifically, in [MP10, Corollary 5.3] they calculate the following proposition for the Brownian motion

Proposition 1.8 (Law of the Iterated Logarithm). *Suppose $(B_t)_{t\geq 0}$ is a standard Brownian motion. Then, almost surely,*

$$\limsup_{h\to 0} \frac{|B(h)|}{\sqrt{2h\ \log(\log(1/h))}} = 1.$$

Using Markov's property it can be rewritten and we can see that for $t \in [0,1]$, almost surely,

$$\limsup_{h\to 0^+} \frac{|B(t+h)-B(t)|}{\sqrt{2h\ \log(\log(1/h))}} = 1.$$

This abruptly contradicts the following result (note the absence of the iterated logarithm)

Theorem 1.9 (Theorem 10.1 in [MP10]). *Almost surely, we have*

$$\max_{0\leq t\leq 1} \limsup_{h\to 0^+} \frac{|B(t+h)-B(t)|}{\sqrt{2h\log(1/h)}} = 1.$$

What Theorem 1.9 tells us is that the maximum change a Brownian motion can experience, when $t \in [0,1]$, is of the order of $\sqrt{\log}$. This is faster than what Law of the iterated logarithm indicates, which states that for each $t \in [0,1]$, the rate of change is of the order of $\sqrt{\log\log}$. This makes such $t \in [0,1]$ an exceptional time. To explore how frequent these exceptional times are, we introduce a spectrum of exceptional points. Given $a > 0$, we will call a time $t \in [0,1]$ an a-fast time if

$$\limsup_{h\to 0} \frac{|B(t+h)-B(t)|}{\sqrt{2h\log(1/h)}} \geq a$$

The proof of Theorem 1.9 in [MP10] shows that the set of fast times forms a countable intersection of non-countable open sets in $[0,1]$, making them dense and non-countable. Additionally, according to the law of the iterated logarithm, this set has Lebesgue measure zero. For these reasons, the Hausdorff dimension emerges as the appropriate tool to measure the quantity of a-fast times.

Theorem 1.10 (Orey and Taylor 1974). *Suppose $(B_t)_{t\geq 0}$ is a Brownian motion. The, for every $a \in [0,1]$, we have almost surely,*

$$dim_H\left(\left\{t \in [0,1] : \limsup_{h\to 0^+} \frac{|B(t+h)-B(t)|}{\sqrt{2h\log(1/h)}} \geq a\right\}\right) = 1 - a^2.$$

This result allows us to understand how the Hausdorff dimension can be used to quantify the rarity and structure of these exceptional points in the one-dimensional context.

1.3.3 Extremes Values in Dimension 2

The thick points of a GFF Φ are specific points where Φ reaches exceptionally high or low values. In this section, we will focus on understanding the nature of these points within the context of the GFF. This analysis is crucial as it will provide us with fundamental techniques and strategies, which will be essential for understanding the thick points in the context of the pGFF.

Since Φ cannot be defined pointwise, this set must be determined through regularization. We denote by $\mu_{x,\varepsilon}$ the uniform measure over $\partial B(x,\varepsilon)$, that is,

$$\int f(y)\mu_{x,\varepsilon}(dy) = \mathbb{E}[f(B_{\tau_{B(x,\varepsilon)}})], \tag{1.6}$$

where $f \in \mathcal{C}(D)$ and $\tau_{B(X,\varepsilon)}$ corresponds to the first moment when a Brownian motion launched from x reaches the boundary of the ball $B(x,\varepsilon)$.

For a fixed $x \in D$ and $\varepsilon \geq 0$, we define the circle average process as $F(x,\varepsilon) = \langle \Phi, \mu_{x,\varepsilon} \rangle$. This process is particularly interesting because, as demonstrated in [HMP10, Proposition 2.1], there exists a version of it such that, with probability one, for each $0 < \gamma < 1/2$ and $\delta, \xi > 0$, there is a constant $M = M(\gamma, \delta, \xi)$ such that

$$|F(x,\varepsilon) - F(y,\varepsilon')| \leq M \log(1/\varepsilon)^{\xi} \frac{\|(x,\varepsilon) - (y,\varepsilon)\|^{\gamma}}{\varepsilon^{\gamma+\delta}},$$

In the rest of the paper, we will always consider a GFF that complies with this version of circular mean.

Moreover, we can define the process $(\Phi_t(x))_{t\geq 0}$ as

$$\Phi_t(x) = F(x, e^{-t}),$$

which is shown in [HMP10] to describe a one-dimensional Brownian motion.

Remark. Usually, in the literature, the process $(\Phi_t(x))_{t\geq 0}$ is defined as $\Phi_t(x) = \sqrt{2\pi}F(x, e^{-t})$. In our case, by defining the product of the space H_0^1 as in 1.1, the constant $\sqrt{2\pi}$ is implicitly included within this definition.

In particular, it almost surely holds that

$$\lim_{\varepsilon \to 0} \frac{F(x,\varepsilon)}{\log(1/\varepsilon)} = \lim_{t\to\infty} \frac{\Phi_t(x)}{t} = 0. \tag{1.7}$$

However, this does not rule out the possibility that there exist exceptional points for which this limit is different from zero. These points are called thick points of Φ. Therefore, a natural definition that arises for $\gamma > 0$ is

Definition 1.9 (Thick Points for the GFF)**.** Let $x \in D$ be a fixed point and Φ a GFF. We will define the set of γ-thick points of Φ as

$$T(\gamma) = \left\{ x \in D : \limsup_{\varepsilon \to 0} \frac{F(x,\varepsilon)}{\log(1/\varepsilon)} = \gamma \right\}.$$

An equivalent definition can be obtained using the equality shown in 1.7. With it, the set is defined by

$$T(\gamma) = \left\{ x \in D : \limsup_{t \to \infty} \frac{\Phi_t(x)}{t} = \gamma \right\}.$$

In Lemma 3.1 of [HMP10], they calculate the upper bound of the Hausdorff dimension of the set defined above. This result is expressed by the following property

Proposition 1.11 (Upper bound of the thick points of the GFF)**.** *If $0 \leq \gamma \leq 2$, then almost surely $dim_H(T(\gamma)) \leq 2 - \gamma^2/2$. If $\gamma > 2$, then $T(\gamma)$ is a.s. empty.*

The proof of this property is very similar to that of Property 4.5, which we will demonstrate in Section 4.4. Thus, we decided not to write this proof here but ask the reader to look at that proof instead.

1.4 Mod-ϕ Convergence

The notion of mod-ϕ convergence was initially introduced in [JKN09] in the context of mod-Gaussian convergence. Later, [DKN11] extended this concept to include other types of convergence within the general framework of mod-ϕ convergence.

The core element of this theory lies in the natural renormalization of the characteristic function of random variables that do not converge in distribution, instead of renormalizing the variables themselves. This strategy allows the sequence of characteristic functions to converge to a non-trivial limit function, facilitating the analysis of situations where classical convergence fails.

In this work, we follow the definition of mod-ϕ convergence proposed in [FMN16], specifically adapting it to the Gaussian case. This adjustment was crucial for deepening the analysis of Non-Gaussian Multiplicative Chaos. Mod-ϕ convergence enabled us to apply proper renormalization to the moment generating function, which facilitated control over this function and, in turn, the study of extreme field fluctuations in this context.

The objective of this section is to formally present the definition of mod-ϕ convergence in the Gaussian case and demonstrate how this notion influences the behavior of the distribution's tails.

Definition 1.10 (Mod-$\mathcal{G}$ Convergence)**.** Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of real-valued random variables, and let us denote by $\varphi_k(z) = \mathbb{E}[e^{zX_k}]$ their moment generating functions, which we assume exist in a interval $(c, d) \subseteq \mathbb{R}$. Suppose for all $z \in (c, d)$ the following convergence holds as $k \to \infty$,

$$\varphi_k(z)e^{-t_k\, z^2/2} \to \psi(z) \tag{1.8}$$

where $(t_k)_{k\in\mathbb{N}}$ is some sequence going to $+\infty$ and ψ is an analytic function. We then say that $(X_k)_{k\in\mathbb{N}}$ converges mod-$\mathcal{G}$ in (c, d), with parameters $(t_k)_{k\in\mathbb{N}}$ and limiting function ψ.

In Theorem 4.3 of [FMN16], the rate of convergence of the tails is shown for the general case of mod-ϕ convergence. Applied to the case where the convergence is Gaussian, this theorem is expressed as follows:

Theorem 1.12. *Take a sequence $(X_k)_{k\in\mathbb{N}}$ that converges mod-$\mathcal{G}$ on a interval (c,d). Then, for $x \in (c,d)$,*

$$\mathbb{P}(X_k \geq t_k x) = \frac{\exp(-t_k \cdot x^2/2)}{x\sqrt{2\pi t_k}}\,\psi(x)(1+o(1))$$

Remark. It is important to highlight that, to demonstrate the mod-$\mathcal{G}$ convergence of the circle average of the pGFF, a result presented in Section 4.4, it was necessary to use the Fourier basis, defined in 1.13. However, it is not ruled out that, by using another type of basis, one could conclude a mod-ϕ convergence that is not necessarily Gaussian.

1.5 Strongly Sub-Gaussianity

A random variable is considered strongly sub-Gaussian when its tails exhibit a sharper decay than those of a Gaussian distribution. In other words, the tails of a sub-Gaussian variable are bounded by those of a Gaussian distribution, implying stronger concentration around its mean. This type of variable is fundamental to our work, as it forms one of the key assumptions regarding the variables used to construct the two-dimensional pGFF. Therefore, in this section, we will present its formal definition and explore its basic properties.

We define Strongly sub-Gaussian variables as

Definition 1.11 (Strongly Subgaussian Random Variable)**.** Let X be a centered random variable with $\sigma^2 = \mathrm{Var}(X)$. X is said strongly subgaussian if

$$\forall t \in \mathbb{R}, \quad \mathbb{E}[\mathrm{e}^{tX}] \leq \mathrm{e}^{\sigma^2 t^2/2}. \tag{1.9}$$

This characteristic influences other aspects of the variable, such as the tail decay or the boundedness of the exponential of its second moment. For this reason, we present the following proposition.

Proposition 1.13. *Let X be a centered random variable with $\sigma^2 = Var(X)$. Each statement bellow implies the next*

(1) Laplace Transform Condition: *X satisfies the definition 1.11, that is, for all $t \in \mathbb{R}$, $\mathbb{E}[e^{tX}] \leq e^{\sigma^2 t^2/2}$.*

(2) Subgaussian Tail Estimate: *For all $\lambda > 0$, $\mathbb{P}(|X| \geq \lambda) \leq 2\exp(-\frac{\lambda^2}{2\sigma^2})$*

(3) Moment Condition: *For any $p \geq 1$, exists $C(p)$ such that $\mathbb{E}[X^{2p}] \leq C(p)\,\sigma^{2p}$.*

(4) Orlicz Condition: *Exists $a < 4$ such that $\mathbb{E}[\exp(\frac{aX^2}{\sigma^2})] < \infty$.*

Proof: Like is usual, we will proof implications downward until the last implies the first.
(1) ⇒ (2) : First, we set that, for all $\lambda > 0$,

$$\mathbb{P}(|X| \geq \lambda) = \mathbb{P}(X \geq \lambda) + \mathbb{P}(-X \leq \lambda).$$

Then, by the Markov's Inequality and for a $t > 0$,

$$\mathbb{P}(X \geq \lambda) = \mathbb{P}(e^{tX} \geq e^{t\lambda}) \leq \mathbb{E}[e^{tx}] \cdot e^{-t\lambda} \leq \exp\left(\frac{\sigma^2 t^2}{2} - t\lambda\right) \leq \exp\left(-\frac{\lambda^2}{2s^2}\right),$$

where in the last inequality, we take $\inf_{s\in\mathbb{R}} \frac{\sigma^2 t^2}{2} - t\lambda = \frac{\lambda}{s^2}$. The idea is the same for $\mathbb{P}(-X \leq \lambda)$ and we conclude.

(2) ⇒ (3) : We have that

$$E[X^{2p}] = \int_0^\infty \mathbb{P}(X^{2p} \geq t)\, dt = \int_0^\infty \mathbb{P}(|X| \geq t^{1/2p})\, dt \leq 2\int_0^\infty e^{-t^{1/p}/(2\sigma^2)}\, dt.$$

In this part, we applied the change of variable $u = t^{1/p}/(2\sigma^2)$ and the calculus continuing with

$$\mathbb{E}[X^{2p}] = 2p\sigma^2 \int_0^\infty (2\sigma^2 u)^{p-1} e^{-u}\, du = 2^p p\sigma^{2p} \int_0^\infty u^{p-1} e^{-u}\, du = 2^p pp!\sigma^{2p} \leq 4^p p!\sigma^{2p}.$$

Taking the $C(p) = 4^p p!$ we conclude.
(3) ⇒ (4) : By the definition of the exponential in power series

$$\mathbb{E}\left[\exp\left(a\frac{X^2}{\sigma^2}\right)\right] = \mathbb{E}\left[\sum_{k\geq 0} a^k \frac{X^{2k}}{\sigma^{2k}} \frac{1}{k!}\right] = \sum_{k\geq 0} \frac{a^k}{k!} \frac{\mathbb{E}[X^{2k}]}{\sigma^{2k}} \leq \sum_{k\geq 0} (a4)^k < \infty$$

□

A key property of strongly subgaussian variables is the stated in [BCG23, Corollary 2.3] which tells us about the necessary condition for an infinite sum of strongly subgaussian variables to be subgaussian

Proposition 1.14. *If $\sum_{k\geq 1} Var(X_k) < \infty$ for independent, strictly subgaussian summands X_k, then the series $X = \sum_{k\geq 1} X_k$ represents a strictly subgaussian random variable.*

After exploring the definition and basic properties of strongly sub-Gaussian variables, it is useful to consider some examples that allow for a new construction of the pGFF. These examples are detailed in [BCG23].

The first example, although basic, is the standard normal variable. It is evident from its definition that this variable is strongly sub-Gaussian. A somewhat more interesting example, used in our simulations, is the symmetric Bernoulli variable with support in $\{-1, 1\}$. This variable satisfies $\mathbb{E}[e^{tX}] = \cosh(t)$. A final example, slightly more elaborate, is a variable X whose density $p(x)$ is given by $p(x) = x^2\varphi(x)$, where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. In this case, $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] = 3$ and

$$\mathbb{E}[e^{tX}] = (1 + t^2)e^{t^2/2} \leq e^{3t^2/2}.$$

Remark. In this context, there is another type of variable worth highlighting: subgaussian variables. The difference between these and strongly subgaussian variables lies in the fact that the former satisfy Definition 1.9 with a constant $c > 0$, instead of $\sigma^2 = \mathrm{Var}(X)$.

1.6 Basis

In Section 1.1.1, we demonstrated how the GFF can be constructed using a basis of a separable Hilbert space, which inspired the creation of the pGFF. In this chapter, we will present the basis of the space $H_0^1(D)$ that we will use in constructing our object of study. These basis are crucial because the correct choice not only facilitates the construction of the pGFF but also allows for a more efficient verification of the key properties of the object, thereby optimizing our analysis.

We will begin by defining the basis that we will primarily use when working with the object in a one temporal dimension, which we will refer to as the Lévy Basis. We will analyze its indexing form and highlight its most important properties. Subsequently, we will introduce the Fourier basis that will be predominantly used in the two-dimensional context and then present its most relevant properties.

1.6.1 Levy Basis

The basis used in the Levy's construction for the construction of Brownian Motion, explained in [MP10] and [Gal18], is indexed through the dyadic sets defined as $\mathcal{D} = \cup_{n\in\mathbb{N}}\mathcal{D}_n$, with

$$\mathcal{D}_n = \left\{k \cdot 2^{-n} : k \in \{0, \dots, 2^n\}\right\}.$$

That is, an element $d \in \mathcal{D}$ is of the form $d = k \cdot 2^{-n}$.

In this work, we will change the indexing of the basis and use the set $\{(n, m) \in \mathbb{N} \times I_n\}$, with $I_n = \{1, \dots 2^n\}$. Additionally, through this enumeration, we define the Levy basis as

Definition 1.12 (Levy's Basis)**.** We will call Levy's basis the family of functions $\{\ell_{n,m}\}_{n\in\mathbb{N},m\in I_n}$ defined by

$$\ell_{n,m}(x) = \begin{cases} 2\pi \cdot 2^{n/2} \cdot (x - (2m-2) \cdot 2^{-(n+1)}), & \text{if } x \in [(2m-2)2^{-(n+1)}, (2m-1)2^{-(n+1)}) \\ 2\pi \cdot -2^{n/2} \cdot (x - 2m \cdot 2^{-(n+1)}), & \text{if } x \in [(2m-1)2^{-(n+1)}, 2m \cdot 2^{-(n+1)}) \\ 0, & \text{else.} \end{cases}$$

Proposition 1.15. *This family is an orthonormal basis of $H_0^1([0, 1])$.*

Proof: We have that

$$\|\ell_{n,m}\|_\nabla = \frac{1}{2\pi}\langle \ell_{n,m}, \ell_{n,m}\rangle_\nabla = (2\pi)^{-1}\|\ell'_{n,m}\|_2$$

Since,

$$\ell'_{n,m}(x) = \begin{cases} 2\pi \cdot 2^{n/2}, & \text{if } x \in [(2m-2)2^{-(n+1)}, (2m-1)2^{-(n+1)}) \\ 2\pi \cdot -2^{n/2}, & \text{if } x \in [(2m-1)2^{-(n+1)}, 2m \cdot 2^{-(n+1)}) \end{cases}$$

Then

$$\|\ell'_{n,m}\|_2^2 = \int \ell'_{n,m}(x)^2 \, dx = 4\pi^2 \cdot 2^n \cdot 2^{-(n+1)} + 4\pi^2 \cdot 2^n \cdot 2^{-(n+1)} = 4\pi^2$$

So we can conclude that $\|\ell_{n,m}\|_\nabla = 1$ □

Graphically, the elements of the aforementioned family look like this

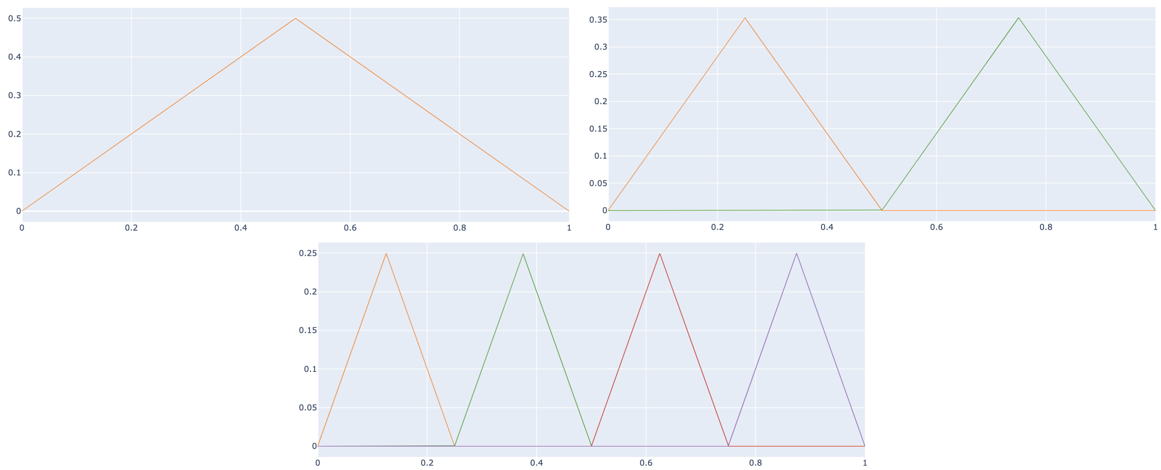


Figure 1.1: Simulations of Levy Basis when $n = 0$ and $m = 1$ in the figure up-left. In figure up-right $n = 1$ and $m \in \{(1, 2\}$. In the figure below $n = 2$ and $m \in \{1, 2, 3, 4\}$

1.6.2 Fourier Basis

Definition 1.13 (Fourier's Basis)**.** We will call Fourier basis the family of functions $\{f_{n,m}\}_{m,n\in\mathbb{N}}$ defined for $x = (x_1, x_2) \in [0, 1]^2$ by

$$f_{m,n}(x) = \frac{2\sin(2\pi n x_1)\sin(2\pi m x_2)}{\sqrt{n^2 + m^2}}$$

Proposition 1.16. *This family is an orthonormal basis of $H_0^1([0, 1]^2)$.*

Proof: We have that

$$\|\nabla f_{n,m}\|_2^2 = \int_0^1 \int_0^1 \|\nabla f_{n,m}(x_1, x_2)\|_2^2 \, dx_1 \, dx_2 = 4\pi^2$$

Hence,

$$\|f_{n,m}\|_\nabla = \frac{1}{2\pi}\|\nabla f_{n,m}\|_2 = 1$$

□

1.7 Gaussian Multiplicative Chaos

The Theory of Gaussian Multiplicative Chaos (GMC) was initially introduced by Kahane in 1985 in his article [Kah85]. In simple words, this theory allows for the study of mass concentration phenomena and singularities in physical and mathematical systems modeled by Gaussian fields. Specifically, GMC is interpreted as a measure that captures the extreme fluctuations of the GFF by applying an exponential transformation to the field. Furthermore, this theory is used to calculate the lower bound of the dimension of the thick points of the GFF.

En forma general, el Gaussian Multiplicative Chaos busca estudiar medidas sobre un dominio $D \subseteq \mathbb{R}^d$ de la forma

$$\mathcal{M}(dz) = \exp(\gamma\Phi(x) - \mathbb{E}[\Phi(x)^2])\sigma(dz), \tag{1.10}$$

donde Φ es una Gaussian free field, $\gamma > 0$ es un parámetro y σ es una medida de referencia.

Debido a la complejidad de evaluar el GFF en un punto, la definición de la medida en 1.10 no resulta inmediatamente clara. Para abordar esta dificultad, se introduce un kernel regularizador $\theta_{x,\varepsilon}$, con $x \in D$ y $\varepsilon > 0$. Bajo este enfoque, al definir $\Phi_\varepsilon(x) = \langle \Phi, \theta_{x,\varepsilon} \rangle$, resulta razonable estudiar la convergencia de la medida descrita en 1.10 a través del análisis de la medida

$$\mathcal{M}_\varepsilon(dz) = \exp(\gamma \Phi_\varepsilon(x) - \mathbb{E}[\Phi_\varepsilon(x)^2])\sigma(dz),$$

En el caso de la Gravedad Cuántica de Liouville bidimensional, se toma la medida de Lebesgue como σ, y la elección natural para el kernel θ es la distribución uniforme en el círculo unitario. En dimensiones más generales, como se muestra en [Ber17], se analiza la convergencia de esta construcción, demostrando que converge a la medida definida en 1.10 cuando $\gamma < \sqrt{2d}$. Además, se establece que esta convergencia es independiente del kernel de regularización utilizado. Este resultado se formaliza en el siguiente teorema

Theorem 1.17. *Assume that $\gamma < 2$. Then $\mathcal{M}_\varepsilon(S)$ converges in probability and in L^1 to a limit $\mathcal{M}(S)$. Furthermore, the random variable $\mathcal{M}(S)$ does not depend on the choice of the regularising kernel σ subject to the above assumptions. Additionally, $\mathcal{M}$ defines a Borel measure on D, and $\mathcal{M}_\varepsilon$ converges in probability towards $\mathcal{M}$ for the topology of weak convergence of measures on D.*

Este resultado es de gran relevancia, ya que en el contexto de la Gravedad Cuántica de Liouville bidimensional, Berestycki y Powell demuestran en [BP24] que, en un dominio acotado, cualquier punto $z \in D$ muestreado según la medida 1.10 es un γ-thick point. De manera más formal, este resultado puede entenderse en el siguiente teorema

Theorem 1.18. *Suppose D is bounded. Let z be a point sampled according to the Liouville measure $\mathcal{M}$. Then, almost surely,*

$$\limsup_{\varepsilon \to 0} \frac{\Phi_\varepsilon(x)}{\log(1/\varepsilon)} = \gamma. \tag{1.11}$$

Chapter 2

The pseudo Gaussian Free Field

In this section, we will introduce the abstract object we are working with and check that its definition is coherent. We will also present its main properties, regardless of the dimension in which it is defined. We will begin by introducing its definition and then prove that it is independent of the basis in which it is formulated. Next, we will demonstrate that the object we are studying belongs to the distribution space $H_0^{d/2-1-\varepsilon}$, for $\varepsilon > 0$. Finally, we will outline the main property of the object, highlighting its ability to represent log-correlated fields in the two-dimensional case.

Definition 2.1 (Pseudo Gaussian Free Field)**.** Let $(\alpha_k)_{k\in\mathbb{N}}$ be random variables with the correlation structure $\mathbb{E}[\alpha_k] = 0$ and $\mathbb{E}[\alpha_n\alpha_m] = \mathbb{1}_{n=m}$ and $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $H_0^1(D)$. We define the pGFF as the following limit in $L^2(\Omega)$

$$\Upsilon \doteq \lim_{N\to\infty} \sum_{k=1}^{N} \alpha_k e_k = \sum_{k\geq 1} \alpha_k e_k. \tag{2.1}$$

To make sure that the above definition is consistent we must prove that it does not depend on the base from which it is defined. That is why we present the following proposition:

Proposition 2.1. *Let $(e_k)_{k\in\mathbb{N}}$ and $(\tilde{e}_k)_{n\in\mathbb{N}}$ be two bases of the space $H_0^1(D)$. Let Υ be a pGFF defined from the variables $(\alpha_k)_{k\in\mathbb{N}}$ and the base $(e_k)_{k\in\mathbb{N}}$. Then, there exists a modification of Υ, $\tilde{\Upsilon}$ defined from $(\tilde{e}_k)_{n\in\mathbb{N}}$. By modification we mean that for any $f \in C_0^\infty$, it holds almost surely that $\langle \Upsilon, f\rangle = \langle \tilde{\Upsilon}, f\rangle$.*

Proof: Consider the bases $(e_k)_{k\in\mathbb{N}}$ and $(\tilde{e}_k)_{n\in\mathbb{N}}$ of the space $H_0^1(D)$, and let Υ be the pGFF defined from the variables $(\alpha_k)_{k\in\mathbb{N}}$ and the first base. Now define the modification $\tilde{\Upsilon}$, which is expressed as

$$\Upsilon = \sum_{k\geq 0} \tilde{\alpha}_k \tilde{e}_k, \tag{2.2}$$

where $\alpha_k = \sum_n \alpha_n \langle e_k, \tilde{e}_n\rangle$.

First, we will prove that $\tilde{\Upsilon}$ is a pGFF and then show that it is a modification. To establish the former, it is sufficient to verify that $\mathbb{E}[\tilde{\alpha}_k\tilde{\alpha}_j] = \mathbb{1}_{k=j}$, as they clearly satisfy the expected value condition. Thus,

$$\mathbb{E}[\tilde{\alpha}_k\tilde{\alpha}_j] = \mathbb{E}\left[\sum_{n\geq 0}\alpha_n\langle e_k,\tilde{e}_n\rangle\sum_{m\geq 0}\alpha_m\langle e_j,\tilde{e}_m\rangle\right] = \sum_{n\geq 0}\sum_{m\geq 0}\langle e_k,\tilde{e}_n\rangle\langle e_j,\tilde{e}_m\rangle\mathbb{E}[\alpha_n\alpha_m] = \langle e_k, e_j\rangle = \mathbb{1}_{k=j}. \tag{2.3}$$

Finally, let us show that $\tilde{\Upsilon}$ is a modification of Υ. Let $f \in \mathcal{C}_0^\infty(D)$, then

$$\langle \Upsilon, f\rangle = \sum_{k\geq 0}\alpha_k\langle e_k, f\rangle = \sum_{k\geq 0}\sum_{n\geq 0}\alpha_k\langle e_k, f\rangle\langle\tilde{e}_n, f\rangle = \sum_{n\geq 0}\sum_{k\geq 0}\alpha_k\langle e_k, f\rangle\langle\tilde{e}_n, f\rangle = \sum_{n\geq 0}\tilde{\alpha}_n\langle\tilde{e}_n, f\rangle = \langle\tilde{\Upsilon}, f\rangle. \tag{2.4}$$

Thus, we conclude that $\tilde{\Upsilon}$ is a modification, and hence, the property is satisfied. □ This last proposition tells us that the definition of $pGFF$ is independent of the base space being used to construct it. This allows us to conclude that, like Proposition 1.5

Proposition 2.2. *For $\varepsilon > 0$ and $D \subseteq \mathbb{R}^d$ the sum described in (2.1) a.s. belongs to the space $H_0^{d/2-1-\varepsilon}$.*

The proof of this property is identical to the one presented in Proposition 1.5. This makes it clear that the regularity of the GFF is not an intrinsic property of the Gaussianity of the variables with which it is defined.

The key proposition of the work and the one that allowed us to deepen the study of correlated log fields is as follows

Proposition 2.3. *Let $(\Psi_f)_{f\in\mathcal{C}_0^\infty}$ be a collection of centred random variables such that*

$$\mathbb{E}\left[\langle\Psi, f\rangle\langle\Psi, g\rangle\right] = \iint_{D\times D} f(x)G^D(x,y)g(y)dxdy.$$

There exist a pseudo GFF $\tilde{\Psi}$ that is a modification of Ψ.

The proof of the previous property relies on the following steps: we start with a collection of centered variables $(\Psi_f)_{f\in\mathcal{C}_0^\infty}$ that satisfy the required property and define a modification of it. We then show that this modification constitutes a pGFF, and finally, we confirm that it is indeed a modification.

Proof: Given a collection of variables $(\Psi_f)_{f\in\mathcal{C}_0^\infty}$ such that

$$\mathbb{E}\left[\langle\Psi, f\rangle\langle\Psi, g\rangle\right] = \iint_{D\times D} f(x)G^D(x,y)g(y)dxdy,$$

we define the variables $\alpha_k = \langle\Psi, e_k\rangle$, where $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of the space $H_0^1(D)$. From these, consider the modification given by

$$\tilde{\Psi} = \sum_{k\geq 0}\alpha_k e_k.$$

To see that $\tilde{\psi}$ is a pGFF let us prove that the variables $(\alpha_k)_{k\in\mathbb{N}}$ from which it is defined fulfill the required correlation structure. Trivially one has that they are centered since $(\Psi_f)_{f\in\mathcal{C}_0^\infty}$ is centered. Then,

$$\mathbb{E}[\alpha_n \alpha_m] = \mathbb{E}[\langle \Psi, e_n \rangle \langle \Psi, e_m \rangle] = \langle e_n, (-\Delta)^{-1} e_m \rangle = \langle e_n, e_m \rangle_\nabla = \mathbb{1}_{n=m}.$$

Thus, we conclude that $\tilde{\Psi}$ is a PGFF.

Let us now see that this is indeed a modification. To do so, let us show that $\mathbb{E}[(\langle \Psi, f \rangle - \langle \tilde{\Psi}, f \rangle)^2] = 0$ for all $f \in \mathcal{C}_0^\infty$. In effect, we have that

$$\mathbb{E}[(\langle \Psi, f \rangle - \langle \tilde{\Psi}, f \rangle)^2] = \mathbb{E}[\langle \Psi, f \rangle^2] - 2\mathbb{E}[\langle \Psi, f \rangle \langle \tilde{\Psi}, f \rangle] + \mathbb{E}[\langle \tilde{\Psi}, f \rangle^2].$$

If we analyze each of the terms in the previous expression, we will conclude that the expectations of each correspond to $\sum_{k \geq 0} \langle f, e_k \rangle^2$, given that $\mathbb{E}[\langle \Psi, e_k \rangle] = \mathbb{E}[\langle \tilde{\Psi}, e_k \rangle]$. Thus, the terms in the previous expression cancel out, thereby obtaining the required result. □

To conclude the section, we could emphasize the relevance of the recent propositions with the following remark

Remark. It is crucial to highlight the importance of the recently presented propositions, as they justify and support the use of the pGFF as a tool to explore the limits of universality in log-correlated fields. First, Proposition 2.1 grants us the flexibility to work with any chosen basis, allowing the properties derived from one specific basis to be generalized, thus endowing the pGFF with greater versatility. Furthermore, Proposition 2.3 validates the use of the pGFF in the study of log-correlated fields by demonstrating that its correlations are determined by the Green's function. As proven in Section 1.2, in two dimensions, this function has a logarithmic form, which makes the pGFF a log-correlated field in that dimension.

Chapter 3

Pseudo GFF in Dimension 1

The main objective of this thesis is to analyze the behavior of the thick points of log-correlated fields. In particular, we will study these points in the pseudo Gaussian free field. To better understand their behavior, we begin by examining a simpler case: the pseudo Brownian Motion (pBM). The latter, as its name suggests, is a particular case of the pGFF in a one-dimensional domain.

The goal of this section is, first, to ensure that the pBM is well-defined and remains continuous when the random variables in its construction are slightly changed with respect to the Brownian motion. Then, we will study its modulus of continuity to understand whether or not this is a property of universality.

We will begin by proving that our object can be properly defined as a continuous function. In Section 3.2, we will demonstrate that, under the assumption that the variables $(\alpha_k)_{k\in\mathbb{N}}$ have a finite moment $\gamma > 2$, it is possible to establish an upper bound for the modulus of continuity. Then, in Section 3.3, we will show that, by adding the assumption of independence for the variables $(\alpha_k)_{k\in\mathbb{N}}$ and assuming reasonable conditions, a lower bound for the modulus of continuity can be obtained. Finally, in Section 3.4, we will calculate the upper bound of the law of the iterated logarithm for the pBM.

3.1 Continuity

In this section, we will demonstrate that, under certain hypotheses on the variables α_k, the pseudo Brownian motion is well-defined in $[0, 1]$ and, furthermore, is continuous.

From this point on, and only in this section, we will enumerate the variables $(\alpha_k)_{k\in\mathbb{N}}$ using the indexing $(n, m) \in \mathbb{N} \times I_n$, which was used to enumerate the Levy basis presented in section 1.6.

Proposition 3.1 (The pBM can be defined as a continuous function)**.** *Let X be a pBM constructed from the Lévy base and defined with random variables $(\alpha_{n,m})_{n\in\mathbb{N},m\in I_n}$ such that there exists $\gamma > 2$ with $\sup_{n,m} \mathbb{E}[\alpha_{n,m}^\gamma] < \infty$. Then, almost surely, X is well-defined and continuous.*

Proof: Take $\gamma > 2$ such that $\sup_{n,m} \mathbb{E}[a_{n,m}^\gamma] < \infty$. Then, using Markov, inequality for $c > 0$,

$$\mathbb{P}(|\alpha_{n,m}| \geq 2^{nc/2}) \leq \mathbb{E}[|\alpha_{n,m}|^\gamma] \cdot 2^{-nc\gamma/2}.$$

Now, the series

$$\sum_{n\geq 0}\mathbb{P}(\{\exists m\in I_n : |\alpha_{n,m}|\geq 2^{nc/2}\})\leq \sum_{n\geq 0}\sum_{m\in I_n}\mathbb{P}(|\alpha_{n,m}|\geq 2^{nc/2})\leq \sum_{n\geq 0}\mathbb{E}[|\alpha_{n,m}|^{\gamma}]\,2^n\cdot 2^{-nc\gamma/2}.$$

Which converges as soon as $c > 2/\gamma$. For that type of c, by Borel-Cantelli lemma there exists a random $\overline{n}$ such that for all $n > \overline{n}$ and $m \in I_n$ we have $|\alpha_{n,m}| < 2^{nc/2}$. Then, for $t \in [0,1]$ and by the definition of the pBM,

$$|X(t)|\leq \sum_{n=0}^{\infty}\sum_{m=1}^{2^n}|\alpha_{n,m}|\cdot|\ell_{n,m}(t)|\leq \sum_{n=0}^{\infty}2^{nc/2}\sum_{m=1}^{2^n}|\ell_{n,m}(t)|,$$

By how the Lévy basis is defined, with n fixed, the terms of $\ell_{n,m}(t)$ will all be null except for a one value of $m \in I_n$, which depends on t. For this reason,

$$|X(t)|\leq \sum_{n=0}^{\infty}2^{nc/2}\sum_{m=1}^{2^n}|\ell_{n,m}(t)| = \sum_{n=0}^{\infty}2^{nc/2}\cdot|\ell_{n,m(t)}(t)|\leq \sum_{n=0}^{\infty}2^{nc/2}\,2^{-(n+2)/2} \tag{3.1}$$

Which converges as soon as $c < 1$

It remains to verify that it is continuous. For the latter, let us proof is a uniform limit of continuous functions. Le us denote the partial sum as X_N by the form

$$|X_N(t)| = \sum_{n=0}^{N}\sum_{m=1}^{2^n}|\alpha_{n,m}|\cdot|\ell_{n,m}(t)|. \tag{3.2}$$

Then, for the same arguments presented before

$$\|X-X_N\|_{\infty}\leq \sup_{t\in[0,1]}\sum_{n=N+1}^{\infty}\sum_{m=1}^{2^n}|\alpha_{n,m}|\cdot|\ell_{n,m}(t)|\leq \sum_{n=N+1}^{\infty}2^{nc/2}\,2^{-(n+2)/2} \tag{3.3}$$

which tends to 0 as $N\to\infty$ is increasing. With this we conclude that the pBM is a well-defined and continuous function. □

3.2 Upper Bound on the Modulus of Continuity

An important microscopic property of Brownian motion is that its modulus of continuity is not random and can be computed.

Theorem 3.2 (Theorem 1.14 in [MP10])**.** *Let B be a Brownian Motion. Then, almost surely,*

$$\limsup_{h\searrow 0}\sup_{0\leq t\leq 1-h}\frac{|B(t+h)-B(t)|}{\sqrt{2h\log(1/h)}} = 1.$$

In this section, we will focus on establishing an upper bound for the modulus of continuity of the pBM. This analysis is essential for two reasons. First, it will allow us to determine whether such a bound is a universal property of pBM or if it depends on its specific construction. Secondly, we will see that this same bound will be key for the circle average process described in Section

1.3.3, when applied in the context of the pGFF. This will provide us with a solid foundation for better understanding the regularity of this process in more general cases.

To begin the analysis, let's start by presenting the following property

Proposition 3.3 (Upper Bound)**.** *Let X be a pBM constructed from the Lévy base and $(\alpha_{n,m})_{n\in\mathbb{N},m\in I_n}$ such that $\sup_{n,m}\mathbb{E}[\alpha_{n,m}^{\gamma}]<\infty$ for some $\gamma>2$. Then, almost surely,*

$$\limsup_{h\searrow 0}\sup_{0\le t\le 1-h}\frac{|X(t+h)-X(t)|}{h^{1/2-1/\gamma}}\le 1.$$

Proof: Let us take $\gamma>2$ such that $sup_{n,m}E[\alpha_{n,m}^{\gamma}]<\infty$. By the definition of the pBM from the Levy Basis, we can easily check that

$$\|X'_{n,m}\|_\infty\le\frac{2\|X_{n,m}\|_\infty}{2^{-n}}\le C\;2^{n(c+1)/2},\tag{3.4}$$

where $C>0$ is a constant and $c\in(2/\gamma,1)$. Now, for each $t,t+h\in[0,1]$, by the definition of the Lévy base

$$\begin{aligned}|X(t+h)-X(t)|&=|\sum_{n\ge 0}\sum_{m=1}^{2^n}\alpha_{n,m}(\ell_{n,m}(t+h)-\ell_{n,m}(t))|\\&=|\sum_{n\ge 0}\alpha_{n,m(t,h)}(\ell_{n,m(t,h)}(t+h)-\ell_{n,m(t,h)}(t))|\end{aligned}$$

Now, splitting the sum and using the average value theorem

$$\begin{aligned}|X(t+h)-X(t)|\le&\sum_{n=0}^{j}|\alpha_{n,m(t,h)}|\cdot|(\ell_{n,m(t,h)}(t+h)-\ell_{n,m(t,h)}(t))|\\&+\sum_{n=j+1}^{\infty}|\alpha_{n,m(t,h)}|\cdot|(\ell_{n,m(t,h)}(t+h)-\ell_{n,m(t,h)}(t))|\\\le&\,h\sum_{n=0}^{j}\|X'_{n,m}\|_\infty+2\sum_{n=j+1}^{\infty}\|X_{n,m}\|_\infty\end{aligned}$$

Hence, using (3.4), for $j>\overline{n}$,

$$|X(t+h)-X(t)|\le h\sum_{n=0}^{\overline{n}}\|X'_{n,m}\|_\infty+C_1\;h\sum_{n=\overline{n}}^{j}2^{n(c+1)/2}+C_2\sum_{n=j}^{\infty}2^{n(c-1)/2},$$

where $j>\overline{n}$ is chosen such that $h^{(\gamma-2)/(c-1)}\le 2^j\le h^{-(\gamma+2)/(\gamma c+\gamma)}$.

Now, we prove that each term of the sum above can be upper bounded by $\varphi(h)=h^{1/2-(1+\varepsilon)/\gamma}$. The first term can easily be upper bounded by taking $h\ll\varphi(h)$. For the second term, we have that

$$C_1\;h\sum_{n=\overline{n}}^{j}2^{n(c+1)/2}\le\tilde{C}_1\;h\;2^{j(c+1)/2}\le\tilde{C}_1\;h\;h^{-(\gamma+2)/(\gamma c+\gamma)}=\tilde{C}\;h^{1/2-1/\gamma},\tag{3.5}$$

and for the last one, we can compute

$$C_2 \sum_{k=j}^{\infty} 2^{n(c-1)/2} \leq \tilde{C}_2\, 2^{j(c-1)/2} \leq h^{(\gamma-2)(\gamma c-\gamma)\cdot(c-1)/2} = \tilde{C}_2\, h^{1/2-1/\gamma}.$$

With this computation, we conclude that for each t and $t+h \in [0,1]$, $|X(t+h) - X(t)|$ is less than or equal to $h^{1/2-1/\gamma}$ multiplied by a constant that not depends of $\gamma > 2$. □

Remark. In the case that the variables $(\alpha_{n,m})_{n\in\mathbb{N},m\in I_n}$ are bounded by a constant $a > 0$, we will have that

$$\|X'_{n,m}\|_\infty \leq \frac{2\|X_{n,m}\|_\infty}{2^{-n}} \leq a\, 2^{n/2}.$$

Now, following the same calculation we did before and taking $j > \overline{n}$,

$$|X_k(t+h) - X_k(t)| \leq h \sum_{n=1}^{\overline{n}} \|X'_k\|_\infty + C_1 \sum_{n=\overline{n}}^{j} 2^{n/2} + C_2 \sum_{n=j}^{\infty} 2^{-n/2}.$$

Finally, taking $j > \overline{k}$ tal que $h^{-1} \leq 2^j \leq h^{-1/3}$, it follows that

$$|X_{n,m}(t+h) - X_{n,m}(t)| \leq \sqrt{h}.$$

With the calculation just performed, we can negatively answer the question posed at the beginning of the section. The modulus of continuity is not a universal property of pBM. This is because it clearly depends on the moments of the variables with which it is defined.

3.3 Lower Bound on the Modulus of Continuity

In this section, we aim to derive a lower bound that complements the result obtained in the previous section. The order of this new bound matches that of the upper bound, although the associated constant differs significantly. Unlike the upper bound, this new bound explicitly depends on the moments of the random variables α_k. It is worth noting that, to reach this result, it was necessary to add the assumption of independence for the variables $(\alpha_{n,m})_{n\in\mathbb{N},m\in I_n}$ and a slight condition on their tails.

Proposition 3.4 (Lower Bound). *Let X be a pBM constructed from the Levy base and independent random variables $(\alpha_{n,m})_{n\in\mathbb{N},m\in I_n}$ such that*

$$\mathbb{P}(|\alpha_{n,m}| \leq 2^{n/\gamma+1/\gamma-3/2}) > 0. \tag{3.6}$$

Also,

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) = \infty, \tag{3.7}$$

where $E_n = \{\exists m \in I_n : |\alpha_{n,m}| \geq 2^{nc/2}\}$*, for* $c \in (\frac{2}{\gamma}, 1)$*. Then, almost surely,*

$$\limsup_{h \searrow 0} \sup_{0 \leq t \leq 1-h} \frac{|X(t+h) - X(t)|}{(2^{-1/2-1/\gamma} - 2^{-1})(h^{1/2-1/\gamma})} \geq 1.$$

The proof is based on the following strategy: first, we will demonstrate that if the algorithm presented below concludes, the desired lower bound is obtained. Subsequently, we will prove that the algorithm indeed terminates.

Proof: For $\gamma > 2$ let us define the function $\varphi(h) = h^{1/2-1/\gamma}$, with $h > 0$.

Starting whit a $n \in \mathbb{N}$, let us apply the following algorithm

```
(1)  For m in range 2^n:
(2)  |  t = (2m − 1)2^(−n)
(3)  |  h = 2^(−(n+1))
(4)  |  If |X_{N−1}(t + h) − X_{N−1}(t)| ≥ φ(h)/2:
(5)  |  |  If |α_{n,m}| · 2^(−(n+2)/2) ≤ φ(h)/4:
(6)  |  |     Return (t, h)
(7)  |  Elif |α_{n,m}| ≥ 2^(nc/2):
(8)  |     Return (t, h)
(9)  n = n + 1
(10) go to line (1)
```

Let us first prove that if for any initial $n \in \mathbb{N}$ the algorithm finishes in finite steps then, we obtain the bound we are looking for. Fix $n \in \mathbb{N}$ and assume that the algorithm returns (t, h). It means that we either enter in the line (6) or the line (8). In the first case, it indicates that the conditions (4) and (5) are met. Then,

$$\begin{aligned} |X_n(t+h) - X_n(t)| &= |X_{n-1}(t+h) - X_{n-1}(t) + \alpha_{n,m(t)} \cdot \ell_{n,m(t)}(t) - \alpha_{n,m(t,h)} \cdot \ell_{n,m(t,h)}(t+h)| \\ &= |X_{n-1}(t+h) - X_{n-1}(t) - \alpha_{n,m(t,h)} \cdot 2^{-(n+2)/2}|, \end{aligned}$$

where the last equality is due to the choice of t and h that we made in the algorithm. Then,

$$\begin{aligned} |X_n(t+h) - X_n(t)| &\geq |X_{n-1}(t+h) - X_{n-1}(t)| - |\alpha_{n,m(t)}| \cdot 2^{-(n+2)/2} \\ &\geq \frac{\varphi(h)}{2} - \frac{\varphi(h)}{4} \\ &\geq \frac{\varphi(h)}{2}. \end{aligned}$$

On the other hand, if it reaches the line (8), it means that the condition (4) is not met and the condition (7) is met. Then,

$$
\begin{aligned}
|X_n(t+h) - X_n(h)| &= |X_{n-1}(t+h) - X_{n-1}(t) - \alpha_{n,m(t,h)} \cdot 2^{-(n+2)/2}| \\
&\geq |\alpha_{n,m}| \cdot 2^{-(n+2)/2} - |X_{n-1}(t+h) - X_{n-1}(t)| \\
&\geq \varphi(h) \cdot 2^{-1/2-1/\gamma} - \frac{\varphi(h)}{2} \\
&\geq \varphi(h)(2^{-1/2-1/\gamma} - 2^{-1})
\end{aligned}
$$

And we can conclude that of the algorithm finishes we obtain the bound we are looking for.

Now, it remains to prove that the algorithm finishes in finite time for any $n \in \mathbb{N}$. To prove this, we will separate into two disjoint events that will help us in the demonstration. The first event is when (4) occurs finitely many times, and the second is when it occurs infinitely many times. If we are in the first scenario, note that hypothesis (3.4) of property (3.4) coincides with the hypothesis of the divergent Borel-Cantelli Lemma. This latter result gives us that condition (7) occurs infinitely many times. On the other hand, if (4) occurs finitely many times, then $(\overline{4})$[1] occurs infinitely many times. Thus, eventually, when $(\overline{4})$ and (7) coincide, the algorithm will terminate.

In the second scenario, let's define the family of stopping times $\{\tau_k\}_{k\in\mathbb{N}}$ as

$$
\begin{aligned}
&\tau_1 = \inf\{k > 0 : (4)_k \text{ happens}\} \\
&\tau_2 = \inf\{k > \tau_1 : (4)_k \text{ happens}\} \\
&\vdots \\
&\tau_M = \inf\{k > \tau_{M-1} : (4)_k \text{ happens}\}
\end{aligned}
$$

which seeks to count the number of times the condition (4) is fulfilled. Then,

$$
\begin{aligned}
\mathbb{P}(\text{A.D.E}) &= \mathbb{P}(\text{A.D.E},\ \tau_M < \infty) + \mathbb{P}(\text{A.D.E},\ \tau_M = \infty) \\
&= \mathbb{P}(\text{A.D.E},\ \tau_M < \infty) = \mathbb{P}(\text{A.D.E}|\tau_M < \infty)\ \mathbb{P}(\tau_M < \infty).
\end{aligned}
$$

where $\mathbb{P}(\text{A.D.E})$ is the probability that the Algorithm Does not End. Now, note that $\mathbb{P}(\text{A.D.E} \mid \tau_M < \infty)$ is bounded by the probability that a random variable $G \sim \text{Geo}(p)$ is greater than $M \in \mathbb{N}$, where p is the probability that condition (5) is met. With this, we obtain that

$$
\mathbb{P}(\text{A.D.E}) \leq \mathbb{P}(Geo(p) > M) \leq (1-p)^M
$$

and by the hypothesis (3.7) we conclude that this probability tends to 0 when M is large enough.
□

3.4 Law of the Iterated Logarithm

The law of the iterated logarithm describes the magnitude of fluctuations in the trajectories of stochastic processes, offering a precise description of their extreme behavior. In the case of standard Brownian motion, [MP10, Theorem 5.1] establishes the following fundamental prop-

[1]We will use the overline to refer to the non-occurrence of the condition.

erty:

Theorem 3.5 (The law of the iterated logarithm). *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Then, almost surely,*

$$\limsup_{t\to 0} \frac{B(t)}{\sqrt{2t\log(\log(t))}} = 1.$$

In this section, we aim to replicate this idea, but only considering an upper bound, in the case of pseudo Brownian Motion (pBM). This analysis is relevant for the study of thick points in log-correlated fields, as explored in Section 4.3, where pBM plays a key role in describing the behavior of the circle average process. Finding an accurate measure of the fluctuations of this process is crucial for understanding its dynamics and establishing connections with log-correlated fields.

Proposition 3.6. *Let $(Y_t)_{t\geq 0}$ be a pseudo Brownian Motion defined like in Proposition 3.3. Then, almost surely,*

$$\limsup_{h\to 0^+} \frac{Y_h}{h^{1/2-1/\gamma}} \leq 1.$$

Proof: Let us consider the pBM $(X_t)_{t\in[0,1]}$ and note that the process $(tX_{1/t})_{t\in[0,1]}$ describes a pBM on $[1,\infty)$. Thus, we define the process $(Y_t)_{t\in[1,\infty)}$ as

$$Y_t \doteq (tX_{1/t}).$$

Then, the following holds:

$$\limsup_{t\to\infty} \frac{X_t}{f(t)} = \limsup_{s\to 0^+} \frac{sX_{1/s}}{f(1/s)s}.$$

On the other hand, from the calculation in Section 3.4, we have

$$\lim_{h\to 0^+} \sup_{0\leq s\leq 1-h} \frac{Y_{s+h}-Y_s}{h^{1/2-1/\gamma}}.$$

Fixing ω, there exists $h_0 = h_0(\omega)$ such that if $h\in(0,h_0)$, then

$$\frac{Y_{s+h}-Y_s}{h^{1/2-1/\gamma}} \leq 1, \quad \forall s\in[0,1-h].$$

In particular, taking $s=0$,

$$\frac{Y_h}{h^{1/2-1/\gamma}} \leq 1$$

and therefore,

$$\limsup_{h\to 0^+} \frac{Y_h}{h^{1/2-1/\gamma}} \leq 1.$$

□

Chapter 4

Pseudo GFF in Dimension 2

In this section, we will conduct a detailed analysis of the central object of this thesis: the pseudo Gaussian Free Field (pGFF) in two dimensions. As mentioned earlier, this field is a key representative within the class of log-correlated fields in this dimensional context. Throughout this section, we will explore how the pGFF allows us to examine the limits of the universality of this class. In particular, we will focus on the study of the extreme values of these fields, known as thick points, and their Hausdorff dimension.

We will begin in Section 4.1 by defining thick points in this new context and demonstrating the regularity of the circle average process. In Section 4.2, we will prove that the circle average process associated with the pGFF behaves like a pseudo Brownian Motion, which will allow us to leverage the theory developed in Chapter 3. In Section 4.3, we will address the mod-ϕ convergence of the pGFF, a fundamental property for understanding the behavior of our object of study and essential for working with Non-Gaussian Multiplicative Chaos theory. Next, in Section 4.4, we will calculate the upper bound of the Hausdorff dimension of the thick points. Finally, in Section 4.5, we will study the Multiplicative Chaos Theory applied to the pGFF, which will provide crucial tools for calculating the lower bound of the Hausdorff dimension of the thick points, a result that will be demonstrated in Section 4.6.

4.1 Circle Average and pGFF

To study the thick points of the pGFF, it is essential to first formalize the concept of the circle average process in this non-Gaussian context. In this section, we will define this process, enabling us to precisely characterize the set of points under analysis. Additionally, we will demonstrate the process's inherent regularity, which is crucial for the subsequent examination of thick points. Achieving this required introducing additional assumptions on the $(\alpha_k)_{k\in\mathbb{N}}$ variables. Specifically, these variables must not only be i.i.d. but also strongly sub-Gaussian, as defined in 1.11. Therefore, from this point forward, we will always assume the $(\alpha_k)_{k\in\mathbb{N}}$ variables to be strongly sub-Gaussian.

Let us consider Γ a pGFF over a domain $D \subseteq \mathbb{R}^2$ and define the circle average as

$$\Gamma_\varepsilon(x) \doteq \langle \Gamma, \mu_{x,\varepsilon} \rangle = \sum_{k\geq 0} \alpha_k c_k, \tag{4.1}$$

where $c_k(x) = \langle e_k, \mu_{x,\varepsilon} \rangle$ and $(e_k)_{k\in\mathbb{N}}$ is the basis of the space in which we will work and specify later. This notion applied to the new log-correlated model allows us to define the set of thick points as follows

Definition 4.1 (Thick Points for the pGFF)**.** Take $\varepsilon > 0$ and Γ a pGFF. We will define the set of γ-thick points of Γ as

$$T(\gamma) = \left\{ x \in D : \limsup_{\varepsilon \to 0} \frac{\Gamma_\varepsilon(x)}{\log(1/\varepsilon)} = \gamma \right\}. \tag{4.2}$$

Similarly, we also define $T_{\geq}(\gamma)$ as the set of points where the aforementioned lim sup is greater than or equal to a.

As described in [HMP10, Proposition 2.1] for the GFF, the circle average process of the $pGFF$ also satisfies some notion of regularity. This is best described by the following property

Proposition 4.1. *Let us consider the family of random variables* $(\alpha_k)_{k\in\mathbb{N}}$ *strictly subgaussian and independent. Then, the circle average process* $\Gamma(z,r)$ *possesses a modification* $\tilde{\Gamma}(z,r)$ *such that for every* $0 < \gamma < 1/2$ *and* $\delta, \xi > 0$ *there exists* $M = M(\gamma, \delta, \xi)$ *such that*

$$|\tilde{\Gamma}(z,r) - \tilde{\Gamma}(w,s)| \leq M \ \log(1/r)^{\xi} \ \frac{\|(z,r)-(w,s)\|^{\gamma}}{r^{\gamma+\delta}}, \tag{4.3}$$

for all $z, w \in D$ *and* $r, s \in (0,1]$ *with* $1/2 \leq r/s \leq 2$.

To perform this proof, it suffices only to verify that the pGFF satisfies the conditions of the [HMP10, Lemma C.1]. That is that for certain $p, q > 0$

$$\mathbb{E}[|\Gamma(z,r) - \Gamma(w,s)|^p] \leq C \left(\frac{\|(z,r)-(w,s)\|}{r \wedge s} \right)^{1+q+2}.$$

Proof: Let us consider Φ a GFF and Γ a pGFF and recall that it is satisfied that

$$\mathbb{E}[|\Gamma(z,r) - \Gamma(w,s)|^2] = \mathbb{E}[|\Phi(z,r) - \Phi(w,s)|^2]$$

Let's take $p = 2e$, with $e \in \mathbb{N}$. Then, since Γ is strongly sub-Gaussian,

$$\mathbb{E}[|\Gamma(z,r) - \Gamma(w,s)|^p] = \mathbb{E}[|\Gamma(z,r) - \Gamma(w,s)|^{2e}] = \mathbb{E}[(|\Phi(z,r) - \Phi(w,s)|^2)^e].$$

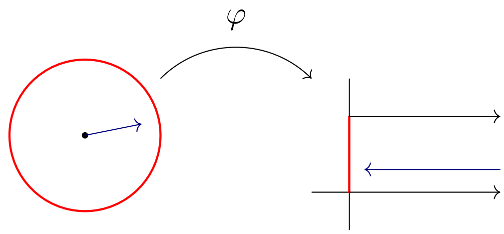
Finally, since Φ satisfies the [HMP10, Lemma C.1] for arbitrarily large values of p, q with p/q arbitrarily close to $1/2$, it satisfies it for these values as well. ☐

It is important to mention that from now on we will always consider a pGFF Γ that satisfies this regularity property.

4.2 pBM inside PGFF

In this section, we will delve deeper into the analysis of the circle average process of the pGFF. Specifically, we will demonstrate that the pBM can naturally integrate into the circle average process of a pGFF. Furthermore, we will prove that given a pBM $(X)_{t\geq 0}$, there exists a pGFF whose circle average process is represented by X. We will begin by constructing a semi-infinite cylinder $\mathcal{C} \subseteq \mathbb{C}$ from the unit ball in the complex plane. Next, we will show that the function space $H_0^1(\mathcal{C})$ can be orthogonally decomposed into two subspaces, P and Q, whose definitions we will detail later. Finally, we will demonstrate that the circle average of a pGFF is entirely described by its projection onto the subspace P.

Let us take the complex unit disk of radius ε_0 and denote it as $\mathbb{D}_0 = \{z \in \mathbb{C} : \|z\| \leq \varepsilon_0\}$. Then, if we denote $z = \varepsilon e^{i\theta}$ as a point in the disk, note that by applying the conformal transformation $\varphi(z) = \ln(\varepsilon^{-1}) + i\theta$ to the entire disk, we transform it into an infinite cylinder.



Let $\mathcal{C}$ denote the cylinder and $r = \ln(\varepsilon^{-1})$. Note that the function space $H_0^1(\mathcal{C})$ can be decomposed orthonormally into

$$P = \{p \in H_0^1(\mathcal{C}) : p(r, \theta) \text{ is constant in } \theta\}$$
$$Q = \{q \in H_0^1(\mathcal{C}) : \forall r > 0, \int q(r, \theta) d\theta = 0\}$$

We will prove this condition in the following lemma:

Lemma 4.2. *The function space $H_0^1(\mathcal{C})$ can be decomposed orthonormally into P and Q.*

Proof: First, let us prove that they are orthogonal subspaces. By density, we can take $p \in P \cap \mathcal{C}_0^\infty$ and $q \in Q$. Then,

$$\begin{aligned}\langle p, q\rangle_\nabla &= \int_{\mathcal{C}} \partial_r p \, \partial_r q \, dr d\theta + \int_{\mathcal{C}} \partial_\theta p \, \partial_\theta q \, dr d\theta \\ &= \int_0^\infty \partial_r p(r, \theta) \, \partial_r \left(\int_0^{2\pi} q(r, \theta) \, d\theta \right) dr \\ &= 0\end{aligned}$$

On the other hand, to verify that these spaces are a direct sum, consider $q \perp P$ and $p \in P$. Then,

$$\begin{aligned}\langle p, q\rangle_\nabla = 0 &\Leftrightarrow \int_0^\infty \partial_r p(r, \theta) \, \partial_r \left(\int_0^{2\pi} q(r, \theta) \, d\theta \right) dr = 0 \\ &\Leftrightarrow \langle p, \int_0^{2\pi} q(\cdot, \theta) \, d\theta \rangle_{\nabla_r} = 0.\end{aligned}$$

Finally, since the above equality holds for all $p \in P$, we conclude $q \in Q$ and hence the sum. $\square$

With this orthonormal decomposition, we can also decompose the *pGFF* into projections onto the respective spaces, i.e., $\Gamma = \Gamma^P + \Gamma^Q$. Then, given the symmetries of the GFF,

$$\langle \Gamma, \mu_{z, e^{-r}} \rangle = \langle \Gamma^P, \mu_{z,r} \rangle + \langle \Gamma^Q, \mu_{z,r} \rangle.$$

However, by the very definition of the subspace Q, the second term in the above sum is zero, so we obtain $\langle \Gamma, \mu_{z,e^{-r}} \rangle = \langle \Gamma^P, \mu_{z,r} \rangle$.

On the other hand, it is easy to see that there is an isometry between the spaces P and $H_0^1(\mathbb{R}_+)$, so the projection onto the real axis of the basis of P will be a basis for $H_0^1(\mathbb{R}_+)$. This analysis allows us to understand the following equality:

$$\Gamma_{e^{-r}}(z) = \sum_{k \geq 0} \alpha_k \langle f_k, \mu_{z,e^{-r}} \rangle = \sum_{k \geq 0} \alpha_k \langle p_k, \mu_{z,r} \rangle,$$

where $p_k = \pi_P(f_k)$ is the projection onto P.

After this analysis, we can easily construct the following proposition:

Proposition 4.3. *Given a pBM $(X_t)_{t \geq 0}$, there exists a pGFF Γ such that X almost surely represents its circle average process.*

Proof: The proof is quite straightforward. Given a pBM $X = (X_t)_{t \geq 0}$, we simply consider a collection of variables β_k that respect the appropriate correlation structure (for example, Gaussian). We then use the circle average process, defined as

$$\Gamma_{e^{-t}}(z) = X_t + \sum_{k \geq 0} \beta_k \langle q_k, \mu_{z,t} \rangle,$$

where $(q_k)_{k \in \mathbb{N}}$ represents a basis of the space Q. □

4.3 Mod-ϕ Convergence of the pGFF

Este resultado lo presentamos en la siguiente propiedad

Proposition 4.4 (Mod-G convergence of the pGFF)**.** *Let $(\alpha_k)_{k \in \mathbb{N}}$ be a sequence of i.i.d. strongly subgaussian random variables. Let Γ be a pGFF defined from these variables and using the base of Fourier. Under these assumptions, for $z \in D$ and $\lambda \in \mathbb{R}$,*

$$\mathbb{E}[\exp(\lambda\, \Gamma_\varepsilon(z))] \exp(-\log(\varepsilon^{-1})\lambda^2/2) \xrightarrow{\varepsilon \to 0} \psi(\lambda),$$

for some analytic function ψ.

Proof: Take $z \in D$, $\lambda \in \mathbb{R}$ and denoting $c_k^\varepsilon(z) = c_k^\varepsilon$, we have that

$$\mathbb{E}[\exp(\lambda\, \Gamma_\varepsilon(z))] = \mathbb{E}\left[\exp\left(\lambda \sum_{k \geq 0} \alpha_k c_k^\varepsilon\right)\right] = \prod_{k \geq 0} \mathbb{E}[e^{\lambda \alpha_k c_k^\varepsilon}] = \exp\left(\log(\prod_{k \geq 0} \mathbb{E}[e^{\lambda c_k^\varepsilon \alpha}])\right)$$
$$= \exp\left(\sum_{k \geq 0} \log(\mathbb{E}[e^{\lambda c_k^\varepsilon \alpha}])\right).$$

If we define $\kappa_{i,k}^\varepsilon$ as the i-th cumulant of the variable $c_k^\varepsilon \alpha$, we can denote $\kappa_{i,k}^\varepsilon = (c_k^\varepsilon)^i \beta_i$, where

β_i is the i-th cumulant of α. Now, we can write

$$\mathbb{E}[\exp(\lambda\,\Gamma_\varepsilon(z))] = \exp\left(\sum_{k\geq 0}\sum_{i\geq 2}\kappa^\varepsilon_{i,k}\cdot\frac{\lambda^i}{i!}\right) = \exp\left(\sum_{k\geq 0}(c^\varepsilon_k)^2\frac{\lambda^2}{2}+\sum_{k\geq 0}\sum_{i\geq 3}(c^\varepsilon_k)^i\,\beta_i\cdot\frac{\lambda^i}{i!}\right)$$

$$\Longrightarrow \mathbb{E}[\exp(\lambda\,\Gamma_\varepsilon(z))]\;\exp\left(-\frac{\lambda^2}{2}\sum_{k\geq 0}(c^\varepsilon_k)^2\right) = \exp\left(\sum_{k\geq 0}\sum_{i\geq 3}(c^\varepsilon_k)^i\,\beta_i\cdot\frac{\lambda^i}{i!}\right).$$

Finally, as pGFF is a strongly sub-Gaussian variable, by Property 1.14, we can conclude using dominated convergence theorem. □

Remark. It is important to note that this convergence holds for all $\lambda \in \mathbb{R}$ largely due to the sub-Gaussianity assumption. However, we believe that this assumption could be relaxed, which would imply that the convergence might no longer hold for all $\lambda \in \mathbb{R}$, but only on a subset of $\mathbb{R}$.

4.4 Upper Bound on the Hausdorff Dimension of the Thick Points

One of the central results of this work is the calculation of the upper bound of the Hausdorff dimension of the thick points of the pGFF. In this section, we present this key result along with its formal proof. The proof closely follows the methodology used for the GFF, as described in [HMP10], adapting the necessary techniques to address the case of the pGFF.

Proposition 4.5. *Let us considered the family of random variables $(\alpha_k)_{k\in\mathbb{N}}$ strictly sub-gaussian and independents and Γ a pGFF constructed from them. If $0 \leq \gamma \leq 2$, then almost surely $\dim_H(T(\gamma)) \leq 2-\gamma^2/2$ and if $\gamma > 2$, $T(\gamma)$ is empty.*

To demonstrate the previous proposition, we will prove that $\dim_H(T_{\geq}(\gamma)) \leq 2-\gamma^2/2$. Since it is clear that $T(\gamma) \subseteq T_{\geq}(\gamma)$, this will allow us to conclude the desired result. Moreover, for $r > 0$, we will start by showing that it is sufficient to consider the values of $\Gamma(x, r)$ at the sequence $r_n = n^{-K}$, where $K = \varepsilon^{-1}$. In other words, we will focus on studying the discrete radii of the form $r_n = n^{-K}$.

Proof: Let us assume that $0 \leq \gamma \leq 2$. Let $\varepsilon > 0$ be arbitrary, and fix $K = \varepsilon^{-1}$. For each $n \in \mathbb{N}$, define $r_n = n^{-K}$. Taking $\xi \in (0, 1)$ and $\eta \in (0, 1/2)$, set $M = M(\eta, \varepsilon, \xi)$ as described in Proposition 4.1. Now, to verify that it is sufficient to analyze the discrete values of r_n, let's examine the following inequality for a fixed $x \in D$ and $r > 0$ such that $r_{n+1} < r < r_n$. By Property 4.1, the following holds

$$\begin{aligned}|\Gamma(x,r)-\Gamma(x,r_n)| &\leq M\log(1/r_n)^\xi\;\frac{\|(x,e^{-t})-(x,r_n)\|^\eta}{r_n^{\eta+\varepsilon}}\\ &\leq MK^\xi\log(n)^\xi\frac{|e^{-t}-r_n|^\eta}{r_n^{(1+\varepsilon)\eta}}\\ &\leq MK^\xi\log(n)^\xi\frac{|r_{n+1}-r_n|^\eta}{r_n^{(1+\varepsilon)\eta}}\end{aligned}$$

Now, since $|r_{n+1} - r_n|^\eta = |(n+1)^{-K} - n^{-K}|^\eta = O(n^{-(K+1)\eta})$ and $r_n^{(1+\varepsilon)\eta} = n^{-K(1+\varepsilon)\eta} = n^{-K\eta-\eta}$, it follows that

$$|\Gamma(x,r) - \Gamma(x,r_n)| = O((\log n)^\xi). \tag{4.4}$$

Therefore, since the constant M depends only on η, ε, and ξ, by dividing by $\log(r_n^{-1})$ on both sides of the above equation, we obtain that the difference $|\Gamma(x,r) - \Gamma(x,r_n)|$ tends to zero as $n \to \infty$. This leads us to conclude that whether x belongs to $T_\geq(\gamma)$ depends on the circle average over a countable number of rings around x. To finalize this idea, with a very similar calculation, we can conclude that if $\|x-y\| \leq r_n^{1+\varepsilon}$, then

$$|\Gamma(x,r_n) - \Gamma(y,r_n)| = O((\log n)^\xi). \tag{4.5}$$

Now consider a grid over the set $D \subseteq \mathbb{R}^2$ with mesh size $r_n^{1+\varepsilon}$, and denote by z_{nj} the center of each square in the grid. With this and the inequality described in (4.5), we have that

$$|\Gamma(x,r_n) - \Gamma(z_{nj},r_n)| = O((\log n)^\xi). \tag{4.6}$$

Let us define $\delta(n) = C\ \log(n)^{\xi-1}$ and

$$\mathcal{I}_n \doteq \{j : |\Gamma(z_{nj},r_n)| \geq (\gamma - \delta(n))\log(r_n^{-1})\},$$

which corresponds to the set of indices of the centers of squares that are nearly thick points. With this, we define the covering set of $T_\geq(\gamma)$ as

$$I(\gamma,N) = \bigcup_{n\geq N}\{B(z_{nj},r_n^{1+\varepsilon}) : j \in \mathcal{I}_n\}. \tag{4.7}$$

Let us prove that the set defined in (4.7) indeed covers $T_\geq(\gamma)$. For $x \in T_\geq(\gamma)$, by the definition of the grid we used, there will exist a $z_{nj} \in D$ such that $\|z_{nj} - x\| \leq r_n^{1+\varepsilon}$. Then, from the calculation in (4.6), we have that

$$\begin{aligned} -\Gamma(z_{nj},r_n) \leq \tilde{C}\log(n)^\xi - \Gamma(x,r_n) &\iff \Gamma(z_{nj},r_n) \geq \Gamma(x,r_n) - \tilde{C}\log(n)^\xi \\ &\iff \frac{\Gamma(z_{nj},r_n)}{\log(r_n^{-1})} \geq \frac{\Gamma(x,r_n)}{\log(r_n^{-1})} - C\log(n)^{\xi-1}. \end{aligned}$$

Thus, for sufficiently large $n > N$, we obtain that

$$|\Gamma(z_{nj},r_n)| \geq (\gamma - \delta(n))\log(r_n^{-1}),$$

and therefore $j \in \mathcal{I}_n$. With this, we conclude that $B(z_{nj},r_n^{1+\varepsilon}) \in I(\gamma,N)$ and that $I(\gamma,N)$ covers $T_\geq(\gamma)$.

Now it remains to calculate the Hausdorff measure to obtain a bound for the dimension. We have that

$$\begin{aligned} \mathbb{E}[H^b(I(\gamma,N))] &\leq \mathbb{E}\left[\sum_{n\geq N}\sum_{\{j\in\mathcal{I}_n\}} \operatorname{diam}(B(z_{nj},r_n^{1+\varepsilon}))^b\right] \\ &= \sum_{n\geq N}(2r_n^{1+\varepsilon})^b\ \mathbb{E}[|\mathcal{I}_n|]. \end{aligned}$$

To calculate the term $\mathbb{E}[|\mathcal{I}_n|]$, note that $\mathbb{E}[|\mathcal{I}_n|] = \sum_j 1 \cdot \mathbb{P}(j \in \mathcal{I}_n)$. Then,

$$\begin{aligned}\mathbb{P}(j \in \mathcal{I}_n) &= \mathbb{P}\left(\frac{|\Gamma(z_{nj}, r_n^{-1})|}{\sqrt{\log(r_n^{-1})}} \geq (\gamma - \delta(n))\sqrt{\log(r_n^{-1})}\right) \\ &\leq \exp\left(-\frac{1}{2}(\gamma - \delta(n))^2 \log(r_n^{-1})\right) \\ &= O(r_n^{\gamma^2/2+o(1)}),\end{aligned}$$

where the first inequality holds because $\Gamma(z_{nj}, r_n^{-1})$ is strongly sub-Gaussian. With this, we obtain that

$$\mathbb{E}[|\mathcal{I}_n|] \leq o\left(\frac{r_n^{\gamma^2/2+o(1)}}{r_n^{2(1+\varepsilon)}}\right) = o(r_n^{\gamma^2/2-2(1+\varepsilon)+o(1)}). \tag{4.8}$$

Returning to the previous calculation,

$$\mathbb{E}[H^b(I(\gamma, N))] \leq \sum_{n \geq N} (2r_n^{1+\varepsilon})^b \, \mathbb{E}[|\mathcal{I}_n|] \leq 2^b \sum_{n \geq N} r_n^{b(1+\varepsilon)+\gamma^2/2-2(1+\varepsilon)+o(1)}.$$

Thus, this sum is null if $b > \frac{2(1-2\varepsilon)}{(1+\varepsilon)} - \frac{\gamma^2}{2(1+\varepsilon)}$, and by taking the $\inf\{b > \frac{2(1-2\varepsilon)}{(1+\varepsilon)} - \frac{\gamma^2}{2(1+\varepsilon)} : \varepsilon > 0\}$, we conclude that $\dim_H(T_{\geq}(\gamma)) \leq 2 - \gamma^2/2$ and therefore $\dim_H(T(\gamma)) \leq 2 - \gamma^2/2$. Furthermore, from equation (4.8), we can deduce that for $\gamma > 2$, $\mathbb{E}[|\mathcal{I}_n|] \to 0$ as $n \to \infty$. □

4.5 Non Gaussian Multiplicative Chaos

In this section, we will delve into the construction of the Liouville Quantum Measure following a similar approach to that proposed by Berestycki in [Ber17]. This theory is essential for estimating the lower bound of the dimension of thick points in the GFF. Although the pGFF is not strictly Gaussian, we will demonstrate that it is possible to obtain an equivalent lower bound in this context, thus highlighting the flexibility of multiplicative chaos even in non-Gaussian settings. An alternative and complementary approach can be found in the work of Junnila in [Jun16], where non-Gaussian multiplicative chaos is also examined under different hypotheses, though with results that could be equivalent to ours regarding the convergence of the measure.

In this section, we will first introduce the fundamental definitions that support our analysis. Subsequently, we will present the central proposition and proceed to prove the crucial lemmas that underpin this proposition. Finally, we will conclude by integrating all elements to reach the desired conclusion.

Let us define the measure σ as

$$\sigma(dx) = e^{\Gamma(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\Gamma(x)^2]} dx, \tag{4.9}$$

where Γ represents a pGFF under certain construction conditions and $\gamma > 0$ is a parameter of which we will give more details later. Since Γ is not defined pointwise, we approximate it using the circle average. Thus, we define the approximate measure σ_ε as

$$\sigma_\varepsilon(dx) = e^{\Gamma_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\Gamma_\varepsilon(x)^2]} \, dx. \tag{4.10}$$

Our goal of this section is demonstrate that σ_ε converges to the measure σ defined in (4.9). More formally, the objective is to prove the following proposition:

Proposition 4.6. *For $\gamma < 2$, $\sigma_\varepsilon(D)$ converges in probability and in L^1 to a limit $\sigma(D)$.*

To prove this proposition, we will follow the approach outlined in [Ber17]. In particular, we will consider a parameter $\kappa > \gamma$ and classify the points $x \in D$ as bad points if their thickness exceeds κ, and as good points otherwise (we will elaborate on this classification later). We will then show that the contribution in L^1 of the bad points is negligible, while that of the good points is Cauchy in L^2, which implies their convergence.

Throughout this analysis, we will verify that our object satisfies the same properties presented by Berestycki in his article if we assume the necessary hypotheses to ensure mod-$\mathcal{G}$ convergence. Moreover, since our object shares the same correlations as the GFF, we will be able to reproduce the estimates made by Berestycki in Lemma 3.5 of his article.

Lemma 4.7 (Lemma 3.5 in [Ber17])**.** *Let $(\alpha_k)_{k\in\mathbb{N}}$ be strongly sub-Gaussian random variables and Γ a pGFF constructed from them and the Fourier basis. For $x, y \in D$, we have the following estimate:*

$$Cov(\Gamma_\varepsilon(x), \Gamma_r(y)) = \log 1/(|x-y| \vee r \vee \varepsilon) + O(1). \tag{4.11}$$

Moreover, if $\eta > 0$ and $|x-y| \geq \eta$, then

$$Cov(\Gamma_\varepsilon(x), \Gamma_\delta(y)) = \log(1/|x-y|) + g(x,y) + o(1), \tag{4.12}$$

where $o(1)$ tends to 0 as $\delta, \varepsilon \to 0$, uniformly in $|x-y| \geq \eta$.

Throughout this section, we will always consider Γ as a pGFF constructed from strongly sub-Gaussian random variables and the Fourier basis. We will consider $\kappa > \gamma$ and introduce the following notation: for $r > 0$, we define,

$$\overline{r} \doteq \inf\{e^{-m} : m \in \mathbb{N},\ r < e^{-m}\}, \tag{4.13}$$

and for $x \in D$, we define a good event as

$$G_\varepsilon^\kappa(x) \doteq \{\Gamma_{\overline{r}}(x) \leq \kappa \log(1/\overline{r}),\ \forall r \in [\varepsilon, \varepsilon_0]\}, \tag{4.14}$$

with $\varepsilon_0 \leq 1$. Finally, for ease of notation, we will denote $\overline{\Gamma}_\varepsilon(x) = \gamma\Gamma_\varepsilon(x) - (\gamma^2/2)\mathbb{E}[\Gamma_\varepsilon(x)^2]$ and $I_\varepsilon = \sigma_\varepsilon(D)$.

The first objective is to prove the following proposition

Proposition 4.8. *I_ε is uniform integrable.*

To develop this proof, we will first need the following lemmas followed by their proofs.

Lemma 4.9. *For any $\kappa > 0$ and uniformly over $x \in D$, we have that $\mathbb{P}(G_\varepsilon^\kappa(x)) \geq 1 - p(\varepsilon_0)$, where the function p may depend on κ. Also, for fixed $\kappa > \gamma$, $p(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$.*

Proof:

Let us consider the process $\Upsilon_t = \Gamma_\varepsilon(x)$, where $\varepsilon = e^{-t}$. Take $m \in \mathbb{N}$ and by the strong sub-Gaussianity of Γ,

$$\mathbb{P}(\Upsilon_m(x) \geq \kappa m) \leq 2\ \exp\left(-\frac{\kappa^2 m^2}{2\log(e^m)}\right) \leq Ce^{-\lambda m},$$

for C, λ strictly positive constants.

By other side,

$$\mathbb{P}(G_\varepsilon^\gamma(x)^c) = \mathbb{P}(\exists r \in [\varepsilon, \varepsilon_0] : \Gamma_{\overline{r}}(x) \geq \kappa \log(1/\overline{r})) \leq \mathbb{P}(\exists m \geq \overline{m}_0 : \Upsilon_m \geq \kappa m),$$

where $\overline{m}_0$ has to be found. Then,

$$\mathbb{P}(\exists m \geq m_0 : |\Gamma_m(x)| \geq \kappa m) \leq \sum_{m \geq m_0} \mathbb{P}(|\Gamma_m(x)| \geq \kappa m) \leq \tilde{C}e^{-\lambda m_0}.$$

Finally, taking $m_0 = \lceil \log(1/\varepsilon_0) \rceil$ and calling $p(\varepsilon_0)$ on the right-hand side, we conclude as requested. □

Another important result allows us to conclude that the Liouville points are no more than γ-thick is the following lemma

Lemma 4.10. *For $\kappa > \gamma$, we have that $\mathbb{P}(G_\varepsilon^\kappa(x)) \geq 1 - C\, p(\varepsilon_0)$, with C a positive constant.*

Proof: Let us first note that

$$\mathbb{E}[e^{\overline{\Gamma}_\varepsilon(x)} \mathbb{1}_{G_\varepsilon^\kappa(x)}] = \tilde{\mathbb{P}}(G_\varepsilon^\kappa(x)),$$

where $\tilde{\mathbb{P}}$ is the measure defined from the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{e^{\overline{\Gamma}_\varepsilon(x)}}{\mathbb{E}[e^{\overline{\Gamma}_\varepsilon(x)}]}. \tag{4.15}$$

Then,

$$\tilde{\mathbb{P}}(G_\varepsilon^\kappa(x)^c) \leq \tilde{\mathbb{P}}(\exists \overline{r} \in [\varepsilon, \varepsilon_0] : \Gamma_{\overline{r}}(x) \geq \kappa \log(1/r))$$

Now, if we index by $m \in \mathbb{N}$ to the values of $\overline{r} \in [\varepsilon, \varepsilon_0]$, we have that

$$\tilde{\mathbb{P}}(G_\varepsilon^\kappa(x)^c) \leq \sum_{m \geq m_0} \tilde{\mathbb{P}}(\Gamma_m \geq \kappa m),$$

where $m_0 = \lceil \log(1/\varepsilon_0) \rceil$. Now, multiplying and dividing by $e^{-(\lambda^2/2 + \lambda\gamma)m}$ we have that

$$\tilde{\mathbb{P}}(G_\varepsilon^\kappa(x)^c) \leq \sum_{m \geq m_0} \frac{\mathbb{E}[e^{(\lambda+\gamma)\Gamma_m}]e^{-(\lambda+\gamma)/2m}}{\mathbb{E}[e^{\overline{\Gamma}_m}]} \cdot e^{(\lambda^2/2 + \lambda\gamma - \lambda\kappa)m}.$$

Due to the convergence of mod-$\mathcal{G}$, the ratio in the equation above converges to a bounded function $\psi(\lambda, \gamma)$, as dictated by the convergence, for all $\lambda > 0$.

Finally by Lemma 4.9,

$$\tilde{\mathbb{P}}(G_\varepsilon^\kappa(x)) \geq 1 - \tilde{C}_1 e^{(\lambda^2/2 + \lambda\gamma - \lambda\kappa)m_0} \geq 1 - \tilde{C}_2 p(\varepsilon_0)$$

and taking $\lambda = \kappa > \gamma$ we can conclude. □

We will now see that points which are more than γ-thick do not contribute significantly to I_ε in expectation and can therefore be safely removed. Fix $\kappa > \gamma$, let us write $G_\varepsilon(x)$ as $G_\varepsilon^\kappa(x)$ and introduce

$$J_\varepsilon = \int_D e^{\overline{\Gamma}_\varepsilon(x)} \mathbb{1}_{G_\varepsilon(x)} dx,$$

which corresponds to the contribution on I_ε of the good points and let us denote by J_ε^c the contribution of bad points.

Lemma 4.11. *For $\kappa > \gamma$ sufficiently close to γ, J_ε is bounded in L^2 and hence uniformly integrable.*

Proof: By Fubini's theorem and the strongly sub-Gaussianity of Γ,

$$\begin{aligned}\mathbb{E}[J_\varepsilon^2] &= \int_{D^2} \mathbb{E}[e^{\overline{\Gamma}_\varepsilon(x)+\overline{\Gamma}_\varepsilon(y)} \mathbb{1}_{G_\varepsilon(x)\cap G_\varepsilon(y)}] dxdy \\ &\leq \int_{D^2} e^{\gamma^2 \mathbb{E}[\Gamma_\varepsilon(x)\Gamma_\varepsilon(y)]} \tilde{\mathbb{P}}(G_\varepsilon(x) \cap G_\varepsilon(y)) dxdy \end{aligned} \tag{4.16}$$

where $\tilde{\mathbb{P}}$ is the probability measure defined from

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{e^{\overline{\Gamma}_\varepsilon(x)+\overline{\Gamma}_\varepsilon(y)}}{\mathbb{E}\big[e^{\overline{\Gamma}_\varepsilon(x)+\overline{\Gamma}_\varepsilon(y)}\big]}.$$

The idea from now on will be to find a good upper bound for probability $\tilde{\mathbb{P}}(G_\varepsilon(x) \cap G_\varepsilon(y))$.

Now, if we take $\varepsilon \leq \varepsilon_0$ y $|x-y| \leq \varepsilon_0$, then

$$\tilde{\mathbb{P}}(G_\varepsilon(x) \cap G_\varepsilon(y)) \leq \tilde{\mathbb{P}}(\Gamma_r(x) \leq \kappa \log(1/r)), \tag{4.17}$$

where $r = \overline{|x-y| \vee \varepsilon}$ (recall the notation $\overline{r} = \inf\{e^{-n} : n \in \mathbb{N},\ r < e^{-n}\}$). Then,

$$\tilde{\mathbb{P}}(\Gamma_r(x) \leq \kappa \log(1/r)) = \mathbb{P}(e^{-\lambda\Gamma_r(x)} \geq e^{-\lambda\kappa\log(1/r)}) \leq \tilde{\mathbb{E}}[e^{-\lambda\Gamma_r(x)}]\, e^{\lambda\kappa\log(1/r)}. \tag{4.18}$$

Let us now seek to bound the expectation of the above equation. We have that

$$\tilde{\mathbb{E}}[e^{-\lambda\Gamma_r(x)}] = \frac{\mathbb{E}\big[e^{-\lambda\Gamma_r(x)+\gamma\Gamma_r(x)+\Gamma_r(y)}\big]}{\mathbb{E}\big[e^{\overline{\Gamma}_\varepsilon(x)+\overline{\Gamma}_\varepsilon(y)}\big]}.$$

Multiplying by a suitable one, we can form the exact term for the convergence mod-$\mathcal{G}$ of both the numerator and denominator of the fraction. This implies that for ε_0 sufficiently small,

$$\tilde{\mathbb{E}}[e^{-\lambda\Gamma_r(x)}] \leq C\ \exp\left(\lambda^2/2 - 2\lambda\gamma\right)\log(1/r).$$

With this computation, we can bound Equation (4.18) and with it the Equation 4.17 to obtain that

$$\tilde{\mathbb{P}}(G_\varepsilon(x) \cap G_\varepsilon(y)) \leq C \exp\left((\lambda^2/2 - 2\lambda\gamma + \lambda\kappa)\log(1/r)\right) \tag{4.19}$$

Using the Lemma 4.7 and the just calculated coordinate we obtain that

$$\begin{aligned}\mathbb{E}[J_\varepsilon^2] &\leq O(1)\int_{D^2}(|x-y|\vee\varepsilon)^{-\gamma^2}\,\tilde{\mathbb{P}}(G_\varepsilon(x)\cap G_\varepsilon(y))dxdy \\ &\leq O(1)\int_{D^2}(|x-y|\vee\varepsilon)^{-\gamma^2}\,r^{-\lambda^2/2+2\lambda\gamma-\lambda\kappa}\,dxdy\end{aligned} \tag{4.20}$$

Finally, for this integral to be finite, it must be satisfied that

$$-\gamma^2-\frac{\lambda^2}{2}+2\lambda\gamma-\lambda\kappa>-2$$

and taking λ and κ as arbitrarily close to γ, it is concluded that

$$-\gamma^2-\frac{\gamma^2}{2}+2\gamma^2-\gamma^2>-2 \text{ or } \gamma<2 \tag{4.21}$$

□

Remark. Let us note that in this last section, we have worked with considerable flexibility regarding the parameter $\lambda\in\mathbb{R}$. This is possible because, under the assumption of strong sub-Gaussianity in the variables $(\alpha_k)_{k\in\mathbb{N}}$, the mod-$\phi$ convergence holds for any value of this parameter. However, if this hypothesis were relaxed, we believe that it would be necessary to optimize λ within a specific subset of $\mathbb{R}$ where the convergence is still preserved.

Since we already have all the lemmas necessary to prove the main proposition of this section, we can proceed to its formal proof.

Proof(Proposition 4.8): First, we observe that $I_\varepsilon=J_\varepsilon+J_\varepsilon^c$. From Lemma 4.10, we know that $E[J_\varepsilon^c]\leq Cp(\varepsilon_0)$, which implies that the contribution of the bad points becomes negligible. On the other hand, Lemma 4.11 establishes that J_ε^2 is bounded in L^2 for a fixed ε_0 and uniformly in ε. Therefore, we conclude that I_ε is uniformly integrable. □

As a final result of the section, we will seek to demonstrate that I_ε converges. Since $\mathbb{E}[J_\varepsilon^c]$ can be made arbitrarily small by choosing ε_0 sufficiently small, it is enough to show that J_ε converges in probability and in L^1. In fact, we will show that it converges in L^2, from which the convergence will follow. To do this, we will show that J_ε forms a Cauchy sequence in L^2 and will start by writing

$$\mathbb{E}[(J_\varepsilon-J_\delta)^2]=\mathbb{E}[J_\varepsilon^2]+\mathbb{E}[J_\delta^2]-2\mathbb{E}[J_\varepsilon J_\delta]. \tag{4.22}$$

The idea in this part is basically to find a better lower bound estimate for $\mathbb{E}[J_\varepsilon^2]$ and an upper bound estimate for $\mathbb{E}[J_\varepsilon J_\delta]$. We will start by proving the following lemma:

Lemma 4.12. *We have*

$$\limsup_{\varepsilon\to0}\mathbb{E}[J_\varepsilon^2]\leq\int_{D^2}\frac{e^{\gamma^2 g(x,y)}}{|x-y|^{\gamma^2}}f_\kappa(x,y)dx\,dy$$

where $f_\kappa(x,y)$ is a non negative function depending on κ,ε_0 and γ such that the above *integral is finite.*

Proof: From Equation (4.16) we have

$$\mathbb{E}[J_\varepsilon^2] \le \int_{D^2} e^{\gamma^2 \mathbb{E}[\Gamma_\varepsilon(x)\Gamma_\varepsilon(y)]}\, \tilde{\mathbb{P}}(G_\varepsilon(x) \cap G_\varepsilon(y))\, dx\, dy. \tag{4.23}$$

The idea, then, is to find a better estimate for $\tilde{\mathbb{P}}(G_\varepsilon(x) \cap G_\varepsilon(y))$ when $|x-y| > \eta$, with $\eta > 0$ arbitrarily small. This is the relevant case, since if $|x-y| \le \eta$, we can apply the same upper bound as in (4.20). In such a situation, the contribution of these points can be bounded by a function $f(\eta)$, where $f(\eta) \to 0$ as $\eta \to 0$, uniformly in ε.

Consider a finite set $\{r_m\}_{m=1}^N$ of e-adic approximations in $(0, \varepsilon_0]$ (recall the notation $\overline{r} = \inf\{e^{-n} : n \in \mathbb{N}, r < e^{-n}\}$). Let us also denote by $\Gamma_m(x) = \Gamma_{r_m}(x)$ and take the vector $\lambda = (\lambda_m)_{m=1}^N \in \mathbb{R}^N$. Now take the expectation

$$\tilde{\mathbb{E}}[e^{\sum_{m=1}^N \lambda_m \Gamma_m(x)}] = \mathbb{E}[e^{\sum_{m=1}^N \lambda_m \Gamma_m(x) + \overline{\Gamma}_\varepsilon(x) + \overline{\Gamma}_\varepsilon(y)}].$$

and it is not difficult to verify that multiplying and dividing by the term

$$\exp\left(-\sum_{m=1}^N \frac{\lambda_m}{2} \log(r_m^{-1}) - 2\gamma^2 \log(\varepsilon^{-1}) - \sum_{i,j=1, i\neq j}^N \lambda_i \lambda_j \mathbb{E}[\Gamma_i(x)\Gamma_i(x)] \right)$$

this expression converges, as $\varepsilon \to 0$, in mod-$\mathcal{G}$ to some bounded function $\psi(\lambda, \gamma)$. This allows us to conclude that, under $\tilde{\mathbb{P}}$, the process $(\Gamma_r(x))_{r \le \varepsilon_0}$ converges to the process $(\tilde{\Gamma}_r(x))_{r \le \varepsilon_0}$ as $\varepsilon \to 0$. Repeating this reasoning, we obtain that the joint law of $(\Gamma_r(x), \Gamma_r(y))_{r \le \varepsilon_0}$ converges, under $\tilde{\mathbb{P}}$ and as $\varepsilon \to 0$, to the process $(\tilde{\Gamma}_r(x), \tilde{\Gamma}_r(y)) r \le \varepsilon_0$.

Since we have the convergence of the joint law $(\Gamma_r(x), \Gamma_r(y))_{r \le \varepsilon_0}$, we can define $\tilde{G}(x) = \{\tilde{\Gamma}_r(x) \le \kappa \log(1/r), \text{ for all } r \in (0, \varepsilon_0]\}$ and verify that

$$\tilde{\mathbb{P}}(G_\varepsilon(x) \cap G_\varepsilon(y)) \to \mathbb{P}(\tilde{G}_\varepsilon(x) \cap \tilde{G}_\varepsilon(y)) \doteq f_\kappa(x, y), \quad (\varepsilon \to 0). \tag{4.24}$$

Consequently, applying Lemma 4.7 and (4.20) to justify the dominated convergence, we obtain that

$$\int_{D^2 : |x-y| > \eta} e^{\gamma^2 \mathbb{E}[\Gamma_\varepsilon(x)\Gamma_\varepsilon(y)]}\, \tilde{\mathbb{P}}(G_\varepsilon(x) \cap G_\varepsilon(y))\, dx\, dy \to \int_{D^2 : |x-y| > \eta} \frac{e^{\gamma^2 g(x,y)}}{|x-y|^{\gamma^2}} f_\kappa(x, y)\, dx\, dy \tag{4.25}$$

Finally, to verify that the right side remains bounded as $\eta \to 0$, we can use the fact that, in inequality (4.19), we established that $\tilde{\mathbb{P}}(G_\varepsilon(x) \cap G_\varepsilon(y)) \le O(1)|x-y|^{-\lambda^2/2 + 2\lambda\gamma + \lambda\kappa}$. Therefore, this inequality should also apply to $f_\kappa(x, y)$, which allows us to conclude in a manner similar to (4.21). □

In a very similar manner and following a very similar scheme, we can conclude the following result

Lemma 4.13. *We have*

$$\liminf_{\varepsilon, \delta \to 0} \mathbb{E}[J_\varepsilon J_\delta] \ge \int_{D^2} \frac{e^{\gamma^2 g(x,y)}}{|x-y|^{\gamma^2}} f_\kappa(x, y)\, dx\, dy,$$

where $f_\kappa(x,y)$ is a non negative function depending on κ, ε_0 and γ such that the above integral is finite.

Proof: The proof is the same as in Lemma 4.12 except that in this case, we will use the measure $\tilde{\mathbb{P}}$ defined through $e^{\overline{\Gamma}_\varepsilon(x)+\overline{\Gamma}_\delta(y)}$. These changes do not alter any argument, so (4.24) and (4.25) remain valid. Since we obtain a lower bound by restricting to $|x-y|>\eta$, we immediately deduce that

$$\liminf_{\varepsilon,\delta\to 0} \mathbb{E}[J_\varepsilon J_\delta] \geq \int_{D^2} \frac{e^{\gamma^2 g(x,y)}}{|x-y|^{\gamma^2}} f_\kappa(x,y)dx\,dy$$

□

Finally, to close this section, we present the following proposition:

Proposition 4.14. *J_ε is Cauchy in L^2.*

Proof: With the results obtained in Lemmas 4.12, 4.13 and equation (4.22), we can conclude that J_ε is Cauchy in L^2. □

This proposition will allow us to formalize the behavior of J_ε and advance in the analysis of the object of study."

4.6 Lower Bound on the Hausdorff Dimension of the Thick Points

In this section, we will calculate the lower bound of the Hausdorff dimension of the thick points using the theory developed in the previous section. We will start by proving that the limiting measure σ provided by Proposition 4.6 is supported on the set of γ-thick points. Then, we will apply the Energy Method [MP10, Theorem 4.27], through which we will conclude that $\dim_H(T(\gamma)) \leq 2-\gamma^2/2$ for $\gamma \in (0,2)$ and thus equality.

Let us begin by proving that the measure σ defined in 1.10 is supported on γ-thick points through the following lemma. The proof follows the line for the Gaussian case as in [Aru20b] Section 2.6

Lemma 4.15. *Suppose D is bounded. Let Γ be a pGFF defined by strongly sub-Gaussian variables, and let z be a point sampled according to the limiting measure σ. Then, almost surely,*

$$\limsup_{\varepsilon\to 0} \frac{\Phi_\varepsilon(x)}{\log(1/\varepsilon)} = \gamma. \tag{4.26}$$

Proof: This proof follow by a classical argument that goes as follows

1. Define the probability measure $\tilde{\mathbb{P}}_\epsilon$ on pairs (x,Γ) where $x\in D$ and Γ is a distribution as

$$\tilde{\mathbb{P}}_\epsilon(dx,d\Gamma) = \frac{\exp(\gamma\Gamma_\epsilon(x))}{\mathbb{E}\left[\exp(\gamma\Gamma_\epsilon(x))\right]}dx\mathbb{P}(d\Gamma),$$

   where $\mathbb{P}$ is the measure of the pGFF.

2. It is clear that $\tilde{\mathbb{P}}_\varepsilon$ converges weakly as $\epsilon\to 0$ to the measure σ

$$\tilde{\mathbb{P}}(dx,d\Gamma) = p(x)\sigma^\gamma(dx)\mathbb{P}(d\Gamma),$$

where σ^γ is the Liouville measure of the Γ with parameter γ and

$$p(x) = \lim_{\epsilon\to 0} \frac{e^{\gamma^2 \log(1/\epsilon)}}{\mathbb{E}\left[\exp(\gamma\Gamma_\epsilon(x))\right]} < \infty.$$

3. The marginal law of x under $\tilde{\mathbb{P}}_\epsilon$ is that of a uniform in D. Thus the maginal law of x under $\tilde{\mathbb{P}}$ is a uniform in D.

4. The marginal law of Γ under $\tilde{\mathbb{P}}_\epsilon$ has Radon-Nykodim derivative with respect to $\mathbb{P}$ given by $\int p(x)\sigma_\epsilon^\gamma(dx) \to \int p(x)\sigma^\gamma(dx)$. Thus Γ under $\tilde{\mathbb{P}}$ is absolutely continuous with respect to the pGFF.

5. The conditional law of x given Γ under $\tilde{\mathbb{P}}_\epsilon$ is taken proportionally to $\sigma_\epsilon^\gamma(dx)$. Thus, under $\tilde{\mathbb{P}}$, given Γ, x is taken proportionally to its Liouville measure $\sigma^\gamma(dx)$.

6. Finally, the conditional law of Γ given x under $\tilde{\mathbb{P}}_\epsilon$ is proportional to $e^{\gamma\Gamma_\epsilon}\mathbb{P}(d\Gamma)$. For $\delta > \epsilon$ and $\gamma < \gamma^+$

$$\begin{aligned}
\tilde{\mathbb{P}}_\epsilon(\Gamma_\delta(x) \geq \gamma^+ \log(1/\delta)) &\leq \frac{\mathbb{E}\left[\exp(\lambda\Gamma_\delta(x) + \gamma\Gamma_\epsilon(x) - \lambda\gamma^+ \log(1/\delta))\right]}{\mathbb{E}\left[\exp(\gamma\Gamma_\epsilon(x))\right]} \\
&\leq K \exp\left(\frac{\lambda^2}{2}\log(1/\delta) + \lambda\gamma\log(1/\delta) - \lambda\gamma^+\log(1/\delta)\right) \\
&\leq K \exp(\frac{\gamma}{2}(\gamma - \gamma^+)\log(1/\delta)),
\end{aligned}$$

where we use $\lambda = \gamma/2$. Doing the same for $\gamma^- < \gamma$

$$\begin{aligned}
\tilde{\mathbb{P}}_\epsilon(\Gamma_\delta(x) \leq \gamma^- \log(\delta)) &= \tilde{\mathbb{P}}_\epsilon(-\Gamma_\delta(x) \geq -\gamma^- \log(1/\delta)) \\
&\leq \frac{\mathbb{E}\left[\exp(-\lambda\Gamma_\delta(x) + \gamma\Gamma_\epsilon(x) - \lambda\gamma^+ \log(1/\delta))\right]}{\mathbb{E}\left[\exp(\gamma\Gamma_\epsilon(x))\right]} \\
&\leq K \exp\left(\frac{\lambda^2}{2}\log(1/\delta) - \lambda\gamma\log(1/\delta) + \lambda\gamma^-\log(1/\delta)\right) \\
&\leq K \exp(\frac{\gamma}{2}(\gamma^- - \gamma)\log(1/\delta)).
\end{aligned}$$

As the bounds do not depend on ϵ, we know that this bounds are true for Γ under $\tilde{\mathbb{P}}$.

To conclude we note that as Γ is absolutely continuous with respecto to the pGFF, (4.4) together with Borel-Cantelli, implies that $x \sim \sigma^\gamma(dx)$ is a thick point of Γ.

□

With the result of the previous Lemma, to compute the lower bound of the thick points it is sufficient to apply the Energy Method. To introduce this technique, let us first recall Definition 1.6 and present the following notation

$$I_\alpha(\mu) = \iint \frac{\mu(dx)\,\mu(dy)}{d(x,y)^\alpha} \tag{4.27}$$

where μ is a mass distribution on a metric space $(E, d(\cdot,\cdot))$. Now, the Energy Method corresponds to the following theorem

Theorem 4.16 (Energy Method). *Let $\alpha \geq 0$ and μ be a mass distribution on a metric space $(E, d(\cdot,\cdot))$. Then, for every $\varepsilon > 0$, we have*

$$H^\alpha_\varepsilon(E) \geq \frac{\mu(E)^2}{\iint_{d(x,y)<\varepsilon} \frac{\mu(dx)\,\mu(dy)}{d(x,y)^\alpha}}$$

Hence, if $\iint I_\alpha(\mu) < \infty$ then $H^\alpha(E) = \infty$ and, in particular $dim_H(E) \leq \alpha$.

In particular, in order to show for a random set E that $\dim_H(E) \geq \alpha$, almost surely, it suffices to show that $\mathbb{E}[I_\alpha(\mu)] < \infty$ for a (random) measure on E. In our case, we will seek to demonstrate that for $\delta > 0$,

$$\mathbb{E}\left[\int_{D^2} \frac{1}{|x-y|^{2-\gamma^2/2-\delta}}\, \sigma(dx)\,\sigma(dy)\right] < \infty$$

Now that we understand the outline we will follow, we present the main result of the section

Proposition 4.17. *Let Γ be a pGFF defined by strongly sub-Gaussian variables, and let $\gamma \in [0,2]$. Then, almost surely, $dim_H(T(\gamma)) \geq 2 - \gamma^2/2$.*

Proof: Take $\delta > 0$ and $\gamma \in [0,2]$. For $N \in \mathbb{N}$ define the function $f^N : \mathbb{R}^2 \to \mathbb{R}$ as

$$f^N(x,y) = \frac{1}{|x-y|^{2-\gamma^2/2-\delta}} \wedge N.$$

By Fatou's lemma and the convergence given by 4.6, we have that

$$\mathbb{E}\left[\int_{D^2} f^N(x,y)\,\sigma(dx)\,\sigma(dy)\right] \leq \liminf_{\varepsilon\to 0}\ \mathbb{E}\left[\int_{D^2} f^N(x,y)\mathbb{1}_{G_\varepsilon(x),G_\varepsilon(y)}\ \sigma_\varepsilon(dx)\,\sigma_\varepsilon(dy)\right].$$

Also, we have that

$$\begin{aligned}\mathbb{E}\left[\int_{D^2} f^N(x,y)\mathbb{1}_{G_\varepsilon(x)\cap G_\varepsilon(y)}\ \sigma_\varepsilon(dx)\,\sigma_\varepsilon(dy)\right] &= \int_{D^2} f^N(x,y)\mathbb{E}[\mathbb{1}_{G_\varepsilon(x)\cap G_\varepsilon(y)} e^{\overline{\Gamma}_\varepsilon(x)+\overline{\Gamma}_\varepsilon(y)}]\,dx\,dy \\ &\leq \int_{D^2} f^N(x,y)\ \tilde{\mathbb{P}}(G_\varepsilon(x),G_\varepsilon(y))\ \mathbb{E}[e^{\overline{\Gamma}_\varepsilon(x)+\overline{\Gamma}_\varepsilon(y)}]\,dx\,dy,\end{aligned}$$

where $\tilde{\mathbb{P}}$ is a new probability measure obtained by the Radon-Nikodyn derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{e^{\overline{\Gamma}_\varepsilon(x)+\overline{\Gamma}_\varepsilon(y)}}{\mathbb{E}[e^{\overline{\Gamma}_\varepsilon(x)+\overline{\Gamma}_\varepsilon(x)}]}.$$

By the Inequation (4.19), we have that

$$\begin{aligned}\mathbb{E}\left[\int_{D^2} f^N(x,y)\mathbb{1}_{G_\varepsilon(x)\cap G_\varepsilon(y)}\ \sigma_\varepsilon(dx)\,\sigma_\varepsilon(dy)\right] &\leq O(1)\int_{D^2} |x-y|^{\gamma^2/2-2+\delta-\gamma^2-\gamma^2/2+2\gamma-\gamma^2}\,dx\,dy \\ &= O(1)\int_{D^2} |x-y|^{-2+\delta}\,dx\,dy\end{aligned}$$

Finally, by Theorem 4.16 we can conclude that $\dim_H(T(\gamma)) < 2 - \gamma^2/2$. □

To finish this section and with this the result of the thesis, through the calculations made in the previous sections, we obtain the following result

Theorem 4.18. *Let* Γ *be a pGFF defined by strongly sub-Gaussian variables, and let* $\gamma \in [0,2]$*. Then, almost surely,* $dim_H(T(\gamma)) = 2 - \gamma^2/2$*.*

Proof: Applying Proposition 4.17 and 4.5 we conclude this property. □

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