Facultad de Ciencias

Two problems on group actions on varieties

Tesis

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En 2004 ingresé al Instituto Nacional Gral. José Miguel Carrera donde tuve mis primeros encuentros con la matemática. Además, comencé a participar en competencias escolares. Algunos años después, en 2011, ingresé a la carrera de Licenciatura en Ciencias con Mención en Matemáticas en la Facultad de Ciencias de la Universidad de Chile. En un confuso salto mortal, un año más tarde, terminé estudiando ingeniería en la misma casa de estudios, pasando incluso por un año de intercambio en la École Centrale de Paris (hoy en día llamada Centrale Supelec). Al volver, acepto el hecho de que volverme ingeniero no es lo que quería. Así, en 2017, vuelvo a la Facultad de Ciencias para terminar la licenciatura. En 2019 entro al programa de Magister en la facultad de Ciencias de la Universidad de Chile y realizo una tesis bajo dirección de Robert Auffarth y Giancarlo Lucchini. Al terminar el Magister, ingreso al programa de Doctorado en la misma casa de estudios.

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to my grandparents

Resumen

Como reza en el título, este trabajo consiste en dos problemas independientes. En lo que sigue se presentan sus respectivos resúmenes.

Descomposición de jacobianas. Se presenta una descomposición de la variedad jacobiana de una curva de Fermat generalizada. Esto extiende un resultado obtenido por Auffarth, Lucchini Arteche y Rojas sobre las curvas de Humbert-Edge, que constituyen un caso particular de las curvas de Fermat generalizadas. Además, se proporcionan ejemplos de curvas de Fermat generalizas con jacobianas completamente descomponible, así como una cota inferior para el número de curvas elípticas que aparecen en la descomposición de las jacobianas de curvas de Fermat generalizadas de tipo $(n, 3)$.

Acciones de toros. Se provee una descripción algebro-combinatorial de variedades normales geométricamente integrales provistas de una acción efectiva de un toro algebraico sobre cuerpos de característica cero, generalizando el trabajo de Altmann, Hausen y Süss. Esta descripción se obtiene en términos de divisores polyhedrales propios y abanicos divisoriales junto una acción semilineal del grupo de Galois relacionado a la extensión que escinde al toro.

Abstract

As stated in the title, this work consists of two independent problems. Their respective abstract are given below.

Descomposition of jacobians. A decomposition of the Jacobian variety of a generalized Fermat curve is given. This extends a result obtained by Auffarth, Lucchini Arteche and Rojas on Humbert-Edge curves, which are a particular case of generalized Fermat curves. Moreover, examples of generalized Fermat curves with completely decomposable jacobians are provided, as well as a lower bound for the number of elliptic curves appearing in the decomposition of the jacobians of generalized Fermat curves of type $(n, 3)$.

Torus actions. An algebro-combinatorial description of geometrically integral normal varieties endowed with an effective action of an algebraic torus over fields of characteristic zero is provided, generalizing the work of Altmann, Hausen and Süss. This description is achieved in terms of proper polyhedral divisors and divisorial fans with a Galois semilinear action.

Contents

Chapter 1

Introduction

This thesis has two parts and each part consist in an independent problem.

Jacobian decomposition. Ekedahl and Serre [ES93] gave some examples of smooth projective curves of genus $g \in \mathbb{N}$ having a *completely decomposable* Jacobian, i.e. the Jacobian is isogenous to a product of g elliptic curves. The curve with the biggest genus having this property is 1279, and it was found by Ekedahl and Serre in the same article, leaving many gaps in between. In [ES93], Ekedahl and Serre make two questions concerning the set of genera of smooth projective curves having completely decomposable Jacobians: Does there exist, for every $g \in \mathbb{N}$, a smooth projective curve of genus g whose Jacobian variety is completely decomposable? Is the set of genera having such a property bounded?

The latter question could be answered by studying families of curves with variable genus. For example, Auffarth, Lucchini-Arteche, and Rojas in [ALAR21] explore the case of Humbert-Edge curves. In this thesis, we explore a larger family: generalized Fermat curves. This family includes the Humbert-Edge curves.

We get the following result concerning the decomposition of the jacobian variety of a generalized Fermat curve and its proof corresponds to Theorem 3.2.3, Proposition 3.3.2 and Theorem 3.5.1.

Theorem A. *Let $X_{(n,p)}$ be a generalized Fermat curve of type (n,p) and $E_{(n,p)} \leq \mathrm{Aut}(X_{(n,p)})$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$ with generators $\sigma_0, \ldots, \sigma_n \in E_{(n,p)}$ such that each of the σ_i has a degree p fixed point and $\sigma_0 \cdots \sigma_n = 1$. Denote $S := \{\sigma_0, \ldots, \sigma_n\}$ and, for $T \subset S$, define S_T as the image of S in*

E_T and $\mathcal{H}_T^S(p) := \{H \leq E_T \mid [E_T : H] = p,\, H \cap S_T = \emptyset\}$. Then,

$$J(X_{(n,p)}) \sim \bigoplus_{\substack{T \subset S \\ n-|T| \geq 2}} \bigoplus_{H \in \mathcal{H}_T^S(p)} \pi_H^* J(X_T/H),$$

where $\pi_H : X_{(n,p)} \to X_T/H$ denotes the natural projection. Moreover,

1. $\dim \pi_H^* J(X_T/H) = \frac{(n-|T|-1)(p-1)}{2}$,
2. *the number of factor of dimension $\frac{(n-|T|-1)(p-1)}{2}$ appearing in the decomposition is*

$$\binom{n+1}{|T|} \frac{(p-1)^{n-|T|} - (-1)^{n-|T|}}{p}$$

and

3. *If $p \geq 5$ the subvarieties $\pi_H^* J(X_{(n,p)}/H)$ are not Prym-Tyurin varieties for $X_{(n,p)}$.*

Moreover, for certain generalized Fermat curves, we are able to give a lower bound on the number of elliptic curves appearing in the decomposition above.

Proposition B. *The jacobian of a generalized Fermat curve of type $(n, 3)$ has at least*

$$\frac{n(n+1)(n-1)(3n-4)}{12}$$

elliptic curves.

The latter result corresponds to Proposition 4.1.7.

Normal T-varieties Let k be an algebraically closed field of characteristic zero, Y a normal semiprojective variety over k, and $\omega \subset N_{\mathbb{Q}}$ a pointed cone, where N is a lattice. Denote $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. A proper polyhedral divisor (abbreviated as pp-divisor) is a finite sum

$$\mathfrak{D} := \sum \Delta_D \otimes D,$$

where the Δ_D's are polyhedra in $N_{\mathbb{Q}}$ with tail cone ω, and the D's are irreducible and effective divisors in $\mathrm{CaDiv}_{\mathbb{Q}}(Y)$.

Given a pp-divisor $\mathfrak{D}$, we can associate with it a *piecewise linear map* $\mathfrak{h}_{\mathfrak{D}} : \omega^{\vee} \cap M \to \mathrm{CaDiv}_{\mathbb{Q}}(Y)$. Based on this construction, Altmann and Hausen defined the following k-algebra:

$$A[Y, \mathfrak{D}] := \bigoplus_{m \in \omega^{\vee} \cap M} H^0(Y, \mathscr{O}_Y(\mathfrak{h}_{\mathfrak{D}}(m))) \subset k(Y)[M],$$

and proved that it is finitely generated. Consequently, the scheme $X(\mathfrak{D}) := \mathrm{Spec}(A[Y, \mathfrak{D}])$ is a normal affine variety over k endowed with an effective action of $T := \mathrm{Spec}(k[M])$. Moreover, they showed that every normal affine T-variety arises in this manner.

Theorem 1.0.1. *[AH06, Theorems 3.1 and 3.4] Let k be an algebraically closed field of characteristic zero.*

i) The scheme $X(\mathfrak{D})$ is a normal k-variety with an effective action of $T := \mathrm{Spec}(k[M])$.

ii) Let X be a normal affine k-variety with an effective T-action. Then, there exists a pp-divisor $\mathfrak{D}$ such that $X \cong X(\mathfrak{D})$ as T-varieties.

Nonaffine normal T-varieties are covered by affine T-stable open subvarieties by Sumihiro's Theorem Theorem 6.3.3. Every element of the covering is encoded by a pp-divisor. Similarly as in the toric case, all these pp-divisors can be put together into a *divisorial fan.*

Theorem 1.0.2. *[AHS08, Theorem 5.6] Let k be an algebraically closed field of characteristic zero. Up to equivariant isomorphism, every normal variety endowed with an effective algebraic torus action arises from a divisorial fan on a normal semiprojective variety Y.*

In this work we generalized these results over any field of characteristic zero and for split torus actions by applying the same technique used by Altmann and Hausen. The proof of Theorem C is found in Section 7.2.3 and corresponds to Proposition 7.2.7 and Proposition 7.2.19.

Theorem C. *Let k be a field of characteristic zero.*

i) The scheme $X(\mathfrak{D})$ is a geometrically integral normal variety over k with an effective action of $T := \mathrm{Spec}(k[M])$.

ii) Let X be a geometrically integral normal affine variety over k with an effective action of a split algebraic torus T. Then, there exists a pp-divisor $\mathfrak{D}$ such that $X \cong X(\mathfrak{D})$ as T-varieties.

Moreover, any normal T-variety over k with a split torus action arises from a *divisorial fan* (cf. Definition 7.3.6). Theorem D corresponds to Theorem 7.3.12.

Theorem D. *Let k be a field of characteristic zero and T be a split torus over k. Up to isomorphism, every normal T-variety over k arises from a divisorial fan $(\mathfrak{S}, Y)$ over k.*

To classify normal T-varieties, Galois descent data is formulated using *Galois semilinear equivariant actions*. This approach allows for the classification of normal T-varieties encoded by divisorial fans, with additional conditions ensuring effectiveness. Theorem E is proved in Section 9.8.

Theorem E. *Let k be a field of characteristic zero and L/k be a finite Galois extension with Galois group Γ.*

a) Let T be a split algebraic torus over L and X be a normal T-variety over L. If there exists a divisorial fan $(\mathfrak{S}_L, Y_L)$ for X admitting a Γ-semilinear action such that

the subvariety $X(\mathfrak{S}(\mathfrak{D}, \Gamma))$ is quasi-projective for every $\mathfrak{D} \in \mathfrak{S}_L$,

then there exists an algebraic torus T' over k and a normal T'-variety X' over k such that $X'_L \cong X$ as T varieties over L.

b) Let T be an algebraic torus over k that splits over L. Let X be a normal variety endowed with an effective T-action over k. Then, there exists a divisorial fan $(\mathfrak{S}_L, Y_L)$ admitting a Γ-semilinear action such that

the subvariety $X(\mathfrak{S}(\mathfrak{D}, \Gamma))$ is quasi-projective for every $\mathfrak{D} \in \mathfrak{S}_L$,

and $X_L \cong X(\mathfrak{S}_L)$ as T_L-varieties.

In the previous theorem $\mathfrak{S}(\mathfrak{D}, \Gamma)$ stands for the *sub divisorial fan generated generated by $\mathfrak{D}$ and* Γ, cf. Definition 9.7.9. Besides, the variety $X(\mathfrak{S}(\mathfrak{D}, \Gamma))$ correspond then to the Galois orbit of the affine variety $X(\mathfrak{D})$.

Structure of the thesis

We give a brief overview in the contents of this thesis.

Part I is concerned with the first problem, decomposition of Jacobians of generalized Fermat curves, which is divided into three chapters.

In Chapter 2 we start with an introduction, where we give the context of the problem and its state of the art. In the following sections, we give some preliminaries on jacobian varieties, Prym-Tyurin varieties, generalized Fermat curves, and jacobians of Humbert-Edge curves. In the last section, we present the main results of this part of the work.

Chapter 3 is devoted to prove Theorem A. The first two sections lead to the proof of the decomposition part of Theorem A, and we compute the dimension of each factor in such decomposition. In Section 3.3, we compute the number of factors having the same dimension that appear in the decomposition. In Section 3.5, we prove the remaining part of Theorem A, i.e. we prove that when $p \geq 5$ is a prime number, none of the factors of the decomposition stated in Theorem 2.5.1 is a Prym-Tyurin variety.

In Chapter 4 we prove Proposition B. Besides, we give some examples of generalized Fermat curves having completely decomposable jacobian.

Part II is concerned with the second problem, normal varieties with torus action. This part is divided into six chapters.

In Chapter 5 we give a historical introduction, where we also talk about the recent works around this theory. This chapter finishes with the main results of this part of this work.

The preliminaries are given in Chapter 6. We recall the definition of algebraic torus, we give some preliminaries on convex geometry (*cones*, *fans* and *polyhedra*) and we talk about toric varieties.

Chapter 7 is devoted to prove Theorem C and Theorem D. We start by recalling the definition of proper polyhedral divisors. As well as of other definitions related to proper polyhedral divisors and some properties. In Section 7.2, we prove Theorem C and we prove Theorem D in Section 7.3.

In Chapter 8 we study the functoriality of the Altmann-Hausen construction and we explore *semilinear* morphisms.

Chapter 9 is devoted to prove Theorem D. In Section 9.5, we establish a correspondence between equivariant automorphism of finite order of a normal variety endowed with an effective action of a torus and a family of morphism of proper polyhedral divisors of a divisorial fan for the variety. Using this correspondence we give a definition of G-stable divisorial fans, for G a finite group.

In Section 9.8, we prove Theorem E and show how Theorem 5.0.4 and Huruguen's Theorem are obtained from it.

In Chapter 10, we study the complexity one case and give some applications of Theorem E.

Part III is devoted to present a series of research projects that could be followed after this thesis. These projects concern both problems of this work.

Part I

Decomposition of jacobians of generalized Fermat curves

Chapter 2

Introduction and preliminaries

Throughout this part of the thesis we work over the complex numbers.

2.1 Ekedahl-Serre questions

Let C be a smooth projective curve. The jacobian variety $J(C)$ of C is a principally polarized abelian variety, equipped with a canonical morphism $X \to J(X)$ that can be defined by a universal property as follows. The jacobian variety $J(C)$ of a smooth curve C is called completely decomposable if it is isogenous to a product of elliptic curves. Ekedahl and Serre were interested in knowing whether, for every $g \in \mathbb{N}$, there exists a curve C of genus g whose jacobian variety $J(C)$ is completely decomposable. In [ES93], the authors gave several examples of such curves, but only for certain values of g, all of them bounded by 1279 and leaving several gaps in between. Some of these gaps have been filled by several authors by considering the action of the group of automorphisms of the curve C on $J(C)$. This method is known as the "group algebra decomposition" of $J(C)$. The most recent results in this direction were achieved by Paulhus and Rojas in [PR17]. The problem of decomposing jacobians of curves with group actions has been a matter of interest before the work of Ekedahl and Serre. Examples of this are the results by Kani and Rosen in [KR89] and the work of Aoki in [Aok91].

The answer to the question of Ekedahl and Serre remains elusive and many mathematicians have been involved in it. For example, in [RCR20], Reyes-Carocca and Rodriguez prove that, if $\pi : C \to C'$ is a regular covering map, the group algebra decomposition of $J(C)$ can be lifted to $J(C')$ in an equivariant way. Another example, in [CLR21], Carocca, Lange and Rodriguez study a weaker question and prove that given any positive integer N, there exists a smooth projective curve X whose jacobian variety $J(C)$ is isogenous to the product of $m \geq N$ jacobian varieties of the same dimension. In [CLZ17], Chen, Lu and Zuo reformulate the question in terms of a problem by Coleman and Oort related to Shimura varieties.

A natural way of trying to answer the Ekedahl-Serre questions is studying families of curves with variable genera. This strategy could lead us to find a non bounded set of genera admitting a smooth projective curve of such a genus with a completely decomposable jacobian. For example, Barraza and Rojas in [BR15] obtained the group algebra decomposition of the jacobian variety of Fermat curves. In other example, Auffarth, Lucchini-Arteche and Rojas in [ALAR21] give a decomposition of the jacobian variety of Humbert-Edge curves, of which we give more details here in Section 2.4.

2.2 Jacobians and Prym-Tyurin varieties

Let V be a complex vector space of dimension g and $\Lambda \subset V$ be a lattice of rank $2g$. The quotient V/Λ is called a *complex torus* and its dimension is g. By [BL04, Corollary A.7], a complex torus is a connected complex manifold. Besides, given that Λ is of rank $2g$, V/Λ is compact. Notice that a complex torus has a natural structure of abelian group.

Let k be an algebraically closed field. An abelian variety is a connected smooth projective algebraic group. Over the complex numbers, abelian varieties can be characterized as complex tori admitting a polarization, i.e. a closed immersion to some projective space. A pair (A, L), where A is an abelian variety and L is a polarization, is called a *polarized abelian variety*. Abelian varieties and polarized abelian varieties are compared under *isogenies*, morphisms of abelian varieties that are surjective and have finite kernel.

An important family within polarized abelian varieties are *jacobian varieties*.

2.2.1 Jacobian varieties

Given a smooth projective curve C, its jacobian variety $(J(C), \Theta)$ is a principally polarized abelian variety with a canonical mophism $C \to J(C)$ satisfying the following universal property: if $C \to A$ is a morphism to an abelian variety, then there exists a unique factorization

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ J(C) \arrow[rr] & & A. \end{tikzcd}$$

From the universal property, for any morphism of smooth projective curves $\pi : C \to C'$ we have a canonical map $\pi^* : J(C') \to J(C)$, which is also a morphism of polarized abelian varieties. We can understand the image of this morphism when $C' = C/G$, where $G \leq \mathrm{Aut}(C)$.

Lemma 2.2.1. *Let C be a projective curve and $G \leq \mathrm{Aut}(C)$ acting on C in the obvious way and $\pi : C \to C/G$ the natural projection. Then*

$$\pi^* J(C/G) = (J(C)^G)^0.$$

Proof. Let $p_G : J(C) \to J(C)$ be given by $p_G(x) := \sum_{g \in G} g(x)$. By [CR06, Proposition 5.2] we have $\mathrm{im}(p_G) = \pi^*(J(C/G))$. The image of p_G is $(J(C)^G)^0$ because it is connected, $p_G|_{J(C)^G}$ is mutiplication by $|G|$, and $\mathrm{im}(p_G) \subset J(C)^G$. Then, $(J(C)^G)^0 = \pi_G^*(J(C/G))$. $\square$

By [BL04, Proposition 11.4.3], the kernel of $\pi^* : J(C/G) \to G$ is nontrivial if and only if π factors through an étale quotient. And in these particular cases, the kernel of $\pi^* : J(C/G) \to G$ can be explicitly computed.

Lemma 2.2.2. *Let C be a smooth projective curve and $G \leq \mathrm{Aut}(C)$ a cyclic finite subgroup acting with no fixed points. If $\pi : C \to C/G$ is the natural projection and $\pi^* : J(C/G) \to J(C)$ the induced morphism between their jacobians, then $\ker(\pi^*) \cong G$.*

Proof. This lemma follows from the proof of [BL04, Proposition 11.4.3]. $\square$

This lemma can be extended to finite abelian groups

Lemma 2.2.3. *Let C be a projective curve and $G \leq \mathrm{Aut}(C)$ a finite subgroup acting with no fixed points. If $\pi : C \to C/G$ is the natural projection, then there exists an injective morphism $\ker(\pi^*) \to G^{ab}$, where G^{ab} denotes the abelianization of G.*

Proof. By the Hochschild-Serre spectral sequence [Mil80, Theorem III.2.20], we deduce the exact sequence

$$0 \longrightarrow H^1(G, \mathbb{C}^*) \longrightarrow H^1(C/G, \mathbb{C}) \longrightarrow H^1(C, \mathbb{C})^G ,$$

which can be written as

$$0 \longrightarrow \mathrm{Hom}(G, \mathbb{C}^*) \longrightarrow \mathrm{Pic}(C/G) \longrightarrow \mathrm{Pic}(C)^G ,$$

where the third arrow is the restriction morphism π^*. And since $J(C) \cong \mathrm{Pic}^0(C)$, we see that $\ker(\pi^*) \subset \mathrm{Hom}(G, \mathbb{C}^*) \cong G^{\mathrm{ab}}$. □

Proposition 2.2.4. *Let C be a projective curve and $G \leq \mathrm{Aut}(C)$ a finite abelian subgroup acting with no fixed points. If $\pi : C \to C/G$ is the natural projection, then $\ker(\pi^*) \cong G$.*

Proof. Since G is an abelian group, $G \cong H_1 \times \cdots \times H_r$ with H_i cyclic. Then, we have the following sequence

$$C \xrightarrow{\pi_1} C/H_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_{r-1}} C/(H_1 \times \cdots \times H_{r-1}) \xrightarrow{\pi_r} C/G ,$$

which induces the next sequence on the jacobians

$$J(C/G) \xrightarrow{\pi_r^*} J(C/(H_1 \times \cdots \times H_{r-1})) \xrightarrow{\pi_{r-1}^*} \cdots \xrightarrow{\pi_2^*} J(C/H_1) \xrightarrow{\pi_1^*} J(C) .$$

Then, by Lemma 2.2.2, $\ker(\pi_i^*) \cong H_i$ for every $i \in \{1, \ldots, n\}$. Therefore, given that $\pi^* = \pi_r^* \circ \cdots \circ \pi_1^*$, it follows that

$$|\ker(\pi^*)| = |\pi_r^* \circ \cdots \circ \pi_1^*| = |H_r| \cdots |H_1| = |G|.$$

Finally, by Lemma 2.2.3, there exists an injective morphism $\ker(\pi^*) \to G$, hence we have $\ker(\pi^*) \cong G$. □

On the decomposability of the jacobian we have the following results.

Proposition 2.2.5. *Let $\pi : C' \to C$ be a Galois covering of smooth projective curves. If $J(C')$ is completely decomposable, then $J(C)$ is completely decomposable.*

Proof. Let us suppose that C' is completely decomposable. Given that $J(C)$ is isogenous to an abelian subvariety of $J(C')$, by the Poincaré Reducibility Theorem, we have that each simple factor appearing in $J(C)$ is an elliptic curve. □

Proposition 2.2.6. *[CHQ16, Lemma 6.1] Let C be a smooth projective curve of genus two. If C admits an automorphism of degree three with four fixed points, then $J(C)$ is completely decomposable.*

2.2.2 Prym-Tyurin varieties

A principally polarized abelian variety (A, Ξ) is said to be a *Prym-Tyurin variety* if there exists a smooth projective curve C such that A is an abelian subvariety of $J(C)$ such that

$$\Theta|_A \equiv e\Xi,$$

where Θ is the canonical polarization of $J(C)$ and e is the *exponent of the Prym-Tyurin*. If the curve C is given, then we say that (A, Ξ) is a *Prym-Tyurin variety for C*.

Proposition 2.2.7. *[BL04, Lemma 12.3.1] Let C and C' be two smooth projective curves of genus ≥ 1. Let $f : C \to C'$ be a morphism of degree n. Denote by (J, Θ) and (J', Θ') respectively the corresponding jacobians. Then,*

$$f^*\Theta \equiv n\Theta'.$$

2.3 Generalized Fermat curves

In this section, we consider the family of *generalized Fermat curves*, which were introduced in [GDHL09]. Over the complex numbers, for n and k in $\mathbb{N}$, a generalized Fermat curve of type (n, k), denoted by $X_{(n,k)}$, is a non-singular irreducible projective algebraic curve such that $\mathrm{Aut}(X_{(n,k)})$ has a subgroup E isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$ and $X_{(n,k)}/E$ is isomorphic to the projective line with exactly $(n+1)$ cone points, each one of order k. This group E is called a generalized Fermat group of type (n, k).

The following results about generalized Fermat curves can be found in [GDHL09].

Proposition 2.3.1. *[GDHL09, Preliminaries] A generalized Fermat curve $X_{(n,k)}$ of type (n, k) is of genus*

$$g_{(n,k)} = \frac{2 + k^{n-1}((n-1)(k-1) - 2)}{2}.$$

Let $S := \{\sigma_0, \dots, \sigma_n\}$ be a set of generators of E such that each σ_i is of degree k and $\sigma_0 \cdot \sigma_1 \cdots \sigma_0 = 0$. Any power of an element σ in S fixes a point in $X_{(n,k)}$. Moreover, they are the only elements in E fixing a point in $X_{(n,k)}$.

Proposition 2.3.2. *[GDHL09, Corollary 2] If $a \in E$ fixes a point of $X_{(n,k)}$, then $a = \sigma^l$ for some $\sigma \in S$ and $l \in \mathbb{N}$.*

An interesting property of generalized Fermat curves is the following: let $X_{(n,k)}$ be a generalized Fermat curve of type (n,k) with E a generalized Fermat group of the same type and $S := \{\sigma_0, \dots, \sigma_n\}$ as above. For $\sigma_i \in S$, the curve $X_{(n,k)}/\langle \sigma_i \rangle$ is a generalized Fermat curves of type $(n-1,k)$, whose respective generalized Fermat group is $E/\langle \sigma_i \rangle$. In general, for $T \subset S$, the quotient $X_T := X_{(n,k)}/\langle T \rangle$ is a generalized Fermat curve of type $(n-|T|,k)$ and $E/\langle T \rangle$ is a generalized Fermat group for it.

2.4 Jacobians of Humbert-Edge curves

A generalized Fermat curve of type $(n,2)$ is a Humbert-Edge curve of type n. For any $T \subset S$, $X_T = X_{(n,2)}/\langle T \rangle$ is a Humbert-Edge curve of type $n-|T|$ and structural group $E/\langle T \rangle$. In such a case, by [FMZ18, Proposition 3.5], there exists a decomposition

$$J(X_T) \sim A \oplus JX_T^-,$$

where A is an abelian subvariety of $J(X_T)$, determined by $S \setminus T$, and JX_T^- is the neutral connected compotent of

$$\{x \in J(X_T) \mid \overline{\sigma_i}(x) = -x \text{ for } \sigma_i \in S \setminus T\},$$

where $\overline{\sigma_i}$ is the image of σ_i in $E/\langle T \rangle$. In [ALAR21], the authors study the jacobian variety of Humbert-Edge curves and give the following decomposition:

Theorem 2.4.1. *[ALAR21, Theorem 2.1, Theorem 4.1] Let X_n be a Humbert-Edge curve of type $n \geq 3$ and genus g_n. Then we have the following decomposition of $J(X_n)$:*

$$\varphi : \bigoplus_{\substack{T \subset S \\ |T| \leq n-3}} \pi_T^* JX_T^- \to J(X_n),$$

where $\pi_T : X_n \to X_T$ is the natural projection and φ is an isogeny. Moreover,

1. *If $n-|T|$ is even, then JX_T^- is trivial; and if $n-|T| = 2m+1$, then $\dim JX_T^- = m$.*
2. *For $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, there are exactly $\binom{n+1}{2m+2}$ summands of dimension m.*

3. *Each factor is a Prym-Tyurin variety of exponent* 2^{n-3} *with respect to* X_n.

4. *The kernel of the isogeny* φ *is of order* $(2^{n-3})^{g_n}$.

In the following chapters we study which of these properties on Humbert-Edge curves can be extended to generalized Fermat curves of type (n, k), for k a prime number. We give an extension of the decomposition part of 2.4.1 for such curves, cf. Theorem 2.5.1 here below. In contrast, we prove that part (3) of Theorem 2.4.1 does not hold for $k \geq 5$ a prime number. Indeed, we prove that in such a case none of the factors in the decomposition is a Prym-Tyurin variety.

A decomposition comparable to the one we give in Theorem 2.5.1 was presented by Carvacho, Hidalgo and Quispe in [CHQ16, Theorem 4.4]. We compare these two results below.

2.5 Main results

Let p be a prime number, let $X_{(n,p)}$ be a generalized Fermat curve of type (n, p) and let $E_{(n,p)} \leq \mathrm{Aut}(X_{(n,p)})$ be isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$ with generators $\sigma_0, \cdots, \sigma_n \in E_{(n,p)}$ such that $\sigma_0 \cdots \sigma_n = 1$ and each of the σ_i fixes a point of degree p. Denote $S := \{\sigma_0, \ldots, \sigma_n\}$. Recall that a generalized Fermat curve $X_{(n,p)}$ of type (n, p) is of genus

$$g_{(n,p)} = \frac{2 + p^{n-1}((n-1)(p-1) - 2)}{2}$$

and the quotient of $X_{(n,p)}$ by any σ_i is a generalized Fermat curve of type $(n-1, p)$ whose structural group is $E_{(n,p)}/\langle\sigma_i\rangle$. There are natural morphisms

$$\pi_i : X_{(n,p)} \to (X_{(n,p)}/\langle\sigma_i\rangle) \qquad \text{and} \qquad \pi_i^* : J(X_{(n,p)}/\langle\sigma_i\rangle) \to J(X_{(n,p)}).$$

In general, if $T \subset S$, $X_T := X_{(n,p)}/\langle T\rangle$ is a generalized Fermat curve of type $(n - |T|, p)$ and structural group $E_T := E_{(n,p)}/\langle T\rangle$, with natural morphisms

$$\pi_T : X_{(n,p)} \to X_T \qquad \text{and} \qquad \pi_T^* : J(X_T) \to J(X_{(n,p)}).$$

The main result of this part of this thesis generalizes the decomposition of [ALAR21] and proves that none of the factors of the decomposition is a Prym-Tyurin variety for $p \geq 5$, concluding that $p = 2$ is a particular case.

Theorem 2.5.1. *Let $X_{(n,p)}$ be a generalized Fermat curve of type (n,p) and $E_{(n,p)} \leq \mathrm{Aut}(X_{(n,p)})$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$ with generators $\sigma_0, \ldots, \sigma_n \in E_{(n,p)}$ such that each of the σ_i has a degree p fixed point and $\sigma_0 \cdots \sigma_n = 1$. Denote $S := \{\sigma_0, \ldots, \sigma_n\}$ and, for $T \subset S$, define S_T as the image of S in E_T and $\mathcal{H}_T^S(p) := \{H \leq E_T \mid [E_T : H] = p,\, H \cap S_T = \emptyset\}$. Then,*

$$J(X_{(n,p)}) \sim \bigoplus_{\substack{T \subset S \\ n-|T| \geq 2}} \bigoplus_{H \in \mathcal{H}_T^S(p)} \pi_H^* J(X_T/H),$$

where $\pi_H : X_{(n,p)} \to X_T/H$ denotes the natural projection. Moreover,

1. $\dim \pi_H^* J(X_T/H) = \frac{(n-|T|-1)(p-1)}{2}$,
2. *the number of factor of dimension $\frac{(n-|T|-1)(p-1)}{2}$ appearing in the decomposition is*
$$\binom{n+1}{|T|} \frac{(p-1)^{n-|T|} - (-1)^{n-|T|}}{p}$$
and
3. *If $p \geq 5$ the subvarieties $\pi_H^* J(X_{(n,p)}/H)$ are not Prym-Tyurin varieties for $X_{(n,p)}$.*

In the case of $n = 2$, Theorem 2.5.1 gives a decomposition of the jacobian of a classical Fermat curve. Another decomposition of this particular jacobian was given in [BR15, Theorem 4.5].

Concerning the decomposition given in [CHQ16, Theorem 4.4], their result gives an explicit description of the curves X_T/H of Theorem 2.5.1. However, it is not clear from the statement how many factors of fixed dimension are and how they are related to the subgroups of $E_{(n,k)}$. Moreover, our approach is recursive and group theoretical, which gives a different insight on this question, complementing the more analytic approach from [CHQ16].

As an application, we can give a lower bound on the number of elliptic curves appearing in the decomposition of the jacobian variety of a generalized Fermat of type $(n, 3)$.

Proposition 2.5.2. *The jacobian of a generalized Fermat curve of type $(n,3)$ has at least*

$$\frac{n(n+1)(n-1)(3n-4)}{12}$$

elliptic curves.

Chapter 3

Decomposition of jacobians of generalized Fermat curves

In the first part of this chapter, we give a decomposition of generalized Fermat curve of type (n,k), with $n,k \in \mathbb{N}$ and $n \geq 3$ and $k \geq 2$. In the second part, we focus on generalized Fermat curves of type (n,p) with p a prime number, which is given by étale quotient of generalized Fermat curves. The last part of this chapter compares the results about Humbert-Edge curves in [ALAR21] and Theorem 2.5.1.

3.1 A decomposition of the Jacobian of generalized Fermat curves of type (n,k)

Let $k \geq 2$, let $X_{(n,k)}$ be a generalized Fermat curve and $E_{(n,k)} \leq \mathrm{Aut}(X_{(n,k)})$ isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$, with generators $\sigma_0, \dots, \sigma_n \in E_{(n,k)}$ such that $\sigma_0 = \sigma_1^{-1} \cdots \sigma_n^{-1}$ and each of the σ_i fixes a point of degree k. The group $E_{(n,k)}$ acts naturally on the jacobian of the curve and, therefore, on the tangent space $T_0(J(X_{(n,k)}))$. This induced action of $E_{(n,k)}$ on $T_0(J(X_{(n,k)}))$ gives the following decomposition

$$T_0(J(X_{(n,k)})) = \bigoplus_{\chi \in \mathcal{C}} V_\chi, \tag{3.1}$$

where $\mathcal{C}$ denotes the set of characters of $E_{(n,k)}$. More precisely, for $\chi \in \mathcal{C}$ we have

$$V_\chi := \{v \in T_0(J(X_{(n,k)})) \mid \sigma \cdot v = \chi(\sigma) v \text{ for all } \sigma \in E_{(n,k)}\},$$

and, for $v \in V$, we have $v = \sum_{\chi \in \mathcal{C}} v_\chi$ with

$$v_\chi := \frac{1}{|E_{(n,k)}|} \sum_{\sigma \in E_{(n,k)}} \chi(\sigma)^{-1} \sigma \cdot v \in V_\chi.$$

This decomposition is closely related with the index d subgroups of $E_{(n,k)}$, for d dividing k, as a consequence of the following lemma.

Lemma 3.1.1. *Let $\chi : E_{(n,k)} \to \mu_k$ be a character and $H := \ker(\chi)$. Then*

$$\bigoplus_{\substack{\chi \in \mathcal{C} \\ \ker \chi = H}} V_\chi = T_0(J(X_{(n,k)}))^H.$$

Proof. Let us denote

$$W := \bigoplus_{\substack{\chi \in \mathcal{C} \\ \ker \chi = H}} V_\chi.$$

On the one hand, if $v \in W$ and $\sigma \in H$, then

$$\sigma \cdot v = \sum_{\substack{\chi \in \mathcal{C} \\ \ker \chi = H}} \chi(\sigma) v_\chi = \sum_{\substack{\chi \in \mathcal{C} \\ \ker \chi = H}} v_\chi = v.$$

Thus, $v \in T_0(J(X_{(n,k)}))^H$.

On the other hand, suppose that $v \in T_0(J(X_{(n,k)}))^H$. We have

$$v = \sum_{\chi \in \mathcal{C}} v_\chi = \frac{1}{|E_{(n,k)}|} \sum_{\chi \in \mathcal{C}} \sum_{\sigma \in E_{(n,k)}} \chi(\sigma)^{-1} \sigma \cdot v.$$

Let $\chi \in \mathcal{C}$ be a character such that $\ker \chi \neq H$ and $\tau \in H$ such that $\chi(\tau) \neq 1$. Due to $\tau \cdot v = v$, we have

$$\tau \cdot \sum_{\sigma \in E_{(n,k)}} \frac{1}{|E_{(n,k)}| \chi(\sigma)} \sigma \cdot v = \sum_{\sigma \in E_{(n,k)}} \frac{1}{|E_{(n,k)}| \chi(\sigma)} \sigma \cdot v.$$

Furthermore, as $\tau \cdot v_\chi = \chi(\tau) v_\chi$, we have $(1 - \chi(\tau)) v_\chi = 0$, and hence $v_\chi = 0$ for all $\chi \in \mathcal{C}$ such that $\ker \chi \neq H$. Thus, $v \in W$ and therefore

$$\bigoplus_{\substack{\chi \in \mathcal{C} \\ \ker \chi = H}} V_\chi = W = T_0(J(X_{(n,p)}))^H.$$

$\square$

A direct consequence of this lemma is that the subspace of $E_{(n,k)}$-invariant vectors is the trivial one.

Corollary 3.1.2. *Let $\chi_0 : E_{(n,k)} \to \mu_k$ be the trivial character. Then, the weight space V_0 associated to χ_0 is trivial, i.e.*

$$V_0 = \{0\}.$$

Proof. By Lemma 3.1.1 it follows

$$V_0 = \bigoplus_{\substack{\chi \in \mathcal{C} \\ \ker \chi = E_{(n,k)}}} V_\chi = T_0(J(X_{(n,k)}))^{E_{(n,k)}}.$$

Given that $T_0((J(X_{(n,k)})^{E_{(n,k)}})^0) = T_0(J(X_{(n,k)}))^{E_{(n,k)}}$, we have

$$V_0 = T_0((J(X_{(n,k)})^{E_{(n,k)}})^0).$$

By Lemma 2.2.1, $T_0((J(X_{(n,k)})^{E_{(n,k)}})^0) = T_0(\pi_{E_{(n,k)}} J(X_{(n,k)}/E_{(n,k)}))$ and by definition $X_{(n,k)}/E_{(n,k)} \cong \mathbb{P}^1$. Thus,

$$T_0(\pi_{E_{(n,k)}} J(X_{(n,k)}/E_{(n,k)})) = T_0(J(\mathbb{P}^1)) = \{0\}$$

and therefore $V_0 = \{0\}$. □

The decomposition in (3.1) for the tangent space $T_0(J(X_{(n,k)}))$ descends to a decomposition of the jacobian variety $J(X_{(n,k)})$ of the generalized Fermat curve into abelian subvarieties. Such a decomposition is parametrized by all the subgroups $H \leq E_{(n,d)}$ of index d with $d|k$ and $d > 1$.

Proposition 3.1.3. *Let $X_{(n,k)}$ be a generalized Fermat curve of type (n,k) and $E_{(n,k)}$ as above. Denote $\mathcal{H}(d) := \{H \leq E_{(n,k)} \mid \left[H : E_{(n,k)}\right] = d\}$. Then,*

$$J(X_{(n,k)}) \sim \bigoplus_{\substack{d|k,\, d>1 \\ H \in \mathcal{H}(d)}} \pi_H^*(J(X_{(n,k)}/H)),$$

where $\pi_H : X_{(n,k)} \to \left(X_{(n,k)}/H\right)$ denotes the natural projection.

Proof. By Lemma 3.1.1 and Corollary 3.1.2 we have

$$T_0(J(X_{(n,k)})) = \bigoplus_{\substack{d|k,\, 1<d \\ H \in \mathcal{H}(d)}} T_0(J(X_{(n,k)}))^H,$$

because all the factors in the decomposition given by Lemma 3.1.1 are parametrized by the index $d|k$ subgroups of $E_{(n,k)}$.

Let $H \in \mathcal{H}(d)$ and $\pi_H : X_{(n,k)} \to \big(X_{(n,k)}/H\big)$ be the natural projection. Given that $T_0((J(X_{(n,k)})^H)^0) = T_0(J(X_{(n,k)}))^H$ we have

$$T_0(J(X_{(n,k)})) = \bigoplus_{\substack{d|k,\, 1<d \\ H\in\mathcal{H}(d)}} T_0((J(X_{(n,k)})^H)^0)$$

and, by Lemma 2.2.1,

$$T_0(J(X_{(n,k)})) = \bigoplus_{\substack{d|k,\, 1<d \\ H\in\mathcal{H}(d)}} T_0(\pi_H^* J(X_{(n,k)}/H)).$$

Finally, since the $\pi_H^*(J(X_{(n,k)}))$ are abelian subvarieties, the proposition holds. □

If k is a prime number, the factors have a nicer behavior. In such a case, the factors are given by the jacobians of étale quotients of the curve.

3.2 Decomposition by jacobians of étale quotients

Let p be a prime number, let $X_{(n,p)}$ be a generalized Fermat curve of type (n,p) and $E_{(n,p)} \leq \mathrm{Aut}(X_{(n,p)})$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$ with generators $\sigma_0, \ldots, \sigma_n \in E_{(n,p)}$ such that $\sigma_0 \cdots \sigma_n = 1$ and each of them fixes a point of degree p. Denote $S := \{\sigma_0, \ldots, \sigma_n\}$. According to Proposition 3.1.3, the tangent space $T_0(J(X_{(n,p)}))$ can be written as a direct sum of abelian subvarieties $\pi_H^* J(X_{(n,p)}/H)$, where $H \in \mathcal{H}(p)$ and $\pi_H : X_{(n,p)} \to X_{(n,p)}/H$ is the natural projection. If $T \subset S$, then $X_T := X_{(n,p)}/\langle T\rangle$ is also a generalized Fermat curve with structural group $E_T := E_{(n,p)}/\langle T\rangle$. Therefore, we can define $\mathcal{H}_T(p)$ for all $T \subset S$ as

$$\mathcal{H}_T(p) := \{H \leq E_T \mid [E_T : H] = p\},$$

and the subset

$$\mathcal{H}_T^S(p) := \{H \leq E_T \mid [E_T : H] = p \quad \text{and} \quad H \cap S_T = \emptyset\},$$

where S_T is the image of S on $E_{(n,p)}/\langle T\rangle$.

For $T \subset S$ and $H \in \mathcal{H}_T^S(p)$, is easy to compute the dimension of the quotient X_T/H using the Riemann-Hurwitz formula and the fact that the quotient X_T/H is étale since all the ramification is condensed in the $\overline{\sigma_i} \in S_T$.

Proposition 3.2.1. *If $H \in \mathcal{H}_T^S(p)$, then X_T/H is of genus $\frac{(n-|T|-1)(p-1)}{2}$.*

Proof. Let $g = \dim J(X_T/H)$ be the genus of X_T/H. By the Riemann-Hurwitz formula and Proposition 2.3.2 applied to $\pi_H : X_T \to (X_T/H)$ we have

$$2 - 2g_{(n-|T|-1,p)} = p^{n-|T|-1}(2-2g),$$

where $g_{(n-|T|-1,p)}$ is the genus of X_T, as stated in (2.3.1). Then, by a direct computation, we have that X_T/H has genus $g = \frac{(n-|T|-1)(p-1)}{2}$. □

Lemma 3.2.2. *Let $T \subset S$. If $H \in \mathcal{H}_T(p)$, then $|H \cap S_T| < n - |T|$.*

Proof. Let $H \in \mathcal{H}_T(p)$ and let us suppose that $|H \cap S_T| \geq n - |T|$. Given that S generates $E_{(n,p)}$ with n of its elements, it follows that S_T generates $E_T := E_{(n,p)}/\langle T\rangle$ with $n-|T|$ of its elements. This implies that $\langle H \cap S_T\rangle = E_T$, so $E_T \leq H$. This is a contradiction, because H is a proper subgroup of E_T. □

The following results correspond to the part (1) of Theorem 2.5.1.

Theorem 3.2.3. *Let $X_{(n,p)}$ be a generalized Fermat curve of type (n,p) and $E_{(n,p)}$ as above. Then, with the notation above,*

$$J(X_{(n,p)}) \sim \bigoplus_{\substack{T\subset S\\ n-|T|\geq 2}} \bigoplus_{H\in\mathcal{H}_T^S(p)} \pi_H^* J(X_T/H),$$

where $\pi_H : X_{(n,p)} \to X_T/H$ denotes the natural projection, and $\pi_H^* J(X_T/H)$ is of dimension $\frac{(n-|T|-1)(p-1)}{2}$.

Proof. Let $n \in \mathbb{N}$ such that $n \geq 2$. By Proposition 3.1.3,

$$J(X_{(n,p)}) \sim \bigoplus_{H\in\mathcal{H}(p)} \pi_H^*(J(X_{(n,p)}/H)).$$

This can be split into summands of the form

$$J_k := \bigoplus_{\substack{H\in\mathcal{H}(p)\\ |H\cap S|=k}} \pi_H^* J(X_{(n,p)}/H),$$

for $k \in \{0, \ldots, n-1\}$, by Lemma 3.2.2. Then, denoting by $T := H \cap S$, we have $X_{(n,p)}/H \cong X_T/(H/\langle T\rangle)$ with $H/\langle T\rangle \leq E_T$ and $[E_T : H/\langle T\rangle] = p$. This implies that

$$J_k = \bigoplus_{\substack{T\subset S,\, |T|=k\\ H\in\mathcal{H}_T^S(p)}} \pi_H^* J(X_T/H).$$

Then,

$$J(X_{(n,p)}) \sim \bigoplus_{\substack{T\subset S,\ n-|T|\geq 1\\ H\in\mathcal{H}_T^S(p)}} \pi_H^* J(X_T/H).$$

By Proposition 3.2.1, the quotient X_T/H is of genus $\frac{(n-|T|-1)(p-1)}{2}$ and, therefore, $\pi_H^* J(X_T/H)$ is of dimension $\frac{(n-|T|-1)(p-1)}{2}$. Notice that, when $n-|T|=1$, the subvariety $\pi_H^* J(X_T/H)$ is of dimension 0. Thus,

$$J(X_{(n,p)}) \sim \bigoplus_{\substack{T\subset S,\ n-|T|\geq 2\\ H\in\mathcal{H}_T^S(p)}} \pi_H^* J(X_T/H).$$

$\square$

3.3 On the number of factors of each dimension

In the following, we compute how many factors having the same dimension appear in the decomposition of the jacobian of some generalized Fermat curves.

Notice that, by definition, $\left|\mathcal{H}_T^S(p)\right| = \left|\mathcal{H}_{T'}^{S'}(p)\right|$ if $|S|-|T| = |S'|-|T'|$. In particular, if two subsets T and T' of S have the same cardinality, then $\left|\mathcal{H}_T^S(p)\right| = \left|\mathcal{H}_{T'}^S(p)\right|$. Therefore, we write $\left|\mathcal{H}_{|T|}^S(p)\right|$ when we do not need to specify the subset we are considering.

In order to avoid confusions, we denote by S_n the set S when the generalized Fermat curve is of type (n,p).

By Theorem 3.2.3, for a generalized Fermat curve of type (n,p), we have the following relation:

$$\sum_{|T|=0}^{n-2} \binom{n+1}{|T|} \left|\mathcal{H}_{|T|}^{S_n}(p)\right| \left(\frac{(n-|T|-1)(p-1)}{2}\right) = g_{(n,p)}, \tag{3.2}$$

where $\binom{n+1}{|T|}\left|\mathcal{H}_{|T|}^{S_n}(p)\right|$ is the number of factors of dimension $\frac{(n-|T|-1)(p-1)}{2}$ appearing in the decomposition. Thus, in order to get the number of factors appearing in the decomposition we need to compute the cardinality of the sets $\mathcal{H}_{|T|}^{S_n}(p)$, which have the same cardinality as the set $\mathcal{H}_0^{S_{n-|T|}}(p)$.

Proposition 3.3.1. *Let $n\in\mathbb{N}$. If $n\geq 2$, then* $\left|\mathcal{H}_0^{S_n}(p)\right| = \dfrac{(p-1)^n-(-1)^n}{p}$.

Proof. By definition

$$\mathcal{H}_0^{S_n}(p) = \{H \leq E \mid [E : H] = p \text{ and } H \cap S = \emptyset\}.$$

Notice that, given that E is a $\mathbb{F}_p$-vector space of dimension n, and the elements $\sigma_1, \ldots, \sigma_n$ form a basis for E, the cardinality h_n of $\mathcal{H}_0^{S_n}(p)$ is equal to the number of hyperplanes in E that do not contain any of the lines $\mathbb{F}_p\sigma_i$, with $i \in \{0, \ldots, n\}$. Dualizing the problem, we need to determine the number of lines that are not orthogonal to any $\mathbb{F}_p\sigma_i$, with $i \in \{0, \ldots, n\}$. Denote by $H_{\sigma_i} = \{f \in E^* \mid f(\sigma_i) = 0\}$. Thus,

$$h_n = \frac{1}{p-1}\left(|E^*| - \left|\bigcup_{i=0}^{n} H_{\sigma_i}\right|\right).$$

Notice that, for every proper subset T of S, the elements of T are linearly independent. This implies that

$$\left|\bigcap_{\sigma_i \in T} H_{\sigma_i}\right| = p^{n-|T|},$$

for any $T \subset S$ a proper subset, otherwise the cardinality is 1. Thus, given that $E \cong E^*$, we have that

$$\begin{aligned} h_n &= \frac{1}{p-1}\left(p^n - \sum_{i=1}^{n+1}(-1)^{i-1}\sum_{\substack{T\subset S\\ |T|=i}}\left|\bigcap_{\sigma_i\in T} H_{\sigma_i}\right|\right) \\ &= \frac{1}{p-1}\left(p^n + \sum_{i=1}^{n}(-1)^i\binom{n+1}{i}p^{n-i} + (-1)^{n+1}\binom{n+1}{n+1}\right) \end{aligned}$$

Thus,

$$h_n = \frac{1}{p-1}\left(\sum_{i=0}^{n}(-1)^i\binom{n+1}{i}p^{n-i} + (-1)^{n+1}\right).$$

This implies that

$$\begin{aligned} p(p-1)h_n - (p-1)(-1)^{n+1} &= \sum_{i=0}^{n}(-1)^i\binom{n+1}{i}p^{n+1-i} + (-1)^{n+1}\binom{n+1}{n+1} \\ &= \sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}p^{n+1-i} = (p-1)^{n+1}. \end{aligned}$$

Then, $p(p-1)h_n = (p-1)^{n+1} + (p-1)(-1)^{n+1}$ and therefore

$$h_n = \frac{(p-1)^n - (-1)^n}{p}.$$

□

Then we have following result, which corresponds to part (2) of Theorem 2.5.1

Proposition 3.3.2. *Let $X_{(n,p)}$ be a generalized Fermat curve of type (n,p). The number or factors of dimension $\frac{(n-|T|-1)(p-1)}{2}$ appearing in the decomposition of Theorem 3.2.3 is*

$$\binom{n+1}{|T|}\frac{(p-1)^{n-|T|}-(-1)^{n-|T|}}{p}.$$

Thus, we have the following consequences.

Corollary 3.3.3. *Let p be a prime number. In a generalized Fermat curve of type $(3,p)$ there are*

- *$4(p-2)$ factors of dimension $(p-1)/2$ and*
- *(p^2-3p+3) factors of dimension $p-1$.*

Proof. By Proposition 3.3.2, the number of factors of dimension $(p-1)/2$ is

$$\binom{4}{1}\frac{(p-1)^2-(-1)^2}{p} = 4\frac{p^2-2p}{p} = 4(p-2)$$

and the number of factors of dimension $p-1$ is

$$\binom{4}{0}\frac{(p-1)^3-(-1)^3}{p} = \frac{p^3-3p^2+3p}{p} = p^2-3p+3.$$

□

Corollary 3.3.4. *Let p be a prime number. In a generalized Fermat curve of type $(4,p)$ there are*

- *$10(p-2)$ factors of dimension $(p-1)/2$,*
- *$5(p^2-3p+3)$ factors of dimension $(p-1)$ and*
- *(p^3-4p^2+6p-4) factors of dimension $3(p-1)/2$.*

Proof. By Proposition 3.3.2, the number of factors of dimension $(p-1)/2$ is

$$\binom{5}{2}\frac{(p-1)^2-(-1)^2}{p} = 10\frac{p^2-2p}{p} = 10(p-2),$$

the number of factors of dimension $p-1$ is

$$\binom{5}{1}\frac{(p-1)^3-(-1)^3}{p} = 5\frac{p^3-3p^2+3p}{p} = 5(p^2-3p+3)$$

and the number of factors of dimension $3(p-1)/2$ is

$$\binom{5}{0}\frac{(p-1)^4-(-1)^4}{p} = \frac{p^4-4p^3+6p^2-4p}{p} = p^3-4p^2+6p-4.$$

□

3.4 About Humbert-Edge curves

A generalized Fermat curve of type $(n,2)$ is a Humbert-Edge curve of type n and, by [FMZ18, Proposition 3.5] and [ALAR21, Lemma 4.2], its jacobian has an isogenous decomposition

$$J(X_{(n,2)}) \sim A \oplus \pi^*_{H_{(n,2)}} J\left(X_{(n,2)}/H_{(n,2)}\right), \tag{3.3}$$

where

$$A = \sum_{i=0}^{n} \pi_i^* J\left(X_{(n,2)}/\langle\sigma_i\rangle\right),$$

and

$$H_{(n,2)} = \langle \sigma_i\sigma_0^{-1} \mid 1 \le i \le n\rangle \le E_{(n,2)}.$$

The morphisms $\pi_i : X_{(n,2)} \to \left(X_{(n,2)}/\langle\sigma_i\rangle\right)$ and $\pi_{H_{(n,2)}} : X_{(n,2)} \to \left(X_{(n,2)}/H_{(n,2)}\right)$ are the natural projections. By [ALAR21, Lemma 3.1] (see also [FMZ18, Remark 3.6]), the image $\pi^*_{H_{(n,p)}} J(X_{(n,2)}/H_{(n,2)})$ is trivial when n is even and is not trivial when n is odd. The decomposition in Proposition 3.1.3 refines (3.3) in the following sense: the first factor in (3.3) satisfies

$$A = \bigoplus_{\substack{H \le E_{(n,2)} \\ [E_{(n,2)}:H]=2 \\ H\cap\{\sigma_0,\ldots,\sigma_n\}\neq\emptyset}} \pi^*_H J(X_{(n,2)}/H)$$

and the other one

$$\pi^*_{H_{(n,2)}} J(X_{(n,2)}/H_{(n,2)}) = \bigoplus_{\substack{H \le E_{(n,2)} \\ [E_{(n,2)}:H]=2 \\ H \cap \{\sigma_0,\ldots,\sigma_n\}=\emptyset}} \pi^*_H J(X_{(n,2)}/H),$$

because, in the case of a Humbert-Edge curve, there is at most one subgroup of index 2 such that $H \cap \{\sigma_0, \ldots, \sigma_n\} = \emptyset$, and it appears just in the case when n is odd. This is compatible with Proposition 3.3.2

3.5 On the factors of the decomposition

Let p be a prime number. Recall that, when $p = 2$, it was proved in [ALAR21] that the jacobian variety of $X_{(n,2)}$ is a direct sum of Prym-Tyurin varieties. However, this property does not hold for generalized Fermat curves of type (n, p) with $p \ge 5$.

Let $X_{(n,p)}$ be a generalized Fermat curve of type (n, p) and $E_{(n,p)} \le \mathrm{Aut}(X_{(n,p)})$ its structural group, which is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$. Let $S := \{\sigma_0, \sigma_1, \ldots, \sigma_n\} \subset E_{(n,p)}$ be a set of generators, with $\sigma_0 \cdots \sigma_n = 1$ and each of them with p fixed points. For $T \subset S$, we define $X_T := X_{(n,p)}/\langle T\rangle$, $E_T := E_{(n,p)}/\langle T\rangle$ and the set

$$\mathcal{H}^S_T(p) := \{H \le E_T \mid [E_T : H] = p \text{ and } H \cap S_T = \emptyset\}$$

as in Section 3.2. Furthermore, we have the morphisms $\pi_H : X_{(n,p)} \to X_T/H$ given by the compositions of the natural projections $X_{(n,p)} \to X_T$ and $X_T \to X_T/H$.

Theorem 3.5.1. *Let $X_{(n,p)}$ be a generalized Fermat curve of type (n, p), with p a prime number, and $T \subset S$. If $p \ge 5$, then for $H \in \mathcal{H}^S_T(p)$ the abelian subvariety $\pi^*_H J(X_T/H)$ is not a Prym-Tyurin variety for $X_{(n,p)}$.*

This result is a consequence of Proposition 3.5.5 here below and corresponds to part (3) of Theorem 2.5.1. In order to prove it, we need the following results.

Lemma 3.5.2. *Let A and B be two abelian varieties of dimension g, $f : A \to B$ an isogeny of prime exponent p and Θ a principal polarization of B. If $f^*\Theta$ is a multiple of a principal polarization of A then $|\ker(f)| \in \{p^g, p^{2g}\}$.*

Proof. Let $f : A \to B$ be an isogeny and suppose that $f^*\Theta = k\Psi$, where Ψ is a principal polarization of A and $k \in \mathbb{Z}$. On the one hand, there exists an isogeny $\tilde{f} : B \to A$ such that $f \circ \tilde{f} = p$ and $\deg(f)\deg(\tilde{f}) = p^{2g}$. On the other hand, by [BL04, Corollary 3.6.2] and [BL04, Corollary 3.6.6], it follows that $\deg(f) = k^g$. This implies that $k^g \deg(\tilde{f}) = p^{2g}$ and, as p is a prime number, the result holds. $\square$

Proposition 3.5.3. *Let C be a projective curve, $G \leq \mathrm{Aut}(C)$ an abelian group of prime exponent p acting with no fixed points and $\pi : C \to C/G$ the natural projection. If g is the genus of C/G and $|G| < p^g$, then $\pi^*J(C/G)$ is not a Prym-Tyurin variety for C.*

Proof. The morphism $\pi^* : J(C/G) \to J(C)$ can be factorized as

$$\begin{tikzcd} \pi^*J(C/G) \arrow[r, "\iota"] & J(C) \\ & J(C/G) \arrow[ul, "f"] \arrow[u, "\pi^*"'] \end{tikzcd} ,$$

where ι is the inclusion of the image of $J(C/G)$ by π^* and f is an isogeny.

Let us suppose that $\pi^*J(C/G)$ is a Prym-Tyurin variety for C, then there exist principal polarizations Θ and Ψ of $J(C)$ and $\pi^*J(C/G)$, respectively, such that $\iota^*\Theta \equiv m\Psi$. On the one hand, $\pi^*\Theta \equiv k\Xi$ with Ξ a principal polarization of $J(C/G)$ by [BL04, Lemma 12.3.1]. On the other hand, $(f \circ \iota)^*\Theta \equiv f^* \circ \iota^*\Theta \equiv f^*(m\Psi) \equiv mf^*\Psi$. Then

$$k\Xi \equiv mf^*\Psi,$$

and therefore $f^*\Psi$ is a multiple of a principal polarization. Then, by Lemma 3.5.2, it follows that $|\ker(f)| \in \{p^g, p^{2g}\}$. However, by Proposition 2.2.4, $|\ker(f)| = |G|$ and, therefore, $|G| \in \{p^g, p^{2g}\}$, which is a contradiction. Finally $\pi^*J(C/G)$ is not a Prym-Tyurin variety for C. $\square$

A direct consecuence of this proposition is

Corollary 3.5.4. *Assume that $p \geq 5$. For any $T \subset S$ and $H \in \mathcal{H}_T^S(p)$, the image of $J(X_T/H)$ in $J(X_T)$ under the natural projection $X_T \to X_T/H$ is not a Prym-Tyurin variety for X_T.*

Proof. Let $T \subset S$ and $H \in \mathcal{H}_T^S(p)$. By Proposition 3.2.1, X_T/H is of genus $\frac{(n-|T|-1)(p-1)}{2}$. Given that $p \geq 5$ we have

$$|H| = p^{(n-|T|-1)} < p^{\frac{(n-|T|-1)(p-1)}{2}},$$

and by Proposition 3.5.3 it follows that $\pi_H^* J(X_T/H)$ is not a Prym-Tyurin variety for X_T. $\square$

Actually, this result is even more general.

Proposition 3.5.5. *Assume that $p \geq 5$. For any $T' \subset T \subset S$ and $H \in \mathcal{H}_T^S(p)$, the image of $J(X_T/H)$ in $J(X_{T'})$ under the natural projection is not a Prym-Tyurin variety for $X_{T'}$.*

Note that Theorem 3.5.1 is a direct consequence of Proposition 3.5.5 for any $T \subset S$ and $T' = \emptyset$.

Proof. Let $T' \subset T \subset S$ and $\pi_H : X_{T'} \to X_T/H$ the morphism given by the composition of the natural projections $X_{T'} \to X_T$ and $p_H : X_T \to X_T/H$. This morphism induces a morphism $\pi_H^* : J(X_T/H) \to J(X_{T'})$, which can be factorized as

$$J(X_T/H) \xrightarrow{f} \pi_H^* J(X_T/H) \xrightarrow{\iota} J(X_{T'}) \, , \tag{3.4}$$

where f is an isogeny and ι is an embedding. Let us suppose $\pi_H^* J(X_T/H)$ is a Prym-Tyurin variety for $X_{T'}$, so for a principal polarization Θ of $J(X_{T'})$ we have $\iota^*\Theta \equiv m\Psi$, with Ψ a principal polarization of $\pi_H^* J(X_T/H)$ and $m \in \mathbb{N}$. By [BL04, Lemma 12.3.1], $\pi_H^*\Theta \equiv l\Xi$, with Ξ a principal polarization of $J(X_T/H)$ and $l \in \mathbb{N}$. Thus,

$$l\Xi \equiv \pi_H^*\Theta \equiv f^* \circ \iota^*\Theta \equiv f^*(m\Psi) \equiv mf^*\Psi.$$

This implies that $f^*\Psi$ is a multiple of a principal polarization. Notice that (3.4) can be also written as

$$J(X_T/H) \xrightarrow{f_T} p_H^* J(X_T/H) \xrightarrow{\iota_1} J(X_T) \xrightarrow{\iota_2} J(X_{T'}) \, ,$$

where ι_1 and ι_2 are injective. The injectivity of $\iota_2 : J(X_T) \to J(X_{T'})$ comes from the fact that ι_2 is a composition of injective morphisms. Indeed, let $\{\sigma_{i_1}, \ldots, \sigma_{i_r}\} = T \setminus T'$. The projections

$$p_r : X_{T'} \to X_{T'}/\langle\sigma_{i_r}\rangle \quad \text{and} \quad p_j : X_{T'}/\langle\sigma_{i_{j+1}}, \ldots, \sigma_{i_r}\rangle \to X_{T'}/\langle\sigma_{i_j}, \ldots, \sigma_{i_r}\rangle, \text{ for } j < r,$$

induce morphisms p_j^* between their jacobians such that $\iota_2 = p_r^* \circ \cdots \circ p_1^*$ and all the p_j^* are injective by [BL04, Proposition 11.4.3] because all the elements in S, and their powers, fix points. Then, $\iota_1 \circ \iota_2$ is injective and $|\ker(f)| =$

$|\ker(f_T)|$. However, $|\ker(f_T)| = |H| = p^{(n-|T|-1)}$, by Proposition 2.2.4, so $|\ker(f)| = p^{(n-|T|-1)}$ and

$$|\ker(f)| = p^{(n-|T|-1)} < p^{\frac{(n-|T|-1)(p-1)}{2}} = p^g,$$

with g the genus of X_T/H by Proposition 3.2.1. Finally, by Lemma 3.5.2, a contradiction arises, thus the image of $J(X_T/H)$ is not a Prym-Tyurin variety for $X_{T'}$. $\square$

Chapter 4

On the Ekedahl-Serre questions

In this chapter we prove Proposition 2.5.2.

4.1 Counting elliptic curves in the decomposition

Let p be a prime number and n be an integer > 3. In the decomposition of a generalized Fermat curve of type (n, p), given by Theorem 3.2.3, we see as direct factors the decompositions of some generalized Fermat curves of type (i, p), with $i \in \{3, \ldots, n-1\}$, which are given by subsets of S. Thus, by applying Proposition 2.2.5, we have the following result.

Proposition 4.1.1. *Let p be a prime number and n be an integer > 3. Let $X_{(n,p)}$ be a generalized Fermat curve of type (n, p). If $J(X_{(n,p)})$ is completely decomposable, then there exist generalized Fermat curves of type (i, p) such that $J(X_{(i,p)})$ is completely decomposable, for each $i \in \{3, \ldots, n-1\}$.*

Proof. By Theorem 3.2.3, for a generalized Fermat curve of type (n, p) we have

$$J(X_{(n,p)}) \sim \bigoplus_{\substack{T \subset S \\ n-|T| \geq 2}} \bigoplus_{H \in \mathcal{H}_T^S(p)} \pi_H^* J(X_T/H).$$

For each $T \subset S$ with $n-|T| \geq 2$, we have a morphism $\pi_T : X_{(n,p)} \to X_T$. Recall that X_T is a generalized Fermat curve of type $(n - |T|, p)$. Then, Proposition 2.2.5, the jacobian of X_T is completely decomposable. This holds for every $|T| \in \{0, \ldots, n-3\}$. This proves the assertion. $\square$

Then, completely decomposability goes down in the type of a generalized Fermat curve. In particular, if a generalized Fermat curve of type $X_{(3,p)}$ is completely decomposable, it is worth to seek further. For example, given that $g_{(3,2)} = 1$, the jacobian $J(X_{(3,2)})$ is an elliptic curve.

For generalized Fermat curves of type $(n, 2)$ a particular phenomenon might occur. When n is odd, complete decomposability could go up one step.

Proposition 4.1.2. *Let $n \geq 3$ be an integer. If n is odd and every generalized Fermat curve of type $(n, 2)$ have a completely decomposable jacobian, then also every generalized Fermat curve of type $(n+1, 2)$ have a completely decomposable jacobian.*

Proof. Let $X_{(n+1,2)}$ be a generalized Fermat curve of type $(n + 1, 2)$. By Theorem 3.2.3, the jacobian of $X_{(n+1,2)}$ decomposes as follows

$$J(X_{(n+1,2)}) \sim \bigoplus_{\substack{T \subset S \\ n+1-|T| \geq 2}} \bigoplus_{H \in \mathcal{H}_T^S(2)} \pi_H^* J(X_T/H).$$

When $T = \emptyset$ and $(n+1)$ is even, from Section 3.4 and by Theorem 2.4.1, the set $\mathcal{H}_T^S(2) = \emptyset$. Then, given that $|T| \geq 1$, the jacobian $J(X_{(n+1,2)})$ is the direct sum of decomposition of jacobian of generalized Fermat curves of type $(n, 2)$. Hence, since every generalized Fermat curve of type $(n, 2)$ have a completely decomposable jacobian, it follows that $X_{(n+1,2)}$ has a completely decomposable jacobian. Thus, the assertion holds. ☐

This proposition has the following consequence.

Corollary 4.1.3. *Every generalized Fermat curve of type $(4, 2)$ has a completely decomposable jacobian and it is isogenous to a product of 5 elliptic curves.*

Proof. Given that the jacobian of every generalized Fermat curve of type $(3, 2)$ is an elliptic curve, by Proposition 4.1.2, every generalized Fermat curve of type $(4, 2)$ is completely decomposable. Moreover, by Proposition 2.3.1, is a product of 5 elliptic curves. ☐

Asking for every generalized Fermat curve of certain type to have a completely decomposable jacobian is a strong condition. However, it is just needed that all the generalized Fermat curves of smaller type appearing in the decomposition have a completely decomposable jacobian.

By [CHQ16, Example 5.7], there exists a generalized Fermat curve of type $(6, 2)$ having a completely decomposable jacobian that is a product of 49 elliptic curves. Then, by Proposition 4.1.1, we have the following.

Proposition 4.1.4. *There exists a generalized Fermat curve of type* $(5, 2)$ *having a completely decomposable jacobian and it is a product of 17 elliptic curves.*

Proof. Let $X_{(6,2)}$ be a generalized Fermat curve of type $(6, 2)$ having a completely decomposable jacobian. By Proposition 4.1.1, the jacobian variety of every generalized Fermat curve of type $(5, 2)$ appearing in the decomposition given by Theorem 3.2.3 has a completely decomposable jacobian. Hence, there exist generalized Fermat curves of type $(5, 2)$ having completely decomposable jacobians. □

By Theorem 3.2.3, for a generalized Fermat curve of type $(n, 3)$ we have the following.

Proposition 4.1.5. *Let* $X_{(n,3)}$ *be a generalized Fermat curve of type* $(n, 3)$ *and* $E_{(n,3)}$ *as above. Then, with the notation above,*

$$J(X_{(n,3)}) \sim \bigoplus_{\substack{T \subset S \\ n-|T| \geq 2}} \bigoplus_{H \in \mathcal{H}_T^S(3)} \pi_H^* J(X_T/H),$$

where $\pi_H : X_{(n,3)} \to X_T/H$ *denotes the natural projection, and* $\pi_H^* J(X_T/H)$ *is of dimension* $n - |T| - 1$.

When $n = 3$, the factors of highest dimension are of dimension 2 and all the other ones are elliptic curves. Let $H \in \mathcal{H}_T^S(3)$ such that X_T/H is a genus two curve. This curve has an action of $E_{(3,3)}/H$, which is a cyclic group of order three, and acts with exactly four fixed points. Then, by Proposition 2.2.6, $J(X_T/H)$ is isogenous to a product of elliptic curves. Therefore, we have the following result, which is also obtained in [CHQ16].

Proposition 4.1.6. *A generalized Fermat curve of type* $(3, 3)$ *is completely decomposable and is isogenous to a product of 10 elliptic curves.*

Proof. By Corollary 3.3.3, the jacobian of a generalized Fermat curve of type $(3, 3)$ is isogenous to a product of four elliptic curves and three abelian surfaces. Hence, the jacobian of a generalized Fermat curve is isogenous to a product of ten elliptic curves, therefore, is completely decomposable. This proves the assertion. □

For a generalized Fermat curve of type $(n, 3)$, with $n \geq 4$, we can give a lower bound on the number of elliptic curves appearing in the decomposition of the jacobian of such curve. The following result corresponds to Proposition 2.5.2.

Proposition 4.1.7. *The jacobian of a generalized Fermat curve of type* $(n, 3)$ *has at least*

$$\frac{n(n+1)(n-1)(3n-4)}{12}$$

elliptic curves.

Proof. By Theorem 2.5.1, the number of factors of dimension 1 corresponds to the summand with $|T| = n - 2$. Therefore, there are

$$\binom{n+1}{n-2}\frac{2^2 - (-1)^2}{3} = \frac{(n+1)(n)(n-1)}{6}$$

elliptic curves arising from the one dimensional factors.

By Proposition 2.2.6, we know that the factors of dimension two are isogenous to a product of elliptic curves. The number of abelian surfaces appearing in the decomposition corresponds to the summand with $|T| = n - 3$. Therefore, there are

$$\binom{n+1}{n-3}\frac{2^3 - (-1)^3}{3} = \frac{(n+1)(n)(n-1)(n-2)}{8}$$

abelian surfaces in the decomposition. Hence, from the abelian surfaces, we have

$$\frac{(n+1)(n)(n-1)(n-2)}{4}$$

elliptic curves.

The number of elliptic curves arising from the factors of dimension 1 and 2 are

$$\frac{(n+1)(n)(n-1)}{6} + \frac{(n+1)(n)(n-1)(n-2)}{4} = \frac{(n+1)(n)(n-1)(3n-4)}{12}.$$

This proves the assertion. $\square$

Part II

Normal varieties with algebraic torus action

Chapter 5

A historical introduction and statement of the main results

Since the work of Demazure in [Dem70], normal varieties endowed with an effective torus action have been extensively studied. In that work, toric varieties naturally emerged, and the author provided a combinatorial description of the smooth ones. At the time, these varieties were referred to as *toroidal embeddings*, as seen in [KKMSD73] and [Oda78]. A foundational survey based on earlier works was presented by Danilov in [Dan78], where these varieties were referred to as *toric varieties* for the first time.[1]

In general, normal toric varieties can be understood in terms of *cones* or *fans* (for modern references, see [CLS11] or [Ful93]). In the words of Fulton, *toric varieties have provided a remarkably fertile testing ground for general theories*. Furthermore, toric varieties have found numerous applications in physics and computational fields.

Throughout this century, new results have emerged regarding toric varieties. Almost all the works mentioned above were developed over algebraically closed fields, as all algebraic tori are *split* in that context. For non-split toric varieties, i.e. when the algebraic torus acting is not split, achieving an algebro-combinatorial description is not possible because the group of cocharacters does not fully manifest. However, since every algebraic torus splits over a finite Galois extension, it is possible to obtain an algebro-combinatorial description accompanied by a Galois action (see [Hur11] and [ELFST14]). Moreover, various taxonomies can be applied to classify non-split toric varieties, depending on the definition of toric varieties and the

[1] Danillov called them *toral* in russian, but in the english traduction appeared as *toric*. See [CLS11, Appendix A] for a brief historical overview of toric varieties.

types of morphisms considered [Dun16].

A toric variety contains a *torsor* of an algebraic torus as a dense open subvariety, and their dimensions coincide. For a variety endowed with an effective torus action (referred to as a T-variety for brevity), the *complexity* is defined as the difference between the dimensions of the variety and the torus. Thus, a toric variety is a T-variety of complexity zero.

For normal T-varieties of complexity one, Mumford [KKMSD73] provided a description in terms of *toroidal fans*[2]. Unfortunately, such a combinatorial description does not extend to higher complexities, even for complexity two. Furthermore, the works of Pinkham [Pin77] and Flenner and Zaidenberg [FZ03], both focused on complexity one surfaces and restricted to the complex numbers.

It was not until 2006 that an algebro-combinatorial description for normal affine T-varieties over algebraically closed fields of characteristic zero was achieved for arbitrary complexity. The object encoding the data of a normal affine T-variety was called a *proper polyhedral divisor* by Altmann and Hausen [AH06].

Let k be an algebraically closed field of characteristic zero, Y a normal semiprojective variety over k, and $\omega \subset N_{\mathbb{Q}}$ a pointed cone, where N is a lattice. Denote $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. A proper polyhedral divisor (abbreviated as pp-divisor) is a finite sum

$$\mathfrak{D} := \sum \Delta_D \otimes D,$$

where the Δ_D's are polyhedra in $N_{\mathbb{Q}}$ with tail cone ω, and the D's are irreducible and effective divisors in $\operatorname{CaDiv}_{\mathbb{Q}}(Y)$.

Given a pp-divisor $\mathfrak{D}$, we can associate with it a *piecewise linear map* $\mathfrak{h}_{\mathfrak{D}} : \omega^\vee \cap M \to \operatorname{CaDiv}_{\mathbb{Q}}(Y)$. Based on this construction, Altmann and Hausen defined the following k-algebra:

$$A[Y, \mathfrak{D}] := \bigoplus_{m \in \omega^\vee \cap M} H^0(Y, \mathscr{O}_Y(\mathfrak{h}_{\mathfrak{D}}(m))) \subset k(Y)[M],$$

and proved that it is finitely generated. Consequently, the scheme $X(\mathfrak{D}) := \operatorname{Spec}(A[Y, \mathfrak{D}])$ is a normal affine variety over k endowed with an effective action of $T := \operatorname{Spec}(k[M])$. Moreover, they showed that every normal affine T-variety arises in this manner.

Theorem 5.0.1. *[AH06, Theorems 3.1 and 3.4] Let k be an algebraically closed field of characteristic zero.*

[2] This is modern terminology. In [KKMSD73], what we now call fans were referred to as *finite rational partial polyhedral decompositions.*

i) The scheme $X(\mathfrak{D})$ is a normal k-variety with an effective action of $T := \operatorname{Spec}(k[M])$.

ii) Let X be a normal affine k-variety with an effective T-action. Then, there exists a pp-divisor $\mathfrak{D}$ such that $X \cong X(\mathfrak{D})$ as T-varieties.

Vollmert [Vol10] makes a correspondence between Mumford's toroidal fans and pp-divisors for complexity one normal affine T-varieties.

On the non affine case, over the same asumptions on the field k, Altmann, Hausen and Süss [AHS08] extended the well known results in toric geometry, stating that normal toric varieties are determined by fans on the cocharacter lattice of the torus (see for instance: [Ful93]). The objects extending fans in toric geometry are refer to as *divisorial fans*.

Definition 5.0.2. [AHS08, Definition 5.1] Let k be an algebraically closed field of characteristic zero. Let N be a $\mathbb{Z}$-module and Y be a normal semiprojective variety. A *divisorial fan* in (Y, N) is a set $\mathfrak{S}$ of pp-divisors $\mathfrak{D}$ with tail cones $\sigma_{\mathfrak{D}} \subset N_{\mathbb{Q}}$ such that, for any pair $\mathfrak{D}, \mathfrak{D}' \in \mathfrak{S}$, the intersection

$$\mathfrak{D} \cap \mathfrak{D}' := \sum (\Delta_D \cap \Delta'_D) \otimes D$$

is a face of both $\mathfrak{D}$ and $\mathfrak{D}'$ and, also belongs to $\mathfrak{S}$.

For two pp-divisors $\mathfrak{D}' = \sum \Delta'_D \otimes D$ and $\mathfrak{D} = \sum \Delta_D \otimes D$ over the same normal semiprojective variety Y over k, whose tail cones $\omega_{\mathfrak{D}'}$ and $\omega_{\mathfrak{D}}$ are defined over the same lattice N, an inclusion $\Delta'_D \subset \Delta_D$ induces a T-equivariant morphism of varieties $X(\mathfrak{D}') \to X(\mathfrak{D})$. The face relation is equivalent to the condition that the corresponding morphism of varieties induced by the face relation is an open embedding.

By Sumihiro's Theorem [Sum74], a normal T-variety, where T is a split torus, has a T-stable affine open covering. For each T-stable affine open subvariety, by Theorem 5.0.1, there exists a pp-divisor on a normal semiprojective variety; however, this normal semiprojective variety may not necessarily be the same for all the pp-divisors. Nonetheless, Altmann, Hausen, and Süss [AHS08] proved that all these pp-divisors can be modified to reside on the same normal semiprojective variety. Under this modification, the pp-divisors fit together into a divisorial fan. They further demonstrated that, from such a divisorial fan, the normal T-variety can be fully recovered.

Theorem 5.0.3. *[AHS08, Theorem 5.6] Let k be an algebraically closed field of characteristic zero. Up to equivariant isomorphism, every normal variety endowed with an effective algebraic torus action arises from a divisorial fan on a normal semiprojective variety Y.*

Once this theory was developed, numerous contributions have bloomed. For example, Hausen and Süss [HS10] studied the Cox ring of normal T-varieties; Petersen and Süss [PS11] focused on T-invariant divisors; Ilten and Süss [IS11] explored polarized T-varieties of complexity one. Many other results can be found in the survey [AIP$^+$12].

When k is no longer algebraically closed, the combinatorial framework vanishes for non-split algebraic tori over k, similar to toric geometry. However, when the algebraic torus is split, the combinatorial structure reappears. Specifically, Theorem 5.0.1 holds for split normal affine T-varieties over k, as shown in [Gil22, Proposición 4.10].

Every algebraic torus over k splits after a finite Galois extension. Thus, the combinatorial framework exists over such extensions, and Galois descent theory provides a mechanism to *bring it back* to the ground field. That is, with additional data describing the combinatorial structure over the extension, it is possible to describe the variety over the ground field. This idea was first implemented by Dubouloz and Liendo [DL22], who classified normal affine varieties endowed with an action of $\mathbb{S}^1 := \mathrm{Spec}(\mathbb{R}[x, y]/(x^2 + y^2 - 1))$ using the language of $\mathbb{R}$-*group structure*. This work was later generalized by Gillard [Gil22] to any field of characteristic zero and any algebraic torus over k, also using the framework of k-*structure* and k-*group structure*.

Let k be a field of characteristic zero, and L/k a finite Galois extension with Galois group $\Gamma := \mathrm{Gal}(L/k)$. If X is a variety over k, then $X_L := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$ has a canonical k-structure given by $\sigma := \mathrm{id} \times \mathrm{Spec}(\gamma)$. This construction defines a functor between the category of pairs (Y, σ), where Y is a quasi-projective variety over L and σ is a k-structure, and the category of quasi-projective varieties over k. Moreover, this functor defines an equivalence of categories (cf. Proposition 9.1.6). A similar statement holds for the category of pairs (G, τ), where G is an algebraic group over L and τ is a k-group structure, and the category of algebraic groups over k. Thus, a normal variety over k with an action of a torus T over k can be studied over any Galois extension by considering the pairs (X_L, σ) and (T_L, τ), via the equivalence of categories.

In this context, it is possible to obtain a pp-divisor $\mathfrak{D}$ over L and construct the M-graded L-algebra

$$A[Y, \mathfrak{D}] := \bigoplus_{m \in \omega \cap M} H^0(Y, \mathscr{O}_Y(\mathfrak{D}(m))) \subset L(Y)[M].$$

However, this data alone is insufficient to describe all the combinatorial-arithmetic information of the torus action on the variety, as the variety

$X(\mathfrak{D}) := \operatorname{Spec}(A[Y, \mathfrak{D}])$ over L may lack a compatible k-structure. The additional data and conditions required are presented in the following result:

Theorem 5.0.4. *[Gil22, Theorem A] Let k be a field of characteristic zero, L be a finite Galois extension with Galois group Γ. Let T be a split algebraic torus over L and (T, τ) be a k-torus.*

1. *Let $\mathfrak{D}$ be a pp-divisor over L. If there exists a k-structure σ_Y over Y and a function $h : \Gamma \to \operatorname{Hom}(\omega^\vee \cap M, \overline{k}(Y)^*)$ such that*

   a) for every $m \in \omega^\vee \cap M$ and every $\gamma \in \Gamma$,

   $$\sigma^*_{Y_\gamma}(\mathfrak{D}(m)) = \mathfrak{D}(\tilde{\tau}_\gamma(m)) + \operatorname{div}_Y(h_\gamma(\tilde{\tau}_\gamma(m)));$$

   b) for every $m \in \omega \cap M$ and every $\gamma_1, \gamma_2 \in \Gamma$,

   $$h_{\gamma_1}(m)\sigma^{\#}_{Y_{\gamma_1}}(h_{\gamma_2}(\tilde{\tau}^{-1}_{\gamma_1}(m))) = h_{\gamma_1\gamma_2}(m),$$

   then $X(\mathfrak{D})$ admits a k-structure $\sigma_{X(\mathfrak{D})}$ such that (T, τ) acts faithfully on $(X(\mathfrak{D}), \sigma_{X(\mathfrak{D})})$.

2. *Let (X, σ) be a normal affine variety endowed with a faithful action of (T, τ). Then, there exists a pp-divisor $\mathfrak{D}$ over L, a k-structure σ_Y over Y and a function $h : \Gamma \to \operatorname{Hom}(\omega^\vee \cap M, \overline{k}(Y)^*)$ satisfying the conditions above such that $(X, \sigma) \cong (X(\mathfrak{D}), \sigma_{X(\mathfrak{D})})$ as (T, τ)-varieties.*

It is worth to mention that there is an analog of this theorem for complexity one normal affine T-varieties over arbitrary fields by Langlois [Lan15].

Main results

When k is no longer an algebraically closed field and T is a split algebraic torus over k, we prove that every geometrically integral normal T-variety over k arises from a pp-divisor over k, recovering [Gil22, Proposition 3.2] by using other methods.

Theorem 5.0.5. *Let k be a field of characteristic zero.*

i) The scheme $X(\mathfrak{D})$ is a geometrically integral normal variety over k with an effective action of $T := \operatorname{Spec}(k[M])$.

ii) Let X be a geometrically integral normal affine variety over k with an effective action of a split algebraic torus T. Then, there exists a pp-divisor $\mathfrak{D}$ such that $X \cong X(\mathfrak{D})$ as T-varieties.

For a split algebraic torus T over k, Sumiriho's Theorem [Sum74] states that every normal T-variety over k has a T-stable affine open covering. This allows us to prove the following result, generalizing Theorem 5.0.3.

Theorem 5.0.6. *Let k be a field of characteristic zero and T be a split torus over k. Up to isomorphism, every normal T-variety over k arises from a divisorial fan $(\mathfrak{S}, Y)$ over k.*

Not every divisorial fan $(\mathfrak{S}, Y)$ gives rise to a normal T-variety. Indeed, some cases are just prevarieties. The divisorial fans that encode a normal T-variety are the *separated* ones (cf. Definition 7.3.10).

In order to classify normal T-varieties over a non algebraically closed field of characteristic zero, we need to develop an appropriate language.

Galois descent data can be formulated in term of a *Galois semilinear equivariant action* or a *Galois semilinear action* (cf. Section 9.1), depending on whether the variety is equipped with an action of an algebraic group or not.

On the one hand, a *Galois semilinear action over a divisorial fan* $(\mathfrak{S}, Y)$ induces a Galois semilinear equivariant action over $X(\mathfrak{S})$, therefore, a Galois descent data over $X(\mathfrak{S})$, the normal T-variety encoded by the divisorial fan. On the other hand, not every Galois descent data over $X(\mathfrak{S})$ induces a Galois semilinear action over the divisorial fan $(\mathfrak{S}, Y)$. But, if the Galois group is finite, there always exists a divisorial fan $(\mathfrak{S}', Y')$ such that $(\mathfrak{S}', Y')$ admits a compatible Galois semilinear action and $X(\mathfrak{S}) \cong X(\mathfrak{S}')$ for the given Galois descent data. Not every Galois descent data is effective, so we need to ask for an extra condition. Thus, we prove the following result, which is the main theorem of this part of the thesis.

Theorem 5.0.7. *Let k be a field of characteristic zero and L/k be a finite Galois extension with Galois group Γ.*

a) Let T be a split algebraic torus over L and X be a normal T-variety over L. If there exists a divisorial fan $(\mathfrak{S}_L, Y_L)$ for X admitting a Γ-semilinear action such that

the subvariety $X(\mathfrak{S}(\mathfrak{D}, \Gamma))$ is quasi-projective for every $\mathfrak{D} \in \mathfrak{S}_L$,

then there exists an algebraic torus T' over k and a normal T'-variety X' over k such that $X'_L \cong X$ as T varieties over L.

b) Let T be an algebraic torus over k that splits over L. Let X be a normal variety endowed with an effective T-action over k. Then, there exists a divisorial fan $(\mathfrak{S}_L, Y_L)$ admitting a Γ-semilinear action such that

$$\textit{the subvariety } X(\mathfrak{S}(\mathfrak{D}, \Gamma)) \textit{ is quasi-projective for every } \mathfrak{D} \in \mathfrak{S}_L,$$

and $X_L \cong X(\mathfrak{S}_L)$ as T_L-varieties.

In Theorem 5.0.7, $\mathfrak{S}(\mathfrak{D}, \Gamma)$ stands for the *sub divisorial fan generated generated by $\mathfrak{D}$ and Γ*, cf. Definition 9.7.9. Besides, the variety $X(\mathfrak{S}(\mathfrak{D}, \Gamma))$ correspond then to the Galois orbit of the affine variety $X(\mathfrak{D})$.

Chapter 6

Convex geometry and toric varieties

This chapter is devoted to summarize some known facts about convex geometry and toric varieties. We start with *algebraic tori* and some of their properties. We continue with convex geometry, recalling the definitions of *cones* and *fans*. We present also the notion of *polyhedra*. In the subsequent section, we talk about *toric varieties*. This chapter is split into two subparts. The first one is about *split* toric varieties and the last one is about non split toric varieties.

6.1 Algebraic tori

Throughout this section k stands for an arbitrary field and $\bar{k}$ for an algebraic closure of k. An *algebraic torus over* k is a linear algebraic group T over k such that for some finite Galois extension $k \subset L \subset \bar{k}$ we have

$$T_L := T \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L) \cong (\mathbb{G}_{\mathrm{m},L})^n ,$$

where $n = \dim(T)$. If this isomorphism holds over k, we say that the algebraic k-torus is *split*. A way to construct a split algebraic torus of dimension n over an arbitrary field k is the following: Let M be a free $\mathbb{Z}$-module of rank n. The group algebra $k[M]$ is a finitely generated k-algebra, which is isomorphic to

$$k[x_1, y_1, x_2, y_2, \ldots, x_n, y_n]/(x_1y_1 - 1, x_2y_2 - 1, \ldots, x_ny_n - 1)$$

as k-algebras. Then, we have the following k-algebra isomorphism

$$k[M] \cong k[x_1, y_1]/(x_1y_1 - 1) \otimes k[x_2, y_2]/(x_2y_2 - 1) \otimes \cdots \otimes k[x_n, y_n]/(x_ny_n - 1).$$

Hence, by taking the spectrum it follows that

$$\mathrm{Spec}(k[M]) \cong \mathbb{G}_{\mathrm{m},k} \times \mathbb{G}_{\mathrm{m},k} \times \cdots \times \mathbb{G}_{\mathrm{m},k} \cong (\mathbb{G}_{\mathrm{m},k})^n .$$

The group of characters of a split torus T is defined as

$$\chi^* (T) := \{\chi : T \to \mathbb{G}_{\mathrm{m},k} \mid \chi \text{ is a } k\text{-group homomorphism}\},$$

which will be denoted as M, and its group of cocharacters is defined as

$$\chi_* (T) := \{\lambda : \mathbb{G}_{\mathrm{m},k} \to T \mid \lambda \text{ is a } k\text{-group homomorphism}\},$$

which will be denoted by N. Both, the group of characters and the group of cocharacters of a split algebraic k-torus, are free $\mathbb{Z}$-modules of finite rank. Notice that if we compose $\chi \in M$ and $\lambda \in N$, we get a k-group morphism $\chi \circ \lambda : \mathbb{G}_{\mathrm{m},k} \to \mathbb{G}_{\mathrm{m},k}$. Given that $\mathrm{End}_{gr}(\mathbb{G}_{\mathrm{m},k}) \cong \mathbb{Z}$, we have a map

$$\begin{aligned} \langle , \rangle : M \times N &\to \mathbb{Z}, \\ (\chi, \lambda) &\mapsto \chi \circ \lambda, \end{aligned}$$

which defines a perfect pairing, as stated in the following result.

Proposition 6.1.1. *Let k be a field and T be a split algebraic torus of dimension n. Then,*

1. $M := \chi^*(T) \cong \mathbb{Z}^n$,
2. $N := \chi_*(T) \cong \mathrm{Hom}_{\mathbb{Z}}(\chi^*(T), \mathbb{Z}) \cong \mathbb{Z}^n$ *and*
3. $T \cong \mathrm{Spec}(k[M])$ *as algebraic groups.*

6.2 Preliminaries on Convex geometry

Let N be a lattice of rank n and $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ be the $\mathbb{Q}$-vector space associated to N by scalar extension. Let $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice of N, which has the same rank as N. The vector space $M_{\mathbb{Q}}$ is canonically isomorphic to $\mathrm{Hom}_{\mathbb{Q}}(N_{\mathbb{Q}}, \mathbb{Q})$, the dual of $N_{\mathbb{Q}}$ as a $\mathbb{Q}$-vector space. The lattices N and M can be considered contained in $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$ respectively.

The natural morphism $\langle , \rangle : M \times N \to \mathbb{Z}$, given by $\langle m, n\rangle := m(n)$, defines a perfect pairing between N and M. This morphism extends to a perfect pairing $\langle , \rangle : M_{\mathbb{Q}} \times \mathbb{N}_{\mathbb{Q}} \to \mathbb{Q}$.

6.2.1 Cones and fans

The definition and results presented in this section can be found in [Ful93] and [CLS11], for instance.

Definition 6.2.1. Let N be a lattice of rank n and $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual lattice. A *convex polyhedral cone on* $N_{\mathbb{Q}}$ is a subset ω of $N_{\mathbb{Q}}$ of the form

$$\omega = \mathrm{cone}(v_1, \dots, v_r) = \left\{ \sum_{i=1}^{k} r_i v_i \mid r_i \in \mathbb{Q}_{\geq 0} \right\},$$

for some $v_1, \dots, v_r \in N_{\mathbb{Q}}$.

Notice that convex polyhedral cones are convex. The *dimension of* ω, denoted by $\dim(\omega)$, is the dimension of the smallest subspace $V \subset N_{\mathbb{Q}}$ containing ω. The *dual cone* $\omega^{\vee}$ of ω is the cone

$$\omega^{\vee} := \{m \in M_{\mathbb{Q}} \mid \langle m, n \rangle \geq 0 \text{ for all } n \in \omega\}.$$

Definition 6.2.2. Let N be a lattice of rank n and $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual lattice. A *face of a convex polyhedral cone* $\omega \subset N_{\mathbb{Q}}$ is a subset τ of ω of the form

$$\tau = \omega \cap m^{\perp} = \{u \in \omega \mid \langle m, u \rangle = 0\},$$

with $m \in \omega^{\vee} \cap M$. The face relation is denoted by $\tau \preceq \omega$.

Notice that for any convex polyhedral cone $\omega \subset N_{\mathbb{Q}}$ we have $\omega \preceq \omega$. A faces τ of ω is called *proper* when $\tau \neq \omega$. Every face of a convex polyhedral cone is a convex polyhedral cone and the intersection of two faces of a convex polyhedral cone is also a face. Other important property is that the face relation is transitive.

Definition 6.2.3. Let N be a lattice of rank n and $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual lattice. A polyhedral cone on $N_{\mathbb{Q}}$ is said to be *pointed* if for every $V \subset N_{\mathbb{Q}}$ subvector space such that $V \subset \omega$, we have $V = \{0\}$.

From now on, by a *cone* in N we mean a pointed convex polyhedral cone in $N_{\mathbb{Q}}$.[1]

Definition 6.2.4. Let N be a lattice. A *fan* in $N_{\mathbb{Q}}$ is a finite set Σ of cones in $N_{\mathbb{Q}}$ such that, for any $\omega \in \Sigma$, if $\tau \preceq \omega$ we have $\tau \in \Sigma$ and, for any pair $\omega, \omega' \in \Sigma$, the intersection $\omega \cap \omega'$ is in Σ and $\omega \cap \omega' \preceq \omega, \omega'$. If the cones on Σ are not necessarily pointed, then we say that Σ is a *quasifan*.

[1] In classical references, we mean [Ful93] and [CLS11], we ask for rationality on the cones, but this is due to the definition is given over real vector spaces.

Definition 6.2.5. Let N be a lattice. A fan Σ on $N_{\mathbb{Q}}$ is said to be quasiprojective if there exists a family of linear forms $(l_\omega)_{\omega\in\Sigma}$ such that the following hold:

i) for every ω and ω' in Σ, we have that $l_\omega = l_\omega$ over $\omega \cap \omega'$ and

ii) for every ω and ω' in Σ, we have that $l_\omega(u) > l_{\omega'}(u)$ for every $u \in \operatorname{relint}(\omega)$.

Let Σ be a fan on $N_{\mathbb{Q}}$. An *automorphism of* Σ is an element $F \in \mathrm{GL}(N)$ such that $F(\omega) \in \Sigma$ for every $\omega \in \Sigma$ and if $\tau \preceq \omega$ in Σ, then $F(\tau) \preceq F(\omega)$. Denote by $\mathrm{Aut}(\Sigma)$ the group of automorphisms of Σ. The following result proves that to be an automorphism it is enough to ask for the first condition.

Proposition 6.2.6. *Let Σ be a fan on $N_{\mathbb{Q}}$ and $F \in \mathrm{GL}(N)$. If $F(\omega) \in \Sigma$ for every $\omega \in \Sigma$, then $F \in \mathrm{Aut}(\Sigma)$.*

Proof. Given that $\tau \preceq \omega$, there exists $m \in \omega^\vee \cap M_{\mathbb{Q}}$ such that $\tau = \omega \cap m^\perp$. Let $F^* : M \to M$ be the dual morphism. Notice that by definition

$$(F^{-1*}(m))^\perp = \{v \in N_{\mathbb{Q}} \mid \langle F^{-1*}(m), v\rangle = 0\}.$$

Then, as $\langle F^{-1*}(m), v\rangle = F^{-1*}(m)(v) = m(F^{-1}(v)) = \langle m, F^{-1}(v)\rangle$, we have

$$\begin{aligned}(F^{-1*}(m))^\perp &= \{v \in N_{\mathbb{Q}} \mid \langle m, F^{-1}(v)\rangle = 0\}\\ &= \{F(w) \in N_{\mathbb{Q}} \mid \langle m, w\rangle = 0\}\\ &= F(m^\perp).\end{aligned}$$

Then, given that F is a bijection,

$$F(\tau) = F(\omega \cap m^\perp) = F(\omega) \cap F(m^\perp) = F(\omega) \cap (F^{-1*}(m))^\perp.$$

Notice that $F^{-1*}(m) \in F^{-1*}(\omega^\vee) = F(\omega)^\vee$. $\square$

Definition 6.2.7. Let N be a lattice and Σ be a fan on $N_{\mathbb{Q}}$. For $G \le \mathrm{Aut}(\Sigma)$ and $\omega \in \Sigma$, we denote by $\Sigma(\omega, G)$ *the orbit sub fan of ω with respect to G*, which is defined as the smallest fan contained in Σ such that $g(\omega) \in \Sigma(\omega, G)$ for every $g \in G$.

6.2.2 Polyhedra

A convex polyhedron in $N_{\mathbb{Q}}$ is the intersection of finitely many closed affine half spaces in $N_{\mathbb{Q}}$. The set of all polyhedra in $N_{\mathbb{Q}}$ comes with a natural semigroup structure under the *Minkowski sum*: for any pair of polyhedra Δ_1 and Δ_2 in $N_{\mathbb{Q}}$

$$\Delta_1 + \Delta_2 := \{v_1 + v_2 \mid v_i \in \Delta_i\}.$$

A *polytope* $\Pi \subset N_{\mathbb{Q}}$ is the convex hull of finitely many points. Every polyhedron Δ in $N_{\mathbb{Q}}$ has a Minkowski decomposition $\Delta = \Pi + \omega$, with Π a polytope in $N_{\mathbb{Q}}$ and ω a cone in $N_{\mathbb{Q}}$. This cone is called the *tail cone* of Δ, or *recession cone* of Δ, and is given by

$$\omega = \{v \in N_{\mathbb{Q}} \mid v' + tv \in \Delta \text{ for all } v' \in \Delta \text{ and } t \in \mathbb{Q}_{\geq 0}\}.$$

Definition 6.2.8. Let ω be a cone in $N_{\mathbb{Q}}$.

1. A *ω-tailed polyhedron* (or *ω-polyhedron* for short) in $N_{\mathbb{Q}}$, is a polyhedron Δ in $N_{\mathbb{Q}}$ having tail cone ω. The set of all ω-polyhedra in $N_{\mathbb{Q}}$ is denoted by $\mathrm{Pol}^+_\omega(N_{\mathbb{Q}})$.

2. $\Delta \in \mathrm{Pol}^+_\omega(N_{\mathbb{Q}})$ is called *integral* if $\Delta = \Pi + \omega$ holds with a polytope $\Pi \subset N_{\mathbb{Q}}$ having its vertices in N. The set of all integral ω-polyhedra in $N_{\mathbb{Q}}$ in denoted by $\mathrm{Pol}^+_\omega(N)$.

The Minkowski sum of two ω-polyhedra is also an ω-polyhedron, then $\mathrm{Pol}^+_\omega(N_{\mathbb{Q}})$ is a monoid having $\omega \in \mathrm{Pol}^+_\omega(N_{\mathbb{Q}})$ as neutral element. This holds also for $\mathrm{Pol}^+_\omega(N)$, because the sum of two integral ω-polyhedra is an integral ω-polyhedron. Denote by $\mathrm{Pol}_\omega(N_{\mathbb{Q}})$ and $\mathrm{Pol}_\omega(N)$ their respective Grothendieck groups.

Recall that the *support function* associated to a convex set $\Delta \subset N_{\mathbb{Q}}$ is given by

$$\begin{aligned} h_\Delta : M_{\mathbb{Q}} &\to \mathbb{Q} \cup \{-\infty\}, \\ m &\mapsto \inf_{v \in \Delta} \langle m, v\rangle \end{aligned}$$

and its support is $\mathrm{Supp}(h_\Delta) := \{m \in M_{\mathbb{Q}} \mid h_\Delta(m) > -\infty\}$. For an ω-polyhedron Δ and $m \in M_{\mathbb{Q}}$, we define

$$\lambda_m := \{m' \in M_{\mathbb{Q}} \mid h_\Delta(m + m') = h_\Delta(m) + h_\Delta(m')\}.$$

The set $\lambda_\Delta := \{\lambda_m \mid m \in M_{\mathbb{Q}}\}$ is finite. Define $\Lambda(\Delta)$ as the set generated by all the finite intersections of elements in λ_Δ. Each element in $\Lambda(\Delta)$ is a cone, not necessarily pointed. The set $\Lambda(\Delta)$ is called *the normal quasifan of* Δ.

In the following we present some properties that can be found in [AH06, Section 1].

Lemma 6.2.9. *Let $\omega \in N_{\mathbb{Q}}$ a pointed cone, $\Delta \in \mathrm{Pol}^+_\omega(N_{\mathbb{Q}})$ and $h_\Delta : M_{\mathbb{Q}} \to \mathbb{Q} \cup \{-\infty\}$ its respective support function. Then, the following hold.*

i) The support of h_Δ is $\omega^\vee$ and it is linear on each cone of the normal quasifan $\Lambda(\Delta)$.

ii) The function h_Δ is convex, i.e. for every m_1 and m_2 in $M_{\mathbb{Q}}$ we have

$$h_\Delta(m_1 + m_2) \leq h_\Delta(m_1) + h_\Delta(m_2).$$

Moreover, the strict inequality holds if and only if m_1 and m_2 do not belong to the same maximal cone of $\Lambda(\Delta)$.

Let $\Delta \in \mathrm{Pol}^+_\omega(N_{\mathbb{Q}})$ and h_Δ its support function. We say that h_Δ is *piecewise linear* if there is a quasifan Λ having $\omega^\vee$ as its support such that h_Δ is linear on each $\lambda \in \Lambda$. Denote $\mathrm{CPL}_{\mathbb{Q}}(\omega)$ the set of convex piecewise linear functions $h : M_{\mathbb{Q}} \to \mathbb{Q} \cup \{-\infty\}$ having $\omega^\vee$ as its support.

Proposition 6.2.10. *Let $\omega \subset N_{\mathbb{Q}}$ a cone. The set $\mathrm{CPL}_{\mathbb{Q}}(\omega)$ is a semigroup and the map*

$$\begin{aligned} \mathrm{Pol}^+_\omega(N_{\mathbb{Q}}) &\to \mathrm{CPL}_{\mathbb{Q}}(\omega), \\ \Delta &\mapsto h_\Delta \end{aligned}$$

is a semigroup isomorphism.

Proposition 6.2.11. *Let $\omega \in N_{\mathbb{Q}}$ a cone. Then, the following statements hold.*

i) There is a commutative diagram of canonical, injective homomorphisms of monoids

$$\begin{tikzcd} \mathrm{Pol}^+_\omega(N) \arrow[r] \arrow[d] & \mathrm{Pol}^+_\omega(N_{\mathbb{Q}}) \arrow[d] \\ \mathrm{Pol}_\omega(N) \arrow[r] & \mathrm{Pol}_\omega(N_{\mathbb{Q}}). \end{tikzcd}$$

ii) The multiplication of elements $\Delta \in \mathrm{Pol}^+_\omega(N_{\mathbb{Q}})$ by positive rational numbers $\alpha \in \mathbb{Q}^+$ defined as

$$\alpha \cdot \Delta := \{\alpha v \mid v \in \Delta\}$$

extends to a unique $\mathbb{Q}$-action over $\mathrm{Pol}_\omega(N_{\mathbb{Q}})$.

iii) The group $\mathrm{Pol}_\omega(N)$ *of integral* ω*-polyhedra is a free abelian group and we have a canonical isomorphism*

$$\mathrm{Pol}_\omega(N_\mathbb{Q}) \cong \mathbb{Q} \otimes_\mathbb{Z} \mathrm{Pol}_\omega(N).$$

iv) For every element $m \in \omega^\vee$*, there is a unique linear evaluation functional* $\mathrm{eval}_m : \mathrm{Pol}_\omega(N_\mathbb{Q}) \to \mathbb{Q}$ *satisfying*

$$\mathrm{eval}_m(\Delta) = \min_{v\in\Delta}\langle m, v\rangle,$$

for $\Delta \in \mathrm{Pol}^+_\omega(N)$*.*

v) Two elements Δ_1 *and* Δ_2 *in* $\mathrm{Pol}_\omega(N_\mathbb{Q})$ *coincide if and only if* $\mathrm{eval}_m(\Delta_1) = \mathrm{eval}_m(\Delta_2)$ *holds for every* $m \in \omega^\vee$*.*

vi) An element $\Delta \in \mathrm{Pol}_\omega(N_\mathbb{Q})$ *is integral if and only if* $\mathrm{eval}_m(\Delta) \in \mathbb{Z}$ *for every* $m \in \omega^\vee \cap M$*.*

Definition 6.2.12. The *face of* $\Delta \in \mathrm{Pol}^+_\omega(N)$ *defined by* $m \in \omega^\vee \cap M$ is

$$\mathrm{face}(\Delta, m) := \{v \in \Delta \mid \langle m, v\rangle \le \langle m, v'\rangle \text{ for all } v' \in \Delta\} \in \mathrm{Pol}^+_{\omega\cap m^\perp}(N).$$

6.3 Toric varieties

This section is based on [CLS11] and [Ful93], for algebraically closed fields, and on [Hur11] and [Dun16], for general fields.

Let k be a field. By a k-variety we mean a separated geometrically integral scheme of finite type over k. Roughly speaking, a toric variety is a variety that has an open orbit isomorphic to an algebraic torus, such that the action of the torus on the open orbit extends to the whole variety. Depending on the definition of a toric variety, we could get different classifications. The main difference relies on whether the torus action is fixed. For example, the affine space $\mathbb{A}^2_k$ has several actions of $\mathbb{G}^2_{\mathrm{m},k}$: $(\lambda, \mu)\cdot(x, y) = (\lambda x, \mu y)$ and $(\lambda, \mu)\cdot(x, y) = (\mu^{-1}x, \lambda\mu^{-1}y)$ are two of them. For us, a toric variety comes with a fixed action of an algebraic torus. In particular, both toric varieties given in the previous example are different between them although isomorphic as abstract varieties. Let us introduce a formal definition of a toric variety.

Definition 6.3.1. Let k be a field and $\bar{k}$ be an algebraic closure. Let T be an algebraic k-torus. A normal k-variety X endowed with an effective action of T is said to be a toric variety if there exists a T-stable zariski open subset $U \subset X$ such that $U(\bar{k})$ is a principal homogeneous space of $T(\bar{k})$. If T is a split algebraic k-torus, then X is said to be a *split toric* k*-variety.*

6.3.1 Split toric varieties

Let T be a split algebraic k-torus, M be its module of characters and N be its module of cocharacters. Let $\omega \subset N_{\mathbb{Q}}$ be a cone in the sense of Section 6.2.1. It is possible to associate to ω a k-variety $X_\omega := \operatorname{Spec}(k[\omega^\vee \cap M])$. Such a k-variety is endowed with an effective action of T, which makes it an affine T-toric variety. Indeed, as can be seen in [CLS11], [Ful93] and [Hur11], every affine split toric k-variety arises in that way.

Proposition 6.3.2. *Let k be a field and T be a split algebraic torus over k. If X is an affine T-toric variety, then there exists a cone $\omega \subset N_{\mathbb{Q}}$ such that $X \cong \operatorname{Spec}(k[\omega^\vee \cap M])$ as T-varieties.*

Non-affine toric varieties can be classified by using the following theorem, due to Sumihiro [Sum74].

Theorem 6.3.3 (Sumihiro's Theorem, split version)**.** *Let k be a field and T be a split algebraic k-torus. Let X be a normal k-variety endowed with an effective action of T, then X has a T-stable affine open covering.*

By Sumihiro's Theorem, every split toric k-variety is covered by affine toric k-varieties. We can consider that the covering is stable under intersection and each element of the covering arises from a cone on $N_{\mathbb{Q}}$ by Proposition 6.3.2. Hence, we can prove that those cones fit together into a fan. In the other direction, we have that from every fan we can build a split toric k-variety. Let us briefly recall the construction. Let Σ be a fan in $N_{\mathbb{Q}}$. For each $\omega \in \Sigma$, Proposition 6.3.2, we have an affine toric variety X_ω. Recall that if $\tau \preceq \omega$ in Σ, then we have a T-equivariant open embedding $X_\tau \to X_\omega$. In particular, the torus $T \cong X_{\{0\}}$ has a T-equivariant open embedding to every X_ω. Consider the scheme

$$\tilde{X} := \bigsqcup_{\omega \in \Sigma} X_\omega.$$

Notice that, for every $\omega \in \Sigma$, there exists a canonical T-equivariant open embedding $X_\omega \to \tilde{X}$ that is compatible with the T-equivariant open embeddings

$$X_{\omega'} \xleftarrow{\iota'} X_{\omega' \cap \omega} \xrightarrow{\iota} X_\omega \ ,$$

for every $\omega' \in \Sigma$. Over the algebraic closure, we define the following relation on $\tilde{X}$: for x and y in $\tilde{X}$ we say that $x \sim y$ if and only if

$x \in X_\omega$, $y \in X_{\omega'}$ and there exists $z \in X_{\omega' \cap \omega}$ such that $\iota(z) = x$ and $\iota'(z) = y$.

The quotient $X_\Sigma := X_\Sigma / \sim$ defines a split T-toric variety over k, with $T \cong \mathrm{Spec}(k[M])$. Moreover, it comes with natural T-equivariant open embeddings $X_\omega \to X_\Sigma$ for every $\omega \in \Sigma$ such that the following diagram commutes

$$\begin{tikzcd} X_{\omega'} \arrow[r] & X_\Sigma \\ X_{\omega' \cap \omega} \arrow[u] \arrow[ur] \arrow[r] & X_\omega \arrow[u] \end{tikzcd}.$$

It is not possible to follow the construction given above over any field k. However, it is possible to construct X_Σ in a more general way. The fan Σ is a partially ordered set under the face relation and, for $\tau \preceq \omega$ in Σ, we have a T-equivariant open embedding $\psi_{\tau\omega} : X_\tau \to X_\omega$. Moreover, if $\nu \preceq \tau \preceq \omega$ in Σ, we have the following commutative diagram

$$\begin{tikzcd} X_\tau \arrow[r, "\psi_{\tau\omega}"] & X_\omega \\ X_\nu . \arrow[u, "\psi_{\nu\tau}"] \arrow[ur, "\psi_{\nu\omega}"'] & \end{tikzcd}$$

Thus, the normal affine T-varieties define a direct system. Then,

$$X_\Sigma := \varinjlim_{\omega \in \Sigma} X_\omega.$$

Proposition 6.3.4. *Let k be a field and T be a split algebraic torus over k. If X is a T-toric variety, then there exists a fan Σ in $N_\mathbb{Q}$ such that $X \cong X_\Sigma$ as T-varieties.*

6.3.2 Nonsplit toric varieties

Non split toric varieties have no combinatorial description over the ground field, because the group of cocharacters of a non-split algebraic torus does not fully appear. Hence, we need to go up to a finite Galois extension, because every T-toric variety X over k is split by a finite Galois extension, i.e. there exists a finite Galois extension L/k such $X_L := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$ is a split T_L-toric variety over L.

Before going beyond in the explanation, we recall the definition of *morphism of toric varieties.* A morphism between two toric varieties X and X', with respective tori T and T', is a pair (φ, f) such that

$$\begin{tikzcd} T \times X \arrow[r, "\mu"] \arrow[d, "(\varphi,f)"'] & X \arrow[d, "f"] \\ T' \times X' \arrow[r, "\mu'"'] & X' \end{tikzcd}$$

commutes, where μ and μ' are the respective actions, $\varphi : T \to T'$ is a morphism of k-algebraic groups and $f : X \to X'$ is a morphism of k-varieties. We call a morphism of toric varieties an isomorphism if f is an isomorphism.

Let X be a T-toric variety over k and $\mathrm{Aut}_T(X)$ be its group of automorphisms as a T-toric variety. When X is split and Σ is its corresponding fan, it is known that there exists a canonical group isomorphism

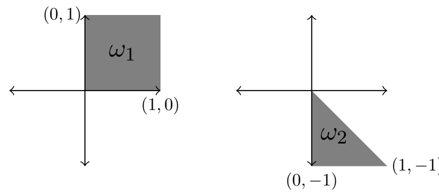
$$\mathrm{Aut}_T(X) \cong \mathrm{Aut}(\Sigma).$$

Let X be a toric variety over k. Let L/k be a finite Galois extension with Galois group Γ such that X_L is a split T_L-toric variety over L. The Galois group acts over X_L, not under toric automorphisms but *toric semilinear automorphisms* (cf. [Hur11, Definition 1.14] and Section 8.4 for general semilinear morphisms). This toric semilinear action on the variety X_L induces an action of the Galois group Γ over $N_{\mathbb{Q}}$ given by $\tilde{F} : \Gamma \to \mathrm{Aut}(N_{\mathbb{Q}})$.

Theorem 6.3.5. *[Hur11, Theorem 1.22] Let k be a field and L/k be a finite Galois extension with Galois group Γ. Let T be an algebraic torus over k. Let X be a T-toric variety over k split by L. If Σ on $N_{\mathbb{Q}}$ is the fan of X_L and $\tilde{F} : \Gamma \to \mathrm{Aut}(N_{\mathbb{Q}})$ is the respective Galois action induced by the toric semilinear action of Γ, then $\tilde{F}$ induces an action $F : \Gamma \to \mathrm{Aut}(\Sigma)$ such that, for every $\omega \in \Sigma$, $\Sigma(\omega, \Gamma)$ is a quasi-projective fan.*

Actually, Huruguen proves that the pairs (Σ, Γ) satisfying the conditions of Theorem 6.3.5 classify nonsplit toric varieties.

Let us see other perspective. As we said above, depending on how you expect to classify toric varieties you could have several taxonomies. For example, consider again the affine space $\mathbb{A}^2_k$ with the following actions of $\mathbb{G}^2_{\mathrm{m},k}$: $(\lambda, \mu) \cdot (x, y) = (\lambda x, \mu y)$ and $(\lambda, \mu) \cdot (x, y) = (\mu^{-1}x, \lambda\mu^{-1}y)$. Following Definition 6.3.1, these are two different toric varieties and their respective cones are $\omega_1 := \mathrm{cone}((1, 0), (0, 1))$ and $\omega_2 := \mathrm{cone}((0, -1), (1, -1))$



However, as varieties, both are the affine plane over k.

Duncan [Dun16] explores this question and gives two definitions on toric varieties.

Definition 6.3.6. Let k be a field and T be an algebraic torus over k. A toric T-variety X is a normal variety over k with an effective T-action and a dense open T-orbit X_0. A toric T-variety X is said to be neutral if there exists a T-equivariant isomorphism $T \to X_0$. A toric T-variety X is said to be split over k if T is a split torus over k.

Over algebraically closed fields, every toric T-variety is neutral and split. Therefore, these considerations are meaningful over non-algebraically closed fields.

Notice that the T-equivariant isomorphism $T \to X_0$ is not fixed in the definition above.

Definition 6.3.7. Let k be a field. We say that X is a toric variety if there exists an algebraic torus T over k with an action on X giving X the structure of a toric T-variety. We say X is neutral if T can be chosen such that X is neutral as a toric T-variety. We say that X is split if T can be chosen such that X is split as a toric T-variety.

The difference between both definitions is whether the algebraic torus is fixed or not.

By [VK85, Proposition 4], if a smooth toric variety X over k has a rational point, then the open orbit also has a rational point. In particular, if X is a smooth toric variety over k with a rational point, then X is neutral. Indeed, for projective smooth toric varieties we have an equivalence.

Proposition 6.3.8. *[Dun16, Proposition 3.7] A smooth projective toric variety X is neutral if and only if X has a rational k-point.*

In particular, a Severi-Brauer variety is neutral if and only if it is a projective space.

Duncan introduces three different categories of toric varieties. One of the main differences between them are the kind of morphisms that are allowed.

a) The category $\mathcal{R}$: the objects are toric varieties with morphisms of varieties.

b) The category $\mathcal{N}$: the objects are pairs (T, X) of toric T-varieties with equivariant morphisms of varieties.

c) The category $\mathcal{W}$: the objects are triples (T, X, ι) where T is a torus, X is a neutral toric T-variety and $\iota : T \to X$ is an isomorphism with the dense open orbit and a morphism from (T, X, ι) to (T', X', ι') is a pair (g, f), where $g : T \to T'$ is a group homomorphism and $f : X \to X'$ is a morphism of varieties such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] & X' \\ T \arrow[u, hook, "\iota"] \arrow[r, "g"'] & T'. \arrow[u, hook, "\iota'"'] \end{tikzcd}$$

Duncan compares these three categories and study how the k-forms change according to the category.

In the following chapters, we consider a generalization of the category $\mathcal{N}$. The objects of this new category are pairs (X, T) of normal varieties X endowed with an effective action of an algebraic torus. When the dimension of X and T coincide, the pair (X, T) represents a toric T-variety.

Chapter 7

Normal varieties with split torus actions

In this chapter we study the algebro-combinatoric of normal varieties endowed with an effective action of a split algebraic torus over a non-algebraically closed field of characteristic zero. In the first section we recall the notion of *proper polyhedral divisors* and we study their properties in the non-algebraically closed case. These objects play the role of cones for affine toric varieties. We generalize [AH06, Theorem 3.1 and 3.4] and recover [Gil22, Theorem A] using a different strategy.

In the remaining sections of this chapter we recall the definition of *divisorial fan* and generalize [AHS08, Theorem 5.6] over non-algebraically closed field of characteristic zero.

7.1 The category of proper polyhedral divisors

Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. As said in Section 6.3, split affine toric k-varieties arise from cones in $N_{\mathbb{Q}}$. The main goal of this section is to present the combinatorial objects that generalize cones for any affine normal k-variety endowed with an effective action of a split algebraic k-torus. These objects were introduced by Altmann and Hausen [AH06] for algebraically closed fields of characteristic zero. However, the definitions work over any field.

7.1.1 Proper polyhedral divisors

Let N be a lattice of finite rank and $\omega \subset N_{\mathbb{Q}}$ be a cone. As stated in Section 6.2.2, the set of all ω-tailed polyhedra $\mathrm{Pol}^+_\omega(N_{\mathbb{Q}})$ is a semigroup, whose neutral element is ω. The same holds for the set of integral ω-tailed polyhedra $\mathrm{Pol}^+_\omega(N) \subset \mathrm{Pol}^+_\omega(N_{\mathbb{Q}})$. Moreover, both admit the construction of a Grothendieck group, denoted by $\mathrm{Pol}_\omega(N_{\mathbb{Q}})$ and $\mathrm{Pol}_\omega(N)$ respectively. These groups are abelian.

Let k be a field and Y be a variety over k. Given that $\mathrm{Pol}_\omega(N_{\mathbb{Q}})$ and $\mathrm{Pol}_\omega(N)$ are abelian groups, we can take the tensor products

$$\mathrm{Pol}_\omega(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathrm{CaDiv}(Y) \quad \text{and} \quad \mathrm{Pol}_\omega(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathrm{CaDiv}(Y).$$

Besides, if Y is normal, we can also consider $\mathrm{Pol}_\omega(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathrm{Div}(Y)$ and $\mathrm{Pol}_\omega(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathrm{Div}(Y)$. These groups are called the group of *rational (resp. integral) polyhedral Cartier divisors* and the group of *rational (resp. integral) Weil divisors.*

Definition 7.1.1. Let k be a field. Let Y be a normal variety over k, N be a lattice and $\omega \subset N_{\mathbb{Q}}$ be a pointed cone:

1. The group of *rational polyhedral Weil divisors* and *rational polyhedral Cartier divisors* of Y with respect to $\omega \subset N_{\mathbb{Q}}$ are

$$\mathrm{Div}_{\mathbb{Q}}(Y, \omega) := \mathrm{Pol}_\omega(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathrm{Div}(Y),$$

$$\mathrm{CaDiv}_{\mathbb{Q}}(Y, \omega) := \mathrm{Pol}_\omega(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathrm{CaDiv}(Y).$$

2. The group of *integral polyhedral Weil divisors* and *integral polyhedral Cartier divisors* of Y with respect to $\omega \subset N_{\mathbb{Q}}$ are

$$\mathrm{Div}(Y, \omega) := \mathrm{Pol}_\omega(N) \otimes_{\mathbb{Z}} \mathrm{Div}(Y),$$

$$\mathrm{CaDiv}(Y, \omega) := \mathrm{Pol}_\omega(N) \otimes_{\mathbb{Z}} \mathrm{CaDiv}(Y).$$

Recall that, for a normal variety Y over k there is a canonical embedding

$$\mathrm{CaDiv}(Y) \to \mathrm{Div}(Y),$$

which allows us to consider $\mathrm{CaDiv}(Y) \subset \mathrm{Div}(Y)$ and, therefore,

$$\mathrm{CaDiv}_{\mathbb{Q}}(Y, \omega) \subset \mathrm{Div}_{\mathbb{Q}}(Y, \omega)$$

for any cone $\omega \subset N_{\mathbb{Q}}$. In particular, we can ask $D \in \mathrm{CaDiv}(Y)$ to be effective and irreducible. This being said, note that we can always write an element in any of these groups as $\mathfrak{D} = \sum_D \Delta_D \otimes D$, where the sum runs through the irreducible divisors D of Y and the Δ_D's are elements in $\mathrm{Pol}_\omega(N)$ or $\mathrm{Pol}_\omega(N_\mathbb{Q})$.

We are now ready to introduce the objects of the category of proper polyhedral divisors. In the following, by a *polyhedral divisor* we mean a rational one.

The sheaf of sections $\mathscr{O}_Y(D)$ of a rational Weil divisor D on a normal variety Y over k is defined as, for every open $U \subset Y$,

$$\mathscr{O}_Y(D)(U) := \{f \in k(Y) \mid \mathrm{div}(f|_U) + D|_U \geq 0\}.$$

Definition 7.1.2. Let Y be a normal k-variety, N be a lattice and $\omega \subset N_\mathbb{Q}$ a cone. A polyhedral divisor $\mathfrak{D} = \sum_D \Delta_D \otimes D \in \mathrm{CaDiv}_\mathbb{Q}(Y, \omega)$ is called *proper* if

1. all the $D \in \mathrm{Div}(Y)$ are effective, irreducible divisors and the Δ_D are in $\mathrm{Pol}^+_\omega(N_\mathbb{Q})$;
2. for every $m \in \mathrm{relint}(\omega^\vee) \cap M$, the evaluation
$$\mathfrak{D}(m) := \sum h_{\Delta_D}(m) D \in \mathrm{CaDiv}_\mathbb{Q}(Y)$$
is a big divisor on Y, i.e. for some $n \in \mathbb{N}$ there exists a section $f \in H^0(Y, \mathscr{O}_Y(\mathfrak{D}(nm)))$ such that Y_f is affine;
3. for every $m \in \omega^\vee \cap M$, the evaluation $\mathfrak{D}(m) \in \mathrm{CaDiv}_\mathbb{Q}(Y)$ is semi-ample, i.e. it admits a basepoint-free multiple. Otherwise stated, for some $n \in \mathbb{N}$ the sets Y_f cover Y, where $f \in H^0(Y, \mathscr{O}_Y(\mathfrak{D}(nm)))$.

The semigroup of proper polyhedral divisors (pp-divisors for short) is denoted by $\mathrm{PPDiv}_\mathbb{Q}(Y, \omega)$ and $\mathrm{tail}(\mathfrak{D}) := \omega$ is called the *tail cone* of $\mathfrak{D}$. The semigroup is partially ordered as follows: if $\mathfrak{D} = \sum_D \Delta_D \otimes D$ and $\mathfrak{D}' = \sum_D \Delta'_D \otimes D$, then $\mathfrak{D}' \leq \mathfrak{D}$ if and only if $\Delta_D \subset \Delta'_D$ for every D.

Definition 7.1.3. Let k be a field. Let Y be a normal variety over k. Let $\mathfrak{D} \in \mathrm{PPDiv}_\mathbb{Q}(Y, \omega)$ and $\mathfrak{D}' \in \mathrm{PPDiv}_\mathbb{Q}(Y, \omega')$ be pp-divisors.

i) We define the intersection of $\mathfrak{D}$ and $\mathfrak{D}'$ as
$$\mathfrak{D} \cap \mathfrak{D}' := \sum (\Delta_D \cap \Delta'_D) \otimes D.$$

ii) For $y \in Y$, we define the *fiber polyhedron at* y as

$$\Delta_y := \sum_{y \in D} \Delta_D.$$

Let $m \in \omega^\vee \cap M$ and $f \in H^0(Y, \mathscr{O}(\mathfrak{D}(m)))$. Denote $Z(f) = \mathrm{Supp}(\mathrm{div}(f) + \mathfrak{D}(m))$ its zero set and $Y_f = Y \smallsetminus Z(f)$ its principal set. If $m \in \mathrm{relint}(\omega^\vee)$, Y_f is an affine open subvariety of Y. Then, the Cartier divisors D appearing in $\mathfrak{D}$ induce Cartier divisors $D|_{Y_f}$ of Y_f once we restrict them. Thus, we can define the following pp-divisor in $\mathrm{PPDiv}_{\mathbb{Q}}(Y_f, \omega \cap m^\perp)$ by taking the faces defined by m of the polyhedra Δ_D:

$$\mathfrak{D}' := \sum \mathrm{face}(\Delta_D, m) \otimes D|_{Y_f}.$$

We would like to have both pp-divisors over the same base, in order to get a nice definition of being a *face* (cf. Definition 7.3.2) in the context of pp-divisors. Then, we will accept pp-divisors with $\emptyset$ coefficients satisfying

$$\emptyset + \Delta := \emptyset \text{ and } 0 \cdot \emptyset := \omega.$$

Thus, we will always assume that $\bigcup_{\Delta_D = \emptyset} \mathrm{Supp}(D)$ is the support of a semi-ample and effective divisor and by a pp-divisor $\mathfrak{D} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega)$ we mean $\mathfrak{D}_{|\mathrm{Loc}(\mathfrak{D})} \in \mathrm{PPDiv}_{\mathbb{Q}}(\mathrm{Loc}(\mathfrak{D}), \omega)$, where $\mathrm{Loc}(\mathfrak{D})$ is defined here below.

Definition 7.1.4. Let Y be a normal k-variety. Let $\mathfrak{D} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega)$ and $m \in \mathrm{relint}(\omega^\vee) \cap M$.

i) We define the *locus of* $\mathfrak{D}$ as

$$\mathrm{Loc}(\mathfrak{D}) := Y \smallsetminus \bigcup_{\Delta_i = \emptyset} \mathrm{Supp}(D_i).$$

ii) The *localization* of $\mathfrak{D}$ by $f \in H^0(Y, \mathscr{O}(\mathfrak{D}(m)))$ is

$$\mathfrak{D}_f := \sum \mathrm{face}(\Delta_D, m) \otimes D|_{Y_f} = \emptyset \otimes (\mathrm{div}(f) + \mathfrak{D}(m)) + \sum \mathrm{face}(\Delta_D, m) \otimes D.$$

As mentioned before, a pp-divisor $\mathfrak{D} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega)$ defines a map $\mathfrak{h}_{\mathfrak{D}} : \omega^\vee \to \mathrm{CaDiv}_{\mathbb{Q}}(Y)$ given by $\mathfrak{h}_{\mathfrak{D}}(m) := \mathfrak{D}(m)$. This map satisfies certain properties summarized in the following definition.

Definition 7.1.5. Let Y be a normal k-variety; let M be a lattice, and let $\omega^\vee \subset M_\mathbb{Q}$ be a cone of full dimension. We say that a map

$$\mathfrak{h} : \omega^\vee \to \mathrm{CaDiv}_\mathbb{Q}(Y)$$

is

i) *convex* if $\mathfrak{h}(m) + \mathfrak{h}(m') \leq \mathfrak{h}(m + m')$ holds for any two elements $m, m' \in \omega^\vee$,

ii) *piecewise linear* if there is a quasifan Λ in $M_\mathbb{Q}$ having $\omega^\vee$ as its support such that $\mathfrak{h}$ is linear on the cones of Λ,

iii) *strictly semiample* if $\mathfrak{h}(m)$ is semiample for all $m \in \omega^\vee$ and if for all $m \in \mathrm{relint}(\omega^\vee)$ is big.

The set of all convex, piecewise linear and strictly semiample maps $\mathfrak{h} : \omega^\vee \to \mathrm{CaDiv}_\mathbb{Q}(Y)$ is denoted by $\mathrm{CPL}_\mathbb{Q}(Y, \omega)$.

To each $\mathfrak{D} \in \mathrm{PPDiv}_\mathbb{Q}(Y, \omega)$ we can associate a convex, piecewise linear and strictly semiample map $\mathfrak{h}_\mathfrak{D} \in \mathrm{CPL}_\mathbb{Q}(Y, \omega)$. Thus, we have a natural map

$$\begin{aligned} \mathrm{PPDiv}_\mathbb{Q}(Y, \omega) &\to \mathrm{CPL}_\mathbb{Q}(Y, \omega), \\ \mathfrak{D} &\mapsto \mathfrak{h}_\mathfrak{D}. \end{aligned}$$

Proposition 7.1.6. *[AH06, Proposition 2.11] Let k be an algebraically closed field of characteristic zero. Let Y be a normal k-variety, N be a lattice, and $\omega \subset N_\mathbb{Q}$ be a pointed cone. Then the set $\mathrm{CPL}_\mathbb{Q}(Y, \omega)$ is a semigroup and the canonical map $\mathrm{PPDiv}_\mathbb{Q}(Y, \omega) \to \mathrm{CPL}_\mathbb{Q}(Y, \omega)$ given by $\mathfrak{D} \mapsto \mathfrak{h}_\mathfrak{D}$ is an isomorphism. Moreover, the integral polyhedral divisors correspond to maps $\mathfrak{h} : \omega^\vee \to \mathrm{CaDiv}_\mathbb{Q}(Y)$ such that $\mathfrak{h}(\omega^\vee \cap M) \subset \mathrm{CaDiv}(Y)$.*

In Section 7.1.3 we will see that this proposition holds for any field of charateristic 0.

7.1.2 Morphisms of proper polyhedral divisors

We have introduced the objects above. In order to construct a category, we need to expose how the objects are related. The morphisms are given by three pieces of data. Among them, there is one called *plurifunction*, whose definition is given below.

Definition 7.1.7. [AH06, Definition 8.2] Let Y be a normal k-variety, N be a lattice and $\omega \subset N_{\mathbb{Q}}$ a pointed cone.

a) A *plurifunction* with respect to the lattice N is an element of

$$k(Y,N)^* := N \otimes_{\mathbb{Z}} k(Y)^*.$$

b) For $m \in M := \mathrm{Hom}(N,\mathbb{Z})$, the *evaluation* of a plurifunction $\mathfrak{f} = \sum v_i \otimes f_i$ with respect to N is

$$\mathfrak{f}(m) := \prod f_i^{\langle m, v_i\rangle} \in k(Y)^*.$$

c) The *polyhedral principal divisor* with respect to $\omega \subset N_{\mathbb{Q}}$ of a plurifunction $\mathfrak{f} = \sum v_i \otimes f_i$ with respect to N is

$$\mathrm{div}(\mathfrak{f}) := \sum (v_i + \omega) \otimes \mathrm{div}(f_i) \in \mathrm{CaDiv}(Y,\omega).$$

Remark 7.1.8. Notice that the map $k(N,Y)^* \to \mathrm{CaDiv}(Y,\omega)$, given by $\mathfrak{f} \mapsto \mathrm{div}(\mathfrak{f})$, is a group homomorphism. For a plurifunction $\mathfrak{f} := \sum v_i \otimes f_i$, the inverse of $\mathrm{div}(\mathfrak{f})$ corresponds to $\mathrm{div}(\sum -v_i \otimes f_i)$.

A morphism of lattices $F : N \to N'$ induces a morphism between the groups $F_* : k(N,Y)^* \to k(N',Y)^*$ given by

$$F_*\left(\sum v_i \otimes f_i\right) = \sum F(v_i) \otimes f_i.$$

A morphism $\psi : Y \to Y'$ induces a morphism $\psi^* : k(N,Y')^* \to k(N,Y)^*$ given by

$$\psi^*\left(\sum v_i \otimes f_i\right) = \sum v_i \otimes \psi^*(f_i).$$

Recall that $\mathrm{PPDiv}_{\mathbb{Q}}(Y,\omega)$ is a partially ordered semigroup with $\mathfrak{D}' \leq \mathfrak{D}$ if and only if $\Delta_D \subset \Delta'_D$ for every D.

Definition 7.1.9. [AH06, Definition 8.3] Let Y and Y' be normal k-varieties, N and N' be lattices and $\omega \subset N$ and $\omega' \subset N'$ be pointed cones. Let us consider

$$\mathfrak{D} = \sum \Delta_i \otimes D_i \in \mathrm{PPDiv}_{\mathbb{Q}}(Y,\omega) \text{ and } \mathfrak{D}' = \sum \Delta'_i \otimes D'_i \in \mathrm{PPDiv}_{\mathbb{Q}}(Y',\omega')$$

two pp-divisors.

a) For morphisms $\psi : Y \to Y'$ such that none of the supports $\mathrm{Supp}(D'_i)$ contains $\psi(Y)$, we define the (not necessarily proper) *polyhedral pullback* as

$$\psi^*(\mathfrak{D}') := \sum \Delta'_i \otimes \psi^*(D'_i) \in \mathrm{CaDiv}_{\mathbb{Q}}(Y, \omega').$$

b) For linear maps $F : N \to N'$ with $F(\omega) \subset \omega'$, we define the (not necessarily proper) *polyhedral pushforward* as

$$F_*(\mathfrak{D}) := \sum (F(\Delta_i) + \omega') \otimes D'_i \in \mathrm{CaDiv}_{\mathbb{Q}}(Y, \omega').$$

c) A map $\mathfrak{D} \to \mathfrak{D}'$ is a triple $(\psi, F, \mathfrak{f})$ with a *dominant* morphism $\psi : Y \to Y'$, F a linear map as in b) and a plurifunction $\mathfrak{f} \in k(Y, N')^*$ such that

$$\psi^*(\mathfrak{D}') \leq F_*(\mathfrak{D}) + \mathrm{div}(\mathfrak{f}).$$

The identity map $\mathfrak{D} \to \mathfrak{D}$ for a pp-divisor is the triple $(\mathrm{id}, \mathrm{id}_N, 1)$. The composition of two morphisms of pp-divisors $(\psi, F, \mathfrak{f})$ and $(\psi', F', \mathfrak{f}')$ is defined as

$$(\psi', F', \mathfrak{f}') \circ (\psi, F, \mathfrak{f}) = (\psi' \circ \psi, F' \circ F, F'_*(\mathfrak{f}) \cdot \psi^*(\mathfrak{f}')).$$

The composition of two morphisms of pp-divisors is a morphism of pp-divisors. Thus, we have the following result.

Proposition 7.1.10. *Let k be a field. The proper polyhedral divisors of normal k-varieties with the morphisms of pp-divisors form a category $\mathfrak{PPDiv}$.*

Recall that every proper polyhedral divisor $\mathfrak{D}$ in $\mathfrak{PPDiv}$ has a tail cone defined on some $N_{\mathbb{Q}}$, with N a lattice. Furthermore, by fixing a lattice we are fixing a split k-torus, as stated in Section 6.1.

Definition 7.1.11. Let N be a lattice. We denote by $\mathfrak{PPDiv}_N$ the full subcategory of $\mathfrak{PPDiv}$ whose objects are the proper polyhedral divisors $\mathfrak{D}$ such that $\mathrm{Tail}(\mathfrak{D})$ is defined on $N_{\mathbb{Q}}$.

7.1.3 Base change for proper polyhedral divisors

The definitions above are given over any field of characteristic zero. However, the results are stated over algebraically closed fields. In this section we will see that such results holds over non algebraically closed fields and are stable under base change.

Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let Y be a geometrically integral normal k-variety. Recall that there is a canonical map $\mathrm{Div}(Y) \to \mathrm{Div}(Y_{\bar{k}})$, which induces a canonical map

$$\mathrm{CaDiv}_{\mathbb{Q}}(Y, \omega) \to \mathrm{CaDiv}_{\mathbb{Q}}(Y_k, \omega);$$
$$\mathfrak{D} = \sum \Delta_D \otimes D \mapsto \mathfrak{D}_{\bar{k}} := \sum \Delta_D \otimes D_{\bar{k}}.$$

The divisors $D_{\bar{k}}$ might not be irreducible, but they can be written as a sum of irreducible effective divisors.

This map turns out to be a group monomorphism. In particular, every pp-divisor on Y induces a rational polyhedral divisor on $Y_{\bar{k}}$, which is a pp-divisor.

Lemma 7.1.12. *Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let N be a lattice, $\omega \subset N_{\mathbb{Q}}$ be a pointed cone, Y be a geometrically integral normal k-variety. If $\mathfrak{D} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega)$, then $\mathfrak{D}_{\bar{k}} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}}, \omega)$.*

Proof. Let $\mathfrak{D} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega)$ with $\mathfrak{D} = \sum \Delta_D \otimes D$ and $\mathfrak{D}_{\bar{k}} = \sum \Delta_D \otimes D_{\bar{k}} \in \mathrm{CaDiv}_{\mathbb{Q}}(Y_{\bar{k}}, \omega)$ as above. Given that the $D \in \mathrm{Div}(Y)$ are effective, all the $D_{\bar{k}} \in \mathrm{Div}(Y_{\bar{k}})$ are effective.

Let $m \in \omega^{\vee} \cap M$ and $n \in \mathbb{N}$. The morphisms $Y_{\bar{k}} \to Y$ and $\mathrm{CaDiv}_{\mathbb{Q}}(Y, \omega) \to \mathrm{CaDiv}_{\mathbb{Q}}(Y_{\bar{k}}, \omega)$ define a morphism

$$\varphi_n : H^0(Y, \mathscr{O}(n\mathfrak{D}(m))) \to H^0(Y_{\bar{k}}, \mathscr{O}(n\mathfrak{D}_{\bar{k}}(m))).$$

This implies that $\mathfrak{D}_{\bar{k}}(m)$ is semiample, because $\mathfrak{D}(m)$ is semiample. Indeed, there exists $n \in \mathbb{N}$ such that Y_f cover Y where $f \in H^0(Y, \mathscr{O}(n\mathfrak{D}(m)))$. Thus, the $(Y_{\bar{k}})_{\varphi_n(f)}$ cover $Y_{\bar{k}}$. Therefore, the $(Y_{\bar{k}})_f$ cover $Y_{\bar{k}}$ for $f \in H^0(Y_{\bar{k}}, \mathscr{O}(n\mathfrak{D}_{\bar{k}}(m)))$. Hence, $\mathfrak{D}_{\bar{k}}(m)$ is semiample for $m \in \omega^{\vee} \cap M$.

If $m \in \mathrm{relint}(\omega^{\vee})$, by definition $\mathfrak{D}(m)$ is big. Then, for some $n \in \mathbb{N}$ there exists a section $f \in H^0(Y, \mathscr{O}(n\mathfrak{D}(m)))$ such that Y_f is affine. Let $f_{\bar{k}} \in H^0(Y_{\bar{k}}, \mathscr{O}(n\mathfrak{D}_{\bar{k}}(m)))$ given by $f_{\bar{k}} = \varphi_n(f)$. Given that $(Y_{\bar{k}})_{f_{\bar{k}}} = (Y_f)_{\bar{k}}$, we have that $f_{\bar{k}}$ has an affine non-vanishing locus. Hence, $\mathfrak{D}_{\bar{k}}(m)$ is big for every $m \in \mathrm{relint}(\omega^{\vee})$.

This proves that $\mathfrak{D}_{\bar{k}} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}}, \omega)$. □

The group homomorphism $\mathrm{CaDiv}_{\mathbb{Q}}(Y, \omega) \to \mathrm{CaDiv}_{\mathbb{Q}}(Y_{\bar{k}}, \omega)$ induces a semigroup homomorphism

$$\mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega) \to \mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}}, \omega).$$

Clearly, this map is not surjective. First, given that $\mathrm{Div}(Y_{\bar{k}})$ has a natural action of $\Gamma := \mathrm{Gal}(\bar{k}/k)$, then $\mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}}, \omega)$ has a natural structure of Γ-module. Then, the image of $\mathrm{PPDiv}_{\mathbb{Q}}(Y,\omega) \to \mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}},\omega)$ lies on $\mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}},\omega)^{\Gamma}$ when Y is semiprojective, i.e. when the global sections $H^0(Y,\mathscr{O}_Y)$ form a finitely generated k-algebra and Y is projective over $\mathrm{Spec}(H^0(Y,\mathscr{O}_Y))$. Actually, the image of $\mathrm{PPDiv}_{\mathbb{Q}}(Y,\omega) \to \mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}},\omega)$ coincides with $\mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}},\omega)^{\Gamma}$.

Proposition 7.1.13. *Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure with Galois group Γ. Let Y be a geometrically integral normal variety over k. Let N be a lattice and $\omega \subset N_{\mathbb{Q}}$ be a pointed cone. If Y is semiprojective, then the image of* $\mathrm{PPDiv}_{\mathbb{Q}}(Y,\omega) \to \mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}},\omega)$ *is* $\mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}},\omega)^{\Gamma}$.

Proof. Clearly, the image of $\mathrm{PPDiv}_{\mathbb{Q}}(Y,\omega) \to \mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}},\omega)$ is contained in $\mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}},\omega)^{\Gamma}$. Let us prove the other inclusion. Let

$$\tilde{\mathfrak{D}} := \sum \Delta_{\tilde{D}} \otimes \tilde{D}$$

in $\mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}},\omega)^{\Gamma}$. Given that the pp-divisor is Galois invariant, we have that $\Delta_{\tilde{D}} = \Delta_{\gamma(\tilde{D})}$ for every $\tilde{D}$ appearing in $\mathfrak{D}$ and $\gamma \in \Gamma$. Therefore, for each $\tilde{D}$ appearing in $\tilde{\mathfrak{D}}$, we have that

$$Z'_{\tilde{D}} := \bigcup_{\Delta_{\tilde{D}} = \Delta_{\tilde{D}'}} \mathrm{supp}(\tilde{D}')$$

is a Galois stable closed subvariety of $Y_{\bar{k}}$. Therefore, it descends to a closed subvariety $Z_{\tilde{D}} \subset Y$. Thus, by taking the irreducible components of $Z_{\tilde{D}}$ for every $\tilde{D}$, we can construct a polyhedral divisor

$$\mathfrak{D} := \sum \Delta_D \otimes D \in \mathrm{CaDiv}_{\mathbb{Q}}(Y,\omega)$$

such that $\Delta_D = \Delta_{\tilde{D}}$ when $\mathrm{supp}(D) \subset Z_{\tilde{D}}$. In order to prove that $\mathfrak{D}$ is a pp-divisor, we need to prove that the $\mathfrak{D}(m)$ is semiample for every $m \in \omega^\vee \cap M$ and big for $m \in \mathrm{relint}(\omega^\vee) \cap M$. First notice that $\mathfrak{D}_{\bar{k}}(m) = \tilde{\mathfrak{D}}(m)$ and recall that the morphism $Y_{\bar{k}} \to Y$ induces morphisms

$$\varphi_n : H^0(Y,\mathscr{O}(n\mathfrak{D}(m))) \to H^0(Y_{\bar{k}},\mathscr{O}(n\mathfrak{D}_{\bar{k}}(m))),$$

for every $n \in \mathbb{N}$.

Given that $\mathfrak{D}_{\bar{k}}(m)$ is big, for $m \in \mathrm{relint}(\omega^\vee) \cap M$, there exist $n \in \mathbb{N}$ and $f \in H^0(Y_{\bar{k}},\mathscr{O}(n\mathfrak{D}_{\bar{k}}(m)))$ such that $(Y_{\bar{k}})_f$ is affine. The Galois group Γ acts on $H^0(Y_{\bar{k}},\mathscr{O}(n\mathfrak{D}_{\bar{k}}(m)))$, because the divisor is Galois stable. Hence, we can consider the orbit of f in $H^0(Y_{\bar{k}},\mathscr{O}(n\mathfrak{D}_{\bar{k}}(m)))$, which is finite. Denote by

$\prod_\Gamma(f) := f_1 \cdots f_l$, the product of the elements in the orbit of f. Thus, for $n' = l \cdot n$, we have that $\prod_\Gamma(f) \in H^0(Y_{\bar{k}}, \mathscr{O}(n'\mathfrak{D}_{\bar{k}}(m)))$. Given that $\prod_\Gamma(f)$ is Galois stable, there exists $g \in H^0(Y, \mathscr{O}(n'\mathfrak{D}(m)))$ such that $\varphi_{n'}(g) = \prod_\Gamma(f)$. We claim that Y_g is affine. On the one hand, for every $i \in \{1, \dots, l\}$, there exists $\gamma_i \in \Gamma$ such that $\gamma_i((Y_{\bar{k}})_f) = (Y_{\bar{k}})_{f_i}$. This implies that each $(Y_{\bar{k}})_{f_i}$ is affine. Thus, the non-zero locus of $\prod_\Gamma(f)$ is affine because is the intersection of affine open subvarieties over $\bar{k}$

$$(Y_{\bar{k}})_{\prod_\Gamma(f)} = \bigcap_{i=1}^{l} (Y_{\bar{k}})_{f_i}.$$

On the other hand, $(Y_g)_{\bar{k}} = (Y_{\bar{k}})_{\prod_\Gamma(f)}$ is affine. Then, Y_g is affine. This implies that $\mathfrak{D}(m)$ is big for every $m \in \operatorname{relint}(\omega^\vee) \cap M$.

Let us prove now that $\mathfrak{D}(m)$ is semiample for all $m \in \operatorname{relint}(\omega^\vee \cap M)$. Let $\mathscr{F}$ be a coherent $\mathscr{O}_Y$-module. Given that $\mathfrak{D}_{\bar{k}}(m)$ is semiample and $Y_{\bar{k}}$ is semiprojective, by [Sch01, Theorem 1.1],

$$\bigoplus_{n \in \mathbb{N}} H^p(Y_{\bar{k}}, \mathscr{F}_{\bar{k}} \otimes n\mathscr{O}(\mathfrak{D}_{\bar{k}}(m)))$$

is a finitely generated $H^0(Y_{\bar{k}}, \mathscr{O}_{Y_{\bar{k}}})$-module for every $p \geq 0$. In particular, is a finitely generated $\bar{k}$-algebra. Then, by [Sta18, Tag 02KZ],

$$\bigoplus_{n \in \mathbb{N}} H^p(Y, \mathscr{F} \otimes n\mathscr{O}(\mathfrak{D}(m)))$$

is a finitely generated $H^0(Y, \mathscr{O}_Y)$-module for every $p \geq 0$. In particular, a finitely generated k-algebra. Hence, by [Sch01, Theorem 1.1], $\mathfrak{D}(m)$ is semiample. This proves the assertion. □

The morphism of base change defined above is stable on the fiber polyhedra.

Lemma 7.1.14. *Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let N be a lattice, $\omega \subset N_{\mathbb{Q}}$ be a pointed cone, Y be a normal variety over k and $\mathfrak{D} \in \operatorname{PPDiv}_{\mathbb{Q}}(Y, \omega)$. Then $\operatorname{Loc}(\mathfrak{D})_{\bar{k}} = \operatorname{Loc}(\mathfrak{D}_{\bar{k}})$ and $\Delta_{\bar{y}} = \Delta_y$ for $\bar{y} \in \{y\}_{\bar{k}}$.*

Proof. The first part of the assertion is clear from the construction of $\mathfrak{D}_{\bar{k}}$. The second part of the assertion follows from the fact that if $y \in D$, then $\bar{y} \in D_{\bar{k}}$. □

Denote by $\mathfrak{PPDiv}(k)$ (resp. $\mathfrak{PPDiv}(\bar{k})$) the category of pp-divisors over k (resp. $\bar{k}$). Let $\mathfrak{D}$ and $\mathfrak{D}'$ be objects in $\mathfrak{PPDiv}(k)$ and $(\psi, F, \mathfrak{f}) : \mathfrak{D}' \to \mathfrak{D}$ a morphism in $\mathfrak{PPDiv}(k)$. By base change we have a morphism of pp-divisors $(\psi_{\bar{k}}, F, \mathfrak{f}_{\bar{k}}) : \mathfrak{D}'_{\bar{k}} \to \mathfrak{D}_{\bar{k}}$ in $\mathfrak{PPDiv}(\bar{k})$. This construction is compatible with the composition law defined above. Thus, this data and the one given by $\mathfrak{D} \mapsto \mathfrak{D}_{\bar{k}}$ define a covariant functor $\mathfrak{PPDiv}(k) \to \mathfrak{PPDiv}(\bar{k})$.

Proposition 7.1.15. *The functor* $\mathfrak{PPDiv}(k) \to \mathfrak{PPDiv}(\bar{k})$ *is faithful.*

Proof. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be objects in $\mathfrak{PPDiv}(k)$. Let $(\psi, F, \mathfrak{f})$ and $(\psi', F', \mathfrak{f}')$ be morphisms in $\mathrm{Mor}_{\mathfrak{PPDiv}(k)}(\mathfrak{D}', \mathfrak{D})$ such that $(\psi_{\bar{k}}, F_{\bar{k}}, \mathfrak{f}_{\bar{k}}) = (\psi'_{\bar{k}}, F'_{\bar{k}}, \mathfrak{f}'_{\bar{k}})$. After the base change, we have $F = F_{\bar{k}}$ and $F' = F'_{\bar{k}}$. Then $F = F'$. Given that $\psi_{\bar{k}} = \psi'_{\bar{k}}$, they coincide in the Galois stable open subvariety of $Y_{\bar{k}}$ and therefore $\psi = \psi'$. If $\mathfrak{f}_{\bar{k}} = \mathfrak{f}'_{\bar{k}}$, then $\mathrm{div}(\mathfrak{f}_{\bar{k}}) = \mathrm{div}(\mathfrak{f}'_{\bar{k}})$. Hence, $\mathrm{div}(\mathfrak{f}) = \mathrm{div}(\mathfrak{f}')$. This implies that $f'_i = c_i f_i$ with $c_i \in k^*$ for every f_i and f'_i appearing in $\mathfrak{f}$ and $\mathfrak{f}'$ respectively. Now, for every $m \in M$ we have that

$$\mathfrak{f}_{\bar{k}}(m) = \mathfrak{f}'_{\bar{k}}(m) = \mathfrak{f}_{\bar{k}}(m) \prod c_i^{\langle m, v_i \rangle}.$$

Then,

$$\prod c_i^{\langle m, v_i \rangle} = 1$$

for every $m \in M$, and therefore all the constants must satisfy $c_i = 1$. Hence, $\mathfrak{f} = \mathfrak{f}'$. Then, we have that the functor $\mathfrak{PPDiv}(k) \to \mathfrak{PPDiv}(\bar{k})$ is faithful. □

Corollary 7.1.16. *Let* N *be a lattice. The induced functor*

$$\mathfrak{PPDiv}_N(k) \to \mathfrak{PPDiv}_N(\bar{k})$$

is faithful.

As in the algebraically closed case, pp-divisors can be understood as elements of $\mathrm{CPL}_{\mathbb{Q}}(Y, \omega)$. The following proposition helps us to prove one of the main theorems of Section 7.2.

Proposition 7.1.17. *Let* k *be a field. Let* Y *be a geometrically integral normal* k*-variety,* N *be a lattice, and* $\omega \subset N_{\mathbb{Q}}$ *be a pointed cone. Then, the set* $\mathrm{CPL}_{\mathbb{Q}}(Y, \omega)$ *is a semigroup and the canonical map* $\mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega) \to \mathrm{CPL}_{\mathbb{Q}}(Y, \omega)$ *given by* $\mathfrak{D} \mapsto \mathfrak{h}_{\mathfrak{D}}$ *is an isomorphism. Moreover, the integral polyhedral divisors correspond to maps* $\mathfrak{h} : \omega^\vee \to \mathrm{CaDiv}_{\mathbb{Q}}(Y)$ *such that* $\mathfrak{h}(\omega^\vee \cap M) \subset \mathrm{CaDiv}(Y)$.

Proof. Consider the following commutative diagram

$$\begin{tikzcd} \mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}}, \omega) \arrow[r, "\cong"] & \mathrm{CPL}_{\mathbb{Q}}(Y_{\bar{k}}, \omega) \\ \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega) \arrow[r] \arrow[u] & \mathrm{CPL}_{\mathbb{Q}}(Y, \omega)\,. \arrow[u] \end{tikzcd}$$

Given that the arrow on the top is injective, by Proposition 7.1.6, and the one on the left is injective by Proposition 7.1.15, the arrow on the bottom is also injective.

It remains to prove surjectivity. Let $\mathfrak{h} \in \mathrm{CPL}_{\mathbb{Q}}(Y, \omega)$. Given that $\omega^\vee \subset M_{\mathbb{Q}}$ is generated by finitely many elements of $\omega^\vee$, there exist finitely many divisors $D_1, \ldots, D_r$ in $\mathrm{Div}(Y)$ such that

$$\mathfrak{h}(m) = \sum_{i=1}^{r} h_i(m) D_i$$

for every $m \in \omega^\vee$, where the $h_i : \omega^\vee \to \mathbb{Q}$ are convex and piecewise linear functions. Otherwise stated, all the h_i are in $\mathrm{CPL}_{\mathbb{Q}}(\omega)$. Then, by Proposition 6.2.10, for every h_i there exists $\Delta_i \in \mathrm{Pol}^+_\omega(N_{\mathbb{Q}})$ such that $h_{\Delta_i} = h_i$. Therefore, the pp-divisor

$$\mathfrak{D} := \sum_{i=1}^{r} \Delta_i \otimes D_i$$

satisfies that $\mathfrak{h}_{\mathfrak{D}} = \mathfrak{h}$. Thus, the assertion holds. □

7.2 Affine normal varieties and pp-divisors

Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Altmann and Hausen proved that any affine normal variety endowed with an effective action of an algebraic torus over $\bar{k}$ arises from a pp-divisor over a semiprojective $\bar{k}$-variety (cf. Theorem 5.0.1).

Few years later, Altmann, Hausen and Süss introduced the the concept of *divisorial fan* and proved that any normal $\bar{k}$-variety endowed with an effective torus action arises from such a combinatorial data on some semiprojective $\bar{k}$-variety (cf. Theorem 5.0.3).

In the first part of this section we generalize Theorem 5.0.1 by proving the following.

Theorem 7.2.1. *Let k be a field of characteristic zero. Let T be a split k-torus and N be its module of cocharacters.*

i) Let $\mathfrak{D} \in \mathfrak{PPDiv}_N(k)$ be a pp-divisor on a normal semiprojective variety Y over k, then the scheme $X[Y,\mathfrak{D}] := \operatorname{Spec}(A[Y,\mathfrak{D}])$ is a geometrically integral normal k-variety with an effective T-action.

ii) Let X be a geometrically integral normal affine k-variety with an effective T-action. Then, there exists $\mathfrak{D} \in \mathfrak{PPDiv}_N(k)$ over a normal semiprojective Y variety over k such that $X \cong X[Y,\mathfrak{D}]$ as T-varieties.

7.2.1 Semiprojective varieties

Let k be a field. A variety Y over k such that the morphism $Y \to \operatorname{Spec}(H^0(Y,\mathscr{O}_Y))$ is proper is called *semiaffine* (cf. [GL73]). By definition, a variety Y over k is semiprojective if it is a semiaffine variety, its ring of global sections $H^0(Y,\mathscr{O}_Y)$ is a finitely generated k-algebra and $Y \to \operatorname{Spec}(H^0(Y,\mathscr{O}_Y))$ is quasi-projective.

An arbitrary product of semiprojective variety is not necessarily semiprojective, because the ring of global sections might not be a finitely generated k-algebra. However, it is a semiaffine variety.

Proposition 7.2.2. *Let k be a field. If $\{Y_i\}_{i\in I}$ is a set of semiaffine varieties over k, then the product $\prod_{i\in I} Y_i$ is semiaffine.*

Proof. Denote $Y := \prod_{i\in I} Y_i$. Notice that $H^0(Y,\mathscr{O}_Y) \cong \bigotimes H^0(Y_i,\mathscr{O}_{Y_i})$. Therefore, we have the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "p_i"] \arrow[d, "\alpha"'] & Y_i \arrow[d, "\alpha_i"] \\ \operatorname{Spec}(H^0(Y,\mathscr{O}_Y)) \arrow[r] & \operatorname{Spec}(H^0(Y_i,\mathscr{O}_{Y_i})), \end{tikzcd}$$

for every $i \in I$.

Let A be a discrete valuation ring and K be its fraction field. By the valuative criterion of properness, for each $i \in I$, we have the following commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r] \arrow[d] & Y \arrow[r, "p_i"] \arrow[d, "\alpha"] & Y_i \arrow[d, "\alpha_i"] \\ \operatorname{Spec}(A) \arrow[r] \arrow[rru, dashed] & H^0(Y,\mathscr{O}_Y) \arrow[r] & H^0(Y_i,\mathscr{O}_{Y_i}). \end{tikzcd}$$

Given that we have a unique morphism $\mathrm{Spec}(A) \dashrightarrow Y_i$ for each $i \in I$, we have a unique morphism $\mathrm{Spec}(A) \dashrightarrow Y$ fitting into the the following commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & Y \arrow[d, "\alpha"'] \\
\mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & H^0(Y, \mathscr{O}_Y).
\end{tikzcd}$$

Then, by the valuative criterion of properness, the morphism $Y \to H^0(Y, \mathscr{O}_Y)$ is proper. $\square$

However, semiprojective varieties are stable under finite product.

Proposition 7.2.3. *Let k be a field. If $\{Y\}_{i\in I}$ a finite set of semiprojective varieties over k, then the product $\prod_{i\in I} Y_i$ is semiprojective.*

Proof. Denote $Y := \prod_{i\in I} Y_i$. Given that $H^0(Y, \mathscr{O}_Y) \cong \bigotimes H^0(Y_i, \mathscr{O}_{Y_i})$, the global sections $H^0(Y, \mathscr{O}_Y)$ form a finitely generated k-algebra, because is the tensor product of finitely many k-algebras of finite type.

Let us prove the projectiveness of $Y \to \mathrm{Spec}(H^0(Y, \mathscr{O}_Y))$. By induction, it is enough to prove for the product of two of them. Let X and Z be two semiprojective varieties over k and denote by $X_0 := \mathrm{Spec}(H^0(X, \mathscr{O}_X))$ and $Z_0 := \mathrm{Spec}(H^0(Z, \mathscr{O}_Z))$. Notice that

$$\mathrm{Spec}(H^0(X \times Z, \mathscr{O}_{X\times Z})) = \mathrm{Spec}(H^0(X, \mathscr{O}_X)) \times \mathrm{Spec}(H^0(Z, \mathscr{O}_Z)) = X_0 \times Z_0$$

and, therefore, we have $(X \times Z)_0 = X_0 \times Z_0$.

We have the following commutative diagram

$$\begin{tikzcd}
X \times Z \arrow[r, "s_1"] \arrow[d, "r_1"'] & X_0 \times Z \arrow[r, "s_2"] \arrow[d, "r_2"'] & Z \arrow[d, "\alpha_Z"] \\
X \times Z_0 \arrow[r, "q_1"] \arrow[d, "p_1"'] & X_0 \times Z_0 \arrow[r, "q_2"] \arrow[d, "p_2"'] & Z_0 \arrow[d, "\beta_Z"] \\
X \arrow[r, "\alpha_X"'] & X_0 \arrow[r, "\beta_X"'] & k.
\end{tikzcd}$$

We claim that $q_1 \circ r_1 = \alpha_{X\times Z} : X \times Z \to X_0 \times Z_0$ is projective. Consider the following commutative diagram

$$
\begin{tikzcd}
W \arrow[drrr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & & \\
& X \times Z \arrow[r, "s_1"] \arrow[d, "r_1"'] & X_0 \times Z \arrow[r, "s_2"] \arrow[d, "r_2"'] & Z \arrow[d, "\alpha_Z"] \\
& X \times Z_0 \arrow[r, "q_1"] \arrow[d, "p_1"'] & X_0 \times Z_0 \arrow[r, "q_2"] \arrow[d, "p_2"'] & Z_0 \arrow[d, "\beta_Z"] \\
& X \arrow[r, "\alpha_X"'] & X_0 \arrow[r, "\beta_X"'] & k,
\end{tikzcd}
$$

where $\beta_X \circ \alpha_X$ and $\beta_Z \circ \alpha_Z$ are the structural morphisms. Given that

$$\beta_X \circ \alpha_X \circ p_1 \circ f = \beta_Z \circ \alpha_Z \circ g,$$

by the universal property of fibered product, there exists a unique morphism $h : W \to X \times Z$ such that $s_2 \circ s_1 \circ h = g$ and $p_1 \circ r_1 \circ h = p_1 \circ f$. Besides, we have

$$q_2 \circ q_1 \circ f = \alpha_Z \circ g = \alpha_Z \circ s_2 \circ s_1 \circ h = q_2 \circ q_1 \circ r_1 \circ h.$$

Given that p_1 is the projection on the first coordinate and $q_2 \circ q_1$ is the projection on the second coordinate, it follows that $r_1 \circ h = f$. Thus, we have that the rectangle at the top is cartesian. Hence, by [Sta18, Tag 02V6], the projectivity of α_Z implies the the projectivity of r_1.

Now, we need to prove that q_1 is projective. As in the previous case, it is enough to prove that the square at the bottom over $\alpha_X : X \to X_0$ is cartesian. Consider the following commutative diagram

$$
\begin{tikzcd}
W \arrow[drr, bend left, "g"] \arrow[dr, bend right, "f"'] & & & \\
& X \times Z_0 \arrow[r, "q_1"] \arrow[d, "p_1"'] & X_0 \times Z_0 \arrow[r, "q_2"] \arrow[d, "p_2"'] & Z_0 \arrow[d, "\beta_Z"] \\
& X \arrow[r, "\alpha_X"'] & X_0 \arrow[r, "\beta_X"'] & k.
\end{tikzcd}
$$

By the universal property of the fibered product, we have that there exists a morphism $u : W \to X \times Z_0$ such that $p_1 \circ u = f$ and $q_2 \circ g = q_2 \circ q_1 \circ u$. Besides, $p_2 \circ q_1 \circ u = p_2 \circ g$. Given that p_1 is the projection on the first coordinate and q_2 is the projection on the second coordinate, we have that $q_1 \circ u = g$. This implies that the square is cartesian. Hence, q_1 is projective by [Sta18, Tag 02V6].

Finally, given that $X_0 \times Z_0$ is separated and quasicompact, we have that $q_1 \circ r_1$ is projective by [Sta18, Tag 0C4P]. Then the assertion holds. □

The following results are useful properties on semiprojective varieties.

Lemma 7.2.4. *Let k be a field. Let Y be a semiprojective k-variety and Y' be a k-variety with $f : Y' \to Y$ a projective morphism. Then Y' is semiprojective.*

Proof. Denote $Y_0 := \operatorname{Spec}(H^0(Y, \mathscr{O}_Y))$ and $Y_0' := \operatorname{Spec}(H^0(Y', \mathscr{O}_{Y'}))$. We have the following commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "f"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Y_0' \arrow[r, "h"'] & Y_0. \end{tikzcd}$$

Given that Y_0 is a k-variety, by [Sta18, Tag 0C4P], we have that $g \circ f : Y' \to Y_0$ is projective. Given that $h : Y_0' \to Y_0$ is separated and $h \circ g' = g \circ f$ is projective, by [Sta18, Tag 0C4Q], we have that g' is projective. Then, Y' is semiprojective. □

Proposition 7.2.5. *Let k be a field. Let W, Y and Z be normal semiprojective varieties over k with birational maps satisfying*

$$\begin{tikzcd} W & Y \arrow[l, dashed, "\alpha"'] \arrow[r, dashed, "\beta"] & Z. \end{tikzcd}$$

Then, there exists a normal semiprojective variety $\tilde{Y}$ with birational morphisms $\tilde{Y} \to W, Y, Z$ such that the diagram

$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "\kappa_W"'] \arrow[d, "\kappa_Y"] \arrow[dr, "\kappa_Z"] & \\ W & Y \arrow[l, dashed, "\alpha"] \arrow[r, dashed, "\beta"'] & Z \end{tikzcd}$$

commutes.

Proof. Let $U_W \subset Y$ be the open subvariety where $\alpha|_{U_W} : U_W \to W$ is defined and $U_Z \subset Y$ be the open subvariety where $\alpha|_{U_Z} : U_Z \to Z$ is defined. Denote $U := U_W \cap U_Z$. Let Y_1 be the normalization of the closure of the graph of $\beta|_U : U \to Y$ on $Y \times Z$. Then, we have the following diagram

$$\begin{tikzcd} & Y_1 \arrow[d, "\kappa_1"'] \arrow[dr, "\kappa_2"] & \\ W & Y \arrow[l, dashed, "\alpha"] \arrow[r, dashed, "\beta"'] & Z, \end{tikzcd}$$

where κ_1 and κ_2 are the projections, which are also birational. Now, consider the rational map $\alpha \circ \kappa_1 : Y_1 \dashrightarrow W$. Notice that this map is defined over $\kappa_1^{-1}(U)$. Then, as before, let $\tilde{Y}$ be the normalization of the closure of the graph of $\alpha \circ \kappa_1 : \kappa_1^{-1}(U) \to W$ on $W \times Y$. Thus, we have the following commutative diagram

$$\begin{tikzcd} & \tilde{Y} \arrow[d, "\kappa_3"] \arrow[ddl, "\kappa_W"'] & \\ & Y_1 \arrow[d, "\kappa_1"'] \arrow[dr, "\kappa_2"] \arrow[dl, dashed] & \\ W & Y \arrow[l, dashed, "\alpha"] \arrow[r, dashed, "\beta"'] & Z, \end{tikzcd}$$

where κ_W and κ_3 are the projections which are also birational. Then, κ_W, $\kappa_Y := \kappa_3 \circ \kappa_1$ and $\kappa_Z := \kappa_3 \circ \kappa_2$ are the desired morphisms.

Let us prove now the semiprojectiveness of $\tilde{Y}$. By Proposition 7.2.3, $Y \times Z$ is semiprojective and, therefore, $\overline{\mathrm{Graph}(\beta_U)}$ is semiprojective. Hence, given that the normalization is a finite morphism, we conclude that $\tilde{Y}$ is semiprojective by Lemma 7.2.4. Now, by Proposition 7.2.3, $Y_1 \times W$ is semiprojective and, therefore, $\overline{\mathrm{Graph}(\alpha \circ \kappa_1(k^{-1}(U))}$ is semiprojective. Hence, given that the normalization is a finite morphism, we conclude that $\tilde{Y}$ is semiprojective by Lemma 7.2.4. □

The morphism $\kappa_Y : \tilde{Y} \to Y$ might not be projective. For example, let $\beta : \mathbb{A}^2_k \dashrightarrow \mathbb{A}^2_k$ be given by $(x, y) \to (x, y/x)$. The graph of β in $\mathbb{A}^2_k \times \mathbb{A}^2_k = \mathrm{Spec}(k[x, y, v, w])$ is $\{x = v, \, xw = y\}$, which is isomorphic to $\mathbb{A}^2_k$. In this case, we have the following resolution of indeterminacy:

$$\begin{tikzcd} & \mathbb{A}^2_k \arrow[dl, "\kappa_Y"'] \arrow[dr, "\kappa_Z"] & \\ \mathbb{A}^2_k \arrow[rr, dashed] & & \mathbb{A}^2_k, \end{tikzcd}$$

where κ_Y is given by $(x, w) \to (x, xw)$. This morphism is not projective. However, under some extra hypothesis, we can ensure the projectiveness of κ_Y.

Proposition 7.2.6. *Let k be a field. Let W, Y and Z be normal semiprojective varieties over k with birational maps satisfying*

$$\begin{tikzcd} W \arrow[d, "f_W"'] & Y \arrow[l, dashed, "\alpha"'] \arrow[r, dashed, "\beta"] \arrow[d, "f_Y"'] & Z \arrow[d, "f_Z"] \\ W_0 & Y_0 \arrow[l, "\alpha_0"] \arrow[r, "\beta_0"'] & Z_0, \end{tikzcd}$$

where $W_0 := \operatorname{Spec}(H^0(W, \mathscr{O}_W))$, $Y_0 := \operatorname{Spec}(H^0(Y, \mathscr{O}_Y))$ and $Z_0 := \operatorname{Spec}(H^0(Z, \mathscr{O}_Z))$ are the spectrums of the respective global sections and $f_W : W \to W_0$, $f_Y : Y \to Y_0$ and $f_Z : Z \to Z_0$ are the canonical maps. Then, there exists a normal semiprojective variety $\tilde{Y}$ with birational morphisms $\kappa_W, \kappa_Y, \kappa_Z : \tilde{Y} \to W, Y, Z$ such that the diagram

$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "\kappa_W"'] \arrow[d, "\kappa_Y"] \arrow[dr, "\kappa_Z"] & \\ W & Y \arrow[l, dashed, "\alpha"] \arrow[r, dashed, "\beta"'] & Z \end{tikzcd}$$

commutes and κ_Y is projective.

Proof. By Proposition 7.2.5, we have the existence of the normal semiprojective variety $\tilde{Y}$ over k and the birational maps κ_W, κ_Y and κ_Z. We claim that these morphisms are projective. We keep the constructions made in the proof of Proposition 7.2.5 with all the notations. We have the following commutative diagram

$$\begin{tikzcd} \overline{\operatorname{Graph}(\beta|_U)} \arrow[r, "\iota"] \arrow[d, "f"'] & Y \times Z \arrow[d, "f_{Y\times Z}"] \\ \overline{\operatorname{Graph}(\beta_0)} \arrow[r, "\iota_0"'] & Y_0 \times Z_0. \end{tikzcd}$$

Since ι and ι_0 are closed embeddings, they are finite and, therefore, projective. Moreover, given that $f_{Y\times Z}$ is projective and $\iota \circ f_{Y\times Z} = f \circ \iota_0$, we have that f is projective. Hence, it follows that

$$\overline{\operatorname{Graph}(\beta_0)} = \operatorname{Graph}(\beta_0) \cong Y_0,$$

because Z_0 is a separated scheme. Thus, $\overline{\mathrm{Graph}(\beta|_U)}$ is projective over Y_0 and, therefore, $\kappa_1 \circ f_Y : Y_1 \to Y_0$ is projective. Then, $\kappa_1 : Y_1 \to Y$ is projective. In this case we have the following commutative diagram

$$\begin{tikzcd}
\overline{\mathrm{Graph}(\alpha \circ \kappa_1(k^{-1}(U))} \arrow[r, "i"] \arrow[d, "g"'] & Y_1 \times W \arrow[d, "(\kappa_1 \circ f_Y) \times f_W"] \\
\overline{\mathrm{Graph}(\alpha_0)} \arrow[r, "i_0"'] & Y_0 \times W_0.
\end{tikzcd}$$

The maps i and i_0 are projective, because they are closed embeddings. Hence, given that $(\kappa_1 \circ f_Y) \times f_W$ is projective and $i \circ ((\kappa_1 \circ f_Y) \times f_W) = g \circ i_0$, we have that g is projective. Thus, $\kappa_3 \circ \kappa_1 \circ f_Y$ is projective and, therefore, κ_Y is projective. This proves the projectivity of κ_Y. □

7.2.2 From pp-divisors to affine normal varieties

Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let $\mathfrak{D}$ be a pp-divisor over a normal semiprojective variety Y over k and a cone $\omega \subset N_{\mathbb{Q}}$, where N is a lattice. From this data we can construct the following M-graded k-algebra

$$A[Y, \mathfrak{D}] := \bigoplus_{m \in \omega^\vee \cap M} H^0(Y, \mathscr{O}_Y(\mathfrak{D}(m))) \subset k(Y)[M]$$

and the scheme $X[Y, \mathfrak{D}] := \mathrm{Spec}(A[Y, \mathfrak{D}])$. The following result states that such a scheme is indeed a normal variety endowed with an effective action of $T = \mathrm{Spec}(k[M])$.

Proposition 7.2.7. *Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let Y be a normal semiprojective variety, N be a lattice, M be its dual lattice, $\omega \subset N_{\mathbb{Q}}$ be a pointed cone and $\mathfrak{D} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega)$ be a pp-divisor. Then, the scheme $X[Y, \mathfrak{D}] := \mathrm{Spec}(A[Y, \mathfrak{D}])$ is a geometrically integral normal k-variety with an effective action of $T := \mathrm{Spec}(k[M])$.*

Proof. The graduation induces an action of T over $X[Y, \mathfrak{D}]$. By [AH06, Theorem 3.1], we know that $T_{\bar{k}}$ acts effectively over

$$X[Y, \mathfrak{D}] \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k}) \cong X[Y_{\bar{k}}, \mathfrak{D}_{\bar{k}}],$$

which is a normal $\bar{k}$-variety. Then, the action of T over $X[Y, \mathfrak{D}]$ is effective and, by [Sta18, Tag 02KZ], we know that $X[Y, \mathfrak{D}]$ is a k-variety. Moreover, given that $X[Y_{\bar{k}}, \mathfrak{D}_{\bar{k}}]$ is integral, $X[Y, \mathfrak{D}]$ is geometrically integral. Finally, normality comes from [Sta18, Tag 037Z] and the fact that $X[Y_{\bar{k}}, \mathfrak{D}_{\bar{k}}]$ is normal and k is a field of characteristic zero. □

This proposition proves (ii) of Theorem 7.2.1.

Remark 7.2.8. Semiampleness of the evaluations $\mathfrak{D}(m)$ is used to ensure that $A[Y, \mathfrak{D}]$ is a finitely generated $H^0(Y, \mathscr{O}_Y)$-module and semiprojectiveness implies that it is a finitely generated k-algebra.

7.2.3 From affine normal varieties to pp-divisors

Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let T be a split algebraic k-torus and $X := \operatorname{Spec}(A)$ be an affine normal k-variety on which T acts effectively. Let M be the character lattice of T and $N := M^*$ be the cocharacter lattice. It is known that A has an M-graduation from the torus action:

$$A = \bigoplus_{m \in M} A_m.$$

Since A is a finitely generated k-algebra, the set $\{m \in M \mid A_m \neq 0\}$ forms a finitely generated semigroup and generates a cone $\omega^\vee \subset M_{\mathbb{Q}}$ called the *weight cone*.

Let L be a T-linearized line bundle over X. A T-linearization of L induces an action of T over the space of sections $H^0(X, L)$ as follows: for $s \in H^0(X, L)$ we have

$$(t \cdot s)(x) := t \cdot s(t^{-1}x).$$

By definition, the space of *semistable points* associated to L, denoted by $X^{\mathrm{ss}}(L)$, is the set of $x \in X$ such that for some $n \in \mathbb{N}$ there exists a T-invariant section $s \in H^0(X, L^n)$ such that $s(x) \neq 0$. By [MFK94, Theorem 1.10], the geometric quotient $X^{\mathrm{ss}}(L) /\!\!/ T$ exists. Notice that the space of semistable points depends on the T-linearization. Two T-linearized line bundles L and L' are called *GIT-equivalent* if $X^{\mathrm{ss}}(L) = X^{\mathrm{ss}}(L')$.

For each $m \in M$ there exists a T-linearization of the trivial bundle L given by

$$t \cdot (x, r) \to (tx, \chi^m(t)r), \tag{7.1}$$

where χ^m denotes the character associated to m. Denote by $X^{\mathrm{ss}}(m) := X^{\mathrm{ss}}(L)$ the space of semistable points associated to L with respect to $m \in M$ and by $Y_m := X^{\mathrm{ss}}(m) /\!\!/ T$ its respective geometric quotient. The main idea of Altmann and Hausen in [AH06] is to glue all these quotients Y_m for $m \in \omega^\vee \cap M$. But before gluing all these quotients, we need to establish which ones among them are GIT-equivalent. This was studied by Bertchtold and Hausen in [BH06] when k is an algebraically closed field. The main definitions and results can be summarized in the following.

Definition 7.2.9. Let $x \in X_{\bar{k}}$ be a closed point.

i) The *orbit monoid* associated to $x \in X_{\bar{k}}$ is the submonoid $S(x) \subset M$ consisting of all $m \in M$ that admit an $f \in A_m$ with $f(x) \neq 0$.

ii) The *orbit cone* associated to $x \in X_{\bar{k}}$ is the convex cone $\omega(x)^\vee \subset M_{\mathbb{Q}}$ generated by the orbit monoid.

iii) The *orbit lattice* associated to $x \in X_{\bar{k}}$ is the sublattice $M(x) \subset M$ generated by the orbit monoid.

The orbit cones are polyhedral and they are contained in $\omega^\vee$.

Proposition 7.2.10. *Let $x \in X_{\bar{k}}$ be a closed point.*

i) The orbit lattice $M(x)$ consists of all $m \in M$ that admit an m-homogeneous function $f \in \bar{k}(X)$ that is defined and invertible near x.

ii) The isotropy group $T_x \subset T$ of the point $x \in X_{\bar{k}}$ is the diagonalizable group given by $T_x = \mathrm{Spec}(\bar{k}[M/M(x)])$.

iii) The orbit closure $T \cdot x$ is isomorphic to $\mathrm{Spec}(\bar{k}[S(x)])$; it comes along with an equivariant open embedding of the torus $T/T_x = \mathrm{Spec}(\bar{k}[M(x)])$.

iv) The normalization of the orbit closure $T \cdot x$ is the toric variety corresponding to the cone $\omega(x)$ in $\mathrm{Hom}(M(x), \mathbb{Z})$.

In terms of the orbit cones, there is a simple description of the set $X^{\mathrm{ss}}_{\bar{k}}(m)$ of semistable points. Namely, by [BH06, Lemma 2.7], we have

$$X^{\mathrm{ss}}_{\bar{k}}(m) = \{x \in X_{\bar{k}} \mid m \in \omega(x)^\vee\}.$$

Definition 7.2.11. The *GIT-cone* associated to $m \in \omega^\vee \cap M$ is the intersection of all orbit cones containing m:

$$\lambda(m)^\vee := \bigcap_{x \in X^{\mathrm{ss}}_{\bar{k}}(m)} \omega(x)^\vee.$$

The main result of [BH06] is the following.

Theorem 7.2.12. *Let A be an integral affine $\bar{k}$-algebra graded by a lattice M. Let $T := \mathrm{Spec}(\bar{k}[M])$ be a $\bar{k}$-torus acting on $X := \mathrm{Spec}(A)$. Then, the following statements hold:*

i) The GIT-cones $\lambda(m)^\vee$, where $m \in M$, form a quasifan Λ in $M_{\mathbb{Q}}$.

ii) The support of the quasifan Λ is the weight cone $\omega^\vee \subset M_{\mathbb{Q}}$.

iii) For any two elements $m_1, m_2 \in \omega^\vee \cap M$, we have

$$X^{\mathrm{ss}}_{\bar{k}}(m_1) \subset X^{\mathrm{ss}}_{\bar{k}}(m_2) \Longleftrightarrow \lambda(m_2)^\vee \subset \lambda(m_1)^\vee.$$

In particular, the equality holds if and only if $\lambda(m_2)^\vee = \lambda(m_1)^\vee$.

We prove that this theorem also holds in the non algebraically closed case, for a split torus.

Proposition 7.2.13. *Let k be a field of characteristic zero. Let A be a geometrically integral affine k-algebra graded by a lattice M. Let $T := \mathrm{Spec}(k[M])$ be a k-torus acting on $X := \mathrm{Spec}(A)$. Then, for any two elements $m_1, m_2 \in \omega^\vee \cap M$, we have*

$$X^{\mathrm{ss}}(m_1) \subset X^{\mathrm{ss}}(m_2) \Longleftrightarrow \lambda(m_2)^\vee \subset \lambda(m_1)^\vee.$$

In particular, the equality holds if and only if $\lambda(m_2)^\vee = \lambda(m_1)^\vee$.

Proof. By [MFK94, Proposition 1.14], we have that

$$(X^{\mathrm{ss}}(m_i)) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k}) = X^{\mathrm{ss}}_{\bar{k}}(m_i).$$

On the one hand, by Theorem 7.2.12, if $\lambda(m_2)^\vee \subset \lambda(m_2)^\vee$ we have $X^{\mathrm{ss}}_{\bar{k}}(m_1) \subset X^{\mathrm{ss}}_{\bar{k}}(m_2)$. Then, $X^{\mathrm{ss}}(m_1) \subset X^{\mathrm{ss}}(m_2)$. On the other hand, if $X^{\mathrm{ss}}(m_1) \subset X^{\mathrm{ss}}(m_2)$, then $X^{\mathrm{ss}}_{\bar{k}}(m_1) \subset X^{\mathrm{ss}}_{\bar{k}}(m_2)$ and, by Theorem 7.2.12, we have that $\lambda(m_2)^\vee \subset \lambda(m_2)^\vee$. □

The sets of semistable points of a T-linearization are T-stable open subvarieties of $X = \mathrm{Spec}(A)$ that admit a geometric quotient for the T-action. For the T-linearization (7.1), by [AH06, Section 5], we have that

$$(Y_{\bar{k}})_m = \mathrm{Proj}\left(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bar{A}_{nm}\right)$$

and $(Y_{\bar{k}})_m$ is projective over $(Y_{\bar{k}})_0 = \mathrm{Spec}(\bar{A}_0)$. Then, we have that

$$Y_m = \mathrm{Proj}\left(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{nm}\right)$$

and Y_m is projective over $Y_0 = \mathrm{Spec}(A_0)$.

Let us see how the normal semiprojective variety Y and the pp-divisor over Y are constructed from the action of T over X. Let Λ be the quasifan in $M_{\mathbb{Q}}$ of Theorem 7.2.12. For every $\lambda \in \Lambda$ and any $m_1, m_2 \in \mathrm{relint}(\lambda)$, the sets of semistable points $X^{\mathrm{ss}}(m_1)$ and $X^{\mathrm{ss}}(m_2)$ are equal by Proposition 7.2.13. Now, denote by W_λ the set of semistable points of any $m \in \mathrm{relint}(\lambda)$ and denote by $q_\lambda : W_\lambda \to Y_\lambda$ the corresponding geometric quotients, which are all normal by [MFK94, Section 0.2]. Notice that $W_0 = X$ and it comes with a natural morphism $q_0 : W_0 \to Y_0 = \mathrm{Spec}(A_0)$. Given that for $\gamma \preceq \lambda$ we have an open embedding $W_\lambda \subset W_\gamma$, the open subschemes W_λ, with $\lambda \in \Lambda \cup \{0\}$, form a filtered inverse system. Let us denote by

$$W := \varprojlim W_\lambda = \bigcap_{\lambda \in \Lambda} W_\lambda,$$

the inverse limit of the sets of semistable points, which is an open subvariety of X. The open embeddings $W_\lambda \subset W_\gamma$ induce morphisms $p_{\lambda\gamma} : Y_\lambda \to Y_\gamma$. Denote by Y' the inverse limit of the Y_λ through the morphism $p_{\lambda\gamma}$, which exists as a scheme because is a finite system. There is a canonical map $q' : W \to Y'$. The scheme Y' might not be reduced, but it has a canonical reduced component, which is the schematic closure of $q'(W)$ in Y'_{red}. This holds because W is reduced. Hence, by taking the normalization of $\overline{q'(W)}$, we obtain a normal variety

$$Y := \overline{q'(W)}^{\nu}.$$

Moreover, by the universal property of the normalization, there exists a morphism $q : W \to Y$. We claim that Y is projective over Y_0. Given that the quasifan Λ is a finite set, we have that $\prod_{\lambda \in \Lambda} Y_\lambda$ is semiprojective by Proposition 7.2.3. The inverse limit $\varprojlim Y_\lambda \subset \prod_{\lambda \in \Lambda} Y_\lambda$ is a closed subscheme and therefore projective over Y_0, because of the following commutative diagram

$$\begin{tikzcd} \varprojlim Y_\lambda \arrow[r] \arrow[d] & \prod_{\lambda \in \Lambda} Y_\lambda \arrow[d] \\ Y_0 \arrow[r] & \prod_{\lambda \in \Lambda} Y_0 \end{tikzcd}$$

and by [Sta18, Tag 0C4Q]. Hence, $\overline{q(W)}$ is also projective over Y_0. Given that $\nu : Y \to \overline{q(W)}$ is finite, is projective by [Sta18, Tag 0B3I]. This implies that Y is projective over Y_0.

Remark 7.2.14. It is not true that the inverse limit of a finite inverse system of normal varieties is normal, even for a filtrant system. For example, consider the filtrant inverse system induced by

$$\begin{tikzcd} & k[z] \arrow[dr, "z\to x^2"] & \\ k[u] \arrow[ur, "u\to z"] \arrow[dr, "u\to w"'] & & k[x,y] \\ & k[w] \arrow[ur, "w\to y^3"'] & \end{tikzcd}$$

The inverse limit of this inverse system is the cuspidal curve, which is not normal.

In general, we have the following result.

Proposition 7.2.15. *Let k be a field. Let $\{Y_i\}$ be a finite inverse system of varieties over k, where all transition maps are dominant, and denote by Y_i^ν the normalization of each Y_i. Then $\{Y_i^\nu\}$ forms a finite inverse system and*

$$\left(\varprojlim Y_i\right)^\nu \cong \left(\varprojlim Y_i^\nu\right)^\nu .$$

Proof. The first assertion follows from the universal property of normalization, every morphism $f_{ij} : Y_j \to Y_i$ induces a morphism $f_{ij}^\nu : Y_j^\nu \to Y_i^\nu$ satisfying the condition of compatibility.

Let $\pi_i : \varprojlim Y_i \to Y_i$ be the projection and $\pi_i^\nu : \left(\varprojlim Y_i\right)^\nu \to Y_i$ be the composition of the projection π_i and the morphism of normalization $\left(\varprojlim Y_i\right)^\nu \to \varprojlim Y_i$. By the universal property of normalization, the π_i induce morphisms

$$\pi_i^\nu : \left(\varprojlim Y_i\right)^\nu \to Y_i^\nu$$

such that $f_{ij}^\nu \circ \pi_j^\nu = \pi_i^\nu$ for every $f_{ij} : Y_j^\nu \to Y_j^\nu$. Hence, by the universal property of inverse limit, we have the following commutative diagram

$$\begin{tikzcd} & \left(\varprojlim Y_i\right)^\nu \arrow[d, "g"] \arrow[ddl, "\pi_j^\nu"'] \arrow[ddr, "\pi_i^\nu"] & \\ & \varprojlim Y_i^\nu \arrow[dl, "p_j"] \arrow[dr, "p_i"'] & \\ Y_j^\nu \arrow[rr, "f_{ij}^\nu"'] & & Y_i^\nu . \end{tikzcd}$$

By the universal property of normalization, there exists a morphism

$$g^\nu : \left(\varprojlim Y_i\right)^\nu \to (\varprojlim Y_i^\nu)^\nu.$$

Simirlarly, by the universal property of normalization, we have a morphism

$$h^\nu : (\varprojlim Y_i^\nu)^\nu \to \left(\varprojlim Y_i\right)^\nu$$

that fits in the following commutative diagram

$$\begin{tikzcd}
 & \left(\varprojlim Y_i^\nu\right)^\nu \arrow[dl, bend right, "h^\nu"'] \arrow[d, "\alpha"] \arrow[dd, bend left, "h"] \\
\left(\varprojlim Y_i\right)^\nu \arrow[ur, bend right, "g^\nu"'] \arrow[r, "g"'] \arrow[dr, "\gamma"'] & \varprojlim Y_i^\nu \arrow[d, "\beta"] \\
 & \varprojlim Y_i.
\end{tikzcd}$$

The morphisms α and γ are birational, since they are normalization morphisms. The morphism β is also birational, because it comes from the birational morphisms $Y_i^\nu \to Y_i$ and the system is finite. Hence, h^ν is birational. Then, by Zariski's main Theorem, we have that h^ν is an isomorphism and, therefore, the second part of the assertion holds. □

Let us study the morphisms p_λ and p_0. Consider the following commutative diagram

$$\begin{tikzcd}
W \arrow[r, "\iota_\lambda"] \arrow[d, "q"'] & W_\lambda \arrow[r, "\iota_{\lambda\gamma}"] \arrow[d, "q_\lambda"'] & W_\gamma \arrow[r, "\iota_{\gamma 0}"] \arrow[d, "q_\gamma"'] & W_0 \arrow[dd, "q_0"'] \\
Y \arrow[r, "p_\lambda"] \arrow[drrr, "p_0"'] & Y_\lambda \arrow[r, "p_{\lambda\gamma}"] \arrow[drr, "p_{\lambda 0}"] & Y_\gamma \arrow[dr, "p_{\gamma 0}"] & \\
 & & & Y_0
\end{tikzcd} \tag{7.2}$$

Proposition 7.2.16. *The morphisms $p_\lambda : Y \to Y_\lambda$ and $p_{\lambda\gamma} : Y_\lambda \to Y_\gamma$ are projective surjections with geometrically connected fibers. Moreover, if* $\dim(Y_\lambda) = \dim(X) - \dim(T)$*, for example if λ intersects* $\operatorname{relint}(\omega^\vee)$*, then the morphism $p_\lambda : Y \to Y_\lambda$ is birational.*

Proof. Recall that the morphisms $p_{\lambda 0} : Y_\lambda \to Y_0$ are projective, because

$$Y_\lambda = \operatorname{Proj}\left(\bigoplus_{n\in\mathbb{Z}_{\geq 0}} A_{nm}\right)$$

for any $m \in \operatorname{relint}(\lambda) \cap M$. Hence, given that $p_{\lambda 0} = p_{\gamma 0} \circ p_{\lambda\gamma}$ is projective and $p_{\gamma 0}$ is separated, we have that $p_{\lambda\gamma}$ is projective by [Sta18, Tag 0C4Q].

By [Sta18, Tag 0C4Q], the morphisms $p_\lambda : Y \to Y_\lambda$ are projective. Since every Y_λ is dominated by W, all morphisms $p_\lambda : Y \to Y_\lambda$ are dominant. Together with properness, this implies surjectivity of each p_λ. The same holds for the morphisms $p_{\lambda\gamma}$.

Let λ and γ in Λ such that $\dim(Y_\lambda) = \dim(Y_\gamma) = \dim(X) - \dim(T)$. If $\gamma \preceq \lambda$, then $p_{\lambda\gamma} : Y_\lambda \to Y_\gamma$ is birational and, therefore, induces the identity between the field of rational functions $k(Y_\lambda) = k(X)^T = k(Y_\gamma)$. Given that Y can be constructed just by taking the subsystem Y_λ with $\lambda \cap \operatorname{relint}(\omega^\vee) \neq \emptyset$, where all the morphisms $p_{\lambda\gamma}$ are birational, we have that p_λ is birational.

The morphisms $p_\lambda : Y \to Y_\lambda$ are proper and surjective, then the generic point of Y goes to the generic point of Y_λ. Let us take the Stein factorization

$$\begin{tikzcd} Y \arrow[r, "f"'] \arrow[rr, bend left, "p_\lambda"] & Y'_\lambda \arrow[r, "g"'] & Y_\lambda, \end{tikzcd}$$

where Y'_λ is the relative normalization of Y_λ in Y, g is an integral finite morphim and f is a proper surjective morphism with geometrically connected fibers. Given that p_λ is surjective, we have that g is also surjective. Let $\nu : Y'^{\nu}_\lambda \to Y_\lambda$ be the normalization morphism. The morphism $h := g \circ \nu : Y'^{\nu}_\lambda \to Y_\lambda$ is integral, because is the composition of two integral morphisms. By [Sta18, Tag 035I], there exists a morphism $r : Y'_\lambda \to Y'^{\nu}_\lambda$ that fits into the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f^\nu"] \arrow[d, "f"'] & Y'^{\nu}_\lambda \arrow[d, "h:=g\circ\nu"] \arrow[dl, bend left, "\nu"] \\ Y'_\lambda \arrow[ur, bend left, "r"] \arrow[r, "g"'] & Y_\lambda \end{tikzcd}$$

and is the normalization of Y'_λ in Y_λ. Thus, $Y'_\lambda = Y'^{\nu}_\lambda$ is normal and $g : Y'_\lambda \to Y_\lambda$ is a finite (integral) morphism. Given that p_λ is birational and surjective, then g is birational. By [Sta18, Tag 0AB1], we have that g is an isomorphism. Thus, it follows that p_λ has geometrically connected fibers.

□

Thus, the normal k-variety Y is semiprojective. The construction above tells us how to construct a normal semiprojective k-variety from an affine normal k-variety X endowed with an effective action of a split k-torus T. In the following, we present some results that will help us to construct a pp-divisor $\mathfrak{D} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega)$, where $\omega^\vee \subset M_{\mathbb{Q}}$ is the weight cone associated to the T-action over X.

Let us give some context before. Recall that, for $\lambda \in \Lambda$ the quasifan associated to $\omega^\vee$ in Theorem 7.2.12, we have

$$Y_\lambda = \mathrm{Proj}(A_{(m)}), \text{ where } A_{(m)} := \bigoplus_{n \in \mathbb{N}} A_{nm}$$

and m any element in $\mathrm{relint}(\lambda) \cap M$. Thus, we can associate to m a sheaf $\mathscr{A}_{\lambda,m}$ on Y_λ given by

$$\mathscr{A}_{\lambda,m} := (q_\lambda)_*(\mathscr{O}_{W_\lambda})_m,$$

where $(\mathscr{O}_{W_\lambda})_m$ denotes the sheaf of semiinvariants with respect to the T-linearization with respect to m. The following results are in [AH06, Section 6] and their proofs follow directly in this context.

Lemma 7.2.17. *[AH06, Lemma 6.3] Let $\lambda \in \Lambda$ and $m \in \mathrm{relint}(\lambda) \cap M$. For $f \in A_{nm}$, let $Y_{\lambda,f} := q_\lambda(X_f)$ be the corresponding affine chart of Y_λ.*

i) On $Y_{\lambda,f}$, the sheaf $\mathscr{A}_{\lambda,m}$ is the coherent $\mathscr{O}_{Y_\lambda}$-module associated to the $(A_f)_0$-module $(A_f)_m$.

ii) If m is saturated, i.e. the ring $A_{(m)}$ is generated in degree one, then $\mathscr{A}_{\lambda,m}$ is an ample invertible sheaf on Y_λ. On the charts $Y_{\lambda,f}$, where $f \in A_m$, we have

$$\mathscr{A}_{\lambda,m} = f \cdot (A_f)_0 = f \cdot \mathscr{O}_{Y_\lambda}.$$

iii) If $g \in \mathrm{Quot}(A)$ and $n \in \mathbb{N}$, then $g^n \in \mathscr{A}_{\lambda,nm}$ implies $g \in \mathscr{A}_{\lambda,m}$.

iv) The global sections of $\mathscr{A}_{\lambda,m}$ are $H^0(Y_\lambda, \mathscr{A}_{\lambda,m}) = A_m$.

For each $\lambda \in \Lambda$ and $m \in \mathrm{relint}(\lambda)$, we have a coherent sheaf $\mathscr{A}_m := p_\lambda^* \mathscr{A}_{\lambda,m}$ with $p_\lambda : Y \to Y_\lambda$. Thus, for each $m \in \omega^\vee \cap M$, we have the coherent sheaf $\mathscr{A}_m$ over Y. These sheaves satisfy the following.

Lemma 7.2.18. *[AH06, Lemma 6.4] Let $m, m' \in \omega^\vee \cap M$.*

i) We have $k(Y) = \mathrm{Quot}(A)_0$, and the natural transformation $p_\lambda^* q_{\lambda *} \to q_* j_\lambda^*$ sends $\mathscr{A}_m$ into $\mathrm{Quot}(A)_m$.

ii) *Let m be saturated. Then $\mathscr{A}_m$ is a globally generated invertible sheaf. On the (not necessarily affine) sets $Y_f := p_\lambda^{-1}(Y_{\lambda,f})$ with $f \in A_m$, we have*

$$\mathscr{A}_m = f \cdot \mathscr{O}_Y \subset f \cdot k(Y) = \mathrm{Quot}(A)_m.$$

Moreover, for the global sections of $\mathscr{A}_m$, we obtain $H^0(Y, \mathscr{A}_m) = A_m$.

iii) *If m, m' and $m + m'$ are saturated, then $\mathscr{A}_m\mathscr{A}_{m'} \subset \mathscr{A}_{m+m'}$. If, moreover, m and m' lie in a common cone of Λ, then the equality holds.*

Now we are ready to prove [AH06, Theorem 3.4] for every affine normal k-variety endowed with an effective action of a split k-torus over any field k of characteristic zero.

Proposition 7.2.19. *Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let $T := \mathrm{Spec}(k[M])$ be a split k-torus and $X := \mathrm{Spec}(A)$ be a geometrically integral normal affine k-variety endowed with an effective T-action. Then, there exists a pp-divisor $\mathfrak{D}$ in $\mathfrak{PPDiv}_N(k)$ such that $X \cong X[Y, \mathfrak{D}]$ as T-varieties.*

Proof. The cone ω corresponds to the dual of the weight cone $\omega^\vee$ induced by the M-graduation and Y is constructed as above. The construction of the pp-divisor follows from a construction of a map $\mathfrak{h} \in \mathrm{CPL}_\mathbb{Q}(Y, \omega)$ as in [AH06, Section 6]. First, choose a homomorphism $s : M \to \mathrm{Quot}(A)^*$ such that for every $m \in M$ $s(m)$ is homogeneous of degree m. This choice is non-canonical and always exists because T acts effectively on X. For each saturated $m \in \omega^\vee \cap M$, there exists a unique Cartier divisor $\mathfrak{h}(m) \in \mathrm{CaDiv}(Y)$ such that

$$\mathscr{O}_Y(\mathfrak{h}(m)) = \frac{1}{s(m)} \cdot \mathscr{A}_m \subset k(Y),$$

whose local equation on Y_f, for $f \in A_m$, is $s(m)/f$. If $m \in \omega^\vee \cap M$ is not saturated, choose a saturated multiple nm (such a saturated multiple always exists by [Bou06, Proposition III.1.3]) and define

$$\mathfrak{h}(m) := \frac{1}{n} \cdot \mathfrak{h}(nm) \in \mathrm{CaDiv}_\mathbb{Q}(Y).$$

This definition does not depend on the choice of $n \in \mathbb{N}$.

Let Λ be the quasifan of Theorem 7.2.12. By Lemma 7.2.18, the map is convex and piecewise linear on Λ. Moreover, given that for $m \in \mathrm{relint}(\lambda) \cap M$ the sheaves $\mathscr{A}_m$ are big, then the $\mathfrak{h}(m)$ are big. Then $\mathfrak{h} \in \mathrm{CPL}_\mathbb{Q}(Y, \omega)$ and, by Proposition 7.1.17, there exists a pp-divisor $\mathfrak{D} \in \mathrm{PPDiv}_\mathbb{Q}(Y, \omega)$ such that

$\mathfrak{h}_{\mathfrak{D}} = \mathfrak{h}$. By Lemma 7.2.18, we have that $H^0(Y, \mathscr{A}_m) = A_m$, therefore if $m \in \omega^\vee \cap M$ is saturated

$$s(m) \cdot H^0(Y, \mathscr{O}_Y(\mathfrak{D}(m))) = H^0(Y, \mathscr{A}_m) = A_m.$$

If $m \in \omega^\vee \cap M$ is not saturated and nm is a saturated multiple, we have

$$g \in H^0(Y, \mathscr{O}_Y(\mathfrak{D}(m))) \Leftrightarrow g^n \in H^0(Y, \mathscr{O}_Y(\mathfrak{D}(nm))) \Leftrightarrow (gs(m))^n \in A_{nm}.$$

Given that A is normal, $g \in H^0(Y, \mathscr{O}_Y(\mathfrak{D}(m)))$ if and only if $gs(m) \in A_m$. This defines an isomorphism of M-graded k-algebras

$$A[Y, \mathfrak{D}] := \bigoplus_{m \in \omega^\vee \cap M} H^0(Y, \mathscr{O}_Y(\mathfrak{D}(m))) \to \bigoplus_{m \in \omega^\vee \cap M} A_m = A.$$

Finally we have that there exists a triple $(\omega, Y, \mathfrak{D})$ such that

$$X = \operatorname{Spec}(A) \cong \operatorname{Spec}(A[Y, \mathfrak{D}]) = X[Y, \mathfrak{D}].$$

This proves the assertion. □

This proposition proves (i) of Theorem 7.2.1.

Every affine normal variety endowed with an effective action of a split algebraic torus arises from a pp-divisors on some normal semiprojective variety. There are many pp-divisors encoding the same pair, the variety and the action. For example, let $\Delta := [1, +\infty] \subset \mathbb{Q}$, the action

$$\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^2 \to \mathbb{A}^2,$$
$$(\lambda, (x, y)) \mapsto (\lambda x, y)$$

is encoded both by $\mathfrak{D}_1 := \Delta \otimes \{0\}$ on $\mathbb{A}^1$ and $\mathfrak{D}_2 := \Delta \otimes \{0\} + \emptyset \otimes \{\infty\}$ on $\mathbb{P}^1$. However, there is notion of *minimality* for pp-divisors. The following is part of [AH06, Section 8], but by Galois descent tools the following holds for a non algebraically closed field of characteristic zero. Let $\mathfrak{D} \in \operatorname{PPDiv}_{\mathbb{Q}}(Y, \omega)$ be a pp-divisor. Given that $\mathfrak{D}(m)$ is semiample for every $m \in \omega^\vee \cap M$, we have natural morphisms

$$\vartheta_m : Y \to Y_m := \operatorname{Proj}\left(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} H^0(Y, \mathscr{O}(\mathfrak{D}(nm)))\right)$$

that are contraction maps. Moreover, they are birational if $m \in \operatorname{relint}(\omega^\vee) \cap M$.

Denote $X := X(\mathfrak{D})$. We can prove that all the Y_m correspond to the GIT-quotients of the semistable subvarieties for the respective linearization of the trivial bundle. Then, all the spaces $Y_\lambda := Y_m$, with $m \in \mathrm{relint}(\lambda)$ and $\lambda \in \Lambda$ the quasifan in Theorem 7.2.12, can be put into an inverse system compatible with the morphisms $\vartheta_\lambda : Y \to Y_\lambda$. Hence, we have a projective and birational morphism

$$\vartheta : Y \to \varprojlim Y_\lambda.$$

The scheme $\varprojlim Y_\lambda$ comes with a canonical reduced component, which is the schematic image of $q : W \to \varprojlim Y_\lambda$ for W the intersection of all semistable subvarieties. The schematic image of $\vartheta : Y \to \varprojlim Y_\lambda$ lies on $q(W)$.

Definition 7.2.20. A pp-divisor $\mathfrak{D} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega)$ is said to be minimal if the morphism $\vartheta : Y \to \varprojlim Y_\lambda$ is the normalization of the canonical reduced component of $\varprojlim Y_\lambda$.

In particular, the pp-divisor constructed in Proposition 7.2.19 are minimal.

Proposition 7.2.21. *Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let $\mathfrak{D}$ be an object in $\mathfrak{PPDiv}(k)$. The pp-divisor $\mathfrak{D}$ is minimal if and only if $\mathfrak{D}_{\bar{k}}$ is minimal.*

Proof. By definition, $\mathfrak{D} \in \mathrm{PPDiv}(Y, \omega)$, with Y a normal semiprojective variety over k and $\omega \subset N_{\mathbb{Q}}$ a cone. The varieties $X(\mathfrak{D})$ and $X(\mathfrak{D}_{\bar{k}})$ have the same quasifan decomposition Λ for $\omega^\vee$. Then, we have the following commutative diagram

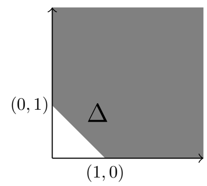
$$\begin{tikzcd} \bar{Y} \arrow[r, "\bar{\vartheta}"] \arrow[d] & \varprojlim \bar{Y}_\lambda \arrow[d] \\ Y \arrow[r, "\vartheta"'] & \varprojlim Y_\lambda, \end{tikzcd}$$

where the vertical arrows correspond to the the base change. Denote by Y' (respectively $\bar{Y}'$) the canonical reduced component of $\varprojlim Y_\lambda$ (respectively $\varprojlim \bar{Y}_\lambda$) Given that $\bar{Y}' = (Y')_{\bar{k}}$, the morphism $\bar{\vartheta} : \bar{Y} \to \bar{Y}'$ is the normalization of $\bar{Y}'$ if and only if $\vartheta : Y \to Y'$ is the normalization of Y'. $\square$

Example 7.2.22. Let k be a field of characteristic zero. The algebraic torus $\mathbb{G}_{\mathrm{m},k}^2$ acts over the tree dimensional affine space $\mathbb{A}_k^3$. Let us consider the action given by

$$(\lambda, \mu) \cdot (x, y, z) = (\lambda x, \mu y, \lambda\mu z).$$

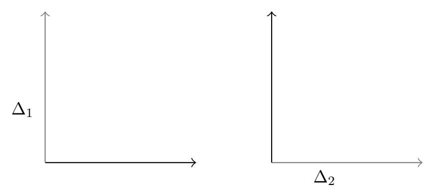
This action is encoded by the pp-divisor $\mathfrak{D} := \Delta \otimes \{\infty\}$ over $\mathbb{P}^1_k$, where Δ is the polyhedron



Example 7.2.23. Let k be a field of characteristic zero. The algebraic group $\mathrm{SL}_{2,k}$ is a normal variety over k with a $\mathbb{G}^2_{\mathrm{m},k}$-structure. Let us consider the action

$$(\lambda, \mu) \cdot (x, y, z, w) = (\lambda x, \mu y, \mu^{-1} z, \lambda^{-1} w).$$

This action is encoded by the pp-divisor $\mathfrak{D} := \Delta_1 \otimes [0] + \Delta_2 \otimes [1]$, where the polyhedra are $\Delta_1 := \mathrm{cone}(0, 1)$ and $\Delta_2 := \mathrm{cone}(1, 0)$ as shown in the following picture.



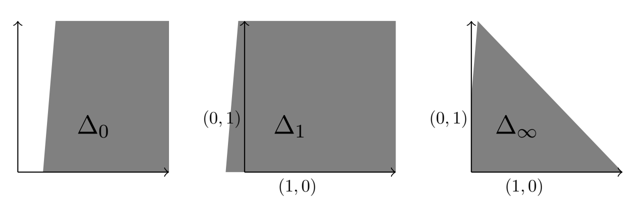
Example 7.2.24. [AH06, Example 11.1] Let k be a field of characteristic zero. The affine threefold $X := \mathrm{Spec}(k[x, y, z, w]/(x^3 + y^4 + zw))$ in $\mathbb{A}^4_k$ with the action of $\mathbb{G}^2_{\mathrm{m},k}$ given by

$$(\lambda, \mu) \cdot (x, y, z, w) = (\lambda^4 x, \lambda^3 y, \mu z, \lambda^{12} \mu^{-1} w)$$

is encoded by the pp-divisor $\mathfrak{D} := \Delta_0 \otimes \{0\} + \Delta_1 \otimes \{1\} + \Delta_\infty \otimes \{\infty\}$, where

$$\Delta_0 = \left(\frac{1}{3}, 0\right) + \omega, \quad \Delta_1 = \left(-\frac{1}{4}, 0\right) + \omega, \quad \Delta_\infty = (\{0\} \times [0, 1]) + \omega$$

and $\omega = \mathrm{cone}((1, 0), (1, 12))$.



Example 7.2.25. Let k be a field of characteristic zero. The affine space $\mathbb{A}^3_k$ with the action of $\mathbb{G}_{\mathrm{m},k}$ given by

$$\lambda \cdot (x, y, z) = (\lambda x, \lambda y, \lambda^{-1} z)$$

is encoded by the pp-divisor

$$\mathfrak{D} := \{1\} \otimes D_{(1,0)} + \{0\} \otimes D_{(0,1)} + [0,1] \otimes D_{(1,1)} \in \mathrm{PPDiv}_{\mathbb{Q}}(\mathrm{Bl}_0(\mathbb{A}^2_k), \omega)$$

where $D_{(1,0)}$, $D_{(0,1)}$ and $D_{(1,1)}$ are the toric invariant divisor of $\mathrm{Bl}_0(\mathbb{A}^2_k)$ associated to the rays $\mathrm{cone}(0,1)$ and $\mathrm{cone}(1,1)$, respectively, and $\omega = \mathrm{cone}(0)$.

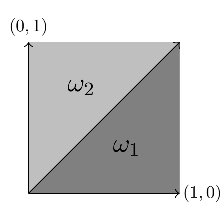
Remark 7.2.26. Since all the examples above are computed by following [AH06, Section 11], they are all minimal over the algebraic closure. Thus, they are minimal over the ground field by Proposition 7.2.21. The latter is of complexity two, so we prove its minimality by following the construction given in [AH06, Section 11]. As a toric variety, $\mathbb{A}^3_k$ under the action of $\mathbb{G}^3_{\mathrm{m},k}$ coordinatewise is given by the cone

$$\omega = \mathrm{cone}((1,0,0), (0,1,0), (0,0,1)).$$

The action of $\mathbb{G}_{\mathrm{m},k}$ on $\mathbb{A}^3_k$ in Example 7.2.25 follows from the embedding $\lambda \to (\lambda, \lambda, \lambda^{-1})$ of the respective tori. This embedding, in terms of their module of cocharacters, is equivalent to the morphism $\mathbb{Z} \to \mathbb{Z}^3$ given by $a \mapsto (a, a - a)$. This latter morphisms fits into the following exact sequence of $\mathbb{Z}$-modules:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "F"] & \mathbb{Z}^3 \arrow[l, bend left, "s"] \arrow[r, "P"] & \mathbb{Z}^2 \arrow[r] & 0 \end{tikzcd},$$

where $F(a) = (a, a, -a)$, $P(a,b,c) = (a + c, b + c)$ and $s(a,b,c) = (a)$. This latter map is a section of F, which can be chosen. Therefore, it is not canonical. Now, we look for the images of the rays of ω by P, which are $P(1,0,0) = (1,0)$, $P(0,1,0) = (0,1)$ and $P(0,0,1) = (1,1)$. The smallest fan in $\mathbb{Z}^2$ admitting $(1,0)$, $(0,1)$ and $(1,1)$ as rays is



This fan correspond to GIT-quotient constructed in Proposition 7.2.19 for the $\mathbb{G}_{\mathrm{m},k}$-action. Besides, this fan corresponds to $\mathrm{Blow}_0(\mathbb{A}^2_k)$.

Each ray corresponds to a toric invariant divisor $D_{(1,0)}$, $D_{(0,1)}$ and $D_{(1,1)}$ of $\mathrm{Blow}_0(\mathbb{A}^2_k)$.

Let us now compute the polyhedra. The exact sequence of the cocharacter modules extend to exact sequence of $\mathbb{Q}$-vector spaces, and so the morphisms.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Q} \arrow[r, "F"] & \mathbb{Q}^3 \arrow[l, bend left, "s"] \arrow[r, "P"] & \mathbb{Q}^2 \arrow[r] & 0 . \end{tikzcd}$$

The polyhedron associated to each toric divisor are compute as

$$\Delta_{(i,j)} := s\left(\omega \cap P^{-1}(i,j)\right),$$

for $i, j \in \{0,1\}$. Thus,

$$\Delta_{(1,0)} = s\left(\{(1-c, -c, c) \mid -c \geq 0 \text{ and } c \geq 0\}\right) = \{1\},$$
$$\Delta_{(0,1)} = s\left(\{(-c, 1-c, c) \mid -c \geq 0 \text{ and } c \geq 0\}\right) = \{0\},$$

and

$$\Delta_{(1,1)} = s\left(\{(1-c, 1-c, c) \mid 1-c \geq 0 \text{ and } c \geq 0\}\right) = [0,1].$$

Thus, the corresponding pp-divisor

$$\mathfrak{D} = \{1\} \otimes D_{(1,0)} + \{0\} \otimes D_{(0,1)} + [0,1] \otimes D_{(1,1)} \in \mathrm{PPDiv}_{\mathbb{Q}}(\mathrm{Bl}_0(\mathbb{A}^2_k), \omega)$$

is minimal.

An advantage of this point of view is that results from [AH06] and [AHS08] work in the setting of a nonalgebraically closed field and a split torus. This allows us to generalized such results to more general morphisms (see: Section 9.5), which gives a relation between *semilinear* equivariant isomorphisms of affine normal T-varieties and *semilinear* morphisms of pp-divisors.

7.3 Normal varieties and split tori

Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. When $k = \bar{k}$, every normal T variety arises from a divisorial fan $(\mathfrak{S}, Y)$ by Theorem 5.0.3. The goal of this section is to prove that this holds when k is no longer algebraically closed and T is a split k-torus. We recall the definitions of *face* in the context of pp-divisors and of *divisorial fan*.

In the last part of the section, we generalize Theorem 5.0.3 for any field of characteristic zero and a split torus.

Theorem 7.3.1. *Let k be a field of characteristic zero and T be a split k-torus. Up to isomorphism, every normal k-variety with an effective T-action arises from a divisorial fan $(\mathfrak{S}, Y)$.*

7.3.1 Divisorial fans

Definition 7.3.2. Let N be a lattice, $\omega, \omega' \subset N_{\mathbb{Q}}$ be pointed cones, Y be a normal variety and consider two pp-divisors on Y:

$$\mathfrak{D}' = \sum \Delta'_D \otimes D \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega'), \quad \mathfrak{D} = \sum \Delta_D \otimes D \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega).$$

We call $\mathfrak{D}'$ a *face* of $\mathfrak{D}$ (written $\mathfrak{D}' \preceq \mathfrak{D}$) if $\Delta'_D \subset \Delta_D$ holds for all D and for any $y \in \mathrm{Loc}(\mathfrak{D}')$ there are $m_y \in \omega^\vee \cap M$ and a D_y in the linear system $|\mathfrak{D}(m_y)|$ with

(i) $y \notin \mathrm{Supp}(D_y)$,

(ii) $\Delta'_y = \mathrm{face}(\Delta_y, m_y)$,

(iii) $\mathrm{face}(\Delta'_z, m_y) = \mathrm{face}(\Delta_z, m_y)$ for every $z \in Y \setminus \mathrm{Supp}(D_y)$.

Recall that the relation $\mathfrak{D}' \leq \mathfrak{D}$ between two pp-divisors means that $\Delta_D \subset \Delta'_D$ for every prime divisor $D \in \mathrm{CaDiv}(Y)$. Then, if $\mathfrak{D}' \preceq \mathfrak{D}$ we have $\Delta'_D \subset \Delta_D$ and, therefore, $\mathfrak{D} \leq \mathfrak{D}'$. Hence, for any pair of pp-divisors $\mathfrak{D}'$ and $\mathfrak{D}$ in $\mathfrak{PPDiv}(k)$ such that $\mathfrak{D}' \preceq \mathfrak{D}$ we have that the triple $(\mathrm{id}, \mathrm{id}, 1)$ defines a morphism of pp-divisors $\mathfrak{D}' \to \mathfrak{D}$.

Let $\mathfrak{D}'$ and $\mathfrak{D}$ in $\mathfrak{PPDiv}(k)$ be such that $\mathfrak{D}' \preceq \mathfrak{D}$ and let $(\mathrm{id}, \mathrm{id}, 1) : \mathfrak{D}' \to \mathfrak{D}$ be its corresponding morphism of pp-divisors. Given that $\mathfrak{D} \leq \mathfrak{D}'$, we have that $\mathfrak{D}(m) \leq \mathfrak{D}'(m)$ for every $m \in \omega^\vee \cap M$. Then, $H^0(Y, \mathscr{O}(\mathfrak{D}(m))) \subset H^0(Y, \mathscr{O}(\mathfrak{D}'(m)))$ for every $m \in \omega'^\vee \cap M$. These inclusions induce an inclusion of M-graded k-algebras

$$A[Y, \mathfrak{D}] = \bigoplus_{m \in \omega^\vee \cap M} H^0(Y, \mathscr{O}(\mathfrak{D}(m))) \subset \bigoplus_{m \in \omega'^\vee \cap M} H^0(Y, \mathscr{O}(\mathfrak{D}'(m))) = A[Y, \mathfrak{D}'],$$

which induces a T-equivariant inclusion

$$X(\mathfrak{D}') := \mathrm{Spec}(A[Y, \mathfrak{D}']) \subset \mathrm{Spec}(A[Y, \mathfrak{D}]) =: X(\mathfrak{D})$$

between the respective affine normal T-varieties over k,. Thus, from the morphism of pp-divisors $(\mathrm{id}, \mathrm{id}, 1) : \mathfrak{D}' \to \mathfrak{D}$ we have a T-equivariant inclusion $\iota : X(\mathfrak{D}') \to X(\mathfrak{D})$. Every T-equivariant inclusion $X(\mathfrak{D}') \subset X(\mathfrak{D})$, of pp-divisors over Y, arises from a morphism of pp-divisors $(\mathrm{id}, \mathrm{id}, 1) : \mathfrak{D}' \to \mathfrak{D}$, but not necessarily from a face relation.

Proposition 7.3.3. *Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let N be a lattice, ω and ω' in $N_{\mathbb{Q}}$ be pointed cones and Y be a geometrically integral normal semiprojective variety over k. Let $\mathfrak{D} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega)$ and $\mathfrak{D}' \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega')$, then*

$$\mathfrak{D}' \preceq \mathfrak{D} \Leftrightarrow \mathfrak{D}'_{\bar{k}} \preceq \mathfrak{D}_{\bar{k}}.$$

Proof. Let $\mathfrak{D}' \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega')$ and $\mathfrak{D} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega)$. By Lemma 7.1.12, we have $\mathfrak{D}'_{\bar{k}} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}}, \omega')$ and $\mathfrak{D}_{\bar{k}} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y_{\bar{k}}, \omega)$. First, notice that $\Delta'_D \subset \Delta_D$ for every effective divisor $D \in \mathrm{Div}(Y)$ if and only if $\Delta'_{\tilde{D}} \subset \Delta_{\tilde{D}}$ for every effective divisor $\tilde{D} \in \mathrm{Div}(Y_{\bar{k}})$.

Suppose that $\mathfrak{D}' \preceq \mathfrak{D}$. Let $\bar{y} \in \mathrm{Loc}(\mathfrak{D}_{\bar{k}})$ and $y \in \mathrm{Loc}(\mathfrak{D})$ with $\bar{y}$ lying above y. By the assumption, we have that there exist $m_y \in \omega^\vee \cap M$ and $D_y \in |\mathfrak{D}(m_y)|$ satisfying the condition of Definition 7.3.2. Now, by Lemma 7.1.14, we have that

$$\Delta'_{\bar{y}} = \Delta'_y = \mathrm{face}(\Delta_y, m_y) = \mathrm{face}(\Delta_{\bar{y}}, m_y).$$

So, by taking $m_{\bar{y}} := m_y$ we have point (ii) of Definition 7.3.2. For the remaining parts it is enough to take $D_{\bar{y}}$ as the corresponding divisor associated to D_y under the morphism $\mathrm{Div}(Y) \to \mathrm{Div}(Y_{\bar{k}})$ and apply Lemma 7.1.14 again. Thus, $\mathfrak{D}' \preceq \mathfrak{D}$ implies that $\mathfrak{D}'_{\bar{k}} \preceq \mathfrak{D}_{\bar{k}}$.

Now suppose that $\mathfrak{D}'_{\bar{k}} \preceq \mathfrak{D}_{\bar{k}}$. Then, $\bar{\iota} : X(\mathfrak{D}'_{\bar{k}}) \to X(\mathfrak{D}_{\bar{k}})$ is an open immersion by [AHS08, Proposition 3.4]. Hence, by [Sta18, Tag 02L3], the morphism $\iota : X(\mathfrak{D}') \to X(\mathfrak{D})$ is also an open immersion. Consider the $\mathscr{O}_Y$-algebras

$$\mathscr{A} := \bigoplus_{m \in \omega^\vee \cap M} \mathscr{O}(\mathfrak{D}(m)) \quad \text{and} \quad \mathscr{A}' := \bigoplus_{m \in \omega'^\vee \cap M} \mathscr{O}(\mathfrak{D}'(m))$$

and the $\mathscr{O}_{Y_{\bar{k}}}$-algebras

$$\mathscr{A}_{\bar{k}} := \bigoplus_{m \in \omega^\vee \cap M} \mathscr{O}(\mathfrak{D}_{\bar{k}}(m)) \quad \text{and} \quad \mathscr{A}'_{\bar{k}} := \bigoplus_{m \in \omega'^\vee \cap M} \mathscr{O}(\mathfrak{D}'_{\bar{k}}(m)).$$

Denote $\tilde{X} = \mathrm{Spec}_Y(\mathscr{A})$, $\tilde{X}' = \mathrm{Spec}_Y(\mathscr{A}')$, $\tilde{X}_{\bar{k}} = \mathrm{Spec}_{Y_{\bar{k}}}(\mathscr{A}_{\bar{k}})$ and $\tilde{X}'_{\bar{k}} = \mathrm{Spec}_{Y_{\bar{k}}}(\mathscr{A}'_{\bar{k}})$ and consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Loc}(\mathfrak{D}'_{\bar{k}}) \arrow[d] & \tilde{X}'_{\bar{k}} \arrow[l, "\bar{\pi}'"'] \arrow[r, "\bar{r}'"] \arrow[d] & X(\mathfrak{D}'_{\bar{k}}) \arrow[r, "\bar{\iota}"] \arrow[d] & X(\mathfrak{D}_{\bar{k}}) \arrow[d] \\
\mathrm{Loc}(\mathfrak{D}') & \tilde{X}' \arrow[l, "\pi'"'] \arrow[r, "r'"] & X(\mathfrak{D}') \arrow[r, "\iota"] & X(\mathfrak{D}).
\end{tikzcd}$$

Let $y \in \mathrm{Loc}(\mathfrak{D}')$ be a closed point. By [Spr09, Lemma 2.3.3], there exists a closed orbit $T \cdot z' \subset (\pi')^{-1}(y)$. Moreover, by [AHS08, Lemma 4.5], such an orbit is unique. Given that $\iota : X(\mathfrak{D}') \to X(\mathfrak{D})$ is an open immersion, there exists $m_y \in \omega^\vee \cap M$ and $f \in A_{m_y}$ such that $f(r'(z')) \neq 0$ and $f_{|X(\mathfrak{D}) \smallsetminus X(\mathfrak{D}')} = 0$, with $A := A[Y, \mathfrak{D}]$. Let $\{z_1, \dots, z_l\} \subset Y_{\bar{k}}$ be the points lying above z and $\{y_1, \dots, y_l\} \subset Y_{\bar{k}}$ be the points lying above y such that $T \cdot z_i' \subset (\bar{\pi}')^{-1}(y_i)$ are the respective closed orbits. The number of points lying over z and y is equal to the degree of the extensions $k(z)$ and $k(y)$ over k because they are closed points. After base change, setting $\bar{A} := A[Y_{\bar{k}}, \mathfrak{D}_{\bar{k}}]$, we have that $f \in \bar{A}_{m_y}$ satisfies $f(\bar{r}'(z_i')) = f(r'(z')) \neq 0$, for every $i \in \{1, \dots, l\}$, and $f_{|X(\mathfrak{D}_{\bar{k}}) \smallsetminus X(\mathfrak{D}'_{\bar{k}})} = 0$. Hence, by [AHS08, Proposition 3.4], the divisor $\bar{D} := \operatorname{div}(f) + \mathfrak{D}_{\bar{k}}(m_y) \in |\mathfrak{D}_{\bar{k}}(m_y)|$ satisfies

- $y_i \notin \operatorname{Supp}(\bar{D})$ for every $i \in \{1, \dots, l\}$,
- $\Delta'_{y_i} = \operatorname{face}(\Delta_{y_i}, m_y)$ for every $i \in \{1, \dots, l\}$, and
- $\operatorname{face}(\Delta'_z, m_y) = \operatorname{face}(\Delta_z, m_y)$ for every $z \in Y_{\bar{k}} \smallsetminus \operatorname{Supp}(\bar{D})$.

Let $D_y := \operatorname{div}(f) + \mathfrak{D}(m_y)$ in $|\mathfrak{D}(m_y)|$. By construction, $y \notin \operatorname{Supp}(D_y)$, hence (i) of Definition 7.3.2. By Lemma 7.1.14, we have that $\Delta_{y_i} = \Delta_y$, then (ii) of Definition 7.3.2 holds too. Finally, also by Lemma 7.1.14, if $z \in Y \smallsetminus \operatorname{Supp}(D_y)$ and $z_1 \in Y_{\bar{k}} \smallsetminus \operatorname{Supp}(\bar{D})$ lies over z, we have that

$$\operatorname{face}(\Delta'_z, m_y) = \operatorname{face}(\Delta'_{z_1}, m_y) = \operatorname{face}(\Delta_{z_1}, m_y) = \operatorname{face}(\Delta_z, m_y),$$

therefore (iii) of Definition 7.3.2 holds. Thus, $\mathfrak{D}' \preceq \mathfrak{D}$.

□

The T-equivariant inclusions $\iota : X(\mathfrak{D}') \to X(\mathfrak{D})$ that arise from a face relation are exactly those that are open immersions.

Corollary 7.3.4. *Let k be a field of characteristic zero. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two pp-divisors such that $\mathfrak{D}' \leq \mathfrak{D}$. Then $\mathfrak{D}' \preceq \mathfrak{D}$ if and only if $\iota : X(\mathfrak{D}') \to X(\mathfrak{D})$ is an open immersion.*

Proof. Let $\bar{k}$ be an algebraic closure. By [Sta18, Tag 02L3], $X(\mathfrak{D}') \to X(\mathfrak{D})$ is an open immersion if and only if $X(\mathfrak{D}'_{\bar{k}}) \to X(\mathfrak{D}_{\bar{k}})$ is an open immersion. By [AHS08, Proposition 3.4], we have that $X(\mathfrak{D}'_{\bar{k}}) \to X(\mathfrak{D}_{\bar{k}})$ is an open immersion if and only if $\mathfrak{D}'_{\bar{k}} \preceq \mathfrak{D}_{\bar{k}}$. Then, by Proposition 7.3.3, the assertion holds. □

Remark 7.3.5. Let $\mathfrak{D} \in \operatorname{PPDiv}_{\mathbb{Q}}(Y, \omega)$ and $\mathfrak{D}' \in \operatorname{PPDiv}_{\mathbb{Q}}(Y', \omega')$ be pp-divisors and denote by $T_{\mathfrak{D}'}$ the split algebraic torus acting on $X(\mathfrak{D}')$ associated to $\mathfrak{D}'$. Assume that $X(\mathfrak{D}') \subset X(\mathfrak{D})$ is a $T_{\mathfrak{D}'}$-equivariant open embedding. If $\mathfrak{D}'$ and $\mathfrak{D}$ are minimal pp-divisors, it does not imply that there is an open immersion $Y' \to Y$. For example, consider the context of

Example 7.2.25 Let k be a field of characteristic zero. The affine space $\mathbb{A}^3_k$ with the action of $\mathbb{G}_{\mathrm{m},k}$ given by

$$\lambda \cdot (x, y, z) = (\lambda x, \lambda y, \lambda^{-1} z)$$

is encoded by the pp-divisor

$$\mathfrak{D} := \{1\} \otimes D_{(1,0)} + \{0\} \otimes D_{(0,1)} + [0, 1] \otimes D_{(1,1)} \in \mathrm{PPDiv}_{\mathbb{Q}}(\mathrm{Bl}_0(\mathbb{A}^2_k), \omega).$$

The $\mathbb{G}_{\mathrm{m},k}$-stable affine open subvariety $\mathbb{A}^2_k \times \mathbb{G}_{\mathrm{m},k}$ of $\mathbb{A}^3_k$ is encode by the following minimal pp-divisor

$$\mathfrak{D}' := \{1\} \otimes \tilde{D}_{(1,0)} + \{0\} \otimes \tilde{D}_{(0,1)} \in \mathrm{PPDiv}_{\mathbb{Q}}(\mathbb{A}^2_k, \omega).$$

Let $\pi : \mathrm{Bl}_0(\mathbb{A}^2_k) \to \mathbb{A}^2_k$. The pp-divisor $\mathfrak{D}'$ is not a face of $\mathfrak{D}$ because they do not *live* over the same semiprojective variety ($\mathrm{Blow}_0(\mathbb{A}^2_k)$ is not affine). However, after pulling back $\mathfrak{D}'$ by π, the pp-divisor $\pi^*\mathfrak{D}'$ is a face of $\mathfrak{D}$ and $X(\mathfrak{D}') \cong X(\pi^*(\mathfrak{D}))$.

Definition 7.3.6. Let N be a lattice and Y semiprojective normal k-variety. A *divisorial fan* in (Y, N) is a finite set $\mathfrak{S}$ of pp-divisors $\mathfrak{D}$ in $\mathfrak{PPDiv}_N(k)$ with base Y such that, for any pair $\mathfrak{D}, \mathfrak{D}' \in \mathfrak{S}$, the intersection $\mathfrak{D} \cap \mathfrak{D}'$ is in $\mathfrak{S}$ and $\mathfrak{D} \cap \mathfrak{D}' \preceq \mathfrak{D}, \mathfrak{D}'$.

Proposition 7.3.7. *Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let N be a lattice and Y be a normal semiprojective variety. A finite set of pp-divisors $\mathfrak{S}$ on (Y, N) is a divisorial fan if and only if $\mathfrak{S}_{\bar{k}}$ is a divisorial fan on $(Y_{\bar{k}}, N)$.*

Proof. This is a direct consequence of Proposition 7.3.3. □

7.3.2 From divisorial fans to normal T-varieties

We will see that Theorem 5.0.3 holds for normal varieties endowed with an effective action of a split torus over any field of characteristic zero.

Let $\mathfrak{S}$ be a divisorial fan over a semiprojective k-variety Y (we will use the pair $(\mathfrak{S}, Y)$ to refer to this data). If $\mathfrak{D}^i$ and $\mathfrak{D}^j$ are in $\mathfrak{S}$, then $\mathfrak{D}^i \cap \mathfrak{D}^j \in \mathfrak{S}$ is a face of both. Therefore, by Corollary 7.3.4, we have T-equivariant open embeddings

$$X(\mathfrak{D}^i) \xleftarrow{\eta_{ij}} X(\mathfrak{D}^i \cap \mathfrak{D}^j) \xrightarrow{\eta_{ji}} X(\mathfrak{D}^j) .$$

Let us set $X_{ij} := \mathrm{im}(\eta_{ij}) \subset X_i$ and $X_{ji} := \mathrm{im}(\eta_{ji}) \subset X_j$. Then, we have T-equivariant isomorphisms $\varphi_{ij} := \eta_{ji} \circ \eta_{ij}^{-1} : X_{ij} \to X_{ji}$. When k is algebraically closed, we construct the following quotient space by a relation over the points:

$$X(\mathfrak{S}) := \left(\bigsqcup_{\mathfrak{D} \in \mathfrak{S}} X(\mathfrak{D}) \right) / \sim,$$

where the relation is given by $x \sim y$ if and only if for some φ_{ij} we have $\varphi_{ij}(x) = y$.

Over nonalgebraically closed fields, we can also construct a scheme $X(\mathfrak{S})$. First, notice that a divisorial fan $(\mathfrak{S}, Y)$ is a partially ordered set under the face relation, as in the toric case with the *regular* fans. The T-equivariant open embeddings $\eta_{\mathfrak{E}\mathfrak{D}} : X(\mathfrak{E}) \to X(\mathfrak{D})$, for $\mathfrak{E} \preceq \mathfrak{D}$ in $\mathfrak{S}$, define a direct system satisfying the compatibility conditions: $\eta_{\mathfrak{F}\mathfrak{D}} = \eta_{\mathfrak{F}\mathfrak{E}} \circ \eta_{\mathfrak{E}\mathfrak{D}}$, for every $\mathfrak{F} \preceq \mathfrak{E} \preceq \mathfrak{D}$ in $\mathfrak{S}$. Thus, the scheme $X(\mathfrak{S})$ can be defined as

$$X(\mathfrak{S}) := \varinjlim_{\mathfrak{D} \in \mathfrak{S}} X(\mathfrak{D}).$$

Besides, over an algebraic closure $\bar{k}$, there is a canonical isomorphism between the two constructions.

Proposition 7.3.8. *Let k be a field of characteristic zero and Y be a normal semiprojective variety over k. Let $(\mathfrak{S}, Y)$ be a divisorial fan. The space $X(\mathfrak{S})$ is a prevariety over k with affine diagonal $X(\mathfrak{S}) \to X(\mathfrak{S}) \times X(\mathfrak{S})$, and it comes with a (unique) T-action such that all the canonical maps $X(\mathfrak{D}) \to X(\mathfrak{S})$ are T-equivariant.*

Proof. By Proposition 7.3.7, $(\mathfrak{S}_{\bar{k}}, Y_{\bar{k}})$ is a divisorial fan and, by [AHS08, Theorem 5.3], the assertion holds for $X_{\bar{k}}(\mathfrak{S}_{\bar{k}})$. Finally, by [Sta18, Tag 02L3] and [Gro65, Proposition 2.7.1], all the morphisms and their properties descend. Hence, the assertion holds for $X(\mathfrak{S})$. □

Separated divisorial fans Let $(\mathfrak{S}, Y)$ be a divisorial fan over k with Y a semiprojective k-variety. The scheme $X(\mathfrak{S})$ is not necessarily separated. In order to establish when this is the case, notice that, by Hironaka's Theorem, there exists a resolution of singularities $\pi : \tilde{Y} \to Y$ and the schemes $X(\mathfrak{S})$ and $X(\pi^*\mathfrak{S})$ are T-equivariantly isomorphic. Thus, we can always assume that Y is smooth.

A valuation μ on a variety Y over k is a valuation on its field of rational functions $\mu : k(Y)^* \to \mathbb{Q}$ with $\mu = 0$ along k. We say that $y \in Y$ is the (unique) center of μ if the valuation ring $(\mathcal{O}_\mu, \mathfrak{m}_\mu)$ dominates the local ring

$(\mathscr{O}_y, \mathfrak{m}_y)$, which means that $\mathscr{O}_y \subset \mathcal{O}_\mu$ and $\mathfrak{m}_y = \mathfrak{m}_\mu \cap \mathscr{O}_y$ hold. For a valuation μ on Y with center $y \in Y$, we have a well-defined group homomorphism

$$\begin{aligned} \mu : \mathrm{CaDiv}(Y) &\to \mathbb{Q}, \\ D &\mapsto \mu(f), \end{aligned}$$

where $D = \mathrm{div}(f)$ near y with $f \in k(Y)$. Besides, for Y smooth, this provides a weight function $\mu : \{\text{prime divisors on } Y\} \to \mathbb{Q}$.

Definition 7.3.9. Let k be a field of characteristic zero. Let N be a lattice, $\omega \subset N_\mathbb{Q}$ a cone and $\mathfrak{D} \in \mathrm{PPDiv}_\mathbb{Q}(Y, \omega)$, with Y a normal variety over k. If $\mu : \{\text{prime divisors of } Y\} \to \mathbb{R}$ is any map, then we define the associated *weighted sum of polyhedral coefficients* as

$$\mu(\mathfrak{D}) = \sum \mu(D)\Delta_D \in \mathrm{Pol}_\omega(N).$$

Definition 7.3.10. Let k be a field of characteristic zero. Let Y be a smooth semiprojective variety over k and let $(\mathfrak{S}, Y)$ a divisorial fan with polyhedral coefficients defined over $N_\mathbb{Q}$, with N a lattice. We say that $(\mathfrak{S}, Y)$ is *separated*, if for any two pp-divisors $\mathfrak{D}$ and $\mathfrak{D}'$ in $\mathfrak{S}_{\bar{k}}$ and any valuation μ on $Y_{\bar{k}}$, we have that $\mu(\mathfrak{D} \cap \mathfrak{D}') = \mu(\mathfrak{D}) \cap \mu(\mathfrak{D}')$.

Separated divisorial fans induce normal varieties endowed with an effective action of a split algebraic torus.

Proposition 7.3.11. *Let k be a field of characteristic zero. Let Y be a geometrically integral smooth semiprojective variety over k and $(\mathfrak{S}, Y)$ be a divisorial fan. Then, $X(\mathfrak{S})$ is separated if and only if $(\mathfrak{S}, Y)$ is separated.*

Proof. Let $\bar{k}$ be an algebraic closure. By [AHS08, Theorem 7.5], $X(\mathfrak{S}_{\bar{k}})$ is separated if and only if $(\mathfrak{S}_{\bar{k}}, Y_{\bar{k}})$ is separated. Let us suppose that $(\mathfrak{S}, Y)$ is separated. By definition, this means that $(\mathfrak{S}_{\bar{k}}, Y_{\bar{k}})$ is separated. Hence, by [AHS08, Theorem 7.5], $X(\mathfrak{S}_{\bar{k}})$ is separated. Then, by [Sta18, Tag 02KZ], we have that $X(\mathfrak{S})$ is separated.

Let us suppose that $X(\mathfrak{S})$ is separated. This implies that $X(\mathfrak{S}_{\bar{k}})$ is separated. Then, by [AHS08, Theorem 7.5], $(\mathfrak{S}_{\bar{k}}, Y_{\bar{k}})$ is separated. Then, by definition $(\mathfrak{S}, Y)$ is separated. This proves the assertion. $\square$

In the following, by a divisorial fan we mean a separated one.

7.3.3 From normal T-varieties to divisorial fans

A normal k-variety endowed with an effective action of a split algebraic k-torus T has a T-stable affine open covering by Sumihiro's Theorem. Such a covering can be taken stable under intersections. By Theorem 5.0.5, each T-stable affine open subvariety arises from a pp-divisors. However, all these pp-divisors could be defined over different normal semiprojective varieties and a divisorial fan is a set of pp-divisors defined over the same semiprojective variety. Then, we need to construct a normal semiprojective that allows us to *transfer* all the pp-divisors to compare them. Once this is done, we can prove the following.

Theorem 7.3.12. *Let k be a field of characteristic zero and T be a split torus over k. Up to isomorphism, every normal T-variety over k arises from a divisorial fan $(\mathfrak{S}, Y)$ over k.*

Proof. Let us fix an algebraic closure $\bar{k}$ of k. We will make the construction over the base field and use some of the results above.

By Sumihiro's Theorem, X has a T-stable affine open covering $\{X_i\}$. Such a covering can be considered stable under intersection. Then, by Theorem 7.2.1, there exist pp-divisors $\mathfrak{D}^i \in \mathrm{PPDiv}_{\mathbb{Q}}(\omega_i, \tilde{Y}^i)$ such that $X_i \cong X(\mathfrak{D}^i)$. Let us consider $X(\mathfrak{D}^{ij}) = X(\mathfrak{D}^i) \cap X(\mathfrak{D}^j)$. Let $\tilde{Y}^i \subset Y^i$ be a projective closure such that $Y^i \smallsetminus \tilde{Y}^i$ is the support of a semiample divisor, for every $\tilde{Y}^i$. By adding empty coefficients, we extend canonically $\mathfrak{D}^i$ to a pp-divisor over Y^i. We will abusively write $\mathfrak{D}^i$ for the extension of the pp-divisor to Y^i.

The natural embeddings $X(\mathfrak{D}^{ij}) \to X(\mathfrak{D}^i)$ and $X(\mathfrak{D}^{ij}) \to X(\mathfrak{D}^j)$ induce birational maps $Y^i \dashrightarrow Y^j$. Let us take the pp-divisor $\tilde{\mathfrak{D}} \in \mathrm{PPDiv}_{\mathbb{Q}}(\tilde{\omega}, \tilde{Y})$ such that $X(\tilde{\mathfrak{D}}) = \cap X(\mathfrak{D}^i)$. Then, there exist birational maps $\tilde{Y} \dashrightarrow Y^i$ for every semiprojective variety Y^i. Now, by Proposition 7.2.6, there exists a normal semiprojective variety Y with birational morphisms $\kappa_i : Y \to Y^i$. Define $(\mathfrak{S}, Y) := \{\kappa_i^* \mathfrak{D}^i \mid X(\mathfrak{D}^i) \cong X_i\}$, the set of pullbacks of all the pp-divisors. By [AHS08, Proof of Theorem 5.6], we have that $(\mathfrak{S}_{\bar{k}}, Y_{\bar{k}})$ is a divisorial fan over $\bar{k}$. Hence, by Proposition 7.3.7, we have that $(\mathfrak{S}, Y)$ is a divisorial fan over k.

By the universal property of the disjoint union, we have a T-equivariant morphism

$$\bigsqcup_{\mathfrak{D} \in \mathfrak{S}} X(\mathfrak{D}) \to X,$$

which induce a morphism of k-prevarieties

$$\left(\bigsqcup_{\mathfrak{D}\in\mathfrak{S}} X(\mathfrak{D})\right) / \sim := X(\mathfrak{S}) \to X.$$

By [AHS08, Proof of Theorem 5.6], after base change to $\bar{k}$, this morphism is an isomorphism of $\bar{k}$-varieties. Then, by Proposition 7.3.11, the theorem holds. $\square$

As a consequence of the proof we have the following corollary.

Corollary 7.3.13. *Let k be a field of characteristic zero and T be a split torus over k. Let X be a normal T-variety over k and $\mathcal{U}$ be a finite T-stable affine open covering of X that is stable under intersections, then there exists a divisorial fan $(\mathfrak{S}, Y)$ over k such that each pp-divisor in $\mathfrak{S}$ corresponds to an element of $\mathcal{U}$.*

Remark 7.3.14. Let k be a field of characteristic zero. If the normal T-variety X is of complexity one, i.e. $\mathrm{tr.deg}(k(X)^T) = 1$, then all the pp-divisors $\mathfrak{D}$ can be taken over a smooth complete curve. In such a case, Proposition 7.2.6 is no longer needed, and all the pp-divisors can be set over the same curve.

Remark 7.3.15. A normal T-variety does not have a unique T-stable affine open covering. Therefore it can arise from several divisoral fans. For example, $\mathbb{G}_{\mathrm{m}}$ acting on $\mathbb{A}^1 \times \mathbb{P}^1$ by multiplication on the first coordinate

$$\begin{aligned} \mathbb{G}_{\mathrm{m}} \times \left(\mathbb{A}^1 \times \mathbb{P}^1\right) &\to \left(\mathbb{A}^1 \times \mathbb{P}^1\right) \\ (\lambda, (x, p)) &\mapsto (\lambda x, p). \end{aligned}$$

By taking any two pairs of affine coverings of $\mathbb{P}^1$, for example

$$\{\mathbb{P} \smallsetminus \{0\}, \mathbb{P}^1 \smallsetminus \{\infty\}\} \text{ and } \{\mathbb{P}^1 \smallsetminus \{p\}, \mathbb{P}^1 \smallsetminus \{q\}\}$$

with $p, q \in \mathbb{P}^1 \smallsetminus \{0, \infty\}$, we have two different T-stable affine open coverings of $\mathbb{A}^1 \times \mathbb{P}^1$. Denote $\Delta := [1, +\infty[$. The divisorial fans are

$$\mathfrak{S}_1 := \{\emptyset \otimes \{0\} + \Delta \otimes \{\infty\}, \Delta \otimes \{0\} + \emptyset \otimes \{\infty\}, \emptyset \otimes \{0\} + \emptyset \otimes \{\infty\}\}$$

and

$$\mathfrak{S}_2 := \{\emptyset \otimes \{p\} + \Delta \otimes \{q\}, \emptyset \otimes \{q\} + \Delta \otimes \{q\}, \emptyset \otimes \{p\} + \emptyset \otimes \{q\}\},$$

which are different.

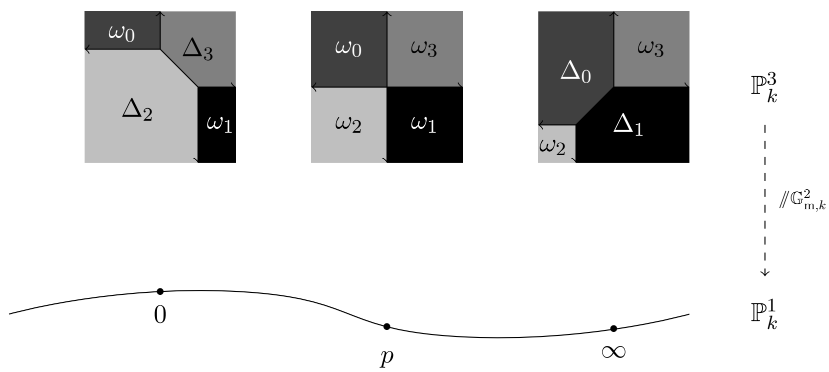
Example 7.3.16. Let us consider $\mathbb{P}^3_k$ with the action of $\mathbb{G}_{\mathrm{m},k}$ given by

$$(\lambda, \mu) \cdot [x_0 : x_1 : x_2 : x_3] = [\lambda x_0 : \mu x_1 : \lambda\mu x_2 : x_3].$$

The affine open covering given by $U_i := \{x_i \neq 0\}$ is $\mathbb{G}_{\mathrm{m},k}$-stable. All the pp-divisors are defined over the quotient $\mathbb{P}^3_k \dashrightarrow \mathbb{P}^1_k$ given by

$$[x_0 : x_1 : x_2 : x_3] \mapsto [x_0x_1 : x_2x_3].$$

Denote by $\mathfrak{D}_i$ the pp-divisor corresponding to U_i. Notice that U_3, with the respective induced action of $\mathbb{G}_{\mathrm{m},k}$, corresponds to Example 7.2.22, but the nontrivial polyhedron is supported over $\{0\}$ instead of $\{\infty\}$ in $\mathbb{P}^1_k$. Thus, this T-variety is encoded by the following divisorial fan



Such a diagrams states that almost all the fibers belong to a embedding of $\mathbb{G}^2_{\mathrm{m},k}$ isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

Chapter 8

Functoriality and semilinear morphisms

In Section 8.3 of this chapter, we present the notion of *semilinear morphisms of pp-divisors*. Then we focus in Section 8.4 on how these morphisms are related to the semilinear equivariant morphisms between their respective varieties.

In order to do this we study first the functoriality of the Altmann-Hausen construction in Section 8.1. And for the convenience of the reader, we recall the definition of semilinear morphisms in Section 8.2.

8.1 Functoriality of the Altmann-Hausen construction

Let k be a field of characteristic zero. As stated in Section 7.1, proper polyhedral divisors form a category. Besides, by Theorem 5.0.5, there is an assignation $\mathfrak{D} \mapsto X(\mathfrak{D})$ from pp-divisors to normal affine varieties endowed with an effective torus action. This assignation actually defines a functor $X : \mathfrak{PPDiv}(k) \to \mathcal{E}(k)$, where $\mathcal{E}(k)$ stands for the category of normal affine varieties endowed with an effective action of a split algebraic torus over k and whose morphisms are equivariant morphisms of varieties over k. In order to prove this statement, we need to exlain how the assignation works on morphisms.

Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two objects in $\mathfrak{PPDiv}(k)$ and $(\psi, F, \mathfrak{f}) : \mathfrak{D}' \to \mathfrak{D}$ be a morphism of pp-divisors over k. This morphism induces a morphisms of modules given by

$$\begin{aligned} H^0(Y, \mathscr{O}(\mathfrak{D}(m))) &\to H^0(Y', \mathscr{O}(\mathfrak{D}'(F^*(m)))), \\ h &\mapsto \mathfrak{f}(m)\psi^*(h), \end{aligned}$$

compatible with the $H^0(Y, \mathscr{O}_Y)$ and $H^0(Y', \mathscr{O}_{Y'})$-module structures. Hence, all these morphisms fit together into a graded morphism

$$A[Y, \mathfrak{D}] = \bigoplus_{m \in \omega^\vee \cap M} H^0(Y, \mathscr{O}(\mathfrak{D}(m))) \to \bigoplus_{m \in \omega'^\vee \cap M'} H^0(Y', \mathscr{O}(\mathfrak{D}'(m))) = A[Y', \mathfrak{D}'],$$

which turns into an equivariant morphism

$$X(\psi, F, \mathfrak{f}) := (\varphi, f) : X(\mathfrak{D}) \to X(\mathfrak{D}'),$$

where $\varphi : T' \to T$ is determined by $F : N' \to N$.

Proposition 8.1.1. *Let k be a field of characteristic zero. The assignation $\mathfrak{D} \mapsto X(\mathfrak{D})$ defines a faithful covariant functor $X : \mathfrak{PPDiv}(k) \to \mathcal{E}(k)$.*

Proof. It remains to prove the compatibility with compositions. Let $\mathfrak{D}$, $\mathfrak{D}'$ and $\mathfrak{D}''$ be objects in $\mathfrak{PPDiv}(k)$. Let $(\psi, F, \mathfrak{f}) : \mathfrak{D}' \to \mathfrak{D}$ and $(\psi', F', \mathfrak{f}') : \mathfrak{D}'' \to \mathfrak{D}'$ be morphisms of pp-divisors. By definition, the composition in $\mathfrak{PPDiv}(k)$ is given by

$$(\psi, F, \mathfrak{f}) \circ (\psi', F', \mathfrak{f}') = (\psi \circ \psi', F \circ F', F_*(\mathfrak{f}') \cdot \psi'^*(\mathfrak{f})).$$

The equivariant morphism $X(\psi, F, \mathfrak{f})$ corresponds to the morphism of modules given by

$$\begin{aligned} H^0(Y, \mathscr{O}(\mathfrak{D}(m))) &\to H^0(Y', \mathscr{O}(\mathfrak{D}'(F^*(m)))), \\ h &\mapsto \mathfrak{f}(m)\psi^*(h), \end{aligned}$$

and $X(\psi', F', \mathfrak{f}')$ corresponds to

$$\begin{aligned} H^0(Y', \mathscr{O}(\mathfrak{D}'(m))) &\to H^0(Y'', \mathscr{O}(\mathfrak{D}''(F'^*(m)))), \\ h &\mapsto \mathfrak{f}'(m)\psi'^*(h). \end{aligned}$$

Therefore, the composition induces the following morphisms on the modules

$$\begin{array}{ccccc} H^0(Y, \mathcal{O}(\mathfrak{D}(m))) & \to & H^0(Y', \mathcal{O}(\mathfrak{D}'(F^*(m)))) & \to & H^0(Y'', \mathcal{O}(\mathfrak{D}''(F'^*(F^*(m))))) \\ h & \mapsto & \mathfrak{f}(m)\psi^*(h) & \mapsto & \mathfrak{f}'(F^*(m))\psi'^*(\mathfrak{f}(m)\psi^*(h)) \\ & & & = & [\mathfrak{f}'(F^*(m))\psi'^*(\mathfrak{f}(m))]\psi'^*(\psi^*(h)) \\ & & & = & [F_*(\mathfrak{f}') \cdot \psi'^*(\mathfrak{f})](m)(\psi \circ \psi')^*(h), \end{array}$$

which coincides with the morphism induced by $(\psi \circ \psi', F \circ F', F_*(\mathfrak{f}') \cdot \psi'^*(\mathfrak{f}))$. Hence, both define the same graded morphisms between the graded algebras $A[Y'', \mathfrak{D}'']$ and $A[Y, \mathfrak{D}]$ and, therefore,

$$\begin{aligned} X((\psi, F, \mathfrak{f}) \circ (\psi', F', \mathfrak{f}')) &= X(\psi \circ \psi', F \circ F', F_*(\mathfrak{f}') \cdot \psi'^*(\mathfrak{f})) \\ &= X(\psi, F, \mathfrak{f}) \circ X(\psi', F', \mathfrak{f}'). \end{aligned}$$

This proves that the assignation is a covariant functor. It remains to prove that it is faithful.

Let $\bar{k}$ be an algebraic closure of k. Let $\mathfrak{PPDiv}(\bar{k})$ be the category of pp-divisors over $\bar{k}$ and $\mathcal{E}(\bar{k})$ be the category of normal affine varieties over $\bar{k}$ endowed with an effective action of an algebraic torus over $\bar{k}$. By Lemma 7.1.12, we have a covariant functor $\mathfrak{PPDiv}(k) \to \mathfrak{PPDiv}(\bar{k})$ that fits into the following commutative diagram

$$\begin{array}{ccc} \mathfrak{PPDiv}(\bar{k}) & \xrightarrow{X_{\bar{k}}} & \mathcal{E}(\bar{k}) \\ \uparrow & & \uparrow \\ \mathfrak{PPDiv}(k) & \xrightarrow[X]{} & \mathcal{E}(k), \end{array}$$

where $\mathcal{E}(k) \to \mathcal{E}(\bar{k})$ is the base change functor. This functor is faithful and covariant.

By [AH06, Proposition 8.6], the functor $X_{\bar{k}}$ is faithful and covariant. Hence, given that $\mathfrak{PPDiv}(k) \to \mathfrak{PPDiv}(\bar{k})$ is faithful and covariant by Corollary 7.1.16, the composite functor is faithful, and so must be X. Then, the assertion holds. □

Let T be a split algebraic torus over k and N be its module of cocharacters. Denote by $\mathfrak{PPDiv}_N(k)$ the full subcategory of all pp-divisors over k whose tail cone is defined on $N_{\mathbb{Q}}$ and by $\mathcal{E}_T(k)$ the full subcategory of all normal affine T-varieties. By Theorem 7.2.1, for $\mathfrak{D}$ an object in $\mathfrak{PPDiv}_N(k)$ we have that $X(\mathfrak{D}) := \mathrm{Spec}(A[Y, \mathfrak{D}])$ is a normal affine T-variety over k. Then, the functor $X : \mathfrak{PPDiv}(k) \to \mathcal{E}(k)$ induces a functor

$$\begin{aligned} X : \mathfrak{PPDiv}_N(k) &\to \mathcal{E}_T(k), \\ \mathfrak{D} &\mapsto X(\mathfrak{D}). \end{aligned}$$

Corollary 8.1.2. *Let k be a field of characteristic zero. The functor*

$$X : \mathfrak{PPDiv}_N(k) \to \mathcal{E}_T(k)$$

is faithful and covariant.

The following result states that a morphism of pp-divisors contains the information of the corresponding morphisms between the faces.

Proposition 8.1.3. *Let k be a field of characteristic zero. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be objects in $\mathfrak{PPDiv}(k)$ and $(\psi, F, \mathfrak{f}) : \mathfrak{D} \to \mathfrak{D}'$ be a morphism of pp-divisors. If $\mathfrak{D}^a \preceq \mathfrak{D}$ and $\mathfrak{D}^b \preceq \mathfrak{D}'$ are faces such that $X(\psi, F, \mathfrak{f})(X(\mathfrak{D}^a)) = X(\mathfrak{D}^b)$, then the morphism of pp-divisors associated to $X(\psi, F, \mathfrak{f})|_{X(\mathfrak{D}^a)}$ is $(\psi, F, \mathfrak{f}|_{\mathfrak{D}^a})$.*

Proof. Let $\mathfrak{D}^a \preceq \mathfrak{D}$ and $\mathfrak{D}^b \preceq \mathfrak{D}'$ be such that $X(\psi, F, \mathfrak{f})(X(\mathfrak{D}^a)) = X(\mathfrak{D}^b)$. Then, by restriction of $X(\psi, F, \mathfrak{f})$ to $X(\mathfrak{D}^a)$ we get an equivariant isomorphism $(\varphi, f) : X(\mathfrak{D}^a) \to X(\mathfrak{D}^b)$. Now, consider the following commutative diagram

$$\begin{tikzcd} \mathfrak{D} \arrow[r, "{(\psi,F,\mathfrak{f})}"] & \mathfrak{D}' \\ \mathfrak{D}^a \arrow[u, "{(\mathrm{id},\mathrm{id},1)}"] \arrow[r, "{(\psi,F,\mathfrak{f})}"'] & \mathfrak{D}^b . \arrow[u, "{(\mathrm{id},\mathrm{id},1)}"'] \end{tikzcd}$$

By applying the functor $X : \mathfrak{PPDiv}_N \to \mathcal{E}_T$ we get

$$\begin{tikzcd} X(\mathfrak{D}) \arrow[r, "{X(\psi,F,\mathfrak{f})}"] & X(\mathfrak{D}') \\ X(\mathfrak{D}^a) \arrow[u, hook, "{X(\mathrm{id},\mathrm{id},1)}"] \arrow[r, "{(\varphi,f)}"'] & X(\mathfrak{D}^b). \arrow[u, hook, "{X(\mathrm{id},\mathrm{id},1)}"'] \end{tikzcd}$$

By Proposition 8.1.1, X is faithful, so we get that $X(\psi, F, \mathfrak{f})|_{X(\mathfrak{D}^a)}$ corresponds to $(\psi, F, \mathfrak{f}) : \mathfrak{D}^a \to \mathfrak{D}^b$. ☐

As stated in Proposition 8.1.1, the functor $X : \mathfrak{PPDiv}(k) \to \mathcal{E}(k)$ is faithful, but it is not full. For example, let $\mathfrak{D} \in \mathrm{PPDiv}_{\mathbb{Q}}(\mathbb{P}^2_k, \omega)$ be any pp-divisor and $\kappa : \mathbb{F}_r \to \mathbb{P}^2_k$ a birational morphism from the Hirzebruch surface to the projective plane. By pulling back, we have $\kappa^*\mathfrak{D} \in \mathrm{PPDiv}_{\mathbb{Q}}(\mathbb{F}_r, \omega)$. Both pp-divisors define the same normal T-variety, then we have the identity map $(\mathrm{id}_T, \mathrm{id}) : X(\mathfrak{D}) \to X(\psi^*\mathfrak{D})$. However, this map does not arise from a morphism of pp-divisors, because that would imply that there exists a non constant morphism $\tilde{\kappa} : \mathbb{P}^2_k \to \mathbb{F}_r$ such that

$$(\kappa, \mathrm{id}, \imath) \circ (\tilde{\kappa}, \mathrm{id}, \imath) = (\mathrm{id}_{\mathbb{P}^2_k}, \mathrm{id}, \imath),$$

which gives a contradiction. Thus, not every dominant equivariant morphism between two fixed normal affine varieties endowed with an effective action of a split algebraic torus arises from a morphism of a pair of fixed pp-divisors.

The morphism above arises rather from a pair of morphisms

$$\mathfrak{D} \xleftarrow{(\kappa,\mathrm{id}_N,1)} \kappa^*\mathfrak{D} \xrightarrow{(\mathrm{id},\mathrm{id}_N,1)} \kappa^*\mathfrak{D}.$$

Let us call a morphism of pp-divisors $(\psi, F, \mathfrak{f})$ *dominating* if $X(\psi, F, \mathfrak{f})$ is dominant. By [AH06, Theorem 8.8], dominant equivariant morphisms of normal affine varieties arise from localized dominating morphisms of pp-divisors over $\bar{k}$, i.e. from a data

$$\mathfrak{D} \xleftarrow{(\kappa,\mathrm{id}_N,1)} \kappa^*\mathfrak{D} \xrightarrow{(\psi,F,\mathfrak{f})} \kappa^*\mathfrak{D} \ ,$$

where $(\psi, F, \mathfrak{f})$ is a dominating morphism of pp-divisors and κ is a projective birational morphism from a normal semiprojective variety. In the following we will prove a more general result involving *semilinear* morphisms. These morphisms form a larger family than morphisms of varieties over k.

8.2 Semilinear morphism of varieties

Semilinear morphisms seem to be the right language to deal with Galois descent problems. These morphisms have been used, for instance, by Huruguen [Hur11] and Borovoi [Bor20].

Definition 8.2.1. Let k be a field, L/k be a Galois extension with Galois group Γ. Let Y and Z be varieties over L and $\gamma \in \Gamma$. A *semilinear morphism with respect to* γ is a morphism of schemes $h_\gamma : Y \to Z$ satisfying the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "h_\gamma"] \arrow[d] & Z \arrow[d] \\ \mathrm{Spec}(L) \arrow[r, "\gamma^\natural"'] & \mathrm{Spec}(L), \end{tikzcd}$$

where $\gamma^\natural := \mathrm{Spec}(\gamma^{-1})$. Moreover, we say that h_γ is a *semilinear isomorphism* if h_γ is an isomorphism of schemes.

Clearly, any morphism of varieties over L is a semilinear morphism with respect to the neutral element of the Galois group. Then, if we denote by

$\mathrm{SAut}(Y)$ the group of semilinear automorphisms of a variety Y over L, there is an exact sequence

$$1 \to \mathrm{Aut}(Y) \to \mathrm{SAut}(Y) \to \Gamma. \tag{8.1}$$

We say that a semilinear morphism h_γ is *dominant* if h_γ is dominant as a morphism of schemes.

Let k be a field, $\bar{k}$ be an algebraic closure and $k \subset L \subset \bar{k}$ a Galois extension with Galois group Γ. Let G and G' be algebraic groups over L and $\gamma \in \Gamma$. A *semilinear group homomorphism with respect to* γ that is a morphism of group schemes $\varphi_\gamma : G \to G'$ is also a semilinear morphism. Moreover, we say that φ_γ is a *semilinear group isomorphism* if φ_γ is an isomorphism of schemes. We denote by $\mathrm{SAut}_{\mathrm{gp}}(G)$ the group of such automorphisms for a fixed group-scheme G.

Let G and G' be algebraic groups over L, X be a G-variety and X' be a G'-variety. Let $\gamma \in \mathrm{Gal}(L/k)$. A *semilinear equivariant morphism with respect to* γ is a pair $(\varphi_\gamma, f_\gamma)$ such that $\varphi_\gamma : G \to G'$ is a semilinear group homomorphism, $f_\gamma : X \to X'$ is a semilinear morphism, both with respect to γ, and the following diagram of semilinear morphisms commutes

$$\begin{tikzcd} G \times X \arrow[r, "\mu"] \arrow[d, "(\varphi_\gamma, f_\gamma)"'] & X \arrow[d, "f_\gamma"] \\ G' \times X' \arrow[r, "\mu'"'] & X', \end{tikzcd}$$

where μ and μ' are the respective actions of G o X and G' on X'.

The group of semilinear equivariant automorphisms over L is denoted by $\mathrm{SAut}_G(X)$, which naturally contains $\mathrm{Aut}_G(X)$. Define $\mathrm{SAut}(G; X)$ as the subgroup of $\mathrm{SAut}_{\mathrm{gp}}(G) \times \mathrm{SAut}_G(X)$ defined as the preimage of the diagonal inclusion $\Gamma \to \Gamma \times \Gamma$. We have then the following exact sequence

$$1 \to \mathrm{Aut}_{\mathrm{gp}}(G) \times \mathrm{Aut}_G(X) \to \mathrm{SAut}(G; X) \to \Gamma.$$

Definition 8.2.2. Let k be a field and L/k be a Galois extension with Galois group Γ. Let G be an algebraic group over L and X be a G-variety over L. Let H be an abstract group. A *semilinear equivariant action of* H *over* X is a group homomorphism $\varphi : H \to \mathrm{SAut}(G; X)$. If $H = \Gamma$ and φ is a section of the exact sequence above, then φ is a *Galois semilinear equivariant action.*

8.3 Semilinear morphisms of pp-divisors

In the algebraically closed case, dominant equivariant morphisms arise from localized dominating morphisms of pp-divisors. In this section we see that this is also true when the field is not algebraically closed and the algebraic torus is split over L. In fact, we prove a more general statement concerning semilinear equivariant morphisms.

Definition 8.3.1. Let L/k be a Galois extension with Galois group $\Gamma := \mathrm{Gal}(L/k)$. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be in $\mathfrak{PPDiv}(L)$, the category of pp-divisors over L. A *semilinear morphism of pp-divisors* is a triple $(\psi_\gamma, F, \mathfrak{f}) : \mathfrak{D} \to \mathfrak{D}'$, where $\psi_\gamma : Y \to Y'$ is a semilinear dominant morphism, $F : N \to N'$ is a morphism of lattices such that $F(\mathrm{Tail}(\mathfrak{D})) \subset \mathrm{Tail}(\mathfrak{D}')$ and $\mathfrak{f} \in L(N', Y)^*$ is a plurifunction such that

$$\psi_\gamma^*(\mathfrak{D}') \leq F^*(\mathfrak{D}) + \mathrm{div}(\mathfrak{f}).$$

Let k be a field of characteristic zero and L/k be a Galois extension with Galois group Γ. Let $(\psi_\gamma, F, \mathfrak{f}) : \mathfrak{D} \to \mathfrak{D}'$ be a semilinear morphism of pp-divisors over L. For every $m \in M'$, we have morphisms of modules (notice that in this case it is only k-linear)

$$\begin{gathered} H^0(Y', \mathscr{O}_{Y'}(\mathfrak{D}'(m))) \to H^0(Y, \mathscr{O}_Y(\mathfrak{D}(F^*(m)))), \\ h \mapsto \mathfrak{f}(m)\psi_\gamma^*(h) \end{gathered}$$

that fit together into a morphism of M-graded L-algebras satisfying the following commutative diagram

$$\begin{tikzcd} A[Y', \mathfrak{D}'] \arrow[r] & A[Y, \mathfrak{D}] \\ L \arrow[u] \arrow[r, "\gamma^{-1}"'] & L \arrow[u] \end{tikzcd},$$

which gives a semilinear equivariant morphism

$$\begin{tikzcd} X(\mathfrak{D}) \arrow[rrr, "X(\psi_\gamma, F, \mathfrak{f}) = (\varphi_\gamma, f_\gamma)"] & & & X(\mathfrak{D}') \\ L \arrow[u] \arrow[rrr, "\gamma^\natural"'] & & & L \arrow[u] \end{tikzcd}.$$

Thus, semilinear morphisms of pp-divisors induce semilinear equivariant morphisms of normal affine varieties with a split torus action over L. As in the case of morphisms of pp-divisors, let us call *dominating* those semilinear morphisms of pp-divisors inducing dominant semilinear equivariant morphisms. Denote by $\mathfrak{PPDiv}(L/k)$ the category of pp-divisors over L with dominating semilinear morphisms and by $\mathcal{E}(L/k)$ the category of normal affine varieties over L endowed with an effective torus action and whose morphisms are dominant semilinear equivariant morphisms of varieties over L. In this setting, there is also a functor $X : \mathfrak{PPDiv}(L/k) \to \mathcal{E}(L/k)$, sending semilinear morphisms of pp-divisors to semilinear equivariant morphisms.

Proposition 8.3.2. *Let k be a field of characteristic zero and L/k be a Galois extension with Galois group Γ. The assignation $\mathfrak{D} \mapsto X(\mathfrak{D})$ induces a faithful covariant functor $X : \mathfrak{PPDiv}(L/k) \to \mathcal{E}(L/k)$.*

Proof. The proof that the assignation is a functor is analogous to the proof of Proposition 8.1.1 and the functor is covariant by construction.

Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two objects in $\mathfrak{PPDiv}(L/k)$. Let $(\psi_{\gamma,1}, F_1, \mathfrak{f}_1)$ and $(\psi_{\eta,2}, F_2, \mathfrak{f}_2)$ be semilinear morphisms of pp-divisors from $\mathfrak{D}' \to \mathfrak{D}$ such that $X(\psi_{\gamma,1}, F_1, \mathfrak{f}_1) = X(\psi_{\eta,2}, F_2, \mathfrak{f}_2) = (\varphi_\gamma, f_\gamma)$. First, given that both define the same semilinear equivariant morphism, it follows that $\gamma = \eta$.

Notice that if $\psi^*_{\gamma,1}, \psi^*_{\gamma,2} : L(Y) \to L(Y')$ are equal, then $\psi_{\gamma,1} = \psi_{\gamma,2}$. Given that $L(Y) = L(X)^T$, a function $f \in L(Y)$ is written as a quotient of g and h in $H^0(Y, \mathscr{A}_m)$ for some $m \in M$. Hence,

$$\psi^*_{\gamma,1}(f) = \psi^*_{\gamma,1}\left(\frac{g}{h}\right) = \frac{\mathfrak{f}_1(m)\psi^*_{\gamma,1}(g)}{\mathfrak{f}_1(m)\psi^*_{\gamma,1}(h)} = \frac{\mathfrak{f}_2(m)\psi^*_{\gamma,2}(g)}{\mathfrak{f}_2(m)\psi^*_{\gamma,2}(h)} = \psi^*_{\gamma,2}\left(\frac{g}{h}\right) = \psi^*_{\gamma,2}(f),$$

where the central equality follows from the fact that both morphisms define the same morphism between the graded algebras. Thus, it follows that $\psi_{\gamma,1} = \psi_{\gamma,2}$.

Given that $(\psi_{\gamma,1}, F_1, \mathfrak{f}_1)$ and $(\psi_{\eta,2}, F_2, \mathfrak{f}_2)$ define the same morphism of graded algebras and we know that $\psi^*_{\gamma,1} = \psi^*_{\gamma,2}$, we have that $\mathfrak{f}_1(m) = \mathfrak{f}_2(m)$ for every $m \in \omega^\vee \cap M$. Hence, $\mathfrak{f}_1 = \mathfrak{f}_2$.

In order to prove $F_1 = F_2$, it suffices to find a point in $x \in X$ such that $f_\gamma(x) \in X'$ has a trivial isotropy group, i.e. $T'_{f_\gamma(x)} = \{1_{T'}\}$. This last part of the assertion can be proved by assuming that L is algebraically closed. Let $x' \in X'$ such that its isotropy group is trivial, for example a generic orbit whose orbit cone is $\omega_{\mathfrak{D}'}$. By Proposition 7.2.10, we have that $T'_{x'} = \{1_{T'}\}$ is equivalent to $M(x') = M'$, where $M(x')$ is the orbit lattice of x'. Let $\{m_1, \ldots, m_r\} \subset S(x')$ be a set of generators of the orbit monoind $S(x')$.

By definintion, for every $i \in \{1, \dots, r\}$, there exists $f_{m_i} \in A_{m_i}$ such that $f_{m_i}(x') \neq 0$. Define

$$U := \bigcap_{i=1}^{r} D_{f_{m_i}}.$$

Notice that, for every $x'' \in U$, we have that $S(x') \subset S(x'')$. Then, we have that $M(x') \subset M(x'') \subset M'$. This implies that $M(x'') = M'$. Otherwise stated, all the elements of U have trivial isotropy group. Finally, given that f_γ is dominant and $U \subset X'$ is open, we have that there exists $x \in X$ such that $f_\gamma(x)$ has a trivial isotropy group. Then, the assertion holds. $\square$

We have also a semilinear version of Proposition 8.1.3.

Proposition 8.3.3. *Let k be a field of characteristic zero, L/k be a finite Galois extesion with Galois group Γ and $\gamma \in \Gamma$. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be objects in $\mathfrak{PPDiv}(L/k)$ and $(\psi_\gamma, F, \mathfrak{f}) : \mathfrak{D} \to \mathfrak{D}'$ be a semilinear morphism of pp-divisors. If $\mathfrak{D}^a \preceq \mathfrak{D}$ and $\mathfrak{D}^b \preceq \mathfrak{D}'$ are faces such that $X(\psi_\gamma, F, \mathfrak{f})(X(\mathfrak{D}^a)) = X(\mathfrak{D}^b)$, then the semilinear morphism of pp-divisors associated to the restriction $X(\psi_\gamma, F, \mathfrak{f})|_{X(\mathfrak{D}^a)}$ is $(\psi_\gamma, F, \mathfrak{f}|_{\mathfrak{D}^a})$.*

Proof. Let $\mathfrak{D}^a \preceq \mathfrak{D}$ and $\mathfrak{D}^b \preceq \mathfrak{D}'$ be such that $X(\psi_\gamma, F, \mathfrak{f})(X(\mathfrak{D}^a)) = X(\mathfrak{D}^b)$. Then, by restriction of $X(\psi_\gamma, F, \mathfrak{f})$ to $X(\mathfrak{D}^a)$ we get an equivariant semilinear isomorphism $(\varphi_\gamma, f_\gamma) : X(\mathfrak{D}^a) \to X(\mathfrak{D}^b)$. Now, consider the following commutative diagram

$$\begin{tikzcd} \mathfrak{D} \arrow[r, "{(\psi_\gamma,F,\mathfrak{f})}"] & \mathfrak{D}' \\ \mathfrak{D}^a \arrow[u, "{(\mathrm{id},\mathrm{id},1)}"] \arrow[r, "{(\psi_\gamma,F,\mathfrak{f})}"'] & \mathfrak{D}^b. \arrow[u, "{(\mathrm{id},\mathrm{id},1)}"'] \end{tikzcd}$$

By applying the functor $X : \mathfrak{PPDiv}(L/k) \to \mathcal{E}(L/k)$ we get

$$\begin{tikzcd} X(\mathfrak{D}) \arrow[r, "{X(\psi_\gamma,F,\mathfrak{f})}"] & X(\mathfrak{D}') \\ X(\mathfrak{D}^a) \arrow[u, hook, "{X(\mathrm{id},\mathrm{id},1)}"] \arrow[r, "{(\varphi_\gamma,f_\gamma)}"'] & X(\mathfrak{D}^b). \arrow[u, hook, "{X(\mathrm{id},\mathrm{id},1)}"'] \end{tikzcd}$$

By Proposition 8.3.2, X is faithful, so we get that $X(\psi_\gamma, F, \mathfrak{f})|_{X(\mathfrak{D}^a)}$ corresponds to $(\psi_\gamma, F, \mathfrak{f}|_{\mathfrak{D}^a}) : \mathfrak{D}^a \to \mathfrak{D}^b$. $\square$

8.4 Semilinear equivariant morphisms

As morphisms of pp-divisors induce equivariant morphisms of affine normal varieties endowed with effective torus actions, semilinear morphisms of pp-divisors similarly induce semilinear equivariant morphisms of affine normal varieties endowed with effective torus actions. However, not every dominant semilinear equivariant morphism of affine normal varieties arises from a semilinear morphism of pp-divisors.

In the following we will prove that dominant semilinear equivariant morphisms between affine normal varieties endowed with an effective torus action arise from localized dominating semilinear morphisms of pp-divisors. The next results are intermediary steps that will help us to achieve our goal.

Proposition 8.4.1. *Let k be a field of characteristic zero, L/k be a finite Galois extension with Galois group $\Gamma := \mathrm{Gal}(L/k)$ and $\gamma \in \Gamma$. Let Y and Y' be normal semiprojective varieties over L. Let $h_\gamma : Y \dashrightarrow Y'$ be a rational semilinear morphism with respecto to γ, then there exists a normal semiprojective variety $\tilde{Y}$ over L satisfying*

$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "\kappa"'] \arrow[dr, "\psi_\gamma"] & \\ Y \arrow[rr, dashed] & & Y', \end{tikzcd}$$

where κ is a projective morphism of varieties over L and ψ_γ is a projective semilinear morphism with respect to γ.

Proof. Consider the diagram corresponding to the semilinear rational map

$$\begin{tikzcd} Y \arrow[r, dashed, "h_\gamma"] \arrow[d] & Y' \arrow[d] \\ L \arrow[r, "\gamma^\natural"'] & L. \end{tikzcd}$$

Denote by $Y'' := \gamma^{-1*}Y'$ the variety over L corresponding to the composition

$$Y' \longrightarrow L \xrightarrow{(\gamma^{-1})^\natural} L.$$

Then, h_γ is a rational morphism of varieties over L between Y and Y''. By Proposition 7.2.6, there exists a normal semiprojective variety $\tilde{Y}$ over L with projective morphisms κ and ψ_γ satisfying the following

$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "\kappa"'] \arrow[dr, "\psi_\gamma"] & \\ Y \arrow[rr, dashed] & & Y''. \end{tikzcd}$$

Then, we have that the following diagram commutes

$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "\kappa"'] \arrow[dr, "\psi_\gamma"] & \\ Y \arrow[rr, dashed, "h_\gamma"'] \arrow[d] & & Y' \arrow[d] \\ L \arrow[rr, "\gamma^\natural"'] & & L. \end{tikzcd}$$

Given that κ is a morphism of varieties over L, we have that ψ_γ is semilinear with respect to γ. Then, the assertion holds. $\square$

Lemma 8.4.2. *Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let Y be a geometrically integral normal k-variety. If D and D' in $\mathrm{CaDiv}_{\mathbb{Q}}(Y)$ are semiample and $H^0(Y, \mathscr{O}(nD)) \subset H^0(Y, \mathscr{O}(nD'))$ holds for infinitely many $n > 0$, then $D \leq D'$.*

Proof. If D and D' are semiample, then $D_{\bar{k}}$ and $D'_{\bar{k}}$ are semiample and also $H^0(Y_{\bar{k}}, \mathscr{O}(nD_{\bar{k}})) \subset H^0(Y_{\bar{k}}, \mathscr{O}(nD'_{\bar{k}}))$ holds for infinitely many $n > 0$. Then, by [AH06, Lemma 9.1], we have that $D_{\bar{k}} \leq D'_{\bar{k}}$. This implies the assertion. $\square$

The following lemma corresponds to [AH06, Lemma 9.2], which holds in the non algebraically closed case.

Lemma 8.4.3. *Let k be a field of characteristic zero and T be a split k-torus. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be objects in $\mathfrak{PPDiv}(k)$, the category of pp-divisors, defining the same normal T-variety. If $\mathfrak{D}$ is constructed as in Proposition 7.2.19, then there exists a plurifunction $\mathfrak{f} \in k(N, Y')^*$ such that $\mathfrak{D}' = \vartheta^*\mathfrak{D} + \mathrm{div}(\mathfrak{f})$, where $\vartheta : Y' \to Y$ is the canonical morphism.*

Proof. Denote

$$\mathscr{A}' := \bigoplus_{m \in \omega^\vee_{\mathfrak{D}'} \cap M} \mathscr{O}_{Y'}(\mathfrak{D}'(m))$$

the $\mathscr{O}_{Y'}$-algebra associated to $\mathfrak{D}'$, $\tilde{X}' := \mathrm{Spec}_{Y'}(\mathscr{A}')$, $A' := H^0(Y', \mathscr{A}')$ and $X' := \mathrm{Spec}(A')$.

On the one hand, there is a natural map $r' : \tilde{X}' \to X'$, which fits into the following commutative diagram

$$\begin{tikzcd}
r'^{-1}(X'^{\mathrm{ss}}(m)) \arrow[r] \arrow[d] & X'^{\mathrm{ss}}(m) \arrow[d] \\
Y' \arrow[r, "\vartheta_m"] \arrow[dr, "\vartheta"'] & Y'_m \\
& Y. \arrow[u, "p_m"']
\end{tikzcd}$$

On the other hand, by construction in the proof of Proposition 7.2.19, we have that

$$\mathscr{O}_Y(\mathfrak{D}(m)) = \frac{1}{s(m)}\mathscr{A}_m \subset k(Y)^*,$$

where $s : M \to k(X)^*$ is a section of the degree map and $\mathscr{A}_m$ is a sheaf such that $H^0(Y, \mathscr{A}_m) = A'_m$, the elements of degree m of A'.

After pulling back $\mathfrak{D}(m)$ by $\vartheta : Y' \to Y$, we have that

$$H^0(Y', \mathcal{O}_{Y'}(\vartheta^*\mathfrak{D}(m))) = \frac{1}{s(m)}A'_m \subset k(Y').$$

Given that $X' = X(\mathfrak{D}')$, we have that $H^0(Y', \mathscr{O}_{Y'}(\mathfrak{D}'(m))) \subset k(Y')$. Hence, by forgetting the grading, we have a multiplicative map

$$\bigcup_{m \in \omega_{\mathfrak{D}'} \cap M} H^0(Y', \mathscr{O}_{Y'}(\mathfrak{D}'(m))) \to k(Y')$$
$$f_m \mapsto f_m.$$

This map extends to the multiplicative system of rational homogeneous functions on X'. This allows us to see the morphisms $s(m)$ as elements in $k(Y')$ and therefore we can consider $\mathrm{div}(s(m)) \in \mathrm{CaDiv}(Y')$. Thus,

$$\begin{aligned}
H^0(Y', \mathscr{O}_{Y'}(\vartheta^*\mathfrak{D}(m))) &= \frac{1}{s(m)}A_{m'} \\
&= \frac{1}{s(m)}H^0(Y', \mathscr{O}_{Y'}(\mathfrak{D}'(m))) \\
&= H^0(Y', \mathscr{O}_{Y'}(\mathfrak{D}'(m) - \mathrm{div}(s(m)))).
\end{aligned}$$

This holds for every nm, for $n \in \mathbb{N}$. Then, by Lemma 8.4.2, we have that $\vartheta^*\mathfrak{D}(m) = \mathfrak{D}' + \mathrm{div}(s(m))$. Hence, defining $\mathfrak{f} \in k(N, Y')^*$ as the plurifunction such that $\mathrm{div}(\mathfrak{f})(m) = s(m)$, we have that $\vartheta^*\mathfrak{D} = \mathfrak{D}' + \mathrm{div}(\mathfrak{f})$. Then, the assertion holds. ☐

Now, we present one of the main result of this section.

Theorem 8.4.4. *Let k be a field of characteristic zero, L/k be a Galois extension with Galois group $\Gamma := \mathrm{Gal}(L/k)$ and $\gamma \in \Gamma$. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two objects in $\mathfrak{PPDiv}(L/k)$. Let $(\varphi_\gamma, f_\gamma) : X(\mathfrak{D}) \to X(\mathfrak{D}')$ be a dominant semilinear equivariant morphism. Then, there exists a normal semiprojective variety $\tilde{Y}$ over L, a projective birational morphism $\kappa : \tilde{Y} \to Y$ of varieties over L and a semilinear morphism of pp-divisors $(\psi_\gamma, F, \mathfrak{f}) : \kappa^*\mathfrak{D} \to \mathfrak{D}'$ such that following diagram commutes*

$$\begin{tikzcd}
 & X(\kappa^*\mathfrak{D}) \arrow[dl, "X(\kappa,\mathrm{id}_N,1)"', "\cong"] \arrow[dr, "X(\psi_\gamma,F,\mathfrak{f})"] & \\
X(\mathfrak{D}) \arrow[rr, "(\varphi_\gamma,f_\gamma)"'] & & X(\mathfrak{D}').
\end{tikzcd}$$

In particular, if $(\varphi_\gamma, f_\gamma)$ is a semilinear isomorphism and $\mathfrak{D}'$ is minimal, then κ can be taken as the identity and $F : N \to N'$ is an isomorphism such that $F(\omega_{\mathfrak{D}}) = \omega_{\mathfrak{D}'}$. Moreover, if $\mathfrak{D}$ is also minimal, then $\psi_\gamma : Y \to Y'$ is a semilinear isomorphism.

Proof. Denote $X := X(\mathfrak{D})$ and $X' := X(\mathfrak{D}')$. Let $F : N \to N'$ be the lattice morphism corresponding to $\varphi_\gamma : T \to T'$ and $F^* : M' \to M$ its dual morphism. Let us consider the case where $\mathfrak{D}$ and $\mathfrak{D}'$ are minimal pp-divisors. Given that $(\varphi_\gamma, f_\gamma) : X \to X'$ is dominant, we have that $f_\gamma^{-1}(X'^{ss}(m)) \subset X^{\mathrm{ss}}(F^*(m))$ is not empty for every $m \in \omega_{\mathfrak{D}'}^\vee \cap M'$. Therefore, we have the following data

$$X^{\mathrm{ss}}(F^*(m)) \xleftarrow{\quad\iota\quad} f_\gamma^{-1}(X'^{ss}(m)) \xrightarrow{(\varphi_\gamma,f_\gamma)} X'^{ss}(m),$$

where ι is the natural embedding. Now, we can take the respective quotients and we get

$$\begin{tikzcd}
X^{\mathrm{ss}}(F^*(m)) \arrow[d] & f_\gamma^{-1}(X'^{ss}(m)) \arrow[l, "\iota"'] \arrow[r, "(\varphi_\gamma,f_\gamma)"] \arrow[d] & X'^{ss}(m) \arrow[d] \\
Y_{F^*(m)} & f_\gamma^{-1}(X'^{ss}(m)) /\!\!/ T \arrow[l] \arrow[r, "(h_\gamma)_m"'] & Y'_m,
\end{tikzcd}$$

where $(h_\gamma)_m$ is a γ-semilinear morphism, which defines a rational γ-semilinear morphism

$$(h_\gamma)_m : Y_{F^*(m)} \dashrightarrow Y'_m.$$

Thus, for $\lambda' \in \Lambda'$ we have rational γ-semilinear morphisms

$$(h_\gamma)_\lambda : Y_{F^*(\lambda)} \dashrightarrow Y'_{\lambda'},$$

where $F^*(\lambda') \in \Lambda$. Hence, we have a rational γ-semilinear morphism between the limits

$$h_\gamma : Y \dashrightarrow Y'.$$

Then, by Proposition 8.4.1, there exists a semilinear resolution of indeterminancies

$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "\kappa"'] \arrow[dr, "\psi_\gamma"] & \\ Y \arrow[rr, dashed] & & Y', \end{tikzcd}$$

such that $\tilde{Y}$ is normal and semiprojective and ψ_γ and κ are projective. Consider the homomorphisms $s : M \to L(X)$ and $s' : M' \to L(X')$ of the proof of Proposition 7.2.19. Then we have the following commutative diagram

$$\begin{tikzcd} A_{F^*(m)} & & A'_m \arrow[ll, "f_\gamma^*"'] \\ H^0(\tilde{Y}, \mathscr{O}_{\tilde{Y}}(\kappa^*\mathfrak{D}(F^*(m)))) \arrow[u, "\cdot s(F^*(m))"] & & H^0(Y', \mathscr{O}_{Y'}(\mathfrak{D}'(m))) \arrow[u, "\cdot s(m)"'] \arrow[dl, "\psi_\gamma^*"] \\ & H^0(\tilde{Y}, \mathscr{O}_{\tilde{Y}}(\psi_\gamma^*\mathfrak{D}'(m))). \arrow[ul, "\cdot \frac{f_\gamma^*(s'(m))}{s(F^*(m))}"] & \end{tikzcd}$$

From the commutative diagram we have a group homomorphism

$$\begin{aligned} M' &\to L(\tilde{Y})^* \\ m &\mapsto f_\gamma^*(s'(m))/s(F^*(m)), \end{aligned}$$

which, by (b) of Definition 7.1.7, defines a plurifunction $\mathfrak{f} \in L(N', \tilde{Y})^*$ such that

$$\mathfrak{f}(m) = f_\gamma^*(s'(m))/s(F^*(m)),$$

for every $m \in M'$ (consider a $\mathbb{Z}$-basis of M and take the f^i as the image of the elements of such a base, for instance). Notice that if $(\varphi_\gamma, f_\gamma)$ is an isomorphism, then no resolution of indeterminancies is needed and, therefore, $\psi_\gamma : Y \to Y'$ is a semilinear isomorphism. We claim that the triple $(\psi_\gamma, F, \mathfrak{f}) : \kappa^*\mathfrak{D} \to \mathfrak{D}'$ is a semilinear morphism of pp-divisors with respect to γ that fits into the commutative triangle of the statement. In order to do this, it suffices to prove that

$$\psi_\gamma^*\mathfrak{D}'(m) \leq \kappa^*\mathfrak{D}(F^*(m)) + \mathrm{div}(\mathfrak{f})(m),$$

for every $m \in \omega_{\mathfrak{D}'}^\vee \cap M'$. Since the morphism

$$H^0(\tilde{Y}, \mathscr{O}_{\tilde{Y}}(\psi_\gamma^*\mathfrak{D}'(m))) \xrightarrow{\frac{f_\gamma^*(s'(m))}{s(F^*(m))}} H^0(\tilde{Y}, \mathscr{O}_{\tilde{Y}}(\kappa^*\mathfrak{D}(F^*(m))))$$

defines an inclusion

$$H^0(\tilde{Y}, \mathscr{O}_{\tilde{Y}}(\psi_\gamma^*\mathfrak{D}'(m) - \mathrm{div}(\mathfrak{f})(m))) \subset H^0(\tilde{Y}, \mathscr{O}_{\tilde{Y}}(\kappa^*\mathfrak{D}(F^*(m)))),$$

the claim holds by Lemma 8.4.2. Therefore, the assertion holds for $\mathfrak{D}$ and $\mathfrak{D}'$ minimal pp-divisors.

Suppose now that only $\mathfrak{D}'$ is minimal and $\mathfrak{D}$ is not. Let $\mathfrak{D}_1$ be a minimal pp-divisor such that $X(\mathfrak{D}) \cong X(\mathfrak{D}_1)$, which exists by the construction made in Section 7.2.3. On the one hand, by Lemma 8.4.3, there exists a plurifunction $\mathfrak{f}_1 \in L(N, Y)$ such that $\mathfrak{D} = \vartheta^*\mathfrak{D}_1 + \mathrm{div}(\mathfrak{f}_1)$, where $\vartheta : Y \to Y_1$ is the canonical morphism. On the other hand, given that $\mathfrak{D}_1$ and $\mathfrak{D}'$ are minimal pp-divisors, the theorem holds. Hence, there exists $\tilde{Y}_1$ a normal semiprojective L-variety, a projective birational morphism $\kappa_1 : \tilde{Y}_1 \to Y_1$ and a semilinear morphism of pp-divisors $(\psi_\gamma, F, \mathfrak{f}) : \kappa_1^*\mathfrak{D}_1 \to \mathfrak{D}'$ such that the following diagram commutes

$$\begin{tikzcd}
& X(\kappa_1^*\mathfrak{D}_1) \arrow[dl, "X(\kappa_1, \mathrm{id}_N, 1)"', "\cong"] \arrow[dr, "(\psi_\gamma, F, \mathfrak{f})"] & \\
X(\mathfrak{D}_1) \arrow[rr, "(\varphi_\gamma, f_\gamma)"'] & & X(\mathfrak{D}').
\end{tikzcd}$$

Now, consider the fiber product

$$\begin{tikzcd}
Y \times_{Y_1} \tilde{Y}_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & \tilde{Y}_1 \arrow[d, "\kappa_1"] \\
Y \arrow[r, "\vartheta"'] & Y_1.
\end{tikzcd}$$

The morphisms ϑ et κ_1 are birational, then there exist open subvarieties of Y and $\tilde{Y}_1$, respectively, isomorphic to open subvarieties of Y_1. Hence, there exists an open subvariety $U \subset \tilde{Y}_1 \times_{Y_1} Y$ isomorphic to open subvarieties of $\tilde{Y}_1$ and Y_1 under the canonical projections π_1 and π_2. Let $\tilde{Y} := \overline{U}^{\nu}$ be the normalization of the closure of U, $p_1 : \tilde{Y} \to \tilde{Y}_1$ the restriction of π_1 and $\kappa_2 : \tilde{Y} \to Y$ the restriction of π_2. Then, we have the following commutative diagram

$$\begin{tikzcd}
\tilde{Y} \arrow[r, "p_1"] \arrow[d, "\kappa_2"'] & \tilde{Y}_1 \arrow[d, "\kappa_1"] \arrow[dr, "\psi_\gamma"] & \\
Y \arrow[r, "\vartheta"'] & Y_1 & Y'.
\end{tikzcd}$$

Notice that the morphisms of the square are morphisms of varieties over L. Then $\psi_\gamma \circ p_1$ is γ-semilinear.

We need to construct a morphism of pp-divisors $\kappa_2^*\mathfrak{D} \to \mathfrak{D}$ from the data above. From the fact that $(\psi_\gamma, F, \mathfrak{f}) : \kappa_1^*\mathfrak{D}_1 \to \mathfrak{D}$ is a semilinear morphism of pp-divisors and applying p_1^* we have

$$\begin{aligned}
(\psi_\gamma \circ p_1)^*\mathfrak{D}' &= p_1^*\psi_\gamma^*\mathfrak{D}' \\
&\le p_1^*F_*\kappa_1^*\mathfrak{D}_1 + \operatorname{div}(p_1^*\mathfrak{f}) \\
&= F_*p_1^*\kappa_1^*\mathfrak{D}_1 + \operatorname{div}(p_1^*\mathfrak{f}),
\end{aligned}$$

and by the commutative of the diagram above and the identity $\mathfrak{D} = \vartheta^*\mathfrak{D}_1 + \operatorname{div}(\mathfrak{f}_1)$,

$$\begin{aligned}
(\psi_\gamma \circ p_1)^*\mathfrak{D}' &\le F_*p_1^*\kappa_1^*\mathfrak{D}_1 + \operatorname{div}(p_1^*\mathfrak{f}) \\
&= F_*(\kappa_1 p_1)^*\mathfrak{D}_1 + \operatorname{div}(p_1^*\mathfrak{f}) \\
&= F_*(\vartheta\kappa_2)^*\mathfrak{D}_1 + \operatorname{div}(p_1^*\mathfrak{f}) \\
&= F_*\kappa_2^*\vartheta^*\mathfrak{D}_1 + \operatorname{div}(p_1^*\mathfrak{f}) \\
&= F_*\kappa_2^*\mathfrak{D} - \operatorname{div}(F_*\kappa_2^*\mathfrak{f}_1) + \operatorname{div}(p_1^*\mathfrak{f}).
\end{aligned}$$

By Remark 7.1.8, there exists a plurifunction $\mathfrak{f}_2$ such that $\operatorname{div}(\mathfrak{f}_2) = -\operatorname{div}(F_*\kappa_2^*\mathfrak{f}_1)$. Then, if we denote $\tilde{\mathfrak{f}} = \mathfrak{f}_2 \cdot p_1^*\mathfrak{f}$, we have

$$(\psi_\gamma \circ p_1)^*\mathfrak{D}' \le F_*\kappa_2^*\mathfrak{D} + \operatorname{div}(\tilde{\mathfrak{f}}).$$

This implies that $(\psi_\gamma \circ p_1, F, \tilde{\mathfrak{f}}) : \kappa_2^*\mathfrak{D} \to \mathfrak{D}'$ is a semilinear morphism of pp-divisors that fits by construction into the commutative triangle of the statement

$$\begin{tikzcd}
 & X(\kappa_2^*\mathfrak{D}) \arrow[dl, "X(\kappa_2{,}\mathrm{id}_N{,}\mathfrak{1})"'] \arrow[d, "X(p_1{,}\mathrm{id}_N{,}\mathfrak{f}_2)"] \arrow[ddr, bend left, "X(\psi_\gamma\circ p_1{,}F{,}\tilde{\mathfrak{f}})"] & \\
X(\mathfrak{D}) \arrow[d, "X(\vartheta{,}\mathrm{id}_N{,}\mathfrak{f}_1)"'] & X(\kappa_1^*\mathfrak{D}_1) \arrow[dl, "\cong"', "X(\kappa_1{,}\mathrm{id}_N{,}\mathfrak{1})"] \arrow[dr, "X(\psi_\gamma{,}F{,}\mathfrak{f})"] & \\
X(\mathfrak{D}_1) \arrow[rr, "(\varphi_\gamma{,}f_\gamma)"'] & & X(\mathfrak{D}'),
\end{tikzcd}$$

where $X(\vartheta, \mathrm{id}_N, \mathfrak{f}_1)$ is the identity map. Now, If (ψ_γ, f_γ) is a semilinear isomorphism with respect to γ, then κ_1 can be considered as the identity map and, therefore, $\tilde{Y}_1 \times_{Y_1} Y = Y$. Then, in this case $\tilde{Y} = \overline{U} = Y$, which implies that κ_2 is the identity. This proves the theorem in the case where $\mathfrak{D}$ is not minimal and $\mathfrak{D}'$ is minimal.

Suppose now that we are in the most general case. The strategy is the same as the previous case, but we have to be careful with the fiber product part. Let $\mathfrak{D}'_2$ be a minimal pp-divisor such that $X(\mathfrak{D}') = X(\mathfrak{D}'_2)$. On the one hand, by Lemma 8.4.3, there exists a plurifunction $\mathfrak{f}_2 \in L(N', Y')$ such that $\mathfrak{D}' = \vartheta^*\mathfrak{D}'_2 + \mathrm{div}(\mathfrak{f}_2)$, where $\vartheta : Y' \to Y'_2$ is the canonical morphism. On the other hand, by what we have so far, we know that the theorem holds for $\mathfrak{D}$ and $\mathfrak{D}'_2$. Then, there exists a normal semiprojective variety $\tilde{Y}_2$ over L, a projective birational morphism $\kappa_2 : \tilde{Y}_2 \to Y$ and a semilinear morphism of pp-divisors $(\psi_\gamma, F, \mathfrak{f}) : \kappa_2^*\mathfrak{D} \to \mathfrak{D}'_2$ such that

$$\begin{tikzcd}
 & X(\kappa_2^*\mathfrak{D}) \arrow[dl, "X(\kappa_2{,}\mathrm{id}_N{,}1)"', "\cong"] \arrow[dr, "(\psi_\gamma{,}F{,}\mathfrak{f})"] & \\
X(\mathfrak{D}) \arrow[rr, "(\varphi_\gamma{,}f_\gamma)"'] & & X(\mathfrak{D}'_2).
\end{tikzcd}$$

In this case we have the following commutative diagram

$$\begin{tikzcd}
\tilde{Y}_2 \arrow[r, "\psi_\gamma"] \arrow[d] & Y'_2 \arrow[d] & Y' \arrow[l, "\vartheta"'] \arrow[d] \\
L \arrow[r, "\gamma^\natural"'] & L & L, \arrow[l, "\mathrm{id}"]
\end{tikzcd}$$

then we can not just take the fiber product because ψ_γ is not a morphism of L-varieties. Denote by $\tilde{Y}''_2$ the L-variety given by the composition

$$\tilde{Y}_2 \xrightarrow{\psi_\gamma} L \xrightarrow{\gamma^\natural} L$$

and by $h : \tilde{Y}_2'' \to Y$ the corresponding morphism of varieties over L. Note that $\tilde{Y}_2 = \tilde{Y}_2''$ as schemes. Consider the fiber product $\tilde{Y}_2'' \times_{Y_2'} Y'$. By following the arguments above, let $\tilde{Y}$ be the normalization of the closure of an open subvariety of $\tilde{Y}_2'' \times_{Y_2'} Y'$ isomorphic to some open subvarieties of each of the factors. Then, we have the following commutative diagram of varieties over L.

$$\begin{tikzcd} \tilde{Y} \arrow[r, "p_1"] \arrow[d, "p_2"'] & Y' \arrow[d, "\vartheta"] \\ \tilde{Y}_2'' \arrow[r, "h"'] & Y_2', \end{tikzcd}$$

where the morphisms p_1 and p_2 are induced by the canonical projections of fiber product. Then, we have the following diagram

$$\begin{tikzcd} & \tilde{Y} \arrow[r, "p_1"] \arrow[d, "p_2"'] & Y' \arrow[d, "\vartheta"] \\ Y & \tilde{Y}_2 \arrow[l, "\kappa_2"] \arrow[r, "\psi_\gamma"'] & Y_2', \end{tikzcd}$$

where p_1 is a projective dominant semilinear morphism with respect to γ and p_2 is a morphism of varieties over L. We denote $\kappa := \kappa_2 \circ p_2$. We claim that the triple $(p_1, F, p_2^*\mathfrak{f} \cdot p_1^*\mathfrak{f}_2)$ is a morphism of pp-divisors $\kappa^*\mathfrak{D} \to \mathfrak{D}'$. Indeed, since $(\psi_\gamma, F, \mathfrak{f})$ is a semilinear morphism of pp-divisors, we have

$$\begin{aligned} p_1^*\mathfrak{D}' &= p_1^*(\vartheta^*\mathfrak{D}_2' + \mathrm{div}(\mathfrak{f}_2)) \\ &= p_1^*\vartheta^*\mathfrak{D}_2' + \mathrm{div}(p_1^*\mathfrak{f}_2) \\ &= p_2^*\psi_\gamma^*\mathfrak{D}_2' + \mathrm{div}(p_1^*\mathfrak{f}_2) \\ &\le p_2^*F_*\kappa_2^*\mathfrak{D} + p_2^*\mathrm{div}(\mathfrak{f}) + \mathrm{div}(p_1^*\mathfrak{f}_2) \\ &= F_*p_2^*\kappa_2^*\mathfrak{D} + \mathrm{div}(p_2^*\mathfrak{f}) + \mathrm{div}(p_1^*\mathfrak{f}_2) \\ &= F_*\kappa^*\mathfrak{D} + \mathrm{div}(p_2^*\mathfrak{f} \cdot p_1^*\mathfrak{f}_2). \end{aligned}$$

The triples $(\kappa, \mathrm{id}, \imath) : \kappa^*\mathfrak{D} \to \mathfrak{D}$ and $(p_1, F, p_2^*\mathfrak{f} \cdot p_1^*\mathfrak{f}_2) : \kappa^*\mathfrak{D} \to \mathfrak{D}'$ are the semilinear morphisms of pp-divisors that satisfy the assertion

$$\begin{tikzcd}
 & X(\kappa^*\mathfrak{D}) \arrow[ddl, "X(\kappa,\mathrm{id}_N,\mathfrak{1})"', "\cong"] \arrow[d, "X(p_2,\mathrm{id}_N,\mathfrak{f}_2)"] \arrow[dr, bend left, "X(p_1,F,p_2^*\mathfrak{f}\cdot p_1^*\mathfrak{f}_2)"] & \\
 & X(\kappa_2^*\mathfrak{D}) \arrow[dl, "X(\kappa_2,\mathrm{id}_N,\mathfrak{1})"] \arrow[dr, "X(\psi_\gamma,F,\mathfrak{f})"] & X(\mathfrak{D}') \arrow[d, "X(\vartheta,\mathrm{id}_N,\mathfrak{f}_2)"] \\
X(\mathfrak{D}) \arrow[rr, "(\varphi_\gamma,f_\gamma)"'] & & X(\mathfrak{D}'_2),
\end{tikzcd}$$

where $X(\vartheta, \mathrm{id}_N, \mathfrak{f}_2)$ is the identity map. □

Remark 8.4.5. Notice that Theorem 8.4.4 generalizes [AH06, Theorem 8.8]. It suffices to consider the semilinear morphisms with γ the neutral element of the Galois group.

Let T be a split algebraic torus over L and N be its cocharacter lattice. Let $\mathfrak{D}$ be an object in $\mathfrak{PPDiv}_N(L/k)$. Consider the set

$$S(\mathfrak{D}) := \{(\psi_\gamma, F, \mathfrak{f}) : \mathfrak{D} \to \mathfrak{D} \mid X(\psi_\gamma, F, \mathfrak{f}) \text{ in } \mathrm{SAut}_T(X(\mathfrak{D}))\}.$$

For a general $\mathfrak{D}$, the set $S(\mathfrak{D})$ has a structure of semigroup, having $(\mathrm{id}, \mathrm{id}, 1)$ as the neutral element, but not necessarily a group structure because of the discussion given in Section 8.1. However, for a minimal pp-divisor, $S(\mathfrak{D})$ has a group structure by Theorem 8.4.4. In such a case, we denote by $\mathrm{SAut}(\mathfrak{D}) := S(\mathfrak{D})$ the group of semilinear automorphisms of pp-divisors of $\mathfrak{D}$. Thus, a direct consequence of Theorem 8.4.4 is the following.

Corollary 8.4.6. *Let k be a field of characteristic zero and L/k be a Galois extension. Let $\mathfrak{D}$ be an object in $\mathfrak{PPDiv}(L/k)$ that is minimal. Then,*

$$\mathrm{SAut}(\mathfrak{D}) \cong \mathrm{SAut}_T(X(\mathfrak{D}))$$

as groups, where $T := T(\mathfrak{D})$ is the corresponding split L-torus acting on $X(\mathfrak{D})$ and $\mathrm{SAut}_T(X(\mathfrak{D}))$ stand for the semilinear equivariant automorphisms of $X(\mathfrak{D})$.

A more precise statement over the semilinear equivariant automorphisms of a minimal pp-divisor is the following.

Corollary 8.4.7. *Let k be a field of characteristic zero, L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let $\mathfrak{D}$ be a minimal pp-divisor in $\mathfrak{PPDiv}(L/k)$. Then the semilinear equivariant automorphisms*

$(\varphi_\gamma, f_\gamma) : X(\mathfrak{D}) \to X(\mathfrak{D})$ correspond to the semilinear morphisms of pp-divisors $(\psi_\gamma, F, \mathfrak{f})$ such that $\psi_\gamma^*(\mathfrak{D}) = F_*(\mathfrak{D}) + \operatorname{div}(\mathfrak{f})$. In particular, if $\varphi_\gamma = \operatorname{id}_T$ we have $X(\psi_\gamma, \operatorname{id}_N, \mathfrak{f}) = (\operatorname{id}_T, f)$ and $\psi_\gamma^*(\mathfrak{D}) = \mathfrak{D} + \operatorname{div}(\mathfrak{f})$.

In the toric case, since the only basis for pp-divisors turns out to be $Y = \operatorname{Spec}(L)$, Theorem 8.4.4 yields the following.

Corollary 8.4.8. *Let k be a field of characteristic zero, L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let X_ω and $X_{\omega'}$ be two affine normal toric varieties over L and $(\varphi_\gamma, f_\gamma) : X_\omega \to X_{\omega'}$ be a semilinear equivariant isomorphism. Then, there exists a triple $(\psi_\gamma, F, \mathfrak{f})$, where $\psi_\gamma = \gamma^\natural : \operatorname{Spec}(L) \to \operatorname{Spec}(L)$, $F : N \to N'$ is an isomorphism of lattices such that $F(\omega) = \omega'$ and $\mathfrak{f} \in N \otimes L^*$, such that $(\varphi_\gamma, f_\gamma) = X(\psi_\gamma, F, \mathfrak{f})$.*

Remark 8.4.9. Notice that in the toric case the plurifunction $\mathfrak{f}$ can be identified with an L-point of T, because there is an identification $T(L) \cong N \otimes_{\mathbb{Z}} \mathbb{G}_{\mathrm{m}}(L)$.

We can always consider that the pp-divisors are defined over complete varieties, by Nagata's compactification Theorem. If we restrict the functor $X(\bullet)$ to the full subcategory of $\mathfrak{PPDiv}_N(L/k)$ whose objects are pp-divisors over smooth complete curves, denoted by $\mathfrak{PPDiv}_N^{\mathrm{smooth}}(L/k)$, then we get an equivalence of categories with the category of complexity one normal T-varieties.

Corollary 8.4.10. *The functor $X : \mathfrak{PPDiv}_N(L/k) \to \mathcal{E}_T(L/k)$ turns to be an equivalence of category when we restrict the category $\mathfrak{PPDiv}_N$ to the subcategory $\mathfrak{PPDiv}_N^{\mathrm{smooth}}$ whose objects are pp-divisors over smooth complete curves and $\mathcal{E}_T$ is restricted to complexity one T-varieties.*

Chapter 9

Normal T-varieties

This chapter is devoted to the proof of Theorem 5.0.7, which we recall below for the convenience of the reader.

We start with Section 9.1, where we establish a parallelism between Galois semilinear actions and equivariant Galois descent data.

Through the subsequent sections, we prove Theorem 5.0.7 under stronger hypothesis. In Section 9.2, we prove it for the affine case when the combinatorial datum is given by a minimal pp-divisor. In Section 9.4, we prove it for the affine case and for any pp-divisor. From Section 9.5 to Section 9.7, we develop the notion of *semilinear morphism of divisorial fans* and *semilinear actions* and study theirs properties to finally prove Theorem 5.0.7 in Section 9.8.

Theorem 9.0.1. *Let k be a field of characteristic zero and L/k be a finite Galois extension with Galois group Γ.*

a) Let T be a split algebraic torus over L and X be a normal T-variety over L. If there exists a divisorial fan $(\mathfrak{S}_L, Y_L)$ for X admitting a Γ-semilinear action such that

$$\textit{the subvariety } X(\mathfrak{S}(\mathfrak{D},\Gamma)) \textit{ is quasi-projective for every } \mathfrak{D}\in\mathfrak{S}_L,$$

then there exists an algebraic torus T' over k and a normal T'-variety X' over k such that $X'_L \cong X$ as T varieties over L.

b) Let T be an algebraic torus over k that splits over L. Let X be a normal variety endowed with an effective T-action over k. Then, there exists a divisorial fan $(\mathfrak{S}_L, Y_L)$ admitting a Γ-semilinear action such that

$$\textit{the subvariety } X(\mathfrak{S}(\mathfrak{D},\Gamma)) \textit{ is quasi-projective for every } \mathfrak{D}\in\mathfrak{S}_L,$$

and $X_L \cong X(\mathfrak{S}_L)$ as T_L-varieties.

9.1 Galois descent via semilinear morphisms

In this section, we establish a correspondence between Galois descent data and Galois semilinear equivariant actions. This allows us to give a combinatorial description of the Galois descent data.

Definition 9.1.1. Let k be a field and L/k be any field extension. Let S be an L-scheme. A *k-model* of S is a pair (S', h) such that S' is a scheme over k and $h : S'_L \to S$ is an isomorphism of schemes over L.

Let k be a field and L be a Galois extension with Galois group Γ. Let S be a scheme over L and $\gamma \in \Gamma$. The automorphism $\gamma : L \to L$ induces a morphism of schemes $\gamma^* : \operatorname{Spec} L \to \operatorname{Spec} L$. Note that γ^* and $\gamma^\natural$ are inverses of each other as morphisms of schemes. We define γS as the fiber product

$$\begin{tikzcd} \gamma S \arrow[r, "\alpha_\gamma"] \arrow[d] & S \arrow[d] \\ \operatorname{Spec} L \arrow[r, "\gamma^*"'] & \operatorname{Spec} L\,, \end{tikzcd}$$

where $S \to \operatorname{Spec} L$ is the structural morphism. Moreover, if S' is another scheme over L and $f : S' \to S$ is a morphism of schemes over L, we denote by $\gamma f : \gamma S' \to \gamma S$ the pullback of the morphism by γ^*, which satisfies

$$\alpha_\gamma \circ \gamma f = f \circ \alpha_\gamma.$$

The morphisms α_γ satisfy

$$\alpha_{\tau\gamma} = \alpha_\tau \circ \tau\alpha_\gamma,$$

for γ and τ in Γ. A *Galois descent system over* S is a family $\{h_\gamma\}_{\gamma\in\Gamma}$ of isomorphisms $h_\gamma : \gamma S \to S$ of varieties over L satisfying the cocycle condition given by the following commutative diagram

$$\begin{tikzcd} \gamma_2\gamma_1 S \arrow[d, "\gamma_2 h_{\gamma_1}"'] \arrow[dr, "h_{\gamma_2\gamma_1}"] & \\ \gamma_2 S \arrow[r, "h_{\gamma_2}"'] & S\,, \end{tikzcd}$$

for every γ_1 and γ_2 in Γ. A *Galois descent datum over* S is a Galois descent system $\{h_\gamma\}_{\gamma\in\Gamma}$ admitting a k'-model (S', h') such that $k \subset k' \subset L$ is a finite

Galois extension over k and the following diagram commutes

$$\begin{tikzcd} \gamma S \arrow[drr, "h_\gamma"] & & \\ \gamma S'_L = S'_L \arrow[u, "\gamma h'"] \arrow[rr, "h'"'] & & S, \end{tikzcd}$$

for every $\gamma \in \Gamma$. We say that a Galois descent datum $\{h_\gamma\}_{\gamma\in\Gamma}$ over S is *effective* if there exists a k-model. Notice that $\gamma S'_L$ has a canonical identification with S'_L. Every Galois descent system over a variety is a Galois descent datum by [Bor20, Proposition 5.6], which might not be effective. For quasi-projective schemes over L, every Galois descent datum is effective (see for instance: [Mil24, Corollary 7.3]). Moreover, by [Gro65, Proposition 2.7.1], if the scheme is a variety over L, the k-model is a variety over k.

Let S_1, S_2 be L-schemes equipped with effective Galois descent data $\{h_{1,\gamma}\}_{\gamma\in\Gamma}$ and $\{h_{2,\gamma}\}_{\gamma\in\Gamma}$ respectively. A morphism $f : S_1 \to S_2$ such that $h'_\gamma \circ f = f \circ h_\gamma$ for all $\gamma \in \Gamma$, descends to a morphism $f' : S'_1 \to S'_2$, where S'_1 and S'_2 are the respective k-models (see [Bor20, Proposition 5.6] and étale descent on morphisms).

All of this can be summarized in the following result.

Proposition 9.1.2. *Let k be a field of characteristic zero and L be a Galois extension with Galois group Γ. Then there is an equivalence of categories between the category of quasi-projective schemes over L equipped with an effective Galois descent datum and the category of quasi-projective k-schemes.*

Let $\{h_\gamma\}_{\gamma\in\Gamma}$ be a Galois descent datum over a scheme S over L. For every $\gamma \in \Gamma$ we define the following semilinear morphism

$$\begin{tikzcd} S \arrow[r, "\alpha_\gamma^{-1}"] \arrow[rr, bend left, "g_\gamma"] \arrow[d] & \gamma S \arrow[r, "h_\gamma"] \arrow[d] & S \arrow[d] \\ \operatorname{Spec} L \arrow[r, "\gamma^\natural"'] & \operatorname{Spec} L \arrow[r, "\mathrm{id}"'] & \operatorname{Spec} L, \end{tikzcd}$$

where $\gamma^\natural := \operatorname{Spec}(\gamma^{-1})$. This construction induces a map $g : \Gamma \to \operatorname{SAut}(S)$.

Lemma 9.1.3. *Let k be a field of characteristic zero and L be a Galois extension with Galois group Γ. Let S be a scheme over L. The map*

$$\begin{aligned} g : \Gamma &\to \operatorname{SAut}(S) \\ \gamma &\mapsto g(\gamma) := g_\gamma \end{aligned}$$

is a group homomorphism that defines a section of (8.1)*. In particular, it is a monomorphism.*

Proof. Let γ and τ be in Γ. By definition we have

$$g_{\tau\gamma} = h_{\tau\gamma} \circ \alpha_{\tau\gamma}^{-1} = h_\tau \circ \tau h_\gamma \circ (\tau\alpha_\gamma)^{-1} \circ \alpha_\tau^{-1}.$$

Given that $(\tau\alpha_\gamma)^{-1} = \tau\alpha_\gamma^{-1}$, we have that

$$g_{\tau\gamma} = h_\tau \circ \tau h_\gamma \circ \tau\alpha_\gamma^{-1} \circ \alpha_\tau^{-1} = h_\tau \circ \tau(h_\gamma \circ \alpha_\gamma^{-1}) \circ \alpha_\tau^{-1}.$$

Then, by the relation $\alpha_\tau \circ \tau f = f \circ \alpha_\tau$, it follows

$$g_{\tau\gamma} = h_\tau \circ \tau(h_\gamma \circ \alpha_\gamma^{-1}) \circ \alpha_\tau^{-1} = h_\tau \circ \alpha_\tau^{-1} \circ h_\gamma \circ \alpha_\gamma^{-1} = g_\tau g_\gamma.$$

Finally, since g_γ is γ-semilinear, g defines a section. Thus, the assertion holds. □

Definition 9.1.4. Let k be a field and L/k a Galois extension with Galois group Γ. Let S be a scheme over L. Let G be an abstract group. A *semilinear action of G over S*, or a *G-semilinear action over S*, is a group homomorphism $\varphi : G \to \mathrm{SAut}(S)$. A *Galois semilinear action* is a G-semilinear action when $G = \Gamma$ and φ is a section of (8.1).

Lemma 9.1.3 tells us that a Galois descent system induces a Galois semilinear action. Moreover, every Galois semilinear action arises from a Galois descent system.

Lemma 9.1.5. *Let k be a field of characteristic zero and L be a Galois extension with Galois group Γ. Let S be a scheme over L and $g : \Gamma \to \mathrm{SAut}(S)$ be a Γ-semilinear action over S. Then, there exists a Galois descent system $\{h_\gamma\}_{\gamma\in\Gamma}$ over S, such that $g(\gamma) = g_\gamma$.*

Proof. For every $\gamma \in \Gamma$, define $h_\gamma := g(\gamma) \circ \alpha_\gamma$. Recall that, for γ and τ in Γ, we have that

$$\alpha_{\tau\gamma} = \alpha_\tau \circ \tau\alpha_\gamma.$$

Hence,

$$h_{\tau\gamma} = g(\tau\gamma) \circ \alpha_{\tau\gamma} = g(\tau) \circ g(\gamma) \circ \alpha_\tau \circ \tau\alpha_\gamma.$$

The relation $\alpha_\tau \circ \tau g(\gamma) = g(\gamma) \circ \alpha_\tau$ implies

$$h_{\tau\gamma} = g(\tau) \circ g(\gamma) \circ \alpha_\tau \circ \tau\alpha_\gamma = g(\tau) \circ \alpha_\tau \circ \tau g(\gamma) \circ \tau\alpha_\gamma.$$

Then, given that $\tau(g(\gamma) \circ \alpha_\gamma) = \tau g(\gamma) \circ \tau\alpha_\gamma$, we have

$$h_{\tau\gamma} = g(\tau) \circ \alpha_\tau \circ \tau g(\gamma) \circ \tau\alpha_\gamma = g(\tau) \circ \alpha_\tau \circ \tau(g(\gamma) \circ \alpha_\gamma) = h_\tau \circ \tau h_\gamma,$$

which is the cocycle condition. Thus, the set $\{h_\gamma\}_{\gamma\in\Gamma}$ forms a Galois descent system. Moreover, for every $\gamma \in \Gamma$, we have that $g_\gamma = h_\gamma \circ \alpha_\gamma^{-1} = g(\gamma)$. This proves the assertion. □

Then, we say that a Galois semilinear action over a variety is *effective* if its respective Galois descent datum is effective. Thus, we have the following result, which is a direct of consequence of Proposition 9.1.2, Lemma 9.1.3 and Lemma 9.1.5.

Proposition 9.1.6. *Let k be a field and L be a finite Galois extension with Galois group Γ. There exists an equivalence of categories between the category of quasi-projective varieties over k and the category of quasi-projective varieties over L endowed with a Γ-semilinear action.*

Let G be an algebraic group over L. Given that G is quasi-projective, every Galois descent datum is effective. In this case, we are considering just the Galois descent data given by semilinear group homomorphisms, or equivalently, by Proposition 9.1.6, a Galois semilinear action $\Gamma \to \mathrm{SAut}_{\mathrm{gr}}(G)$. This is because we are interested in the k-models that are also algebraic groups.

For a G-variety X over L, an *equivariant Galois descent system* is a pair of a Galois descent system $\{\sigma_\gamma\}_{\gamma\in\Gamma}$ over G and a Galois descent system $\{h_\gamma\}_{\gamma\in\Gamma}$ over X such that the following diagram commutes

$$\begin{tikzcd} \gamma G \times \gamma X \arrow[r, "\gamma\mu"] \arrow[d, "(\sigma_\gamma, h_\gamma)"'] & \gamma X \arrow[d, "h_\gamma"] \\ G \times X \arrow[r, "\mu"'] & X, \end{tikzcd}$$

where $\mu : G \times X \to X$ is the action. An equivariant Galois descent system is an *equivariant Galois descent datum* if for some finite extension $k \subset k' \subset L$ there exist a k'-model (G', ψ') of G, a k'-model (X', h') with X' a G'-action such that $(\psi', h') : G'_L \times X'_L \to G \times X$ is an equivariant isomorphism. We say that an equivariant Galois descent datum is *effective* if both Galois descent data are effective with k-models G_0 of G and X_0 of X with X_0 a G_0-variety. By [Bor20, Proposition 5.6], every equivariant Galois descent system has an equivariant model over some finite Galois extension. Otherwise stated, every equivariant Galois descent system is an equivariant Galois descent datum.

By Proposition 9.1.6, an equivariant Galois descent datum is equivalent to a Galois semilinear equivariant action as defined in Definition 8.2.2. In particular, it is a group homomorphism

$$g : \Gamma \to \mathrm{SAut}(G; X) \subset \mathrm{SAut}_{\mathrm{gr}}(G) \times \mathrm{SAut}(X),$$

such that the following diagram commutes

$$\begin{tikzcd} G \times X \arrow[r, "\mu"] \arrow[d, "g(\gamma)"'] & X \arrow[d, "g(\gamma)"] \\ G \times X \arrow[r, "\mu"'] & X . \end{tikzcd}$$

The Galois descent datum for G is effective, then it always has a k-model G_0. In particular, both pieces of descent data are effective when X is a quasi-projective variety over L, which does not directly imply that the equivariant Galois descent datum is effective. However, the action also descends (see for instance: [Bor20, Lemma 5.4]).

Proposition 9.1.7. *Let k be a field of characteristic zero and L be a finite Galois extension with Galois group Γ. Let G be an algebraic group over L and X be a G-variety over L. Let $g : \Gamma \to \mathrm{SAut}(G; X)$ be a Γ-semilinear equivariant action over X and G_0 be the k-model of G. If X is quasi-projective, then the descent is effective as a G_0-variety over k.*

Let G and G' be algebraic groups over L. Let X be a G-variety and X' be a G'-variety, both over L. Let g and g' be effective Γ-semilinear equivariant actions on X and X', respectively. Denote by (G_0, X_0) the k-model of the pair (G, X) and by (G'_0, X'_0) the k-model of the pair (G', X'). An equivariant morphism $(\varphi, f) : X \to X'$ such that $g(\gamma) \circ (\varphi, f) = (\varphi, f) \circ g'(\gamma)$ for all $\gamma \in \Gamma$, descends to an equivariant morphism $(\varphi_0, f_0) : X_0 \to X'_0$ (see [Bor20, Proposition 5.6]). Then, we have the following result.

Proposition 9.1.8. *Let k be a field and L be a finite Galois extension with Galois group Γ. Let G be an algebraic group over L and X be a G-variety over L. Let $g : \Gamma \to \mathrm{SAut}(G; X)$ be a Galois semilinear equivariant action. If X is covered by G-stable and Γ-stable quasi-projective open subvarieties, then the Galois semilinear equivariant action is effective.*

Proof. Let $\mathcal{U} := \{X_i\}$ be a finite G-stable and Γ-stable quasi-projective open covering, which can be considered stable under intersections because the intersection of quasi-projective varieties is quasi-projective. Given that each quasi-projective subvariety X_i is G-stable and Γ-stable, the Galois semilinear equivariant action $g : \Gamma \to \mathrm{SAut}(G; X)$ induces Galois semilinear equivariant actions $g_i : \Gamma \to \mathrm{SAut}(G; X_i)$. By Proposition 9.1.7, each triple (G, X_i, g_i) has an effective descent $(G_{0,i}, X_{0,i}, (\psi_i, h_i))$. Given that each g_i induces the same Galois semilinear action over G, we have that $G_0 = G_{0,i}$ and $\psi = \psi_i$ for each X_i. Then, the k-models are of the form $(G_0, X_{0,i}, (\psi, h_i))$ for each (G, X_i, g_i).

Let us see that these G_0-varieties have a gluing data. For the intersection $X_{ij} := X_i \cap X_j$, we have canonical G-equivariant open embeddings $\iota_{ij} : X_{ij} \to X_i$ and $\iota_{ji} : X_{ij} \to X_j$ which are compatible with the Galois semilinear equivariant actions g_i, g_j and g_{ij}. These morphisms descend to G_0-equivariant open embeddings $\eta_{ij} : X_{0,ij} \to X_{0,i}$ and $\eta_{ji} : X_{0,ij} \to X_{0,j}$ that satisfy the following commutative diagram

$$\begin{tikzcd}
X_{0,i} \times_k L \arrow[r, "{(\psi,h_i)}"] & X_i \\
X_{0,ij} \times_k L \arrow[u, "\eta_{ij}\times_k \mathrm{id}_L"] \arrow[r, "{(\psi,h_{ij})}"'] & X_{ij}. \arrow[u, "\iota_{ij}"']
\end{tikzcd}$$

From the morphisms η_{ij} and η_{ji}, we have G_0-equivariant isomorphisms $\varphi_{ij} := \eta_{ji} \circ \eta_{ij}^{-1} : \mathrm{Im}(\eta_{ij}) \to \mathrm{Im}(\eta_{ji})$. Let us consider the following quotient space:

$$X_0 := \left(\bigsqcup_{X_{0,i} \in \mathcal{U}_0} X_{0,i} \right) / \sim,$$

where the relation is given by $x \sim y$ if and only if for some φ_{ij} we have $\varphi_{ij}(x) = y$. The canonical G_0-equivariant embeddings $X_{0,i} \to X_0$ fit into the following commutative diagram

$$\begin{tikzcd}
X_{0,i} \arrow[r] & X_0 \\
X_{0,ij} \arrow[u, "\eta_{ij}"] \arrow[ur] \arrow[r, "\eta_{ji}"'] & X_{0,j}\,. \arrow[u]
\end{tikzcd}$$

Also, notice that there is a canonical G_0-equivariant isomorphism

$$X_0 \times_k L \cong \left(\bigsqcup_{X_{0,i} \in \mathcal{U}_0} X_{0,i} \times_k L \right) / \sim$$

where the relation is given by $x \sim y$ if and only if for some $\varphi_{ij} \times_k \mathrm{id}_L$ we have $\varphi_{ij} \times_k \mathrm{id}_L(x) = y$. Now, let us take

$$(\psi, \tilde{h}) : \bigsqcup_{X_{0,i} \in \mathcal{U}_0} X_{0,i} \times_k L \to \bigsqcup_{X_i \in \mathcal{U}} X_i,$$

the morphism induced by the $(\psi, h_i) : X_{0,i} \times_k L \to X_i$. Notice that if for x and y there exists $(\varphi_{ij} \times_k \mathrm{id}_L)(x) = y$, then there exists $z \in X_{0,ij}$ such that $(\eta_{ij} \times_k \mathrm{id}_L)(z) = x$ and $(\eta_{ji} \times_k \mathrm{id}_L)(z) = y$. Thus,

$$\tilde{h}(x) = h_i(x) = h_i((\eta_{ij} \times_k \mathrm{id}_L)(z)) = h_j((\eta_{ji} \times_k \mathrm{id}_L)(z)) = h_j(y) = \tilde{h}(y).$$

This implies that $(\psi, \tilde{h})$ induces a morphism $(\psi, h) : X_0 \times_k L \to X$, which is indeed an equivariant isomorphism. Hence, $(X_0, G_0, (\psi, h))$ is a k-model for (X, G). Given that X is a variety over L, we have that X_0 is a variety over k by [Gro65, Proposition 2.7.1]. Thus, the Galois semilinear equivariant action is effective and the assertion holds. □

This allows us to prove the following result.

Proposition 9.1.9. *Let k be a field and L be a finite Galois extension with Galois group Γ. Let G be a connected algebraic group over L. Then, there exists an equivalence of categories between the category of normal varieties with effective G'-actions, where G' is a k-model of G, and the category of normal varieties over L with effective G-actions endowed with Galois semilinear equivariant actions, which are covered by G-stable and Γ-stable quasi-projective subvarieties.*

Remark 9.1.10. The reader should be warned that morphisms in these categories are given by pairs of morphisms (φ, f), where φ is a morphism of algebraic groups and f is a morphism of varieties. In particular, even if we fix a group G, a morphism might not be the identity on G, so the latter is not a subcategory of the category of G-varieties with G-equivariant morphisms. This is actually crucial in order to let Γ act semilinearly on it.

Proof. We give the equivalence at the level of objects. The equivalence at the level of morphisms will follow from Proposition 9.1.6.

Let (G', ψ) be a k-model of G and X' be a normal G'-variety over k. By [Bri17, Theorem 1], X' is covered by G'-stable quasi-projective open subvarieties over L. Hence, $X_L := X' \times_k L$ is a normal G'_L-variety over L covered by Γ-stable quasi-projective subvarieties. Then, X_L has a compatible structure of G-variety under the isomorphism $\psi : G'_L \to G$.

The other direction is given by Proposition 9.1.8. □

9.2 Affine case and minimal pp-divisors

Let k be a field of characteristic zero, L/k be a finite Galois extension with Galois group Γ. Let $\mathfrak{D}$ be a minimal pp-divisor over L. In this section, we define *semilinear actions over* minimal pp-divisors and get a new proof of Gillard's Theorem (cf. Theorem 5.0.4).

Definition 9.2.1. Let k be a field and L/k be a Galois extension. Let G be a group. Let $\mathfrak{D}$ be a minimimal pp-divisor in $\mathfrak{PPDiv}(L/k)$. A *G-semilinear action over* $\mathfrak{D}$ is a group homomorphism $\varphi : G \to \mathrm{SAut}(\mathfrak{D})$.

Let G be an abstract group. A G-semilinear action $\varphi : G \to \mathrm{SAut}(\mathfrak{D})$ induces a G-semilinear equivariant action (recall Definition 8.2.2)

$$X(\varphi) : G \to \mathrm{SAut}(T; X(\mathfrak{D})),$$

via the functor $X : \mathfrak{PPDiv}(L/k) \to \mathcal{E}(L/k)$. Given that $\mathfrak{D}$ is a minimal pp-divisor, every G-semilinear equivariant action $\rho : G \to \mathrm{SAut}(T; X(\mathfrak{D}))$ arises from a G-semilinear action of pp-divisors by Corollary 8.4.6. Actually, this defines a bijection between the set of semilinear actions over $\mathfrak{D}$ and the set of semilinear equivariant actions over $X(\mathfrak{D})$.

Proposition 9.2.2. *Let k be a perfect field and L be a Galois extension. Let $\mathfrak{D}$ be an object in $\mathfrak{PPDiv}(L/k)$ that is minimal. Then, there exists a bijection between the set of semilinear actions over $\mathfrak{D}$ and the set of semilinear equivariant actions over $X(\mathfrak{D})$.*

Proof. This is consequence of Corollary 8.4.6, because it implies that the following commutative diagram can be always completed in a unique way

$$\begin{tikzcd} & G \arrow[dl, "\varphi"'] \arrow[dr, "\rho"] & \\ \mathrm{SAut}(\mathfrak{D}) \arrow[rr, "\cong", "X"'] & & \mathrm{SAut}(T; X(\mathfrak{D})). \end{tikzcd}$$

Otherwise stated, having φ we can construct a unique ρ and having ρ there exists a unique φ. $\square$

Let $\mathfrak{PPDiv}(\Gamma)$ the category of pairs $(\mathfrak{D}, g)$, where $\mathfrak{D}$ is a minimal pp-divisor over L and $g : \Gamma \to \mathrm{SAut}(\mathfrak{D})$ is a Galois semilinear action. A morphism in this category is a morphism of pp-divisors $(\psi, F, \mathfrak{f}) : \mathfrak{D} \to \mathfrak{D}'$ such that

$$g'_\gamma \circ (\psi, F, \mathfrak{f}) = (\psi, F, \mathfrak{f}) \circ g_\gamma$$

for every $\gamma \in \Gamma$. Let $(\mathfrak{D}, g)$ be an object in $\mathfrak{PPDiv}(\Gamma)$. By Theorem 7.2.1, $X(\mathfrak{D})$ is a geometrically integral normal $T_{\mathfrak{D}}$-variety over L, where $T_{\mathfrak{D}}$ denote its respective torus action. Moreover, by Proposition 9.2.2, $X(\mathfrak{D})$ comes with a Galois semilinear equivariant automorphisms

$$X(g) : \Gamma \to \mathrm{SAut}(T_{\mathfrak{D}}; X(\mathfrak{D})).$$

Then, by Proposition 9.1.9, there exists a geometrically integral normal T-variety $X := X(\mathfrak{D}, g)$ over k such that $X_L \cong X(\mathfrak{D})$ as $T_{\mathfrak{D}}$-varieties over L. This proves the first part of the following theorem.

Theorem 9.2.3. *Let k be a field of characteristic zero, L/k be a finite Galois extension with Galois group Γ.*

a) Let $(\mathfrak{D}_L, g)$ be an object in $\mathfrak{PPDiv}(\Gamma)$. Then, $X(\mathfrak{D}_L, g)$ is a geometrically integral normal affine variety endowed with an effective action of an algebraic torus T over k such that T splits over L and $X(\mathfrak{D}_L, g)_L \cong X(\mathfrak{D}_L)$ as $T_{\mathfrak{D}_L}$-varieties over L.

b) Let X be a geometrically integral normal affine variety over k endowed with an effective T-action such that T_L is split. Then, there exists an object $(\mathfrak{D}_L, g)$ in $\mathfrak{PPDiv}(\Gamma)$ such that $X \cong X(\mathfrak{D}_L, g)$ as T-varieties.

Proof. Let us prove part (b), the remaining part of the theorem. Let X be a geometrically integral normal variety over k endowed with an effective T-action. By Proposition 9.1.9, as a T-variety over k, X is equivalent to a pair (X_L, g'), where X_L is a geometrically integral normal T_L-variety, with T_L split over L, and a Γ-semilinear equivariant action g'. By Proposition 7.2.19, there exists a pp-divisor $\mathfrak{D}$ such that $X_L \cong X(\mathfrak{D})$ as T_L-varieties over L. This pp-divisor, by the proof of Proposition 7.2.19, can be chosen minimal. Now, by Proposition 9.2.2, we have that the Γ-semilinear equivariant action on $X(\mathfrak{D}_L)$ induces a unique Γ-semilinear action g on $\mathfrak{D}_L$. Then, the pair $(\mathfrak{D}, g)$ encodes the pair (X_L, g'). Hence, there exists a pair $(\mathfrak{D}, g)$ in $\mathfrak{PPDiv}(\Gamma)$ such that $X \cong X(\mathfrak{D}, g)$ as T-varieties. □

By Theorem 9.2.3, every pair $(\mathfrak{D}, g)$ corresponds to a geometrically integral normal affine variety $X(\mathfrak{D}, g)$ endowed with a torus action over k that is split over L. This construction induces a functor

$$\begin{aligned} X : \mathfrak{PPDiv}(\Gamma) &\to \mathcal{E}(k, L); \\ (\mathfrak{D}, g) &\mapsto X(\mathfrak{D}, g), \end{aligned}$$

where $\mathcal{E}(k, L)$ is the category of affine normal varieties over k endowed with an effective action of an algebraic torus over k that is split over L. This functor is the composition of the functor $(\mathfrak{D}, g) \mapsto (X(\mathfrak{D}), X(g))$, from the category $\mathfrak{PPDiv}(\Gamma)$ to the category of geometrically integral normal affine varieties endowed with an effective action of a split algebraic torus over L and a Γ-semilinear equivariant action, and the equivalence of categories of Proposition 9.1.9. Given that the first functor is faithful, covariant and essentially surjective, we have the following.

Proposition 9.2.4. *Let k be a field of characteristic zero and L/k be a finite Galois extension with Galois group Γ. The functor $X : \mathfrak{PPDiv}(\Gamma) \to \mathcal{E}(k, L)$ is covariant, faithful and essentially surjective.*

Recovering Gillard's Theorem. Let $(\mathfrak{D}, g)$ be an object in $\mathfrak{PPDiv}(\Gamma)$ such that $\mathfrak{D}$ is a minimal pp-divisor. Recall that for every $\gamma \in \Gamma$, $g_\gamma := (\psi_\gamma, F_\gamma, \mathfrak{f}_\gamma) : \mathfrak{D} \to \mathfrak{D}$ is a semilinear automorphism of pp-divisors and

$$g_{\gamma_2\gamma_1} = (\psi_{\gamma_2\gamma_1}, F_{\gamma_2\gamma_1}, \mathfrak{f}_{\gamma_2\gamma_1}) = (\psi_{\gamma_2}\psi_{\gamma_1}, F_{\gamma_2}F_{\gamma_1}, F_{\gamma_2 *}(\mathfrak{f}_{\gamma_1})\psi^*_{\gamma_1}(\mathfrak{f}_{\gamma_2})) = g_{\gamma_2}g_{\gamma_1},$$

for every $\gamma_1, \gamma_2 \in \Gamma$. If we define $h_\gamma := \mathfrak{f}_\gamma \circ F^*_{\gamma^{-1}}$, where we view $\mathfrak{f}_\gamma$ as a morphism $M \to L(Y)^*$ and $F^*_\gamma : M \to M$ is the dual map of F_γ, we have

$$\begin{aligned}
h_{\gamma_2\gamma_1} \circ F^*_{\gamma_2\gamma_1} &= \mathfrak{f}_{\gamma_2\gamma_1} \\
&= F_{\gamma_2 *}(\mathfrak{f}_{\gamma_1}) \cdot \psi^*_{\gamma_1}(\mathfrak{f}_{\gamma_2}) \\
&= (\mathfrak{f}_{\gamma_1} \circ F^*_{\gamma_2}) \cdot \psi^*_{\gamma_1}(\mathfrak{f}_{\gamma_2}) \\
&= (h_{\gamma_1} \circ F^*_{\gamma_1} \circ F^*_{\gamma_2}) \cdot \psi^*_{\gamma_1}(h_{\gamma_2} \circ F^*_{\gamma_2}) \\
&= (h_{\gamma_1} \circ F^*_{\gamma_2\gamma_1}) \cdot \psi^*_{\gamma_1}(h_{\gamma_2} \circ F^*_{\gamma_1^{-1}} \circ F^*_{\gamma_2\gamma_1}) \\
&= (h_{\gamma_1} \cdot \psi^*_{\gamma_1}(h_{\gamma_2} \circ F^*_{\gamma_1^{-1}})) \circ F^*_{\gamma_2\gamma_1}.
\end{aligned}$$

Thus, the maps $h_\gamma : M \to L(Y)^*$ satisfy

$$h_{\gamma_2\gamma_2} = h_{\gamma_1} \cdot \psi^*_{\gamma_1}(h_{\gamma_2} \circ F^*_{\gamma_1^{-1}}),$$

for every $\gamma_1, \gamma_2 \in \Gamma$. This condition corresponds to the condition (1b) of Theorem 5.0.4. The other condition is fulfilled by Corollary 8.4.7. Then we recover Gillard's Theorem.

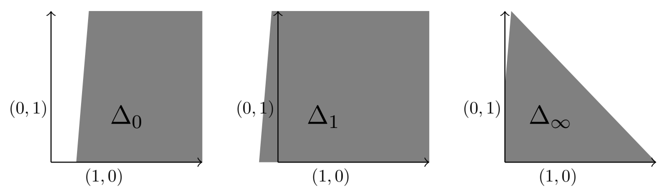
Example 9.2.5 (Example 7.2.24 revisited)**.** Let k be a field of characteristic zero and L/k be a quadratic extension with Galois group Γ. The affine threefold $X := \mathrm{Spec}(L[x,y,z,w]/(x^3+y^4+zw))$ in $\mathbb{A}^4_L$ with the action of $\mathbb{G}^2_{\mathrm{m},L}$ given by

$$(\lambda,\mu)\cdot(x,y,z,w) = (\lambda^4 x, \lambda^3 y, \mu z, \lambda^{12}\mu^{-1} w)$$

is encoded by the pp-divisor $\mathfrak{D} := \Delta_0 \otimes \{0\} + \Delta_1 \otimes \{1\} + \Delta_\infty \otimes \{\infty\}$, where

$$\Delta_0 = \left(0, \frac{1}{3}\right) + \omega, \quad \Delta_1 = \left(-\frac{1}{4}, 0\right) + \omega, \quad \Delta_\infty = (\{0\} \times [0,1]) + \omega$$

and $\omega = \mathrm{cone}((1,0),(1,12))$.



We claim that this affine normal T-variety has no nontrivial k-forms. Let X' be a k form of X as a T-variety, it means that X' is endowed with and effective action of T' a k-form of T. By Theorem 9.2.3, there exists a Galois semilinear action $\Gamma \to \mathrm{SAut}(\mathfrak{D})$ given by $(\psi_\gamma, F, \mathfrak{f})$, where γ is the nontrivial element of Γ. Since $(\psi_\gamma, F, \mathfrak{f})$ is a semilinear automorphism of $\mathfrak{D}$, it holds that $F(\omega) = \omega$.

Let us prove our claim. It is known that the k-forms of $\mathbb{G}^2_{\mathrm{m},L}$ are

$$(\mathbb{G}_{\mathrm{m},k})^2, \quad \mathbb{G}_{\mathrm{m},k} \times \mathrm{R}^1_{L/k}(\mathbb{G}_{\mathrm{m},L}), \quad \left(\mathrm{R}^1_{L/k}(\mathbb{G}_{\mathrm{m},L})\right)^2 \quad \text{and} \quad \mathrm{R}_{L/k}(\mathbb{G}_{\mathrm{m},L}),$$

where $\mathrm{R}_{L/k}(\mathbb{G}_{\mathrm{m},L})$ is the Weil restriction and $\mathrm{R}^1_{L/k}(\mathbb{G}_{\mathrm{m},L})$ is its respective norm one subtorus. Their respective Galois descent data $\Gamma \to \mathrm{SAut}(\mathbb{G}^2_{\mathrm{m},L})$ are encode by one the following group homomorphisms $F : \Gamma \to \mathrm{Aut}(N)$:

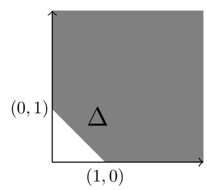
$$F(\gamma) \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

The only one that preserves $\omega = \mathrm{cone}((1,0),(1,12))$ is $F = \mathrm{id}_N$. Thus, the Galois semilinear action is given by $(\psi_\gamma, \mathrm{id}_N, \mathfrak{f})$. This implies that T' is split, then X' comes from a pp-divisor $\mathfrak{D}'$ over k, which can be considered minimal. After a base change, we have that $\mathfrak{D}'_L \cong \mathfrak{D}$. Since all the polyhedra are different between them, the divisors defining $\mathfrak{D}'$ remains irreducible. This implies that ψ_γ must to fix divisors defining $\mathfrak{D}$. Then, $\psi_\gamma([x:y]) = [\gamma(x) : \gamma(y)]$ in $\mathbb{P}^1_k$. This implies that this proves the claim.

Example 9.2.6 (Example 7.2.22 revisited)**.** Let k be a field of characteristic zero and L/k be a quadratic extension with Galois group Γ. The affine space $\mathbb{A}^3_L$ endowed the action of $\mathbb{G}^2_{\mathrm{m},L}$ given by

$$(\lambda, \mu) \cdot (x, y, z) = (\lambda x, \mu y, \lambda\mu z)$$

arises from the pp-divisor $\mathfrak{D} := \Delta \otimes \{\infty\}$ over $\mathbb{P}^1_L$, where Δ is the polyhedron



The quotient map $\mathbb{A}^3_L \dashrightarrow \mathbb{P}^1_L$ is given by $(x, y, z) \mapsto (z, xy)$. Let us consider the following Galois semilinear equivariant action on $\mathbb{A}^3_L$:

$$\begin{aligned} \mathbb{A}^3_L &\to \mathbb{A}^3_L \\ (x, y, z) &\mapsto (\gamma(y), \gamma(x), \gamma(z)). \end{aligned}$$

In the torus, the Galois semilinear action is given by $(\lambda, \mu) \mapsto (\gamma(\mu), \gamma(\lambda))$. In terms of the pp-divisor, the Galois semilinear action is given by $(\psi_\gamma, F, \mathfrak{f})$, with $\psi_\gamma([v : w]) = [\gamma(w) : \gamma(v)]$, $F(a, b) = (b, a)$ and $\mathfrak{f} = \mathbb{1}$. Notice that

$$\Delta \otimes \{\infty\} = \psi^*_\gamma \mathfrak{D} = F_* \mathfrak{D} = \Delta \otimes \{\infty\}.$$

Then the decent as a T-variety is effective by Theorem 9.2.3. Now, the semilinear equivariant action over $\mathbb{A}^3_L$ is given by an equivariant semilinear action in $\mathbb{A}^2_L$ and anotherone over $\mathbb{A}^1_L$. Given that only separeble k-forms of $\mathbb{A}^2_L$ are the affine plane by [Kam75, Theorem 3], the corresponding k-form of $\mathbb{A}^3_L$ is $\mathbb{A}^3_k$. For the torus action, the respective k-form is $\mathrm{Res}_{L/k}(\mathbb{G}_{\mathrm{m},L})$.

9.3 A localized category of pp-divisors

The functor $X : \mathfrak{PPDiv}(L/k) \to \mathcal{E}(L/k)$ is faithful by Proposition 8.1.1. However, it does not induce an equivalence of categories. The only problem relies on the surjectivity on morphisms. A way to solve this problem would be to *localize* the category $\mathfrak{PPDiv}(L/k)$. In other words, we can pump up our category in order to get an equivalence of categories with the category $\mathcal{E}(L/k)$. In this section, we build such a category and prove that it satisfies the universal property of a localization.

A stuffed category The category $\mathfrak{PPDiv}(L/k)$ does not have enough morphisms. Our new category, denoted by $\mathfrak{PPDiv}_S(L/k)$, has the same objects as $\mathfrak{PPDiv}(L/k)$. Let us now define the morphisms. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two objects in $\mathfrak{PPDiv}_S(L/k)$, i.e. two pp-divisors over L. Consider the set $M_{\mathfrak{D},\mathfrak{D}'}$ as the set of triangles

$$\begin{tikzcd} & \kappa^*\mathfrak{D} \arrow[dl, "{(\kappa,\mathrm{id}_N,\mathfrak{1})}"'] \arrow[dr, "{(\psi_\gamma,F,\mathfrak{f})}"] & \\ \mathfrak{D} & & \mathfrak{D}', \end{tikzcd}$$

where $\tilde{Y}$ is a normal semiprojective variety over L, $\kappa : \tilde{Y} \to Y$ is a projective and surjective morphism and $(\psi_\gamma, F, \mathfrak{f}) : \kappa^*\mathfrak{D} \to \mathfrak{D}'$ is a dominating semilinear morphism of pp-divisors. In order to lighten the notation, the elements of $M_{\mathfrak{D},\mathfrak{D}'}$ will be denoted by $(\kappa, \psi_\gamma, F, \mathfrak{f})$. In $M_{\mathfrak{D},\mathfrak{D}'}$ we define the following relation:

$$(\kappa, \psi_\gamma, F, \mathfrak{f}) \sim (\kappa', \psi'_{\gamma'}, F', \mathfrak{f}')$$

if and only if there exists a normal semiprojective variety $\hat{Y}$ and projective surjective morphisms $\hat{\kappa} : \hat{Y} \to \tilde{Y}$ and $\hat{\kappa}' : \hat{Y} \to \tilde{Y}'$ such that

$$(\kappa \circ \hat{\kappa}, \psi_\gamma \circ \hat{\kappa}, F, \hat{\kappa}^*\mathfrak{f}) = (\kappa' \circ \hat{\kappa}', \psi'_{\gamma'} \circ \hat{\kappa}', F', \hat{\kappa}'^*\mathfrak{f}').$$

This relation is clearly reflexive and symmetric.

Lemma 9.3.1. *Let k be a field of characteristic zero and L/k be a Galois extension. Let $(\kappa, \psi_\gamma, F, \mathfrak{f})$ and $(\kappa', \psi'_{\gamma'}, F', \mathfrak{f}')$ in $M_{\mathfrak{D},\mathfrak{D}'}$. Then,*

$$(\kappa, \psi_\gamma, F, \mathfrak{f}) \sim (\kappa', \psi'_{\gamma'}, F', \mathfrak{f}')$$

if and only if $X(\psi_\gamma, F, \mathfrak{f}) = X(\psi'_{\gamma'}, F', \mathfrak{f}')$.

Proof. If $(\kappa, \psi_\gamma, F, \mathfrak{f}) \sim (\kappa', \psi'_{\gamma'}, F', \mathfrak{f}')$ we have the following commutative diagram

$$\begin{tikzcd} & & \kappa^*\mathfrak{D} \arrow[dl, "{(\kappa,\mathrm{id}_N,\mathfrak{1})}"'] \arrow[dr, "{(\psi_\gamma,F,\mathfrak{f})}"] & \\ \hat{\kappa}^*\kappa^*\mathfrak{D} = \hat{\kappa}'^*\kappa'^*\mathfrak{D} \arrow[urr, bend left, "{(\hat{\kappa},\mathrm{id}_N,\mathfrak{1})}"] \arrow[drr, bend right, "{(\hat{\kappa}',\mathrm{id}_N,\mathfrak{1})}"'] & \mathfrak{D} & & \mathfrak{D}' \\ & & \kappa'^*\mathfrak{D} \arrow[ul, "{(\kappa',\mathrm{id}_{N'},\mathfrak{1})}"] \arrow[ur, "{(\psi'_{\gamma'},F',\mathfrak{f}')}"'] & . \end{tikzcd}$$

This diagram induces the following in terms of normal affine varieties

$$\begin{tikzcd}
 & & X(\kappa^*\mathfrak{D}) \arrow[dl, "{X(\kappa,\mathrm{id}_N,\imath)}"'] \arrow[dd, "\mathrm{id}"] \arrow[dr, "{X(\psi_\gamma,F,\mathfrak{f})}"] & \\
X(\hat{\kappa}^*\kappa^*\mathfrak{D}) = X(\hat{\kappa}'^*\kappa'^*\mathfrak{D}) \arrow[urr, bend left, "{X(\hat{\kappa},\mathrm{id}_N,\imath)}"] \arrow[r, "\mathrm{id}"] \arrow[drr, bend right, "{X(\hat{\kappa}',\mathrm{id}_N,\imath)}"'] & X(\mathfrak{D}) & & X(\mathfrak{D}') \\
 & & X(\kappa'^*\mathfrak{D}) \arrow[ul, "{X(\kappa',\mathrm{id}_{N'},\imath)}"] \arrow[ur, "{X(\psi'_{\gamma'},F',\mathfrak{f}')}"'] &
\end{tikzcd}.$$

Given that, except for $X(\psi_\gamma, F, \mathfrak{f})$ and $X(\psi'_{\gamma'}, F', \mathfrak{f}')$, all the morphisms in the diagram are identities, we have that $X(\psi_\gamma, F, \mathfrak{f}) = X(\psi'_{\gamma'}, F', \mathfrak{f}')$.

In the other direction, if we assume $X(\psi_\gamma, F, \mathfrak{f}) = X(\psi'_{\gamma'}, F', \mathfrak{f}')$, we have the following commutative diagram

$$\begin{tikzcd}
 & X(\kappa^*\mathfrak{D}) \arrow[dl, "{X(\kappa,\mathrm{id}_N,\imath)}"'] \arrow[dd, "\mathrm{id}"] \arrow[dr, "{X(\psi_\gamma,F,\mathfrak{f})}"] & \\
X(\mathfrak{D}) & & X(\mathfrak{D}') \\
 & X(\kappa'^*\mathfrak{D}) \arrow[ul, "{X(\kappa',\mathrm{id}_{N'},\imath)}"] \arrow[ur, "{X(\psi'_{\gamma'},F',\mathfrak{f}')}"'] &
\end{tikzcd}.$$

By applying Theorem 8.4.4 on the indentity $\mathrm{id} : X(\kappa^*\mathfrak{D}) \to X(\kappa'^*\mathfrak{D})$, we obtain the following commutative diagram

$$\begin{tikzcd}
 & & \kappa^*\mathfrak{D} \arrow[dl, "{(\kappa,\mathrm{id}_N,\imath)}"'] \arrow[dr, "{(\psi_\gamma,F,\mathfrak{f})}"] & \\
\hat{\kappa}^*\kappa^*\mathfrak{D} = \hat{\kappa}'^*\kappa'^*\mathfrak{D} \arrow[urr, bend left, "{(\hat{\kappa},\mathrm{id}_N,\imath)}"] \arrow[drr, bend right, "{(\hat{\kappa}',\mathrm{id}_N,\imath)}"'] & \mathfrak{D} & & \mathfrak{D}' \\
 & & \kappa'^*\mathfrak{D} \arrow[ul, "{(\kappa',\mathrm{id}_{N'},\imath)}"] \arrow[ur, "{(\psi'_{\gamma'},F',\mathfrak{f}')}"'] &
\end{tikzcd}.$$

Hence, we have that $(\kappa, \psi_\gamma, F, \mathfrak{f}) \sim (\kappa', \psi'_{\gamma'}, F', \mathfrak{f}')$. Then, the assertion holds. $\square$

By Lemma 9.3.1, the relation defined above is transitive. This implies that the relation is an equivalence relation. Thus, we define

$$\mathrm{Mor}_{\mathfrak{PPDiv}_S(L/k)}(\mathfrak{D}, \mathfrak{D}') := M_{\mathfrak{D},\mathfrak{D}'}/\sim .$$

Proposition 9.3.2. *Let k be a field of characteristic zero and L/k be a Galois extension. The class $\mathfrak{PPDiv}_S(L/k)$ defines a category.*

Proof. We need to prove that the composition is well defined. Let $\mathfrak{D}$, $\mathfrak{D}'$ and $\mathfrak{D}''$ be objects in $\mathfrak{PPDiv}_S(L/k)$ and $(\kappa, \psi_\gamma, F, \mathfrak{f}) : \mathfrak{D} \to \mathfrak{D}'$ and $(\kappa', \psi'_{\gamma'}, F', \mathfrak{f}') : \mathfrak{D}' \to \mathfrak{D}''$ be respectively in $\mathrm{Mor}_{\mathfrak{PPDiv}_S(L/k)}(\mathfrak{D}, \mathfrak{D}')$ and $\mathrm{Mor}_{\mathfrak{PPDiv}_S(L/k)}(\mathfrak{D}', \mathfrak{D}'')$. These morphisms form the following diagram

$$\begin{tikzcd}
& \kappa^*\mathfrak{D} \arrow[dl, "{(\kappa,\mathrm{id}_N,\mathbb{1})}"'] \arrow[dr, "{(\psi_\gamma,F,\mathfrak{f})}"] & & \kappa'^*\mathfrak{D}' \arrow[dl, "{(\kappa',\mathrm{id}_{N'},\mathbb{1})}"] \arrow[dr, "{(\psi'_{\gamma'},F',\mathfrak{f}')}"] & \\
\mathfrak{D} & & \mathfrak{D}' & & \mathfrak{D}''.
\end{tikzcd}$$

The morphism $(\kappa, \psi_\gamma, F, \mathfrak{f}) : \mathfrak{D} \to \mathfrak{D}'$ induces a dominant equivariant semilinear morphism $X(\kappa, \psi_\gamma, F, \mathfrak{f}) : X(\mathfrak{D}) \to (\mathfrak{D}')$, which corresponds to

$$X(\psi_\gamma, F, \mathfrak{f}) : X(\kappa^*\mathfrak{D}) \to X(\mathfrak{D}').$$

Hence, given that $X(\mathfrak{D}) = X(\kappa^*\mathfrak{D})$ and $X(\mathfrak{D}') = X(\kappa'^*\mathfrak{D}')$, we have the following morphism $X(\kappa, \psi_\gamma, F, \mathfrak{f}) : X(\kappa^*\mathfrak{D}) \to X(\kappa'^*\mathfrak{D}')$. Then, by Theorem 8.4.4, there exists a morphism $(\tilde{\kappa}, \tilde{\psi}_\gamma, \tilde{F}, \tilde{\mathfrak{f}})$ in $\mathrm{Mor}_{\mathfrak{PPDiv}_S(L/k)}(\kappa^*\mathfrak{D}, \kappa'^*\mathfrak{D}')$ such that $X(\tilde{\kappa}, \tilde{\psi}_\gamma, \tilde{F}, \tilde{\mathfrak{f}}) = X(\kappa, \psi_\gamma, F, \mathfrak{f})$. Notice that $\tilde{F} = F$ because both represent the same morphisms $\varphi_\gamma : T \to T'$. All these three localized dominating semilinear morphisms of pp-divisors fit into the following commutative diagram

$$\begin{tikzcd}
& & \tilde{\kappa}^*\kappa^*\mathfrak{D} \arrow[dl, "{(\tilde{\kappa},\mathrm{id}_N,\mathbb{1})}"'] \arrow[dr, "{(\tilde{\psi}_\gamma,F,\tilde{\mathfrak{f}})}"] & & \\
& \kappa^*\mathfrak{D} \arrow[dl, "{(\kappa,\mathrm{id}_N,\mathbb{1})}"'] \arrow[dr, "{(\psi_\gamma,F,\mathfrak{f})}"] & & \kappa'^*\mathfrak{D}' \arrow[dl, "{(\kappa',\mathrm{id}_N,\mathbb{1})}"] \arrow[dr, "{(\psi'_{\gamma'},F',\mathfrak{f}')}"] & \\
\mathfrak{D} & & \mathfrak{D}' & & \mathfrak{D}''.
\end{tikzcd}$$

Then, the composition gives

$$(\kappa, \psi_\gamma, F, \mathfrak{f}) \circ (\kappa', \psi'_{\gamma'}, F', \mathfrak{f}') = (\tilde{\kappa} \circ \kappa, \psi'_{\gamma'} \circ \tilde{\psi}_\gamma, F' \circ F, \tilde{\psi}^*_\gamma(\mathfrak{f}) \cdot F'_*(\tilde{\mathfrak{f}}))$$

in $\mathrm{Mor}_{\mathfrak{PPDiv}_S(L/k)}(\mathfrak{D}, \mathfrak{D}'')$. Then, the morphisms can be composed and, therefore, $\mathfrak{PPDiv}_S(L/k)$ defines a category. $\square$

Remark 9.3.3. Notice that if κ and κ' are identities, we have that the composition turns to be

$$(\mathrm{id}_Y, \psi_\gamma, F, \mathfrak{f}) \circ (\mathrm{id}_{Y'}, \psi'_{\gamma'}, F', \mathfrak{f}') = (\mathrm{id}_Y, \psi'_{\gamma'} \circ \psi_\gamma, F' \circ F, \psi^*_\gamma(\mathfrak{f}) \cdot F'_*(\mathfrak{f})),$$

since the semilinear morphisms of pp-divisors $(\kappa, \mathrm{id}_N, \imath)$ and $(\kappa', \mathrm{id}_{N'}, \imath)$ are isomorphisms.

The category $\mathfrak{PPDiv}_S(L/k)$ has the same objects as $\mathfrak{PPDiv}(L/k)$. We prove now that the latter can be see as a subcategory of the former.

The assignation $Q : \mathfrak{PPDiv}(L/k) \to \mathfrak{PPDiv}_S(L/k)$, defined as the identity on objects and by $Q(\psi_\gamma, F, \mathfrak{f}) = (\mathrm{id}_Y, \psi_\gamma, F, \mathfrak{f})$ in morphisms, satisfies the following

$$\begin{aligned} Q(\psi_\gamma, F, \mathfrak{f}) \circ Q(\psi'_{\gamma'}, F', \mathfrak{f}') &= (\mathrm{id}_Y, \psi_\gamma, F, \mathfrak{f}) \circ (\mathrm{id}_{Y'}, \psi'_{\gamma'}, F', \mathfrak{f}') \\ &= (\mathrm{id}_Y, \psi'_{\gamma'} \circ \psi_\gamma, F' \circ F, \psi_\gamma^*(\mathfrak{f}) \cdot F'_*(\mathfrak{f})) \\ &= Q(\psi'_{\gamma'} \circ \psi_\gamma, F' \circ F, \psi_\gamma^*(\mathfrak{f}) \cdot F'_*(\mathfrak{f})). \end{aligned}$$

Then, $Q : \mathfrak{PPDiv}(L/k) \to \mathfrak{PPDiv}_S(L/k)$ defines a functor.

Lemma 9.3.4. *Let k be a field of characteristic zero and L/k be a Galois extension. The functor $Q : \mathfrak{PPDiv}(L/k) \to \mathfrak{PPDiv}_S(L/k)$ is faithful.*

Proof. This is consequence of Lemma 9.3.1. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be objects in $\mathfrak{PPDiv}(L/k)$ and $(\psi_\gamma, F, \mathfrak{f})$ and $(\psi'_{\gamma'}, F', \mathfrak{f}')$ be two morphisms of pp-divisors such that
$$Q(\psi_\gamma, F, \mathfrak{f}) = Q(\psi'_{\gamma'}, F', \mathfrak{f}').$$
By Lemma 9.3.1, we have that this is equivalent to

$$X(\psi_\gamma, F, \mathfrak{f}) = X(\psi'_{\gamma'}, F', \mathfrak{f}'),$$

which implies that $(\psi_\gamma, F, \mathfrak{f}) = (\psi'_{\gamma'}, F', \mathfrak{f}')$ by Proposition 8.1.1. □

According to [KS06, Definition 7.1.1], given a collection of morphisms S in a category $\mathcal{C}$, a localization $\mathcal{C}_S$ is a category satisfying a certain universal property that "makes elements in S invertible". Under suitable conditions on S, one can ensure the existence of the category $\mathcal{C}_S$. However, the family of morphisms that we need to make invertible does not form a *right multiplicative system*, in the sense of [KS06, Definition 7.1.5]. The axioms $S3$ and $S4$ in [KS06, Definition 7.1.5] both fail.

Nevertheless, we prove here below that, if S is the collection of morphisms in $\mathfrak{PPDiv}(L/k)$ of the form $(\kappa, \mathrm{id}, \imath)$, then $\mathfrak{PPDiv}_S(L/k)$ is a localization with respect to S. We denote $s_\kappa := (\kappa, \mathrm{id}, \imath)$.

Proposition 9.3.5. *The category $\mathfrak{PPDiv}_S(L/k)$ is a localization of the category $\mathfrak{PPDiv}(L/k)$ with respect to S.*

Proof. In order to prove this, we verify the universal property described in [KS06, Definition 7.1.1]. Let $\mathcal{C}$ be a category and $G : \mathfrak{PPDiv}(L/k) \to \mathcal{C}$ be a functor such that $G(s)$ is invertible in $\mathrm{Mor}_{\mathcal{C}}$ for every $s \in S$. Let us define the functor $G_S : \mathfrak{PPDiv}_S(L/k) \to \mathcal{C}$ given by $G_S(\mathfrak{D}) := \mathfrak{D}$ and

$$G_S(\kappa, \psi_\gamma, F, \mathfrak{f}) := G(\psi_\gamma, F, \mathfrak{f}) \cdot G(\kappa, \mathrm{id}, \mathfrak{1})^{-1}.$$

This assignation does not depend on the representatives. Indeed, if $(\kappa, \psi_\gamma, F, \mathfrak{f}) \sim (\kappa', \psi'_\gamma, F', \mathfrak{f}')$ we have the following commutative diagram

$$\begin{tikzcd}
 & & \kappa^*\mathfrak{D} \arrow[dl, "{(\kappa,\mathrm{id}_N,\mathfrak{1})}"'] \arrow[dr, "{(\psi_\gamma,F,\mathfrak{f})}"] & \\
\hat{\kappa}^*\kappa^*\mathfrak{D} = \hat{\kappa}'^*\kappa'^*\mathfrak{D} \arrow[urr, bend left, "{(\hat{\kappa},\mathrm{id}_N,\mathfrak{1})}"] \arrow[drr, bend right, "{(\hat{\kappa}',\mathrm{id}_N,\mathfrak{1})}"'] & \mathfrak{D} & & \mathfrak{D}' \\
 & & \kappa'^*\mathfrak{D} \arrow[ul, "{(\kappa',\mathrm{id}_{N'},\mathfrak{1})}"] \arrow[ur, "{(\psi'_\gamma,F',\mathfrak{f}')}"'] & .
\end{tikzcd}$$

Under the functor G, this diagram yields the following

$$G(\psi_\gamma, F, \mathfrak{f}) \cdot G(s_{\hat{\kappa}}) \cdot G(s_{\hat{\kappa}})^{-1} \cdot G(s_\kappa)^{-1} = G(\psi'_\gamma, F', \mathfrak{f}') \cdot G(s_{\hat{\kappa}'}) \cdot G(s_{\hat{\kappa}'})^{-1} \cdot G(s'_\kappa)^{-1},$$

and then

$$\begin{aligned}
G_S(\kappa, \psi_\gamma, F, \mathfrak{f}) &= G(\psi_\gamma, F, \mathfrak{f}) \cdot G(s_\kappa)^{-1} \\
&= G(\psi'_\gamma, F', \mathfrak{f}') \cdot G(s_{\kappa'})^{-1} \\
&= G_S(\kappa', \psi'_\gamma, F', \mathfrak{f}').
\end{aligned}$$

Let us denote by $Q : \mathfrak{PPDiv}(L/k) \to \mathfrak{PPDiv}_S(L/k)$, given by $Q(\mathfrak{D}) := \mathfrak{D}$ and $Q(\psi_\gamma, F, \mathfrak{f}) := (\mathrm{id}, \psi_\gamma, F, \mathfrak{f})$, the localization functor. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be objects of $\mathfrak{PPDiv}(L/k)$ and $(\psi_\gamma, F, \mathfrak{f}) : \mathfrak{D} \to \mathfrak{D}'$ be a morphism in $\mathfrak{PPDiv}(L/k)$. Notice that

$$G_S \circ Q(\psi_\gamma, F, \mathfrak{f}) = G_S(Q(\psi_\gamma, F, \mathfrak{f})) = G_S(\mathrm{id}, \psi_\gamma, F, \mathfrak{f}) = G(\psi_\gamma, F, \mathfrak{f}) \cdot G(s_{\mathrm{id}})^{-1}.$$

Given that $G(s_{\mathrm{id}}) = \mathrm{id}$, the following diagram commutes

$$\begin{tikzcd}
G(\mathfrak{D}) \arrow[r, "{G(\psi_\gamma,F,\mathfrak{f})}"] & G(\mathfrak{D}') \\
G_S \circ Q(\mathfrak{D}) \arrow[u, "\mathrm{id}"] \arrow[r, "{G_S \circ Q(\psi_\gamma,F,\mathfrak{f})}"'] & G_S \circ Q(\mathfrak{D}'). \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Thus, this data gives rise to a natural transformation $\eta : G \Rightarrow G_S \circ Q$ that is an isomorphism in the category of functors.

Let G_1 and G_2 be elements of $\mathbf{Fct}(\mathfrak{PPDiv}_S(L/k), \mathcal{C})$, the category of functors between $\mathfrak{PPDiv}_S(L/k)$ and $\mathcal{C}$. There is a natural map

$$\varrho : \mathrm{Hom}_{\mathbf{Fct}(\mathfrak{PPDiv}_S(L/k),\mathcal{C})}(G_1, G_2) \to \mathrm{Hom}_{\mathbf{Fct}(\mathfrak{PPDiv}(L/k),\mathcal{C})}(G_1 \circ Q, G_2 \circ Q).$$

This map is injective since Q is the identity on objects. Let us prove that it is also surjective. Let $\eta : G_1 \circ Q \Rightarrow G_2 \circ Q$ be a natural transformation. Given that $\mathfrak{PPDiv}(L/k)$ and $\mathfrak{PPDiv}_S(L/k)$ have the same objects, η defines maps $\eta_{S,\mathfrak{D}} := \eta_{\mathfrak{D}} : G_1(\mathfrak{D}) \to G_2(\mathfrak{D})$, for each pp-divisor $\mathfrak{D}$ in $\mathfrak{PPDiv}_S(L/k)$. In order to prove that η_S defines a natural transformation between $\mathfrak{PPDiv}_S(L/k)$ and $\mathcal{C}$, it suffices to prove that the following diagram commutes

$$\begin{tikzcd} G_1(\mathfrak{D}) \arrow[r, "G_1(\kappa,\psi_\gamma,F,\mathfrak{f})"] \arrow[d, "\eta_{S,\mathfrak{D}}"'] & G_1(\mathfrak{D}') \arrow[d, "\eta_{S,\mathfrak{D}'}"] \\ G_2(\mathfrak{D}) \arrow[r, "G_2(\kappa,\psi_\gamma,F,\mathfrak{f})"'] & G_2(\mathfrak{D}'), \end{tikzcd}$$

for every pair of pp-divisors $\mathfrak{D}$ and $\mathfrak{D}'$ in $\mathfrak{PPDiv}_S(L/k)$. In $\mathfrak{PPDiv}_S(L/k)$, we have that

$$\begin{tikzcd} & \kappa^*\mathfrak{D} \arrow[dl, "Q(\kappa,\mathrm{id},\imath)"'] \arrow[dr, "Q(\psi_\gamma,F,\mathfrak{f})"] & \\ \mathfrak{D} \arrow[rr, "(\kappa,\psi_\gamma,F,\mathfrak{f})"'] & & \mathfrak{D}'. \end{tikzcd}$$

Otherwise stated, $(\kappa, \psi_\gamma, F, \mathfrak{f}) = Q(\psi_\gamma, F, \mathfrak{f}) \cdot Q(\kappa, \mathrm{id}, \imath)^{-1}$. Then,

$$\begin{aligned} G_i(\kappa, \psi_\gamma, F, \mathfrak{f}) &= G_i \left(Q(\psi_\gamma, F, \mathfrak{f}) \cdot Q(\kappa, \mathrm{id}, \imath)^{-1} \right) \\ &= G_i \left(Q(\psi_\gamma, F, \mathfrak{f}) \right) \cdot G_i \left(Q(\kappa, \mathrm{id}, \imath)^{-1} \right) \\ &= \left((G_i \circ Q)(\psi_\gamma, F, \mathfrak{f}) \right) \cdot \left((G_i \circ Q)(\kappa, \mathrm{id}, \imath) \right)^{-1}, \end{aligned}$$

for every $i \in \{1, 2\}$. Besides, we have following diagram, where the two rectangles in the back are commutative:

$$\begin{tikzcd}
 & (G_1 \circ Q)(\kappa^*\mathfrak{D}) \arrow[dl, "(G_1\circ Q)(\kappa,\mathrm{id},\imath)"'] \arrow[dr, "(G_1\circ Q)(\psi_\gamma,F,\mathfrak{f})"] \arrow[dd, "\eta_{\kappa^*\mathfrak{D}}"'] & \\
(G_1 \circ Q)(\mathfrak{D}) \arrow[rr, "G_1(\kappa,\psi_\gamma,F,\mathfrak{f})"'] \arrow[dd, "\eta_{\mathfrak{D}}"'] & & (G_1 \circ Q)(\mathfrak{D}') \arrow[dd, "\eta_{\mathfrak{D}'}"] \\
 & (G_2 \circ Q)(\kappa^*\mathfrak{D}) \arrow[dl, "(G_2\circ Q)(\kappa,\mathrm{id},\imath)"'] \arrow[dr, "(G_2\circ Q)(\psi_\gamma,F,\mathfrak{f})"] & \\
(G_2 \circ Q)(\mathfrak{D}) \arrow[rr, "G_2(\kappa,\psi_\gamma,F,\mathfrak{f})"'] & & (G_2 \circ Q)(\mathfrak{D}').
\end{tikzcd}$$

And since $(G_i \circ Q)(\kappa, \mathrm{id}, \imath)$ is invertible in $\mathcal{C}$, we have that the rectangle in the front is commutative. Thus

$$\eta_{S,\mathfrak{D}'} \cdot G_1(\kappa, \psi_\gamma, F, \mathfrak{f}) = G_2(\kappa, \psi_\gamma, F, \mathfrak{f}) \cdot \eta_{S,\mathfrak{D}}.$$

Thus, η_S is natural transformation and $\varrho(\eta_S) = \eta$. Hencefore, $\mathfrak{PPDiv}_S(L/k)$ is a localization of $\mathfrak{PPDiv}(L/k)$ with respect to S. $\square$

Equivalence of categories By the universal property of the localization, the functor $X : \mathfrak{PPDiv}(L/k) \to \mathcal{E}(L/k)$ fits into the following commutative diagram

$$\begin{tikzcd}
\mathfrak{PPDiv}_S(L/k) \arrow[r, "\mathcal{X}"] & \mathcal{E}(L/k) \\
\mathfrak{PPDiv}(L/k). \arrow[u, "Q"] \arrow[ur, "X"'] &
\end{tikzcd}$$

Proposition 9.3.6. *Let k be a field of characteristic zero and L/k be a Galois extension. The functor $\mathcal{X} : \mathfrak{PPDiv}_S(L/k) \to \mathcal{E}(L/k)$ is an equivalence of categories.*

Proof. The functor $\mathcal{X}$ is essentially surjective, since X is essentially surjective and the diagram above is commutative. The fullness is a consequence of Theorem 8.4.4. The faithfulness follows from Lemma 9.3.1 and the fact that

$$\mathcal{X}(\kappa, \psi_\gamma, F, \mathfrak{f}) = X(\psi_\gamma, F, \mathfrak{f}) \circ X(\kappa, \mathrm{id}, \imath)^{-1} = X(\psi_\gamma, F, \mathfrak{f}).$$

$\square$

Remark 9.3.7. Notice that Proposition 9.3.6 generalizes [AH06, Corollary 8.14], whose proof is omitted in *loc. cit.* Indeed, it suffices to consider the semilinear morphisms with γ the neutral element of the Galois group.

9.4 Affine case and general pp-divisors

Let k be a field of characteristic zero and L/k be a finite Galois extension with Galois group Γ. Let T be an algebraic torus over k that splits over L. Let X be a geometrically integral normal affine variety over L. By Proposition 9.1.9, X can be written as a pair (X_L, g'), where X_L is the base change and g' is a Γ-semilinear equivariant action. Moreover, by Theorem 9.2.3, X is encoded by a pair $(\mathfrak{D}_L, g)$, where $\mathfrak{D}_L$ is a minimal pp-divisor over L and g is a Γ-semilinear action over $\mathfrak{D}_L$. However, such a pair might not exist for a non-minimal pp-divisor $\mathfrak{D}'$ such that $X(\mathfrak{D}) \cong X(\mathfrak{D}')$ as $T_{\mathfrak{D}}$-varieties over L, since we are not always able to construct a Γ-semilinear action over $\mathfrak{D}'$. In such a case, we need to consider a wider family of morphisms. By Theorem 8.4.4, for every dominant semilinear equivariant morphism $(\varphi_\gamma, f_\gamma) : X(\mathfrak{D}) \to X(\mathfrak{D}')$, there exists a normal semiprojective variety $\tilde{Y}$ over L, a projective birational morphism $\kappa : \tilde{Y} \to Y$ of varieties over L and a semilinear morphism of pp-divisors $(\psi_\gamma, F, \mathfrak{f}) : \kappa^*\mathfrak{D} \to \mathfrak{D}'$ such that following diagram commutes

$$\begin{tikzcd}
 & X(\kappa^*\mathfrak{D}) \arrow[dl, "X(\kappa,\mathrm{id}_N,1)"', "\cong"] \arrow[dr, "X(\psi_\gamma,F,\mathfrak{f})"] & \\
X(\mathfrak{D}) \arrow[rr, "(\varphi_\gamma,f_\gamma)"'] & & X(\mathfrak{D}').
\end{tikzcd}$$

As we explained in Section 9.3, we may consider then the pair

$$\begin{tikzcd}
 & \kappa^*\mathfrak{D} \arrow[dl, "(\kappa,\mathrm{id}_N,1)"'] \arrow[dr, "(\psi_\gamma,F,\mathfrak{f})"] & \\
\mathfrak{D} & & \mathfrak{D}'
\end{tikzcd}$$

as a morphism in the localization $\mathfrak{PPDiv}_S(L/k)$.

For every pp-divisor $\mathfrak{D}$ in $\mathfrak{PPDiv}_S(L/k)$, we denote by $\mathrm{SAut}_{\mathrm{loc}}(\mathfrak{D})$ its group of *localized* semilinear automorphisms, i.e. its automorphisms in the category $\mathfrak{PPDiv}_S(L/k)$. The normal subgroup of id_Γ-linear automorphisms is denoted by $\mathrm{Aut}_{\mathrm{loc}}(\mathfrak{D})$.

Definition 9.4.1. Let k be a field and L/k be a Galois extension with Galois group Γ. Let $\mathfrak{D}$ be an object in the localization $\mathfrak{PPDiv}_S(L/k)$. Let G be an abstract group. A *localized semilinear action of G over $\mathfrak{D}$*, or a *G-localized semilinear action*, is a group homomorphism $\varphi : G \to \mathrm{SAut}_{\mathrm{loc}}(\mathfrak{D})$. A *Galois localized semilinear action* is a Γ-localized semilinear action when $G = \Gamma$ and φ is a section of the sequence

$$1 \to \mathrm{Aut}_{\mathrm{loc}}(\mathfrak{D}) \to \mathrm{SAut}_{\mathrm{loc}}(\mathfrak{D}) \to \Gamma.$$

Let G be an abstract group. A G-localized semilinear action

$$\varphi : G \to \mathrm{SAut}_{\mathrm{loc}}(\mathfrak{D})$$

induces a G-semilinear equivariant action (recall Definition 8.2.2)

$$\mathcal{X}(\varphi) : G \to \mathrm{SAut}(T; X(\mathfrak{D}))$$

via the functor $\mathcal{X} : \mathfrak{PPDiv}_S(L/k) \to \mathcal{E}(L/k)$. Given that $\mathcal{X}$ is an equivalence of categories by Proposition 9.3.6, every G-semilinear equivariant action $\rho : G \to \mathrm{SAut}(T; X(\mathfrak{D}))$ arises from a G-localized semilinear action of pp-divisors. Actually, this defines a bijection between the set of localized semilinear actions over $\mathfrak{D}$ and the set of semilinear equivariant actions over $X(\mathfrak{D})$. In other words, we have the following immediate result.

Proposition 9.4.2. *Let k be a field of characteristic zero and L/k be a Galois extension. Let $\mathfrak{D}$ be an object in $\mathfrak{PPDiv}_S(L/k)$. Then, there exists a bijection between the set of localized semilinear actions over $\mathfrak{D}$ and the set of semilinear equivariant actions over $X(\mathfrak{D})$.*

Let k be a field of characteristic zero and L/k be a Galois extension with Galois group Γ. Let $\mathfrak{PPDiv}_S(\Gamma)$ be the category of pairs $(\mathfrak{D}, g)$, where $\mathfrak{D}$ is a pp-divisor over L and $g : \Gamma \to \mathrm{SAut}_{\mathrm{loc}}(\mathfrak{D})$ is a Galois localized semilinear action. A morphism in this category is a morphism $(\psi, F, \mathfrak{f}) : \mathfrak{D} \to \mathfrak{D}'$ in $\mathfrak{PPDiv}_S(L/k)$ such that

$$g'_\gamma \circ (\psi, F, \mathfrak{f}) = (\psi, F, \mathfrak{f}) \circ g_\gamma$$

for every $\gamma \in \Gamma$. Let $(\mathfrak{D}, g)$ be an object in $\mathfrak{PPDiv}_S(\Gamma)$. By Theorem 7.2.1, $\mathcal{X}(\mathfrak{D})$ is a geometrically integral normal $T_{\mathfrak{D}}$-variety over L, where $T_{\mathfrak{D}}$ denotes its respective torus. Moreover, by Proposition 9.4.2, $\mathcal{X}(\mathfrak{D})$ comes with a Galois semilinear equivariant action $\mathcal{X}(g) : G \to \mathrm{SAut}(T_{\mathfrak{D}}; X(\mathfrak{D}))$. Then, by Proposition 9.1.9, there exists a geometrically integral normal T-variety $X := \mathcal{X}(\mathfrak{D}, g)$ over k such that $X_L \cong \mathcal{X}(\mathfrak{D})$ as $T_{\mathfrak{D}}$-varieties over L. This proves the first part of the following theorem.

Theorem 9.4.3. *Let k be a field of characteristic zero, L/k be a finite Galois extension with Galois group Γ.*

a) Let $(\mathfrak{D}_L, g)$ be an object in $\mathfrak{PPDiv}_S(\Gamma)$. Then, $\mathcal{X}(\mathfrak{D}_L, g)$ is a geometrically integral normal affine variety endowed with an effective action of an algebraic torus T over k such that T splits over L and $\mathcal{X}(\mathfrak{D}_L, g)_L \cong X(\mathfrak{D}_L)$ as $T_{\mathfrak{D}_L}$-varieties over L.

b) *Let X be a geometrically integral normal affine variety over k endowed with an effective T-action such that T_L is split. Let $\mathfrak{D}_L$ be a pp-divisor such that $X_L \cong X(\mathfrak{D}_L)$. Then, there exists a Γ-localized semilinear action $g : \Gamma \to \mathrm{SAut}(\mathfrak{D}_L)$ such that $X \cong \mathcal{X}(\mathfrak{D}_L, g)$ as T-varieties.*

Proof. Let us prove part (b), the remaining part of the theorem. Let X be a geometrically integral normal variety over k endowed with an effective T-action and $\mathfrak{D}_L$ be a pp-divisor such that $X_L \cong X(\mathfrak{D}_L) = \mathcal{X}(\mathfrak{D}_L)$ as T_L-varieties over L. By Proposition 9.1.9, as a T-variety over k, X is equivalent to a pair (X_L, g'), where X_L is a geometrically integral normal T_L-variety, with T_L split over L, and a Γ-semilinear equivariant action g'. Now, by Proposition 9.4.2, we have that the Γ-semilinear equivariant action on $X(\mathfrak{D}_L)$ induces a unique Γ-localized semilinear action g on $\mathfrak{D}_L$. Then, the pair $(\mathfrak{D}_L, g)$ encodes the pair (X_L, g'). □

By Theorem 9.2.3, every pair $(\mathfrak{D}, g)$ in $\mathfrak{PPDiv}_S(\Gamma)$ corresponds to a geometrically integral normal affine variety $\mathcal{X}(\mathfrak{D}, g)$ endowed with a torus action over k that is split over L. This construction induces a functor

$$\begin{aligned} \mathcal{X} : \mathfrak{PPDiv}_S(\Gamma) &\to \mathcal{E}(k, L); \\ (\mathfrak{D}, g) &\mapsto X(\mathfrak{D}, g), \end{aligned}$$

where $\mathcal{E}(k, L)$ is the category of affine normal varieties over k endowed with an effective action of an algebraic torus over k that is split over L. This functor is the composition of the functor $(\mathfrak{D}, g) \mapsto (\mathcal{X}(\mathfrak{D}), \mathcal{X}(g))$, from the category $\mathfrak{PPDiv}_S(\Gamma)$ to the category of geometrically integral normal affine varieties endowed with an effective action of a split algebraic torus over L and a Γ-semilinear equivariant action, and the equivalence of categories of Proposition 9.1.9. Given that the first functor is faithful, covariant and essentially surjective, we have the following.

Proposition 9.4.4. *Let k be a field of characteristic zero and let L/k be a finite Galois extension with Galois group Γ. Then the functor $\mathcal{X}$: $\mathfrak{PPDiv}_S(\Gamma) \to \mathcal{E}(k, L)$ is an equivalence of categories. In particular the functor $X : \mathfrak{PPDiv}(\Gamma) \to \mathcal{E}(k, L)$ is faithful, covariant and essentially surjective.*

9.5 Semilinear morphisms of divisorial fans

Let k be a field of characteristic zero. In this section we present the definition of semilinear morphisms of divisorial fans.

Definition 9.5.1. Let k be a field of characteristic zero. Let $\mathfrak{D}$ be an object in $\mathfrak{PPDiv}(k)$ and $\mathfrak{f} = \sum v_i \otimes f_i$ a plurifunction in $k(N,Y)^*$. We define *the restriction of* $\mathfrak{f}$ *to* $\mathfrak{D}$ as

$$\mathfrak{f}|_{\mathfrak{D}} := \sum_{\mathrm{div}(f_i) \subset \mathrm{Supp}(\mathfrak{D})} v_i \otimes f_i.$$

Definition 9.5.2. Let k be a field of characteristic zero, L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let $(\mathfrak{S}, Y)$ and $(\mathfrak{S}', Y')$ be two divisorial fans over L. A *semilinear morphism of divisorial fans with respect to* γ is a triple $(\psi_\gamma, F, \mathfrak{f})$ such that

i) for every $\mathfrak{D} \in \mathfrak{S}$ there exists $\mathfrak{D}' \in \mathfrak{S}'$ such that the triple $(\psi_\gamma, F, \mathfrak{f}|_{\mathfrak{D}}) : \mathfrak{D} \to \mathfrak{D}'$ is a semilinear morphism of pp-divisors and

ii) for every $\mathfrak{D}, \mathfrak{D}' \in \mathfrak{S}$ as in (**??**), if $\mathfrak{E} \in \mathfrak{S}$ is a face of $\mathfrak{D}$, then there exists $\mathfrak{E}' \in \mathfrak{S}$ such that $(\psi_\gamma, F, \mathfrak{f}|_{\mathfrak{E}}) : \mathfrak{E} \to \mathfrak{E}'$ is a semilinear morphism of pp-divisors and $\mathfrak{E}'$ is a face of $\mathfrak{D}'$.

We cannot ask for uniqueness of $\mathfrak{D}'$ in Definition 9.5.2 part (**??**), because if $\mathfrak{D}'$ is a proper face of some other pp-divisor in $\mathfrak{S}'$, then we have a map from $\mathfrak{D}$ to that pp-divisor under the composition. However, this is the only obstruction to the uniqueness. In the following we denote $g_\gamma := (\psi_\gamma, F, \mathfrak{f})$ and $g_{\mathfrak{D},\gamma} := (\psi_\gamma, F, \mathfrak{f}|_{\mathfrak{D}})$ for simplicity.

From a semilinear morphism of divisorial fans $g_\gamma : (\mathfrak{S}, Y) \to (\mathfrak{S}', Y')$, we get equivariant semilinear morphisms $X(g_{\mathfrak{D},\gamma}) : X(\mathfrak{D}) \to X(\mathfrak{D}')$ for each $\mathfrak{D} \in \mathfrak{S}$. From Definition 9.5.2 part (**??**), we have the following commutative diagram of equivariant semilinear morphisms of pp-divisors

$$\begin{tikzcd}
X(\mathfrak{D}) \arrow[r, "X(g_{\mathfrak{D},\gamma})"] & X(\mathfrak{D}') \\
X(\mathfrak{D} \cap \mathfrak{E}) \arrow[u, "X(\mathrm{id}_Y, \mathrm{id}_N, 1)"] \arrow[r, "X(g_{\mathfrak{D}\cap\mathfrak{E},\gamma})"'] \arrow[d, "X(\mathrm{id}_Y, \mathrm{id}_N, 1)"'] & X(\mathfrak{D}' \cap \mathfrak{E}') \arrow[u, "X(\mathrm{id}_{Y'}, \mathrm{id}_{N'}, 1)"'] \arrow[d, "X(\mathrm{id}_{Y'}, \mathrm{id}_{N'}, 1)"] \\
X(\mathfrak{E}) \arrow[r, "X(g_{\mathfrak{E},\gamma})"'] & X(\mathfrak{E}').
\end{tikzcd}$$

Then, $X(g_{\mathfrak{D},\gamma})|_{X(\mathfrak{D}\cap\mathfrak{E})} = X(g_{\mathfrak{E},\gamma})|_{X(\mathfrak{D}\cap\mathfrak{E})}$. This implies that all the semilinear morphisms $X(g_{\mathfrak{D},\gamma})$ fit into a semilinear equivariant morphism

$$X(g_\gamma) : X(\mathfrak{S}) \to X(\mathfrak{S}').$$

Unfortunately, it is not always possible to associate a semilinear morphism of divisorial fans to every dominant equivariant semilinear morphism $(\varphi_\gamma, f_\gamma) : X(\mathfrak{S}) \to X(\mathfrak{S}')$. For example, when $\mathfrak{S} := \{\mathfrak{D}\}$ and $\mathfrak{S}' := \{\mathfrak{D}'\}$ and none of the pp-divisors is minimal. In this manner, we need to consider a wider family of morphisms of divisorial fans. Recall that a localized semilinear morphism of pp-divisors $\mathfrak{D} \in \mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega_{\mathfrak{D}})$ and $\mathfrak{D}' \in \mathrm{PPDiv}_{\mathbb{Q}}(Y', \omega_{\mathfrak{D}'})$ is represented by a pair of semilinear morphisms of pp-divisors

$$\begin{tikzcd} & \kappa^*\mathfrak{D} \arrow[dl, "{(\kappa,\mathrm{id}_N,\mathfrak{1})}"'] \arrow[dr, "{(\psi_\gamma,F,\mathfrak{f})}"] & \\ \mathfrak{D} & & \mathfrak{D}', \end{tikzcd}$$

where $\kappa : \tilde{Y} \to Y$ is a projective morphism of L-varieties, $\psi_\gamma : \tilde{Y} \to Y'$ is a projective semilinear morphism, $\tilde{Y}$ is a normal semiprojective variety over L and the morphism $(\psi_\gamma, F, \mathfrak{f})$ is dominating. In order to simplify the notation we will denote by $(\kappa, \psi_\gamma, F, \mathfrak{f}) : \mathfrak{D} \to \mathfrak{D}'$ the localized semilinear morphism of pp-divisors.

Definition 9.5.3. Let k be a field of characteristic zero, L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let $(\mathfrak{S}, Y)$ and $(\mathfrak{S}', Y')$ be two divisorial fans over L. A *localized semilinear morphism of divisorial fans with respect to* γ is a family of localized semilinear morphisms of pp-divisors $\mathcal{M}_\gamma$ such that

i) for every $\mathfrak{D} \in \mathfrak{S}$ and $\mathfrak{D}' \in \mathfrak{S}'$ there exists at most one localized semilinear morphism of pp-divisors $(\kappa, \psi_\gamma, F, \mathfrak{f}) : \mathfrak{D} \to \mathfrak{D}'$ in $\mathcal{M}_\gamma$, denoted by $\mathcal{M}_{\mathfrak{D},\gamma}$, and

ii) let $\mathfrak{E}, \mathfrak{D} \in \mathfrak{S}$ and $\mathfrak{E}', \mathfrak{D}' \in \mathfrak{S}'$ such that $\mathfrak{E} \preceq \mathfrak{D}$ and $\mathfrak{E}' \preceq \mathfrak{D}'$. If there exists morphisms $\mathcal{M}_{\mathfrak{D},\gamma} : \mathfrak{D} \to \mathfrak{D}'$ and $\mathcal{M}_{\mathfrak{E},\gamma} : \mathfrak{E} \to \mathfrak{E}'$ in $\mathcal{M}_\gamma$, then the following diagram commutes

$$\begin{tikzcd} \mathfrak{D} \arrow[r, "\mathcal{M}_{\mathfrak{D},\gamma}"] & \mathfrak{D}' \\ \mathfrak{E} \arrow[u, "{(\mathrm{id}_Y,\mathrm{id}_N,\mathfrak{1})}"] \arrow[r, "\mathcal{M}_{\mathfrak{E},\gamma}"'] & \mathfrak{E}'. \arrow[u, "{(\mathrm{id}_{Y'},\mathrm{id}_{N'},\mathfrak{1}')}"'] \end{tikzcd}$$

With every localized semilinear morphism of divisorial fans $\mathcal{M}_\gamma : (\mathfrak{S}, Y) \to (\mathfrak{S}, Y')$ we can associate a semilinear equivariant morphism $\mathcal{X}(\mathcal{M}_\gamma) : \mathcal{X}(\mathfrak{S}) \to$

$\mathcal{X}(\mathfrak{S}')$, constructed in the same way as for semilinear morphisms of divisorial fans, by Lemma 9.3.1. Notice that, $\mathcal{X}(\mathfrak{S}) = X(\mathfrak{S})$, since both functors coincides on objects.

Theorem 9.5.4. *Let k be a field of characteristic zero, L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let $(\mathfrak{S}, Y)$ and $(\mathfrak{S}', Y')$ be two divisorial fans over L. Let $(\varphi_\gamma, f_\gamma) : \mathcal{X}(\mathfrak{S}) \to \mathcal{X}(\mathfrak{S}')$ be a dominant semilinear equivariant morphism. If for every $\mathfrak{D} \in \mathfrak{S}$ there exists $\mathfrak{D}' \in \mathfrak{S}'$ such that $f_\gamma(X(\mathfrak{D})) \subset X(\mathfrak{D}')$, then there exists a localized semilinear morphism of divisorial fans $\mathcal{M}_\gamma : (\mathfrak{S}, Y) \to (\mathfrak{S}', Y')$ such that $\mathcal{X}(\mathcal{M}_\gamma) = (\varphi_\gamma, f_\gamma)$.*

Proof. First notice that, for every $\mathfrak{D} \in \mathfrak{S}$ and $\mathfrak{D}' \in \mathfrak{S}'$ such that $f_\gamma(\mathcal{X}(\mathfrak{D})) \subset \mathcal{X}(\mathfrak{D}')$, we have that the induced semilinear equivariant morphism

$$(\varphi_\gamma, f_\gamma)|_{\mathcal{X}(\mathfrak{D})} : \mathcal{X}(\mathfrak{D}) \to \mathcal{X}(\mathfrak{D}')$$

is dominant. Let us prove this last assertion. Let $U \subset \mathcal{X}(\mathfrak{D}')$ be an open subvariety, then $U \subset \mathcal{X}(\mathfrak{S}')$ is an open subvariety. Given that $(\varphi_\gamma, f_\gamma)$ is dominant, $f_\gamma^{-1}(U)$ is a nonempty subvariety of $\mathcal{X}(\mathfrak{S})$. This implies that $\mathcal{X}(\mathfrak{D}) \cap f_\gamma^{-1}(U)$ is not empty and, therefore, $f_\gamma(\mathcal{X}(\mathfrak{D})) \cap U$ is not empty. Then, the restrictions $(\varphi_\gamma, f_\gamma)|_{\mathcal{X}(\mathfrak{D})} : \mathcal{X}(\mathfrak{D}) \to \mathcal{X}(\mathfrak{D}')$ are dominant for every $\mathfrak{D} \in \mathfrak{S}$.

Given that, for every $\mathfrak{D} \in \mathfrak{S}$, the restriction map $(\varphi_\gamma, f_\gamma)|_{\mathcal{X}(\mathfrak{D})} : \mathcal{X}(\mathfrak{D}) \to \mathcal{X}(\mathfrak{D}')$ is a dominant semilinear equivariant morphism, there exists a localized semilinear morphism of pp-divisors $\mathcal{M}_{\mathfrak{D},\gamma} : \mathfrak{D} \to \mathfrak{D}'$ such that $\mathcal{X}(\mathcal{M}_{\mathfrak{D},\gamma}) = (\varphi_\gamma, f_\gamma)|_{X(\mathfrak{D})}$ by Proposition 9.3.6. Denote $\mathcal{M}_\gamma := \{\mathcal{M}_{\mathfrak{D},\gamma}\}$. By construction, $\mathcal{M}_\gamma$ satisfies part (i) of Definition 9.5.3. In order to prove part (ii) of Definition 9.5.3, it suffices to do it on the respective varieties by Proposition 9.3.6 and this is obvious from the commutative diagram

$$\begin{tikzcd}
\mathcal{X}(\mathfrak{D}) \arrow[r, "\mathcal{X}(\mathcal{M}_{\mathfrak{D},\gamma})"] & \mathcal{X}(\mathfrak{D}') \\
\mathcal{X}(\mathfrak{E}) \arrow[u, "\mathcal{X}(\mathrm{id}_Y, \mathrm{id}_N, 1)"] \arrow[r, "\mathcal{X}(\mathcal{M}_{\mathfrak{E},\gamma})"'] & \mathcal{X}(\mathfrak{E}'). \arrow[u, "\mathcal{X}(\mathrm{id}_{Y'}, \mathrm{id}_{N'}, 1')"']
\end{tikzcd}$$

Finally, we that that $\mathcal{X}(\mathcal{M}_\gamma) = (\varphi_\gamma, f_\gamma)$, since $\mathcal{X}(\mathcal{M}_\gamma)$ is the gluing on the restriction maps $(\varphi_\gamma, f_\gamma)|_{\mathcal{X}(\mathfrak{D})}$. Then, the assertion holds. □

A semilinear equivariant morphism between two normal varieties endowed with an effective torus action does not necessarily satisfy the hypothesis of Theorem 9.5.4. However, given that such varieties have several divisorial fans, we can always find a pair of divisorial fans satisfying the condition of the theorem.

The following is a tool that allows us to construct suitable divisorial fans.

Lemma 9.5.5. *Let k be a field of characteristic zero, L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let T and T' be split algebraic tori over L. Let X be a normal T-variety and X' be a normal T'-variety, both over L. If $(\varphi_\gamma, f_\gamma) : X \to X'$ is a semilinear equivariant morphism, then there exist divisorial fans $(\mathfrak{S}, Y)$ and $(\mathfrak{S}', Y')$ over L such that $X \cong \mathcal{X}(\mathfrak{S})$ and $X' \cong \mathcal{X}(\mathfrak{S}')$ satisfying that for every $\mathfrak{D} \in \mathfrak{S}$ there exists $\mathfrak{D}' \in \mathfrak{S}'$ satisfying $f_\gamma(\mathcal{X}(\mathfrak{D})) \subset \mathcal{X}(\mathfrak{D}')$.*

Proof. By Sumihiro's Theorem, X' has a T'-stable affine open covering $\mathcal{U}'$. Given that, for every $U \in \mathcal{U}'$, we have that $f_\gamma^{-1}(U)$ is a T-stable normal open subvariety, using Sumihiro's Theorem on each $f_\gamma^{-1}(U)$ we get a T-stable affine open covering $\mathcal{U}$ of X such that for every $U \in \mathcal{U}$ there exists $U' \in \mathcal{U}'$ satisfying $f_\gamma(U) \subset U'$. Then, applying Corollary 7.3.13 to these affine open coverings, we get the divisorial fans satisfying the conditions of the lemma. □

Corollary 9.5.6. *Let k be a field of characteristic zero, L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let T and T' be split algebraic tori over L. Let X be a normal T-variety and X' be a normal T'-variety, both over L. Let $(\varphi_\gamma, f_\gamma) : X \to X'$ be a dominant semilinear equivariant morphism. Then, there exist divisorial fans $(\mathfrak{S}, Y)$ and $(\mathfrak{S}', Y')$ such that $X \cong \mathcal{X}(\mathfrak{S})$, $X' \cong \mathcal{X}(\mathfrak{S}')$ and a localized semilinear morphism of divisorial fans $\mathcal{M}_\gamma$ such that $\mathcal{X}(\mathcal{M}_\gamma) = (\varphi_\gamma, f_\gamma)$.*

Proof. By Lemma 9.5.5, there exist two divisorial fans $(\mathfrak{S}, Y)$ and $(\mathfrak{S}', Y')$ over L satisfying the hyphotesis of Theorem 9.5.4. Then, there exists a localized semilinear morphism of divisorial fans $\mathcal{M}_\gamma : (\mathfrak{S}, Y) \to (\mathfrak{S}', Y')$ such that $\mathcal{X}(\mathcal{M}_\gamma) = (\varphi_\gamma, f_\gamma)$. Then, the assertion holds. □

Toric case. For normal toric varieties, from a semilinear equivariant ismorphisms of toric varieties, we get a semilinear isomorphism of fans.

Corollary 9.5.7. *Let k be a field of characteristic zero, L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let X_Δ and $X_{\Delta'}$ be two normal split toric varieties over L and $(\varphi_\gamma, f_\gamma) : X_\Delta \to X_{\Delta'}$ be a semilinear equivariant isomorphism. Then, there exists a triple $(\psi_\gamma, F, \mathfrak{f})$, where $\psi_\gamma = \gamma^\natural : \mathrm{Spec}(L) \to \mathrm{Spec}(L)$, $F : N \to N'$ is an isomorphism of lattices* *such that* $F(\omega) = \omega'$ *for* $\omega \in \Delta$ *and* $\omega' \in \Delta'$ *satisfying* $f_\gamma(X_\omega) = X_{\omega'}$ *and* $\mathfrak{f} \in N \otimes L^*$ *such that* $(\varphi_\gamma, f_\gamma) = X(\psi_\gamma, F, \mathfrak{f})$.

Proof. This results follows from Theorem 9.5.4 applied to the coverings given by the fans. The last part comes from the fact that $\mathfrak{f}$ restricted to any cone (seen as a pp-divisor) is itself. □

9.6 The group of semilinear automorphisms of a divisorial fan

In the following, we will focus on the case where $(\mathfrak{S}, Y)$ and $(\mathfrak{S}', Y')$ are the same divisorial fan.

Definition 9.6.1. Let k be a field, L be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let $(\mathfrak{S}, Y)$ be a divisorial fan over L. A *semilinear automorphism with respect to* γ *of* $(\mathfrak{S}, Y)$ is a semilinear morphism of divisorial fans $g_\gamma : (\mathfrak{S}, Y) \to (\mathfrak{S}, Y)$ such that, for every $\mathfrak{D} \in \mathfrak{S}$, there exists a unique $g_\gamma(\mathfrak{D}) \in \mathfrak{S}$ such that $g_{\mathfrak{D},\gamma} : \mathfrak{D} \to g_\gamma(\mathfrak{D})$ is a semilinear isomorphism of pp-divisors. If γ is the neutral element of Γ, then we say that $g := g_\gamma$ is an *automorphism of divisorial fans.*

Let g_γ and h_β be semilinear automorphisms of $(\mathfrak{S}, Y)$. Then the composition $h_\beta \circ g_\gamma$ is a semilinear automorphism $r_{\beta\gamma}$ satisfying:

a) For every $\mathfrak{D} \in \mathfrak{S}$, $r_{\beta\gamma}(\mathfrak{D}) = h_\beta(g_\gamma(\mathfrak{D}))$ and

b) $r_{\mathfrak{D},\beta\gamma} = h_{g_\gamma(\mathfrak{D}),\beta} \circ g_{\mathfrak{D},\gamma}$.

Thus, the set of all semilinear automorphisms $\mathrm{SAut}(\mathfrak{S}, Y)$ comes with a semi-group structure. Even in the affine case, $\mathrm{SAut}(\mathfrak{D})$ does not necessarily have a group structure. This problem is solved by considering the localized category $\mathfrak{PPDiv}_S(L/k)$.

Definition 9.6.2. Let k be a field of characteristic zero, L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let $(\mathfrak{S}, Y)$ be a divisorial fans over L. A *localized semilinear automorphism of divisorial fans with respect to* γ is a localized semilinear morphism of divisorial fans $\mathcal{M}_\gamma : (\mathfrak{S}, Y) \to (\mathfrak{S}, Y)$ with respect to γ, such that each $\mathcal{M}_{\mathfrak{D},\gamma}$ is a localized semilinear isomorphism.

The set of localized semilinear automorphisms $\mathrm{SAut}_{\mathrm{loc}}(\mathfrak{S}, Y)$, has a group structure and the functor $\mathcal{X} : \mathfrak{PPDiv}_S(L/k)$ induces a map

$$\begin{aligned} \mathcal{X} : \mathrm{SAut}_{\mathrm{loc}}(\mathfrak{S}, Y) &\to \mathrm{SAut}(T_{\mathfrak{S}}; \mathcal{X}(\mathfrak{S})), \\ \mathcal{M}_\gamma &\mapsto \mathcal{X}(\mathcal{M}_\gamma), \end{aligned}$$

where $T_{\mathfrak{S}}$ denotes the split algebraic torus over L associated with $(\mathfrak{S}, Y)$.

Proposition 9.6.3. *Let k be a field of characteristic zero and L/k be a Galois extension. Let $(\mathfrak{S}, Y)$ be a divisorial fan over L. The map $\mathrm{SAut}_{\mathrm{loc}}(\mathfrak{S}, Y) \to \mathrm{SAut}(T_{\mathfrak{S}}; \mathcal{X}(\mathfrak{S}))$ is a monomorphism of abstract groups.*

Proof. For each $\mathcal{M}_\gamma \in \mathrm{SAut}_{\mathrm{loc}}(\mathfrak{S}, Y)$ there exists an equivariant semilinear automorphism $\mathcal{X}(\mathcal{M}_\gamma)$. Thus, we need to prove that

$$\mathcal{X}(\mathcal{N}_\beta \circ \mathcal{M}_\gamma) = \mathcal{X}(\mathcal{N}_\beta) \circ \mathcal{X}(\mathcal{M}_\gamma)$$

for all $\mathcal{N}_\beta, \mathcal{M}_\gamma \in \mathrm{SAut}_{\mathrm{loc}}(\mathfrak{S}, Y)$.

Let $\mathfrak{D}$ be in $\mathfrak{S}$. By restricting $\mathcal{X}(\mathcal{N}_\beta \circ \mathcal{M}_\gamma)$ to $\mathcal{X}(\mathfrak{D})$ we have

$$\begin{aligned}
\mathcal{X}(\mathcal{N}_\beta \circ \mathcal{M}_\gamma)|_{\mathcal{X}(\mathfrak{D})} &= \mathcal{X}((\mathcal{N} \circ \mathcal{M})_{\mathfrak{D}, \beta \cdot \gamma}) \\
&= \mathcal{X}(\mathcal{N}_{\mathcal{M}_\gamma(\mathfrak{D}), \beta} \circ \mathcal{M}_{\mathfrak{D}, \gamma}) \\
&= \mathcal{X}(\mathcal{N}_{\mathcal{M}_\gamma(\mathfrak{D}), \beta}) \circ \mathcal{X}(\mathcal{M}_{\mathfrak{D}, \gamma}) \\
&= \mathcal{X}(\mathcal{N}_\beta)|_{\mathcal{X}(\mathcal{M}_\gamma(\mathfrak{D}))} \circ \mathcal{X}(\mathcal{M}_\gamma)|_{\mathcal{X}(\mathfrak{D})} \\
&= (\mathcal{X}(\mathcal{N}_\beta) \circ \mathcal{X}(\mathcal{M}_\gamma))|_{\mathcal{X}(\mathfrak{D})}.
\end{aligned}$$

Given that $\mathcal{X}((\mathcal{N} \circ \mathcal{M})_{\beta \cdot \gamma})|_{\mathcal{X}(\mathfrak{D})} = (\mathcal{X}(\mathcal{N}_\beta) \circ \mathcal{X}(\mathcal{M}_\gamma))|_{\mathcal{X}(\mathfrak{D})}$ for every $\mathfrak{D}$ and the $\mathcal{X}(\mathfrak{D})$ cover $\mathcal{X}(\mathfrak{S})$, we have that $\mathcal{X}((\mathcal{N} \circ \mathcal{M})_{\beta \cdot \gamma}) = \mathcal{X}(\mathcal{N}_\beta) \circ \mathcal{X}(\mathcal{M}_\gamma)$. Then, the map $\mathrm{SAut}_{\mathrm{loc}}(\mathfrak{S}, Y) \to \mathrm{SAut}(T_{\mathfrak{S}}; (\mathcal{X}(\mathfrak{S}))$ is a group homomorphism. The injectivity follows from Proposition 9.3.6. Then, the proposition holds. $\square$

Remark 9.6.4. Notice that $\mathrm{Aut}_{\mathrm{loc}}(\mathfrak{S}, Y)$ is a subgroup of $\mathrm{SAut}_{\mathrm{loc}}(\mathfrak{S}, Y)$.

In the following, if it yields no confusion, the group of localized semilinear autmorphisms of a divisorial fan will be referred to as $\mathrm{SAut}(\mathfrak{S}, Y)$.

9.7 Semilinear actions of finite groups on divisorial fans

A semilinear equivariant automorphism that is in the image of the group homomorphism $\mathcal{X} : \mathrm{SAut}(\mathfrak{S}, Y) \to \mathrm{SAut}(\mathcal{X}(\mathfrak{S}))$ corresponds to a semilinear equivariant automorphism that induces an action on the set $\{\mathcal{X}(\mathfrak{D}) \mid \mathfrak{D} \in \mathfrak{S}\}$. Therefore, this homomorphism is far from being surjective, even if k is algebraically closed. For example, consider the case of the $\mathbb{G}_{\mathrm{m},k}$-variety $X := \mathbb{A}^1_k \times E$, with E an elliptic curve, where $\mathbb{G}_{\mathrm{m},k}$ acts in the obvious way on $\mathbb{A}^1_k$ and trivially on E. Let $a \in E$ be a torsion free element. The semilinear equivariant automorphism

$$\begin{aligned}(\mathrm{id}_{\mathbb{G}_{\mathrm{m},k}}, f_a) : X &\to X,\\ (x,y) &\mapsto (x, y+a)\end{aligned}$$

has no finite T-stable affine open cover that is also f-stable, i.e. $f^r(U) \in \mathcal{U}$ for every $U \in \mathcal{U}$ and $r \in \mathbb{N}$. Indeed, let $\mathcal{U}$ be such a covering. Notice that each $U \in \mathcal{U}$ can be of the form $U = \mathbb{A}^1_k \times V$ or $U = \mathbb{G}_{\mathrm{m},k} \times V$, with $V \subset E$ an affine open subset. Since $\{0\} \times E \subset X$ is $\mathbb{G}_{\mathrm{m},k}$-stable, at least one of the elements in $\mathcal{U}$ must be of the form $U = \mathbb{A}^1_k \times V$. Fix such an open U. For some $n \in \mathbb{N}$ we have $U = f^n(U)$. Then $U^c = f^n(U^c)$ and therefore $U^c = f^{mn}(U^c)$ for every $m \in \mathbb{N}$. This is a contradiction because $U^c = \mathbb{A}^1_k \times V^c$ with V^c a finite set and a is a torsion free element. Then, there exists no divisorial fan $(\mathfrak{S}, Y)$ for X such that $(\mathrm{id}_{\mathbb{G}_{\mathrm{m},k}}, f_a)$ is in the image of $\mathcal{X} : \mathrm{SAut}(\mathfrak{S}, Y) \to \mathrm{SAut}(\mathbb{G}_{\mathrm{m},k}; X)$.

A semilinear equivariant automorphism of finite order in $\mathrm{SAut}(T_{\mathfrak{S}}; \mathcal{X}(\mathfrak{S}))$ may not be in the image of $\mathcal{X} : \mathrm{SAut}(\mathfrak{S}, Y) \to \mathrm{SAut}(T_{\mathfrak{S}}; \mathcal{X}(\mathfrak{S}))$ either. However, we will see that for finite subgroups of $\mathrm{SAut}(T_{\mathfrak{S}}; \mathcal{X}(\mathfrak{S}))$ we can always find another divisorial fan $(\mathfrak{S}', Y')$ such that $\mathcal{X}(\mathfrak{S}') \cong \mathcal{X}(\mathfrak{S})$ and the group is in the image of $\mathcal{X} : \mathrm{SAut}(\mathfrak{S}', Y') \to \mathrm{SAut}(T_{\mathfrak{S}'}; \mathcal{X}(\mathfrak{S}'))$.

Proposition 9.7.1. *Let k be a field of characteristic zero, L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let T be a split algebraic torus over L. Let X be a normal T-variety over L. Let $(\varphi_\gamma, f_\gamma) : X \to X$ be a semilinear equivariant automorphism. If X has a $(\varphi_\gamma, f_\gamma)$-stable finite affine open covering, then there exists a divisorial fan $(\mathfrak{S}, Y)$ and $\mathcal{M}_\gamma \in \mathrm{SAut}(\mathfrak{S}, Y)$ such that $X \cong X(\mathfrak{S})$ and $X(\mathcal{M}_\gamma) = (\varphi_\gamma, f_\gamma)$.*

Proof. Let $\mathcal{U}$ be $(\varphi_\gamma, f_\gamma)$-stable affine open covering of X. Notice that $\mathcal{U}$ is also T-stable. Let $(\mathfrak{S}, Y)$ be the divisorial fan associated to $\mathcal{U}$ (cf. Corollary 7.3.13). The existence of $\mathcal{M}_\gamma \in \mathrm{SAut}(\mathfrak{S}, Y)$ is ensured by Theorem 9.5.4, therefore, the assertion holds. $\square$

Not every semilinear equivariant automorphism $(\varphi_\gamma, f_\gamma) : X \to X$ has such a covering. However, semilinear equivariant automorphisms of finite order, i.e. $(\varphi_\gamma, f_\gamma)^n = (\varphi_\gamma^n, f_\gamma^n) = (\mathrm{id}_T, \mathrm{id}_X)$, do have one.

Lemma 9.7.2. *Let k be a field of characteristic zero, L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let T be a split algebraic torus over L. Let X be a normal T-variety and $(\varphi_\gamma, f_\gamma) : X \to X$ be a semilinear equivariant automorphism of order $n \in \mathbb{N}$. There exists a T-stable affine open covering $\mathcal{U}$ of X that is $(\varphi_\gamma, f_\gamma)$-stable.*

Proof. By Sumihiro's Theorem X has a T-stable affine open covering $\mathcal{U}'$. For every $U \in \mathcal{U}'$, $f_\gamma(U)$ is an affine open subvariety and also T-stable. Indeed, let $x \in f_\gamma(U)$ and $t \in T$. There exist $x' \in U$ and $t' \in T$ such that $x = f_\gamma(x')$ and $t = \varphi_\gamma(t')$. Thus, $t \cdot x = \varphi_\gamma(t') \cdot f_\gamma(t') = f_\gamma(t' \cdot x') \in f_\gamma(U)$. Therefore, $f_\gamma(U)$ is T-stable. Finally, the set $\mathcal{U}$ consisting of all the (finitely many) intersections of the elements of $\{f_\gamma^k(U) \mid U \in \mathcal{U}' \text{ and } k \in \mathbb{N}\}$ is $(\varphi_\gamma, f_\gamma)$-stable. ☐

Then, we have the following result.

Corollary 9.7.3. *Let k be a field of characteristic zero, L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let T be a split algebraic torus over L. Let X be a normal T-variety and $(\varphi_\gamma, f_\gamma)$ be a semilinear equivariant automorphism of order $n \in \mathbb{N}$. Then, there exists a divisorial fan $(\mathfrak{S}, Y)$ and $\mathcal{M}_\gamma \in \mathrm{SAut}(\mathfrak{S}, Y)$ such that $X \cong \mathcal{X}(\mathfrak{S})$ and $(\varphi_\gamma, f_\gamma) = \mathcal{X}(\mathcal{M}_\gamma)$.*

Proof. By Lemma 9.7.2, X has a $(\varphi_\gamma, f_\gamma)$-stable finite affine open covering. Then, by Proposition 9.7.1, the assertion holds. ☐

A normal T-variety X can have many divisorial fans and not every divisorial fan admits an action induced by a semilinear equivariant automorphism of X. However, Corollary 9.7.3 states that for every finite cyclic subgroup of $\mathrm{SAut}(T; X)$, there exists a divisorial fan $(\mathfrak{S}, Y)$ such that $X \cong \mathcal{X}(\mathfrak{S})$ and the cyclic group is in the image of $\mathcal{X} : \mathrm{SAut}(\mathfrak{S}, Y) \to \mathrm{SAut}(T; X)$. Before we prove such an assertion for any finite subgroup, we present the notion of *semilinear action* for divisorial fans.

Definition 9.7.4. Let k be a field, L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let G be a group and $(\mathfrak{S}, Y)$ be a divisorial fan over L. A *semilinear action of G over* $(\mathfrak{S}, Y)$ is a group homomorphism $\varphi : G \to \mathrm{SAut}(\mathfrak{S}, Y)$. If the image of G lies on $\mathrm{Aut}(\mathfrak{S}, Y)$, it said to be an *action of G over* $(\mathfrak{S}, Y)$. A *Galois semilinear action over* $(\mathfrak{S}, Y)$ is a semilinear action of Γ over $(\mathfrak{S}, Y)$ with φ a section of the sequence

$$1 \to \mathrm{Aut}(\mathfrak{S}, Y) \to \mathrm{SAut}(\mathfrak{S}, Y) \to \Gamma.$$

Proposition 9.7.5. *Let k be a field of characteristic zero and L/k be a Galois extension. Let T be a split algebraic torus over L. Let X be a normal T-variety and G be a finite group of semilinear equivariant automorphisms of X. Then, there exists a T-stable affine open covering $\mathcal{U}$ of X that is G-stable.*

Proof. By Sumihiro's Theorem, X has a T-stable affine open covering $\mathcal{U}'$. As in Lemma 9.7.2, each $g(U)$ is a T-stable affine subvariety. Let us define $R := \{g(U) \mid U \in \mathcal{U}' \text{ and } g \in G\}$. Then the set $\mathcal{U}$, consisting of all possible (finitely many) intersections of elements in R, is G-stable. □

Proposition 9.7.6. *Let k be a field of characteristic zero and L/k be a Galois extension. Let X be a normal L-variety endowed with an effective action of a split algebraic k-torus and G be a finite group of semilinear equivariant automorphisms. Then, there exists a divisorial fan $(\mathfrak{S}, Y)$, such that $X \cong \mathcal{X}(\mathfrak{S})$ and that admits a semilinear action of G.*

Proof. By Proposition 9.7.5, there exists a T-stable affine open covering, which is also G-stable. Let $(\mathfrak{S}, Y)$ be the divisorial fan associated to such an open covering by Corollary 7.3.13. By Theorem 9.5.4, for every $(\varphi_\gamma, f_\gamma) \in G$ there exists $\mathcal{M}_\gamma \in \mathrm{SAut}(\mathfrak{S}, Y)$ such that $\mathcal{X}(\mathcal{M}_\gamma) = (\varphi_\gamma, f_\gamma)$. Hence, given that $\mathrm{SAut}(\mathfrak{S}, Y) \to \mathrm{SAut}(T; X)$ is a group homomorphism and that G is in the image of $\mathrm{SAut}(\mathfrak{S}, Y) \to \mathrm{SAut}(T; X)$, we have a group homomorphism of $G \to \mathrm{SAut}(\mathfrak{S}, Y)$. Then, the assertion holds. □

Both results, Proposition 9.6.3 and Proposition 9.7.6, can be summarized in the following theorem.

Theorem 9.7.7. *Let k be a field of characteristic zero and L/k be a Galois extension. Let G be a finite group and X be a normal variety over L endowed with an effective action of a split torus T over L. There exists a semilinear equivariant algebraic action of G over X if and only if there exists a divisorial fan $(\mathfrak{S}, Y)$ with a semilinear action of G such that $X \cong \mathcal{X}(\mathfrak{S})$.*

Definition 9.7.8. Let k be a field of characteristic zero, L be a Galois extension. Let N be a lattice and $G \leq \mathrm{Aut}(N)$ be a finite group. We say that a divisorial fan $(\mathfrak{S}, Y, N)$ is *G-stable* if there exists a semilinear action of G over $(\mathfrak{S}, Y)$. In this case, for every $g \in G$ and $\mathfrak{D}$, we denote by $g(\mathfrak{D})$ the corresponding pp-divisor in part (**??**) of Definition 9.5.2. If the semilinear action factors through $\mathrm{Aut}(\mathfrak{S}, Y)$, then it is said to be *G-stable*.

Definition 9.7.9. Let k be a field of characteristic zero, L/k be a Galois extension. Let $(\mathfrak{S}, Y)$ be a G-stable divisorial fan over L. Let $\mathfrak{D} \in \mathfrak{S}$, we denote by $\mathfrak{S}(\mathfrak{D}, G)$ to the *sub divisorial fan generated generated by $\mathfrak{D}$ and G*, which is defined as the smallest divisorial fan containing the set $\{g(\mathfrak{D}) \mid g \in G\}$ in $(\mathfrak{S}, Y)$.

9.8 Proof of the main theorem

In this section, we prove Theorem 5.0.7. For the convenience of the reader we recall the statement.

Theorem 9.8.1. *Let k be a field of characteristic zero and L/k be a finite Galois extension with Galois group Γ.*

a) Let T be a split algebraic torus over L and X be a normal T-variety over L. If there exists a divisorial fan $(\mathfrak{S}_L, Y_L)$ for X admitting a Γ-semilinear action such that

the subvariety $X(\mathfrak{S}(\mathfrak{D}, \Gamma))$ is quasi-projective for every $\mathfrak{D} \in \mathfrak{S}_L$,

then there exists an algebraic torus T' over k and a normal T'-variety X' over k such that $X'_L \cong X$ as T varieties over L.

b) Let T be an algebraic torus over k that splits over L. Let X be a normal variety endowed with an effective T-action over k. Then, there exists a divisorial fan $(\mathfrak{S}_L, Y_L)$ admitting a Γ-semilinear action such that

the subvariety $X(\mathfrak{S}(\mathfrak{D}, \Gamma))$ is quasi-projective for every $\mathfrak{D} \in \mathfrak{S}_L$

and $X_L \cong X(\mathfrak{S}_L)$ as T_L-varieties.

Now, we have almost all the tools to prove Theorem 5.0.7. The last result that we need to present is a general version of Sumihiro's Theorem (cf. Theorem 6.3.3), because for non split algebraic tori there might be no T-stable affine open coverings.

Theorem 9.8.2 (Sumihiro's Theorem, general version [Sum74])**.** *Let k be a field and T be an algebraic k-torus. Let X be a normal variety over k endowed with an action of T, then X has a T-stable quasi-projective open covering.*

Remark 9.8.3. In [Sum74], Sumihiro states that the hypothesis of normality cannot be relaxed. A counter example can be found in [CLS11].

We are ready to prove the main theorem of the second part of this thesis.

Proof of Theorem 5.0.7. Let us prove first part (a) of Theorem 5.0.7. Let $(\mathfrak{S}_L, Y_L)$ be a divisorial fan such that $X_L \cong X(\mathfrak{S}_L)$ and that admits a Γ-semilinear action such that $X(\mathfrak{S}(\mathfrak{D}, \Gamma))$ is quasi-projective for every $\mathfrak{D} \in \mathfrak{S}_L$. By Theorem 9.7.7, it is equivalent to a Γ-semilinear equivariant action over

$X(\mathfrak{S}_L)$ with a T_L-stable and Γ-stable quasi-projective covering. Hence, by Proposition 9.1.9, X_L has a k-form as a T-variety.

Now, let us prove part (b) of Theorem 5.0.7. By Theorem 9.8.2, X can be covered by T-stable quasi-projective open subvarieties $\{X_i\}_{i\in I}$. Given that $T_L := T \times_{\mathrm{Spec}(k)} L$ is split, each $X_{i,L} := X_i \times_{\mathrm{Spec}(k)} L$ can be covered by T_L-stable open subvarieties $\{U_{ij}\}_{j\in J_i}$, by Theorem 6.3.3. Notice that each $X_{i,L}$ is Γ-stable and quasiprojective. Denote by $(\varphi_\gamma, f_\gamma)$ the semilinear equivariant morphism induced by the base change, where $f_\gamma := \mathrm{id}_X \times \gamma^\natural$. For each $\gamma \in \Gamma$, the subvariety $f_\gamma(U_{ij}) \subset X_{i,L} \subset X_L$ is T-stable. Therefore, the set

$$\mathcal{U} := \{f_\gamma(U_{ij}) \mid \gamma \in \Gamma \text{ and } U_{ij} \subset X_{i,L}\}$$

form a T_L-stable affine open covering such that is Γ-stable and for any $U \in \mathcal{U}$ the union $\cup_{\gamma\in\Gamma}(f_\gamma(U))$ is quasi-projective. Moreover, the covering can be considered stable under intersections. Let $(\mathfrak{S}_L, Y_L)$ be the divisorial fan associated to $\mathcal{U}$ (cf. Corollary 7.3.13). By Proposition 9.3.6, for every $\gamma \in \Gamma$, there exists a localized semilinear automorphism of divisorial fans $\mathcal{M}_\gamma \in \mathrm{SAut}(\mathfrak{S}_L, Y_L)$ such that $X(\mathcal{M}_\gamma) = (\varphi_\gamma, f_\gamma)$. This gives the following commutative diagram of abstract groups

$$\begin{tikzcd} & \Gamma \arrow[dl] \arrow[dr] & \\ \mathrm{SAut}(\mathfrak{S}_L, Y_L) \arrow[rr] & & \mathrm{SAut}(T_L; X(\mathfrak{S}_L)). \end{tikzcd}$$

Hence, there exists a semilinear action of Γ over $(\mathfrak{S}_L, Y_L)$. Moreover, for every $\mathfrak{D} \in \mathfrak{S}_L$ we have that $X(\mathfrak{S}(D, \Gamma))$ is a quasi-projective variety. Therefore, $(\mathfrak{S}_L, Y_L)$ is a Γ-stable divisorial fan and $X_L \cong X(\mathfrak{S}_L)$ as T_L-varieties. □

Remark 9.8.4. The quasi-projectivity condition in Theorem 5.0.7, by [PS11, Corollary 3.28], can be tested on the divisorial fan when X is a T-variety of complexity one.

Toric case. There is a difference between a divisorial fan and a classical fan. On the one hand, a fan must contain all the faces of all the cones in the set. On the other hand, for a divisorial fan we just ask that the set must be stable under pairwise intersections. This is because a pp-divisor can have infinitely many faces if the complexity is greater or equal than one. However, in the complexity zero case, a pp-divisor is just a cone over a point. This being said, a divisorial fan of a complexity zero normal variety is just a set of cones that is stable under intersections. Notice that this does not imply that the set is stable under faces. For example, take any affine normal toric variety: as a pp-divisor it forms a divisorial fan, because it is stable under intersections, but as a cone it does not form a fan unless we add all its faces. Thus, in order to turn a divisorial fan into a classical fan we need to add its faces.

Let k be a field of characteristic zero, T be an algebraic torus over k and L/k a finite Galois extension with Galois group Γ that splits T. Let X_Σ be a T-toric variety over k. By Theorem 5.0.7, there exists a divisorial fan $(\tilde{\mathfrak{S}}_L, \operatorname{Spec}(L))$ over L such that admits a Γ-semilinear action, for each $\mathfrak{D} \in \tilde{\mathfrak{S}}_L$ the subvariety $X(\mathfrak{S}(\mathfrak{D},\Gamma))$ is quasiprojective and $X_L \cong X(\tilde{\mathfrak{S}}_L)$ as toric varieties. We consider the following divisorial fan

$$\mathfrak{S} := \{\omega \otimes \operatorname{Spec}(L) \mid \omega \preceq \omega_{\mathfrak{D}} \text{ for some } \mathfrak{D} \in \tilde{\mathfrak{S}}_L\}.$$

The Galois semilinear action over $(\tilde{\mathfrak{S}}_L, \operatorname{Spec}(L))$ extends to a Galois semilinear action over $(\mathfrak{S}, \operatorname{Spec}(L))$. Notice that $X(\mathfrak{S}) \cong X_\Sigma$ and $\operatorname{Tail}(\mathfrak{S}) = \Sigma$, since all the $\omega_{\mathfrak{D}}$ are in Σ. Moreover, $X(\mathfrak{D}) = X_{\omega_{\mathfrak{D}}}$ and $X(\mathfrak{S}(\mathfrak{D},\Gamma))$ is a Γ-stable quasiprojective variety for every $\mathfrak{D} \in \mathfrak{S}$. Given that $X(\mathfrak{S}(\mathfrak{D},\Gamma)) = X_{\Sigma(\omega_{\mathfrak{D}},\Gamma)}$, for every $\mathfrak{D} \in \mathfrak{S}$, we have that $\Sigma(\omega_{\mathfrak{D}},\Gamma)$ is a quasiprojective fan (cf. Definition 6.2.5) for every $\omega_{\mathfrak{D}}$ by [Hur11, Proposition 1.9].

The Galois semilinear action over $(\mathfrak{S}, \operatorname{Spec}(L))$ induces a group homomorphism $\Gamma \to \operatorname{Aut}(N)$, given by $\gamma \mapsto F_\gamma$, such that $F_\gamma(\Sigma) = \Sigma$ for every $\gamma \in \Gamma$. Then, by Proposition 6.2.6, we have a group homomorphism $\Gamma \to \operatorname{Aut}(\Sigma)$ also denoted by F. Hence, we have a Γ-stable fan such that the fan $\Sigma(\omega,\Gamma)$ is quasiprojective for all $\omega \in \Sigma$.

Corollary 9.8.5. *[Hur11, Theorem 1.22] Let k be a field of characteristic zero and L/k be a finite Galois extension with Galois group Γ. Let N be a lattice and $M \cong \operatorname{Hom}(N,\mathbb{Z})$ its dual lattice. Let $T := \operatorname{Spec}(L[M])$ be a split algebraic torus over L and X_Σ be the normal T-toric variety associated to a fan Σ in $N_{\mathbb{Q}}$. If there exists a group homomorphism $F : \Gamma \to \operatorname{Aut}(N)$ such that Δ is Γ-stable and the fans $\Sigma(\omega,\Gamma)$ are quasiprojective for all $\omega \in \Sigma$, then X_Σ has a k-form as a toric variety.*

Counting k-forms. A natural question that arises after Theorem 5.0.7 and it is how many k-forms does a normal T-variety have? For example, it is known that toric varieties have finitely many k-forms. This no longer the case for complexity strictly greater than one.

Proposition 9.8.6. *For every integer $c \geq 2$, there exists normal T-varieties X over $\mathbb{C}$ of complexity c and infinitely many $\mathbb{R}$-forms.*

Proof. There exist smooth surfaces with infinitely many non isomorphic real forms [Bot23]. Let S be one of those surfaces and define the complexity two T-variety $X := T \times S$, where T is an algebraic torus over $\mathbb{C}$. This variety has infinitely many non isomorphic real forms as T-variety, because such a form is given datum over the torus and a datum over S. For $c > 2$, it is enough to take $Y := S \times \mathbb{A}^n_{\mathbb{C}}$. $\square$

Lucchini-Arteche and Terpereau recently answered the following question in an unpublished work. We'll keep you in suspense. (Now published: [LAT25])

Question 9.8.7. How many real forms have a complexity one normal T-varieties over $\mathbb{C}$?

Chapter 10

Complexity one and Applications

This chapter has two parts. The first one is a restatement of Theorem 5.0.7 for complexity one normal T-varieties. In order to get a semilinear action over a divisorial fan we need to consider a localisation of the category $\mathfrak{PPDiv}(L/k)$. Nevertheless, it is possible to avoid this problem for complexity one normal T-varieties. Moreover, we can describe the galois descent data purely in terms of the divisorial fan.

Theorem 10.0.1. *Let T be an algebraic k-torus, $k \subset L \subset \overline{k}$ a finite Galois extension that split the torus and $\Gamma := \mathrm{Gal}(L/k)$.*

a) Let X_L be a complexity one T_L-variety over L. If there exists a divisorial fan $(\mathfrak{S}_L, Y_L)$ over a complete curve admitting a Γ-semilinear action such that $X_L \cong X(\mathfrak{S}_L)$ and for every $\mathfrak{D} \in \mathfrak{S}_L$ the divisorial subfan $\mathfrak{S}(\mathfrak{D}, \Gamma)$ is quasi-projective, then X_L has a k-form X as a T-variety.

b) Let X be a complexity one normal T-variety over k. Then, there exists a divisorial fan $(\mathfrak{S}_L, Y_L)$ over a complete curve admitting a Γ-semilinear action such that $X_L \cong X(\mathfrak{S}_L)$ as T_L-varieties over L and for every $\mathfrak{D} \in \mathfrak{S}_L$ the divisorial subfan $\mathfrak{S}(\mathfrak{D}, \Gamma)$ is quasi-projective.

In the last part of this chapter we compute the group of equivariant automorphisms of the Hirzebruch surface and we prove that such a variety has a nontoric k-form, which is a k-form as a T-variety.

Proposition 10.0.2. *Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let $k \subset L \subset \bar{k}$ be a finite Galois extension with Galois* *group $\Gamma \cong S_4$. If -1 is a sum of two squares in k, then the Hirzebruch surface $\mathbb{F}_r$ over L has a k-form as a normal T-variety that is nontoric.*

10.1 Complexity one

Let k be a field of characteristic zero and T be a split algebraic torus over k. Let X be a complexity one normal affine T-variety over k. In this setting, every complexity one normal T-variety over k arises from a divisorial fan over a smooth complete curve (see Remark 7.3.14). From now on, every divisorial fan will be considered over a smooth complete curve.

Let L/k be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let T be a split algebraic torus over L. Let $X(\mathfrak{D})$ and $X(\mathfrak{D}')$ be two complexity one normal affine T-varieties over L and $(\varphi_\gamma, f_\gamma) : X(\mathfrak{D}) \to X(\mathfrak{D}')$ be a semilinear equivariant dominant morphism. By Theorem 8.4.4, this morphism arises from a triangle of semilinear morphism of pp-divisors as follows

$$\mathfrak{D} \xleftarrow{(\kappa,\mathrm{id}_N,1)} \kappa^*\mathfrak{D} \xrightarrow{(\psi_\gamma,F,\mathfrak{f})} \mathfrak{D}',$$

where $\kappa : \tilde{Y} \to Y$ is a projective birational map and $\tilde{Y}$ is a smooth semiprojective variety that is actually projective. However, given that Y is a smooth complete curve $\kappa : \tilde{Y} \to Y$ can be considered as the identity. Then, in this setting, Section 9.3 is not needed. Thus, the complexity one version of Theorem 8.4.4 is the following.

Theorem 10.1.1. *Let k be a field of characteristic zero, L/k be a finite Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two pp-divisors over complete curves and $(\varphi_\gamma, f_\gamma) : X(\mathfrak{D}) \to X(\mathfrak{D}')$ be a dominant semilinear equivariant morphism. Then, there exists a dominating semilinear morphism of pp-divisors $(\psi_\gamma, F, \mathfrak{f}) : \mathfrak{D} \to \mathfrak{D}'$ such that $(\varphi_\gamma, f_\gamma) = X(\psi, F, \mathfrak{f})$. Moreover, If $(\varphi_\gamma, f_\gamma)$ is a semilinear equivariant isomorphism, then ψ_γ is a semilinear isomorphism of varieties over L, F is a lattice isomorphism such that $F(\omega_{\mathfrak{D}}) = \omega_{\mathfrak{D}'}$ and $\psi_\gamma^*\mathfrak{D}' = F_*\mathfrak{D} + \mathrm{div}(\mathfrak{f})$.*

If we restrict the category $\mathfrak{PPDiv}(L/k)$ to the full subcategory of pp-divisors over smooth complete curves, the functor $X : \mathfrak{PPDiv}(L/k) \to \mathcal{E}(L/k)$ becomes an equivalence of categories with the category of complexity one normal affine varieties endowed with a split algebraic torus action and dominant semilinear equivariant morphisms.

Proposition 10.1.2. *Let k be a field of characteristic zero and L/k be a finite Galois extension. The functor $\mathfrak{D} \to X(\mathfrak{D})$ is an equivalence of categories between the category of pp-divisors over complete curves and whose morphisms are dominant semilinear morphism of pp-divisors, with the category of complexity one normal varieties endowed with a split algebraic torus action over L and dominant semilinear equivariant morphisms.*

As we saw in Section 9.5, for a semilinear equivariant automorphism of finite order $(\varphi_\gamma, f_\gamma) : X \to X$ there exists a $\langle(\varphi_\gamma, f_\gamma)\rangle$-stable divisorial fan $(\mathfrak{S}_L, Y_L)$ (cf. Corollary 9.7.3). For higher complexity, the data of $\langle(\varphi_\gamma, f_\gamma)\rangle$ over $(\mathfrak{S}_L, Y_L)$ could not be given by a semilinear morphism of divisorial fans. However, we have the following version of Theorem 9.5.4.

Proposition 10.1.3. *Let k be a field of characteristic zero, L be a Galois extension with Galois group Γ and $\gamma \in \Gamma$. Let X be a normal L-variety endowed with an effective actions of a split algebraic torus T over L. Let $(\varphi_\gamma, f_\gamma) : X \to X$ be a semilinear equivariant automorphism. Then, for every $\langle(\varphi_\gamma, f_\gamma)\rangle$-stable divisorial fan $(\mathfrak{S}_L, Y_L)$ over a smooth complete curve Y_L such that $X \cong X(\mathfrak{S}_L)$, there exists a semilinear automorphism of divisorial fans $(\psi_\gamma, F, \mathfrak{f}) : (\mathfrak{S}, Y_L) \to (\mathfrak{S}_L, Y_L)$ such that $X(\psi_\gamma, F, \mathfrak{f}) = (\varphi_\gamma, f_\gamma)$.*

Proof. First notice that, for every $\mathfrak{D} \in \mathfrak{S}$ and $\mathfrak{D}' \in \mathfrak{S}'$ such that $f_\gamma(X(\mathfrak{D})) \subset X(\mathfrak{D}')$, we have that the induced semilinear equivariant morphism

$$(\varphi_\gamma, f_\gamma)|_{X(\mathfrak{D})} : X(\mathfrak{D}) \to X(\mathfrak{D}')$$

is dominant. Let us prove this last assertion. Let $U \subset X(\mathfrak{D}')$ be an open subvariety, then $U \subset X(\mathfrak{S}')$ is an open subvariety. Given that $(\varphi_\gamma, f_\gamma)$ is dominant, $f_\gamma^{-1}(U)$ is a nonempty subvariety of $X(\mathfrak{S})$. This implies that $X(\mathfrak{D}) \cap f_\gamma^{-1}(U)$ is not empty and, therefore, $f_\gamma(X(\mathfrak{D})) \cap U$ is not empty. Then, the restrictions $(\varphi_\gamma, f_\gamma)|_{X(\mathfrak{D})} : X(\mathfrak{D}) \to X(\mathfrak{D}')$ are dominant for every $\mathfrak{D} \in \mathfrak{S}$.

Given that, for every $\mathfrak{D} \in \mathfrak{S}$, the restriction map $(\varphi_\gamma, f_\gamma)|_{X(\mathfrak{D})} : X(\mathfrak{D}) \to X(\mathfrak{D}')$ is a dominant semilinear equivariant morphism, there exists a semilinear morphism of pp-divisors $(\psi_{\mathfrak{D},\gamma}, F_{\mathfrak{D}}, \mathfrak{f}_{\mathfrak{D}}) : \mathfrak{D} \to \mathfrak{D}'$, denoted by $g_{\mathfrak{D},\gamma}$, such that $X(g_{\mathfrak{D},\gamma}) = (\varphi_\gamma, f_\gamma)|_{X(\mathfrak{D})}$ by Proposition 10.1.2. By Proposition 8.3.3, we have that, for every $\mathfrak{D}$ and $\mathfrak{D}'$ in $\mathfrak{S}$,

$$(\psi_{\mathfrak{D},\gamma}, F_{\mathfrak{D}}, \mathfrak{f}_{\mathfrak{D}}|_{\mathfrak{D}\cap\mathfrak{D}'}) = (\psi_{\mathfrak{D}',\gamma}, F_{\mathfrak{D}'}, \mathfrak{f}_{\mathfrak{D}'}|_{\mathfrak{D}\cap\mathfrak{D}'}).$$

Then, we can set $\psi_\gamma := \psi_{\mathfrak{D},\gamma}$ and $F := F_{\mathfrak{D}}$, for any $\mathfrak{D}$ in $\mathfrak{S}$. In order to define a semilinear autmorphism of divisorial fans, it is enough to define the plurifuction. Let $\{\mathfrak{D}_1, \ldots, \mathfrak{D}_r\} \subset \mathfrak{S}$ be the set of *maximal* pp-divisors, which means that they are just faces of themselves. First, we set $\mathfrak{f}_1 := \mathfrak{f}_{\mathfrak{D}_1}$ and

$$\mathfrak{f}_j := \mathfrak{f}_{j-1} + \mathfrak{f}_{\mathfrak{D}_j} - \mathfrak{f}_j|_{\mathfrak{D}_j},$$

for $j \geq 2$. Notice that, for any $j \geq 1$ and any $\mathfrak{D}$ in $\mathfrak{S}$, we have that $\mathfrak{f}_j|_{\mathfrak{D}} = \mathfrak{f}_{\mathfrak{D}}$ by Proposition 8.3.3. Thus, we define $g_\gamma := (\psi_\gamma, F, \mathfrak{f})$, with $\mathfrak{f} := \mathfrak{f}_r$. Given that $g_{\mathfrak{D},\gamma} = (\psi_\gamma, F, \mathfrak{f}|_{\mathfrak{D}}) = (\psi_\gamma, F, \mathfrak{f}_{\mathfrak{D}})$, it satisfies Definition 9.5.2 part (i). In order to prove part (ii) of Definition 9.5.2, it suffices to do it on the respective varieties by Proposition 9.3.6 and this is obvious from the commutative diagram

$$\begin{tikzcd}
X(\mathfrak{D}) \arrow[r, "X(g_{\mathfrak{D},\gamma})"] & X(\mathfrak{D}') \\
X(\mathfrak{E}) \arrow[u, "X(\mathrm{id}_Y, \mathrm{id}_N, 1)"] \arrow[r, "X(g_{\mathfrak{E},\gamma})"'] & X(\mathfrak{E}'). \arrow[u, "X(\mathrm{id}_{Y'}, \mathrm{id}_{N'}, 1')"']
\end{tikzcd}$$

Finally, we that that $X(g_\gamma) = (\varphi_\gamma, f_\gamma)$, since $X(x_\gamma)$ is the gluing on the restriction maps $(\varphi_\gamma, f_\gamma)|_{X(\mathfrak{D})}$. Then, the assertion holds. □

10.1.1 Quasi-projective divisorial fans

In Theorem 5.0.7, the quasi-projectivity condition is given in terms of the variety induced by the divisorial fan. For complexity one T-varieties there exists a characterization of quasi-projetivity for a divisorial fan, which extends the characterization known in toric geometry. In the following, we present some definitions introduced in [PS11].

Definition 10.1.4. [PS11, Definition 3.2] Let $\Sigma := \mathrm{tail}(\mathfrak{S})$ be a subdivision of $N_{\mathbb{Q}}$. A continuous function $h : |\Sigma| \to \mathbb{Q}$, which is affine on every polyhedron $\Delta \in \Sigma$, is called a $\mathbb{Q}$-support function, or merely a support function, if it has integer slope and integer translation, i.e. for $v \in |\Sigma|$ and $l \in \mathbb{N}$ such that lv is a lattice point we have $lh(v) \in \mathbb{Z}$. The group of support functions on Σ is denoted by $\mathrm{SF}(\Sigma)$.

Definition 10.1.5. [PS11, Definition 3.3] Let $\Sigma := \mathrm{tail}(\mathfrak{S})$ be a subdivision of $N_{\mathbb{Q}}$ and $h : |\Sigma| \to \mathbb{Q}$ be a support function. Let $\Delta \in \Sigma$ be a polyhedron with tailcone ω. We define a linear function h_t^{Δ} on ω by setting $h_t^{\Delta}(v) := h(p+v) - h(p)$ for some $p \in \Delta$. As h_t^{Δ} is induced by h we call it the *linear part* of $h|_{\Delta}$, or $\mathrm{lin}\, h|_{\Delta}$ for short.

Definition 10.1.6. [PS11, Definition 3.4] Let k be a field. Let $(\mathfrak{S}, Y)$ be a divisorial fan over k with Y a smooth projective curve. We define $\mathrm{SF}(\mathfrak{S})$ to be the group of all collections

$$(h_p)_{p\in Y} \in \prod_{p\in Y} \mathrm{SF}(\mathfrak{S}_p)$$

such that

1. all h_p have the same linear part h_t, i.e. for polytopes $\Delta \in \mathfrak{S}_p$ and $\Delta' \in S_{p'}$ with the same tailcone ω we have that $\operatorname{lin} h_p|_\Delta = \operatorname{lin} h_p|_{\Delta'} = h_t|_\omega$.

2. h_p differs from h_t for only finitely many $p \in Y$.

We call $\mathrm{SF}(\mathfrak{S})$ the group of divisorial supports functions on $\mathfrak{S}$.

Remark 10.1.7. We may restrict an element $h \in \mathrm{SF}(\mathfrak{S})$ to a *divisorial subfan* or even to a pp-divisor $\mathfrak{D} \in \mathfrak{S}$. The restriction will be denoted by $h|_\mathfrak{D}$.

Definition 10.1.8. Let $(\mathfrak{S}, Y)$ be a complexity one divisorial fan and Σ its tail fan. For a cone $\omega \in \Sigma(n)$ of maximal dimension and a point $p \in Y$ we get exactly one polyhedron $\Delta_p^\omega \in \mathfrak{S}_p$ having tail cone ω. For a given support function $h = (h_p)_{p\in Y} \in \mathrm{SF}(\mathfrak{S})$ we have

$$h_p|_{\Delta_p^\omega} = \langle u^h(\omega), \cdot\rangle + a_p^h(\omega).$$

The constant part gives a divisor on Y:

$$h|_\omega(0) := \sum_{p\in Y} a_p^h(\omega)p.$$

Definition 10.1.9. A divisorial fan $(\mathfrak{S}, Y)$ over a complete curve is said to be quasi-projective if there exists $h \in \mathrm{CaSF}(\mathfrak{S})$ such that h_p are strictly concave and, for all tail cones ω belonging to a pp-divisor $\mathfrak{D} \in \mathfrak{S}$ with affine locus, $\deg h|_\omega(0) = \sum_p a_p^h(\omega) < 0$, i.e. $-h|_\omega(0)$ is ample.

From all definition we can present the following proposition.

Proposition 10.1.10. *Let $(\mathfrak{S}, Y)$ a divisorial fan over a smooth complete curve. Then, $X(\mathfrak{S})$ is a quasi-projective variety if and only of $(\mathfrak{S}, Y)$ is a quasi-projective divisorial fan.*

Proof. This is a direct consequence of [PS11, Corollary 3.28]. □

10.1.2 Descent in complexity one

In the complexity one case, Theorem 5.0.7 can be stated in terms of quasi-projective divisorial fans:

Theorem 10.1.11. *Let T be an algebraic k-torus, $k \subset L \subset \overline{k}$ a finite Galois extension that split the torus and $\Gamma := \mathrm{Gal}(L/k)$.*

a) Let X_L be a complexity one T_L-variety over L. If there exists a divisorial fan $(\mathfrak{S}_L, Y_L)$ over a complete curve admitting a Γ-semilinear action such that $X_L \cong X(\mathfrak{S}_L)$ and for every $\mathfrak{D} \in \mathfrak{S}_L$ the divisorial subfan $\mathfrak{S}(\mathfrak{D}, \Gamma)$ is quasi-projective, then X_L has a k-form X as a T-variety.

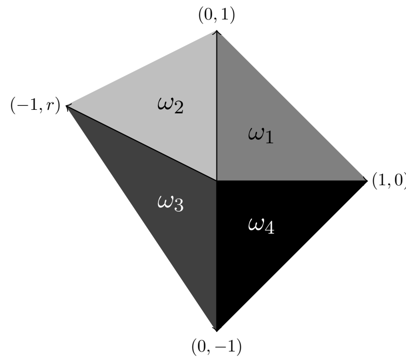
b) Let X be a complexity one normal T-variety over k. Then, there exists a divisorial fan $(\mathfrak{S}_L, Y_L)$ over a complete curve admitting a Γ-semilinear action such that $X_L \cong X(\mathfrak{S}_L)$ as T_L-varieties over L and for every $\mathfrak{D} \in \mathfrak{S}_L$ the divisorial subfan $\mathfrak{S}(\mathfrak{D}, \Gamma)$ is quasi-projective.

10.2 Applications

Among the applications of Theorem 5.0.7 we can find the classification of smooth real Fano varieties with torus action. For example, Hendrik Süss in [SÏ4] studies complex Fano threefolds with two dimensional torus action that admit a Kähler-Einstein metric.

Hirzebruch surface Let k be a field of characteristic zero and L/k be a quadratic extension with Galois group $\Gamma := \mathrm{Gal}(L/k)$. Let T be a split algebraic torus over k such that $T_L \cong (\mathbb{G}_{\mathrm{m},L})^n$. All k-tori that are split by a quadratic extension are products of the following algebraic tori $\mathrm{R}_{L/k}(\mathbb{G}_{\mathrm{m},L})$, $\mathrm{R}^1_{L/k}(\mathbb{G}_{\mathrm{m},L})$ and $\mathbb{G}_{\mathrm{m},k}$.

Let $k \subset L \subset \bar{k}$ be a finite Galois extension with Galois group Γ. Let $r \in \mathbb{N}$ and $\mathbb{F}_r$ be the toric L-variety generated by the fan Σ given in following figure.



Such a variety is known as Hirzebruch surface. If L is a quadratic extension, by [Hur11, Theorem 1.25], $\mathbb{F}_r$ has a toric k-form if and only if the fan Δ is Γ-stable.

Proposition 10.2.1. *The group of automorphisms of Δ is isomorphic to C_2 and is generated by*

$$F = \begin{pmatrix} -1 & 0 \\ r & 1 \end{pmatrix}.$$

Proof. The group of automorphisms of Δ is completely determined by the image of ω_1. Let $F \in GL_2(\mathbb{Z})$ be an automorphism of Δ, then $F(\omega_1) = \omega_i$ for $i \in \{1, 2, 3, 4\}$. In each case there are two options, because each cone has two rays. If $F(\omega_1) = \omega_1$, the first one corresponds to the identity map, and the other one the permutation $(1, 0) \mapsto (0, 1) \mapsto (1, 0)$. However, in such a case, $F(\omega_3) \neq \omega_i$ because $F(-1, r) = (r, -1)$ is not a ray of any ω_i.

If $F(\omega_1) = \omega_2$ the first case is $F(1, 0) = (0, 1)$ and $F(0, 1) = (-1, r)$, then $F(-1, r) = (-r, r^2 - 1)$. Given that $F(-1, r)$ is not a ray of Δ, then this case does not hold. The other case corresponds to $F(1, 0) = (-1, r)$ and $F(0, 1) = (0, 1)$. In such a case,

$$F(\omega_3) = \langle F(-1, r), F(0, -1)\rangle = \langle (1, 0), (0, -1)\rangle = \omega_4.$$

Hence, $F \in \mathrm{GL}_2(\mathbb{Z})$ such that $F(1, 0) = (-1, r)$ and $F(0, 1) = (0, 1)$ is an automorphism of Δ and $F^2 = \mathrm{id}$.

If $F(\omega_1) = \omega_3$, the first case is $F(1, 0) = (-1, r)$ and $F(0, 1) = (0, -1)$. However, in such a case $F(-1, r) = (1, -2r)$ is not a ray of Δ. In the other case, $F(1, 0) = (0, -1)$ and $F(0, 1) = (-1, r)$, we have $F(-1, r) = (-r, 1+r^2)$ which is not a ray of Δ.

If $F(\omega_1) = \omega_4$, the first case is $F(1, 0) = (0, -1)$ and $F(0, 1) = (1, 0)$. However, in such a case $F(-1, r) = (r, 1)$ is not a ray of Δ. In the other case, $F(1,0) = (1,0)$ and $F(0,1) = (0,-1)$, we have $F(-1,r) = (-1,-r)$ which is not a ray of Δ.

Finally, the group of automorphims of Δ is isomorphic to C_2 and is generated by

$$F = \begin{pmatrix} -1 & 0 \\ r & 1 \end{pmatrix}.$$

□

Proposition 10.2.2. *Let $k \subset L \subset \bar{k}$ be a quadratic extension with Galois group $\Gamma := \mathrm{Gal}(L/k)$. There are two non isomorphic toric k-forms of the Hirzebruch surface $\mathbb{F}_r$.*

Proof. By [Hur11, Theorem 1.25], the fan Δ has to be Γ-stable. By Proposition 10.2.1, the group of automorphisms of Δ is generated by

$$F = \begin{pmatrix} -1 & 0 \\ r & 1 \end{pmatrix}.$$

Then, there are two possibilities. On the one hand, Γ acts trivially over Δ and the corresponding k-form is the Hirzebruch surface over k. On the other hand, if $\gamma \in \Gamma$ is the non trivial element, then $F_\gamma = F$. Let $\mathbb{F}'_r$ be the respective k-form of $\mathbb{F}_r$ and T' be the respective k-form of $\mathbb{G}^2_{\mathrm{m},L}$. The respective Γ-action on the torus is the following

$$\begin{aligned} \mathbb{G}^2_{\mathrm{m},L} &\to \mathbb{G}^2_{\mathrm{m},L}, \\ (t_1,t_2) &\mapsto (\gamma(1)^{-1}\gamma(t_2)^r, \gamma(t_2)). \end{aligned}$$

Then,

$$(\mathbb{G}^2_{\mathrm{m},L})^\Gamma := \{(t_1,t_2) \in \mathbb{G}^2_{\mathrm{m},L} \mid t_1\gamma(t_1) = t_2^r \quad \text{and} \quad t_2 = \gamma(t_2)\}.$$

Notice that $S = \{(t_1,1) \in \mathbb{G}^2_{\mathrm{m},L} \mid t_1\gamma(t_1) = 1\} \leq (\mathbb{G}^2_{\mathrm{m},L})^\Gamma$. This implies that

$$\mathrm{R}^1_{L/k}(\mathbb{G}_{\mathrm{m},L}) \leq T'.$$

In particular, $\mathbb{F}'_r$ is an equivariant compactification of the Weil restriction $\mathrm{R}_{L/k}(\mathbb{G}_{\mathrm{m},L})$. □

Proposition 10.2.3. *Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let $k \subset L \subset \bar{k}$ be a finite Galois extension with Galois group $\Gamma := \mathrm{Gal}(L/k)$ such that $|\Gamma|$ is odd, then the Hirzebruch surface $\mathbb{F}_r$ over L has a unique toric k-form, the trivial one.*

Proof. By [Hur11, Theorem 1.25], the fan Δ has to be Γ-stable. By Proposition 10.2.1, the group of automorphisms of Δ is C_2. Then, there exists a morphism of groups $\Gamma \to C_2$. Given that $|\Gamma|$ is odd, the only group homomorphism is the trivial one. Finally, $\mathbb{F}_r$ has a unique k-form, the trivial one. □

Let $k \subset L \subset \bar{k}$ be a finite Galois extension with Galois group Γ. Let $\mathbb{F}_r$ be the Hirzebruch surface over L with torus T' and $T \leq T'$ be a one dimensional subtorus. Then, T acts effectively over $\mathbb{F}_r$. By Theorem 7.3.1, $\mathbb{F}_r$ arises from a divisorial fan $(\mathfrak{S}, Y)$ over a complete curve.

Proposition 10.2.4. *Let T' be a split algebraic torus and X be a T'-toric variety. Let $T \leq T'$ be a codimension 1 subtorus and $(\mathfrak{S}, Y)$ be a divisorial fan such that $X \cong X(\mathfrak{S})$ as T-varieties and Y is a smooth projective curve, then $Y \cong \mathbb{P}^1$.*

Proof. Let $t \in T'$ be an element such that $[t] \in T'/T$ is not trivial. This element defines a T-equivariant automorphism $t : X \to X$ and, by Proposition 10.1.3, there exists an isomorphism $\psi_t : Y \to Y$. Such morphisms satisfy $\psi_{tt'} = \psi_t \psi_{t'}$. Then we have an action of T'/T over Y. This action is effective, because if $\psi_t = \mathrm{id}_Y$ we have that $t \in T$ by Theorem 10.1.1. Given that Y is a smooth projective curve with an effective action of T'/T, we have that $Y \cong \mathbb{P}^1$. □

This result is a bit more general.

Proposition 10.2.5. *Let k be a field of characteristic zero and L/k be a finite Galois extension with Galois group Γ. Let T be an algebraic torus over k that splits over L. Let X_L be a complexity one projective normal T_L-variety and $(\mathfrak{S}_L, Y_L)$ be a divisorial fan over a smooth complete curve Y_L such that $X_L \cong X(\mathfrak{S}_L)$ as T_L-varieties. Let $N_{T_L}(\mathfrak{S}_L) \leq \mathrm{SAut}^0(X_L)$ be the subgroup of the normalizer of T_L in $\mathrm{SAut}^0(X_L)$ whose elements induce actions on $\mathfrak{S}_L$, then there is a group homomorphism*

$$N_{T_L}(\mathfrak{S}_L) \to \mathrm{SAut}(Y_L)$$

whose kernel contains T_L.

Proof. For each $g \in N_{T_L}(\mathfrak{S}_L)$, there exists a semilinear isomorphism $\psi_g : Y \to Y$ by Proposition 10.1.3. These morphisms satisfy $\psi_{gg'} = \psi_g \psi_{g'}$, for every g and g' in $N_{T_L}(\mathfrak{S}_L)$. Then, we have a group homomorphism $\psi : N_{T_L}(\mathfrak{S}_L) \to \mathrm{SAut}(Y)$. Given that for every element $t \in T \leq N_{T_L}(\mathfrak{S}_L)$, we have that $\psi_t = \mathrm{id}_Y$. This proves the assertion. □

From Proposition 10.2.4, there exists a divisorial fan $(\mathfrak{S}, \mathbb{P}^1_L)$ for the Hirzebruch surface $\mathbb{F}_r$. By Proposition 10.1.3, the semilinear equivariant action of Γ over $\mathbb{F}_r$ induces a semilinear equivariant action of Γ over $\mathbb{P}^1_L$.

By [Bea10, Proposition 1.1], we know that $\mathrm{PGL}_2(L)$ contains S_4 if and only if -1 is a sum of two squares in L.

Proposition 10.2.6. *Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let $k \subset L \subset \bar{k}$ be a finite Galois extension with Galois group $\Gamma \cong S_4$. If -1 is a sum of two squares in k, then $\mathbb{P}^1_L$ has a non toric k-form.*

Proof. Given that -1 is a sum of two squares in k, by [Bea10, Proposition 1.1], we have that $S_4 \in \mathrm{PGL}_2(k)$. Then, $\mathrm{Hom}(\Gamma, \mathrm{PGL}_2(k))$ has an injective morphism $\alpha : \Gamma \to \mathrm{PGL}_2(k)$. There is an injective map

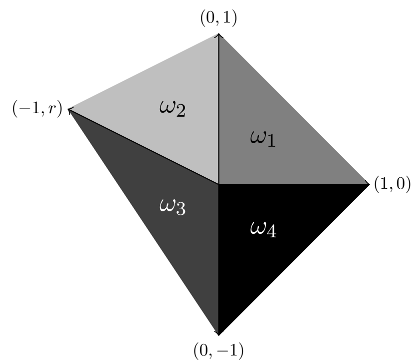
$$\mathrm{Hom}(\Gamma, \mathrm{PGL}_2(k))/\sim \hookrightarrow H^1(\Gamma, \mathrm{PGL}_2(L)) \ ,$$

where $\sim$ denotes the relation given by conjugacy. Let $[\alpha]$ be the corresponding class of α under this map. This implies that there exists a semilinear action of Γ over $\mathbb{P}^1_L$ and, therefore, a group homomorphism $\Gamma \to \mathrm{Aut}(\Sigma)$, where Σ is the fan of $\mathbb{P}^1_L$. However, given that $\mathrm{Aut}(\Sigma) \cong \mathbb{Z}/2\mathbb{Z}$, this implies that there is no semilinear action of Γ over $\mathbb{P}^1_L$ that is compatible with its toric structure. Thus, the assertion holds. $\square$

A variety over L, admitting a toric structure, can have a k-form with a nontoric structure but as a T-variety.

Proposition 10.2.7. *Let k be a field of characteristic zero and $\bar{k}$ be an algebraic closure. Let $k \subset L \subset \bar{k}$ be a finite Galois extension with Galois group $\Gamma \cong S_4$. If -1 is a sum of two squares in k, then the Hirzebruch surface $\mathbb{F}_r$ over L has a k-form as a normal T-variety that is nontoric.*

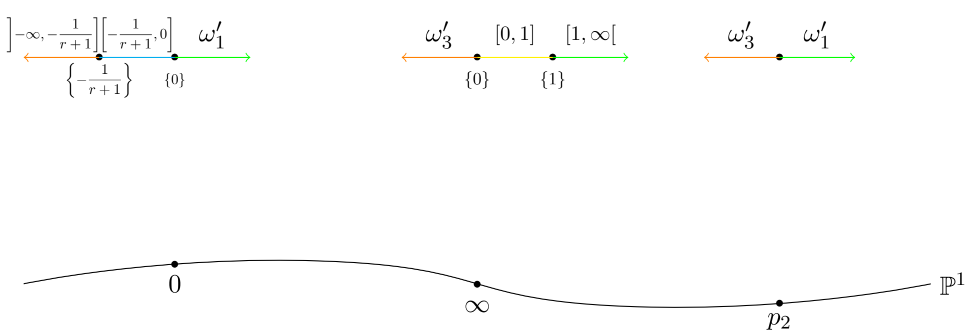
Proof. Let us consider the Hirzebruch surface $\mathbb{F}_r$ and let us consider the $\mathbb{G}_{\mathrm{m},L}$-structure given by the diagonal inclusion $\mathbb{G}_{\mathrm{m},L} \to \mathbb{G}^2_{\mathrm{m},L}$. As a toric variety, $\mathbb{F}_r$ is encoded by the following fan Σ:



The open subvarieties associated to the cones in Σ are $\mathbb{G}_{\mathrm{m},L}$-stable. Thus, $\{X_\omega \mid \omega \in \Sigma\}$ is a $\mathbb{G}_{\mathrm{m},L}$-stable affine open covering of $\mathbb{F}_r$. The respective divisorial fan $(\mathfrak{S}_L, \mathbb{P}^1_L)$ associated to that covering is generated by the pp-divisors:

- $\mathfrak{D}_{\omega_1} := [1, \infty[\otimes \{\infty\} + \omega'_1 \otimes \{0\}$,
- $\mathfrak{D}_{\omega_2} := [0,1] \otimes \{\infty\} + \emptyset \otimes \{0\}$,
- $\mathfrak{D}_{\omega_3} := \omega'_3 \otimes \{\infty\} + \left]-\infty, -\dfrac{1}{r+1}\right] \otimes \{0\}$ and
- $\mathfrak{D}_{\omega_4} := \emptyset \otimes \{\infty\} + \left[-\dfrac{1}{r+1}, 0\right] \otimes \{0\}$,

where $\omega'_1 := \mathrm{Tail}(\mathfrak{D}_{\omega_1})$ and $\omega'_3 := \mathrm{Tail}(\mathfrak{D}_{\omega_3})$. The divisorial fan can be put together in the following figure.



Let $g : \Gamma \to \mathrm{SAut}(\mathbb{G}_{\mathrm{m},L}; \mathbb{F}_r) \subset \mathrm{SAut}_{\mathrm{gp}}(\mathbb{G}_{\mathrm{m},L}) \times \mathrm{SAut}(\mathbb{F}_r)$ be a Γ-semilinear equivariant action. Assume that g is as in Proposition 10.2.6. For every $\gamma \in \Gamma$, $g(\gamma) = (\varphi_\gamma, f_\gamma)$.

The divisorial fan $(\mathfrak{S}_L, \mathbb{P}^1_L)$ defines the same open covering of $\mathbb{F}_r$ that the one defined by the fan Σ. This implies that the divisorial fan is not Γ-stable. However, in contrast to toric varieties, we can construct a Γ-stable divisorial fan from $(\mathfrak{S}_L, \mathbb{P}^1_L)$. Denote

$$\mathcal{U} := \left\{ \bigcap_{\mathfrak{D}\in I, \gamma\in J} f_\gamma(X(\mathfrak{D})) \,\middle|\, I \subset \mathfrak{S}_L \text{ and } J \subset \Gamma \right\}.$$

The set $\mathcal{U}$ is a $\mathbb{G}_{\mathrm{m},L}$-stable affine open covering of $\mathbb{F}_r$. Thus, the Γ-stable divisorial fan induced by $(\mathfrak{S}_L, \mathbb{P}^1_L)$ is the divisorial fan $(\mathfrak{S}'_L, \mathbb{P}^1_L)$ obtained from $\mathcal{U}$ (cf. Corollary 7.3.13). Notice that the divisorial fan $(\mathfrak{S}'_L, \mathbb{P}^1_L)$ has 96 maximal pp-divisors, 24 for each maximal pp-divisor of $(\mathfrak{S}_L, \mathbb{P}^1_L)$. Given that $\mathbb{F}_r$ is projective, for every $\mathfrak{D} \in \mathfrak{S}'_L$, the orbit $\mathfrak{S}(\mathfrak{D}, \Gamma)$ is quasi-projective. Then, by Theorem 10.1.11, $\mathbb{F}_r$ has a k-form Z as T-variety, where T is an algebraic torus over k such that $T_L \cong \mathbb{G}_{\mathrm{m},L}$.

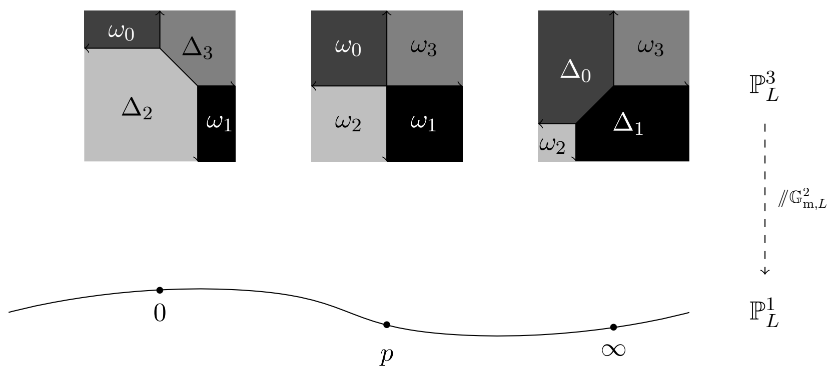
Let us suppose that Z is toric for some $T \leq T'$. This implies that $\mathbb{F}_r \cong Z_L$ has a T'_L-stable divisorial fan

The Γ-semilinear equivariant action $g : \Gamma \to \mathrm{SAut}(\mathbb{G}_{\mathrm{m},L}; \mathbb{F}_r)$ induces a Γ-semilinear action $\psi_g : \Gamma \to \mathrm{SAut}(\mathbb{P}^1_L)$, by Theorem 10.1.1. Then, by Proposition 10.2.6, this Galois semilinear action corresponds to a nontoric k-form of $\mathbb{P}^1_L$. ☐

Projective space $\mathbb{P}^3$ Let k be a field of characteristic zero and L/k be a quadratic extension with Galois group Γ. Denote by $\gamma \in \Gamma$ the nontrivia element of the Galois group. Let us consider $\mathbb{P}^3_L$ with the action of $\mathbb{G}_{\mathrm{m},L}$ given by

$$(\lambda, \mu) \cdot [x_0 : x_1 : x_2 : x_3] = [\lambda x_0 : \mu x_1 : \lambda\mu x_2 : x_3],$$

as Example 7.3.16 where all the pp-divisors are defined over the quotient $\mathbb{P}^3_L \dashrightarrow \mathbb{P}^1_L$ given by $[x_0 : x_1 : x_2 : x_3] \mapsto [x_0x_1 : x_2x_3]$. Recall the divisorial fan is as follows



Let us consider the following Galois semilinear equivariant action given by

$$\varphi_\gamma(\lambda,\mu) = (\gamma(\mu),\gamma(\lambda)) \text{ and } f_\gamma([x_0 : x_1 : x_2 : x_3]) = [\gamma(x_1) : \gamma(x_0) : \gamma(x_2) : \gamma(x_3)].$$

In terms of divisorial fans this action is given by automorphism of the divisorial fan given $g_\gamma := (\psi_\gamma, F, \mathfrak{f})$, where $\psi_\gamma([z : w]) = [\gamma(z) : \gamma(w)]$, $F(a,b) = (-b,-a)$ and $\mathfrak{f} = \imath$. Notice that $g_\gamma \mathfrak{D}_0 = \mathfrak{D}_1$ and $g_\gamma \mathfrak{D}_2 = \mathfrak{D}_3$. This combinatorial datum encodes an action of $\mathrm{Res}_{L/k}(\mathbb{G}_{\mathrm{m},k})$ over $\mathbb{P}^3_k$.

Part III

Further questions

Chapter 11

Further questions

11.1 Counting elliptic curves on the jacobian of a generalized Fermat curve of type $(n, 6)$

By Proposition 3.1.3, we have that the jacobian of a generalized Fermat curve of type $(n, 6)$ have the following decomposition

$$J(X_{(n,6)}) \sim \bigoplus_{\substack{d \in \{2,3,6\} \\ H \in \mathcal{H}(d)}} \pi_H^*(J(X_{(n,6)}/H)).$$

Thus, this decomposition can be studied by *blocks*. These blocks are

$$B(d) := \bigoplus_{H \in \mathcal{H}(d)} \pi_H^*(J(X_{(n,6)}/H)),$$

for $d \in \{2, 3, 6\}$.

Recall that $E_{(n,6)} \cong (\mathbb{Z}/6\mathbb{Z})^n$. Let $\{\sigma_0, \ldots, \sigma_n\} \in E_{(n,6)}$ be a set of generators such that $\sigma_0 = \sigma_1^{-1} \cdots \sigma_n^{-1}$ and each of the σ_i fixes a point of degree 6. Denote by $E_2 \leq E_{(n,6)}$ the subgroup generated by $\{\sigma_0^3, \ldots, \sigma_n^3\}$ and, similarly, by $E_3 \leq E_{(n,6)}$ the subgroup generated by $\{\sigma_0^2, \ldots, \sigma_n^2\}$.

Blocks with $d \in \{2, 3\}$. Notice that if $H \in \mathcal{H}(2)$, then $\sigma_i^3 \in H$ for every $i \in \{0, \ldots, n\}$. Thus, for every $H \in \mathcal{H}(2)$ we have that $E_2 \leq H$. Hence, the quotient $X_{(n,6)}/H \cong (X_{(n,6)}/E_2)/(H/E_2)$ and $X_{(n,6)}/E_2$ is a generalized Fermat curve of type $(n, 2)$ with generalized Fermat group $E_{(n,6)}/E_2$. This implies that the factors appearing in the block $B(2)$ are factors appearing in the decomposition of the jacobian variety of a generalized Fermat curve of type $(n, 2)$. It is worth mentioning that these jacobians are completely decomposable for $n \geq 6$.

The previous argument is completely symmetric with respect to $d = 3$. In such a case, the block $B(3)$ is a sum of factors appearing in the decomposition of the jacobian variety of a generalized Fermat curve of type $(n, 3)$ and we know that the jacobian of a generalized Fermat curve of type $(3, 3)$ is completely decomposable.

Block with $d = 6$. A factor of this block satisfies one of the following conditions: $H \cap S \neq \emptyset$ or $H \cap S = \emptyset$. If we are in the first condition, we have to determine whether such a factor is isogenous to a product of elliptic curves. In the latter condition, denote $T := H \cap S$, the problem is reduced to the previous case by considering the group $H/\langle T \rangle$ on a generalized Fermat curve of type $(n - |T|, 6)$.

Let $X_{(3,6)}$ be generalized Fermat curve of type $(3, 6)$. The factors appearing in blocks $B(2)$ and $B(3)$ are isogenous to elliptic curves. **Are the factors appearing in $B(6)$ isogenous to products of elliptic curves?**

A positive answer to this question could lead to a completely decomposable jacobian of a curve of genus 145 by Proposition 2.3.1. Moreover, this would open the question whether the jacobian of a generalized Fermat curve of type $(4, 6)$ is completely decomposable, which is a curve of genus 1405.

11.2 The jacobians of Klein surfaces

The Torelli principle states that two smooth projective complex curves having isomorphic jacobian varieties, they are isomorphic as curves. There are different generalizations for higher dimensional varieties, for example the albanese variety or the intermediate jacobians. In [GALMVL24], the authors look for a Torelli principle for *Klein hypersurfaces*. A Klein hypersurface of degree d and dimension n is a variety given by

$$X := \{x_0^{d-1}x_1 + x_1^{d-1}x_2 + \cdots + x_n^{d-1}x_{n+1} + x_{n+1}^{d-1}x_0 = 0\} \subset \mathbb{P}^{n+1}.$$

The authors prove that there is a Torelli principle for some of these hypersurfaces and compute the full group of automorphisms of these varieties. Besides, it is also know which of those intermediate jacobians are simple as abelian varieties. In particular, it is known when a Klein curve of degree d, i.e. a Klein hypersuface of dimension $n = 1$, has a simple jacobian variety. Notice that a Klein curve of degree d is of genus $g = \frac{d-1}{2}$. Then, the set of genera of Klein curves is unbounded. **Is the set of genera of**

Klein curves having a completely decomposable jacobian variety unbounded? If the answer to this question is negative, it is worth to answer about **an upper bound**. Recall that the biggest example is a curve of genus 1279. Otherwise stated, **What is the biggest posible genus for Klein curve having a completely decomposable jacobian variety?**

11.3 Projective normal T-varieties

Let k be a field of characteristic zero. The covariant functor $X : \mathfrak{PPDiv}(k) \to \mathcal{E}(k)$, where $\mathcal{E}(k)$ stands for the category of normal affine varieties endowed with an effective split torus action and dominant equivariant morphisms ([AH06] and Proposition 8.3.2). This functor is faithful and essentially surjective, but not full. Given a normal affine variety X endowed with an effective action of an algebraic torus T, a pp-divisor $\mathfrak{D}$ over a semiprojective variety Y yielding X can be constructed using Geometric Invariant Theory (GIT) [MFK94]. The pp-divisors constructed using GIT tools are called minimal. Minimal pp-divisors have the following property: for every minimal pp-divisor $\mathfrak{D}$, the functor $X : \mathfrak{PPDiv}(k) \to \mathcal{E}(k)$ induces an isomorphism between $\mathrm{Aut}(\mathfrak{D})$ and $\mathrm{Aut}_{T_{\mathfrak{D}}}(X(\mathfrak{D}))$, the group of equivariant automorphisms of $X(\mathfrak{D})$, where $T_{\mathfrak{D}}$ is the algebraic torus acting on $X(\mathfrak{D})$ that encodes the pp-divisor $\mathfrak{D}$ (Corollary 8.4.6).

Nonaffine normal varieties with an effective action of an algebraic torus over k arise from a *divisorial fan* $(\mathfrak{S}, Y)$. Some of the pp-divisors in a divisorial fan are not minimal. The existence of nonminimal pp-divisors in a divisorial fan $(\mathfrak{S}, Y)$ makes it difficult to understand the group of equivariant automorphisms of $X(\mathfrak{S})$, the normal variety associated to the fan, in terms of automorphisms of the divisorial fan. **This problem is a consequence of the nonfullness of the functor $X : \mathfrak{PPDiv}(k) \to \mathcal{E}(k)$ and carries some difficulties in the description of normal varieties endowed with an effective action of an algebraic torus over nonalgebraically closed fields**.

For normal affine varieties endowed with a nonsplit algebraic torus, the language of semilinear morphisms is used to encode the Galois descent data. Let L/k be a finite Galois extension with Galois group Γ. In order to study the non-split case, the following two larger categories are considered: $\mathfrak{PPDiv}(L/k)$, whose objects are the objects of $\mathfrak{PPDiv}(L)$ and whose morphisms are semilinear morphisms of pp-divisors, and $\mathcal{E}(L/k)$, whose objects are the objects of $\mathcal{E}(L)$ and whose morphisms are the dominant semilinear equivariant ones. **There is a faithful covariant functor**

$X : \mathfrak{PPDiv}(L/k) \to \mathcal{E}(L/k)$, **which is not full**. Minimal pp-divisors also exist, and they have the following property: for every minimal pp-divisor $\mathfrak{D}$, the functor $X : \mathfrak{PPDiv}(L/k) \to \mathcal{E}(L/k)$ induces an isomorphism between $\mathrm{SAut}(\mathfrak{D})$ and $\mathrm{SAut}_{T_{\mathfrak{D}}}(X(\mathfrak{D}))$, the group of semilinear equivariant automorphisms of $X(\mathfrak{D})$. This property allows us to translate the Galois descent data for minimal pp-divisors, which exist for every X in $\mathcal{E}(L/k)$. Then, if we denote by $\mathfrak{PPDiv}(\Gamma)$ the category of pairs $(\mathfrak{D}, g)$, where $\mathfrak{D}$ is a minimal pp-divisor and $g : \Gamma \to \mathrm{SAut}(\mathfrak{D})$ is a Galois semilinear action, there is a covariant functor $X : \mathfrak{PPDiv}(\Gamma) \to \mathcal{E}(k, L)$, where $\mathcal{E}(k, L)$ stands for the category of normal affine varieties endowed with an effective action of an algebraic torus over k that splits over L. This functor is actually an equivalence of categories. **However, if we consider a category of pairs $(\mathfrak{D}, g)$, with $\mathfrak{D}$ a non-minimal pp-divisor, then the respective functor is no longer an equivalence of categories. In order to obtain an equivalence of categories, a "*localization*" process must be constructed.**

There is no notion of *minimal divisorial fans*, and the pp-divisors in a divisorial fan are not necessarily minimal; thus, the study of normal varieties with an effective action of a non-split torus can be hard to manage in practical terms. The Galois descent data is given in terms of morphisms of pp-divisors in a "*localization*" of $\mathfrak{PPDiv}(L/k)$. When Y is a curve, this problem does not arise because Y can be considered as a smooth projective curve, and the Galois descent data can be given in terms of a Galois semilinear action over the divisorial fan. For higher dimensions, we do not have such a description since there are several models of smooth projective varieties having the same field of functions. Thus, **How can we choose, canonically, the normal semiprojective variety Y?** For projective varieties, it might be possible to give a positive answer. Any projective normal variety endowed with an effective action of an algebraic torus T has a T-linearized ample line bundle. Thus, **the GIT tools could be applied in this context to construct a canonical model and, therefore, provide a nice description of projective normal varieties endowed with an effective action of a non-split algebraic torus.**

11.4 Infinitesimal Subgroups of the Cremona Group

Let k be a perfect field of characteristic $p > 0$. The Cremona group of $\mathbb{P}^n_k$ is the group of birational transformations of $\mathbb{P}^n_k$, denoted by $\mathrm{Cr}_n := \mathrm{Bir}(\mathbb{P}^n_k)$. We denote by μ_p the infinitesimal algebraic group $\mathrm{Spec}(k[x]/(x^p))$, which appears as the kernel of the Frobenius endomorphism $\mathbb{G}_{\mathrm{m},k} \to \mathbb{G}_{\mathrm{m},k}$, $t \mapsto t^p$.

We aim to **determine the conjugacy classes of μ_p in** Cr_1. By Weil's regularization theorem, combined with recent results of Brion (see [Bri24a]), there exists a projective curve C such that $\mu_p \subset \mathrm{Aut}^0(C)$ in Cr_1, where the morphism $C \to C/\mu_p \simeq \mathbb{P}^1_k$ is a μ_p-torsor over a dense open subset of $\mathbb{P}^1_k$.

Moreover, the group μ_p is naturally contained in $\mathbb{G}_{\mathrm{m},k}$. We can then consider the *twisted product* $X^\# := \mathbb{G}_{\mathrm{m},k} \times^{\mu_p} C$, which is a $\mathbb{G}_{\mathrm{m},k}$-variety of complexity one. Studying a μ_p-normal curve C is equivalent to studying the $\mathbb{G}_{\mathrm{m},k}$-variety $X^\#$ equipped with a morphism $\varphi : X^\# \to \mathbb{G}_{\mathrm{m},k}/\mu_p$, whose fiber over the base point is geometrically integral (see [Bri24b, Remark 4.3]).

Thus, a μ_p-normal curve is described by a divisorial fan and additional data on the torus. This approach, based on the combinatorial description of Altmann-Hausen-Süß, should allow us to determine all conjugacy classes of μ_p in Cr_1. **Once this case is well understood, we can consider extending this method to higher dimensions (Cr_n with $n \geq 2$) or to other infinitesimal groups**.

11.5 Affine normal schemes with torus actions over discrete valuation rings

Let R be a ring and $\mathbb{G}_{\mathrm{m},R}$ be the multiplicative group over R. Let $T := \mathbb{G}^n_{\mathrm{m},R}$ be a *split torus over* R.

The understanding of moduli spaces of geometric objects could lead to the study of schemes over discrete valuation rings. For example: deformation theory. Developing a generalization of Altmann-Hausen theory over these rings might help to understand those kinds of spaces, for instance. Assume that R is a discrete valuation ring. Thus, the aim of this part of the research project is **to seek a generalization of Theorem 7.2.1 for affine normal schemes over R endowed with an effective action of a split algebraic torus over R**. It would be possible to achieve this problem since the work of Seshadri [Ses77] and Schroer [Sch01]. These works would allow us to follow mostly the same techniques applied over fields.

The nonsplit case would be treated as the algebraic case. Tori over rings split over étale extensions, then a classification could be completed by using étale descent. Thus, it could be possible to reach the last part of this problem: **to generalize Theorem 5.0.7 for affine normal schemes over R endowed with an effective action of an algebraic torus over R (not necessarily split)**.

Further question from this objective: There is a even more general version of Theorem 9.8.2, also due to Sumihiro [Sum75].

Theorem 11.5.1. *Let S be normal noetherian scheme and G be a smooth locally diagonalizable group scheme over S with connected fibers. Let X be a normal scheme over S and on which G acts regularly. Then X is covered by G-stable open subscheme which are affine over S.*

Thus, once the affine case have been solved, it would be possible to work over the nonaffine case.

11.6 Nonnormal affine schemes with torus actions over perfect fields

Nonnormal affine toric varieties are encoded in terms of semi-groups and this family is larger than the family of normal ones. For example, one dimensional normal affine toric varieties are just $\mathbb{A}^1_k$ with different actions of $\mathbb{G}_{\mathrm{m},k}$, but it is just one variety. However, there are infinitely many nonisomorphics nonnormal affine toric varieties of dimension one. Hence, the aim would be **to determine a combinatorial description for nonnormal affine T-varieties**.

11.7 Classification of Real Fano Varieties

Mori and Mukai classified smooth complex Fano varieties of dimension three. Among all the families they found, some varieties admit an action of a two-dimensional torus. They can therefore be described by divisorial fans. These were studied by Süß in [S1̈4], where he describes which ones admit a Kähler-Einstein metric.

On the other hand, real Fano varieties appear in the works of Kuznetsov and Prokhorov. For example, in [KP23] and [KP24], the authors studied the rationality of certain families of real Fano varieties arising from Mori and Mukai's classification. Furthermore, Kóllar developed the minimal model theory for real varieties. Another application of real Fano varieties is K-stability, as seen in the recent work of Abban, Cheltsov, Kishimoto, and Mangolte [ACKM24].

Real Fano varieties appear in numerous studies, making their classification desirable. **We seek to determine smooth real Fano varieties of complexity one using the Altmann-Hausen-Süß theory**. Moreover, Shepherd-Barron [SB97] proved that, over fields of positive characteristic, Fano varieties are determined by the same families. Consequently, **we could extend this classification to base fields of positive characteristic via the Altmann-Hausen-Süß theory developed during my doctorate**.

11.8 On the Realization of Algebraic Groups as Automorphism Groups of Projective Varieties

Given a Galois extension, there exists an associated group called the Galois group of the extension. One may ask whether, given a finite group, this group is the Galois group of some extension. This problem has been widely studied and has fascinated many mathematicians.

An equivalent problem exists for algebraic groups: given an algebraic group G, can this group be realized as the automorphism group of some projective variety? This question has been studied by Brion in [Bri14] and by the same author with Schröer [BS22] and Blanc [BB23] in independent works. One of my projects is to work on this problem for semiabelian varieties. Semi-abelian varieties G fit into an exact sequence of algebraic groups:

$$0 \to T \to G \to A \to 0,$$

where T is an algebraic torus and A is an abelian variety.

This question has been resolved for abelian varieties in [Flo23a] and for linear groups in [Flo23b], particularly for algebraic tori. This suggests that these groups should also satisfy the property of being automorphism groups of certain projective varieties. One approach to this problem would be to use the theory developed by Altmann, Hausen, and Süß. For example, if X_A is a projective variety such that $\mathrm{Aut}(X_A) \simeq A$, then the semiabelian variety associated with the algebraic group G is a T-variety, implying that there exists Altmann-Hausen-Süß data where the quotient Y is A.

The idea would then be to use this algebraic-geometric data to construct one on X_A, which would correspond to a normal T-variety and could have G as its automorphism group.

11.9 $\mathbb{G}_{\mathrm{a}}^n$-actions on complexity one T-varieties

Arzhantsev and Romaskevich [AR17] study complete toric varieties that also have an additive variety structure. They prove that every complete toric variety that admits an additive action, i.e. an additive variety structure, then it admits an additive action that is normalized by the torus. The aim of this part of the project is to explore this question for complexity one complete normal T-varieties. **Does a complete complexity one normal T-variety, admitting an additive variety structure, admits an additive variety structure normalized by T?**. The arguments used by Arzhantsev and Romaskevich [AR17] yields over the Theory of Cox Ring of complete toric varieties and Demazure roots. There are some analogue for complexity one T-varieties (see: [HS10] and [AHHL14]).

Once the first problem is solved, a natural question arises: **When two of this normalized additive structures are isomorphics?**. This problem is solved in the toric case in the same paper, so it would be natural to explore it in the complexity one case.

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