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UN PROBLEMA EXTREMAL DE VALORES PROPIOS PARA UN CONDUCTOR DE DOS FASES EN UNA BOLA

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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UN PROBLEMA EXTREMAL DE VALORES PROPIOS PARA UN CONDUCTOR DE DOS FASES EN UNA BOLA.

El tema que trata esta memoria de título es minimizar el primer valor propio de un conductor compuesto por dos materiales homogéneos, que son distribuidos en proporciones fijas dentro de un dominio.

Los trabajos pioneros de F. Murat y L. Tartar [26] muestran que esta clase de problemas del cálculo de variaciones podrían tener existencia de minimizadores sólo en una clase más grande, llamada clase de materiales homogenizados o con micro-estructura, excluyendo a priori distribuciones clásicas de material como soluciones optimales. Para dominios en una dimensión, M. G. Kreĭn [22] probó la existencia de una solución clásica. En dimensiones más altas, cuando el problema se restringe a una bola, A. Alvino, P. L. Trombetti y P. L. Lions [4] probaron que se pueden obtener soluciones clásicas radialmente simétricas. Sin embargo, estos resultados han sido vistos como excepcionales, atribuidos a la completa simetría del dominio. Cox y Lipton [11], sólo estudiaron condiciones para un diseño óptimo del problema asumiendo soluciones homogenizadas. Aún es desconocido si en dominios con simetría parcial es posible o no obtener una solución clásica que respete la simetría del dominio.

Esperamos revivir el interés a esta pregunta dando una nueva prueba del resultado en una bola. Creemos además que, en este caso, distribuir el material de mayor conductividad en el centro es una solución óptima.

En los primeros capítulos se introduce el problema y se hace un resumen crítico del estado del arte en lo que se refiere a la existencia de un minimizador, incluyendo algunas referencias clásicas que plantean la no existencia de solución para problemas similares. Luego se describen las principales herramientas utilizadas en el desarrollo de esta tesis. Se da un énfasis particular a los re-arreglos de funciones. En el capítulo cuarto se describe el problema general y en el quinto un análisis exhaustivo del problema en una dimensión. En el capítulo sexto se desarrolla el caso de una bola N dimensional, otorgando una nueva prueba de la existencia de una solución clásica radialmente simétrica. En el capítulo séptimo se desarrolla el cálculo de la derivada con respecto al dominio del primer valor propio, y en el octavo se muestran experiencias numéricas asociadas al problema, en el caso de un disco en $\mathbb{R}^2$. En el capítulo noveno se genera un análisis del signo de la derivada para el caso de una bola N dimensional, otorgando resultados, con los cuales se espera concluir, en un futuro próximo, que la solución del problema para este tipo de dominios, se encuentra disponiendo el material de más alta conductividad en el centro.

Dedicado a :
Mis padres

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Chapter 1

Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ which is to be called the design region. Let m be a positive number, $0 < m < |\Omega|$, where $|\Omega|$ is the total volume (Lebesgue measure) of the region Ω. Two materials with conductivities α and β $(0 < \alpha < \beta)$ are distributed in arbitrary disjoint measurable subsets A and B, respectively, of Ω so that $A \cup B = \Omega$ and $|B| = m$. For any such distribution the **first eigenvalue** in the spectral problem

$$\begin{cases} -div((\alpha\chi_A + \beta\chi_B)\nabla u) & = & \lambda u & \text{in} & \Omega \\ u & = & 0 & \text{on} & \partial\Omega \end{cases}. \tag{0.1}$$

is obtained by minimizing the Rayleigh quotient (2.47) as below

$$\lambda(B) := \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega (\alpha\chi_A + \beta\chi_B) |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx} \tag{0.2}$$

Let $\mathcal{A} := \{B : B \subset \Omega, B \text{ measurable}, |B| = m\}$ be the class of admissible domains for the material with conductivity β. We are interested in the following eigenvalue minimization problem

$$\inf \{\lambda(B) : B \in \mathcal{A}\}. \tag{\textbf{GP}}$$

In this thesis we mainly study problem (**GP**) when the domain is a ball in $\mathbb{R}^N$. Since the geometry involved has radial symmetry, the adopted strategy consists in using rearrangement techniques. In fact, in this work we give a new simpler proof of an existence result of a radially symmetric admissible domain for the problem (**GP**) in a ball which is originally due to Alvino, Trombetti and Lions [4]. We expect that these results could be generalized to domains with less symmetries, such as squares or stars.

One of the problems of interest is to characterize the solution. We conjecture that the optimal solution consists in placing the material of conductivity β in the center of the ball. A way to prove this conjecture is using the shape derivative of the eigenvalue functional which is explicitly calculated. We later analyze this for certain configurations to substantiate our conjecture. We give further evidence to this conjecture through some numerical results obtained in a disc.

The numerical experiments that we carry out here are mainly in the disc and square in $\mathbb{R}^2$ for greater simplicity.

Introducción

Sea Ω un dominio acotado de $\mathbb{R}^N$ y m un número positivo, $0 < m < |\Omega|$, donde Ω es el volumen total (medida de Lebesgue) de la región Ω. Dos materiales de conductividad α y β $(0 < \alpha < \beta)$ son distribuidos en subconjuntos arbitrarios de Ω disjuntos A y B respectivamente, de tal manera que $A \cup B = \Omega$ y $|B| = m$. Para cada una de estas formas de distribuir los materiales α y β, analicemos el **primer valor propio** del problema espectral

$$\begin{cases} -div((\alpha\chi_A + \beta\chi_B)\nabla u) &= \lambda u \qquad \text{en} \quad \Omega \\ u &= 0 \qquad \text{sobre} \quad \partial\Omega \end{cases}, \tag{0.3}$$

que se expresa mediante el cuociente de Rayleigh, por

$$\lambda_1(B) := \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega (\alpha\chi_A + \beta\chi_B) |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx} \tag{0.4}$$

Sea $\mathcal{A} := \{B : B \subset \Omega, B \text{ medible}, |B| = m\}$ la clase de dominios admisibles asociado a esta configuración. Nos interesa el problema de minimizar el primer valor propio de (0.3), esto es,

$$\inf \{\lambda_1(B) : B \in \mathcal{A}\}. \tag{\textbf{GP}}$$

En esta memoria, nos interesa principalmente estudiar el problema (**GP**) cuando el dominio es una bola en $\mathbb{R}^N$. En virtud de la geometría radial del problema, utilizaremos las técnicas de rearreglos como estrategia para resolverlo. En efecto, en este trabajo se desarrolla una demostración más simple de un resultado de existencia debido a Alvino, Trombetti y Lions [4], el cual dice que, cuando el dominio es una bola, existe una solución radial para (**GP**) en el conjunto de dominios admisibles $\mathcal{A}$. Esperamos que estos resultados sean generalizables a dominios con menos simetrías, como es el caso de cuadrados o estrellas.

Uno de los problemas que nos interesa es el de caracterizar la solución. Se conjetura que la solución óptima consiste en distribuir todo el material de conductividad β en el centro de la bola. Se espera poder potenciar esta conjetura en base a un estudio numérico riguroso que incluye el cálculo de derivada de λ_1 con respecto a la geometría, en conjunto con el análisis del signo de esta derivada, para algunos casos interesantes en $\mathbb{R}^N$.

Por simplicidad, los experimentos numéricos fueron realizados en discos y cuadrados planos, y se buscó confirmar la conjetura antes mencionada, así como describir el comportamiento del primer valor propio en algunos dominios con menos simetrías.

Chapter 2

Background

In this section we show the background research that is currently in the literature concerning the problem (**GP**) or (**GP**).

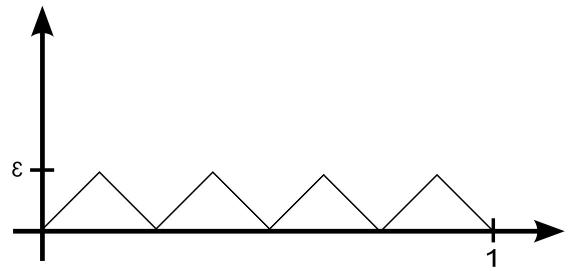
The pioneering work of F. Murat and L. Tartar [26] go a long way showing, in general, some problems of optimal design may not admit solutions if microstructural designs are excluded from consideration. Therefore, assuming, tacitly, that the problem of minimizing the first eigenvalue of a two-phase conducting material with the conducting phases to be distributed in a fixed proportion in a given domain, has no true solution in general domains, S. Cox and R. Lipton only study conditions for an optimal microstructural design [11]. Although, the problem in one dimension has a classical solution (cf. Kreĭn [22]) and, in higher dimensions, the problem set in a ball can be deduced to have a radially symmetric solution (cf. Alvino et. al. [4]), these existence results have been regarded so far as being exceptional owing to complete symmetry.

2.1 A classical example of non existence

Let us consider the following minimization problem: We want to minimize the functional

$$J(x) = \int_0^1 (1 + x^2)(1 + (\dot{x}^2 - 1)^2)dt. \tag{1.1}$$

for $x = x(t)$ *smooth*; $x(0) = x(1) = 0$. Examining the J functional, we observe that $x \equiv 0$ gives the value 2 to the J objective function ($x \equiv 0$ minimizes properly the factor $(1 + x^2)$, but it does not adapt well to the factor with $\dot{x}$). If we consider *broken* functions such as the function in the figure (2.1) we see it is possible to improve initial guess for the minimum



value of J. In fact, in this broken line we have that $\dot{x}$ takes the values 1 and -1 alternately, over sequent subintervals which divide the domain $(0,1)$ in a pair number of subintervals with the same length. If we consider the function x that takes values between 0 and ϵ we have

$$J(x) \leq 1 + \epsilon^2.$$

On the other hand, for all admissible curves $x(\cdot)$, it is clear that $J(x) \geq 1$ and $J(x) = 1$ if and only if $x = 0$ and $\dot{x} = \pm 1$. Then, the infimum is 1, and is not reached by any admissible function and the minimizing sequence can be considered as oscillating functions in subintervals that tend to zero.

2.2 Murat Tartar

The minimizing sequences related to the problem of optimal distribution of two conducting materials with fixed proportions, in general, can develop micro-structures, that is, the value that accomplish the infimum is an homogenized value. We show some examples where it is natural to find a homogenized limit for the minimizing sequence, but finding a classical solution (without homogenization) is not really natural.

2.2.1 First example

Let

$$K = \left\{ a \in L^\infty(\Omega) \;\mid\; 0 < \alpha \leq a(x) \leq \beta \text{ a.e } x \in \Omega \right\}. \tag{2.2}$$

where Ω is an open bounded set in $\mathbb{R}^N$. For each $a \in K$, we set $A = aI$ with I the identity matrix in $\mathbb{R}^N \times \mathbb{R}^N$. Consider the unique solution of

$$\left\{ \begin{array}{rcll} -div(a(x)\nabla u) & = & f & \text{in} \quad \Omega \\ u & = & g & \text{on} \quad \partial\Omega \end{array} \right. . \tag{2.3}$$

where $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$ are given functions. We are interested in find the *best* $a \in K$, in the sense a functional of the form

$$J(a) = \int_\Omega |u - \eta|^2 \, dx$$

is minimum, where $\eta \in L^2(\Omega)$ is also a given function.
The problem

$$\inf_{a \in K} J(a). \tag{2.4}$$

is solve, theoretically, in a simple manner. In fact, let $a^\epsilon \in K$ be a minimizing sequence, namely, such that

$$J(a^\epsilon) \longrightarrow \gamma_0 = \inf_{a \in K} J(a)$$

Denote u^ϵ the solution of (2.3) with $a = a^\epsilon$. It is clear that u^ϵ is bounded in $H^1(\Omega)$ and then, excepting a subsequence,

$$u^\epsilon \longrightarrow u^0 \quad \text{in } H^1(\Omega) \text{ weak, } L^2(\Omega) \text{ strong}$$

and then

$$\int_\Omega |u^\epsilon - \eta|^2 \, dx = J(a^\epsilon) \longrightarrow \int_\Omega \left|u^0 - \eta\right|^2 dx.$$

The question that naturally appears is if u^0 is solution of a problem of the form (2.3), namely, if there is a coefficient $a(x) \in L^\infty(\Omega)$ (or a matrix $A \in L^\infty(\Omega)^{N \times N}$, with $A = a(x)I$) such that

$$\begin{cases} -div(a(x)\nabla u^0) & = & f & \text{in} & \Omega \\ u^0 & = & g & \text{on} & \partial\Omega \end{cases}. \tag{2.5}$$

Usually, we don't have a reliable answer to this question.

The theory of homogenization [26] shows that u^0 is the solution of a more general problem of the form

$$\begin{cases} -div(A^0(x)\nabla u^0) & = & f & \text{in} & \Omega \\ u^0 & = & g & \text{on} & \partial\Omega \end{cases}.. \tag{2.6}$$

but where $A^0(x)$ is a matrix which lives in a different set than K. Due to Spagnolo theorem (3.5.3), A^0 lives in $[\alpha;\beta]_s$, so there is no indication that A^0 is necessarily a scalar matrix. Furthermore, there are examples []where we don't have the existence of $a^0 \in K$, such that $J(a^0) = \inf_{a \in K} J(a)$, but as can be shown with the H-convergence theory

$$J(A^0) = \inf_{a \in K} J(a).$$

This problem is of those of calculus of variations which don't have an optimum and need to be relaxed in order to have a solution. In this case, a possible relaxation is

$$\inf_{A \in \bar{K}} \tilde{J}(A)$$

where $\bar{K}$ is the adherence of K in $[\alpha;\beta]_s$ with respect to the H-convergence. Unfortunately, it is not easy to find a simple characterization of $\bar{K}$.

2.2.2 Second example

Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set with regular boundary. We search for a set $\Omega_1 \subset \Omega$ such that

$$a(x) = \begin{cases} \alpha & \text{in } \Omega_1 \\ \beta & \text{in } \Omega \backslash \Omega_1 \end{cases}$$

and if u is a solution of the problem

$$\begin{cases} -div(a(x)\nabla u) & = & f & \text{in} & \Omega \\ f & = & 0 & \text{on} & \partial\Omega \end{cases}. \tag{2.7}$$

then u minimizes

$$J(a) = \int_\Omega F(x, u(x))dx.$$

Without smooth properties of Ω_1, it could be impossible to find a solution.

Let a^ϵ be a minimizing sequence. Then $a^\epsilon \in [\alpha;\beta]_s$ and we can redo the analysis given in the first example, showing that in this case

$$a^\epsilon \quad H\text{-converges to } A^0$$

where in general A^0 neither takes only the values α and β nor is a scalar matrix.

2.3 Cox Lipton

Roughly speaking, if c denotes the conductivity of some material, κ the specific heat, and ρ its density, in *Extremal eigenvalue problems for a two phase conductors* Steven Cox and

Robert Lipton [11] were interested in the study some extremal problems like

$$\inf_{c,\varrho=\kappa\rho\in ad_\gamma} \lambda_k(c,\varrho) \tag{3.8}$$

Where $\lambda = \lambda_k(c,\varrho)$ is the k^{th} eigenvalue of the problem

$$\begin{cases} -div(c\nabla v) &= \lambda\varrho v \quad \text{in} \quad \Omega \\ v &= 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

and ad_γ is an admissible set of conductivities and capacities.

In great detail, they define the admissible set

$$ad_\gamma = \left\{\chi \in L^\infty(\Omega) \ \mid \ \chi(x) \in \{0,1\} \text{ a.e., } \int_\Omega \chi dx = \gamma\right\}$$

and if χ_ω denotes the indicator function of the subset $\omega \subset \Omega$, they define the capacity

$$\varrho_+(\chi_\omega) = \varrho_2 + \chi_\omega(\varrho_1 - \varrho_2)$$

and the conductivity of the material by

$$c_+(\chi_\omega) = c_2 + \chi_\omega(c_1 - c_2)$$

In the section 2 of their article [11], they established the existence of solutions of the relax formulation of (3.8) associated to the parameters $c_+(\chi)$ and $\varrho_+(\chi)$. They relaxed the problem in the sense that they considered a bigger admissible set which is defined by the weak-$*$ limits in ad_γ regarding to $L^\infty(\Omega)$,

$$\theta \in ad_\gamma^* = \left\{\vartheta \in L^\infty(\Omega) \ \mid \ 0 \leq \vartheta(x) \leq 1 \text{ a.e., } \int_\Omega \vartheta dx = \gamma\right\}$$

and they allow c to range over the class of matrix valued functions M_θ given in [26, Proposition 10]. This M_θ set is defined in the following way. For a matrix $A^* \in [\alpha;\beta]_s$ we consider for every $x \in \mathbb{R}^N$ the eigenvalues $(\mu_1(x),\ldots,\mu_N(x))$ of the matrix $A^*(x)$ and we define the values

$$\begin{cases} \mu_+(\theta) &= \theta\alpha + (1-\theta)\beta \\ \mu_-(\theta) &= \dfrac{1}{\theta/\alpha + (1-\theta)/\beta} \end{cases} \tag{3.9}$$

Then $A^* \in M_\theta$ is equivalent to $(\mu_1, \ldots, \mu_N) \in K_\theta$, where K_θ is defined by

$$\begin{cases} \mu_-(\theta) & \leq \mu_j \leq \mu_+(\theta) & j = 1, \ldots, N \\ \sum_{j=1}^{N} \frac{1}{\mu_j - \alpha} & \leq & \dfrac{1}{\mu_-(\theta) - \alpha} + \dfrac{N-1}{\mu_+(\theta) - \alpha} \\ \sum_{j=1}^{N} \frac{1}{\beta - \mu_j} & \leq & \dfrac{1}{\beta - \mu_-(\theta)} + \dfrac{N-1}{\beta - \mu_+(\theta)} \end{cases} \tag{3.10}$$

The new relaxed admissible set is then

$$\mathcal{G} = \left\{ (\theta, c) \;\mid\; \theta \in ad_\gamma^*,\ c \in M_\theta \right\}$$

Cox and Lipton showed, using the H convergence defined in (3.5.2) [26], that actually there exists a relaxed solution, namely

$$\inf_{\chi \in ad_\gamma} \lambda_k(c_+(\chi), \varrho_+(\chi)) = \min_{(\theta,c)\in\mathcal{G}} \lambda_k(c, \varrho_+(\theta))$$

2.4 Kreĭn

The one-dimensional version of the problem **GP** admits a classical solution as shown by Kreĭn [22]. He exploits the equivalence of the original problem and a similar problem for

$$\lambda(B) := \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega (\alpha\chi_A + \beta\chi_B) |u|^2} dx.$$

a vibrating membrane involving the objective functional
In chapter (5) we show in detail this development, giving a complete characterization of the solution.

It is important to remark that the equivalence with the vibrating membrane problem is not hold in higher dimensions. The work of Cox and McLaughlin [12, 13] show that the vibrating problem, in any dimension, has a true solution.

2.5 Alvino Trombetti Lions

A. Alvino, G. Trombetti and P.L.Lions in the paper *On Optimization Problems With Prescribed Rearrangements* review some results concerning functions with prescribed distributions and related optimization problems, giving also new applications of these tools.

Concerning to us, they study a minimization problem of eigenvalues which is very related to our problem. In fact, in (6.1) we reformulate the general problem (**GP**) in terms of the notation given in [4] and with a straightforward application of [4, Corollary 3.2] we get a classical spherical solution for the case when Ω is a ball in $\mathbb{R}^N$. Let us enunciate the results of our interest.

Let $A > 0$ and $1 < p < \infty$. Let $1 \leq q \leq Np/(N-p)$. Let $\varphi \in L^\infty(0, A)$ such that $\varphi = \varphi^*$ (namely, φ is a decreasing rearrangement) such that $\varphi \geq \alpha$ a.e. Let Ω be an open set in $\mathbb{R}^N$ with $|\Omega| = A$ and $\nu \in L^\infty_+(\Omega)$ defined by $\nu^* = \varphi$.

Alvino et. al. consider in [4, Corollary 3.2] the functional $F : W_0^{1,p} \backslash \{0\} \longrightarrow \mathbb{R}$ such that

$$F(u) = \frac{\int_\Omega \nu \, |\nabla u|^p}{\left(\int_\Omega |u|^q\right)^{p/q}}$$

and they were interested in the quantity:

$$\lambda(\nu, \Omega) = \inf_{u \neq 0} F(u) \tag{5.11}$$

In the following propositions B is going to be the N-dimensional centered at the origin ball with measure $|B| = A$. The following corollaries concern our work.

Corollary 2.5.1. *Let Ω be an open set in $\mathbb{R}^N$ with Lebesgue measure $|\Omega| = A$ and let $\nu \in L^\infty_+(\Omega)$. Then, there exists $\tilde{\nu} \in L^\infty_+(\Omega)$ spherical symmetrical such that $\tilde{\nu}^* = \nu^*$ y $\lambda(\tilde{\nu}, B) \leq \lambda(\nu, \Omega)$.*

Corollary 2.5.2. *There exists $\bar{\nu} \in L^\infty_+(B)$ spherical symmetrical such that $\bar{\nu}^* = \varphi$ and $\lambda(\bar{\nu}, B) \leq \lambda(\nu, \Omega)$ for all $\nu \in L^\infty_+(\Omega)$, with $|\Omega| = A$ and $\nu^* = \varphi$.*

In order to prove those corollaries Alvino et. al. needed to prove the following theorem which we have tried to generalize unsuccessfully to other type of domains B with partial symmetries like the case of Steiner symmetries.

Theorem 2.5.3. *Let $f \geq 0$ in $L^p(\Omega)$, Alvino et. al. use that there exists a non-negative maximal solution $u \in W^{1,p}(\Omega)$ if*

$$|Du| \leq f \qquad a.e \ in \ \Omega \ , \quad u \in W_0^{1,p}\Omega)$$

Then they choose $q \in \{1, Np/(N-p)\}$ where $p < N$, they consider the following maximization problem:

$$I(\Omega) = sup\left\{ \|u\|_{L^q(\Omega)} / f \in C(\varphi) \right\}$$

Then, they prove that there exist functions u and f defined in B spherically symmetrical such that $f^* = \varphi$, $I(B) = \|u\|_{L^p(\Omega)}$, u is of the form

$$u(|x|) = \int\limits_{|x|}^{R} f(s)ds$$

and

$$|Du| = f \quad a.e \text{ in } B \ , \quad u \in W^{1,p}(B) \quad u \geq 0 \text{ in } B$$

2.6 Alvino Trombetti

In the paper [3] *A lower bound for the first eigenvalue of an elliptic operator* Alvino and Trombetti proved the following inequality which permits a better understanding of our problem and to give a more meaningful proof of the main result (6.4.5). We remark in (6.5.1) we have given another proof of the same inequality.

Lemma 2.6.1 (Lemma 1.2 [3]). *If $u \in \hat{H}^1(\nu)$, we have*

$$\int_\Omega \nu(x) |\nabla u|^2 dx \geq \int_{\Omega \sharp} \tilde{\nu}(w_N |x|^N) \left|\nabla u^\sharp\right|^2 dx. \tag{6.12}$$

where $\hat{H}^1(\nu)$ denote the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{\hat{H}^1(\nu)} = \left(\int_\Omega \nu(x) |\nabla u|^2\right)^2 dx,$$

$u^\sharp$ is the Schwarz rearrangement of u, w_N is the volume of the unitary ball in $\mathbb{R}^N$ and $\tilde{\nu}$ is a radially symmetric function defined through the relation,

$$\int\limits_{\{u \geq c\}^\sharp} \frac{1}{\tilde{\nu}(x)} dx = \int\limits_{\{u \geq c\}} \frac{1}{\nu(x)} dx \quad \forall \ c \in \mathbb{R}. \tag{6.13}$$

2.7 Continuity of eigenvalues

Our problem (**GP**) or (**GP**) relates to continuity properties of the first eigenvalue depending of a specific parameter, in this case, where the β material is placed. In the following section we give a large background of problems concerning continuity properties of different classes of eigenvalues and parameters. This summary was made by Henrot in his book *Extremum* *problems for Eigenvalues Of Elliptic Operators* [16]. We just rewrite some of the propositions written there.

In order to prove existence of minimizers or maximizers for eigenvalues or functions of eigenvalues, we obviously need continuity of eigenvalues with respect to the variable. We give some references of eigenvalues depending:

- either on the domain
- or on the coefficients of the operator.

The latter is simpler and classical. The former is less classical and it is related with the so-called $\gamma-$convergence. Let us start with a classical result of eigenvalues of operators.

Theorem 2.7.1. *Let T_1 and T_2 be two self-adjoint, compact and positive operators on a separable Hilbert space H. Let $\mu_k(T)$ and $\mu_k(T_2)$ be their k-th respective eigenvalues. Then*

$$|\mu_k(T_1) - \mu_k(T_2)| \leq \|T_1 - T_2\| := \sup_{f \in H} \frac{\|(T_1 - T_2)(f)\|}{\|f\|}. \tag{7.14}$$

An immediate consequence of Theorem (2.7.1) is that strong convergence of operators implies convergence of eigenvalues. We are now going to see that, in our particular context, thanks to compactness properties of embeddings $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, actually simple convergence of resolvant operators implies convergence of eigenvalues.

We are concerned with Dirichlet boundary conditions. If L is any elliptic operator given by (7.16) but considering constant coefficients, we denote by A_L (or A_L^Ω when we want to emphasize dependence on the domain Ω) its resolvant operator, namely the operator from $L^2(\Omega)$ into $L^2(\Omega)$ such that $A_L(f)$ is the solution of the Dirichlet problem $u \in H_0^1(\Omega)$, $Lu = f$. When we consider a sequence of domains Ω_n included in a fixed domain D, we decide to extend the operators A_L^Ω to $L^2(D)$ by setting

$$\begin{array}{rccc} A_L^\Omega : & L^2(\Omega) & \longrightarrow & L^2(\Omega) \\ & f & \longmapsto & \tilde{u} \end{array}, \tag{7.15}$$

where $u \in H_0^1(\Omega)$ is the solution of $Lu = f$ and $\tilde{u}$ is its extension by zero outside Ω. For sake of simplicity, we go on denoting by u the extension (instead of $\tilde{u}$).

Theorem 2.7.2. *Let A_n, A be a sequence of resolvant operators from $L^2(D)$ to $L^2(D)$, corresponding to a sequence of uniformly elliptic operators with Dirichlet boundary conditions. We assume that, for every $f \in L^2(D)$, $A_n(f)$ converges to $A(f)$ in $L^2(D)$. Then A_n converges* *to A strongly (i.e for the operator norm). In particular, the eigenvalues of A_n converge to the corresponding eigenvalues of A.*

Continuity with variable coefficients

We can now state the continuity result for eigenvalues when the coefficients on the elliptic operator vary. We consider a sequence of elliptic operators L_n defined by:

$$L_n u := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a^n_{i,j}(x)\frac{\partial u}{\partial x_j}\right) + a^n_0(x)u. \tag{7.16}$$

where the bounded functions $a^n_{i,j}$ are assumed to satisfy an uniformly ellipticity condition, namely, the positive ellipticity constant can be chosen independently of n.

Theorem 2.7.3. *Let L_n be a sequence of uniformly elliptic operators defined on an open set D by (7.16). We assume that for fixed i, j, the sequence $a^n_{i,j}$ is bounded in L^∞ and converge almost everywhere to a function $a_{i,j}$; we also assume that the sequence a^n_0 is bounded in L^∞ and converges weakly-$*$ in L^∞ to a function a_0. Let L be the (elliptic) operator defined on D as in (7.16) by the functions $a_{i,j}$ and a_0. Then each eigenvalue of L_n converges to the corresponding eigenvalue of L.*

In one dimension, we can prove the same continuity result with weaker assumptions on the convergence of the $a_{i,j} = a_{11} = \sigma(x)$. Actually weak-$*$ convergence of the inverse is enough in this case:

Theorem 2.7.4. *Let $\Omega = (0, L)$, $0 < \alpha \leq \beta$, and $\sigma_n(x)$ be a sequence of functions satisfying $\alpha \leq \sigma_n(x) \leq \beta$. We denote by $\lambda_k(\sigma)$ the eigenvalues of the operator $-\frac{\mathrm{d}}{\mathrm{d}x}\left(\sigma(x)\frac{\mathrm{d}}{\mathrm{d}x}\right)$. Then, if $1/\sigma_n$ converges weak-$*$ in $L^\infty(\Omega)$ to $1/\sigma$, each eigenvalue $\lambda_k(\sigma_n)$ converges to $\lambda_k(\sigma)$ and the corresponding eigenfunctions converge weakly in $H^1(\Omega)$ and strongly in L^Ω.*

Remark 2.7.5. From the min Rayleigh formulae, we see that $\sigma \longmapsto \lambda_k(\sigma)$ is upper-semi continuous for the weak-$*$ convergence (as infimum of continuous functions), but the previous theorem shows that it is not continuous in general.

Continuity with variable domains (Dirichlet case)

Definition 2.7.6 (γ-convergence)**.** Let D be a fixed ball, $\Omega_n \subseteq D$ a sequence or open set. We say Ω_n γ-converges to Ω (and we write $\Omega_n \xrightarrow{\gamma} \Omega$) if, for every $f \in L^2(\Omega)$, the solution

$u^f_{\Omega_n}$ of the Dirichlet problem for the Laplacian

$$\begin{cases} Lu &= f \quad \text{in} \quad \Omega_n \\ u &= 0 \quad \text{on} \quad \partial\Omega_n \end{cases}$$

conver ges (strongly) in $L^2(D)$ to u^f_Ω, the solution on Ω (as usual, every function in $H^1_0(\Omega)$ is extended by zero outside Ω_n).

In other words, using the notation (7.15) , $\Omega_n \xrightarrow{\gamma} \Omega$, if $\forall\, f \in L^2(\Omega)$, $A^D_\triangle(\Omega_n)(f) \longrightarrow A^D_\triangle(\Omega)(f)$ in $L^2(\Omega)$. We gather in the following theorem different characterization of the γ-convergence. See the references in [16].

Theorem 2.7.7. *The following properties are equivalent.*

(i) Ω_n *γ-converges to* Ω.

(ii) ***Sverak:*** $A^D_\triangle(\Omega_n)(1) \longrightarrow A^D_\triangle(\Omega)(1)$ *(i.e the convergence takes place for* $f = 1$*).*

(iii) ***Mosco Convergence.*** $H^1_0(\Omega_n)$ *converges in the sense of Mosco to* $H^1_0(\Omega)$ *i.e*

- *(M1) For every* $v \in H^1_0(\Omega)$*, there exists a sequence* $v_n \in H^1_0(\Omega_n)$ *such that* $v_n \longrightarrow v$ *(strong convergence in* $H^1_0(D)$*).*
- *(M2) For every sub-sequence* v_{n_k} *of functions in* $H^1_0(\Omega_{n_k})$ *which converges weakly to a function* $v \in H^1_0(D)$*, then* $v \in H^1_0(\Omega)$.

(iv) ***Distance to*** $\mathbf{H^1_0}$***:*** $\forall\, \varphi \in H^1_0(D)$, $d(\varphi, H^1_0(\Omega)) = \lim_{n\longrightarrow+\infty} d(\varphi, H^1_0(\Omega_n))$ *(where* $d(\varphi, X)$ *denotes, as usual, the distance of* φ *to the convex set* X*).*

(v) ***Projection on*** $\mathbf{H^1_0}$***:*** $\forall\, \varphi \in H^1_0(D)$, $proj_{H^1_0(\Omega)}(\varphi) = \lim_{n\longrightarrow+\infty} proj_{H^1_0(\Omega_n)}\varphi$ *(where* $proj_X(\varphi)$ *denotes the projection of* φ *on the convex set* X*).*

(vi) $\boldsymbol{\Gamma}$***-Convergence:*** J_{Ω_n} *Γ-converges to* J_Ω *where, for any open set* $\omega \subset D$, J_ω *is defined by*

$$J_\omega = \frac{1}{2}\int_D |\nabla v(x)|^2\, dx - \int_D f v(x) dx + \begin{cases} 0 & \text{if } v \in H^1_0(\omega) \\ +\infty & \text{else} \end{cases}$$

and the Γ*-convergence means:*

- *(G1)* $\forall\, v_n \longrightarrow v \quad J_\Omega(v) \leq \operatorname{liminf} J_{\Omega_n}(v_n)$.
- *(G2)* $\exists\, v_n \longrightarrow v \quad J_\Omega(v) \geq \operatorname{limsup} J_{\Omega_n}(v_n)$.

(vii) ***(Strong) Convergence resolvant operators:*** $\left\| A^D_\triangle(\Omega_n) - A^D_\triangle \right\|(\Omega) \longrightarrow 0.$

Applying Theorem (2.7.2), we have:

Corollary 2.7.8. *If any of the above items (i)-(vii) is true, then* $\lambda(\Omega_n) \longrightarrow \lambda(\Omega)$.

Chapter 3

Tools

In this chapter we show the main mathematical tools we use in this work. We give some references and definitions for classical theorems, some of them with their proof. Also, we explain in a more detailed fashion the results that are not really known.

In the section (3.1) we give detailed information about symmetrization through rearrangements, giving the according definition in different dimensions. We also give the proof of basic theorems such as equimeasurability (3.1.21) and Hardy-Littlewood Inequality (3.1.44) and some other rearrangements results. Due to the complexity of the proof, for the important Pólya and Szegö theorem (3.1.45) we just give the definition.

In the sections (3.2), (3.3), (3.4), we mainly show classic results of PDE, measure, and convex analysis theory. We wrote this propositions in order to make a self contained book. In (3.5) we give the definition of H-convergence and we declare the Spagnolo theorem. In (3.6) we give some theorems written by Simon in [31] but without a very strict description of their hypothesis.

3.1 Rearrangements

Symmetrization is a tool that can be useful in some minimization problems, in which it is expected that the possible minimizer, if exists, has some kind of symmetry.

Roughly speaking, a rearrangement operation maps a measurable function into a new one that is distributed in a similar way enjoying some additional symmetry properties. This kind of property is also called symmetrization [19].

The common feature of all rearrangements is that if a given function f is transformed into a new function f^*, we want f^* has some desired symmetry property. This is done by a *rearrangement* of the level sets of f. Therefore the rearrangement of a function f is closely tied to the rearrangement of the level-set of the function f.

Notation 3.1.1. Given a Lebesgue measurable set D we write its rearrangement as D^*.

The symbol D^* will slightly vary its definition according to the required type of symmetrization. We remark the space where it is defined is not mentioned, so nothing prevents the rearrangement set lies in a different vector space. Nevertheless, D^* has common properties regardless the type of the definition of rearrangement:

R 1) If $D = \phi$, then $D^* = \phi$.

R 2) If $D \neq \phi$, then D^* is a Lebesgue measurable set having the same Lebesgue measure than D.

R 3) If $D_1 \subseteq D_2$, then

$$D_1^* \subseteq D_2^*. \tag{1.1}$$

Definition 3.1.2. Let Ω be a bounded domain in $\mathbb{R}^N$ and c a real number. Consider a function $f : \Omega \longrightarrow \mathbb{R}$. We define the level set $\Omega_{f,c}$ of f as

$$\Omega_{f,c} = \{f > c\} = \{x \in \Omega \ \mid \ f(x) > c\}\,. \tag{1.2}$$

The next step is defining the rearrangement of functions. From this moment on the supreme of a function will always be understood as the essential supreme with the Lebesgue measure.

Definition 3.1.3. Given the function $f : \Omega \longrightarrow \mathbb{R}$, we define its rearrangement as the function $f^* : \Omega^* \longrightarrow \mathbb{R}$ given by:

$$f^*(x) = \sup\left\{c \ \mid \ x \in \Omega^*_{f,c}\right\}. \tag{1.3}$$

There are several types of rearrangements in the literature [19] depending of the wanted symmetries and the number of variables of the rearrangement. In this document we want to focus in the main properties of the two most well-known, the unidimensional and the Schwarz symmetrization rearrangement.

Remark 3.1.4. In the literature [9, 19, 27] related to this research the level sets sometimes are defined slightly different, with different type of inequalities. In [9] we define the level set with the non strict inequality,

$$\Omega_{f,c} = \{x \in \Omega \ \mid \ f(x) \geq c\}.$$

Thus, the rearrangement f^* defined in (1.3) could give different values. This two definitions with the strict and non-strict sign coincide as we see in the proposition (3.1.41).

Now we define the specific kind of rearrangement we are going to use herein.

Definition 3.1.5 (Schwarz symmetrization of sets)**.** We define the **Schwarz symmetrization** $\Omega^\sharp \subseteq \mathbb{R}^N$ of the set $\Omega \subseteq \mathbb{R}^N$ as the ball centered at the origin which has the same Lebesgue measure of Ω.

Definition 3.1.6 (Decreasing symmetrization of sets)**.** We define the **decreasing rearrangement** $\Omega^* \subseteq \mathbb{R}$ of the set $\Omega \subseteq \mathbb{R}^N$ as the interval $[0, |\Omega|)$, where $|\Omega|$ denotes the Lebesgue measure of the set Ω.

Definition 3.1.7 (Schwarz symmetrization)**.** Let $f : \Omega \longrightarrow \mathbb{R}$ be a measurable function. The **Schwarz rearrangement** of f is the function $f^\sharp : \Omega^\sharp \longrightarrow \mathbb{R}$ defined as:

$$f^\sharp(x) = \sup\left\{c \ \mid \ x \in \Omega^\sharp_{f,c}\right\}. \tag{1.4}$$

Definition 3.1.8 (Decreasing symmetrization)**.** Let $f : \Omega \longrightarrow \mathbb{R}$ be a measurable function. The **(unidimensional) decreasing rearrangement** of f is the function $f^* : \Omega^* \longrightarrow \mathbb{R}$ defined as:

$$f^*(x) = \sup\left\{c \ \mid \ x \in \Omega^*_{f,c}\right\}. \tag{1.5}$$

Notation 3.1.9. From now on if A is a set in some $\mathbb{R}^d$, $|A|$ will denote its Lebesgue measure.

It will be important to know some characterizations and relations between the earlier definitions. In this perspective we present the following propositions.

Proposition 3.1.10. *Let $\Omega \subseteq \mathbb{R}^N$ be bounded and let $f : \Omega \longrightarrow \mathbb{R}$ be a measurable function. Then the* ***(unidimensional) decreasing rearrangement*** *of f, is given by*

$$f^*(s) = \begin{cases} \sup\limits_{x\in\Omega} f(x) & \text{if } s = 0 \\ \sup\{t \ \mid \ |\{f > t\}| > s\} & \text{if } s > 0 \end{cases} \tag{1.6}$$

Proof. The point 0 always is in any set rearrangement. So the result follows when $s = 0$. If $s > 0$ we have:

$$\begin{aligned} f^*(s) &= \sup\{c \ | \ s \in \Omega_c^*\} \\ &= \sup\{c \ | \ s \in (0, |\{f > c\}|)\} \\ &= \sup\{c \ | \ |\{f > c\}| > s \ , \ s > 0\} \\ &= \sup\{c \ | \ |\{f > c\}| > s\} \end{aligned}$$

□

From now on w_N will denote the Lebesgue measure of the unitary ball in $\mathbb{R}^N$. By simple computations it can be seen that the Lebesgue measure of the ball with R radio is given by

$$|B(0, R)| = R^N w_N.$$

We give a direct relation between the decreasing and the Schwarz symmetrization.

Proposition 3.1.11. *Let $f : \Omega \longrightarrow \mathbb{R}$, then:*

$$f^\sharp(x) = f^*(w_N |x|^N). \tag{1.7}$$

Proof.

$$\begin{aligned} f^\sharp(x) &= \sup\{c \ | \ x \in \Omega_c^*\} \\ &= \sup\{c \ | \ x \in B(0, R_c)\} \\ &= \sup\{c \ | \ |x| < R_c \ , \ w_N R_c = |\Omega_c|\} \\ &= \sup\left\{c \ | \ w_N |x|^N < w_N R_c^N = |\Omega_c|\right\} \\ &= f^*(w_N |x|^N) \end{aligned}$$

□

We are going to define a function that is very related to the rearrangement of a function. This function can be understood like the inverse of the rearrangement.

Definition 3.1.12. Given the function $f : \Omega \longrightarrow \mathbb{R}$, we define its **distribution function** as

$$\mu_f(t) = |\{f > t\}| \qquad \text{for } t \geq 0 \tag{1.8}$$

where $\{f > t\} = \{x \in \Omega : f(x) > t\}$ and $|\cdot|$ is the Lebesgue measure.

Remark 3.1.13. It is easy to see the function μ_f is decreasing and $\mu_f(t) = 0$ for $t \geq \sup(f)$ and $\mu_f(t) = |\Omega|$ for $t \geq \inf f$. Hence, the range of the distribution function $\mu_f(t)$ is the interval $[0, |\Omega|]$.

3.1.1 Properties of the decreasing rearrangement

In the following pages we are going to see several properties of the decreasing rearrangement we suggest to see more carefully. There are several classical results, some of them are explained in detail and some others are simply mentioned, as the Polya-Szego theorem (3.1.45). Some of the proofs were taken from the Kesavan book [21]. Furthermore, we give some results ad-hoc to our problem we have proved -or we just have rewritten- studying the proofs given in [21].

Proposition 3.1.14. *Let $f : \Omega \longrightarrow \mathbb{R}$ be a function in $L^\infty(\Omega)$. Then f^* is a decreasing function.*

Proof. Let $s_1 < s_2$. We have

$$\{f > s_2\} \subset \{f > s_1\}.$$

Then, if there exists some $t > 0$ such that $|\{f > s_2\}| > t$, we will have $|\{f > s_1\}| > t$. Hence $\{t : |\{f > s_2\}|\} \subset \{t : |\{f > s_1\}|\}$. Taking the supreme we get

$$f^*(s_2) \leq f^*(s_1).$$

□

Proposition 3.1.15. *Let $f : \Omega \longrightarrow \mathbb{R}$ be a function in $L^\infty(\Omega)$. Then*

$$f^* \in L^\infty([0, |\Omega|]) \tag{1.9}$$

Proof. We have:

- $f^*(0) = \sup\{t \mid |\{f > t\}| > 0\} = \sup_\Omega f$,
- $f^*(|\Omega|) = \sup\{t \mid |\{f > t\}| > |\Omega|\} = \inf_\Omega f$.

Since $f \in L^\infty(\Omega)$ and f^* is decreasing it follows that $f^* \in L^\infty(\Omega^*)$, i.e. $f^* \in L^\infty([0, |\Omega|])$. □

Proposition 3.1.16. *Let $f : \Omega \longrightarrow \mathbb{R}$ be a function in $L^\infty(\Omega)$. Then f^* is a right continuous function.*

Proof. Let $\bar{s} \in [0, \Omega]$ and let $\epsilon > 0$. Because f^* is a decreasing function, we must show there exists $h > 0$ such that

$$\forall\, s \in [\bar{s}, \bar{s}+h] \quad f^*(\bar{s}) - f^*(\bar{s}+h) \leq \epsilon.$$

By definition of f^*, there exists $t > 0$ such that $t + \epsilon > f^*(\bar{s})$, $f^*(\bar{s}) > t$ and $\mu_f(t) = |\{f > t\}| > \bar{s}$.

Choosing $h > 0$ such that $|\{f > t\}| > \bar{s}+h$, we have $|\{f > t\}| > s \quad \forall\, s \in [\bar{s}, \bar{s}+h]$ then it follows $f^*(s) \geq t \quad \forall\, s \in [\bar{s}, \bar{s}+h]$, therefore

$$f^*(\bar{s}) - f^*(s) \leq (t+\epsilon) - t = \epsilon.$$

□

Proposition 3.1.17. *$T : f \longmapsto f^*$ is a increasing map, i.e. if $f \leq g$, where f and g are real valued functions on Ω, then $f^* \leq g^*$ [21].*

Proof. Let $f \leq g$ be two real valued functions. Because $\{f > t\} \subset \{g > t\}$, then if $|\{f > t\}| > s$ it follows $|\{g > t\}| > s$. Therefor $\{t \mid |\{f > t\}| > s\} \subset \{t \mid |\{g > t\}| > s\}$. Taking supreme over t we obtain $f^*(s) \leq g^*(s)$. □

Proposition 3.1.18. *Let $f : \Omega \longrightarrow \mathbb{R}$ be a function in $L^\infty(\Omega)$. Let $s = |\{f^* > t\}|$. Then*

$$f^*(s) \leq t \quad \textit{and} \quad f^*(s+\epsilon) < t \quad \forall\, \epsilon > 0. \tag{1.10}$$

Proof. Let $s = |\{f^* > t\}|$. Since f^* is a decreasing function and $f^*(0) = \sup f$, it follows that $\{f^* > t\}$ is an interval in the form $[0, a)$ or $[0, a]$. Hence $a = s$ and $\forall\, \epsilon > 0$ we have

$$f^*(s-\epsilon) \geq t \quad \text{y} \quad f^*(s+\epsilon) \leq t.$$

Using right continuity it follows $f^*(s) \leq t$.

Suppose $f^*(s+\epsilon) < t \quad \forall\, \epsilon > 0$ was not true. Thus, there would exist $\epsilon > 0$ such that $f^*(s+\epsilon) = t$ and since $f^*(s) \leq t$ and f^* is decreasing, we would have $f^*(s+x) = t \quad \forall\, x \in [0, \epsilon]$. Then $|\{f^* > t\}| \geq |\{[0, s+\epsilon]\}| > s$ which is a contradiction. □

The following property shows the inverse relation between the distribution function and the rearrangement of a function.

Proposition 3.1.19. *Let $f : \Omega \longrightarrow \mathbb{R}$ be a function in $L^\infty(\Omega)$. Let $s = |\{f^* > t\}|$ Then*

$$f^*(s) = t. \tag{1.11}$$

Proof. We know from the previous proposition (3.1.18) that

$$f^*(s) \leq s.$$

Let suppose by contradiction that $f^*(s) < t$.

By definition of f^*, we would have $|\{f > t\}| \leq s$. The inequality satisfies strictly because if it holds $|\{f > t\}| = s$, by definition of f^* we could take a sequence $t_n \nearrow t$ such that $|\{f > t_n\}| > s$, which would lead to $f^*(s) \geq t_n \quad \forall\, n$. But the former implies $f^*(s) \geq t$, which is a contradiction. Therefor, supposing $f^*(s) < t$ (our main contradiction hypothesis), we have $|\{f > t\}| < s$.

Now let take $\epsilon > 0$ such that $|\{f > t\}| < s - \epsilon$. It follows that $f^*(s - \epsilon) < t$. If we suppose that $f^*(s-\epsilon) \geq t$, we could take a sequence $t_n \nearrow t$ such that $|\{f > t_n\}| > s - \epsilon$ and since $\{f > t_n\} \searrow \{f > t\}$, by measure continuity (and the finite Lebesgue measure of Ω) the former would lead to $|\{f > t\}| \geq s$. Therefor, supposing $f^*(s) < t$ (our main contradiction hypothesis), we have $f^*(s-\epsilon) < t$. But this is a contradiction because $f^*(s-\delta) \geq t \quad \forall\, \delta > 0$. Thus, our main contradiction hypothesis is wrong, so we concluded that $f^*(s) = t$. ☐

Definition 3.1.20 (Equimeasurable)**.** Two real valued functions (with possibly different domains of definition) are said to be **equimeasurable** if they have the same distribution function. Equimeasurable functions are said to be **rearrangements** of each other (see [21]).

Theorem 3.1.21 (Equimeasurability)**.** *The functions $f : \Omega \longrightarrow \mathbb{R}$ and $f^* : \Omega^* \longrightarrow \mathbb{R}$ are equimeasurable, ie. for all t,*

$$|\{f > t\}| = |\{f^* > t\}| \tag{1.12}$$

Proof. Let $s = |\{f^* > t\}|$. We know from the proposition (3.1.19) that $f^*(s) = t$. It is easy to see that $|\{f > t\}| \geq s$. In fact, taking a sequence $\epsilon \searrow 0$, by definition of f^* it follows $|\{f > t - \epsilon\}| > s$. Since $\{f > t - \epsilon\} \searrow \{f > t\}$, by measure continuity (and the finite measure of Ω) we have $|\{f > t\}| \geq s$.

Because $f^*(s + \epsilon) < t \quad \forall\, \epsilon$, we have $|\{f > t\}| \leq s + \epsilon \quad \forall\, \epsilon$. Therefore,

$$|\{f > t\}| = s = |\{f^* > t\}|.$$

☐

Corollary 3.1.22. *Let $f : \Omega \longrightarrow \mathbb{R}$. We have*

$$\left.\begin{array}{rcl} |\{f > t\}| & = & |\{f^* > t\}| \\ |\{f \geq t\}| & = & |\{f^* \geq t\}| \\ |\{f < t\}| & = & |\{f^* < t\}| \\ |\{f \leq t\}| & = & |\{f^* \leq t\}| \end{array}\right\} \tag{1.13}$$

Proof. The first relation has already been proved. The rest follow easily by complementation and suitable limiting arguments. □

Remark 3.1.23. From the proposition (3.1.19) and the equimeasurability we have $f^*(\mu(t)) = t \quad \forall\, t$

We can extend the equimeasurability (3.1.21) property to the L^p *sense*. The following propositions leads in this direction.

Corollary 3.1.24. *Let $f \in L^p(\Omega)$ with $1 \leq p \leq \infty$. If $f \geq 0$ then $f^* \in L^p([0, |\Omega|])$ and*

$$\|f\|_{p,\Omega} = \|f^*\|_{p,(0,|\Omega|)} \tag{1.14}$$

Proof. If $p = \infty$, then by definition $\|f^*\|_{p,\Omega} = \sup\limits_{s\in[0,|\Omega|]} \{|f^*(s)|\} = f^*(0) = \sup\limits_{t\geq 0} \{\mu_f(t) > 0\} = \|f\|_\infty$.

Let $1 \leq p < \infty$. By equimeasurability, both f and f^* have the same distribution function and as both functions are positive, we can use the *cake slide* theorem (3.3.1) to obtain:

$$\begin{array}{rcl} \|f\|^p_{p,\Omega} & = & \displaystyle\int\limits_\Omega f^p(x)dx \\ & = & \|f^*\|_{p,(0,|\Omega|)}\,. \end{array}$$

□

This result is also true without the non-negativity condition. In fact, as a consequence of the equimeasurability (3.1.21), we have the following general powerful result.

Theorem 3.1.25. *Let $f : \Omega \longrightarrow \mathbb{R}$ be measurable. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a positive Borel measurable function. Then*

$$\int\limits_\Omega F(f(x))dx = \int\limits_0^{|\Omega|} F(f^*(s))ds \tag{1.15}$$

Proof. Let $E = [t, \infty)$ and set $F(\xi) = \chi_E(\xi)$ where χ_E is the characteristic function of E. Then

$$\int_\Omega F(f(x))dx = |\{f > t\}| = |\{f^* > t\}| = \int_0^{|\Omega|} F(f^*(s))ds$$

Similarly, the result holds for $F = E$ where E is any interval and hence if E in any Borel set, by standard arguments. If F is any non-negative Borel function, it can be expressed as the limit of an increasing sequence $\{F_n\}$ of non-negative simple functions F. Thus, for each n we have

$$\int_\Omega F_n(f(x))dx = \int_0^{|\Omega|} F_n(f^*(s))ds.$$

We can pass to the limit as $n \longrightarrow \infty$ to get (1.15) using the monotone convergence theorem. □

Corollary 3.1.26. *Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a Borel function and let $f : \Omega \longrightarrow \mathbb{R}$ be such that $F(f) = F \circ f \in L^1(\Omega)$. Then $F(f^*) \in L^1((0, |\Omega|))$ and (1.15) is still valid.*

Proof. We write $F = F^+ - F^-$ and both F^+ and F^- are non-negative Borel functions and so (1.15) holds for each of then in place. If $F(f) \in L^1(\Omega)$, then both $\int_\Omega F^+(f(x))dx$ and $\int_\Omega F^-(f(x))dx$ are finite and we can subtract the relation for F^- form that of F^+ to get (1.15). □

Corollary 3.1.27. *Let $f \in L^p(\Omega)$ for $1 \leq p \leq \infty$. Then $f^* \in L^p((0, |\Omega|))$ and the corresponding L^p norms are equal.*

Proof. If $p \neq \infty$ take $F(t) = |t|^p$ in the preceding theorem. If $p = \infty$ use corollary (3.1.24) for the function $|f|$. □

Remark 3.1.28. Since the proofs of Theorem (3.1.25) and its consequences depended only on the equimeasurability, these results also hold for other types of rearrangements which have the equimeasurable property.

Now we prove another important property of the decreasing rearrangement which is a consequence of Theorem 3.1.21.

Lemma 3.1.29. *Let $l > 0$ and $f : [0, l] \longrightarrow \mathbb{R}$ be a decreasing function. Then $f = f^*$ a.e.*

Proof. Let $s \in [0, l]$. Since f is a decreasing function we have

$$|\{f > f(s)\}| \geq s.$$

Hence, by definition of f^*

$$f^*(s) \geq f(s).$$

Let now $s \in [0, l]$ and t such that $|\{f > t\}| \geq s$. By equimeasurability (3.1.21), this holds for $t = f^*(s)$. Since f is decreasing we have

$$f(x) > f^*(s) \quad \forall\, x \in [0, s),$$

particularly for $x = s - h$, with h small enough. Since f is monotone, there are at most a numerable set of discontinuity points, so they have null Lebesgue measure , therefore

$$f(s) \geq f^*(s) \quad a.s.$$

$\square$

Corollary 3.1.30. *Let $f : \Omega \longrightarrow \mathbb{R}$ and $g : \tilde{\Omega} \longrightarrow \mathbb{R}$ be equimeasurable functions with $\Omega \subseteq \mathbb{R}^N$, $\tilde{\Omega} \subseteq R^M$ and $|\Omega| = |\tilde{\Omega}|$. Then*

$$f^* = g^* \tag{1.16}$$

Proof. From equimeasurability we have f^* and g^* are decreasing real equimeasurable valued functions. Since $f^* = (f^*)^*$ the result follows from the previous lemma (3.1.29) applied to the rearrangements f^* and g^*. $\square$

Proposition 3.1.31. *Let v , w : $[0, l] \longrightarrow \mathbb{R}$ be two equimeasurable decreasing functions. Then $v = w$ a.s.*

Proof. By the preceding lemma (3.1.29) $v = v^*$ y $w = w^*$ a.s. Since they are equimeasurable

$$|\{v^* > t\}| = |\{w^* > t\}|.$$

From the corollary (3.1.30) we obtain $v^* = w^*$. $\square$

The following lemma is very useful. It says an increasing function goes through the rearrangement.

Lemma 3.1.32. *Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function. Consider $f : \Omega \longrightarrow \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is a bounded set. Then*

$$\psi(f^*) = (\psi(f))^* \quad a.s\ . \tag{1.17}$$

Proof. Due to the previous lemmas and propositions it is enough to verify $\psi(f^*)$ and $(\psi(f))^*$ are increasing equimeasurable functions in $[0, |\Omega|]$. Since the decreasing rearrangement map and ψ preserve the order, the increasing property follows straightforward.

Let $F(y) = \chi_{(t,\infty)} \circ \psi(y) = \chi_{\{\psi(f(y)>t)\}}$, which is a positive and measurable function. Hence

$$|\{\psi(f^*) > t\}| = \int_0^{|\Omega|} \chi_{\{\psi(f^*)>t\}}(s)ds,$$

by the equi-integrability theorem (3.1.25)

$$\int_0^{|\Omega|} \chi_{\{\psi(f^*)>t\}}(s)ds = \int_\Omega \chi_{\{\psi(f)>t\}}(x)dx = |\{\psi(f) > t\}|,$$

and by the equimeasurability property (3.1.21)

$$|\{\psi(f) > t\}| = |\{(\psi(f))^* > t\}|,$$

concluding that $\psi(f^*)$ and $(\psi(f))^*$ are equimeasurable. □

3.1.2 Some other properties of rearrangements

Here we want to give some other properties of the rearrangements of non general knowledge which are ad hoc for our work.

Proposition 3.1.33. *Let $c : \Omega \longrightarrow \mathbb{R}$ be a constant function in Ω (let say c), then*

$$c^*(s) = c \quad \forall\ s \in [0, |\Omega|]. \tag{1.18}$$

Proof. For $s \in [0, |\Omega|]$

$$c^*(s) = \sup\{t \ \ | \quad |\{c > t\}| > s\} = c$$

□

The following property will be very important in chapter 6, one the main chapters of this thesis.

Proposition 3.1.34. *If $\Omega = A \cup B$, $A \cap B = \phi$ and $\alpha < \beta$, then*

$$(\alpha\chi_A + \beta\chi_B)^*(s) = \beta\chi_{[0,|B|)} + \alpha\chi_{[|B|,|\Omega|)}. \tag{1.19}$$

Reciprocally, if $f^* = \beta\chi_{[0,b)} + \alpha\chi_{[b,|\Omega|)}$*, then*

$$f = \alpha\chi_A + \beta\chi_B. \tag{1.20}$$

with $A \cup B = \Omega$*,* $A \cap B = \phi$ *and* $|B| = b$*.*

Proof. From equimeasurability, since

$$|\{f \geq \beta\}| = |\{f = \beta\}| = |\{f^* \geq \beta\}|$$

and

$$|\{f > \beta\}| = 0 = |\{f^* > \beta\}|,$$

we have

$$|\{f = \beta\}| = |\{f^* = \beta\}|$$

Doing a similar analysis for the level set $\{f \leq \alpha\}$ we obtain

$$|\{f = \alpha\}| = |\{f^* = \alpha\}|$$

Since f has only 2 different levels of values, using the equalities showed above we deduce that f^* also has only two different values, namely f^* is a step function with the values of f and preserving the measure of the places where is α and β. □

Sometimes it will be useful to symmetrize but considering an increasing function.

Definition 3.1.35. Let $\Omega \subset \mathbb{R}^n$ be a bounded set and $f : \Omega \longrightarrow \mathbb{R}$ a bounded measurable function. The **(unidimensional) increasing rearrangement** of f is the function $f_* : [0, |\Omega|] \longrightarrow \mathbb{R}$ defined by:

$$f_*(s) = \begin{cases} \sup(f) & \text{if s } = |\Omega| \\ \inf\{t \ \mid \ s \in \{f < t\}^*\} & \text{if s } < |\Omega|. \end{cases} \tag{1.21}$$

It is not so difficult to prove similar properties we have stayed for the decreasing rearrangement. The following lemma resumes the properties of the increasing rearrangement. We recommend to see [21, 19] for further information.

Lemma 3.1.36. *Given a function* $f : \Omega \longrightarrow \mathbb{R}$*, the increasing rearrangement* f_* *is a increasing function and is equimeasurable with the function* f*.*

In the next lines, we give two properties that relates the decreasing with the increasing rearrangement.

Proposition 3.1.37.

$$f^* = -\left(-f\right)_* \quad a.s. \tag{1.22}$$

Proof. We have

$$\begin{aligned}
\left|\{f^* > t\}\right| &= \left|\{f > t\}\right| \\
&= \left|\{-f < -t\}\right| \\
&= \left|\{\left(-f\right)_* < -t\}\right| \\
&= \left|\{-\left(-f\right)_* > t\}\right|.
\end{aligned}$$

Then both f^* and $-\left(-f\right)_*$ are decreasing and equimeasurable functions, due to the proposition (3.1.31) it follows they are equals a.s. ☐

Proposition 3.1.38.

$$\left(f_*\right)^* = f^* \quad a.s. \tag{1.23}$$

Proof. We have

$$\begin{aligned}
\left|\{\left(f_*\right)^* > t\}\right| &= \left|\{f_* > t\}\right| \\
&= \left|\{-\left(-f\right)^* > t\}\right| \\
&= \left|\{\left(-f\right)^* < -t\}\right| \\
&= \left|\{-\left(-f\right)^* > t\}\right| \\
&= \left|\{-f < -t\}\right| \\
&= \left|\{f > t\}\right| \\
&= \left|\{f^* > t\}\right|
\end{aligned}$$

Then both f^* and $\left(f_*\right)^*$ are decreasing and equimeasurable functions, due to the proposition (3.1.31) it follows they are equals a.s. ☐

The following proposition will be demanding in the important proposition (6.3.1) which is a kind of generalization of the Pólya Szegö inequality (3.1.45).

Proposition 3.1.39. *Let $\Omega \subseteq \mathbb{R}^N$ be bounded and let $f : \Omega \longrightarrow \mathbb{R}$ be an integrable function. Let $E \subseteq \Omega$ be a measurable subset. Then*

$$\int_E f(x)dx \leq \int_0^{|E|} f^*(s)ds \tag{1.24}$$

and equality holds if, and only if,

$$(f|_E)^* = f^*|_{[0,|E|]} \ a.e.$$

Proof. Here we rewrite the proof given in [21, Proposition 1.2.2]. Let $g = f|_E$. If $s \in [0, |E|]$ and if $|\{f > t\}| < s$, then

$$|\{g > t\}| = |\{f > t\} \cap E| < s.$$

Thus

$$\{t \ | \ |\{f > t\}| < s\} \subseteq \{t \ | \ |\{g > t\}| < s\}$$

and so $g^*(s) < f^*(s)$. Thus

$$\int_E f(x)dx = \int_0^{|E|} g^*(s)ds \leq \int_0^{|E|} f^*(s), \tag{1.25}$$

which proves (1.24). If equality holds in (1.24), then we have equality throughout in (1.25) and this is possible if, and only if, $g^* = f^*$ a.e. in E the result is proved. □

Proposition 3.1.40. *Let $\Omega \subseteq \mathbb{R}^N$ be bounded and let $f : \Omega \longrightarrow \mathbb{R}$ be an integrable function. Then*

$$\int_0^r f^* = \max\left\{\int_A f \ \middle| \ A \textit{ is a Borelian Set }, A \subset \Omega \ , \ |A| = r\right\}. \tag{1.26}$$

Proof. From the previous proposition (3.1.39)

$$\int_A f \leq \int_0^r f^*.$$

Thus,

$$\max\left\{\int_A f \ \middle| \ A \text{ is a Borelian set, } A \subset \Omega \ , \ |A| = r\right\} \leq \int_0^r f^*$$

$$|A| = |\{f^* > r\}| = r.$$

Define

$$g = \begin{cases} f & \text{if } f > f^*(r) \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\int_{\Omega} g = \int_{A} f,$$
$$\int_{\Omega} g = \int_{0}^{r} g^*$$

and

$$g^* = (g|_A)^* = f^*|_{[o,r]}.$$

□

The next result shows in the definition (3.1.3) of function rearrangements we can consider level sets with an $\geq$ sign instead of the strict inequality.

Lemma 3.1.41. *Let $f : \Omega \longrightarrow \mathbb{R}$. Let $s \in [0, |s|]$. Then*

$$\sup\{c \mid s \in \{f \geq c\}^*\} = \sup\{c \mid s \in \{f > c\}^*\}. \tag{1.27}$$

In other words

$$f^*(s) = \sup\{c \mid s \in \{f \geq c\}^*\}. \tag{1.28}$$

Proof. Define

$$l(s) = \sup\{c \mid s \in \{f \geq c\}^*\}$$
$$g(s) = \sup\{c \mid s \in \{f > c\}^*\}$$

Since

$$\{f > c\} \subseteq \{f \geq c\}$$

By the monotonicity property (1.1) of the rearrangement of sets we have

$$\{f > c\}^* \subseteq \{f \geq c\}^*$$

Taking supreme over c we have

$$g(s) \leq l(s)$$

Let $c_n \nearrow l(s)$ such that $s \in \{f \geq c_n\}^*$. We can redefine c_n such that

$$s \in \{f > c_n\}^* \text{ and } c_n \nearrow l(s)$$

Then, for all $\epsilon > 0$ there exists some n_0 such that for all $n \geq n_0$ we have

$$l(s) \leq c_n + \epsilon \text{ and } s \in \{f > c_n\}^*$$

Hence, since $c_n \leq g(s)\ \forall\ n$

$$l(s) \leq g(s) + \epsilon \quad \forall\ \epsilon > 0$$

and therefore

$$l(s) = g(s)$$

□

Lemma 3.1.42. *Let $f : \Omega \longrightarrow \mathbb{R}$ with $f(x) \neq 0$ a.e . Then*

$$\left(f^{-1}\right)^* = (f_*)^{-1}. \tag{1.29}$$

Proof. Let $s \in [0, |\Omega|]$. We have

$$\begin{aligned}
\left(f^{-1}\right)^*(s) &= \sup\left\{c \;\mid\; s \in \left\{f^{-1} > c\right\}^*\right\} \\
&= \sup\left\{c \;\mid\; s \in \{1/c > f\}^*\right\} \\
&= \left(\inf\left\{\delta \;\mid\; s \in \{f < \delta\}^*\right\}\right)^{-1} \\
&= f_*^{-1}(s)
\end{aligned}$$

□

The following property will be strongly required in the proof of the main theorem showed in section (6.4).

Corollary 3.1.43. *Let $f : \Omega \longrightarrow \mathbb{R}$ with $f(x) \neq 0$ a.e . If $\theta = f^*$,then*

$$\left(f^{-1}\right)^* = \left(\theta^{-1}\right)^*. \tag{1.30}$$

Proof. From the proposition (3.1.38) we have $f^* = (f_*)^*$ and also is easy to see that

$$f_* = (f^*)_*$$

using the preceding lemma (3.1.42) we have

$$\begin{aligned}\left(\theta^{-1}\right)^{*} &= \left(\theta_{*}\right)^{-1}\\ &= \left(f_{*}\right)^{-1}\\ &= \left(f^{-1}\right)^{*}\end{aligned}$$

□

3.1.3 Main theorems of rearrangements

Here we show the classical inequality properties of rearrangements such as Hardy-Littlewood and Pólya - Szegö inequalities and some isoperimetric inequalities that were useful in this work. This properties leaded us to get new proofs of the properties from Alvino and Trombetti [3, Lemma 1.2] and the Alvino, Trombetti and Lions [4, Theorem 3.1]. We redo this propositions in order to enlighten or get a better understanding of the general problem (**GP**) when we are not in a fully symmetrical case (such as the case when Ω is a ball). We expect this tools give us some knowledge of Steiner symmetries, for instance, the case of square domains. See [21, 19, 8] for further information.

Theorem 3.1.44 (Hardy- Littlewood)**.** *Let Ω be a bounded set, p, $q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p(\Omega)$, $g \in L^q(\Omega)$. Then*

$$\int_{\Omega} fg \leq \int_{0}^{|\Omega|} f^{*} g^{*}. \tag{1.31}$$

And

$$\int_{\Omega} fg \geq \int_{0}^{|\Omega|} f_{*} g^{*}. \tag{1.32}$$

Proof. The proof of the first equation can be checked in [21, Theorem 1.2.2]. We will prove the second inequality, supposing valid the first one.

Since $f_{*} = -\left(-f\right)^{*}$, using Hardy-Littlewood inequality (1.31) for $-f$ and g we obtain

$$\int_{0}^{|\Omega|} \left(-f\right)^{*} g^{*} \geq \int_{\Omega} -fg$$

hence

$$\int_0^{|\Omega|} f_* g^* \leq \int_\Omega fg$$

□

Theorem 3.1.45 (Pólya - Szegö). *[21, Theorem 2.3.1][27]. Let $1 \leq p < \infty$, Ω a bounded domain and $u \in W_0^{1,p}(\Omega)$ such that $u \geq 0$. Then*

$$\int_{\Omega^\sharp} \left|\nabla u^\sharp\right|^p dx \leq \int_\Omega |\nabla u|^p dx. \tag{1.33}$$

In particular, the Schwarz symmetrization $u^\sharp$ is in $W_0^{1,p}\left(\Omega^\sharp\right)$.

The following definitions and propositions concerns the framework of isoperimetric inequalities.

Definition 3.1.46 (The de Giorgi Perimeter). [21, Chapter 2] Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $E \subseteq \Omega$ be a measurable set. **The de Giorgi perimeter of E with respect to Ω** , denoted $\mathbf{P_\Omega(E)}$, is defined as the total variation of the characteristic function χ_E of E. In mathematical terms,

$$P_\Omega(E) = \left.\begin{array}{c} \sup\left\{\frac{\langle \nabla \chi_E, \phi\rangle}{\|\phi\|} \;\middle|\; \phi \in (\mathcal{D}(\Omega))^N \;\;, \phi \neq 0\right\} \\ \\ \sup\left\{\frac{\int_E div(\phi)}{\|\phi\|} \;\middle|\; \phi \in (\mathcal{D}(\Omega))^N \;\;, \phi \neq 0\right\} \end{array}\right\} \tag{1.34}$$

where

$$\|\phi\|^2 = \max_{x \in \overline{\Omega}} \left\{ \sum_{i=1}^N |\phi_i(x)|^2 \;\middle|\; \phi = (\phi_1, \ldots, \phi_N) \right\}. \tag{1.35}$$

If $\Omega = \mathbb{R}^N$ we write $P(E)$.

One could also interpret $P_\Omega(E)$ as the surface area of that part of the boundary of E contained in Ω where a normal can be unambiguously defined. If Ω were smooth, then, by the divergence theorem,

$$\int_\Omega div(\phi) dx = \int_{\partial\Omega} \phi \cdot n dS$$

where n is the outer normal on $\partial\Omega$. It clear that

$$\int_{\partial\Omega} \phi \cdot n dS \leq \|\phi\| \int_{\partial\Omega} dS$$

and since we can choose $\phi = n$ in the arguments of the supreme defined in (1.34) we have

$$P_{\mathbb{R}^N}(\Omega) = P(\Omega) = |\partial\Omega|. \tag{1.36}$$

and thus $P(\Omega)$ would be the *usual surface area of* $\partial\Omega$.

Theorem 3.1.47 (Isoperimetric Inequality)**.** *The Schwarz rearrangement $u^\sharp$ of the function u satisfies*

$$P(\{u \geq t\}) \geq P(\{u^\sharp \geq t\}). \tag{1.37}$$

and

$$P(\{u^\sharp \geq t\}) = N w_N^{\frac{1}{N}} \mu_u(t)^{1-\frac{1}{N}}. \tag{1.38}$$

which correspond to the perimeter of the ball $\{u^\sharp \geq t\}$, being $\mu_u(t)$ the distribution function of u defined in (3.1.12) and w_N the volume of the unit sphere in $\mathbb{R}^N$

Theorem 3.1.48 (Fleming - Rischell)**.** *[21, Theorem 2.2.1] Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in W^{1,1}(\Omega)$. Then*

$$\int_\Omega |\nabla u| = \int_{-\infty}^{\infty} P_\Omega\left(\{u > t\}\right) dt. \tag{1.39}$$

Theorem 3.1.49 (Co-area formula)**.** *[21, formula (2.2.1)]*

$$\int_\Omega g(x)\,|\nabla u(x)|\,dx = \int_{-\infty}^{\infty} \int_{u=s} g(x) d\sigma(x) ds. \tag{1.40}$$

where $\sigma(x)$ stands for the integration with respect to an appropriate $(N-1)-$ dimensional measure on the level set $\{u = t\}$.

Theorem 3.1.50. *[21, Theorem 2.2.3] Let $u \in \mathcal{D}(\Omega)$ be such that $u \geq 0$. Let μ denote the distribution function of u. Then, for almost every t in the range of u, we have*

$$-\mu'(t) = \int_{\{u=t\}} \frac{d\sigma}{|\nabla u|} = \int_{\{u^\sharp=t\}} \frac{d\sigma}{|\nabla u^\sharp|} \tag{1.41}$$

where $u^\sharp$ is the Schwarz symmetrization of u defined in (3.1.7).

3.2 Eigenvalue value problem for second order linear elliptic operators

Let Ω be a domain in $\mathbb{R}^n$ and consider the the elliptic operator in the divergence form

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} \tag{2.42}$$

where $a^{ij} \in L^{\infty}(\Omega)$. We suppose the operator is uniformly elliptic and it satisfies the symmetry property:

$$a^{ij} = a^{ji} \quad \forall\ i, j \tag{2.43}$$

The operator L is symmetric, hence, in particular, the bilinear form $B[,]$ satisfies

$$B[u,v] = B[v,u] \quad \forall\ u, v \in H_0^1(\Omega)$$

where

$$B[u,v] = \int_{\Omega} \sum_{i,j=1}^{n} \left(a^{ij} u_{x_i} v_{x_j}\right) \tag{2.44}$$

Theorem 3.2.1 (Eigenvalues of elliptic operators [14])**.** *We have the following properties:*

i. Each eigenvalue of L is real.

ii. Besides, if we repeat the eigenvalue according to its multiplicity (which is finite), we have

$$\Sigma = \{\lambda_k\}_{k=1}^{\infty}$$

where

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$$

and

$$\lambda_k \longrightarrow \infty \quad \text{when } k \longrightarrow \infty$$

iii. Finally, there exists an orthonormal base $\{w_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$, where $w_k \in H_0^1(\Omega)$ is an eigenfunction corresponding to the eigenvalue λ_k, namely, the base satisfies

$$\begin{cases} Lw_k = \lambda_k w_k & \text{in} \quad \Omega \\ w_k = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{2.45}$$

for each $k = 1, 2, \ldots$

Proof. We know the unique weak solution $u \in H_0^1(\Omega)$ of the problem

$$B[u,v] = \int_\Omega fv \quad \forall\, v \in H_0^1(\Omega) \tag{2.46}$$

satisfies

$$\begin{aligned} \lambda \|u\|^2_{H_0^1(\Omega)} &\leq B[u,u] \\ &\leq C\|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\ &\leq C\|f\|_{L^2(\Omega)} \|u\|_{H_0^1(\Omega)} \end{aligned}$$

Hence,

$$\|u\|_{H_0^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

Using the compact embedding (3.2.2) of $H_0^1(\Omega)$ in $L^2(\Omega)$ and defining the operator $K : L^2(\Omega) \longrightarrow L^2(\Omega)$ which associates to each f the only weak solution u of the problem (2.46), but considering it as a function in $L^2(\Omega)$, it follows that K is a compact operator.

Furthermore, K is selfadjoint and injective:

- **Selfadjoint:**

$$\begin{aligned} \langle K(f), g\rangle_{L^2(\Omega)} &= \int_\Omega K(f)g \\ &= B[f,g] \\ &= B[g,f] \\ &= \int_\Omega K(g)f \\ &= \langle K(g), f\rangle_{L^2(\Omega)} \\ &= \langle f, K(g)\rangle_{L^2(\Omega)} \end{aligned}$$

- **Injective:**
  If $K(f) = 0 \Rightarrow B[K(f), v] = \int_\Omega fv = 0 \forall\, v \in H_0^1(\Omega)$.
  Since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$ it follows $f = 0$

If we apply the spectral theory of compact selfadjoint operators to the operator K it follows there is an orthonormal base $\{w_k\}$ of $L^2(\Omega)$ and $\{\mu_k\}$ such that:

- $K(w_k) = \mu_k w_k$.

- $\mu_k \longrightarrow 0$, $\mu_k > 0 \quad \forall\, k$.
- Each μ_k has finite multiplicity.

Then

$$\begin{aligned} B\left[K(w_k), v\right] &= \int_\Omega w_k v \\ &\Rightarrow \\ B\left[w_k, v\right] &= \frac{1}{\mu_k}\int_\Omega w_k v \end{aligned}$$

Which is equivalent to

$$\begin{cases} Lw_k = \dfrac{1}{\mu_k} w_k & \text{in} \quad \Omega \\ w_k = 0 & \text{on} \quad \partial\Omega \end{cases}$$

$\square$

Theorem 3.2.2 (Rellich)**.** *We have the following well known inclusion property.*

- *For any bounded open set Ω, the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*
- *If Ω is a bounded open set with Lipschitz boundary, the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*

Theorem 3.2.3 (Positiveness of the first eigenfunction)**.** *Let us assume that Ω is a regular connected open set. Then the first eigenvalue $\lambda(L, \Omega)$ of the elliptic operator given in (2.42) is simple and the first eigenfunction u has a constant sign on Ω. Usually, we chose it to be positive on Ω.*

Actually, the previous theorem is a consequence of the Kreĭn-Rutman Theorem which is an abstract result we recall here.

Theorem 3.2.4 (Kreĭn Rutman)**.** *Let E be a Banach space and C be a closed convex cone in E with vertex at O, non-empty interior $Int(C)$ satisfying $C \cap (-C) = \{O\}$. Let T be a compact operator in E which satisfies $T(C\setminus\{O\}) \subseteq Int(C)$; then the greatest eigenvalue of T is simple, and the corresponding eigenvector is in $Int(C)$ (or in $-Int(C)$).*

The following classic result a very useful characterization of the first eigenvalue.

Theorem 3.2.5 (Rayleigh quotient)**.** *Let us define the* ***Rayleigh quotient*** *of the operator L to be:*

$$R_L[v] := \frac{\sum_{i,j=1}^{n} \int_\Omega a^{ij} v_{x_i} v_{x_j} dx}{\int_\Omega v(x)^2 dx}. \tag{2.47}$$

Then, the first eigenvalue satisfies

$$\lambda(L,\Omega) = \min_{v \in H_0^1(\Omega)} R_L[v]. \tag{2.48}$$

And the minimum is achieved by the corresponding eigenfunction.

3.3 Measure theory

Theorem 3.3.1. *Let ν be a measure over the borel sets in the real positive line $[0,\infty)$ such that*

$$\phi(t) := \nu([0,t]) \tag{3.49}$$

is finite for all $t > 0$. (Realize that $\phi(0) = 0$ and ϕ, since it is monotone, is Boreal measurable.) Let (Ω, Σ, μ) be a measure space and f be a non-negative measurable function over Ω. Then

$$\int_\Omega \phi(f(x))\, \mu(dx) = \int_0^\infty \mu\{f > t\}\, \nu(dt). \tag{3.50}$$

In particular, taking $\nu(dt) = pt^{p-1}dt$ for $p > 0$, it follows

$$\int_\Omega f(x)^p\, \mu(dx) = p \int_0^\infty t^{p-1} \mu\{f > t\}\, dt$$

Choosing $p = 1$ and μ as the Dirac measure in a point $x \in \mathbb{R}^n$, we have

$$f(x) = \int_0^\infty \chi_{f>t}(x) dt$$

Definition 3.3.2. Let Ω be a bounded domain in $\mathbb{R}^N$ and $f \in L^1(\Omega)$ a non-negative function such that $1/f \in L^1(\Omega)$. We define the **harmonic mean of f** as

$$\frac{|\Omega|}{\displaystyle\int_\Omega \frac{1}{f(x)}dx}. \tag{3.51}$$

Proposition 3.3.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ and $f \in L^1(\Omega)$ a non-negative function such that $1/f \in L^1(\Omega)$. Then*

$$\frac{\displaystyle\int_\Omega f(x)dx}{|\Omega|} \geq \left(\frac{\displaystyle\int_\Omega \frac{1}{f(x)}dx}{|\Omega|} \right)^{-1}. \tag{3.52}$$

That is, the arithmetic mean is greater than the harmonic mean.

Sketch of the proof. We have

$$|\Omega|^2 = \left(\int_\Omega \sqrt{f(x)} \frac{1}{\sqrt{f(x)}} \right)^2 \leq \int_\Omega f(x) \frac{1}{f(x)} \leq \int_\Omega f(x) \int_\Omega \frac{1}{f(x)}$$

Where in the next to last inequality we have used Jensen inequality and in the last one Cauchy-Schwartz. □

Definition 3.3.4 (Absolute continuity of functions)**.** Let (X, d) be a metric space and let I be an interval in the real line $\mathbb{R}$. A function $f : I \longrightarrow X$ is **absolutely continuous** on I if for every positive number ϵ, there is a positive number δ such that whenever a sequence of pairwise disjoint sub-intervals $[x_k, y_k]$ of I, $k = 1, 2, \ldots, n$ satisfies

$$\sum_{k=1}^{n} |y_k - x_k| < \delta$$

then

$$\sum_{k=1}^{n} d\left(f(y_k), f(x_k)\right) < \epsilon.$$

3.4 Convex analysis

In this section we give some basic ideas and well known theorems of convex analysis. Some of this concepts were rewritten from chapter 32 of [28]. We can say -in advance of the main theorem of this thesis- we are going to require the treatment of extreme points from certain convex compact sets. In addition, we will deal with minimizing sequences of a concave functional, so is important to have a background knowledge in convex functions and sets which could permit us to extract some information of convex sets, its extreme points and minimizing sequences.

The theory of the maximum of a convex function relative to a convex set has an entirely different character from the theory of the minimum [28]. In this perspective we give the following definitions and propositions.

Definition 3.4.1 (Affine hull)**.** [28]. Given any $S \subseteq \mathbb{R}^n$, we define the **affine hull** of S, as the affine set given by the intersection of the collection of affine sets M such that $M \supset S$.

Definition 3.4.2 (Relative interior)**.** [28]. The relative interior of a convex set $C \in \mathbb{R}^n$, which we denote by $ri\ C$, is defined as the interior which results when C is regarded as a subset of the affine hull of C.

Theorem 3.4.3. *Let f be a convex function, and let C be a convex set contained in $dom f$. If f attains its supremum relative to C at some point of $ri\ C$, then f is actually constant throughout C.*

Proof. [28]. Suppose the relative supremum is attained at a point $z \in riC$. Let x be a point of C other than z. We must show that $f(x) = f(z)$. Since $z \in ri\ C$, there is a real number $\mu > 1$ such that the point $y = (1-\mu)x + \mu z$ belongs to C. For $\lambda = \mu^{-1}$, one has

$$z = (1-\lambda)x + \lambda y, \quad 0 < \lambda < 1$$

and the convexity of f implies that

$$f(z) \leq (1-\lambda)f(x) + \lambda f(y)$$

. At the same time, $f(x) \leq f(z)$ and $f(y) \leq f(z)$ because $f(z)$ is the supremum of f relative to C. If $f(x) \neq f(z)$, we would necessarily have $f(z) > f(x)$. Then $f(y)$ would have to be finite in the convexity inequality (since otherwise $f(y) = -\infty$ and $f(z) = -\infty$), and we would deduce the impossible relation

$$f(z) < (1-\lambda)f(z) + \lambda f(z) = f(z)$$

Therefore $f(x) = f(z)$ $\square$

Definition 3.4.4. Given a set $X \subseteq E$ we define its convex hull as the set

$$co(X) = \{x \in E \ \mid \ \exists x_1, x_2 \in X, \exists \lambda \in [0,1] \text{ such that } x = \lambda x_1 + (1 - \lambda x_2)\}. \tag{4.53}$$

Proposition 3.4.5. *Let $f : co(X) \longrightarrow \mathbb{R}$ be a convex function over the set $co(X)$. Then*

$$\sup\{f(x) : x \in X\} = \sup\{f(x) : x \in Co(X)\}. \tag{4.54}$$

Proof. We always have

$$\sup\{f(x) \ \mid \ x \in X\} \leq \sup\{f(x) \ \mid \ x \in Co(X)\}.$$

Since f is convex, its level sets of the form $\{f \leq \lambda\}$ are convex. So f is quasiconvex and then

$$f(\alpha x_1 + (1-\alpha)x_2) \leq max\{f(x_1), f(x_2)\} \quad \forall \ \alpha \in [0,1]$$

Therefore,

$$\sup\{f(x) \ \mid \ x \in co(X)\} = \sup\{f(\alpha x_1 + (1-\alpha)x_2) \ \mid \ x_1 \ , \ x_2 \in X \quad , \alpha \in [0,1]\}$$

$$\leq sup\{max\{f(x_1), f(x_2)\} \ \mid \ x_1, x_2 \in X\} = \sup\{f(x) \ \mid \ x \in X\}$$

□

Theorem 3.4.6 (Kreĭn Milman)**.** *Let X be a locally convex topological vector space, and let K be a compact convex subset of X. Then, K is the closed convex hull of its extreme points.*

Proposition 3.4.7. *Let $f : \mathcal{C} \longrightarrow \mathbb{R}$ be a convex function and $\mathcal{C}$ a convex compact set. Let $\mathbb{E}$ be the set of extreme points of $\mathcal{C}$. Then*

$$\sup\{f(x) : x \in \mathcal{C}\} = \sup\{f(x) : x \in \mathbb{E}\}. \tag{4.55}$$

and in the case of the existence of a maximizing element in either both sides of the previous equality, then

$$\max\{f(x) : x \in \mathcal{C}\} = \max\{f(x) : x \in \mathbb{E}\}. \tag{4.56}$$

Proof. The equation (4.55) holds straightforward from Kreĭn Milmam theorem (3.4.6) and the proposition (3.4.5).

If we have a maximizing value, let suppose by contradiction that the supreme is reached in the interior, but does not in the in $\mathbb{E}$. Let $\bar{x} = \lambda x + (1 - \lambda y)$ element in $\mathcal{C}$ which maximizes

f en $\mathcal{C}$, with $x, y \in \mathbb{E}$. Then $f(x), f(y) < f(\bar{x})$. Due to the convexity of f:

$$f(\bar{x}) \leq \lambda f(x) + (1-\lambda) f(y) < f(\bar{x})$$

which is a contradiction. □

Theorem 3.4.8. *Let $f : \mathcal{C} \longrightarrow \mathbb{R}$ be a convex lower semi-continuous function and $\mathcal{C}$ a convex compact set. Then f reaches its maximum value at an extreme point of $\mathcal{C}$.*

3.5 Homogenization

In this section we restrict ourself to give some basic ideas and results from homogenization theory although we do not use it directly in any of our results. However one can obtain generalized solutions to our problem using this theory. (See [26, 11]). Here we describe homogenization of second order elliptic operators, which appear in connection with conductivity problems. See [2, 26, 25] for more details.

Let $\mathcal{M}_N$ be the linear space of square matrices with real coefficients of order N. For every pair of numbers $0 < \alpha$ and $0 < \beta$, we define the subspace of $\mathcal{M}_N$ of the coercive matrices with inverse coefficients:

$$M_{\alpha,\beta} = \left\{ M \in \mathcal{M}_N : M\xi \cdot \xi \geq \alpha \left|\xi\right|^2 \ , \ M^{-1}\xi \cdot \xi \geq \beta^{-1} \left|\xi\right|^2 \quad \forall\, \xi \in \mathbb{R}^N \right\}. \tag{5.57}$$

A coercive matrix such its inverse is also coercive is bounded. In fact if $M \in \mathcal{M}_{\alpha,\beta}$, writing $\eta = M^{-1}\xi$ we deduce from the definition (5.57)

$$\beta^{-1} \left|M\eta\right|^2 \leq M\eta \cdot \eta.$$

Applying the Cauchy-Schwarz inequality, we have

$$|M\eta| \leq \beta\eta \quad \forall\, \eta \in \mathbb{R}^N. \tag{5.58}$$

Similarly, we have

$$\left|M^{-1}\eta\right| \leq \alpha^{-1}\eta \quad \forall\, \eta \in \mathbb{R}^N. \tag{5.59}$$

Remark 3.5.1. From (5.57) and (5.58) it follows that a necessarily and sufficient condition for a matrix M belongs to the space $\mathcal{M}_{\alpha,\beta}$ is that $\alpha \left|\xi\right|^2 \leq M\xi \cdot \xi \leq \beta \left|\xi\right|^2$ for every vector ξ. Then, the set $\mathcal{M}_{\alpha,\beta}$ is no-empty if and only if the positive constants α y β satisfy $\alpha\beta^{-1} \leq 1$. From now on, we assume always the have this condition.

Let Ω be an open bounded set in $\mathbb{R}^N$. We define the **space of admissible coefficients** in Ω by

$$L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}) := [\alpha; \beta].$$

and if the matrices are symmetric we write the symbol $[\alpha;\beta]_s$.

Denoting by $\epsilon > 0$ a real sequence of numbers which tends to zero, we will consider the sequence of matrices $A^\epsilon(x) \in [\alpha;\beta]$.

For a given source $f \in H^{-1}(\Omega)$, we consider the following elliptic equation of second order with Dirichlet condition:

$$\begin{cases} -div(A^\epsilon \nabla u_\epsilon) = f & \text{in} \quad \Omega \\ u_\epsilon = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{5.60}$$

The former equations admits the following varational formulation. For every $\phi \in H_0^1(\Omega)$ we have

$$\int_\Omega A^\epsilon \nabla u_\epsilon \cdot \nabla\phi = \langle f, \phi \rangle_{H^{-1}, H_0^1(\Omega)} \tag{5.61}$$

Replacing ϕ by u_ϵ in the previous equation, the norm of the solution u_ϵ can be a priori estimated using the coercivity of A^ϵ:

$$\begin{aligned} \alpha \|\nabla u_\epsilon\|^2_{L^2(\Omega)} &\leq \int_\Omega A^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon \\ &\leq \langle f, u_\epsilon \rangle_{H^{-1}, H_0^1(\Omega)} \\ &\leq \|f\|_{H^{-1}} \|u_\epsilon\|_{H_0^1(\Omega)} \end{aligned}$$

Since Ω is a bounded domain, using the Poincare inequality, namely, we know there exists a positive constant C which depends only of Ω such that

$$\|\phi\|_{L^2(\Omega)} \leq C \|\nabla\phi\|_{L^2(\Omega)^N} \quad \forall\, \phi \in H_0^1(\Omega)$$

Hence, we have

$$\|\nabla u_\epsilon\|_{L^2(\Omega)^N} \leq \frac{C}{\alpha} \|f\|_{H^{-1}(\Omega)} \tag{5.62}$$

The a priori bound found in the former equation implies that the sequence of solutions u_ϵ is bounded in $H_0^1(\Omega)$, independently of ϵ. Since the bounded sets are relatively compact for the weak topology in $H_0^1(\Omega)$, there is a subsequence, which we still write as u_ϵ, that converges weakly in $H_0^1(\Omega)$ to a limit $u \in H_0^1(\Omega)$.

Introducing the flux

$$\sigma_\epsilon = A^\epsilon \nabla u_\epsilon,$$

it follows from (5.58) for the operator A^ϵ that the sequence σ_ϵ is also bounded in $L^2(\Omega)^N$. Then, there is a subsequence of σ_ϵ, called it in the same way, that weakly converges to a limit $\sigma \in L^2(\Omega)^N$.

From the equation (5.60) σ_ϵ satisfies

$$-div\ \sigma_\epsilon = f \quad in \quad \Omega$$

Therefore, taking the limits in the previous equation we have that

$$-div\ \sigma = f$$

The very interesting question is if $\sigma = A^* u$, for some matrix A, and if in that case u solves the problem

$$\begin{cases} -div(A\nabla u) &= f \qquad \text{in} \quad \Omega \\ u &= 0 \qquad \text{on} \quad \partial\Omega \end{cases}$$

The following definition formalizes the notion involved in the above question.

Definition 3.5.2 (H-Convergence)**.** Given a sequence of matrices $A^\epsilon \in [\alpha;\beta]$, we say it **converges in the homogenization sense**, or simply **H-converges**, to a homogenized limit, or H-limit $\mathbf{A}^* \in [\alpha;\beta]$, if for every given right side $f \in H^{-1}(\Omega)$, the solution u_ϵ of the equation (5.60) satisfies

$$\begin{cases} u_\epsilon \rightharpoonup u \text{ weakly in } H_0^1(\Omega) \\ A^\epsilon \nabla u_\epsilon \rightharpoonup A^* \nabla u \text{ weakly in } L^2(\Omega)^N \end{cases} \tag{5.63}$$

where u is the solution of the homogenized equation

$$\begin{cases} -div A^* \nabla u &= f \qquad \text{in} \quad \Omega \\ u &= 0 \qquad \text{on} \quad \partial\Omega \end{cases} \tag{5.64}$$

The next theorem shows always we can extract subsequences that H converges.

Theorem 3.5.3 (Spagnolo)**.** *Let $A^\epsilon \in [\alpha;\beta]_s$ be a sequence of symmetric matrices that satisfies (5.60) for all ϵ. Then there is a subsequence A^η of A^ϵ and there is a symmetric* *matrix* $A \in [\alpha;\beta]_s$ *such that*

$$A^\eta \quad H\text{-converges to } A^*.$$

3.6 Shape derivative

The shape derivative is a tool which permits to understand the variation of quantities which depend on the domain (cf. Simon [31]). This is widely used in the study of shape optimization, front tracking, image segmentation problems etc. In this chapter we only give the basic definitions.

Definition 3.6.1. Let $\omega \subseteq \mathbb{R}^N$. Let $t > 0$. If θ is a map from $\mathbb{R}^N$ to $\mathbb{R}^N$, we say that $\omega + t\theta$ **is a perturbation of the domain** ω in the direction θ with scalar t. We write

$$x \in \omega + t\theta \Leftrightarrow x + t\theta(x) \in \omega. \tag{6.65}$$

.

Let us consider a functional F which depends on the domain ω (shape functional). For a variation of the domain ω by a fairly smooth **perturbative** vector field θ, which has its support in a neighborhood of $\partial\omega$, we are interested in the variation of the functional F.

Definition 3.6.2. The **total derivative** of F in the direction θ is defined as

$$\dot{F}(\omega;\theta) = \lim_{t \longrightarrow 0} \frac{F(\omega + t\theta) \circ (I + t\theta) - F(\omega)}{t}. \tag{6.66}$$

Definition 3.6.3. The **local derivative** of F in the direction θ is defined as

$$F'(\omega;\theta) = \lim_{t \longrightarrow 0} \frac{F(\omega + t\theta) - F(\omega)}{t}. \tag{6.67}$$

It is useful to recall the following important relation between the total and local derivatives

Remark 3.6.4. The following relation usually holds

$$u'(x) = \dot{u}(x) - \theta \cdot \nabla u. \tag{6.68}$$

Chapter 4

Some General Aspects

In the introduction we said that $\Omega \subset \mathbb{R}^n$ corresponds to the shape of a conductor which is composed of two materials, one of conductivity $\alpha > 0$ and the other of conductivity $\beta > \alpha$. The quantity of the material of conductivity β is limited. Let us call b the available volume for this material, which we suppose is smaller than the volume of Ω.

If A is the region where is located the material of conductivity α and $B = \Omega \backslash A$ the region where is placed the other material, the conductivity coefficient of the material in the whole region is given by:

$$\sigma = \alpha\chi_A + \beta\chi_B. \tag{0.1}$$

And the conductivity equation which rules the phenomena is given by

$$\begin{cases} -div\,(\sigma\nabla u) &= f & \text{in} & \Omega \\ u &= 0 & \text{on} & \partial\Omega \end{cases}$$

where $f \in H^{-1}(\Omega)$ is given source. Hence, for every σ of the form (0.1) we have a second order elliptic equation, and since $0 < \alpha \leq \sigma \leq \beta$ is easy to see is a uniformly elliptic family with uniform ellipticity constant α and continuity constant β (the coefficient σ can be seen in $[\alpha; \beta]$ defined in (5.57)).

From the theorem (3.2.1) we know there is a orthogonal basis in $H_0^1(\Omega)$ such that each element of the base is an eigenvalue of the operator $T_\sigma : H_0^1(\Omega) \longrightarrow L^2(\Omega)$ defined by

$$T_\sigma u = -div\,(\sigma\nabla u) \tag{0.2}$$

We know from the theorem (3.2.1) and (3.2.3) that **the first eigenvalue** of T_σ, which we call $\lambda(\sigma)$, is strictly positive, the eigenfunction is simple and can be taken positive.

From the Rayleigh formula (2.48) we know that

$$\lambda(\sigma) = \inf_{\mathbf{u}\in\mathbf{H_0^1(\Omega)},\mathbf{u}\neq\mathbf{0}} \frac{\int_{\boldsymbol{\Omega}} \sigma\,|\nabla\mathbf{u}|^{\mathbf{2}}}{\int_{\boldsymbol{\Omega}} \mathbf{u^2}} \tag{0.3}$$

In terms of σ, we can rewrite the constant volume constraint as $\int\limits_{\Omega} \sigma = \alpha\left(|\Omega| \setminus b\right) + \beta b := c$. Hence the integral of σ is constant independently of the distribution of A and B.

Let Θ be the feasible set

$$\Theta = \left\{ \sigma = \alpha\chi_A + \beta\chi_B \;\mid\; A,\, B \subset \Omega \quad \text{and} \int\limits_{\Omega} \sigma = c \right\}. \tag{0.4}$$

The main question this document is based upon is if we can find $\bar{A}$ y $\bar{B}$ such that $\bar{\sigma} = \chi_{\bar{A}} + \chi_{\bar{B}}$ and:

$$\lambda\left(\bar{\sigma}\right) = \inf_{\sigma\in\boldsymbol{\Theta}} \lambda\left(\sigma\right) \tag{\textbf{GP}}$$

The general problem (**GP**) is very difficult to handle without homogenization. Besides, if we use this tools, it is still very difficult to find a non-homogenized solution, so it is even harder to find a classical one which could live in (0.4). As we have seen in the background several researchers such as Murat, Tartar, Cox and Lipton could not find -or even did not try to find- a classical solution to this kind of problem.

In order to obtain at least partial friendlier results, we have simplified the problem. The main problem we are going to manage in this thesis is solving (**GP**) in $\mathbb{R}^N$ when Ω is the unit ball, namely, when Ω is

$$\Omega = D = \left\{ x \in \mathbb{R}^N \;\mid\; \|x\| \leq 1 \right\}$$

We want to find a classical solution and, if possible, obtain a partial characterization of this solution. We believe the characterization is going to be strongly related with the symmetrical properties of the disc Ω. It is expected in a spherically symmetrical domain the solution has spherically symmetrical level sets. Indeed, this is happening and can be checked after a reformulation of the problem with the arguments shown by Alvino. et. al in [4].

Our intuition tells us a classical solution is not only spherically symmetrical but it also happens when the whole material β is placed in the center. We are giving numerical experiments that could make stronger our intuition, reaffirming this conjecture numerically speaking.

Despite the thesis problem is *strongly bounded* since we have restrict us to the unitary disc in $\mathbb{R}^2$, we will give more general results when possible. Most of the properties we have proved also hold for a unitary ball in $\mathbb{R}^N$. In addition, we expect some of the properties we have proved, not only hold in the spherically symmetrical case, but also when the domain satisfies other types of symmetries, such as squares or stars. We encourage the readers and ourselves to give new results for this type of symmetries in future researches.

4.1 Structural properties of the objective functional

In order to find a solution to the problem (**GP**) is necessary to obtain good properties of the eigenvalue functional (0.3). Unfortunately, in the general case we cannot say much. We only deduced the following result.

Proposition 4.1.1. *The objective function $\lambda(\sigma)$ is concave and upper semi-continuous in σ.*

Proof. The proof follows immediately from the definition of λ through (**GP**). In fact, λ is the infimum of the linear -so concave- continuous functionals

$$\sigma \longmapsto \frac{\displaystyle\int_\Omega \sigma \left|\nabla u\right|^2 dx}{\displaystyle\int_\Omega \left|u\right|^2 dx}$$

as u ranges over $H_0^1(\Omega)$ and thus, concavity and upper semi-continuity hold immediately. $\square$

4.2 Conclusions

The infimum in a minimization problem will be attained, by the direct methods of the calculus of variation, if it happens that the objective functional is lower semi-continuous and the constraint set is compact for some topology.

The constraint set (0.4) is relatively compact for the weak-$*$ topology as a subset of $L^\infty(\Omega)$ as bounded sets in $L^\infty(\Omega)$ are weak-$*$ compact.

However the objective functional is only upper semi-continuous for this topology and so we cannot obtain immediately a solution to our problem (**GP**).

In general, in order to calculate the infimum, at first, the closure of the constraint set needs to be calculated with respect to a suitable topology and then, the lower semi-continuous envelope of the objective functional with respect to the same topology. In our problem, this is hard to achieve without the consideration of micro-structural designs and, the results of Cox and Lipton [11] are in this spirit but lead further away from the study of a classical solution.

Chapter 5

Unidimensional Case

5.1 Introduction

Kreĭn [22, 16] has shown the minimum value in the unidimensional case for the general problem (**GP**) is actually reached and it has a classical representation, that is, without micro-structure or homogenized coefficients. He also characterized the solution, proving that an optimal solution is to place all the material β in the center. In this section we rewrite the proof showed in Henrot's book [16] explaining deeply all the necessary steps.

In section (5.2) we set the problem in the unidimensional case and we rewrite the equations in order to get more manageable equations. This new formulation belongs to the PDE class of membrane systems and thanks to the rearrangement tools it can be solved successfully giving a complete characterization of the solution. Luckily the treatment with homogenization techniques is completely avoidable, which allows giving a step further from Murat-Tartar [26] and Cox-Lipton [11] researches.

The new vibrating membrane problem involves the objective functional

$$\lambda^1(\rho) = \inf_{v \in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2}{\displaystyle\int_\Omega \rho\, |v|^2}. \tag{1.1}$$

which represents the first eigenvalue of the membrane equation

$$\begin{cases} \triangle v &= \lambda \rho v \qquad \text{in} \quad \Omega \\ v &= 0 \qquad\;\; \text{on} \quad \partial\Omega \end{cases}$$

Cox and McLaughlin [12, 13] exhibited the former problem has a true solution in any dimension, but unfortunately, this equivalence does not hold in higher dimensions. [1]

In this new formulation, a lower semi-continuous concave functional over a compact convex set is minimized. A classical result from convex analysis (3.4.8) tell us that the solution is an extreme point. In the section (5.4) we find out the characterization of this points.

With the rearrangement machinery assistance, in section (5.6) we exhibit the solution of the membrane problem can be chosen with symmetric properties. Afterward it is proved the unidimensional conductivity problem (2.3) has a solution with the material of higher conductivity in the center.

5.2 Setting the problem

Studying the limit of minimizer sequences in this type of elliptic problems (**GP**), even in the unidimensional case, is difficult without homogenization tools. [2]

We want to find a coefficient $\sigma = \alpha\chi_A + \beta\chi_B$ which hopefully satisfies de minimum condition (**GP**). The first step is change the feasible set (0.4) into a new convex one. We enlarge the conditions for the functions σ considering the set:

$$\mathcal{C} := \left\{ \sigma \in L^\infty([0,1]) / \quad 0 < \alpha \leq \sigma \leq \beta \text{ and } \int_0^1 \sigma(x)dx = c \right\}. \tag{2.2}$$

Given $\sigma \in \mathcal{C}$, the conductivity equation in the unidimensional case is given by

$$\begin{aligned} -\frac{d}{dx}\left(\sigma(x)\frac{du}{dx}\right) &= \lambda u(x) \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned} \tag{2.3}$$

5.2.1 Equivalence with the membrane problem

We are going to see that the statement given above (2.3) is equivalent to a membrane problem. This membrane problem is in fact tractable with rearrangements and classical tools of

[1] Nevertheless, the improve for the unidimensional case using rearrangement tools motivated us to find classical solutions in any dimension when we consider domains with some kind of symmetries, such as the case of a symmetrically spherical Ω, namely, a ball. See chapter 6 for more details.

[2] There is a classical unidimensional example which motivates the theory of homogenization [26], showing this *undesirable* behavior.

functional analysis, without using homogenization techniques.

Scheme. The first step is to define the transformation $y : [0,1] \longrightarrow \mathbb{R}$ as

$$y(x) = \int_0^x \sigma^{-1}(t)dt. \tag{2.4}$$

Since $y'(\cdot) > 0$, we have that y is a bijective map of $[0,1]$ into $[0,l]$, with $l = y(1)$. Integrating, the equation (2.3) can be written as:

$$\sigma(x)\frac{du}{dx} = -\lambda \int_0^x u(t)dt. \tag{2.5}$$

Now we define the auxiliary functions which will permit us obtain a membrane equivalent problem. We define v, ρ and z as

$$v(y) = u(x(y)) \tag{2.6}$$

$$\rho(y) = \sigma(x(y)) \tag{2.7}$$

$$z = \int_0^t \sigma^{-1}(l)l \tag{2.8}$$

With this notations the integral equation (2.5) can be rewritten as

$$\rho(y)\frac{dv}{dy}\frac{dy}{dx} = -\lambda \int_0^{y(x)} u(t(z))\rho(z)dz. \tag{2.9}$$

The last equation leads to

$$\frac{dv}{dy} = -\lambda \int_0^y v(z)\rho(z)dz$$

The derivative of the above equation reads

$$\frac{d^2v}{dy^2} = -\lambda\rho v$$

Now the condition

$$c = \int_0^1 \sigma(x)dx$$

say that

$$\int_0^l \rho(y)dy = \int_0^1 \sigma \cdot y^{-1'} dt = 1.$$

Defining

$$\tilde{\mathcal{C}} := \left\{ \rho \in L^\infty([0,l]) / \quad 0 < \alpha \leq \rho \leq \beta \text{ y } \int_0^l \rho(y)dy = 1 \right\} \tag{2.10}$$

the conductivity problem (2.3) for $\sigma \in C$ is equivalent to the following membrane problem for $\rho \in \tilde{\mathcal{C}}$:

$$\begin{aligned} -\frac{d^2v}{dy} &= \lambda\rho(y)v \\ v(0) &= 0 \\ v(l) &= 0 \end{aligned} \tag{2.11}$$

Thus, we have the equivalence. $\square$

5.3 Properties of the eigenvalue membrane functional

Let $\sigma \in \mathcal{C}$ and $\lambda(\sigma)$ the first (positive) eigenvalue of the problem (2.3). The Rayleigh quotient says that σ satisfies

$$\lambda(\sigma) = \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_0^1 \sigma \left(\frac{du}{dx}\right)^2 dx}{\int_0^1 u^2 dx} \tag{3.12}$$

In the previous subsection we have shown that the conductivity problem (2.3) is equivalent to the membrane problem (2.11). By straightforward calculations it can be seen that the eigenvalues of both problems are the same and if $u(x)$, λ is an eigen-pair of the problem (2.3), then $v(y) = u \circ (y^{-1}(y))$ is an eigenfunction of the problem (2.11) with eigenvalue λ.

The first eigenvalue for the membrane problem also satisfies the Rayleigh quotient

$$\tilde{\lambda}(\rho) = \inf_{v \in H_0^1([0,1]), v \neq 0} \frac{\int_0^1 \left(\frac{dv}{dy}\right)^2 dy}{\int_0^1 \rho v^2 dy} \tag{3.13}$$

We are interested in lower semi-continuity properties of this functional with a certain topology which hopefully assures at the same time compactness of the feasible set. If this two condition are satisfied, applying the theorem (3.4.8) we can obtain the existence of a minimizer element for the membrane problem.

Theorem 5.3.1. *The eigenvalue functional defined for the membrane problem (2.11) is a lower semi-continuous application with the week* $*$ *topology in* $L^\infty(\Omega)$.

Proof. Let (λ_n, u_n) be sequence of eigenvalues and eigenfunctions associated to the membrane parameter $\rho_n \in \tilde{\mathcal{C}}$ with $\|u_n\|_{H_0^1} = 1$. The Rayleigh quotient (3.13) says

$$\tilde{\lambda}(\rho_n) = \frac{\int_0^l \left(\frac{du_n}{dy}\right)^2}{\int_0^l \rho_n u_n^2}$$

Since u_n is a bounded sequence in H_0^1 by Rellich-Kondrachov theorem (3.2.2) there exists a subsequence (which we called in the same way) such that $u_n \longrightarrow u$ in L^2 and $u_n \rightharpoonup u$ in H_0^1.

By lower semi-continuity of the norm we have

$$\liminf \int \left(\frac{du_n}{dy}\right)^2 \geq \int \left(\frac{du}{dy}\right)^2$$

Since ρ_n is bounded we can extract a subsequence (which we called in the same way) that weakly-$*$ converges in L^∞ to some $\rho \in L^\infty$.

Then

$$\langle \rho_n, u_n^2 \rangle_{L^1, L^\infty} \longrightarrow \langle \rho, u^2 \rangle_{L^1, L^\infty}$$

Namely,

$$\int \rho_n u_n^2 \longrightarrow \int \rho u^2$$

Then

$$\text{liminf}\,\tilde{\lambda}(\rho_n) = \frac{\text{liminf} \displaystyle\int \left(\frac{du_n}{dy}\right)^2}{\displaystyle\int \rho_n u^2} \geq \tilde{\lambda}(\rho)$$

which shows the lower semi-continuity of $\tilde{\lambda}$. □

5.4 Properties of the feasible set of the membrane problem

In the previous section we have shown the continuity properties of the functional that is required to be minimized. Besides it is important to have manageable properties - like compactness - of the feasible set. In this section we show properties in that direction: the feasible set $\tilde{\mathcal{C}}$ is a convex compact set which their extreme points are of the form $\sigma = \alpha\chi_A + \beta\chi_B$.

Proposition 5.4.1. *The feasible set $\tilde{\mathcal{C}}$ defined in (2.10) for the membrane equation (2.11) is a compact convex set.*

Proof. Let $\rho_1,\ \rho_2 \in \tilde{\mathcal{C}}$ and $\eta \in [0,1]$. It is clear that

$$\alpha \leq \eta\rho_1 + (1-\eta)\,\rho_2 \leq \beta$$

and

$$\int_0^1 \eta\rho_1 + (1-\eta)\,\rho_2 = \eta\int_0^1 \rho_1 + (1-\eta)\int_0^1 \rho_2 = \eta 1 + (1-\eta)\,1 = 1$$

Then $\tilde{\mathcal{C}}$ is a convex set. By standard argument we can see $\tilde{C}$ is bounded and closed for the weakly-$*$ convergence. This follows from the fact that the integral constraint $\int_0^1 \rho_n = 1$ is satisfied in the limit because the function $\chi_\Omega \in L^1(\Omega)$ and the compactness in $\mathbb{R}$ of the inequality constraint which defined $\tilde{\mathcal{C}}$. □

Proposition 5.4.2. *The extreme points of the set $\tilde{\mathcal{C}}$ defined in (2.10) are of the form*

$$\alpha\chi_\omega + \beta\chi_\omega. \tag{4.14}$$

Proof. Let $\mathcal{A} = \{\alpha\chi_\omega + \beta\chi_\omega \ \mid\ \omega \subset \Omega\}$ and $\mathbb{E}$ the set of extreme points of $\tilde{\mathcal{C}}$.

We are going to prove first that $\mathbb{E} \subseteq \mathcal{A}$, which means $\tilde{\mathcal{C}} \backslash \mathcal{A}$ does not contain any extreme point.

Let $f \in \mathcal{C}\backslash\mathcal{A}$. There exists some $\epsilon > 0$ and a set Θ of non zero Lebesgue measure that

$$\alpha + \epsilon < f(x) < \beta - \epsilon \quad \forall\, x \in \Theta.$$

Let Θ_1, Θ_2 disjoint sets which have the same Lebesgue measure such that $\Theta = \Theta_1 \cup \Theta_2$. We define the function g such that

$$g(x) = \begin{cases} \frac{\epsilon}{2} & \text{if } x \in \Theta_1 \\ -\frac{\epsilon}{2} & \text{if } x \in \Theta_2 \\ 0 & \text{in other case.} \end{cases}$$

We have:

- $\displaystyle\int g = \frac{\epsilon}{2}\,|\Theta_1| - \frac{\epsilon}{2}\,|\Theta_2| = 0.$
- $f + g \begin{cases} = f + \frac{\epsilon}{2} \leq \beta - \frac{\epsilon}{2} \text{ in } \Theta_1 \\ = f - \frac{\epsilon}{2} \geq \alpha + \frac{\epsilon}{2} \text{ in } \Theta_2 \end{cases}$
- $f - g \begin{cases} = f + \frac{\epsilon}{2} \leq \beta - \frac{\epsilon}{2} \text{ en } \Theta_2 \\ = f - \frac{\epsilon}{2} \geq \alpha + \frac{\epsilon}{2} \text{ en } \Theta_1 \end{cases}$

Then $f + g \in \tilde{\mathcal{C}}$, $f - g \in \tilde{\mathcal{C}}$ and

$$f = \frac{f+g}{2} + \frac{f-g}{2},$$

hence f is not an extreme point of $\tilde{\mathcal{C}}$.

Let us prove now that $\mathcal{A} \subseteq \mathbb{E}$, namely, if f is not an extreme point, then $f \notin \mathcal{A}$. We prove this by contradiction. Suppose we have that $f \notin \mathbb{E}$ and $f \in \mathcal{A}$.

There exist $f_1, f_2 \in \mathbb{E}$, $f_1 \neq f_2$ such that $\frac{f_1+f_2}{2} = f$. But the former proved inclusion shows that $f_1, f_2 \in \mathcal{A}$, in particular, f_1 y f_2 are of the form

$$\begin{aligned} f_1 &= \alpha\chi_{\omega_1} + \beta\chi_{\Omega\backslash\omega_1} \\ f_2 &= \alpha\chi_{\omega_2} + \beta\chi_{\Omega\backslash\omega_2} \end{aligned}$$

Hence

$$f = \begin{cases} \alpha & \text{in } \omega_1 \cap \omega_2 \\ \frac{\alpha+\beta}{2} & \text{in } (\omega_1 \cap \Omega\backslash\omega_2) \cup (\omega_2 \cap \Omega\backslash\omega_1) \\ \beta & \text{in } (\Omega\backslash\omega_1) \cup (\Omega\backslash\omega_2) \end{cases}$$

Therefore, f does not belong to the set $\mathcal{A}$, which is a contradiction. □

5.5 Existence for the membrane problem

In the previous subsection 5.4 we have shown the set $\tilde{\mathcal{C}}$ is compact and convex. Since we are minimizing $\tilde{\lambda}(\rho)$ over the compact set $\tilde{\mathcal{C}}$ and the theorem (5.3.1) and the concavity indicated in (4.1.1) show that ρ is a lower semi-continuous concave application, applying the theorem (3.4.8), we see the infimum is achieved at an extreme point of $\tilde{\mathcal{C}}$.

5.6 Characterization of the membrane problem

Using rearrangement tools, we are going to show the solution of the membrane problem (2.11) is symmetric.

Lemma 5.6.1. *For the membrane problem (2.11) the minimization of the first eigenvalue has a solution which can be taken symmetrical, namely, can be consider with the coefficient of higher elasticity in the middle of the interval.*

Proof. Let $\bar{\rho}$ be the limit of a minimizing sequence of $\tilde{\lambda}$. Let $\bar{v}$ be an eigenfunction of the (2.11) membrane problem associated to $\bar{\rho}$. We know from the preceding subsection 5.5 that

$$\tilde{\lambda}(\bar{\rho}) = \inf_{u\in H_0^1([0,l]),v\neq 0} \frac{\int_0^l \left(\frac{dv}{dy}\right)^2 dy}{\int_0^l \rho v^2 dy}$$

If we employ the Schwarz rearrangement in the unidimensional case displaced in $l/2$ it is clear that $\bar{v}^*(0) = \bar{v}^*(l) = 0$. Furthermore by the Pólya - Szegö inequality (3.1.45)

$$\int_0^l \frac{d\bar{v}^*}{dy}^2 \leq \int_0^l \frac{d\bar{v}}{dy}^2 \tag{6.15}$$

Applying the Hardy-Littlewood inequality (1.31) we have

$$\int_0^l \rho^*(\bar{v}^2)^* \geq \int_0^l \rho\bar{v}^2 \tag{6.16}$$

Since $\bar{v} \geq 0$, from the lemma (3.1.32)

$$(\bar{v}^2)^* = (\bar{v}^*)^2$$

thus, the pair $\bar{v}^*$-ρ^* must be the minimum, which shows the minimum is reached in symmetrical parameters. In other words, the solution ρ^* has the coefficient of higher elasticity in the middle of the interval . □

5.7 Characterization of the conductivity problem

In the previous lines we have proved that the solution of the membrane equation (2.11) is of the form

$$\rho = \alpha \chi_A + \beta \chi_B$$

where B is an interval who is centered with respect to $[o, l]$, that is, its center is $l/2$. Now, we want to show that the solution of the original conductivity equation (2.3), which we called σ, has a similar symmetry.

Characterization of the conductivity problem. Realize that $\sigma(x) = \rho \circ y(x) = \alpha \chi_{y^{-1}(A)} + \beta \chi_{y^{-1}(B)}$. Since the transformation y is an homeomorphism and B is connected it follows $y^{-1}(B)$ is connected. Since A has two connected components, $y^{-1}(A)$ has two connected components.

Besides, $0, l$ belong to different connected components of A and since $y(0) = 0$ and $y(1) = l$, it follows that $y^{-1}(B)$ is in the interior of the interval $[0, 1]$. Let suppose the interval limits of $y^{-1}(B)$ are $x_1 < x_2$.

We want to prove this interval is in the middle of the $[0, 1]$ interval. It is enough to show that the distance of its boundaries x_1, x_2 to 0 y 1 respectively is the same. In other words, it is enough to show that $|x_1| = |1 - x_2|$, namely, $x_1 + x_2 = 1$.

Since

$$\frac{dy}{dx} = \frac{1}{\rho}$$

then

$$x = \int_0^y \rho$$

. Let $y_1 < y_2$ the boundaries of the interval B. We have

$$x_2 = \int_0^{y_2} \rho = \int_0^{y_1} \rho + \int_{y_1}^{y_2} \rho$$

But $\rho|_{(0,y_1)} = \rho|_{(y_2,l)}$. Then

$$x_2 = \int_{y_2}^{l} \rho + \int_{y_1}^{y_2} \rho$$

Hence

$$x_1 + x_2 = \int_0^{y_1} + \int_{y_2}^{l} \rho + \int_{y_1}^{y_2} \rho = \int_0^{l} \rho = 1.$$

□

Chapter 6

N Dimensional Case: Spherical Symmetry

6.1 A First reformulation

Since homogenization tools only lead us far away from classical solutions, if we want to get one it is imperative to think of the problem in different terms, so with this new conception of the problem. Perhaps taking a different view of the problem it is easier to get closer to a solution.

The key point is the (3.1.34) rearrangement property, which allows to reformulate the feasible set (0.4) given in chapter 4.

Proposition 6.1.1 (First reformulation)**.** *If A is the place where we put the material α, given a decreasing unidimensional function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ of the form*

$$\varphi = \alpha\chi_{[0,|A|)} + \beta\chi_{[|A|,|\Omega|)}$$

the set of classical solutions (0.4) given in chapter 4 can be recast as

$$\mathcal{C} = \{\sigma \in L^\infty(\Omega) \ \mid \ \sigma^* = \varphi\}. \tag{1.1}$$

*Hence, the general problem (**GP**) can be reformulated in terms of $\mathcal{C}$ given above, namely (**GP**) is equivalent to*

$$\inf_{\sigma\in\mathcal{C}} \lambda(\sigma). \tag{1.2}$$

Proof. Since the feasible set (0.4) given in chapter 4 is the set of indicator functions with a prescribe measure of the only two non trivial level sets, the proof follows directly form the proposition (3.1.34) and the equimeasurability property (3.1.21). ☐

It is important to endow the feasible set $\mathcal{C}$ with some topology that hopefully makes $\mathcal{C}$ compact and at the same time assures lower semi continuity of the functional $\lambda(\sigma)$. Due to the concavity (4.1.1), we might guarantee the infimum of (1.2) is still reached at $\mathcal{C}$.

Mainly, what we are going to do is to give some properties related to new topologies and the closure of sets that have similar structures with (1.1). We found some of this results in [4, 3]. In addition, we are going to study the properties of the extension of the functional $\lambda(\sigma)$ related to the new closures of the feasible regions, and some other properties of this type of sets which can lead us to find the solution in the original feasible set.

6.2 Prescribed rearrangement Sets

Let $f \in L^p(\Omega)$, $\Omega \in \mathbb{R}^N$ be a bounded domain. Suppose also that $f \geq 0$. Recall the distribution function of f defined in (3.1.12) is given by

$$\mu_f(t) = |\{f > t\}|$$

and the decreasing rearrangement of f is given by

$$f^*(s) = \sup\{t > 0 : \mu_f(t) > s\}$$

Definition 6.2.1. For $f_0 \geq 0$ in $L^p(\Omega)$ we define the set of equimeasurable functions of f_0 as

$$C(f_0) = \{f \geq 0 \mid f^* = f_0^*\ ,\ f \in L^p(\Omega)\} \tag{2.3}$$

By definition, we see all the functions in $\mathcal{C}(f_0)$ are equimeasurable. Hence, this set defines an equivalence class over the functions having the same rearrangement. In order to understand better this set equivalence relation we define the following ordering relation in $L^1(\Omega)$.

Definition 6.2.2. Given two functions $f, g \in L^1(\Omega)$, we write $f \prec g$ and we say that f **has**

less measure than g, if

$$\begin{cases} \displaystyle\int_0^t f^* ds & \leq \displaystyle\int_0^t g^* ds \quad \forall\, t \in [0, |\Omega|] \\ \displaystyle\int_0^{|\Omega|} f^* ds & = \displaystyle\int_0^{|\Omega|} g^* ds \end{cases} \tag{2.4}$$

Proposition 6.2.3. *Let* $1 \leq p \leq \infty$ *and* $f_0 \in L^p(\Omega)$*. Then* $C(f_0)$ *is relatively weak compact for* $1 \leq p < \infty$ *and relatively weak* $*$ *compact for* $p = \infty$*.*

Proof. For $1 \leq p \leq \infty$, thanks to the equimeasurability property (3.1.21) , for all $f \in C(f_0)$ we have

$$\|f\|_{L^p(\Omega)} = \|f_0\|_{L^p(\Omega)} .$$

Then result follows for $1 < p \leq \infty$. We skip the proof for $p = 1$. □

Notation 6.2.4. Let $1 \leq p \leq \infty$ and $f_0 \in L^p(\Omega)$. We denote the weak closure (weak * closure if $p = \infty$) of $C(f_0)$ as the set $\mathcal{K}(f_0)$, namely

$$\mathcal{K}(f_0) = \overline{C\,(f_0)}^{L^p(\Omega) \text{ weak}} \tag{2.5}$$

The following proposition shows a relation between the ordering relation and the set $\mathcal{C}(f_0)$.

Proposition 6.2.5. *Let* $1 \leq p \leq \infty$ *and* $f_0 \in L^p(\Omega)$*. Then*

$$f \in C(g) \Leftrightarrow f \prec g \text{ and } g \prec f. \tag{2.6}$$

Proof. The necessary condition follows straightforward. In order to prove the sufficient condition it is enough to show

$$\int_a^b f^* ds = \int_a^b g^* ds \quad \forall\, a, b \in [0, |\Omega|] .$$

and this holds because

$$\begin{aligned} \int_a^b f^* ds &= \int_0^b f^* - \int_0^a f^* ds \\ &= \int_0^b g^* ds - \int_0^a g^* ds \\ &= \int_a^b g^* ds \end{aligned}$$

□

Notation 6.2.6. We denote $L^1(\Omega)_+$ as the set of functions in $L^1(\Omega)$ that are not negative.

Now we give several characterizations of the first condition in of the ordering relation $\prec$ which could make more tractable this relation.

Proposition 6.2.7. *Let $f,\, g \in L^1_+(\Omega)$. The following properties are equivalent:*

(i) $\forall\, t \in [0, |\Omega|] \quad \int_0^t f^* \le \int_0^t g^*.$

(ii) $\forall\, \varphi \in L^1_+(\Omega) \quad \int_\Omega f\varphi \le \int_0^{|\Omega|} g^*\varphi^*.$

(iii) $\forall\, \varphi \in L^1_+(\Omega) \quad \int_0^{|\Omega|} f^*\varphi^* \le \int_0^{|\Omega|} g^*\varphi^*.$

(iv) $\int_\Omega F(f) \le \int_\Omega F(g)$ *For all functions F convex, non negative such that $F(0) = 0$ and F Lipschitz.*

Remark 6.2.8. If $f,\, g \in L^p(\Omega)$ we take $\varphi \in L^q(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$, $p,\, q \ge 1$ and in (iv) we change F' bounded with $F(t) \le C\,(1 + |t|^p)$.

Proof. We are going to prove **iii) ⇒ ii), ii) ⇒ i), i) ⇒ iii), iii) ⇒ iv) and iv) ⇒ i)**.

iii) ⇒ ii) Using Hardy-Littlewood inequality (3.1.44)

$$\int_\Omega f\varphi \le \int_0^{|\Omega|} f^*\varphi^* \le \int_0^{|\Omega|} g^*\varphi^*$$

ii) ⇒ i) Applying the proposition (3.1.40), if ii) holds we have

$$\int_\Omega f\varphi \le \int_\Omega g^*\varphi^*.$$

Taking $\varphi = \chi_A$,

$$\int_A f \le \int_0^{|A|} f^* \quad \forall\, A \text{ such that } |A| = r.$$

Taking supreme over A, using the proposition (3.1.40) we have

$$\int_0^r f^* \leq \int_0^r g^*.$$

i) ⇒ iii) We integrate by parts to obtain

$$\int_o^{|\Omega|} f^*\varphi^* ds = -\int_0^{|\Omega|}\left[\int_0^t f^*(s)ds\right] d\varphi^*(t) + \left[\int_0^t f^*(s)ds\right]\varphi^*(t)|_0^{|\Omega|}.$$

$$\int_0^{|\Omega|} g^*\varphi^* - f^*\varphi^* = \int_0^{|\Omega|} (g^* - f^*)\,\varphi^*$$

$$= -\int_0^{|\Omega|}\int\left[\int_0^t (g^* - f^*)\,(s)ds\right] d\varphi^*(t) + \int_0^{|\Omega|} (g^* - f^*)\,(s)ds\varphi^*|\Omega|$$

$$\geq -\int_0^{|\Omega|}\left[\int_0^t (g^* - f^*)\,(s)ds\right] d\varphi^*(t)$$

Since φ^* is a decreasing function, the differential $d\varphi^*(t)$ is negative for all t, hence it follows the result.

iii) ⇒ iv) We will prove $F \in \mathcal{C}^1$ is convex. We have F' is decreasing which implies

$$(F'(f))^* = F'(f^*).$$

Using the hypothesis

$$\int_0^{|\Omega|} f^*F'(f^*) \leq \int_0^{|\Omega|} g^*F'(f^*)$$

and due to the convexity property

$$\int_0^{|\Omega|} F(g^*) - F(f^*) \geq \int_0^{|\Omega|} F'(f^*)\,(g^* - f^*) \geq 0.$$

iv) ⇒ i) By contradiction, suppose it holds iv) but i) does not. The exists $t > 0$ such that

$$\int_0^t g^* < \int_0^t f^*$$

.

Let $[\underline{r}, \overline{r}]$ be the maximal interval where the former inequality holds for t.

Hence

$$\int_0^{\underline{r}} f^* - g^* ds = \int_0^{\overline{r}} f^* - g^* ds = 0 \quad 0 \leq \underline{r} < \overline{r} \leq |\Omega| \, .$$

Let $r_1 \in]\underline{r}, \overline{r}[$ tal que $f^*(r_1) > g^*(r_1)$. Let $F(t) = (t - g^*(r_1))^+$. We have that F is convex, $F(0) = 0$ y and is Lipschitz.

Since F is convex

$$\int_\Omega F(f) = \int_0^{|\Omega|} F(f^*) = \int_0^{|\Omega|} (f^*(t) - g^*(r_1))^+$$

$$\begin{aligned}
&\geq \int_0^{r_1} \left(f^*(t) - g^(r_1)\right)^+ \\
&\geq \int_0^{r_1} (f^*(t) - g^*(r_1)) \qquad \text{(adding non-negative numbers)} \\
&> \int_0^{r_1} g^*(t) - g^*(r_1) dt \\
&= \int_0^{r_1} (g^*(t) - g^*(r_1))^+ \qquad \text{(since the integrand is positive)} \\
&= \int_0^{r_1} F(g^*) \\
&= \int_0^{r_1} F(g)
\end{aligned}$$

Which is a contradiction with iv).

$\square$

The following propositions show the interesting fact the ordering relation $\prec$ is in indeed a characterization of the weak-$*$ limits of $\mathcal{C}(f_0)$, namely is a characterization of $\mathcal{K}(f_0)$.

Proposition 6.2.9. *If $f \in \mathcal{K}(f_0)$ then $f \prec f_0$.*

Proof. It is enough to prove that

$$\forall\ \varphi \in L_+^\infty(\Omega) \quad \int_\Omega f\varphi \le \int_{|\Omega|} f_0^* \varphi^*$$

and

$$\int_0^{|\Omega|} f^* = \int_0^{|\Omega|} f_0^*.$$

Let $f_n \in C(f_0)$ such that $f_n \longrightarrow f$ weakly.
Since $f_n \prec f_0$ y $f_0 \prec f_n$ we have

$$\int_\Omega f_n\varphi \le \int_0^{|\Omega|} f_n^*\varphi^* = \int_0^{|\Omega|} f_0^*\varphi^*.$$

Taking the limit it follows

$$\int_\Omega f\varphi \le \int_0^{|\Omega|} f_0^*\varphi^*.$$

On the other hand

$$\int_0^{|\Omega|} f_n^* = \int_0^{|\Omega|} f_0^*$$

and

$$\int_0^{|\Omega|} f_n^* = \int_\Omega f_n \longrightarrow \int_\Omega f = \int_0^{|\Omega|} f^*.$$

Then

$$\int_0^{|\Omega|} f^* = \int_0^{|\Omega|} f_0^*.$$

□

The converse relation is proved in the paper of Alvino, Trombetti and Lions [4].

Proposition 6.2.10. *If $f \prec f_0$ then $f \in \mathcal{K}(f_0)$.*

Proof. See [4] for a complete proof. □

The following corollary will be used in the proof of a very important result.

Corollary 6.2.11. *Let $\varphi = \varphi^\sharp$, namely, φ is a radially symmetric decreasing function. The set $\mathcal{K}(\varphi)$ is a weak-* compact convex set characterized by the relation*

$$\mathcal{K}(\varphi) = \left\{ f \in L^\infty(\Omega) \;\middle|\; \int_{B(0,r)} f(x)dx \leq \int_{B(0,r)} \varphi(x)dx \quad \forall\, r, \quad \int_\Omega f(x)dx = \int_\Omega \varphi(x)dx \right\}. \tag{2.7}$$

Proof. Using the equivalence ii) of (6.2.7) that induces the order relation $f \prec \varphi$, we have that

$$f \in \mathcal{K}(\varphi) \Leftrightarrow \forall\, \psi \in L^1_+(\Omega) \quad \int_\Omega f\psi \leq \int_0^{|\Omega|} \varphi^*\psi^* \quad \text{and} \int_\Omega f(x)dx = \int_\Omega \varphi(x)dx. \tag{2.8}$$

Taking $\psi = \chi_{B(0,r)}$, we have $\psi^* = \chi_{[0,t]}$ where $t = |B(0,r)|$. Hence, the inequality of (2.8) implies

$$\int_{B(0,r)} f \leq \int_0^t \varphi^*$$

Since the property (3.1.11) says that $\varphi^\sharp(x) = f^*(w_N |x|^N)$, and $\varphi^\sharp = \varphi$, the former inequality, after a change of variables, reads

$$\int_{B(0,r)} f(x)dx \leq \int_{B(0,r)} \varphi(x)dx$$

And thus, the relation (2.7). □

Now we are going to see other properties of the set $\mathcal{K}(f_0)$.

Proposition 6.2.12. *$\mathcal{K}(f_0)$ is a convex set.*

Proof. Let f_1,$f_2 \in \mathcal{K}(f_0)$. There exist subsequences (f_n^1), $(f_n^2) \subset C(f_0)$ such that $f_n^1 \longrightarrow f_1$, $f_n^2 \longrightarrow f_2$ weakly to f_1 y f_2 respectively. For $\lambda \in [0,1]$, let $f_n = \lambda f_n^1 + (1-\lambda) f_n^2$. Let F be a convex function satisfying the requirements in the characterization of $\mathcal{K}(f_0)$ (See (6.2.7) point iv) or [4]).

Clearly

$$\int\limits_\Omega F(f_n) \leq \lambda \int\limits_\Omega F(f_n^1) + (1-\lambda) \int\limits_\Omega F(f_n^2),$$

and using the characterization iv), since f_n^1, $f_n^2 \prec f_0$ we obtain that

$$f_n \prec f_0.$$

Since

$$f_n \longrightarrow \lambda f_1 + (1-\lambda) f_2$$

we obtain

$$\lambda f_1 + (1-\lambda) f_2 \in \mathcal{K}(f_0).$$

□

The following theorem provides a good result which will lead to the finding of the classical solution of the problem (**GP**).

Theorem 6.2.13. *$C(f_0)$ are the extreme points of $\mathcal{K}(f_0)$*

We only give a proof of one of the inclusions. The other one can be found in [4].

Proposition 6.2.14. *Let $\mathbb{E}(f_0)$ the set of extreme points of $\mathcal{K}(f_0)$. Then,*

$$C(f_0) \subset \mathbb{E}(f_0). \tag{2.9}$$

Proof. By contrapositive, if f is not an extreme point of $\mathcal{K}(f_0)$, then there exist f_1, $f_2 \in \mathbb{E}$, $f_1 \neq f_2$, $\lambda \in (0,1)$ such that

$$f = \lambda f_1 + (1-\lambda) f_2.$$

Taking an strictly convex function F (for example $F(t) = \sqrt{1+t^2}$) we have

$$\int\limits_\Omega F(f) < \lambda \int\limits_\Omega +(a-\lambda) \int\limits_\Omega F(f_2)$$

since $f_1,\ f_2 \in \mathcal{K}(f_0)$, using the characterization iv), it follows

$$\int\limits_{\Omega} F(f) < \int\limits_{\Omega} F(f_0)$$

which says that

$$f \notin C(f_0)$$

$\square$

Proposition 6.2.15. *We have*

$$\mathcal{K}(f_0) = \overline{convC(f_0)}. \tag{2.10}$$

Proof. $\mathcal{K}(f_0)$ is close convex set which contains $C(f_0)$. Then

$$K \supset \overline{ConvC(f_0)}.$$

On the other hand

$$\overline{ConvC(f_0)} \supset C(f_0)$$

then

$$\overline{ConvC(f_0)} \supset \overline{C(f_0)} = \mathcal{K}(f_0).$$

$\square$

We believe the solution to our problem (**GP**) in the radially symmetric case has indeed a spherical symmetry. In order to prove this believe we will need the following simple observation.

Proposition 6.2.16. *Let Ω be a ball in $\mathbb{R}^N$. Let φ_i be a sequence of bounded radially symmetric functions which converges weak-$*$ in $L^\infty(\Omega)$ to a function φ. Then φ is a radially symmetric function.*

Proof. Let $T : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a rotation transformation. Then is an orthonormal linear transformation, invertible, $|\det T| = 1$ and $T^{-1}(\Omega) = \Omega$.It is enough to show that

$$\varphi(x) = \varphi(Tx) \quad a.e \in \Omega$$

The last equation holds for all φ_i since there are spherically symmetrical. Let $g \in L^1(\Omega)$.

We have

$$\int_{\Omega} (\varphi_i(Tx) - \varphi(Tx))\, g(x)dx = \int_{T(\Omega)} (\varphi_i(y) - \varphi(y))\, g(T^{-1}y) \left|\det T^{-1}\right| dy$$

So, since $g \circ T^{-1} \in L^1(\Omega)$ the result holds. ☐

The property proved above encourage us to study the prescribe rearrangement sets with symmetry constraints. In that spirit are the following definitions and propositions.

Definition 6.2.17. For $f_0 \geq 0$ in $L^{\infty}(\Omega)$ we define the set of **radially symmetric equimeasurable functions** of f_0 as

$$C_s(f_0) = \{f \geq 0 \ \mid \ f^* = f_0^* \ , \ f \in L^{\infty}(\Omega) \ \text{radially symmetric}\} \tag{2.11}$$

Notation 6.2.18. For $f_0 \geq 0$ in $L^{\infty}(\Omega)$ we denote the set $\mathcal{K}_s(f_0)$ as the set

$$\mathcal{K}_s(f_0) = \{f \in \mathcal{K}(f_0) \ \mid \ f \text{ is radially symmetric } \} \tag{2.12}$$

As we expect, we have the following proposition.

Proposition 6.2.19. *We have*

$$\overline{Co\,(C_s(f_0))}^{\,weak\,*} = K_s\,(f_0)\,. \tag{2.13}$$

Proof. This follows directly from the theorem (3.4.6) and the next proposition. ☐

Proposition 6.2.20. $C_s(f_0)$ *are the extreme points of* $\mathcal{K}_s(f_0)$

Proof. Let f be an extreme point of $\mathcal{K}_s(f_0)$ Let $f \in C_s(f_0) \subseteq \mathcal{C}(f_0)$. Since the theorem (6.2.13) says $\mathcal{C}(f_0)$ is the set of extreme points of $\mathcal{K}(f_0)$, if $g_1,\, g_2 \in \mathcal{K}(f_0)$, then

$$f \notin (g_1, g_2)\,.$$

In particular, this is satisfied if $g_1,\, g_2 \in \mathcal{K}_s(f_0) \subseteq \mathcal{K}(f_0)$, which says f is an extreme point of $\mathcal{K}_s(f_0)$. Reciprocarly, to show the extreme points of $K_s(f_0)$ are points of $C_s(f_0)$, it is enough to show the extreme points of $\tilde{K}(f_0)$ are in $\tilde{C}(f_0)$ where this sets represent $\mathcal{K}$ and $\mathcal{C}$ when in dimension $N = 1$. Thus, due to (6.2.13) with $N = 1$ this is satisfied. ☐

6.3 Properties of λ

In this section we construct good properties for the eigenvalue functional (0.3) of chapter 4 that we wish to minimize. In order to find a solution of the problem (**GP**) when the domain Ω is a ball, we desire to create an extension of the functional that appears in (**GP**) which with a certain topology could be lower semi continuous and at the same time assures compactness of the feasible region.

We give new formulations or extensions of the eigenvalue functional with properties which can assure the existence of a minimum.

We gladly show with the re-formulation of the problem along with the new eigenvalue functional that it is possible to find a classical solution, and even better, the parameters of this solution can be taken spherically symmetrical.

The following results are crucial for the proof of our main theorem functions. The first of these is a consequence of a result form Alvino and Trombetti [3, Lemma 1.2] which we shall use, instead of a finer result [4, Theorem 3.1], to limit our search for minimizers among radially symmetric functions. The theorem 3.1 proved in [4] could be more complicated to extend to domains with partial symmetry compared to this simpler result.

Proposition 6.3.1. *Given any $\nu \in \mathcal{C}(\theta)$ and any $u \in H_0^1(\Omega)$, there exists a $\tilde{\nu}$ which is radially symmetric with $\tilde{\nu}^{-1} \in K((\theta^{-1})^*)$ such that*

$$\int_\Omega \nu \left|\nabla u\right|^2 dx \geq \int_\Omega \tilde{\nu} \left|\nabla u^\sharp\right|^2 dx. \tag{3.14}$$

Proof. With the same hypothesis as in this proposition, the Lemma 1.2 in Alvino et. al. [3] says that (3.14) holds for the radially symmetric function $\tilde{\nu}(z) = \xi(w_N \left|z\right|^N)$ for ξ defined below through the relation

$$\int_0^{|u\leq c|} \frac{1}{\xi(r)} dr := \int_{\{u\leq c\}} \frac{1}{\nu(x)} dx. \tag{3.15}$$

which holds for all $c \in \mathbb{R}$. This gives the relation

$$\int_{\Omega^*_{u,c}} \frac{1}{\tilde{\nu}(x)} dx = \int_{\Omega_{u,c}} \frac{1}{\nu(x)} dx. \tag{3.16}$$

for all c real, where we recall that $\Omega_{u,c}$ is the level set of u at the level c and $\Omega^*_{u,c}$ is a ball centered at the origin having the same measure as $\Omega_{u,c}$. In particular the above identity hold on the full domain Ω. So due to the property (3.1.43) we have $(\nu^{-1})^* = (\theta^{-1})^*$, using the formula (3.1.25) it follows

$$\int_{\Omega} \frac{1}{\tilde{\nu}(x)} dx = \int_{\Omega} (\theta^{-1})^*(x) dx. \tag{3.17}$$

Once again as $(\nu^{-1})^* = (\theta^{-1})^*$, form the property (3.1.39) we obtain

$$\int_{\Omega_{u,c}} \frac{1}{\nu(x)} dx \leq \int_{\Omega^*_{u,c}} (\theta^{-1})^* dx.$$

The above inequality combined with (3.16) gives the relation

$$\int_{\Omega^*_{u,c}} \frac{1}{\tilde{\nu}(x)} dx \leq \int_{\Omega^*_{u,c}} (\theta^{-1})^*(x) dx. \tag{3.18}$$

for all c real. We then note that the two relations (3.17) and (3.18), by the characterization (6.2.11), imply that

$$\tilde{\nu}^{-1} \in K((\theta^{-1})^*)$$

□

Now we prove a symmetry result which is very intuitive.

Proposition 6.3.2. *If ν is a non-negative, radially symmetric measurable function bounded form below and above by positive constants defined on the unit ball, then any eigenfunction corresponding to the first eigenvalue $\lambda(\nu)$ is radially symmetric. So, we have*

$$\lambda(\nu) = \inf_{u \in H_0^1(\Omega)} \frac{\displaystyle\int_{\Omega} \nu \left|\nabla u\right|^2 dx}{\displaystyle\int_{\Omega} \left|u\right|^2 dx} = \inf_{\substack{u \in H_0^1(\Omega) \\ u \text{ radially symmetric}}} \frac{\displaystyle\int_{\Omega} \nu \left|\nabla u\right|^2 dx}{\displaystyle\int_{\Omega} \left|u\right|^2 dx}. \tag{3.19}$$

Proof. By the Kreĭn-Rutman theorem (3.2.4), the first eigenvalue $\lambda(\nu)$ is simple and any eigenfunction is of constant sign. So, it is enough to prove the result for a normalized eigenfunction u which can be taken to be positive and having L^2 norm equal to 1.

Let, now, T be any orthogonal transformation on the unit ball and let $v = v \circ T$. Since T is a linear transformation, we have

$$\nabla_y v(y) = T^t \nabla_x (u(Ty))$$

and since T is a rotation, it has unit norm, then

$$|\nabla_x u(Ty)| = |\nabla_y v(y)|\,.$$

Using change of variables we obtain

$$\frac{\displaystyle\int_\Omega \nu(x)\,|\nabla_x u(x)|^2\,dx}{\displaystyle\int_\Omega |u(x)|^2\,dx} = \frac{\displaystyle\int_\Omega \nu(Ty)\,|\nabla_x u(Ty)|^2\,|\det(T)|\,dy}{\displaystyle\int_\Omega |u(Ty)|^2\,|\det(T)|\,dy} = \frac{\displaystyle\int_\Omega \nu(y)\,|\nabla_y v(y)|^2\,dy}{\displaystyle\int_\Omega |v(y)|^2\,dy}$$

The last equality hold being ν radially symmetric, since in this case $|\det(T)| = 1$. So the function v is also a positive minimizer for $\lambda(\nu)$ having the same L^2 norm as u So, by the Kreĭn- Rutman theorem, we have $u = v$, that is

$$u = u(Tx) \quad a.e$$

an this holds for any orthogonal transformation T. This proves the radial symmetry of the function u and establishes the identity (3.19). □

The following proposition establishes the continuity of the first eigenvalue with respect to weak-* convergence of the reciprocal of the coefficients, for radially symmetric coefficients. A much more restrictive convergence result having a similar flavor is proved in [4, Corollary 3.2], which assures convergence only to minimizing sequences.

Proposition 6.3.3. *Let ν_i be a sequence of radially symmetric functions in $K(\varphi)$ such that ν_i^{-1} converges weakly-$*$ to a function ν^{-1} as i tends to ∞. Then, we have $\lambda(\nu_i)$ converges to $\lambda(\nu)$ as i tends to ∞.*

Proof. Let the sequence ν_i and the function ν satisfy the hypotheses of the proposition. We write

$$\nu_i(x) = \frac{1}{\xi_i(|x|)}$$

and

$$\nu(x) = \frac{1}{\xi(|x|)}.$$

Then, by the hypotheses it follows that ξ_i weak-* converges to ξ in $L^\infty(0,1)$.

Now, if u_i gives the minimum value in the definition of $\lambda(\nu_i)$ then, by proposition (6.3.2) it is radially symmetric. We may also assume that u_i is non-negative and further, normalize it so that it's L^2 norm is 1. The Euler equation corresponding to the minimizing property of

u_i reads

$$- \, div\,(v_i \nabla u_i) = \lambda\,(\nu_i)\,u_i. \tag{3.20}$$

It can be checked from this that the sequence u_i is bounded in $H_0^1\,(\Omega)$ and a subsequence can be extracted converging weakly in $H_0^1\,(\Omega)$ to a radial function $u(x) = v(|x|)$. A further subsequence, indexed by i_k, may be extracted so that

$$\lambda\,(\nu_{i_k}) \text{ converges to some } \lambda \quad \text{as } k \longrightarrow \infty$$

Now, writing

$$u_{i_k}(x) = v_k(|x|),$$

the Euler equation (3.20) in radial co-ordinates, for this subsequence, reads

$$-\left(r^{n-1}\frac{1}{\xi_{i_k}}v_k'(r)\right)' = \lambda(\nu_{i_k})r^{n-1}v_k(r). \tag{3.21}$$

By integration, we obtain

$$r^{n-1}\frac{1}{\xi_{i_k}}(r)v'(r) = -\lambda(\nu_{i_k})\int_0^r s^{n-1}v_k(s)ds. \tag{3.22}$$

Since u_{i_k} is bounded in $H_0^1\,(\Omega)$, it can be checked that the sequence v_k converges weakly in $H_0^1\,(0,1)$,. So after transferring ξ_{i_k} to the right hand side of (3.22), it is possible to pass to the limit therein as $k \longrightarrow \infty$ to obtain the relation

$$r^{n-1}v'(r) = -\overline{\lambda}\xi(r)\int_0^r s^{n-1}v(s)ds.$$

We then divide by $\xi(r)$, differentiate with respect to r and write the equation that we obtain in the original co-ordinates as

$$- \, div\,(\nu\nabla u) = \overline{\lambda}u. \tag{3.23}$$

The function u is non-zero as its L^2 norm is 1 and thus, is an eigenfunction and, being the limit o f non-negative functions, is itself non-negative. So, by the Kreĭn -Rutman theorem, $\overline{\lambda}$ is the first eigenvalue in the above spectral problem. By the uniqueness of the limit,

$$\overline{\lambda} = \lambda(\nu)$$

Therefore, it follows that the entire sequence $\lambda(\nu_i)$ converges to $\lambda(\nu)$. □

Remark 6.3.4. If we consider positive the first eigenvalue of the Euler equation (3.20), the radial co-ordinates equation (3.21) says that $\nu \dfrac{\partial u}{\partial n}$ is decreasing in each region where ν is constant and the integral equation (3.22) also says that u is decreasing.

6.4 Proof of the main theorem

This proof is given in several steps.

Step 1: Reformulation

Let us recall that the constraint in the original problem can be written as

$$\nu \in \mathcal{C}\left(\theta\right).$$

Thanks to the corollary (3.1.43) this constrain can be recast as

$$\nu^{-1} \in \mathcal{C}\left(\left(\theta^{-1}\right)^*\right). \tag{4.24}$$

So the minimization problem reads

$$\inf\left\{\lambda\left(\nu\right) \ \mid \ \nu^{-1} \in \mathcal{C}\left(\left(\theta^{-1}\right)^*\right)\right\}. \tag{4.25}$$

Proposition 6.4.1. *We have the following problem equivalence*

$$\inf\left\{\lambda\left(\nu\right) \ \mid \ \nu^{-1} \in \mathcal{C}\left(\left(\theta^{-1}\right)^*\right)\right\} = \inf\left\{\lambda\left(\nu\right) \ \mid \ \nu^{-1} \in \mathcal{K}_s\left(\left(\theta^{-1}\right)^*\right)\right\}. \tag{4.26}$$

Proof. Following the proposition (6.2.19) we have that $\mathcal{K}_s\left(\left(\theta^{-1}\right)^*\right)$ is the closed convex hull of $\mathcal{C}_s\left(\left(\theta^{-1}\right)^*\right)$ for the weak-$*$ topology. So applying the continuity property in proposition (6.3.3) and the proposition (3.4.5) we have

$$\inf\left\{\lambda\left(\nu\right) \ \mid \ \nu^{-1} \in \mathcal{C}_s\left(\left(\theta^{-1}\right)^*\right)\right\} = \inf\left\{\lambda\left(\nu\right) \ \mid \ \nu^{-1} \in \mathcal{K}_s\left(\left(\theta^{-1}\right)^*\right)\right\}.$$

So it readily follows that

$$\inf\left\{\lambda\left(\nu\right) \ \mid \ \nu^{-1} \in \mathcal{K}_s\left(\left(\theta^{-1}\right)^*\right)\right\} \geq \inf\left\{\lambda\left(\nu\right) \ \mid \ \nu^{-1} \in \mathcal{C}\left(\left(\theta^{-1}\right)^*\right)\right\}. \tag{4.27}$$

To prove the reverse inequality, let $\nu^{-1} \in \mathcal{C}\left(\left(\theta^{-1}\right)^*\right)$ be arbitrary and let u be the corresponding minimizer in the definition of $\lambda\left(\nu\right)$. Considering $\tilde{\nu}^{-1} \in \mathcal{K}_s\left(\left(\theta^{-1}\right)^*\right)$ and u^* associated to the pair (ν, u) as given by proposition (6.3.1) and using the property (3.1.24) we obtain

$$\lambda(\nu) = \frac{\int_\Omega \nu |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx} \geq \frac{\int_\Omega \tilde{\nu} |\nabla u^*|^2 \, dx}{\int_\Omega |u^*|^2 \, dx} \geq \lambda(\tilde{\nu}) \geq \inf\left\{\lambda(\nu) \mid \nu^{-1} \in \mathcal{K}_s\left(\left(\theta^{-1}\right)^*\right)\right\}.$$

□

By the arbitrariness of ν the reverse inequality to (4.27) follows.

Step 2: The reformulation achieves the infimum

We define a topology on the set

$$\tilde{\mathcal{K}} := \left\{\nu \mid \nu^{-1} \in \mathcal{K}_s\left(\left(\theta^{-1}\right)^*\right)\right\}.$$

Definition 6.4.2. We write

$$\nu_{\mathbf{i}} \stackrel{\tau}{\rightharpoonup} \nu$$

Saying that ν_i **Tau inverse converges to** ν, if and only if

$$\nu_i^{-1} \text{ converges to } \nu^{-1}$$

weakly-$*$ in $L^\infty(\Omega)$.

Proposition 6.4.3. *The infimum of the right hand side of (4.26) is in fact a minimum, that is to say, the infimum value is achieved.*

Proof. It is clear from (6.2.3) and (6.2.15) that $\tilde{\mathcal{K}}$ is a compact set for the weak-$*$ topology on $L^\infty(\Omega)$. It follows that $\tilde{\mathcal{K}}$ is a compact set for the topology defined above. In fact let be $\nu_n \in \tilde{\mathcal{K}}$ a sequence. There exists a subsequence $\nu_{n_k}^{-1}$ of ν_n^{-1} which converges to $\nu^{-1} \in \mathcal{K}_s\left(\left(\theta^{-1}\right)^*\right)$, hence by definition $\nu_{n_k} \stackrel{\tau}{\rightharpoonup} \nu \in \tilde{\mathcal{K}}$.

Besides, by proposition (6.3.3) we know that λ restricted to $\tilde{\mathcal{K}}$ is continuous for the above topology. Thus, our thesis follows. □

Step 3: The problem has a classical solution

In the previous step, we have been able to show that the minimization problem admits a solution in a slightly enlarged class. Although, the functional λ is concave, it is not clear whether the constraint set

$$\left\{\nu : \nu^{-1} \in \mathcal{K}_s\left(\left(\theta^{-1}\right)^*\right)\right\}$$

is convex. If this were so it is immediate (by proposition (3.4.7)) to obtain a solution in the original class as, whenever a concave function admits a minimum over a compact convex set there is a minimizer which is an extreme point.

So, in this problem, in order to show that there is a solution in the original class, we shall have to do differently as is done in Alvino et. al [4]. We have the following technical proposition showed in [4].

Proposition 6.4.4. *The map*

$$J : \nu^{-1} \longrightarrow (\lambda(\nu))^{-1}$$

when is restricted to $\mathcal{K}_s\left(\left(\theta^{-1}\right)^*\right)$ *is a convex application.*

Sketch. Indeed, in [4, Corollary 3.2] it is shown that

$$J(\mu) = \max\left\{\int_\Omega \mu\left(|x|^{N-1}\int_0^{|x|} s^{N-1}v(s)ds\right)^2 dx \;\mid\; v \in L^2(\Omega), \quad \int_\Omega v^2(x)dx = 1, \quad v \text{ radial}\right\}.$$

showing that $J(\mu)$ is a supremum of linear functionals. □

Due to the previous proposition the minimization problem on the right hand side of (4.26) is equivalent to maximizing the reciprocal functional J. The above mentioned convexity guarantees that there is a maximizer of J which is an extreme point of the compact convex set $\mathcal{K}_s\left(\left(\theta^{-1}\right)^*\right)$. By proposition (6.2.20) it has to belong to $\mathcal{C}_s\left(\left(\theta^{-1}\right)^*\right)$. This permits us to conclude that the infimum in (4.25) is achieved for a radially symmetric function. □

Finally, we are able to announce the following main theorem.

Theorem 6.4.5. *Let Ω denote a ball in $\mathbb{R}^N$. The problem (**GP**) of minimizing the first eigenvalue, defined by (0.3), given two conducting materials with conductivities α and β, in a given ratio, admits a radially symmetric solution.*

6.5 Remarks

We remark that we only require Lemma 1.2 [3] in the form state below for our applications. Now, we give a more flexible alternate proof of the same.

Proposition 6.5.1. *Given any $\nu \in \mathcal{C}(\theta)$ an any non-negative $u \in H_0^1(\Omega)$, for $\tilde{\nu}$ radially symmetric defined through the relation,*

$$\int_{\Omega^*_{u,c}} \frac{1}{\tilde{\nu}(x)} dx = \int_{\Omega_{u,c}} \frac{1}{\nu(x)} dx \tag{5.28}$$

we have

$$\int_{\Omega} \nu \left|\nabla u\right|^2 \geq \int_{\Omega} \tilde{\nu} \left|\nabla u^*\right| dx. \tag{5.29}$$

Proof. We shall make repeated use of the co-area formula (3.1.49). Applying (1.40) given in (3.1.49), we obtain the identity

$$\int_{\{u \geq t\}} \nu(x) \left|\nabla u(x)\right|^2 dx = \int_t^{\infty} \int_{\{u=s\}} \nu(x) \left|\nabla u(x)\right| d\sigma(x) dt.$$

Therefore, it follows that,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\{u \geq t\}} \nu(x) \left|\nabla u(x)\right|^2\right) = \int_{\{u=t\}} \nu(x) \left|\nabla u(x)\right| d\sigma(x). \tag{5.30}$$

We apply the fact that the arithmetic mean of a non-negative function is always grater than the harmonic mean (3.3.3), to the function $\nu \left|\nabla u\right|$ on the surface $\{u = t\}$ equipped with its surface measure, to conclude that

$$
\begin{aligned}
\int\limits_{\{u=t\}} \nu\,|\nabla u(x)|\,d\sigma(x) &= \frac{\displaystyle\int\limits_{\{u=t\}} \nu(x)\,|\nabla u(x)|\,d\sigma(x)}{\displaystyle\int\limits_{\{u=t\}} d\sigma(x)} \int\limits_{\{u=t\}} d\sigma(x) \\
&\geq \frac{\displaystyle\int\limits_{\{u=t\}} d\sigma(x)}{\displaystyle\int\limits_{\{u=t\}} \frac{1}{\nu(x)\,|\nabla u(x)|} d\sigma(x)} \int\limits_{\{u=t\}} d\sigma(x) \\
&= \left(\int\limits_{\{u=t\}} \frac{1}{\nu(x)\,|\nabla u(x)|} d\sigma(x) \right)^{-1} \left(P(\{u \geq t\})\right)^2 \\
&\geq \left(\int\limits_{\{u=t\}} \frac{1}{\nu(x)\,|\nabla u(x)|} d\sigma(x) \right)^{-1} \left(P(\{u^* \geq t\})\right)^2 \qquad (5.31)
\end{aligned}
$$

The last inequality above is due to the isoperimetric inequality (3.1.47). Therefore, from (5.30) and (5.31) we have

$$
-\frac{\mathrm{d}}{\mathrm{d}t} \left(\int\limits_{\{u\geq t\}} \nu(x)\,|\nabla u(x)|^2\,dx \right) \geq \left(\int\limits_{\{u=t\}} \frac{1}{\nu(x)\,|\nabla u(x)|} d\sigma(x) \right)^{-1} \left(P(\{u^* \geq t\})\right)^2. \qquad (5.32)
$$

We remember that $\{u^* \geq t\}$ form a continuum of concentric balls, having radius r_t, whose union over $t \geq 0$ is the ball Ω. Observe that, as u^* is a radially symmetric function, $\nabla u^*(x)$ depends only on $|x|$. So, we may define a radially symmetric function $\tilde{\nu}$ as follows.

$$
\tilde{\nu}(x) := \frac{\displaystyle\int\limits_{\{u^*=t\}} d\sigma(y)}{\left(\displaystyle\int\limits_{\{u=t\}} \frac{1}{\nu(y)\,|\nabla u(y)|} d\sigma(y) \right) |\nabla u^*(x)|} \qquad \text{for any } x,\ |x| = r_t. \qquad (5.33)
$$

We check, first, that $\tilde{\nu}$ satisfies (5.28). To see this we use the co-area formula. We have

$$\begin{aligned}
\int\limits_{\{u^*\geq t\}} \frac{1}{\tilde{\nu}(x)} dx &= \int\limits_t^\infty \int\limits_{\{u^*=t\}} \frac{1}{\tilde{\nu}(x)\,|\nabla u^*(x)|} d\sigma(x) ds \\
&= \int\limits_t^\infty \int\limits_{\{u=t\}} \frac{1}{\nu(x)\,|\nabla u(x)|} d\sigma(x) ds \\
&= \int\limits_{\{u\geq t\}} \frac{1}{\nu(x)} dx
\end{aligned}$$

where in the penultimate expression we have plugged in (5.33). Then, (5.32) may be rewritten using $\tilde{\nu}$ as

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\limits_{\{u\geq t\}} \nu(x)\,|\nabla u(x)|^2\, dx \right) &\geq \tilde{\nu}(x)\,|\nabla u^*(x)|\, P(\{u^* \geq t\}) \qquad \text{for any } x,\ |x| = r_t. \\
&= \int\limits_{\{u^*=t\}} \tilde{\nu}(x)\,|\nabla u^*(x)|\, d\sigma(x) \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\limits_{\{u^*\geq t\}} \tilde{\nu}(x)\,|\nabla u^*(x)|^2\, dx \right).
\end{aligned}$$

Integrating the last equation we obtain the needful. $\square$

Chapter 7

Shape Derivative of the Eigenvalue Functional

7.1 Introduction

Let ω be given the region where we put the material β. Let u and λ be the first eigenfunction and eigenvalue of the problem (0.1) given in Chapter 1. The questions that immediately appear are: *Is this configuration optimal? Can we diminish the eigenvalue changing the configuration?* In order to have an answer to these questions we analyze the sensitivity of the first eigenvalue when we slightly perturb the distribution of the materials.

The shape derivative defined in Section 3.6 will be our tool to address these questions. We shall calculate the shape derivative of the eigenvalue formally without worrying much about the hypothesis necessary for everything to make complete sense.

7.2 Some results

Recall the spectral problem (**GP**) for the first eigenvalue functional is given by

$$\left\{ \begin{array}{rcllc} -div\,(\sigma\nabla u) & = & \lambda u & \text{in} & \Omega \\ u & = & 0 & \text{on} & \partial\Omega \end{array} \right. . \tag{2.1}$$

where the first eigenvalue $\lambda_1 = \lambda = \lambda(\omega)$ depends of the material distribution and $\sigma = \alpha\chi_{\Omega\setminus\omega} + \beta\chi_{\omega}$.

We can suppose the eigenfunction is normalized in order to satisfy

$$\int_\Omega u^2 dx = 1. \tag{2.2}$$

Let us define the set

$$\omega_t \equiv \omega + t\theta = \left\{ x \in \mathbb{R}^N \ \mid \ x \in \omega \ , \ x + t\theta(x) \in \omega \right\}. \tag{2.3}$$

Definition 7.2.1. Suppose for this context that $\omega = \omega_0 \subset\subset \Omega$. The **admissible perturbations** of ω are of the form $\omega + t\theta$ where θ is a sufficiently smooth vector field such that $\omega + t\theta \subset\subset \Omega$ and such that

$$|\omega + t\theta| = |\omega|. \tag{2.4}$$

From now on we consider the same eigenvalue problem given in (2.1) but now when the material with conductivity β is placed in the region w_t. If σ_t, u_t, λ_t are the corresponding symbols for the new problem, we write:

$$\left\{ \begin{array}{rcll} -div\,(\sigma_t \nabla u_t) & = & \lambda_t u_t & \text{in} \quad \Omega \\ u_t & = & 0 & \text{on} \quad \partial\Omega \end{array} \right. . \tag{2.5}$$

where λ_t is the fist eigenvalue regarding this equation.

Hypothesis 7.2.2. The shape derivative of λ exists and the shape derivative of u exists and belongs to a suitable space. We also assume all the additional hypothesis necessary for our calculations to makes sense.

Lemma 7.2.3. *Assuming the hypothesis (7.2.2) we have*

$$-\,div\,(\sigma' \nabla u + \sigma \nabla u') = \lambda' u + \lambda u' \qquad in\ \Omega \tag{2.6}$$

Remark 7.2.4. The earlier equation is very formal. Is not very clear in this equation what the term σ' means or how to interpret $\sigma' \nabla u$. In spite of all this the formula that we shall obtain for (**LP**) can be rigorously justified. See appendix 8.16(b).

Sketch. From the equations (2.1) and (2.5) we have

$$\begin{array}{rcl} -div\,(\sigma \nabla u) & = & \lambda u \\ -div\,(\sigma_t \nabla u_t) & = & \lambda_t u_t \end{array}$$

Subtracting both equation and adding a zero term, we have

$$-div\left(\left(\sigma_t-\sigma\right)\nabla u_t+\sigma\left(\nabla u_t-\nabla u\right)\right)=\left(\lambda_t-\lambda\right)u_t+\lambda\left(u_t-u\right)$$

Dividing by t, and making t tends to zero, applying the hypothesis (7.2.2) we obtain (2.6). ☐

It is a standard fact that the volume constraint (2.4) is equivalent to the following incompressibility condition

Proposition 7.2.5. *From the constraint* $|\omega_t|=|\omega|$ *follows*

$$\int\limits_{\omega}\operatorname{div}\theta dx=\int\limits_{\partial\omega}\theta\cdot ndS=0. \tag{2.7}$$

Sketch. We have

$$\begin{aligned} 0 &= \int\limits_{\omega_t}dx-\int\limits_{\omega}dx \\ &= \int\limits_{\omega_t\cap\omega}dx+\int\limits_{\omega_t\setminus\omega}dx-\int\limits_{\omega\cap\omega_t}dx-\int\limits_{\omega\setminus\omega_t}dx \\ &= \int\limits_{\omega_t\setminus\omega}dx-\int\limits_{\omega\setminus\omega_t}dx \end{aligned}$$

If t tends to zero, noting $o(t)$ a "little o" of t (assuming the hypothesis (7.2.2)), from the last equation we have (see the figure (7.1))

$$\int\limits_{\partial\omega}t\theta\cdot ndS+o(t)=0$$

Dividing by t, , and tending t to zero it follows (2.7). ☐

Lema 7.2.1. The normalization constraint (2.2) leads to the following orthogonal relation in $L^2(\Omega)$ between u and u':

$$\int\limits_{\Omega}uu'=0. \tag{2.8}$$

Sketch. We have

$$\lim_{t\longrightarrow 0}\frac{|u_t|^2-|u|^2}{t}=u'u$$

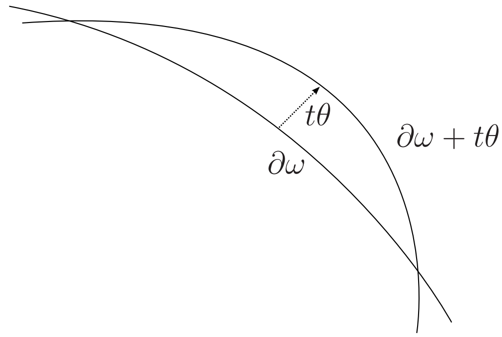


Figure 7.1: A drawing of ω and ω_t

And

$$\int_\Omega \left(|u_t|^2 - |u|^2\right) = 0$$

Assuming (7.2.2)

$$\lim_{t\longrightarrow 0} \int_\Omega \frac{|u_t|^2 - |u|^2}{t} = 0$$

which implies (2.8). □

Proposition 7.2.6. *The function*

$$\sigma_t = \alpha\chi_{\Omega\backslash\omega_t} + \beta\chi_{\omega_t}$$

Satisfies

$$\dot{\sigma} = \lim_{t\longrightarrow 0} \frac{\sigma_t \circ (I + t\theta) - \sigma_0}{t} = 0 \quad a.e \ in \ \Omega. \tag{2.9}$$

Sketch. If $x \in \partial\omega$, then $x + t\theta(x) \in \partial\omega_t$, hence

$$\sigma_t(I + t\theta)(x) = \sigma_0(x) \quad \forall\ t.$$

If $x \in \omega \cup int\,(\Omega\backslash\omega)$ we have the same relation showed above. Therefore the result follows. □

7.3 Variation of the first Eigenvalue

In the construction of the derivative of the eigenvalue functional we will need the following proposition.

Proposition 7.3.1. *For functions* $\varphi \in \mathcal{C}^\infty(\overline{\omega}) \cap \mathcal{C}^\infty\left(\overline{\Omega\backslash\omega}\right)$ *we have*

$$\begin{aligned} -\int_\Omega \theta \cdot \nabla\sigma\, |\nabla u|^2 &:= \langle -\theta\cdot\nabla\sigma\ ,\ \varphi\rangle &(3.10)\\ &= \int_\Omega \sigma div(\theta)\varphi + \int_\Omega \sigma\theta\cdot\nabla\varphi &(3.11)\\ &= \int_{\partial\omega} [\sigma\varphi]\,\theta\cdot ndS &(3.12)\\ &= \int_{\partial\omega} (\beta\varphi_\beta - \alpha\varphi_\alpha)\,\theta\cdot ndS &(3.13)\end{aligned}$$

Where $\varphi_\beta(x) = \lim\limits_{\omega x_n \longrightarrow x} \varphi(x)$, $\varphi_\beta(x) = \lim\limits_{\Omega\backslash\omega x_n\longrightarrow x}\varphi(x)$.

Proof. Let $\varphi \in C^\infty(\Omega)$,

$$\begin{aligned} \langle -\theta\cdot\nabla\sigma, \varphi\rangle &= \sum_{i=1}^N \left\langle -\theta_i \frac{\partial\sigma}{\partial x_i}, \varphi\right\rangle &(3.14)\\ &= \sum_{i=1}^N -\int_\Omega \theta_i \frac{\partial\sigma}{\partial x_i}\varphi &(3.15)\\ &= \sum_{i=1}^N \int_\Omega \frac{\partial\theta_i\varphi}{\partial x_i} - \sum_{i=1}^N \int_{\partial\Omega} \theta_i\varphi\sigma n_i dS &(3.16)\\ &= \sum_{i=1}^N \left(\int_\Omega \sigma\frac{\partial\theta_i}{\partial x_i}\varphi + \int_\Omega \sigma\theta_i\frac{\partial\varphi}{\partial x_i}\right) - \int_{\partial\Omega}\sigma\varphi\theta\cdot ndS &(3.17)\\ &= \int_\Omega \sigma div\,(\theta)\,\varphi + \int_\Omega \sigma\theta\cdot\nabla\varphi &(3.18)\end{aligned}$$

but $\sigma = \alpha\chi_{\Omega\backslash\omega} + \beta\chi_\omega$, then

$$\begin{aligned} \langle -\theta\cdot\nabla\sigma, \varphi\rangle &= \int_\Omega \sigma div\,(\theta)\,\varphi - \alpha\int_{\Omega\backslash\omega} div\,(\theta)\,\varphi - \int_{\partial(\Omega\backslash\omega)} \theta\cdot n\varphi - \beta\int_\omega div\,(\theta)\,\varphi - \int_{\partial\omega}\theta\cdot n\varphi\\ &= \alpha\int_{\partial\Omega\backslash\omega} \theta\cdot n\varphi + \beta\int_{\partial\omega}\theta\cdot n\varphi \end{aligned}$$

where n corresponds to the exterior normal in each domain $\Omega\backslash\omega$ and ω. Since $\sigma \equiv \alpha$ in $\Omega\backslash\omega$

and $\sigma \equiv \beta$ in ω, it follows that

$$\langle -\theta \cdot \nabla\sigma, \varphi \rangle = \int\limits_{\partial\omega} [\sigma\varphi]\, \theta \cdot n dS$$

□

Theorem 7.3.2. *The shape derivative of λ, given an admissible perturbation θ, reads as follows*

$$\lambda'(\omega;\theta) = \int_{\partial\omega} \left[\sigma \left|\nabla u\right|^2\right] \theta \cdot n dS. \tag{\textbf{LP}}$$

where $[\varphi]$ is the jump of φ across $\partial\omega$, that is,

$$[\varphi]\,(x) = (\varphi\lfloor_{\partial\omega^-} - \varphi\lfloor_{\partial\omega^+})\,(x). \tag{3.19}$$

with $\varphi\lfloor_{\partial\omega^-}$ and $\varphi\lfloor_{\partial\omega^+}$ denoting, respectively the inner and outer trace of φ on $\partial\omega$.

Proof. We denote $\lambda(\omega)$ quite simply by λ. By standard calculations in shape derivative calculus (see Simon [31] and (7.2.3)), we have

$$\left\{ \begin{array}{rcl} -div\,(\sigma'\nabla u + \sigma\nabla u') & = & \lambda' u + \lambda u' \qquad \text{in} \quad \Omega \\ u' & = & -\theta \cdot n\dfrac{\partial u}{\partial n} \qquad \text{on} \quad \partial\Omega \end{array} \right. . \tag{3.20}$$

We multiply the equation (2.6) by u and integrate by parts on Ω to obtain:

$$\begin{aligned} \int\limits_{\Omega} -div\,(\sigma'\nabla u + \sigma\nabla u')\, u =& \lambda' \int\limits_{\Omega} u^2 + \lambda \int\limits_{\Omega} u'u \\ \int\limits_{\Omega} (\sigma'\nabla u + \sigma\nabla u') \cdot \nabla u - \int\limits_{\partial\Omega} \frac{\partial\,(\sigma'\nabla u + \sigma\nabla u')}{\partial n} u =& \lambda' + \lambda \int\limits_{\Omega} u'u \\ \int\limits_{\Omega} \sigma' \left|\nabla u\right|^2 + \int\limits_{\Omega} \sigma\nabla u' \cdot \nabla u - \int\limits_{\partial\Omega} \frac{\partial\,(\sigma'\nabla u + \sigma\nabla u')}{\partial n} u =& \lambda' + \lambda \int\limits_{\Omega} u'u \\ \int\limits_{\Omega} \sigma' \left|\nabla u\right|^2 + \int\limits_{\Omega} \sigma\nabla u' \cdot \nabla u =& \lambda' + \lambda \int\limits_{\Omega} u'u \qquad (u = 0 \text{ on } \partial\Omega) \end{aligned} \tag{3.21}$$

We have

$$\int\limits_{\Omega} \sigma\nabla u \cdot \nabla u' = \int\limits_{\Omega} -div\,(\sigma\nabla u)\, u' + \int\limits_{\partial\Omega} \frac{\partial\sigma\nabla u}{\partial n} u'. \tag{3.22}$$

and by the formula (6.67)

$$\dot{u}(x) = \lim_{t \longrightarrow 0} \frac{u_t \circ (I + t\theta)(x) - u(x)}{t}.$$

Since the perturbation $\theta \equiv 0$ in a neighborhood of $\partial\Omega$, we have

$$x + t\theta(x) = x \quad \forall x \in \partial\Omega.$$

Since u y u_t are solutions of the Dirichlet problems in Ω we have

$$u_t \circ (I + t\theta)(x) = u_t(x) = 0 \quad \text{ and } u(x) = 0 \quad \forall\, x \in \partial\Omega$$

Hence, the relation of the total and local derivative (6.68) implies

$$u' = -\theta \cdot \nabla u \text{ on } \partial\Omega$$

But again, using that $\theta \equiv 0$ on $\partial\Omega$

$$u' \equiv 0 \quad \text{ on } \partial\Omega$$

It is possible to show (see appendix for further details)

$$u'\lfloor_{\Omega\setminus\omega} \in H^1(\Omega\setminus\omega) \quad \text{ and } \quad u'\lfloor_{\omega} \in H^1(\omega). \tag{3.23}$$

Then, the last equation with integrals (3.22) the boundary integral is zero, thus

$$\int_\Omega -div\,(\sigma\nabla u)\, u' = \int_\Omega \sigma\nabla u \cdot \nabla u'$$

Since u is an eigenfunction, the week formulation says

$$\int_\Omega \sigma\nabla u \cdot \nabla u' = \lambda \int_\Omega u u'$$

Then, from the equation (3.21) we have

$$\lambda' = \int_{\Omega} \sigma' |\nabla u|^2 \tag{3.24}$$

In (7.2.6) we showed easily that $\dot{\sigma} = 0$ in Ω. Thus, thanks to the formula (6.68)

$$\sigma' = \dot{\sigma} - \theta \cdot \nabla\sigma$$

therefore

$$\lambda' = -\int_{\omega} \theta \cdot \nabla\sigma\, |\nabla u|^2 \tag{3.25}$$

$$= \int_{\partial\omega} \left[\sigma\, |\nabla u|^2\right] \theta \cdot n dS \tag{3.26}$$

where in the last equivalence we have used the relation (3.12) of the proposition (7.3.1) □

Chapter 8

Desarrollo Numérico

8.1 Implementación

En esta sección se mostrará la implementación numérica de la derivada con respecto al dominio del valor propio obtenida en la fórmula (**LP**).

La implementación se realizó inicialmente con el programa Matlab[1]. Se eligió este programa en particular ya que posee el paquete de programas **PDE Toolbox**, el cual resuelve EDPs con una interfaz intuitiva para el usuario, con botones y ventanas, para una gran variedad de geometrías del dominio. Esta característica permite revisar los resultados obtenidos del cálculo de valor propio, comparando los valores del algoritmo con los obtenidos mediante la interfaz gráfica del toolbox de Matlab. [2]

La implementación se puede subdividir en 3 subpartes:

1) Implementación de la geometría del dominio.

2) Cálculo del primer valor propio y de la primera función propia.

3) Cálculo de la integral de contorno dada por la fórmula (**LP**).

En las próximas subseciones mostraremos las etapas necesarias para implementar los puntos 1), 2) y 3) explicados anteriormente.

[1]En un trabajo posterior de investigación se implementó también con el programa freeFem. En esta tesis explicaremos el detalle del código solamente para el programa Matlab.

[2]En virtud de comparar los resultados con otro programas, en un trabajo de investigación que va un poco más allá que esta tesis, se está implementando el mismo código en *freefem*.

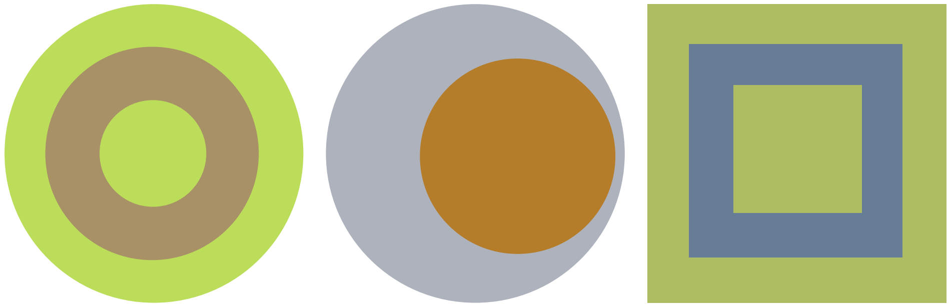


Figure 8.1: Configuraciones de geometría del dominio. Anillos concéntricos, bola desplazada y cuadrados. concentricos. EL material β se encuentra en la región café, naranja y azul respectívamente.

8.1.1 Geometría del dominio

En los experimentos numéricos que realizaremos, nos interesan tres tipos de geometrías para el dominio:

a) **Tres Anillos concéntricos.** En este caso considereramos Ω el disco unitario en $\mathbb{R}^2$. La región donde incorporamos el material con conductividad β corresponde a un anillo concéntrico en el interior del disco, y por lo tanto, el material de conductividad α en los anillos concéntricos aledaños.

b) **Disco desplazado dentro del disco unitario.** En este caso consideramos Ω el disco unitario en $\mathbb{R}^2$. La región donde incorporamos el material con conductividad β corresponde a un disco $\omega \subseteq \Omega$, el cual se encuentra ubicado en alguna región de Ω, no necesariamente en el centro.

c) **Tres Cuadrados concéntricos.** En este caso Ω es un cuadrado en $\mathbb{R}^2$. La región donde incorporamos el material con conductividad β corresponde a un marco rectángular concéntrico en el interior del rectángulo más grande Ω, donde los bordes son cuadrados centrados en el centro de Ω. El material de conductividad α se ubica en el interior de los marcos aledaños a la region del material β.

En cada una de estas geometrías nos interesa generar las condiciones que delimitan las regiones donde se encuentran los materiales. Las funciones implementadas par tal efecto fueron *prefix*DecomposedGeometryMatrix y *prefix*BoundaryCondition, donde *prefix* puede ser la palabra *ring*, *ball*, *squares* según sea la geometría a), b) o c) respectívamente.

La entrada de la función *prefixDecomposedGeometryMatrix* corresponde a los parámetros de configuración siguientes:

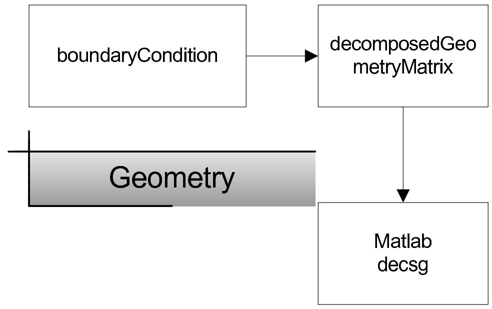


Figure 8.2: Esquema que representa la implementación de la geometría del dominio.

i) Cantidad de volumen ocupada por la region de conductividad β.

ii) Parámetro de ubicación de la región de conductividad β: radio menor de los anillos o cuadrados concéntricos o desplazamiento de la bola según sea la geometría del dominio.

Con la ayuda de la función *decsg* de matlab, se obtiene una completa descripción de la geometría del problema, que puede ser utilizada por otras funciones del *PDE toobox*.

En *prefixBoundaryCondition* se generan las condiciones de borde asociadas a la geometría para nuestro problema. Se generan las condiciones Dirichlet en el borde de Ω y las condiciones de transmisión en la frontera de las subregiones donde se encuentran los materiales. Además, para simplificar los cálculos posteriores entrega la salida de *prefixDecomposedGeometryMatrix*.

8.1.2 Malla

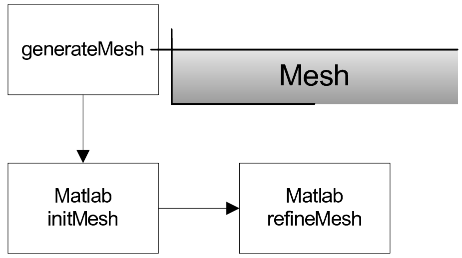
Dada la geometría del dominio entregada por la función *prefixBoundaryCondition*, la función generateMesh genera la malla de la geometría con el refinamiento y propiedades dadas en la entrada.

En términos generales, generamos 2 tipos de mallado:

i) **Mallado Homogéneo.** En este tipo de mallado nos preocupamos que los triángulos generados sean lo más similares posibles, esto es, en promedio dejamos los triangulos con la misma área. Utilizamos principalmente las funciones *initmesh*, *refinemesh y jigglemesh* y el parámetro *Hgrad* de Matlab con el fin de generar tales efectos. [1]

• **Mallado Simple.** En este tipo de mallado, en términos generales, dejamos la triangulación con las opciones por defecto que entregan las funciones de matlab *initmesh* y *refinemesh*.[1]

[1]Para ver mas detalles, ver el código comentado en el programa.



En las figuras (8.3) se muestran los tipos de mallados para distintas situaciones.

Cuando utilizamos el mallado simple y tenemos geometrías que tienen zonas muy delgadas con un tipo de material, la triangulación que se forma tiende a generar muchísmos más triángulos en las zonas delgadas. Para tener una medida más uniforme utilizamos la triangulación homogénea, la cual tiene como fin homogenizar la cantidad de triángulos en las regiones, independiente de el espesor de las zonas.

8.1.3 Poniendo el material

Para poner los materiales conductores en las diferentes regiones implementamos las funciones findBetaSubdomain*Postfix* y putMaterial*Postfix*, donde *Postfix* es la palabra *Ring*, *Square* o *Ball* según corresponda la geometría.

En findBetaSubdomain*Postfix* obtenemos el indice de la región donde está colocado el material de conductividad β.

En putMaterial*Postfix* colocamos los materiales en las distintas regiones del dominio, según lo especificado por los índices de los subdominios.

8.1.4 Valor propio

Para calcular el valor propio y la función propia asociada a una configuración específica, utilizamos la función *pdeeig* del PDE toolbox de Matlab. A esta función entregamos como parámetro las siguientes variables:

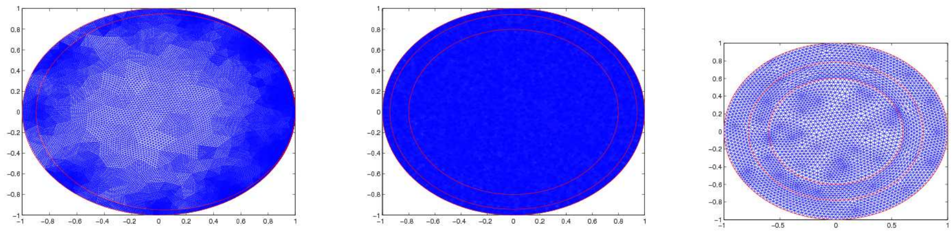
i. Descripción de las condiciones de borde asociadas a la geometría.

ii. Malla de la geometría.

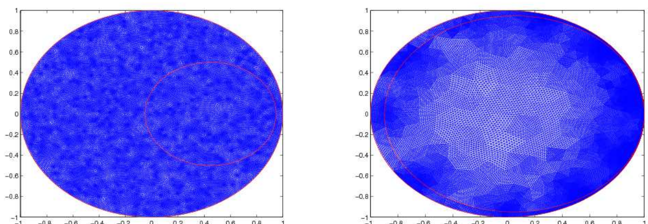
iii. Coeficientes de los materiales.

iv. Intervalo donde el algoritmo buscará valores propios.

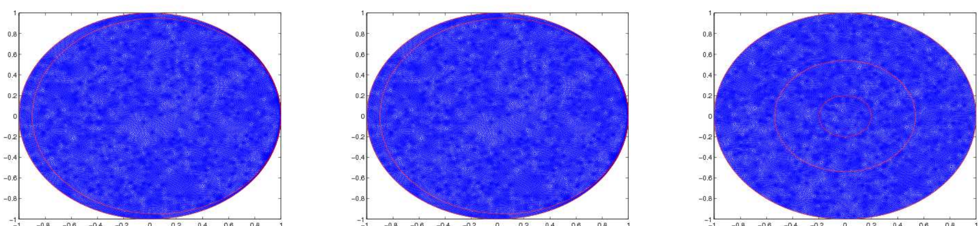
Para tener más detalles de la función sugerimos ver la ayuda de matlab.



(a)



(b)



(c)

Figure 8.3: Ejemplos de Mallados

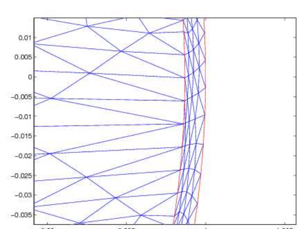
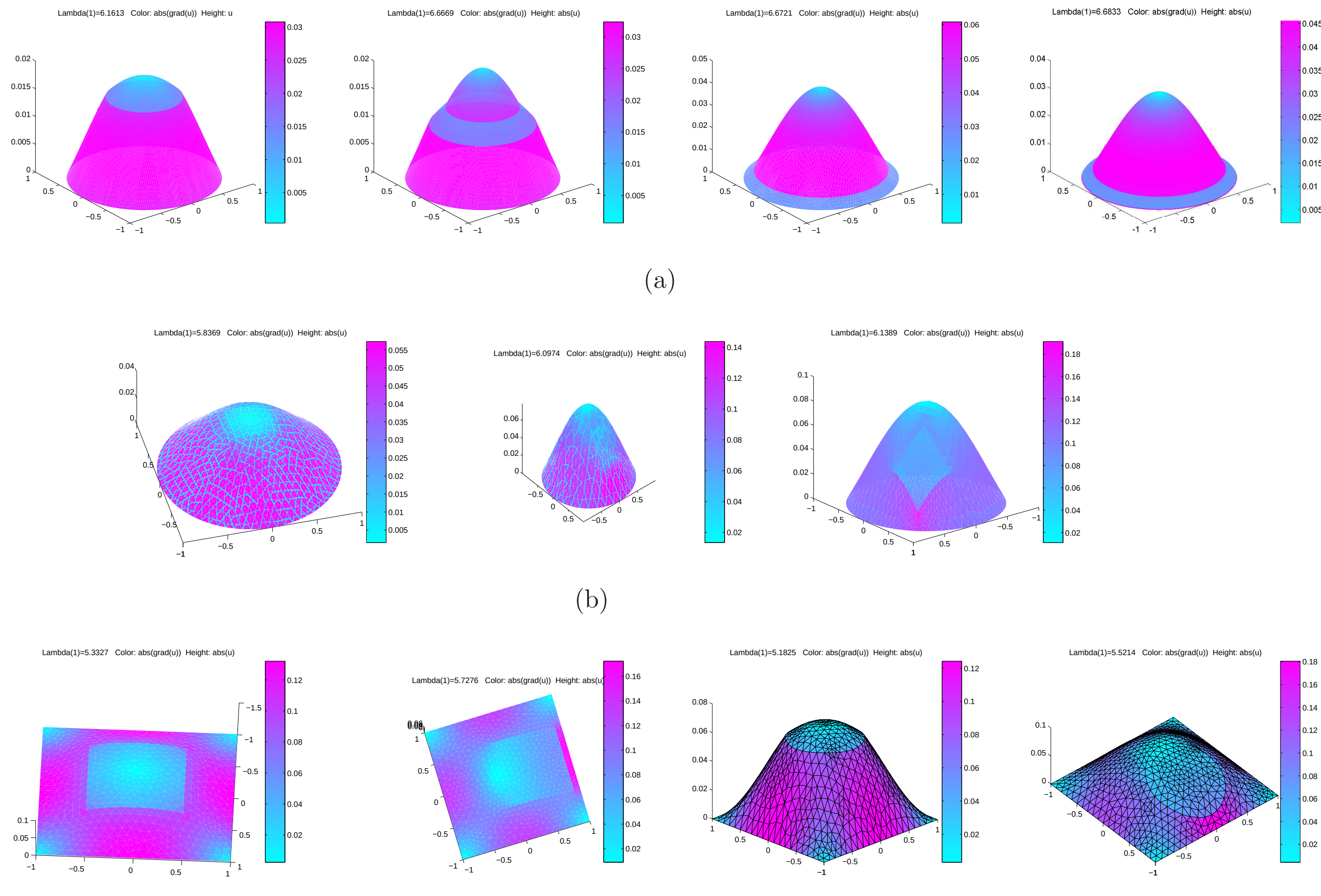


Figure 8.4: Cambio de tamaño en los triángulos cuando la diferencia de porte en las regiones es considerable.



(c)

Figure 8.5: Valores propios para distintas configuraciones. Los colores representan la magnitud del gradiente.

En las figuras (8.5) mostramos distintas funciones propias generadas para distintas geometrías.

8.1.5 Integral de contorno

Esta función es la que implementa el cálculo de integral de borde dada por la fórmula (**LP**). Los parámetros de entrada son:

i. Los arcos de la frontera de la región con material β.

ii. El índice de la subregión con material β.

iii. La malla de la geometría.

iv. Los coeficientes α y β.

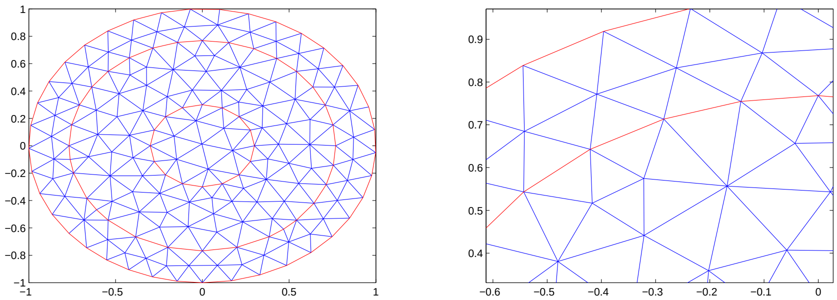


Figure 8.6: El mallado de la geometría permite discretizar la frontera de las subregiones con algunos arcos de la triangulación. Con la función *findBetaSubdomain* recuperamos estos arcos y con *findTrianglesFromEdge* obtenemos los triangulos limítrofes.

v. La primera función propia asociada a la configuración.

vi. La perturbación del dominio.

En lo que sigue explicaremos con un poco de más detalle esta función.

i. Para obtener un valor del gradiente del valor propio, generamos lo aproximamos generando una interpolación de éste en los puntos medio de cada triángulo con la función de matlab *pdegrad*.

ii. Encontramos los triangulos asociados a los arcos de la frontera de la región con la función *findTrianglesFromEdge*. Cada uno de estos triángulos está una región diferente.

iii. Discretizamos la integral de contorno (**LP**) utilizando una suma de Riemman sobre los arcos del contorno. Aproximamos la integral por una suma sobre los arcos de la frontera de la región. En la figura (8.6) se puede apreciar la discretización del borde de las subregiones.

iv. En cada uno de los arcos del contorno calculamos el salto $\left[\sigma \left|\nabla u\right|^2\right]$ utilizando los valores de σ y $\left|\nabla u\right|^2$ en el par de triángulos que comparten el arco. Cada uno de estos triángulos pertence a una región con distinto material.

v. Aproximamos la normal en el contorno por la normal a cada uno de los arcos del contorno.

vi. Multiplicamos la normal por el vector θ (que perturba la región ω) pasado por parámetro.

8.1.6 λ'

Para implementar la derivada con respecto al dominio dada por (**LP**) obtenemos las componentes conexas del borde de la región con material de conductividad β. En cada una de estas regiones calculamos las integrales de contornos entregadas por la funcion *contourIntegral*. Luego sumamos las cantidades (o las restamos, si consideramos el mismo signo para la normal) obtiendo la discretización de la cantidad (**LP**).

8.2 Resultados numéricos

En esta sección mostramos los resultados numericos obtenidos del cálculo de la derivada con respecto al dominio del primer valor propio de la ecuación de conductividad encontrada en la formula (**LP**).

Se realizaron experimentos concerniendo 3 tipos de geometría diferente:

a) **Tres Anillos concéntricos.** En este caso considereramos Ω el disco unitario en $\mathbb{R}^2$. La región donde incorporamos el material con conductividad β corresponde a un anillo concéntrico en el interior del disco, y por lo tanto, el material de conductividad α en los anillos concéntricos aledaños.

b) **Disco desplazado dentro del disco unitario.** En este caso consideramos Ω el disco unitario en $\mathbb{R}^2$. La región donde incorporamos el material con conductividad β corresponde a un disco $\omega \subseteq \Omega$, el cual se encuentra ubicado en alguna región de Ω, no necesariamente en el centro.

c) **Tres Cuadrados concéntricos.** En este caso Ω es un cuadrado en $\mathbb{R}^2$. La región donde incorporamos el material con conductividad β corresponde a un marco rectángular concéntrico en el interior del rectángulo más grande Ω, donde los bordes son cuadrados centrados en el centro de Ω. El material de conductividad α se ubica en el interior de los marcos aledaños a la region del material β.

Para cada una de estas geometrías (ver figura (8.1)) se hicieron cálculos del primer valor propio (ver fórmula (0.3) del capítulo 4) y la derivada de forma (**LP**). Cada dupla valor propio-derivada está en función del parámetro libre según la geometría del dominio. Este parámetro corresponde a:

- El radio interno del anillo donde se encuentra el material β, para el caso de los anillos concéntricos.

β	proporción
2	0.1
2	0.5
2	0.9
200	0.1
200	0.5
200	0.9

Table 8.1: Distintos valores de β y su proporción en los experimentos realizados.

- El desplazamiento del centro del disco donde se encuentra material β, para el caso de los discos desplazados.
- El semi-ancho del cuadrado interno que correponde a la frontera interna donde se encuentra el material β, para el caso de los cuadrádos concéntricos.

En todos los experimentos se fijó el valor $\alpha = 1$. Se generaron datos para los valores de β y proporción con respecto al área total dados en la tabla (8.1).

En las figuras (8.7) y (8.8) mostramos los resultados de los experimentos para el caso de los anillos concéntricos. En las figuras (8.9) y (8.10) mostramos los resultados de los experimentos para el caso de discos desplazados. En las figuras (8.11) y (8.12) mostramos los resultados de los experimentos para el caso de los cuadrados concéntricos. En este último caso no implementamos el cálculo de la derivada.

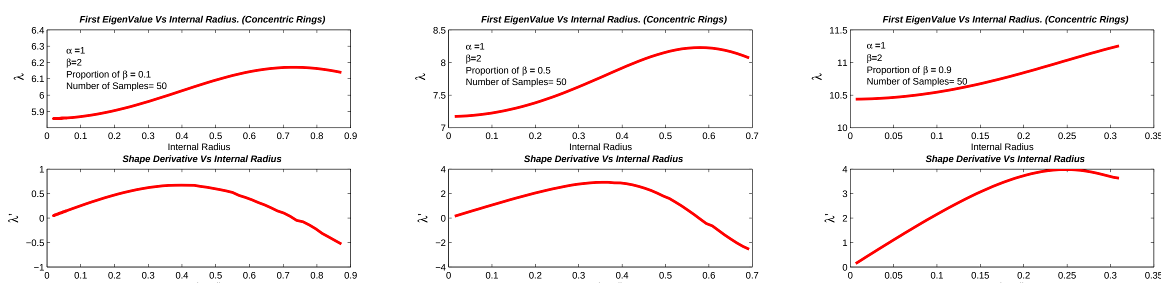


Figure 8.7: Discos concéntricos para $\beta = 2$

8.3 Análisis numérico

La generación de datos mostrada en la sección anterior tiene el propósito de validar las siguientes conjeturas:

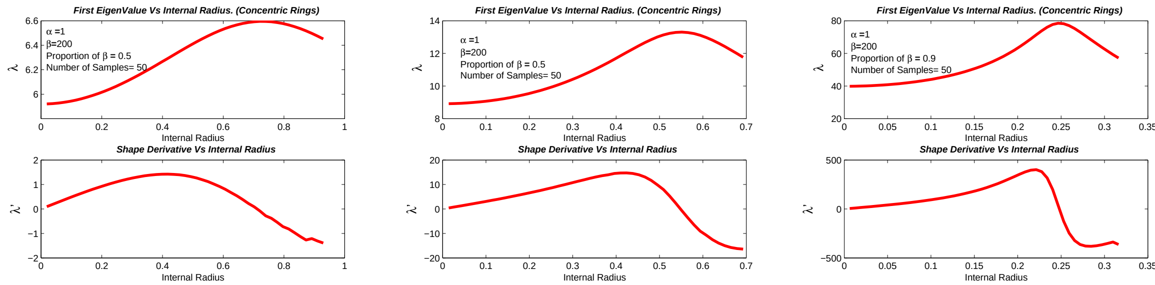


Figure 8.8: Discos concéntricos para $\beta = 200$

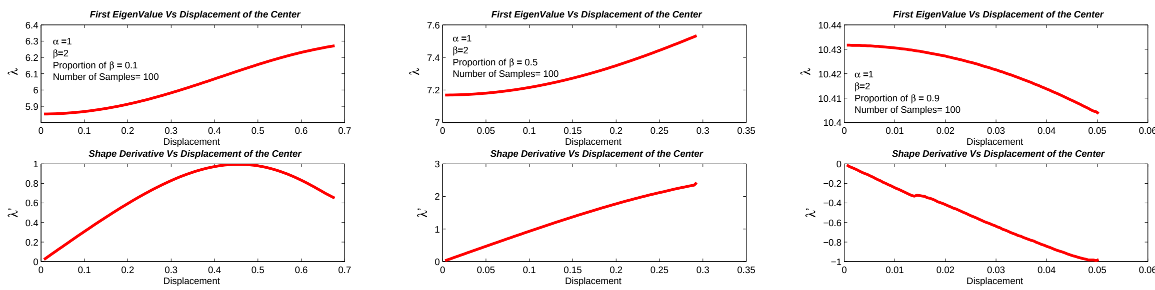


Figure 8.9: Discos No-Concéntricos para $\beta = 2$

Conjectura 8.3.1. *La solución óptima de (**GP**), en el caso del disco unitario en $\mathbb{R}^2$ o en la bola unitaria en $\mathbb{R}^N$, se encuentra poniendo el material de conductividad β en el medio.*

Conjectura 8.3.2. *La fórmula formal de la derivada con respecto al dominio dada por (**LP**) es correcta.*

Los experimentos relevantes son en virtud de las conjeturas anteriores son:

- El caso de anillos concéntricos.
- El caso de la bola desplazada del origen.

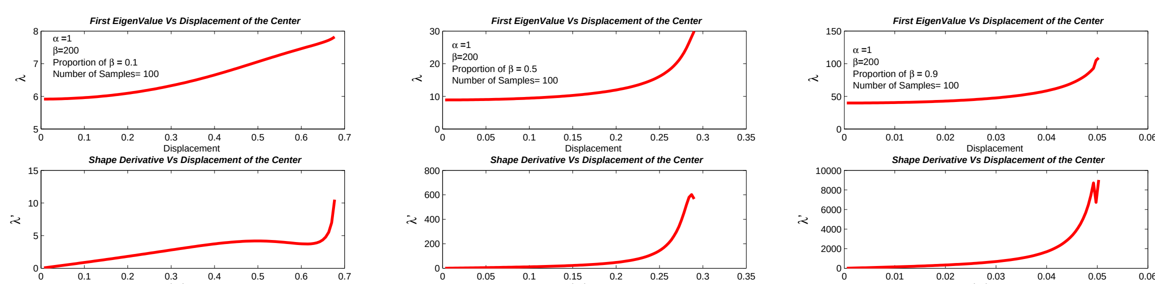


Figure 8.10: Discos No-Concéntricos para $\beta = 200$

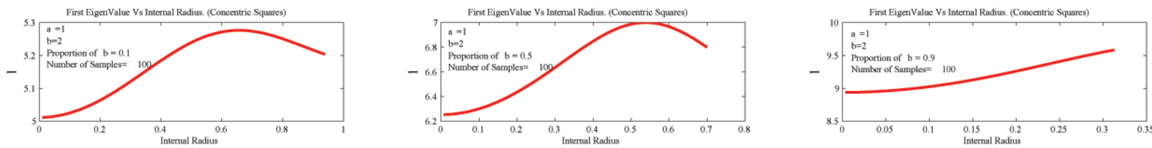


Figure 8.11: Cuadrados concéntricos para $\beta = 2$

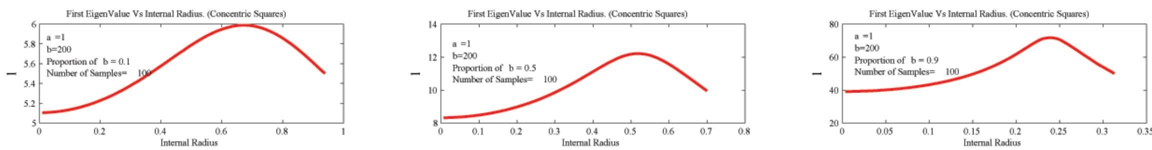


Figure 8.12: Cuadrados concéntricos para $\beta = 200$

El experimento en cuadrados tiene la función de dar una señal al estudio posterior (no considerado en esta tesis) del problema (**GP**) en el caso de un dominio con otro tipo de geometrías, como es el caso de cuadrados o estrellas.

Los datos numéricos resumidos en los gráficos (8.7), (8.8), (8.9), (8.10) muestran que en todos los experimentos, y en todas la geometrías, la fórmula (**LP**) tiene el mismo signo de la variación del primer valor propio con respecto al parámetro de la geometría. En otras palabras, la derivada con respecto al dominio (**LP**) pareciera, al menos en sentido numérico, ser correcta.

Pareciera que los datos numéricos comprueban la conjetura (8.3.1), sin embargo, en el caso de la bola desplazada (8.9), cuando la proporción de β es 0,9, el resultado numérico da a entender que el mínimo no se encuentra en la region esperada.

Cabe mencionar que el experimento anterior es bastante crítico. Apriori, sin analizar en detalle el experimento, podemos apreciar que la proporción de material β en este caso es muy grande. También, la variación numérica del valor propio es muy pequeña comparatívamente todos los otros experimentos, del orden de un centésimo.

En virtud de los resultados *espurios* del experimento anterior, se generaron más datos en posibles configuraciones indeseables (que no satisfagan ni (8.3.1) ni (8.3.2)). En la próxima sección se analiza lo más en detalle posible este tipo de configuraciones.

8.4 Errores numéricos

Nos dimos cuenta que en el caso de la bola desplazada (8.9), cuando la proporción de material β es grande, la solución óptima numéricamente pareciera que no se encuentra distribuyendo el material de conductividad β en el medio, si no que dejando la bola desplazada lo más al borde posible.

Creemos fuertementemente que esto es solo un error numérico. En esta sección buscaremos más pruebas, que nos permitan decir con más convicción que esto es un error numérico y que, por lo tanto, no se relaciona con los resultados esperados.

Para entender cuál es el fenómeno de por medio se realizaron más experimentos con $\beta = 2$, cambiando la proporción del material. En la figura (8.13) se muestra el cambio de la proporción en multiplos de 10. Se observa que antes de tener el 70% de la región ocupada con material β los resultados concuerdan con las conjeturas (8.3.1) y (8.3.2).

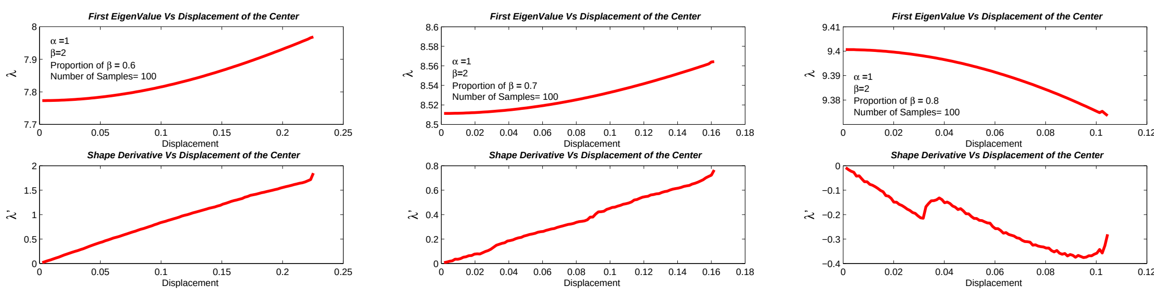


Figure 8.13: Comportamiento en el experimento de discos desplazados. El error numérico comienza entre las proporciones 0.7 y 0.8 para β.

Probando en distintas configuraciones, nos dimos cuenta de la siguiente observación.

Remark 8.4.1 (Configuración Critica)**.** El conjunto de configuraciones críticas a analizar que arrojaron la mayor cantidad de errores numéricos, se encuentran en torno al punto de configuración siguiente:

- $\beta = 2$
- Proporcion de β: 0.75 o 0.753
- Numero de nodos en la frontera del disco unitario: 100.
- Número de nodos en la frontera del disco interno: 100.

Algunas configuraciones se muestran en la figura (8.14).

Comparando el grupo de gráficos de las figuras (8.13) y (8.14) se puede apreciar que, en los resultados que están de acuerdo con las conjeturas (8.3.1) y (8.3.2), el valor numérico del valor propio varía según el parámetro de la geometría en un orden ya sea de centésismos o décimos. Sin embargo, en los experimimentos que no concuerdan con nuestras suposiciones, las variaciones son del orden de los milésimos y centésimos. Es bastante claro que el orden

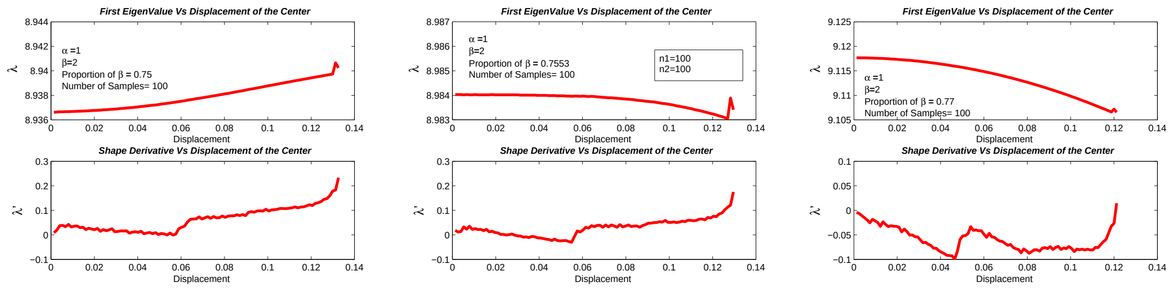


Figure 8.14: Comportamiento en el experimento de discos desplazados. Algunas configuraciones críticas.

de la variación tiene mucha relación con lo poco esperado de los resultados: la variación del valor propio es muy pequeña en estos casos.

Nos interesa ahora observar el comportamiento cuando refinamos la malla. Denotamos por n_1 y n_2 la cantidad de nodos en los bordes del disco exterior e interior respectívamente. Estudiamos el caso para $\beta = 2$, proporción 0.773 para distintos valores de n_1 y n_2. En la figura (8.15) se muestran los resultados. Para el valor propio, se observan variaciones del orden de un milésimo. El valor propio decrece muy poco y el cálculo de la derivada es incongruente.

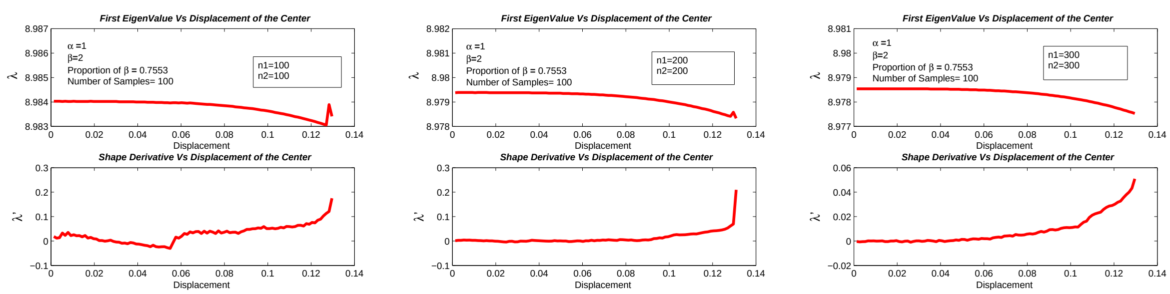
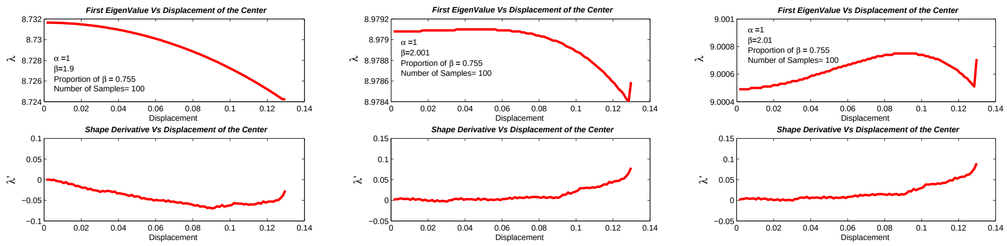


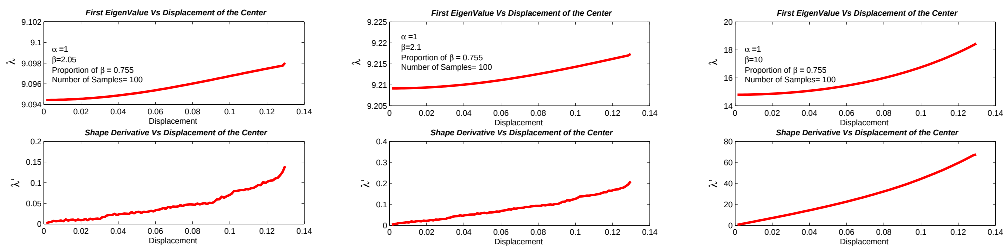
Figure 8.15: Comportamiento en el experimento de discos desplazados cambiando el refinamiento de la malla.

Ahora nos interesa estudiar los casos problemáticos dados en la observación (8.4.1) cuando perturbamos el valor de β.

En los experimentos mostrados en el bloque de figuras (8.16) dejamos fijo la proporción de β en el valor 0.755 y la cantidad de nodos en las fronteras externa e interna los fijamos en 200 cada uno. Se aprecia que para las configuraciones con β ligeramente diferentes de 2, el comportamiento no refleja claramente lo esperado por las conjeturas (8.3.1) y (8.3.2). A partir de $\beta = 2.05$ en adelante, tanto (8.3.1) como (8.3.2) se condicen con los resultados numéricos.



(a) Bad configurations near $\beta = 2$



(b) From $\beta = 2.05$ on the configurations behave correctly

Figure 8.16: Behavior in the displaced disc experiment changing the value of β.

8.5 Conclusiones numéricas

El comportamiento numérico que se observa en las configuraciones mostradas en la sección de errores anterior (8.4), es de una alta inestabilidad, tanto para el primer valor propio como para la fórmula (**LP**) de la derivada del primer valor propio con respecto a la geometría.

En estas experiencias numéricas, el cambio del valor propio numérico no supera el orden 10^{-3}, salvo en pocos casos como en el que la proporción de β es igual a 0.9, donde el orden es de 10^{-2}.

A excepción de lo anterior, los experimentos se comportan según nuestras conjeturas (8.3.1) y (8.3.2) y, en estas configuraciones, los cambios del valor propio son mayores a 10^{-2}.

Además, en el grupo de gráficos (8.16) apreciamos que, aumentando levemente el valor de β, desde 2.05 en adelante, el cálculo para las configuraciones *problemáticas* se regulariza, entregando valores según las conjeturas (8.3.1) y (8.3.2). Claramente el valor la magnitud de la diferencia entre los coeficientes α y β tiene relevancia para el desempeño numérico.

Por otro lado, debido a que el problema tiene una condición Dirichlet, el computador penaliza con valores muy altos los nodos del borde, otorgando, para la matriz de esfuerzos, un condicionamiento muy grande, del orden de 10^{28}.

Concluimos que los resultados espurios se deben a turbulencias numéricas. Los cálculos del valor propio y de la derivada con respecto a la forma se tornan inestables, poco analizables, y además, los órdenes de magnitud del valor propio son muy pequeños. Estos valores no son relevantes para el análisis debido al mal condicionamiento de la matriz .

Finalmente, en términos generales, concluimos que el desempeño numérico confirma las conjeturas (8.3.1) y (8.3.2).

Chapter 9

Characterization of the Solution

In the theorem (6.4.5) we have proved the existence of a classical radially symmetric solution of the problem (**GP**) when the domain Ω is a ball in $\mathbb{R}^N$, which in other words, assures the two materials are distributed in measurable rings, and so there is no homogenization zone. Nevertheless, nothing tell us that it is not possible to have a zone of positive measure with empty interior where the materials are distributed. Thus in the practice it couldn't be possible to distinguish accurately where the materials are placed.

The previous situation is not very comforting, at least practically speaking. We can ask ourself several questions that partially relieve us from a situation like this.

- How many connected components can there be of the different materials?
- What can be said about the perimeter of each set where are placed the materials? Is it finite?
- Is it possible to find connected components having only material α or β?

In particular we strongly think that in the case of a ball in $\mathbb{R}^N$, the solution consists in placing the material β in the middle of the ball.

9.1 Optimal distribution in a ball

We know by the results of the paper Alvino et. al. that the minimum is attained for a radially symmetric distribution of the materials α and β which means that the materials could be distributed in various spherical shells.

We have the following conjecture

Conjecture 9.1.1. *When Ω is a ball, among all radially symmetric distributions of β with fixed volume m the configuration where all the material β is in the middle gives the lowest value to the first eigenvalue.*

We now give some arguments which partially justify this conjecture. The procedure consists in showing, systematically, using the shape derivative calculated in the previous section, that whenever there is a layer of α preceding a layer of β (as we move radially outward) $\lambda'(\omega;\theta) < 0$ for the radially symmetric perturbation θ which moves the layer of β inwards while conserving the volumes of α and β.

Theorem 9.1.2. *Assuming that the formula (**LP**) holds, given an annulus configuration w_0 where we put the material β in a annulus of non empty interior, for perturbation θ of this annulus which moves it outwards while preserving the volume, we have that the shape derivative of the first eigenvalue satisfies*

$$\lambda'(w_0) \geq 0.$$

This indicates that is better to place the material β in the middle.

Proof. Denote the reference configuration by σ and let u be the normalized first eigenfunction, which we know to be radially symmetric (see for instance [9]). Let us concentrate on a layer ω_0 of β which follows a layer of α and let us write its boundary as $S_1 \cup S_2$ where S_1 and S_2 are, respectively, the inner and outer boundaries. We may consider a radially symmetric perturbation θ which is zero outside ω_0 and conserves the volume of ω_0. The conservation of mass condition (2.4) gives the relation

$$(\theta \cdot n)\lfloor_{S_1} \mathrm{per}(S_1) + (\theta \cdot n)\lfloor_{S_2} \mathrm{per}(S_2) = 0 \tag{1.1}$$

where $\mathrm{per}(S)$ is the perimeter in $\mathbb{R}^{N-1}$ defined in (3.1.46) of S, for instance, if ω_0 is an annulus in $\mathbb{R}^2$ and S is one of its boundaries, $\mathrm{per}(S) = 2\pi r$ with an appropriate radius r. Now, from the equation (**LP**) we have

$$\begin{aligned}
\lambda' &= \int_{S_1} \left[\sigma \left|\nabla u\right|^2\right] \theta \cdot n dS + \int_{S_2} \left[\sigma \left|\nabla u\right|^2\right] \theta \cdot n dS \\
&= \left(\left[\sigma \left|\nabla u\right|^2\right] \theta \cdot n\right) \lfloor_{S_1} \mathrm{per}(S_1) + \left(\left[\sigma \left|\nabla u\right|^2\right] \theta \cdot n\right) \lfloor_{S_2} \mathrm{per}(S_2) \\
&= \left(\left[\sigma \left|\nabla u\right|^2\right] \lfloor_{S_2} - \left[\sigma \left|\nabla u\right|^2\right] \lfloor_{S_1}\right) (\theta \cdot n)\lfloor_{S_2} \mathrm{per}(S_2)
\end{aligned} \tag{1.2}$$

where we in the last to equations we have used that $\left[\sigma \left|\nabla u\right|^2\right]$ is constant in each boundary. Let us analyze (1.2) for its sign. Denoting by S_i^- and S_i^+ the inner and outer surfaces of S_i

with respect to w_0, for $i = 1, 2$ (see figure 9.1), the transmission condition on S_i reads

$$(\sigma \nabla u \cdot n)_{S_i^-} = (\sigma \nabla u \cdot n)_{S_i^+} \quad i = 1, 2. \tag{1.3}$$

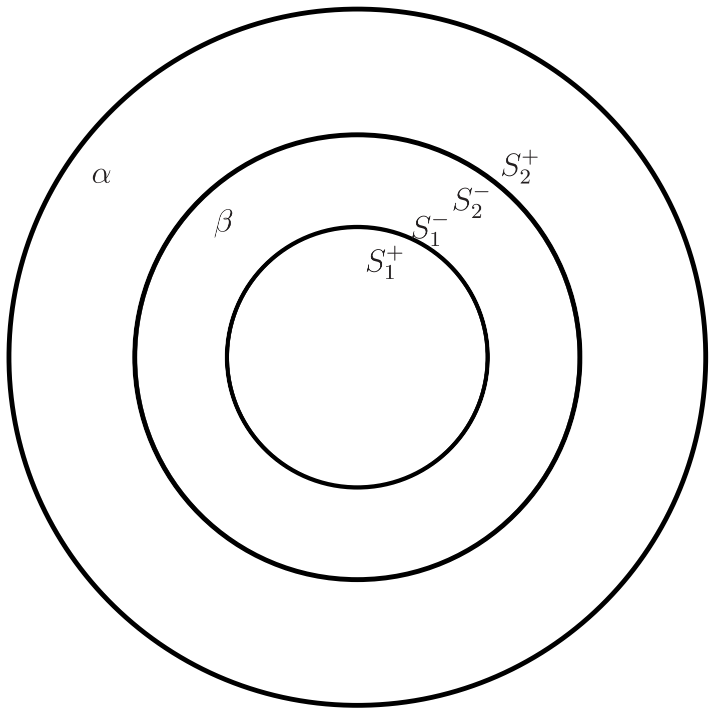


Figure 9.1: Symbols for the different boundaries.

In view of the fact that u is radially symmetric, we can write $n = \frac{\nabla u}{|\nabla u|}$, and, therefore, from the above relation we conclude that

$$\sigma |\nabla u| \lfloor_{S_i^-} = \sigma |\nabla u| \lfloor_{S_i^+} \quad i = 1, 2. \tag{1.4}$$

This allows to write the jumps in $\sigma |\nabla u|^2$ as follows

$$\begin{aligned} \left[\sigma |\nabla u|^2\right] &= \left(|\nabla u| \lfloor_{S_i^+} - |\nabla u| \lfloor_{S_i^-}\right) (\sigma |\nabla u|) \lfloor_{S_i^-} \\ &= \left(\frac{\beta}{\alpha} - 1\right) |\nabla u| \lfloor_{S_i^-} (\sigma |\nabla u|) \lfloor_{S_i^-} \end{aligned}$$

Therefore, the shape derivative in (1.2), can be rewritten as

$$\lambda' = \left(\frac{1}{\alpha} - \frac{1}{\beta}\right) \left((\sigma |\nabla u| \lfloor_{S_2^-})^2 - (\sigma |\nabla u| \lfloor_{S_1^-})^2\right) (\theta \cdot n) \lfloor_{S_2} \mathrm{per}(S_2). \tag{1.5}$$

Recalling the remark (6.3.4) says that u is a decreasing function and that $\sigma \dfrac{\partial u}{\partial n}$ is a decreasing function in each region where σ is constant, it follows that $\sigma\,|\nabla u|$ is non-decreasing in the radial direction on ω_0. Therefore, λ' assumes a positive sign if θ is such that $\theta \cdot n$ is positive on S_2 and consequently $\theta \cdot n$ is positive on S_1. This means that λ increases if shell is moved outwards. This concludes the proof of the theorem.

$\square$

Chapter 10

Conclusions

We have been able to give a new proof of the fact that the problem (**GP**), in a spherically symmetrical domain in $\mathbb{R}^N$ has a radially symmetrical solution. We expect very soon to extend the existence result to the case of domains with less symmetries, such as squares and stars, by using other forms of symmetrization.

In Chapter 7 we calculated the shape derivative of the eigenvalue functional. Using this and also through some numerical experiments made in planar domains we gathered evidence for our conjecture that the minimum value in (**GP**) is obtained by placing the material of conductivity β in the middle. One of our aims is to rigorously establish this conjecture based on a careful analysis of the shape derivative.

Conclusiones

Hemos podido dar una nueva prueba del hecho que el problema (**GP**), restringido a un dominio esféricamente simétrico en $\mathbb{R}^N$, posee una solución radialmente simétrica. Con los elementos matemáticos introducidos esperamos, en un futuro próximo de investigación, tener un mejor entendimiento de este problema, lo que nos permitiría en particular, extender el resultado de existencia a dominios con menos simetrías, como cuadrados o estrellas.

Las herramientas de derivación con respecto al dominio introducidas en la sección (3.6) permitieron obtener la derivada del primer valor propio con respecto a la distribución de materiales. Usando lo anterior, en conjunto con algunos experimentos numéricos realizados en dominios planos, fue posible obtener evidencia del hecho que, cuando el dominio es una bola en $\mathbb{R}^N$, es conveniente distribuir el material β en el centro.

Nuestra intención, en un trabajo de investigación ulterior, es confirmar rigurosamente esta conjetura, analizando cuidadosamente la derivada del primer valor propio con respecto al dominio.

Appendix A

First Paper

A.1 Abstract

The pioneering works of F. Murat and L. Tartar [26] go a long way in showing, in general, that problems of optimal design may not admit solutions if microstructural designs are excluded from consideration. Therefore, assuming, tactitly, that the problem of minimizing the first eigenvalue of a two-phase conducting material with the conducting phases to be distributed in a fixed proportion in a given domain has no true solution in general domains, S. Cox and R. Lipton only study conditions for an optimal microstructural design [11]. Although, the problem in one dimension has a solution (cf. Kreĭn [22]) and, in higher dimensions, the problem set in a ball can be deduced to have a radially symmetric solution (cf. Alvino et. al. [4]), these existence results have been regarded so far as being exceptional owing to complete symmetry. It is still not clear why the same problem in domains with partial symmetry should fail to have a solution which does not develop microstructure and respecting the symmetry of the domain. We hope to revive interest in this question by giving a new proof of the result in a ball using a simpler symmetrization result from A. Alvino and G. Trombetti [3].

A.2 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ which is to be called the design region. Let m be a positive number, $0 < m < |\Omega|$, where $|\Omega|$ is the total volume (Lebesgue measure) of the region Ω. Two materials with conductivities α and β $(0 < \alpha < \beta)$ are distributed in arbitrary disjoint measurable subsets A and B, respectively, of Ω so that $A \cup B = \Omega$ and $|B| = m$. For any such distribution, it is well known (cf. [10, 6, 21]) that the first eigenvalue in the spectral problem

$$\begin{cases} -\mathrm{div}\,((\alpha\chi_A + \beta\chi_B)\nabla u) &= \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Omega \end{cases} \tag{2.1}$$

is given by

$$\lambda^1(B) := \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega (\alpha\chi_A + \beta\chi_B)|\nabla u|^2 dx}{\int_\Omega |u|^2 dx}. \tag{2.2}$$

Let $\mathcal{A} := \{B : B \subset \Omega, B \text{ measurable}, |B| = m\}$ be the class of admissible domains for the material with conductivity β. We are interested in the following eigenvalue minimization problem

$$\inf\left\{\lambda^1(B) : B \in \mathcal{A}\right\}. \tag{2.3}$$

Starting from the works of Murat and Tartar on a control problem involving immiscible fluids [26] it is well known that, generally speaking, optimal design problems may not always have a solution if the development of microstructures is not taken into consideration. However, if microstructures are allowed as admissible designs then the infimum is reached corresponding to some microstructure. In the case of our problem such an approach was followed by Cox and Lipton [11] and a characterization of the optimal microstrucure has been established. Nevertheless, the original problem in the one-dimensional case and, in the case of a ball admit true solutions with symmetry as has been shown by Kreĭn [22] and Alvino et. al. [4], respectively. The one-dimensional problem was solved by Kreĭn [22, 16] by exploiting the equivalence between the original problem and a similar problem for a vibrating membrane involving the objective functional

$$\lambda^1(B) := \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega (\alpha\chi_A + \beta\chi_B)|u|^2 dx} \tag{2.4}$$

although this equivalence does not hold in higher dimensions. The works of Cox and McLaughlin [12, 13] show that the latter problem, in any dimension, has a true solution. It still remains to answer the question whether the original problem admits a minimum. Our aim is to revive interest in this question by giving an almost self-contained and a vastly simplified treatment of the existence result in a ball originally due to Alvino et. al. [4]. The main result of our paper reads as follows.

Theorem *Let Ω denote a ball in $\mathbb{R}^n$. The problem (2.3) of minimizing the first eigenvalue, defined by (2.1), given two conducting materials with conductivities α, β, in given ratio, admits a radially symmetric solution.*

It is worth observing that this is a kind of Faber-Krahn inequality for non-homogeneous elliptic operators. The paper of Alvino et. al. [4] treats many other problems of this kind. We refer also to Burton [7] for some results on problems of a similar nature.

Our proof of the above theorem will be based on a symmetrization result from Alvino and Trombetti [3, Lemma 1.2] whereas the original proof given in [4] is based on a more fine comparison result for the solutions of Hamilton-Jacobi equations [4, Theorem3.1].

Plan of the paper: In the next section we shall introduce some notations and recall, briefly, the Schwarz symmetrization and some basic results on the Schwarz symmetrization. The problem will then be reformulated in a way that makes it possible to apply symmetrization techniques. Subsequently, we shall discuss some of the structural properties of the objective functional and the constraint set provided in [4, 3]. We then recall a symmetrization result [3, Lemma 1.2] of which we give a different but formal proof (see Appendix) which could be adapted to domains with partial symmetry.

In Section 3, we shall give a proof of the main theorem (cf. Corollaries 3.1 and 3.2 [4]) with the help of the above symmetrization result and some basic properties discussed in [4, 3] instead of the more intricate [4, Theorem 3.1].

A.3 Notations and Preliminaries

As the results of this article concern a ball, henceforth, Ω will refer to $B(0,1)$, the n-dimensional unit ball in $\mathbb{R}^n$ centered at the origin. We shall use f^{-1} to denote the reciprocal of a non-vanishing real valued function f. Given a measurable function $f:\Omega\to\mathbb{R}$ and a real number c, $\Omega_{f,c}$ will denote the level set

$$\Omega_{f,c} := \{x\in\Omega : f(x)\geq c\} \tag{3.5}$$

which is a measurable subset of Ω and, of course, depends on the function f. We denote by $\Omega^*_{f,c}$ a ball concentric to Ω and having the same Lebesgue measure as $\Omega_{f,c}$.

Schwarz symmetrization: The Schwarz symmetrization of the function f is a radially symmetric decreasing function f^* defined on Ω through the relation

$$f^*(z) := \sup\{c\in\mathbb{R} : z\in\Omega^*_{f,c}\}\,. \tag{3.6}$$

It follows from the very definition of f^* that $\{f^*\geq c\}=\Omega^*_{f,c}$ and therefore, that the functions

f and its Schwarz symmetrization f^* are equimeasurable in the sense that

$$|\{x \in \Omega : f(x) \geq c\}| = |\{z \in \Omega : f^*(z) \geq c\}| \,. \tag{3.7}$$

Remark A.3.1. The $\geq$ sign in (3.7) can be changed to $\leq$ without changing any of the consequences. As a consequence, the relation in (3.7) holds with the $=$ sign replacing $\geq$, but this cannot be taken as a characterization of equi-measurability, except when we deal with simple functions.

The equimeasurability property has several important consequences such as, for any measurable function, $h : \mathbb{R} \to \mathbb{R}$, we have:

$$\int_\Omega h(f(x))\, dx = \int_\Omega h(f^*(z))\, dz \,. \tag{3.8}$$

In particular, one has

$$\int_\Omega |f(x)|^2\, dx = \int_\Omega |f^*(z)|^2\, dz \,. \tag{3.9}$$

The following inequality is also fundamental (cf. [21, Proposition 1.2.2]) :

$$\int_E f(x)\, dx \leq \int_{E^*} f^*(x)\, dx \tag{3.10}$$

for all measurable subsets $E \subset \Omega$. Another fundamental property of the Schwarz symmetrization is the iso-perimetric inequality

$$P(\{f \geq c\}) \geq P(\{f^* \geq c\}) \tag{3.11}$$

where $P(C)$ denotes the perimeter of a subset C in Ω, when it is defined.

Suitable forms of the above properties are also true of various other forms of symmetrization. An extensive treatment of the various forms of symmetrizations and their applications may be found in the monographs [19, 21, 27].

A reformulation of the minimization problem: Let us begin by considering λ^1, defined in (2.1), as a function of $\nu := \alpha\chi_{\Omega\setminus B} + \beta\chi_B$ instead of looking at it as a set function while writing $\lambda^1(\nu)$ for $\lambda^1(B)$. Let $\theta := \alpha\chi_{\Omega\setminus B_0} + \beta\chi_{B_0}$ be the simple function where B_0 is a ball centered at 0 having Lebesgue measure m. Note that θ is a radially symmetric and decreasing function.

Proposition A.3.2. *The minimization problem (2.3) can be recast as*

$$\inf\left\{\lambda^1(\nu) : \nu^* = \theta\right\} . \tag{3.12}$$

Proof: It is clear that if $\nu := \alpha\chi_{\Omega\setminus B} + \beta\chi_B$ for some $B \in \mathcal{A}$ then it's radially symmetric decreasing rearrangement is the function θ. We would like to establish the converse now. By the last part of Remark A.3.1 in the previous section, as θ is a simple function, if we have $\nu^* = \theta$ then ν is a simple function taking the same values as θ on sets of equal measure. In particular, $|\{x \in \Omega : \nu(x) = \beta\}| = |B_0| = m$. So, the one-one correspondence between the constraints in (2.3) and (3.12) is established. □

In the same way, if we set $\eta(\xi) = \lambda^1\left(\xi^{-1}\right)$, the minimization problem can also be written as

$$\inf\left\{\eta(\xi) : \xi^* = \left(\theta^{-1}\right)^*\right\} . \tag{3.13}$$

The infimum in a minimization problem will be attained, by the direct methods of the calculus of variation, if it happens that the objective functional is lower semi-continuous and the constraint set is compact for some topology.

The constraint set in either formulation (3.12) or (3.13) is of the form

$$C(\varphi) = \{f : f^* = \varphi\} \tag{3.14}$$

given φ which is a non-negative, bounded, measurable, radially symmetric decreasing function on the ball Ω. This set is relatively compact for the weak-$*$ topology as a subset of $L^\infty(\Omega)$ as all $f \in C(\varphi)$ have the same L^∞ norm as φ, being equimeasurable with φ and, as bounded sets in $L^\infty(\Omega)$ are weak-$*$ compact. However, this is not closed as, in the first place, weak-$*$ limits of simple functions need not be simple whereas, we have seen, in the arguments given in the proof of Proposition A.3.2, that the Schwarz symmetrization of a simple function is also a simple function.

Remark A.3.3. In general, in order to calculate the infimum, at first, the closure of the constraint set needs to be calculated with respect to a suitable topology and then, the lower semicontinuous envelope of the objective functional with respect to the same topology. In our problem, this is hard to achieve without the consideration of micro-structural designs and, the results of Cox and Lipton [11] are in this spirit but lead further away from the study of a classical solution.

We now put together some observations which highlight some of the structure of the problem leading to the determination of a classical solution to our problem. A characterization of the weak-$*$ closure of this set in $L^\infty(\Omega)$, to be denoted by $K(\varphi)$, was given by Migliaccio [24].

Proposition A.3.4. *The set $K(\varphi)$ is a weak-$*$ compact convex set characterized by the relation*

$$K(\varphi) = \left\{ f \in L^\infty(\Omega) : \int_{B(0,r)} f(x)\,dx \leq \int_{B(0,r)} \varphi(z)\,dz \;\forall r, \int_\Omega f(x)\,dx = \int_\Omega \varphi(z)\,dz \right\}. \tag{3.15}$$

Proposition A.3.5. *The set $C(\varphi)$ is the set of extreme points of $K(\varphi)$.*

These results can be found in Alvino et. al. [4, Section 2]. Let us now make the following simple observation.

Remark A.3.6. It is quite easy to see that the above propositions continue to hold if we consider $C^s(\varphi)$ and $K^s(\varphi)$ consisting of the radially symmetric functions in $C(\varphi)$ and $K(\varphi)$, respectively.

The following proposition establishes the continuity of the first eigenvalue with respect to weak-$*$ convergence of the reciprocals of the coefficients, for radially symmetric coefficients. A similar convergence result is proved in [4, Corollary 3.2] but for minimizing sequences of the functional λ^1. It is worth mentioning here that the objective functional λ^1 is not lower semi-continuous for the weak-$*$ convergence of the coefficients.

Proposition A.3.7. *Let ν_i be a sequence of radially symmetric functions in $K(\varphi)$ such that ν_i^{-1} converges weakly-$*$ to a function ν^{-1} as i tends to ∞. Then, we have $\lambda^1(\nu_i)$ converges to $\lambda^1(\nu)$ as i tends to ∞.*

Proof: Let the sequence ν_i and the function ν satisfy the hypotheses of the proposition. We write $\nu_i(x) = \dfrac{1}{\xi_i(|x|)}$ and $\nu(x) = \dfrac{1}{\xi(|x|)}$. Then, by the hypothesis it follows that ξ_i weak-$*$ converges to ξ in $L^\infty(0,1)$. Now, if u_i gives the minimum value in the definition of $\lambda^1(\nu_i)$ then it can be argued, using the Kreĭn-Rutman theorem, that this is radially symmetric. We may also assume that u_i is non-negative and further, normalize it so that it's L^2 norm is 1. The Euler equation corresponding to the minimizing property of u_i reads

$$-\text{div}\,(\nu_i \nabla u_i) = \lambda^1(\nu_i) u_i\,. \tag{3.16}$$

It can be checked from this that the sequence u_i is bounded in $H_0^1(\Omega)$ and a subsequence can be extracted converging weakly in $H_0^1(\Omega)$ to a radial function $u(x) = v(|x|)$. A further subsequence, indexed by i_k, may be extracted so that $\lambda^1(\nu_{i_k})$ converges to some λ as $k \to \infty$. Now, writing $u_{i_k}(x) = v_k(|x|)$, the Euler equation (3.16) in radial co-ordinates, for this subsequence, reads

$$-\left(r^{n-1}\frac{1}{\xi_{i_k}(r)}v_k'(r)\right)' = \lambda^1(\nu_{i_k})r^{n-1}v_k(r)\,. \tag{3.17}$$

By integration, we obtain

$$r^{n-1}\frac{1}{\xi_{i_k}(r)}v_k'(r) = -\lambda^1(\nu_{i_k})\int_0^r s^{n-1}v_k(s)\,ds\,. \tag{3.18}$$

It can be checked that the sequence v_k converges weakly in $L^2(0,1)$ to the function v. So, after transferring ξ_{i_k} to the right hand side of (3.18), it is possible to pass to the limit therein as $k \to \infty$ to obtain the relation

$$r^{n-1}v'(r) = -\lambda\xi(r)\int_0^r s^{n-1}v(s)\,ds\,. \tag{3.19}$$

We then divide by $\xi(r)$, differentiate with respect to r and write the equation that we obtain in original co-ordinates as

$$-\mathrm{div}\,(\nu\nabla u) = \lambda u\,. \tag{3.20}$$

The function u is non-zero as it's L^2 norm is 1 and thus, is an eigenfunction and, being the limit of non-negative functions, is itself non-negative. So, by the Kreĭn-Rutman theorem, λ is the first eigenvalue in the above spectral problem. By the uniqueness of the limit, $\lambda = \lambda^1(\nu)$ it follows that the entire sequence $\lambda^1(\nu_i)$ converges to $\lambda^1(\nu)$. $\square$

Next, we make the observation that the objective functional λ^1 is concave in ν being, by its definition, the infimum of linear functionals. It is interesting to know whether it is strictly concave in ν.

In the proof of our main theorem, we shall employ the following symmetrization result, based on [3, Lemma 1.2] to limit our search for minimizers among radially symmetric functions. The older proof by Alvino et. al. [4] achieves the same while it is based on a finer comparison result based for solutions of Hamilton-Jacobi equations [4, Theorem 3.1].

Proposition A.3.8. *Given any $\nu \in C(\theta)$ and any $u \in H_0^1(\Omega)$, there exists a $\widetilde{\nu}$ which is radially symmetric with $\widetilde{\nu}^{-1} \in K\left((\theta^{-1})^*\right)$ such that*

$$\int_\Omega \nu|\nabla u|^2\,dx \geq \int_\Omega \widetilde{\nu}|\nabla u^*|^2\,dx\,. \tag{3.21}$$

Proof: With the same hypothesis as in this proposition, the Lemma 1.2 in Alvino et. al.

[3] says that (3.21) holds for the radially symmetric function $\widetilde{\nu}(z) = \varphi(C_n|z|^n)$ for φ defined below through the relation

$$\int_0^{|\{u\geq c\}|} \frac{1}{\varphi(r)}\, dr := \int_{\{u\geq c\}} \frac{1}{\nu(x)}\, dx \tag{3.22}$$

which holds for all $c \in \mathbb{R}$. This gives the relation

$$\int_{\Omega^*_{u,c}} \frac{1}{\widetilde{\nu}(x)}\, dx = \int_{\Omega_{u,c}} \frac{1}{\nu(x)}\, dx \tag{3.23}$$

for all c real, where we recall that $\Omega_{u,c}$ is the level set of u at the level c and $\Omega^*_{u,c}$ is a ball centred at the origin having the same measure as $\Omega_{u,c}$. In particular the above identity holds on the full domain Ω. So, as $\left(\nu^{-1}\right)^* = \left(\theta^{-1}\right)^*$, by using the formula (3.8) we have,

$$\int_\Omega \frac{1}{\widetilde{\nu}(x)}\, dx = \int_\Omega \left(\theta^{-1}\right)^* (x)\, dx\,. \tag{3.24}$$

Once again as $\left(\nu^{-1}\right)^* = \left(\theta^{-1}\right)^*$, from the property (3.10) we obtain

$$\int_{\Omega_{u,c}} \frac{1}{\nu(x)}\, dx \leq \int_{\Omega^*_{u,c}} \left(\theta^{-1}\right)^* (x)\, dx\,.$$

The above inequality combined with (3.23) gives the relation

$$\int_{\Omega^*_{u,c}} \frac{1}{\widetilde{\nu}(x)}\, dx \leq \int_{\Omega^*_{u,c}} \left(\theta^{-1}\right)^* (x)\, dx \tag{3.25}$$

for all c real. We then note that the two relations (3.24) and (3.25), by the characterization (3.15), imply that $\widetilde{\nu}^{-1} \in K\left((\theta^{-1})^*\right)$. □

A.4 Proof of the Main Theorem

The proof of the main theorem is given in several steps.

STEP 1: Let us recall that the constraint in the original problem can be written as $\nu \in C(\theta)$ or equivalently, as $\nu^{-1} \in C\left((\theta^{-1})^*\right)$. So the minimization problem reads

$$\inf\left\{\lambda^1(\nu) : \nu^{-1} \in C\left((\theta^{-1})^*\right)\right\}\,. \tag{4.26}$$

We shall denote by $C^s\left((\theta^{-1})^*\right)$ and $K^s\left((\theta^{-1})^*\right)$ the subset of radially symmetric functions in $C\left((\theta^{-1})^*\right)$ and $K\left((\theta^{-1})^*\right)$, respectively. We use Proposition A.3.8 above to show that

$$\inf\left\{\lambda^1(\nu) : \nu^{-1} \in C\left((\theta^{-1})^*\right)\right\} = \inf\left\{\lambda^1(\nu) : \nu^{-1} \in K^s\left((\theta^{-1})^*\right)\right\} . \tag{4.27}$$

Following Remark A.3.6 we deduce that $K^s\left((\theta^{-1})^*\right)$ is the closed convex hull of $C^s\left((\theta^{-1})^*\right)$ for the weak-$*$ topology. So, applying the continuity property in Proposition A.3.7, we obtain first that

$$\inf\left\{\lambda^1(\nu) : \nu^{-1} \in C^s\left((\theta^{-1})^*\right)\right\} = \inf\left\{\lambda^1(\nu) : \nu^{-1} \in K^s\left((\theta^{-1})^*\right)\right\} .$$

So, it readily follows that

$$\inf\left\{\lambda^1(\nu) : \nu^{-1} \in K^s\left((\theta^{-1})^*\right)\right\} \geq \inf\left\{\lambda^1(\nu) : \nu^{-1} \in C\left((\theta^{-1})^*\right)\right\} . \tag{4.28}$$

To prove the reverse inequality, let $\nu^{-1} \in C\left((\theta^{-1})^*\right)$ be arbitrary and let u be the corresponding minimizer in the definition of $\lambda^1(\nu)$. Considering a $\widetilde{\nu}^{-1} \in K^s\left((\theta^{-1})^*\right)$ and u^* associated to the pair (ν, u) as given by Proposition A.3.8 and using the property (3.9) we obtain

$$\lambda^1(\nu) = \frac{\int_\Omega \nu|\nabla u|^2\,dx}{\int_\Omega |u|^2\,dx} \geq \frac{\int_\Omega \widetilde{\nu}|\nabla u^*|^2\,dx}{\int_\Omega |u^*|^2\,dx} \geq \lambda^1(\widetilde{\nu}) \;\geq\; \inf\left\{\lambda^1(\nu) : \nu^{-1} \in K^s\left((\theta^{-1})^*\right)\right\} . \tag{4.29}$$

By the arbitrariness of ν the reverse inequality to (4.28) follows.

STEP 2: The inf on the right hand side of (4.27) is in fact a minimum, that is to say, the infimum value is achieved. To see this let us define a topology on the set $K := \left\{\nu : \nu^{-1} \in K^s\left((\theta^{-1})^*\right)\right\}$ by saying that ν_i converges to ν if and only if ν_i^{-1} converges weakly-$*$ to ν^{-1} in $L^\infty(\Omega)$. Then, with the knowledge that $K^s\left((\theta^{-1})^*\right)$ is a compact set for the weak-$*$ topology on $L^\infty(\Omega)$ as announced by Proposition A.3.4, it follows that K is a compact set for the topology defined above. Besides, by Proposition A.3.7, we know that λ^1 restricted to K is continuous for the above topology. Thus, our thesis follows.

STEP 3: In the previous step, we have been able to show that the minimization problem admits a solution in a slightly enlarged class. Although, the functional λ^1 is concave, it is not clear whether the constraint set $\left\{\nu : \nu^{-1} \in K^s\left((\theta^{-1})^*\right)\right\}$ is convex. If this were so it is immediate to obtain a solution in the original class as, whenever a concave function admits a minimum over a compact convex set there is a minimizer which is an extreme point. So, in this problem, in order to show that there is a solution in the original class, we shall have to do differently as is done in Alvino et. al [4]. It can be shown that $J : \nu^{-1} \mapsto \left(\lambda^1(\nu)\right)^{-1}$ is a convex map when restricted to $K^s\left((\theta^{-1})^*\right)$ (cf. [4, Corollary 3.2]). Indeed, it is shown that

$$J(\mu) = \max\left\{\int_\Omega \mu\left(|x|^{n-1}\int_0^{|x|} s^{n-1}v(s)\,ds\right)^2 dx : v \in L^2(\Omega), \int_\Omega v^2(x)\,dx = 1, v \text{ radial}\right\}. \tag{4.30}$$

So, as the minimization problem on the right hand side of (4.27) is equivalent to maximizing the reciprocal functional J, the above mentioned convexity guarantees that there is a maximizer of J which is an extreme point of the compact convex set $K^s\left((\theta^{-1})^*\right)$ which, by Proposition A.3.5 and Remark A.3.6, has to belong to $C^s\left((\theta^{-1})^*\right)$. This permits us to conclude that the infimum in (4.26) is achieved for a radially symmetric function. □

A.5 Appendix

We remark that we only require Lemma 1.2 [3] in the form stated below for our applications. Now, we give a more flexible alternate proof of the same.

Proposition A.5.1. *Given any $\nu \in C(\theta)$ and any non-negative $u \in H_0^1(\Omega)$, for $\widetilde{\nu}$ defined through the relation,*

$$\int_{\Omega^*_{u,c}} \frac{1}{\widetilde{\nu}(x)}\,dx = \int_{\Omega_{u,c}} \frac{1}{\nu(x)}\,dx \tag{5.31}$$

we have

$$\int_\Omega \nu|\nabla u|^2\,dx \geq \int_\Omega \widetilde{\nu}|\nabla u^*|^2\,dx\,. \tag{5.32}$$

Proof: We shall make repeated use of the co-area formula (cf. formula (2.2.1) Kesavan [20])

$$\int_\Omega g(x)\,|\nabla u(x)|\;dx = \int_{-\infty}^{\infty}\int_{u=s} g(x)\,d\sigma(x)\,ds \tag{5.33}$$

where $d\sigma(x)$ is the surface element on the level surface $u = s$ at the point x. Applying (5.33), we obtain the identity

$$\int_{\{u\geq t\}} \nu(x)\,|\nabla u(x)|^2\;dx = \int_t^{\infty}\int_{\{u=s\}} \nu(x)\,|\nabla u(x)|\;d\sigma(x)\,ds \tag{5.34}$$

Therefore, it follows that,

$$-\frac{d}{dt}\left(\int_{\{u\geq t\}} \nu(x)\,|\nabla u(x)|^2\;dx\right) = \int_{\{u=t\}} \nu(x)\,|\nabla u(x)|\;d\sigma(x)\,. \tag{5.35}$$

We apply the fact that the arithmetic mean of a non-negative function is always greater than the harmonic mean, to the function $\nu\,|\nabla u|$ on the surface $\{u=t\}$ equipped with it's surface measure, to conclude that

$$\begin{aligned}
\int_{u=t} \nu(x)\,|\nabla u(x)|\;d\sigma(x) &= \left(\frac{\int_{\{u=t\}} \nu(x)\,|\nabla u(x)|\;d\sigma(x)}{\int_{\{u=t\}} d\sigma(x)}\right)\left(\int_{\{u=t\}} d\sigma(x)\right) \\
&\geq \left(\frac{\int_{\{u=t\}} d\sigma(x)}{\int_{\{u=t\}} \dfrac{1}{\nu(x)\,|\nabla u(x)|}\,d\sigma(x)}\right)\left(\int_{\{u=t\}} d\sigma(x)\right). \\
&= \left(\int_{\{u=t\}} \frac{1}{\nu(x)\,|\nabla u(x)|}\,d\sigma(x)\right)^{-1} \left(P\left(\{u\geq t\}\right)\right)^2 \qquad (5.36) \\
&\geq \left(\int_{\{u=t\}} \frac{1}{\nu(x)\,|\nabla u(x)|}\,d\sigma(x)\right)^{-1} \left(P\left(\{u^*\geq t\}\right)\right)^2. \qquad (5.37)
\end{aligned}$$

The last inequality above is due to the iso-perimetric inequality (3.11). Therefore, from (5.35) and (5.37) we have

$$-\frac{d}{dt}\left(\int_{\{u\geq t\}} \nu(x)\,|\nabla u(x)|^2\;dx\right) \geq \left(\int_{\{u=t\}} \frac{1}{\nu(x)\,|\nabla u(x)|}\,d\sigma(x)\right)^{-1} \left(P\left(\{u^*\geq t\}\right)\right)^2. \qquad (5.38)$$

We remember that $\{u^*\geq t\}$ for $t\geq 0$ form a continuum of concentric balls, having radius r_t, whose union over $t\geq 0$ is the ball Ω. Observing that u^* is a radially symmetric function and consequently, so is $\nabla u^*(x)$, we may define a radially symmetric function $\widetilde{\nu}$ as follows.

$$\widetilde{\nu}(|x|) := \frac{\int_{\{u^*=t\}} d\sigma(x)}{\left(\int_{\{u=t\}} \frac{1}{\nu(x)|\nabla u(x)|}\,d\sigma(x)\right)\,|\nabla u^*(x)|} \qquad \text{for any } x, |x| = r_t\,. \qquad (5.39)$$

We check, first, that $\widetilde{\nu}$ satisfies (5.31). To see this we use the co-area formula. We have

$$\begin{aligned}
\int_{\{u^*\geq t\}} \frac{1}{\widetilde{\nu}(x)}\,dx &= \int_t^\infty \int_{\{u^*=s\}} \frac{1}{\widetilde{\nu}(x)\,|\nabla u^*(x)|}\,d\sigma(x)\,ds \\
&= \int_t^\infty \int_{\{u=s\}} \frac{1}{\nu(x)\,|\nabla u(x)|}\,d\sigma(x)\,ds \\
&= \int_{\{u\geq t\}} \frac{1}{\nu(x)}\,dx
\end{aligned}$$

where in the penultimate expression we have plugged in (5.39). Then, (5.38) may be rewritten using $\widetilde{\nu}$ as

$$\begin{aligned}
-\frac{d}{dt}\left(\int_{\{u\geq t\}} \nu(x)\,|\nabla u(x)|^2\ dx\right) &\geq \int_{\{u^*=t\}} \widetilde{\nu}(x)\,|\nabla u^*(x)|\ d\sigma(x)\\
&= -\frac{d}{dt}\left(\int_{\{u^*\geq t\}} \widetilde{\nu}(x)\,|\nabla u^*(x)|^2\ dx\right). \qquad (5.40)
\end{aligned}$$

Integrating (5.40) we obtain the needful. □

Remark A.5.2. The definition (5.39) of the rearranged coefficient can be written entirely in terms of the coefficient ν, the function u and the derivative of the corresponding distribution function $\mu_u(t) = |\{u \geq t\}| = \mu_{u^*}(t)$ as

$$\widetilde{\nu}(|x|) := \frac{-\left(\mu_u\right)'(t)}{\left(\int_{\{u=t\}} \frac{1}{\nu(x)|\nabla u(x)|}\,d\sigma(x)\right)} \qquad \text{for any } x, |x| = r_t\,. \qquad (5.41)$$

This is due to the fact that, by using the co-area formula, we have (see also [8, Lemma 4.1] for a similar result in the case of Steiner symmetrization)

$$\left(\mu_u\right)'(t) = \left(\mu_{u^*}\right)'(t) = -\int_{\{u^*=t\}} \frac{1}{|\nabla u^*(x)|}\,d\sigma(x)\,.$$

It is worthwhile to note from the above that the gradient of the rearranged function u^* can be written in terms of the distribution function μ_u and it's derivative as

$$|\nabla u^*(x)| = -\frac{nC_n^{\frac{1}{n}}\,\mu_u(t)^{1-\frac{1}{n}}}{\left(\mu_u\right)'(t)} \qquad \text{for any } x, |x| = r_t \qquad (5.42)$$

since $P\left(\{u^* = t\}\right) = nC_n^{\frac{1}{n}}\,\mu_u(t)^{1-\frac{1}{n}}$, being C_n the volume of the unit sphere in $\mathbb{R}^n$.

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Appendix B

Second paper (In Revision)

B.1 Abstract

This article deals with the minimization of the first eigenvalue of a two-phase conducting medium problem. Although, in general, this minimization problem may have to be relaxed to include microstructural designs, it, nevertheless, admits a true solution in the one-dimensional case [22] and in balls in any dimension [4]. In the light of these and other recent results obtained in [9] we are led to believe that a classical minimizer (that is, without microstructures) exists not only in balls but also in other domains, possibly having lesser symmetry. Our conjecture is that, in such domains, optimal distribution of the material for this problem requires placing the material with higher conductivity in the middle. In this article, we show the existence and then give an expression for the shape derivative of the eigenvalue functional, which is an important tool for understanding the sensitivity of the eigenvalue with respect to domain variations. We gather evidence for our conjecture by analyzing this derivative for certain initial annular configurations in a ball. Numerical results obtained in discs and squares also give more substance to our conjecture.

B.2 Introduction

Given the design region $\Omega \subset \mathbb{R}^n$ and two conducting materials with conductivities α and β $(0 < \alpha < \beta)$ which are to be distributed in Ω so that the volume of the region ω occupied by β is a fixed number m $(0 < m < |\Omega|)$, we are required to minimize the first eigenvalue of a

Dirichlet problem given by

$$\lambda^1(\omega) := \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega (\alpha\chi_{\Omega\setminus\omega} + \beta\chi_\omega)|\nabla u|^2 dx}{\int_\Omega |u|^2 dx}\,. \tag{2.1}$$

In general, such shape optimization problems may not have a solution as shown by Murat and Tartar in the case of a control problem involving immiscible fluids [26]. One may need to consider also micro-structures in addition to classical shapes. In the above problem, characterizations of optimal designs involving micro-structures have been given by Cox and Lipton [11]. As compared to this, in the one-dimensional case Kreĭn [22, 16] has shown that the solution consists in placing the material with higher conductivity in the middle and when the domain is a ball in any dimension, Alvino et. al. [4] have shown that the minimum is attained for a classical design (that is, without micro-structures) having radial symmetry. However, this does not fully resolve the problem as it remains to characterize the radially symmetric minimizer.

Our conjecture is that, in such domains, optimal distribution of the material for this problem requires placing the material with higher conductivity in the middle. A tool which can be used to analyze possible local or global minimizers and to develop some algorithms for the numerical search of such minimizers is the derivative of the objective functional with respect to variations of the domain. In line with this idea, the main results of this paper are Theorem B.3.2, where we show the existence of the shape derivative of the two-phase eigenvalue problem, and Theorem B.3.4 wherein we obtain an explicit formula for it.

There are very few results of the shape derivative calculus for two-phase conductivity problems which can also be seen as transmission problems. The shape derivatives in an inverse conductivity problem with two conducting phases was first calculated in Hettlich and Rundell [18] and later established rigorously in Afraites et. al. [1]. Discussions of the shape derivative of one phase eigenvalue problems for different operators and boundary conditions can be found in [5, 29, 30, 31, 32, 16, 17]. Our results on the shape derivative of a two-phase eigenvalue problem (see Theorem B.3.2 and B.3.4) seem to be the first of its kind and should be interesting in themselves.

We then use the calculated shape derivative to analyze the sensitivity of the first eigenvalue for domain variations of certain annular configurations in a ball. The Theorem B.4.1 shows that when there are a finite number of annuli in which to distribute the two materials one can always obtain smaller values for the eigenvalues by moving the material with conductivity β more to the centre. We also study, numerically, the variation of the first eigenvalue with respect to the position of the domain containing the material having higher conductivity for certain annular and disc-like configuration inside a disc. The results that we obtain point towards the veracity of our conjecture.

The methods which we used to show the existence of a classical minimizer to the problem in a ball in [9] lead us to believe that in domains with only a few planes of symmetry there is once again a classical minimizer. The results from a numerical experiment conducted in a square domain where the materials are arranged in three concentric squares seem to indicate, once again, that the material with higher conductivity should be placed in the middle.

The layout of the article is as follows. In the next section we prove Theorems B.3.2 and B.3.4 which are about the shape derivative of the eigenvalue functional. In Section 4, we prove Theorem B.4.1 and we present the numerical results in a disc providing more substance to our conjecture. In Section 5, we give the numerical results in a square and in the final section, conclude by proposing some future directions of work.

B.3 Shape derivative of the eigenvalue functional

The shape derivative is a tool which permits to understand the variation of quantities which depend on the domain (cf. Simon [31]). This is widely used in the study of shape optimization, front tracking, image segmentation problems etc. It is defined in the following way. Let us consider a functional F which depends on the domain ω (shape functional). For a variation of the domain ω by a fairly smooth perturbative vector field θ, which has its support in a neighbourhood of $\partial\omega$, the infinitesimal variation of F in the direction u is defined as

$$F'(\omega;\theta) = \lim_{t \longrightarrow 0} \frac{F(\omega + t\theta) - F(\omega)}{t}\,. \tag{3.1}$$

Now F itself may depend on a function u defined on ω. So, if u_t is the corresponding function when ω changes to $\omega_t := (id + t\theta)(\omega)$, the local derivative (also called shape derivative)and the total derivative (also called material derivative) of u are defined, respectively, to be $u'(x) = \lim_{t\to 0} \dfrac{u_t(x) - u(x)}{t}$ and $\dot{u}(x) = \lim_{t\to 0} \dfrac{u_t(x + t\theta) - u(x)}{t}$.

Remark B.3.1. An important part of the shape derivative calculus is to rigorously establish the existence of the shape derivatives. This requires some careful analysis as it is usually hard to explicit the dependence of the quantities on the perturbation field θ or to say whether even this dependence is continuous, Lipshitzian etc. An alternate way is to use some form of the implicit function theorem as the quantities depend on the perturbation field usually in an implicit way. We shall use the latter approach in this article as it turns out to be quite simple for this problem.

We recall the setting of the spectral problem for the first eigenvalue functional before stating the existence theorem of the shape derivative. Let ω be a reference configuration with smooth boundary where the material β is given. Given the distribution $\sigma = \alpha\chi_{\Omega\setminus\omega} + \beta\chi_\omega$ with $\omega \subset\subset \Omega$, the eigenvalue problem reads as follows:

$$\left\{ \begin{array}{rcl} -\mathrm{div}\,(\sigma(\omega)\nabla u) & = & \lambda(\omega)\; u \text{ in } \Omega \\ u & = & 0 \text{ on } \partial\Omega \end{array} \right. \tag{3.2}$$

Let $\lambda_1(\omega)$ be the first eigenvalue. It is simple and the first eigenfunction is characterized by its constant sign [23]. We normalize the first eigenfunction by assuming it to be non-negative and taken to satisfy

$$\int_\Omega |u|^2\, dx = 1\,. \tag{3.3}$$

The admissible perturbations of ω are the images ω_t of ω under transformations $id + t\theta$ where θ is a smooth vector field with its support inside a neighbourhood of $\partial\omega$ such that $\omega_t \subset\subset \Omega$ and such that $|\omega_t| = |\omega|$.

Theorem B.3.2. *The material derivative $\dot{u}$ of the normalized first eigenfunction u exists and $\dot{u} \in H_0^1(\Omega)$. Its shape derivative u' also exists and is such that its restrictions to ω and $\Omega \setminus \overline{\omega}$ belong to $H^1(\omega)$ and $H^1(\Omega \setminus \overline{\omega})$ respectively. In addition, the shape derivative of λ, denoted by $\lambda_1'(\omega;\theta)$, exists.*

Proof: We prove this result using an argument based on the Implicit Function Theorem following an established procedure which is well explained in the text [17]. The existence of the material derivative will be obtained as an existence result of a smooth family of solutions after rewriting the perturbed eigenvalue problem in a suitable way. The perturbed eigenvalue problem is

$$\left\{ \begin{array}{rcl} -\mathrm{div}\,(\sigma(\omega_t)\nabla u_t) & = & \lambda(\omega_t)\; u_t \text{ in } \Omega \\ u_t & = & 0 \text{ on } \partial\Omega \end{array} \right. \tag{3.4}$$

where $\omega_t = (id + t\theta)\,(\omega)$ under the assumptions made above on the perturbative vector field θ and u_t is the first eigenfunction normalized as above. For small t the smooth change of variables $\Phi_t := (id + t\theta)$ on Ω is invertible while it maps ω onto ω_t. The problem (3.4) transported to the inverse image of Ω may be rewritten using this change of coordinates as

$$\left\{ \begin{array}{rcl} -\mathrm{div}\,((\sigma(\omega_t)\circ\Phi_t)\, A_t\nabla\,(u_t\circ\Phi_t)) & = & \lambda(\omega_t)\,(\;(u_t\circ\Phi_t)\, J(\Phi_t))\;\text{ in } \Omega \\ u_t\circ\Phi_t & = & 0 \text{ on } \partial\Omega \end{array} \right. \tag{3.5}$$

where $A_t := D\Phi_t^{-1}\left(D\Phi_t^{-1}\right)^T J(\Phi_t)$ and $J(\Phi_t)$ is the Jacobian of the transformation Φ_t. We also observe that the function $u_t \circ \Phi_t$ remains non-negative and satisfies the normalization condition

$$\int_\Omega |u_t \circ \Phi_t|^2 \, J(\Phi_t)dx = 1 \,. \tag{3.6}$$

We refer to [1, 17] for the details. Note that by the preceding discussion $(\lambda(\omega_t)\,,\, u_t \circ \Phi_t)$ satisfies the equations (3.5) and (3.6) if and only if $(\lambda(\omega_t)\,,\, u_t)$ is a normalized eigenpair of (3.4). Let $(\lambda_1(\omega_t)\,,\, u_t)$ be a normalized eigenpair of (3.4). The existence of the material derivative of u_t and the existence of the shape derivative of $\lambda_1(\omega_t)$ can be drawn as a consequence of the existence of a smooth curve of zeros for the following function in a neighbourhood of $(0, \lambda_1(\omega), u_0)$:

$$\begin{aligned} F(t,\lambda,v) \;&:= \left(-\mathrm{div}\left((\sigma(\omega_t)\circ\Phi_t)\, A_t\nabla v\right) - \lambda v\,,\, \int_\Omega |v|^2 \, J(\Phi_t)dx - 1\right) \\ &= \left(-\mathrm{div}\left(\sigma(\omega)A_t\nabla v\right) - \lambda v\,,\, \int_\Omega |v|^2 \, J(\Phi_t)dx - 1\right). \end{aligned} \tag{3.7}$$

Note that the last equality is due to the fact that $\sigma(\omega_t) \circ \Phi_t \equiv \sigma(\omega)$ (indeed, as Φ_t maps ω onto ω_t and the coefficient $\sigma(\omega_t)$ has the value β on ω_t and the value α elsewhere on Ω while $\sigma(\omega)$ takes the values β and α, respectively, on the regions ω and $\Omega \setminus \omega$).

We now obtain the existence of a smooth curve of zeros for the function F defined above by verifying the hypotheses of the implicit function theorem. As Φ_t is a smooth function of t we deduce that the maps $t \mapsto D\Phi_t$, and $t \mapsto A_t$ are smooth functions of t. Consequently, $F : \mathbb{R} \times \mathbb{R} \times H_0^1(\Omega) \to H^{-1}(\Omega) \times \mathbb{R}$ is a smooth function of t as also in the varibales λ and v being linear or quadratic in those. Now we check that $F_{\lambda,v}(0, \lambda_1(\omega), u_0) : \mathbb{R} \times H_0^1(\Omega)\,, \to H^{-1}(\Omega) \times \mathbb{R}$ is invertible. As

$$\langle F_{\lambda,v}(0,\lambda_1(\omega),u_0)\,,\,(\lambda,v)\rangle = \left(-\mathrm{div}\left(\sigma(\omega)\nabla v\right) - \lambda_1(\omega)v - \lambda u_0\,,\, 2\int_\Omega v u_0 \, dx\right) \tag{3.8}$$

we now solve

$$\begin{aligned} -\mathrm{div}\left(\sigma(\omega)\nabla v\right) - \lambda_1(\omega)v - \lambda u_0 \;&=\; f \\ 2\textstyle\int_\Omega v u_0 \, dx \;&=\; c\,. \end{aligned} \tag{3.9}$$

The first of these equations has a solution, by the Fredholm alternative, if and only if

$$\langle f + \lambda u_0\,,\, u_0\rangle = 0\,.$$

Thus $\lambda = \langle f\,,\, u_0\rangle$. Let w be a particular solution of the first equation for this value of λ. The solution space is one dimensional and all the solutions are of the form $w + ku_0$. Plugging this in the second equation in (3.9) we have

$$2\int_{\Omega}(w+ku_0)\,u_0\;dx = c\;.$$

This determines k uniquely. Thus, the system (3.9) admits a unique solution $(\lambda\,,\,w+ku_0)$ which shows that the operator $F_{\lambda,v}(0,\lambda_1(\omega),u_0)$ is bijective. The continuity of the inverse follows from the Banach-Steinhaus open mapping theorem. So, by applying the Implicit Function Theorem we obtain a smooth curve of zeros $t\mapsto(t,\lambda_t,v_t)$ to F in a neighbourhood of $(0,\lambda_1(\omega),u_0)$.

We now reach the conclusions that the shape derivative of the first eigenvalue and the material derivative of the first eigenfuntion exist as follows. By our earlier observation, $\left(\lambda_t, v_t\circ\Phi_t^{-1}\right)$ is a normalized eigenpair for (3.4). So, indeed, $\lambda_t=\lambda_1(\omega_t)$ and so it follows that the shape derivative of λ_1 exists. Writing $u_t=v_t\circ\Phi_t^{-1}$ we have that $u_t\circ\Phi_t$ is a smooth function of t. However, by the definition of the material derivative $\dot{u}=\dfrac{d}{dt}_{|t=0}(u_t\circ\Phi_t)$ and it exists due to the differentiability of v_t.

Finally, we conclude that the shape derivative of u exists from the the following simple but important relation between the local and total derivatives (see Simon [31])

$$u'(x)=\dot{u}(x)-\theta\cdot\nabla u(x) \tag{3.10}$$

where u is the function on the unperturbed domain. On the one hand we have seen that $\dot{u}\in H_0^1(\Omega)$. On the other hand, as ω is a smooth domain and on each of ω and $\Omega\setminus\overline{\omega}$, u satisfies an elliptic eigenvalue problem with smooth coefficients, by standard regularity theory, it is smooth in each of these domains and consequently also ∇u (see Gilbarg and Trudinger [15]). However, we have only the continuity of $\sigma\dfrac{\partial u}{\partial n}$ across the boundary $\partial\omega$ and as σ is discontinuous across $\partial\omega$ so must be ∇u. Thus, from the relation (3.10) we can only conclude that $u'\lfloor_{\omega}\in H^1(\Omega)$ and $u'\lfloor_{\Omega\setminus\overline{\omega}}\in H^1(\Omega\setminus\omega)$. $\square$

Remark B.3.3. The above theorem shows the Gateaux differentiablity of the first eigenfunction u in the direction of the perturbative field θ. The same proof modified, while considering the deformations $id+\theta$ for sufficiently small θ, will show that the first eigenfunction is Frechêt differentiable with respect to θ. $\square$

Theorem B.3.4. *The shape derivative of λ, given an admissible perturbation θ, reads as follows*

$$\lambda_1'(\omega;\theta)=\int_{\partial\omega}\left[\sigma\left|\nabla u\right|^2\right]\theta\cdot n dS \tag{3.11}$$

where $[\varphi]$ *is the jump of* φ *across* $\partial\omega$*, that is,* $[\varphi](x) = (\varphi\lfloor_{\partial\omega^-} - \varphi\lfloor_{\partial\omega^+})(x)$ *with* $\varphi\lfloor_{\partial\omega^-}$ *and* $\varphi\lfloor_{\partial\omega^+}$ *denoting, respectively the inner and outer trace of* φ *on* $\partial\omega$*.*

Proof: The variational formulation of the equation (3.5) is

$$\int_\Omega \sigma(\omega) A_t \left(\nabla u_t \circ \Phi_t\right) \cdot \nabla w \; dx = \int_\Omega \lambda(\omega_t) \left(u_t \circ \Phi_t\right) w \; dx \,. \tag{3.12}$$

The integrands are continuosly differentiable with respect to the variable t and thus we are allowed to differentiate under the integral sign with respect to t at $t = 0$. Doing so, we obtain

$$\begin{aligned} \int_\Omega \sigma(\omega) \nabla \dot{u} \cdot \nabla w \; dx + \int_\Omega \sigma(\omega) \left(\operatorname{div}\theta \, I - \left((D\theta)^T + D\theta\right)\right) \nabla u \cdot \nabla w \; dx \\ = \int_\Omega \lambda_1' u w \; dx + \int_\Omega \lambda_1 \dot{u} w \; dx + \int_\Omega \lambda_1 u w \operatorname{div}\theta \; dx \,. \end{aligned} \tag{3.13}$$

Similarly, differentiating the relations (3.6) and the volume constraint $|\omega_t| = |\omega|$, written as $\int_\omega J(\Phi_t) \; dx = 1$, with respect to t, we have, respectively,

$$\int_\Omega 2u\dot{u} \; dx + \int_\Omega u^2 \operatorname{div}\theta \; dx = 0 \tag{3.14}$$

$$\int_\omega \operatorname{div}\theta \; dx = \int_{\partial\omega} \theta \cdot n \; dS = 0 \,. \tag{3.15}$$

Now, we shall use the above relations to deduce the expression for the shape derivative of λ. To begin with, we take $w = u$ in (3.13) and use $\dot{u}$ as a test function in (3.2) to obtain

$$\int_\Omega \sigma(\omega) \nabla \dot{u} \cdot \nabla u \; dx + \int_\Omega \sigma(\omega) \left(\operatorname{div}\theta \;\; |\nabla u|^2 - 2D\theta \nabla u \cdot \nabla u\right) \; dx = \lambda_1' + \int_\Omega \lambda_1(\omega) \dot{u} u \; dx + \int_\Omega \lambda_1 u^2 \operatorname{div}\theta \; dx \tag{3.16}$$

$$\int_\Omega \sigma(\omega) \nabla u \cdot \nabla \dot{u} \; dx = \lambda_1(\omega) \int_\Omega u\dot{u} \; dx \,. \tag{3.17}$$

Subtracting (3.17) from (3.16) we get

$$\int_\Omega \sigma(\omega) \left(\operatorname{div}\theta \;\; |\nabla u|^2 - 2D\theta \nabla u \cdot \nabla u\right) \; dx = \lambda_1' + \int_\Omega \lambda_1 u^2 \operatorname{div}\theta \; dx \tag{3.18}$$

As ∇u is smooth in each of ω and $\Omega \setminus \overline{\omega}$ we have the following identity (see [1, Theorem 3.1 equation (3.10)])

$$\operatorname{div}\theta \;\; |\nabla u|^2 - 2D\theta \nabla u \cdot \nabla u = -\operatorname{div}\left(2\theta \cdot \nabla u \nabla u - |\nabla u|^2 \theta\right) + 2\theta \cdot \nabla u \Delta u \tag{3.19}$$

while we also have straightaway that

$$\operatorname{div}\left(2\theta \cdot \nabla u \nabla u\right) = 2\theta \cdot \nabla u \Delta u + 2\nabla u \cdot \nabla\left(\theta \cdot \nabla u\right). \tag{3.20}$$

So, from (3.19) and (3.20) we have

$$\operatorname{div}\theta \ |\nabla u|^2 - 2D\theta\nabla u \cdot \nabla u = \operatorname{div}\left(|\nabla u|^2 \theta\right) - 2\nabla u \cdot \nabla\left(\theta \cdot \nabla u\right). \tag{3.21}$$

This allows us to rewrite (3.18) as

$$\lambda_1' = \int_\Omega \sigma(\omega)\operatorname{div}\left(|\nabla u|^2 \theta\right) - 2\nabla u \cdot \nabla\left(\theta \cdot \nabla u\right)\ dx - \int_\Omega \lambda_1 u^2 \operatorname{div}\theta\ dx\,. \tag{3.22}$$

Now, if we take $\theta \cdot \nabla u$ as a test function in (3.2) and use the fact that θ is identically zero near $\partial\Omega$, we shall obtain

$$\begin{aligned}
\int_\Omega \nabla u \cdot \nabla\left(\theta \cdot \nabla u\right)\ dx &= \lambda_1(\omega)\int_\Omega u\theta \cdot u\ dx \\
&= \frac{1}{2}\int_\Omega \theta \cdot \nabla\left(u^2\right)\ dx \\
&= -\frac{1}{2}\int_\Omega u^2 \operatorname{div}\theta\ dx\,.
\end{aligned} \tag{3.23}$$

So, we conclude from (3.22) and (3.23) that

$$\lambda_1' = \int_\Omega \sigma(\omega)\operatorname{div}\left(|\nabla u|^2 \theta\right)\ dx\,. \tag{3.24}$$

The expression (3.11) follows by an integration parts on each of the domains ω and $\Omega \setminus \overline{\omega}$ while using the fact that σ is constant on each of these subdomains. $\square$

B.4 Minimizing distribution in a ball

We know by the results of [4] and [9] that there exists minimizing configurations in a ball which are radially symmetric. This means that the materials are to be distributed in various spherical shells. We prove the following theorem by using the shape derivative calculated in the previous section.

Theorem B.4.1. *Whenever there is a layer of α preceding a layer of β (as we move radially outward) $\lambda'(\omega;\theta) < 0$ for the radially symmetric perturbation θ which moves the layer of β*

inwards while conserving the volumes of α and β.

Proof: Denote the reference configuration by σ and let u be the normalized first eigenfunction, which we know to be radially symmetric (see for instance [9]). Let us concentrate on a layer ω_0 of β which follows a layer of α and let us write its boundary as $S_1 \cup S_2$ where S_1 and S_2 are, respectively, the inner and outer boundaries. We may consider a radially symmetric perturbation θ which is zero outside ω_0 and conserves the volume of ω_0. The conservation of mass condition (3.15) gives the relation

$$(\theta \cdot n) \lfloor_{S_1} \times \mathrm{per}(S_1) + (\theta \cdot n) \lfloor_{S_2} \times \mathrm{per}(S_2) = 0 \,. \tag{4.25}$$

Now, we have

$$\begin{aligned} \lambda' &= \int_{\partial S_1} \left[\sigma \left|\nabla u\right|^2\right] \theta \cdot n dS + \int_{\partial S_2} \left[\sigma \left|\nabla u\right|^2\right] \theta \cdot n dS \\ &= \left(\left[\sigma \left|\nabla u\right|^2\right] \theta \cdot n\right) \lfloor_{S_1} \mathrm{per}(S_1) + \left(\left[\sigma \left|\nabla u\right|^2\right] \theta \cdot n\right) \lfloor_{S_2} \mathrm{per}(S_2) \\ &= \left(\left[\sigma \left|\nabla u\right|^2\right] \lfloor_{S_2} - \left[\sigma \left|\nabla u\right|^2\right] \lfloor_{S_1}\right) (\theta \cdot n) \lfloor_{S_2} \mathrm{per}(S_2) \end{aligned} \tag{4.26}$$

Let us analyze (4.26) for it's sign. Denoting by S_i^- and S_i^+ the inner and outer surfaces of S_i with respect to ω_0, for $i = 1, 2$, the transmission condition on S_i reads

$$(\sigma \nabla u \cdot n)_{S_i^-} = (\sigma \nabla u \cdot n)_{S_i^+} \;,\; i = 1, 2 \,.$$

In view of the fact that u is radially symmetric, we can write $n = \dfrac{\nabla u}{|\nabla u|}$, and therefore, from the above relation we conclude that

$$\sigma \left|\nabla u\right| \lfloor_{S_i^-} = \sigma \left|\nabla u\right| \lfloor_{S_i^+} , i = 1, 2 \,. \tag{4.27}$$

This allows to write the jumps in $\sigma \left|\nabla u\right|^2$ as follows

$$\begin{aligned} \left[\sigma \left|\nabla u\right|^2\right] \lfloor_{S_i} &= \left(\left|\nabla u\right| \lfloor_{S_i^+} - \left|\nabla u\right| \lfloor_{S_i^-}\right) (\sigma \left|\nabla u\right|) \lfloor_{S_i^-} \\ &= \left(\frac{\beta}{\alpha} - 1\right) \left|\nabla u\right| \lfloor_{S_i^-} (\sigma \left|\nabla u\right|) \lfloor_{S_i^-} \,. \end{aligned}$$

Therefore, the shape derivative in (4.26), can be written as

$$\lambda' = \left(\frac{1}{\alpha} - \frac{1}{\beta}\right) \left\{ \left((\sigma \left|\nabla u\right|) \lfloor_{S_2^-}\right)^2 - \left((\sigma \left|\nabla u\right|) \lfloor_{S_1^-}\right)^2 \right\} (\theta \cdot n) \lfloor_{S_2} \mathrm{per}(S_2) \,. \tag{4.28}$$

Observing that u is a decreasing function (see equation (3.18) in appendix A)) and that $\sigma\dfrac{\partial u}{\partial n}$ (see equation (3.17) n appendix A)) is a decreasing function in each region where σ is constant, it follows that $\sigma\,|\nabla u|$ is non-decreasing in the radial direction on ω_0 Therefore, λ' assumes a negative sign if θ is such that $\theta \cdot n$ is negative on S_2 and consequently $\theta \cdot n$ is positive on S_1. This means that θ is opposite to the radial direction on S_2 and follows the external normal on the boundary surface S_1 of ω_0, that is once again in the direction opposite to the radial direction. This means that λ can be decreased by moving exterior shells where there is β towards the centre. This concludes the proof of the theorem. $\square$

We now give further evidence to our conjecture in a ball by plotting numerically the eigenvalues for some two-dimensional configurations in a disk.

In the first of these experiments, we consider a domain which is a disk of unit radius and we assume that the material β to be placed in an annular region having internal and external radius r_1 and r_2 respectively within a disk and the material α, in the complement of this annulus within the unit disk. Let m be the proportion of the total volume that the material β occupies so that $m = (r_2^2 - r_1^2)$.

In figure B.1 we plot the first eigenvalue and the shape derivative as a function of the internal radius taking $\alpha = 1$, $\beta = 2$ and proportions $m = 0.1$, 0.5, and 0.9. We have do the same for $\alpha = 1$, $\beta = 200$ in figure B.2. We make the followings the observations.

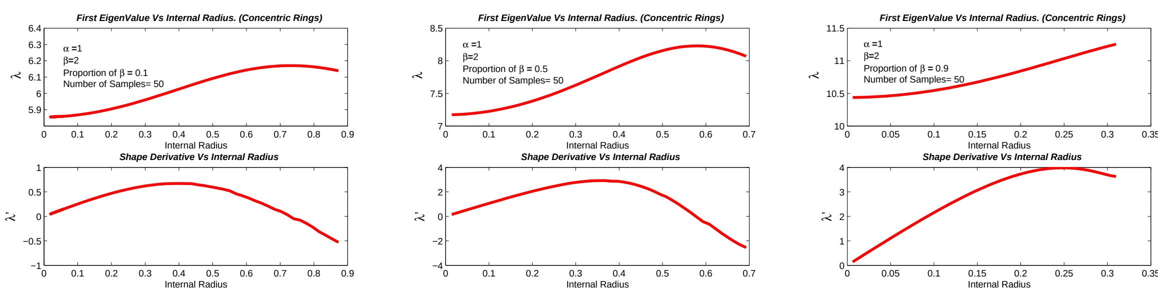

Figure B.1: Concentric disks for $\beta = 2$

In all the cases we see that the first eigenvalue is the smallest for $r_1 = 0$ which corresponds to taking the material β in the middle.

The only significant information which is contained in graph of the shape derivative function is the sign of this function. The shape derivative is a directional derivative which depends on the perturbation θ and its sign is independent of the magnitude of θ. In plotting the graph of the shape derivative function we have chosen as θ, the radial perturbation towards the centre which respects the area constraint. At those points r_1 where the shape derivative function is negative we conclude, therefore, that the first eigenvalue can be reduced

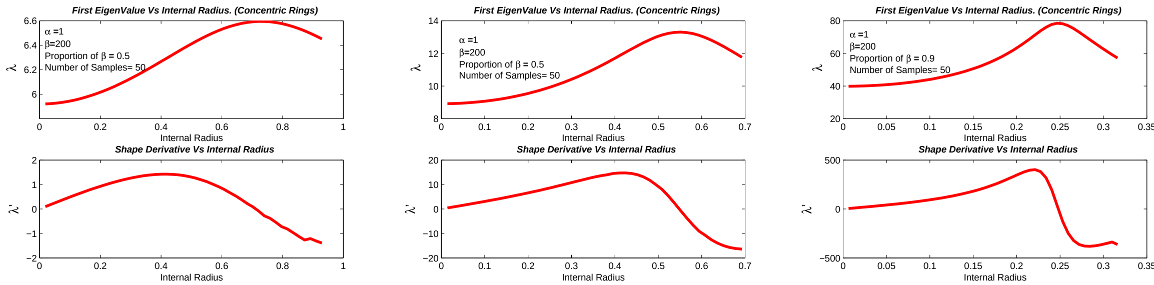

Figure B.2: Concentric disks for $\beta = 200$

by perturbing the ring inwards. This is in concordance with what is observed in the graph of the first eigenvalue function.

In the second of these experiments, we consider a domain which is a disk of unit radius and we put the material β in a smaller disk inside occupying a fraction m of the total area. The centre of the smaller disk is displaced from the center of the domain. We can assume the centre of the smaller disk is on the horizontal axis.

In figure B.3 we plot the first eigenvalue and the shape derivative as a function of the displacement taking $\alpha = 1$, $\beta = 2$ and proportions $m = 0.1$, 0.5, and 0.9. We have do the same for $\alpha = 1$, $\beta = 200$ in figure B.4.

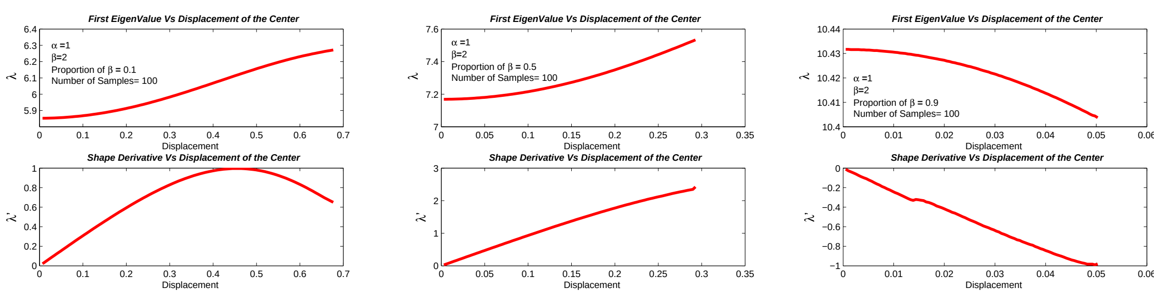

Figure B.3: Non-concentric disks configurations for $\beta = 2$

In plotting the graph of the shape derivative function we have chosen as θ, the perturbation of the disk towards the centre of the domain. We see once again that the sign of the shape derivative is in accordance with the behavior of the eigenvalue with respect to the displacement. This is in concordance with what is observed in the graph of the first eigenvalue function. In general we see that we obtain smaller values for the eigenvalue when the inner disk is near the centre of the domain. In the case when the inner disk occupies a large proportion of the area the range of displacement is small and the eigenvalues vary very little with the displacement. Although for coefficients of the same order displaced configurations

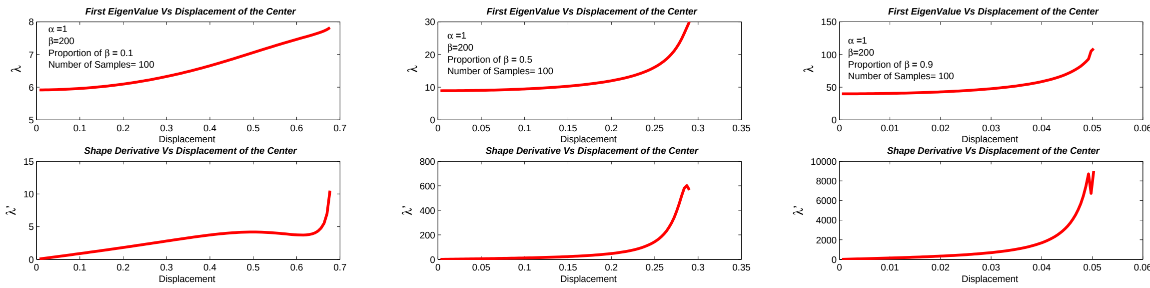

Figure B.4: Non-concentric disks configurations for $\beta = 200$

seem to have smaller eigenvalues than when the disk is in the centre, we believe that this does not have much significance because the eigenvalues vary very little and the numerical result could be spurious because of numerical errors.

B.5 Minimal configurations in domains with less symmetry

We are of the firm belief that the arguments that we have used to show the existence of a classical minimizer in a ball can be extended using the Steiner symmetrization to show the existence of a classical solution in a domain with Steiner symmetry. If that be case, one might be interested in knowing once again what are the distributions which give the minimal value to our problem in such domains. We provide partial answers this by numerically studying the behaviour of the eigenvalue for certain concentric configurations in a square-domain.

We consider a square domain and the material β is to be placed in the middle region of three concentric squares. We plot the first eigenvalue as a function of the inner radius as a function of the inner radius (where by inner radius, we half the length of the inner square) for different proportions of α and β and for different orders of magnitude. The length of the middle square is set considering the measure constraint. In the figure B.5, we have plotted the first eigenvalue against the inner radius for $\alpha = 1$ and $\beta = 2$ for the proportions $0.1, 0.5$ and 0.9 of the total volume occupied by the material β. In figure B.6, we have done the same for $\alpha = 1$ and $\beta = 200$.

We see the same behaviour of the eigenvalue function as in the first experiment. In all the cases we see that the first eigenvalue is the smallest when all the β is placed inside.

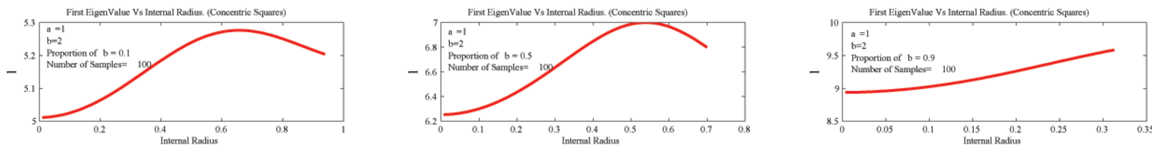

Figure B.5: Square configurations for $\beta = 2$

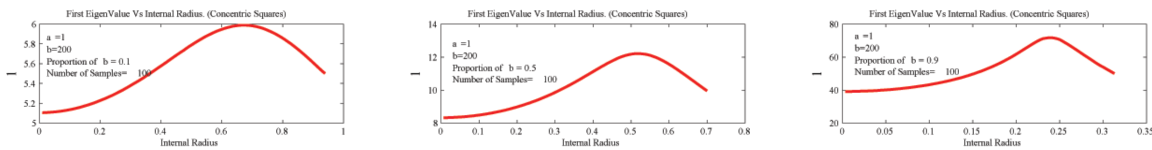

Figure B.6: Square configurations for $\beta = 200$

B.6 Conclusions

On the basis of the above numerical results we reach the conclusion that in domains like balls or squares it may be better to place the material with higher conductivity in the middle in order to minimize the first eigenvalue. A task that we set for ourselves is to verify the truth of this conjecture using the shape derivative calculus and to devise algorithms for automatic discovery of the optimal shape.

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