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SINGULAR LIMITS IN LIOUVILLE TYPE EQUATIONS WITH EXPONENTIAL NEUMANN DATA

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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En este trabajo de memoria se demostró un teorema de existencia para la ecuación de Liouville con condición de borde no lineal:

$$\begin{cases} \Delta u + \epsilon^2 e^u = 0 \quad \text{en } \Omega \\ \dfrac{\partial u}{\partial \nu} = \epsilon e^{\frac{u}{2}} \quad \text{en } \Gamma_1 \\ u = 0 \quad \text{en } \Gamma_2. \end{cases}$$

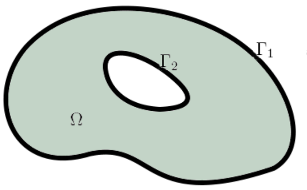
El primer paso en esta demostración consiste en la aproximación del problema original usando un *ansatz* de la solución u_ϵ, que explota en m puntos $\bar{\xi}_j \in \Gamma_1,\ j = 1, \ldots, m$ cuando el parámetro $\epsilon \to 0$, más un término de corrección ϕ, sobre el cual se obtienen un conjunto de ecuaciones que van a caracterizar la solución del problema principal. En el capítulo 4 se analizó el operador lineal asociado a estas ecuaciones y se encontró un resultado de solubilidad al modificar la ecuación con términos aditivos de coeficientes $c_j,\ j = 1, \ldots, m$. A continuación se estableció la existencia de una solución al problema no lineal con la modificación aditiva y se estudió su comportamiento en función de los puntos singulares $\bar{\xi} = (\bar{\xi}_1, \ldots, \bar{\xi}_m)$. Se demostró que la solución del problema principal, dada por el hecho de encontrar un conjunto de puntos $\bar{\xi}$ tales que $c_j(\bar{\xi}) = 0,\ \forall\, j$, puede ser reducida al análisis de los puntos críticos de una función $\varphi_m(\bar{\xi})$ en $\{(\bar{\xi}_1, \ldots, \bar{\xi}_m) \in \Gamma_1^m : \bar{\xi}_i \neq \bar{\xi}_j \text{ if } i \neq j\}$. En el capítulo final se mostró que existen al menos dos de estos puntos críticos y en consecuencia al menos dos soluciones del problema principal que explotan en m puntos.

Contents

Chapter 1

Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth external boundary Γ_1 and internal boundary Γ_2.



In this work we are going to study the equation

$$\begin{cases} \Delta u + \epsilon^2 e^u = 0 \quad \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \epsilon e^{\frac{u}{2}} \quad \text{on } \Gamma_1 \\ u = 0 \quad \text{on } \Gamma_2 \end{cases} \tag{1.1}$$

where ν is the outer unit normal vector to Γ_1 and $\epsilon > 0$ is a small parameter.

Equation (1.1) has a long history since already in 1853, J. Liouville [Lio53] has derived a representation formula for all solutions of $\Delta u + \epsilon^2 e^u = 0$ which are defined in all $\mathbb{C}$. This problem and similar ones have attracted great attention over the last decades because of the many physical and geometrical applications which are described by this equation.

In a two dimensional space, elliptic equations with this kind on nonlinearity on the domain as well as in the boundary condition arise in conformal geometry, in the problem of prescribing the Gaussian curvature of the domain and the curvature of the boundary. To be more precise let us consider two smooth surfaces with boundaries S_1 and S_2 and a conformal mapping $h : S_1 \to S_2$, that is h satisfies the property $\langle dh(v), dh(w)\rangle =$

$\lambda^2\langle v, w\rangle$, where λ is a positive function called the conformal factor. Writing $\lambda = e^{\phi}$ one has the following formulas

$$K_2 = e^{-2\phi}(K_1 - \Delta\phi)$$

where K_i is the Gauss curvature of the surface S_i, and

$$k_2 = e^{-\phi}(k_1 + \frac{\partial\phi}{\partial\nu})$$

where k_i is the geodesic curvature of the boundary of S_i. If we consider S_1 as the domain Ω, with curvature $K_1 = 0$, and the problem of finding a conformal surface with Gauss curvature ϵ^2 and geodesic curvature of the boundary ϵ, we arrive to the problem

$$\begin{cases} \Delta\phi + \epsilon^2 e^{2\phi} \ = \ 0 \quad \text{in } \Omega \\ \dfrac{\partial\phi}{\partial\nu} \ = \ \epsilon e^{\phi} - k_1 \quad \text{on } \partial\Omega \end{cases} \tag{1.2}$$

Equation (1.1) can be regarded as a simplified model of the previous geometric problem.

An analogous version of problem (1.1) with Dirichlet boundary condition

$$\begin{cases} \Delta u + \epsilon^2 e^{u} \ = \ 0 \quad \text{in } \Omega \\ u \ = \ 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.3}$$

has been well studied. The behavior of solutions to problem (1.3) that blow up is now well understood thanks to the works of H. Brezis and F. Merle [BM91], Y-Y. Li and I. Shafrir [LS94], K. Nagasaki T. Suzuki [NS90], and T. Suzuki [Suz92]. They proved that if u_ϵ is an unbounded family of solutions for which $\epsilon^2 \int_\Omega e^{u_\epsilon}$ remains uniformly bounded, then there is an integer $m \geq 1$ such that necessarily

$$\lim_{\epsilon\to 0} \epsilon^2 \int_\Omega e^{u_\epsilon} \ = \ 8m\pi \tag{1.4}$$

Moreover there are m distinct points ξ_j, $j = 1, \ldots, m$ in Ω, separated uniformly from each other and from $\partial\Omega$ as $\epsilon \to 0$, such that u_ϵ peaks to infinity in each one of them, and remains bounded away from them. The location of the blow-up points $\xi_1, \ldots, \xi_m$ is such that they, after passing to a subsequence, converge to a critical point of the function

$$\mathcal{F}(\xi_1, \ldots, \xi_m) \ = \ -\sum_{j=1}^{m} H_d(\xi_j, \xi_j) - \sum_{i\neq j} G_d(\xi_i, \xi_j) \tag{1.5}$$

where $G_d(x, y)$ is the Green function of the problem

$$\begin{cases} -\Delta_x G_d(x, y) = 8\pi\delta_y \quad \text{in } \Omega \\ G_d(x, y) = 0 \text{ for } x \in \partial\Omega \end{cases}$$

and H_d its regular part defined as

$$H_d(x,y) = G_d(x,y) + 4\log|x-y|.$$

In the analysis of the asymptotic behavior of solutions of equation (1.3), after an appropriate scaling of the solution around one of the local maxima, one is lead to consider the same equation in all of $\mathbb{R}^2$. The classification of solutions to this problem has been well studied in the work of W. Chen and C. Li in [CL91].

The reciprocal question of the existence of solutions to problem (1.3) with property (1.4) it is also been studied carefully. Construction of singular solutions to Liouville's equation has been done by S. Baraket and F. Pacard in [BP98] where they have shown that given a non-degenerate critical point of the function $\mathcal{F}$, there is a sequence u_ϵ of solutions of the Dirichlet problem that converges to a function u^* which pikes in m points of Ω. A similar study has been done simultaneously by P. Esposito, M. Grossi, A. Pistoia, and M. del Pino, M. Kowalczyk, M. Musso in [EGP05] [dPKM05] where they analyzed this problem and proved that in multiply connected domains, there exist solutions with many point of blow-up, without assuming the non-degeneracy of the critical points of $\mathcal{F}$.

In the case of equation (1.1) some work has been done by Y. Li and M. Zhu in [LZ95] where they have classified the solutions in a half space, under some integrability assumptions. No previous work has been found regarding the asymptotic behavior of solutions of (1.1), although there are some papers that deal with a linear equation with similar Neumann exponential data, such as [DdPM05] and [Wei08, WW08].

A method for the construction of blowing-up solutions is presented in this thesis. As in [dPKM05] and [DdPM05] we are going to prove the existence of solutions to equation (1.1) via the approximation of a basic function u_ϵ that explodes as $\epsilon \to 0$ in m points contained in Γ_1.

Teorema 1.0.1. *Given any $m \geq 1$, there is $\epsilon_0 > 0$ such that, for $0 < \epsilon \leq \epsilon_0$ there exists two solutions u_ϵ of equation (1.1) such that for each one, there is a family of m different points $\bar{\xi}_1, \ldots, \bar{\xi}_m$ located in Γ_1, separated by uniformly positive distance δ from each other, for which the function u_ϵ remains uniformly bounded on $\Omega \setminus \bigcup_{j=1}^{m} B(\bar{\xi}_j, \rho)$, and*

$$\sup_{B(\bar{\xi}_j,\rho)} u_\epsilon \to +\infty$$

for any $\rho > 0$.

The location of the blowing-up points in Γ_1 is characterized by the critical point of the functional

$$\varphi_m(\bar{\xi}_1, \ldots, \bar{\xi}_m) = a\pi \sum_{j=1}^{m} H(\bar{\xi}_j, \bar{\xi}_j) - b\pi \sum_{l \neq j} G(\bar{\xi}_j, \bar{\xi}_l) \tag{1.6}$$

where $a > 0,\ b > 0$ *are constants,* $G(x, y)$ *is the Green function of the problem*

$$\begin{cases} \Delta G(x, y) = 0 \ \ \textit{in}\ \Omega \\ \dfrac{\partial G}{\partial \nu} = 2\pi \delta_y \ \ \textit{on}\ \Gamma_1 \\ G(x, y) = 0 \ \ \textit{on}\ \Gamma_2 \end{cases}$$

and H *its regular part defined as*

$$H(x, y) = G(x, y) + 2 \log |x - y|.$$

In the subsequent chapters we will present a proof for this theorem. We start with the construction of a basic approximation in chapter 2. Solutions are found as a small additive perturbation of these initial approximations and the analysis of the equations satisfied by the perturbation is carried out in chapters 4 to 5. This procedure leads to a finite dimensional reduction (chapter 6). To solve this reduced problem we need to adjust the location of the concentration points, for which we use a variational argument. The final proof of the theorem is established in chapter 8, and finally, in chapter 9, we present the conclusions of this thesis and some possible future work.

Chapter 2

A first approximation of the solution

2.1 Basic solution

In this chapter we will provide the first approximation to the solutions of problem (1.1). For this purpose we will use the radially symmetrical solutions of

$$\begin{cases} \Delta u + e^u = 0, \ \text{in } \mathbb{R}^2, \\ u(x) \to -\infty, \ \text{as} |x| \to \infty, \end{cases} \tag{2.1}$$

which are given by the one-parameter family of functions

$$w_\mu(r) = \log \frac{8\mu^2}{(\mu^2 + r^2)^2},$$

where μ is any positive number.

Let m be a positive integer and choose m distinct points in Γ_1, say $\bar{\xi}_1, \ldots, \bar{\xi}_m$. Let $\mu_j,\ j = 1, \ldots, m$ be positive numbers. We observe that the function

$$u_j(x) = \log \frac{8\mu_j^2}{(\mu_j^2 \epsilon^2 + |x - \xi_j|^2)^2} = w_{\mu_j}(\frac{|x - \xi_j|}{\epsilon}) + 4 \log \frac{1}{\epsilon}$$

where $\xi_j = \bar{\xi}_j + \sqrt{2}\mu_j \epsilon \nu(\bar{\xi}_j)/2$, satisfies in entire $\mathbb{R}^2$

$$\Delta u_j + \epsilon^2 e^{u_j} = 0.$$

And in the line $\{x_2 = \xi_j^2 + \sqrt{2}\mu_j \epsilon/2\}$

$$\frac{\partial u_j}{\partial \nu} = \epsilon e^{u_j/2}.$$

We would like to take $\sum_{j=1}^{m} u_j$ as a first approximation, but we need to modify it in order to satisfy the boundary conditions. We consider $H_j(x)$ solution of

$$\begin{cases} \Delta H_j = 0 \ \text{ in } \Omega, \\ \dfrac{\partial H_j}{\partial \nu} = \epsilon e^{u_j/2} - \dfrac{\partial u_j}{\partial \nu} \ \text{ on } \Gamma_1 \\ H_j = -u_j \ \text{ on } \Gamma_2 \end{cases} \tag{2.2}$$

and the approximated solution is

$$U(x) = \sum_{j=1}^{m} u_j(x) + H_j(x) \tag{2.3}$$

Lemma 2.1.1. *For any* $0 < \alpha < 1$

$$H_j(x) = 2H(x, \bar{\xi}_j) - \log 8\mu_j^2 + O(\epsilon^\alpha) \tag{2.4}$$

uniformly in $\bar{\Omega}$*, where* H *is the regular part of the Green function defined by*

$$\begin{cases} \Delta G(x,y) = 0 \ \textit{ in } \Omega \\ \dfrac{\partial G}{\partial \nu} = 2\pi \delta_y \ \textit{ on } \Gamma_1 \\ G(x,y) = 0 \ \textit{ on } \Gamma_2 \end{cases} \tag{2.5}$$

i.e.

$$H(x,y) = G(x,y) + 2\log|x-y|$$

Proof. Let us define the difference

$$z_\epsilon(x) = H_j(x) + \log 8\mu_j^2 - 2H(x, \bar{\xi}_j)$$

which satisfies

$$\begin{cases} \Delta z_\epsilon = 0 \ \text{ in } \Omega \\ \dfrac{\partial z_\epsilon}{\partial \nu} = \dfrac{\partial H_j}{\partial \nu} - \dfrac{4(x-\bar{\xi}_j)\cdot \nu}{|x-\bar{\xi}_j|^2} \ \text{ on } \Gamma_1 \\ z_\epsilon = -u_j + \log 8\mu_j^2 - 4\log|x-\bar{\xi}_j| \ \text{ on } \Gamma_2 \end{cases} \tag{2.6}$$

First let us find a bound for the boundary term

$$\frac{\partial H_j}{\partial \nu} - \frac{4(x-\bar{\xi}_j)\cdot \nu}{|x-\bar{\xi}_j|^2} \tag{2.7}$$

Notice that

$$\begin{aligned} \frac{\partial H_j}{\partial \nu} &= \epsilon e^{u_j/2} - \frac{\partial u_j}{\partial \nu} \\ &= \epsilon \frac{\sqrt{8}\mu_j}{\mu_j^2\epsilon^2 + |x-\xi_j|^2} + \frac{4(x-\xi_j)\cdot\nu}{\mu_j^2\epsilon^2 + |x-\xi_j|^2} \end{aligned}$$

So 2.7 is

$$
\begin{aligned}
&= \frac{4(\sqrt{2}\mu_j\epsilon/2 + (x-\xi_j)\cdot\nu(x))}{\mu_j^2\epsilon^2 + |x-\xi_j|^2} - \frac{4(x-\bar{\xi}_j)\cdot\nu(x)}{|x-\bar{\xi}_j|^2} \\
\textit{remembering that} \quad & \xi_j = \bar{\xi}_j + \frac{\sqrt{2}}{2}\mu_j\epsilon\nu(\bar{\xi}_j) \\
&= \frac{2\sqrt{2}\mu_j\epsilon(1-\nu(\bar{\xi}_j)\cdot\nu(x))}{\mu_j^2\epsilon^2 + |x-\xi_j|^2} \\
&\quad + \frac{4(x-\bar{\xi}_j)\cdot\nu(x)}{\mu_j^2\epsilon^2 + |x-\xi_j|^2}\left(\frac{|x-\bar{\xi}_j|^2 - \mu_j^2\epsilon^2 - |x-\xi_j|^2}{|x-\bar{\xi}_j|^2}\right) \\
&= \frac{2\sqrt{2}\mu_j\epsilon(1-\nu(\bar{\xi}_j)\cdot\nu(x))}{\mu_j^2\epsilon^2 + |x-\xi_j|^2} \\
&\quad + \frac{4(x-\bar{\xi}_j)\cdot\nu(x)}{\mu_j^2\epsilon^2 + |x-\xi_j|^2}\left(\frac{(x-\bar{\xi}_j)\cdot\nu(\bar{\xi}_j)\sqrt{2}\mu_j\epsilon - 3\mu_j^2\epsilon^2/2}{|x-\bar{\xi}_j|^2}\right)
\end{aligned}
$$

We will use the following bounds,

$$
|1-\nu(\bar{\xi}_j)\cdot\nu(x)| \le C|x-\bar{\xi}_j|^2, \;\; |(x-\bar{\xi}_j)\cdot\nu(x)| \le C|x-\bar{\xi}_j|^2, \; \forall x \in \Gamma_1 \tag{2.8}
$$

that we extracted from [DdPM05].

Using all this together,

$$
\begin{aligned}
\left|\frac{\partial H_j}{\partial \nu} - \frac{4(x-\bar{\xi}_j)\cdot\nu}{|x-\bar{\xi}_j|^2}\right| &\le C\epsilon\frac{|x-\bar{\xi}_j|^2}{\mu_j^2\epsilon^2 + |x-\xi_j|^2} + C\frac{(\sqrt{2}\mu_j\epsilon(x-\bar{\xi}_j)\cdot\nu(\bar{\xi}_j) - \frac{3}{2}\mu_j^2\epsilon^2)}{\mu_j^2\epsilon^2 + |x-\xi_j|^2} \\
&\le C\epsilon + C\frac{(\sqrt{2}\mu_j\epsilon(x-\bar{\xi}_j)\cdot\nu(\bar{\xi}_j) - \frac{3}{2}\mu_j^2\epsilon^2)}{\mu_j^2\epsilon^2 + |x-\xi_j|^2}
\end{aligned}
$$

Fix $\rho > 0$ small. Then, for $|x-\bar{\xi}_j| \ge \rho$, $x \in \Gamma_1$ we have

$$
\left|\frac{\partial H_j}{\partial \nu} - \frac{4(x-\bar{\xi}_j)\cdot\nu}{|x-\bar{\xi}_j|^2}\right| \le C\epsilon \tag{2.9}
$$

and now, let $p > 1$. Changing variables $\mu_j\epsilon y = x - \bar{\xi}_j$ we have that

$$
\begin{aligned}
\int_{B(\bar{\xi}_j,\rho)\cap\Gamma_1} \left|\frac{\sqrt{2}\mu_j\epsilon(x-\bar{\xi}_j)\cdot\nu(\bar{\xi}_j) - \frac{3}{2}\mu_j^2\epsilon^2}{\mu_j^2\epsilon^2 + |x-\xi_j|^2}\right|^p dx &= \mu_j\epsilon \int_{B(0,\rho/\mu_j\epsilon)\cap\Gamma_1^{\mu_j\epsilon}} \left|\frac{\sqrt{2}y\cdot\nu(0) - 3/2}{(1+|y-\sqrt{2}\nu(0)/2|^2)}\right|^p dy \\
&\le C\epsilon\int_0^{\rho/\mu_j\epsilon}\frac{1}{(1+s)^p}\,ds \\
&\le C\epsilon
\end{aligned}
$$

It can be easily checked because of the definition of u_j that

$$\|z_\epsilon\|_{L^p(\Gamma_2)} \leq C\epsilon^{\frac{1}{p}}$$

By L^p theory

$$\|z_\epsilon\|_{W^{1+s,p}(\Omega)} \leq C(\|\frac{\partial z_\epsilon}{\partial \nu}\|_{L^p(\Gamma_1)} + \|\Delta z_\epsilon\|_{L^p(\Omega)} + \|z_\epsilon\|_{L^p(\Gamma_2)}) \leq C\epsilon^{\frac{1}{p}}$$

for any $0 < s < \frac{1}{p}$. By the Morrey embedding we obtain

$$\|z_\epsilon\|_{C^\gamma(\bar{\Omega})} \leq C\epsilon^{\frac{1}{p}}$$

for any $0 < \gamma < \frac{1}{2} + \frac{1}{p}$, which proves the result with $\alpha = \frac{1}{p}$. □

2.2 Scaled equation

It will be convenient to work with the scaling of u given by

$$v(y) = u(\epsilon y) + 4\log\epsilon$$

where now $y \in \Omega_\epsilon$ the extended domain (Ω/ϵ).
If u is a solution to 1.1 then v satisfies

$$\begin{aligned} \Delta_y v(y) &= \epsilon^2 \Delta u(\epsilon y) = -\epsilon^2\epsilon^2 e^{u(\epsilon y)} \\ &= -e^{u(\epsilon y)+4\log\epsilon} \\ &= -e^{v(y)} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial v}{\partial \nu} &= \epsilon\frac{\partial u(\epsilon y)}{\partial \nu} = \epsilon^2 e^{u(\epsilon y)/2} \\ &= e^{\frac{u(\epsilon y)}{2}+\frac{4\log\epsilon}{2}} \\ &= e^{v(y)/2} \end{aligned}$$

So v finally satisfies

$$\begin{cases} \Delta v + e^v = 0 & \text{in } \Omega_\epsilon \\ \dfrac{\partial v}{\partial \nu} = e^{v(y)/2} & \text{on } \Gamma_1^\epsilon \\ v = 4\log\epsilon & \text{on } \Gamma_2^\epsilon \end{cases} \tag{2.10}$$

With this scaling u_j transforms into

$$v_j(y) = u_j(\epsilon y) + 4\log\epsilon = \log\frac{8\mu_j^2}{(\mu_j^2 + |y-\xi_j'|^2)^2}$$

where we have defined $\xi_j' = \xi/\epsilon$.

We will search for solutions of the form

$$v = V + \phi$$

with $V(y) = U(\epsilon y) + 4\log\epsilon$.
Replacing in 2.10 we find

$$\begin{aligned} & \Delta V + \Delta\phi + e^{V+\phi} = 0 \\ \Leftrightarrow\quad & \Delta V + \Delta\phi + e^{V}(e^{\phi} - 1 - \phi) + e^{V} + e^{V}\phi = 0 \\ \Leftrightarrow\quad & \Delta\phi + e^{V}\phi = -\left[\Delta V + e^{V} + e^{V}(e^{\phi} - 1 - \phi)\right] \end{aligned}$$

Let us call

$$R_1(y) = \Delta V + e^{V(y)} \tag{2.11}$$

$$N_1(\phi, y) = e^{V(y)}(e^{\phi} - 1 - \phi) \tag{2.12}$$

$$W_1(y) = e^{V(y)} \tag{2.13}$$

Now replacing in the boundary condition of Γ_1^ϵ

$$\begin{aligned} & \frac{\partial V}{\partial\nu} + \frac{\partial\phi}{\partial\nu} = e^{V/2}e^{\phi/2} \\ \Leftrightarrow\quad & \frac{\partial V}{\partial\nu} + \frac{\partial\phi}{\partial\nu} = e^{V/2}(e^{\phi/2} - 1 - \frac{\phi}{2}) + e^{V/2} + \frac{e^{V/2}\phi}{2} \\ \Leftrightarrow\quad & \frac{\partial\phi}{\partial\nu} = e^{V/2}\frac{\phi}{2} + e^{V/2}(e^{\phi/2} - 1 - \frac{\phi}{2}) + \left[e^{V/2} - \frac{\partial V}{\partial\nu}\right] \end{aligned}$$

Let us call

$$R_2(y) = e^{V/2} - \frac{\partial V}{\partial\nu} \tag{2.14}$$

$$N_2(\phi, y) = e^{V/2}(e^{\phi/2} - 1 - \frac{\phi}{2}) \tag{2.15}$$

$$W_2(y) = e^{V/2} \tag{2.16}$$

And now in Γ_2^ϵ

$$\begin{aligned} & V + \phi = 4\log\epsilon \\ \Leftrightarrow\quad & \phi = 4\log\epsilon - V \\ \Leftrightarrow\quad & \phi = -U(\epsilon y) \end{aligned}$$

But $U \equiv 0$ in Γ_2^ϵ.
In summary:

$$\begin{cases} \Delta\phi + W_1\phi = -R_1(y) - N_1(\phi, y) & \text{in } \Omega_\epsilon \\ \dfrac{\partial\phi}{\partial\nu} = W_2\dfrac{\phi}{2} + R_2(y) + N_2(\phi, y) & \text{on } \Gamma_1^\epsilon \\ \phi = 0 & \text{on } \Gamma_2^\epsilon \end{cases} \tag{2.17}$$

2.3 Error bounds

Let us find bounds for the terms R_1 and R_2.

2.3.1 Estimate of R_1

Notice that

$$e^{V(y)} = \epsilon^4 e^{U(\epsilon y)} = \epsilon^4 e^{U(x)}$$

Now, we know that $u_j(x)$ blows up in $x = \xi_j$, but far from ξ_j is a smooth decreasing function. Let us fix a small number $\delta > 0$ and observe that for $|x-\xi_j| > \delta \, \forall j = 1, \ldots, m$:

$$|u_j(x)| < M \;\Rightarrow\; \epsilon^4 e^{U(x)} < O(\epsilon^4)$$

Also in this region

$$\begin{aligned} u_j(x) &= \log \frac{8\mu_j^2}{(\mu_j^2\epsilon^2 + |x-\xi_j|^2)^2} \\ &= \log 8\mu_j^2 - 4\log|x-\xi_j| - 2\log\,(1 + \frac{\mu_j^2\epsilon^2}{|x-\xi_j|^2}) \end{aligned}$$

Using Taylor's approximation formula

$$u_j(x) = \log 8\mu_j^2 - 4\log|x-\xi_j| + O(\mu_j^2\epsilon^2) \;\text{ for } |x-\xi_j| > \delta. \tag{2.18}$$

With this estimate and lemma 2.1.1 we have that

$$\begin{aligned} u_j(x) + H_j(x) &= \cancel{\log 8\mu_j^2} - 4\log|x-\xi_j| + O(\mu_j^2\epsilon^2) + 2H(x,\xi_j) - \cancel{\log 8\mu_j^2} + O(\mu_j^2\epsilon^2) \\ &= \cancel{-4\log|x-\xi_j|} + 2G(x,\xi_j) + \cancel{4\log|x-\xi_j|} + O(\mu_j^2\epsilon^2) \\ &= 2G(x,\xi_j) + O(\mu_j^2\epsilon^2) \end{aligned}$$

And so, using that G solves 2.5

$$\Delta U = \Delta(u_j + H_j) = \cancel{2\Delta G(x,\xi_j)} + O(\mu_j^2\epsilon^2) = O(\mu_j^2\epsilon^2)$$

$$\Rightarrow \Delta V = \epsilon^2 \Delta U = O(\epsilon^4) \;\text{ for } |x-\xi_j| > \delta \,\forall\, j$$

$$\Rightarrow R_1(y) = O(\epsilon^4) \;\text{ for } |y-\xi_j'| > \frac{\delta}{\epsilon} \,\forall\, j$$

Now, if for any j, $|y-\xi'_j| < \frac{\delta}{\epsilon}$, lets find a bound for $e^{V(y)}$
We write $y = \bar{\xi}'_j + z$, so

$$\begin{aligned}
e^{V(y)} &= e^{U(\epsilon y)+4\log\epsilon} \\
&= \epsilon^4 e^{\sum_{l=1}^{m} u_l + H_l} = \epsilon^4 e^{u_j} e^{H_j + \sum_{l\neq j} u_l + H_l} \\
&= \epsilon^4 \frac{8\mu_j^2}{(\mu_j^2\epsilon^2 + |\epsilon y - \xi_j|^2)^2} exp\left(H_j(\epsilon y) + \sum_{l\neq j} u_l + H_l\right) \\
&= \frac{8\mu_j^2}{(\mu_j^2 + |z - \sqrt{2}\mu_j\nu(0)/2|^2)^2} exp\Big(H_j(\bar{\xi}_j + \epsilon z) \\
&\quad + \sum_{l\neq j} \log \frac{8\mu_l^2}{(\mu_l^2\epsilon^2 + |\bar{\xi}_j - \bar{\xi}_l + \epsilon z + \sqrt{2}\epsilon/2(\mu_j\nu(0) - \mu_l\nu(\bar{\xi}_j - \bar{\xi}_l))|^2)^2} + H_l(\bar{\xi}_j + \epsilon z)\Big)
\end{aligned}$$

Let us expand the term inside the exponential

$$H_j(\bar{\xi}_j+\epsilon z)+\sum_{l\neq j} \log \frac{8\mu_l^2}{(\mu_l^2\epsilon^2 + |\bar{\xi}_j - \bar{\xi}_l + \epsilon z + \sqrt{2}\epsilon/2(\mu_j\nu(0) - \mu_l\nu(\bar{\xi}_j - \bar{\xi}_l))|^2)^2}+H_l(\bar{\xi}_j+\epsilon z) \tag{2.19}$$

Using the approximation of H in Lemma 2.1.1, we have that

$$H_l(\epsilon y) = 2H(\epsilon y, \bar{\xi}_l) - \log 8\mu_l^2 + O(\mu_l^2\epsilon^2)$$

And using a Taylor expansion around $\bar{\xi}_j$,

$$H_l(\epsilon y) = 2H(\bar{\xi}_j, \bar{\xi}_l) - \log 8\mu_l^2 + O(\mu_l^2\epsilon^2) + O(\epsilon|y - \xi'_j|)$$

Replacing in 2.19

$$\begin{aligned}
&= 2H(\bar{\xi}_j, \bar{\xi}_j) - \log 8\mu_j^2 + O(\mu_j^2\epsilon^2) + O(\epsilon|y-\xi'_j|) \\
&\quad + \sum_{l\neq j}\Big[\log \frac{8\mu_l^2}{(\mu_l^2\epsilon^2 + |\bar{\xi}_j - \bar{\xi}_l + \epsilon z + \sqrt{2}\epsilon/2(\mu_j\nu(0) - \mu_l\nu(\bar{\xi}_j - \bar{\xi}_l))|^2)^2} \\
&\quad + 2H(\bar{\xi}_j, \bar{\xi}_l) - \log 8\mu_l^2 + O(\mu_l^2\epsilon^2) + O(\epsilon|y-\xi'_j|)\Big] \\
&= 2H(\bar{\xi}_j, \bar{\xi}_j) - \log 8\mu_j^2 + \sum_{l\neq j}\Big[-2\log\left(\mu_l^2\epsilon^2 + |\bar{\xi}_j - \bar{\xi}_l + \epsilon z + \sqrt{2}\epsilon/2(\mu_j\nu(0) - \mu_l\nu(\bar{\xi}_j - \bar{\xi}_l))|^2\right) \\
&\quad +2H(\bar{\xi}_j, \bar{\xi}_l)\Big] + O(\epsilon^2) + O(\epsilon|y-\xi'_j|) \\
&= 2H(\bar{\xi}_j, \bar{\xi}_j) - \log 8\mu_j^2 + \sum_{l\neq j}(-4\log|\bar{\xi}_j - \bar{\xi}_l| + 2H(\bar{\xi}_j, \bar{\xi}_l)) + O(\epsilon|y-\xi'_j|) + O(\epsilon^2) \\
&= 2H(\bar{\xi}_j, \bar{\xi}_j) - \log 8\mu_j^2 + \sum_{l\neq j} 2G(\bar{\xi}_j, \bar{\xi}_l) + O(\epsilon|y-\xi'_j|) + O(\epsilon^2)
\end{aligned}$$

Remembering that the basic solution that we considered has the free parameter μ, we could fix it in order to eliminate the error above.
We set

$$\log 8\mu_j^2 = 2H(\bar{\xi}_j, \bar{\xi}_j) + \sum_{l\neq j} 2G(\bar{\xi}_j, \bar{\xi}_l) \tag{2.20}$$

So, 2.19 remains as only $O(\epsilon|y-\xi_j'|) + O(\epsilon^2)$.
And then we have that

$$e^{V(y)} = \frac{8\mu_j^2}{(\mu_j^2 + |y-\xi_j'|^2)^2}[1 + O(\epsilon|y-\xi_j'|) + O(\epsilon^2)] \text{ for } |y-\xi_j'| < \delta/\epsilon. \tag{2.21}$$

Also in this region

$$\begin{aligned} \Delta V(y) &= \sum_{l=1}^{m} \epsilon^2(\Delta u_l + \cancel{\Delta H_l}) = \epsilon^2 \Delta u_j + \sum_{l\neq j} \epsilon^2 \Delta u_l \\ &= -\epsilon^4 \frac{8\mu_j^2}{(\mu_j^2\epsilon^2 + |x-\xi_j|^2)^2} - \sum_{l\neq j} \epsilon^4 \frac{8\mu_l^2}{(\mu_l^2\epsilon^2 + |x-\xi_l|^2)^2} \\ &= -\frac{8\mu_j^2}{(\mu_j^2 + |y-\xi_j'|^2)^2} - \sum_{l\neq j} \frac{8\mu_l^2}{|y-\xi_l'|^4(1 + \underbrace{\frac{\mu_l^2}{|y-\xi_l'|^2}}_{O(\epsilon^2/\delta^2)})^2} \\ &= -\frac{8\mu_j^2}{(\mu_j^2 + |y-\xi_j'|^2)^2} + O(\epsilon^4) \end{aligned}$$

and combining both results together we have

$$\Delta V + e^{V(y)} = O(\frac{1}{1 + |y-\xi_j'|^4}) + O(\frac{\epsilon|y-\xi_j'|}{1 + |y-\xi_j'|^4})$$

In summary,

$$\text{For } |y-\xi_j'| > \frac{\delta}{\epsilon}\ \forall j \quad \Rightarrow \quad R_1(y) = O(\epsilon^4) \tag{2.22}$$

$$\text{For } |y-\xi_j'| < \frac{\delta}{\epsilon} \text{ for some } j \quad \Rightarrow \quad R_1(y) = O(\frac{\epsilon|y-\xi_j'|}{1 + |y-\xi_j'|^4}). \tag{2.23}$$

2.3.2 Estimate of R_2

Analogously to the estimation of R_1, R_2 is

$$\text{For } |y-\xi_j'| > \frac{\delta}{\epsilon}\ \forall j \quad \Rightarrow \quad R_2(y) = O(\epsilon^2) \tag{2.24}$$

$$\text{For } |y-\xi_j'| < \frac{\delta}{\epsilon} \text{ for some } j \quad \Rightarrow \quad R_2(y) = O(\frac{\epsilon|y-\xi_j'|}{(\mu_j^2 + |y-\xi_j'|^2)}).$$

Chapter 3

Non degeneracy

We will prove the non degeneracy of the solution to:

$$\begin{cases} \Delta u + e^u = 0 \text{ in } \{(x,y) \in \mathbb{R}^2 : y > \frac{\sqrt{2}}{2}\} \\ \frac{\partial u}{\partial \nu} = e^{u/2} \text{ on } \{(x,y) \in \mathbb{R}^2 : y = \frac{\sqrt{2}}{2}\}, \end{cases} \tag{3.1}$$

linearized around the radial solution

$$u \;=\; log(\frac{8\mu}{(\mu^2 + |z|^2)^2}), \tag{3.2}$$

with respect to the natural invariances of the equation under dilatations and translations in the horizontal axis.

3.1 Study of the linearized equation

Let us suppose that $u + v$ is a solution of the equation 3.1

$$\Delta(u+v) + e^{u+v} = \Delta u + \Delta v + e^u e^v \;=\; 0$$

Retaining only the linear terms in v, and using that u is a solution of the original equation we find that,

$$\Delta v + \frac{8\mu^2}{(\mu^2 + |z|^2)^2} v = 0$$

Analogously with the boundary condition we find

$$\frac{\partial v}{\partial \nu} - \frac{\sqrt{2}\mu}{(\mu^2 + |z|^2)} v = 0$$

We now study the bounded solutions to this problem

$$\begin{cases} \Delta v + \dfrac{8\mu^2}{(\mu^2+|z|^2)^2} v = 0 \text{ in } \{z=(x,y)\in\mathbb{R}^2 : y \geq \dfrac{\sqrt{2}}{2}\} \\ \dfrac{\partial v}{\partial \nu} - \dfrac{\sqrt{2}\mu}{(\mu^2+|z|^2)} v = 0 \text{ on } \{z=(x,y)\in\mathbb{R}^2 : y = \dfrac{\sqrt{2}}{2}\}, \end{cases} \tag{3.3}$$

Lemma 3.1.1. *The only bounded solutions to problem 3.3 are*

$$z_0(z) = \frac{\partial}{\partial s}(u(sz)+2\log s)|_{s=1} = \frac{-2z_1}{\mu^2+|z|^2}$$

$$z_1(z) = \frac{\partial}{\partial \zeta_1} u(z+\zeta_1)|_{\zeta_1=0} = \frac{2(\mu^2-|z|^2)}{\mu^2+|z|^2}$$

Proof. Let us consider the following transformations:

$$\begin{aligned} T_1 : D_2 &\longrightarrow D_1 \\ (x,y) &\longrightarrow (x, y - \frac{\mu(\sqrt{3}-1)}{\sqrt{2}}) \\ T_2 : D_3 &\longrightarrow D_2 \\ (x,y) &\longrightarrow (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}) \end{aligned}$$

With

$$D_1 = \{(x,y)\in\mathbb{R}^2 | y \geq \frac{\sqrt{2}\mu}{2}\} \tag{3.4}$$

$$D_2 = \{(x,y)\in\mathbb{R}^2 | y \geq \frac{\sqrt{6}\mu}{2}\} \tag{3.5}$$

$$D_3 = \{(x,y)\in\mathbb{R}^2 | x^2 + (y - \frac{1}{\mu\sqrt{6}})^2 \leq \frac{1}{6\mu^2}\} \tag{3.6}$$

We will define $\hat{v}$ over the disk D_3 as

$$\begin{aligned} \hat{v} = v\circ T_1 \circ T_2 : D_3 &\longrightarrow \mathbb{R} \\ (x,y) &\longrightarrow \hat{v}(x,y) = v(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} - \frac{\mu(\sqrt{3}-1)}{\sqrt{2}}) \end{aligned}$$

And using 3.3 we find that $\hat{v} \in C^\infty(\bar{D}_3)$ is a solution of:

$$\begin{cases} \Delta \hat{v} + \dfrac{8\mu^2}{(\mu^2(3-\sqrt{3})(r^2-\frac{1}{6\mu^2})+1)^2} \hat{v} = 0 \text{ in } D_3 \\ \dfrac{\partial \hat{v}}{\partial \nu} - \sqrt{2}\mu\hat{v} = 0 \text{ on } \partial D_3, \end{cases} \tag{3.7}$$

Which have radial symmetry in the disk D_3, i.e. r is the distance to the center of D_3. We write solution $\hat{v}$ in fourier series:

$$\hat{v}(r,\theta) = \sum_{k\in\mathbb{Z}} w_k(r)e^{ik\theta}$$

with $w_k(r) = \int_0^{2\pi} \hat{v}(r,\theta)e^{-ik\theta}\,d\theta$ wich is also bounded in D_3.
So equation 3.7 is

$$\frac{1}{r}\frac{\partial w_k}{\partial r} + \frac{\partial^2 w_k}{\partial r^2} + \frac{8\mu^2}{(\mu^2(3-\sqrt{3})(r^2 - \frac{1}{6\mu^2}) + 1)^2} w_k(r) = \frac{k^2}{r^2} w_k(r) \tag{3.8}$$

With the boundary condition

$$\frac{\partial w_k}{\partial r} = \sqrt{2}\mu\, w_k \text{ on } r = \frac{1}{\sqrt{6}\mu}$$

Equation 3.8 can be rewritten as

$$\frac{1}{r}\frac{\partial w_k}{\partial r} + \frac{\partial^2 w_k}{\partial r^2} + \frac{8\mu^2}{\frac{(3+\sqrt{3})}{6^2}(\frac{\mu^2 6(3-\sqrt{3})r^2}{3+\sqrt{3}} + 1)^2} w_k(r) = \frac{k^2}{r^2} w_k(r)$$

Let us call $a^2 = \dfrac{\mu^2 6(3-\sqrt{3})}{(3+\sqrt{3})}$, and changing variables $R = ar$ the equation becomes

$$\frac{1}{R}\frac{\partial w_k}{\partial R} + \frac{\partial^2 w_k}{\partial R^2} - \frac{k^2}{R^2} w_k + \frac{8}{(R^2+1)^2} w_k = 0$$

The only solution bounded in $0 < R < \frac{a}{\sqrt{6}\mu}$ is (for $|k| \geq 1$)

$$w_k(R) = \frac{(k+1) + (k-1)R^2}{(1+R^2)} R^k \tag{3.9}$$

Let us find the values of k replacing the solution 3.9 into the boundary condition:

$$a\frac{\partial w_k}{\partial R} = \sqrt{2}\mu w_k \text{ on } R = \frac{a}{\sqrt{6}\mu}$$

$$\Leftrightarrow a\left[\frac{2((k-1)R^{k+1}}{(1+R^2)} + \frac{(k+1+(k-1)R^2)R^{k-1}k}{(1+R^2)} - \frac{2(k+1+(k-1)R^2)R^{k+1}}{(1+R^2)^2}\right] = \sqrt{2}\mu\frac{(k+1)R^k + (k-1)R^{k+2}}{1+R^2}$$

Multiplying by $\dfrac{(1+R^2)^2}{R^{k-1}}$

$$
\begin{aligned}
&a\left[2(k-1)R^2+2(k-1)R^4+k(k+1)+k(k+1)R^2+(k-1)R^2k+\right.\\
&\left.(k-1)kR^4-2(k+1)R^2-2(k-1)R^4\right]=\\
&\sqrt{2}\mu\left[(k+1)R+(k-1)R^3+(k+1)R^3+(k-1)R^5\right]
\end{aligned}
$$

Evaluating in $R=\dfrac{a}{\sqrt{6}\mu}$ and collecting terms in orders of k

$$
k^2\left(1+\frac{2a^2}{6\mu^2}+\frac{a^4}{36\mu^4}\right)+k\left(1-\frac{a^4}{36\mu^4}-\frac{1}{\sqrt{3}}-\frac{2a^2}{6\sqrt{3}\mu^2}-\frac{a^4}{36\sqrt{3}\mu^4}\right)+
\left(\frac{-2a^2}{3\mu^2}-\frac{1}{\sqrt{3}}+\frac{a^4}{36\sqrt{3}\mu^4}\right) = 0
$$

and replacing the value of a

$$
k^2\left(1+2(2-\sqrt{3})+(2-\sqrt{3})^2\right)+k\left(1-(2-\sqrt{3})^2-\frac{1}{\sqrt{3}}-\frac{2(2-\sqrt{3})}{\sqrt{3}}-\frac{(2-\sqrt{3})^2}{\sqrt{3}}\right)+
\left(-4(2-\sqrt{3})-\frac{1}{\sqrt{3}}+\frac{(2-\sqrt{3})^2}{\sqrt{3}}\right)=0
$$

$$
\begin{aligned}
k^2(12-6\sqrt{3})-12+6\sqrt{3}&=0\\
(k^2-1)(12-6\sqrt{3})&=0\\
k&=\pm 1
\end{aligned}
$$

This implies that the only bounded solutions in the domain for this equation with $|k|\geq 1$ are,

$$
\begin{aligned}
w_1&=\frac{2R}{1+R^2}\\
w_{-1}&=\frac{-2R}{1+R^2}
\end{aligned}
$$

Now for $k=0$ the equation is

$$
\frac{1}{R}\frac{\partial w_0}{\partial R}+\frac{\partial^2 w_0}{\partial R^2}+\frac{8w_0}{(1+R^2)^2}=0
$$

which has as the only bounded solution in D_3, the function

$$
w_0=\frac{1-R^2}{1+R^2}
$$

We need to study for $k = 0, 1, -1$ the linearly independent solutions provided by the Liouville's formula

$$\begin{aligned}
\tilde{w}_1 &= Cw_1 \int \frac{1}{w_1^2} exp\left(-\int \frac{1}{R}\, dR\right) \\
&= Cw_1 \int \frac{(1+R^2)^2}{4R^3}\, dR \\
&= \frac{-C}{4R(1+R^2)} + \frac{CR\log R}{1+R^2} + \frac{CR^3}{4(1+R^2)}
\end{aligned}$$

We see that for $R \longrightarrow 0$, $|\tilde{w}_1| \longrightarrow +\infty$, so we can discard this solution.

Now for w_0,

$$\begin{aligned}
\tilde{w}_0 &= Cw_0 \int \frac{1}{w_0} exp\left(-\int \frac{1}{R}\, dR\right) dR \\
&= C\frac{(1-R^2)}{1+R^2} \int \frac{(1+R^2)^2}{(1-R^2)^2 R}\, dR \\
&= C\left(\frac{2}{1+R^2} + \frac{(1-R^2)\log R}{1+R^2}\right)
\end{aligned}$$

again for $R \longrightarrow 0$, $|\tilde{w}_0| \longrightarrow +\infty$, and we discard this solution as well.
Finally we keep as the only bounded solutions in the domain,

$$w_0 = \frac{1-R^2}{1+R^2} \tag{3.10}$$

$$w_1 = \frac{2R}{1+R^2} \tag{3.11}$$

Which after the inverse transformation becomes z_0 and z_1 respectively, constant except.
This finshes the proof since if v is a bounded solution to 3.3, we showed that, after changing variables, v necessarily will take the form of a linear combination of w_0 and w_1. □

Chapter 4

Analysis of the linearized operator

A main step into solving problem 2.17 is that of a solvability theory for the linear operator. In developing this theory we will take into account the invariance, under translations and dilatations, of the problem $\Delta u + e^u = 0$ in $\mathbb{R}^2$. In this Chapter we will prove the bounded invertibility of the equation in this sense using weighted L^∞-norms naturally attached to the setting of the problem.

For convenience let us define

$$L(\phi) = \Delta\phi + W_1(y)\phi \tag{4.1}$$

Notice that, in a region close to ξ_j', operator 4.1 formally aproaches as $\epsilon \longrightarrow 0$ to

$$L(\phi) \to \Delta\phi + \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2}\phi$$

Which is exactly the same operator that appears when the equation $\Delta v + e^v = 0$ is linearized around the basic solution $v_j(z) = \log \frac{8\mu_j^2}{(\mu_j^2+|z|^2)^2}$.

As we prove in chapter 3 this equation with its boundary condition has only two bounded solutions, and we will see that they correspond to the translation and dilatation of the basic function v_j respectively. With this information it is natural to believe that the solution of the homogeneous equation (2.17) will be somehow similar to these functions in a neighborhood of $\bar{\xi}_j$ with corrections in orders of ϵ, so if we remove this functions from the space of solutions, then we would expect that the remaining solution would be small as we will see in the following lemmas.

Definition 4.0.2. Let us call

$$\begin{aligned} z_{1j}(z) &= \frac{\partial}{\partial \zeta_1} v_j(z+\zeta)|_{\zeta=0} \\ z_{0j}(z) &= \frac{\partial}{\partial s}(v_j(sz) + 2\log s)|_{s=1} \end{aligned} \tag{4.2}$$

the elements of the kernel of L asociated to the natural invariances of the equation 3.1.

It will be very usefull in the following calculations to have a transformation of the domain to simplify the analysis.

Around each point $\bar{\xi}_j' \in \Gamma_1^\epsilon$, we consider a smooth change of variables

$$F_j^\epsilon(y) = \frac{1}{\epsilon} F_j(\epsilon y)$$

where $F_j : B(\bar{\xi}_j, \rho) \longrightarrow M$ with M an open neighborhood of the origin such that $F_j(\Omega \cap B(\bar{\xi}_j, \rho)) = \mathbb{R}^2_+ \cap M$, $F_j(\Gamma_1 \cap B(\bar{\xi}_j, \rho)) = \partial\mathbb{R}^2_+ \cap M$. We can select F_j so it preserves area.

Define also

$$Z_{ij} = z_{ij}(F_j^\epsilon(y)) \;\; i = 0, 1 \;\; j = 1, \ldots, m. \tag{4.3}$$

To keep integrals bounded in the following let us define a cut function.

We choose a large but fix number $R_0 > 0$ and a nonnegative function $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ so that $\chi(r) = 1$ for $r \leq R_0$ and $\chi(r) = 0$ for $r \geq R_0 + 1$ $(0 \leq \chi \leq 1)$. Then set

$$\chi_j(y) = \chi(|F_j^\epsilon(y)|)$$

The main result of this chapter is the solvability of the following linear problem: given $h_1,\ h_2$, find $\phi,\ c_1, \ldots, c_m$ such that

$$\begin{cases} L(\phi) = h_1 + \sum\limits_{j=1}^{m} c_j \chi_j Z_{1j} \quad \text{in } \Omega_\epsilon \\ \dfrac{\partial \phi}{\partial \nu} - W_2(y) \dfrac{\phi}{2} = h_2 \quad \text{on } \Gamma_1^\epsilon \\ \phi = 0 \quad \text{on } \Gamma_2^\epsilon \\ \int\limits_{\Omega_\epsilon} \chi_j Z_{1j} \phi = 0 \;\; \forall\, j = 1, \ldots, m. \end{cases} \tag{4.4}$$

4.1 First a priori estimate

The first step is to prove uniform a priori estimates for the problem 4.4 when ϕ satisfies additionally orthogonality with respect to Z_{0j}.

Lemma 4.1.1. *There are $R_0 > 0$ and $\epsilon_0 > 0$, so that for $0 < \epsilon < \epsilon_0$ and any solution ϕ of*

$$\begin{cases} L(\phi) = h_1 \quad \text{in } \Omega_\epsilon \\ \dfrac{\partial \phi}{\partial \nu} - W_2(y)\dfrac{\phi}{2} = h_2 \quad \text{on } \Gamma_1^\epsilon \\ \phi = -U(\epsilon y) \quad \text{on } \Gamma_2^\epsilon \end{cases} \tag{4.5}$$

satisfying

$$\int_{\Omega_\epsilon} \chi_j Z_{ij} \phi = 0 \ \forall\, i = 0,1,\ j = 1, \dots, m \tag{4.6}$$

we have

$$\|\phi\|_{L^\infty(\Omega_\epsilon)} \le C(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon})$$

where C is independent of ϵ, and

$$\begin{aligned} \|h_1\|_{*,\Omega_\epsilon} &= \sup_{y \in \Omega_\epsilon} \frac{|h_1(y)|}{\epsilon^2 + \sum_{j=1}^m (1 + |y - \xi_j'|)^{-(2+\sigma)}} \\ \|h_2\|_{*,\Gamma_1^\epsilon} &= \sup_{y \in \Gamma_1^\epsilon} \frac{|h_2(y)|}{\epsilon + \sum_{j=1}^m (1 + |y - \xi_j'|)^{-(1+\sigma)}} \end{aligned} \tag{4.7}$$

with $0 < \sigma < 1$ fixed, although the precise choice will be made later on.

In order to prove this Lemma some preliminary steps are required.
We need a maximum principle for L, and to prove this principle we will use the following technical lemma.

Lemma 4.1.2. *For $\epsilon > 0$ small enough there exists $R_1 > 0$, and $\psi : \Omega_\epsilon \setminus \bigcup_{j=1}^m B(\bar{\xi}_j', R_1) \longrightarrow \mathbb{R}$ smooth and positive so that*

$$\begin{aligned} L(\psi) &\le -\left(\sum_{j=1}^m \frac{1}{|y - \bar{\xi}_j'|^{2+\sigma}} + \epsilon^2 \right) \quad \text{in } \Omega_\epsilon \setminus \bigcup_{j=1}^m B(\bar{\xi}_j', R_1) \\ \frac{\partial \psi}{\partial \nu} - W_2 \frac{\psi}{2} &\ge \sum_{j=1}^m \frac{1}{|y - \bar{\xi}_j'|^{1+\sigma}} + \epsilon \quad \text{on } \Gamma_1^\epsilon \setminus \bigcup_{j=1}^m B(\bar{\xi}_j', R_1) \\ \psi &\ge 1 \quad \text{on } \Omega_\epsilon \cap \bigcup_{j=1}^m \partial B(\bar{\xi}_j', R_1) \\ \psi &\ge 1 \quad \text{on } \Gamma_2^\epsilon \setminus \bigcup_{j=1}^m B(\bar{\xi}_j', R_1) \\ \psi &> 0 \quad \text{in } \Omega_\epsilon \setminus \bigcup_{j=1}^m B(\bar{\xi}_j', R_1) \end{aligned}$$

The constants $R_1 > 0$, $C > 0$ can be chosen independently of ϵ, and ψ is bounded uniformly

$$0 < \psi \leq C \quad \textit{in } \Omega_\epsilon \setminus \bigcup_{j=1}^{m} B(\bar{\xi}_j', R_1)$$

Proof. We take

$$\psi_{1j}(y) = \frac{(y - \bar{\xi}_j') \cdot \nu(\bar{\xi}_j')}{r^{1+\sigma}} \tag{4.8}$$

with $r = |y - \xi_j'|$.

Let us calculate

$$\Delta\psi_{1j} = \frac{-2(y - \xi_j') \cdot \nu(\bar{\xi}_j')(1+\sigma)}{r^{3+\sigma}} - \frac{2(1+\sigma)(y - \bar{\xi}_j') \cdot \nu(\bar{\xi}_j)}{r^{3+\sigma}} + \frac{(1+\sigma)(3+\sigma)\left[(y - \bar{\xi}_j') \cdot \nu(\bar{\xi}_j')\right]}{r^{3+\sigma}}$$

and using the bounds 2.8

$$\Delta\psi_{1j} = O(r^{-(2+\sigma)})$$

Now let us calculate $\dfrac{\partial\psi_{1j}}{\partial\nu}$:
if $\delta > 0$ is small but fixed, and $R_1 > 0$ is large and fixed, then

$$\frac{\partial\psi_{1j}}{\partial\nu} \geq \frac{C}{r^{1+\sigma}} \quad \text{for } R_1 < r < \frac{\delta}{\epsilon}$$

To prove the last assetion we may suppose that $\bar{\xi}_j'$ is at the origin and assume that the normal vector at $\bar{\xi}_j'$ is $(0, -1)$.

Then

$$\psi_{1j}(y) = \frac{-y_2}{r^{1+\sigma}}$$

Let us write Γ_1^ϵ near $\bar{\xi}_j'$ as the graph $\{(y_1, y_2) : \; y_2 = G_\epsilon(y_1)\}$ with $G_\epsilon(y_1) = \frac{1}{\epsilon}G(\epsilon y_1)$ and G a smooth function such that $G(0) = 0$ and $G'(0) = 0$.
Fix $\delta > 0$ small. Then for $R_1 < r < \frac{\delta}{\epsilon}$ we have that r is comparable to y_1, $G_\epsilon'(y_1) = O(\epsilon r)$ and $G_\epsilon(y_1) = O(\epsilon r^2)$

Then

$$\frac{\partial\psi_{1j}}{\partial\nu} = \nabla\psi_{1j} \cdot \nu$$

with

$$\begin{aligned}
\nu &= \left(\frac{1}{\sqrt{\frac{1}{G'_\epsilon(y_1)^2}+1}}, \frac{-\frac{1}{G'_\epsilon(y_1)}}{\sqrt{\frac{1}{G'_\epsilon(y_1)^2}+1}} \right) \\
&= \left(\frac{G'_\epsilon(y_1)}{\sqrt{G'_\epsilon(y_1)^2+1}}, -\frac{1}{\sqrt{G'_\epsilon(y_1)^2+1}} \right)
\end{aligned}$$

and

$$\nabla\psi_{1j} = \left(\frac{y_2(1+\sigma)y_1}{r^{3+\sigma}}, \frac{-r^2+y_2(1+\sigma)(y_2+\sqrt{2}\mu_j\epsilon/2)}{r^{3+\sigma}} \right)$$

So,

$$\begin{aligned}
\frac{\partial\psi_{1j}}{\partial\nu} &= \frac{1}{\sqrt{G'_\epsilon(y_1)^2+1}} \left(\frac{G'_\epsilon(y_1)G_\epsilon(y_1)y_1(1+\sigma)}{r^{3+\sigma}} + \frac{1}{r^{1+\sigma}} \right. \\
&\quad \left. - \frac{(1+\sigma)G_\epsilon(y_1)(G_\epsilon(y_1)+\sqrt{2}\mu_j\epsilon/2)}{r^{3+\sigma}} \right)
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\partial\psi_{1j}}{\partial\nu} &= \frac{1}{\sqrt{G'_\epsilon(y_1)^2+1}} \left(\frac{1}{r^{1+\sigma}} + \frac{O(\epsilon^2 r^4)}{r^{3+\sigma}} \right) \\
&= \frac{1}{\sqrt{O(\delta^2)+1}} \left(\frac{1}{r^{1+\sigma}} + \frac{O(\delta^2)}{r^{1+\sigma}} \right) \quad \text{for } R_1 < r < \frac{\delta}{\epsilon}
\end{aligned}$$

and if δ is sufficiently small

$$\frac{\partial\psi_{1j}}{\partial\nu} \geq \frac{C}{r^{1+\sigma}} \quad \text{for } R_1 < r < \frac{\delta}{\epsilon}$$

which proves the assertion.

Consider also

$$\psi_{2j}(r) = 1 - \frac{1}{r^\sigma}$$

So,

$$\begin{aligned}\nabla\psi_{2j} &= (\frac{\sigma r^{\sigma-1}y_1}{r^{2\sigma+1}}, \frac{\sigma r^{\sigma-1}(y_2+\sqrt{2}\mu_j\epsilon/2)}{r^{2\sigma+1}}) \\ &= \frac{\sigma}{r^{\sigma+2}}(y_1, y_2+\sqrt{2}\mu_j\epsilon/2) \\ \text{and} \\ \nu &= \frac{1}{\sqrt{G_\epsilon'(y_1)^2+1}}(G_\epsilon'(y_1),-1) \\ \Rightarrow \frac{\partial\psi_{2j}}{\partial\nu} &= \frac{\sigma}{r^{2+\sigma}\sqrt{G_\epsilon'(y_1)^2+1}}(y_1\underbrace{G_\epsilon'(y_1)}_{O(\epsilon r)}-y_2-\sqrt{2}\mu_j\epsilon/2) \\ &= \frac{O(\epsilon r^2)}{r^{2+\sigma}} - \frac{O(\epsilon)}{r^{2+\sigma}} \\ &= \frac{O(\epsilon)}{r^\sigma} \quad \forall\, R_1 < r < \frac{\delta}{\epsilon}\end{aligned}$$

In this region

$$W(y) = e^{V(y)} = O(\frac{1}{r^4}) + O(\frac{\epsilon}{r^3})$$

then

$$\begin{aligned}\Rightarrow L(\psi_{1j}) &= \Delta\psi_{1j} + W\psi_{1j} \\ &= O(\frac{1}{r^{2+\sigma}}) + O(\frac{y_2}{r^4 r^{1+\sigma}}) + O(\frac{\epsilon y_2}{r^3 r^{1+\sigma}}) \\ &= O(\frac{1}{r^{2+\sigma}})\end{aligned}$$

Now

$$\begin{aligned}L_2(\psi_{1j}) &:= \frac{\partial\psi_{1j}}{\partial\nu} - W_2\psi_{1j} \\ &\geq \frac{C}{r^{1+\sigma}} + O(\frac{1}{r^2})\frac{y_2}{r^{1+\sigma}} \\ &\geq O(\frac{1}{r^{1+\sigma}})\end{aligned}$$

$$\begin{aligned}L(\psi_{2j}) &= -\frac{\sigma^2}{r^{2+\sigma}} + W(1-\frac{1}{r^\sigma}) \\ &= -\frac{\sigma^2}{r^{2+\sigma}} + O(\frac{1}{r^4}) + O(\frac{\epsilon}{r^3}) - O(\frac{1}{r^{+\sigma}}) - O(\frac{\epsilon}{r^{3+\sigma}}) \\ &= \frac{-\sigma^2}{r^{2+\sigma}} + \frac{\alpha}{r^4} + \frac{\beta\epsilon}{r^3}\end{aligned}$$

with α and β some positive constants.

Also

$$\begin{aligned} L_2(\psi_{2j}) &= O(\frac{\epsilon}{r^\sigma}) - O(\frac{1}{r^2})(1-\frac{1}{r^\sigma}) \\ &= O(\frac{\epsilon}{r^\sigma}) - O(\frac{1}{r^2}) + O(\frac{1}{r^{2+\sigma}}) \end{aligned}$$

Now let $\psi_{3j} = \psi_{1j} + C\psi_{2j}$,

$$\begin{aligned} \Rightarrow L(\psi_{3j}) &= L(\psi_{1j}) + CL(\psi_{2j}) \\ &= O(\frac{1}{r^{2+\sigma}}) + C\left(\frac{-\sigma^2}{r^{2+\sigma}} + \frac{\alpha}{r^4} + \frac{\beta\epsilon}{r^3}\right) \end{aligned}$$

Then taking C sufficiently large but independent of ϵ we have that

$$L(\psi_{3j}) \leq -\frac{\sigma^2}{r^{2+\sigma}}\tilde{C} \quad \forall\, R_1 < r < \frac{\delta}{\epsilon},\ \tilde{C} > 0$$

Thus

$$\begin{aligned} \frac{\partial \psi_{3j}}{\partial \nu} - W_2\psi_{3j} &= L_2(\psi_{1j}) + CL_2(\psi_{2j}) \\ &\geq O(\frac{1}{r^{1+\sigma}}) - C\left(O(\frac{\epsilon}{r^\sigma}) - O(\frac{1}{r^2}) + O(\frac{1}{r^{2+\sigma}})\right) \\ &\geq O(\frac{1}{r^{1+\sigma}}) - CO(\frac{1}{r^2}) \\ &\geq \frac{C'}{r^{1+\sigma}} \end{aligned}$$

we can choose R_1 larger if necessary.

Let $\eta_j \ \in C_0^\infty(\bar{\Omega}_\epsilon)$ such that $0 \leq \eta_j \leq 1,\ \eta_j \equiv 1$ in $\Omega_\epsilon \cap B(\bar{\xi}'_j, \delta/2\epsilon),\ \eta_j \equiv 0$ in $\Omega_\epsilon \setminus B(\bar{\xi}'_j, \delta/\epsilon),\ |\nabla\eta_j| \leq C\epsilon$ in $\Omega_\epsilon,\ |\Delta\eta_j| \leq C\epsilon^2$ in Ω_ϵ.

Let $\psi_0(y) = \tilde{\psi}(\epsilon y)$, where $\tilde{\psi}$ is solution to

$$\begin{cases} \Delta\tilde{\psi} \ = \ -1 \quad \text{in } \Omega \\ \dfrac{\partial\tilde{\psi}}{\partial\nu} \ = \ 1 \quad \text{on } \Gamma_1 \\ \tilde{\psi} \ = \ 1 \quad \text{on } \Gamma_2 \end{cases}$$

so that $\Delta\psi_0 = -\epsilon^2$ in Ω_ϵ, $\frac{\partial\psi_0}{\partial\nu} = \epsilon$ on Γ_1^ϵ, and $\psi_0 = 1$ on Γ_2^ϵ. In particular ψ_0 is uniformly bounded in Ω_ϵ.

The function $\psi = \sum_{j=1}^{m} \eta_j \psi_{3j} + C\psi_0$, with C a sufficiently large constant, meets the requirements of the lemma.

Indeed, for $R_1 < r_j = |y - \xi_j'| < \frac{\delta}{\epsilon}$

$$\begin{aligned}\Delta\psi + W\psi &= \sum_{j=1}^{m}(\Delta\eta_j\psi_{3j} + 2\nabla\eta_j\nabla\psi_{3j} + \eta_j\Delta\psi_{3j}) + W\eta_j\psi_{3j} + C(\Delta\psi_0 + W\psi_0) \\ &\leq \Delta\eta_j\psi_{3j} + 2\nabla\eta_j\nabla\psi_{3j} + \eta_j(\frac{-\tilde{C}\sigma^2}{r_j^{2+\sigma}}) - C\epsilon^2 + \frac{C}{r_j^4}\psi_0\end{aligned}$$

Then, choosing $\tilde{C}$ larger if necessary, we have for $R_1 < r_j < \frac{\delta}{2\epsilon}$

$$\Rightarrow\ L(\psi) \leq \frac{-\tilde{C}\sigma^2}{r_j^{2+\sigma}} - C\epsilon^2 \tag{4.9}$$

Now in $\frac{\delta}{2\epsilon} < r_j < \frac{\delta}{\epsilon}$, we use that $|\nabla\psi_{3j}| = O(\frac{1}{r_j^{1+\sigma}})$, so in this region

$$L(\psi) \leq \tilde{C}\epsilon^2\psi_{3j} + \tilde{C}\epsilon\nabla\psi_{3j} + \eta_j\frac{\tilde{C}\sigma^2}{r_j^{2+\sigma}} - C\epsilon^2$$

if C is big enough

$$\leq -C\epsilon^2 + O(\frac{\epsilon}{r_j^{1+\sigma}}) - O(\frac{1}{r_j^{2+\sigma}})$$

but in this region $\frac{2\epsilon}{\delta} > \frac{1}{r_j} > \frac{\epsilon}{\delta}$, and so, for ϵ small enough

$$C\epsilon^2 > (\frac{2\epsilon}{\delta})^{2+\sigma} > \frac{1}{r_j^{2+\sigma}} > (\frac{\epsilon}{\delta})^{2+\sigma}$$

$$\Rightarrow\ L(\psi)\ \leq\ -\frac{C}{r_j^{2+\sigma}}\ \ R_1 < r_j < \frac{\delta}{\epsilon}$$

And in an analogous way,

$$\frac{\partial\psi}{\partial\nu} - W_2\psi\ \geq\ \frac{C}{r_j^{1+\sigma}} + C\epsilon\quad R_1 < r_j < \frac{\delta}{\epsilon}$$

If now we consider that $\forall j = 1, \ldots, m,\ r_j > \frac{\delta}{\epsilon}$, then

$$\begin{aligned}
\psi &= C\psi_0 \\
\Rightarrow L(\psi) &= -C\epsilon^2 + W\psi_0 \ = \ -C\epsilon^2 + O(\frac{1}{r_j^4})\psi_0 \\
&\leq -C\epsilon^2 \\
L_2(\psi) &= C\epsilon + CW_2\psi_0 \ = \ C\epsilon + O(\frac{1}{r_j^2})\psi_0 \\
&\geq C\epsilon \quad \text{on } \Gamma_1^\epsilon, r_j > \frac{\delta}{\epsilon}\forall j \\
\psi &= C\psi_0 \ = \ C \ \geq \ 1 \quad \text{on } \Gamma_2^\epsilon
\end{aligned}$$

proving the existence of the function given by the Lemma. □

With this barrier the following maximum principle holds:
Let $R_1 > 0$, if $\phi \in H^1(\Omega_\epsilon \setminus \bigcup_{j=1}^m B(\bar{\xi}_j', R_1))$ satisfies

$$\begin{aligned}
L(\phi) &\leq 0 \ \text{ in } \Omega_\epsilon \setminus \bigcup_{j=1}^m B(\bar{\xi}_j', R_1) \\
\frac{\partial \phi}{\partial \nu} - W_2(y)\frac{\phi}{2} &\geq 0 \ \text{ on } \Gamma_1^\epsilon \setminus \bigcup_{j=1}^m B(\bar{\xi}_j', R_1) \\
\phi &\geq 0 \ \text{ on } \Gamma_2^\epsilon \setminus \bigcup_{j=1}^m B(\bar{\xi}_j', R_1) \\
\phi &\geq 0 \ \text{ in } \Omega_\epsilon \bigcap (\bigcup_{j=1}^m \partial B(\bar{\xi}_j', R_1))
\end{aligned}$$

Then $\phi \geq 0$ in $\Omega_\epsilon \setminus \bigcup_{j=1}^m B(\bar{\xi}_j', R_1)$.

4.1.1 Proof of the first a priori estimate

Proof. Let $h_1,\ h_2$ be bounded, and ϕ solution to 4.5. We first claim that $\|\phi\|_{L^\infty(\Omega_\epsilon)}$ can be controlled in terms of $\|h_1\|_{*,\Omega_\epsilon},\ \|h_2\|_{*,\Gamma_1^\epsilon}$ and the following inner norm of ϕ:

$$\|\phi\|_i = \sup_{\Omega_\epsilon \cap (\bigcup_{j=1}^m B(\bar{\xi}_j', R_1))} |\phi|$$

In deed, if we set $\tilde{\phi} = C_1\psi(\|\phi\|_i + \|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon})$, with ψ the barrier function mentioned in the previous technical lemma 4.1.2, and then taking $\tilde{\phi} - \phi$, we have that in

$\Omega_\epsilon \setminus \bigcup_{j=1}^{m} B(\bar{\xi}'_j, R_1)$

$$\begin{aligned} L(\tilde{\phi}-\phi) &= C_1 L(\psi)(\|\phi\|_1 + \|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) - \underbrace{L(\phi)}_{=h_1} \\ &\leq C_1 L(\psi)\|\phi\|_i + C_1 L(\psi)\|h_2\|_{*,\Gamma_1^\epsilon} - C_1\left(\sum_{j=1}^{m}\frac{1}{|y-\bar{\xi}'_j|^{2+\sigma}} + \epsilon^2\right)\|h_1\|_{*,\Omega_\epsilon} - h_1 \\ &= \underbrace{C_1 L(\psi)(\|\phi\|_i + \|h_2\|_{*,\Gamma_1^\epsilon})}_{\leq 0} \\ &\quad -C_1\left(\sum_{j=1}^{m}\frac{1}{|y-\bar{\xi}'_j|^{2+\sigma}} + \epsilon^2\right)\underbrace{\left(\|h_1\|_{*,\Omega_\epsilon} + \frac{h_1}{\epsilon^2 + \sum_{j=1}^{m}|y-\bar{\xi}'_j|^{-(2+\sigma)}}\right)}_{\geq 0} \\ \Rightarrow L(\tilde{\phi}-\phi) &\leq 0 \end{aligned}$$

Now $L_2(\tilde{\phi}-\phi) = \dfrac{\partial\tilde{\phi}-\phi}{\partial\nu} - W_2(\tilde{\phi}-\phi)$, and we have that in $\Gamma_1^\epsilon \setminus \bigcup_{j=1}^{m} B(\bar{\xi}'_j, R_1)$:

$$L_2(\tilde{\phi}-\phi) = C_1 L_2(\psi)(\|\phi\|_i + \|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) - h_2$$

Then analogously

$$L_2(\tilde{\phi}-\phi) \geq 0$$

Now in $\Omega_\epsilon \cap (\bigcup_{j=1}^{m} \partial B(\bar{\xi}'_j, R_1))$, we use that the balls are closed and so ϕ is bounded by $\|\phi\|_i$ and so

$$\begin{aligned} \tilde{\phi}-\phi &= C_1 \underbrace{\psi}_{\geq 1}(\|\phi\|_i + \|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) - \phi \\ \Rightarrow \tilde{\phi}-\phi &\geq 0 \end{aligned}$$

Then by the maximum principle we have that $\tilde{\phi}-\phi \geq 0$, and analogously, $\tilde{\phi}+\phi \geq 0$ in $\Omega_\epsilon \setminus \bigcup_{j=1}^{m} B(\bar{\xi}'_j, R_1)$, so

$$|\phi| \leq \tilde{\phi} = C_1\psi(\|\phi\|_i + \|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) \tag{4.10}$$
$$\Rightarrow \|\phi\|_{L^\infty(\Omega_\epsilon)} \leq C(\|\phi\|_i + \|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) \tag{4.11}$$

because ψ is bounded uniformly.

Now let us resume the proof of the a priori bound.

By contradiction, let us assume the existence of a secuence $\epsilon_n \longrightarrow 0$, points $\bar{\xi}_j \in \Gamma_1^\epsilon$ which satisfy (falta la descripcion de los puntos), functions $h_{1n},\ h_{2n}$ with $\|h_{1n}\|_{*,\Omega_\epsilon} \longrightarrow 0$ and $\|h_{2n}\|_{*,\Gamma_1^\epsilon} \longrightarrow 0$, and ϕ_n with $\|\phi_n\|_\infty = 1$ such that

$$\begin{cases} L(\phi_n) \ = \ h_{1n} \ \text{ in } \Omega_\epsilon \\ \dfrac{\partial \phi_n}{\partial \nu} - W_2 \dfrac{\phi_n}{2} \ = \ h_{2n} \ \text{ on } \Gamma_1^\epsilon \\ \phi_n \ = \ 0 \ \text{ on } \Gamma_2^\epsilon. \end{cases}$$

By 4.10

$$\begin{aligned} \|\phi_n\|_\infty &\leq C(\|\phi_n\|_i + \|h_{1n}\|_{*,\Omega_\epsilon} + \|h_{2n}\|_{*,\Gamma_1^\epsilon}) \\ 1 &\leq C\|\phi_n\|_i + C\underbrace{(\|h_{1n}\|_{*,\Omega_\epsilon} + \|h_{2n}\|_{*,\Gamma_1^\epsilon})}_{\to 0} \end{aligned}$$

Then we see that $\|\phi_n\|_i$ stays away from zero.

For one of the indices, say j, we can assume that $\sup\limits_{B(\bar{\xi}_j',R_1)} |\phi_n| \geq C > 0$ for all n, because, since there is a finite number of $j = 1,\ldots,m$, then we could consider the subsequence that concentrate in one j.

Consider $\hat{\phi}_n = \phi_n(z - \bar{\xi}_j')$ and let us translate and rotate Ω_{ϵ_n} so that it approaches the upper half plane $\mathbb{R}^2_+$ and $\xi_j' = 0$.

Using elliptic estimates, $\hat{\phi}_n$ converges uniformly on compact sets to a non trivial solution of 3.3. By the non degeneracy of this problem, the limit $\hat{\phi}$ is a linear combination of z_{0j} and z_{1j}. But taking to the limit the orthogonality relations, and observing that limits of the functions Z_{ij} are just rotations and translations of z_{ij},

$$\int\limits_{\mathbb{R}^2_+} \chi \hat{\phi} z_{ij} = 0 \ \ i = 0,1$$

$$\Rightarrow \ \hat{\phi} = 0$$

which is a contradiction with the fact that $\hat{\phi} \not\equiv 0$, and so the proof is completed. □

4.2 Second a priori estimate

We will establish next an a priori estimate for solutions to problem 4.5 that satisfy orthogonality conditions with respect to Z_{1j} only.

Lemma 4.2.1. *For ϵ sufficiently small, if ϕ solves*

$$\begin{cases} \Delta\phi + W\phi \;=\; h_1 \;\; \text{in } \Omega_\epsilon \\ \dfrac{\partial\phi}{\partial\nu} - W_2\dfrac{\phi}{2} \;=\; h_2 \;\; \text{on } \Gamma_1^\epsilon \\ \phi \;=\; 0 \;\; \text{on } \Gamma_2^\epsilon \end{cases} \tag{4.12}$$

and satisfies

$$\int\limits_{\Omega_\epsilon} Z_{1j}\chi_j\phi = 0 \;\; \forall\, j = 1,\ldots,m. \tag{4.13}$$

Then

$$\|\phi\|_\infty \le C\log\frac{1}{\epsilon}(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) \tag{4.14}$$

where C is independent of ϵ.

Proof. First we will modify ϕ to satisfy the orthogonality relations 4.6, and for this purpose we consider modifications with compact support of the function Z_{0j}.

Let $R > R_0 + 1$ be large and fixed. Set

$$\widehat{Z_{0j}}(y) = \psi Z_{0j} \tag{4.15}$$

where $\psi(y) = h(|F_j^\epsilon(y)|)$, and $h(x) = \dfrac{\log\delta/\epsilon - \log|x|}{\log\delta/\epsilon - \log R}$.

Note that h is just the solution to

$$\begin{cases} \Delta h \;=\; 0 \;\; \text{in } B(0,\delta/\epsilon)\setminus\bar{B}(0,R) \\ h \;=\; 1 \;\; \text{on } |x| = R \\ h \;=\; 0 \;\; \text{on } |x| = \delta/\epsilon \end{cases}$$

Let $\bar{\eta}_{1j},\ \bar{\eta}_{2j}$ be radial smooth cut-off funtions on $\mathbb{R}^2$ so that $0 \le \bar{\eta}_{1j} \le 1,\ |\nabla\bar{\eta}_{1j}| \le C$ in $\mathbb{R}^2$, $\bar{\eta}_{1j} \equiv 1$ in $B(0,R),\ \bar{\eta}_{1j} \equiv 0$ in $\mathbb{R}^2\setminus B(0,R+1)$, and $\bar{\eta}_{2j} \equiv 1$ in $B(0,\delta/4\epsilon),\ \bar{\eta}_{2j} \equiv 0$ in $\mathbb{R}^2\setminus B(0,\delta/3\epsilon)$, $0 \le \bar{\eta}_{2j} \le 1,\ |\nabla\bar{\eta}_{2j}| \le C\frac{\epsilon}{\delta},\ |\nabla^2\bar{\eta}_{2j}| \le C\frac{\epsilon^2}{\delta^2}$

Write $\eta_{1j} = \bar{\eta}_{1j}(F_j^\epsilon(y)),\ \eta_{2j}(y) = \bar{\eta}_{2j}(F_j^\epsilon(y))$, and now define

$$\widetilde{Z}_{0j} = \eta_{1j}Z_{0j} + (1-\eta_{1j})\eta_{2j}\widehat{Z_{0j}} \tag{4.16}$$

And let

$$\tilde{\phi} = \phi + \sum_{j=1}^{m} d_j\widetilde{Z}_{0j} \tag{4.17}$$

we choose d_j so that $\tilde{\phi}$ satisfies the orthogonality conditions, i.e:

$$\begin{aligned}
&\int\limits_{\Omega_\epsilon} \chi_j Z_{0j} \tilde{\phi} = 0 \\
\Leftrightarrow \quad &\int\limits_{\Omega_\epsilon} \chi_j Z_{0j} \phi + \sum_{k=1}^{m} d_j \int\limits_{\Omega_\epsilon} \chi_j Z_{0j} \widetilde{Z}_{0k} = 0 \\
\Leftrightarrow \quad &\int\limits_{\Omega_\epsilon} \chi_j Z_{0j} \phi + d_j \int\limits_{\Omega_\epsilon} \chi_j |Z_{0j}|^2 = 0
\end{aligned}$$

then

$$d_j = \frac{-\int\limits_{\Omega_\epsilon} \chi_j Z_{0j} \phi}{\int\limits_{\Omega_\epsilon} \chi_j |Z_{0j}|^2} \tag{4.18}$$

Estimate 4.14 is a direct consequence of:

Claim 4.2.2.

$$|d_j| \leq C \log \frac{1}{\epsilon} \left(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon} \right) \tag{4.19}$$

We start proving this by observing that

$$L(\tilde{\phi}) = h_1 + \sum_{j=0}^{m} d_j L(\widetilde{Z}_{0j}) \quad \text{in } \Omega_\epsilon \tag{4.20}$$

and

$$L_2(\tilde{\phi}) = h_2 + \sum_{j=0}^{m} d_j L_2(\widetilde{Z}_{0j}) \quad \text{on } \Gamma_1^\epsilon \tag{4.21}$$

Then by the previous lemma

$$\begin{aligned} \|\tilde{\phi}\|_\infty &\leq C(\|h_1 + \sum_{j=1}^{m} d_j L(\widetilde{Z}_{0j})\|_{*,\Omega_\epsilon} + \|h_2 + \sum_{j=1}^{m} d_j L_2(\widetilde{Z}_{0j})\|_{*,\Gamma_1^\epsilon}) \\ &\leq C(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) + \\ &\quad +C\sum_{j=1}^{m} |d_j|(\|L(\widetilde{Z}_{0j})\|_{*,\Omega_\epsilon} + \|L_2(\widetilde{Z}_{0j})\|_{*,\Gamma_1^\epsilon}) \end{aligned} \tag{4.22}$$

Multiplying equation 4.20 by $\widetilde{Z}_{0k}$ and integrating, we have:

$$\begin{aligned} \int_{\Omega_\epsilon} L(\tilde{\phi})\widetilde{Z}_{0k} &= \int_{\Omega_\epsilon} h_1\widetilde{Z}_{0k} + \sum_{j=1}^{m} d_j \int_{\Omega_\epsilon} L(\widetilde{Z}_{0j})\widetilde{Z}_{0k} \\ \int_{\Omega_\epsilon} \Delta\tilde{\phi}\widetilde{Z}_{0k} + \int_{\Omega_\epsilon} W\tilde{\phi}\widetilde{Z}_{0k} &= \int_{\Omega_\epsilon} h_1\widetilde{Z}_{0k} + \sum_{j=1}^{m} d_j \int_{\Omega_\epsilon} (\Delta\widetilde{Z}_{0j} + W\widetilde{Z}_{0j})\widetilde{Z}_{0k} \end{aligned}$$

Using the Green theorem in the left side of the equation and noting that the integrands are cero in Γ_2^ϵ

$$\begin{aligned} \text{Left side} &= \int_{\Omega_\epsilon} \tilde{\phi}\Delta\widetilde{Z}_{0k} + \int_{\Gamma_1^\epsilon} (\widetilde{Z}_{0k}\frac{\partial\tilde{\phi}}{\partial\nu} - \frac{\partial\widetilde{Z}_{0k}}{\partial\nu}\tilde{\phi}) + \int_{\Omega_\epsilon} W\tilde{\phi}\widetilde{Z}_{0k} \\ &= \int_{\Omega_\epsilon} \tilde{\phi}L(\widetilde{Z}_{0k}) + \int_{\Gamma_1^\epsilon} \widetilde{Z}_{0k}L_2(\tilde{\phi}) - \int_{\Gamma_1^\epsilon} \tilde{\phi}L_2(\widetilde{Z}_{0k}) \\ &= \int_{\Omega_\epsilon} \tilde{\phi}L(\widetilde{Z}_{0k}) - \int_{\Gamma_1^\epsilon} \tilde{\phi}L_2(\widetilde{Z}_{0k}) + \int_{\Gamma_1^\epsilon} \widetilde{Z}_{0k}h_2 + \sum_{j=1}^{m}\int_{\Gamma_1^\epsilon} \widetilde{Z}_{0k}L_2(\widetilde{Z}_{0j}) \end{aligned}$$

Replacing and putting together all terms with d_j we have

$$\sum_{j=1}^{m} d_j \left[\int_{\Gamma_1^\epsilon} \widetilde{Z}_{0k}L_2(\widetilde{Z}_{0j}) - \int_{\Omega_\epsilon} \widetilde{Z}_{0k}L(\widetilde{Z}_{0j})\right] = \int_{\Omega_\epsilon} h_1\widetilde{Z}_{0k} - \int_{\Gamma_1^\epsilon} \widetilde{Z}_{0k}h_2 + \int_{\Gamma_1^\epsilon} \tilde{\phi}L_2(\widetilde{Z}_{0k}) - \int_{\Omega_\epsilon} \tilde{\phi}L(\widetilde{Z}_{0k}) \tag{4.23}$$

It only survives the $j = k$ term because the supports are disjoints

$$d_k \left[\int_{\Gamma_1^\epsilon} \widetilde{Z}_{0k} L_2(\widetilde{Z}_{0k}) - \int_{\Omega_\epsilon} \widetilde{Z}_{0k} L(\widetilde{Z}_{0k}) \right] = \int_{\Omega_\epsilon} h_1 \widetilde{Z}_{0k} - \int_{\Gamma_1^\epsilon} \widetilde{Z}_{0k} h_2 + \int_{\Gamma_1^\epsilon} \tilde{\phi} L_2(\widetilde{Z}_{0k}) - \int_{\Omega_\epsilon} \tilde{\phi} L(\widetilde{Z}_{0k}) \tag{4.24}$$

Now,

$$\begin{aligned} \int_{\Omega_\epsilon} h_1 \widetilde{Z}_{0k} &\leq \|h_1\|_{*,\Omega_\epsilon} \int_{\Omega_\epsilon} |\widetilde{Z}_{0k}| (\epsilon^2 + \sum_{j=1}^{m} (1 + |y - \xi_j'|)^{-(2+\sigma)}) \\ &\leq C \|h_1\|_{*,\Omega_\epsilon} \end{aligned}$$

and analogously,

$$\int_{\Gamma_1^\epsilon} h_2 \widetilde{Z}_{0k} \leq C \|h_2\|_{*,\Gamma_1^\epsilon}$$

Then,

$$\begin{aligned} |d_k| \left[\int_{\Gamma_1^\epsilon} \widetilde{Z}_{0k} L_2(\widetilde{Z}_{0k}) - \int_{\Omega_\epsilon} \widetilde{Z}_{0k} L(\widetilde{Z}_{0k}) \right] &\leq C \|h_1\|_{*,\Omega_\epsilon} + C \|h_2\|_{*,\Gamma_1^\epsilon} \\ &\quad + C \|\tilde{\phi}\|_\infty \|L_2(\widetilde{Z}_{0k})\|_{*,\Gamma_1^\epsilon} + C \|\tilde{\phi}\|_\infty \|L(\widetilde{Z}_{0k})\|_{*,\Omega_\epsilon} \end{aligned}$$

Now we use 4.22

$$\begin{aligned} |d_k| \left[\int_{\Gamma_1^\epsilon} \widetilde{Z}_{0k} L_2(\widetilde{Z}_{0k}) - \int_{\Omega_\epsilon} \widetilde{Z}_{0k} L(\widetilde{Z}_{0k}) \right] &\leq C(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) + C(\|L_2(\widetilde{Z}_{0k})\|_{*,\Gamma_1^\epsilon} + \\ &\quad + \|L(\widetilde{Z}_{0k})\|_{*,\Omega_\epsilon}) \cdot (\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon} + \\ &\quad + \sum_{j=1}^{m} |d_j| (\|L_2(\widetilde{Z}_{0k})\|_{*,\Gamma_1^\epsilon} + \|L(\widetilde{Z}_{0k})\|_{*,\Omega_\epsilon})) \\ &\leq C(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) \Big(1 + \|L_2(\widetilde{Z}_{0k})\|_{*,\Gamma_1^\epsilon} + \|L(\widetilde{Z}_{0k})\|_{*,\Omega_\epsilon} \Big) \\ &\quad + C \sum_{j=1}^{m} |d_j| (\|L_2(\widetilde{Z}_{0k})\|^2_{*,\Gamma_1^\epsilon} + \|L(\widetilde{Z}_{0k})\|^2_{*,\Omega_\epsilon}) \end{aligned}$$

To achieve the claim, we must prove the following estimates:

For some constant $C > 0$ independent of ϵ we have:

$$\int_{\Gamma_1^\epsilon} \widetilde{Z}_{0j} L_2(\widetilde{Z}_{0j}) - \int_{\Omega_\epsilon} \widetilde{Z}_{0j} L(\widetilde{Z}_{0j}) \geq \frac{1}{C \log \frac{1}{\epsilon}} \tag{4.25}$$

$$\|L(\widetilde{Z}_{0j})\|_{*,\Omega_\epsilon} \leq \frac{C}{\log \frac{1}{\epsilon}} \tag{4.26}$$

$$\|L_2(\widetilde{Z}_{0j})\|_{*,\Gamma_1^\epsilon} \leq \frac{C}{\log \frac{1}{\epsilon}} \tag{4.27}$$

4.2.1 Proof of 4.25

We write

$$\int_{\Omega_\epsilon} L(\widetilde{Z}_{0j})\widetilde{Z}_{0j} = I_0 + I_1 + I_2 + I_3$$

where

$$I_l = \int_{R_l} L(\widetilde{Z}_{0j})\widetilde{Z}_{0j}$$

and,

$$R_0 = (F_j^\epsilon)^{-1}(\{r < R\} \cap \mathbb{R}^2_+) \tag{4.28}$$

$$R_1 = (F_j^\epsilon)^{-1}(\{R < r < R+1\} \cap \mathbb{R}^2_+) \tag{4.29}$$

$$R_2 = (F_j^\epsilon)^{-1}(\{R+1 < r < \frac{\delta}{4\epsilon}\} \cap \mathbb{R}^2_+) \tag{4.30}$$

$$R_3 = (F_j^\epsilon)^{-1}(\{\frac{\delta}{4\epsilon} < r < \frac{\delta}{3\epsilon}\} \cap \mathbb{R}^2_+) \tag{4.31}$$

Estimate of I_0

$$I_0 = \int_{R_0} L(\widetilde{Z}_{0j})\widetilde{Z}_{0j}(y)dy$$

We change variables $x = F_j^\epsilon(y)$ and recall that this map preserves area, so

$$I_0 = \int_{\{r<R\}\cap\mathbb{R}^2_+} \tilde{L}(\tilde{z}_{0j})\tilde{z}_{0j}(y)dy$$

with $\widetilde{z}_{0j}(x) = \widetilde{Z}_{0j}((F_j^\epsilon)^{-1}(x)) = \bar{\eta}_{1j} z_{0j} + (1-\bar{\eta}_{1j})\bar{\eta}_{2j} h(|x|) z_{0j}$, and

$$\tilde{L} \;=\; \Delta_x + O(\epsilon|x|)\nabla^2 + O(\epsilon)\nabla + W((F_j^\epsilon)^{-1}(x)) \tag{4.32}$$

$$\begin{aligned}
\Rightarrow\; \tilde{L}(\widetilde{Z}_{0j}) &= \Delta(\bar{\eta}_{1j} z_{0j} + (1-\bar{\eta}_{1j})\bar{\eta}_{2j} h(|x|) z_{0j}) + O(\epsilon) + W\widetilde{z}_{0j} \\
&= \Delta\bar{\eta}_{1j} z_{0j} + \Delta(1-\bar{\eta}_{1j})\bar{\eta}_{2j} h(|x|) z_{0j} + \nabla\bar{\eta}_{1j}\nabla z_{0j} - \nabla\bar{\eta}_{1j}\bar{\eta}_{2j} h(|x|) z_{0j} \\
&\quad + \bar{\eta}_{1j}\Delta z_{0j} + (1-\bar{\eta}_{1j})\Delta(\bar{\eta}_{2j} h(|x|) z_{0j}) + O(\epsilon) + W z_{0j} \\
\text{in} \quad & \{r < R\} \cap \mathbb{R}^2_+ \\
&= \Delta z_{0j} + W z_{0j} + O(\epsilon) \\
&= \Delta z_{0j} + \frac{8\mu_j^2}{(\mu_j^2 + |x-\xi_j'|^2)^2} z_{0j} + O(\epsilon) + O(\frac{\epsilon}{r^3}) + \\
&\quad \sum_{k\neq j}^{m} \frac{8\mu_k^2}{(\mu_k^2 + |x-\xi_k'|^2)^2}(1+\Theta_\epsilon(x))
\end{aligned}$$

with $|\Theta_\epsilon(x)| \leq C\epsilon \sum_{j=1}^{m}(|x-\xi_j'|+1)$

$$\Rightarrow\; \tilde{L}(\widetilde{Z}_{0j}) \;=\; O(\epsilon) + O(\frac{\epsilon}{r^3})$$

$$\Rightarrow\; I_0 \;=\; \int_{R_0} L(\widetilde{Z}_{0j})\widetilde{Z}_{0j} \;=\; \int_{\{r<R\}\cap\mathbb{R}^2_+} \tilde{L}(\widetilde{Z}_{0j})\widetilde{Z}_{0j} \;=\; O(R\epsilon) + O(\frac{\epsilon}{R})$$

$$I_0 \;=\; O(R\epsilon) \tag{4.33}$$

Estimate of I_3

$$I_3 = \int_{R_3} L(\widetilde{Z}_{0j})\widetilde{Z}_{0j}$$

Again we change variables $x = F_j^\epsilon(y)$

$$I_3 = \int_{\{\frac{\delta}{4\epsilon}<r<\frac{\delta}{3\epsilon}\}\cap\mathbb{R}^2_+} \tilde{L}(\widetilde{z}_{0j})\widetilde{z}_{0j}$$

In this region $\widetilde{z}_{0j} = \bar{\eta}_{2j} h(|x|) z_{0j}$.

We know that $|\nabla\bar{\eta}_{2j}| \leq C\frac{\epsilon}{\delta}$ and $|\nabla^2\bar{\eta}_{2j}| \leq C\frac{\epsilon^2}{\delta^2}$.

Also $h(x) = \dfrac{\log\delta/\epsilon - \log|x|}{\log\delta/\epsilon - \log R}$, so with $|x| \geq \frac{\delta}{4\epsilon}$

$$\begin{aligned} h(x) &\leq \frac{\cancel{\log\delta/\epsilon} - \cancel{\log\delta/\epsilon} + \log 4}{\log\delta/\epsilon - \log R} \\ h(x) &= O(\frac{1}{\log\delta/\epsilon}) \end{aligned}$$

which implies that $\widetilde{z}_{0j} = O(\dfrac{1}{\log\delta/\epsilon})$.

Now

$$\begin{aligned} \Delta\widetilde{z}_{0j} &= \Delta\bar{\eta}_{2j}h(x)z_{0j} + 2\nabla\bar{\eta}_{2j}\nabla(hz_{0j}) + \bar{\eta}_{2j}\Delta(hz_{0j}) \\ &= \Delta\bar{\eta}_{2j}h(x)z_{0j} + 2\nabla\bar{\eta}_{2j}\nabla hz_{0j} + 2\nabla\bar{\eta}_{2j}h\nabla z_{0j} + 2\bar{\eta}_{2j}\nabla h\nabla z_{0j} + \bar{\eta}_{2j}h\Delta z_{0j} \end{aligned}$$

Let us remember that $z_{0j} = \dfrac{2\mu_j^2 - 2|z|^2}{\mu_j^2 + |z|^2}$

$$\begin{aligned} \frac{\partial z_{0j}}{\partial x}(x,y) &= \frac{-8x(\mu_j^2 + \cancel{x^2+y^2}) + 8(\cancel{x^2+y^2})x}{(\mu_j^2 + x^2 + y^2)^2} \\ &= \frac{-8\mu_j^2 x}{(\mu_J^2 + x^2 + y^2)^2} \\ &= O(\frac{1}{r^3}) \\ \text{and } \frac{\partial h}{\partial r}(r) &= \frac{-1}{(\log\delta/\epsilon - \log R)r} \\ &= O(\frac{1}{r\log\delta/\epsilon}) \end{aligned}$$

Using that $\Delta z_{0j} = \dfrac{-8\mu_j^2}{(\mu_j^2 + |z|^2)^2} z_{0j}$ and since $\frac{\delta}{4\epsilon} < r < \frac{\delta}{3\epsilon}$, then

$$\begin{aligned} \Delta\widetilde{z}_{0j} &= O(\frac{\epsilon^2}{\delta^2\log\delta/\epsilon}) + O(\frac{\epsilon}{\delta\log\delta/\epsilon}\frac{1}{r}) + O(\frac{\epsilon}{\delta\log\delta/\epsilon}\frac{1}{r^3}) + O(\frac{1}{\log\delta/\epsilon}\frac{1}{r^4}) + O(\frac{1}{\log\delta/\epsilon}\frac{1}{r^4}) \\ &= O(\frac{\epsilon^2}{\delta^2\log\delta/\epsilon}) + O(\frac{\epsilon^4}{\delta^4\log\delta/\epsilon}) \end{aligned}$$

$$\Delta\widetilde{z}_{0j} = O(\frac{\epsilon^2}{\delta^2\log\delta/\epsilon}) \tag{4.34}$$

Next,

$$
\begin{aligned}
\nabla^2\widetilde{z}_{0j} &= \nabla^2(\bar{\eta}_{2j}h(|x|)z_{0j}) \\
&= \nabla^2\bar{\eta}_{2j}h(|x|)z_{0j} + 2\nabla(\bar{\eta}_{2j})\nabla(h(|x|)z_{0j}) + \bar{\eta}_{2j}\nabla^2(hz_{0j}) \\
&= \nabla^2\bar{\eta}_{2j}hz_{0j} + 2\nabla(\bar{\eta}_{2j})\nabla hz_{0j} + 2\nabla(\bar{\eta}_{2j})h\nabla z_{0j} + \bar{\eta}_{2j}\nabla^2 hz_{0j} + 2\bar{\eta}_{2j}\nabla h\nabla z_{0j} + \\
&\quad \bar{\eta}_{2j}h\nabla^2 z_{0j} \\
&= O(\frac{\epsilon^2}{\delta^2}\frac{1}{\log\delta/\epsilon}) + O(\frac{\epsilon}{\delta}\frac{1}{r\log\delta/\epsilon}) + O(\frac{\epsilon}{\delta}\frac{1}{\log\delta/\epsilon}\frac{1}{r^3}) + O(\frac{1}{r^2}\frac{1}{\log\delta/\epsilon}) \\
&\quad + O(\frac{1}{r}\frac{1}{\log\delta/\epsilon}\frac{1}{r^3}) + O(\frac{1}{\log\delta/\epsilon}\frac{1}{r^4})
\end{aligned}
$$

Since $\frac{\delta}{4\epsilon} < r < \frac{\delta}{3\epsilon}$

$$
\Rightarrow \quad \nabla^2\widetilde{z}_{0j} = O(\frac{\epsilon^2}{\delta^2\log\delta/\epsilon}) \tag{4.35}
$$

Now,

$$
\begin{aligned}
\nabla\widetilde{z}_{0j} &= \nabla(\bar{\eta}_{2j}hz_{0j}) \\
&= \nabla\bar{\eta}_{2j}hz_{0j} + \bar{\eta}_{2j}\nabla hz_{0j} + \bar{\eta}_{2j}h\nabla z_{0j} \\
&= O(\frac{\epsilon}{\delta}\frac{1}{\log\delta/\epsilon}) + O(\frac{1}{r\log\delta/\epsilon}) + O(\frac{1}{\log\delta/\epsilon}\frac{1}{r^3}) \\
&= O(\frac{\epsilon}{\delta\log\delta/\epsilon})
\end{aligned}
$$

And finally,

$$
\begin{aligned}
W((F_j^\epsilon)^{-1}(x))\widetilde{z}_{0j} &= \frac{8\mu_j^2}{(\mu_j^2+|x|^2)^2}\bar{\eta}_{2j}hz_{0j} + \frac{8\mu_j^2}{(\mu_j^2+|x|^2)^2}\bar{\eta}_{2j}hz_{0j}\Theta_\epsilon(x) \\
&\quad + \sum_{k\neq j}^{m}\frac{8\mu_k^2}{(\mu_k^2+|x-\xi_k'|^2)^2}\bar{\eta}_{2j}hz_{0j}(1+\Theta_\epsilon(x)) \\
&= O(\frac{1}{r^4\log\delta/\epsilon}) + O(\frac{1}{r^4\log\delta/\epsilon}) + O(\frac{\epsilon^4}{\delta^4\log\delta/\epsilon})
\end{aligned}
$$

Since $\dfrac{\delta}{4\epsilon} < r < \dfrac{\delta}{3\epsilon}$,

$$
W((F_j^\epsilon)^{-1}(x))\widetilde{z}_{0j} = O(\frac{\epsilon^4}{\delta^4}) \tag{4.36}
$$

So, using 4.34, 4.35 and 4.36,

$$
\begin{aligned}
\tilde{L}(\widetilde{z}_{0j}) &= \Delta\widetilde{z}_{0j} + O(\epsilon|x|)\nabla^2\widetilde{z}_{0j} + O(\epsilon)\nabla\widetilde{z}_{0j} + W\widetilde{z}_{0j} \\
&= O(\frac{\epsilon^2}{\delta^2\log\delta/\epsilon}) + O(\frac{\epsilon^4}{\delta^4})
\end{aligned}
$$

And then,

$$\begin{aligned} I_3 &= \int\limits_{\{\frac{\delta}{4\epsilon}<r<\frac{\delta}{3\epsilon}\}\cap\mathbb{R}^2_+} \tilde{L}(\widetilde{z}_{0j})\widetilde{z}_{0j} \\ &= \int\limits_{\{\frac{\delta}{4\epsilon}<r<\frac{\delta}{3\epsilon}\}\cap\mathbb{R}^2_+} O(\frac{\epsilon^2}{\delta^2\log\delta/\epsilon})O(\frac{1}{\log\delta/\epsilon}) \\ &= O(\frac{\epsilon^2}{\delta^2(\log\delta/\epsilon)^2})O(\frac{\delta^2}{\epsilon^2}) \end{aligned}$$

$$I_3 = O(\frac{1}{(\log\delta/\epsilon)^2}) \tag{4.37}$$

Estimate of I_1

$$I_1 = \int\limits_{R_1} L(\widetilde{Z}_{0j})\widetilde{Z}_{0j}$$

we change variables again $x = F_j^\epsilon(y)$,

$$I_1 = \int\limits_{\{R<r<R+1\}\cap\mathbb{R}^2_+} \tilde{L}(\widetilde{z}_{0j})\widetilde{z}_{0j}$$

In this region $\bar{\eta}_{2j} \equiv 1$, so

$$\widetilde{z}_{0j}(x) = \bar{\eta}_{1j}(x)z_{0j} + (1-\bar{\eta}_{1j})h(|x|)z_{0j}$$

Let us calculate $\tilde{L}(\widetilde{z}_{0j})$, first

$$\begin{aligned} \nabla^2\widetilde{z}_{0j} &= \nabla^2\bar{\eta}_{1j}z_{0j} + 2\nabla\bar{\eta}_{1j}\nabla z_{0j} + \bar{\eta}_{1j}\nabla^2 z_{0j} \\ &\quad -\nabla^2\bar{\eta}_{1j}h(|x|)z_{0j} - 2\nabla\bar{\eta}_{1j}\nabla(hz_{0j}) + (1-\bar{\eta}_{1j})\nabla^2(hz_{0j}) \end{aligned}$$

z_{0j} is bounded in this region and so are his derivatives and h.
Then

$$|\nabla^2\widetilde{z}_{0j}| \le C$$

.

Now,

$$\begin{aligned} \nabla\widetilde{z}_{0j} &= \nabla\bar{\eta}_{1j}z_{0j} + \bar{\eta}_{1j}\nabla z_{0j} - \nabla\bar{\eta}_{1j}hz_{0j} + (1-\bar{\eta}_{1j})\nabla(hx_{0j}) \\ &= O(1) \end{aligned}$$

$$\Rightarrow \tilde{L}(\widetilde{z}_{0j})\widetilde{z}_{0j} = \Delta\widetilde{z}_{0j}\widetilde{z}_{0j} + O(\epsilon R) + W((F_j^{\epsilon})^{-1}(x))\widetilde{z}_{0j}\widetilde{z}_{0j}$$

Let us compute

$$\begin{aligned}
\Delta\widetilde{z}_{0j} &= \Delta((\bar{\eta}_{1j} + (1-\bar{\eta}_{1j})h)z_{0j}) \\
&= (\bar{\eta}_{1j} + (1-\bar{\eta}_{1j})h)\Delta z_{0j} + (2\nabla\bar{\eta}_{1j} - 2\nabla\bar{\eta}_{1j}h + 2(1-\bar{\eta}_{1j})\nabla h)\nabla z_{0j} \\
&\quad +(\Delta\bar{\eta}_{1j}(1-h) - 2\nabla\bar{\eta}_{1j}\nabla h + \cancel{(1-\bar{\eta}_{1j})\Delta h})z_{0j} \\
&= (\bar{\eta}_{1j} + (1-\bar{\eta}_{1j})h)\Delta z_{0j} + 2\nabla\bar{\eta}_{1j}\nabla((1-h)z_{0j}) + 2(1-\bar{\eta}_{1j})\nabla h\nabla z_{0j} + \\
&\quad \Delta\bar{\eta}_{1j}(1-h)z_{0j}
\end{aligned}$$

The first term will vanish with $W((F_j^{\epsilon})^{-1}(x))\widetilde{z}_{0j}$ leaving some small term that we will calculate later on.

Let us calculate $\nabla h\nabla z_{0j}$

$$\begin{aligned}
\frac{\partial z_{0j}}{\partial x}(x,y) &= \frac{-8\mu_j^2 x}{(\mu_j^2 + x^2 + y^2)^2} \\
\frac{\partial h}{\partial x} &= \frac{-x}{r^2(\log\delta/\epsilon - \log R)} \\
\Rightarrow \nabla h\nabla z_{0j} &= \frac{8\mu_j^2\cancel{r^2}}{\cancel{r^2}(\mu_j^2 + r^2)^2(\log\delta/\epsilon - \log R)} \\
&\geq 0
\end{aligned}$$

In fact that term is of order $O(\frac{1}{r^4\log\delta/\epsilon})$, but this inequality is more useful for the proof of the claim.

$$\Rightarrow \Delta\widetilde{z}_{0j} + W\widetilde{z}_{0j} \leq 2\nabla\bar{\eta}_{1j}\nabla((1-h)z_{0j}) + \Delta\bar{\eta}_{1j}(1-h)z_{0j} + O(\frac{\epsilon}{R^3})$$

It follows that

$$I_1 \leq \int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} 2\nabla\bar{\eta}_{1j}\nabla((1-h)z_{0j})\widetilde{z}_{0j} + \Delta\bar{\eta}_{1j}(1-h)z_{0j}\widetilde{z}_{0j} + O(R\epsilon) + O(\frac{\epsilon}{R^2})$$

we integrate by parts the second term in the right hand

$$\begin{aligned}
I_1 \leq & \int_{R_1} 2\nabla\bar{\eta}_{1j}\nabla((1-h)z_{0j})\widetilde{z}_{0j} - \int_{R_1} 2\nabla\bar{\eta}_{1j}\nabla h z_{0j}\widetilde{z}_{0j} + \int_{\partial R_1} \cancel{\nabla\bar{\eta}_{1j}\cdot\hat{n}}(1-h)z_{0j}\widetilde{z}_{0j} \\
& - \int_{R_1} \nabla\bar{\eta}_{1j}\nabla((1-h)z_{0j})\widetilde{z}_{0j} - \int_{R_1} \nabla\bar{\eta}_{1j}(1-h)z_{0j}\nabla\widetilde{z}_{0j} + O(\epsilon R)
\end{aligned}$$

where the third term vanishes because $\bar{\eta}_{1j}$ is radial.

$$-I_1 \geq -\int\limits_{R_1} \nabla\bar{\eta}_{1j}\nabla z_{0j}(1-h)\widetilde{z}_{0j} + \int\limits_{R_1} \nabla\bar{\eta}_{1j}\nabla h z_{0j}\widetilde{z}_{0j} + \int\limits_{R_1} \nabla\bar{\eta}_{1j}(1-h)z_{0j}\nabla\widetilde{z}_{0j} + O(\epsilon R)$$

Let us find a bound for the first integral

$$\int\limits_{R_1} |\nabla\bar{\eta}_{1j}\nabla z_{0j}(1-h)\widetilde{z}_{0j}|$$

We have that

$$\begin{aligned} |1-h| &= \frac{|\log r - \log R|}{|\log\delta/\epsilon - \log R|} \\ &\leq \frac{C}{R\log\delta/\epsilon} \\ \nabla z_{0j} &= \frac{-8\mu_j^2}{(\mu_j^2+r^2)^2}\overrightarrow{x} \\ |\nabla z_{0j}| &= O(\frac{1}{R^3}) \end{aligned}$$

So,

$$\int\limits_{R_1} |\nabla\bar{\eta}_{1j}\nabla z_{0j}(1-h)\widetilde{z}_{0j}| = O(\frac{1}{R^3\log\delta/\epsilon}) + O(\frac{R}{R^3\log\delta/\epsilon}) = O(\frac{1}{R^2\log\delta/\epsilon}) \quad (4.38)$$

The third integral is similar since in the region $\nabla h = O(\frac{1}{R\log\delta/\epsilon})$ and hence

$$\begin{aligned} \nabla\widetilde{z}_{0j} &= \nabla(\bar{\eta}_{1j}z_{0j}(1-h) + hz_{0j}) \\ &= \nabla\bar{\eta}_{1j}(1-h)z_{0j} - \bar{\eta}_{1j}\nabla h z_{0j} + \bar{\eta}_{1j}(1-h)\nabla z_{0j} + \nabla h z_{0j} + h\nabla z_{0j} \\ \Rightarrow \nabla\widetilde{z}_{0j} &= O(\frac{\log r}{\log\delta/\epsilon}) + O(\frac{1}{R\log\delta/\epsilon}) + O(\frac{1}{R^3}) \end{aligned}$$

Integrating,

$$\begin{aligned} \int\limits_{R_1} \nabla\bar{\eta}_{1j}\nabla\widetilde{z}_{0j}(1-h)z_{0j} &= \int\limits_{R_1} O(\frac{(\log r)^2}{(\log\delta/\epsilon)^2}) + O(\frac{\log r}{R(\log\delta/\epsilon)^2}) + O(\frac{\log r}{R^3\log\delta/\epsilon}) \\ &= O(\frac{R^2}{(\log\delta/\epsilon)^2}) + O(\frac{1}{R^2(\log\delta/\epsilon)^2}) + O(\frac{1}{R^4\log\delta/\epsilon}) \end{aligned}$$

$$\int_{R_1} \nabla\bar{\eta}_{1j}\nabla\widetilde{z}_{0j}(1-h)z_{0j} \;=\; O(\frac{R^2}{\log\delta/\epsilon^2}) + O(\frac{1}{R^4\log\delta/\epsilon}) \tag{4.39}$$

In the second integral, z_{0j} and $\widetilde{z}_{0j}$ have lower bounds independent of $\epsilon,\ \delta,\ R$ and $|\nabla h| = \frac{1}{r(\log\delta/\epsilon-\log R)}$. Hence

$$\int_{R_1} \nabla\bar{\eta}_{1j}\nabla h z_{0j}\widetilde{z}_{0j} \;\geq\; \frac{\bar{c}}{(\log\delta/\epsilon-\log R)} \;\geq\; \frac{\bar{c}}{\log\delta/\epsilon} \tag{4.40}$$

with $\bar{c}>0$ independent of $\epsilon,\ \delta,\ R$.

Finally putting 4.38, 4.39, and 4.40 together

$$-I_1 \;\geq\; \frac{\bar{c}}{\log\delta/\epsilon} + O(R\epsilon) + O(\frac{1}{R^2\log\delta/\epsilon}) + O(\frac{R^2}{\log\delta/\epsilon^2}) \tag{4.41}$$

Estimate of I_2

$$I_2 = \int_{R_2} L(\widetilde{Z}_{0j})\widetilde{Z}_{0j} = \int_{\{R+1<r<\frac{\delta}{4\epsilon}\}\cap\mathbb{R}^2_+} \tilde{L}(\widetilde{z}_{0j})\widetilde{z}_{0j}$$

changing variables as before.

In the region $R+1<r<\frac{\delta}{4\epsilon}$ $\widetilde{z}_{0j} = hz_{0j}$, so

$$\begin{aligned}
\Delta\widetilde{z}_{0j} &= \Delta h z_{0j} + 2\nabla h\nabla z_{0j} + h\Delta z_{0j} \\
\nabla^2\widetilde{z}_{0j} &= \nabla^2 h z_{0j} + 2\nabla h\nabla z_{0j} + h\nabla^2 z_{0j} \\
\Rightarrow |\nabla^2\widetilde{z}_{0j}| &\leq O(\frac{1}{r^2\log\delta/\epsilon}) + O(\frac{1}{r^4\log\delta/\epsilon}) + O(\frac{1}{r^4}) \\
&\text{also} \\
|\nabla\widetilde{z}_{0j}| &\leq |\nabla h||z_{0j}| + |h\nabla z_{0j}| \\
&= O(\frac{1}{r\log\delta/\epsilon}) + O(\frac{1}{r^3})
\end{aligned}$$

Hence,

$$\begin{aligned}
\tilde{L}(\widetilde{z}_{0j}) &= \Delta\widetilde{z}_{0j} + O(\epsilon|x|)\nabla^2\widetilde{z}_{0j} + O(\epsilon)\nabla\widetilde{z}_{0j} + W((F_j^e)^{-1}(x))\widetilde{z}_{0j} \\
&= 2\nabla h\nabla z_{0j} + h\Delta z_{0j} + O(\epsilon r)\left[O(\frac{1}{r^2\log\delta/\epsilon}) + O(\frac{1}{r^4\log\delta/\epsilon}) + O(\frac{1}{r^4})\right] \\
&\quad + O(\epsilon)\left[O(\frac{1}{r\log\delta/\epsilon}) + O(\frac{1}{r^3})\right] + hW((F_j^e)^{-1}(x))z_{0j} \\
&= O(\frac{1}{r^4\log\delta/\epsilon}) + O(\frac{\epsilon}{r\log\delta/\epsilon}) + O(\frac{\epsilon}{r^3\log\delta/\epsilon}) + O(\frac{\epsilon}{r^3}) + O(\frac{\epsilon}{r\log\delta/\epsilon}) + O(\frac{\epsilon}{r^3}) + \\
&\quad O(\frac{\epsilon}{r^6}) + O(\frac{\epsilon}{r^3}) \\
&= O(\frac{1}{r^4\log\delta/\epsilon}) + O(\frac{\epsilon}{r\log\delta/\epsilon}) + O(\frac{\epsilon}{r^3}) + O(\frac{\epsilon}{r^6})
\end{aligned}$$

This yields

$$\begin{aligned}
\int_{\{R+1<r<\frac{\delta}{4\epsilon}\}\cap\mathbb{R}^2_+} \tilde{L}(\widetilde{z}_{0j})\widetilde{z}_{0j} &= \left[O(\frac{1}{\log\delta/\epsilon}\frac{1}{r^2}) + O(\frac{\epsilon}{\log\delta/\epsilon}r) + O(\epsilon\frac{1}{r}) + O(\epsilon\frac{1}{r^4})\right]_{R+1}^{\frac{\delta}{4\epsilon}} \\
&= O(\frac{\epsilon^2}{\delta^2\log\delta/\epsilon}) + O(\frac{1}{R^2\log\delta/\epsilon}) + O(\frac{\delta}{\log\delta/\epsilon}) + O(\frac{\epsilon R}{\log\delta/\epsilon}) \\
&\quad + O(\frac{\epsilon^2}{\delta}) + O(\frac{\epsilon}{R}) + O(\frac{\epsilon^5}{\delta^4}) + O(\frac{\epsilon}{R^4})
\end{aligned}$$

And finally

$$I_2 = O(\frac{\delta}{\log\delta/\epsilon}) + O(\frac{\epsilon R}{\log\delta/\epsilon}) + O(\frac{\epsilon}{R^4}) \tag{4.42}$$

Gathering all the terms 4.33,4.41,4.42 and 4.37 we have that

$$-\int_{\Omega_\epsilon} L(\widetilde{Z}_{0j})\widetilde{Z}_{0j} \geq O(\epsilon R) + O(\frac{R^2}{\log\delta/\epsilon^2}) + \frac{\bar{c}}{\log\delta/\epsilon} + O(\frac{\epsilon}{R^4}) + O(\frac{\delta}{\log\delta/\epsilon}) + O(\frac{\epsilon R}{\log\delta/\epsilon}) \tag{4.43}$$

Estimate of $\int_{\Gamma_1^\epsilon} \widetilde{Z}_{0j}L_2(\widetilde{Z}_{0j})$

We change variables through the map F_j^ϵ:

$$\int_{\Gamma_1^\epsilon} \widetilde{Z}_{0j}(\frac{\partial}{\partial\nu} - W_2)\widetilde{Z}_{0j} = \int_{\partial\mathbb{R}^2_+} \widetilde{z}_{0j}(B(\widetilde{z}_{0j}) - W_2((F_j^\epsilon)^{-1}(x))\widetilde{z}_{0j})b(x)$$

with $b(x)$ a function that arises from the change of variables and it is positive and uniformly bounded in ϵ. B is a differential operator of oder one on $\partial\mathbb{R}^2_+$,

$$B = -\frac{\partial}{\partial x_2} + O(\epsilon|x|)\nabla$$

First let us estimate the integral in the region $|x| < R$.

$$\begin{aligned}
\widetilde{z}_{0j} &= z_{0j} \\
\nabla z_{0j} &= \frac{-8\mu_j^2}{(\mu_j^2 + r^2)^2}\overrightarrow{x} \\
\Rightarrow B(\widetilde{z}_{0j}) &= \frac{-\partial z_{0j}}{\partial x_2} + O(\frac{\epsilon|x|^2}{(1+|x|^2)^2})
\end{aligned}$$

Since we have the expansion $(F_j^\epsilon)^{-1}(x) = x + \bar{\xi}_j' + O(\epsilon|x|)$, we find

$$W_2((F_j^\epsilon)^{-1}(x)) = W_2(\bar{\xi}_j' + x + O(\epsilon|x|))$$

So, in $|x| < R,\ x \in \partial\mathbb{R}^2_+$,

$$W_2((F_j^\epsilon)^{-1}(x)) = \frac{2\sqrt{2}\mu_j}{\mu_j^2 + (x - \sqrt{2}\mu_j\nu(\bar{\xi}_j')/2)^2} + O(\epsilon)$$

$$\Rightarrow B(\widetilde{z}_{0j}) - W_2((F_j^\epsilon)^{-1}(x))\widetilde{z}_{0j} = \underbrace{-\frac{\partial z_{0j}}{\partial x_2} - \frac{2\sqrt{2}\mu_j}{\mu_j^2 + (x - \sqrt{2}\mu_j\nu(\bar{\xi}_j')/2)^2}}_{=0} + O(\epsilon)$$

$$\Rightarrow \int\limits_{\partial\mathbb{R}^2_+\cap\{|x|<R\}} \widetilde{z}_{0j}(B(\widetilde{z}_{0j}) - W_2((F_j^\epsilon)^{-1}(x))\widetilde{z}_{0j})b(x) = O(\epsilon R) \tag{4.44}$$

Next in the region $R < |x| < R + 1$

$$
\begin{aligned}
\widetilde{z}_{0j} &= \bar{\eta}_{1j}(1-h)z_{0j} + hz_{0j} \\
\nabla\widetilde{z}_{0j} &= \nabla\bar{\eta}_{1j}(1-h)z_{0j} - \nabla h\bar{\eta}_{1j}z_{0j} + \bar{\eta}_{1j}(1-h)\nabla z_{0j} + \\
&\quad \nabla h z_{0j} + h\nabla z_{0j} \\
&= O(\frac{C}{R\log\delta/\epsilon}) + \bar{\eta}_{1j}(1-h)\nabla z_{0j} + h\nabla z_{0j} \\
\Rightarrow B(\widetilde{z}_{0j}) &= -\frac{\partial\widetilde{z}_{0j}}{\partial x_2} + O(\epsilon|x|)\nabla\widetilde{z}_{0j} \\
&= -\frac{\partial\bar{\eta}_{1j}}{\partial x_2}(1-h)z_{0j} + \bar{\eta}_{1j}\frac{\partial h}{\partial x_2}z_{0j} - \bar{\eta}_{1j}(1-h)\frac{\partial z_{0j}}{\partial x_2} - \\
&\quad \frac{\partial h}{\partial x_2}z_{0j} - h\frac{\partial z_{0j}}{\partial x_2} + \\
&\quad O(\epsilon R)\left(O(\frac{C}{R\log\delta/\epsilon}) + \bar{\eta}_{1j}(1-h)\nabla z_{0j} + h\nabla z_{0j}\right)
\end{aligned}
$$

because these are radial functions

$$
= -\bar{\eta}_{1j}(1-h)\frac{\partial z_{0j}}{\partial x_2} - h\frac{\partial z_{0j}}{\partial x_2} + O(\frac{\epsilon}{\log\delta/\epsilon}) + O(\frac{\epsilon}{R^2})
$$

And,

$$
\begin{aligned}
W_2((F_j^\epsilon)^{-1}(x))\widetilde{z}_{0j} &= \left(\frac{2\sqrt{2}\mu_j}{\mu_j^2 + (x-\sqrt{2}\mu_j\nu(\bar{\xi}_j')/2)^2} + O(\frac{\epsilon^2}{R^2}) + O(\frac{\epsilon}{R})\right)(\bar{\eta}_{1j}(1-h)z_{0j} + hz_{0j}) \\
&= \frac{2\sqrt{2}\mu_j}{\mu_j^2 + (x-\sqrt{2}\mu_j\nu(\bar{\xi}_j')/2)^2}(\bar{\eta}_{1j}(1-h)z_{0j} + hz_{0j}) + O(\frac{\epsilon^2}{R^2}) \\
&\quad + O(\frac{\epsilon^2}{R^2\log\delta/\epsilon}) + O(\frac{\epsilon}{R})
\end{aligned}
$$

$$
\Rightarrow B(\widetilde{z}_{0j}) - W_2((F_j^\epsilon)^{-1}(x))\widetilde{z}_{0j} = O(\frac{\epsilon}{\log\delta/\epsilon}) + O(\frac{\epsilon}{R})
$$

It follows that

$$
\int_{\partial\mathbb{R}^2_+\cap\{R<|x|<R+1\}} \widetilde{z}_{0j}(B(\widetilde{z}_{0j}) - W_2((F_j^\epsilon)^{-1}(x))\widetilde{z}_{0j})b(x) = O(\frac{\epsilon}{\log\delta/\epsilon}) + O(\frac{\epsilon}{R}) \quad (4.45)
$$

Next, in the region $R+1<|x|<\frac{\delta}{4\epsilon}$

$$
\widetilde{z}_{0j} = hz_{0j}
$$

$$
\begin{aligned}
\Rightarrow\ B(\widetilde{z}_{0j}) &= -h\frac{\partial z_{0j}}{\partial x_2} + O(\epsilon|x|)(\nabla h z_{0j} + h\nabla z_{0j}) \\
&= -h\frac{\partial z_{0j}}{\partial x_2} + O(\frac{\epsilon}{\log\delta/\epsilon}) + O(\frac{\epsilon}{r^2}) \\
\Rightarrow\ B(\widetilde{z}_{0j}) - W_2((F_j^\epsilon)^{-1}(x))\widetilde{z}_{0j} &= -h\frac{\partial z_{0j}}{\partial x_2} - \frac{2\sqrt{2}\mu_j}{\mu_j^2 + (x - \sqrt{2}\mu_j\nu(\bar{\xi}_j')/2)^2} h z_{0j} + O(\frac{\epsilon}{\log\delta/\epsilon}) \\
&\quad + O(\frac{\epsilon}{r^2}) + O(\frac{\epsilon^2}{r^2}) + O(\frac{\epsilon}{r}) \\
&= O(\frac{\epsilon}{\log\delta/\epsilon}) + O(\frac{\epsilon}{r^2}) + O(\frac{\epsilon}{r})
\end{aligned}
$$

And we conclude

$$
\int_{\partial\mathbb{R}^2_+\cap\{R+1<|x|<\frac{\delta}{4\epsilon}\}} \widetilde{z}_{0j}(B(\widetilde{z}_{0j}) - W_2((F_j^\epsilon)^{-1}(x))\widetilde{z}_{0j})b(x) = O(\frac{\delta}{\log\delta/\epsilon}) + O(\frac{\epsilon^2}{\delta}) + O(\epsilon\log\delta/\epsilon) \tag{4.46}
$$

Finally we consider $\frac{\delta}{4\epsilon} < |x| < \frac{\delta}{3\epsilon}$,

$$
\widetilde{z}_{0j} = \bar{\eta}_{1j} h z_{0j}
$$

$$
h = O(\frac{1}{\log\delta/\epsilon})
$$

$$
\begin{aligned}
B(\widetilde{z}_{0j}) &= B(\bar{\eta}_{2j}) h z_{0j} + \bar{\eta}_{2j} B(h z_{0j}) \\
&= -\frac{\partial\bar{\eta}_{2j}}{\partial x_2} h z_{0j} + O(\epsilon r)\nabla\bar{\eta}_{2j} h z_{0j} + \bar{\eta}_{2j} B(h z_{0j}) \\
&= O(\frac{\epsilon^2 r}{\delta\log\delta/\epsilon}) + \bar{\eta}_{2j} B(h z_{0j}) \\
&= O(\frac{\epsilon^2 r}{\delta\log\delta/\epsilon}) - \bar{\eta}_{2j} h\frac{\partial z_{0j}}{\partial x_2} + O(\epsilon r)(\nabla h z_{0j} + h\nabla z_{0j}) \\
&= -\bar{\eta}_{2j} h\frac{\partial z_{0j}}{\partial x_2} + O(\frac{\epsilon^2 r}{\delta\log\delta/\epsilon}) + O(\frac{\epsilon}{\log\delta/\epsilon}) + O(\frac{\epsilon}{r^2}) \\
&= O(\frac{1}{r^3}) + O(\frac{\epsilon}{r^2}) + O(\frac{\epsilon^2 r}{\delta\log\delta/\epsilon}) + O(\frac{\epsilon}{\log\delta/\epsilon})
\end{aligned}
$$

Integrating this term alone we have

$$
\int_{\partial\mathbb{R}^2_+\cap\{\frac{\delta}{4\epsilon}<|x|<\frac{\delta}{3\epsilon}\}} \widetilde{z}_{0j} B(\widetilde{z}_{0j}) b(x) = O(\frac{\epsilon^2}{\delta^2\log\delta/\epsilon}) + O(\frac{\epsilon^2}{\delta\log\delta/\epsilon}) + O(\frac{\delta}{\log\delta/\epsilon^2})
$$

And, since

$$
W_2((F_j^\epsilon)^{-1}(x)) = O(\frac{1}{r^2}) + O(\frac{\epsilon^2}{r^2}) + O(\frac{\epsilon}{r})
$$

$$\Rightarrow \int\limits_{\partial\mathbb{R}^2_+\cap\{\frac{\delta}{4\epsilon}<|x|<\frac{\delta}{3\epsilon}\}} \widetilde{z}_{0j} W_2((F_j^\epsilon)^{-1}(x))\widetilde{z}_{0j} b(x) = O(\frac{\epsilon}{\delta \log \delta/\epsilon^2}) + O(\frac{\epsilon \cancel{\log \delta/\epsilon}}{\log \delta/\epsilon^{\cancel{2}}})$$

Then

$$\begin{aligned}\int\limits_{\partial\mathbb{R}^2_+\cap\{\frac{\delta}{4\epsilon}<|x|<\frac{\delta}{3\epsilon}\}} \widetilde{z}_{0j}(B(\widetilde{z}_{0j}) - W_2\widetilde{z}_{0j})b(x) &= O(\frac{\epsilon^2}{\delta^2 \log \delta/\epsilon}) + O(\frac{\delta}{\log \delta/\epsilon^2}) \\ &\quad + O(\frac{\epsilon}{\delta \log \delta/\epsilon^2}) + O(\frac{\epsilon}{\log \delta/\epsilon^2})\end{aligned} \tag{4.47}$$

Putting it all together (4.44, 4.45, 4.46 and 4.47)

$$\begin{aligned}\int\limits_{\Gamma_1^\epsilon} \widetilde{Z}_{0j}(\frac{\partial}{\partial\nu} - W_2)\widetilde{Z}_{0j} &= O(R\epsilon) + O(\frac{\epsilon}{R}) + O(\frac{\epsilon}{\log \delta/\epsilon}) + O(\frac{\delta}{\log \delta/\epsilon}) \\ &\quad + O(\frac{\epsilon^2}{\delta}) + O(\epsilon \log \delta/\epsilon) + O(\frac{\epsilon^2}{\delta^2 \log \delta/\epsilon}) + O(\frac{\delta}{\log \delta/\epsilon^2}) \\ &\quad + O(\frac{\epsilon}{\delta \log \delta/\epsilon^2}) + O(\frac{\epsilon}{\log \delta/\epsilon})\end{aligned}$$

So,

$$\int\limits_{\Gamma_1^\epsilon} \widetilde{Z}_{0j}(\frac{\partial}{\partial\nu} - W_2)\widetilde{Z}_{0j} = O(\frac{\delta}{\log \delta/\epsilon}) \tag{4.48}$$

And with all together, 4.43 and 4.48

$$\int\limits_{\Gamma_1^\epsilon} \widetilde{Z}_{0j}(\frac{\partial}{\partial\nu} - W_2)\widetilde{Z}_{0j} - \int\limits_{\Omega_\epsilon} L(\widetilde{Z}_{0j})\widetilde{Z}_{0j} \geq O(\epsilon R^4) + \frac{\bar{c}}{\log \delta/\epsilon} + O(\frac{\delta}{\log \delta/\epsilon})$$

Choosing $\delta > 0$ small, then 4.25

$$\int\limits_{\Gamma_1^\epsilon} \widetilde{Z}_{0j}(\frac{\partial}{\partial\nu} - W_2)\widetilde{Z}_{0j} - \int\limits_{\Omega_\epsilon} L(\widetilde{Z}_{0j})\widetilde{Z}_{0j} \geq \frac{C}{\log \frac{1}{\epsilon}}$$

holds for $\epsilon > 0$ small enough.

4.2.2 Proof of 4.26

We have that

$$\begin{aligned} L(\widetilde{Z}_{0j}) &= O(\epsilon) \quad \text{for } r < R \\ L(\widetilde{Z}_{0j}) &= O(\frac{\epsilon^2}{\delta^2 \log \delta/\epsilon}) \quad \text{for } \frac{\delta}{4\epsilon} < r < \frac{\delta}{3\epsilon} \\ L(\widetilde{Z}_{0j}) &= O(\frac{1}{r^4 \log \delta/\epsilon}) + O(\frac{\epsilon}{r \log \delta/\epsilon}) + O(\frac{\epsilon}{r^3}) + O(\frac{\epsilon}{r^3})\widetilde{Z}_{0j} \quad \text{for } R+1 < r < \frac{\delta}{4\epsilon} \end{aligned}$$

Now in $R < r < R+1$

$$\begin{aligned} L(\widetilde{Z}_{0j}) &= \tilde{L}(\widetilde{z}_{0j}) = \Delta\widetilde{z}_{0j} + W((F_j^\epsilon)^{-1}(x))\widetilde{z}_{0j} + O(\epsilon|x|) \\ &= 2\nabla\bar{\eta}_{1j}\nabla((1-h)z_{0j}) + \Delta\bar{\eta}_{1j}(1-h)z_{0j} + 2(1-\bar{\eta}_{1j})\nabla h\nabla z_{0j} + O(\frac{\epsilon}{R^2}) \\ &= -2\nabla\bar{\eta}_{1j}\nabla h z_{0j} + 2\nabla\bar{\eta}_{1j}(1-h)\nabla z_{0j} + \Delta\bar{\eta}_{1j}(1-h)z_{0j} + 2(1-\bar{\eta}_{1j})\nabla h\nabla z_{0j} + \\ &\quad O(\frac{\epsilon}{R^2}) \\ &= O(\frac{1}{R\log\delta/\epsilon}) + O(\frac{\epsilon}{R^2}) \\ &= O(\frac{1}{R\log\delta/\epsilon}) \end{aligned}$$

Now let us call

$$a = \frac{|L(\widetilde{Z}_{0j})|}{\epsilon^2 + \sum_{j=1}^{m}(1+|y-\xi_j'|)^{-2-\sigma}}$$

and using the previous calculations we have that

$$\begin{aligned} a &= O(\epsilon) \quad \text{in } R_0 \\ a &= O(\frac{R^{1+\sigma}}{\log\delta/\epsilon}) \quad \text{in } R_1 \\ a &= O(\frac{1}{R\log\delta/\epsilon}) \quad \text{in } R_2 \\ a &= O(\frac{1}{\log\delta/\epsilon^2}) \quad \text{in } R_3 \end{aligned}$$

So

$$\|L(\widetilde{Z}_{0j})\|_{*,\Omega_\epsilon} = O(\frac{1}{\log\delta/\epsilon}) \leq C\frac{1}{\log\frac{1}{\epsilon}} \tag{4.49}$$

which proves 4.27.

4.2.3 Proof of 4.27

We have that

$$\begin{aligned} L_2(\widetilde{Z}_{0j}) &= O(\epsilon) \quad \text{for } y \in \Gamma_1^\epsilon, |y| < R \\ L_2(\widetilde{Z}_{0j}) &= O(\frac{\epsilon}{\log\delta/\epsilon}) + O(\frac{\epsilon}{R}) \quad \text{for } y \in \Gamma_1^\epsilon, R < |y| < R+1 \\ L_2(\widetilde{Z}_{0j}) &= O(\frac{\epsilon}{\log\delta/\epsilon}) + O(\frac{\epsilon}{r^2}) + O(\frac{\epsilon}{r}) \quad \text{for } y \in \Gamma_1^\epsilon, R+1 < |y| < \frac{\delta}{4\epsilon} \end{aligned}$$

And for $y \in \Gamma_1^\epsilon$, $\frac{\delta}{4\epsilon} < |y| < \frac{\delta}{3\epsilon}$

$$\begin{aligned} L_2(\widetilde{Z}_{0j}) &= O(\frac{1}{r^2}) + O(\frac{1}{r^3}) + O(\frac{\epsilon}{r^2}) + O(\frac{\epsilon^2 r}{\delta \log \delta/\epsilon}) \\ &+ O(\frac{\epsilon}{\log \delta/\epsilon}) + O(\frac{\epsilon^2}{r^2}) + O(\frac{\epsilon}{r}) \\ &= O(\frac{1}{r^2}) + O(\frac{\epsilon}{r^2}) + O(\frac{\epsilon^2 r}{\delta \log \delta/\epsilon}) + O(\frac{\epsilon}{\log \delta/\epsilon}) + O(\frac{\epsilon}{r}) \end{aligned}$$

Now let us call

$$b = \frac{|L_2(\widetilde{Z}_{0j})|}{\epsilon + \sum_{j=1}^{m}(1 + |y - \xi_j'|)^{-1-\sigma}}$$

and using the previous calculations we have that

$$\begin{aligned} b &= O(\epsilon) \quad \text{in } R_0 \\ b &= O(\epsilon R^\sigma) \quad \text{in } R_1 \\ b &= O(\frac{1}{\log \delta/\epsilon}) \quad \text{in } R_2 \\ b &= O(\frac{1}{\log \delta/\epsilon}) \quad \text{in } R_3 \end{aligned}$$

So

$$\|L_2(\widetilde{Z}_{0j})\|_{*,\Gamma_1^\epsilon} = O(\frac{1}{\log \delta/\epsilon}) \leq \frac{C}{\log \delta/\epsilon} \tag{4.50}$$

which proves 4.27.

With this estimates we are now ready to complete the proof of the claim.

We had that

$$|d_j| \left[\int_{\Gamma_1^\epsilon} \widetilde{Z}_{0j} L_2(\widetilde{Z}_{0j}) - \int_{\Omega_\epsilon} \widetilde{Z}_{0j} L(\widetilde{Z}_{0j}) \right] \leq C(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) \Big(1 + \|L_2(\widetilde{Z}_{0j})\|_{*,\Gamma_1^\epsilon} + \|L(\widetilde{Z}_{0j})\|_{*,\Omega_\epsilon} \Big) + C \sum_{k=1}^{m} |d_k| (\|L_2(\widetilde{Z}_{0j})\|^2_{*,\Gamma_1^\epsilon} + \|L(\widetilde{Z}_{0j})\|^2_{*,\Omega_\epsilon})$$

using 4.25, 4.26, and 4.27

$$|d_j|\frac{C}{\log\frac{1}{\epsilon}} \leq C(\|h_1\|_{*,\Omega_\epsilon}+\|h_2\|_{*,\Gamma_1^\epsilon})(1+\frac{C}{\log\delta/\epsilon})+C\sum_{k=1}^{m}|d_k|\frac{C}{\log\frac{1}{\epsilon}^2}$$

$$\Rightarrow |d_j| \leq C\log\frac{1}{\epsilon}(\|h_1\|_{*,\Omega_\epsilon}+\|h_2\|_{*,\Gamma_1^\epsilon})$$

and the claim holds.

Continuing with the proof of the a priori bound we had the inequality 4.22

$$\begin{aligned}\|\tilde{\phi}\|_\infty &\leq C(\|h_1\|_{*,\Omega_\epsilon}+\|h_2\|_{*,\Gamma_1^\epsilon})+C\sum_{j=1}^{m}|d_j|(\|L(\widetilde{Z}_{0j})\|_{*,\Omega_\epsilon}+\|L_2(\widetilde{Z}_{0j})\|_{*,\Gamma_1^\epsilon})\\ &\leq C(\|h_1\|_{*,\Omega_\epsilon}+\|h_2\|_{*,\Gamma_1^\epsilon})+C\log\frac{1}{\epsilon}(\|h_1\|_{*,\Omega_\epsilon}+\|h_2\|_{*,\Gamma_1^\epsilon})\\ &\leq C\log\frac{1}{\epsilon}(\|h_1\|_{*,\Omega_\epsilon}+\|h_2\|_{*,\Gamma_1^\epsilon})\end{aligned}$$

Then,

$$\|\phi\|_\infty \leq \|\tilde{\phi}\|_\infty+\sum_{j=1}^{m}|d_j|\|\widetilde{Z}_{0j}\|_\infty$$

$$\|\phi\|_\infty \leq C\log\frac{1}{\epsilon}(\|h_1\|_{*,\Omega_\epsilon}+\|h_2\|_{*,\Gamma_1^\epsilon})$$

which ends the proof of the second a priori bound. □

4.3 Solvability of the linear operator

We now prove the solvability of the problem 4.4

Proposition 4.3.1. *Let $\delta > 0$ be fixed. There exist positive numbers ϵ_0 and C, such that for any family of points $\bar{\xi}_j, j = 1, \dots, m$ in Γ_1 with*

$$|\bar{\xi}_i - \bar{\xi}_j| \geq \delta \ \textit{for}\ i \neq j, \tag{4.51}$$

there is,for all $\epsilon < \epsilon_0$, a unique solution to the problem:

Given $h_1,\ h_2$, find ϕ and scalars $c_j,\ j = 1, \dots, m$, such that

$$\begin{cases} L(\phi) \ = \ h_1 + \sum_{j=1}^{m} c_j \chi_j Z_{ij} \ \ \textit{in}\ \Omega_\epsilon \\ L_2(\phi) \ = \ h_2 \ \ \textit{on}\ \Gamma_1^\epsilon \\ \phi \ = \ 0 \ \ \textit{on}\ \Gamma_2^\epsilon \end{cases} \tag{4.52}$$

and

$$\int_{\Omega_\epsilon} \chi_j Z_{1j} \phi \ = \ 0\ \forall\, j = 1, \dots, m. \tag{4.53}$$

Moreover

$$\|\phi\|_{L^\infty(\Omega_\epsilon)} \ \leq \ C \log \frac{1}{\epsilon} (\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}). \tag{4.54}$$

Proof. First, we must prove that the inequality holds.

By the previous lemma (4.2.1)

$$\begin{aligned} \|\phi\|_\infty \ &\leq \ C \log \frac{1}{\epsilon} (\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon} + \sum_{j=1}^{m} |c_j| \|\chi_j Z_{1j}\|_{*,\Omega_\epsilon}) \\ &\leq \ C \log \frac{1}{\epsilon} (\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon} + \sum_{j=1}^{m} |c_j|) \end{aligned}$$

Therefore it is enough to prove that $|c_j| \leq C(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon})$.

Let η_{2j} be the cut-off function defined earlier and multiply equation 4.52 by $\eta_{2k} Z_{1k}$ and integrate.

$$\Rightarrow \int \Delta\phi \eta_{2k} Z_{1k} + W \phi \eta_{2k} Z_{1k} \ = \ \int h_1 \eta_{2k} Z_{1k} + \sum_{j=1}^{m} c_j \chi_j Z_{1j} \eta_{2k} Z_{1k}$$

Integrating by parts

$$\int_{\partial\Omega_\epsilon} \frac{\partial\phi}{\partial\nu}\eta_{2k}Z_{1k} - \int_{\Omega_\epsilon} \nabla\phi\nabla(\eta_{2k}Z_{1k}) + \int_{\Omega_\epsilon} W\phi\eta_{2k}Z_{1k} = \int_{\partial\Omega_\epsilon} h_1\eta_{2k}Z_{1k} + c_k\int_{\Omega_\epsilon}\chi_k Z_{1k}^2\eta_{2k}$$

$$\int_{\Gamma_1^\epsilon} \frac{\partial\phi}{\partial\nu}\eta_{2k}Z_{1k} + \int_{\Omega_\epsilon} \phi\Delta(\eta_{2k}Z_{1k}) - \int_{\Gamma_1^\epsilon} \phi\frac{\partial}{\partial\nu}(\eta_{2k}Z_{1k}) + \int_{\Omega_\epsilon} W\phi\eta_{2k}Z_{1k} = \int_{\Omega_\epsilon} h_1\eta_{2k}Z_{1k} + c_k\int_{\Omega_\epsilon}\chi_k Z_{1k}^2\eta_{2k}$$

Using the equation

$$\int_{\Gamma_1^\epsilon} h_2\eta_{2k}Z_{1k} + \int_{\Gamma_1^\epsilon} W_2\phi\eta_{2k}Z_{1k} + \int_{\Omega_\epsilon}\phi(\Delta(\eta_{2k}Z_{1k}) + W\eta_{2k}Z_{1k}) - \int_{\Gamma_1^\epsilon}\phi\frac{\partial\eta_{2k}}{\partial\nu}Z_{1k} - \int_{\Gamma_1^\epsilon}\phi\eta_{2k}\frac{\partial Z_{1k}}{\partial\nu}$$

$$= \int_{\Omega_\epsilon} h_1\eta_{2k}Z_{1k} + c_k\int_{\Omega_\epsilon}\chi_k Z_{1k}^2\eta_{2k}$$

$$\begin{aligned}
\Rightarrow\ c_k\int_{\Omega_\epsilon}\chi_k Z_{1k}^2\eta_{2k} &= \int_{\Gamma_1^\epsilon} h_2\eta_{2k}Z_{1k} - \int_{\Omega_\epsilon} h_1\eta_{2k}Z_{1k} \\
&\quad + \int_{\Omega_\epsilon}\phi(\Delta(\eta_{2k}Z_{1k}) + W\eta_{2k}Z_{1k}) \\
&\quad - \int_{\Gamma_1^\epsilon}\phi\eta_{2k}\left(\frac{\partial Z_{1k}}{\partial\nu} - W_2Z_{1k}\right) \\
&\quad - \int_{\Gamma_1^\epsilon}\phi\frac{\partial\eta_{2k}}{\partial\nu}Z_{1k} \qquad (4.55)
\end{aligned}$$

Let us compute $\Delta(\eta_{2k}Z_{1k})$. First notice that

$$\begin{aligned}
Z_{1k} &= \frac{-4x_1}{(\mu_j^2 + |x|^2)} \\
\nabla Z_{1k} &= \begin{pmatrix} -4\mu_j^2 + 4x_1^2 - 4x_2^2 \\ -8x_1x_2 \end{pmatrix}\frac{1}{(\mu_j^2+|x|^2)^2} \\
\nabla^2 Z_{1k} &= O\left(\frac{r^3(1+2r^2)}{(1+r^2)^4}\right) \\
\nabla\eta_{2k} &= O(\epsilon) \\
\nabla^2\eta_{2k} &= O(\epsilon^2)
\end{aligned}$$

So,

$$\begin{aligned}
\Delta(\eta_{2k}Z_{1k}) &= O(\frac{\epsilon^2 r}{1+r^2}) + O(\frac{\epsilon(1-2r^2)}{(1+r^2)^2}) + \eta_{2k}\Delta Z_{1k} \\
\Rightarrow \; \Delta(\eta_{2k}Z_{1k}) + W\eta_{2k}Z_{1k} &= \eta_{2k}L(Z_{1k}) + O(\frac{\epsilon^2 r}{1+r^2}) + O(\frac{\epsilon(1-2r^2)}{(1+r^2)^2}) \\
&= \eta_{2k}(\Delta z_{1k} + O(\epsilon r)\nabla^2 z_{1k} + O(\epsilon)\nabla z_{1k} + W z_{1k} + \\
&\quad O(\frac{\epsilon}{(1+r^2)^3}) z_{1k} \\
&\quad + O(\frac{\epsilon(1+r)}{(1+r^2)^2}) z_{1k}) + O(\frac{\epsilon^2 r}{1+r^2}) + O(\frac{\epsilon(1-2r^2)}{(1+r^2)^2}) \\
&= \eta_{2k}(O(\frac{\epsilon r^4(1+2r^2)}{(1+r^2)^4}) + O(\frac{\epsilon(1-2r^2)}{(1+r^2)^2}) + O(\frac{\epsilon r}{(1+r^2)^4}) \\
&\quad + O(\frac{\epsilon r(1+r)}{(1+r^2)^5})) + O(\frac{\epsilon^2 r}{1+r^2}) + O(\frac{\epsilon(1-2r^2)}{(1+r^2)^2}) \\
\text{Simplifying} & \\
&= \eta_{2k} O(\frac{\epsilon(r^2+2)(2r^6+3r^4-r^2-r-1)}{(1+r^2)^5}) + O(\frac{\epsilon^2 r}{1+r^2})
\end{aligned}$$

Integrating

$$\begin{aligned}
\int_{\Omega_\epsilon} \Delta(\eta_{2k}Z_{1k}) + W\eta_{2k}Z_{1k} &= O(\epsilon) + O(\epsilon \log \frac{\delta^2}{\epsilon^2} + 1) \\
&= O(\epsilon) + O(\epsilon \log \frac{1}{\epsilon}) \\
&= O(\epsilon \log \frac{1}{\epsilon}) \qquad (4.56)
\end{aligned}$$

Now let us compute

$$\frac{\partial Z_{1k}}{\partial \nu} - W_2 Z_{1k} = B(z_{1k}) - W_2((F_k^\epsilon)^{-1}(x)) z_{1k}$$

With $B = -\frac{\partial}{\partial x_2} + O(\epsilon|x|)\nabla$, the differential operator that comes from the change of variables.

$$\begin{aligned}\frac{\partial Z_{1k}}{\partial \nu} - W_2 Z_{1k} &= -\cancel{\frac{\partial z_{1k}}{\partial x_2}} + O(\frac{\epsilon r(1-2r^2)}{(1+r^2)^2})\cancel{-W_2 z_{1k}} + O(\frac{\epsilon^2 r}{(1+r^2)^2}) + O(\frac{\epsilon^2 r^2}{(1+r^2)^2}) + \\ & O(\frac{\epsilon r}{(1+r^2)^3}) + O(\frac{\epsilon^2 r^2}{(1+r^2)^3})\end{aligned}$$

Simplifying

$$= O(\frac{\epsilon r(-2r^4 + r^3 - r^2 + r + 2)}{(r^2+1)^3}) + O(\frac{\epsilon^2 r(r^2+r+1)}{(r^2+1)^3})$$

And integrating

$$\begin{aligned}\int_{\Gamma_1^\epsilon} \frac{\partial Z_{1k}}{\partial \nu} - W_2 Z_{1k} &= O(\epsilon^2) + O(\epsilon) + O(\epsilon \log \frac{1}{\epsilon}) \\ &= O(\epsilon \log \frac{1}{\epsilon}) \qquad (4.57)\end{aligned}$$

Then,

$$|c_k| \int_{\Omega_\epsilon} \chi_k Z_{1k}^2 \eta_{2k} \;\le\; C(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon} + \|\phi\|_\infty \epsilon \log \frac{1}{\epsilon})$$

and using the previous lemma

$$\begin{aligned}|c_k| \int_{\Omega_\epsilon} \chi_k Z_{1k}^2 \eta_{2k} &\le C(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon} + \epsilon \log \frac{1}{\epsilon} C \log \frac{1}{\epsilon}(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon} + \sum_{j=1}^m |c_j|) \\ &\le C(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon} + C\epsilon \log \frac{1}{\epsilon}^2 \sum_{j=1}^m |c_j|)\end{aligned}$$

then, for ϵ sufficiently small

$$|c_k| \;\le\; C(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon})$$

With this we find finally

$$\|\phi\|_\infty \;\le\; C \log \frac{1}{\epsilon}(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon})$$

It only remains to prove the solvability assertion. To this purpose we consider the space

$$H = \left\{ \phi \in H^1(\Omega_\epsilon) : \; \phi \;\equiv\; 0 \; \text{ on } \Gamma_2^\epsilon \; \text{ and } \int_{\Omega_\epsilon} \chi_j Z_{1j} \phi = 0 \;\; \forall j = 1, \ldots, m \right\} \qquad (4.58)$$

endowed with the inner product $< \phi, \psi > = \int\limits_{\Omega_\epsilon} \nabla\phi\nabla\psi + \int\limits_{\Omega_\epsilon} \phi\psi$. Equation 4.52 is equivalent to find $\phi \in H$ such that

$$-\int\limits_{\Omega_\epsilon} \nabla\phi\nabla\psi + \int\limits_{\Omega_\epsilon} W\phi\psi + \int\limits_{\Gamma_1^\epsilon} h_2\psi + \int\limits_{\Gamma_1^\epsilon} W_2\phi\psi + \int\limits_{\Gamma_2^\epsilon} \frac{\partial\phi}{\partial\nu}\psi \ = \int\limits_{\Omega_\epsilon} h_1\psi \ \forall\, \psi \ \in \ H$$

$$\int\limits_{\Omega_\epsilon} \nabla\phi\nabla\psi + \int\limits_{\Omega_\epsilon} \phi\psi \ = \ \int\limits_{\Omega_\epsilon} (W+1)\phi\psi + \int\limits_{\Gamma_1^\epsilon} W_2\phi\psi - \int\limits_{\Omega_\epsilon} h_1\psi + \int\limits_{\Gamma_1^\epsilon} h_2\psi \tag{4.59}$$

where we define the linear operators $l_\phi(\psi)$ and $l_h(\psi)$ over H:

$$l_\phi(\psi) \ = \ \int\limits_{\Omega_\epsilon} (W+1)\phi\psi + \int\limits_{\Gamma_1^\epsilon} W_2\phi\psi \tag{4.60}$$

$$l_h(\psi) \ = \ -\int\limits_{\Omega_\epsilon} h_1\psi + \int\limits_{\Gamma_1^\epsilon} h_2\psi \tag{4.61}$$

Riesz's representation theorem implies that $l_\phi(\psi) \ = < \chi_\phi, \psi >, \ l_h(\psi) \ = < \tilde{h}, \psi >$, with χ_ϕ and $\tilde{h} \ \in H$.

So equation 4.59 is

$$< \phi, \psi > = < \chi_\phi, \psi > + < \tilde{h}, \psi > \quad \forall\, \psi \ \in \ H \tag{4.62}$$

In operator form

$$\phi \ = \ \chi_\phi + \tilde{h} \tag{4.63}$$

We define the operator $K(\phi) = \chi_\phi$ and we need to prove it's compactness.

Let $\{\phi_k\}_{k=1}^{\infty}$ be a bounded sequence in H. We want to prove that $\{K(\phi_k)\}_{k=1}^{\infty}$ has a subsequence that converges in H.

$$< \chi_{\phi_k}, \psi > = \int\limits_{\Omega_\epsilon} (W+1)\phi_k\psi + \int\limits_{\Gamma_1^\epsilon} W_2\phi_k\psi + \int\limits_{\Gamma_2^\epsilon} \frac{\partial\phi_k}{\partial\nu}\psi \ = \ l_{\phi_k}(\psi)$$

$$\|\chi_{\phi_k}\|_H^2 \ = < \chi_{\phi_k}, \chi_{\phi_k} > = \int\limits_{\Omega_\epsilon} (W+1)\phi_k\chi_{\phi_k} + \int\limits_{\Gamma_1^\epsilon} W_2\phi_k\chi_{\phi_k}$$

Using Holder and Poincare's inequalities

$$\|\chi_{\phi_k}\|_H^{\not 2} \leq C\|\phi_k\|_H \not{\|\chi_{\phi_k}\|_H}$$

since ϕ_k is bounded,

$$\Rightarrow \|\chi_{\phi_k}\|_H \leq C.$$

Since H is Hilbert, there exists a subsequence such that $\chi_{\phi_{k_j}} \rightharpoonup \chi_\phi$ (χ_ϕ appears because $\phi_{k_j} \rightharpoonup \phi$).

But,

$$\begin{aligned}
\|\chi_{\phi_{k_j}} - \chi_\phi\|_H^2 &= < \chi_{\phi_{k_j}} - \chi_\phi, \chi_{\phi_{k_j}} - \chi_\phi > \\
&= < \chi_{\phi_{k_j}}, \chi_{\phi_{k_j}} - \chi_\phi > - < \chi_\phi, \chi_{\phi_{k_j}} - \chi_\phi > \\
&= l_{\phi_{k_j}}(\chi_{\phi_{k_j}} - \chi_\phi) - \underbrace{l_\phi(\chi_{\phi_{k_j}} - \chi_\phi)}_{\longrightarrow 0}
\end{aligned}$$

The last term goes to cero because of the weak convergence. Now

$$\begin{aligned}
l_{\phi_{k_j}}(\chi_{\phi_{k_j}} - \chi_\phi) &= \int_{\Omega_\epsilon} (W+1)\phi_{k_j}(\chi_{\phi_{k_j}} - \chi_\phi) + \int_{\Gamma_1^\epsilon} W_2\phi_{k_j}(\chi_{\phi_{k_j}} - \chi_\phi) \\
&\leq C\|\phi_{k_j}\|_{L^2}\|\chi_{\phi_{k_j}} - \chi_\phi\|_{L^2} \\
&\leq C\underbrace{\|\phi_{k_j}\|_H}_{\text{bounded}}\|\chi_{\phi_{k_j}} - \chi_\phi\|_{L^2}
\end{aligned}$$

Rellich - Kondrachov theorem implies that the inyection of H^1 into L^2 is compact, and so $\chi_{\phi_{k_j}}$ converges in L^2.

$$\begin{aligned}
\Rightarrow \|\chi_{\phi_{k_j}} - \chi_\phi\|_H^2 &\longrightarrow 0 \text{ when } k_j \to \infty \\
\Rightarrow \chi_{\phi_{k_j}} &\longrightarrow \chi_\phi
\end{aligned}$$

$\therefore$ K is compact.

So we have $\phi = K(\phi) + \tilde{h}$, with K a compact operator.
Fredholm's alternative guarantees unique solvability for all $h_1,\ h_2$ provided that the homogeneous equation $\phi = K(\phi)$ has only the zero solution in H, which is obtained from the previous a priori estimate.
This finishes the proof. □

The previous result implies that the unique solution $\phi = T(h_1, h_2)$ of 4.52 defines a continuous linear map from the Banach space C_* of all functions $(h_1, h_2) \in L^\infty(\Omega_\epsilon) \times L^\infty(\Gamma_1^\epsilon)$ for which $\|h_1\|_{*,\Omega_\epsilon} < \infty$ and $\|h_2\|_{*,\Gamma_1^\epsilon} < \infty$, into L^∞.

It is important for later purposes to understand the differentiability of the operator T with respect to the variables $\bar{\xi}'_j$. Fix $(h_1, h_2) \in C_*$ and let $\phi = T(h_1, h_2)$. We want to compute derivatives of ϕ with respect to, say $\bar{\xi}'_k$. Formally $Z = \partial_{\bar{\xi}'_k} \phi$ should satisfy in Ω_ϵ the equation

$$
\begin{aligned}
\partial_{\bar{\xi}'_k}(\Delta\phi + W\phi) &= \partial_{\bar{\xi}'_k}(h_1 + \sum_{j=1}^{m} c_j \chi_j Z_{1j}) \\
\Leftrightarrow \ \Delta Z + \partial_{\bar{\xi}'_k} W\phi + WZ &= \sum_{j=1}^{m} (\partial_{\bar{\xi}'_k} c_j) \chi_j Z_{1j} + c_k \partial_{\bar{\xi}'_k}(\chi_k Z_{1k})
\end{aligned}
$$

Denominate $d_j = \partial_{\bar{\xi}'_k}(c_j)$

$$
\Rightarrow \ \Delta Z + WZ = -\partial_{\bar{\xi}'_k} W\phi + c_k \partial_{\bar{\xi}'_k}(\chi_k Z_{1k}) + \sum_{j=1}^{m} d_j \chi_j Z_{1j} \tag{4.64}
$$

Now on Γ_1^ϵ

$$
\begin{aligned}
\partial_{\bar{\xi}'_k}\left(\frac{\partial\phi}{\partial\nu} - W_2\phi\right) &= \partial_{\bar{\xi}'_k} h_2 = 0 \\
\frac{\partial Z}{\partial\nu} - W_2 Z &= \partial_{\bar{\xi}'_k} W_2 \phi
\end{aligned}
$$

and on Γ_2^ϵ

$$
\begin{aligned}
\partial_{\bar{\xi}'_k}\phi &= -\partial_{\bar{\xi}'_k} U(\epsilon y) \\
Z &= -\partial_{\bar{\xi}'_k} U(\epsilon y)
\end{aligned}
$$

The orthogonality conditions now become

$$
\begin{aligned}
\partial_{\bar{\xi}'_k} \int_{\Omega_\epsilon} \chi_j Z_{1j} \phi &= 0 \\
\Leftrightarrow \int_{\Omega_\epsilon} \chi_j Z_{1j} Z &= 0 \quad j \neq k \\
\int_{\Omega_\epsilon} \partial_{\bar{\xi}'_k}(\chi_k Z_{1k}) \phi &= -\int_{\Omega_\epsilon} \chi_k Z_{1k} Z
\end{aligned}
$$

Let us write $\widetilde{Z} = Z + b_k \chi_k Z_{1k}$, where

$$b_k \int_{\Omega_\epsilon} \chi_k^2 |Z_{1k}|^2 = \int_{\Omega_\epsilon} \phi \partial_{\bar{\xi}'_k} (\chi_k Z_{1k})$$

Hence,

$$\begin{aligned}
\int_{\Omega_\epsilon} \widetilde{Z} \chi_j Z_{1j} &= \int_{\Omega_\epsilon} Z \chi_j Z_{1j} + b_k \int_{\Omega_\epsilon} \chi_k Z_{1k} \chi_j Z_{1j} \\
&= \int_{\Omega_\epsilon} Z \chi_j Z_{1j} = 0 \ \text{ if } j \neq k \\
&= \int_{\Omega_\epsilon} Z \chi_j Z_{1j} + b_k \int_{\Omega_\epsilon} \chi_k^2 |Z_{1k}|^2 \\
&= \int_{\Omega_\epsilon} Z \chi_j Z_{1j} + \int_{\Omega_\epsilon} \partial_{\bar{\xi}'_k} (\chi_k Z_{1k}) \phi \\
&= 0 \ \text{ if } j = k
\end{aligned}$$

And,

$$\begin{aligned}
\Delta \widetilde{Z} + W \widetilde{Z} &= \Delta Z + WZ + b_k (\Delta(\chi_k Z_{1k}) + W \chi_k Z_{1k}) \\
&= -\partial_{\bar{\xi}'_k} W \phi + c_k \partial_{\bar{\xi}'_k} (\chi_k Z_{1k}) + b_k (\Delta(\chi_k Z_{1k}) + W \chi_k Z_{1k}) + \sum_{j=1}^{m} d_j \chi_j Z_{1j}
\end{aligned}$$

Define

$$a = -\partial_{\bar{\xi}'_k} W \phi + c_k \partial_{\bar{\xi}'_k} (\chi_k Z_{1k}) + b_k (\Delta(\chi_k Z_{1k}) + W \chi_k Z_{1k})$$

So

$$\Delta \widetilde{Z} + W \widetilde{Z} = a + \sum_{j=1}^{m} d_j \chi_j Z_{1j}$$

and

$$\frac{\partial \widetilde{Z}}{\partial \nu} - W_2 \widetilde{Z} = \frac{\partial Z}{\partial \nu} + b_k \left(\frac{\partial}{\partial \nu} (\chi_k Z_{1k}) - W_2 \chi_k Z_{1k} \right)$$

Definig

$$b = \partial_{\bar{\xi}'_k} W_2 \phi + b_k \left(\frac{\partial}{\partial \nu} (\chi_k Z_{1k}) - W_2 \chi_k Z_{1k} \right)$$

$$\Rightarrow \frac{\partial \widetilde{Z}}{\partial \nu} - W_2 \widetilde{Z} = b$$

We need bounds for a and b.

The first term $\partial_{\bar{\xi}'_k} W\phi$,

$$\begin{aligned}
\|\partial_{\bar{\xi}'_k} W\phi\|_{*,\Omega_\epsilon} &= \sup_{y\in\Omega_\epsilon} \frac{|\partial_{\bar{\xi}'_k} W||\phi|}{\epsilon^2 + \sum(1+|y-\xi'_j|)^{-2-\sigma}} \\
&\leq \|\phi\|_\infty \|\partial_{\bar{\xi}'_k} W\|_{*,\Omega_\epsilon} \\
&\leq C\log\frac{1}{\epsilon}(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon})\|\partial_{\bar{\xi}'_k} W\|_{*,\Omega_\epsilon}
\end{aligned}$$

The term $\|\partial_{\bar{\xi}'_k} W\|_{*,\Omega_\epsilon}$ is clearly bounded because $W(\bar{\xi}'_k) = O(\frac{1}{(1+|\bar{\xi}'_k|^2)^2})$, so

$$\partial_{\bar{\xi}'_k} W = O(\frac{1}{(1+r^2)^3} + \frac{r^2}{(1+r^2)^3})$$

and multiplied by the weight of the $\|\|_{*,\Omega_\epsilon}$ norm

$$= O(\frac{1}{(1+r^2)^3(\epsilon^2 + \frac{1}{(1+r)^{2\sigma}})} + \frac{r^2}{(1+r^2)^3(\epsilon^2 + \frac{1}{(1+r)^{2\sigma}})}) = O(\frac{1}{r^{3-\sigma}} + \frac{1}{r^{1-\sigma}})$$

So it is bounded.

$$\|\partial_{\bar{\xi}'_k} W\|_{*,\Omega_\epsilon} \leq C\log\frac{1}{\epsilon}(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon})$$

For the second term $c_k\partial_{\bar{\xi}'_k}(\chi_k Z_{1k})$ we have $|c_k| \leq C(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon})$.

$$\partial_{\bar{\xi}'_k}(\chi_k Z_{1k}) = \partial_{\bar{\xi}'_k}\chi_k Z_{1k} + \chi_k \partial_{\bar{\xi}'_k} Z_{1k}$$

but $\partial_{\bar{\xi}'_k}\chi_k \neq 0$ only in $R_0 < r < R_0 + 1$ around y.

$$\begin{aligned}
Z_{1k}(\vec{x} - \xi'_k) &= \frac{-4x_1}{\mu_k^2 + |x-\xi'_k|^2} \\
\partial_{\bar{\xi}'_k} Z_{1k} &= \frac{-8x_1(x-\xi'_k)}{(\mu_k^2 + |x-\xi'_k|^2)^2} + O(\epsilon)
\end{aligned}$$

So,

$$\begin{aligned}
\|c_k\partial_{\bar{\xi}'_k}(\chi_k Z_{1k})\|_{*,\Omega_\epsilon} &\leq C(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon})\underbrace{\|\partial_{\bar{\xi}'_k}(\chi_k Z_{1k})\|_{*,\Omega_\epsilon}}_{\text{bounded}} \\
&\leq C(\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon})
\end{aligned}$$

Let us find an estimate now for b_k.

$$
\begin{aligned}
b_k \int\limits_{\Omega_\epsilon} \chi_k^2 |Z_{1k}|^2 &= \int\limits_{\Omega_\epsilon} \phi \partial_{\bar{\xi}_k'} (\chi_k Z_{1k}) \\
&\leq C \log \frac{1}{\epsilon} (\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) \\
\Rightarrow \ |b_k| &\leq C \log \frac{1}{\epsilon} (\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon})
\end{aligned}
$$

And so the third term

$$
b_k(\Delta(\chi_k Z_{1k}) + W \chi_k Z_{1k}) \ \leq \ C \log \frac{1}{\epsilon} (\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon})
$$

obtaining

$$
\|a\|_{*,\Omega_\epsilon} \ \leq \ C \log \frac{1}{\epsilon} (\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) \tag{4.65}
$$

And in a similar fashion

$$
\|b\|_{*,\Gamma_1^\epsilon} \ \leq \ C \log \frac{1}{\epsilon} (\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) \tag{4.66}
$$

With these results and the previous lemma we have

$$
\begin{aligned}
\|\widetilde{Z}\|_{L^\infty(\Omega_\epsilon)} &\leq C \log \frac{1}{\epsilon} (\|a\|_{*,\Omega_\epsilon} + \|b\|_{*,\Gamma_1^\epsilon}) \\
\Rightarrow \ \|Z\|_{L^\infty(\Omega_\epsilon)} &\leq C \log \frac{1}{\epsilon}^2 (\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) - \|b_k \chi_k Z_{1k}\|_{L^\infty(\Omega_\epsilon)}
\end{aligned}
$$

$$
\therefore \ \|\partial_{\bar{\xi}_k'} \phi\|_{L^\infty(\Omega_\epsilon)} \ \leq \ C \log \frac{1}{\epsilon}^2 (\|h_1\|_{*,\Omega_\epsilon} + \|h_2\|_{*,\Gamma_1^\epsilon}) \tag{4.67}
$$

Chapter 5

The nonlinear problem

Consider the nonlinear equation

$$\begin{cases} \Delta\phi + W\phi \;=\; -R_1(y) + N_1(\phi) + \sum_{j=1}^{m} c_j\chi_j Z_{1j} \;\; \text{in } \Omega_\epsilon \\ \dfrac{\partial\phi}{\partial\nu} - W_2\dfrac{\phi}{2} \;=\; R_2(y) + N_2(\phi) \;\; \text{on } \Gamma_1^\epsilon \\ \phi \;=\; 0 \;\; \text{on } \Gamma_2^\epsilon \\ \int\limits_{\Omega_\epsilon} \chi_j Z_{1j}\phi \;=\; 0 \;\forall j = 1,\ldots,m. \end{cases} \tag{5.1}$$

Lemma 5.0.2. *Let $m > 0,\ d > 0$. Then there exist $\epsilon_0 > 0,\ C > 0$ such that for $0 < \epsilon < \epsilon_0$ and any $\bar\xi_1,\ldots,\bar\xi_m \;\in \Gamma_1^\epsilon$ satisfying*

$$|\bar\xi_i - \bar\xi_j| \geq \delta \;\; \forall i \neq j$$

the problem 5.1 admits a unique solution $\phi,\ c_1,\ldots,\ c_m$ such that

$$\|\phi\|_{L^\infty(\Omega_\epsilon)} \;\leq\; C\epsilon^\alpha \tag{5.2}$$

where α is any number in the interval $(0,1)$.
Furthermore, the function $\bar\xi \;\rightarrow\; \phi(\bar\xi) \in C(\bar\Omega_\epsilon)$ is C^1 and

$$\|D_{\bar\xi'}\phi\|_{L^\infty(\Omega_\epsilon)} \;\leq\; C\epsilon^\alpha \tag{5.3}$$

Proof. In terms of the operator T defined in the previous section, problem 5.1 becomes

$$\phi \;=\; T(-R_1 + N_1(\phi), R_2 + N_2(\phi)) \;\equiv\; A(\phi)$$

For a given number $\gamma > 0$, let us consider the region

$$F_\gamma \;\equiv\; \{\phi \in C(\bar\Omega_\epsilon) :\; \|\phi\|_{L^\infty(\Omega_\epsilon)} \leq \gamma\epsilon^\alpha\} \tag{5.4}$$

From proposition 4.3.1, we get

$$\begin{aligned}\|A(\phi)\|_{L^\infty(\Omega_\epsilon)} = \|\phi\|_{L^\infty(\Omega_\epsilon)} &\leq C\log\frac{1}{\epsilon}(\|R_1+N_1(\phi)\|_{*,\Omega_\epsilon}+\|R_2+N_2(\phi)\|_{*,\Gamma_1^\epsilon})\\ &\leq C\log\frac{1}{\epsilon}(\|R_1\|_{*,\Omega_\epsilon}+\|N_1(\phi)\|_{*,\Omega_\epsilon}+\|R_2\|_{*,\Gamma_1^\epsilon}+\|N_2(\phi)\|_{*,\Gamma_1^\epsilon})\end{aligned}$$

we have that $|R_1(y)| \leq C\epsilon\sum_{j=1}^m \frac{1}{1+|y-\xi_j'|^3}$, and so

$$\begin{aligned}\|R_1\|_{*,\Omega_\epsilon} &= \sup_{y\in\Omega_\epsilon}\frac{|R_1|}{\epsilon^2+\sum_{j=1}^m\frac{1}{(1+|y-\xi_j'|)^{2+\sigma}}}\\ &\leq C\epsilon\end{aligned}$$

Also the definition of N_1 inmediately yields $\|N_1(\phi)\|_{*,\Omega_\epsilon} \leq C\|\phi\|^2_{L^\infty(\Omega_\epsilon)}$, and for $\phi_1,\ \phi_2 \in F_\gamma$

$$\begin{aligned}N_1(\phi_1)-N_1(\phi_2) &= e^V(e^{\phi_1}-1-\phi_1-e^{\phi_2}+1+\phi_2)\\ &\leq Ce^V(\phi_1^2-\phi_2^2)\\ &\leq Ce^V(\phi_1+\phi_2)(\phi_1-\phi_2)\\ \Rightarrow \|N_1(\phi_1)-N_1(\phi_2)\|_{*,\Omega_\epsilon} &\leq C\gamma\epsilon^\alpha\|\phi_1-\phi_2\|_{L^\infty(\Omega_\epsilon)}\end{aligned}$$

Analogously,

$$\|R_2\|_{*,\Gamma_1^\epsilon} \leq C\epsilon^{1-\sigma}$$

and from the definition of N_2

$$\|N_2(\phi)\|_{*,\Gamma_1^\epsilon} \leq C\|\phi\|^2_{L^\infty(\Omega_\epsilon)}$$

also for $\phi_1,\ \phi_2 \in F_\gamma$

$$\begin{aligned}N_2(\phi_1)-N_2(\phi_2) &= e^{V/2}(e^{\phi_1/2}-1-\phi_1/2-e^{\phi_2/2}+\\ &\quad 1+\phi_2/2)\\ &\leq Ce^{V/2}(\frac{\phi_1^2}{4}-\frac{\phi_2^2}{4})\\ \Rightarrow \|N_2(\phi_1)-N_2(\phi_2)\|_{*,\Gamma_1^\epsilon} &\leq C\gamma\epsilon^\alpha\|\phi_1-\phi_2\|_{L^\infty(\Omega_\epsilon)}\end{aligned}$$

Hence, choosing σ so that $1-\sigma>\alpha$,

$$\begin{aligned}\|A(\phi)\|_{L^\infty(\Omega_\epsilon)} &\leq C\log\frac{1}{\epsilon}(C\epsilon+C\|\phi\|^2_{L^\infty(\Omega_\epsilon)}+C\epsilon^{1-\sigma}+C\|\phi\|^2_{L^\infty(\Omega_\epsilon)})\\ &\leq C\log\frac{1}{\epsilon}(\epsilon^{1-\sigma}+\gamma^2\epsilon^{2\alpha})\\ &\leq C\epsilon^\alpha\log\frac{1}{\epsilon}(\epsilon^{1-\sigma-\alpha}+\gamma^2\epsilon^\alpha)\\ &\leq \gamma\epsilon^\alpha\end{aligned}$$

Now $\phi_1 - \phi_2$ satisfy

$$\begin{aligned} \Delta(\phi_1-\phi_2)+W(\phi_1-\phi_2) &= N_1(\phi_1)-N_1(\phi_2) \quad \text{in } \Omega_\epsilon \\ \frac{\partial(\phi_1-\phi_2)}{\partial\nu}-W_2(\phi_1-\phi_2) &= N_2(\phi_1)-N_2(\phi_2) \quad \text{on } \Gamma_1^\epsilon \\ \phi_1-\phi_2 &= 0 \quad \text{on } \Gamma_2^\epsilon \\ \int_{\Omega_\epsilon}\chi_j Z_{1j}(\phi_1-\phi_2) &= 0 \ \ \forall j=1,\dots,m. \end{aligned}$$

$$\begin{aligned} \|A(\phi_1)-A(\phi_2)\|_{L^\infty(\Omega_\epsilon)} &\le C\log\frac{1}{\epsilon}(\|N_1(\phi_1)-N_1(\phi_2)\|_{*,\Omega_\epsilon}+\|N_2(\phi_1)-N_2(\phi_2)\|_{*,\Gamma_1^\epsilon}) \\ &\le C\log\frac{1}{\epsilon}(\gamma\epsilon^\alpha\|\phi_1-\phi_2\|_{L^\infty(\Omega_\epsilon)}+\gamma\epsilon^\alpha\|\phi_1-\phi_2\|_{L^\infty(\Omega_\epsilon)}) \\ &\le \gamma\epsilon^\alpha|\log\epsilon|\|\phi_1-\phi_2\|_{L^\infty(\Omega_\epsilon)} \end{aligned}$$

It follows that for all sufficiently small ϵ we get that A is a contraction mapping of F_γ, and therefore there exists a unique fixed point of A in this region.

Let us now discuss the differentiability of ϕ.

Since $R_1,\ R_2$ depends continuously (in the * norm) on the m-tuple $\bar{\xi}' = (\bar{\xi}'_1,\dots,\bar{\xi}'_m)$, the fixed point characterization yields so for the map $\bar{\xi}' \longrightarrow \phi$.

Then, formally,

$$\begin{aligned} \partial_{\bar{\xi}'_k}N_1(\phi) &= \partial_{\bar{\xi}'_k}(W(e^\phi-1-\phi)) \\ &= \partial_{\bar{\xi}'_k}W(e^\phi-1-\phi)+W(e^\phi-1)\partial_{\bar{\xi}'_k}\phi \end{aligned}$$

Since $\|\partial_{\bar{\xi}'_k}W\|_{*,\Omega_\epsilon}$ is uniformly bounded, we conclude

$$\begin{aligned} \Rightarrow \ \|\partial_{\bar{\xi}'_k}N_1(\phi)\|_{*,\Omega_\epsilon} &\le C(\|\phi\|^2_{L^\infty(\Omega_\epsilon)}+\|\phi\|_{L^\infty(\Omega_\epsilon)}\|\partial_{\bar{\xi}'_k}\phi\|_{L^\infty(\Omega_\epsilon)}) \\ &\le C(\epsilon^{2\alpha}+\epsilon^\alpha\|\partial_{\bar{\xi}'_k}\phi\|_{L^\infty(\Omega_\epsilon)}) \end{aligned} \tag{5.5}$$

Similarly,

$$\begin{aligned} \partial_{\bar{\xi}'_k}N_2(\phi) &= \partial_{\bar{\xi}'_k}(W_2(e^{\phi/2}-1-\frac{\phi}{2})) \\ &= \partial_{\bar{\xi}'_k}W_2(e^{\phi/2}-1-\frac{\phi}{2})+\frac{W_2}{2}(e^{\phi/2}-1)\partial_{\bar{\xi}'_k}\phi \\ \Rightarrow \ \|\partial_{\bar{\xi}'_k}N_2(\phi)\|_{*,\Gamma_1^\epsilon} &\le \|\partial_{\bar{\xi}'_k}W_2\|_{*,\Gamma_1^\epsilon}\|\phi\|^2_{L^\infty(\Omega_\epsilon)}+C\|W_2\|_{*,\Gamma_1^\epsilon}\|\phi\|_{L^\infty(\Omega_\epsilon)}\|\partial_{\bar{\xi}'_k}\phi\|_{*,\Gamma_1^\epsilon} \\ &\le C(\|\phi\|_{L^\infty(\Omega_\epsilon)}+\|\partial_{\bar{\xi}'_k}\phi\|_{*,\Gamma_1^\epsilon})\|\phi\|_{L^\infty(\Omega_\epsilon)} \\ &\le C(\epsilon^\alpha+\|\partial_{\bar{\xi}'_k}\phi\|_{*,\Gamma_1^\epsilon})\epsilon^\alpha \end{aligned} \tag{5.6}$$

And observe that we have

$$\partial_{\bar{\xi}'_k}\phi \;=\; \partial_{\bar{\xi}'_k}(T(-R_1+N_1(\phi),R_2+N_2(\phi)))$$

but $T \;=\; T(\bar{\xi}'_k,(-R_1+N_1(\phi),R_2+N_2(\phi)))$, then

$$\begin{aligned}\partial_{\bar{\xi}'_k}\phi \;=\;& (\partial_{\bar{\xi}'_k}T)(-R_1+N_1(\phi),R_2+N_2(\phi))+T(\partial_{\bar{\xi}'_k}(-R_1+N_1(\phi)),R_2+N_2(\phi))\\ &+T(-R_1+N_1(\phi),\partial_{\bar{\xi}'_k}(R_2+N_2(\phi)))\end{aligned}$$

For the first term we will use the bound 4.67 and for the second and the third we will use proposition 4.3.1.

$$\begin{aligned}\|\partial_{\bar{\xi}'_k}\phi\|_{L^\infty(\Omega_\epsilon)} \;\le\;& C\log\epsilon^2(\|-R_1+N_1(\phi)\|_{*,\Omega_\epsilon}+\|R_2+N_2(\phi)\|_{*,\Gamma_1^\epsilon})\\ &+C\log\frac{1}{\epsilon}(\|\partial_{\bar{\xi}'_k}(-R_1+N_1(\phi))\|_{*,\Omega_\epsilon}+\|R_2+N_2(\phi)\|_{*,\Gamma_1^\epsilon})\\ &+C\log\frac{1}{\epsilon}(\|-R_1+N_1(\phi)\|_{*,\Omega_\epsilon}+\|\partial_{\bar{\xi}'_k}(R_2+N_2(\phi))\|_{*,\Gamma_1^\epsilon})\end{aligned}$$

for ϵ small enough

$$\begin{aligned}\|\partial_{\bar{\xi}'_k}\phi\|_{L^\infty(\Omega_\epsilon)} \;\le\;& C\log\frac{1}{\epsilon}[\log\frac{1}{\epsilon}(\|-R_1+N_1(\phi)\|_{*,\Omega_\epsilon}+\|R_2+N_2(\phi)\|_{*,\Gamma_1^\epsilon})\\ &+\|\partial_{\bar{\xi}'_k}R_1\|_{*,\Omega_\epsilon}+\|\partial_{\bar{\xi}'_k}N_1(\phi)\|_{*,\Omega_\epsilon}\\ &+\|\partial_{\bar{\xi}'_k}R_2\|_{*,\Gamma_1^\epsilon}+\|\partial_{\bar{\xi}'_k}N_2(\phi)\|_{*,\Gamma_1^\epsilon}]\end{aligned}$$

Since it is easily checked that $\|\partial_{\bar{\xi}'_k}R_1\|_{*,\Omega_\epsilon} \;\le\; C\epsilon$ and $\|\partial_{\bar{\xi}'_k}R_2\|_{*,\Gamma_1^\epsilon} \;\le\; C\epsilon^\alpha$

$$\begin{aligned}\Rightarrow\;\|\partial_{\bar{\xi}'_k}\phi\|_{L^\infty(\Omega_\epsilon)} \;\le\;& C\log\frac{1}{\epsilon}(\epsilon\log\frac{1}{\epsilon}+\log\frac{1}{\epsilon}\|\phi\|^2_{L^\infty(\Omega_\epsilon)}+\log\frac{1}{\epsilon}\epsilon^\alpha+\\ &+\log\frac{1}{\epsilon}\|\phi\|^2_{L^\infty(\Omega_\epsilon)}+\epsilon+\epsilon^\alpha\|\partial_{\bar{\xi}'_k}\phi\|_{L^\infty(\Omega_\epsilon)})\\ \le\;& C\log\frac{1}{\epsilon}(\epsilon^\alpha\log\frac{1}{\epsilon}+\epsilon^\alpha\|\partial_{\bar{\xi}'_k}\phi\|_{L^\infty(\Omega_\epsilon)})\\ (1-C\epsilon^\alpha\log\frac{1}{\epsilon})\|\partial_{\bar{\xi}'_k}\phi\|_{L^\infty(\Omega_\epsilon)} \;\le\;& C\epsilon^\alpha\log\frac{1}{\epsilon}^2\\ \Rightarrow\;\|\partial_{\bar{\xi}'_k}\phi\|_{L^\infty(\Omega_\epsilon)} \;\le\;& C\epsilon^\alpha\log\frac{1}{\epsilon}^2\end{aligned}$$

The computation can be made rigorous by means of the implicit function theorem.
Let us consider the differentiable function

$$\begin{aligned}F:\;(\mathbb{R}^2)^m\times C(\Omega_\epsilon) \;&\longrightarrow\; C(\Omega_\epsilon)\\ (\bar{\xi},\phi) \;&\longrightarrow\; \phi-A_{\bar{\xi}}(\phi)\end{aligned}$$

Taking the solution ϕ from the previous lemma we have that

$$F(\bar{\xi}, \phi_{\bar{\xi}}) = 0$$

So we only need the invertibility of $\partial_\phi F(\phi_{\bar{\xi}})$ to justify the bound of $\partial_{\bar{\xi}'_k}\phi$.

$$\begin{aligned}
\partial_\phi F(\phi_{\bar{\xi}})[h] &= \partial_\phi(\phi - A_{\bar{\xi}}(\phi))[h] \\
&= h - T(W(e^{\phi} - 1)h, R_2 + N_2(\phi)) - T(-R_1 + N_1(\phi), \frac{W_2}{2}(e^{\phi/2} - 1)h) \\
&= h - \bar{T}(h) \\
&= (I - \bar{T})[h]
\end{aligned}$$

by the previous estimates $\|\bar{T}(h)\|_{L^\infty(\Omega_\epsilon)} \leq C\epsilon^{2\alpha} \log \frac{1}{\epsilon} \|h\|_{L^\infty(\Omega_\epsilon)}$, so taking ϵ sufficiently small

$$\|\bar{T}\|_{L(C(\Omega_\epsilon), C(\Omega_\epsilon))} < 1$$

and $\partial_\phi F(\phi_{\bar{\xi}})$ is invertible. □

Chapter 6

Variational Reduction

After the nonlinear problem has been solved we will find solutions to the full problem if we manage to adjust the m-tuple $\bar{\xi}$ in such a way that $c_j(\bar{\xi}) = 0$ for all $j = 1, \ldots, m$.

In view of lemma 5.0.2, given $\bar{\xi} = (\bar{\xi}_1, \ldots, \bar{\xi}_m) \in (\Gamma_1^\epsilon)^m$ satisfying $|\bar{\xi}_i - \bar{\xi}_j| \geq \delta \; \forall \, i \neq j$, we define $\phi(\bar{\xi})$ and $c_j(\bar{\xi})$ to be the unique solution to the nonlinear problem and the corresponding bound.

We write $U(\bar{\xi}) \; = \; \sum_{j=1}^{m} u_j(x) + H_j^\epsilon(x)$ as the ansatz previously used.

Let us define the energy functional in $H^1(\Omega)$

$$J_\epsilon(u) \;=\; \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \epsilon^2 \int_\Omega e^u \, dx - 2\epsilon \int_{\Gamma_1} e^{u/2} \, ds \tag{6.1}$$

J_ϵ is a C^1 functional in $H^1(\Omega)$ since the two dimensional Moser-Trudinger inequality implies the following bounds

For any $\alpha \in \mathbb{R}$ there exist two constants, $C > 0$ and $\beta > 0$ such that

$$\int_\Omega e^{\alpha v} \, dx \;\leq\; C e^{\beta \|v\|^2_{H^1(\Omega)}}$$

$$\int_{\partial\Omega} e^{\alpha v} \, d\sigma \;\leq\; C e^{\beta \|v\|^2_{H^1(\Omega)}}$$

An outline for the proof of this results can be found in [KV03].

Set

$$F_\epsilon(\bar{\xi}) \;=\; J_\epsilon(U(\bar{\xi}) + \tilde{\phi}(\bar{\xi})) \tag{6.2}$$

where $\tilde{\phi}(\bar{\xi})(x) = \phi(\bar{\xi})(\frac{x}{\epsilon})\ \ x \in \Omega$.

Lemma 6.0.3. *If $\bar{\xi} = (\bar{\xi}_1, \ldots, \bar{\xi}_m) \in (\Gamma_1^\epsilon)^m$ satisfying 4.51 is a critical point of F_ϵ then $u = U(\bar{\xi}) + \tilde{\phi}(\bar{\xi})$ is a critical point of J_ϵ, that is, a solution to 1.1.*

Proof. Let

$$I_\epsilon(v) \;=\; \frac{1}{2}\int_{\Omega_\epsilon} |\nabla v|^2\,dy - \int_{\Omega_\epsilon} e^v\,dy - 2\int_{\Gamma_1^\epsilon} e^{v/2}\,dy \tag{6.3}$$

Then $F_\epsilon(\bar{\xi}) = J_\epsilon(U(\bar{\xi}) + \tilde{\phi}(\bar{\xi})) = I_\epsilon(V(\bar{\xi}') + \phi(\bar{\xi}'))$. Therefore

$$\begin{aligned}
\frac{\partial F_\epsilon(\bar{\xi})}{\partial \bar{\xi}_k} &= \frac{1}{\epsilon}\frac{\partial}{\partial \bar{\xi}_k'} I_\epsilon(V(\bar{\xi}') + \phi(\bar{\xi}')) \\
&= \frac{1}{\epsilon} DI_\epsilon(V(\bar{\xi}') + \phi(\bar{\xi}'))[\frac{\partial V(\bar{\xi}')}{\partial \bar{\xi}_k'} + \frac{\partial \phi(\bar{\xi}')}{\partial \bar{\xi}_k'}] \\
&= \frac{1}{\epsilon}\sum_{j=1}^m \int_{\Omega_\epsilon} c_j \chi_j Z_{1j}(\frac{\partial V}{\partial \bar{\xi}_k'} + \frac{\partial \phi}{\partial \bar{\xi}_k'})
\end{aligned} \tag{6.4}$$

with $\dfrac{\partial}{\partial \bar{\xi}_k}$ a tangential derivative on the boundary Γ_1.

Let us remember that $V(y) = U(\epsilon y) + 4\log\epsilon$

$$\begin{aligned}
\Rightarrow\ V(y) &= \sum_{j=1}^m u_j(\epsilon y) + H_j(\epsilon y) + 4\log\epsilon \\
&\text{also} \\
z_{1j}(y) &= \frac{\partial}{\partial \zeta_1} v_j(z - \zeta_1)|_{\zeta_1=0} \quad \text{with } z = y - \xi_j' \\
&= \frac{\partial}{\partial \zeta_1}[\log \frac{8\mu_j^2}{(\mu_j^2 + (z_1+\zeta_1)^2 + z_2^2)^2}]|_{\zeta_1=0}
\end{aligned}$$

and $Z_{1k} \;=\; z_{1k}(F_j^\epsilon(y))$

So,

$$\begin{aligned}
\frac{\partial V(\bar{\xi})}{\partial \bar{\xi}_k'} &= \sum_{j=1}^m \frac{\partial v_j(y)}{\partial \bar{\xi}_k'} + \frac{\partial H_j(\epsilon y)}{\partial \bar{\xi}_k'} \\
&= \frac{\partial v_k(y)}{\partial \bar{\xi}_k'} + \frac{\partial H_k(\epsilon y)}{\partial \bar{\xi}_k'} \\
&= \pm Z_{1k}(y) + \frac{\partial H_k(\epsilon y)}{\partial \bar{\xi}_k'}
\end{aligned}$$

What is the order of $\frac{\partial H_k(\epsilon y)}{\partial \bar{\xi}'_k}$?

$$
\begin{aligned}
\frac{\partial H_k(\epsilon y)}{\partial \bar{\xi}'_k} &= \frac{\partial H_k(\epsilon y)}{\partial \bar{\xi}_k}\frac{\partial \bar{\xi}_k}{\partial \bar{\xi}'_k} \\
&= \epsilon \underbrace{\frac{\partial H_k(\epsilon y)}{\partial \bar{\xi}_k}}_{regular function}
\end{aligned}
$$

$$
\Rightarrow \frac{\partial V(\bar{\xi})}{\partial \bar{\xi}'_k} = \pm Z_{1k}(y) + \epsilon O(1)
$$

We use also from lemma 5.0.2 that

$$
\|\partial_{\bar{\xi}'_k} \phi\|_{L^\infty(\Omega_\epsilon)} \leq C\epsilon^\alpha
$$

And so, since $\bar{\xi}$ is a critical point of F_ϵ we found from equation 6.4 that

$$
\sum_{j=1}^{m} \int_{\omega_\epsilon} c_j \chi_j Z_{1j}(\pm Z_{1k} + \epsilon O(1)) = 0 \quad \forall k = 1, \ldots, m,
$$

which, if ϵ is small enough, is a diagonal dominant system of equations for c_j.

This implies that $c_j = 0 \, \forall \, i = 1, \ldots, m$. □

So now we have to study critical points of F_ϵ, which is $F_\epsilon(\bar{\xi}) = J_\epsilon(U(\bar{\xi}) + \tilde{\phi}(\bar{\xi}))$.

We notice that F_ϵ is a perturbation by $\tilde{\phi}$ of $J_\epsilon(U)$ and we will analyze in the following lemma that relationship.

Lemma 6.0.4. *The following expansion holds*

$$
F_\epsilon(\bar{\xi}) = J_\epsilon(U) + \Theta_\epsilon(\bar{\xi})
$$

where $|\Theta_\epsilon| + |\nabla\Theta_\epsilon| \longrightarrow 0$ *uniformly on points satisfying the constrain 4.51.*

Proof. Let $\tilde{\Theta}_\epsilon(\bar{\xi}') = I_\epsilon(V + \phi) - I_\epsilon(V)$. Remember that $DI_\epsilon(V + \phi)[\phi] = 0$.

$$
I_\epsilon(V + \phi) = I_\epsilon(V) + \cancel{DI_\epsilon(V + \phi)[\phi]} + \int_0^1 D^2 I_\epsilon(V + t\phi)[\phi]^2(1 - t)\, dt
$$

Let us calculate $D^2 I_\epsilon(v)[h_1, h_2]$:

$$
\begin{aligned}
I_\epsilon(v) &= \frac{1}{2}\int\limits_{\Omega_\epsilon} |\nabla v|^2\, dy - \int\limits_{\Omega_\epsilon} e^v\, dy - 2\int\limits_{\Gamma_1^\epsilon} e^{v/2}\, dy \\
DI_\epsilon(v)[h] &= \int\limits_{\Omega_\epsilon} \nabla v \nabla h\, dy - \int\limits_{\Omega_\epsilon} e^v h\, dy - \int\limits_{\Gamma_1^\epsilon} e^{v/2} h\, dy \\
D^2 I_\epsilon(v)[h_1, h_2] &= \int\limits_{\Omega_\epsilon} \nabla h_1 \nabla h_2\, dy - \int\limits_{\Omega_\epsilon} e^v h_1 h_2\, dy - \int\limits_{\Gamma_1^\epsilon} e^{v/2} h_1 \frac{h_2}{2}\, dy \\
&= -\int\limits_{\Omega_\epsilon} \Delta h_1 h_2\, dy + \int\limits_{\partial\Omega_\epsilon} \frac{\partial h_1}{\partial \nu} h_2\, dy - \int\limits_{\Omega_\epsilon} e^v h_1 h_2\, dy - \int\limits_{\Gamma_1^\epsilon} e^{v/2} h_1 \frac{h_2}{2}\, dy
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\ I_\epsilon(V+\phi) - I_\epsilon(V) &= \int_0^1 \Bigg(-\int\limits_{\Omega_\epsilon} (\Delta\phi\phi + e^{V+t\phi}\phi^2) + \int\limits_{\partial\Omega_\epsilon} \frac{\partial \phi}{\partial \nu}\phi - \\
&\qquad \int\limits_{\Gamma_1^\epsilon} e^{\frac{V+t\phi}{2}} \frac{\phi^2}{2} \Bigg) (1-t)\, dt \\
&= \int_0^1 \Bigg(-\int\limits_{\Omega_\epsilon} (\Delta\phi + W\phi)\phi\, dy - \int\limits_{\Omega_\epsilon} (e^{V+t\phi} - e^V)\phi^2 + \\
&\qquad \int\limits_{\Gamma_1^\epsilon} (\frac{\partial \phi}{\partial \nu} - W_2 \frac{\phi}{2})\phi + \int\limits_{\Gamma_2^\epsilon} \frac{\partial \phi}{\partial \nu}\phi \\
&\qquad - \int\limits_{\Gamma_1^\epsilon} (e^{\frac{V+t\phi}{2}} - e^{V/2}) \frac{\phi^2}{2} \Bigg) (1-t)\, dt \\
&= \int_0^1 \Bigg(-\int\limits_{\Omega_\epsilon} (N_1(\phi) - R_1)\phi\, dy - \int\limits_{\Omega_\epsilon} e^V (e^{t\phi} - 1)\phi^2\, dy \\
&\qquad + \int\limits_{\Gamma_1^\epsilon} (R_2 + N_2(\phi))\phi\, dy - \int\limits_{\Gamma_1^\epsilon} e^{V/2}(e^{t\phi/2} - 1)\frac{\phi^2}{2}\, dy + \\
&\qquad \int\limits_{\Gamma_2^\epsilon} \frac{\partial \phi}{\partial \nu}\phi \Bigg) (1-t)\, dt
\end{aligned}
$$

We know that $\|\phi\|_{L^\infty(\Omega_\epsilon)} \leq C\epsilon^\alpha |\log\epsilon|$, so

$$= \int_0^1 \left(-\int\limits_{\Omega_\epsilon} (N_1(\phi) - R_1)\phi\, dy + \int\limits_{\Gamma_1^\epsilon} (R_2 + N_2(\phi))\phi\, dy \right)(1-t)\, dt + O(\epsilon^{2\alpha}|\log\epsilon|^3)$$

Let us see the bounds for $N_1(\phi),\ R_1,\ N_2(\phi)$ and R_2

$$\begin{aligned} |R_1(y)| &\leq C\epsilon \sum_{j=1}^m \frac{1}{1+|y-\xi_j'|^3} \\ |R_2(y)| &\leq C\epsilon \sum_{j=1}^m \frac{1}{1+|y-\xi_j'|} \\ \|N_1(\phi)\|_{*,\Omega_\epsilon} &\leq C\|\phi\|^2_{L^\infty(\Omega_\epsilon)} \\ \|N_2(\phi)\|_{*,\Gamma_1^\epsilon} &\leq C\|\phi\|^2_{L^\infty(\Omega_\epsilon)} \end{aligned}$$

Then

$$\begin{aligned} I_\epsilon(V+\phi) - I_\epsilon(V) \ &\leq \int_0^1 \left(\int\limits_{\Omega_\epsilon} f(y)\phi^3 + g(y)\epsilon\phi - \int\limits_{\Gamma_1^\epsilon} \epsilon g(y)\phi + f(y)\phi^3 \right)(1-t)\, dt + \\ &+O(\epsilon^{2\alpha}\log\epsilon^3) \end{aligned}$$

with f and g regular functions that comes from the weights of the * norms.

$$= \ O(\epsilon^{2\alpha}\log\epsilon^3)$$

Let us differentiate with respect to $\bar{\xi}_k'$

$$\begin{aligned} \partial_{\bar{\xi}_k'}[I_\epsilon(V+\phi) - I_\epsilon(V)] \ &= \int_0^1 \Bigg(-\int\limits_{\Omega_\epsilon} \partial_{\bar{\xi}_k'}[(N_1(\phi) - R_1)\phi] - \int\limits_{\Omega_\epsilon} \partial_{\bar{\xi}_k'}[e^V(1-e^{t\phi})\phi^2] \\ &+ \int\limits_{\Gamma_1^\epsilon} \partial_{\bar{\xi}_k'}[(R_2 + N_2(\phi))\phi] - \int\limits_{\Gamma_1^\epsilon} \partial_{\bar{\xi}_k'}[e^{V/2}(e^{t\phi/2}-1)\frac{\phi^2}{2}] + \\ &\int\limits_{\Gamma_2^\epsilon} \partial_{\bar{\xi}_k'}[\frac{\partial\phi}{\partial\nu}\phi] \Bigg)(1-t)dt \end{aligned}$$

Using the bounds for $\|\partial_{\bar{\xi}_k'} N_1(\phi)\|_{*,\Omega_\epsilon} \leq C(\epsilon^2\log\epsilon^2 + \epsilon\log\epsilon\|\partial_{\bar{\xi}_k'}\phi\|_{L^\infty(\Omega_\epsilon)})$, $\|\partial_{\bar{\xi}_k'} N_2(\phi)\|_{*,\Gamma_1^\epsilon} \leq C(\epsilon|\log\epsilon| + \|\partial_{\bar{\xi}_k'}\phi\|_{L^\infty(\Omega_\epsilon)})\epsilon|\log\epsilon|$ and $\|\partial_{\bar{\xi}_k'}\phi\|_{L^\infty(\Omega_\epsilon)} \ \leq \ C\epsilon^\alpha\log\epsilon^2$, we find similarly

$$\Rightarrow \ \partial_{\bar{\xi}_k'}[I_\epsilon(V+\phi) - I_\epsilon(V)] \ = \ O(\epsilon^{2\alpha}\log\epsilon^4)$$

□

Chapter 7

Expansion of the Energy

Lemma 7.0.5. *Let $\delta > 0$ be a fixed small number and U be the function defined in 2.3. With the choice 2.20 for the parameters μ_j, the following expansion holds for every $0 < \alpha < 1$*

$$J_\epsilon(U) \;=\; (-\lambda\pi + \gamma + \frac{8\beta}{\sqrt{3}})m + (16 - \frac{20}{\sqrt{3}})\pi m \log\frac{1}{\epsilon} + \varphi_m(\bar{\xi}) + O(\epsilon^\alpha) \tag{7.1}$$

where the function φ_m is defined by

$$\varphi_m(\bar{\xi}_1, \ldots, \bar{\xi}_m) \;=\; (-8 + \frac{46}{\sqrt{3}})\pi \sum_{j=1}^{m} H(\bar{\xi}_j, \bar{\xi}_j) + (-9{,}65m + 8)\pi \sum_{l \neq j}^{m} G(\bar{\xi}_j, \bar{\xi}_l) \tag{7.2}$$

and $\lambda,\ \beta,\ \gamma$ are finite constants.

Proof. Let us study $J_\epsilon(U)$.

Define $U_j(x) = u_j(x) + H_j(x)$ so $U = \sum\limits_{j=1}^{m} U_j$, then

$$\begin{aligned} J_\epsilon(U) &= \frac{1}{2}\int\limits_{\Omega} |\sum_{j=1}^{m} \nabla U_j|^2 - \epsilon^2 \int\limits_{\Omega} e^{\sum\limits_{j=1}^{m} U_j} - 2\epsilon \int\limits_{\Gamma_1} e^{\frac{1}{2}\sum\limits_{j=1}^{m} U_j} \\ &= \underbrace{\frac{1}{2}\sum_{j=1}^{m}\int\limits_{\Omega} |\nabla U_j|^2}_{I_A} + \underbrace{\frac{1}{2}\sum_{i\neq j}^{m}\int\limits_{\Omega} \nabla U_i \nabla U_j}_{I_B} \underbrace{- \epsilon^2 \int\limits_{\Omega} e^{\sum\limits_{j=1}^{m} U_j}}_{I_C} \underbrace{- 2\epsilon \int\limits_{\Gamma_1} e^{\frac{1}{2}\sum\limits_{j=1}^{m} U_j}}_{I_D} \end{aligned}$$

Let us analize the behavior of I_A:

$$\int_\Omega |\nabla U_j|^2 = \int_\Omega |\nabla u_j|^2 + 2\nabla u_j \nabla H_j + |\nabla H_j|^2 \tag{7.3}$$

H_j satisfies 2.2, i.e.

$$\begin{cases} \Delta H_j = 0 \ \text{ in } \Omega, \\ \dfrac{\partial H_j}{\partial \nu} = \epsilon e^{u_j/2} - \dfrac{\partial u_j}{\partial \nu} \ \text{ on } \Gamma_1 \\ H_j = -u_j \ \text{ on } \Gamma_2 \end{cases}$$

Multiply by H_j and integrate

$$\begin{aligned} \int_\Omega \Delta H_j H_j &= \int_{\partial\Omega} \frac{\partial H_j}{\partial \nu} H_j - \int_\Omega |\nabla H_j|^2 = 0 \\ \Rightarrow \int_\Omega |\nabla H_j|^2 &= \int_{\partial\Omega} \frac{\partial H_j}{\partial \nu} H_j \\ &= \int_{\Gamma_1} (\epsilon e^{u_j/2} - \frac{\partial u_j}{\partial \nu}) H_j - \int_{\Gamma_2} \frac{\partial H_j}{\partial \nu} u_j \end{aligned}$$

Now multiplying 2.2 by u_j and integrating

$$\begin{aligned} \int_\Omega \Delta H_j u_j &= \int_{\partial\Omega} \frac{\partial H_j}{\partial \nu} u_j - \int_\Omega \nabla H_j \nabla u_j = 0 \\ \Rightarrow \int_\Omega \nabla H_j \nabla u_j &= \int_{\partial\Omega} \frac{\partial H_j}{\partial \nu} u_j \\ &= \epsilon \int_{\Gamma_1} (e^{u_j/2} - \frac{\partial u_j}{\partial \nu}) u_j + \int_{\Gamma_2} \frac{\partial H_j}{\partial \nu} u_j \end{aligned}$$

Combining this together

$$\begin{aligned} \int_\Omega |\nabla U_j|^2 &= \int_\Omega |\nabla u_j|^2 + 2\epsilon \int_{\Gamma_1} e^{u_j/2} u_j - 2\int_{\Gamma_1} \frac{\partial u_j}{\partial \nu} u_j + 2\int_{\Gamma_2} \frac{\partial H_j}{\partial \nu} u_j + \epsilon \int_{\Gamma_1} e^{u_j/2} H_j \\ &\quad - \int_{\Gamma_1} \frac{\partial u_j}{\partial \nu} H_j - \int_{\Gamma_2} \frac{\partial H_j}{\partial \nu} u_j \\ &= \int_\Omega |\nabla u_j|^2 + \epsilon \int_{\Gamma_1} e^{u_j/2}(2u_j + H_j) - \int_{\Gamma_1} \frac{\partial u_j}{\partial \nu}(2u_j + H_j) + \int_{\Gamma_2} \frac{\partial H_j}{\partial \nu} u_j \end{aligned}$$

Remember that u_j satisfies in entire $\mathbb{R}^2$

$$\Delta u_j + \epsilon^2 e^{u_j} = 0$$

multiplying by u_j and integrating we find

$$\begin{aligned} \int_\Omega \Delta u_j u_j + \epsilon^2 \int_\Omega e^{u_j} u_j &= 0 \\ \int_{\partial\Omega} u_j \frac{\partial u_j}{\partial \nu} - \int_\Omega |\nabla u_j|^2 + \epsilon^2 \int_\Omega e^{u_j} u_j &= 0 \end{aligned}$$

so

$$\Rightarrow \int_\Omega |\nabla u_j|^2 = \int_{\Gamma_1} u_j \frac{\partial u_j}{\partial \nu} + \int_{\Gamma_2} u_j \frac{\partial u_j}{\partial \nu} + \epsilon^2 \int_\Omega e^{u_j} u_j$$

and then

$$\int_\Omega |\nabla U_j|^2 = \epsilon^2 \int_\Omega e^{u_j} u_j + \epsilon \int_{\Gamma_1} e^{u_j/2}(2u_j + H_j) - \int_{\Gamma_1} \frac{\partial u_j}{\partial \nu}(u_j + H_j) + \int_{\Gamma_2} u_j \frac{\partial u_j}{\partial \nu} + \int_{\Gamma_2} u_j \frac{\partial H_j}{\partial \nu} \tag{7.4}$$

We will analyze this integral term by term.

Let us study

$$\epsilon^2 \int_\Omega e^{u_j} u_j = \epsilon^2 \int_\Omega \frac{8\mu_j^2}{(\epsilon^2\mu_j^2 + |x - \xi_j|^2)^2} \log \frac{8\mu_j^2}{(\epsilon^2\mu_j^2 + |x - \xi_j|^2)^2}$$

We change variables $\epsilon\mu_j y = x - \bar{\xi}_j$

$$\begin{aligned} &= \int_{\Omega^{\epsilon\mu_j}} \frac{8}{(1 + |y - \sqrt{2}\nu(0)/2|^2)^2} (\log \frac{1}{(1 + |y - \sqrt{2}\nu(0)/2|^2)^2} + \log \frac{8}{\mu_j^2 \epsilon^4})\, dy \\ &= \int_0^\pi \int_0^\infty \frac{r dr d\theta 8}{(3/2 + r^2 + \sqrt{2} r \sin\theta)^2} (\log \frac{1}{(3/2 + r^2 + \sqrt{2} r \sin\theta)^2} + \log \frac{8}{\mu_j^2 \epsilon^4}) + O(\epsilon) \end{aligned}$$

Let us call

$$\gamma = \int_0^\pi \int_0^\infty \frac{r dr d\theta 8}{(3/2 + r^2 + \sqrt{2} r \sin\theta)^2} \log \frac{1}{(3/2 + r^2 + \sqrt{2} r \sin\theta)^2}$$

So,

$$\epsilon^2 \int_\Omega e^{u_j} u_j = \gamma + (4 - 4\sqrt{3}/3)\pi(\log \frac{8}{\mu_j^2} + 4 \log \frac{1}{3}) + O(\epsilon) \tag{7.5}$$

Let us find the asymptotic behavior of

$$\epsilon\int\limits_{\Gamma_1} e^{u_j/2}(2u_j+H_j) = \epsilon\int\limits_{\Gamma_1}\frac{2\sqrt{2}\mu_j}{\mu_j^2\epsilon^2+|x-\bar{\xi}_j-\sqrt{2}\mu_j^2\epsilon\nu(\bar{\xi}_j)|^2}\big(-2\log(\mu_j^2\epsilon^2+|x-\bar{\xi}_j- \sqrt{2}\mu_j^2\epsilon\nu(\bar{\xi}_j)|^2)^2+2H(x,\bar{\xi}_j)+\log 8\mu_j^2+O(\mu_j^2\epsilon^2)\big)$$

Where we have used lemma 2.1.1 to expand H_j.

We change variables $\epsilon\mu_j y = x-\bar{\xi}_j$

$$= 2\sqrt{2}\int\limits_{\Gamma_1^{\mu_j\epsilon}}\frac{1}{1+|y-\sqrt{2}\nu(0)/2|^2}\Big(2\log\frac{1}{(1+|y-\sqrt{2}\nu(0)/2|^2)^2}+2H(\bar{\xi}_j+\epsilon\mu_j y,\bar{\xi}_j) +\log 8\mu_j^2-4\log\mu_j\epsilon+O(\mu_j^2\,e^2)\Big)$$

We will calculate first these two integrals:

$$\begin{aligned}\int\limits_{\Gamma_1^{\mu_j\epsilon}}\frac{1}{1+|y-\sqrt{2}\nu(0)/2|^2} &= \int_{-\infty}^{\infty}\frac{dx}{1+\frac{1}{2}+x^2}+O(\epsilon)\\ &= \sqrt{\frac{2}{3}}\arctan\sqrt{\frac{2}{3}}x\Big]_{-\infty}^{\infty}+O(\epsilon)\\ &= \sqrt{\frac{2}{3}}\pi+O(\epsilon)\end{aligned} \tag{7.6}$$

$$\int\limits_{\Gamma_1^{\mu_j\epsilon}}\frac{1}{1+|y-\sqrt{2}\nu(0)/2|^2}\log\frac{1}{(1+|y-\sqrt{2}\nu(0)/2|^2)^2} = \int_{-\infty}^{\infty}\frac{dx}{1+\frac{1}{2}+x^2}\log\frac{1}{(\frac{3}{2}+x^2)^2}+ +O(\epsilon)$$

change of variables $\sqrt{\frac{2}{3}x}=y$, $dx=\sqrt{\frac{3}{2}}dy$

$$\begin{aligned} &= \int_{-\infty}^{\infty}\frac{\sqrt{\frac{3}{2}}dy}{\frac{3}{2}(1+y^2)}(\log\frac{1}{(1+y^2)^2}-2\log\frac{3}{2})\\ &= \sqrt{\frac{2}{3}}\left[\int_{-\infty}^{\infty}\frac{dy}{1+y^2}\log\frac{1}{(1+y^2)^2}-\int_{-\infty}^{\infty}\frac{dy}{1+y^2}2\log\frac{3}{2}\right]+O(\epsilon)\\ &= 2\sqrt{\frac{2}{3}}(\beta-\pi\log\frac{3}{2})+O(\epsilon)\end{aligned} \tag{7.7}$$

with $\beta = \frac{1}{2}\int_{-\infty}^{\infty} \dfrac{dy}{1+y^2}\log\dfrac{1}{(1+y^2)^2}$.

So, using the taylor expansion for H

$$H(\bar{\xi}_j+\epsilon\mu y,\bar{\xi}_j) = H(\bar{\xi}_j,\bar{\xi}_j) + \underbrace{\frac{\partial H}{\partial x_1}(\bar{\xi}_j,\bar{\xi}_j)\epsilon\mu_j y}_{O(\epsilon|y|)} + O(\epsilon^2\mu_j^2|y|^2)$$

we find:

$$\begin{aligned}\epsilon\int_{\Gamma_1} e^{u_j/2}(2u_j+H_j) &= 4\sqrt{2}\int_{\Gamma_1^{\mu_j\epsilon}} \frac{1}{1+|y-\sqrt{2}\nu(0)/2|^2}\log\frac{1}{(1+|y-\sqrt{2}\nu(0)/2|^2)^2} \\ &+2\sqrt{2}\int_{\Gamma_1^{\mu_j\epsilon}} \frac{1}{1+|y-\sqrt{2}\nu(0)/2|^2}(2H(\bar{\xi}_j,\bar{\xi}_j)+O(\epsilon^\alpha|y|^\alpha)) \\ &+2\sqrt{2}[\log 8\mu_j^2 - 4\log\mu_j\epsilon + O(\epsilon^2\mu_j^2)]\int_{\Gamma_1^{\mu_j\epsilon}} \frac{1}{1+|y-\sqrt{2}\nu(0)/2|^2}\end{aligned}$$

$$\Rightarrow \epsilon\int_{\Gamma_1} e^{u_j/2}(2u_j+H_j) = \frac{16\beta}{\sqrt{3}} - \frac{4}{\sqrt{3}}\pi\log\frac{128}{81\mu_j^2} - \frac{16}{\sqrt{3}}\pi\log\epsilon + \frac{8\pi}{\sqrt{3}}H(\bar{\xi}_j,\bar{\xi}_j) + O(\epsilon^\alpha) \tag{7.8}$$

In an analogous way

$$\begin{aligned}\int_{\Gamma_1}\frac{\partial u_j}{\partial\nu}(u_j+H_j) &= \int_{\Gamma_1}\frac{-4(x-\xi_j)\cdot\nu(x)}{\mu_j^2\epsilon^2+|x-\xi_j|^2}(\log\frac{1}{(\mu_j^2\epsilon^2+|x-\xi_j|^2)^2}+2H(x,\bar{\xi}_j) \\ &+\log 8\mu_j^2 + O(\mu_j^2\epsilon^2))\end{aligned}$$

$$\begin{aligned}\Rightarrow \int_{\Gamma_1}\frac{\partial u_j}{\partial\nu}(u_j+H_j) &= \frac{8}{\sqrt{3}}(\beta-\pi\log\frac{3}{2}) + 2\sqrt{\frac{2}{3}}\pi\log 8\mu_j^2 + 2\sqrt{\frac{2}{3}}H(\bar{\xi}_j,\bar{\xi}_j) \\ &+\frac{32}{\sqrt{3}}\pi\log\frac{1}{\mu_j\epsilon} + O(\epsilon^\alpha) + O(\epsilon\log\epsilon)\end{aligned} \tag{7.9}$$

Now $\int_{\Gamma_2} u_j\dfrac{\partial H_j}{\partial\nu}$.

We will use that,

$$\begin{aligned}\int_{\Omega_\epsilon}\Delta H_j u_j &= \int_{\partial\Omega}\frac{\partial H_j}{\partial\nu}u_j - \int_{\Omega}\nabla H_j\nabla u_j \\ \int_{\Omega_\epsilon}\Delta u_j H_j &= \int_{\partial\Omega}\frac{\partial u_j}{\partial\nu}H_j - \int_{\Omega}\nabla H_j\nabla u_j\end{aligned}$$

So,

$$\begin{aligned}
0 &= \int\limits_{\Gamma_2} \frac{\partial H_j}{\partial \nu} u_j + \int\limits_{\Gamma_1} \frac{\partial H_j}{\partial \nu} u_j + \int\limits_{\Omega} H_j \Delta u_j - \int\limits_{\partial\Omega} \frac{\partial u_j}{\partial \nu} H_j \\
\Rightarrow \int\limits_{\Gamma_2} \frac{\partial H_j}{\partial \nu} u_j &= -\epsilon \int\limits_{\Gamma_1} e^{u_j/2} u_j + \int\limits_{\Gamma_1} \frac{\partial u_j}{\partial \nu} u_j + \epsilon^2 \int\limits_{\Omega} H_j e^{u_j} + \int\limits_{\partial\Omega} \frac{\partial u_j}{\partial \nu} H_j \\
&= -\frac{2}{\sqrt{3}} \pi \log \frac{8}{\mu_j^2 \epsilon^4} + \frac{16}{\sqrt{3}} \pi \log \frac{1}{\mu_j \epsilon} + \frac{\sqrt{2}}{3} \pi \log 8\mu_j^2 + \int\limits_{\Gamma_1} \frac{\partial u_j}{\partial \nu} H_j \\
&\quad - \int\limits_{\Gamma_2} \frac{\partial u_j}{\partial \nu} u_j + \epsilon^2 \int\limits_{\Omega} H_j e^{u_j} \\
&= \frac{8\pi}{\sqrt{3}} \log \frac{1}{\mu_j \epsilon} - \frac{(2-\sqrt{2})\pi}{\sqrt{3}} \log 8\mu_j^2 + O(\epsilon \log \epsilon) + 2\sqrt{\frac{2}{3}} \pi H(\bar{\xi}_j, \bar{\xi}_j) \\
&\quad + O(\epsilon^\alpha) + \epsilon^2 \int\limits_{\Omega} H_j e^{u_j} - \int\limits_{\Gamma_2} \frac{\partial u_j}{\partial \nu} u_j
\end{aligned}$$

Let us calculate $\chi_j = \int\limits_{\Gamma_2} u_j \frac{\partial u_j}{\partial \nu}$

$$= \int\limits_{\Gamma_2} \log \frac{8\mu_j^2}{(\mu_j^2 \epsilon^2 + |x-\xi_j|^2)^2} \frac{-4(x-\xi_j)\cdot \nu(x)}{(\mu_j^2 \epsilon^2 + |x-\xi_j|^2)}\, dx$$

Change of variables $\mu_j \epsilon^2 y = x - \bar{\xi}_j$

$$\begin{aligned}
&= \int\limits_{(\Gamma_2 - \bar{\xi}_j)/\mu_j \epsilon} \frac{-4 \log 8\mu_j^2 (\mu_j \epsilon y - \sqrt{2}\mu_j \epsilon \nu(0)) \cdot \nu(x) \mu_j \epsilon\, dy}{\mu_j^4 \epsilon^4 (1 + |y - \sqrt{2}\nu(0)/2|^2)^2 \mu_j^2 \epsilon^2 (1 + |y - \sqrt{2}\nu(0)/2|^2)} \\
&= \frac{-4 \log 8\mu_j^2}{\mu_j^4 \epsilon^4} \int\limits_{(\Gamma_2 - \bar{\xi}_j)/\mu_j \epsilon} \frac{(y - \sqrt{2}\nu(0)/2) \cdot \nu(\bar{\xi}_j + \mu_j \epsilon y)}{(1 + |y - \sqrt{2}\nu(0)/2|^2)^2} dy
\end{aligned}$$

Notice that $|y| = O(\frac{1}{\epsilon})$, so

$$\begin{aligned}
&= \frac{-4 \log 8\mu_j^2}{\mu_j^4 \epsilon^4} \int\limits_{(\Gamma_2 - \bar{\xi}_j)/\mu_j \epsilon} \frac{O(\frac{1}{\epsilon})}{\frac{1}{\epsilon^6}} \\
\chi_j &= O(\epsilon) \qquad (7.10)
\end{aligned}$$

Now

$$\epsilon^2 \int\limits_{\Omega} H_j e^{u_j} = \epsilon^2 \int\limits_{\Omega} H_j \frac{8\mu_j^2}{(\mu_j^2 \epsilon^2 + |x-\xi_j|^2)^2}$$

Change of variables $\mu_j \epsilon y = x - \bar{\xi}_j$

$$
\begin{aligned}
&= \int\limits_{\Omega^{\mu_j \epsilon}} H_j(\bar{\xi}_j + \mu_j \epsilon y) \frac{8}{(1+|y-\sqrt{2}\nu(0)/2|^2)^2} \\
&= 8 \int\limits_{\Omega^{\mu_j \epsilon}} \frac{2H(\bar{\xi}_j, \bar{\xi}_j) + O(\epsilon|y|) - \log 8\mu_j^2 + O(\epsilon^\alpha)}{(1+|y-\sqrt{2}\nu(0)/2|^2)^2} \\
&= -\frac{64}{3}(3-\sqrt{3})\pi \sum_{l \neq j} G(\bar{\xi}_j, \bar{\xi}_l) + O(\epsilon^\alpha)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int\limits_{\Gamma_2} \frac{\partial H_j}{\partial \nu} u_j \;=\;& \frac{8\pi}{\sqrt{3}} \log \frac{1}{\mu_j \epsilon} - \frac{(2-\sqrt{2})\pi}{\sqrt{3}} \log 8\mu_j^2 + O(\epsilon \log \epsilon) + 2\sqrt{\frac{2}{3}} \pi H(\bar{\xi}_j, \bar{\xi}_j) \\
& -\frac{64}{3}(3-\sqrt{3})\pi \sum_{l \neq j} G(\bar{\xi}_j, \bar{\xi}_l) + O(\epsilon^\alpha) \qquad (7.11)
\end{aligned}
$$

Putting it all together now,

$$
\begin{aligned}
\int\limits_{\Omega} |\nabla U_j|^2 \;=\;& [4\pi(3-\sqrt{3}) \log 64/3 + 4\pi \log 81/1024/\sqrt{3} - 24\pi \log \sqrt{8}/\sqrt{3} \\
& +8\pi \log 3/2/\sqrt{3} + \gamma + 8\beta/\sqrt{3}] + [-24\pi/\sqrt{3} + 16(3-\sqrt{3})\pi/3 + 16\pi] \log \frac{1}{\epsilon} \\
& +[2\pi\sqrt{3}/3 + 64(3-\sqrt{3})\pi/3 + 36\pi(-2+\sqrt{3})] H(\bar{\xi}_j, \bar{\xi}_j) \\
& +9\pi(-2+\sqrt{3})4 \sum_{l \neq j} G(\bar{\xi}_j, \bar{\xi}_l) + O(\epsilon^\alpha)
\end{aligned}
$$

So

$$
\begin{aligned}
I_A \;=\;& (-5,68102\pi + \gamma + \frac{8}{\sqrt{3}}\beta)m + (16 - \frac{20}{\sqrt{3}})\pi m \log \frac{1}{\epsilon} + (-4 + \frac{23}{\sqrt{3}})\pi \sum_{j=1}^{m} H(\bar{\xi}_j, \bar{\xi}_j) \\
& +36(-2+\sqrt{3})\pi m \sum_{l \neq j}^{m} G(\bar{\xi}_j, \bar{\xi}_l) + O(\epsilon^\alpha) \qquad (7.12)
\end{aligned}
$$

Now let us analize the behavior of I_B

$$
\int\limits_{\Omega} \nabla U_i \nabla U_j \quad \text{with } i \neq j \qquad (7.13)
$$

Notice that $\Delta U_i = \Delta u_i + \cancel{\Delta H_i} = -\epsilon^2 e^{u_i}$, and also

$$\begin{aligned} U_i &\equiv 0 \quad \text{on } \Gamma_2 \\ \frac{\partial U_i}{\partial \nu} &= \frac{\partial u_i}{\partial \nu} + \frac{\partial H_i}{\partial \nu} \\ &= \cancel{\frac{\partial u_i}{\partial \nu}} + \epsilon e^{u_j/2} - \cancel{\frac{\partial u_i}{\partial \nu}} = \epsilon e^{u_j/2} \quad \text{on } \Gamma_1 \end{aligned}$$

So,

$$\begin{aligned} \int_\Omega \Delta U_i U_j &= \int_{\partial\Omega} \frac{\partial U_i}{\partial \nu} U_j - \int_\Omega \nabla U_i \nabla U_j \\ &= \epsilon \int_{\Gamma_1} e^{u_i/2} U_j - \int_\Omega \nabla U_i \nabla U_j \\ \Rightarrow \int_\Omega \nabla U_i \nabla U_j &= \epsilon^2 \int_\Omega e^{u_i} U_j + \epsilon \int_{\Gamma_1} e^{u_i/2} U_j \\ &= \epsilon^2 \int_\Omega e^{u_i}(u_j + H_j) + \epsilon \int_{\Gamma_1} e^{u_i/2}(u_j + H_j) \end{aligned}$$

The first term

$$\epsilon^2 \int_\Omega e^{u_i}(u_j + H_j) = \int_\Omega \frac{8\mu_i^2\epsilon^2}{(\mu_i^2\epsilon^2 + |x - \xi_i|^2)^2}(w_j(\frac{|x-\xi_j|}{\epsilon}) + \log\frac{1}{\epsilon^4} + H_j(x))dx$$

change of variables $\mu_i \epsilon y = x - \bar{\xi}_i$

$$\begin{aligned} &= \int_{\frac{(\Omega - \bar{\xi}_i)}{\mu_i \epsilon}} \frac{8}{(1 + |y - \sqrt{2}\nu(0)/2|^2)^2}(\log\frac{1}{(\mu_j^2\epsilon^2 + |\epsilon\mu_i y + \bar{\xi}_i - \bar{\xi}_j|^2)^2} + \log 8{\mu_j}^2 + H_j(\bar{\xi}_i + \mu_i\epsilon y)) \\ &= \int_{\frac{(\Omega - \bar{\xi}_i)}{\mu_i \epsilon}} \frac{8}{(1 + |y - \sqrt{2}\nu(0)/2|^2)^2}[\log\frac{1}{(\mu_j^2\epsilon^2 + |\epsilon\mu_i y + \bar{\xi}_i - \bar{\xi}_j|^2)^2} - 4\log\frac{1}{|\bar{\xi}_i - \bar{\xi}_j|}] \\ &\quad + \int_{\frac{(\Omega - \bar{\xi}_i)}{\mu_i \epsilon}} \frac{8}{(1 + |y - \sqrt{2}\nu(0)/2|^2)^2}[H_j(\bar{\xi}_i + \mu_j\epsilon y) - H_j(\bar{\xi}_i)] \\ &\quad + \int_{\frac{(\Omega - \bar{\xi}_i)}{\mu_i \epsilon}} \frac{8}{(1 + |y - \sqrt{2}\nu(0)/2|^2)^2}[H_j(\bar{\xi}_i) - 2H(\bar{\xi}_j, \bar{\xi}_i) + \log 8\mu_j^2] \\ &\quad + \int_{\frac{(\Omega - \bar{\xi}_i)}{\mu_i \epsilon}} \frac{8}{(1 + |y - \sqrt{2}\nu(0)/2|^2)^2}[2H(\bar{\xi}_j, \bar{\xi}_i) + 4\log\frac{1}{|\bar{\xi}_i - \bar{\xi}_j|}] \end{aligned}$$

$$\Rightarrow \epsilon^2 \int_\Omega e^{u_i}(u_j + H_j) = \frac{8}{3}(3-\sqrt{3})\pi G(\bar{\xi}_i, \bar{\xi}_j) + O(\epsilon^\alpha) \tag{7.14}$$

Now the second term

$$\epsilon \int_{\Gamma_1} e^{u_j/2}(u_j + H_j) = \epsilon \int_{\Gamma_1} \frac{\sqrt{8}\mu_i}{(\mu_i^2\epsilon^2 + |x-\xi_i|^2)}(\log \frac{1}{(\mu_i^2\epsilon^2 + |x-\xi_i|^2)^2} + H_j + \log 8\mu_j^2)$$

change variables $\mu_i \epsilon y = x - \bar{\xi}_i$

$$\begin{aligned}
&= \int_{\Gamma_1^{\mu_i\epsilon} - \bar{\xi}_i} \frac{\sqrt{8}}{(1+|y-\sqrt{2}\nu(0)/2|^2)}(\log \frac{1}{(\mu_j^2\epsilon^2 + |\mu_i\epsilon y + \bar{\xi}_i - \bar{\xi}_j - \sqrt{2}\mu_j\epsilon\nu(\bar{\xi}_j)|^2)} \\
&\quad + H_j(\bar{\xi}_i + \mu_i\epsilon y) + \log 8\mu_j^2) \\
&= \int_{\Gamma_1^{\mu_i\epsilon} - \bar{\xi}_i} \frac{\sqrt{8}}{(1+|y-\sqrt{2}\nu(0)/2|^2)}[\log \frac{1}{(\mu_j^2\epsilon^2 + |\mu_i\epsilon y - \sqrt{2}\mu_j\epsilon\nu(\bar{\xi}_j) + \bar{\xi}_i - \bar{\xi}_j|^2)} - 4\log \frac{1}{|\bar{\xi}_i - \bar{\xi}_j|}] \\
&\quad + \int_{\Gamma_1^{\mu_i\epsilon} - \bar{\xi}_i} \frac{\sqrt{8}}{(1+|y-\sqrt{2}\nu(0)/2|^2)}[H_j(\bar{\xi}_i + \mu_i\epsilon y) - H_j(\bar{\xi}_i)] \\
&\quad + \int_{\Gamma_1^{\mu_i\epsilon} - \bar{\xi}_i} \frac{\sqrt{8}}{(1+|y-\sqrt{2}\nu(0)/2|^2)}[H_j(\bar{\xi}_i) - 2H(\bar{\xi}_i, \bar{\xi}_j) + \log 8\mu_j^2] \\
&\quad + \int_{\Gamma_1^{\mu_i\epsilon} - \bar{\xi}_i} \frac{\sqrt{8}}{(1+|y-\sqrt{2}\nu(0)/2|^2)}[2H(\bar{\xi}_i, \bar{\xi}_j) + 4\log \frac{1}{|\bar{\xi}_i - \bar{\xi}_j|}]
\end{aligned}$$

$$\Rightarrow \epsilon \int_{\Gamma_1} e^{u_j/2}(u_j + H_j) = \frac{8}{\sqrt{3}}\pi G(\bar{\xi}_i, \bar{\xi}_j) + O(\epsilon^\alpha) \tag{7.15}$$

So, with 7.14 and 7.15 together we have,

$$\begin{aligned}
\int_\Omega \Delta U_i U_j &= (\frac{8}{3}(3-\sqrt{3})\pi + \frac{8}{\sqrt{3}}\pi)G(\bar{\xi}_i, \bar{\xi}_j) + O(\epsilon^\alpha) \\
&= 8\pi G(\bar{\xi}_i, \bar{\xi}_j) + O(\epsilon^\alpha)
\end{aligned} \tag{7.16}$$

Now let us analyze the behavior of I_C

$$\epsilon^2 \int_\Omega e^{\sum_{j=1}^m U_j} = \epsilon^2[\sum_{j=1}^m \int_{B(\bar{\xi}_j, \tilde{\delta})\cap\Omega} e^U \, dx] + \epsilon^2\Theta_\epsilon(\bar{\xi})$$

with Θ_ϵ a uniformly bounded function as $\epsilon \to 0$.

Now,

$$\epsilon^2 \int\limits_{B(\bar{\xi}_j,\tilde{\delta})\cap\Omega} e^U\, dx = \epsilon^2 \int\limits_{B(\bar{\xi}_j,\tilde{\delta})\cap\Omega} e^{U_j} e^{\sum\limits_{i\neq j}^{m} U_i}\, dx$$

$$= \epsilon^2 \int\limits_{B(\bar{\xi}_j,\tilde{\delta})\cap\Omega} \frac{8\mu_j^2 e^{H_j}}{(\mu_j^2\epsilon^2 + |x-\xi_j|^2)^2} e^{\sum\limits_{i\neq j}^{m}(\log \frac{8\mu_i^2}{(\mu_i^2\epsilon^2+|x-\xi_i|^2)^2} + H_i(x))}$$

$$= \epsilon^2 \int\limits_{B(\bar{\xi}_j,\tilde{\delta})\cap\Omega} \frac{e^{\log 8\mu_j^2 + H_j}}{\mu_j^4\epsilon^4(1+(\frac{|x-\xi_j|}{\epsilon\mu_j})^2)^2} e^{\sum\limits_{i\neq j}^{m}\log \frac{1}{(\mu_i^2\epsilon^2+|x-\xi_i|^2)^2} + \log 8\mu_i^2 + H_i(x)}$$

$$= \frac{1}{\epsilon^2\mu_j^4} \int\limits_{B(\bar{\xi}_j,\tilde{\delta})\cap\Omega} \frac{e^{2H(x,\bar{\xi}_j)+O(\epsilon^\alpha)}}{(1+(\frac{|x-\xi_j|}{\epsilon\mu_j})^2)^2} e^{\sum\limits_{i\neq j}^{m}\log \frac{1}{(\mu_i^2\epsilon^2+|x-\xi_i|^2)^2} + 2H(x,\bar{\xi}_i)+O(\epsilon^\alpha)}$$

change variables $\epsilon\mu_j y = x - \bar{\xi}_j$

$$= \frac{1}{\mu_j^2} \int\limits_{B(0,\frac{\tilde{\delta}}{\epsilon\mu_j})\cap\Omega_{\epsilon\mu_j}} \frac{e^{2H(\bar{\xi}_j+\epsilon\mu_j y,\bar{\xi}_j)+O(\epsilon^\alpha)}}{(1+|y-\sqrt{2}\nu(0)/2|^2)^2} exp\{\sum_{i\neq j}^{m}[-\log(\mu_i^2\epsilon^2 + |\bar{\xi}_j+\epsilon\mu_j y - \bar{\xi}_i - \sqrt{2}\mu_i\epsilon\nu(\bar{\xi}_i)/2|^2)^2 + 2H(\bar{\xi}_j+\mu_j\epsilon y,\bar{\xi}_i) + O(\epsilon^\alpha)]\}$$

We have that

$$e^{2H(\bar{\xi}_j+\epsilon\mu_j y,\bar{\xi}_j)+O(\epsilon^\alpha)} = e^{2H(\bar{\xi}_j,\bar{\xi}_j)} + O(\epsilon^\alpha|y|^\alpha)$$

and

$$exp\{\sum_{i\neq j}^{m}[\log\frac{1}{(\mu_i^2\epsilon^2+|\bar{\xi}_j+\epsilon\mu_j y-\bar{\xi}_i-\sqrt{2}\mu_i\epsilon\nu(\bar{\xi}_i)/2|^2)^2} + 2H(\bar{\xi}_j+\mu_j\epsilon y,\bar{\xi}_i)+O(\epsilon^\alpha)]\}$$

$$= exp\{\sum_{i\neq j}^{m}[\log\frac{1}{|\bar{\xi}_j-\bar{\xi}_i|^4} + 2H(\bar{\xi}_j,\bar{\xi}_i)]\} + O(\epsilon^\alpha|y|^\alpha)$$

$$= e^{\sum\limits_{i\neq j}^{m} 2G(\bar{\xi}_j,\bar{\xi}_i)} + O(\epsilon^\alpha|y|^\alpha)$$

So,

$$\epsilon^2 \int\limits_{B(\bar{\xi}_j,\tilde{\delta})\cap\Omega} e^U\,dx \;=\; \frac{1}{\mu_j^2} \int\limits_{B(0,\frac{\tilde{\delta}}{\epsilon\mu_j})\cap\Omega_{\epsilon\mu_j}} \frac{[e^{2H(\bar{\xi}_j,\bar{\xi}_j)+2\sum\limits_{i\neq j}^{m} G(\bar{\xi}_j,\bar{\xi}_i)} + O(\epsilon^\alpha|y|^\alpha)]}{(1+|y-\sqrt{2}\nu(0)/2|^2)^2}$$

$$= \frac{1}{\mu_j^2} e^{2H(\bar{\xi}_j,\bar{\xi}_j)+2\sum\limits_{i\neq j}^{m} G(\bar{\xi}_j,\bar{\xi}_i)} \frac{-4}{3}(-3+\sqrt{3})\pi + O(\epsilon^\alpha)$$

but

$$\log 8\mu_j^2 \;=\; 2H(\bar{\xi}_j,\bar{\xi}_j) + 2\sum_{i\neq j}^{m} G(\bar{\xi}_j,\bar{\xi}_i)$$

Then

$$\epsilon^2 \int\limits_{B(\bar{\xi}_j,\tilde{\delta})\cap\Omega} e^U\,dx \;=\; \frac{32}{3}(3-\sqrt{3})\pi + O(\epsilon^\alpha)$$

and therefore,

$$I_C \;=\; \frac{32}{3} m\pi(3-\sqrt{3}) + O(\epsilon^\alpha) \tag{7.17}$$

Finally let us analize I_D:

$$\frac{1}{2}\epsilon \int\limits_{\Gamma_1} e^{\frac{1}{2}\sum\limits_{i=1}^{m} U_i} \;=\; \frac{1}{2}\epsilon \sum_{j=1}^{m} \int\limits_{\Gamma_1\cap B(\bar{\xi}_j,\tilde{\delta})} e^{\frac{1}{2}\sum\limits_{i=1}^{m} U_i} + \epsilon\Theta_\epsilon(\bar{\xi})$$

Now

$$\epsilon \int\limits_{\Gamma_1\cap B(\bar{\xi}_j,\tilde{\delta})} e^{\frac{1}{2}\sum\limits_{i=1}^{m} u_i+H_i} \;=\; \epsilon \int\limits_{\Gamma_1\cap B(\bar{\xi}_j,\tilde{\delta})} e^{\frac{1}{2}(u_j+H_j)} e^{\frac{1}{2}\sum\limits_{i\neq j}^{m} u_i+H_i}$$

$$= \epsilon \int\limits_{\Gamma_1\cap B(\bar{\xi}_j,\tilde{\delta})} exp\{\frac{1}{2}(\log\frac{1}{(\mu_j^2\epsilon^2+|x-\xi_j|^2)^2} + 2H(x,\bar{\xi}_j)$$

$$+O(\epsilon^\alpha))\} exp\{\frac{1}{2}\sum_{i\neq j}^{m} \log\frac{1}{(\mu_i^2\epsilon^2+|x-\xi_i|^2)^2} + H_i + \log 8\mu_i^2\}$$

$$= \epsilon \int\limits_{\Gamma_1\cap B(\bar{\xi}_j,\tilde{\delta})} \frac{e^{H(x,\bar{\xi}_j)+O(\epsilon^\alpha)}}{(\mu_j^2\epsilon^2+|x-\xi_j|^2)} exp\{\frac{1}{2}\sum_{i\neq j}^{m} \log\frac{1}{(\mu_i^2\epsilon^2+|x-\xi_i|^2)^2}$$

$$+2H(x,\bar{\xi}_i) + O(\epsilon^\alpha)\}$$

change variables $\mu_j \epsilon y = x - \bar{\xi}_j$

$$
\begin{aligned}
&= \frac{1}{\mu_j} \int_{\frac{1}{\mu_j\epsilon}(\Gamma_1 \cap B(\bar{\xi}_j,\tilde{\delta}))-\bar{\xi}_j} \frac{1}{(1+|y-\sqrt{2}\nu(0)/2|^2)} exp\{H(\bar{\xi}_j+\mu_j\epsilon y,\bar{\xi}_j)+ \\
&\quad O(\epsilon^\alpha)\}exp\{\frac{1}{2}\sum_{i\neq j}^{m}(\log\frac{1}{|\bar{\xi}_i-\bar{\xi}_j|^4}+O(\epsilon^\alpha|y|^\alpha)+2H(\bar{\xi}_j,\bar{\xi}_i))\} \\
&= \frac{1}{\mu_j} \int_{\Gamma_1^{\mu_j\epsilon}\cap B(0,\frac{\tilde{\delta}}{\mu_j\epsilon})} \frac{e^{H(\bar{\xi}_j,\bar{\xi}_j)+\sum_{i\neq j}^{m}G(\bar{\xi}_i,\bar{\xi}_j)}+O(\epsilon^\alpha|y|^\alpha)}{(1+|y-\sqrt{2}\nu(0)/2|^2)} \\
&= \sqrt{8}(\sqrt{\frac{2}{3}}\pi+O(\epsilon))+O(\epsilon^\alpha) \\
&= \frac{4}{\sqrt{3}}\pi+O(\epsilon^\alpha)
\end{aligned}
$$

So

$$I_D = \frac{1}{2}\sum_{j=1}^{m}\frac{4}{\sqrt{3}}\pi+O(\epsilon^\alpha)+\epsilon\Theta(\bar{\xi}) = \frac{2\pi m}{\sqrt{3}}+O(\epsilon^\alpha) \tag{7.18}$$

So thanks to 7.12, 7.16, 7.17 and 7.18

$$
\begin{aligned}
J_\epsilon(U) &= I_A+I_B-I_C-I_D \\
&= (-20,36\pi+\gamma+\frac{8\beta}{\sqrt{3}})m+(16-\frac{20}{\sqrt{3}})\pi m\log\frac{1}{\epsilon}+(-4+\frac{23}{\sqrt{3}})\pi\sum_{j=1}^{m}H(\bar{\xi}_j,\bar{\xi}_j) \\
&\quad +(-9,65m+8)\pi\sum_{l\neq j}^{m}G(\bar{\xi}_j,\bar{\xi}_l)+O(\epsilon^\alpha)
\end{aligned}
$$

□

Chapter 8

Proof of the final theorem

Proof. We will consider

$$\hat{\Gamma}_1^m \ = \ \{(\bar{\xi}_1, \ldots, \bar{\xi}_m) \in \Gamma_1^m : \bar{\xi}_i \neq \bar{\xi}_j \ \text{ if } i \neq j\}$$

According to lemma 6.0.3, we have a solution to problem 1.1 if we adjust the points $\bar{\xi}$ so that it is a critical point of F_ϵ defined by 6.2.

This is equivalent of finding a critical point of

$$\tilde{F}_\epsilon(\bar{\xi}) \ = \ F_\epsilon(\bar{\xi}) - B \log \frac{1}{\epsilon} - A \tag{8.1}$$

with A and B defined by

$$A \ = \ (-20, 36\pi + \gamma + \frac{8\beta}{\sqrt{3}})m$$
$$B \ = \ (16 - \frac{20}{\sqrt{3}})\pi m$$

From lemma 6.0.4 and 7.0.5 we have that for $\bar{\xi} \in \hat{\Gamma}_1^m$ such that $|\bar{\xi}_i - \bar{\xi}_j| \geq \delta \ \ \forall i \neq j$, namely $\bar{\xi} \in \tilde{\Gamma}_1^m$,

$$\tilde{F}_\epsilon(\bar{\xi}) \ = \ \varphi_m(\bar{\xi}) + \epsilon^\alpha \Theta_\epsilon(\bar{\xi}) \tag{8.2}$$

where

$$\varphi_m(\bar{\xi}) \ = \ (-4 + \frac{23}{\sqrt{3}})\pi \sum_{j=1}^m H(\bar{\xi}_j, \bar{\xi}_j) + (-9, 65m + 8)\pi \sum_{j \neq i}^m G(\bar{\xi}_i, \bar{\xi}_j)$$

and Θ_ϵ and $\nabla_{\bar{\xi}}\Theta_\epsilon$ are uniformly bounded in the considered region as $\epsilon \to 0$.

We will show that φ_m has at least two distinct critical points in $\tilde{\Gamma}_1^m$.

The function φ_m is C^1 bounded from above in $\hat{\Gamma}_1^m$ because H is a regular function, and $G(\bar{\xi}_i, \bar{\xi}_j)$ goes to infinity only at the frontier of $\hat{\Gamma}_1^m$ where $\bar{\xi}_i \to \bar{\xi}_j$ for some $i \neq j$ so

$$\varphi_m(\bar{\xi}_1, \ldots, \bar{\xi}_m) \to -\infty \text{ as } |\bar{\xi}_i - \bar{\xi}_j| \to 0 \text{ for some } i \neq j$$

Since $\delta > 0$ is arbitrarily small, then φ_m has an absolute maximum M in $\tilde{\Gamma}_1^m$.

The existence of the second critical point will be assured by the Ljusternik-Schnirelmann theory. It establish that the number of critical points for φ_m can be estimated from below by the Ljusternik-Schnirelmann category of $\tilde{\Gamma}_1^m$ relative to $\tilde{\Gamma}_1^m$:

$cat(\tilde{\Gamma}_1^m) \equiv$ Minimal number of closed and contractible sets in $\tilde{\Gamma}_1^m$ whose union covers $\tilde{\Gamma}_1^m$.

As proven in [DdPM05] page 40, $cat(\tilde{\Gamma}_1^m) \geq 2$ for any $m \geq 1$, and so, using Ljusternik-Schnirelmann theory we conclude that φ_m has at least two distinct critical points in $\tilde{\Gamma}_1^m$.
These critical points persist under small C^0-perturbations of the function, so $\tilde{F}_\epsilon$ has at least two distinct critical points in $\tilde{\Gamma}_1^m$, and since δ is arbitrarily small, $\tilde{F}_\epsilon$ has two critical points in $\hat{\Gamma}_1^m$ and hence problem 1.1 has at least two different solutions. □

Chapter 9

Conclusions

In this thesis work we demostrated an existence theorem for the Liouville equation with non linear boundary conditions by dividing the problem in several steps.

In the first step we demostrated the non degeneracy of the equation in $\mathbb{R}^2_+$ linearized around the radial solution u 3.2 with respect to the invariance of the equation under dilatation and translations in the horizontal direction. In order to achieve this, we found the linearized equation 3.3 and solve it directly by transforming the domain into a disk and separating variables. After this procedure we realized that the only bounded solutions found were the dilatation and the translation in the horizontal direction of u. This step was determining whether it was possible or not to solve equation 1.1 by this method because the non degeneracy plays a key role in the proof of the first a priori bound 4.1.1.

The second step in this sequence was carried on the chapters 2 and 4, and consisted in the transformation of the original problem using a basic ansatz for the solution, namely U_j, and a correction term ϕ, to obtain finally the set of equations for ϕ 2.17 that will characterize the solution of the main problem. We analize in chapter 4 the linear operator and found at the end a solvability result for the linear problem with the introduction of the coefficients c_j who will play a fundamental role in the following chapters. Besides direct calculation, only classical theory of functional analysis was used to complete this proof.

In the next step we established the existence of a solution ϕ to the complete non linear problem with the addition of the c_j terms on the right hand side 5.0.2. The previous results on the linear operator and Banach's fix point theorem allow us to demostrate this result in lemma 5.0.2. Moreover we observate the behavior of the solution ϕ as a function of the singular points $\bar{\xi}$ and realized that it is a well defined function of class C^1 whose derivative is bounded by 5.3.

At this point of the work we realize that if we manage to adjust the points $\bar{\xi}$ so that

$c_j(\bar{\xi}) = 0$ for all $j = 1, \ldots, m$ we will have a solution to the main problem. In step four of our methodology we demostrated in chapter 6 that this fact can be reduced to the analysis of the critical points of the functional $F_\epsilon(\bar{\xi})$, and that this in turn can be described by the energy functional $J_\epsilon(U)$ associated to the original equation. Also in chapter 7 we give an expansion in terms of the function $\varphi_m(\bar{\xi})$ of such functional which takes us to the final stage of the main result:

The existence of critical points of $F_\epsilon(\bar{\xi})$ through the study of the critical points of $\varphi_m(\bar{\xi})$.

We found by direct analysis, using the regularity of the function $\varphi_m(\bar{\xi})$, the existence of one critical point given by its absolute maximum. The existence of at least one more critical point is demostrated using Ljusternik-Schnirelmann cathegory theory, whose applicability to this problem is derived from one of our guiding references [DdPM05].

In summary, we found at least two different sets of points $\bar{\xi}$ for which the coefficients $c_j(\bar{\xi}) = 0$ and so our solvability results for ϕ gives a solution to the complete problem for each set $\bar{\xi}$.

As a side note one could have considered the more general problem:

$$\begin{cases} \Delta u + \epsilon^2 K_2(x) e^u = 0 \quad \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \epsilon k(x)^{\frac{1}{2}} e^{\frac{u}{2}} - k_1 \quad \text{on } \Gamma_1 \\ u = 0 \quad \text{on } \Gamma_2 \end{cases} \tag{9.1}$$

with $K_2(x)$ and $k_1(x)$ functions that would represent the Gauss curvature of S_2 and the geodesic curvature of the boundary of S_1 in the geometrical problem (1.2), but we considered $K_2 \equiv 1$ and $k_1 \equiv 0$ for simplicity, so the introduction of these functions is left for future work.

Finally the characterization of the critical points of φ_m is also left for future work, and we can see a proposed approach in [DMK] for a similar system of equations with Robin boundary condition.

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