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APORTES AL ESTUDIO DE OPERADORES ELÍPTICOS NO LINEALES

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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La primera parte de la presente memoria busca encontrar la sucesión completa de valores propios asociados a funciones propias con simetría radial para el problema

$$\begin{aligned} H(u'', u', x) + \langle b(x), |\nabla u|^{\alpha} \nabla u \rangle + c(x)|u|^{\alpha} u = -\lambda |u|^{\alpha} u \quad & \text{en } B_R(0), \\ u = 0 \qquad\qquad & \text{en } \partial B_R(0), \end{aligned}$$

donde H es un operador elíptico $(\alpha+1)$-homogéneo y H, b y c presentan simetría radial. Para el caso unidimensional la elipticidad permite reformular este problema como un problema cuasilineal del tipo $(\alpha+2)$-Laplaciano. Esta reformulación permite usar argumentos de ecuaciones diferenciales ordinarias para encontrar el primer valor propio en un intervalo. Posteriormente un argumento tipo Nehari, basado en teoría del grado, posibilita localizar los k ceros de la k-ésima función propia, construida al tomar la primera función propia entre dos ceros consecutivos. Esta operación puede hacerse unívocamente gracias a un principio del máximo ad hoc. Finalmente, cotas apropiadas para las soluciones en dimensiones mayores permiten emplear los mismos argumentos del caso unidimensional.

La segunda parte está enfocada a resolver una ecuación con no linealidad no Lipschitziana y un operador integral:

$$(-\Delta)^{\alpha} u = u^p - u^q \quad \text{en } \mathbb{R}^N, \qquad \lim_{|x|\to\infty} u(x) = 0,$$

donde $u > 0$, $\alpha \in (0,1)$, $0 < q < 1 < p < \frac{N+2\alpha}{N-2\alpha}$ y $N \geq 3$. Una técnica basada en el principio variacional de Ekeland y el teorema del paso de la montaña permite demostrar la existencia de soluciones débiles en $H^{\alpha}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$. Mediante una iteración basada en la teoría L^p, el uso del núcleo de Bessel (al sumar u a ambos lados de la ecuación) y un argumento de localización de Silvestre se prueba la regularidad de las soluciones en $H^{\alpha}(\mathbb{R}^N)$; en particular, que $(-\Delta)^{\alpha} u$ puede evaluarse en cada punto de $\mathbb{R}^N$.

El uso de subsoluciones y supersoluciones apropiadas permite encontrar la tasa de decaimiento de las soluciones clásicas del problema. Finalmente, empleando un resultado de simetría de Terracini para un problema con condición de borde Neumann en el semiespacio, junto al trabajo de Caffarelli y Silvestre, se muestra la simetría radial de las soluciones del problema.

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Contents

Capítulo 1

Introducción y resumen en castellano

1.1. Un problema de valores propios

Una herramienta muy importante en el análisis de ecuaciones no lineales es la comprensión del problema de valores propios asociado. Para ilustrar este punto, consideremos el problema de valores propios

$$-\Delta u = \lambda u \quad \text{en } \Omega, \tag{1.1.1}$$

$$u = 0 \quad \text{en } \partial\Omega, \tag{1.1.2}$$

donde $\Omega \subseteq \mathbb{R}^N$ es un dominio acotado. Es sabido [39, §8.12] que existe una sucesión creciente y divergente a $+\infty$ de valores propios λ_n, los cuales son aislados (esto es, no hay otros valores propios en una vecindad). Adicionalmente, el primer valor propio es simple (el espacio de funciones propias asociadas a λ_1 tiene dimensión 1) y la función propia asociada es positiva en Ω. Más aún, el conjunto de funciones propias para este problema es una base de Hilbert de $H_0^1(\Omega)$. Algunas de las propiedades de la sucesión de valores propios son conocidas para otros operadores, como el p-Laplaciano unidimensional en el trabajo de Del Pino y Manásevich [29]. El lector también puede consultar [31] y las referencias allí mencionadas.

1.1.1. Algunos comentarios sobre trabajos previos

En el contexto de ecuaciones totalmente no lineales el problema de encontrar el conjunto completo de valores propios es un problema abierto. El problema

$$F(D^2u, Du, u, x) = -\lambda u \quad \text{en } \Omega, \tag{1.1.3}$$

$$u = 0 \qquad \text{en } \partial\Omega, \tag{1.1.4}$$

donde F es un operador elíptico 1-homogéneo, fue primeramente abordado por P.L. Lions, [48, 49], usando argumentos probabilísticos. Posteriormente el problema fue estudiado usando métodos propios de ecuaciones en derivadas parciales; más específicamente, el método de soluciones viscosas [26, 42] permitió encontrar el primer valor propio del problema (1.1.3)–(1.1.4); por ejemplo, en el trabajo de Armstrong [3], Quaas y Sirakov [54], Ishii y Yoshimura [43] y Patrizi [53]. Resultados similares han sido obtenidos para operadores que generalizan el p-Laplaciano:

$$\begin{aligned} F(D^2u, Du, x) &= -\lambda |u|^\alpha u \quad \text{en } \Omega,\ \alpha > -1, \\ u &= 0 \qquad\qquad \text{en } \partial\Omega, \end{aligned} \tag{1.1.5}$$

por Birindelli and Demengel en una serie de publicaciones: [4], [5], [6], [7] y [8]. La noción de solución para problemas como (1.1.5) fue definida por Chen, Giga y Goto en [21]. Es interesante también mencionar publicaciones relacionadas a este problema por Juutinen [44], Imbert [41] y Dávila, Felmer y Quaas [27, 28].

La mayoría de los resultados citados usan argumentos relativamente complicados basados en la teoría de soluciones viscosas, pero la simetría radial permite usar argumentos más sencillos, que además requieren hipótesis más débiles. Resultados iniciales en esta dirección fueron obtenidos por Busca, Esteban y Quaas en [10], donde se obtiene la sucesión completa de valores propios y funciones propias para el operador de Pucci. Más recientemente Allendes y Quaas [1] extendieron este resultado a una clase más general de operadores extremales 1-homogéneos, y Demengel [30] demostró que un resultado similar se tiene para una clase de operadores $(\alpha+1)$-homogéneos, incluyendo operadores singulares ($\alpha \in (-1, 0)$) y degenerados ($\alpha \in (0, +\infty)$). Si bien la teoría provee la sucesión completa de valores propios, caracterizada por el número de ceros de las respectivas funciones propias, en los trabajos citados los resultados sólo se tienen para operadores autónomos. Esto se debe a que la herramienta base para los resultados de existencia es el método del disparo, para posteriormente reescalar y así obtener una solución en la bola o anillo deseado. Una excepción es el trabajo de Esteban, Felmer y Quaas [33], basado en una aproximación tipo

Nehari a la que nos referiremos más adelante.

En este contexto la noción de valor propio ha sido extendida para un operador como el del problema (1.1.5) en la forma de dos *semivalores propios* λ^+ y λ^- asociados a funciones propias ϕ^+ y ϕ^-, que son respectivamente positiva y negativa en el dominio, por ejemplo, por Armstrong [3] y por Quaas y Sirakov en [54]. Para ilustrar este punto, consideremos el siguiente operador en el caso $N = 1$: escribamos $x^+ := \max\{x, 0\}$, $x^- := \max\{-x, 0\}$ y definamos

$$F(m) := (m)^+ - \gamma(m)^-,$$

con $\gamma \in (0, 1)$. Para el problema de valores propios

$$F(u'') = -\lambda u \quad \text{en } (0, 1), \qquad u(0) = u(1) = 0,$$

tenemos una función propia positiva $\phi_+(t) = \sin(\pi t)$ asociada al valor propio $\lambda^+ = \gamma\pi^2$ y la función propia negativa $\phi_-(t) = -\sin(\pi t)$ asociada al valor propio $\lambda_- = \pi^2$. Notemos además que cada una de estas funciones propias lo sigue siendo si ponderamos por un real positivo, pero no si ponderamos por un real negativo. Ello se debe a que F es un operador positivamente homogéneo.

El trabajo previo para nuestro caso $(\alpha + 1)$-homogéneo presenta fundamentalmente dos desventajas:

- los argumentos de viscosidad requieren una teoría relativamente compleja, innecesaria para el caso radial, y
- argumentos más sencillos, como en el trabajo de Demengel [30], son válidos sólo en el caso de un operador autónomo.

1.1.2. El resultado obtenido

El problema de valores propios que tratamos en el capítulo 3 consiste en encontrar la sucesión completa de *valores propios* (en el sentido de los semivalores propios) del problema

$$F(D^2u, Du, u, x) = -\lambda |u|^\alpha u \quad \text{en } B_R, \tag{1.1.6}$$

$$u = 0 \quad \text{en } \partial B_R, \tag{1.1.7}$$

donde B_R es la bola de radio R en $\mathbb{R}^N$ centrada en 0 y F es un operador elíptico $(\alpha+1)$-positivamente homogéneo, $\alpha > -1$. Denotando $\varphi(s) := |s|^\alpha s$, el operador que nos interesa es

$$F(M,p,u,x) := H(M,p,x) + b(|x|)\varphi(p) + c(|x|)\varphi(u),$$

donde b, $c \in C([0,R])$, $\|b\|_\infty$, $\|c\|_\infty \leq \gamma$ y H cumple las siguientes hipótesis:

(H1) para todo $t \in \mathbb{R} \setminus \{0\}$, $\mu \geq 0$, $x \in \Omega$, $p \in \mathbb{R}^N \setminus \{0\}$, $X \in \mathcal{S}(N)$ se tiene

$$H(\mu X, tp, x) = |t|^\alpha \mu H(X,p,x),$$

(H2) existen $0 < \lambda \leq \Lambda$ tales que para todo $x \in \Omega$, $p \in \mathbb{R}^N \setminus \{0\}$, $M_1 \in \mathcal{S}(N)$, $M_2 \in \mathcal{S}(N)$,

$$|p|^\alpha \mathcal{M}^-_{\lambda,\Lambda}(M_2) \leq H(M_1 + M_2, p, x) - H(M_1, p, x) \leq |p|^\alpha \mathcal{M}^+_{\lambda,\Lambda}(M_2),$$

(H3) y el operador F tiene simetría radial; esto es,

$$H\left(\frac{p}{r}I + \big(m - \frac{p}{r}\big)\frac{x \otimes x}{r^2}, \frac{p}{r}x, x\right)$$

sólo depende de x a través de $r = |x|$.

Cabe mencionar que la hipótesis de elipticidad (H2) implica que Δ es un operador elíptico, pero no $-\Delta$, lo cual es subsanado poniendo $-\lambda$ al lado derecho de (1.1.6). Un operador sencillo, pero totalmente no lineal, que cumple las hipótesis es

$$H(M,p,x) = |p|^\alpha \mathcal{M}^+_{\lambda,\Lambda}\big(a(|x|)M\big),$$

donde $a : [0,R] \to \mathbb{R}$ es una función continua y acotada. Si $\lambda = \Lambda$, obtenemos el $(\alpha+2)$-Laplaciano, y si $a \equiv 1$, el operador es $|p|^\alpha \mathcal{M}^+_{\lambda,\Lambda}(M)$, que es la generalización totalmente no lineal más sencilla del $(\alpha+2)$-Laplaciano, tratada por Demengel en [30].

En primer lugar tratamos el caso unidimensional. El resultado inicial, fundamental para el trabajo posterior, es

Lema 1.1.1. *Bajo las condiciones (H1)–(H2) existe una función* $G : \mathbb{R}^2 \times (\mathbb{R} \setminus \{0\}) \times [a,b] \to \mathbb{R}$ *tal que*

$$F(m,p,u,t) = q \iff |p|^\alpha m = G(p,u,q,t)$$

si $p \neq 0$*, donde* G *es Lipschitz en* $(q, \varphi(u), \varphi(p))$ *y estrictamente creciente en* q*, y en* u *cuando* $c < 0$*.*

La demostración de este lema se basa en la condición (H2) de elipticidad, que muestra que $F(\cdot, p, u, t)$ toma valores a todo $\mathbb{R}$. Este lema permite que las soluciones del problema

$$F[u] = f, \quad u(a) = A, \quad u'(a) = B$$

puedan ser entendidas de la siguiente forma:

Definición 1.1.1. *u se dice solución si $u \in C^1([a,b])$, $\varphi(u')$ es absolutamente continua en cada subconjunto compacto de (a,b) y u resuelve c.t.p. la ecuación*

$$\varphi(u')' = (\alpha+1)G(u', u, f, t) \text{ en } (a,b), \quad u(a) = A, \quad u'(a) = B.$$

Similarmente entenderemos una solución del problema de condiciones de borde. También entenderemos $F[u] \geq F[v]$ como que existen funciones continuas f_u, f_v tales que $F[u] = f_u$, $F[v] = f_v$ y $f_u \geq f_v$. Aquí denotamos $F[u] = H(u'', u', t) + b\varphi(u') + c\varphi(u)$.

Posteriormente probamos un principio de comparación:

Corolario 1.1.1. *Supongamos que F satisface (H2), $\kappa > \gamma$, y que u, v son tales que*

$$F[v] + \kappa\varphi(v) \geq F[u] + \kappa\varphi(u) \quad \text{en } (a,b),$$

$v(a) \leq u(a)$ y $v(b) \leq u(b)$. Entonces $v \leq u$ en (a,b).

El siguiente resultado, indispensable para poder reunir las funciones propias de manera única, es el Corolario 3.2.3: si $F[u] = \mu\varphi(u) + f$, $u(a) = 0$ y $\mu \geq \gamma$, entonces $u'(a) \neq 0$.

En la Sección 3.3 algunos otros resultados auxiliares permiten probar la unicidad del problema de condición inicial (Lema 3.3.2)

$$\begin{aligned} \varphi(u')' &= (\alpha+1)G(u', u, f+\kappa\varphi(u), t) \quad \text{en } (a,b), \\ u(T) &= A, \quad u'(T) = B \end{aligned}$$

para el caso singular, y del problema (Lema 3.3.3)

$$\begin{aligned} \varepsilon u'' + \varphi(u')' &= (\alpha+1)G(u', u, f+\kappa\varphi(u), t) \quad \text{en } (a,b), \\ u(T) &= A, \quad u'(T) = B \end{aligned}$$

en el caso degenerado. De aquí se obtiene en el Teorema 3.3.1 la existencia y unicidad de solución para el problema de condición de borde

$$F[u] - \kappa\varphi(u) = f(t) \quad \text{en } (a,b),\ u(a) = u(b) = 0,$$

usando el método del disparo y el principio de comparación para el caso singular, y argumentos de compacidad para el caso degenerado, tras primero probar unicidad para el problema perturbado

$$\varepsilon u'' + \varphi(u')' = (\alpha+1)G(u', u, f + \kappa\varphi(u), t), \quad u(a) = u(b) = 0.$$

El Teorema 3.3.2 permite obtener resultados de unicidad en el caso radial.

En la Sección 3.4 probamos el Teorema 3.4.2, que da la existencia del primer valor propio. Más específicamente,

Teorema 1.1.1. *Bajo las hipótesis (H1) y (H2) el problema*

$$\begin{aligned} F[u] &= -\mu\varphi(u) \quad \text{en } (a,b), \\ u(a) &= u(b) = 0 \end{aligned}$$

tiene una solución (u^+, λ^+) *tal que* $u^+ > 0$ *en* (a,b)*, y otra solución* (u^-, λ^-) *tal que* $u_- < 0$ *en* (a,b)*. Más aún, toda solución positiva (resp. negativa) de este problema es un múltiplo de* u^+ *(resp.* u^-*).*

Para la demostración de este teorema usamos un argumento de punto fijo y el teorema VIII.1 en [57], que permite encontrar soluciones a nuestro problema en el contexto de espacios de Banach mediante teoría del grado, de manera similar a [33]. Una vez encontrado el primer valor propio en el intervalo (a,b) estudiamos la continuidad y monotonía del primer valor propio en $(a_1, b_1) \subsetneq (a,b)$.

En la siguiente sección encontramos la sucesión completa de valores propios para el problema unidimensional. Para ello, un argumento de teoría del grado (Lema 3.5.1) permite encontrar la ubicación de los ceros de la función propia. Usando el Corolario 3.2.3, citado más arriba, podemos pegar las funciones propias en cada subintervalo entre dos ceros ponderando apropiadamente, gracias a que en los extremos la derivada es no nula. La unicidad (salvo constante multiplicativa positiva) de la función propia sigue el mismo argumento del Teorema 4.1 de [33].

Una vez estudiado el problema unidimensional nos abocamos al problema radial. El problema que debemos enfrentar en este caso es la singularidad en $r = 0$. En primer lugar, la existencia de resultados iniciales análogos al Lema 1.1.1 permiten definir la noción de solución para el problema en N dimensiones como la solución del problema unidimensional asociado. Posteriormente, el problema aproximado

$$\mathcal{F}[u] - \kappa\varphi(u) = f \quad \text{en } (\varepsilon, R), \tag{1.1.8}$$

$$u'(\varepsilon) = u(R) = 0 \tag{1.1.9}$$

(donde $\mathcal{F}$ es el operador radial escrito en términos de variables unidimensionales) es resuelto obteniendo cotas independientes de ε (Lema 3.6.4) para la solución de (1.1.8)–(1.1.9), lo que permite tomar el límite $\varepsilon \to 0$, posibilitando encontrar el primer valor propio en el caso radial (Teorema 3.6.2). También es necesaria la desigualdad de Alexandrov-Bakelman-Pucci, probada para un caso como el nuestro en [27], de la cual exponemos una versión simplificada en el caso unidimensional en la Proposición 3.2.4, mientras que el caso radial y la demostración para ambos casos se encuentran en la Proposición 3.6.1.

Repitiendo el argumento tipo Nehari podemos concluir la existencia de la sucesión de valores propios en el Teorema 3.1.1:

Teorema 1.1.2. *Bajo las hipótesis (H1), (H2) y (H3), el problema de valores propios* (1.1.6) *tiene una sucesión de soluciones c.t.p.* $\{(\lambda_n^\pm, u_n^\pm)\}$, *donde* u_n^+, u_n^- *tienen exactamente* n *ceros interiores* $0 < r_1 < \ldots < r_n < R$, *y* u_n^+ *(respectivamente* u_n^-*) es positiva (resp. negativa) en* $(0, r_1)$. *La sucesión* $(\lambda_n^\pm)_n$ *es creciente en* n, *y la sucesión* $\{(\lambda_n^\pm, u_n^\pm)\}$ *es completa en el sentido que no hay otros pares de valor propio y función propia para* (1.1.6) *que tengan simetría radial.*

1.2. Una ecuación involucrando el Laplaciano fraccionario

La segunda parte de esta memoria estudia la ecuación

$$\begin{cases} (-\Delta)^\alpha u + u^q = u^p \quad \text{en } \mathbb{R}^N, \\ u > 0 \quad \text{en } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.2.1}$$

donde $\alpha \in (0,1)$, $0 < q < 1 < p < \frac{N+2\alpha}{N-2\alpha}$ y $N \geq 3$. El operador $(-\Delta)^\alpha$ puede ser definido de varias maneras:

Definición 1.2.1. *Sea* $v \in H^\alpha(\mathbb{R}^N)$, *y denotemos la transformada de Fourier como* $\mathcal{F}$.

Definimos el laplaciano fraccionario de v como

$$\mathcal{F}\big((-\Delta)^{\alpha}v\big) := |\xi|^{2\alpha}\mathcal{F}(v).$$

Si v es suficientemente regular; en particular, si v está en la clase de Schwarz, la siguiente definición es equivalente a la primera:

Definición 1.2.2. *Sean $v \in \mathcal{S}(\mathbb{R}^N)$ y $x \in \mathbb{R}^N$. Entonces*

$$(-\Delta)^{\alpha}v(x) = C_{N,\alpha}P.V. \int_{\mathbb{R}^N} \frac{v(x)-v(y)}{|x-y|^{N+2\alpha}}dy,$$

donde $P.V.$ es el valor principal de Cauchy.

La demostración de la equivalencia se encuentra en [45]. La tercera definición fue recientemente obtenida por Caffarelli y Silvestre [17]:

Definición 1.2.3. *Sea $\mathrm{E}_{\alpha}(u)$ la extensión α-armónica de u una función regular; esto es,*

$$\begin{aligned} \nabla\cdot(y^{1-2\alpha}\nabla \mathrm{E}_{\alpha}(u)) = 0 \quad & en\ \mathbb{R}^{N+1}_{+} \\ \mathrm{E}_{\alpha}(u)(x,0) = u \quad & en\ \mathbb{R}^{N}. \end{aligned}$$

Entonces

$$(-\Delta)^{\alpha}u(x) = -C_{N,\alpha}\lim_{y\to 0^+} y^{1-2\alpha}\frac{\partial \mathrm{E}_{\alpha}(u)}{\partial y}(x,y).$$

El problema que buscamos estudiar en este capítulo resulta ser de interés por la no Lipschitzianidad del lado derecho, un hecho relevante tanto en las cotas precisas que debemos obtener como en los métodos a emplear para obtener la existencia, regularidad, decaimiento y simetría de las soluciones de (1.2.1).

1.2.1. Resultados previos

El laplaciano fraccionario es un operador integral de orden 2α, y puede ser visto como el generador infinitesimal de un proceso de Lévy, lo que ha demostrado tener interesantes aplicaciones (por ejemplo, [16, 22, 32], [62] y las referencias allí mencionadas). Recientemente estos fenómenos han atraído gran interés, y problemas de condición de borde, transición de fase, frontera libre y otros han sido estudiados; por ejemplo, por Brändle, Colorado y de Pablo [9], Cabré y Roquejoffre [11], Cabré y Sire [12], Cabré y Solà Morales [13], Cabré y Tan [14], Caffarelli y Silvestre [17, 18], Capella, Dávila, Dupaigne y Sire [19], Silvestre [58], y Sire y Valdinoci [59].

Cuando $\alpha = 1$ y $q = 1$, obtenemos la clásica ecuación

$$-\Delta u + u = u^p \quad \text{en } \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N),$$

muy estudiada en las últimas décadas. Si $\alpha = 1$, pero $0 < q < 1$, la ecuación obtenida es

$$-\Delta u = u^p - u^q \quad \text{en } \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N),$$

estudiada, por ejemplo, por Cortázar, Elgueta y Felmer [23, 24]. En estos trabajos los autores muestran la existencia de soluciones en $H^1(\mathbb{R}^N)$ a soporte compacto; más aún, cada componente conexa de una solución en $H^1(\mathbb{R}^N)$ es una bola, y la solución es radial en dicha bola. También demuestran que la solución de mínima energía es única, salvo traslaciones.

Si $\alpha < 1$ y $q = 1$, la ecuación es

$$(-\Delta)^\alpha u + u = u^p \text{ en } \mathbb{R}^N,\ u \in H^\alpha(\mathbb{R}^N),$$

tratada por Felmer, Quaas y Tan en [37]. Las soluciones de esta ecuación son clásicas (en un sentido a precisar), su tasa de decaimiento en infinito es $-N - 2\alpha$, y son radialmente simétricas. El resultado puede extenderse a un lado derecho un poco más general.

1.2.2. El resultado obtenido

Primeramente debemos definir la noción de solución para (1.2.1):

Definición 1.2.4. $u \in H^\alpha(\mathbb{R}^N)$ *se dice solución débil de* (1.2.1) *si*

$$\int_{\mathbb{R}^N} |\xi|^{2\alpha} |\hat{u}| \hat{v} d\xi = \int_{\mathbb{R}^N} (u^p - u^q) v dx$$

se tiene para toda $v \in H^\alpha(\mathbb{R}^N)$.

En caso que u sea suficientemente regular (en particular, si $u \in C^2$ y es acotada), podemos emplear la definición 1.2.2, que puede ser reescrita como

$$(-\Delta)^\alpha u(x) = -\int_{\mathbb{R}^N} \frac{\delta(u)(x,y)}{|y|^{N+2\alpha}} dy, \tag{1.2.2}$$

donde $\delta(u)(x, y) := u(x + y) + u(x - y) - 2u(x)$.

Definición 1.2.5. *$u \in C(\mathbb{R}^N)$ se dice solución clásica de (1.2.1) si $(-\Delta)^\alpha u$ puede escribirse como en (1.2.2) y (1.2.1) se cumple puntualmente en $\mathbb{R}^N$.*

El primer resultado que obtuvimos es

Teorema 1.2.1.

(I) *El problema (1.2.1) tiene una solución débil $u \in H^\alpha(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$, y $u \geq 0$ c.t.p. en $\mathbb{R}^N$.*

(II) *Si $u \in H^\alpha$ es solución débil de (1.2.1), entonces u es una solución clásica. En particular, $u > 0$ en $\mathbb{R}^N$.*

Es interesante notar el contraste con el caso $\alpha = 1$ y $0 < q < 1$, en que las soluciones presentan soporte compacto.

La demostración de este teorema consiste en encontrar puntos críticos del funcional

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\xi|^{2\alpha}\hat{u}^2 d\xi - \int_{\mathbb{R}^N}\left(\frac{u_+^{p+1}}{p+1} - \frac{|u|^{q+1}}{q+1}\right) dx.$$

definido para $u \in H^\alpha(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$ mediante el principio variacional de Ekeland. La demostración de positividad de una solución débil está basada en la fórmula

$$\int_{\mathbb{R}^N} (-\Delta)^\alpha u \cdot \varphi dx = C \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{N+2\alpha}} dxdy,$$

de la cual se deduce la positividad al usar u_- como función test.

Para demostrar la regularidad de las soluciones débiles, podemos reescribir la ecuación como

$$(-\Delta)^\alpha u + u = u^p + u - u^q \quad \text{en } \mathbb{R}^N, \tag{1.2.3}$$

nuevamente con $u > 0$ y $\lim_{|x|\to\infty} u(x) = 0$. Esta formulación permite emplear el núcleo de Bessel $\mathcal{K}$; esto es, si $u = \mathcal{K} * f$, entonces $(-\Delta)^\alpha u + u = f$. Los Teoremas 4.3.1 y 4.3.3 proveen las propiedades del núcleo y las inclusiones tipo Sobolev necesarias para nuestro trabajo.

La primera parte de la demostración es el Lema 4.3.1, que prueba que una solución débil de (1.2.1) pertenece a $C^{0,\mu}(\mathbb{R}^N)$, para algún $\mu > 0$, aplicando la iteración usual de la teoría L^p junto con un argumento de localización de Silvestre [58] que permite evitar la singularidad del núcleo en torno a 0.

Demostración del Teorema 1.2.1, parte II. Sea u_1 tal que $(-\Delta)^\alpha u_1 + u_1 = \eta_1 h$ en $\mathbb{R}^N$, donde $\eta_1 \in C_0^\infty(\mathbb{R}^N, [0,1])$ y $\eta \equiv 1$ en B_1. Usando que $g(s) = s^p - s^q$ es σ-Hölder continua, concluimos, como en el lema de regularidad, que $u_1 \in C^{0,\sigma_0}$ para cierto $\sigma_0 \in (0, \sigma)$. Por lo tanto, si w es una solución de

$$-\Delta w = -u_1 + \eta_1 h,$$

entonces $w \in C^{2,\sigma_0}$. Cuando $2\alpha + \sigma_0 > 1$, $(-\Delta)^{1-\alpha} w \in C^{1,2\alpha+\sigma_0-1}$; en caso contrario, $(-\Delta)^{1-\alpha} w \in C^{0,2\alpha+\sigma_0}$. Pero

$$(-\Delta)^\alpha \big(u_1 - (-\Delta)^{1-\alpha} w\big) = 0,$$

lo que implica que $u_1 - (-\Delta)^{1-\alpha} w$ es armónica; luego, u_1 tiene la misma regularidad que $(-\Delta)^{1-\alpha} w$. Repitiendo el procedimiento (como en [37]), obtenemos la misma regularidad para u en $\mathbb{R}^N$ (dependiendo del signo de $2\alpha + \sigma_0 - 1$), y en ambos casos la representación (1.2.2) del laplaciano fraccionario se cumple. La demostración de $u > 0$ en $\mathbb{R}^N$ se basa en (1.2.2) y en que $g(s) < 0$ cerca de 0. □

El segundo resultado obtenido es

Teorema 1.2.2. *Sea u una solución clásica de (1.2.1). Entonces existe una constante $C > 0$ tal que*

$$u(x) \le C|x|^{\frac{-(N+2\alpha)}{q}},$$

y para todo $\eta < -(N+2\alpha)/q$, existe $L_0 > 0$ tal que

$$u(x) \ge |x|^\eta \quad \textit{si } |x| > L_0.$$

La demostración de este teorema es hecha en dos partes. En primer lugar, usando que, cerca de $z = 0$, se tiene

$$\delta(w)(x,z) = \left\langle z, \left(\int_0^1 [D^2 w(x+tz) + D^2 w(x-tz)](1-t)dt \right) z \right\rangle$$

podemos demostrar que si $w(x) := \beta |x|^\eta$, para $\eta < -\frac{N+2\alpha}{q}$, entonces existe R_0 tal que

$$(-\Delta)^\alpha w(x) + w(x)^q \le 0 \quad \text{si } |x| \ge R_0.$$

La segunda parte de la demostración consiste en probar que si

$$w(x) = C|x|^{\frac{-(N+2\alpha)}{q}}$$

para $|x|$ suficientemente grande, entonces

$$(-\Delta)^{\alpha} w + \frac{1}{2} w^q \geq 0.$$

Nuestro tercer teorema es

Teorema 1.2.3. *Sea u solución clásica de (1.2.1). Entonces u es radialmente simétrica en torno a algún punto de $\mathbb{R}^N$.*

Para probar este teorema necesitamos un resultado de simetría de Terracini [61]:

Theorem 1.2.1. *Sea $\varphi \in L^{\delta}(\mathbb{R}^{N+1}_{+})$, $\varphi \geq 0$ solución de*

$$\begin{cases} -\Delta \varphi = f(x, y, \varphi) & \text{en } \mathbb{R}^{N+1}_{+}, \\ -\dfrac{\partial \varphi}{\partial y} = g(x, \varphi) & \text{en } \mathbb{R}^{N}, \end{cases}$$

donde $x \in \mathbb{R}^N$, $y > 0$, y asumamos las siguientes hipótesis:

- *f y g son no decrecientes en la dirección x_1, para $x_1 < 0$.*

- *Existen μ_1, $\mu_2 > 0$, $\sigma_1 \in L^{(N+1)/2}(\mathbb{R}^{N+1}_{+})$, $\sigma_2 \in L^N(\mathbb{R}^N)$, ρ_1 y ρ_2 tales que, para todo $t > s > 0$,*

$$\frac{f(x, y, s) - f(x, y, t)}{s - t} \leq \sigma_1(x, y) + \rho_1(x, y) t^{\mu_1},$$
$$\frac{g(x, s) - g(x, t)}{s - t} \leq \sigma_2(x) + \rho_1(x) t^{\mu_1}.$$

- *$\rho_1 \varphi^{\mu_1} \in L^{(N+1)/2}(\mathbb{R}^{N+1}_{+})$ y $\rho_2 \varphi(x, 0)^{\mu_2} \in L^N(\mathbb{R}^N)$.*

Sea (x_λ, y) la reflexión de (x, y) con respecto al hiperplano $\{(x, y) \in \mathbb{R}^{N+1}_{+} : x_1 = \lambda\}$ y $\varphi^\lambda(x, y) = \varphi(x_\lambda, y)$. Entonces se tiene que $\varphi^\lambda > u$ en Σ^λ para todo $\lambda < 0$, o bien existe $\lambda^* < 0$ tal que $\varphi^{\lambda^*} = \varphi$ en Σ^{λ^*}.

Una adaptación de este resultado permite obtener una conclusión análoga para $\mathrm{E}_\alpha(u)$, la extensión armónica de u; vale decir, $\mathrm{E}_\alpha(u)$ resuelve

$$\begin{aligned} \nabla \cdot (y^{1-2\alpha} \nabla \mathrm{E}_\alpha(u)) = 0 \quad & \text{en } \mathbb{R}^{N+1}_+ \\ \mathrm{E}_\alpha(u)(x,0) = u \quad & \text{en } \mathbb{R}^N, \end{aligned}$$

lo que permite concluir por el resultado de Caffarelli y Silvestre [17].

Una vez demostrada la simetría de las soluciones de (1.2.1), analizamos brevemente las dificultades asociadas a intentar aplicar directamente el método de los planos móviles a u.

Chapter 2

Introduction

During all this thesis N is a positive integer, $B_r(x)$ is the ball of radius $r > 0$ centered at $x \in \mathbb{R}^N$, $\langle \cdot, \cdot \rangle$ denotes the usual dot product in $\mathbb{R}^N$, $\nabla \cdot$ is the divergence operator, $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$ and C is a constant which may change during a computation, even when appearing twice in the same line. We may also omit the variable with respect to which an integral is taken.

2.1. An eigenvalue problem

A very important tool for the analysis of nonlinear equations is the understanding of the associated eigenvalue problem. To illustrate this, consider the eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \tag{2.1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2.1.2}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain. It is well known [39, §8.12] that there is an increasing sequence λ_n of eigenvalues, which are isolated (that is, there are no other eigenvalues in a neighbourhood of λ_1). Additionaly, the first eigenvalue is simple (the dimension of the subspace of H_0^1 generated by the eigenfunctions associated to λ_1 is 1) and the associated eigenfunction is positive in Ω. Moreover, the set of eigenfunctions for this problem is a Hilbert basis for $H_0^1(\Omega)$. Some of the properties of the sequence of eigenvalues are also known for other operators, such as the p-Laplacian in one dimension as was found by Del Pino and Manásevich in [29]. We also refer the reader to [31] and the references therein.

2.1.1. Previous results

Within the context of fully nonlinear equations the problem of finding the whole sequence of eigenvalues remains an open problem. The problem

$$F(D^2u, Du, u, x) = -\lambda u \quad \text{in } \Omega, \tag{2.1.3}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{2.1.4}$$

where F is a 1-homogeneous elliptic operator, was first addressed by P.L. Lions [48, 49] by probabilistic arguments. Afterwards the problem was studied using purely partial differential equations arguments; namely, the method of viscosity solutions [26, 42] allowed to find the first eigenvalue of (2.1.3)–(2.1.4); for example, the work of Armstrong [3], Quaas and Sirakov [54], Ishii and Yoshimura [43] and Patrizi [53]. Similar results have been obtained for problems generalising the p-Laplacian:

$$\begin{aligned} F(D^2u, Du, u, x) &= -\lambda|u|^\alpha u \quad \text{in } \Omega,\ \alpha > -1, \\ u &= 0 \qquad\qquad \text{on } \partial\Omega, \end{aligned} \tag{2.1.5}$$

by Birindelli and Demengel in a series of papers: [4], [5], [6], [7] and [8]. The notion of solution for problems such as (2.1.5) was defined by Chen, Giga and Goto in [21]. It is also interesting to mention related works by Juutinen [44], Imbert [41] and Dávila, Felmer and Quaas [27, 28].

Most of the results cited use rather complicated arguments based in the theory of viscosity solutions, but radial symmetry allows to use simpler arguments, also requiring weaker hypotheses. First results in this direction were obtained by Busca, Esteban and Quaas in [10], where the whole sequence of eigenvalues and eigenfuctions was obtained for Pucci's operator. More recently Allendes and Quaas [1] extended this result to a more general class of 1-homogeneous extremal operators, and Demengel [30] was able to prove that a similar result holds for a class of $(1+\alpha)$-homogeneous operators, including singular ($\alpha \in (-1, 0)$) and degenerate ($\alpha \in (0, \infty)$) operators. While the theory provides infinitely many eigenvalues and eigenfunctions characterised by their number of zeros, the results in these works apply only to autonomous operators. This is because the basic tool for existence theory is the shooting method to get oscillatory solutions and scaling to match the given size of the ball (domain). An exception is the work by Esteban, Felmer and Quaas [33], based on a Nehari type approach, which we explain later on.

Within this context the notion of eigenvalue has been extended for an operator such as the one in problem (2.1.5) as two *semi eigenvalues* λ^+ y λ^- associated to eigenfunctions ϕ^+ and ϕ^-, which are respectively positive and negative in Ω; for instance, see Armstrong [3] and Quaas and Sirakov in [54]. To illustrate this idea, consider the following operator in the case $N = 1$: let $x^+ := \max\{x, 0\}$, $x^- := \max\{-x, 0\}$, and define

$$F(m) := (m)^+ - \gamma(m)^-,$$

where $\gamma \in (0, 1)$. For the eigenvalue problem

$$F(u'') = -\lambda u \quad \text{in } (0,1), \quad u(0) = u(1) = 0,$$

we can find a positive eigenfunction $\phi_+(t) = \sin(\pi t)$ associated to the eigenvalue $\lambda^+ = \gamma\pi^2$, and a negative eigenfunction $\phi_-(t) = -\sin(\pi t)$ associated to the eigenvalue $\lambda_- = \pi^2$. Notice also that these eigenfunctions can only be scaled by a positive factor, due to F being positively homogeneous.

Previous work in our case $(\alpha + 1)-$homogeneous presents two main disadvantages:

- arguments using viscosity solutions theory become unnecessary complex for the radial case, and
- simpler methods, such as those used by Demengel in [30] are valid only if the operator is autonomous.

2.1.2. Contributions of this thesis

The first part of this thesis is devoted to solve the eigenvalue problem (2.1.5) for $\alpha > -1$, $\Omega = B_R(0)$ and F being a radially symmetric $(\alpha + 1)$-homogeneous operator. We can summarise the technique as follows: in the first place, problem (2.1.5) is rewritten as a p-Laplacian boundary problem using the ellipticity of the operator. Afterwards this ODE problem is uniquely solved (directly in the singular case, by means of a perturbation in the degenerate case), yielding the first eigenvalue. Then the n-th eigenvalue is obtained putting together n eigenfunctions, choosing appropriately the $n - 1$ interior zeros, using a Nehari type approach. Finally, an approximate problem is solved in the radial case, passing to the limit using bounds not depending on the approximation parameter.

This scheme is essentially the one used in [33], but in that work the fully nonlinear problem is proven to be equivalent to an ODE on u'' with Lipschitz continuous right hand side. The singular case is somewhat similar, but uniqueness for the initial value problem cannot be obtained directly for the degenerate case. In both non uniformly elliptic cases comparison principles are also considerably more difficult to prove, because the operator is not sublinear; hence, the comparison principles obtained are weaker than when uniform ellipticity holds.

2.2. An equation involving the fractional Laplacian

In the second part of this thesis we deal with the equation

$$\begin{cases} (-\Delta)^{\alpha} u + u^q = u^p \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{2.2.1}$$

where $\alpha \in (0,1)$, $0 < q < 1 < p < \frac{N+2\alpha}{N-2\alpha}$ and $N \geq 3$. There are three ways to define the fractional Laplacian $(-\Delta)^{\alpha}$: one is based on its Fourier transform, the second is a pointwise formula, and the third relies on the so called Dirichlet to Neumann operator. These definitions are given in detail in Chapter 4.

2.2.1. Previous results

The fractional Laplacian is an integral operator of order 2α, and it can be seen as the infinitesimal generator of a Lévy process, which has been found to be of interesting applications (e.g. [16, 22, 32], [62] and the references therein). Recently such phenomena have attracted much interest, and boundary value, phase transition, free boundary and other problems have been addressed; for example, by Brändle, Colorado and de Pablo [9], Cabré and Roquejoffre [11], Cabré and Sire [12], Cabré and Solà Morales [13], Cabré and Tan [14], Caffarelli and Silvestre [17, 18], Capella, Dávila, Dupaigne and Sire [19], Silvestre [58], and Sire and Valdinoci [59].

When $\alpha = 1$ and $q = 1$, we recover the classical equation

$$-\Delta u + u = u^p \quad \text{in } \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N),$$

extensively studied in the last decades. If $\alpha = 1$ but $0 < q < 1$, the equation becomes

$$-\Delta u = u^p - u^q \quad \text{in } \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N),$$

studied, for example, by Cortázar, Elgueta and Felmer [23, 24]. In these works the authors proved the existence of solutions in $H^1(\mathbb{R}^N)$ which are compactly supported; moreover, each connected component of a solution in $H^1(\mathbb{R}^N)$ is a ball, and the solution is radial in that ball. They also prove the solution of minimal energy is unique up to translations.

If $\alpha < 1$ and $q = 1$, the equation becomes

$$(-\Delta)^\alpha u + u = u^p \text{ in } \mathbb{R}^N, \; u \in H^\alpha(\mathbb{R}^N),$$

treated by Felmer, Quaas and Tan in [37]. Solutions for this equation are classical (in a sense to be precised), have a precise decay rate and are radially symmetric. This result can be extended for a slightly more general right hand side.

2.2.2. Contributions of this thesis

In Chapter 4 we study problem (2.2.1). This problem turns out to be interesting due to the right hand side being non Lipschitzian, a fact which is of relevance both on the precise bounds to be obtained and in the methods to be used for our conclusions. More precisely, we obtain the following results:

- There exists a weak solution of (2.2.1) in $H^\alpha(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$.
- If $u \in H^\alpha(\mathbb{R}^N)$ is a weak solution of (2.2.1), then u is a classical solution, decays as $|x|^{-(N+2\alpha)/q}$ at infinity, and is radially symmetric.

We can summarise the techniques as follows: to prove the existence of a weak solution we use variational arguments, an appropriate representation of $(-\Delta)^\alpha u$ tested with $\varphi \in H^\alpha(\mathbb{R}^N)$ and the properties of the kernel of the equation when rewritten as

$$(-\Delta)^\alpha u + u = u^p + u - u^q \quad \text{in } \mathbb{R}^N.$$

The decay rate is established using a sub-supersolution argument, while the symmetry result is a consequence of a symmetry result by Terracini [61] and the extension formula devised by Caffarelli and Silvestre [17]. Proving that solutions cannot have compact support is straightforward for classical solutions.

Chapter 3

An eigenvalue problem

3.1. Introduction and notation

In this chapter we are interested in the existence of nontrivial solutions (λ, u) of the boundary value problem

$$F(D^2u, Du, u, x) = -\lambda |u|^\alpha u \quad \text{in } B_R, \tag{3.1.1}$$

$$u = 0 \qquad \text{on } \partial B_R, \tag{3.1.2}$$

where $B_R = B_R(0) \subseteq \mathbb{R}^N$ and F is a positively $(\alpha+1)$-homogeneous fully nonlinear elliptic operator.

The purpose of this Chapter is to prove the existence of a sequence of eigenvalues and eigenfunctions for (3.1.1)–(3.1.2) in the radially symmetric case, based on arguments from elementary ODE theory and degree theory in a considerably self-contained fashion. Our main arguments are inspired from the 1-homogeneous case analysed by Esteban, Felmer and Quaas in [33]. This construction is based on the existence of two *semi-eigenvalues* (see, e.g., [3, 54]) associated to positive and negative eigenfunctions in the ball and in concentric annuli, put together via degree theory through a Nehari type approach [52]. In some of our proofs we also need to use the ABP inequality in the radial case, which in this setting can be easily proved.

Now we need to precise some definitions and fix notation before stating our main theorem.

Let $H : \mathcal{S}(N) \times (\mathbb{R}^N \setminus \{0\}) \times B_R \to \mathbb{R}$ and $b, c : \bar{B}_R \to \mathbb{R}$ be continuous functions, where $\mathcal{S}(N)$ is the space of $N \times N$ symmetric matrices with real coefficients. In particular, these hypotheses imply $b, c \in L^\infty(\bar{B}_R)$, and we shall write $\gamma := \sup\{\|c\|_\infty, \|b\|_\infty\}$. Next we define the operator

$$F(M, p, u, x) := H(M, p, x) + J(p, u, x),$$

where $J : \mathbb{R}^N \times \mathbb{R} \times \bar{B}_R \to \mathbb{R}$ is defined as

$$J(p, u, x) = \langle b(x), |p|^\alpha p \rangle + c(x)|u|^\alpha u.$$

For notational convenience we will often write

$$F[u] := F(D^2u, Du, u, x),\ H[u] := H(D^2u, Du, x) \quad \text{and} \quad J[u] := J(Du, u, x).$$

Concerning H and J we assume the following hypotheses:

(H1) For all $t \in \mathbb{R} \setminus \{0\}$, $\mu \geq 0$, $x \in \Omega$, $p \in \mathbb{R}^N \setminus \{0\}$, $X \in \mathcal{S}(N)$ we have

$$H(\mu X, tp, x) = |t|^\alpha \mu H(X, p, x).$$

(H2) There exist $0 < \lambda \leq \Lambda$ such that for all $x \in \Omega$, $p \in \mathbb{R}^N \setminus \{0\}$, $M_1 \in \mathcal{S}(N)$, $M_2 \in \mathcal{S}(N)$,

$$|p|^\alpha \mathcal{M}^-_{\lambda,\Lambda}(M_2) \leq H(M_1 + M_2, p, x) - H(M_1, p, x) \leq |p|^\alpha \mathcal{M}^+_{\lambda,\Lambda}(M_2).$$

(H3) The operator F is radially invariant; that is,

$$H\left(\frac{p}{r} I + \left(m - \frac{p}{r}\right)\frac{x \otimes x}{r^2}, \frac{p}{r}x, x\right) \quad \text{and} \quad J\left(\frac{p}{r}x, u, x\right)$$

only depend on x through $r = |x|$. Here $(x \otimes y)_{ij} = x_i y_j$.

In particular, (H3) implies that $b = b(|x|)$, $c = c(|x|)$.

In order to illustrate these hypotheses we provide two examples of operators satisfying (H1), (H2) and (H3).

Example 3.1.1. *The simplest example is*

$$H(M, p, x) = |p|^\alpha \mathcal{M}^+_{\lambda,\Lambda}\big(a(|x|)M\big),$$

where $a : [0, R] \to \mathbb{R}$ is a continuous, bounded function. If $\lambda = \Lambda$, we recover the $(\alpha + 2)$-Laplacian, and if $a \equiv 1$ we obtain $|p|^\alpha \mathcal{M}^+_{\lambda,\Lambda}(M)$, which is the operator treated by Demengel *in [30]. It is clear that this operator satisfies the proposed hypotheses.*

Example 3.1.2. *This example is an Isaacs type operator. Define*

$$L_{\rho\sigma}(M,p,t) := tr\,(A^{\rho\sigma}(t)M|p|^{\alpha})\,,$$

where $A^{\rho\sigma}:\mathbb{R}_+\to\mathcal{S}(N)$ *for any* $\rho\in R$, $\sigma\in S$, *and take*

$$H(M,p,x) := \inf_{\rho\in R}\sup_{\sigma\in S} L_{\rho\sigma}(M,p,|x|).$$

This operator clearly satisfies (H1) and (H3), and it also satisfies (H2) provided $A^{\rho\sigma}\in\mathcal{S}(N)$ *and* $\lambda I\le A^{\rho\sigma}(t)\le\Lambda I$ *for all* $(\rho,\sigma)\in R\times S$ *and* $t\in\mathbb{R}_+$.

Now we are in a position to state our main theorem.

Theorem 3.1.1. *Under hypotheses (H1), (H2) and (H3), the eigenvalue problem (3.1.1)–(3.1.2) has a sequence of solutions (in a sense to be precised)* $\{(\lambda_n^{\pm},u_n^{\pm})\}$, *where* u_n^+ *and* u_n^- *have exactly* n *interior zeros* $0<r_1<\ldots<r_n<R$, *and* u_n^+ *(respectively* u_n^-*) is positive (resp. negative) in* $(0,r_1)$. *The sequence* $(\lambda_n^{\pm})_n$ *is increasing and unbounded in* n, *and the sequence* $\{(\lambda_n^{\pm},u_n^{\pm})\}$ *is complete in the sense that there are no other radially symmetric eigenpairs of (3.1.1)–(3.1.2).*

For the proof of this theorem we analyse the eigenvalue problem (3.1.1)–(3.1.2) in concentric annuli. All these problems are one dimensional. This preliminary part of our work allows to prove Theorem 3.5.1, analogous to Theorem 3.1.1, in the one dimensional case, which we think is interesting by itself. Then we address the study of the eigenvalue problem in a ball by taking limit of the one dimensional case.

This Chapter is organised in the following way. In Section 3.2 we prove some auxiliary results for the one dimensional case. In particular we show that our problem can be rewritten as a quasilinear equation with a p-Laplacian term, allowing for a very simple notion of solution. Then we study the associated boundary value problem in the one dimensional case in Section 3.3, and we construct the first eigenvalue and eigenfunction in the one dimensional case in Section 3.4. Section 3.5 is devoted to prove Theorem 3.5.1 on the existence of the entire sequence of eigenvalues and eigenfunctions to the one dimensional eigenvalue problem. In Section 3.6 we make the necessary adaptations to conclude in the radial case, and we also prove the Alexandrov-Bakelman-Pucci inequality for radially symmetric operators, combining arguments from [33] and [27].

3.2. Preliminaries

In this section we prove some results which are on the basis of our analysis of the 1-dimensional problem: first, in Lemma 3.2.1 we obtain a p-Laplacian formulation of our fully nonlinear problem, then we prove various comparison results. It is interesting to notice that, unlike the 1-homogeneous case [33], comparison results here do not rely on the ABP inequality, because the operator is not sublinear, resulting in weaker comparison results. However, the qualitative maximum principle obtained through ABP will still be necessary in obtaining the domain dependence of the eigenvalue for small domains (intervals). We state the ABP inequality and postpone its proof to Section 3.6.

During all of this Chapter, except for Section 3.6, we assume $N = 1$.

Lemma 3.2.1. *Under conditions (H1)–(H2), there exists a function $G : \mathbb{R}_* \times \mathbb{R}^2 \times [a, b] \to \mathbb{R}$ such that*

$$F(m, p, u, t) = q \iff |p|^\alpha m = G(p, u, q, t)$$

if $p \neq 0$, where G is Lipschitz continuous in $(\varphi(p), \varphi(u), q)$. Moreover, G is strictly increasing in q and, when $c < 0$, it is also strictly increasing in u.

Proof. For the proof of this lemma we assume that $p \neq 0$. Using (H2) we have

$$|p|^\alpha \left(\lambda m^+ - \Lambda m^-\right) \leq F(m, p, u, t) - F(0, p, u, t) \leq |p|^\alpha \left(\Lambda m^+ - \lambda m^-\right);$$

therefore, for (p, u, t) given, $F(\cdot, p, u, t)$ is onto $\mathbb{R}$. On the other hand, if m, m' are such that

$$F(m, p, u, t) = F(m', p, u, t),$$

then $H(m, p, t) = H(m', p, t)$ and thus (H2) yields

$$|p|^\alpha \left(\lambda(m - m')^+ - \Lambda(m - m')^-\right) \leq 0 \leq |p|^\alpha \left(\Lambda(m - m')^+ - \lambda(m - m')^-\right),$$

from where $m = m'$. Since $H(m, p, t) = H(|p|^\alpha m, 1, t)$ by (H1), there is a unique $w \in \mathbb{R}$ such that $H(w, 1, t) + J(p, u, t) = q$, and we denote this w as $G(p, u, q, t)$.

Suppose now that

$$q = F(m, p, u, t),\ q' = F(m', p', u', t);$$

then, by (H1), we have

$$q = H(|p|^\alpha m, 1, t) + J(p, u, t),\ q' = H(|p'|^\alpha m', 1, t) + J(p', u', t).$$

Using (H2), if $|p|^\alpha m \geq |p'|^\alpha m'$, then

$$q - q' \geq \lambda(|p|^\alpha m - |p'|^\alpha m') + J(p, u, t) - J(p', u', t),$$

obtaining

$$0 \leq G(p, u, q, t) - G(p', u', q', t) \leq \frac{1}{\lambda}|q - q'| + \frac{\gamma}{\lambda}\left|\varphi(u) - \varphi(u')\right| + \frac{\gamma}{\lambda}\left|\varphi(p) - \varphi(p')\right|.$$

If $|p|^\alpha m < |p'|^\alpha m'$, we have $(|p|^\alpha m - |p'|^\alpha m')^- = |p'|^\alpha m' - |p|^\alpha m$, and

$$q - q' \leq -\lambda(|p'|^\alpha m' - |p|^\alpha m) - J(p, u, t) + J(p', u', t),$$

hence,

$$0 \leq G(p', u', q', t) - G(p, u, q, t) \leq \frac{1}{\lambda}|q - q'| + \frac{\gamma}{\lambda}\left|\varphi(u) - \varphi(u')\right| + \frac{\gamma}{\lambda}\left|\varphi(p) - \varphi(p')\right|,$$

concluding that G is Lipschitz continuous in q and in $\varphi(u)$, $\varphi(p)$.

Now let $q < q'$ and w, w' be such that $w = G(p, u, q, t)$, $w' = G(p', u, q, t)$. Then $H(w, 1, t) + J(p, u, t) = q < q' = H(w', 1, t) + J(p, u, t)$, from where

$$-\Lambda(w - w')^- + \lambda(w - w')^+ \leq H(w, 1, t) - H(w', 1, t) = q - q' < 0,$$

which implies $w < w'$, concluding that G is strictly increasing in q. For the monotonicity in u, suppose $c < 0$ and let u, u' be such that $w = G(p, u, q, t)$, $w' = G(p, u', q, t)$. Then

$$H(w, 1, t) + J(p, u, t) = H(w', 1, t) + J(p, u', t).$$

Using again (H2),

$$-\Lambda(w - w')^- + \lambda(w - w')^+ \leq -c(\varphi(u) - \varphi(u')) \leq \Lambda(w - w')^+ - \lambda(w - w')^-.$$

If $u > u'$, we see that $c(\varphi(u) - \varphi(u')) < 0$, and then from the right hand side we conclude that $w > w'$. □

This lemma allows us to make a precise definition of the notion of solution to the problem (3.1.1)–(3.1.2) in the one dimensional case.

Definition 3.2.1. *u is said to be a solution of the equation*

$$F(u'', u', u, t) = f \quad \text{in } (a, b) \tag{3.2.1}$$
$$u(a) = u(b) = 0 \tag{3.2.2}$$

if $u \in C^1([a, b])$, $\varphi(u')$ is absolutely continuous in (a, b) and u satisfies a.e. in (a, b)

$$\varphi(u')' = (\alpha + 1)G(u', u, f, t) \quad \text{in } (a, b), \tag{3.2.3}$$
$$u(a) = u(b) = 0. \tag{3.2.4}$$

Similarly, we define the notion of solution for the initial value problem that we will use in our analysis. Recall that, for notational convenience, we will usually write $F[u] = F(u'', u', u, t)$.

Definition 3.2.2. *u is said to be a solution of the the initial value problem in (a, b)*

$$F[u] = f, \quad u(a) = A, \quad u'(a) = B \tag{3.2.5}$$

if $u \in C^1([a, b])$, $\varphi(u')$ is absolutely continuous in (a, b) and u satisfies a.e. in (a, b)

$$\varphi(u')' = (\alpha + 1)G(u', u, f, t), \quad u(a) = A, \quad u'(a) = B. \tag{3.2.6}$$

It is known that these solutions are in $C^{1,\beta}(a, b)$ for all $\beta \in (0, 1)$ and they are of class C^2 at those points where u' does not vanish (e.g. [7]).

Remark 3.2.1. *When we write $F[u] \geq F[v]$, we mean that for certain continuous functions f_u and f_v such that u and v solve the equations $F[u] = f_u$ and $F[v] = f_v$, respectively, we have $f_u \geq f_v$.*

Using Lemma 3.2.1 we now prove a comparison principle, similar to the one found in [34].

Proposition 3.2.1. *Suppose F satisfies (H1), (H2), $\kappa > \gamma$ and u, v are such that*

$$F[v] - \kappa\varphi(v) = f \leq g = F[u] - \kappa\varphi(u) \quad \text{in } (a, b)$$

and $v(a) \leq u(a)$, $v(b) \leq u(b)$. Then $v \leq u$ in $[a, b]$.

Proof. From now on we denote by G_κ the function G —given by Lemma 3.2.1— associated to $F[u] - \kappa\varphi(u)$. We have

$$\varphi(u')' \leq (\alpha+1) G_\kappa\big(u', u, f(t), t\big),$$
$$\varphi(v')' \geq (\alpha+1) G_\kappa\big(v', v, g(t), t\big).$$

By the hypothesis on κ and Lemma 3.2.1 we have that G_κ is strictly increasing in the second and third variables and φ-Lipschitz continuous in the first variable. Let $M > 1$ be such that G_κ is Lipschitz continuous in its first three variables with constant $M(\alpha+1)^{-1}$. Subtracting and using the Lipschitz continuity and the assumption on f and g,

$$\begin{aligned}
\big[\varphi(u') - \varphi(v')\big]' &\leq (\alpha+1)\,\big\{G_\kappa\big(u', u, f(t), t\big) - G_\kappa\big(v', v, g(t), t\big)\big\} \\
&= (\alpha+1)\,\big\{G_\kappa\big(u', u, f(t), t\big) - G_\kappa\big(v', u, f(t), t\big) + \\
&\qquad + G_\kappa\big(v', u, f(t), t\big) - G_\kappa\big(v', v, f(t), t\big) + \\
&\qquad + G_\kappa\big(v', v, f(t), t\big) - G_\kappa\big(v', v, g(t), t\big)\big\} \\
&\leq M\gamma|\varphi(u') - \varphi(v')| + (\alpha+1)\{G_\kappa\big(v', u, f(t), t\big) - G_\kappa\big(v', v, f(t), t\big)\}.
\end{aligned} \tag{3.2.7}$$

Suppose $w := u - v$ is such that $\min\{w(t) : t \in (a,b)\} = w(t_0) < 0$. For $0 < \eta \leq \eta_0 := -w(t_0)/2$, let

$$\phi_\eta := (w - w(t_0) - \eta)^-$$

and write $\Sigma_\eta := \operatorname{supp}\,\phi_\eta \subseteq (a,b)$. Define also

$$c_\eta := \inf\big\{[G_\kappa\big(v', v, f(t), t\big) - G_\kappa\big(v', u, f(t), t\big)] : t \in \Sigma_\eta\big\},$$

and notice that $c_\eta \geq c_{\eta_0} > 0$ for all $\eta \in (0, \eta_0]$, since G_κ is strictly increasing in the second variable.

Taking $e^{-\int_a^t b(s)ds}\phi_\eta(t)$ (which we will simply denote by $e^{-\int b}\phi_\eta$) as a test function in (3.2.7),

$$\begin{aligned}
&\Big(\big[\varphi(u') - \varphi(v')\big]' - M\gamma|\varphi(u') - \varphi(v')|\Big)\, e^{-\int b}\phi_\eta \\
&\qquad\qquad \leq (\alpha+1)\,\big[G_\kappa\big(v', u, f(t), t\big) - G_\kappa\big(v', v, f(t), t\big)\big]\, e^{-\int b}\phi_\eta.
\end{aligned}$$

Integrating by parts and reordering,

$$
\begin{aligned}
-\int_{\Sigma_\eta} \big[\varphi(u') - \varphi(v')\big] e^{-\int b}\phi_\eta' \, dt - 2M\gamma \int_{\Sigma_\eta} |\varphi(u') - \varphi(v')| \, e^{-\int b}\phi_\eta \, dt \\
\leq (\alpha+1) \int_{\Sigma_\eta} \big[G_\kappa\big(v', u, f(t), t\big) - G_\kappa\big(v', v, f(t), t\big)\big] \, e^{-\int b}\phi_\eta \, dt,
\end{aligned}
$$

where we have used that $\|b\|_\infty \leq \gamma$ and $M > 1$ to discard the other term obtained when integrating by parts. Since the first term is always positive, we obtain

$$
\begin{aligned}
-2M\gamma \int_{\Sigma_\eta} |\varphi(u') - \varphi(v')| e^{-\int b}\phi_\eta \, dt \\
\leq (\alpha+1) \int_{\Sigma_\eta} \big[G_\kappa\big(v', u, f(t), t\big) - G_\kappa\big(v', v, f(t), t\big)\big] \, e^{-\int b}\phi_\eta \, dt. \quad (3.2.8)
\end{aligned}
$$

Now let $E := \{t \in (a, b) : w(t) = w(t_0)\} \subseteq \Sigma_\eta$ for all $\eta \in (0, \eta_0]$. Also notice that $\mathrm{dist}(E, \partial\Sigma_\eta) \to 0$ as $\eta \to 0$. Since $w'(t) = u'(t) - v'(t) = 0$ in E and the left hand side of (3.2.8) is continuous in t, there exists $\bar\eta \in (0, \eta_0]$ small enough so that

$$
\frac{4M\gamma}{\alpha+1} |\varphi(u') - \varphi(v')| \leq c_{\eta_0} \quad \text{for all } t \in \Sigma_{\bar\eta}.
$$

Using $e^{-\int b}\phi_{\bar\eta}$ as a test function in this inequality and using (3.2.8),

$$
\begin{aligned}
\frac{c_{\eta_0}}{2} \int_{\Sigma_{\bar\eta}} e^{-\int b}\phi_{\bar\eta} \, dt \geq \int_{\Sigma_{\bar\eta}} \big[G_\kappa\big(v', v, f(t), t\big) - G_\kappa\big(v', u, f(t), t\big)\big] \, e^{-\int b}\phi_\eta \, dt \\
\geq c_{\bar\eta} \int_{\Sigma_{\bar\eta}} e^{-\int b}\phi_{\bar\eta} \, dt,
\end{aligned}
$$

which is a contradiction, since $c_{\bar\eta} \geq c_{\eta_0}$. □

A simple, yet relevant, corollary of Proposition 3.2.1 is this:

Corollary 3.2.1. *Suppose F satisfies (H1), (H2), $f \in C([a, b])$ and $\kappa > \gamma$. Then the problem*

$$
F[u] - \kappa\varphi(u) = f(t) \quad \text{in } (a, b), \; u(a) = u(b) = 0
$$

has at most one solution.

Proposition 3.2.1 also holds for a perturbation of the differential operator. We will need this result in dealing with the the case $\alpha > 0$.

Corollary 3.2.2. *Let G and G_κ be as in Proposition 3.2.1, $\kappa > \gamma$, u, v and $\varepsilon > 0$ be such that*

$$\varepsilon u'' + \varphi(u')' = (\alpha+1)G_\kappa(u', u, f, t),$$
$$\varepsilon v'' + \varphi(v')' = (\alpha+1)G_\kappa(v', v, g, t)$$

where $f \leq g$, $\kappa > \gamma$ and $v(a) \leq u(a)$, $v(b) \leq u(b)$. Then $v \leq u$ in $[a, b]$.

Proof. We prove the corollary by checking step by step the proof of Proposition 3.2.1 using that $\varphi(u') - \varphi(v')$ and $\varepsilon(u' - v')$ have the same sign, while ϕ'_η has the opposite sign. □

The following proposition is a comparison principle which will be relevant in the radial context:

Proposition 3.2.2. *Suppose F satisfies (H2) and $\kappa > \gamma$. If u, v satisfy*

$$F[u] - \kappa\varphi(u) \geq F[v] - \kappa\varphi(u) \quad \text{in } (a, b)$$

and $u'(a) \geq v'(a) \wedge u(b) \leq v(b)$, or $u(a) \leq v(a) \wedge u'(b) \leq v'(b)$, then $u \leq v$ in $[a, b]$.

Proof. Suppose that $u'(a) \geq v'(a)$, $u(b) \leq v(b)$ (the other case is similar). In case $u(a) \leq v(a)$, we conclude using Proposition 3.2.1, so we assume that $u(a) > v(a)$. According to Remark 2.1, by hypothesis, there exist functions f and g such that $f(t) \geq g(t)$ for all $t \in (a, b)$ and

$$\varphi(u')' = (\alpha+1)G_\kappa(u', u, f, t) \quad \text{and} \quad \varphi(v')' = (\alpha+1)G_\kappa(v', v, g, t).$$

Defining $\tilde{v}(t) := v(t) + u(a) - v(a) > v(t)$, we easily see that for the function $\tilde{g} := g - (c(t) - \kappa)(\varphi(v) - \varphi(\tilde{v}))$ we have

$$\varphi(\tilde{v}')' = (\alpha+1)G_\kappa(\tilde{v}', \tilde{v}, \tilde{g}, t);$$

that is, using that $\kappa > \gamma$,

$$F[u] - \kappa\varphi(u) > F[\tilde{v}] - \kappa\varphi(\tilde{v}) \quad \text{in } (a, b),$$

so we can apply Proposition 3.2.1 to find that $u \leq \tilde{v}$ and therefore $u'(a) = \tilde{v}'(a) = v'(a)$. On the other hand, we have that

$$\varphi(u')' - \varphi(\tilde{v}')' = (\alpha+1)\left[G_\kappa\big(u', u, f, t\big) - G_\kappa\big(\tilde{v}', \tilde{v}, g, t\big)\right] > 0$$

in the interval $(a, a+\varepsilon)$, for certain $\varepsilon > 0$, since G_κ is continuous, $u(a) > v(a)$, $u'(a) = v'(a)$ and $f \geq g$. Consequently, by integrating near a we have

$$\varphi(u') - \varphi(\tilde{v}') > 0, \quad \text{in } (a, a+\varepsilon),$$

and this implies $u > \tilde{v}$ in $(a, a+\varepsilon)$, providing a contradiction. □

Remark 3.2.2. *The same procedure allows to find an analogous comparison principle for the perturbed equation in the degenerate case.*

The next proposition corresponds to a sort of Hopf lemma that will be very important in the analysis of sign changing eigenfunctions.

Proposition 3.2.3. *Suppose F satisfies (H1) and (H2), and $u \not\equiv 0$ is a solution of*

$$F[u] = \kappa\varphi(u) + f, \; u(a) = 0,$$

where $\kappa \geq \gamma$ and $f \leq 0$ is a continuous, bounded function. Then $u'(a) > 0$.

Proof. Let u be a nontrivial solution of

$$F[u] - \kappa\varphi(u) = f \quad \text{in } (a, b),$$

with $u(a) = 0$, and let $T \in [a, b)$ be defined as

$$T = \sup\{s \in [a, b) : \; u(t) = 0 \text{ for all } t \in [a, s]\}.$$

Now, by using the Peano local existence theorem, there exists $\varepsilon > 0$ and a solution $\bar{u}$ of

$$F[\bar{u}] - \kappa\varphi(\bar{u}) = 0 \quad \text{in } (T, T+\varepsilon),$$

such that $\bar{u}(T) = 0$, $\bar{u}'(T) > 0$ and $\bar{u}(t) > 0$ in $(T, T+\varepsilon)$.

By definition of T, there is $\bar{\varepsilon} \in (0, \varepsilon)$ such that $u(T+\bar{\varepsilon}) > 0$. Noting that (H1) implies that, for all $\sigma > 0$, the function $\sigma\bar{u}$ solves the same equation as $\bar{u}$, we can find $\sigma > 0$ so that $u(T+\bar{\varepsilon}) > \sigma\bar{u}(T+\bar{\varepsilon})$. Using Proposition 3.2.1, we find that $\sigma\bar{u}(t) \leq u(t)$ in $[T, T+\bar{\varepsilon}]$) and hence we have $u'(T) > 0$. This last conclusion implies finally that $T = a$ and $u'(a) > 0$. □

We end this section stating the Alexandrov-Bakelman-Pucci inequality for this one dimensional case. Certainly this case is a consequence of the general case proved in [27],

[41] and [51]. We will provide a proof of this theorem in the radial case, greatly simplifying the proof given in the general case.

Proposition 3.2.4. *Let u be a solution of*

$$\Lambda\left[\varphi(u')'\right]^{+} - \lambda\left[\varphi(u')'\right]^{-} + \gamma|\varphi(u')| \geq -f^{-} \quad \text{in } \{u > 0\}, \tag{3.2.9}$$

with $u(a), u(b) \leq 0$. Then

$$\sup_{(a,b)} u^{+} \leq B\|f^{-}\|_{L^1([a,b])}^{\frac{1}{\alpha+1}}. \tag{3.2.10}$$

In a similar way, if $u(a), u(b) \geq 0$ and

$$\lambda\left[\varphi(u')'\right]^{+} - \Lambda\left[\varphi(u')'\right]^{-} - \gamma|\varphi(u')| \geq f^{-} \quad \text{in } \{u > 0\}, \tag{3.2.11}$$

then

$$\sup_{(a,b)} u^{-} \leq B\|f^{+}\|_{L^1([a,b])}^{\frac{1}{\alpha+1}}. \tag{3.2.12}$$

Here a solution of (3.2.9) (resp. (3.2.11)) is a function $u \in C^1$ such that $\varphi(u')$ is absolutely continuous and satisfies (3.2.9) (resp. (3.2.11)) almost everywhere.

3.3. The boundary value problem in one dimension

The purpose of this section is to prove an existence theorem for the boundary value problem. This theorem will be on the basis of our construction of a positive eigenfunction via degree theory. Here is the precise statement of our result.

Theorem 3.3.1. *Suppose F satisfies (H1) and (H2). Then for all $\kappa > \gamma$ and $f \in C([a,b])$, $f \leq 0$, the problem*

$$F[u] - \kappa\varphi(u) = f(t) \quad \text{in } (a,b),\ u(a) = u(b) = 0 \tag{3.3.1}$$

has a unique solution.

We prove this theorem via the shooting method. For this purpose we need to analyse the singular case, $\alpha \in (-1, 0)$, and the degenerate case, $\alpha > 0$, separately. In the singular case we prove uniqueness for the initial value problem in a straightforward way, while in the degenerate case, we first study a perturbed initial value problem, which yields the existence for the perturbed boundary value problem. Finally, we take the limit as the parameter defining the perturbation vanishes.

In what follows we prove an existence and uniqueness result for the initial value problem, a key result to use the shooting method. We start with the existence part.

Lemma 3.3.1. *Suppose F satisfies (H1) and (H2), and let $\alpha \in (-1, +\infty)$ and G_κ be the function G given by Lemma 3.2.1 for the operator $F[u] - \kappa\varphi(u)$. Then, for every $T \in [a, b]$, $A, B \in \mathbb{R}$,*

(a) *if $\alpha \in (-1, 0]$, then there exists a solution in $[a, b]$ of*

$$\begin{aligned} \varphi(u')' &= (\alpha+1)G_\kappa(u', u, f, t) \quad \text{in } (a, b), \\ u(T) &= A, \quad u'(T) = B. \end{aligned} \tag{3.3.2}$$

(b) *if $\varepsilon > 0$ and $\alpha > 0$, then there exists a solution in $[a, b]$ of*

$$\begin{aligned} \varepsilon u'' + \varphi(u')' &= (\alpha+1)G_\kappa(u', u, f, t) \quad \text{in } (a, b), \\ u(T) &= A, \quad u'(T) = B. \end{aligned} \tag{3.3.3}$$

Proof. The existence of a local solution to both problems is a consequence of the Peano existence theorem. Suppose $T < b$ and the maximal existence interval is $[T, \tilde{T})$, where $\tilde{T} < b$ (the argument to the left of T is similar). Using the Fundamental Theorem of Calculus and the inequality $|a+b|^s \le C_s(|a|^s + |b|^s)$, valid for all $s > 0$ and $a, b \in \mathbb{R}$ with a certain constant C_s, we have that for all $t \in [T, \tilde{T})$ and $\alpha > -1$

$$\begin{aligned} |\varphi(u(t))| = |u(t)|^{\alpha+1} &\le C_\alpha \left(\left| \int_T^t u'(s)ds \right|^{\alpha+1} + |u(T)|^{\alpha+1} \right) \\ &\le C_\alpha \left((t-T)\|u'\|^{\alpha+1}_{L^\infty([T,t])} + |u(T)|^{\alpha+1} \right). \end{aligned}$$

Then, using that G_k is Lipschitz continuous as given in Lemma 3.2.1; say, with constant $M(\alpha+1)^{-1}$, and the equation satisfied by u, we have

$$\begin{aligned} |u'(t)|^{\alpha+1} = |\varphi(u'(t))| &= \left| \varphi(B) - (\alpha+1) \int_T^t G_\kappa(u', u, f, s)ds \right| \\ &\le M \int_T^t |\varphi(u'(s))| + |\varphi(u(s))| ds + |\varphi(B)| + M \int_T^t |f(s)|ds \\ &\le M \int_T^t (1 + C_\alpha(s-T))\|u'\|^{\alpha+1}_{L^\infty([T,s])} ds + \\ &\quad + |\varphi(B)| + M \int_T^t |f(s)|ds + MC_\alpha(t-T)|A|^{\alpha+1}. \end{aligned}$$

This implies

$$\|u'\|_{L^\infty([T,t])}^{\alpha+1} \leq C + M \int_T^t (1 + C_\alpha(s-T)) \|u'\|_{L^\infty([T,s])}^{\alpha+1} ds,$$

and using Gronwall's inequality we get

$$\|u'\|_{L^\infty([T,t])}^{\alpha+1} \leq C \exp\left(M \int_T^t (1 + C_\alpha(s-T)) ds \right).$$

From here we conclude that u' is bounded in $[T, \tilde{T})$ and, hence, u is bounded in $[T, \tilde{T})$, contradicting the maximality of the existence interval. This proves statement (a).

In order to prove statement (b) we note that the same procedure yields

$$|\varepsilon u'(t)| + |u'(t)|^{\alpha+1} \leq C + M \int_T^t (1 + C_\alpha(s-T)) \|u'\|_{L^\infty([T,s])}^{1+\alpha} ds,$$

from where we obtain the same conclusion applying again Gronwall's inequality. □

Now we present a lemma about uniqueness of the initial value problem in the singular case.

Lemma 3.3.2. *Assume that* $\alpha \in (-1, 0]$, $\kappa > \gamma$ *and* $f \leq 0$ *is a continuous, bounded function. Then, for* A, $B \in \mathbb{R}$ *and* $T \in [a, b]$, (3.3.2) *has a unique solution in* (a, b).

Proof. It is clear that the global uniqueness result is a consequence of local uniqueness, and global existence was proven in Lemma 3.3.1. For notational convenience we assume that $T = 0$ and we analyse the interval $(0, \varepsilon)$ (the interval $(-\varepsilon, 0)$ is totally analogous). We divide our proof in three cases.

Case 1. Suppose $A \neq 0$, $B \in \mathbb{R}$. Defining $x = u$, $y = \varphi(u')$, the equation can be written as the following ODE system:

$$\begin{aligned} x' &= \varphi^{-1}(y), \\ y' &= (\alpha + 1) G(\varphi^{-1}(y), x, f, t), \end{aligned}$$

with initial condition $(x(0), y(0)) = (A, \varphi(B))$. Since the right hand side is Lipschitz continuous close to $(A, \varphi(B))$, the solution is unique in an interval of the form $(0, \varepsilon)$, for $\varepsilon > 0$ small enough.

Case 2. Now we assume that $u(0) = 0$ and $u'(0) = B \neq 0$ (without loss of generality, $B > 0$). The proof of this case is inspired on the uniqueness result given by Li in [46] for the p-Laplacian. Let v be another solution with $v(0) = 0$ and $v'(0) = B$ and let $0 < k \leq K$ be such that

$$k \leq u'(t) \leq K \quad \text{and} \quad k \leq v'(t) \leq K \qquad \text{for all } t \in [0, \varepsilon]. \tag{3.3.4}$$

Such a k exists if we choose $\varepsilon > 0$ small enough, since $B > 0$. We observe that we also have $u,\ v > 0$ in $(0, \varepsilon)$. Regarding the function φ we have that for all $x, y \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned} |\varphi(x) - \varphi(y)| \leq& (\alpha + 1) \min\{|x|, |y|\}^{\alpha} |x - y| \quad \text{and} \\ |\varphi^{-1}(x) - \varphi^{-1}(y)| \leq& (q + 1) \max\{|x|, |y|\}^{q} |x - y|, \end{aligned}$$

where $q := -\alpha(1 + \alpha)^{-1}$. Then for all $t \in (0, \varepsilon)$ we have

$$\begin{aligned} &|\varphi(u(t)) - \varphi(v(t))| \leq t^{\alpha+1} \min\left\{\frac{u(t)}{t}, \frac{v(t)}{t}\right\}^{\alpha} \left|\frac{u(t) - v(t)}{t}\right| \leq t^{\alpha+1} k^{\alpha} \|u' - v'\|_{L^\infty([0,\varepsilon])}, \\ &|\varphi(u'(t)) - \varphi(v'(t))| \leq k^{\alpha} \|u' - v'\|_{L^\infty([0,\varepsilon])}, \end{aligned} \tag{3.3.5}$$

where we used (3.3.4) and the Mean Value Theorem. For notational convenience, in what follows we write $G(u', u, f, s) = G[u]$ and similarly for v. Taking ε smaller if necessary, we find that

$$\left|\varphi(B) - (\alpha + 1) \int_0^{\varepsilon} G[u] ds\right| \leq \frac{3}{2} \varphi(B) \tag{3.3.6}$$

and a similar bound for v. Considering the equation satisfied by u and v we obtain, for all $0 \leq t \leq \varepsilon$,

$$\begin{aligned} |u'(t) - v'(t)| &= \left|\varphi^{-1}\left(\varphi(B) - (\alpha + 1) \int_0^t G[u] ds\right) - \varphi^{-1}\left(\varphi(B) - (\alpha + 1) \int_0^t G[v] ds\right)\right| \\ &\leq \left(\frac{3}{2} \varphi(B)\right)^q \int_0^{\varepsilon} |G[u] - G[v]| ds \\ &\leq M\gamma \left(\frac{3}{2} \varphi(B)\right)^q \int_0^{\varepsilon} \left(\left|\varphi(u') - \varphi(v')\right| + \left|\varphi(u) - \varphi(v)\right|\right) ds, \end{aligned}$$

where we used the Lipschitz continuity of G and (3.3.6). From here, using (3.3.5) and writing $C = M\gamma \left(\frac{3}{2}\varphi(B)\right)^q$, we find that

$$|u'(t) - v'(t)| \leq C k^{\alpha} \|u' - v'\|_{L^\infty([0,\varepsilon])} \int_0^{\varepsilon} (1 + s^{\alpha+1}) ds \leq \frac{1}{2} \|u' - v'\|_{L^\infty([0,\varepsilon])},$$

if ε is chosen small enough. This inequality provides a contradiction if u and v are different in the interval $(0, \varepsilon)$.

Case 3. If $u(0) = u'(0) = 0$, Proposition 3.2.3 implies that $u \equiv 0$, so uniqueness holds. $\square$

The proposition that follows corresponds to Theorem 3.3.1 when $\alpha \in (-1, 0]$.

Proposition 3.3.1. *Under the assumptions of Theorem 3.3.1, and additionally assuming $\alpha \in (-1, 0]$, the boundary value problem (3.3.1) has a unique solution.*

Proof. Consider the initial value problem

$$
\begin{aligned}
F[u] - \kappa\varphi(u) = f \quad \text{in } (a, b), \\
u'(a) = d, \quad u(a) = 0,
\end{aligned}
\tag{3.3.7}
$$

for $d \in \mathbb{R}$. By Lemma 3.3.2 this problem has a unique solution, which depends continuously on d. We denote this solution by $u(d, t)$.

Now we consider two constants $M_- < 0 < M_+$ such that

$$
G_\kappa(0, M_+, f, t) \geq 0 \quad \text{and} \quad G_\kappa(0, M_-, f, t) \leq 0.
$$

We claim there exist $d_1 \in \mathbb{R}$ and $t_1 \in (a, b)$ such that $u(d_1, t) \geq M_+$ for $t \in (t_1, b]$ and, similarly, there exist $d_2 \in \mathbb{R}$ and $t_2 \in (a, b)$ such that $u(d_2, t) \leq M_-$ for all $t \in (t_2, b]$. Notice that, in particular, $u(d_1, b) > 0$ and $u(d_2, b) < 0$, so that by continuity, there exists d such that $u(d, b) = 0$, and hence the claim implies the existence of a solution to (3.3.1) when $\alpha \in (-1, 0]$.

We shall now prove our claim. In the first place we notice that u satisfies

$$
\begin{aligned}
\varphi(u')' = (\alpha + 1) G_\kappa(u', u, f, t), \\
u(a) = 0, \quad u'(a) = d.
\end{aligned}
$$

Using that G is Lipschitz continuous we find a constant L such that

$$
(\alpha + 1)|G_\kappa(u', u, f, t)| \leq L(|u'(t)|^{\alpha+1} + |u(t)|^{\alpha+1} + 1).
$$

If we consider $d \geq 1$ such that $|u(t)| \leq 2d$ and $|u'(t)| \leq 2d$, with $t \in (0, t_1)$, we have that

$$
(\alpha + 1)\left|G_\kappa(u', u, f, t)\right| \leq 3L(2d)^{\alpha+1}.
$$

Since $u = u(d, \cdot)$ is a solution of the equation, integrating for $t \leq t_1$ we have

$$\begin{aligned}\left|\varphi(u')(t) - \varphi(u')(a)\right| &\leq (\alpha+1)\int_a^t \left|G_\kappa\big(u'(s), u(s), f(s), s\big)\right| \, ds \\ &\leq 3(\alpha+1)L(t_1 - a)(2d)^{\alpha+1}.\end{aligned}$$

Taking $t_1 := a + 1/(3 \cdot 4^{\alpha+1}(\alpha+1)L)$, we obtain

$$\left|\varphi(u') - d^{\alpha+1}\right| \leq \frac{d^{\alpha+1}}{2^{\alpha+1}},$$

that is, $u'(t) \geq d/2$ for all $t \in (a, t_1)$. Choosing $d_1 = d$ large enough, $d_1 t_1 > 2M_+$, and hence $u(t_1) > M_+$. Using the comparison principle in Proposition 3.2.2, we conclude that $u \geq M_+$ in $(t_1, b]$. The proof concerning M_- is similar, completing the existence part of the proof. The uniqueness of the solution is a consequence of Corollary 3.2.1 □

Now we consider the proof of Theorem 3.3.1 in the case $\alpha > 0$; that is, the degenerate case. We will use a perturbation approach in order to avoid the degeneracy. We start with the analysis of the initial value problem for the perturbed equation.

Lemma 3.3.3. *Assume that $\alpha > 0$, $\kappa > \gamma$ and $f \leq 0$ is a continuous, bounded function and consider $\varepsilon > 0$. Then, for A, $B \in \mathbb{R}$ and $T \in [a, b]$, (3.3.3) has a unique solution.*

Proof. For notational convenience we write $\psi(s) = \varepsilon s + \varphi(s)$. We have

$$\psi'(s) = \varepsilon + (\alpha+1)|s|^\alpha \geq \varepsilon > 0.$$

Therefore, ψ has a C^1 inverse and both ψ and ψ^{-1} are locally Lipschitz continuous. Our equation can be written as

$$\psi(u')'(t) = (\alpha+1)G_\kappa(u', u, f, s).$$

Defining $x = u$ and $y = \psi(u')$ the equation is equivalent to the system

$$\begin{aligned}x' &= \psi^{-1}(y),\\ y' &= (\alpha+1)G_k(\psi^{-1}(y), x, f, t),\end{aligned}$$

with initial condition $x(T) = A$ and $y(T) = \psi(B)$. The local Lipschitz continuity of ψ^{-1}, G, φ allows to use the contraction mapping theorem to obtain local existence and uniqueness. The global uniqueness comes from A and B being arbitrary, while the global existence was proven in Lemma 3.3.1. □

Remark 3.3.1. *Note that φ^{-1} is not locally Lipschitz; then, if $\varepsilon = 0$ this proof does not work. This is a well known difficulty for the initial value problem involving the p-Laplacian in the degenerate case.*

The following is a preparation lemma, allowing to prove the Theorem 3.3.1 by passing ε to 0.

Lemma 3.3.4. *Under the hypotheses of Theorem 3.3.1 and $\alpha > 0$, let u_ε be the solution of*

$$\varepsilon u'' + \varphi(u')' = (\alpha+1)G_\kappa(u', u, f, t) \text{ in } (a,b), \quad u(a) = u(b) = 0, \tag{3.3.8}$$

for $\varepsilon > 0$. Then $u_0 := \lim_{\varepsilon\to 0} u_\varepsilon$ exists and it is a solution of

$$\varphi(u')' = (\alpha+1)G_\kappa(u', u, f, t) \text{ in } (a,b), \quad u(a) = u(b) = 0. \tag{3.3.9}$$

Proof. Suppose $\beta_\varepsilon := \|u_\varepsilon\|_1$ is unbounded, where $\|u\|_1 = \|u\|_{L^\infty([a,b])} + \|u'\|_{L^\infty([a,b])}$. Let $v_\varepsilon := u_\varepsilon/\beta_\varepsilon$, and note that $\|v_\varepsilon\|_1 = 1$. The homogeneity of G implies v_ε solves

$$\varepsilon v_\varepsilon'' + \varphi(v_\varepsilon')' = (\alpha+1)G_\kappa\big(v_\varepsilon', v_\varepsilon, \frac{f}{\beta_\varepsilon^{\alpha+1}}, t\big) \text{ in } (a,b), \quad v_\varepsilon(a) = v_\varepsilon(b) = 0,$$

and the Lipschitz continuity of G implies

$$\left\|\varepsilon v_\varepsilon'' + \varphi(v_\varepsilon')'\right\|_\infty \le M\left(\left\|f/\beta_\varepsilon^{\alpha+1}\right\|_\infty + (\gamma+\kappa)\|u\|_1^{\alpha+1}\right) \le C.$$

Since φ is an increasing function, v_ε'' and $\varphi(v_\varepsilon')'$ have the same sign; hence, $\|\varphi(v_\varepsilon')'\|_\infty$ is bounded. This implies that v_ε' and v_ε are equicontinuous; therefore, using Arzelà-Ascoli we conclude that for a sequence $\varepsilon_n \to 0$, $v_n = v_{\varepsilon_n}$ converges in C^1 to $v_0 \in C^1$.

Now consider the weak formulation of (3.3.8): for $w \in C_0^\infty$,

$$-\varepsilon\int_a^b v_\varepsilon' w' dt - \int_a^b \varphi(v_\varepsilon')w' dt = (\alpha+1)\int_a^b G_\kappa\big(v_\varepsilon', v_\varepsilon, \frac{f}{\beta_\varepsilon^{\alpha+1}}, t\big)w\, dt.$$

Using that v_ε' is bounded, the first term vanishes when $\varepsilon \to 0$, and then the equation for v_0 is

$$-\int_a^b \varphi(v_0')w' dt = (\alpha+1)\int_a^b G_\kappa\big(v_0', v_0, 0, t\big)w\, dt.$$

From here, the regularity of v_0 follows and v_0 solves (3.3.9) in the sense of Definition 3.2.1 with $f \equiv 0$. But $u \equiv 0$ also solves the same equation, and using Corollary 3.2.1 we conclude $v_0 \equiv 0$, contradicting $\|v_\varepsilon\|_1 = 1$. Therefore, $\|v_\varepsilon\|_1$ is bounded. We complete the proof of the lemma repeating the convergence argument for u_ε and using Corollary 3.2.1 again. □

Proposition 3.3.2. *Let $\kappa > \gamma$, $f \leq 0$ be a continuous function, and $\alpha > 0$. Then the boundary value problem (3.3.1) has a unique solution.*

Proof. We first prove that (3.3.8) has a unique solution, following the proof of Proposition 3.3.1 step by step, using Lemma 3.3.3 instead of Lemma 3.3.2 and Remark 3.2.2. Then we pass to the limit using Lemma 3.3.4 to obtain a solution to (3.3.1). Uniqueness is again a consequence of Corollary 3.2.1. □

Proof of Theorem 3.3.1. This theorem is a direct consequence of Proposition 3.3.1 and Proposition 3.3.2. □

The following result is not directly related to the 1-dimensional theory, but will be useful in the radial case. Since its proof is very similar to the one of Theorem 3.3.1, it is convenient to mention it here.

Theorem 3.3.2. *Suppose F satisfies (H1) and (H2). Then for all $\kappa > \gamma$, $T \in [a, b)$ and $f \in C[a, b]$, $f \leq 0$, the equation*

$$\begin{aligned} F[u] - \kappa\varphi(u) &= f \quad \text{in } (a, b) \\ u'(T) = u(b) &= 0 \end{aligned} \tag{3.3.10}$$

has a unique solution.

Proof. For $\alpha \in (-1, 0]$, consider the initial value problem

$$\begin{aligned} &\varphi(u')' = (\alpha + 1)G_\kappa(u', u, f, t) \quad \text{in } (a, b), \\ &u'(T) = 0, \;\; u(T) = d, \end{aligned}$$

for a given $d \in \mathbb{R}$. Due to Lemma 3.3.2 the corresponding solution is unique. We denote it as $u(d, t)$ and notice that $d \mapsto u(d, b)$ is continuous. Choose $M_- < 0 < M_+$ such that

$$G_\kappa(0, M_+, f, t) \geq 0 \quad \text{and} \quad G_\kappa(0, M_-, f, t) \leq 0 \quad \text{in } (a, b)$$

We shall verify now that whenever $d_1 > M_+$ the function $u_1(t) := u(d_1, t)$ satisfies $u_1 >$

M_+ in (T,b). For this, let

$$F_T(x,y,z,t) := \begin{cases} F(x,y,z,t) & \text{if } t \in [T,b], \\ F(x,y,z,2T-t) & \text{if } t \in [2T-b,T], \end{cases}$$

and similarly define f_T and the solution $u_T(d,t)$. We see that

$$F_T[u_T] - \kappa\varphi(u_T) \geq F(0,0,M_+,t) - \kappa\varphi(M_+).$$

If $u_1(b) \leq M_+$, there exists $\tau \in (T,b]$ such that $u_1(\tau) = M_+$. Using this equality and considering the function u_T in $(2T-\tau,\tau)$, we obtain from Proposition 3.2.1 that $u_T \leq M_+$ in $(2T-\tau,\tau)$, which contradicts $u_T(T) = u_1(T) = d_1 > M_+$. In a similar way we obtain that for $d_2 < M_-$ we have that $u_2 < M_-$ in (T,b). This implies $u_1(b) \geq M_+$, $u_2(b) \leq M_-$, and using the continuity of $d \mapsto u(d,b)$ we deduce the result.

The case $\alpha > 0$ can be analysed in the same way as Proposition 3.3.2, using an intermediate lemma similar to Lemma 3.3.4, but with boundary conditions $u'(c) = u(b) = 0$, and then passing to the limit as $\varepsilon \to 0$. Afterwards this theorem is a consequence of Corollary 3.2.1 and the reflection argument just given above. □

3.4. The first eigenvalue in one dimension

This section is devoted to obtain the first eigenvalue and eigenfunction of (3.1.1)–(3.1.2) in the 1-dimensional case. We begin with a compactness lemma, which will be useful in the proof of the result of this section.

Lemma 3.4.1. *Under hypotheses (H1) and (H2), let u_n be the solution of (3.3.1), where $\alpha \in (-1,+\infty)$, with right hand side f_n, where $\{f_n\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $C\big([a,b],(-\infty,0]\big)$. Then there exists C independent of n such that*

$$\|u_n\|_\infty + \|u_n'\|_\infty + \|\varphi(u_n')'\|_\infty \leq C$$

Proof. The arguments used in Lemma 3.3.4 can be adapted to this case. □

In what follows we shall use Corollary 1 of theorem VIII.1 in [57], inspired by [33]:

Theorem 3.4.1. *Let E be a Banach space and K be a closed cone of E with a vertex at 0. Let $T : \mathbb{R}^+ \times K \to K$ be a compact operator such that $T(0,u) = 0$ for all $u \in E$. Then there exists an unbounded connected component $\mathcal{C} \subseteq \mathbb{R}^+ \times K$ of solutions of $u = T(\mu,u)$ containing $(0,0)$.*

Theorem 3.4.2. *Under assumptions (H1) and (H2) the problem*

$$
\begin{aligned}
F[u] &= -\mu\varphi(u) \quad \textit{in } (a,b) \\
u(a) &= u(b) = 0
\end{aligned}
\tag{3.4.1}
$$

has a solution (u^+,λ^+) such that $u^+>0$ in (a,b), and another solution (u^-,λ^-) satisfying $u_-<0$ in (a,b). Furthermore, every positive (respectively negative) solution of (3.4.1) is a multiple of u^+ (resp. u^-).

Proof. Let $K := \{u \in C[a,b] : u \geq 0,\ u(a)=u(b)=0\}$, and let $\kappa > \gamma$. We can use Theorem 3.3.1 to find the unique solution to the problem

$$
\begin{aligned}
F[u] - \kappa\varphi(u) &= -\varphi(g(t)) \quad \text{in } (a,b), \\
u(a) &= u(b) = 0,
\end{aligned}
$$

where $g \in K$. Denote this solution as $\mathcal{L}(g)$ and define $T : \mathbb{R}^+ \times K \to K$ as $T(\mu,f) = \mu\mathcal{L}(f)$. As a consequence of Propositions 3.2.1 and 3.2.3, $T(\mu,f)>0$ whenever $f \in K \setminus \{0\}$ and $\mu > 0$. Lemma 3.4.1 shows T is compact, and clearly $T(0,g)=0$ for all $g \in K$; hence, T satisfies the conditions of Theorem 3.4.1. Let $u_0 \in K \setminus \{0\}$. Knowing that $\mathcal{L}(u_0)'(a) > 0$ and $\mathcal{L}(u_0)'(b) < 0$ (by Lemma 3.2.3) we deduce the existence of M such that $M\mathcal{L}(u_0) \geq u_0$. Now let $\mathcal{L}_\varepsilon : \mathbb{R}^+ \times K \to K$ be defined as $\mathcal{L}_\varepsilon(\mu,u) := \mu\mathcal{L}(u) + \mu\varepsilon\mathcal{L}(u_0)$, for $\varepsilon > 0$. Theorem 3.4.1 allows us to obtain the existence of an unbounded connected set $\mathcal{C}_\varepsilon$ of solutions of $\mathcal{L}_\varepsilon(\mu,u) = u$. Furthermore, $\mathcal{C}_\varepsilon \subseteq [0,M] \times K$. Indeed, let $(\mu,u) \in \mathcal{C}_\varepsilon$. Then

$$
u = \mu\mathcal{L}(u) + \mu\varepsilon\mathcal{L}(u_0).
$$

Since $\mathcal{L}(u) \geq 0$, $u \geq \mu\varepsilon\mathcal{L}(u_0) \geq \frac{\mu}{M}\varepsilon u_0$. Applying $\mathcal{L}$ again,

$$
\mathcal{L}(u) \geq \frac{\mu}{M}\varepsilon\mathcal{L}(u_0) \geq \frac{\mu}{M^2}\varepsilon u_0
$$

and $u \geq \mu\mathcal{L}(u)$ yields $u \geq \left(\frac{\mu}{M}\right)^2 \varepsilon u_0$. Repeating the same procedure we get

$$
(\forall n \in \mathbb{N} \setminus \{0\}) \quad u \geq \left(\frac{\mu}{M}\right)^n \varepsilon u_0,
$$

allowing us to conclude $\mu \leq M$. Taking this into account and the unboundedness of $\mathcal{C}_\varepsilon$ we obtain that there exists $(\mu_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon$ such that $\|u_\varepsilon\|_\infty = 1$. The uniform bound for the solutions given by Lemma 3.4.1 permits us to pass to the limit $\varepsilon \to 0$, obtaining $\mu^+ \in [0,M]$ and $u^+ > 0$ such that $u^+ = \mu^+\mathcal{L}(u^+)$. Using that $u^+>0$ we deduce $\mu^+>0$, and we write $\lambda^+ := -\kappa + \mu^+$.

For the isolation of this eigenvalue and its simplicity, let $h \in K \setminus \{0\}$ and μ be an eigenpair solving (3.4.1). Propositions 3.2.2 and 3.2.3 allow us to find s such that $h < su^+$ in (a,b). Let

$$\tau := \inf\{s > 0 : h \le su^+ \text{ in } (a,b)\}.$$

If $\mu < \mu^+$, then either $h = \tau u^+$ (and therefore $\mu = \mu^+$, a contradiction) or $h < \tau u^+$, contradicting the definition of τ. Indeed, if for some $T \in (a,b)$ holds $h(T) = \tau u^+(T)$, then $h'(T) = \tau(u^+)'(T)$ (since the contrary is not compatible with the previous inequality). Using the comparison principle and $\mu < \mu^+$ we obtain $F[h] = f_1 > f_2 = F[\tau u^+]$, which, by means of Lemma 3.2.1 and integration, implies, for $t > T$,

$$h'(t) - \tau(u^+)'(t) = \int_T^t G(h', h, f_1, s) - G\big(\tau(u^+)', \tau u^+, f_2, s\big)\; ds > 0,$$

contradicting $h \le \tau u^+$. The case $\mu = \mu^+$ can be discarded similarly, because $h \le \tau u^+$, $h \not\equiv \tau u^+$ implies $h < \tau u^+$ by Corollary 3.2.2, both h and τu^+ solving the same problem. Analogous arguments allow us to discard the case $\mu > \mu^+$.

In order to obtain (u^-, λ^-) we can apply the same scheme to $-F(-m, -p, -u, t)$. □

Now we mention two corollaries of this theorem. From now on we shall denote as $\lambda^+(t_1, t_2)$ (resp. $\lambda^-(t_1, t_2)$) the first eigenvalue associated to the positive (resp. negative) eigenfunction in the interval $(t_1, t_2) \subseteq (a,b)$. We point out that the proof of the second corollary requires the ABP inequality.

Corollary 3.4.1. *$(a_1, b_1) \subseteq (a,b)$ and $(a_1, b_1) \neq (a,b)$ implies $\lambda^\pm(a_1, b_1) > \lambda^\pm(a,b)$.*

Proof. Consider the eigenpair (λ_1^+, u_1^+) on the interval (a_1, b_1), where $\mu_1^+ = \kappa + \lambda_1^+$ as in Theorem 3.4.2. Let $\bar{u}$ be the function obtained by extending u_1^+ as zero to the whole interval $[a,b]$, and define $\tilde{u} := \mathcal{L}\big(\varphi^{-1}(\lambda_1^+ \bar{u})\big)$. Using the comparison principle and Corollary 3.2.1, $\tilde{u} > u_1^+$ in $[a_1, b_1]$ and, therefore, $\tilde{u} > \bar{u}$ in (a,b). Defining $w := \mathcal{L}(\bar{u})$ and $v := \mathcal{L}(\tilde{u})$,

$$F[w] - \kappa\varphi(w) = -\varphi(\bar{u}) > -\varphi(\tilde{u}) = F[v] - \kappa\varphi(v),$$

and therefore the comparison principle again implies $w < v$; that is,

$$\tilde{u} = \mu_1^+ \mathcal{L}(\bar{u}) < \mu_1^+ \mathcal{L}(\tilde{u}).$$

We can replace μ_1^+ by a slightly smaller value $M < \mu_1^+$ without changing the strict inequality. We can repeat now the arguments of the proof of Theorem 3.4.2 with $u_0 = \tilde{u}$

and $M < \mu_1^+$ in order to obtain

$$\mu^+ := \lambda^+(a,b) + \kappa \leq M < \mu_1^+$$

obtaining the conclusion for λ^+. The proof for λ^- is similar. □

Corollary 3.4.2. *The functions $\lambda^+, \lambda^- : \{(t_1,t_2) : a \leq t_1 < t_2 \leq b\}$ are continuous and*

$$\lim_{t_2-t_1\to 0^+} \lambda^+(t_1,t_2) = \lim_{t_2-t_1\to 0^+} \lambda^-(t_1,t_2) = +\infty$$

Proof. The continuity of these functions is a consequence of the uniqueness of the eigenvalues for positive (negative) eigenfunctions, while the limit is a consequence of Proposition 3.2.4. Indeed, writing $\mu^+ = \lambda + \kappa$, with $\kappa > \gamma$, and using the right hand inequality in (H2), we obtain

$$\sup_{(t_1,t_2)} u^+ \leq B\mu^+ \|\varphi(u^+)\|_{L^1([t_1,t_2])}^{\frac{1}{\alpha+1}} \leq B(t_2-t_1)\mu^+ \sup_{(t_1,t_2)} u^+,$$

which completes the proof. □

3.5. The one dimensional eigenvalue problem

We shall now analyse the problem of finding the whole sequence of eigenvalues and eigenfunctions. The result we pursue is the following:

Theorem 3.5.1. *Under assumptions (H1), (H2), the eigenvalue problem*

$$F[u] = -\mu\varphi(u) \quad \text{in } (a,b), \qquad (3.5.1)$$
$$u(a) = u(b) = 0,$$

has two solving sequences $(\lambda_n^\pm, u_n^\pm)_{n\in\mathbb{N}}$ such that u_n^+ and u_n^- have exactly n interior zeros and u_n^+ (respectively u_n^-) is positive (resp. negative) in (a,t_1), negative (resp. positive) in (t_1,t_2) and so on. Moreover, the sequence $(\lambda_n^\pm)_{n\in\mathbb{N}}$ is increasing in n and the sequence $(\lambda_n^\pm, u_n^\pm)_{n\in\mathbb{N}}$ is complete in the sense that there are no eigenpairs of (3.5.1) outside these sequences.

In order to prove this theorem the following result is necessary:

Lemma 3.5.1. *Let $n \in \mathbb{N} \setminus \{0\}$ and define*

$$\Delta_n := \{(t_1,\ldots,t_n) : a < t_1 < \ldots < t_n < b\},\ t_0 := a,\ t_{n+1} := b$$

and the function $V : \Delta_n \to \mathbb{R}^n$ *as*

$$V_i(t) := \lambda^{(-1)^i}(t_{i-1}, t_i) - \lambda^{(-1)^{i+1}}(t_i, t_{i+1})$$

where $\lambda^{\pm 1}$ *is notation for* $\lambda^{\pm}$*. Under hypotheses (H1), (H2),*

$$(\forall n \in \mathbb{N})(\exists t \in \Delta_n)\ V(t) = 0.$$

Proof. Let us consider a point $\vec{t} \in \partial\Delta_n$. Then there are $0 \le k < \ell \le n+1$ such that $t_k = t_{k+1} = \cdots = t_\ell$, and that they additionally satisfy $k = 0$ or $t_{k-1} < t_k$, and $\ell = n+1$ or $t_\ell < t_{\ell+1}$. We further assume that k is the smallest integer for which the situation described occurs. We observe that simultaneously we cannot have $k = 0$ and $\ell = n+1$.

In what follows we denote by $\{e_1, e_2, \ldots, e_n\}$ the canonical basis of $\mathbb{R}^n$. If $k = 0$, then we have $t_\ell < t_{\ell+1}$ and we define $T(\vec{t}) = -e_\ell$. If $0 < k < \ell < n+1$, then we define $T(\vec{t}) = e_k - e_\ell$, and if $0 < k$ and $\ell = n+1$, then we define $T(\vec{t}) = e_k$. In this way we have defined $T : \partial\Delta_n \to \mathbb{R}^n$ as a function. We observe that T defines a *normal vector field*, which is not continuous on the edges of $\partial\Delta_n$.

Assume now that we have a sequence $\{\vec{t}_m\} \subseteq \Delta_n$ such that $\vec{t}_m \to \vec{t} \in \partial\Delta_n$, as $m \to \infty$. Then we have

$$\lim_{m\to\infty} \left\langle V(\vec{t}_m), T(\vec{t}) \right\rangle = -\infty. \tag{3.5.2}$$

In order to prove (3.5.2) we have three cases, according to the numbers $k < \ell$ associated to $\vec{t} \in \partial\Delta_n$. First, if $k = 0$ and $\ell < n+1$, then

$$(\vec{t}_m)_\ell - (\vec{t}_m)_{\ell-1} \to 0 \quad \text{and} \quad (\vec{t}_m)_{\ell+1} - (\vec{t}_m)_\ell > c > 0,$$

so that $V_\ell(\vec{t}_m) \to \infty$ as m goes to $+\infty$, proving (3.5.2).

Second, if $k > 0$ and $\ell < n+1$ then we have

$$(\vec{t}_m)_\ell - (\vec{t}_m)_{\ell-1} \to 0 \quad \text{and} \quad (\vec{t}_m)_{\ell+1} - (\vec{t}_m)_\ell > c > 0$$

and also

$$(\vec{t}_m)_{k+1} - (\vec{t}_m)_k \to 0 \quad \text{and} \quad (\vec{t}_m)_k - (\vec{t}_m)_{k-1} > c > 0.$$

Thus, by definition of V we have $V_\ell(\vec{t}_m) \to \infty$ and $V_k(\vec{t}_m) \to -\infty$ as m goes to $+\infty$, proving (3.5.2). The third case, when $k > 0$ and $\ell = n+1$ is similar. This completes the proof of (3.5.2).

Now we consider the point $\vec{t_0}$ defined as

$$(\vec{t_0})_i := \frac{(n-i+1)a+ib}{n+1} \quad \text{for } i=1,\dots,n.$$

Note that $\vec{t_0} \in \Delta_n$ is the average of the vertices of Δ_n. Next we define the field $F : \Delta_n \to \mathbb{R}^n$ as $F(\vec{t}) = -\vec{t} + \vec{t_0}$ and we claim that

$$\big\langle F(\vec{t}), T(\vec{t})\big\rangle < 0 \quad \text{for all } \vec{t} \in \partial\Delta_n. \tag{3.5.3}$$

Indeed, given the numbers $k < \ell$ associated to $\vec{t}$ we have three cases. First, if $k = 0$ and $\ell < n+1$, then

$$\big\langle F(\vec{t}), T(\vec{t})\big\rangle = (\vec{t} - \vec{t_0})_\ell = a - \frac{(n-\ell+1)a+\ell b}{n+1} = \frac{\ell(a-b)}{n+1} < 0.$$

Second, if $k > 0$ and $\ell < n+1$ then we have

$$\big\langle F(\vec{t}), T(\vec{t})\big\rangle = (\vec{t} - \vec{t_0})_\ell - (\vec{t} - \vec{t_0})_k = \frac{(\ell-k)(a-b)}{n+1} < 0.$$

The third case, when $k > 0$ and $\ell = n+1$ is again similar to the previous one. This completes the proof of (3.5.3).

Now we define the (continuous) homotopy $H : \Delta_n \times [0,1] \to \mathbb{R}^n$ as $H(\vec{t}, s) = sV(\vec{t}) + (1-s)F(\vec{t})$. Then we claim that there is $\varepsilon > 0$ so that for all $s \in [0,1]$ and all $\vec{t} \in \Delta_n$ satisfying $\operatorname{dist}(\vec{t}, \partial\Delta_n) < \varepsilon$, we have

$$H(\vec{t}, s) \neq 0.$$

Assuming that the above claim is true, we apply homotopy invariance of the degree, together with $\deg(F, \Delta_n, 0) = (-1)^n$, to get the existence of a zero for V.

In order to prove the claim we assume the contrary. Then there is a sequence $(\vec{t_m}, s_m)$ such that $\vec{t_m} \to \vec{t} \in \partial\Delta_n$ and $s_m \to s \in [0,1]$ as $m \to \infty$ and such that $H(\vec{t_m}, s_m) = 0$ for all m. Thus we have

$$\lim_{m\to\infty} \big\langle H(\vec{t_m}, s_m), T(\vec{t})\big\rangle = 0,$$

contradicting (3.5.2) and (3.5.3).

If we observe the definition of V we see that the first component V_1 is associated to $\lambda^-(t_0, t_1)$ and $\lambda^+(t_1, t_2)$, so that the eigenfunction that we can construct out of solutions of equation (3.5.1) will start being negative. For eigenfunctions starting with positive values in the first interval (t_0, t_1) we need to define the above arguments to the slightly modified function

$$\tilde{V}_i(\vec{t}) = \lambda^{(-1)^{i+1}}(t_{i-1}, t_i) - \lambda^{(-1)^i}(t_i, t_{i+1}), \qquad i = 1, \ldots, n\,.$$

□

Proof of Theorem 3.5.1. Let $u^{\pm}(s,t)$ be the eigenfunction associated to the eigenvalue $\lambda^{\pm}(s,t)$ (for the interval $(s,t) \subseteq (a,b)$). Let $t \in \Delta_n$ be such that $V(t) = 0$. We shall construct the eigenfunction u_n^- as follows: in (a, t_1) we define u_n^- as $u^-(a, t_1)$; in (t_1, t_2) as $\alpha_1 u^+(t_1, t_2)$, where α_1 is chosen to have

$$(u^-)'(a, t_1)(t_1) = \alpha_1 (u^+)'(t_1, t_2)(t_1).$$

The existence of α_1 is a direct consequence of Proposition 3.2.3. Repeating this argument in each interval we obtain a function u_n^- which is $C^1[a,b]$ and C^2 in $(a,b) \setminus \{x \in (a,b) : u'(x) = 0 \vee x = t_k, k \in \{1, \ldots, n\})\}$. The eigenvalue associated to this eigenfunction is $\lambda_n^- = \lambda^-(a, t_1)$.

In order to prove uniqueness, assume there exists a second eigenpair (λ, v) associated with n, such that there exist values $a < s_1 < s_2 < \ldots < s_n < b$, where v changes sign at those points and is negative in the interval (a, s_1). If $\lambda = \lambda_n^-$, then by Corollary 3.4.1, we necessarily have $s_i = t_i$ for all $i = 1, 2, \ldots, n$, and then the simplicity and isolation of the first eigenfunctions proved in Theorem 3.4.2 completes the argument.

Now suppose that $\lambda > \lambda_n^-$. Using Corollary 3.4.1, $s_1 < t_1$ and hence

$$\lambda > \lambda^-(a, t_1). \tag{3.5.4}$$

We either have $1 \leq i \leq n-1$ such that $(t_i, t_{i+1}) \subseteq (s_i, s_{i+1})$ or $s_n \leq t_n$. In the first case, if i is odd $\lambda^+(t_i, t_{i+1}) \geq \lambda$ and if i is even $\lambda^-(t_i, t_{i+1}) \geq \lambda$, contradicting (3.5.4) in both cases. In the second case, $\lambda \leq \lambda^+(t_n, b)$, if n is odd, contradicting (3.5.4) again and similarly if i is even. □

3.6. The radial problem

Our purpose is now to solve the N-dimensional problem, $N > 1$, in $\Omega = B_R$, for radially symmetric functions. First of all we shall introduce more appropriate notation. For $H : \mathcal{S}(N) \times (\mathbb{R}^N \setminus \{0\}) \times B_R \to \mathbb{R}$ define $\mathcal{H} : \mathbb{R}^2 \times \mathbb{R}_* \times [0, R] \to \mathbb{R}$ as

$$\mathcal{H}(m, \ell, p, r) := H(\ell I + (m - \ell)e_1 \otimes e_1, pe_1, re_1),$$

and the operators

$$P^+(a, b) := \Lambda(a^+ + (N-1)b^+) - \lambda(a^- + (N-1)b^-),$$
$$P^-(a, b) := \lambda(a^+ + (N-1)b^+) - \Lambda(a^- + (N-1)b^-).$$

In a similar way $\mathcal{F} : \mathbb{R}^3 \times \mathbb{R}_* \times [0, R] \to \mathbb{R}$ and $\mathcal{J} : \mathbb{R}^2 \times [0, R] \to \mathbb{R}$ can be defined. It is convenient to rewrite hypothesis (H2) for this context:

(H2') there exist $0 < \lambda \leq \Lambda$ such that for all $r \in [0, R]$, $p \in \mathbb{R} \setminus \{0\}$, $m, m' \in \mathbb{R}$, $\ell, \ell' \in \mathbb{R}$,

$$|p|^\alpha P^-(m, \ell) \leq \mathcal{H}(m + m', \ell + \ell', p, r) - \mathcal{H}(m', \ell', p, r) \leq |p|^\alpha P^+(m, \ell).$$

Here $|p|^\alpha m$ stands for $(\alpha + 1)^{-1}\varphi(u')'$ while $|p|^\alpha \ell$ stands for $\varphi(u')/r$. The difference with the one dimensional case is the singularity at $r = 0$, for which we need some extra arguments. From now on we shall assume that hypotheses (H1), (H2') and (H3) hold.

The first result is a lemma similar to Lemma 3.2.1, with similar proof so we omit it.

Lemma 3.6.1.

(I) *There exists a continuous function $\mathcal{G} : \mathbb{R}^3 \times \mathbb{R}_* \times [0, R] \to \mathbb{R}$ such that for $p \neq 0$ we have*

$$\mathcal{F}(m, \ell, p, u, r) = q \iff |p|^\alpha m = \mathcal{G}(|p|^\alpha \ell, p, u, q, r),$$

(II) *and there exists a continuous function $\mathcal{G}_1 : \mathbb{R}^2 \times \mathbb{R}_* \times [0, R] \to \mathbb{R}$ such that for $p \neq 0$ we have*

$$\mathcal{F}(\ell, \ell, p, u, r) = q \iff \ell = \mathcal{G}_1(p, u, q, r).$$

Here $\mathcal{G}$ and $\mathcal{G}_1$ are Lipschitz continuous in $(\varphi(p), \varphi(u), q)$ and additionally $\mathcal{G}$ is Lipschitz continuous in $|p|^\alpha \ell$.

Lemma 3.6.2. *Let $f \in C([0,R])$ and $u : [0,R] \to \mathbb{R}$ be a solution of*

$$\begin{aligned} \mathcal{F}[u] &= f \quad \textit{in } (0,R), \\ u'(0) &= 0,\ u(R) = 0. \end{aligned} \tag{3.6.1}$$

If $\varphi(u')'$ and $\varphi(u')/r$ are bounded in $(0,R)$, then the limit

$$\lim_{r\to 0} \frac{\varphi(u')(r)}{r}$$

exists and, consequently, $\varphi(u')'(0)$ is well defined.

Proof. Lemma 3.6.1 allows us to rewrite the problem as

$$\varphi(u')' = (1+\alpha)\mathcal{G}\big(\frac{\varphi(u')}{r}, u', u, f, r\big).$$

Defining $\phi(r) := \varphi(u')/r$ and using the boundary conditions we get

$$r\phi = (\alpha+1)\int_0^r \mathcal{G}(\phi, u', u, f, s)ds.$$

Taking derivative in this equality,

$$r\phi' + \phi = (\alpha+1)\mathcal{G}(\phi, u', u, f, r). \tag{3.6.2}$$

Suppose ϕ does not converge when $r \to 0^+$, then, being bounded, it has to oscillate. Thus, there exist $a < b$ and two sequences $(r_n^+)_{n\in\mathbb{N}}$, $(r_n^-)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to+\infty} r_n^+ = \lim_{n\to+\infty} r_n^- = 0, \qquad \lim_{n\to+\infty} \phi(r_n^+) = b,\ \lim_{n\to+\infty} \phi(r_n^-) = a$$

and

$$\phi'(r_n^+) = \phi'(r_n^-) = 0, \quad \forall n \in \mathbb{N}.$$

Using Lemma 3.6.1 and (3.6.2) we obtain

$$\phi(r_n^\pm) = (\alpha+1)\mathcal{G}\big(\phi(r_n^\pm), u'(r_n^\pm), u(r_n^\pm), f(r_n^\pm), r_n^\pm\big),$$

and then

$$\phi(r_n^\pm) = (\alpha+1)\mathcal{G}_1\big(u'(r_n^\pm), u(r_n^\pm), f(r_n^\pm), r_n^\pm\big).$$

But f, u, u' and $\mathcal{G}_1$ are continuous at $r = 0$, hence,

$$\lim_{n\to+\infty} \phi(r_n^+) = \lim_{n\to+\infty} \phi(r_n^-)$$

which is a contradiction. □

Remark 3.6.1. *It is noteworthy to relate our notion of radial solution to*

$$\begin{aligned} F(D^2u, Du, u, r) &= f \quad \text{in } B_R, \\ u &= 0 \quad \text{on } \partial B_R \end{aligned} \tag{3.6.3}$$

with the notion of viscosity solution given by Birindelli and Demengel in [4], based on the definition of Chen, Giga and Goto in [21]. If u is a solution of (3.6.1), then the function $w(x) = u(|x|)$ is of class C^1 and, whenever $u'(r) \neq 0$, u is of class C^2 close to r and, hence, w is of class C^2 at $|x| = r$. If, on the other hand, $u'(r) = 0$, then $Dw(x) = 0$ for all $|x| = r$ and we do not test the equation there (see [4]). We observe this is certanly the case of the origin, since $u'(0) = 0$ by definition. Moreover, if u is locally constant, then (3.6.1) is clearly satisfied pointwise, and consequently the equation is also satisfied in the viscosity sense proposed in [4]. Using this we conclude that solutions of (3.6.1) satisfy (3.6.3) in the viscosity sense.

Moreover, our solutions are such that $\varphi(u')$ is differentiable almost everywhere in $(0, R)$; therefore, $|Dw(x)|^\alpha Dw(x)$ is differentiable almost everywhere in B_R, and it is also differentiable at $x = 0$, as we have just proven.

Having this lemma, a solution of

$$\begin{aligned} F[u] &= f \quad \text{in } B_R \\ u &= 0 \quad \text{on } \partial B_R \end{aligned}$$

will be understood as a function $u : B_R \to \mathbb{R}$ such that $u(|\cdot|)$ solves (3.6.1) in the sense of Definition 3.2.1, with its boundary conditions. Notice that at points where u' does not vanish we get a C^2 solution for the radial equation; that is, $F[u]$ can be directly evaluated, and it is a solution in the usual sense thanks to (H3).

Now we state and prove the Alexandrov-Bakelman-Pucci inequality we need.

Proposition 3.6.1. *Let u be such that $\varphi(u')$ is absolutely continuous and a solution of*

$$P^+\big(\varphi(u')', \frac{\varphi(u')}{r}\big) + \gamma|u'|^{\alpha+1} \geq -f^- \quad \text{in } \{u > 0\}$$

where $u(R) \leq 0$ and $u'(0) = 0$. Then

$$\sup_{(0,R)} u^+ \leq B\|f^-\|_{L^N(B_R)}^{\frac{1}{1+\alpha}}. \tag{3.6.4}$$

In a similar way, if u is a solution of

$$P^-\big(\varphi(u')', \frac{\varphi(u')}{r}\big) - \gamma|u'|^{\alpha+1} \leq f^+ \quad \text{in } \{u < 0\}$$

and $u(R) \geq 0$, $u'(0) = 0$, then

$$\sup_{(0,R)} u^- \leq B\|f^+\|_{L^N(B_R)}^{\frac{1}{1+\alpha}}. \tag{3.6.5}$$

The constant B depends on N, λ, γ and R, and B is increasing in R.

Proof. Suppose $\sup_{(0,R)} u > 0$ and write $l_0 := \frac{1}{R}\sup_{(0,R)} u$. Let r_0 be a maximum point of u in $[0, R)$. Since $u(R) \leq 0$ and $u'(0) = 0$, there exists $r_- \in (0, R)$ such that $-u'(r_-) = l_0$ and $-u'(r) \leq l_0$ in (r_0, r_-). Even more, we can find a set I, an union of intervals, such that $\varphi(u')' \leq 0$ in I and $-\varphi(u')(I) = (0, \varphi(l_0))$. Within this set $\varphi(u')$ and $\varphi(u')'$ are not positive; therefore,

$$P^+\big(\varphi(u')', \frac{\varphi(u')}{r}\big) = \lambda\big(\varphi(u')' + (N-1)\frac{\varphi(u')}{r}\big) \quad \text{for all } r \in I.$$

Now let $k > 0$. A change of variable allows us to write

$$\begin{aligned} \ln\big(1 + \frac{\varphi(l_0^N)}{k}\big) &= \int_0^{\varphi(l_0^N)} \frac{dz}{z+k} \\ &\leq \int_I \frac{-N(-\varphi(u'(r))^{N-1}\varphi(u')'(r)dr}{(-\varphi(u')(r))^N + k}, \end{aligned}$$

where the inequality holds because in general u' is not injective in I. Developing this inequality,

$$
\begin{aligned}
\ln\left(1+\frac{\varphi(l_0^N)}{k}\right) &\leq N\int_I \left(\frac{-\varphi(u')(r)}{r}\right)^{N-1}(-\varphi(u')'(r))\frac{r^{N-1}dr}{(-\varphi(u')(r))^N+k}\\
&\leq N^{1-N}\int_I \left(-\varphi(u')'(r)-(N-1)\frac{\varphi(u')(r)}{r}\right)^N \frac{r^{N-1}dr}{(-\varphi(u')(r))^N+k}\\
&\leq \frac{(N/2)^{1-N}}{\lambda^N}\int_I \left(\frac{|f^-|^N}{k}+\gamma^N\right) r^{N-1}dr\\
&\leq \frac{(N/2)^{1-N}}{\lambda^N}\left(\frac{1}{k\omega_N}\|f^-\|^N_{L^N(B_R)}+\frac{\gamma^N R^N}{N}\right).
\end{aligned}
$$

In these computations we have used that the geometric mean is always below the arithmetic mean, and $(a+b)^N \leq 2^{N-1}(a^N+b^N)$. ω_N is the $(N-1)$-dimensional measure of $\mathbb{S}^{N-1}$. This inequality implies that $\|f^-\|_{L^N(B_R)} > 0$ because $k > 0$ is arbitrary. Taking $k = \|f^-\|^N_{L^N(B_R)}$, we obtain

$$\varphi(l_0) \leq C\|f^-\|_{L^N(B_R)},$$

with $C > 0$ depending on N, λ, γ and R. Taking φ^{-1} we get the desired inequality. □

Remark 3.6.2.

(I) *The regularity of u implies that the equalities in the statement of Proposition 3.6.1 hold a.e.; therefore, it is not necessary to use an approximation argument as in [15]. In particular, it is not necessary to separate the degenerate and the singular case as in [27].*

(II) *Note that if $N = 1$ this procedure is the proof of Proposition 3.2.4, after doing the natural change in the domain to take in account a general interval (a, b).*

We also have the following comparison principle, whose proof follows the same arguments of Proposition 3.2.2.

Proposition 3.6.2. *Suppose $\mathcal{F}(m, \ell, p, u, r)$ is nonincreasing in u. Let u, v be such that*

$$\mathcal{F}[u] \geq \mathcal{F}[v] \quad \textit{in } (0, R)$$

and $u(R) \leq v(R)$, $u'(0) = v'(0) = 0$, where $\varphi(u')/r$ and $\varphi(v')/r$ are bounded functions. Then $u \leq v$ in $[0, R]$.

Now we prove the following basic existence and uniqueness theorem.

Theorem 3.6.1. *There exists $\kappa > \gamma$ such that*

$$\mathcal{F}[u] - \kappa\varphi(u) = f \quad \text{in } (0, R), \tag{3.6.6}$$

$$u'(0) = u(R) = 0 \tag{3.6.7}$$

has a unique solution for each continuous function $f \leq 0$.

To prove this result we first see the following lemma, which is a consequence of Theorem 3.3.2:

Lemma 3.6.3. *There exists $\kappa > \gamma$ (independent of ε) such that for all $f \in C([0, R])$, $f \leq 0$ and $\varepsilon > 0$ the problem*

$$\mathcal{F}[u] - \kappa\varphi(u) = f \quad \text{in } (\varepsilon, R), \tag{3.6.8}$$

$$u'(\varepsilon) = u(R) = 0, \tag{3.6.9}$$

has a unique solution u_ε.

The next lemma provides bounds for u_ε which do not depend on ε. The proof is inspired by the one of Lemma 2.2 in [35].

Lemma 3.6.4. *Let u_ε be the solution of (3.6.8)–(3.6.9) given by Lemma 3.6.3. Then there exists C, independent of ε, such that*

$$\left\| \frac{\varphi(u'_\varepsilon)}{r} \right\|_{L^\infty([\varepsilon,R])} + \left\| \varphi(u'_\varepsilon)' \right\|_{L^\infty([\varepsilon,R])} \leq C.$$

Proof. Let us assume, for the moment, that u_ε and u'_ε are uniformly bounded in $[\varepsilon, R]$ (we prove it later). Suppose first there are sequences $\varepsilon_n \to 0^+$, $r_n \in (\varepsilon_n, R]$ such that

$$\lim_{n\to+\infty} \frac{\varphi(u'_n)(r_n)}{r_n} = -\infty, \tag{3.6.10}$$

where we have written $u_n = u_{\varepsilon_n}$. Hypothesis (H2'), (3.6.8) and the bounds over u_ε and u'_ε yield that $\varphi(u'_n)'(r_n) \to +\infty$ as $n \to +\infty$.

If $\varphi(u'_n)' > 0$ for all $r \in (\varepsilon_n, r_n]$, then $u'_n(r_n) > 0$, which is not possible by (3.6.10). Therefore, for all n there exists $\bar{r}_n \in (\varepsilon_n, r_n)$ such that $\varphi(u'_n)'(\bar{r}_n) = 0$ and $\varphi(u'_n)'(r) > 0$ for $r \in (\bar{r}_n, r_n)$. Using that φ is strictly increasing, $u'_n(\bar{r}_n) < u'_n(r_n)$, and so

$$\lim_{n\to+\infty} \frac{\varphi(u'_n)(\bar{r}_n)}{\bar{r}_n} = -\infty, \quad \varphi(u'_n)'(r_n) = 0,$$

which is not possible by (3.6.8), (H2') and the bounds on u_ε and u'_ε. If we assume

$$\lim_{n\to+\infty} \frac{\varphi(u'_n)(r_n)}{r_n} = +\infty,$$

similar arguments lead us to contradictions. Hence, for certain constant C

$$|\frac{\varphi(u'_\varepsilon)(r)}{r}| \le C,$$

for all $r \in [\varepsilon, R]$. To conclude, we use again the equation (3.6.8) and (H2').

Now we prove our first claim by contradiction: if $\beta_\varepsilon := \|u_\varepsilon\|_1$ is unbounded, then the function $v_\varepsilon := u_\varepsilon/\beta_\varepsilon$ is such that v_ε and v'_ε are bounded, and v_ε solves

$$\mathcal{F}[v_\varepsilon] - \kappa\varphi(v_\varepsilon) = \frac{f}{\beta_\varepsilon^{\alpha+1}} \quad \text{in } (\varepsilon, R),$$
$$v'_\varepsilon(\varepsilon) = 0,\ v_\varepsilon(R) = 0.$$

Hence,

$$\left|\frac{\varphi(v'_\varepsilon)(r)}{r}\right| + |\varphi(v'_\varepsilon)'(r)| \le C$$

for all $r \in [\varepsilon, R]$, where C is independent of ε. Using the Arzelà-Ascoli theorem, we find a subsequence $v_{\varepsilon_n} \to v$ uniformly in $C^1([0, R])$, with v solving (3.6.8)–(3.6.9) in $(0, R)$ with right hand side 0. The ABP inequality in Proposition 3.6.1 yields $v \equiv 0$, contradicting $\|v_\varepsilon\|_1 = 1$ for all ε. □

Now we have all the necessary tools to prove Theorem 3.6.1.

Proof of Theorem 3.6.1. Using Proposition 3.6.1 we obtain a sequence of approximate solutions to (3.6.6)–(3.6.7). Using Lemma 3.6.4 we have a bound over these solutions, allowing us to use the Arzelà-Ascoli theorem to find a solution of the problem. □

The following is a compactness lemma whose proof is similar to the ones of Lemma 3.4.1 and Lemma 3.6.4:

Lemma 3.6.5. *Let u_n be the solution (3.6.6)–(3.6.7) with right hand side f_n, where $(f_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $C([0, R])$. Then there exists C independent of n such that*

$$\|u_n\|_\infty + \|u'_n\|_\infty + \|\varphi(u'_n)'\|_\infty \le C.$$

All the previous theory permits us to prove the following

Theorem 3.6.2. *The problem* (3.1.1)–(3.1.2) *has a radial solution* (λ^+, u^+) *which is strictly positive in* B_R*, and a radial solution* (λ^-, u^-) *such that* $u^- < 0$ *in* B_R*. Moreover,*

(I) $\lambda^+ \leq \lambda_-$,

(II) *every positive (resp. negative) solution of* (3.1.1) *is a multiple of* u^+ *(resp.* u^-*),*

(III) *denoting* $\lambda^\pm(R)$ *the eigenvalue associated to* B_R*,* $\lambda^\pm(R) < \lambda^\pm(R')$ *whenever* $R > R'$*, and*

(IV) $\lambda^\pm(R) \to +\infty$ *if* $R \to 0$*.*

Proof. Theorem 3.6.1 and Lemma 3.6.5 allow us to follow the proof of Theorem 3.4.2 yielding the existence of eigenvalues and eigenfunctions. The qualitative results concerning the behaviour of $\lambda^\pm$ can be proven using the same arguments as in the 1-dimensional case. □

Proof of Theorem 3.1.1. The results obtained in this section allow to follow the proof of Theorem 3.5.1 in this case. □

Chapter 4

An equation involving the fractional Laplacian

4.1. Introduction and notation

The purpose of this chapter is to study the problem

$$\begin{cases} (-\Delta)^\alpha u + u^q = u^p \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{4.1.1}$$

where $0 < \alpha < 1$, $N \geq 3$ and $0 < q < 1 < p < \frac{N+2\alpha}{N-2\alpha}$, assumptions that are held for the rest of this Chapter. From now on, we denote $\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times [0, +\infty)$, $\hat{\cdot}$ (or $\mathcal{F}(\cdot)$) stands for the Fourier transform, and the fractional Sobolev space

$$H^\alpha(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2\alpha})|\hat{u}|^2 d\xi < +\infty \right\},$$

is endowed with the norm

$$\|u\|_\alpha^2 := \int_{\mathbb{R}^N} (1 + |\xi|^{2\alpha})|\hat{u}|^2 d\xi.$$

The operator $(-\Delta)^\alpha$ can be defined in several alternative ways:

Definition 4.1.1. *Let $v \in H^\alpha(\mathbb{R}^N)$, and denote the Fourier transform as $\mathcal{F}$. The fractional Laplacian of v is defined as*

$$\mathcal{F}\big((-\Delta)^\alpha v\big) := |\xi|^{2\alpha}\mathcal{F}(v).$$

If v is smooth enough; in particular, if v belongs to the Schwarz class, the following is an equivalent definition:

Definition 4.1.2. *Let $v \in \mathcal{S}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. Then*

$$(-\Delta)^\alpha v(x) = C_{N,\alpha} P.V. \int_{\mathbb{R}^N} \frac{v(x)-v(y)}{|x-y|^{N+2\alpha}} dy,$$

where P.V. stands for the Cauchy principal value.

In [45] it is proven that these definitions are equivalent. The third definition was recently devised by Caffarelli and Silvestre [17]:

Definition 4.1.3. *Let v be the α-harmonic extension of $u \in C^2(\mathbb{R}^N)$; that is,*

$$\nabla \cdot (y^{1-2\alpha} \nabla v) = 0 \quad \text{in } \mathbb{R}^{N+1}_+$$
$$v(x,0) = u \quad \text{on } \mathbb{R}^N.$$

Then

$$(-\Delta)^\alpha u(x) = -C_{N,\alpha} \lim_{y \to 0^+} y^{1-2\alpha} \frac{\partial v}{\partial y}(x,y).$$

The exact value of $C_{N,\alpha}$ in the last two definitions is not relevant for our applications, and hence we take $C_{N,\alpha} = 1$.

When (4.1.1) is written as

$$(-\Delta)^\alpha u + u = u^p + u - u^q \quad \text{in } \mathbb{R}^N, \tag{4.1.2}$$

this equation can be treated using the Bessel kernel

$$\mathcal{K}(x) := \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|^{2\alpha}}\right); \tag{4.1.3}$$

that is, $u = \mathcal{K} * f$ solves

$$(-\Delta)^\alpha u + u = f \quad \text{in } \mathbb{R}^N.$$

The essential properties of this kernel are summarised in Theorem 4.3.1.

Before stating our results, it is necessary to precise the notions of solution to be used for our problem.

Definition 4.1.4. *$u \in H^\alpha(\mathbb{R}^N)$ is said to be a weak solution of (4.1.1) if*

$$\int_{\mathbb{R}^N} |\xi|^{2\alpha} \hat{u}\hat{v} d\xi = \int_{\mathbb{R}^N} (u^p - u^q) v dx$$

holds for all $v \in H^\alpha(\mathbb{R}^N)$.

Definition 4.1.2 applies when u has sufficient regularity (e.g. $u \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$). It is also possible to rewrite it as

$$(-\Delta)^\alpha u(x) = -\int_{\mathbb{R}^N} \frac{\delta(u)(x,y)}{|y|^{N+2\alpha}} dy, \tag{4.1.4}$$

where we have written $\delta(u)(x,y) := u(x+y) + u(x-y) - 2u(x)$.

Definition 4.1.5. *$u \in C(\mathbb{R}^N)$ is said to be a classical solution of (4.1.1) if $(-\Delta)^\alpha u$ can be written as in (4.1.4) and (4.1.1) is satisfied pointwise in $\mathbb{R}^N$.*

4.1.1. Main results

We now state the main results obtained in this Chapter. The first result is

Theorem 4.1.1.

(I) *Problem (4.1.1) has a weak solution $u \in H^\alpha(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$, which satisfies $u \geq 0$ a.e. in $\mathbb{R}^N$.*

(II) *If $u \in H^\alpha$ is a weak solution of (4.1.1), then u is a classical solution. In particular, $u > 0$ in $\mathbb{R}^N$.*

For this theorem we work with the fractional Laplacian via the Fourier transform (Definition 4.1.1), which allows to use the Mountain Pass theorem [2] applied to the corresponding functional and an additional argument accounting for the loss of compactness when taking the whole space as a domain. Afterwards we obtain a suitable weak maximum principle.

In order to obtain the regularity of weak solutions we use the formulation (4.1.4) of the equation that permits the use of a Bessel kernel, combined with the usual iteration from the L^p theory and a localisation idea by Silvestre [58].

The second result is the following.

Theorem 4.1.2. *Let u be a classical solution of (4.1.1). Then there exists a constant $C > 0$ such that*

$$u(x) \leq C|x|^{\frac{-(N+2\alpha)}{q}},$$

and for every $\eta < -(N + 2\alpha)/q$ there exists $L_0 > 0$ such that

$$u(x) \geq |x|^{\eta} \quad \text{if } |x| > L_0.$$

The difference with the case $\alpha = 1$ is enormous, since in that case [23] solutions have compact support. A consequence of this fact was that in [23] the application of the moving planes method [38] was relatively easy, the main problem being the non-Lipschitzian nonlinearity.

The third theorem is a radial symmetry result.

Theorem 4.1.3. *Let u be a classical solution of (4.1.1). Then u is radially symmetric around some point of $\mathbb{R}^N$.*

The proof of this result mixes the extension result by Caffarelli and Silvestre with a result by Terracini [61] about symmetry around the e_N-axis for the problem

$$\begin{aligned} -\Delta u &= f(y, u) \quad \text{in } \mathbb{R}^N_+ \\ -\frac{\partial u}{\partial y_N} &= g(\bar{y}, u) \quad \text{on } \mathbb{R}^{N-1}, \end{aligned}$$

where $y \in \mathbb{R}^N$ is written as $(\bar{y}, y_N)$. The result differs considerably from the usual application of the moving planes method. For instance, a straightforward application of the method is hampered due to the lack of an appropriate maximum principle to conclude; while methods devised for integral equations (e.g. the work of Li [47] and Chen, Li and Ou [20]) rely heavily on the Lipschitz continuity of the nonlinearity. The natural disadvantage of our approach is not knowing the full extent of the class of integral operators for which our method applies.

This Chapter is organised in the following way. Section 4.2 is devoted to prove the first part of Theorem 4.1.1, while Section 4.3 proves the second part. Section 4.4 contains the proof of Theorem 4.1.2, Section 4.5 the proof of the symmetry result (Theorem 4.1.3). Finally, we formulate some remarks on the moving planes method within the context of this work.

4.2. Existence of a weak solution

Define the Banach space

$$E := \left\{ u \in H^{\alpha}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{q+1} dx < +\infty \right\}$$

with norm

$$\|u\|_E := \|u\|_{\alpha} + \left(\int_{\mathbb{R}^N} |u|^{q+1} dx \right)^{\frac{1}{q+1}}.$$

Consider the functional $I : E \to \mathbb{R}$ defined as

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\xi|^{2\alpha} |\hat{u}|^2 d\xi - \int_{\mathbb{R}^N} \left(\frac{u_+^{p+1}}{p+1} - \frac{|u|^{q+1}}{q+1} \right) dx.$$

This functional is well defined and of class C^1, which can be proven using arguments from [25, 23]. Its derivative is

$$I'(u)[\varphi] = \int_{\mathbb{R}^N} |\xi|^{2\alpha} \hat{u} \hat{\varphi} d\xi - \int_{\mathbb{R}^N} (u_+^p - |u|^q \text{sgn } u) \varphi \, dx,$$

where sgn u stands for the sign of u. Note that $u \cdot \text{sgn } u = |u|$. The basic properties of E are summarised in

Lemma 4.2.1. *Let $2 \leq r \leq 2_{\alpha}^* := 2N/(N - 2\alpha)$ and $u \in E$. Then*

$$\|u\|_{L^r(\mathbb{R}^N)} \leq C \|u\|_E$$

Moreover, if r is as before and $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, then every bounded sequence $(u_k)_k \subseteq E$ has a converging subsequence in $L^r(\Omega)$.

We refer the reader to [37] and the references therein for the proof of this Lemma.

The following concentration-compactness principle inspired by Coti-Zelati and Rabinowitz [25] is required as well:

Lemma 4.2.2. *Let $(u_k)_k$ be a bounded sequence in E such that*

$$\lim_{k \to \infty} \sup_{\xi \in \mathbb{R}^N} \int_{B_R(\xi)} |u_k(x)|^{q+1} dx = 0,$$

where $R > 0$. Then $u_k \to 0$ in $L^r(\mathbb{R}^N)$, for $q + 1 < r < 2_{\alpha}^*$.

Proof. Let $q+1<r<2^*_\alpha$, $R>0$ and $\xi\in\mathbb{R}^N$. Using the Hölder inequality we obtain, for every k, that

$$\|u_k\|_{L^r(B_R(\xi))}\le \|u_k\|^{1-\lambda}_{L^{q+1}(B_R(\xi))}\|u_k\|^{\lambda}_{L^{2^*_\alpha}(B_R(\xi))},$$

where $\frac{1-\lambda}{q+1}+\frac{\lambda}{2^*_\alpha}=\frac{1}{r}$. Covering $\mathbb{R}^N$ with balls of radius R in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we deduce

$$\int_{\mathbb{R}^N}|u_k|^r dx\le (N+1)\|u_k\|^{(1-\lambda)r}_{L^{q+1}(B_R(\xi))}\|u_k\|^{\lambda r}_{L^{2^*_\alpha}(B_R(\xi))}.$$

Using Lemma 4.2.1 and the assumption, $u_k\to 0$ in $L^r(\mathbb{R}^N)$. □

Note that if $I'(u)[u]=0$ and $u\not\equiv 0$, then necessarily $u_+\not\equiv 0$, and it is easy to see that if $I'(u)[u]=0$, then $I(u)>0$.

Consider the set of functions

$$\Gamma:=\big\{g\in C\big([0,1],E\big):g(0)=0,\ I(g(1))<0\big\}$$

and define

$$c:=\inf_{g\in\Gamma}\sup_{t\in[0,1]}I(g(t))$$

Using that $I'(u)[u]=0$ implies $u\equiv 0$ or $I(u)>0$ we easily conclude $c>0$.

Theorem 4.2.1. *I has at least one critical value with critical value c.*

Proof. By the Ekeland variational principle [50], there is a sequence u_n such that

$$I(u_n)\to c \quad\text{and}\quad I'(u_n)\to 0.$$

For n large enough we have

$$\begin{aligned}C+(\||\xi|^\alpha\hat{u}_n\|_{L^2}+\|u_n\|_{L^{q+1}})&\ge (p+1)I(u_n)-I'(u_n)[u_n]\\&=\left(\frac{p+1}{2}-1\right)\int_{\mathbb{R}^N}|\xi|^{2\alpha}|hatu_n|^2 d\xi+\\&\quad+\left(\frac{p+1}{q+1}-1\right)\int_{\mathbb{R}^N}|u|^{q+1}dx\\&\ge\frac{p-1}{2}\left(\||\xi|^\alpha\hat{u}_n\|^2_{L^2}+\|u_n\|^{q+1}_{L^{q+1}}\right),\end{aligned}$$

from where u_n is bounded in E. Using Lemma 4.2.1, there is a subsequence of $(u_n)_n$ converging weakly in $H^\alpha(\mathbb{R}^N)$ and L^p_{loc} to $u\in H^\alpha(\mathbb{R}^N)$. Hence, for such a subsequence and any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\lim_{n\to\infty} I'(u_n)[\varphi] = I'(u)[\varphi] = 0.$$

If we prove that $u \not\equiv 0$, then $I'(u) = 0$ and $I(u) \geq c$. On the other hand, for $R > 0$ large enough,

$$\begin{aligned} I(u_n) - \frac{1}{2}I'(u_n)[u_n] &= \frac{p-1}{2(p+1)}\int_{\mathbb{R}^N}(u_+)_n^{p+1}dx + \frac{1-q}{2(1+q)}\int_{\mathbb{R}^N}|u_n|^{q+1}dx \\ &\geq \frac{p-1}{2(p+1)}\int_{B_R(0)}(u_+)_n^{p+1}dx + \frac{1-q}{2(1+q)}\int_{B_R(0)}|u_n|^{q+1}dx. \end{aligned}$$

Taking limit as $n \to +\infty$,

$$c \geq \frac{p-1}{2(p+1)}\int_{B_R(0)}u_+^{p+1}dx + \frac{1-q}{2(1+q)}\int_{B_R(0)}|u|^{q+1}dx.$$

This inequality holds for all R large enough, hence, it also holds in the whole space. Using that $I'(u) = 0$, follows that $I(u) \leq c$.

In order to complete the proof we need to show that u is nontrivial. Using Lemma 4.2.2, it is possible to find a sequence $(y_n)_n \subseteq \mathbb{R}^N$, $R > 0$ and $\beta > 0$ such that

$$\int_{B_R(y_n)}|u_n|^{q+1}dx > \beta$$

for all n. Indeed, if we assume the contrary, we have $u_n \to 0$ in $L^{p+1}(\mathbb{R}^N)$. But then, for n large enough,

$$\begin{aligned} \frac{c}{2} &\leq I(u_n) - \frac{1}{2}I'(u_n)[u_n] \\ &= \frac{p-1}{2(p+1)}\int_{B_R(0)}(u_+)_n^{p+1}dx + \frac{1-q}{2(1+q)}\int_{B_R(0)}|u_n|^{q+1}dx, \end{aligned}$$

which is a contradiction, because $c > 0$.

Now, let $\tilde{u}_n(x) := u_n(x + y_n)$. Using the discussion given above, $u :=$ w-$\lim \tilde{u}_n$ is a nontrivial critical point of I. □

Proof of Theorem 4.1.1, part I. Having the existence of a critical point u of I in E, we just have to prove that $u \geq 0$. For this, we follow Theorem 1.1 in [37], using the fact that

$$\int_{\mathbb{R}^N}(-\Delta)^\alpha u \cdot \varphi \, dx = C\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{N+2\alpha}}dxdy$$

for all $\varphi \in H^\alpha(\mathbb{R}^N)$. Testing with $u_- := \max\{-u, 0\}$,

$$\int_{\mathbb{R}^N} (-\Delta)^\alpha u \cdot u_- dx = \int_{\mathbb{R}^N} u_+^p u_- - |u|^q \text{sgn}\, u \cdot u_- dx = \int_{\mathbb{R}^N} u_-^{q+1} dx.$$

But this cannot occur for $u_- \not\equiv 0$, because

$$\begin{aligned}\int_{\mathbb{R}^N} (-\Delta)^\alpha u \cdot u_- = C \iint_{\{u<0\}\times\{u>0\}} & \frac{[u(x)-u(y)]u_-(x)}{|x-y|^{N+2\alpha}} dxdy + \\ + C \iint_{\{u>0\}\times\{u<0\}} & \frac{[u(x)-u(y)]u_-(y)}{|x-y|^{N+2\alpha}} dxdy + \\ + C \iint_{\{u<0\}\times\{u<0\}} & \frac{[u(x)-u(y)][u_-(x)-u_-(y)]}{|x-y|^{N+2\alpha}} dxdy.\end{aligned}$$

The last term can be written as

$$-C \iint_{\{u<0\}\times\{u<0\}} \frac{[u_-(x)-u_-(y)]^2}{|x-y|^{N+2\alpha}} dxdy,$$

which is strictly negative unless $u_- \equiv 0$ a.e. The other two terms are also negative; hence, $u_- \equiv 0$ and the conclusion follows. □

4.3. Regularity of solutions

Our purpose is to prove the second part of Theorem 4.1.1. This will be done rewriting (4.1.1) as

$$(-\Delta)^\alpha u + u = u^p + u - u^q \quad \text{in } \mathbb{R}^N, \tag{4.3.1}$$

also with $u > 0$ and $\lim_{|x|\to\infty} u(x) = 0$. This equation has a Bessel kernel, as discussed before, whose properties are recalled in the following

Theorem 4.3.1. *Let $\mathcal{K}$ be the Bessel kernel associated to (4.3.1); that is,*

$$\mathcal{K}(x) = \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|^{2\alpha}}\right).$$

Then the following properties hold:

(I) *$\mathcal{K}$ is positive, radially symmetric and smooth in $\mathbb{R}^N \setminus \{0\}$. Moreover, $\mathcal{K}$ is nonincreasing as a function of $|x|$.*

(II) *For appropriate constants* C_1 *and* C_2*,*

$$\mathcal{K}(x) \leq C_1|x|^{-N-2\alpha} \quad \text{if } |x| \geq 1,$$
$$\mathcal{K}(x) \leq C_2|x|^{-N+2\alpha} \quad \text{if } |x| \leq 1.$$

(III) *There exists a constant* C *such that, for* $|x| \geq 1$*,*

$$|\nabla\mathcal{K}(x)| \leq C|x|^{-(N+1+2\alpha)} \quad \text{and} \quad |D^2\mathcal{K}(x)| \leq C|x|^{-(N+2+2\alpha)}.$$

(IV) *If* $r \geq 1$ *and* $s \in \left(N - 2\alpha - \frac{N}{r}, N + 2\alpha - \frac{N}{r}\right)$*, then* $|x|^s\mathcal{K}(x) \in L^r(\mathbb{R}^N)$*.*

(V) $|x|^{N+2\alpha}\mathcal{K}(x) \in L^\infty(\mathbb{R}^N)$*.*

The proof of this Theorem can be found in the Appendix of [37].

Before proceeding with the regularity results, recall the fractional Sobolev spaces for $p \geq 1$ and $\beta > 0$:

$$\mathcal{L}^{\beta,p} := \left\{u \in L^p(\mathbb{R}^N) : \mathcal{F}^{-1}\left[(1+|\xi|^2)^{\beta/2}\hat{u}\right] \in L^p(\mathbb{R}^N)\right\}.$$

Associated to the fractional Laplacian is the space

$$\mathcal{W}^{\beta,p} := \left\{u \in L^p(\mathbb{R}^N) : \mathcal{F}^{-1}\left[(1+|\xi|^\beta)\hat{u}\right] \in L^p(\mathbb{R}^N)\right\}.$$

The following properties are quoted from [60]:

Theorem 4.3.2. *Let* $p \geq 1$ *and* $\beta > 0$*. Then*

(I) $\mathcal{L}^{\beta,p} = \mathcal{W}^{\beta,p}$ *and* $\mathcal{L}^{n,p} = W^{n,p}(\mathbb{R}^N)$ *for* $n \in \mathbb{N}$*.*

(II) *If* $\alpha \in (0,1)$ *and* $2\alpha < \beta$*, then* $(-\Delta)^\alpha : \mathcal{W}^{\beta,p} \to \mathcal{W}^{\beta-2\alpha,p}$*.*

(III) *Let* $\alpha, \gamma \in (0,1)$*. If* $0 < \mu \leq \gamma - 2\alpha$ *and* $2\alpha < \gamma$*, then*

$$(-\Delta)^\alpha : C^{0,\gamma}(\mathbb{R}^N) \to C^{0,\mu}(\mathbb{R}^N),$$

and if $0 \leq \mu \leq 1 + \gamma - 2\alpha$ *and* $2\alpha > \gamma$*, then*

$$(-\Delta)^\alpha : C^{1,\gamma}(\mathbb{R}^N) \to C^{0,\mu}(\mathbb{R}^N).$$

Another Theorem is about embeddings.

Theorem 4.3.3.

(I) *Suppose $s \geq 0$ and either*

$$1 < r \leq t \leq \frac{Nr}{N - sr} < \infty \quad or \quad r = 1 \text{ and } 1 \leq t < \frac{N}{N - s}.$$

Then $\mathcal{L}^{s,r}$ is continuously embedded in $L^t(\mathbb{R}^N)$.

(II) *Assume $0 \leq s \leq 2$ and $s > N/r$. If*

$$s - \frac{N}{r} > 1 \quad and \quad 0 < \mu \leq s - \frac{N}{r} - 1,$$

then $\mathcal{L}^{s,r}$ is continuously embedded in $C^{1,\mu}(\mathbb{R}^N)$. If

$$s - \frac{N}{r} < 1 \quad and \quad 0 < \mu \leq s - \frac{N}{r},$$

then $\mathcal{L}^{s,r}$ is continuously embedded in $C^{0,\mu}(\mathbb{R}^N)$.

Lemma 4.3.1. *Let $u \in H^\alpha(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$ be a weak solution of (4.1.1). Then $u \in C^{0,\mu}(\mathbb{R}^N)$ for some $\mu \in (0, 1)$. Moreover, $u(x) \to 0$ as $|x| \to +\infty$.*

Proof. In this proof we follow the approach introduced in [37], including the localisation idea in [58]. Since u satisfies (4.1.1), then u satisfies (4.3.1) in the sense of distributions. Let $1 = r_0 > r_{1/2} > r_1 > \ldots > 0$, and write $B_i := B_{r_i}(0)$. Define $h(x) := u^p(x) + u(x) - u^q(x)$, and notice that

$$|h(x)| \leq C\big(|u(x)|^q + |u(x)|^p\big). \tag{4.3.2}$$

Writing $q_0 := \frac{2N}{N-2\alpha}$, the Sobolev embedding (Theorem 4.3.3) implies $u \in L^{q_0}(\mathbb{R}^N)$. Let $\eta_1 \in C^\infty(\mathbb{R}^N)$ be such that $0 \leq \eta_1 \leq 1$, $\eta_1 \equiv 1$ in $B_{1/2}$ and $\eta_1 \equiv 0$ outside B_0. Define u_1 as the solution of

$$(-\Delta)^\alpha u_1 + u_1 = \eta_1 h \quad \text{in } \mathbb{R}^N. \tag{4.3.3}$$

Subtracting u and u_1,

$$(-\Delta)^\alpha (u - u_1) + (u - u_1) = (1 - \eta_1)h \quad \text{in } \mathbb{R}^N, \tag{4.3.4}$$

and using the Bessel kernel we obtain

$$u - u_1 = \mathcal{K} * [(1 - \eta_1)h]. \tag{4.3.5}$$

Using Hölder's inequality and part II of Theorem 4.3.1, for all $x \in B_1$ we have

$$|u(x) - u_1(x)| \leq C \Big[\|\mathcal{K}\|_{L^{s_0}(B^c_{1/2})} \|(1-\eta_1)u\|_{L^{q_0}(\mathbb{R}^N)} + \\ + \|\mathcal{K}\|_{L^{s_1}(B^c_{1/2})} \|(1-\eta_1)u\|^p_{L^{q_0}(\mathbb{R}^N)} \Big] \quad (4.3.6)$$

where $s_0 = \frac{q_0}{q_0-1}$ and $s_1 = \frac{q_0}{q_0-p}$. This inequality allows us to focus on u_1. Write $g_1 = -u_1 + \eta_1 h$. The fact that

$$(-\Delta)^\alpha u_1 = g_1 \quad \text{in } \mathbb{R}^N,$$

that $u \in L^{q_0}(\mathbb{R}^N)$ and B_0 being bounded imply that $\eta_1 h \in L^{p_1}(\mathbb{R}^N)$, where $p_1 = q_0/p$. By the definition of $\mathcal{W}^{2\alpha,p_1}$ and u_1 solving (4.3.3), we deduce $u_1 \in \mathcal{W}^{2\alpha,p_1}$, and $\|u_1\|_{\mathcal{W}^{2\alpha,p_1}}$ depends on N, α, q and $\|u\|_{H^\alpha}$. At this point three cases should be acknowledged: 1) $p_1 < N/(2\alpha)$, 2) $p_1 = N/(2\alpha)$, and 3) $p_1 > N/(2\alpha)$.

Suppose case 1) holds. Then we use the Sobolev embedding and (4.3.6) to obtain $u \in L^{q_1}(B_1)$, where $q_1 = \frac{Np_1}{N-2\alpha p_1}$. The procedure can now be repeated, considering now a smooth function η_2 such that $0 \leq \eta_2 \leq 1$, supported in B_1 and satisfying $\eta_2 \equiv 1$ in $B_{3/2}$. Proceeding as above, mutatis mutandis, we get

$$u_2 = \mathcal{K} * (\eta_2 h);$$

hence, $u_2 \in \mathcal{W}^{2\alpha,p_2}$, where $p_2 = q_1/p$. Once again three cases are identified as before.

As long as case 1) holds, we define the sequence q_j as follows:

$$\frac{1}{q_j} = \sum_{i=1}^{j} p^i \left(\frac{1}{q_1} - \frac{1}{q_0} \right) + \frac{1}{q_1}.$$

Using that $1 < p < 2^*_\alpha - 1$, $q_1 > q_0$ and $p^i > 1$, the right hand side eventually becomes negative. Let j be the smallest natural for which the sum is not positive. Then either $p_{j+1} = N/(2\alpha)$ or $p_{j+1} > N/(2\alpha)$.

If $p_{j+1} > N/(2\alpha)$, then $u_{j+1} \in \mathcal{W}^{2\alpha,p_{j+1}}$ so that, by the Sobolev embedding, we can choose

$$0 < \mu < \min\left\{ 2\alpha - \frac{N}{p_{j+1}}, 1 \right\}$$

and then conclude $u_{j+1} \in C^{0,\mu}(\mathbb{R}^N)$. From (4.3.5) and (4.3.6), for u_{j+1} and η_{j+1} instead of u_1 and η_1, the L^∞ estimate for $u - u_{j+1}$ is obtained. Using the convolution formula, we can differentiate $u - u_{j+1}$: for all $x \in B_{j+1}$,

$$\begin{aligned}|\nabla(u-u_{j+1})(x)| &\le \int_{\mathbb{R}^N} |\nabla\mathcal{K}(x-y)|\big|(1-\eta_{j+1}(y))[u^q(y)+u^p(y)]\big|dy \\ &\le C\int_{\mathbb{R}^N\setminus B_{j+1/2}} |\nabla\mathcal{K}(x-y)||u^q(y)|dy + \\ &\quad + C\int_{\mathbb{R}^N\setminus B_{j+1/2}} |\nabla\mathcal{K}(x-y)||u^p(y)|dy.\end{aligned}$$

But $x \in B_{j+1}$, $|x-y| \ge r_{j+1/2} - r_{j+1} > 0$ over the integral. Using this, $u \in L^{q_0}(\mathbb{R}^N)$, Theorem 4.3.1, part III, and Hölder's inequality,

$$|\nabla(u-u_{j+1})(x)| \le C(N,\alpha,p,q)\|u\|_{H^\alpha},$$

for all $x \in B_{j+1}$. This implies $u \in C^{0,\mu}(B_{j+1})$, and the norm of u in this space depends only on N, α, p, q, $\|u\|_{H^\alpha}$ and the finite sequence $r_0, \ldots, r_{j+1}$.

If $p_{j+1} = N/(2\alpha)$, note that for $0 < \tilde{\alpha} < \alpha$ we have $u_{j+1} \in \mathcal{W}^{2\tilde{\alpha},p_{j+1}}$, and then $p_{j+1} < N/(2\tilde{\alpha})$. Iterating again and choosing $\tilde{\alpha}$ close enough to α, $p_{j+2} > N/(2\tilde{\alpha})$ and we can proceed as before.

Now, the ball B_{j+1} or B_{j+2} is centered at the origin, but it can be moved arbitrarily in $\mathbb{R}^N$. From here, $u \in C^{0,\mu}(\mathbb{R}^N)$. This fact and the integrability yield $u(x) \to 0$ as $|x| \to +\infty$, thus completing the proof. □

Proof of Theorem 4.1.1, part II. Let u_1 be the solution of (4.3.3). Proceeding as in the last Lemma and using that $g(s) = s^p - s^q$ is σ-Hölder continuous, we conclude $u_1 \in C^{0,\sigma_0}$ for some $\sigma_0 \in (0,\sigma)$. Therefore, if w is a solution of

$$-\Delta w = -u_1 + \eta_1 h \quad \text{in } \mathbb{R}^N,$$

then $w \in C^{2,\sigma_0}$. When $2\alpha + \sigma_0 > 1$, $(-\Delta)^{1-\alpha}w \in C^{1,2\alpha+\sigma_0-1}$, otherwise, $(-\Delta)^{1-\alpha}w \in C^{0,2\alpha+\sigma_0}$. But

$$(-\Delta)^\alpha\big(u_1 - (-\Delta)^{1-\alpha}w\big) = 0,$$

which allows us to deduce $u_1 - (-\Delta)^{1-\alpha}w$ is harmonic; hence, u_1 has the same regularity of $(-\Delta)^{1-\alpha}w$. Repeating again the procedure (as in [37]), the same regularity for u in $\mathbb{R}^N$ is obtained (depending on the sign of $2\alpha + \sigma_0 - 1$), and in either case the representation (4.1.4) of the fractional Laplacian holds.

Suppose now $u \not\equiv 0$ is a solution of (4.1.1). Knowing that $u \geq 0$, suppose $x_0 \in \mathbb{R}^N$ is a global minimum of u (therefore, $u(x_0) = 0$). Then $u(x_0)^p - u(x_0)^q = 0$, but $(-\Delta)^\alpha u(x_0) < 0$ (unless $u \equiv 0$), thus leading to a contradiction. □

4.4. The decay rate of a classical solution

This Section consists of two lemmas obtaining suitable subsolutions and supersolutions, and the proof of Theorem 4.1.2. This is done by means of appropriate changes to the case $q = 1$ treated in [37].

Lemma 4.4.1. *Let $\beta > 0$, and let $w \in C^2(\mathbb{R}^N)$ be a continuous, positive function such that, for $|x| > 1/3$,*

$$w(x) := \beta |x|^\eta, \quad \text{where } \eta < -\frac{N+2\alpha}{q}.$$

Then there exists R_0 such that

$$(-\Delta)^\alpha w(x) + w(x)^q \leq 0 \quad \text{if } |x| \geq R_0. \tag{4.4.1}$$

Proof. During all this proof we assume $|x| > 1/3$, and let $w(x) = f(|x|) = \beta|x|^\eta$. When analysing the integral defining $(-\Delta)^\alpha w(x)$ near $z = 0$ it is useful to consider the following representation of $\delta(w)$:

$$\delta(w)(x,z) = \left\langle z, \left(\int_0^1 [D^2 w(x+tz) + D^2 w(x-tz)](1-t)dt \right) z \right\rangle.$$

Note that $f''(r) = \eta(\eta-1)\beta r^{\eta-2}$; therefore,

$$\|D^2 w(x+tz) - D^2 w(x-tz)\| \leq C\big(|x+tz|^{\eta-2} + |x-tz|^{\eta-2}\big).$$

Now let $\delta > 0$ and $z \in B_\delta(0)$. It is easy to see that, for x large enough,

$$\left| \int_{B_\delta(0)} \frac{\delta(w)(x,z)}{|z|^{N+2\alpha}} dz \right| \leq C|x|^{\eta-2} \int_{B_\delta(0)} \frac{|z|^2}{|z|^{N+2\alpha}} dz \leq C|x|^{\eta-2}. \tag{4.4.2}$$

Now consider $D_1 := \{z : \delta(w)(x,z) \leq 0\}$ and $D_2 = D_1^c$. Then we have

$$\int_{D_1 \setminus B_\delta(0)} \frac{\delta(w)(x,z)}{|z|^{N+2\alpha}} dz \geq \int_{D_1 \setminus B_\delta(0)} \frac{-2f(|x|)}{|z|^{N+2\alpha}} dz. \tag{4.4.3}$$

On the other hand, $\{\frac{1}{2} \leq |x+z| \leq 1\} \subseteq D_2$ for $|x|$ large enough; hence, for a positive constant C we have

$$\int_{D_2} \frac{\delta(w)(x,z)}{|z|^{N+2\alpha}} dz \geq C|x|^{-N-2\alpha}. \tag{4.4.4}$$

Combining the estimates (4.4.2), (4.4.3) and (4.4.4), we obtain

$$(-\Delta)^\alpha w + w^q \leq C|x|^{\eta-2} + 2|x|^\eta \int_{D_1 \setminus B_\delta(0)} \frac{dz}{|z|^{N+2\alpha}} - C|x|^{-N-2\alpha} + \beta^q |x|^{\eta q},$$

which allows to conclude the desired inequality when $|x|$ is large enough. □

Lemma 4.4.2. *There is a positive function w such that, for large $|x|$,*

$$w(x) = C|x|^{\frac{-N-2\alpha}{q}}$$

and

$$(-\Delta)^\alpha w + \frac{1}{2} w^q \geq 0 \tag{4.4.5}$$

Proof. Let w be a smooth function, nonincreasing in $|x|$, such that $w \equiv 1$ in $B_1(0)$ and $w(x) = |x|^{\sigma/q}$ whenever $|x| \geq 2$, where we write $\sigma = -(N+2\alpha)$. We first want to prove that

$$\int_{B_{|x|/3}(0)} \frac{\delta(w)(x,z)}{|z|^{N+2\alpha}} dz \leq C_1 |x|^\sigma.$$

Note that if $z \in B_{|x|/3}(0)$, then, for $t \in (0,1)$, $|x \pm tz| \geq |x| - |z| \geq \frac{2}{3}|x|$; hence, for $|x|$ sufficiently large,

$$|D^2 w(x \pm tz)| \leq C|x \pm tz|^{\sigma/q-2} \leq C|x|^{\sigma/q-2},$$

and therefore

$$\int_{B_{|x|/3}(0)} \frac{\delta(w)(x,z)}{|z|^{N+2\alpha}} dz \leq C|x|^{\sigma/q-2} \int_{B_{|x|/3}(0)} \frac{|z|^2}{|z|^{N+2\alpha}} dz \leq C|x|^{\sigma/q-2\alpha},$$

thus proving the desired inequality. Now we shall find $C_2 > 0$ such that, considering $|x|$ sufficiently large,

$$\int_{B_{|x|/3}(x)} \frac{\delta(w)(x,z)}{|z|^{N+2\alpha}} dz \leq C_2 |x|^\sigma.$$

The estimate for the integral over $B_{|x|/3}(-x)$ is similar. We have

$$\int_{B_{|x|/3}(x)} \frac{w(x)}{|z|^{N+2\alpha}} dz \leq C \frac{|x|^{\sigma/q}}{|x|^{N+2\alpha}} \left(\frac{|x|}{3}\right)^N \leq C|x|^{\sigma/q-2\alpha}.$$

Now we estimate the integral for $w(x-z)$ for $|x|$ large enough. Let $A(x) := B_{|x|/3}(x) \setminus$

$B_2(x)$, and note that $B_{|x|/3}(x) = A(x) \cup B_2(x)$. Using that $w \le 1$ in $\mathbb{R}^N$,

$$\int_{B_2(x)} \frac{w(x-z)}{|z|^{N+2\alpha}} dz \le C|x|^{\sigma/q} \int_{B_2(x)} dz \le C|x|^{\sigma/q},$$

and

$$\int_{A(x)} \frac{w(x-z)}{|z|^{N+2\alpha}} dz = |x|^{\sigma-2\alpha} \int_{B_{1/3}(e_1)\setminus B_{2/|x|}(e_1)} \frac{|e_1-y|^\sigma}{|y|^{N+2\alpha}} dy \le C|x|^\sigma,$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^N$. Hence,

$$\int_{B_{|x|/3}(x)} \frac{w(x-z)}{|z|^{N+2\alpha}} dz \le C(|x|^\sigma + |x|^{\sigma/q}) \le C|x|^\sigma.$$

If $x \in C(x) := \mathbb{R}^N \setminus \big(B_{|x|/3}(0) \cup B_{|x|/3}(-x) \cup B_{|x|/3}(x)\big)$, then $|z+x|, |z-x|, |z| \ge |x|/3$, and therefore $w(x+z), w(x), w(x-z) \le |x|^{\sigma/q}$. Thus

$$\int_{C(x)} \frac{\delta(w)}{|z|^{N+2\alpha}} dz \le C|x|^{\sigma/q} \int_{\mathbb{R}^N\setminus B_{|x|/3}(0)} \frac{1}{|z|^{N+2\alpha}} dz \le C|x|^{\sigma/q-2\alpha}. \qquad (4.4.6)$$

Combining the results described above, we find $C_0 > 0$ such that, for $|x|$ large enough,

$$(-\Delta)^\alpha w + C_0 w^q \ge 0.$$

The proof is completed by rescaling. □

Proof of Theorem 4.1.2. Given $\theta > 0$ and $\beta = 1$, there exists R_0, given by Lemma 4.4.1, such that the function w from Lemma 4.4.1 satisfies (4.4.1). By continuity of u, there exists $C_1 > 0$ such that $W(x) := u(x) - C_1 w(x) \ge 0$ in $\bar{B}_{R_0}$. Moreover, $W(x) \to 0$ as $|x| \to +\infty$ and $(-\Delta)^\alpha W \ge -u^q + C_1 w^q$ in $B^c_{R_0}$. Supposing that $u^q - C_1 w^q \not\ge 0$ in $B^c_{R_0}$, we obtain the existence of a global negative minimum of W attained at $x_0 \in B^c_{R_0}$, but this cannot occur because it would imply $(-\Delta)^\alpha W(x_0) < 0$. This, in turn, implies that, eventually taking $C_1 > 0$ smaller, $W \ge 0$ in $B^c_{R_0}$. From here we conclude the second inequality.

Now we use again that $u(x) \to 0$ as $|x| \to +\infty$ to find that u satisfies

$$(-\Delta)^\alpha u + \frac{1}{2} u^q \le 0 \quad \text{in } B^c_{R_1}$$

for a certain R_1 large enough. Then we consider the supersolution w found in Lemma 4.4.2, which satisfies (4.4.5). Using similar comparison arguments we conclude the first inequality. □

4.5. Symmetry of positive solutions

This section is devoted to prove that positive, classical solutions of (4.1.1) are radially symmetric around some point of $\mathbb{R}^N$. This is done by means of proving symmetry around the y-axis for the α-harmonic extension of u (where y is the new variable introduced, as in Definition 4.1.3), which in turn implies the radial symmetry for u.

4.5.1. The result

Before proceeding we need to fix notation. For $\lambda \in \mathbb{R}$ and $w : \mathbb{R}^{N+1}_+ \to \mathbb{R}$, let

$$
\begin{aligned}
\Sigma^\lambda &:= \{(x, y) \in \mathbb{R}^{N+1}_+ : x_1 < \lambda\},\\
\frac{\partial w}{\partial y^\alpha}(x) &:= -\lim_{y\to 0} y^{1-2\alpha} \frac{\partial w}{\partial y}(x, y),\\
s^+ &:= \max\{s, 0\},\\
s^- &:= \max\{-s, 0\},
\end{aligned}
$$

so that $s = s^+ - s^-$. Also let $a > 0$ be such that $g(s) = s^p - s^q$ attains its minimum at a. Also recall the trace inequality in this context, e.g., as quoted by Brändle, Colorado and de Pablo [9]:

Theorem 4.5.1. *Let $z : \mathbb{R}^{N+1}_+ \to \mathbb{R}$ be such that*

$$
\int_{\mathbb{R}^{N+1}_+} y^{1-2\alpha} |\nabla z|^2 dxdy < +\infty,
$$

and let $v : \mathbb{R}^N \to \mathbb{R}$ be its trace. Then

$$
\left(\int_{\mathbb{R}^N} |v(x)|^{\frac{2N}{N-2\alpha}} dx\right)^{\frac{N-2\alpha}{N}} \leq S(\alpha, N) \int_{\mathbb{R}^{N+1}_+} y^{1-2\alpha} |\nabla z(x, y)|^2 dxdy
$$

The first result provides the integrability needed later on for the α-harmonic extension.

Lemma 4.5.1. *Let u be a classical solution of (4.1.1) and let $\mathrm{E}_\alpha(u)$ be its α-harmonic extension to $\mathbb{R}^{N+1}_+$; that is, $\mathrm{E}_\alpha(u)$ solves*

$$
\begin{aligned}
\nabla \cdot (y^{1-2\alpha} \nabla \mathrm{E}_\alpha(u)) &= 0 \quad \text{in } \mathbb{R}^{N+1}_+\\
\mathrm{E}_\alpha(u)(x, 0) &= u \quad \text{on } \mathbb{R}^N.
\end{aligned}
$$

Then $y^{(1-2\alpha)/2} \mathrm{E}_\alpha(u) \in L^2(\mathbb{R}^{N+1}_+)$.

Proof. From [17, §3] it is known that

$$\widehat{\mathrm{E}_\alpha(u)}(\xi, y) = \hat{u}(\xi)\phi(|\xi|y),$$

where the Fourier transform is taken on x, and ϕ solves

$$\begin{aligned} -\phi(y) + \frac{1-2\alpha}{y}\phi_y(y) + \phi_{yy}(y) &= 0, \\ \phi(0) &= 1, \\ \lim_{y\to+\infty} \phi(y) &= 0. \end{aligned}$$

In particular, ϕ minimises the functional

$$J(\phi) := \int_0^{+\infty} \big(|\phi|^2 + |\phi'|^2\big) y^{1-2\alpha} dy$$

among ϕ such that $\phi(0) = 1$. From here we have

$$\begin{aligned} \int_{\mathbb{R}^{N+1}_+} y^{1-2\alpha} |\mathrm{E}_\alpha(u)|^2 &= \int_{\mathbb{R}^N} \int_0^{+\infty} |\widehat{\mathrm{E}_\alpha(u)}|^2 y^{1-2\alpha} dy d\xi \\ &= \int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 \int_0^{+\infty} |\phi(|\xi|y)|^2 y^{1-2\alpha} dy d\xi \\ &\leq \int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 |\xi|^{2\alpha} J(\phi) d\xi \\ &\leq C \int_{\mathbb{R}^N} |\hat{u}|^2 |\xi|^{4\alpha} d\xi + C \int_{B_1} |\hat{u}|^2 d\xi \\ &\leq C \int_{\mathbb{R}^N} \big|\widehat{(-\Delta)^\alpha} u(\xi)\big|^2 d\xi + C \\ &\leq C \int_{\mathbb{R}^N} (u^p - u^q)^2 dx + C \end{aligned}$$

from where we conclude using the integrability properties derived from the decay result:

$$\begin{aligned} \int_{\mathbb{R}^N} (u^p - u^q)^2 dx &\leq C \int_{\mathbb{R}^N} u^{2p} + u^{2q} dx \\ &\leq C + C \int_1^{+\infty} r^{\frac{-N-2\alpha}{q}\cdot 2p + N - 1} + r^{\frac{-N-2\alpha}{q}\cdot 2q + N - 1} dr \\ &\leq C + C \int_1^{+\infty} r^{-N-4\alpha-1} + r^{-N-4\alpha-1} dr, \end{aligned}$$

which is finite. □

The following lemma is a maximum principle, based on the work by Terracini [61], needed to obtain the symmetry result:

Lemma 4.5.2. *Suppose w satisfies*

$$\begin{aligned} -\nabla \cdot (y^{1-2\alpha}\nabla w) &\geq 0 && \text{in } \Sigma^\lambda \cap \mathbb{R}^{N+1}_+ \\ \frac{\partial w}{\partial y^\alpha} &\geq b(x)w && \text{on } \Sigma^\lambda \cap \mathbb{R}^N \\ w &= 0 && \text{on } \{\lambda\} \times \mathbb{R}^{N-1} \times \mathbb{R}_+ \end{aligned} \tag{4.5.1}$$

and assume $y^{1-2\alpha}w^\delta \in L^1(\mathbb{R}^{N+1}_+)$ for some $\delta > 1$, $y^{(1-2\alpha)/2}\nabla w \in L^2_{loc}(\mathbb{R}^{N+1}_+)$. Then there is a constant $C = C(N,\delta)$ such that, writing $\Omega^- := \{(x,y) \in \Sigma^\lambda : w(x,y) < 0\}$, we have

$$\text{either } |\Omega^-| = 0, \text{ or } C\|b^+\|_{L^{N/\alpha}(\Omega^- \cap \mathbb{R}^N)} \geq 1$$

Proof. Let $s := (\delta - 2)/2$, $K > 1$ and $\varepsilon > 0$. Fix η such that $\eta(x,y) = 1$ for $|(x,y)| \leq K$, $\eta(x,y) = 0$ if $|(x,y)| \geq 2K$, and $|\nabla\eta| \leq 1$. Write $v := \min\{(w^-)^s, K\}$, and define

$$\varphi := -\eta^2 v^2 w^-, \quad \psi := -\eta v w^-.$$

Then we have

$$\begin{aligned} \int_{\Omega^-} y^{1-2\alpha}|\nabla\psi|^2 &= \int_{\Omega^-} y^{1-2\alpha}\eta^2|\nabla(vw^-)|^2 + 2y^{1-2\alpha}\eta v w^- \left\langle \nabla\eta, \nabla(vw^-)\right\rangle + \\ &\quad + |\nabla\eta|^2(vw^-)^2 \\ &\leq \int_{\Omega^-} (1+\varepsilon)y^{1-2\alpha}\eta^2|\nabla(vw^-)|^2 + C_\varepsilon y^{1-2\alpha}|\nabla\eta|^2(vw^-)^2 \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega^-} y^{1-2\alpha}\left\langle -\nabla w^-, \nabla\varphi\right\rangle &= \int_{\Omega^-} y^{1-2\alpha}\left\langle \nabla w^-, 2\eta(v^2w^-)\nabla\eta + \eta^2\nabla(v^2w^-)\right\rangle \\ &\geq \int_{\Omega^-} y^{1-2\alpha}\eta^2\left\langle \nabla w^-, \nabla(v^2w^-)\right\rangle \\ &\quad - \varepsilon y^{1-2\alpha}\eta^2 v^2|\nabla w^-|^2 - y^{1-2\alpha}C_\varepsilon|\nabla\eta|^2(vw^-)^2 \\ &\geq \int_{\Omega^-} (1-\varepsilon)y^{1-2\alpha}\eta^2\left\langle \nabla w^-, \nabla(v^2w^-)\right\rangle - C_\varepsilon y^{1-2\alpha}|\nabla\eta|^2(vw^-)^2. \end{aligned}$$

Since

$$2\left\langle -\nabla w^-, -\nabla(v^2w^-)\right\rangle \geq \frac{1+2s}{(1+s)^2}|\nabla(vw^-)|^2,$$

we obtain

$$\frac{1-\varepsilon}{1+\varepsilon}\cdot\frac{1+2s}{2(1+s)^2}\int_{\Omega^-} y^{1-2\alpha}|\nabla\psi|^2 \leq \int_{\Omega^-} y^{1-2\alpha}\left\langle -\nabla w^-, \nabla\varphi\right\rangle + C_\varepsilon y^{1-2\alpha}|\nabla\eta|^2(vw^-)^2.$$

On the other hand, testing (4.5.1) with φ (which is negative) and using Hölder's inequality,

$$\begin{aligned}\int_{\Omega^-} y^{1-2\alpha}\left\langle -\nabla w^-, \nabla\varphi\right\rangle &\leq \int_{\Omega^-\cap\mathbb{R}^N} b\psi^2 \\ &\leq \|b^+\|_{L^{N/\alpha}(\Omega^-\cap\mathbb{R}^N)}\|\psi\|^2_{L^{2N/(N-2\alpha)}(\Omega^-\cap\mathbb{R}^N)}.\end{aligned}$$

From here we can use the trace inequality in Theorem 4.5.1, yielding

$$\begin{aligned}&\frac{1-\varepsilon}{1+\varepsilon}\cdot\frac{1+2s}{2(1+s)^2}\int_{\Omega^-} y^{1-2\alpha}|\nabla\psi|^2 \leq \\ &\qquad\leq S_2\|b^+\|_{L^{N/\alpha}(\Omega^-\cap\mathbb{R}^N)}\int_{\Omega^-} y^{1-2\alpha}|\nabla\psi|^2 + \int_{\Omega^-} C_\varepsilon y^{1-2\alpha}|\nabla\eta|^2(vw^-)^2.\end{aligned}$$

Assuming $|\Omega^-| \neq 0$, and dividing the above inequality by $\int_{\Omega^-} y^{1-2\alpha}|\nabla\psi|^2$, the desired inequality is obtained taking $K \to \infty$. Indeed, since $2+2s = \delta$ and $y^{1-2\alpha}(w^-)^\delta$ is integrable, we have

$$\int_{\Omega^-} C_\varepsilon y^{1-2\alpha}|\nabla\eta|^2(vw^-)^2 \leq \int_{\Omega^-\cap\{|(x,y)|\geq K\}} C'_\varepsilon y^{1-2\alpha}(w^-)^{2+2s} \to 0$$

as $K \to \infty$. □

We now quote Theorem 1.1 and Corollary 1.3 of [61]. The notation from the original has been slightly altered to match ours. Recall also $\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0,+\infty)$.

Theorem 4.5.2. *Let $\varphi \in L^\delta(\mathbb{R}^{N+1}_+)$ be a positive solution of*

$$\begin{cases} -\Delta\varphi = f(x,y,\varphi) & \text{in } \mathbb{R}^{N+1}_+, \\ -\dfrac{\partial\varphi}{\partial y} = g(x,\varphi) & \text{in } \mathbb{R}^N, \end{cases}$$

where $x \in \mathbb{R}^N$, $y > 0$, and assume the following hypotheses:

- *f and g are nondecreasing in the x_1 direction for $x_1 < 0$.*
- *There are μ_1, $\mu_2 > 0$, $\sigma_1 \in L^{(N+1)/2}(\mathbb{R}^{N+1}_+)$, $\sigma_2 \in L^N(\mathbb{R}^N)$, ρ_1 and ρ_2 such that, for*

all $t > s > 0$,

$$\frac{f(x,y,s)-f(x,y,t)}{s-t} \le \sigma_1(x,y) + \rho_1(x,y)t^{\mu_1},$$
$$\frac{g(x,s)-g(x,t)}{s-t} \le \sigma_2(x) + \rho_1(x)t^{\mu_1}.$$

- $\rho_1\varphi^{\mu_1} \in L^{(N+1)/2}(\mathbb{R}^{N+1}_+)$ *and* $\rho_2\varphi(x,0)^{\mu_2} \in L^N(\mathbb{R}^N)$.

Let (x_λ, y) be the reflection of (x,y) with respect to the hyperplane $\{(x,y) \in \mathbb{R}^{N+1}_+ : x_1 = \lambda\}$ and $\varphi^\lambda(x,y) = \varphi(x_\lambda, y)$. Then either $\varphi^\lambda > u$ on Σ^λ for every $\lambda < 0$, or there exists $\lambda^* < 0$ such that $\varphi^{\lambda^*} = \varphi$ on Σ^{λ^*}.

Corollary 4.5.1. *Under the assumptions of Theorem 4.5.2, let f and g be independent of (x,y). Then there exists $x \in \mathbb{R}^N$ such that $\varphi\big((x,y)-(x,0)\big)$ is symmetric around the y-axis.*

Proof of Theorem 4.1.3. The proof of Theorem 1.1 and Corollary 1.3 in [61] can be directly followed using Lemma 4.5.2 (setting $\delta = 2$) if we can verify that the hypotheses of Theorem 4.5.2 apply to $\mathrm{E}_\alpha(u)$, the α-harmonic extension of u, which satisfies

$$\nabla \cdot (y^{1-2\alpha}\nabla \mathrm{E}_\alpha(u)) = 0 \quad \text{in } \mathbb{R}^{N+1}_+$$
$$\frac{\partial \mathrm{E}_\alpha(u)}{\partial y^\alpha}(x,0) = u^p - u^q \quad \text{on } \mathbb{R}^N.$$

Indeed, the hypotheses on f are obvious, because $f \equiv 0$. For the hypotheses on g, recall that $g(s) = s^p - s^q$ and a is its minimum, take $\mu_2 = \max\{q, p-1\}$ and use the fact that

$$\frac{g(s)-g(t)}{s-t} \le 0 \quad \text{if } a \ge t > s > 0$$

(because g is decreasing between 0 and a), and $s^{p-1} \le \rho_2 s^{\mu_2}$ for $s \ge a$ and $\rho_2 > 0$ a constant large enough. Moreover, from the decay result it can be deduced that

$$\rho_2 u^{\mu_2} \in L^{N/\alpha}(\mathbb{R}^N).$$

Indeed,

$$\rho_2 \int_{\mathbb{R}^N} u^{N\mu_2/\alpha}dx \le \int_{B_{R_0}(0)} u^{N\mu_2/\alpha}dx + C\int_{R_0}^{+\infty} r^{-\frac{N+2\alpha}{q}\cdot\frac{\mu_2}{\alpha}\cdot N} r^{N-1}dr$$

and, using the definition of μ_2,

$$-\frac{N+2\alpha}{q}\cdot\frac{\mu_2}{\alpha}\cdot N+N-1<-1.$$

Also recall that, since we are not interested in the behaviour of the derivatives of $\mathrm{E}_\alpha(u)$, the hypothesis $\mathrm{E}_\alpha(u)\in C^1(\mathbb{R}^{N+1}_+)$ (which appears in [61]) is not necessary in this case. $\square$

4.5.2. A remark on the moving planes method

As we mentioned before, recent moving planes developments for integral equations rely heavily in the Lipschitz continuity of the nonlinearity. The question is whether the moving planes method can be directly applied to the integral operator developed in this Chapter.

Let $\lambda\in\mathbb{R}$, and for $x=(x_1,\dots,x_N)\in\mathbb{R}^N$, write

$$x^\lambda:=(2\lambda-x_1,x_2,\dots,x_N).$$

Let $u^\lambda(x)=u(x^\lambda)$, define $w_\lambda:=u^\lambda-u$ and

$$\Sigma_\lambda:=\{x\in\mathbb{R}^N: x_1<\lambda\}.$$

Note that

$$(w_\lambda)^\lambda(x)=u^\lambda(x^\lambda)-u(x^\lambda)=u(x)-u^\lambda(x)=-w_\lambda(x).$$

With this equality (basically an antisymmetry result), using $s^p-s^q<0$ for $s\in(0,a)$, and using $u(x)\to 0$ as $|x|\to+\infty$, it can be deduced that $w_\lambda\geq 0$ for $\lambda<0$, with $|\lambda|$ sufficiently large. Moreover, in this case either $w_\lambda\equiv 0$ or $w_\lambda>0$. It is also known that

$$-\infty<\sup\{\lambda: w_\lambda\geq 0 \text{ in } \Sigma_\lambda\}=\lambda^*<+\infty.$$

The method allows to conclude whenever $w_\lambda\equiv 0$ can be proven. This is usually performed by means of a maximum principle. For instance, in second order elliptic PDEs in bounded domains, the maximum principle for small domains is a consequence of the Alexandrov-Bakelman-Pucci inequality. Assume Ω is a bounded domain in $\mathbb{R}^N$, and let u be a solution of

$$\begin{aligned} Lu&=f(u) && \text{in } \Omega,\\ u&=0 && \text{on } \partial\Omega, \end{aligned}$$

where $f(u) \geq 0$ is Lipschitz continuous and is below the critical exponent. If $\lambda^* + \delta > \lambda > \lambda^*$, for $\delta > 0$ small enough, then the set

$$\{x \in \Sigma_\lambda \cap \Omega : x < 0\}$$

has small measure, which contradicts the maximum principle for small domains (because $u = 0$ in $\partial(\Sigma_\lambda \cap \Omega)$). The key for this step is the fact that one can focus in $\Sigma_\lambda \cap \Omega$, not considering the behaviour of u outside this set. To the best of our knowledge, that cannot be done for our integral operator, as illustrated by the ABP inequality [40].

Chapter 5

Conclusions

5.1. An eigenvalue problem

A major difference between the case $\alpha = 0$ and $\alpha \neq 0$ is the difficulty to obtain appropriate comparison principles and existence solutions for the initial value problem. For instance, unlike [33], our comparison principles cannot be directly extended to more general situations (e.g., a right hand side without sign restriction). Also in the case $\alpha = 0$ several results relied on the ABP inequality, while in our case only one result did use it, which occurs because the operator is not sublinear in general. This, in turn, is an explanation to the variety of methods used to prove our comparison principles.

The eigenvalue theory developed in Chapter 3 may be used to build on the existence of positive (or negative) solutions of the equation

$$F(D^2u, Du, u, x) = -\lambda\varphi(u) + f(x, u) \quad \text{in } B_R, \tag{5.1.1}$$

$$u = 0 \quad \text{on } \partial B_R, \tag{5.1.2}$$

by means of bifurcation theory, using the ideas of Rabinowitz [55, 56, 57]. In [10] the authors develop a bifurcation theory for the Pucci operator and in [1] this theory was extended to a larger class of nonlinear extremal operators. In [36] results in another direction are obtained for general fully nonlinear uniformly elliptic operators. All these theories may be extended to include degenerate or singular operators as the ones treated in this thesis. We have not pursued in this direction in this thesis, but it could be considered as an interesting problem for future research.

A relevant open problem, as we have discussed in the introduction, is the general eigenvalue problem

$$\begin{aligned} F(D^2u, Du, u, x) &= -\lambda\varphi(u) \quad \text{in } \Omega \\ u &= 0 \qquad\qquad \text{on } \partial\Omega \end{aligned}$$

for which only the first eigenvalue has been found. A particular case of this is to find all the eigenvalues of the p-Laplacian

$$\begin{aligned} -\nabla \cdot (\varphi(\nabla u)) &= \lambda\varphi(u) \quad \text{in } \Omega, \\ u &= 0 \qquad\quad \text{on } \partial\Omega. \end{aligned}$$

For this problem it was already known that there are infinitely many eigenvalues associated to radial eigenfunctions (when Ω is a ball), but the existence of more eigenvalues is not known in general. We have concluded in this thesis the same situation holds for a more general class of singular or degenerate operators.

5.2. An equation involving the fractional Laplacian

Even though it was known beforehand that solutions for the problem analysed in Chapter 4 could not have compact support, it is interesting to note that the difference in the decay rate between the case $0 < q < 1$ and $q = 1$ is not as significant as when $\alpha = 1$. A consequence of this is that proving symmetry for solutions required a method completely different from previous work for similar equations (e.g. [23, 37]). A natural question is the behaviour of a sequence of solutions as $\alpha \to 1$, but we have not conducted research about this.

Two problems are, in our opinion, interesting for future work. The first is to extend this results to more general non Lipschitzian nonlinearities and more general operators. It is essential, though, to have the nonlinearity to be negative close to 0; otherwise, we fall in a case similar to [9], where considerably different issues have to be addressed. It seems to be possible to extend the results to a class of operators with similar kernels, at least for existence of weak and classical solutions.

The second problem is related to the proof of the symmetry result. A very important question is whether the moving planes method can be directly applied. If it were possible, symmetry results could be obtained for a larger class of integral operators in a relatively straightforward manner, as long as a strong maximum principle type result can be obtained. If it were not possible, our technique could only be applied as long as there exists an appropriate extension to the half space.

Bibliography

[1] Alejandro Allendes and Alexander Quaas. Multiplicity results for extremal operators through bifurcation. *Discrete Continuous Dynamical Systems*, 29(1):51–65, 2011.

[2] Antonio Ambrosetti and Paul H. Rabinowitz. Dual variational methods in critical point theory and applications. *Journal of Functional Analysis*, 14:349–381, 1973.

[3] Scott N. Armstrong. Principal eigenvalues and an anti-maximum principle for homogeneous fully nonlinear elliptic equations. *Journal of Differential Equations*, 246:2958–2987, 2009.

[4] Isabeau Birindelli and Françoise Demengel. Comparison principle and Liouville type results for singular fully nonlinear operators. *Annales de la faculté des sciences de Toulouse*, 13:261–287, 2004.

[5] Isabeau Birindelli and Françoise Demengel. Eigenvalue and maximum principle for fully nonlinear singular operators. *Advances in Differential Equations*, 11(1):91–119, 2006.

[6] Isabeau Birindelli and Françoise Demengel. Eigenvalue and Dirichlet problem for fully nonlinear operators in non smooth domains. *Journal of Mathematical Analysis and its applications*, 352(2):822–835, 2009.

[7] Isabeau Birindelli and Françoise Demengel. Regularity and uniqueness of the first eigenfunction for singular fully nonlinear operators. *Journal of Differential Equations*, 249:1089–1110, 2010.

[8] Isabeau Birindelli and Françoise Demengel. Uniqueness of the first eigenfunction for fully nonlinear equations: the radial case. *Journal for Analysis and its applications*, 29:77–90, 2010.

[9] Cristina Brändle, Eduardo Colorado, and Arturo de Pablo. A concave-convex elliptic problem involving the fractional Laplacian. Preprint.

[10] Jerome Busca, Maria J. Esteban, and Alexander Quaas. Nonlinear eigenvalues and bifurcation problems for Pucci's operator. *Annales de l'Institut H. Poincaré*, 22:187–206, 2005.

[11] Xavier Cabré and Jean-Michel Roquejoffre. Front propagation in Fisher-KPP equations with fractional diffusion. *Comptes Rendus Mathematique*, 347:1361–1366, 2009.

[12] Xavier Cabré and Yannick Sire. Nonlinear equations for fractional Laplacians I: regularity, maximum principles and Hamiltonian estimates. Preprint, 2010.

[13] Xavier Cabré and Juan Solà-Morales. Layer solutions in a half-space for boundary reactions. *Communications on Pure and Applied Mathematics*, 58:1678–1732, 2005.

[14] Xavier Cabré and Jinggang Tan. Positive solutions of nonlinear problems involving the square root of the Laplacian. *Advances in Mathematics*, 224(5):2052–2093, 2010.

[15] Luis Caffarelli, Michael Crandall, Maciej Kocan, and Andrzej Świech. On viscosity solutions of fully nonlinear equations with measurable ingredients. *Communications on Pure and Applied Mathematics*, 49(4):365–398, 1996.

[16] Luis Caffarelli, Jean-Michel Roquejoffre, and Yannick Sire. Variational problems with free boundaries for the fractional Laplacian. *Journal of the European Mathematical Society*, 12(5):1151–1179, 2010.

[17] Luis Caffarelli and Luis Silvestre. An extension problem related to the fractional Laplacian. *Communications in Partial Differential Equations*, 32:1245–1260, 2007.

[18] Luis Caffarelli and Luis Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Communications on Pure and Applied Mathematics*, 62:597–638, 2009.

[19] Antonio Capella, Juan Dávila, Louis Dupaigne, and Yannick Sire. Regularity of radial extremal solutions for some non local semilinear equations. Preprint, 2011.

[20] Wenxiong Chen, Congming Li, and Biao Ou. Classification of solutions for an integral equations. *Communications on Pure and Applied Mathematics*, 5:597–638, 2009.

[21] Yung-Gang Chen, Yoshikazu Giga, and Shun'ichi Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *Journal of Differential Geometry*, 33:749–786, 1991.

[22] Rama Cont and Peter Tankov. *Financial modelling with jump processes.* Chapman & Hall/CRC, 2004.

[23] Carmen Cortázar, Manuel Elgueta, and Patricio Felmer. On a semilinear elliptic problem in $\mathbb{R}^N$ with a non-Lipschitzian nonlinearity. *Advances in Differential Equations*, 1(2):199–218, March 1996.

[24] Carmen Cortázar, Manuel Elgueta, and Patricio Felmer. Symmetry in an elliptic problem and the blow-up set of a quasilinear heat equation. *Communications in Partial Differential Equations*, 21(3&4):507–520, 1996.

[25] Vitorio Coti-Zelati and Paul H. Rabinowitz. Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^n$. *Communications on Pure and Applied Mathematics*, XLV:1217–1269, 1992.

[26] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, 27:1–67, 1992.

[27] Gonzalo Dávila, Patricio Felmer, and Alexander Quaas. Alexandroff-Bakelman-Pucci estimate for singular or degenerate fully nonlinear elliptic equations. *Comptes Rendus Mathematique*, 347:1165–1168, 2009.

[28] Gonzalo Dávila, Patricio Felmer, and Alexander Quaas. Harnack inequality for singular fully nonlinear operators and some existence results. *Calculus of Variations and PDE*, 39:557–578, 2010.

[29] Manuel del Pino and Raúl Manásevich. Global bifurcation from the eigenvalues of the p-Laplacian. *Journal of Differential Equations*, 92:219–245, 1991.

[30] Françoise Demengel. Generalized eigenvalues for fully nonlinear singular or degenerate operators in the radial case. *Advances in Differential Equations*, 14:1127–1154, 2009.

[31] Pavel Drábek and Jaroslav Milota. *Methods of Nonlinear Analysis: Applications to differential equations*. Birkhäuser, 2007.

[32] Georges Duvaut and Jacques-Louis Lions. *Inequalities in mechanics and physics*. Springer-Verlag, Berlin, 1976.

[33] Maria J. Esteban, Patricio Felmer, and Alexander Quaas. Eigenvalues for radially symmetric fully nonlinear operators. *Communications in Partial Differential Equations*, 35(9):1716–1737, 2010.

[34] Patricio Felmer and Alexander Quaas. On the strong maximum principle for quasilinear elliptic equations and systems. *Advances in Differential Equations*, 7(1):25–46, 2002.

[35] Patricio Felmer and Alexander Quaas. Positive radial solutions to a "semilinear" equation involving the Pucci's operator. *Journal of Differential Equations*, 199(2):376–393, 2004.

[36] Patricio Felmer, Alexander Quaas, and Boyan Sirakov. Landesman-Lazer type results for second order Hamilton-Jacobi-Bellman equations. *Journal of Functional Analysis*, 258(258):4154–4182, 2010.

[37] Patricio Felmer, Alexander Quaas, and Jinggang Tan. Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian. Preprint.

[38] Basilis Gidas, Wei-Ming Ni, and Louis Nirenberg. Symmetry and related properties via the maximum principle. *Communications in Mathematical Physics*, 68(3):209–243, 1979.

[39] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer, reprint of the 1998 edition, 2001.

[40] Nestor Guillen and Russell W. Schwab. Aleksandrov-Bakelman-Pucci type estimates for integro-differential equations. Preprint arXiv:1101.0279v3 [math.AP].

[41] Cyril Imbert. Alexandroff-Bakelman-Pucci estimate and Harnack inequality for degenerate fully non-linear elliptic equations. *Journal of Differential Equations*, 250(3):1325–1766, 2011.

[42] Hitoshi Ishii. Viscosity solutions of nonlinear partial differential equations. *Sugaku Expositions*, 9(2):135–152, 1996.

[43] Hitoshi Ishii and Yasuhiro Yoshimura. Demi-eigenvalues for uniformly elliptic Isaacs operators. Preprint.

[44] Petri Juutinen. On the principal eigenvalue of a very badly degenerate equation. *Journal of Differential Equations*, 236:532–550, 2007.

[45] Naum S. Landkof. *Foundations of modern potential theory.* Springer-Verlag, New York, 1972.

[46] Hong-Xu Li. Initial value problems of $p-$Laplacian with a strong singular indefinite weight. *Publicationes Mathematicae Debrecen*, 77:1–11, 2010.

[47] Yanyan Li. Remark on some conformally invariant integral equations: the method of moving spheres. *Journal of the European Mathematical Society*, 6:153–180, 2004.

[48] Pierre-Louis Lions. Bifurcation and optimal stochastic control. *Nonlinear Analysis*, 7(2):177–207, 1983.

[49] Pierre-Louis Lions. Two remarks on Monge-Ampère equations. *Annali di Matematica Pura ed Applicata*, 142(4):263–275, 1985.

[50] Jean Mawhin and Michel Willem. *Critical point theory and Hamiltonian systems.* Springer, Berlin, 1989.

[51] Taisa Junges Miotto. The Aleksandrov-Bakelman-Pucci estimates for singular fully nonlinear operators. *Communications in Contemporary Mathematics*, 12(4):607–627, 2010.

[52] Zeev Nehari. Characteristic values associated with a class of nonlinear second order differential equations. *Acta Math.*, 105:141–175, 1961.

[53] Stefania Patrizi. The Neumann problem for singular fully nonlinear operators. *Journal de Mathématiques Pures et Appliqués*, 90:286–311, 2008.

[54] Alexander Quaas and Boyan Sirakov. Principal eigenvalues and the Dirichlet problem for fully nonlinear elliptic operators. *Advances in Mathematics*, 218:105–135, 2008.

[55] Paul H. Rabinowitz. Some global results for nonlinear eigenvalue problems. *Journal of Functional Analysis*, 7:487–513, 1971.

[56] Paul H. Rabinowitz. Some aspect of nonlinear eigenvalue problem. *Rocky Mountain Journal of Mathematics*, 74(3):487–513, 1973.

[57] Paul H. Rabinowitz. *Théorie du degré topologique et applications à des problèmes aux limites non linéaires.* Lectures Notes Lab. Analyse Numérique, Université Paris VI, 1975.

[58] Luis Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Communications on Pure and Applied Mathematics*, 60:67–112, 2007.

[59] Yannick Sire and Enrico Valdinoci. Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result. *Journal of Functional Analysis*, 2009(6):1842–1864, 256.

[60] Elias M. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, 1970.

[61] Susanna Terracini. Symmetry properties of positive solutions to some elliptic equations with nonlinear boundary conditions. *Differential Integral Equations*, 8(8):1911–1922, 1995.

[62] Enrico Valdinoci. From the long jump random walk to the fractional Laplacian. *Boletín de la Sociedad Española de Matemática Aplicada*, 49:33–44, 2009.