UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

ENTIRE SOLUTIONS TO THE INHOMOGENEOUS ALLEN-CAHN EQUATION IN $\mathbb{R}^2$, WITH A TRANSITION ON A NONCOMPACT CURVE.

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

ANDRÉS JAHIR ZÚÑIGA MUNIZAGA.

PROFESOR GUÍA:
MANUEL DEL PINO MANRESA.

MIEMBROS DE LA COMISIÓN:
JUAN DÁVILA BONCZOS.
MICHAL KOWALCZYK.

SANTIAGO DE CHILE
JULIO 2012

ABSTRACT OF THE REPORT
TO QUALIFY TO THE
MATHEMATICAL ENGINEERING DEGREE.
BY: ANDRÉS ZÚÑIGA MUNIZAGA.
DATE: JULY 17th, 2012.
ADVISOR: MR. MANUEL DEL PINO.

ENTIRE SOLUTIONS TO THE INHOMOGENEOUS ALLEN-CAHN EQUATION IN $\mathbb{R}^2$, WITH A TRANSITION ON A NONCOMPACT CURVE

This engineering thesis work presents a study of the singularly perturbed Allen-Cahn equation with inhomogeneity

$$\varepsilon^2 \operatorname{div}\left(a(x) \cdot \nabla_x u(x)\right) + a(x) f(u(x)) = 0, \quad \text{in} \quad \mathbb{R}^2 \tag{0.1}$$

where $\varepsilon > 0$ is a small parameter, $a(x)$ is a uniformly positive smooth potential, that induces a way of measuring distances between points in $\mathbb{R}^2$, and f is the nonlinearity given by $f(u) = u - u^3$. This study deals with the construction of entire solutions of (0.1), under the condition that u vanishes near some curve $\Gamma \subset \mathbb{R}^2$. The proposed approach assumes that Γ is an unbounded curve, nondegenerate geodesic relative to the weighted arclength $\int_\Gamma a(\vec{x})$, with smooth curvature k_Γ which decays at a polynomial rate.

It is of interest the study of the Allen-Cahn equation in the presence of inhomogeneity term $a(x) \not\equiv 1$, since this entails the study of geodesics for a nontrivial metric in $\mathbb{R}^2$. Besides, is relevant to consider the case where the nodal set of u takes place near an unbounded curve Γ, because it leads the study of a differential equation in non-compact contexts. The main result of this work assures the existence of a solution $u(x)$ of (0.1), which converges exponentially to the constant ± 1 when x departs from Γ. A second result shows examples of a potential $a(x)$ and a curve Γ, for which it is possible to build a solution u with the behavior previously stated.

The proof of this result is based on a technique known as the Lyapunov-Schmidt reduction method, which motivates the choice of a candidate for a solution as $u = w + \phi$, where w in some suitable coordinates is a solution of $w'' + f(w) = 0$, determining the profile of u at main order. Additionally ϕ is a correction function, in order to make u an exact solution of (0.1), forcing ϕ to solve a nonlinear differential equation. From then on, the problem consists in studying the existence and uniqueness of the latter equation on a suitable functional space. This was done, first by analyzing the linearized operator of the Allen-Cahn equation, and then by solving the nonlinear problem using a fixed point scheme. For the solvability, it becomes necessary the adjustment of Γ in a small perturbation h, which amounts to a nonlinear ODE in h involving the second variation of the length $l_{a,\Gamma}[h]$ related to $\int_\Gamma a(\vec{x})$.

Finally, the method employed not only proves the existence of a solution u of (0.1), but also provides a complete characterization for this solution in size, and in the behavior in coordinates associated to the curve Γ.

RESUMEN DE LA MEMORIA
PARA OPTAR AL TÍTULO DE
INGENIERO CIVIL MATEMÁTICO.
POR: ANDRÉS ZÚÑIGA MUNIZAGA.
FECHA: 17 DE JULIO DE 2012.
PROF. GUÍA: SR. MANUEL DEL PINO.

SOLUCIONES ENTERAS DE LA ECUACIÓN INHOMOGÉNEA DE ALLEN-CAHN EN $\mathbb{R}^2$, CON TRANSICIÓN SOBRE UNA CURVA NO COMPACTA

Este trabajo de memoria de título presenta un estudio de la ecuación de perturbación singular de Allen-Cahn con inhomogeneidad:

$$\varepsilon^2 \operatorname{div}(a(x) \cdot \nabla_x u(x)) + a(x) f(u(x)) = 0, \quad \text{en} \quad \mathbb{R}^2 \tag{0.1}$$

donde $\varepsilon > 0$ es un parámetro pequeño, $a(x)$ es un potencial uniformemente positivo y suave, que induce una forma de medir distancias para puntos en $\mathbb{R}^2$, y f es la nolinealidad dada por $f(u) = u - u^3$. El estudio aborda la construcción de soluciones enteras de (0.1), bajo la condición que u se anule cerca de una curva $\Gamma \subset \mathbb{R}^2$. El enfoque propuesto asume que Γ es una curva no acotada, geodésica no-degenerada relativa al funcional de longitud de arco $\int_\Gamma a(\vec{x})$, con curvatura k_Γ suave que decae a una tasa polinomial.

Es de interés el estudio de la ecuación de Allen-Cahn con presencia de un término de inhomogeneidad $a(x) \not\equiv 1$, ya que esto conlleva el estudio de curvas geodésicas para una métrica no trivial de $\mathbb{R}^2$. Además, es relevante considerar que el conjunto nodal de u yace cerca de una curva *no acotada*, pues esto se refleja en el estudio de ecuaciones diferenciales en contextos no compactos. El resultado principal asegura la existencia de una solución de (0.1), la cual converge exponencialmente a ± 1 cuando x se aleja de Γ. Un segundo resultado entrega ejemplos de potenciales $a(x)$ y curvas Γ, para los cuales es posible construir una solución u con el comportamiento antes descrito.

La demostración de este resultado está basada en una técnica conocida como reducción infinito dimensional de Lyapunov-Schmidt, la cual motiva a la elección de un candidato a solución del tipo $u = w + \phi$, donde w en coordenadas adecuadas resuelve $w'' + f(w) = 0$, y determina el perfil de u a orden principal. Además ϕ es una función de corrección, con el fin de convertir a u en solución exacta de (0.1), lo que obliga a ϕ a resolver una ecuación diferencial no lineal. De ahí en más, el problema consiste en estudiar la existencia y unicidad de la última ecuación en un espacio funcional adecuado. Esto se realizó analizando el operador linealizado asociado a la ecuación de Allen-Cahn, y luego el problema no-lineal que es resuelto mediante un esquema de punto fijo. Para el ultimo análisis, fue necesario ajustar Γ en un parámetro de perturbación h, lo que equivale a una EDO no lineal en h donde participa la segunda variación del funcional de largo $l_{a,\Gamma}$ asociado a $\int_\Gamma a(\vec{x})$.

Finalmente, el método utilizado no sólo provee la existencia de una solución u de (0.1), sino que además entrega una caracterizacón completa de ésta, tanto en tamaño como en comportamiento cualitativo en coordenadas relacionadas a la curva Γ.

Dedicado a toda mi familia,
especialmente a mi Tía Cynthia y a mi Tío Gastón.

AGRADECIMIENTOS

En el ámbito personal quisiera agradecer profundamente a mi madre Marjorie, a mi padre Jorge, a mi hermana Natalia por el gran apoyo que me han brindado desde el primer día que entré en esta Facultad, preocupándose hasta del más mínimo detalle necesario para hacer de mi vida sólo un desafío académico. Ellos han sido la base de todos mis logros. Del mismo modo, agradezco a toda mi familia por su constante apoyo y compañía, y porque sé que esto es también importante para ellos.

Igual de importante ha sido la Vale, mi polola, quien ha estado junto a mí durante toda la carrera dándome la seguridad y compañía necesaria para sentirme una persona completa. Gracias por estar a mi lado en los momentos buenos como en los más difíciles, has sido uno de los mejores regalos que esta vida me ha entregado.

En segundo lugar quisiera agradecer afectuosamente a los profesores del DIM quienes han sido una de mis motivaciones más grandes para seguir esta carrera. En particular quisiera nombrar a los académicos Rafael Correa, Servet Martínez y Martín Matamala, los cuales con su entusiasmo y rigurosidad me guiaron en el estudio de la Matemática en el Plan Común. Asimismo, me gustaría destacar a los profesores Carlos Conca, Felipe Álvarez, y Fethi Mahmoudi, quienes han sido un pilar fundamental en mi formación en Analisis, Ecuaciones Diferenciales y Geometría, a los cuales les debo mi pasión por estas áreas de la Matemática.

Muy especialmente quiero dar las gracias a los profesores Manuel del Pino y Juan Dávila, quienes son los Matemáticos que más admiro en lo personal y profesional, a quienes agradezco por todo los consejos y el apoyo brindado, de los cuales he aprendido muchísimo durante la carrera y han sido mis mentores en el área de EDP.

Un gran agredecimiento muy especial merece Oscar Agudelo, por su paciencia y gran disposición, por las horas de estudio invertidas, por enseñarme su manera de enfrentar las adversidades, y por toda la ayuda brindada en mi primera investigación en EDP. Ha sido un verdadero privilegio haber trabajado contigo.

En otro ámbito, quiero agradecer a mis compañeros de la carrera, con quienes compartí la oficina 437. Una mención especial merecen: Mauro “Bambino” Escobar, F.Unda, B.Obando, Benja Palacios, Tonho Lizama, Pastero Godoy, G.Contador, Vale Giacconi y Javi Urrutia, gracias por todos los buenos momentos vividos y por las infinitas sesiones de estudio. Asimismo quiero destacar a compañeros como Nico Tapia, Emilio, Orly Rivera, y Erwin Topp por sus siempre valiosos consejos y su buena compañía.

Doy las gracias muy afectuosamente a Eterin Jaña y Oscar Mori por su siempre buena disposición y simpatía, y en general a todos los funcionarios del DIM.

Finalmente, agradezco a Conicyt, quien a través del proyecto Fondecyt 1110181 financió este trabajo.

Contents

List of Figures

Chapter 1

Introduction

In this thesis work we consider the semilinear elliptic problem

$$\varepsilon^2 \Delta_g u(x) - F'(u) = 0, \quad \text{in} \quad M \tag{1.1}$$

where (M, g) is a smooth Riemannian Manifold, $\varepsilon > 0$ is a small parameter, Δ_g denotes the Laplace Beltrami operator on M, and the function $F : \mathbb{R} \to \mathbb{R}$ is a double-well potential, that is, a function satisfying

$$\forall s \neq \pm 1 : \quad F(s) > 0 \tag{1.2}$$

$$F(-1) = F(+1) = 0 \quad (\ \{-1, 1\} \text{ are global minima}) \tag{1.3}$$

$$\sigma_\pm^2 := F''(\pm 1) > 0 \tag{1.4}$$

conditions that provide function F some particular profile, as sketched below

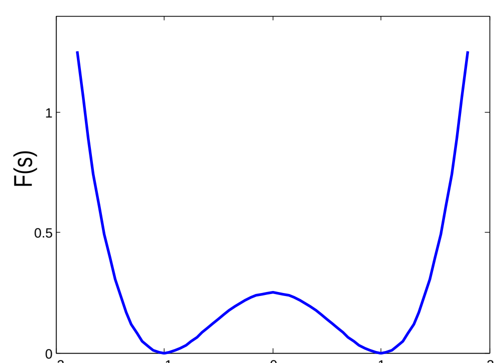

Figure 1: Graph of nonlinearity F, satisfying (1.2) to (1.4).

The typical example for F corresponds to the balanced and bi-stable twin-pit nonlinearity

$$F(u) := \frac{1}{4}(1 - u^2)^2 \tag{1.5}$$

for which

$$-F'(u) = u - u^3$$

Equation (1.1) is known as the singularly perturbed Allen-Cahn equation. It arises in gradient theory of phase transitions [1], where the function u is meant to represent the phase of a material in a given point of the manifold M. In this physical model, there are two different states of a material represented by the values $u = \pm 1$. It is of interest to study nontrivial configurations of the phase, in which two states try to coexist. Hence, the function u represents a smooth realization of the phase, which except for a narrow region, is expected to take values close to ± 1, namely the global minima of F.

1.1. The Allen-Cahn equation and the theory of Minimal Surfaces

Let us consider the energy functional

$$J_\varepsilon(u) = \int_M \left[\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}F(u)\right] dV_g \tag{1.6}$$

We are interested in critical points of (1.6), which correspond to solutions of (1.1). Suppose that $\Lambda \subset M$, then observe that the function

$$u_\Lambda^* := \chi_\Lambda - \chi_{M\setminus\Lambda}, \quad \text{in} \quad M \tag{1.7}$$

minimizes the second term in (1.6), however, it is evident that is not a smooth solution.

By considering an $\varepsilon-$regularization of u_Λ^*, say $u_{\Lambda,\varepsilon}$, it can be checked that

$$J_\varepsilon(u_{\Lambda,\varepsilon}) \approx \int_{\partial\Lambda} 1 dS_g \tag{1.8}$$

for $\varepsilon > 0$ small, where dS_g denotes the area element in $\partial\Lambda$. Relation (1.8) entails that, transitions varying from -1 to $+1$ must be selected, for instance, in such way that the area functional

$$\int_{\partial\Lambda} 1 dS_g \tag{1.9}$$

is minimized when evaluated in $\partial\Lambda$.

In the case that $\partial\Lambda$ is a smooth submanifold of M, it is said that $\partial\Lambda$ is a *minimal submanifold* of M if $\partial\Lambda$ is critical for the area functional (1.9). In particular, it is easy to see that minimizing submanifolds are critical for (1.9), and therefore they are minimal submanifolds of M.

The intuition behind the previous remarks, was first observed by Modica in [12], based upon the fact that when $\partial\Lambda$ is a smooth submanifold of M, then transitions varying from -1 to $+1$ take place along the normal direction of $\partial\Lambda$ in M, having a $1D-$profile in this direction. This profile corresponds to a function w, which is the heteroclinic solution to the ODE

$$\begin{cases} w''(t) - F'(w(t)) = 0, & \text{in} \quad \mathbb{R} \\ w(\pm\infty) = \pm 1 \end{cases} \tag{1.10}$$

connecting the two states. The existence of w is ensured by conditions (1.2)-(1.3)-(1.4) of F. As for the twin pit nonlinearity, we have that

$$w(t) = \tanh(t/\sqrt{2}), \quad t \in \mathbb{R}$$

This intuition gave a great impulse to the Calculus of Variations, and the theory of the $\Gamma-$Convergence in the 70th's. Regarding this matter, it is worth to mention some result by Modica and Mortola. In [12] the authors proved that if $M = \Omega \subset \mathbb{R}^N$ is a smooth Euclidean domain and $\{u_\varepsilon\}_{\varepsilon>0}$ is a family of local minimizers of (1.6) with uniformly bounded energy, then up to subsequence, u_ε converges in a $L^1_{loc}(\Omega)$-sense to some limit u^*_Λ of the form (1.7). Moreover, it was proved the convergence of the energy

$$J_\varepsilon(u_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \left(\int_{\partial\Lambda} 1 dS\right) \int_{\mathbb{R}} \left[\frac{1}{2}|w'(t)|^2 + F(w(t))\right] dt \tag{1.11}$$

where w is determined by (1.10), and where $\partial\Lambda$ minimizes the area functional in (1.9).

The condition of local minimizers can be relaxed to a family of critical points with uniformly bounded energy, as was proved in [11]. In this case, the authors showed that the convergence of the interface remains under an integer multiplicity, which takes into account the possibility of multiple transitions layers converging to the same set of minimal perimeter. For related results involving stronger notions of convergence we refer the reader to [2], [3], and references there in.

There has been a number of important works regarding existence and asymptotic behavior of solutions to (1.1), under a variety of different setting. In [13] Pacard and Ritore studied equation (1.1) in the case that (M, g) is a compact Riemannian Manifold, they construct a family of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ to (1.1) having transition from -1 to $+1$ on a region $\varepsilon-$close to a compact minimal submanifold N, with positive Ricci-curvature $k_N := |A_N|^2 + Ric(\nu_N, \nu_N)$. Under the same conditions del Pino, Kowalczyk, Wei and Yang constructed in [9] a sequence of solutions with multiple clustered layers collapsing onto N. A gap condition is needed, related with the interaction between interfaces.

There are related other results under a similar setting for M and N, regarding the equation

$$\varepsilon^2 \Delta_g u - V(z) F'(u) = 0, \quad \text{in} \quad M$$

done by B.Lai and Z.Du in [14] where a family of solutions with a single transition is constructed. Additionally L.Wang and Z.Du dealt in [16] with the same problem, considering multiple transitions this time. In both works the stationarity and nondegeneracy properties of N, are with respect to the weighted area functional $\int_M V^{1/2}$. In the same line, it is worth to mention another work done a short time ago, due to Z.Du and C.Gui [15] where they build a smooth solution to the Neumann problem

$$\varepsilon^2 \Delta u - V(z)F'(u) = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

having a single transition near a smooth closed curve $\Gamma \subset \Omega$, nondegenerate geodesic relative to the arclength $\int_\Gamma V^{1/2}$. Here, Ω is a smooth bounded domain in $\mathbb{R}^2$, and V is an uniformly positive smooth potential.

As for the noncompact case, recently in [8] del Pino et al. considered equation (1.1) when $M = \mathbb{R}^3$. Here the authors build for any small $\varepsilon > 0$ a family of solutions with transitions close to a non-degenerate complete embedded minimal surface with finite total curvature. In addition, entire solutions with multiple transition layers to (1.1) in $\mathbb{R}^2$ were found in [6]. In this case the nodal set of the solutions consists on multiple noncompact curves, not intersecting with themselves, whose location is governed by the Toda system of ODEs.

Finally, we mentioned that all these works take advantage of a very versatile and powerful tool, namely, the infinite dimensional Lyapunov-Schmidt reduction method, which is in the spirit of the pioneering work [10] due to Floer and Weinstein for the standing wave problem in the cubic Schödinger equation.

1.2. The Main Result

We consider (1.1) in a slightly general form, but we restrict ourselves to dimension $N = 2$. More precisely, let us consider the equation

$$\varepsilon^2 \operatorname{div}(a(x)\nabla u(x)) - a(x)F'(u) = 0, \quad \text{in} \quad \mathbb{R}^2 \tag{1.12}$$

As far as our knowledge goes, little is known about entire solutions to (1.12) in the case that $a(x)$ is not identically constant, having a single transition close to a noncompact curve. In this work we will consider a smooth noncompact curve Γ parametrized by arc-length, with a vector field $\gamma : \mathbb{R} \to \Gamma \subset \mathbb{R}^2$. We denote by $\nu : \Gamma \to \mathbb{R}^2$ a choice of the normal vector to Γ. Points $x \in \mathbb{R}^2$ that are $\delta-$close to this curve, with δ small, can be represented as

$$x = \gamma(s) + z \cdot \nu(s) =: X(s,z), \quad |z| < \delta, \quad s \in \mathbb{R}$$

Thus the map $x \mapsto (s,z)$ defines a local diffeomorphism. Any smooth curve $\delta-$close to Γ in C^m-topology can be parametrized by

$$\gamma_h(s) = \gamma(s) + h(s)\nu(s)$$

where h is a small C^m-function. The weighted length of Γ_h is given by

$$l_\Gamma(h) := \int_{\Gamma_h} a(x)d\vec{r} = \int_{-\infty}^{+\infty} a\left(\gamma_h(s)\right)|\dot{\gamma}_h(s)|ds$$
$$= \int_{-\infty}^{+\infty} a(s,h(s))|\dot{\gamma} + h\dot{\nu} + h'\nu|ds$$

Since $|\dot{\gamma}| = 1$ and $\dot{\nu}(s) = -k(s)\dot{\gamma}(s)$, where k is the signed curvature of Γ, we find that

$$l_h(h) = \int_{-\infty}^{+\infty} a(s,h(s))[(1-kh)^2 + |h'|^2]ds$$

We say that Γ is a stationary curve respect to the function $a(x)$, if and only if,

$$l'_\Gamma[h] = \int_{\Gamma_h} (\partial_z a(s,0) - a(s,0)k(s))h(s)ds$$
$$= 0, \quad \forall h \in C_c^\infty(\mathbb{R})$$

This amounts to

$$\partial_z a(s,0) = k(s)a(s,0), \quad s \in \mathbb{R} \tag{1.13}$$

Regarding the stability properties of the stationary curve Γ, and the second variation of the length functional l_Γ,

$$l''_\Gamma(h,h) = \int_{-\infty}^{+\infty} \left\{a(s,0)|h'(s)|^2 + [\partial_{zz}a(s,0) - 2k^2(s)]h^2(s)\right\} ds$$

it arises the Jacobi operator of Γ, corresponding to

$$\mathcal{J}_{a,\Gamma}(h) = h''(s) + \frac{\partial_s a(s,0)}{a(s,0)}h'(s) - \left[\partial_{zz}a(s,0) - 2k^2(s)\right]h(s) \tag{1.14}$$

We say that the stationary curve is also nondegenerate respect to the potential $a(x)$, if and only if, the bounded kernel of $\mathcal{J}_{a,\Gamma}$ is the trivial one. The nondegeneracy condition basically implies that $\mathcal{J}_{a,\Gamma}$ has an appropriate right inverse.

In order to state the main result, we first assume that the mapping $x = X(s,z)$ provides local coordinates in a region of the form

$$\mathcal{O}_\delta = \left\{x = X(s,z) \Big/ |z| < \delta + c_0|s|\right\}$$

with a small constant $c_0 > 0$. Assume in addition the existence of positive constants α, m, M for which

$$m < a(s,z) \leq M, \quad |\nabla_{s,z}a(s,z)| \leq \frac{C}{(1+|s|)^{1+\alpha/2}}, \quad |D^2a(s,z)| \leq \frac{C}{(1+|s|)^{2+\alpha}} \tag{1.15}$$

and additionally

$$|k_\Gamma(s)| + |k'_\Gamma(s)| + |k''_\Gamma(s)| \leq \frac{C}{(1+|s|)^{1+\alpha/2}} \tag{1.16}$$

Hence, condition (1.16) implies that

$$\dot\gamma_\pm := \lim_{s\to\pm\infty} \dot\gamma(s) \in \mathbb{R}^2$$

In order for the neighborhood $\mathcal{O}_\delta$ to be well defined, we must assume some non-parallelism condition

$$-1 \leq \langle \dot\gamma_+, \dot\gamma_- \rangle < 1$$

The following picture illustrates this geometrical setting

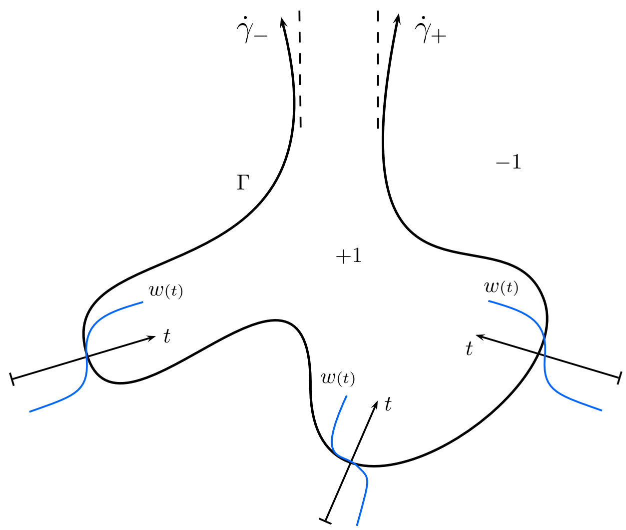

Figure 2: Asymptotic behavior of Γ and the profile of the solution.

Now we proceed to state the main result.

Theorem 1. *Assume that $a(\vec{x})$ is a smooth potential satisfying* (1.13)-(1.15) *and let Γ be a smooth curve with decay* (1.16)*, which is a nondegenerate geodesic respect to $\int_\Gamma a(\vec{x})$. Then for any $\varepsilon > 0$ small enough, there exists a smooth bounded solution u_ε to the inhomogeneous Allen-Cahn equation* (1.12)*, such that*

$$u_\varepsilon(x) = w\left(\frac{z-h(s)}{\varepsilon}\right) + O(\varepsilon^2), \quad \textit{for} \quad x = X(s,z), \quad |z| < \delta \tag{1.17}$$

where the function h satisfies

$$\|h\|_{C^1(\mathbb{R})} \leq C\varepsilon \tag{1.18}$$

This solution is converging to a constant as it moves away from Γ, namely

$$u_\varepsilon(x) \to \pm 1, \quad \textit{as} \quad \varepsilon \to 0 \quad \textit{for } x \notin \mathcal{O}_\delta \tag{1.19}$$

It is worth to mention some important facts related to Theorem 1, which are not direct from the previous statement, but rather are consequences of the scheme performed later on.

Remark 1. *The effectiveness in the demonstration of this result, as mentioned before, relies on an infinite dimensional reduction, for which the choice of a "good" approximation to a solution is of vital importance.*

Remark 2. *Throughout the proof of this theorem, we give an explicit construction of the solution u_ε based on some error terms, for which we get to know a precise qualitative behavior at main order. Accordingly, we obtain specific description for u_ε and its derivatives, than of what was stated in the theorem.*

1.3. Structure of the Exposition

Now we present, in a few words, the structure of this thesis work. On Chapter 2 we deal with the geometrical setting of this problem, and we explain in depth the conditions of stationarity and nondegeneracy for the curve Γ with respect to the arclength $\int_\Gamma a(\vec{x})$. We also give a detailed discussion on the kernel and the invertibility of the Jacobi operator (1.14) in an suitable functional setting. Later on we exhibit a couple of examples regarding the potential $a(\vec{x})$ and the curve Γ, to get a precise idea of the geometrical aspects that we are dealing with. At the end, we announce formulae for the elliptic operator associated to (1.13), to compute in Fermi coordinates.

An equivalent form for the inhomogeneous Allen-Cahn equation in Fermi coordinates is developed in Chapter 3 . In addition, a study is carried out about the connection between Minimal Surfaces and phase transition phenomena, by leading a formal discussion on Modica's result [12]. In Chapter 4 a proof for the main result is developed, while in Chapter 5 there are several demonstrations for auxiliary results needed for the proof of Theorem 1, mainly consisting on important, rather standard techniques. Finally, Chapter 6 contains some concluding remarks, together with a description on the work yet to be done.

Chapter 2

Geometrical Settings

2.1. Geodesic curves and the Jacobi operator

Section 2.1 is intended to establish some necessary conditions satisfied by both, the curve Γ and the potential $a(x, y)$, with the purpose of building a smooth bounded solution $u : \mathbb{R}^2 \to \mathbb{R}$ to the inhomogeneous Allen-Cahn equation, changing sign near a region defined by the curve $\Gamma \subset \mathbb{R}^2$.

This work deals with a cornerstone concept in Differential Geometry. This idea concerns with the notion of *curve of minimum length* in some generalized sense, explained below. Suppose that Γ is a non-compact curve of $\mathbb{R}^2$, let us define the weighted length functional induced by the potential $a : \mathbb{R}^2 \to \mathbb{R}^+$, trough

$$l_{a,\Gamma} : C^1(\mathbb{R}) \to \mathbb{R}, \quad l_{a,\Gamma}(h) := \int_{\Gamma_h} a(x, y)\ d\vec{\gamma}(x, y) \tag{2.1}$$

where Γ_h represents any parametrized curve sufficiently close to Γ, as a result of perturbing each point of the Γ in the normal direction ν_γ by h. Then, a parametrization of Γ_h can be obtained using the function h as the normal distance separating Γ from this new curve, as

$$X_h(\boldsymbol{s}) := \gamma(\boldsymbol{s}) + h(\boldsymbol{s}) \cdot \nu(\boldsymbol{s})$$

It is told in Differential Geometry that functional $l_{a,\Gamma}(h)$ computes the length of the curve Γ_h using *a metric induced by the potential* a. This motivates to define the stationarity of $\Gamma \subset \mathbb{R}^2$ with respect to $l_{a,\Gamma}$, provided sufficiently smoothness, by requiring that this curve eliminates the first variation of $l_{a,\Gamma}$ when is restrained to normal perturbations of Γ, and additionally, asking to the second variation of $l_{a,\Gamma}$ to be *definite positive* when considering the same type of small perturbations.

A relevant aspect arises from the analysis of the length functional, which is the quadratic form associated to the second variation of the $l_{a,\Gamma}$ around Γ, commonly referred as the Jacobi operator $\mathcal{J}_a$ of the curve Γ.

Here and subsequently, it will be of vital importance to get a appropriate description of the Jacobi operator, essentially because we want to solve the nonlinear differential equation (3.32). Particularly, we begin the study of this operator by developing an explicit formula that describes it, in terms of the curve Γ and the potential a. This can be done by computing the Gateaux derivatives of $l_{a,\Gamma}$ up to second order, thus obtaining the desired quadratic form.

To perform these calculations we will use the following notation for the potential a in coordinates associated to the curve Γ:

$$a(\boldsymbol{s},\boldsymbol{t}) := a\left(X(\boldsymbol{s}) + \boldsymbol{t}\cdot\nu(\boldsymbol{s})\right)$$

Then functional (2.1) can be written as

$$\begin{aligned} l_{a,\Gamma}(h) &= \int\limits_{\mathbb{R}} a(\gamma(\boldsymbol{s}) + h(\boldsymbol{s})\nu(\boldsymbol{s}))|\dot\gamma(\boldsymbol{s}) + \dot h(\boldsymbol{s})\nu(\boldsymbol{s}) + h(\boldsymbol{s})\dot\nu(\boldsymbol{s})|d\boldsymbol{s} \\ &= \int\limits_{\mathbb{R}} a(\boldsymbol{s}, h(\boldsymbol{s}))|\dot\gamma(\boldsymbol{s}) + \dot h(\boldsymbol{s})\nu(\boldsymbol{s}) + h(\boldsymbol{s})\dot\nu(\boldsymbol{s})|d\boldsymbol{s} \end{aligned} \tag{2.2}$$

Denoting by

$$\begin{aligned} Q(\boldsymbol{s},\epsilon,h) :&= |\dot\gamma(\boldsymbol{s}) + \epsilon\dot h(\boldsymbol{s})\nu(\boldsymbol{s}) + \epsilon h(\boldsymbol{s})\dot\nu(\boldsymbol{s})| \\ &= \left[1 + \epsilon^2|\dot h(\boldsymbol{s})|^2 + \epsilon^2 h^2(\boldsymbol{s})k^2(\boldsymbol{s}) - 2\epsilon h(\boldsymbol{s})k(\boldsymbol{s})\right]^{1/2} \end{aligned} \tag{2.3}$$

we get that the functional (2.2) is

$$l_{a,\Gamma}(h) = \int\limits_{\mathbb{R}} a(\boldsymbol{s}, h(\boldsymbol{s}))\cdot Q(\boldsymbol{s},1,h)d\boldsymbol{s} \tag{2.4}$$

2.1.1. First variation of $l_{a,\Gamma}$: Necessary condition for geodesics

Set $h \in C_c^\infty(\mathbb{R})$ be a test function, let us compute the Gateaux's derivative of $l_{a,\Gamma}$ at zero:

$$l'_{a,\Gamma}(0)[h] = \lim_{\epsilon\to 0}\frac{l_{a,\Gamma}(0+\epsilon h) - l_{a,\Gamma}(0)}{\epsilon} = \frac{d}{d\epsilon}\int\limits_{\Gamma_{\epsilon h}} a(x,y)d\vec\gamma(x,y)\Bigg|_{\epsilon=0} \tag{2.5}$$

The relation (2.4) implies that

$$l_{a,\Gamma}(\epsilon h) = \int\limits_{\mathbb{R}} a(\boldsymbol{s},\epsilon h(\boldsymbol{s}))\cdot Q(\boldsymbol{s},\epsilon,h)d\boldsymbol{s} \tag{2.6}$$

So

$$\frac{d}{d\epsilon}l_{a,\Gamma}(\epsilon h) = \int\limits_{\mathbb{R}} \left[\frac{d}{d\epsilon}a(\boldsymbol{s},\epsilon h(\boldsymbol{s}))Q(\boldsymbol{s},\epsilon,h) + a(\boldsymbol{s},\epsilon h(\boldsymbol{s}))\frac{\partial}{\partial\epsilon}Q(\boldsymbol{s},\epsilon,h)\right] d\boldsymbol{s}$$

$$= \int\limits_{\mathbb{R}} \partial_{\boldsymbol{t}} a \cdot h \cdot Q(\boldsymbol{s},\epsilon,h)d\boldsymbol{s} + \int\limits_{\mathbb{R}} a \cdot \frac{1}{2\sqrt{Q^2(\boldsymbol{s},\epsilon,h)}} \cdot \frac{\partial}{\partial\epsilon}Q^2(\boldsymbol{s},\epsilon,h)ds \tag{2.7}$$

However, since

$$\frac{\partial}{\partial\epsilon}Q^2(\boldsymbol{s},\epsilon,h) = 2(\epsilon|\dot{h}(\boldsymbol{s})|^2 + \epsilon h(\boldsymbol{s})k^2(\boldsymbol{s}) - h(\boldsymbol{s})k(\boldsymbol{s})) \tag{2.8}$$

and noticing from (2.3) that $Q(\boldsymbol{s},0,h) = 1$, the desired derivative can be computed from (2.7) as follows

$$\left.\frac{d}{d\epsilon}l_{a,\Gamma}(\epsilon h)\right|_{\epsilon=0} = \int\limits_{\mathbb{R}} \partial_{\boldsymbol{t}} a(\boldsymbol{s},0)\cdot h(\boldsymbol{s})\cdot Q(\boldsymbol{s},0,h)d\boldsymbol{s} + \int\limits_{\mathbb{R}} \frac{a(\boldsymbol{s},0)}{2\;Q(\boldsymbol{s},0,h)}\left.\frac{\partial}{\partial\epsilon}Q^2(\boldsymbol{s},\epsilon,h)^2\right|_{\epsilon=0} d\boldsymbol{s}$$

$$= \int\limits_{\mathbb{R}} \partial_{\boldsymbol{t}} a(\boldsymbol{s},0)\cdot h(\boldsymbol{s})d\boldsymbol{s} - \int\limits_{\mathbb{R}} a(\boldsymbol{s},0)\cdot h(\boldsymbol{s})k(\boldsymbol{s})d\boldsymbol{s}$$

$$= \int\limits_{\mathbb{R}} \left[\partial_{\boldsymbol{t}} a(\boldsymbol{s},0) - a(\boldsymbol{s},0)k(\boldsymbol{s})\right] h(\boldsymbol{s})d\boldsymbol{s} \tag{2.9}$$

Imposing that Γ is a critical curve for the weighted arc-length functional amounts to the condition $l'_{a,\Gamma}(0)[h] \equiv 0$. Since h is a smooth and arbitrary function, and $\text{ínf}_{\mathbb{R}^2} a > 0$, we deduce *the criticality condition for* Γ:

$$\boxed{\partial_{\boldsymbol{t}} a(\boldsymbol{s},0) = k(\boldsymbol{s})\cdot a(\boldsymbol{s},0), \quad \text{a.e.} \quad \boldsymbol{s}\in\mathbb{R}} \tag{2.10}$$

Definition 1. *Stationarity*
The curve $\Gamma \subset \mathbb{R}^2$ is said to be **stationary**, *or* **geodesic**, *relative to the weighted arclength $\int_\Gamma a(\vec{x})$ if Γ satisfies the criticality condition* (2.10). *This property amounts to the fact that the operator $l_{a,\Gamma}$ vanishes its first variation around zero.*

2.1.2. Second variation of $l_{a,\Gamma}$: The Jacobi operator

Analogously, let us compute the second Gateaux's derivative of $l_{\Gamma,a}$ at zero. Because of the calculation carried out in (2.7), it follows

$$\frac{d^2}{d\epsilon^2}l_{a,\Gamma}(\epsilon h) = \frac{d}{d\epsilon}\left[\int\limits_{\mathbb{R}} \partial_{\boldsymbol{t}} a\cdot h(\boldsymbol{s})\cdot Q(\boldsymbol{s},\epsilon,h)d\boldsymbol{s} + \int\limits_{\mathbb{R}} \frac{a(\boldsymbol{s},\epsilon h)}{2\sqrt{Q(\boldsymbol{s},\epsilon,h)^2}}\cdot\frac{\partial}{\partial\epsilon}Q^2(\boldsymbol{s},\epsilon,h)d\boldsymbol{s}\right]$$

$$= \int_{\mathbb{R}} \left[\partial_{\boldsymbol{tt}} a \cdot h^2(\boldsymbol{s}) \cdot Q(\boldsymbol{s}, \epsilon, h) + \partial_{\boldsymbol{t}} a \cdot h(\boldsymbol{s}) \cdot \frac{\partial}{\partial t} \sqrt{Q^2(\boldsymbol{s}, \epsilon, h)} \right] d\boldsymbol{s}$$

$$+ \int_{\mathbb{R}} \left[\frac{\partial_{\boldsymbol{t}} a \cdot h(\boldsymbol{s})}{2\, Q(\boldsymbol{s}, \epsilon, h)} \frac{\partial}{\partial \epsilon} Q^2(\boldsymbol{s}, \epsilon, h) - \frac{a(\boldsymbol{s}, \epsilon h) \left| \frac{\partial}{\partial \epsilon} Q^2(\boldsymbol{s}, \epsilon, h) \right|^2}{4 \big(Q(\boldsymbol{s}, \epsilon, h)^2\big)^{3/2}} + \frac{a(\boldsymbol{s}, \epsilon h)}{2\, Q(\boldsymbol{s}, \epsilon, h)} \frac{\partial^2}{\partial \epsilon^2} Q^2(\boldsymbol{s}, \epsilon, h) \right] d\boldsymbol{s} \tag{2.11}$$

But from (2.3) is straightforward that

$$\frac{\partial^2}{\partial \epsilon^2} Q^2(\boldsymbol{s}, \epsilon, h) = 2 \left[|\dot{h}(\boldsymbol{s})|^2 + h^2(\boldsymbol{s}) k^2(\boldsymbol{s}) \right] \tag{2.12}$$

So that by using (2.8) and (2.12), the second derivative is computed using (2.11)

$$\frac{d^2}{d\epsilon^2} l_{a,\Gamma}(\epsilon h) \bigg|_{\epsilon=0} = \int_{\mathbb{R}} \left[\partial_{\boldsymbol{tt}} a \cdot h^2(\boldsymbol{s}) \cdot Q(\boldsymbol{s}, 0, h) + \partial_{\boldsymbol{t}} a \cdot h(\boldsymbol{s}) \cdot \frac{(-2h(\boldsymbol{s})k(\boldsymbol{s}))}{2\, Q(\boldsymbol{s}, 0, h)} \right] d\boldsymbol{s}$$

$$+ \int_{\mathbb{R}} \left[\frac{\partial_{\boldsymbol{t}} a \cdot h(\boldsymbol{s})(-2h(\boldsymbol{s})k(\boldsymbol{s}))}{2\, Q(\boldsymbol{s}, 0, h)} - \frac{a(\boldsymbol{s}, 0)(-2h(\boldsymbol{s})k(\boldsymbol{s}))^2}{4 Q^3(\boldsymbol{s}, 0, h)} + a(\boldsymbol{s}, 0) \frac{2 \left(|\dot{h}(\boldsymbol{s})|^2 + h^2(\boldsymbol{s}) k^2(\boldsymbol{s}) \right)}{2\, Q(\boldsymbol{s}, 0, h)} \right] d\boldsymbol{s} \tag{2.13}$$

Finally, using $Q(\boldsymbol{s}, 0, h) = 1$, we have that expression (2.13) reduces to

$$l''_{a,\Gamma}(0)[h, h] = \int_{\mathbb{R}} \Big(h^2(\boldsymbol{s}) [\partial_{\boldsymbol{tt}} a - 2k(\boldsymbol{s}) \partial_{\boldsymbol{t}} a] - a(\boldsymbol{s}, 0) \left[h^2(\boldsymbol{s}) k^2(\boldsymbol{s}) - |\dot{h}(\boldsymbol{s})|^2 - h^2(\boldsymbol{s}) k^2(\boldsymbol{s}) \right] \Big) d\boldsymbol{s}$$

Then gathering terms in common, we get

$$l''_{a,\Gamma}(0)[h, h] = \int_{\mathbb{R}} \left[a(\boldsymbol{s}, 0) |\dot{h}(\boldsymbol{s})|^2 + \Big(\partial_{\boldsymbol{tt}} a(\boldsymbol{s}, 0) - 2k(\boldsymbol{s}) \partial_{\boldsymbol{t}} a(\boldsymbol{s}, 0) \Big) h^2(\boldsymbol{s}) \right] d\boldsymbol{s}$$

Now we impose in the last expression for $\mathcal{L}''_{a,\Gamma}$, the criticality condition (2.10) for Γ, obtaining:

$$l''_{a,\Gamma}(0)[h, h] = \int_{\mathbb{R}} \left[a(\boldsymbol{s}, 0) |\dot{h}(\boldsymbol{s})|^2 - \Big(2k^2(\boldsymbol{s}) a(\boldsymbol{s}, 0) - \partial_{\boldsymbol{tt}} a(\boldsymbol{s}, 0) \Big) h^2(\boldsymbol{s}) \right] d\boldsymbol{s} \tag{2.14}$$

Integrating by parts and factorizing by $h(\boldsymbol{s})$, the second variation of $l_{a,\Gamma}$ around zero can be written as

$$\begin{aligned} l''_{a,\Gamma}(0)[h, h] &= \int_{\mathbb{R}} \left[-\left(a(\boldsymbol{s}, 0) h'(\boldsymbol{s}) \right)' h(\boldsymbol{s}) - \Big(2k^2(\boldsymbol{s}) a(\boldsymbol{s}, 0) - \partial_{\boldsymbol{tt}} a \Big) h^2(\boldsymbol{s}) \right] d\boldsymbol{s} \\ &= - \int_{\mathbb{R}} \left[\partial_{\boldsymbol{s}} a(\boldsymbol{s}, 0) h'(\boldsymbol{s}) + a(\boldsymbol{s}, 0) h''(\boldsymbol{s}) + \Big(2ak^2(\boldsymbol{s}) - \partial_{\boldsymbol{tt}} a \Big) h(\boldsymbol{s}) \right] h(\boldsymbol{s}) d\boldsymbol{s} \end{aligned}$$

So rearranging the terms of $l''_{a,\Gamma}(0)$ we get the quadratic form associated to (2.14).

$$l''_{a,\Gamma}(0)[h,h] = -\int_{\mathbb{R}} a(\boldsymbol{s},0)\left[h''(\boldsymbol{s}) + \frac{\partial_{\boldsymbol{s}} a}{a} h'(\boldsymbol{s}) + \left(2k^2(\boldsymbol{s}) - \frac{1}{a}\partial_{\boldsymbol{tt}} a\right) h(\boldsymbol{s})\right] h(\boldsymbol{s}) d\boldsymbol{s}$$

Therefore, from this formula we deduce an expression for the Jacobi operator of the curve Γ associated to the potential a:

$$\boxed{\mathcal{J}_a[h](\boldsymbol{s}) := h''(\boldsymbol{s}) + \frac{\partial_{\boldsymbol{s}} a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)} h'(\boldsymbol{s}) + \left(2k^2(\boldsymbol{s}) - \frac{\partial_{\boldsymbol{tt}} a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)}\right) h(\boldsymbol{s})} \tag{2.15}$$

□

Definition 2. *Non-degeneracy*
The curve Γ *will be* nondegenerate*, if the differential Jacobi equation*

$$\mathcal{J}_a[h](s) = 0, \quad \forall s \in \mathbb{R}$$

has $h \equiv 0$ *as the only bounded solution.*

2.2. Invertibility of the Jacobi operator

Once that it has obtained an expression for the Jacobi operator $\mathcal{J}_a$, the next step is to study conditions that ensure its invertibility, in order to apply the infinite dimensional Lyapunov-Schmidt reduction method. The goal is to find adequate conditions both on the curve Γ and on the potential $a(x,y)$ that guarantee the injectivity of the operator $\mathcal{J}_a$ in some adequate functional space.

2.2.1. Study of the Kernel

Section 2.1 provides a formula associated to a and Γ, for the Jacobi operator

$$\mathcal{J}_a[h](s) = h''(s) + \frac{\partial_s a(s,0)}{a(s,0)} h'(s) - Q(s)h(s)$$

where now we adopt the convention

$$Q(s) := \left[\frac{\partial_{\boldsymbol{tt}} a(s,0)}{a(s,0)} - 2k^2(s)\right] \tag{2.16}$$

It will be of interest to consider the auxiliary equation

$$\frac{d}{ds}\left(p(s)\frac{d}{ds}h(s)\right) - q(s)h(s) = 0, \qquad \text{in } \mathbb{R} \tag{AE}$$

where we assume that $p, q : \mathbb{R} \to \mathbb{R}$ satisfy the following

$$p \in C^1[0,+\infty) \cap L^\infty[0,+\infty), \;\; q \in C^1[0,+\infty) \tag{2.17}$$

$$\lim_{s\to\pm\infty} p(s) =: p(\pm\infty) \in \mathbb{R} \setminus \{0\} \tag{2.18}$$

$$p(s) \geq p_0 > 0, \quad \forall s \geq 0 \tag{2.19}$$

$$|p(s)| + (1+|s|)^{2+\alpha}|p'(s)| \leq C, \quad \forall s \geq 0 \tag{2.20}$$

$$|q(s)| + |q'(s)| \leq \frac{C}{1+|s|^{2+\alpha}}, \quad \forall s \geq 0 \tag{2.21}$$

for some constants $\alpha > -1$, $\beta_0 > 0$ and $C > 0$.

The first result concerns the decay for the derivative of a solution to the auxiliary equation, provided that p and q decay sufficiently fast.

Lemma 1. *Suppose $\alpha > -1$, and consider a one-sided bounded solution $h \in L^\infty[0,\infty)$ of (AE), for which functions p and q fulfill* (2.17) *to* (2.21)*. Then there is a constant $C = C(p,q,\alpha,h) > 0$ such that*

$$|h'(s)| \leq \frac{C}{|s|^{1+\alpha}}, \quad \forall s > 0$$

where $C(p,q,\alpha,h) = \|p^{-1}\|_{L^\infty[0,\infty)}\|h\|_{L^\infty[0,\infty)}\|(1+|s|)^{2+\alpha}q\|_{L^\infty[0,\infty)}$.

Proof.-
Observe first that thanks to assumptions (2.17)-(2.18), it holds

$$p(s) = p(+\infty) - \int_s^{+\infty} p'(\xi)d\xi \tag{2.22}$$

Now, since h solves the equation, then for $s_1 > s_2 > 0$ we have

$$\begin{aligned} |p(s_1)h'(s_1) - p(s_2)h'(s_2)| &\leq \int_{s_1}^{s_2} |q(s)h(s)| \\ &\leq \|h\|_{L^\infty[0,\infty)}\|(1+|s|)^{2+\alpha}q\|_{L^\infty[0,\infty)} \left|\int_{s_1}^{s_2} \frac{1}{1+|s|^{2+\alpha}}ds\right| \\ &\leq C(q,h)\left|\frac{1}{|s_1|^{1+\alpha}} - \frac{1}{|s_2|^{1+\alpha}}\right| \end{aligned}$$

where $C(q,h) := C \cdot \|h\|_{L^\infty[0,\infty)} \|(1+|s|)^{2+\alpha} q\|_{L^\infty[0,\infty)} \in \mathbb{R}$ is fixed.
In particular using that $1+\alpha > 0$, it follows that

$$\lim_{s_1 \to +\infty} |p(s_1)h'(s_1)| \leq |p(s_2)h'(s_2)| + C(q,h)\frac{1}{|s_2|^{1+\alpha}} < +\infty$$

which implies that $p(+\infty)h'(\infty) \in \mathbb{R}$. From this, we can rewrite the (AE) in its integral form

$$p(s)h'(s) = p(+\infty)h'(+\infty) - \int_s^{+\infty} q(\xi)h(\xi)d\xi \tag{2.23}$$

but using (2.22), this amounts to

$$p(+\infty)h'(s) - h'(s)\int_s^{+\infty} p'(\xi)d\xi = p(+\infty)h'(+\infty) - \int_s^{+\infty} q(\xi)h(\xi)d\xi$$

and so

$$p(+\infty)h'(s) = p(+\infty)h'(+\infty) + h'(s)\int_s^{+\infty} p'(\xi)d\xi - \int_s^{+\infty} q(\xi)h(\xi)d\xi$$

Integrating again between 0 and s, we obtain an expression for the solution h of (AE)

$$\begin{aligned} p(+\infty)h(s) &= p(+\infty)h(0) + p(\infty)h'(+\infty)s \\ &\quad + \underbrace{\int_0^s h'(\xi)\int_\xi^{+\infty} p'(\tau)d\tau d\xi}_{I} - \underbrace{\int_0^s \int_\xi^{+\infty} q(\tau)h(\tau)d\tau d\xi}_{II} \end{aligned} \tag{2.24}$$

Let us estimate these integrals, but in order to do this, we first need to note that h' is bounded in $[0,+\infty)$ since from (2.23)

$$h'(s) = \left(p(+\infty)h'(+\infty) - \int_s^{+\infty} q(\xi)h(\xi)\right) \cdot p^{-1}(s)$$

and this property will follow from assumptions (2.17)-(2.19) that suppose the boundedness of h, the decay of q, and that p is away from zero.
Now we estimate integral I

$$\begin{aligned} |I| &\leq \int_0^s |h'(\xi)| \int_\xi^{+\infty} |p'(\tau)|d\tau \\ &\leq C\|h'\|_{L^\infty[0,\infty)} \|(1+|s|^{2+\alpha})p'\|_{L^\infty[0,\infty)} \int_0^s \int_\xi^{+\infty} \frac{1}{1+|\tau|^{2+\alpha}} d\tau d\xi \\ &\leq C_{h',p',\alpha} \int_0^s \frac{1}{1+|\xi|^{1+\alpha}} d\xi = O(1+|s|^{-\alpha}) \end{aligned}$$

where $C_{h',p',\alpha} := C\|h'\|_{L^\infty[0,\infty)}\|(1+|s|^{2+\alpha})p'\|_{L^\infty[0,\infty)}$.
In the same way, we estimate II

$$\begin{aligned}|II| &\leq \int_0^s \int_\xi^{+\infty} |q(\tau)|\ |h(\tau)| d\tau d\xi \\ &\leq C\|h\|_{L^\infty[0,\infty)}\|(1+|s|^{2+\alpha})q\|_{L^\infty[0,\infty)} \int_0^s \int_\xi^{+\infty} \frac{d\tau d\xi}{1+|\tau|^{1+\alpha}} \\ &\leq C_{h,q,\alpha}(1+|s|)^{-\alpha}\end{aligned}$$

with $C_{h,q,\alpha} := C\|h\|_{L^\infty[0,\infty)}\|(1+|s|)^{2+\alpha}q\|_{L^\infty[0,\infty)}$.
Therefore, since h is a bounded solution, we deduce from (2.24)

$$O(1) = p(+\infty)h(0) + p(+\infty)h'(+\infty)s + O(1+|s|^{-\alpha}) \tag{2.25}$$

As $\alpha > -1$, then $-\alpha - 1 < 0$, therefore dividing (2.25) by $s >> 0$ and taking the limit $s \to +\infty$, we get:

$$0 = p(+\infty)h'(+\infty)$$

which implies $h'(+\infty) = 0$, as $p(+\infty) \neq 0$ is assumed in (2.18).
In particular, the latter fact together with formula (2.23), imply the desired estimate:

$$\begin{aligned}& p(s)h'(s) = \int_s^\infty q(\xi)h(\xi)d\xi \\ \Rightarrow \quad & |h'(s)| \leq C\|p^{-1}\|_{L^\infty[0,\infty)}\|h\|_{L^\infty[0,\infty)}\|(1+|s|^{2+\alpha})q\|_{L^\infty[0,\infty)} \frac{1}{1+|s|^{1+\alpha}}\end{aligned}$$

which completes the proof. □

Remark 3. *It was not necessary to make use of the sign of $q(s)$, however it was important that p does not get close to zero, plus the fact that the limit $p(+\infty) \neq 0$ is well defined.*

The core of this section is reflected in the next result, which gives a qualitative asymptotic description to the class of solutions of some differential equation associated to the $\mathcal{J}_a$.

Lemma 2. *Let $\alpha > 2$, and suppose function q satisfies (2.17)-(2.21). Then the equation*

$$u''(s) - q(s)u(s) = 0, \qquad in \quad \mathbb{R} \tag{2.26}$$

has two linearly independent smooth solutions $u(s), \tilde{u}(s)$, that behave as $s \to +\infty$ like

$$u(s) = s + O(1) + O(|s|^{1-\alpha}), \qquad \tilde{u}(s) = 1 + O(|s|^{-1} + |s|^{-\alpha}) \tag{2.27}$$

$$u'(s) = 1 + O(|s|^{-1} + |s|^{-\alpha}), \qquad \tilde{u}'(s) = O(|s|^{-1} + |s|^{-1-\alpha}) \tag{2.28}$$

Proof.-
Let us prove first that exists a function u as stated, and then we will focus on the existence of another solution $\tilde{u}$ to be linearly independent to the first one. To begin with, suppose that solution u of (2.26) can be written as $u(s) = sv(s)$, then we have

$$u'(s) = v(s) + sv'(s), \quad u''(s) = sv''(s) + 2v'(s)$$

So, by multiplying equation (2.26) by s, follows the differential equation satisfied by v

$$\frac{d}{ds}\left(s^2 v'(s)\right) - q(s)s^2 v(s) = 0 \tag{2.29}$$

Now, define auxiliary functions

$$x(s) := s^2 v'(s), \quad y(s) := v(s) \tag{2.30}$$

so that equation (2.29) amounts to the linear system of differential equations

$$\begin{cases} x'(s) = q(s)s^2 y(s) \\ y'(s) = \dfrac{1}{s^2}x(s) \end{cases}, \quad \forall s \in [s_0, +\infty) \tag{2.31}$$

We want to prove not only that $y(s)$ is bounded in $[s_0, +\infty)$, but also that $y(s)$ converges polynomially to a constant, as s_0 approaches to infinity.

Integrating this system between s_0 and s we obtain the identities

$$\begin{aligned} y(s) &= y(s_0) + \int_{s_0}^{s} \frac{1}{\xi^2} x(\xi) d\xi \\ x(s) &= x(s_0) + \int_{s_0}^{s} q(\xi)\xi^2 y(\xi) d\xi \end{aligned} \tag{2.32}$$

In particular this allow us to deduce an explicit formula for $y(s)$, depending only of an expression involving y itself and some constants, given by

$$\begin{aligned} y(s) &= y(s_0) + \int_{s_0}^{s} \frac{1}{\xi^2}\left(x(s_0) + \int_{s_0}^{\xi} q(\tau)\tau^2 y(\tau) d\tau\right) d\xi \\ &= y(s_0) + x(s_0)\int_{s_0}^{s} \frac{1}{\xi^2} d\xi + \int_{s_0}^{s}\int_{s_0}^{\xi} \frac{1}{\xi^2} q(\tau)\tau^2 y(\tau) d\tau d\xi \end{aligned}$$

changing the order of integration to $d\xi\ d\tau$ we get

$$\begin{aligned} &= y(s_0) + x(s_0)\left(\frac{1}{s_0} - \frac{1}{s}\right) + \int_{s_0}^{s} y(\tau)q(\tau)\tau^2 \left(\int_{\tau}^{s} \frac{1}{\xi^2} d\xi\right) d\tau \\ &= y(s_0) + x(s_0)\left(\frac{1}{s_0} - \frac{1}{s}\right) + \int_{s_0}^{s} y(\tau)q(\tau)\tau^2 \left(\frac{1}{\tau} - \frac{1}{s}\right) d\tau \end{aligned} \tag{2.33}$$

In this way, we can estimate $y(s)$ for $s \geq s_0$ as

$$|y(s)| \leq |y(s_0)| + |x(s_0)| \left(\frac{1}{s_0} - \frac{1}{s}\right) + \int_{s_0}^{s} |y(\tau)|\ |q(\tau)|\tau \left(1 - \frac{\tau}{s}\right) d\tau$$

but Gronwall's inequality in its integral form gives us the estimate

$$|y(s)| \leq \left(|y(s_0)| + \frac{2|x(s_0)|}{s_0}\right) \exp\left(\int_{s_0}^{s} |q(\tau)|\tau \left(1 - \frac{\tau}{s}\right) d\tau\right) \tag{2.34}$$

However note that for any $s \geq \tau > s_0 :\ \left|\tau\left(1 - \frac{\tau}{s}\right)\right| \leq 2\tau = O(\tau)$. This fact combined with the decay of $q(s)$ will be helpful to study the finiteness of the last integral, because

$$\left|\int_{s_0}^{s} |q(\tau)|\tau \left(1 - \frac{\tau}{s}\right) d\tau\right| \leq C \int_{s_0}^{s} |q(\tau)|\tau d\tau \leq C\|(1+|s|)^{2+\alpha} q\|_{L^\infty[s_0,+\infty)} \int_{s_0}^{s} \frac{1}{|\tau|^{1+\alpha}} d\tau \tag{2.35}$$

This means that the integral is finite for any $s > s_0$, if and only if $1 + \alpha > 1$, condition ensured by imposing $\alpha > 0$. Finally, estimate (2.35) applied to estimate (2.34) gives that

$$|y(s)| \leq C_{q,\alpha}(|y(s_0)| + \frac{2}{s_0}|x(s_0)|)$$

where $C_{q,\alpha} := C\|(1+|s|)^{2+\alpha} q\|_{L^\infty[s_0,+\infty)} \int_{s_0}^{\infty} |\tau|^{-1-\alpha} d\tau$. Which concludes that $y \in L^\infty[s_0, +\infty)$.

On the other hand, we readily see that $y(s)$ tends to a limit as s approaches to infinity, because from (2.33) follows that for any $s_1 > s_2 \geq s_0 > 0$:

$$|y(s_1) - y(s_2)| \leq |x(s_0)| \left(\frac{1}{s_2} - \frac{1}{s_1}\right) + C \int_{s_2}^{s_1} |q(\tau)|\tau d\tau$$

$$\Rightarrow \quad \lim_{s_1 \to \infty} |y(s_1)| \leq |y(s_2)| + \frac{|x(s_0)|}{s_2} + C \int_{s_2}^{+\infty} |q(\tau)|\tau d\tau$$

implying that $y(+\infty) \in \mathbb{R}$. Moreover, same formula (2.33) gives a explicit expression for this number

$$y(+\infty) = y(s_0) + \frac{x(s_0)}{s_0} + \int_{s_0}^{+\infty} y(\tau)q(\tau)\tau^2 \left(\frac{1}{\tau} - 0\right) d\tau$$

which allows $y(s)$ to be written as

$$y(s) - y(+\infty) = -\frac{x(s_0)}{s_0} - \int_{s_0}^{s} y(\tau)q(\tau)\frac{\tau^2}{s} d\tau - \int_{s}^{+\infty} y(\tau)q(\tau)\tau d\tau$$

In particular, by choosing the constants to be $y(+\infty) = 1,\ x(s_0) = 0$, we finally deduce

$$y(s) = 1 - \int_{s_0}^{s} y(\tau)q(\tau)\frac{\tau^2}{s} d\tau - \int_{s}^{+\infty} y(\tau)q(\tau)\tau d\tau \tag{2.36}$$

Additionally, the derivative $y'(s) = v'(s)$ can be obtained from $x(s)$ using relation (2.32), as

$$v'(s) = \frac{x(s)}{s^2} = \frac{0}{s^2} + \frac{1}{s^2}\int_{s_0}^{s} q(\xi)\xi^2 y(\xi)d\xi \tag{2.37}$$

Now that $y(s)$ is bounded in $[s_0, +\infty)$, similar arguments as shown in (2.35) imply the same estimates for the integrals in (2.36)-(2.37), since

$$\left|\int_s^{+\infty} y(\tau)q(\tau)\tau d\tau\right| \leq \|y\|_{L^\infty[s_0,\infty)}\|(1+|\tau|^{2+\alpha})q\|_{L^\infty[s_0,\infty)}\int_s^\infty \frac{d\tau}{|\tau|^{1+\alpha}} \leq \frac{C_{y,q}}{|s|^\alpha} = O(|s|^\alpha)$$

$$\begin{aligned}\left|\int_{s_0}^{s} y(\tau)q(\tau)\frac{\tau^2}{s}d\tau\right| &\leq \|y\|_{L^\infty[s_0,\infty)}\|(1+|\tau|)^{2+\alpha}q\|_{L^\infty[s_0,\infty)}\left(\frac{1}{s}\int_{s_0}^{s}\frac{d\tau}{|\tau|^\alpha}\right)\\ &\leq \frac{C_{y,q}}{s}\left(\frac{1}{|s_0|^{-1+\alpha}} - \frac{1}{|s|^{-1+\alpha}}\right) = O(|s|^{-1} + |s|^{-\alpha})\end{aligned}$$

and then

$$\left|\int_{s_0}^{s} y(\xi)q(\xi)\frac{\xi^2}{s^2}d\xi\right| \leq \frac{C_{y,q}}{s^2}\left(\frac{1}{|s_0|^{-1+\alpha}} - \frac{1}{|s|^{-1+\alpha}}\right) = O(|s|^{-2} + |s|^{-1-\alpha})$$

From all these estimates, we conclude that

$$\begin{aligned} v(s) &= y(s) = 1 + O(|s|^{-1} + |s|^{-\alpha})\\ v'(s) &= O(|s|^{-2} + |s|^{-1-\alpha})\end{aligned}$$

So the asymptotic behavior of the first solution follows, as $\alpha > 0$ and by definition of u:

$$\left.\begin{aligned} u(s) &= s\left(1 + O(|s|^{-1} + |s|^{-\alpha})\right) = s + O(1 + |s|^{1-\alpha})\\ u'(s) &= v(s) + sv'(s) = 1 + O(|s|^{-1} + |s|^{-\alpha})\end{aligned}\right., \quad s >> s_0$$

which finishes the analysis of the profile of the first solution found to equation (2.26).

To conclude, it is possible to choose another solution to the equation (2.26) that is linearly independent to $u(s)$, using reduction of order $\tilde{u}(s) = C(s)u(s)$:

$$\tilde{u}''(s) - q(s)\tilde{u}(s) = 0 \quad \Leftrightarrow \quad u''(s)C(s) + 2u'(s)C'(s) + u(s)C''(s) - q(s)u(s)C(s) = 0$$

multiplying by $u(s)$, and then choosing the constant from integration equal to 1, implies that

$$2u(s)u'(s)C'(s) + u^2(s)C''(s) = 0 \quad \Leftrightarrow \quad \frac{d}{ds}\left(u^2(s)C'(s)\right) = 0 \quad \Leftrightarrow \quad C(s) = \int_s^{+\infty}\frac{1}{u^2(\xi)}d\xi$$

Therefore the second solution is $\tilde{u}(s) = \left(\int_s^\infty u^{-2}(\xi)d\xi\right)\cdot u(s)$.
Note that

$$|C(s)| \leq \int_s^{+\infty}\frac{d\xi}{\xi^2} = O(|s|^{-1}), \quad |C'(s)| = \frac{1}{u^2(s)} = O(|s|^{-2})$$

and this implies the asymptotic behavior of $\tilde{u}$:

$$\begin{aligned} \tilde{u}(s) &= C(s)u(s) = 1 + O(|s|^{-1} + |s|^{-\alpha}) \\ \tilde{u}'(s) &= C'(s)u(s) + C(s)u'(s) = O(|s|^{-1} + |s|^{-2} + |s|^{-1-\alpha}) \end{aligned} \quad , \qquad \text{for } s >> s_0$$

which concludes the proof of Lemma 2. $\square$

Now proceed to state the main result of this section, which characterize the profile of the kernel of the Jacobi operator.

Proposition 1. *Given $\alpha > 0$, let $\Gamma \subset \mathbb{R}^2$ be a non-degenerate curve as in definition 2. Suppose additionally that Γ and $a(s,t)$ satisfy the polynomial decay*

$$|\partial_{\boldsymbol{tt}} a(s,0)| + |\partial_{\boldsymbol{ss}} a(s,0)| \leq \frac{C}{(1+|s|)^{2+\alpha}} \ , \quad |k(s)| + |\partial_{\boldsymbol{s}} a(s,0)| + |\partial_{\boldsymbol{t}} a(s,0)| \leq \frac{C}{(1+|s|)^{1+\alpha/2}} \tag{2.38}$$

and that additionally the potential stabilizes on the curve at infinity, namely

$$a(\pm\infty, 0) := \lim_{s \to \pm\infty} a(s,0) \in \mathbb{R} \setminus \{0\} \tag{2.39}$$

Then there are two linearly independent smooth solutions h_1, h_2 in the kernel of the Jacobi operator, which have the following one-sided asymptotic behavior

$$\begin{aligned} h_i(s) &= |s| + O(1) + O(|s|^{-1} + |s|^{-\alpha}) \\ h_i'(s) &= O(1) + O(|s|^{-1} + |s|^{-1-\alpha}) \end{aligned} \ , \qquad as \quad (-1)^i s \to +\infty \tag{2.40}$$

and they are just bounded functions on the opposite side of $\mathbb{R}$, respectively. Furthermore, in the region where the latter happens, it holds

$$|h_i(s)| + (1 + |s|^{1+\alpha})|h_i'(s)| \leq C, \qquad as \quad (-1)^{i+1} s \to +\infty \tag{2.41}$$

Proof.-
Consider the Jacobi equation

$$h''(s) + \frac{\partial_{\boldsymbol{s}} a(s,0)}{a(s,0)} h'(s) - Q(s)h(s) = 0, \qquad \text{in } \mathbb{R}$$

and choose the function $h(s)$ to be the product $h(s) = a(s,0)^{-1/2} \cdot u(s)$, to eliminate the first-order differential term of $\mathcal{J}_a[h]$. More explicitly, the Jacobi equation amounts to

$$\frac{d^2}{ds^2}(a(s,0)^{-1/2})u(s) + a(s,0)^{-1/2}u''(s) + \left(\frac{-2}{2} \frac{\partial_{\boldsymbol{s}} a(s,0)}{a(s,0)^{3/2}} + \frac{\partial_{\boldsymbol{s}} a(s,0)}{a(s,0)} a(s,0)^{-1/2} \right) u'(s)$$
$$- \frac{1}{2} \left| \frac{\partial_{\boldsymbol{s}} a(s,0)}{a(s,0)} \right|^2 u(s) - Q(s)a(s,0)^{-1/2}u(s) = 0$$

which means that u solves the auxiliary equation

$$u''(s) - \tilde{q}(s)u(s) = 0, \quad \text{in} \quad \mathbb{R} \tag{2.42}$$

with

$$\tilde{q}(s) := \left[\frac{\partial_{tt} a(s,0)}{a(s,0)} - 2k^2(s) + \frac{1}{2}\frac{\partial_{ss} a(s,0)}{a(s,0)} - \frac{1}{4}\left|\frac{\partial_s a(s,0)}{a(s,0)}\right|^2 \right]$$

Now, thanks to the hypothesis (2.38) on $a(s,t)$ and Γ of this proposition, plus the fact that $a(s,0)$ is bounded and strictly positive, it follows that $(1+|s|)^{2+\alpha}|\tilde{q}(s)| \le C$. Therefore we can apply Lemma 2 on the region $[0,+\infty)$ and deduce the existence of two solutions linearly independent of equation (2.42) in $\mathbb{R}$, denoted by $u(s)$ and $\tilde{u}(s)$, which satisfies the right-sided asymptotic behavior as $s \to +\infty$

$$\begin{aligned} u(s) &= s + O(1) + O(|s|^{1-\alpha}), & \tilde{u}(s) &= 1 + O(|s|^{-1} + |s|^{-\alpha}) \\ u'(s) &= 1 + O(|s|^{-1} + |s|^{-\alpha}), & \tilde{u}'(s) &= O(|s|^{-1} + |s|^{-1-\alpha}) \end{aligned} \tag{2.43}$$

Applying Lemma 2 again, but this time on the region $(-\infty,0]$, we obtain two other solutions $v(s)$ and $\tilde{v}(s)$ linearly independent of equation (2.42) in $\mathbb{R}$, that now satisfy the left-sided asymptotic behavior as $s \to -\infty$

$$\begin{aligned} v(s) &= |s| + O(1) + O(|s|^{1-\alpha}), & \tilde{v}(s) &= 1 + O(|s|^{-1} + |s|^{-\alpha}) \\ v'(s) &= 1 + O(|s|^{-1} + |s|^{-\alpha}), & \tilde{v}'(s) &= O(|s|^{-1} + |s|^{-1-\alpha}) \end{aligned} \tag{2.44}$$

The main idea for what comes next, is to build two elements h_1, h_2 in the kernel of the Jacobi operator, just as stated in the beginning of this proposition, by making use of the functions $u(s), \tilde{u}(s)$ whose existence was already proved.

Let us note first that the non-degeneracy property of curve Γ implies that $\tilde{u}(s)$ cannot be bounded on $(-\infty,0]$. This follows as $a(s,z)$ is bounded, and then $\tilde{h}(s) := a(s,0)^{-1/2}\tilde{u}(s)$ would be a globally bounded element of the Kernel of $\mathcal{J}_a$, which would be nontrivial. Therefore $\tilde{u}(s)$ must diverge as s approaches to $-\infty$. However, in order to control the growth rate at which $\tilde{u}(s)$ departs from zero, we will use the following argument.

Recall that $\{u,\tilde{u}\}$ and $\{v,\tilde{v}\}$ represent two different basis of the vector space of solutions to the equation (2.42). We can take advantage of this fact to describe the behavior of the pair $\{u,\tilde{u}\}$ on the other region where there is no information available yet. More precisely, there exists coefficients $\{\alpha_i\}_{i=1}^4 \subset \mathbb{R}$ such that

$$\forall s \in \mathbb{R}: \quad u(s) = \alpha_1 v(s) + \alpha_2 \tilde{v}(s), \quad \tilde{u}(s) = \alpha_3 v(s) + \alpha_4 \tilde{v}(s) \tag{2.45}$$

In particular, this implies the asymptotic behavior of $u(s)$ and $\tilde{u}(s)$ on the portion $(-\infty,0]$

$$\begin{aligned} u(s) &= \alpha_1|s| + \alpha_2 + O(1) + O(|s|^{-1} + |s|^{-\alpha} + |s|^{1-\alpha}) \\ \tilde{u}(s) &= \alpha_3|s| + \alpha_4 + O(1) + O(|s|^{-1} + |s|^{-\alpha} + |s|^{1-\alpha})' \end{aligned} \quad \text{as } s \to -\infty \tag{2.46}$$

and its derivatives

$$\begin{aligned} u'(s) &= \alpha_1 + O(|s|^{-1} + |s|^{-1-\alpha} + |s|^{-\alpha}) \\ \tilde{u}'(s) &= \alpha_3 + O(|s|^{-1} + |s|^{-1-\alpha} + |s|^{-\alpha})' \end{aligned} \quad \text{as } s \to -\infty \tag{2.47}$$

From the previous discussion about $\tilde{u}$, it is straightforward not only that $\tilde{u}$ grows at most at a linear rate on $(-\infty, 0]$, but also that the non-degeneracy property implies $\alpha_3 \neq 0$. Furthermore, as $\mathcal{J}_a$ is a linear, we can find a function $h_1(s) := \alpha_3^{-1} a(s,0)^{-1/2}\tilde{u}(s)$ that belongs to the kernel of this operator, with an asymptotic expansion similar to those of $\tilde{u}$ on $[0, +\infty)$, but with a leading term $1 \cdot |s|$ on the left region.

Likewise, the same argument can be applied to $u(s)$, and begin with an unbounded element of the kernel $h_2(s) := a(s,0)^{-1/2}u(s)$ that diverges at a linear rate on the positive portion of $\mathbb{R}$, due to (2.43). Nonetheless, this time there is no information about the growth rate of h_2 as s approaches to $-\infty$, since the bounded behavior now is permitted. Without loss of generality we can take α_1 to be zero, because if α_1 were not zero in (2.45), we can set $h_2(s) := a(s,0)^{-1/2}u(s) - \alpha_1 h_1(s)$ which gives a solution of $\mathcal{J}_a$ such that $h_2(s) = O(1) + O(|s|^{-1} + |s|^{-\alpha} + |s|^{1-\alpha})$ as $s \to -\infty$ and also $h_2(s) = s + O(|s|^{-1} + |s|^{-\alpha} + |s|^{1-\alpha})$ as $s \to +\infty$.

As a consequence of this argument, follows that the derivatives of functions $h_1(s)$ and $h_2(s)$ satisfies estimates (2.40), inherited from the behavior of u, $\tilde{u}$ and its derivatives, plus the decay of the potential $a(s,0)$ and $\partial_{\boldsymbol{s}} a(s,0)$. More explicitly in the case of h_1, we have by its the definition that

$$h_1'(s) = \frac{-1}{2}\frac{\partial_{\boldsymbol{s}} a(s,0)}{a(s,0)}\frac{\tilde{u}(s)}{\alpha_3} + \frac{1}{\sqrt{a(s,0)}}\frac{\tilde{u}'(s)}{\alpha_3}$$

then using expansions (2.46)-(2.47) for $\tilde{u}, \tilde{u}'$ we deduce for $s << s_0 < 0$ large enough

$$h_1'(s) = O(|s|^{-2-\alpha})(|s| + O(1+|s|^{-1}+|s|^{-\alpha})) + O(1)O(1+|s|^{-1}+|s|^{-1-\alpha}) = O(1+|s|^{-1}+|s|^{-1-\alpha})$$

The case of h_2 for which $s >> s_0 > 0$ is large enough, is completely analogue.

Regardless of how are the profiles of functions $u(s), \tilde{u}(s)$, it is always possible to build two solutions linearly independent $h_1(s), h_2(s)$ of $\mathcal{J}_a[h] = 0$, which behave asymptotically as a non-constant straight line on one side, and as a constant on the opposite side. This implies directly the estimate $|h_i| \leq C$, as $(-1)^{i+1}s \to +\infty$. Additionally, we can deduce some estimates for $h_1'(s)$ and $h_2'(s)$ implied by the expansions (2.43) and (2.44), so it is easy to see that $|h_i'| \leq C$ as $(-1)^{i+1}s \to +\infty$.
Note on the other hand, that every solution to $\mathcal{J}_a[h] = 0$ also solves the following auxiliary equation

$$\frac{d}{ds}\left(a(s,0)\frac{d}{ds}h(s)\right) - a(s,0)Q(s)h(s) = 0$$

In particular, coefficients $p(s) := a(s,0)$ and $q(s) := a(s,0)Q(s)$ satisfies properties (2.17) to (2.21), since a and Γ meet the required hypothesis (2.38)-(2.39).

Finally, given that $\alpha > -1$ and since we are considering the bounded part of h_i on that region, Lemma 1 gives to us some the decay of the derivative by $|h_i'(s)| \leq C|s|^{-1-\alpha}$ on the infinite interval where $(-1)^{i+1}s \to +\infty$. This completes the proof of Proposition 1. □

2.2.2. Invertibility: Proposition 2

Once we describe the kernel of the Jacobi operator, the next step is studying the solvability of the Jacobi equation for a right-hand side decaying polynomially.

Proposition 2. *Invertibility of the Jacobi Operator*
Given $\alpha > 0$, $\lambda \in (0,1)$ and f with $\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} < +\infty$, assume that Γ is a non-degenerate stationary curve with respect to $l_{a,\Gamma}$, as in definition 2. Further, suppose that Γ and $a(s,t)$ fulfil the hypothesis of Proposition 1, so that $|Q(\bar{s})| \leq C(1+|\bar{s}|)^{-2-\alpha}$. Then the Jacobi equation $\mathcal{J}_a[h](\bar{s}) = f(\bar{s})$ in $\mathbb{R}$ has a unique bounded solution. Further, h is given by the variation of parameters formula:

$$h(\bar{s}) = -h_1(\bar{s}) \int_{-\infty}^{\bar{s}} a(\xi,0)h_2(\xi)f(\xi)d\xi - h_2(\bar{s}) \int_{\bar{s}}^{+\infty} a(\xi,0)h_1(\xi)f(\xi)d\xi \tag{2.48}$$

In addition, there is some positive constant $C = C(a,\Gamma,\alpha)$ such that

$$\|h\|_{L^\infty(\mathbb{R})} + \|(1+|s|)^{1+\alpha}h'\|_{L^\infty(\mathbb{R})} + \sup_{s\in\mathbb{R}}(1+|s|)^{2+\alpha}\|h''\|_{C^{0,\lambda}(s-1,s+1)} \leq C\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} \tag{2.49}$$

Proof.-
Let us begin by proving estimate (2.49), assuming that there is a solution to $\mathcal{J}_a[h] = f$. Recall that h_1, h_2 are unbounded solutions in the kernel of $\mathcal{J}_a$ which have the asymptotic behavior described in (2.40)-(2.41) of Proposition 1. As h is given by an explicit expression in terms of f, we can use the formula (2.48) to estimate this function

$$\begin{aligned} |h(\bar{s})| &\leq |h_1(\bar{s})| \int_{-\infty}^{\bar{s}} |a(\xi,0)|\ |h_2(\xi)|\ |f(\xi)|d\xi + |h_2(\bar{s})| \int_{\bar{s}}^{+\infty} |a(\xi,0)|\ |h_1(\xi)|\ |f(\xi)|d\xi \\ &\leq C_{a,f,\alpha}\Bigg(\underbrace{|h_1(\bar{s})| \int_{-\infty}^{\bar{s}} \frac{|h_2(\xi)|}{1+|\xi|^{2+\alpha}}d\xi}_{I} + \underbrace{|h_2(\bar{s})| \int_{\bar{s}}^{+\infty} \frac{|h_1(\xi)|}{1+|\xi|^{2+\alpha}}d\xi}_{II}\Bigg) \end{aligned}$$

with $C_{a,f,\alpha} := \|a(\xi,0)\|_\infty \|(1+|\xi|)^{2+\alpha}f\|_\infty < \infty$. We want to show that I-II remain bounded. Then for $\bar{s} >> \bar{s}_0 > 0$ sufficiently large, we have thanks to (2.40)-(2.41) that

$$|I| \leq O(1)\left(\int_{-\infty}^{\bar{s}_0} \frac{O(1)}{1+|\xi|^{2+\alpha}}d\xi + \int_{\bar{s}_0}^{\bar{s}} \frac{|\xi|+O(1)}{1+|\xi|^{2+\alpha}}d\xi\right) \leq O(1)O(1+|\bar{s}|^{-\alpha}+1) = O(1+|\bar{s}|^{-\alpha})$$

$$|II| \leq (|\bar{s}|+O(1)) \int_{\bar{s}}^{+\infty} \frac{O(1)}{1+|\xi|^{2+\alpha}}d\xi \leq (|\bar{s}|+O(1))O(|\bar{s}|^{-1-\alpha}) = O(|\bar{s}|^{-\alpha}+|\bar{s}|^{-1-\alpha})$$

which are both bounded as $\bar{s} \to +\infty$, since $\alpha > 0$.

In the same way, for $\bar{s} << -\bar{s}_0 < 0$ sufficiently large, it holds that

$$|I| \leq (|\bar{s}| + O(1)) \int_{-\infty}^{\bar{s}} \frac{O(1)}{1+|\xi|^{2+\alpha}} d\xi \leq (|\bar{s}| + O(1)) O(|\bar{s}|^{-1-\alpha}) = O(|\bar{s}|^{-\alpha} + |\bar{s}|^{-1-\alpha})$$
$$|II| \leq O(1) \left(\int_{\bar{s}}^{-\bar{s}_0} \frac{|\bar{s}| + O(1)}{1+|\xi|^{2+\alpha}} d\xi + \int_{-\bar{s}_0}^{+\infty} \frac{O(1)}{1+|\xi|^{2+\alpha}} d\xi \right) \leq O(1) O(1 + |\bar{s}|^{-\alpha} + 1) = O(1 + |\bar{s}|^{-\alpha})$$

whose terms don't diverge as $\bar{s} \to -\infty$, given that $\alpha > 0$.
Additionally, note that $\|(1+|\bar{s}|)^{2+\alpha} f\|_\infty \leq \sup_{\bar{s}\in\mathbb{R}} (1+|\bar{s}|)^{2+\alpha} \|f\|_{C^{0,\gamma}(\bar{s}-1,\bar{s}+1)}$. In summary, all this analysis lead us to deduce that h is bounded, and that its norm L^∞ is less or equal than a constant times the norm $C^{0,\gamma}_{2+\alpha,*}$ of f.

For what comes next, it will be necessary to discuss some properties of locally Hölder functions. In particular, we are interested in bounding the norm $C^{0,\lambda}$ of the product function $u(\bar{s})v(\bar{s})$ where $u, v \in C^{0,\lambda}$ are both locally Hölder functions. Defining $I_{\bar{s}} := (\bar{s}-1, \bar{s}+1)$

$$\begin{aligned}
|u(\bar{s}_1)v(\bar{s}_1) - u(\bar{s}_2)v(\bar{s}_2)| &\leq |u(\bar{s}_1)|\ |v(\bar{s}_1) - v(\bar{s}_2)| + |u(\bar{s}_1) - u(\bar{s}_2)|\ |v(\bar{s}_1)| \\
&\leq (\|u\|_{L^\infty(I_{\bar{s}})}[v]_{0,\gamma,I_{\bar{s}}} + [u]_{0,\gamma,I_{\bar{s}}}\|v\|_{L^\infty(I_{\bar{s}})})|\bar{s}_1 - \bar{s}_2|^\gamma \\
&\leq 2\|u\|_{C^{0,\gamma}(I_{\bar{s}})}\|v\|_{C^{0,\gamma}(I_{\bar{s}})}|\bar{s}_1 - \bar{s}_2|^\gamma
\end{aligned}$$

Thus we deduce

$$\|u \cdot v\|_{C^{0,\gamma}(\bar{s}-1,\bar{s}+1)} \leq 2\|u\|_{C^{0,\gamma}(\bar{s}-1,\bar{s}+1)}\|v\|_{C^{0,\gamma}(\bar{s}-1,\bar{s}+1)} \tag{2.50}$$

Now, from the variation of parameter formula follows that h is locally a Hölder function. Additionally, the Hölder norm of h is bounded by those of f. To see this, let us analyze first

$$\rho_1(\bar{s}) := h_1(\bar{s}) \int_{-\infty}^{\bar{s}} a(\xi, 0) h_2(\xi) f(\xi) d\xi$$

As a consequence of (2.50), we only need to bound the Hölder norm of the integral, in terms of those of f, since Proposition 1 ensures that h_1 is locally Hölder. In fact,

$$\left\| \int_{-\infty}^{\bar{s}} a(\xi,0)h_2(\xi)f(\xi)d\xi \right\|_{L^\infty(I_{\bar{s}})} \leq \|a(\xi,0)\|_\infty \|(1+|\xi|)^{2+\alpha} f\|_\infty \begin{cases} O(1+|\bar{s}|^{-\alpha}) & \text{as } \bar{s} \to +\infty \\ O(|\bar{s}|^{-1-\alpha}) & \text{as } \bar{s} \to -\infty \end{cases}$$

$$\begin{aligned}
\left[\int_{-\infty}^{\bar{s}} a(\xi,0)h_2(\xi)f(\xi)d\xi \right]_{0,\lambda,I_{\bar{s}}} &\leq \sup_{\bar{s}_1,\bar{s}_2 \in I_{\bar{s}}} |\bar{s}_2 - \bar{s}_1|^{-\lambda} \int_{\bar{s}_1}^{\bar{s}_2} a(\xi,0)h_2(\xi)f(\xi)d\xi \\
&\leq 2 \sup_{\bar{s}_1,\bar{s}_2 \in I_{\bar{s}}} |\bar{s}_2 - \bar{s}_1|^{1-\lambda} \cdot \sup_{\tau \in I_{\bar{s}}} a(\tau,0)|h_2(\tau)|\ |f(\tau)| \\
&\leq 2^{2-\lambda} \|a(\tau,0)\|_\infty \|\tau^{-1}h_2\|_\infty \|(1+|\tau|)^{2+\alpha} f\|_\infty O(|\bar{s}|^{-1-\alpha})
\end{aligned}$$

Therefore,

$$\left\| \int_{-\infty}^{\bar{s}} a(\xi,0)h_2(\xi)f(\xi)d\xi \right\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)}$$

$$\leq C_{a,h_2,\alpha} \sup_{\bar{s}\in\mathbb{R}}(1+|\bar{s}|)^{2+\alpha}\|f\|_{C^{0,\gamma}(\bar{s}-1,\bar{s}+1)} \cdot \begin{cases} O(1+|\bar{s}|^{-\alpha}) & \text{as } \bar{s}\to+\infty \\ O(|\bar{s}|^{-1-\alpha}) & \text{as } \bar{s}\to-\infty \end{cases} \tag{2.51}$$

Nonetheless, from Proposition 1 it easy to see that

$$\|h_1\|_{C^{0,\gamma}(\bar{s}-1,\bar{s}+1)} \leq \begin{cases} O(1) & \text{as } \bar{s}\to+\infty \\ O(|\bar{s}|) & \text{as } \bar{s}\to-\infty \end{cases}$$

In this way, using (2.50) we deduce that

$$\|\rho_1(\bar{s})\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)} \leq C_{a,h_2,\alpha}\ Z_{h_1}(\bar{s})\|f\|_{C^{0,\gamma}_{2+\alpha,*}(\mathbb{R})}$$

where $Z_{h_1}(\bar{s}) := \|h_1\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)}\| \int_{-\infty}^{\bar{s}} a(\xi,0)h_2(\xi)f(\xi)d\xi\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)}$, is bounded on $\bar{s}\in\mathbb{R}$.

Repeating the very same arguments lead that function

$$\rho_2(\bar{s}) := h_2(\bar{s}) \int_{\bar{s}}^{\infty} a(\xi,0)h_1(\xi)f(\xi)d\xi$$

is also Hölder on any interval $I_{\bar{s}} = (\bar{s}-1,\bar{s}+1)$, with norm $C^{0,\lambda}$ bounded by those of f. In conclusion, gathering the estimates for ρ_1 and ρ_2, we proved that for any $s\in\mathbb{R}$, it holds

$$\|h_1\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)} \leq C_{a,h_1,h_2,\alpha}(Z_{h_1}(\bar{s})+Z_{h_2}(\bar{s}))\|f\|_{C^{0,\gamma}_{2+\alpha,*}(\mathbb{R})} \leq \tilde{C}_{a,h_1,h_2,\alpha}\|f\|_{C^{0,\gamma}_{2+\alpha,*}(\mathbb{R})}$$

We proceed now to find an bound for the derivative $h'(s)$. Let us note that

$$h'(\bar{s}) = \underbrace{h_1'(\bar{s})\int_{-\infty}^{\bar{s}} a(\xi,0)h_2(\xi)f(\xi)d\xi}_{III} - \underbrace{h_2'(\bar{s})\int_{\bar{s}}^{+\infty} a(\xi,0)h_1(\xi)f(\xi)d\xi}_{IV}$$
$$+ h_1(\bar{s})\cdot a(\bar{s},0)h_2(\bar{s})f(\bar{s}) + h_2(\bar{s})\cdot a(\bar{s},0)h_1(\bar{s})f(\bar{s}) \tag{2.52}$$

so by estimating the terms involving integrals, we obtain:

$$|III| \leq |h_1'(\bar{s})|\ \|a(\bar{s},0)\|_\infty\ \|(1+|\xi|)^{2+\alpha}f\|_\infty \int_{-\infty}^{\bar{s}} \frac{|h_2(\xi)|}{1+|\xi|^{2+\alpha}}d\xi$$
$$|IV| \leq |h_2'(\bar{s})|\ \|a(\bar{s},0)\|_\infty\ \|(1+|\xi|)^{2+\alpha}f\|_\infty \int_{\bar{s}}^{+\infty} \frac{|h_1(\xi)|}{1+|\xi|^{2+\alpha}}d\xi$$

For $\bar{s} >> \bar{s}_0 > 0$ sufficiently large, conditions (2.40)-(2.41) imply

$$|III| \leq O(|\bar{s}|^{-1-\alpha}) \left(\int_{-\infty}^{\bar{s}_0} \frac{O(1)}{1+|\xi|^{2+\alpha}} d\xi + \int_{\bar{s}_0}^{\bar{s}} \frac{|\bar{s}|+O(1)}{1+|\xi|^{2+\alpha}} d\xi \right) = O(|\bar{s}|^{-1-\alpha}) O(1+|\bar{s}|^{-\alpha})$$
$$= O(|\bar{s}|^{-1-\alpha} + |\bar{s}|^{-1-2\alpha})$$
$$|IV| \leq O(|\bar{s}|^{-1} + |\bar{s}|^{-1-\alpha}) \int_{\bar{s}}^{+\infty} \frac{O(1)}{1+|\xi|^{2+\alpha}} d\xi = O(|\bar{s}|^{-1} + |\bar{s}|^{-1-\alpha}) O(|\bar{s}|^{-1-\alpha})$$
$$= O(|\bar{s}|^{-2-\alpha} + |\bar{s}|^{-2-2\alpha})$$
$$|a(\bar{s},0)h_1(\bar{s})h_2(\bar{s})f(\bar{s})| \leq \|a(\bar{s},0)\|_\infty \|(1+|\bar{s}|)^{2+\alpha} f\|_\infty \frac{O(1)(|\bar{s}|+O(1))}{1+|\bar{s}|^{2+\alpha}}$$
$$= O(|\bar{s}|^{-1-\alpha} + |\bar{s}|^{-2-\alpha})$$

Analogously, for $\bar{s} << -\bar{s}_0 < 0$ sufficiently large, it holds that for $\alpha > 0$

$$|III| \leq O(1+|\bar{s}|^{-1} + |\bar{s}|^{-1-\alpha}) \int_{-\infty}^{\bar{s}} \frac{O(1)}{1+|\xi|^{2+\alpha}} d\xi = O(1+|\bar{s}|^{-1} + |\bar{s}|^{-1-\alpha}) O(|\bar{s}|^{-1-\alpha})$$
$$= O(|\bar{s}|^{-1-\alpha} + |\bar{s}|^{-2-\alpha})$$
$$|IV| \leq O(|\bar{s}|^{-1-\alpha}) \left(\int_{\bar{s}}^{-\bar{s}_0} \frac{|\bar{s}|+O(1)}{1+|\xi|^{2+\alpha}} d\xi + \int_{-\bar{s}_0}^{\infty} \frac{O(1)}{1+|\xi|^{2+\alpha}} d\xi \right) = O(|\bar{s}|^{-1-\alpha}) O(|\bar{s}|^{-\alpha} + 1)$$
$$= O(|\bar{s}|^{-1-\alpha} + |\bar{s}|^{-1-2\alpha})$$
$$|a(\bar{s},0)h_1(\bar{s})h_2(\bar{s})f(\bar{s})| \leq \|a(\bar{s},0)\|_\infty \|(1+|\bar{s}|)^{2+\alpha} f\|_\infty \frac{(|\bar{s}|+O(1))O(1)}{1+|\bar{s}|^{2+\alpha}}$$
$$= O(|\bar{s}|^{-1-\alpha} + |\bar{s}|^{-2-\alpha})$$

In brief, from this analysis we deduce not only that h' and $|\bar{s}|^{1+\alpha} h'(s)$ are bounded, but also that its L^∞ norms are less or equal than the respective norm of f.

Now, from formula (2.52) follows that h' is locally a Hölder function. Moreover, we claim that h' is bounded in the Hölder norm by those of f. To justify this, we study each term of h'. The first two of are easily bounded in $C^{0,\lambda}$ norm just like it was done before, when we proved the bound for h. In particular for the first term, just note that from Proposition 1

$$\|h_1'\|_{C^{0,\gamma}(\bar{s}-1,\bar{s}+1)} \leq \begin{cases} O(|s|^{-1-\alpha}) & \text{as } s \to +\infty \\ O(1+|s|^{-1}) & \text{as } s \to -\infty \end{cases}$$

and we also have the validity of estimate (2.51) of the integral. Hence, we deduce

$$(1+|\bar{s}|)^{1+\alpha} \|h_1'\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)} \left\| \int_{-\infty}^{\bar{s}} a(\xi,0) h_2(\xi) f(\xi) d\xi \right\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)} \leq$$
$$C_{a,h_2,\alpha} \sup_{\bar{s}\in\mathbb{R}} (1+|\bar{s}|)^{2+\alpha} \|f\|_{C^{0,\gamma}(\bar{s}-1,\bar{s}+1)} \cdot \begin{cases} O(1+|\bar{s}|^{-\alpha}) & \text{as } \bar{s} \to +\infty \\ O(1+|\bar{s}|^{-1}) & \text{as } \bar{s} \to -\infty \end{cases}$$

The very same argument works to prove that the second term IV of h' exhibits a similar estimate. In addition, the last two terms can be bounded just by iterating property (2.50), in order to write the norm of the product function:

$$(1+|s|)^{1+\alpha}\|a(s,0)h_1(s)h_2(s)f(s)\|_{C^{0,\lambda}(s-1,s+1)} \\ \leq \|a(s,0)\|_{C^{0,\gamma}(\mathbb{R})}\ Z(h_1,h_2)(s)\ \sup_{s\in\mathbb{R}}(1+|s|)^{2+\alpha}\|f\|_{C^{0,\lambda}(s-1,s+1)}$$

where $Z(h_1,h_2)(s) := |s|^{-1}\|h_1\|_{C^{0,\lambda}(s-1,s+1)}\|h_2\|_{C^{0,\lambda}(s-1,s+1)}$ is bounded function in $\mathbb{R}$.
The consequence of all these estimates previously shown, is that

$$(1+|s|)^{1+\alpha}\|h'\|_{C^{0,\lambda}(s-1,s+1)} \leq C_{a,h_1,h_2,\alpha}\|f\|_{C^{0,\gamma}_{2+\alpha,*}(\mathbb{R})}$$

Finally, to conclude the proof of estimate (2.49), we just need to find a locally Hölder bound for the second derivative h'', and prove that it decays polynomially. Note that since h solves the Jacobi equation,

$$h''(s) = -\frac{\partial_{\boldsymbol{s}} a(s,0)}{a(s,0)}h'(s) + Q(s)h(s) + f(s), \quad \forall s\in\mathbb{R}$$

then it follows that h'' is locally Hölder , since the product of two Hölder functions is again a Hölder function, and given that the potential $a(s,t)$ and its derivative $\partial_{\boldsymbol{s}} a(s,0)$, plus $h(s)$-$h'(s)$ and also the right-hand side f, they all satisfy this property. Moreover, we have

$$(1+|\bar{s}|)^{2+\alpha}\|h''\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)} \leq 2(1+|\bar{s}|)\left\|\frac{\partial_{\boldsymbol{s}} a(\bar{s},0)}{a(\bar{s},0)}\right\|_{C^{0,\lambda}(s-1,s+1)}(1+|\bar{s}|)^{1+\alpha}\|h'\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)} \\ + 2(1+|\bar{s}|)^{2+\alpha}\|Q(\bar{s})\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)}\|h\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)} + \|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})}$$

and therefore from the previous estimates found for h and h', we deduce that

$$\sup_{\bar{s}\in\mathbb{R}}(1+|\bar{s}|)^{2+\alpha}\|h''\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)} \leq C_a\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} + C_Q\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} + \|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} \\ \leq C_{a,Q,\alpha}\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})}$$

where $C_a := \sup_{\bar{s}\in\mathbb{R}}(1+|\bar{s}|)\,\|\partial_{\boldsymbol{s}} a/a\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)}\,, C_Q := \sup_{\bar{s}\in\mathbb{R}}(1+|\bar{s}|)^{2+\alpha}\|Q(\bar{s})\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)}$ are finite since $\partial_s a/a$ and Q are smooth functions whose derivatives in s are bounded, which exhibit polynomial decay in $\bar{s}$. This ends the first part of the proof.

On the other hand, to prove the existence and uniqueness of a solution h to the Jacobi equation, note that $\mathcal{J}_a$ is a second-order differential operator that is also linear. Classical theory of ordinary differential equations assures that any solution of $\mathcal{J}_a[h](\bar{s}) = f(\bar{s})$ in $\mathbb{R}$ can be written as the sum

$$h(\bar{s}) = h_H(\bar{s}) + h_P(\bar{s})$$

where h_H corresponds to a solution of the homogeneous equation $\mathcal{J}_a[h](\bar{s}) = 0$, and h_P is any particular solution of the Jacobi equation with right-hand side $f(\bar{s})$.

So the uniqueness property will be a consequence as we are looking for bounded solutions. Since h_H cannot diverge, then the non-degeneracy condition of Γ implies that $h_H(\bar{s}) \equiv 0$. Moreover, the same fact about looking for bounded solution implies that the particular solution h_P will be unique for any $f(\bar{s})$ with $\|(1+|\bar{s}|)^{2+\alpha} f\|_\infty$. Indeed, suppose that there are two solutions $\bar{h}_P$, $\underline{h}_P$ to the equation $\mathcal{J}_a[h] = f$. Then by linearity of $\mathcal{J}_a$ follows that $\hat{h} := \bar{h}_P - \underline{h}_P$ solves the homogeneous equation $\mathcal{J}_a[h] = 0$. However, as both functions are bounded, the non-degeneracy property of Γ ensures that $\hat{h} \equiv 0$, and so $\bar{h}_P = \underline{h}_P$.

The previous argument shows that the Jacobi equation $\mathcal{J}_a[h] = f$, under the non-degeneracy assumption on Γ, possesses at most one bounded solution. So the solvability question reduces to find a globally bounded particular solution h_P.

We only need to exhibit a suitable h_P, but this can be done since a natural candidate arises from the variation of parameters formula. Recall the relation

$$h_P(\bar{s}) = -h_1(\bar{s}) \int_{\bar{s}_0}^{\bar{s}} \frac{h_2(\xi) f(\xi)}{W(h_1,h_2)(\xi)} d\xi + h_2(\bar{s}) \int_{\bar{s}_1}^{\bar{s}} \frac{h_1(\xi) f(\xi)}{W(h_1,h_2)(\xi)} d\xi$$

gives a particular solution to the equation

$$h''(\bar{s}) + \frac{\partial_{\boldsymbol{s}} a(\bar{s},0)}{a(\bar{s},0)} h'(\bar{s}) - Q(\bar{s}) h(\bar{s}) = f(\bar{s})$$

if h_1, h_2 are two linearly independent solutions of the respective homogeneous equation. But using Abel's formula for the Wronskian of h_1, h_2, we see that

$$W(h_1,h_2)(\xi) = \exp\left(-\int \frac{\partial_{\boldsymbol{s}} a(\tau,0)}{a(\tau,0)} d\tau \right) = \exp(-\log a(\xi,0)) = \frac{1}{a(\xi,0)}$$

Then, formula

$$h_P(\bar{s}) = -h_1(\bar{s}) \int_{\bar{s}_0}^{\bar{s}} a(\xi,0) h_2(\xi) f(\xi) d\xi + h_2(\bar{s}) \int_{\bar{s}_0}^{\bar{s}} a(\xi,0) h_1(\xi) f(\xi) d\xi$$

solves the Jacobi equation, for any $-\infty \le \bar{s}_0, \bar{s}_1 \le +\infty$.

Particularly, given that $\|(1+|\bar{s}|)^{2+\alpha} f\|_\infty$ and the linear behavior of the kernel elements h_1, h_2, plus $\alpha > 0$, it follows $\left| \int_{-\infty}^{\infty} a(\xi,0) h_i(\xi) f(\xi) d\xi \right| < +\infty$ for $i = 1,2$. This allow us to choose the integrals to be evaluated at the initials points $\bar{s}_0 = -\infty$, and $\bar{s}_1 = +\infty$.

Finally by estimate (2.49) we have that the particular solution h_p is globally bounded. This completes the proof of Proposition 2. □

2.3. Examples

Is of interest to mention that Theorem 1 relies on a very important fact, whose nature is essentially geometrical. This concerns the existence of a curve $\Gamma \subset \mathbb{R}^2$, given a fixed suitable potential $a(x)$ of the Allen-Cahn equation (1.12), that be a stationary curve with respect to the weighted length functional $l_{a,\Gamma}$ in the sense of 1, and also be a non-degenerate curve as in definition 2. To get a better understanding of the geometrical settings of this problem, it would be useful to present some examples that portray the nature of the curves and of the potentials, and how they interact in order for this configuration to be admissible.

The following section is devoted in giving concrete examples of such curves associated to some nontrivial potential $a(x)$, in a way that they meet all the hypothesis of Theorem 1.

2.3.1. Characterization of Non-degeneracy

Now we state a result that provides precise conditions on a and Γ, where in such case the non-degeneracy property of the curve holds.

Corollary 1. *Non-degeneracy in the minimizing case*
Let $\alpha > -1/2$, Γ be a stationary curve with respect to $l_{a,\Gamma}$ as in (2.10), and let the potential $a(\bar{s}, t)$ along with Γ be such that $Q(\bar{s}) := \partial_{tt} a(\bar{s}, 0)/a(\bar{s}, 0) - 2k^2(\bar{s})$ satisfies the following conditions

$$Q(\bar{s}) \geq 0, \qquad \textit{and} \qquad Q(\bar{s}) \not\equiv 0 \tag{2.53}$$

and the asymptotic polinomial decay

$$|Q(\bar{s})| \leq \frac{C}{(1+|\bar{s}|)^{2+\alpha}}, \qquad \textit{for} \qquad |\bar{s}| >> \bar{s}_0 \tag{2.54}$$

then the curve Γ is non-degenerate, in the sense of 2.

Proof.-
Let h be a bounded element in the kernel of the Jacobi operator, so that $h \in L^\infty(\mathbb{R})$ solves $\mathcal{J}_a[h](\bar{s}) = 0$ in $\mathbb{R}$. Since $h \in L^\infty[0, +\infty) \cap L^\infty(-\infty, 0]$, Lemma 1 assures the existence of a constant $C > 0$ such that

$$|h'(\bar{s})| \leq \frac{C}{1+|\bar{s}|^{1+\alpha}}, \quad \forall \bar{s} \in \mathbb{R} \tag{2.55}$$

This directly implies that $h' \in L^2(\mathbb{R})$ given that $\alpha > -1/2$. Furthermore, condition (2.54) plus $\alpha > -1/2$ imply the integrability of $Q(s)$ in $\mathbb{R}$ since

$$\int_{\mathbb{R}} |Q(\bar{s})| d\bar{s} = \int_{|\bar{s}|>\bar{s}_0} Q(\bar{s}) d\bar{s} + \int_{|\bar{s}|\leq \bar{s}_0} Q(\bar{s}) d\bar{s} \leq C \int_{|\bar{s}|>\bar{s}_0} \frac{d\bar{s}}{1+|\bar{s}|^{2+\alpha}} + \int_{|\bar{s}|\leq \bar{s}_0} Q(\bar{s}) d\bar{s}$$

In particular, the foregoing guarantees that the following expression is well defined

$$\begin{aligned} \|h\|_Q := \langle \mathcal{J}_a[h], h \rangle_{L^2(a(s,0))} &= \int_{\mathbb{R}} [h''(\bar{s}) + \frac{\partial_{\boldsymbol{s}} a(\bar{s}, 0)}{a(\bar{s}, 0)} h'(\bar{s}) - Q(\bar{s}) h(\bar{s})] h(\bar{s}) a(\bar{s}, 0) d\bar{s} \\ &= a(\bar{s}, 0) h'(\bar{s}) h(\bar{s}) \Big|_{\bar{s}=-\infty}^{\bar{s}=+\infty} - \int_{\mathbb{R}} a(\bar{s}, 0) [|h'(\bar{s})|^2 + Q(\bar{s}) h^2(\bar{s})] d\bar{s} \end{aligned}$$

where the boundary value terms at infinity vanishes, as h is bounded and from the decay (2.55). Finally as h solves $\mathcal{J}_a[h] = 0$, the latter together with hypothesis (2.53) imply

$$0 = \int_{\mathbb{R}} a(\bar{s}, 0)[|h'(\bar{s})|^2 + Q(\bar{s})h^2(\bar{s})]d\bar{s} \geq \int_{\mathbb{R}} a(\bar{s}, 0)|h'(\bar{s})|^2 d\bar{s}$$

As $a(\bar{s}, 0) > 0$, we deduce that $h' = 0$ a.e. in $\mathbb{R}$. Moreover, the smoothness of h guarantees that $h' \equiv 0$ in $\mathbb{R}$. Using again the last inequality we get that

$$0 = \int_{\mathbb{R}} Q(\bar{s})h^2(\bar{s})d\bar{s}$$

However, condition (2.53) on Q assures that $Q(\bar{s}) > 0$ on a neighborhood of some point $\bar{s}_0 \in \mathbb{R}$. Therefore, last equality gives that $h(\bar{s}) = 0$ in this neighborhood, but as $h \equiv C$ is a constant function in the entire space, we conclude $h = 0$, which concludes the proof of Corollary 1. □

2.3.2. Weighted Geodesics in Euclidean coordinates

In what follows, we will admit curves that can be represented as the graph of some function. Let us consider a smooth function $f : \mathbb{R} \to \mathbb{R}$, $f = f(\boldsymbol{x})$, and a parametrized curve $\Gamma := \{\gamma(\boldsymbol{x}) / \ \boldsymbol{x} \in \mathbb{R}\} \subset \mathbb{R}^2$ such that

$$\gamma(\boldsymbol{x}) = (\boldsymbol{x}, f(\boldsymbol{x})), \quad \dot{\gamma}(\boldsymbol{x}) = (1, f'(\boldsymbol{x})) \tag{2.56}$$

In addition we choose the normal ν of Γ oriented negatively, meaning that the wedge product $\dot{\gamma}(\boldsymbol{x}) \times \nu(\boldsymbol{x})$ points in the opposite direction than e_3, the generator of the $z-$axis in $\mathbb{R}^3$. This forces

$$\nu(\boldsymbol{x}) = \frac{1}{\sqrt{1 + |f'(\boldsymbol{x})|^2}}(f'(\boldsymbol{x}), -1)$$

Let us also consider a potential defined in Euclidean coordinates $a = a(\boldsymbol{x}, \boldsymbol{y})$, adopting the convention where $(\boldsymbol{x}, \boldsymbol{y}) := (\bar{x}, \bar{y})$, which satisfies all the hypothesis supposed for it in Chapter 1.

Recall from the criticality condition (2.10), that in order for Γ to be a stationary curve with respect to the weighted arc-length $l_{a,\Gamma}$, is necessary that the potential a and the curvature k satisfy the equation

$$\partial_{\boldsymbol{t}} a(\boldsymbol{s}, 0) = k(\boldsymbol{s}) \cdot a(\boldsymbol{s}, 0) \ , \quad \text{a.e. } \boldsymbol{s} \in \mathbb{R} \tag{2.57}$$

Denoting $X(\boldsymbol{x}, \boldsymbol{t}) := \gamma(\boldsymbol{x}) + \boldsymbol{t}\nu(\boldsymbol{x})$, we can now set the potential written in this coordinates as

$$\tilde{a}(\boldsymbol{x}, \boldsymbol{t}) := a \circ X(\boldsymbol{x}, \boldsymbol{t}) = a\left(\boldsymbol{x} + \frac{\boldsymbol{t}f'(\boldsymbol{x})}{\sqrt{1 + |f'(\boldsymbol{x})|^2}}, \ f(\boldsymbol{x}) - \frac{\boldsymbol{t}}{\sqrt{1 + |f'(\boldsymbol{x})|^2}}\right) \tag{2.58}$$

Accordingly, relation (2.58) implies that the criticality condition (2.57) amounts to the following equation in Euclidean coordinates

$$\frac{\partial_{\boldsymbol{x}} a(\boldsymbol{x}, f(\boldsymbol{x}))f'(\boldsymbol{x})}{\sqrt{1+|f'(\boldsymbol{x})|^2}} - \frac{\partial_y a(\boldsymbol{x}, f(\boldsymbol{x}))}{\sqrt{1+|f'(\boldsymbol{x})|^2}} = \frac{f''(\boldsymbol{x})}{(1+|f'(\boldsymbol{x})|^2)^{3/2}} \cdot a(\boldsymbol{x}, f(\boldsymbol{x})) \tag{2.59}$$

where it has been used the classical formula for the curvature of Γ as given in (2.56),

$$k(\boldsymbol{x}) = f''(\boldsymbol{x})(1+|f'(\boldsymbol{x})|^2)^{-3/2}$$

2.3.3. Example 1: The x-axis

For the sake of simplicity, let us find a some particular kind of stationary curve. We will be interested in finding $\Gamma \subset \mathbb{R}^2$ as a straight line on the Euclidean plane, further, we want this line to be the x-axis. Nonetheless, the stationarity of this line must be with respect to some nontrivial potential $a(\boldsymbol{x}, \boldsymbol{y}) \not\equiv 1$ that does not represent the classic Euclidean metric in $\mathbb{R}^2$, case in which all straight lines are trivially known as stationary curves.

With this purpose, let us set the function $f(\boldsymbol{x}) \equiv 0$ in (2.56), implying that $\Gamma = \overrightarrow{0X}$. In particular, adopting the convention $e_i := (\delta_{i1}, \delta_{i2})$ with δ_{ij} denoting the Kronecker delta, we have on the curve that $\nu(\boldsymbol{x}) \equiv e_2$, thus the Fermi coordinates are reduced simply to the Euclidean coordinates, namely $X(\boldsymbol{x}, \boldsymbol{t}) = \boldsymbol{x}e_1 + \boldsymbol{t}e_2 = (\boldsymbol{x}, \boldsymbol{t})$.

In this simplified context, it turns out that the criticality condition (2.59) is reduced to

$$-\frac{\partial_{\boldsymbol{y}} a(\boldsymbol{x}, 0)}{\sqrt{1+0}} = 0, \quad \forall \boldsymbol{x} \in \mathbb{R} \tag{2.60}$$

Therefore, we only need to find a nontrivial potential $\tilde{a}(\boldsymbol{x}, \boldsymbol{t}) = a(\boldsymbol{x}, \boldsymbol{y})$ in such way the x-axis becomes a stationary curve, and also a nondegenerate curve as in the sense of 2.

Claim 1. *Given any $\alpha > 0$, the following potential*

$$a(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{(1+|\boldsymbol{x}|)^{2+\alpha}} \cdot \left(\frac{\boldsymbol{y}^2}{\cosh(\boldsymbol{y})} \right) + 1 \tag{2.61}$$

satisfies all the requirements previously indicated, in relation with the curve $\Gamma = \overrightarrow{0X}$.

Proof.-
Let us note that $a(\boldsymbol{x}, \boldsymbol{y})$ is smooth, globally bounded, and bounded below far away from zero. Further, it is direct that $\overrightarrow{0X}$ is a stationary curve relative to $l_{a,\Gamma}$ since solves equation (2.60)

$$\partial_{\boldsymbol{y}} a(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{(1+|\boldsymbol{x}|)^{2+\alpha}} \left(\frac{2\boldsymbol{y} - \boldsymbol{y}^2 \sinh(\boldsymbol{y})}{\cosh^2(\boldsymbol{y})} \right) \quad \Rightarrow \quad \partial_{\boldsymbol{y}} a(\boldsymbol{x}, 0) = 0, \quad \forall \boldsymbol{x} \in \mathbb{R}$$

Now to see that $\overrightarrow{0X}$ is a nondegenerate curve, just note that the potential achieves it minimum exactly on the region defined by the $\boldsymbol{x}$-axis, and moreover, around this curve the potential is strictly convex in the $\boldsymbol{y}$-direction. The latter translates in the fact that $\partial_{\boldsymbol{yy}}a(\boldsymbol{x},0)>0$, given

$$\partial_{\boldsymbol{yy}}a(\boldsymbol{x},\boldsymbol{y})=\frac{1}{(1+|\boldsymbol{x}|)^{2+\alpha}}\left(\frac{2-2\boldsymbol{y}\sinh(\boldsymbol{y})-\boldsymbol{y}^2\cosh(\boldsymbol{y})}{\cosh^2(\boldsymbol{y})}-\frac{2(2\boldsymbol{y}-\boldsymbol{y}^2\sinh(\boldsymbol{y}))\sinh(\boldsymbol{y})}{\cosh^3(\boldsymbol{y})}\right)$$

$$\Rightarrow \quad \partial_{\boldsymbol{yy}}a(\boldsymbol{x},0)=\frac{2}{(1+|\boldsymbol{x}|)^{2+\alpha}}>0, \quad \forall \boldsymbol{x}\in\mathbb{R}$$

Taking this into account, note that $a(\boldsymbol{x},\boldsymbol{y})$ and $k(\boldsymbol{x})\equiv 0$ are such that term

$$Q(\boldsymbol{x}):=\frac{\partial_{\boldsymbol{yy}}a(\boldsymbol{x},0)}{a(\boldsymbol{x},0)}-2k^2(\boldsymbol{x})$$

fulfills the following conditions

$$Q(\boldsymbol{x})>0, \qquad \text{and} \qquad |Q(\boldsymbol{x})|\le\frac{2}{(1+|\boldsymbol{x}|)^{2+\alpha}}, \quad \forall \boldsymbol{x}\in\mathbb{R}$$

Hence we deduce from Corollary 1 of section 2.2.2, that $\Gamma=\overrightarrow{0X}$ is a nondegenerate curve with respect to the potential $a(\boldsymbol{x},\boldsymbol{y})$ given in (2.61), finishing the proof of Claim 1. □

Using the software MATLAB v2010, we plot the potential on the square $[-10,10]\times[-10,10]$, and we illustrate in color red the respective stationary curve Γ.

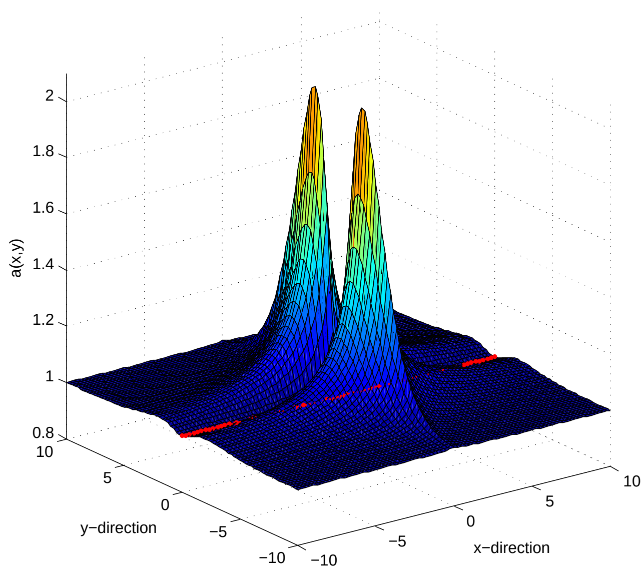

Figure 3: Potential $a(\boldsymbol{x},\boldsymbol{y})$ (2.61) with geodesic $\Gamma=0\vec{X}$, for $\alpha=10^{-2}$.

2.3.4. Example 2: Asymptotic straight line

This time we consider a different type of curve $\Gamma \subset \mathbb{R}^2$. For $\omega \neq 0$, let us set function $f(\boldsymbol{x}) := \sqrt{1+\omega^2 x^2}$, so that Γ converges asymptotically to straight lines as $|\boldsymbol{x}| \to \infty$. We have to exhibit some nontrivial potential $a(\boldsymbol{x},\boldsymbol{y})$ for which Γ be nondegenerate geodesic relative to the arclength $\int_\Gamma a(\vec{x})$. Since this curve is not exactly a straight line, we don't get any simplification of the Fermi coordinates $X(\boldsymbol{x},\boldsymbol{t})$. Therefore, we will assume a weaker dependence of the potential in Euclidean variables, namely $a = a(\boldsymbol{y})$. Note that

$$f'(\boldsymbol{x}) = \frac{\omega^2 \boldsymbol{x}}{\sqrt{1+\omega^2\boldsymbol{x}^2}},\ f''(\boldsymbol{x}) = \frac{\omega^2}{(\sqrt{1+\omega^2\boldsymbol{x}^2})^{3/2}},\ \frac{f''(\boldsymbol{x})}{1+|f'(\boldsymbol{x})|^2} = \frac{\omega^2}{f^3(\boldsymbol{x})+\omega^2 f(\boldsymbol{x})(f^2(\boldsymbol{x})-1)} \tag{2.62}$$

So, given the dependence of a only on $\boldsymbol{y}-$variable, criticality condition (2.59) amounts to

$$\frac{a'(f(\boldsymbol{x}))}{a(f(\boldsymbol{x}))} = \frac{-f''(\boldsymbol{x})}{1+|f'(\boldsymbol{x})|^2} = g(f(x)) \tag{2.63}$$

with $g(\boldsymbol{y}) := -\omega^2[(1+\omega^2)\boldsymbol{y}^3 - \omega^2\boldsymbol{y}]^{-1}$.
We can solve directly this ordinary differential equation (2.63), for a in $\boldsymbol{y}-$variable.

$$\log(a(\boldsymbol{y})) = \int g(\boldsymbol{y})d\boldsymbol{y} + M \quad \Leftrightarrow \quad a(\boldsymbol{y}) = M\exp\left(\int \frac{-\omega^2 d\boldsymbol{y}}{(1+\omega^2)\boldsymbol{y}^3 - \omega^2\boldsymbol{y}}\right)$$

This integral can be computed using partial fraction decomposition, noting the factorization $\boldsymbol{y}^3 - \omega^2/(1+\omega^2)\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{y}-y_+)(\boldsymbol{y}-y_-)$ in which $y_\pm := \pm\omega(\sqrt{1+\omega^2})^{-1}$. Then,

$$\frac{A}{\boldsymbol{y}} + \frac{B}{\boldsymbol{y}-y_+} + \frac{C}{\boldsymbol{y}-y_-} = \frac{(A+B+C)\boldsymbol{y}^2 + (-y_-B - y_+C)\boldsymbol{y} - \omega^2/(1+\omega^2)A}{\boldsymbol{y}^3 - \omega^2/(1+\omega^2)\boldsymbol{y}}$$

which leads to a linear system, solved by $A = 1,\ B = -\frac{1}{2},\ C = -\frac{1}{2}$. Hence we obtain

$$a(\boldsymbol{y}) = M\exp\left(\int\frac{d\boldsymbol{y}}{\boldsymbol{y}} - \int\frac{d\boldsymbol{y}}{2(\boldsymbol{y}-y_+)} - \int\frac{d\boldsymbol{y}}{2(\boldsymbol{y}-y_-)}\right) = \frac{M\boldsymbol{y}}{\sqrt{(\boldsymbol{y}-y_+)(\boldsymbol{y}-y_-)}}$$

For this construction, we will need to consider a slight modification of function a as follows. We say that the potential $\hat{a} : \mathbb{R}^2 \to \mathbb{R}$ is an *admissible left-extension* of function $a(x,y)$, provided that *I*) $\hat{a}$ be smooth bounded function, of at least $C^2(\mathbb{R}^2)$ class. *II*) $\hat{a}(x,y) = a(x,y)$ for points with $y \geq \omega^2/(1+\omega^2)$. *III*) $\hat{a}$ is uniformly positive, bounded below away from zero. We state the following

Claim 2. *Given $|\omega| \leq 1/\sqrt{2}$, any admissible left-extension of the potential given below*

$$a(\boldsymbol{x},\boldsymbol{y}) := \frac{\sqrt{1+\omega^2}\boldsymbol{y}}{\sqrt{(1+\omega^2)\boldsymbol{y}^2 - \omega^2}} \tag{2.64}$$

induces a metric in $\mathbb{R}^2$ for which $\Gamma = \left\{(\boldsymbol{x}, \sqrt{1+\omega^2\boldsymbol{x}^2})\right\}_{\boldsymbol{x}\in\mathbb{R}}$ is a nondegenerate geodesic.

Proof.-
Regardless the value of the parameter $\omega \neq 0$, it can be readily checked that within the region $y \geq 2\omega/\sqrt{1+\omega^2}$, function (2.64) is smooth, bounded, and uniformly positive. Moreover, this potential satisfies the asymptotic stability on the curve Γ, since $f(\boldsymbol{x}) \to +\infty$ as $|\boldsymbol{x}| \to +\infty$ and additionally $\lim_{\boldsymbol{y}\to+\infty} a(\boldsymbol{x},\boldsymbol{y}) = 1$, $\forall \boldsymbol{x} \in \mathbb{R}$. The previous construction of $a(\boldsymbol{x},\boldsymbol{y})$ was intended to build a potential satisfying the criticality condition (2.63) for the curve generated by $f(\boldsymbol{x}) = \sqrt{1+\omega^2 x^2}$. Thus Γ is a geodesic for the arclength $\int_\Gamma a(\vec{x})$. All these features of a ensure that any *admissible left-extension* will provide a potential with the desired properties to induce a smooth metric in $\mathbb{R}^2$, fulfilling hypothesis (1.13) of Theorem 1. Notwithstanding, in order to estimate the derivatives of $\tilde{a}$ we need to compute first

$$\partial_{\boldsymbol{x}}\tilde{a}(\boldsymbol{x},0) = a'(f(\boldsymbol{x}))f'(\boldsymbol{x}), \quad \partial_t\tilde{a}(\boldsymbol{x},0) = -a'(f(\boldsymbol{x}))[1+|f'(\boldsymbol{x})|^2]^{-1}$$
$$\partial_{\boldsymbol{xx}}\tilde{a}(\boldsymbol{x},t) = a''(f(\boldsymbol{x}))|f'(\boldsymbol{x})|^2 + a'(f(\boldsymbol{x}))f''(\boldsymbol{x}), \quad \partial_{\boldsymbol{tt}}\tilde{a}(\boldsymbol{x},t) = (-1)^2 a''(f(\boldsymbol{x}))[1+|f'(\boldsymbol{x})|^2]^{-1}$$
$$\partial_{\boldsymbol{xt}}\tilde{a}(\boldsymbol{x},t) = -a''(f(\boldsymbol{x}))f'(\boldsymbol{x})/[1+|f'(\boldsymbol{x})|^2] - 2a'(f(\boldsymbol{x}))f'(\boldsymbol{x})f''(\boldsymbol{x})/[1+|f'(\boldsymbol{x})|^2]^2$$

Moreover a tedious but simple calculation shows that

$$a'(\boldsymbol{y}) = \frac{-\omega^2\sqrt{1+\omega^2}}{(\boldsymbol{y}^2+\omega^2(\boldsymbol{y}^2-1))^{3/2}}, \quad a''(\boldsymbol{y}) = \frac{3\omega^2(1+\omega^2)^{3/2}\boldsymbol{y}}{[(1+\omega^2)\boldsymbol{y}^2-\omega^2]^{5/2}}$$

Therefore, taking into account the decay (2.62) of $f(\boldsymbol{x})$ and its derivatives, follows that this potential satisfies condition (1.15) of Theorem 1, for $\alpha = 2 > 0$. It only remains to prove the nondegeneracy property of the curve Γ, and to do this, we will make use of Corollary 1. It can be checked the positiveness of the term $Q(\boldsymbol{x})$, in fact

$$2k^2(\boldsymbol{x}) = \frac{2|f''(\boldsymbol{x})|^2}{(1+|f'(\boldsymbol{x})|^2)^{3/2}} = \frac{2\omega^2}{(1+(\omega^2+\omega^4)\boldsymbol{x}^2)^3}, \quad \partial_{\boldsymbol{tt}}\tilde{a}(\boldsymbol{x},0) = a''(f(\boldsymbol{x}))\frac{1+\omega^2\boldsymbol{x}^2}{1+(\omega^2+\omega^4)\boldsymbol{x}^2}$$

so by the definition $Q(\boldsymbol{x}) = \partial_{tt}\tilde{a}(\boldsymbol{x},0)/\tilde{a}(\boldsymbol{x},0) - 2k^2(\boldsymbol{x})$ we obtain

$$Q(\boldsymbol{x}) \geq \min\{1, \|a\|_\infty^{-1}\}\left(\frac{3\omega^2(1+\omega^2)^{3/2}\sqrt{1+\omega^2\boldsymbol{x}^2}}{[(1+\omega^2)(1+\omega^2\boldsymbol{x}^2)-\omega^2]^{5/2}}\cdot\frac{1+\omega^2\boldsymbol{x}^2}{1+(\omega^2+\omega^4)\boldsymbol{x}^2} - \frac{2\omega^2}{(1+(\omega^2+\omega^4)\boldsymbol{x}^2)^3}\right)$$
$$> C_a\left(\frac{3\omega^2(1+\omega^2)^{3/2}(1+\omega^2\boldsymbol{x}^2)^{1/2}}{(1+\omega^2)^{5/2}(1+\omega^2\boldsymbol{x}^2)^{5/2}} - \frac{2\omega^2}{(1+(\omega^2+\omega^4)\boldsymbol{x}^2)^3}\right)$$
$$\geq C_a\left(\frac{3\omega^2}{(1+\omega^2)(1+\omega^2\boldsymbol{x}^2)^2} - \frac{2\omega^2}{(1+\omega^2\boldsymbol{x}^2)^3}\right) = \frac{C_a\omega^2}{(1+\omega^2\boldsymbol{x}^2)^2}\left(\frac{3}{1+\omega^2} - \frac{2}{1+\omega^2\boldsymbol{x}^2}\right)$$

Hence choosing $\omega \in \mathbb{R}\setminus\{0\}$ with $|\omega| \leq 1/\sqrt{2}$, we get that $Q(\boldsymbol{x}) > 0$ in the entire domain $\mathbb{R}$. Finally the term $Q(\boldsymbol{x})$ decays polynomially at a rate $O((1+|\boldsymbol{x}|)^{-4})$ as a consequence of the decay of the potential and the squared curvature, which finishes the proof of Claim 2. □

Remark 4. *We emphasize the fact that the criticality condition for* Γ *and the nondegeneracy property are tested only within the semi-space* $\boldsymbol{y} \geq 1$*, which involve only the part* (2.64) *of the admissible left-extension, since* $\hat{a}(\boldsymbol{x},\boldsymbol{y}) = a(\boldsymbol{y})$ *in this region and the curve complies* $|f(\boldsymbol{x})| \geq 1$.

Using the software MATLAB v2010, we plot an admissible left-extension of the potential on the square $[-10, 10] \times [-10, 10]$, and we illustrate in color red the respective stationary curve $\Gamma_\omega := \left\{ (\boldsymbol{x}, \sqrt{1 + \omega^2 \boldsymbol{x}^2}) : \ \boldsymbol{x} \in \mathbb{R} \right\}$.

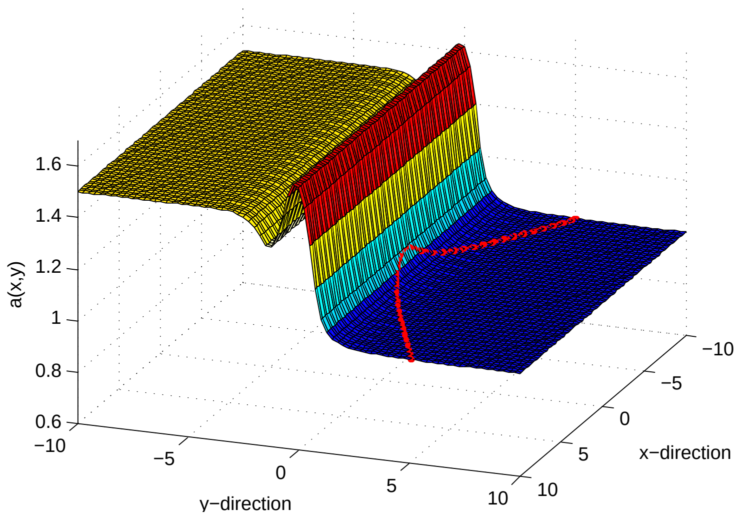

Figure 4: Potential $\hat{a}(\boldsymbol{x}, \boldsymbol{y})$ (2.64) with Γ_ω as nondegenerate geodesic, for $\omega = 1/2$.

2.4. Computing the Laplacian and Gradients in Fermi coordinates

Recall that the inhomogeneous Allen-Cahn equation is given by (1.12). Denoting the nonlinearity by $f(u) := -F'(u)$, and assuming that the potential is uniformly positive, namely $\inf_{\mathbb{R}^2} a \geq \delta_0 > 0$, it follows that this equation amounts to

$$\varepsilon^2 \Delta_{\bar{x}} u + \varepsilon^2 \frac{\nabla_{\bar{x}} a}{a} \cdot \nabla_{\bar{x}} u + f(u) = 0 \quad \text{in} \ \ \mathbb{R}^2$$

Scaling this equation in factor ε^{-1}, we get that $u(x) := u(\varepsilon^{-1} \bar{x})$ solves the differential equation

$$\Delta_x u + \varepsilon \frac{\nabla_{\bar{x}} a}{a} \cdot \nabla_x u + f(u) = 0 \quad \text{in} \ \ \varepsilon^{-1} \mathbb{R}^2 \tag{2.65}$$

This section is mainly oriented to find an expression for the differential operators involved in (2.65), written in some coordinates associated with the curve, known as the Fermi coordinates. These define a local change of variables in a neighborhood of Γ, so our effort will focus on finding an equivalent form for the Allen-Cahn equation in these coordinates for points relatively "close" to Γ.

Let us consider a bounded smooth function $h : \mathbb{R} \to \mathbb{R}$, and define the following local coordinates near $\Gamma_\varepsilon := \varepsilon^{-1}\Gamma$

$$X_{\varepsilon,h}(s,t) = X_\varepsilon(s, t + h(\varepsilon s)) = \frac{1}{\varepsilon}\gamma(\varepsilon s) + (t + h(\varepsilon s)) \cdot \nu(\varepsilon s) \tag{2.66}$$

known as dilated and translated Fermi coordinates, on the region

$$\mathcal{N}_{\varepsilon,h} = \left\{ (x,y) = X_{\varepsilon,h}(s,t) \in \mathbb{R}^2 / \; |t + h(\varepsilon s)| < \frac{\delta}{\varepsilon} + 2c_0|s| \right\} \tag{2.67}$$

for a fixed number $c_0 > 0$, which is a dilated tubular neighborhood of Γ_ε translated in h, in such way $X_{\varepsilon,h}$ defines a local change of variables. The next picture depicts this geometrical setting

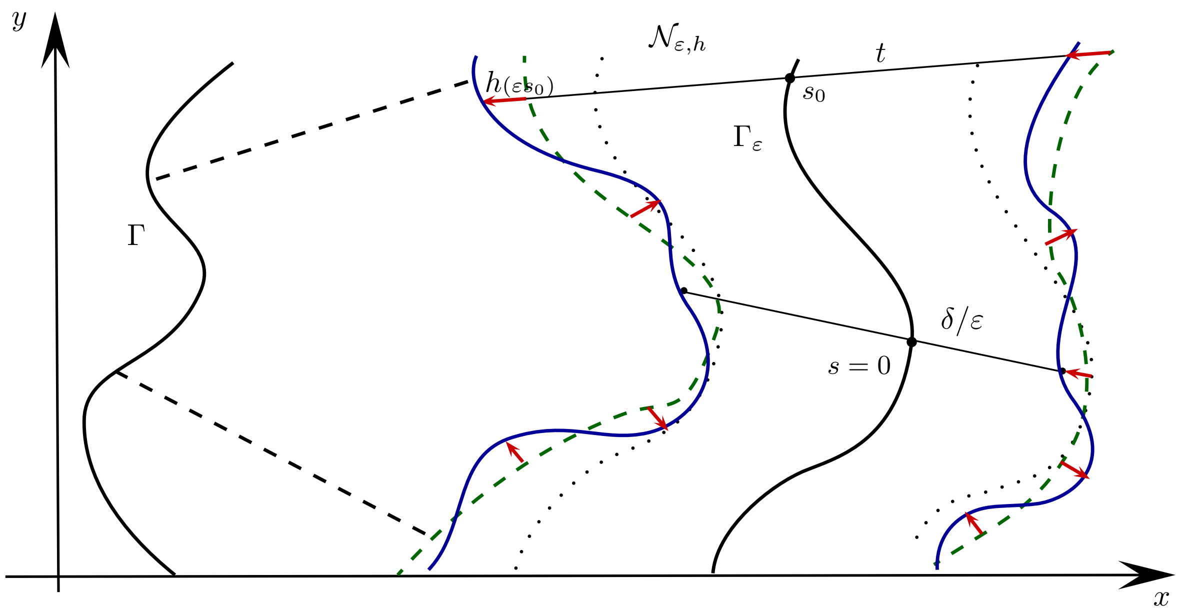

Figure 5: Neighborhood $\mathcal{N}_{\varepsilon,h}$ in Fermi coordinates $X_{\varepsilon,h}(s,t)$

Remark 5. *It is direct from definition* (2.66) *that* $X_\varepsilon(s,t) = X_{\varepsilon,h}(s, t - h(\varepsilon s))$.

2.4.1. The Euclidean Laplacian in coordinates

We have the validity of the following expression for the Euclidean Laplacian $\Delta_{x,y}$ in terms of the dilated and translated Fermi coordinates. A detailed proof of this fact can be found in Section A.1 of the Appendix.

Lemma 3. *On the open neighborhood* (2.67) $\mathcal{N}_{\varepsilon,h} \subset \mathbb{R}^2$ *of* Γ_ε*, the Euclidean Laplacian has the following expression when is computed in the coordinate* $x = X_{\varepsilon,h}(s,t)$ *given in* (2.66)*:*

$$\Delta_{X_{\varepsilon,h}} = \partial_{tt} + \partial_{ss} - 2\varepsilon h'(\varepsilon s)\partial_{st} - \varepsilon^2 h''(\varepsilon s)\partial_t + \varepsilon^2 |h'(\varepsilon s)|^2 \partial_{tt} - \varepsilon[k(\varepsilon s) + \varepsilon(t + h(\varepsilon s))k^2(\varepsilon s)] \cdot \partial_t + D_{\varepsilon,h}(s,t) \quad (2.68)$$

where

$$\begin{aligned} D_{\varepsilon,h}(s,t) := {} & \varepsilon(t+h)A_0(\varepsilon s, \varepsilon(t+h))[\partial_{ss} - 2\varepsilon h'(\varepsilon s)\partial_{ts} - \varepsilon^2 h''(\varepsilon s)\partial_t + \varepsilon^2 |h'(\varepsilon s)|^2 \partial_{tt}] \\ & + \varepsilon^2 (t+h) B_0(\varepsilon s, \varepsilon(t+h))[\partial_s - \varepsilon h'(\varepsilon s)\partial_t] \\ & + \varepsilon^3 (t+h)^2 C_0(\varepsilon s, \varepsilon(t+h))\partial_t \end{aligned} \quad (2.69)$$

for which

$$A_0(\varepsilon s, \varepsilon[t + h(\varepsilon s)]) = 2k(\varepsilon s) + \varepsilon O(|[t + h(\varepsilon s)]k^2(\varepsilon s)|) \quad (2.70)$$

$$B_0(\varepsilon s, \varepsilon[t + h(\varepsilon s)]) = \dot{k}(\varepsilon s) + \varepsilon O(|(t + h(\varepsilon s))\dot{k}(\varepsilon s) \cdot k^2(\varepsilon s)|) \quad (2.71)$$

$$C_0(\varepsilon s, \varepsilon[t + h(\varepsilon s)]) = k^3(\varepsilon s) + \varepsilon O(|(t + h(\varepsilon s))k^4(\varepsilon s)|) \quad (2.72)$$

are smooth functions and these relations can be derived.

2.4.2. Gradients in coordinates

Just like the preceding part, the main idea here is to compute the first-order derivatives $\varepsilon\nabla_{\bar{x}}a/a \cdot \nabla_x u$ of equation (2.65), in terms of the Fermi coordinates. The next result address this fact, for which we left a detailed proof in Section A.2 in the Appendix.

Lemma 4. *On the open neighborhood* (2.67) $\mathcal{N}_{\varepsilon,h} \subset \mathbb{R}^2$ *of* Γ_ε*, the product of the Euclidean Gradients has the following expression when is computed in the coordinate* $x = X_{\varepsilon,h}(s,t)$ *given in* (2.66)*:*

$$\begin{aligned} \varepsilon \frac{\nabla_X a}{a} \cdot \nabla_{X_{\varepsilon,h}} = {} & \varepsilon \frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s, 0)[\partial_s - \varepsilon h'(\varepsilon s) \cdot \partial_t] \\ & + \varepsilon \left[\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s, 0) + \varepsilon(t + h(\varepsilon s)) \left(\frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s, 0) - \left| \frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s, 0) \right|^2 \right) \right] \partial_t \\ & + E_{\varepsilon,h}(s,t) \end{aligned} \quad (2.73)$$

where

$$\begin{aligned} E_{\varepsilon,h}(s,t) := {} & \varepsilon^2 (t + h(\varepsilon s)) D_0(\varepsilon s, \varepsilon(t+h))[\partial_s - \varepsilon h'(\varepsilon s) \cdot \partial_t] \\ & + \varepsilon^3 (t + h(\varepsilon s))^2 F_0(\varepsilon s, \varepsilon(t+h))\partial_t \end{aligned} \quad (2.74)$$

and for which the next functions are smooth

$$D_0(\varepsilon s, \varepsilon(t+h)) = \partial_{\boldsymbol{t}} \left[\frac{\partial_{\boldsymbol{s}} a}{a}\right](\varepsilon s, 0) + \varepsilon O\left((t+h(\varepsilon s))\partial_{\boldsymbol{tt}}\left[\frac{\partial_t a}{a}\right]\right)$$
$$+ A_0(\varepsilon s, \varepsilon(t+h)) \cdot \frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s, \varepsilon(t+h)) \tag{2.75}$$
$$F_0(\varepsilon s, \varepsilon(t+h)) = \frac{1}{2}\partial_{\boldsymbol{tt}} \left[\frac{\partial_t a}{a}\right](\varepsilon s, 0) + \varepsilon O\left((t+h(\varepsilon s))\partial_{\boldsymbol{ttt}}\left[\frac{\partial_t a}{a}\right]\right) \tag{2.76}$$

and where $A_0(\varepsilon s, \varepsilon(t+h))$ *given by* (A.35). *Further, these relations can be derived.*

Chapter 3

Preliminary Remarks on the Allen-Cahn equation

3.1. Transforming the problem in Fermi coordinates

Since every term of the inhomogeneous Allen-Cahn equation has an expression in Fermi coordinates, we are now able to rewrite equation (2.65) in these coordinates, by using formulae stated in Section 2.4.

As we saw, the Allen-Cahn equation scaled in factor ε^{-1} corresponds to

$$\Delta_{x,y}\tilde{u}(x,y)+\varepsilon\frac{\nabla_{\bar{x},\bar{y}}a(\varepsilon x,\varepsilon y)}{\tilde{a}(\varepsilon x,\varepsilon y)}\cdot\nabla_{x,y}\tilde{u}(x,y)+f(\tilde{u}(x,y))=0,\quad\forall(x,y)\in\mathbb{R}^2\tag{3.1}$$

Consider now the dilated and translated Fermi coordinates on the neighborhood $\mathcal{N}_{\varepsilon,h}$, as it was explained in (2.66). Recall that v_u^* represents $\tilde{u}=\tilde{u}(x,y)$ in these new coordinates, throughout the relation

$$v_u^*(s,t):=\tilde{u}\circ X_{\varepsilon,h}(s,t)=\tilde{u}(x,y)$$

Making use of the Lemmas 3 and 4 from the preceding sections, we can write equation (3.1) in coordinates $(x,y)=X_{\varepsilon,h}(s,t)$ as

$$\begin{aligned}\Delta_{X_{\varepsilon,h}}\tilde{u}(x,y)+\varepsilon\frac{\nabla_X a(\varepsilon x,\varepsilon y)}{a(\varepsilon x,\varepsilon y)}\cdot\nabla_{X_{\varepsilon,h}}\tilde{u}(x,y)+f(\tilde{u}(x,y))=&\\ \partial_{tt}v^*(s,t)+\partial_{ss}v^*(s,t)-2\varepsilon h'(\varepsilon s)\partial_{st}v^*(s,t)-\varepsilon^2h''(\varepsilon s)\partial_t v^*(s,t)&\\ +\varepsilon^2|h'(\varepsilon s)|^2\partial_{tt}v^*-\varepsilon[k(\varepsilon s)+\varepsilon(t+h(\varepsilon s))k^2(\varepsilon s)]\partial_t v^*+D_{\varepsilon,h}(s,t)&\\ +\varepsilon\frac{\partial_{s}a}{a}(\varepsilon s,0)[\partial_s v_u^*(s,t)-\varepsilon h'(\varepsilon s)\cdot\partial_t v_u^*(s,t)]&\end{aligned}$$

$$+\varepsilon\left[\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s,0)+\varepsilon(t+h(\varepsilon s))\left(\frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s,0)-\left|\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s,0)\right|^2\right)\right]\partial_t v_u^*(s,t)$$

$$+E_{\varepsilon,h}(\varepsilon s,t)+f(v_u^*(s,t)) \tag{3.2}$$

where functions $D_{\varepsilon,h}$ and $E_{\varepsilon,h}$ are explicitly given in (A.34) and (A.57).

In particular, rearranging the terms of expression (3.2), we get

$$\Delta_{X_{\varepsilon,h}}\tilde{u}(x,y)+\varepsilon\frac{\nabla_X a(\varepsilon x,\varepsilon y)}{a(\varepsilon x,\varepsilon y)}\cdot\nabla_{X_{\varepsilon,h}}\tilde{u}(x,y)+f(\tilde{u}(x,y))=$$

$$\partial_{tt}v_u^*(s,t)+\partial_{ss}v_u^*(s,t)-\varepsilon\left[k(\varepsilon s)-\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s,0)\right]\partial_t v_u^*(s,t)+\varepsilon\frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s,0)\partial_s v_u^*(s,t)$$

$$-\varepsilon^2\left\{h''(\varepsilon s)+h'(\varepsilon s)\frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s,0)+h(\varepsilon s)\left[k^2(\varepsilon s)-\frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s,0)+\left|\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s,0)\right|^2\right]\right\}\partial_t v_u^*(s,t)$$

$$-\varepsilon^2\left[k^2(\varepsilon s)-\frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s,0)+\left|\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s,0)\right|^2\right]t\partial_t v_u^*(s,t)$$

$$-2\varepsilon h'(\varepsilon s)\partial_{st}v_u^*(s,t)+\varepsilon^2|h'(\varepsilon s)|^2\partial_{tt}v_u^*(s,t)$$

$$+\varepsilon(t+h(\varepsilon s))A_0(\varepsilon s,\varepsilon(t+h))[\partial_{ss}v^*-2\varepsilon h'(\varepsilon s)\partial_{ts}v^*-\varepsilon^2 h''(\varepsilon s)\partial_t v^*+\varepsilon^2|h'(\varepsilon s)|^2\partial_{tt}v^*]$$

$$+\varepsilon^2(t+h(\varepsilon s))\tilde{B}_0(\varepsilon s,\varepsilon(t+h))[\partial_s v^*(s,t)-\varepsilon h'(\varepsilon s)\partial_t v^*(s,t)]$$

$$+\varepsilon^3(t+h(\varepsilon s))^2\tilde{C}_0(\varepsilon s,\varepsilon(t+h))\partial_t v^*(s,t)+f(v_u^*(s,t)) \tag{3.3}$$

where

$$\tilde{B}_0(\varepsilon s,\varepsilon(t+h)):=B_0(\varepsilon s,\varepsilon(t+h))+D_0(\varepsilon s,\varepsilon(t+h)) \tag{3.4}$$

$$\tilde{C}_0(\varepsilon s,\varepsilon(t+h)):=C_0(\varepsilon s,\varepsilon(t+h))+F_0(\varepsilon s,\varepsilon(t+h)) \tag{3.5}$$

and the foregoing is valid for any $(x,y)=X_{\varepsilon,h}(s,t)\in\mathcal{N}_{\varepsilon,h}$.

Thus, we can now incorporate to this equation both, the criticality condition of the curve Γ with respect to the weighted length $\int_\Gamma a(\vec{x})$, and the Jacobi operator associated to $a(\vec{x})$ found in 2.1. Suppose that Γ is a stationary with respect to $l_{a,\Gamma}$, then as stated in (2.10), the curve must satisfy

$$\partial_{\boldsymbol{t}} a(\boldsymbol{s},0)=k(\boldsymbol{s})\cdot a(\boldsymbol{s},0),\quad \text{a.e. } \boldsymbol{s}\in\mathbb{R}$$

Remark 6. *It is worth pointing out that the stationarity property of the curve Γ can be incorporated to the expansion of the inhomogeneous Allen-Cahn equation, eliminating the leading term of order $O(\varepsilon)$ in (3.3) multiplying $\partial_t v_u^*(s,t)$. This fact will be essential for the Lyapunov-Schmidt reduction method to succeed, because it reduces the size in powers of ε, of the operators involved in the Allen-Cahn equation.*

Then using condition (2.10) for $\boldsymbol{s} = \varepsilon s$, we get that equation (3.3) amounts to

$$\begin{aligned}
&\Delta_{X_{\varepsilon,h}}\tilde{u} + \varepsilon\frac{\nabla_X a(\varepsilon x, \varepsilon y)}{a(\varepsilon x, \varepsilon y)}\nabla_{X_{\varepsilon,h}}\tilde{u} + f(\tilde{u}(x,y)) = \\
&\quad \partial_{tt}v_u^*(s,t) + \partial_{ss}v_u^*(s,t) + 0 + \varepsilon\frac{\partial_{\boldsymbol{s}}a}{a}(\varepsilon s, 0)\cdot\partial_s v_u^*(s,t) \\
&\quad - \varepsilon^2\left\{h''(\varepsilon s) + h'(\varepsilon s)\frac{\partial_{\boldsymbol{s}}a}{a}(\varepsilon s,0) + h(\varepsilon s)\left[2k^2(\varepsilon s) - \frac{\partial_{\boldsymbol{tt}}a}{a}(\varepsilon s, 0)\right]\right\}\partial_t v_u^*(s,t) \\
&\quad - \varepsilon^2\left[2k^2(\varepsilon s) - \frac{\partial_{\boldsymbol{tt}}a}{a}(\varepsilon s,0)\right]t\partial_t v_u^*(s,t) - 2\varepsilon h'(\varepsilon s)\partial_{st}v_u^*(s,t) + \varepsilon^2|h'(\varepsilon s)|^2\partial_{tt}v_u^*(s,t) \\
&\quad + \varepsilon(t + h(\varepsilon s))A_0(\varepsilon s, \varepsilon(t+h))[\partial_{ss}v^* - 2\varepsilon h'(\varepsilon s)\partial_{ts}v^* - \varepsilon^2 h''(\varepsilon s)\partial_t v^* + \varepsilon^2|h'(\varepsilon s)|^2\partial_{tt}v^*] \\
&\quad + \varepsilon^2(t+h(\varepsilon s))\tilde{B}_0(\varepsilon s, \varepsilon(t+h))[\partial_s v^*(s,t) - \varepsilon h'(\varepsilon s)\partial_t v^*(s,t)] \\
&\quad + \varepsilon^3(t+h(\varepsilon s))^2\tilde{C}_0(\varepsilon s, \varepsilon(t+h))\partial_t v^*(s,t) + f(v_u^*(s,t))
\end{aligned}$$

for any $(x,y) = X_{\varepsilon,h}(s,t) \in \mathcal{N}_{\varepsilon,h}$.

Moreover, making explicit the Jacobi operator (2.15) in the previous equation, we finally obtain the inhomogeneous Allen-Cahn equation for any point $(x,y) = X_{\varepsilon,h}(s,t) \in \mathcal{N}_{\varepsilon,h}$, in dilated and translated Fermi coordinates:

$$\begin{aligned}
&\Delta_{X_{\varepsilon,h}}\tilde{u}(x,y) + \varepsilon\frac{\nabla_X a(\varepsilon x, \varepsilon y)}{a(\varepsilon x, \varepsilon y)}\nabla_{X_{\varepsilon,h}}\tilde{u} + f(\tilde{u}(x,y)) = \\
&\quad \partial_{tt}v_u^*(s,t) + \partial_{ss}v_u^*(s,t) + \varepsilon\frac{\partial_{\boldsymbol{s}}a}{a}(\varepsilon s, 0)\partial_s v_u^*(s,t) - \varepsilon^2\mathcal{J}_a[h](\varepsilon s)\partial_t v_u^*(s,t) \\
&\quad - \varepsilon^2\left[2k^2(\varepsilon s) - \frac{\partial_{\boldsymbol{tt}}a}{a}(\varepsilon s,0)\right]t\partial_t v_u^*(s,t) - 2\varepsilon h'(\varepsilon s)\partial_{st}v_u^*(s,t) + \varepsilon^2|h'(\varepsilon s)|^2\partial_{tt}v_u^*(s,t) \\
&\quad + \varepsilon(t + h(\varepsilon s))A_0(\varepsilon s, \varepsilon(t+h))[\partial_{ss}v^* - 2\varepsilon h'(\varepsilon s)\partial_{ts}v^* - \varepsilon^2 h''(\varepsilon s)\partial_t v^* + \varepsilon^2|h'(\varepsilon s)|^2\partial_{tt}v^*] \\
&\quad + \varepsilon^2(t+h(\varepsilon s))\tilde{B}_0(\varepsilon s, \varepsilon(t+h))[\partial_s v^*(s,t) - \varepsilon h'(\varepsilon s)\partial_t v^*(s,t)] \\
&\quad + \varepsilon^3(t+h(\varepsilon s))^2\tilde{C}_0(\varepsilon s, \varepsilon(t+h))\partial_t v^*(s,t) + f(v_u^*(s,t)) \qquad (3.6)
\end{aligned}$$

where the error operators satisfy

$$A_0(\varepsilon s, \varepsilon[t + h(\varepsilon s)]) = 2k(\varepsilon s) + \varepsilon(t + h(\varepsilon s))\,O(k^2) \qquad (3.7)$$

$$\begin{aligned}
\tilde{B}_0(\varepsilon s, \varepsilon[t+h(\varepsilon s)]) &= B_0(\varepsilon s, \varepsilon[t+h(\varepsilon s)]) + D_0(\varepsilon s, \varepsilon[t+h(\varepsilon s)]) \\
&= \dot{k}(\varepsilon s) + A_0(\varepsilon s, \varepsilon(t+h))\frac{\partial_{\boldsymbol{s}}a}{a}(\varepsilon s, \varepsilon(t+h))
\end{aligned}$$

$$+\,\partial_{\boldsymbol{t}}\left[\frac{\partial_{\boldsymbol{s}} a}{a}\right](\varepsilon s,0)+\varepsilon(t+h(\varepsilon s))\,O\left(\partial_{\boldsymbol{tt}}\left[\frac{\partial_{\boldsymbol{t}} a}{a}\right]+\dot{k}\cdot k^2\right) \tag{3.8}$$

$$\begin{aligned}\tilde{C}_0(\varepsilon s,\varepsilon[t+h(\varepsilon s)]) &= C_0(\varepsilon s,\varepsilon[t+h(\varepsilon s)])+F_0(\varepsilon s,\varepsilon[t+h(\varepsilon s)])\\ &= k^3(\varepsilon s)+\frac{1}{2}\partial_{\boldsymbol{tt}}\left[\frac{\partial_t a}{a}(\varepsilon s,0)\right]+\varepsilon(t+h(\varepsilon s))\,O\left(\partial_{\boldsymbol{ttt}}\left[\frac{\partial_{\boldsymbol{t}} a}{a}\right]+k^4\right)\end{aligned} \tag{3.9}$$

which are smooth functions, and these relations can be derived. □

3.2. Formal Asymptotic Behavior of u^ε

The following Section is based on [4], where del Pino studied the asymptotic behavior of a solution to the Allen-Cahn equation in $\mathbb{R}^N$ in the case without the inhomogeneity term. Let us argue formally, in our case, to get an idea on how a solution to the singularly perturbed inhomogeneous Allen-Cahn equation

$$\varepsilon^2\Delta u^\varepsilon+\varepsilon^2\frac{\nabla a}{a}\cdot\nabla u^\varepsilon+f(u^\varepsilon)=0, \quad \text{in} \quad \mathbb{R}^2 \tag{3.10}$$

should look like near a limiting interface Γ, assuming uniformly bounded energy for u^ε.

Recall that we are assuming $f(s) := -F'(s)$, where F is supposed to be a double-well function satisfying (1.7)-(1.8)-(1.9), as we stated in the Introduction. Let Γ be a smooth curve with a regular parametrization $\gamma : \mathbb{R} \to \Gamma$, and let ν be a choice of the unit normal. Using Fermi coordinates, we can represent points near Γ by

$$x=\gamma(s)+z\cdot\nu(s), \quad s\in\mathbb{R},\ |z|<\delta$$

A well known formula, as proved in [5], states that the Euclidean Laplacian in Fermi coordinates reads as follows

$$\Delta_x=\partial_{zz}+\Delta_{\Gamma^z}-k_{\Gamma^z}\cdot\partial_z$$

where Δ_{Γ^z} designates the Laplace-Beltrami operator on the curve Γ^z, acting on functions of the s-variable. In addition k_{Γ^z} denotes the curvature of Γ^z, which is the curve given by

$$\Gamma^z := \{\gamma(s)+z\cdot\nu(s)/\ s\in\mathbb{R}\}$$

Furthermore, denoting by $k(s)$ the Gaussian curvature of Γ, it holds that k_{Γ^z} satisfies

$$k_{\Gamma^z}(s)=\frac{k(s)}{1-z\cdot k(s)} \tag{3.11}$$

For our purposes, it is reasonable to consider that the solution has uniform smoothness in the $s-$direction, while in the transition direction z, elliptic estimates applied to the transformed equation

$$\Delta u^\varepsilon+\frac{\nabla a}{a}\cdot\nabla u^\varepsilon+f(u^\varepsilon)=0, \quad \text{in} \quad \mathbb{R}^2 \tag{3.12}$$

yield uniform smoothness in the variable $\zeta := \varepsilon^{-1}z$.

Introducing the function $v_\varepsilon(s,\zeta) := u(s,\varepsilon^{-1}z)$, the equation written in half-dilated Fermi coordinates (s,ζ) amounts to

$$\varepsilon^2\Delta_{\Gamma^{\varepsilon\zeta}}v_\varepsilon - \varepsilon k_{\Gamma^{\varepsilon\zeta}}(s)\partial_\zeta v_\varepsilon + \frac{1}{a(s,\varepsilon\zeta)}\left[\frac{\varepsilon^2 k^2_{\Gamma^{\varepsilon\zeta}}(s)}{k^2(s)}\partial_s a\cdot\partial_s v_\varepsilon + \varepsilon\ \partial_z a(s,\varepsilon\zeta)\cdot\partial_\zeta v_\varepsilon\right]$$
$$+\partial_{\zeta\zeta}v_\varepsilon + f(v_\varepsilon(s,\zeta)) = 0 \tag{3.13}$$

where $s\in\mathbb{R},\ |\zeta| < \varepsilon^{-1}\delta$.

We can make two strong assumptions:

1. The zero level set of v_ε lies within $O(\varepsilon^2)$-neighborhood of Γ, that is, on a region in which $|\zeta| = O(\varepsilon)$. Further, we also may assume that $\partial_\zeta v_\varepsilon > 0$ on this nodal set.
2. $v_\varepsilon(s,\zeta)$ can be expanded in powers of ε by means of
$$v_\varepsilon(s,\zeta) = v_0(s,\zeta) + \varepsilon v_1(s,\zeta) + \varepsilon^2 v_2(s,\zeta) + \cdots \tag{3.14}$$
with bounded smooth coefficients, and bounded derivatives.

By substituting expression (3.14) in equation (3.13), we can use the first assumption and let $\varepsilon\to 0$. We obtain

$$\begin{aligned} \partial_{\zeta\zeta}v_0(s,\zeta) + f(v_0(s,\zeta)) = 0\ , &\quad \text{in} \quad \mathbb{R}\times\mathbb{R} \\ v_0(s,0) = 0\ , &\quad \text{for all } s\in\mathbb{R} \\ \partial_\zeta v_0(s,0) \geq 0\ , &\quad \text{for all } s\in\mathbb{R} \end{aligned} \tag{3.15}$$

Furthermore we assume that the energy of the weighted Allen-Cahn equation is uniformly bounded for this family of solutions:

$$\sup_{\varepsilon>0} J_{\varepsilon,a}(u^\varepsilon) = \sup_{\varepsilon>0}\iint_{\mathbb{R}^2}\left[\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{1}{4\varepsilon}F(u^\varepsilon)\right]a(x,y)dxdy < +\infty$$

but since the potential a is bounded below, this implies that

$$\int_{\mathbb{R}}\int_{-\delta/\varepsilon}^{\delta/\varepsilon}\left[\frac{1}{2}|\partial_\zeta v_\varepsilon|^2 + \frac{1}{4}F(v_\varepsilon^2)\right]d\zeta ds \leq \tilde{C}J_{\varepsilon,a}(u^\varepsilon) \leq C$$

Therefore, using the expansion of v_ε and of the nonlinearity F, and letting $\varepsilon\to 0$ follows that

$$\int_{-\infty}^{+\infty}\left[\frac{1}{2}|\partial_\zeta v_0(s,\zeta)|^2 + \frac{1}{4}F(v_0^2(s,\zeta))\right]d\zeta < +\infty \tag{3.16}$$

However it is known for the classical nonlinearity (1.5), that conditions (3.15) and (3.16) force to $v_0(s,\zeta) \equiv w(\zeta)$, where w is the unique solution of the ODE

$$w'' + w - w^3 = 0, \quad w(0) = 0, \quad w(\pm\infty) = \pm 1 \tag{3.17}$$

namely

$$w(\zeta) := \tanh(\zeta/\sqrt{2}) \tag{3.18}$$

To see this, multiply equation (3.15) by $\partial_\zeta v_0(s,\zeta)$ and recognize it as the derivative of some function:

$$\partial_{\zeta\zeta} v_0(s,\zeta) \cdot \partial_\zeta v_0(s,\zeta) + f(v_0(s,\zeta)) \cdot \partial_\zeta v_0(s,\zeta) = 0, \quad \forall (s,\zeta) \in \mathbb{R} \times \mathbb{R}$$

$$\Rightarrow \quad \frac{1}{2}|\partial_\zeta v_0(s,\zeta)|^2 - F(v_0(s,\zeta)) = C(s), \quad \forall s \in \mathbb{R} \tag{3.19}$$

But since (3.16) holds, the positivity of the integrands yields that

$$\lim_{\zeta\to\infty} \frac{1}{2}|\partial_\zeta v_0(s,\zeta)|^2 = 0 \ , \quad \lim_{\zeta\to\infty} F(v_0(s,\zeta)) = 0 \quad \forall s \in \mathbb{R}$$

So if we take limit $\zeta \to \infty$ on the left-hand side of (3.19), we get

$$\lim_{\zeta\to\infty} \frac{1}{2}|\partial_\zeta v_0(s,\zeta)|^2 - F(v_0(s,\zeta)) = 0 \equiv C(s)$$

In this way, an explicit expression for v_0 in terms of the variable ζ can be find, by solving equation (3.19).

$$\partial_\zeta v_0(s,\zeta) = \pm\sqrt{2F(v_0(s,\zeta))} = \pm\frac{1}{\sqrt{2}}(1 - v_0^2(s,\zeta))$$

Nonetheless, the second assumption on the continuity of the derivative $\partial_\zeta v_0(s,\zeta)$, forces the last expression to have only one sign. Further, the first assumption implies that it has to be the positive root. Thus solving the equation by separation of variables for a fixed $s \in \mathbb{R}$, we get

$$\int \frac{dv_0}{1 - v_0^2(s,\zeta)} = \int \frac{d\zeta}{\sqrt{2}} + \hat{C}(s) \quad \Leftrightarrow \quad v_0(s,\zeta) = \tanh(\zeta/\sqrt{2} + \hat{C}(s)) \ , \quad \forall s \in \mathbb{R}$$

Finally, from the initial condition $v_0(s,0) = 0$, it follows $\hat{C}(s) = 0$, and in conclusion we obtain $v_0(s,\zeta) \equiv w(\zeta)$.

On the other hand, analyzing equation (3.13) at order $O(\varepsilon)$ by replacing the expansion of v_ε, using last identity of v_0 and calculating the Taylor expansion of nonlinearity f around

w, we get the equation

$$\varepsilon^2\Delta_{\Gamma^{\varepsilon\zeta}}\Big(w(\zeta)+\varepsilon v_1(s,\zeta)+\varepsilon^2 v_2(s,\zeta)+\cdots\Big)-\varepsilon k_{\Gamma^{\varepsilon\zeta}}(s)\Big(w'(\zeta)+\varepsilon\partial_\zeta v_1(s,\zeta)$$
$$+\varepsilon^2\partial_\zeta v_2(s,\zeta)+\cdots\Big)+\frac{1}{a(s,\varepsilon\zeta)}\Big[\varepsilon^2\frac{k^2_{\Gamma^{\varepsilon\zeta}}(s)}{k^2(s)}\partial_s a\Big(0+\varepsilon\partial_s v_1+\varepsilon^2\partial_s v_2+\cdots\Big)$$
$$+\varepsilon\partial_z a(s,\varepsilon\zeta)\Big(w'(\zeta)+\varepsilon\partial_\zeta v_1+\varepsilon^2\partial_\zeta v_2+\cdots\Big)\Big]+\Big(w''(\zeta)+\varepsilon\partial_{\zeta\zeta}v_1+\varepsilon^2\partial_{\zeta\zeta}v_2+\cdots\Big)$$
$$+f(w)+f'(w)\left(\varepsilon v_1+\varepsilon^2 v_2+\cdots\right)+\frac{f''(w)}{2}\left(0+\varepsilon v_1+\varepsilon^2 v_2+\cdots\right)^2+O(\varepsilon^3)=0$$

So gathering terms with the same order in ε gives

$$w''(\zeta)+f(w(\zeta))+\varepsilon\left(-k_{\Gamma^{\varepsilon\zeta}}(s)w'(\zeta)+\partial_{\zeta\zeta}v_1+\frac{\partial_z a(s,\varepsilon\zeta)}{a(s,\varepsilon\zeta)}w'(\zeta)+f'(w)v_1\right)+O(\varepsilon^2)=0 \tag{3.20}$$

But since w is exact solution of (3.17), we can divide by $\varepsilon>0$ and letting $\varepsilon\to 0^+$ in (3.20), implying that $v_1(s,\zeta)$ must satisfy the equation

$$\partial_{\zeta\zeta}v_1(s,\zeta)+f'(w(\zeta))v_1(s,\zeta)=\left(k(s)-\frac{\partial_z a(s,0)}{a(s,0)}\right)w'(\zeta),\quad \forall(s,\zeta)\in\mathbb{R}\times\mathbb{R}$$

where we have used the smoothness of the potential $a(s,\zeta)$ and of the curve Γ, for which $k_{\Gamma^{\varepsilon\zeta}}(s)\to k(s)$. Then, testing the equation against $w'(\zeta)$ and integrating by parts in ζ, it follows the necessary condition on Γ

$$\int_{\mathbb{R}}(\underbrace{\partial_{\zeta\zeta}(w'(\zeta))+f'(w)w'(\zeta)}_{=0})v_1(s,\zeta)\,d\zeta=\left(k(s)-\frac{\partial_z a(s,0)}{a(s,0)}\right)\underbrace{\int_{\mathbb{R}}|w'(\zeta)|^2d\zeta}_{\neq 0}\quad\forall s\in\mathbb{R}$$

$$\Rightarrow\quad k(s)=\frac{\partial_z a(s,0)}{a(s,0)},\quad\forall s\in\mathbb{R} \tag{3.21}$$

which gives that Γ in a especial curve that satisfies a specific relation with potential $a(s,z)$. Because of criticality condition (3.21), it turns out that $v_1(s,\zeta)$ satisfies the homogeneous equation

$$\partial_{\zeta\zeta}v_1(s,\zeta)+f'(w(\zeta))v_1(s,\zeta)=0\,,\quad\forall(s,\zeta)\in\mathbb{R}\times\mathbb{R} \tag{3.22}$$

To see explicitly $v_1(s,\zeta)$ note that for any $s\in\mathbb{R}$ fixed (3.22) represents an ODE in the variable ζ, so we can write any solution of the homogeneous equation in terms of the two-dimensional kernel of the linearized operator $\partial_{\zeta\zeta}+f'(w(\zeta))$, where the second element is found by using reduction of order:

$$v_1(s,\zeta)=\hat{h}_0(s)w'(\zeta)+\hat{h}_1(s)\varphi_2(\zeta)\quad\text{where}\quad\varphi_2(\zeta)=w'(\zeta)\int_0^\zeta\frac{\exp(-\int 0d\zeta)}{|w'(\zeta)|^2}\,d\zeta \tag{3.23}$$

Nonetheless, we can estimate the growth rate of $\varphi_2(\zeta)$ in the region where $t \to +\infty$ since

$$\varphi_2(\zeta) = O(e^{-\sqrt{2}\zeta}) \int_0^\zeta \frac{1}{O(e^{-2\sqrt{2}\zeta})} d\zeta = O(e^{\sqrt{2}\zeta}) \tag{3.24}$$

and this implies that $\varphi_2(\zeta)$ is not bounded. Thanks to assumption 2, we are looking for bounded solutions $v_\varepsilon(s,\zeta)$, so the only chance for $v_1(s,\zeta)$ to be is

$$v_1(s,t) = -h(s)w'(\zeta), \qquad \text{for a certain function } h(s)$$

In this way we can write v_ε, using (3.14) and the Taylor approximation around w, as follows

$$v_\varepsilon(s,\zeta) = w(\zeta - \varepsilon h(s)) + \varepsilon^2 v_2(s,\zeta) + \varepsilon^3 v_3(s,\zeta) + \cdots$$

Analogously, from the mean curvature expression (3.11) and the fact that Γ is a critical curve, we expand

$$k_{\Gamma^{\varepsilon\zeta}}(s) = k(s) + \varepsilon k^2(s)\zeta + \varepsilon^2 k^3(s)\zeta^2 + \cdots$$

Therefore, replacing in equation (3.13) the expansion for the mean curvature and function v_ε we obtain

$$\begin{aligned}
&\varepsilon^2 \Delta_{\Gamma^{\varepsilon\zeta}} \Big(w(\zeta - \varepsilon h(s)) + \varepsilon^2 v_2(s,\zeta) + \varepsilon^3 v_3(s,\zeta) + \cdots \Big) - \varepsilon \Big(k(s) + \varepsilon k^2(s)\zeta \\
&+\varepsilon^2 k^3(s)\zeta^2 + \cdots \Big) \cdot \Big(w'(\zeta - \varepsilon h(s)) + \varepsilon^2 \partial_\zeta v_2(s,\zeta) + \varepsilon^3 \partial_\zeta v_3(s,\zeta) + \cdots \Big) \\
&+ \frac{1}{a(s,\varepsilon\zeta)} \Bigg[\frac{\varepsilon^2}{k^2(s)} \Big(k^2(s) + \varepsilon^2 (k^2(s))^2 \zeta^2 + \varepsilon^4 (k^3(s))^2 \zeta^4 + 2\varepsilon k(s) k^2(s)\zeta \\
&+ 2\varepsilon^2 k(s) k^3(s) \zeta^2 + \cdots \Big) \cdot \partial_s a(s, \varepsilon(t + \varepsilon h)) \cdot \Big(0 + \varepsilon^2 \partial_s v_2 + \varepsilon^3 \partial_s v_3 + \cdots \Big) \\
&+ \varepsilon \partial_z a(s,\varepsilon\zeta) \Big(w'(\zeta - \varepsilon h(s)) + \varepsilon^2 \partial_\zeta v_2 + \varepsilon^3 \partial_\zeta v_3 + \cdots \Big) \Bigg] \\
&+ \left(w''(\zeta - \varepsilon h(s)) + \varepsilon^2 \partial_{\zeta\zeta} v_2 + \varepsilon^3 \partial_{\zeta\zeta} v_3 + \cdots \right) + f(w) + f'(w) \left(\varepsilon^2 v_2 + \varepsilon^3 v_3 + \cdots \right) \\
&+ \frac{f''(w)}{2} \left(0 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \cdots \right)^2 + O(\varepsilon^4) = 0
\end{aligned}$$

For the following, it will be convenient to write this expansion in terms of the variable $t = \zeta - \varepsilon h(s)$, as

$$v_\varepsilon(s,t) = w(t) + \varepsilon^2 v_2(s,t) + \varepsilon^3 v_3(s,t) + \cdots \tag{3.25}$$

Thus, taking into account $t = \zeta - \varepsilon h(s)$ and using the new expansion (3.25) we deduce from equation (3.13) that

$$\varepsilon^2 \Delta_{\Gamma^{\varepsilon\zeta}} \Big(w(t) + \varepsilon^2 v_2(s, t+\varepsilon h) + \varepsilon^3 v_3(s, t+\varepsilon h) + \cdots \Big) - \varepsilon \Big(k(s) + \varepsilon k^2(s)(t+\varepsilon h)$$
$$+ \varepsilon^2 k^3(s)(t+\varepsilon h)^2 + \cdots \Big) \cdot \Big(w'(t) + \varepsilon^2 \partial_t v_2(s, t+\varepsilon h) + \varepsilon^3 \partial_t v_3(s, t+\varepsilon h) + \cdots \Big)$$
$$+ \frac{1}{a(s, \varepsilon(t+\varepsilon h))} \Bigg[\frac{\varepsilon^2}{k^2(s)} \Big(k^2(s) + \varepsilon^2 k^4(s)(t+\varepsilon h)^2 + \varepsilon^4 (k^3(s))^2 (t+\varepsilon h)^4$$
$$+ 2\varepsilon k(s) k^2(s)(t+\varepsilon h) + 2\varepsilon^2 k(s) k^3(s)(t+\varepsilon h)^2 + \cdots \Big) \cdot \Big[\partial_s a(s, \varepsilon(t+\varepsilon h))$$
$$+\varepsilon \partial_t a(s, \varepsilon(t+\varepsilon h)) h'(s) \Big] \Big(0 + \varepsilon^2 \partial_s v_2 + \varepsilon^3 \partial_t v_2 h'(s) + \varepsilon^3 \partial_s v_3 + \varepsilon^4 \partial_s v_3 h'(s) + \cdots \Big)$$
$$+\varepsilon \partial_z a(s, \varepsilon(t+\varepsilon h)) \big(w'(t) + \varepsilon^2 \partial_t v_2 + \varepsilon^3 \partial_t v_3 + \cdots \big) \Bigg] + \big(w''(t) + \varepsilon^2 \partial_{tt} v_2 + \varepsilon^3 \partial_{tt} v_3 + \cdots \big)$$
$$+ f(w) + f'(w) \big(\varepsilon^2 v_2 + \varepsilon^3 v_3 + \cdots \big) + \frac{f''(w)}{2} \big(0 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \cdots \big)^2 + O(\varepsilon^4) = 0$$

Then, we can rearrange these terms up to order $O(\varepsilon^3)$, so that

$$\begin{aligned} 0 &= \Delta v_\varepsilon + \frac{\nabla a}{a} \cdot \nabla v_\varepsilon + f(v_\varepsilon) \\ &= w''(t) + f(w(t)) + \big[\partial_{tt} + f'(w(t))\big](\varepsilon^2 v_2 + \varepsilon^3 v_3) + \varepsilon^2 \Delta_{\Gamma^{\varepsilon\zeta}} w(t) \\ &- \varepsilon^3 \left(k(s) - \frac{\partial_z a(s, \varepsilon(t+\varepsilon h))}{a(s, \varepsilon(t+\varepsilon h))} \right) \partial_s v_2 - w'(t) \cdot \Bigg\{ \varepsilon \left(k(s) - \frac{\partial_z a(s, \varepsilon(t+\varepsilon h))}{a(s, \varepsilon(t+\varepsilon h))} \right) \\ &+\varepsilon^3 k^3(s) t^2 + \varepsilon^2 k^2(s)(t + \varepsilon h(s)) \Bigg\} + O(\varepsilon^4) \end{aligned} \tag{3.26}$$

However, to analyze this equation up to order $O(\varepsilon^3)$, we must expand the gradient

$$\frac{\partial_z a(s, \varepsilon(t+\varepsilon h))}{a(s, \varepsilon(t+\varepsilon h))} = \frac{\partial_z a(s,0)}{a(s,0)} + \varepsilon \left[\frac{\partial_{zz} a(s,0)}{a(s,0)} - \frac{|\partial_z a(s,0)|^2}{a(s,0)^2} \right] (t+\varepsilon h)$$
$$+ \varepsilon^2 \left[\frac{\partial_{zzz} a(s,0)}{a(s,0)} - \frac{3\partial_{zz} a(s,0) \cdot \partial_z a(s,0)}{a(s,0)^2} + \frac{2(\partial_z a(s,0))^3}{a(s,0)^3} \right] (t+\varepsilon h)^2$$

Now imposing the criticality condition, the term of order $O(\varepsilon)$ becomes

$$\varepsilon \left(k(s) - \frac{\partial_z a(s, \varepsilon(t+\varepsilon h))}{a(s, \varepsilon(t+\varepsilon h))} \right) = (\varepsilon^2 t + \varepsilon^3 h) \left[\frac{|\partial_z a(s,0)|^2}{a(s,0)^2} - \frac{\partial_{zz} a(s,0)}{a(s,0)} \right]$$
$$+ \varepsilon^3 \left[\frac{3\partial_{zz} a(s,0) \cdot \partial_z a(s,0)}{a(s,0)^2} - \frac{\partial_{zzz} a(s,0)}{a(s,0)} - \frac{2(\partial_z a(s,0))^3}{a(s,0)^3} \right] t^2$$

On the other hand, we need to compute the expansion of the Laplacian on $\Gamma^{\varepsilon\zeta}$

$$\varepsilon^2\Delta_{\Gamma^{\varepsilon\zeta}} w(\zeta-\varepsilon h(s)) = \varepsilon^2 \underbrace{\Delta_{\Gamma^{\varepsilon\zeta}} w(\zeta)}_{\equiv 0} - \varepsilon^3 w'(\zeta)\Delta_{\Gamma^{\varepsilon\zeta}} h(s) + \frac{\varepsilon^4}{2} w''(\zeta)\Delta_{\Gamma^{\varepsilon\zeta}} h^2(s) + O(\varepsilon^5)$$

Therefore, by replacing all the previous computation into equation (3.26), we get

$$\begin{aligned}
0 &= \Delta v_\varepsilon + \frac{\nabla a}{a}\nabla v_\varepsilon + f(v_\varepsilon)\\
&= w''(t) + f(w(t)) + \big[\partial_{tt} + f'(w(t))\big](\varepsilon^2 v_2 + \varepsilon^3 v_3) - \varepsilon^3\Bigg\{\left(k(s) - \frac{\partial_z a(s,\varepsilon(t+\varepsilon h))}{a(s,\varepsilon(t+\varepsilon h))}\right)\\
&\cdot \partial_s v_2\Bigg\} - w'(t)\cdot\Bigg\{\varepsilon^2\left(\left[\frac{|\partial_z a(s,0)|^2}{a(s,0)^2} - \frac{\partial_{zz} a(s,0)}{a(s,0)}\right] + k^2(s)\right)t\\
&+ \varepsilon^3\left(\Delta_{\Gamma^{\varepsilon\zeta}} h + k^2(s)h(s) + \left[\frac{|\partial_z a(s,0)|^2}{a(s,0)^2} - \frac{\partial_z^2 a(s,0)}{a(s,0)}\right]h\right)\\
&+ \varepsilon^3\left[\frac{3\partial_{zz} a(s,0)\cdot\partial_z a(s,0)}{a(s,0)^2} - \frac{\partial_{zzz} a(s,0)}{a(s,0)}\right]t^2 + \varepsilon^3\left(k^3(s) - \frac{2(\partial_z a(s,0))^3}{a(s,0)^3}\right)t^2\Bigg\} + O(\varepsilon^4)
\end{aligned}$$

Finally, using again the fact that $w(t)$ is exact solution of the ODE (3.17), we can divide by ε^2 and ε^3 respectively in the latter. Now letting $\varepsilon \to 0$, we found the equations satisfied by v_2 and v_3

$$\partial_{tt} v_2 + f'(w(t))v_2 = -Q(s)\cdot t w'(t) \tag{3.27}$$

$$\partial_{tt} v_3 + f'(w(t))v_3 = \big[\Delta_\Gamma h - Q(s)h\big]w'(t) + M_a(s)\cdot t^2 w'(t) \tag{3.28}$$

where we applied the criticality condition on $k(s)$. Additionally,

$$Q(s) := \left[\frac{\partial_{zz} a(s,0)}{a(s,0)} - 2k^2(s)\right] \tag{3.29}$$

$$M_a(s) := \left[\frac{3\partial_{zz} a(s,0)\cdot\partial_z a(s,0)}{a(s,0)^2} - \frac{\partial_{zzz} a(s,0)}{a(s,0)} - \left(\frac{\partial_z a(s,0)}{a(s,0)}\right)^3\right] \tag{3.30}$$

Applying the variation of parameters formula to equation (3.27), we note the existence of a bounded solution v_2, thanks to the orthogonality $\int_{\mathbb{R}} t|w'(t)|^2 dt = 0$. For any $s \in \mathbb{R}$ fixed, we can write

$$v_2(s,t) = w'(t)\int \frac{Q(s)\cdot t w'(t)\cdot\varphi_2(t)}{W(w',\varphi_2)(t)}dt \; - \varphi_2(t)\int \frac{Q(s)t|w'(t)|^2}{W(w',\varphi_2)(t)}dt$$

but since the Wronskian associated to $\partial_{tt} + f'(w(t))$ is constant due to the Abel's formula

$$W(w',\varphi_2)(t) = \exp\left(-\int 0dt\right) \equiv C$$

we have

$$v_2(s,t) = \frac{Q(s)}{C}\Bigg[w'(t)\int_t^\infty \tau w'(\tau)\left(w'(\tau)\int_0^\tau \frac{1}{|w'(\xi)|^2}d\xi\right)d\tau - \left(w'(t)\int_0^t \frac{1}{|w'(\tau)|^2}d\tau\right)\int_t^{+\infty}\tau|w'(\tau)|^2 d\tau\Bigg] \tag{3.31}$$

here in (3.31) we use the expansion (3.23) of the second element $\varphi_2(t) \in \mathrm{Ker}\,(\partial_t^2 + f'(w))$.

Thus we estimate $v_2(s,t)$ at main order, asymptotically in variable $t \in \mathbb{R}$, for any $s \in \mathbb{R}$ fixed. In the case $t \to -\infty$, the orthogonality condition $\int_{\mathbb{R}} t|w'(t)|^2 dt = 0$ gives

$$\begin{aligned}
|v_2(s,t)| &\leq \tilde{C}|Q(s)|\Bigg[O\left(e^{-\sqrt{2}|t|}\right)\int_t^{+\infty}\left(O\left(e^{-2\sqrt{2}|\tau|}\right)\int_0^\tau O\left(e^{2\sqrt{2}\xi}\right)d\xi\right)d\tau \\
&\quad + O\left(e^{-\sqrt{2}|t|}\right)\int_0^t O\left(e^{2\sqrt{2}|\tau|}\right)d\tau \cdot \left|0 - \int_{-\infty}^t O\left(e^{-2\sqrt{2}\tau}\right)\right|d\tau\Bigg] \\
&\leq C_Q\left[O\left(e^{-\sqrt{2}|t|}\right)\int_t^{+\infty} O(1) + O\left(e^{-\sqrt{2}|t|}\right)O\left(e^{2\sqrt{2}|t|}\right)O\left(e^{-2\sqrt{2}|t|}\right)\right] = O(1)
\end{aligned}$$

In the other case, using similar estimates than before but without the need of the orthogonality condition, we readily see $v_2(s,t) = O(1)$ as $t \to \infty$, for any $s \in \mathbb{R}$.

Notwithstanding, the bounded solvability of the equation for v_3 (3.28) is obtained, if and only if, h solves the following elliptic equation in Γ:

$$\boxed{\mathcal{J}_{a,\Gamma}[h](s) := \Delta_\Gamma h - Q(s)h = cM_a(s), \quad \text{in} \quad \mathbb{R}} \tag{3.32}$$

where $\mathcal{J}_{a,\Gamma}$ is by definition the Jacobi operator of the curve Γ associated with potential $a(x)$, and the constant c is given by $c := -\int_{\mathbb{R}} t^2|w'(t)|^2 dt / \int_{\mathbb{R}} |w'(t)|^2 dt$.

In conclusion, we deal with the problem of constructing entire solutions of equation (3.10) exhibiting the asymptotic behavior described above, around any fixed curve Γ that splits the space $\mathbb{R}^2$ into two components, for which the Fermi coordinates are well defined. A key element for such a construction is precisely the question of solvability of equation (3.32) that determines the deviation of the nodal u^ε from Γ, at main order.

In terms of the original problem (3.12), the issue is to consider a large dilation of Γ, $\Gamma_\varepsilon := \varepsilon^{-1}\Gamma$, and find an entire solution u^ε of (3.12) in such way that for a function h_ε defined on Γ with $\sup_{\varepsilon>0}\|h_\varepsilon\|_{L^\infty(\Gamma)} < +\infty$, we can actually write this exact solution as

$$u^\varepsilon(x) = w(\zeta - \varepsilon h_\varepsilon(\varepsilon s)) + O(\varepsilon^2) \tag{3.33}$$

uniformly for points $x = \gamma(s) + \zeta\nu(\varepsilon s)$ with $|\zeta| \leq \dfrac{\delta}{\varepsilon}$, $s \in \mathbb{R}$, and which exhibits the following asymptotic behavior

$$|u_\varepsilon(x)| \to 1, \quad \text{as} \quad \operatorname{dist}(x, \Gamma_\varepsilon) \to +\infty \tag{3.34}$$

3.3. Approximation of a solution and prior discussion

3.3.1. First candidate for a solution and its projection

Let us consider a function $h : \mathbb{R} \to \mathbb{R}$ that we suppose smooth and bounded. Regard this function $h = h(\varepsilon s)$ as a parameter to be adjusted later on. Assume hereafter, that for a certain constant $\mathcal{K} > 0$ it holds

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} := \|h\|_{L^\infty(\mathbb{R})} + \|(1+|\varepsilon s|)^{1+\alpha} h'\|_{L^\infty(\mathbb{R})} + \sup_{s\in\mathbb{R}} (1+|\varepsilon s|)^{2+\alpha} \|h''\|_{C^{0,\lambda}(\varepsilon s-1, \varepsilon s+1)} \leq \mathcal{K}\varepsilon \tag{3.35}$$

We want to find a smooth solution to the inhomogeneous Allen-Cahn equation

$$S(u) := \varepsilon^2 \Delta_x u + \varepsilon \frac{\nabla_{\bar{x}} a}{a} \nabla_x u + f(u) = 0 \quad \text{in } \mathbb{R}^2 \tag{3.36}$$

So a first attempt for an approximate solution, is choosing the heteroclinic solution (3.18) in the Fermi coordinate t on the region $\mathcal{N}_{\varepsilon,h}$ given by (2.67), that is

$$u_0(x) := w(t) = w(z - h(\varepsilon s)) \tag{3.37}$$

where z designates the normal coordinate to Γ.

Let us evaluate the error of applying u_0 to the equation (3.36). To do this, we will use the characterization of this equation in translated and dilated Fermi coordinates (3.6), thus obtaining

$$\begin{aligned} S(u_0) = & -\varepsilon^2 \mathcal{J}_a[h](\varepsilon s) w'(t) - \varepsilon^2 \left[2k^2(\varepsilon s) - \frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s, 0) \right] t w'(t) + \varepsilon^2 |h'(\varepsilon s)|^2 w''(t) \\ & + \varepsilon (t + h(\varepsilon s)) A_0(\varepsilon s, \varepsilon(t+h)) [-\varepsilon^2 h''(\varepsilon s) w'(t) + \varepsilon^2 |h'(\varepsilon s)|^2 w''(t)] \\ & + \varepsilon^2 (t + h(\varepsilon s)) \tilde{B}_0(\varepsilon s, \varepsilon(t+h)) \left(-\varepsilon h'(\varepsilon s) w'(t)\right) \\ & + \varepsilon^3 (t + h(\varepsilon s))^2 \tilde{C}_0(\varepsilon s, \varepsilon(t+h)) w'(t) \end{aligned} \tag{3.38}$$

with $A_0, B_0, \tilde{C}_0$ are given in (3.7)-(3.8)-(3.9), respectively. We emphasize in the latter, that w is the heteroclinic solution of the ODE $w'' + f(w) = 0$, and that we have broken formula (3.38) into powers of ε, keeping in mind that $h = O(\varepsilon)$.

Since we want u_0 to be as close as possible of a solution to the Allen-Cahn equation (3.36), it would be convenient choosing h in such way that quantity (3.38) be as small as possible. Examining the above expression, it seems that nothing can be done to improve it in absolute terms. However part of this error could be made smaller by adjusting h. Let us consider the L^2-projection onto $w'(t)$ of the error for each fixed $s \in \mathbb{R}$, given by

$$\Pi(s) := \int_{-\infty}^{+\infty} S(u_0)(s,t)w'(t)dt$$

where for simplicity we are assuming that coordinates are defined for all t, since the difference with the integration taken in all the actual domain for t produces only exponentially small terms in ε^{-1}. So computing the projection we find

$$\begin{aligned}\Pi(s) = &-\varepsilon^2 \mathcal{J}_a[h](\varepsilon s)\int_{-\infty}^{+\infty} |w'(t)|^2 dt - \varepsilon^3 h'' \int_{-\infty}^{+\infty} (t+h)A_0(\varepsilon s, \varepsilon(t+h))|w'(t)|^2 dt \\ &+ \varepsilon^3 |h'|^2 \int_{-\infty}^{+\infty} (t+h)A_0(\varepsilon s, \varepsilon(t+h))w''(t)w'(t)dt \\ &- \varepsilon^3 h' \int_{-\infty}^{+\infty} (t+h)\tilde{B}_0(\varepsilon s, \varepsilon(t+h))|w'(t)|^2 dt \\ &+ \varepsilon^3 \int_{-\infty}^{+\infty} (t+h)^2 \tilde{C}_0(\varepsilon s, \varepsilon(t+h))|w'(t)|^2 ds \end{aligned} \tag{3.39}$$

where we have used that $\int_{-\infty}^{+\infty} t|w'(t)|^2dt = 0$, $\int_{-\infty}^{+\infty} w''(t)w'(t)dt = 0$, to get rid of the terms of order ε^2, and function $A_0, \tilde{B}_0, \tilde{C}_0$ are given by (3.7),(3.8) and (3.9).

Making all these projections equal to zero amounts to a nonlinear differential equation for h of the form

$$\mathcal{J}_a[h](\varepsilon s) = h''(\varepsilon s) + \frac{\partial_s a(\varepsilon s, 0)}{a(\varepsilon s, 0)} h'(\varepsilon s) - Q(\varepsilon s)h(\varepsilon s) = G_0[h](\varepsilon s)\ ,\ \forall s \in \mathbb{R} \tag{3.40}$$

where $Q(s) = [\partial_{tt} a(s,0)/a(s,0) - 2k^2(s)]$ and G_0 consists in the remaining terms of (3.39).

Note that G_0 is easily checked to be a shrinking map in h, on the ball of radius $O(\varepsilon)$ with C^2 norm. Here is where *nondegeneracy condition* on the Jacobi operator $\mathcal{J}_a$ plays a fundamental role, since we need to invert it in such way that equation (3.40) can be set as a fixed problem for a contraction mapping of a ball of the form (3.35).

3.3.2. Improvement of the approximation

The previous considerations are not sufficient, since after adjusting optimally h, the error in absolute value does not necessarily decrease. Further, by taking into account that $h = O(\varepsilon)$

in some suitable norm, we readily check using expansion (3.38) of the last section, that the "leading term" in powers of ε of the error $S(u_0)$ is

$$-\varepsilon^2\left[2k^2(\varepsilon s)-\frac{\partial_{\boldsymbol{tt}}a(\varepsilon s,0)}{a(\varepsilon s,0)}\right]tw'(t)$$

which did not contribute to the projection onto w'. Hereinafter, it will be necessary to pay attention to the size of $S(u_0)$ up to $O(\varepsilon^2)$, because the solvability of the nonlinear Jacobi equation (3.32) depends strongly on the fact that the error created is sufficiently small, with the purpose of making h to have a tiny size that is consistent with this equation.

On the other hand, considering again that $h=O(\varepsilon)$, it can be checked that now this new term

$$\varepsilon^3\left[k^3(\varepsilon s)+\frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\varepsilon s,0)\right]t^2w'(t)$$

is actually "leading" in the size of $S(u_0)$, in powers of ε, provided that the previous term is removed. Due to technical reasons, since we are allowing the analytic functional setting to be regular in a Hölder sense, up to second order derivatives, then we must compensate this regularity with some smaller size for the projected error $S(u_0)$ on w', at least of $O(\varepsilon^4)$, in order make the nonlinear functional scheme to work.

Having said the foregoing, we need to reduce the size of this remaining part $O(\varepsilon^2)$ of this error, and the part of $O(\varepsilon^3)$ which has no projection on w'. To do so, we improve the approximation through the following argument. Let us consider the ODE

$$\psi_0''(t)+f'(w(t))\psi_0(t)=tw'(t)$$

which has a unique bounded solution with $\psi_0(0)=0$ given explicitly by the variation of parameters formula

$$\psi_0(t)=w'(t)\int_0^t|w'(s)|^{-2}ds\cdot\int_{-\infty}^t s|w'(s)|^2ds$$

Observe that this function is well defined, smooth and bounded since $\int_{\mathbb{R}}s|w'(s)|^2ds=0$. Moreover this solution satisfies $\psi_0(t)\sim e^{-\sqrt{2}|t|}$ as $|t|\to\infty$. Analogously, consider $g(t):=t^2w'(t)$ and note that we can decompose $g=C_gw'+g_\perp$ where $g_\perp$ denotes the orthogonal projection of g onto w' in $L^2(\mathbb{R})$, given by $g_\perp(t):=t^2w'(t)-\left(\int_{\mathbb{R}}\tau^2|w'(\tau)|^2d\tau/\int_{\mathbb{R}}|w'(\tau)|^2d\tau\right)w'(t)$. Thus by setting

$$\psi_1(t)=w'(t)\int_0^t|w'(s)|^{-2}ds\cdot\int_{-\infty}^t g_\perp(t)\cdot w'(s)ds$$

this formula not only provides a bounded solution of $\psi_1''(t)+f'(w(t))\psi_1(t)=g_\perp(t)$, since $\int_{\mathbb{R}}g_\perp(t)w'(t)dt=0$, but also provides a solution with exponential decay $\psi_1(t)\sim e^{-\sqrt{2}|t|}$ as $|t|\to+\infty$, given that $g_\perp$ exhibits this exponential decay. Accordingly, to get rid of the leading terms in the error $S(u_0)$, the previous considerations motivate the next choice of the approximation

$$u_1(s,t) := u_0(s,t) + \varphi_1(s,t) = w(t) + \varphi_1(s,t) \tag{3.41}$$

where

$$\varphi_1(s,t) := \varepsilon^2 \left[2k^2(\varepsilon s) - \frac{\partial_{\boldsymbol{tt}} a(\varepsilon s,0)}{a(\varepsilon s,0)}\right]\psi_0(t) - \varepsilon^3\left[k^3(\varepsilon s) + \frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\varepsilon s,0)\right]\psi_1(t) \tag{3.42}$$

which can be easily seen to behave like $\varphi_1(s,t) = O(\varepsilon^2(1+|\varepsilon s|)^{-2-\alpha}e^{-\sqrt{2}|t|})$, thanks to the assumptions on $k(s), a(s,t), \partial_{i,j,k}a(s,t)$ for $i,j,k \in \{0,s,t\}$, and to the previous observation on $\psi_0(t),\ \psi_1(t)$.

Now, to analyze the error terms created by the Allen-Cahn equation (3.36) on the second approximation $u_1(s,t)$, note that

$$\begin{aligned} S(u_0+\varphi_1) &= \varepsilon^2\Delta_x u_0 + \varepsilon^2\Delta_x\varphi_1 + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x u_0 + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\varphi_1 + f(u_0+\varphi_1) \\ &= S(u_0) + \varepsilon^2\Delta_x\varphi_1 + f'(u_0)\varphi_1 + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\varphi_1 + N_0(\varphi_1) \end{aligned} \tag{3.43}$$

where

$$N_0(\varphi_1) = f(u_0+\varphi_1) - f(u_0) - f'(u_0)\varphi_1 \tag{3.44}$$

Observe from the definition of φ_1, that

$$\varepsilon^2\partial_{tt}\varphi_1 + f'(u_0)\varphi_1 = \varepsilon^2\left[2k^2(\varepsilon s) - \frac{\partial_{tt}a(\varepsilon s,0)}{a(\varepsilon s,0)}\right]tw'(t) - \varepsilon^3\left[k^3(s) + \frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\varepsilon s,0)\right]g_\perp(t)$$

Hence we get the largest remaining term in the error (3.38), which has no projection onto w', are canceled. Moreover, the other largest term of $O(\varepsilon^3)$ with no projection onto w' is also eliminated. Thus we get

$$\begin{aligned} S(u_1) = S(u_0) &- \left(-\varepsilon^2\left[2k^2(\varepsilon s) - \frac{\partial_{tt}a(\varepsilon s,0)}{a(\varepsilon s,0)}\right]tw'(t) + \varepsilon^3\left[k^3(s) + \frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\varepsilon s,0)\right]g_\perp(t)\right) \\ &+ \varepsilon^2[\Delta_x - \partial_{tt}]\varphi_1 + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\varphi_1 + N_0(\varphi_1) \end{aligned} \tag{3.45}$$

Analyzing the "new error" created by φ_1, we readily check using the expansions for the differential operators (2.68)-(2.73) and the definition (3.44) of N_0, that this error amounts to

$$\varepsilon^2[\Delta_x - \partial_{tt}]\varphi_1 + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\varphi_1 + N_0(\varphi_1) :=$$

$$\left[k(\varepsilon s) - \frac{\partial_{\boldsymbol{t}}a(\varepsilon s,0)}{a(\varepsilon s,0)}\right](-\varepsilon^3 Q(\varepsilon s)\psi_0'(t) + \varepsilon^4 U(\varepsilon s)\psi_1'(t))$$

$$+\varepsilon^4\Big\{(-Q''(\varepsilon s)\psi_0)+\big[\mathcal{J}_a[h](\varepsilon s)-tQ(\varepsilon s)\big]Q(\varepsilon s)\psi_0'$$
$$-\frac{\partial_{\boldsymbol{s}}a(\varepsilon s,0)}{a(\varepsilon s,0)}Q'(\varepsilon s)\psi_0-2h_1'(-Q'(\varepsilon s)\psi_0')-|h_1'|^2Q(\varepsilon s)\psi_0''\Big\}$$
$$+O(\varepsilon^4)w(t)(-Q(\varepsilon s)\psi_0)^2+O(\varepsilon^5(1+|\varepsilon s|)^{-4-\alpha}e^{-\sqrt{2}|t|}) \tag{3.46}$$

where we have adopted the following convention

$$\begin{aligned} Q(s) &:= \left[\frac{\partial_{\boldsymbol{tt}}a(s,0)}{a(s,0)}-2k^2(s)\right], \\ U(s) &:= \left[k^3(s)+\frac{\partial_{ttt}a(s,0)}{a(s,0)}-3\frac{\partial_{tt}a(s,0)\partial_t a(s,0)}{a^2(s,0)}+2\left(\frac{\partial_t a(s,0)}{a(s,0)}\right)^3\right] \end{aligned} \tag{3.47}$$

and have used that the error terms in the differential operator evaluated in φ_1, associated to $A_0(\varepsilon s,\varepsilon(t+h))$, $\tilde{B}_0(\varepsilon s,\varepsilon(t+h))$, $\tilde{C}_0(\varepsilon s,\varepsilon(t+h))$ are basically like $O(\varepsilon^5(1+|\varepsilon s|)^{-4-2\alpha}e^{-\sqrt{2}|t|})$, given that h has a bounded size is εs by (3.35), and since $\varphi_1(s,t)$ has smooth dependence in εs with size $O(\varepsilon^2(1+|\varepsilon s|)^{-2-\alpha}e^{-\sqrt{2}|t|})$.

Then, upon considering that Γ satisfies the criticality condition (2.10), the error (3.46) is reduced to

$$\varepsilon^2[\Delta_x-\partial_{tt}]\varphi_1+\varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\varphi_1+N_0(\varphi_1)=\varepsilon^4Q(\varepsilon s)\psi_0'(t)h''(\varepsilon s)+R_0(\varepsilon s,t,h) \tag{3.48}$$

where the function $R_0=R_0(\varepsilon s,t,h(\varepsilon s),h'(\varepsilon s))$ has Lipschitz dependence in variables h,h' on the ball $\|h\|_{L^\infty(\mathbb{R})}+\|h'\|_{L^\infty(\mathbb{R})}\le\mathcal{K}\varepsilon$. Moreover, it turns out that for any $\lambda\in(0,1)$:

$$\|R_0(\varepsilon s,t,h)\|_{C^{0,\lambda}(B_1(s,t))}\le C\varepsilon^4(1+|\varepsilon s|)^{-4-\alpha}e^{-\sqrt{2}|t|}$$

given the assumptions on $h(s,t),k(s),a(s,t),\partial_{ijk}a(s,t)$, and the observation made on ψ_0.

In consequence, we have eliminated in $S(u_0)$ the h-independent term $O(\varepsilon^2)$ that did not contribute to the projection $\Pi(s)$, and replaced it by one of smaller size and faster decay. In addition, we have also canceled a component of the term $O(\varepsilon^3)$ in $S(u_0)$ that is orthogonal to w', and independent of h, replacing it with an error smaller and with faster decay.

More explicitly, thanks to the decomposition (3.45) of $S(u_1)$, to the expression (3.38) for $S(u_0)$ and to the estimate (3.48), we can compute the error of the second approximation u_1

$$\begin{aligned} S(u_1) &:= \varepsilon^2\Delta_x u_1+\varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x u_1+f(u_1) \\ &= -\varepsilon^2\mathcal{J}_a[h](\varepsilon s)w'(t)+\varepsilon^4Q(\varepsilon s)\psi_0'(t)h''(\varepsilon s)-\varepsilon^3(t+h)A_0(\varepsilon s,\varepsilon(t+h))h''(\varepsilon s)w'(t) \\ &\quad +R_1(\varepsilon s,t,h(\varepsilon s),h'(\varepsilon s)) \end{aligned} \tag{3.49}$$

where the operators A_0 and $\tilde{B}_0$ are given in (3.7)-(3.8).
More explicitly, the new error amounts to

$$
\begin{aligned}
R_1 = {} & \varepsilon^2|h'|^2 w''(t) + R_0(\varepsilon s, t) + \varepsilon^3(t+h)A_0(\varepsilon s, \varepsilon(t+h))|h'|^2 w''(t) \\
& - \varepsilon^3(t+h)\tilde{B}_0(\varepsilon s, \varepsilon(t+h))h'w'(t) + \varepsilon^4(t+h)\,O\left(\partial_{\boldsymbol{ttt}}\left(\frac{\partial_{\boldsymbol{t}} a}{a}\right) + k^4\right) t^2 w'(t)
\end{aligned}
\tag{3.50}
$$

taking into account the expansion (3.9) for $\tilde{C}_0$, with its term h-independent is canceled as a result of this better approximation u_1. Indeed, we have

$$
\begin{aligned}
& \varepsilon^3(t+h)^2\tilde{C}_0(\varepsilon s, \varepsilon(t+h))w'(t) - \varepsilon^3 U(\varepsilon s)t^2 w'(t) = \\
& \qquad \varepsilon^4(t+h)\,O\left(\partial_{\boldsymbol{ttt}}\left(\frac{\partial_{\boldsymbol{t}} a}{a}\right) + k^4\right) t^2 w'(t) + \varepsilon^3(h+h^2)\tilde{C}_0(\varepsilon s, \varepsilon(t+h))\,O(w'(t))
\end{aligned}
$$

Furthermore, $R_1 = R_1(\varepsilon s, t, h(\varepsilon s), h'(\varepsilon s))$ has Lipschitz dependence in variables h, h' on the ball $\|h\|_{L^\infty(\mathbb{R})} + \|h'\|_{L^\infty(\mathbb{R})} \leq \mathcal{K}\varepsilon$, and this error satisfies

$$
|\partial_\imath R_1(\varepsilon s, t, \imath, \jmath)| + |\partial_\jmath R_1(\varepsilon s, t, \imath, \jmath)| + |R_1(\varepsilon s, t, \imath, \jmath)| \leq C\varepsilon^4(1+|\varepsilon s|)^{-2-2\alpha}e^{-\sqrt{2}|t|}
$$

with the constant C above depending on the number $\mathcal{K}$ of condition (3.35).

For the next chapter it will be essential the following remark:

Remark 7. *Assuming that h behaves as stated in* (3.35)*, it can be readily checked that $\mathcal{J}_a[h](\varepsilon s) = O((1+|\varepsilon s|)^{-2-\alpha})$, $Q(\varepsilon s) = O((1+|\varepsilon s|)^{-2-\alpha})$, and also $A_0(\varepsilon s, \varepsilon(t+h)) = O((1+|\varepsilon s|)^{-1-\alpha/2})$. Therefore, incorporating the size of h given in* (3.35)*, plus the behavior of $w(t), \psi_0'(t), \psi_1(t)$ along with its derivatives, we can deduce that this new error behaves like*

$$
S(u_1) - \left[-\varepsilon^2\mathcal{J}_a[h](\varepsilon s)w'(t)\right] = O(\varepsilon^4(1+|\varepsilon s|)^{-2-\alpha}e^{-\sqrt{2}|t|}) \tag{3.51}
$$

3.3.3. The global first approximation

The approximation $u_1(x)$ in (3.41) will be sufficient for our purposes, however, it is defined so far only within a region of the type $\mathcal{N}_\delta$. Since we are assuming that Γ is a connected, simple and oriented curve, that it also possesses two infinite ends departing from each other, it follows that $\mathbb{R}^2 \setminus \Gamma$ consists of precisely two components S_+ and S_-. Let us use the convention that ν points towards S_+. The previous comments allow us to define consistently in $\mathbb{R}^2 \setminus \Gamma$ the function $\mathbb{H}$ as

$$
\mathbb{H}(x) := \begin{cases} +1 & \text{if } x \in S_+ \\ -1 & \text{if } x \in S_- \end{cases} \tag{3.52}
$$

A important point to note here is that $\mathbb{H}$ properly solves the Allen-Cahn equation with inhomogeneity, since vanishes the nonlinearity term $f(\mathbb{H})$ on its two states of minimal energy $f(\pm 1) = -F'(\pm 1) = 0$. This amount to say

$$S(\mathbb{H}) = \Delta_x \mathbb{H} + \frac{\nabla_{\bar{x}} a}{a} \nabla_x \mathbb{H} + f(\mathbb{H}) \equiv 0, \qquad \text{a.e.} \qquad x \in \mathbb{R}^2$$

A direct consequence of the latter, is that $\mathbb{H}$ creates no error in the Allen-Cahn equation within the infinite region $\mathbb{R}^2 \setminus \mathcal{N}_\delta$.

Recall that the main goal is to find a global approximate solution u, generating the least error as possible in the entire space $\mathbb{R}^2$. This makes $\mathbb{H}$ to be a solid candidate of a solution in the outer region $\mathbb{R}^2 \setminus \mathcal{N}_\delta$. The latter, plus the exponential convergence of the inner approximation u_1 to $\mathbb{H}$ as $|t|$ increases, motivates to set the global approximation u simply as the interpolation between u_1 and $\mathbb{H}$ sufficiently well inside in $\mathbb{R}^2 \setminus \Gamma$. This job can be done using a cut-off function depending on the Fermi coordinate $|t|$.

To develop this construction, for $\varepsilon > 0$ define the open set

$$\tilde{\mathcal{N}_\delta} = \left\{ x = X_{\varepsilon,h}(s,t) \in \mathbb{R}^2 / \ |t + h(\varepsilon s)| < \frac{\delta}{\varepsilon} + c_0 |s| =: \rho_\varepsilon(s) \right\} \tag{3.53}$$

where $\delta > 0$ is small and $c_0 > 0$ is a fixed number. Note that coordinates (s,t) are well-defined in $\tilde{\mathcal{N}_\delta}$ for any sufficiently small δ, further, $X_{\varepsilon,h}$ is one to one in this set since o$h = O(\varepsilon)$.

Let $\eta(s)$ be a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$, and define

$$\zeta_3(x) := \begin{cases} \eta(|t + h(\varepsilon s)| - \rho_\varepsilon(s) + 3) & \text{if } x \in \tilde{\mathcal{N}_\delta} \\ 0 & \text{if } x \notin \tilde{\mathcal{N}_\delta} \end{cases} \tag{3.54}$$

where ρ_ε is defined in (3.53). Observe from the definition of η that given any small and fixed $\delta > 0$, and $\varepsilon > 0$ comparatively smaller, it holds that $\text{supp}(\zeta_3)$ is properly embedded in a region of $x \in \mathbb{R}^2$ for which the Fermi coordinates (s,t) are well defined. This function satisfies $\zeta_3 \equiv 1$ for $x = X_{\varepsilon,h}(s,t)$ with $|t + h(\varepsilon s)| < \rho_\varepsilon(s) - 2$, and $\zeta_3 \equiv 0$ for $x = X_{\varepsilon,h}(s,t)$ such $|t + h(\varepsilon s)| > \rho_\varepsilon(s) - 1$, and also for points $x \notin \tilde{N}_\delta$.

Now consider the global approximation $\mathtt{w}(x)$ to be defined as

$$\mathtt{w} := \zeta_3 \cdot u_1 + (1 - \zeta_3) \cdot \mathbb{H} \tag{3.55}$$

where $\mathbb{H}$ is given by (3.52), and $u_1(x)$ by (3.41). In this way we can think as $u_1(s,t)$ to be equal to $\mathbb{H}$ outside the region $\tilde{\mathcal{N}_\delta}$.

Using that $\mathbb{H}$ is an exact solution in $\mathbb{R}^2 \setminus \Gamma$, the error of global approximation can be computed as

$$\begin{aligned} S(\mathtt{w}) &= \Delta_x \mathtt{w} + \frac{\nabla_{\bar{x}} a}{a} \nabla_x \mathtt{w} + f(\mathtt{w}) \\ &= (\Delta_x \zeta_3 u_1 + 2\nabla_x \zeta_3 \nabla_x u_1 + \zeta_3 \Delta_x u_1) + (\Delta_x (1 - \zeta_3)\mathbb{H} + 2\nabla_x (1 - \zeta_3) \nabla_x \mathbb{H} + (1 - \zeta_3)\Delta_x \mathbb{H}) \end{aligned}$$

$$+ u_1 \frac{\nabla_{\bar{x}} a}{a} \nabla_x \zeta_3 + \zeta_3 \frac{\nabla_{\bar{x}} a}{a} \nabla_x (u_1 - \mathbb{H}) + \mathbb{H} \frac{\nabla_{\bar{x}} a}{a} \nabla_x (1 - \zeta_3) + (1 - \zeta_3) \frac{\nabla_{\bar{x}} a}{a} \nabla_x \mathbb{H}$$
$$+ f(\zeta_3 u_1 + (1 - \zeta_3)\mathbb{H})$$
$$= \zeta_3 S(u_1) + E \tag{3.56}$$

where $S(u_1)$ is calculated in (3.49) and whose size is bounded in (3.51).
Additionally, the term E is given by

$$E = \Delta_x \zeta_3 (u_1 - \mathbb{H}) + 2\nabla_x \zeta_3 \nabla_x (u_1 - \mathbb{H}) + (u_1 - \mathbb{H}) \frac{\nabla_{\bar{x}} a}{a} \nabla_x \zeta_3 + f\big(\zeta_3 u_1 + (1 - \zeta_3)\mathbb{H}\big) - \zeta_3 f(u_1) \tag{3.57}$$

It is worth to mention that the divergent form chosen for the neighborhood $\tilde{\mathcal{N}}_\delta$ in (3.53), has the purpose of making the new error terms created in E, to have an exponential decay in variable $|s|$, and moreover, an exponentially small size $O(e^{-\delta/\varepsilon})$. In fact, the convergence of functions $(w - \pm 1), \psi_0, \psi_1$ to 0, at an exponentially rate, forces that

$$u_1 - \mathbb{H} = (w - \pm 1) + \varphi_1(s,t) \sim e^{-\sqrt{2}|t + h(\varepsilon s)|} \text{ within the region } \rho_\varepsilon - 2 < |t + h| < \rho_\varepsilon - 1$$

and therefore

$$|E| \leq C\ e^{-\sqrt{2}|t + h(\varepsilon s)|} \leq C e^{-\sqrt{2}\delta/\varepsilon} \cdot e^{-c|s|}$$

Furthermore, observe that $|t + h(\varepsilon s)| = |z|$ where z is the normal coordinate to Γ, so formula (3.54) implies that ζ_3 does not depend on h. Thus, in particular the term $\Delta_x \zeta_3$ does not involve the second derivative of h.

Chapter 4

The proof of Theorem 1

4.1. Setting the Functional Scheme

We look for a solution u of the inhomogeneous Allen-Cahn equation (2.65) in the form

$$u = \mathtt{w} + \varphi \tag{4.1}$$

where $\mathtt{w}$ is the global approximation defined in (3.55) and φ is small in some suitable sense. The equation that φ needs to solve is the following

$$\Delta_x \varphi + \varepsilon \frac{\nabla_{\bar{x}} a}{a} \nabla_x \varphi + f'(\mathtt{w})\varphi = -S(\mathtt{w}) - N_1(\varphi) \tag{4.2}$$

where

$$S(\mathtt{w}) := \Delta_x \mathtt{w} + \varepsilon \frac{\nabla_{\bar{x}} a}{a} \nabla_x \mathtt{w} + f(\mathtt{w}) \tag{4.3}$$

$$N_1(\varphi) := f(\mathtt{w} + \varphi) - f(\mathtt{w}) - f'(\mathtt{w})\varphi \tag{4.4}$$

This section is aimed to reduce problem (4.2), by solving one qualitatively similar for a function $\phi(s,t)$ defined in the whole space $\mathbb{R} \times \mathbb{R}$.

However before doing this, we need to introduce various norms that will allow us to set up an adequate functional scheme to solve (4.2). Let us consider $\eta(s)$, a cut-off function with $\eta(s) = 1$ for $s < 1$ and $\eta = 0$ for $s > 2$, we define

$$\zeta_n(x) := \begin{cases} \eta\left(|t + h(\varepsilon s)| - \rho_\varepsilon(s) + n\right) & \text{if } x \in \tilde{\mathcal{N}}_\delta \\ 0 & \text{if } x \notin \tilde{\mathcal{N}}_\delta \end{cases} \tag{4.5}$$

where ρ_ε and $\tilde{\mathcal{N}_\delta}$ are defined in (3.53) respectively. Note that $\zeta_n \equiv 1$ for $x = X_{\varepsilon,h}(s,t)$ such that $|t+h| < \rho_\varepsilon - (n-1)$ and also $\zeta_n \equiv 0$ for $x = X_{\varepsilon,h}(s,t)$ with $|t+h| > \rho_\varepsilon - (n-2)$.

For a function $g(x)$ defined in $\mathbb{R}^2$, let us consider numbers $\lambda \in (0,1)$, $b_1, b_2 > 0$, and some weight function $K(x)$ defined for $x = (x_1, x_2)$ as follows

$$K(x) := \zeta_2(x)\left[e^{\sigma|t|/2}(1+|\varepsilon s|)^\mu\right] + (1-\zeta_2(x))e^{b_1|x_1|+b_2|x_2|} \tag{4.6}$$

where we assume that $b_1^2 + b_2^2 < (\sqrt{2}-\tau)/2$ for $\tau > 0$ small but fixed. We agreed

$$\|g\|_{L^\infty_K(\mathbb{R}^2)} := \sup_{x\in\mathbb{R}^2} K(x)\|g\|_{L^\infty(B_1(x))} \tag{4.7}$$

$$\|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} := \sup_{x\in\mathbb{R}^2} K(x)\|g\|_{C^{0,\lambda}(B_1(x))} \tag{4.8}$$

On the other hand, for functions $g(s,t)$ and $\phi(s,t)$ defined in whole $\mathbb{R}\times\mathbb{R}$, and for certain $\mu \geq 0$, $0 < \sigma < \sqrt{2}$, and $\varepsilon > 0$ let

$$\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} := \sup_{(s,t)\in\mathbb{R}\times\mathbb{R}} (1+|\varepsilon s|)^\mu e^{\sigma|t|}\|g\|_{C^{0,\lambda}(B_1(s,t))} \tag{4.9}$$

As for ϕ, we define

$$\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} := \|D^2\phi\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|D\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} + \|\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} \tag{4.10}$$

Consider also for $\lambda \in (0,1)$ and a function defined in $\mathbb{R}$, the norm

$$\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} := \sup_{s\in\mathbb{R}}(1+|s|)^{2+\alpha}\|f\|_{C^{0,\lambda}(s-1,s+1)} \tag{4.11}$$

Assume in what follows that there is a constant $\mathcal{K} > 0$ and $\alpha > 0$, such that the parameter function $h(\bar{s})$ satisfies for any $\lambda \in (0,1)$

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} := \|h\|_{L^\infty(\mathbb{R})} + \|(1+|\bar{s}|)^{1+\alpha}h'\|_{L^\infty(\mathbb{R})} + \sup_{\bar{s}\in\mathbb{R}}(1+|\bar{s}|)^{2+\alpha}\|h''\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)} \leq \mathcal{K}\varepsilon \tag{4.12}$$

4.2. Gluing procedure

The purpose of this section is to find a system of differential equations satisfied by a solution close to u_1, to the inhomogeneous Allen-Cahn equation, which takes into account the fact that this equation must be satisfied in the whole space $\mathbb{R}\times\mathbb{R}$ and where the local approximation creates error terms that may be become large.

We will make use of the technique explained below known as *the gluing procedure*. The idea is to look for a solution $\varphi(x)$ of problem (4.2), that has the following form

$$\varphi(x) = \zeta_3(x)\phi(s,t) + \psi(x), \quad \text{for} \quad (s,t) = X^{-1}_{\varepsilon,h}(x) \tag{4.13}$$

where ϕ is defined in $\mathbb{R}\times\mathbb{R}$, $\psi(x)$ is defined in entire $\mathbb{R}^2$, and $\zeta_3(x)\phi(s,t)$ is understood as zero outside $\tilde{\mathcal{N}}_\delta$. Using that $\zeta_3\cdot\zeta_4=\zeta_4$, we get that this candidate of a solution u satisfies

$$
\begin{aligned}
S(\mathtt{w}+\varphi) &= \Delta_x\varphi+\varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\varphi+f'(\mathtt{w})\varphi+S(\mathtt{w})+N_1(\varphi)\\
&=(\Delta_x\zeta_3\phi+2\nabla_x\zeta_3\nabla_x\phi+\zeta_3\Delta_x\phi)+\Delta_x\psi+\varepsilon\frac{\nabla_{\bar{x}}a}{a}(\nabla_x\zeta_3\phi+\zeta_3\nabla_x\phi+\nabla_x\psi)\\
&+f'(\mathtt{w})(\zeta_3\phi+\psi)+S(\mathtt{w})+N_1(\zeta_3\phi+\psi)\\
&=\zeta_3\left[\Delta_x\phi+\varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\phi+f'(u_1)\phi+\zeta_4[f'(u_1)-f'(H(t))]\psi+\zeta_4N_1(\psi+\phi)+S(u_1)\right]\\
&+\Delta_x\psi+\varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\psi+[(1-\zeta_4)f'(u_1)+\zeta_4f'(H(t))]\psi+(1-\zeta_3)S(\mathtt{w})\\
&+(1-\zeta_4)N_1(\psi+\zeta_3\phi)+2\nabla_x\zeta_3\nabla_x\phi+\phi\Delta_x\zeta_3+\varepsilon\phi\frac{\nabla_{\bar{x}}a}{a}\nabla_x\zeta_3
\end{aligned}
\tag{4.14}
$$

where we have used definitions (3.55) of $\mathtt{w}$, (4.4) of N_1, and that $H(t)$ is some increasing smooth function satisfying

$$
H(t)=\begin{cases}+1 & \text{if } t>1\\ -1 & \text{if } t<-1\end{cases}
\tag{4.15}
$$

In this way, we will have constructed a solution $\varphi=\zeta_3\phi+\psi$ to problem (4.2) if we require that the pair (ϕ,ψ) satisfies the coupled system below

$$
\Delta_x\phi+\varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\phi+f'(u_1)\phi+\zeta_4[f'(u_1)-f'(H(t))]\psi+\zeta_4N_1(\psi+\phi)+S(u_1)=0\quad\text{for } |t|<\frac{\delta}{\varepsilon}
\tag{4.16}
$$

$$
\begin{aligned}
&\Delta_x\psi+\varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\psi+[(1-\zeta_4)f'(u_1)+\zeta_4f'(H(t))]\psi+(1-\zeta_3)S(\mathtt{w})+(1-\zeta_4)N_1(\psi+\zeta_3\phi)\\
&\quad+2\nabla_x\zeta_3\nabla_x\phi+\phi\Delta_x\zeta_3+\varepsilon\phi\frac{\nabla_{\bar{x}}a}{a}\nabla_x\zeta_3=0\quad\text{in}\quad\mathbb{R}^2
\end{aligned}
\tag{4.17}
$$

In order to find a solution, we will extend equation (4.16) to entire $\mathbb{R}\times\mathbb{R}$ in the following way. Let us set

$$
\mathtt{B}(\phi)=\zeta_0\tilde{\mathtt{B}}_0(\phi):=\zeta_0[\Delta_x-\partial_{tt}-\partial_{ss}]\phi
\tag{4.18}
$$

where Δ_x is expressed in $(s,t)-$coordinates using formula (2.68), and $\mathtt{B}(\phi)$ is understood to be zero for $|t+h|>\delta/\varepsilon+2$. The remaining terms of equation (4.16) are simply extended as zero beyond the support of ζ_1. Thus equation (4.16) is extended as

$$
\begin{aligned}
\partial_{tt}\phi+&\partial_{ss}\phi+\mathtt{B}(\phi)+f'(w(t))\phi=-\tilde{S}(u_1)\\
&-\left\{\varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\phi+[f'(u_1)-f'(w)]\phi+\zeta_4[f'(u_1)-f'(H(t))]\psi+\zeta_4N_1(\psi+\phi)\right\}\quad\text{in}\ \ \mathbb{R}^2
\end{aligned}
\tag{4.19}
$$

where the operator $\tilde{S}$ is given by

$$\tilde{S}(u_1)(\varepsilon s,t) = -\varepsilon^2 \mathcal{J}_a[h](\varepsilon s) w'(t) + \varepsilon^4 Q(\varepsilon s)\psi_0'(t)\cdot h''(\varepsilon s) + \zeta_0 \Big\{ \varepsilon^3 (t+h) A_0(\varepsilon s, \varepsilon(t+h)) h''(\varepsilon s)\cdot w''(t) + R_1 \Big\} \tag{4.20}$$

thanks to formula (3.49), and by recalling that

$$R_1 = R_1(\varepsilon s, t, h(\varepsilon s), h'(\varepsilon s))$$

satisfies

$$|\partial_\imath R_1(\varepsilon s,t,\imath,\jmath)| + |\partial_\jmath R_1(\varepsilon s,t,\imath,\jmath)| + |R_1(\varepsilon s,t,\imath,\jmath)| \le C\varepsilon^4 (1+|\varepsilon s|)^{-2-2\alpha} e^{-\sqrt{2}|t|} \tag{4.21}$$

In summary, $\tilde{S}(u_1)$ coincides with $S(u_1)$ in the region where $\zeta_0 \equiv 1$, while outside the support of ζ_0 the terms of $S(u_1)$ which are not defined for all t, are cut off.

To solve the resulting system (4.17)-(4.19), we first need to solve equation (4.17) in ψ for a given small function ϕ in absolute value. Observe that the zeroth order term of this differential operator, $[(1-\zeta_4)f'(u_1) + \zeta_4 f'(H)]$, is uniformly negative and so the operator in (4.17) is qualitatively similar to $\Delta_x + \varepsilon \nabla_{\bar{x}} a/a \cdot \nabla_x - 1$, which is invertible for $\varepsilon > 0$ small under suitable restrictions on the potential $a(\bar{x})$. This allow us to use the contraction mapping principle to find a solution $\psi = \Psi(\phi)$, according to the next result whose detailed proof is carried out in Section 5.

Lemma 5. *Let $\lambda \in (0,1)$, $\sigma \in (0,\sqrt{2})$, $\mu \in (0, 2+\alpha)$. There is $\varepsilon_0 > 0$, such that for any small $\varepsilon \in (0,\varepsilon_0)$ the following holds. Given ϕ with $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le 1$, there is a unique solution $\psi = \Psi(\phi)$ of problem* (4.17) *with*

$$\|\psi\|_X := \|D^2\psi\|_{C^{0,\lambda}_K(\mathbb{R}^2)} + \|D\psi\|_{L^\infty_K(\mathbb{R}^2)} + \|\psi\|_{L^\infty_K(\mathbb{R}^2)} \le Ce^{-\sigma\delta/2\varepsilon} \tag{4.22}$$

Besides, Ψ satisfies the Lipschitz condition

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_X \le Ce^{-\sigma\delta/2\varepsilon} \|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{4.23}$$

where the norms $L^\infty_K, C^{0,\lambda}_K, C^{2,\lambda}_{\mu,\sigma}$ are defined in (4.7)-(4.8)-(4.10).

Thanks to this Lemma we can replace $\psi = \Psi(\phi)$ into the first equation (4.19), and then by setting the nonlinear term

$$\mathtt{N}(\phi) := \mathtt{B}(\phi) + \varepsilon \frac{\nabla_{\bar{x}} a}{a} \nabla_x \phi + [f'(u_1) - f'(w)]\phi + \zeta_4 [f'(u_1) - f'(H(t))]\Psi(\phi) + \zeta_4 N_1(\Psi(\phi)+\phi) \tag{4.24}$$

our problem is reduced to find a solution ϕ to the following nonlinear, nonlocal problem

$$\partial_{tt}\phi + \partial_{ss}\phi + f'(w)\phi = -\tilde{S}(u_1) - \mathtt{N}(\phi) \ \text{ in } \ \mathbb{R}\times\mathbb{R} \tag{4.25}$$

In the remaining of the proof we concentrate our efforts in solving equation (4.25). For this purpose, we will find a solution of the nonlocal problem by considering two steps:

1. Improving the approximation, which basically corresponds to solve the nonlocal equation for ϕ that eliminates the part of the error that does not contribute to the projections onto w', namely $\int_{\mathbb{R}}[\tilde{S}(u_1) + N(\phi)]w'(t)dt$, that amounts to a nonlinear problem in ϕ.
2. Adjusting h in such way that the resulting projection is actually zero.

Let us set up the scheme for the Step 1 in a precise form.

4.3. Eliminating terms not contributing to projections

Consider the problem of finding a function $\phi(s,t)$ such that for a certain function $c(s)$ defined in $\mathbb{R}$, satisfies the following nonlinear projected problem

$$(NPP)\begin{cases} \partial_{tt}\phi + \partial_{ss}\phi + f'(w)\phi = -\tilde{S}(u_1) - \mathtt{N}(\phi) + c(s)w'(t) \ \text{ in } \ \mathbb{R}\times\mathbb{R} \\ \displaystyle\int_{\mathbb{R}} \phi(s,t)w'(t)dt = 0, \quad \forall s\in\mathbb{R} \end{cases} \tag{4.26}$$

Solving this problem for ϕ amounts "eliminating the part of the error that does not contribute to the projection" in equation (4.26). To justify this phrase, let us consider the associated linear projected problem

$$\begin{cases} \partial_{tt}\phi + \partial_{ss}\phi + f'(w)\phi = g(s,t) + c(s)w'(t) \ \text{ in } \ \mathbb{R}\times\mathbb{R} \\ \displaystyle\int_{\mathbb{R}} \phi(s,t)w'(t)dt = 0, \quad \forall s\in\mathbb{R} \end{cases} \tag{4.27}$$

Assuming that the corresponding operations can be carried out, we can multiply the equation by $w'(t)$ and integrate t in $\mathbb{R}$, for fixed s. Then

$$\frac{d^2}{ds^2}\left(\int_{\mathbb{R}}\phi(s,t)w'(t)dt\right) + \int_{\mathbb{R}}\phi(s,t)[w''' + f'(w)w']dt = \int_{\mathbb{R}} g(s,t)w'(t) + c(s)\int_{\mathbb{R}}|w'(t)|^2$$

Noting that the left hand side of the above identity is identically zero, it turns out that

$$c(s) = -\frac{\int_{\mathbb{R}} g(s,t)w'(t)dt}{\int_{\mathbb{R}}|w'(t)|^2dt} \tag{4.28}$$

Hence if ϕ solves problem (4.27), then ϕ is precisely "eliminating" the part of g which does not contribute to the projections in the equation $\Delta_{t,s}\phi + f'(w)\phi = g$. This means that ϕ solves the same equation, but with g replaced by $\tilde{g}$ given by

$$\tilde{g}(s,t) = g(s,t) - \frac{\int_{\mathbb{R}} g(s,\tau)w'(\tau)d\tau}{\int_{\mathbb{R}} |w'(\tau)|^2 d\tau} w'(t) \tag{4.29}$$

Observe that the term $c(s)$ in problem (4.26) has a similar role, except that we cannot find it so explicitly, since this time the PDE in ϕ is nonlinear.

The solvability of problem (4.26) is a consequence of a theory devised to solve the linear problem (4.27), in which we consider a class of right hand sides g that behave qualitatively similar to that of the error $S(u_1)$. As we seen in (4.20), a typical element of this error is of the type $O\left((1+|\varepsilon s|)^{-\mu} e^{-\sqrt{2}|t|}\right)$, and therefore this kind of functions $g(s,t)$ are those that we want to take into account.

Now we show that the linear problem (4.27) has a unique solution ϕ, which respects the size of g in norm (4.9), up to its second derivatives. Concerning this property the next result is established, whose proof is carried out in Section 5:

Proposition 3. *Given $\mu \geq 0$ and $0 < \sigma < \sqrt{2}$, there is a constant $C > 0$ such that for all sufficiently small $\varepsilon > 0$ the following holds. For any g with $\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} < \infty$, the problem* (4.27) *with $c(s)$ defined in* (4.28), *has a unique solution ϕ with $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} < \infty$. Furthermore, this solution satisfies the estimate*

$$\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{4.30}$$

Thanks to this proposition, we can use the contraction mapping principle for a small ϕ, in order to solve problem (4.26). This can be done due to the small Lipschitz character of the terms involved in the operator $\mathtt{N}(\phi)$ in (4.24). In addition, the error terms $\phi-$independent satisfy

$$\|\tilde{S}(u_1) + \varepsilon^2 \mathcal{J}_a[h](\varepsilon s) \cdot w'(t)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon^4 \tag{4.31}$$

Using this, and the fact that $\mathtt{N}(\phi)$ defines a contraction within a ball centered at zero with radius $O(\varepsilon^4)$ in norm C^1, we conclude the existence of a unique small solution of problem (4.26) whose size is $O(\varepsilon^4)$ in this norm. This solution ϕ turns out to define an operator in h, namely $\phi = \Phi(h)$, which exhibits a Lipschitz character in norms $\|\cdot\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$. In precise terms, we have the validity of the next result.

Proposition 4. *Given $\lambda \in (0,1)$, $\mu \in (0, 2+\alpha)$ and $\sigma \in (0,\sqrt{2})$, there exists a constant $K > 0$ such that the nonlinear projected problem* (4.26) *has a unique solution $\phi = \Phi(h)$ with*

$$\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq K\varepsilon^4 \tag{4.32}$$

Besides Φ *has small a Lipschitz dependence on* h *satisfying condition* (3.35), *in the sense*

$$\|\Phi(h_1) - \Phi(h_2)\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon^3 \|h_1 - h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})} \tag{4.33}$$

for any $h_1, h_2 \in C^{2,\lambda}_{loc}(\mathbb{R})$ *with* $\|h_i\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})} \leq \mathcal{K}\varepsilon$.

4.4. Adjusting the nodal set

In order to conclude the proof of Theorem 1 we have to carry out the second step, the adjustment of h within a region of the form (3.35), in such way that the "projections against w'" of the errors in the nonlinear projected problem for ϕ vanishes in whole $\mathbb{R}$. This amounts to make function $c(s)$ found for the solution $\phi = \Phi(h)$ of problem (4.26), identically zero. This projection can be found making use of expression (4.28) for $c(s)$

$$c(s)\int_{\mathbb{R}} |w'(t)|^2 dt = \int_{\mathbb{R}} \tilde{S}(u_1)(\varepsilon s, t) w'(t) dt + \int_{\mathbb{R}} \mathtt{N}(\Phi(h))(s,t) w'(t) dt \tag{4.34}$$

Setting $c_* := \int_{\mathbb{R}} |w'(t)|^2 dt$, using expression (4.20), and carrying out the same computation needed to calculate the projection of $S(u_0)$ (3.39), we readily obtain that the respective projection of $\tilde{S}(u_1)$ is given by

$$\int_{\mathbb{R}} \tilde{S}(u_1)(\varepsilon s, t) w'(t) dt = -c_*\varepsilon^2 \mathcal{J}_a[h](\varepsilon s) + c_*\varepsilon^2 G_1(h)(\varepsilon s) \tag{4.35}$$

where

$$\begin{aligned} c_* G_1(h)(\varepsilon s) :=\ & \varepsilon h''(\varepsilon s)\int_{\mathbb{R}} \zeta_0(t+h) A_0(\varepsilon s, \varepsilon(t+h)) w''(t) w'(t) dt \\ & + \varepsilon^2 Q(\varepsilon s) h''(\varepsilon s)\int_{\mathbb{R}} \psi_0'(t) w'(t) dt + \varepsilon^{-2}\int_{\mathbb{R}} \zeta_0\ R_1(\varepsilon s, t, h, h') w'(t) dt \end{aligned} \tag{4.36}$$

and we recall that R_1 is of size $O(\varepsilon^4)$ in the sense of (4.21). Thus setting

$$c_* G_2(h)(\varepsilon s) := \varepsilon^{-2}\int_{\mathbb{R}} \mathtt{N}(\Phi(h))(s,t) w'(t) dt, \quad \mathtt{G}(h)(\varepsilon s) := G_1(h)(\varepsilon s) + G_2(h)(\varepsilon s) \tag{4.37}$$

it turns out that equation (4.34) is equivalent to

$$c(s)\cdot c_* = -c_*\varepsilon^2 \mathcal{J}_a[h](\varepsilon s) + c_*\varepsilon^2 G_1(h)(\varepsilon s) + c_*\varepsilon^2 G_2(h)(\varepsilon s)$$

Therefore the condition of no projection for the error terms, amounts to the following problem on h

$$\mathcal{J}_a[h](\varepsilon s) = h''(\varepsilon s) + \frac{\partial_s a(\varepsilon s, 0)}{a(\varepsilon s, 0)} h'(\varepsilon s) - Q(\varepsilon s) h(\varepsilon s) = \mathtt{G}[h](\varepsilon s), \quad in \quad \mathbb{R} \tag{4.38}$$

Consequently, we will have proved Theorem 1 if we find a function h in such way it solves equation (4.38), while respecting the restriction (3.35) for suitable $\mathcal{K}$. Nevertheless, this task is not so simple since the operator $\mathcal{J}_a$ may have nontrivial bounded kernel under general assumptions on the potential $a(x, y)$ and on the curve Γ. Notwithstanding, if Γ meets a particular geometrical property related to the potential $a(x, y)$, it can be ensured that this situation on $\mathcal{J}_a$ cannot occur. Assuming the latter as possible, we need to devise a corresponding linear invertibility theory to solve problem (4.38). Then we consider the linear problem

$$\mathcal{J}_a[h](\varepsilon s) = f(\varepsilon s), \quad \text{in} \quad \mathbb{R} \tag{4.39}$$

and we look for suitable conditions on the curve and on the potential a, that guarantees the property already stated. The next result address this matter:

Proposition 5. *Given $\alpha > 0$, $\lambda \in (0,1)$, and a function f with $\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} < \infty$, assume that Γ is a nondegenerate geodesic curve with respect to $l_{a,\Gamma}$, as in definition 2. Further, suppose that Γ and the potential $a(x)$ meet the hypothesis of Proposition 1, so that $|Q(s)| \leq C(1+|s|)^{-2-\alpha}$. Then there exists a unique bounded solution h of problem* (4.39)*, and exists a positive constant $C = C(a, \Gamma, \alpha)$ such that*

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq C\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} \tag{4.40}$$

with the norms defined in (4.11)-(4.12).

Let us note that $\mathtt{G}$ is a small operator of size $O(\varepsilon)$ uniformly on function h satisfying (3.35). Hence Proposition 5 plus the contraction mapping principle yield the next result, which ensures the solvability of the nonlinear Jacobi equation. Its detailed proof can be found in Section 5.3.

Proposition 6. *Given $\alpha > 0$ and $\lambda \in (0,1)$, there exist a positive constant $\mathcal{K} > 0$ such that for any $\varepsilon > 0$ small enough the following holds. There is a unique solution h of* (4.38) *on the region* (4.12)*, namely $\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq \mathcal{K}\varepsilon$.*

4.5. Conclusion

To finalize, we will briefly summarize the scheme performed in the construction of a solution to the inhomogeneous Allen-Cahn equation (1.12).

We looked for a solution $u(x) = \mathtt{w}(x) + \zeta_3(x)\Phi(h)(x) + \Psi(\Phi(h))(x)$, where $\mathtt{w}$ is the global approximation and the other terms are corrections. As we saw, u would be an exact solution of equation (1.12), provided that the pair (ϕ, ψ) solves the coupled gluing system (4.16)-(4.17).

Lemma 5 gave a solution $\psi = \Psi(\phi)$ of the second gluing equation (4.17), requiring that ϕ is small enough in $C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)$-topology.

On the other hand, Proposition 6 provided a small solution h to the nonlinear Jacobi equation, in such way that the error terms of the right-hand side of the equation (4.26), $\tilde{S}(u_1[h]) + \mathtt{N}(\phi[h])$, became *orthogonal to* w' *in* $L^2(\mathbb{R})$. This ensures the solvability of the nonlinear projected problem for a sufficiently small $\phi = \Phi[h]$ as stated in Proposition 4, and in particular this provided a solution to the nonlocal problem (4.25), which corresponds to the extension of the first gluing equation (4.16).

Thus we have proved the existence of the solution pair (ϕ, ψ) to the gluing system. The proof of Theorem 1 is now complete. $\square$

Chapter 5

Auxiliary Results

5.1. The linearized operator

The purpose of the whole section is to give a proof of Proposition 3, which deals with the solvability of the linear projected problem (4.27) for ϕ. This will be done by dividing the demonstration into different parts: First we study the bounded kernel of the second-order differential operator associated to equation (4.27), next we justify the validity of the *a priori estimates* stated in this proposition, and then we finalize with the study of the existence for the linear projected problem.

5.1.1. Studying the Kernel of L

At the core of the proof of the stated a priori estimate, is the fact that the heteroclinic solution $w(t)$ of the ODE (3.17) is *nondegenerate* in a L^∞-sense, meaning that the linearized operator around w defined by

$$L(\phi) := \partial_{ss}\phi + \partial_{tt}\phi + f'(w(t))\phi \ , \quad (s,t) \in \mathbb{R}^2$$

satisfies the property below.

Lemma 6. *Let ϕ be a bounded and smooth solution of the problem*

$$L(\phi) = 0 \quad \text{in} \quad \mathbb{R}^2 \tag{5.1}$$

Then necessarily $\phi(s,t) = Cw'(t)$, with $C \in \mathbb{R}$.

This result asserts that the set $\mathfrak{B}_L$ comprising the smooth bounded kernel of the operator L, is a simple vector space generated by the one-variable heteroclinic solution w'. Hence a full description of the set $\mathfrak{B}_L$ has been reached.

Proof.-
First we recall an important fact about the one-dimensional operator $L_0(\psi) = \psi'' + f'(w)\psi$, through the following coercivity result.

Lemma 7. *There exists a constant $\vartheta > 0$ such that $\forall \psi \in H^1(\mathbb{R})$ satisfying the orthogonality condition $\int_{\mathbb{R}} \psi(t) \cdot w'(t)dt = 0$, it holds*

$$B(\psi) := \int_{\mathbb{R}} [|\psi'(t)|^2 - f'(w)\psi^2(t)]dt \geq \vartheta \int_{\mathbb{R}} [|\psi'(t)|^2 + |\psi(t)|^2]dt \tag{5.2}$$

A detailed proof of this property can be found in Section A.3 in the Annexe.

Now, let ϕ be a bounded solution of (5.1), and define the "orthogonal part of ϕ with respect to $w'(t)$", that is

$$\tilde{\phi}(s,t) := \phi(s,t) - \left(\int_{\mathbb{R}} \phi(s,\zeta) w'(\zeta) d\zeta \right) \frac{w'(t)}{\int_{\mathbb{R}} |w'(\zeta)|^2 d\zeta} \tag{5.3}$$

Observe that $\tilde{\phi}$ satisfies $L(\tilde{\phi})(s,t) = 0$, since

$$\begin{aligned} \partial_{ss}\tilde{\phi}(s,t) &= \partial_{ss}\phi(s,t) + \left(\int_{\mathbb{R}} -\partial_{ss}\phi(s,\zeta) w'(\zeta) d\zeta \right) \frac{w'(t)}{\int_{\mathbb{R}} |w'(\zeta)|^2 d\zeta} \\ &= \partial_{ss}\phi(s,t) + \left(\int_{\mathbb{R}} [\partial_{\zeta\zeta}\phi + f'(w(\zeta))\phi] w'(\zeta) d\zeta \right) \frac{w'(t)}{\int_{\mathbb{R}} |w'(\zeta)|^2 d\zeta} \end{aligned}$$

and integrating by parts in ζ and using that $w' \to 0$ as $|t| \to \infty$, we get

$$\begin{aligned} &= \partial_{ss}\phi(s,t) + \left(\int_{\mathbb{R}} [w'''(\zeta) + f'(w(\zeta)) w'(\zeta)] \phi(s,\zeta) d\zeta \right) \frac{w'(t)}{\int_{\mathbb{R}} |w'(\zeta)|^2 d\zeta} \\ &= \partial_{ss}\phi(s,t) + 0 \end{aligned}$$

given that w' solves the linearized version of ODE (3.17), namely $v'' + f'(w)v = 0$. Furthermore, we readily check that

$$\partial_{tt}\tilde{\phi}(s,t) = \partial_{tt}\phi(s,t) - \left(\int_{\mathbb{R}} \phi(s,\zeta) w'(\zeta) d\zeta \right) \cdot \frac{w'''(t)}{\int_{\mathbb{R}} |w'(\zeta)|^2 d\zeta}$$

So, using that $L(\phi) = 0$ and the fact that $(w'')' + f'(w)w' = 0$, it follows the desired property

$$L(\tilde{\phi})(s,t) = [\partial_{ss}\phi + \partial_{tt}\phi + f'(w)\phi] - \frac{\int_{\mathbb{R}} \phi(s,\zeta) \cdot w'(\zeta) d\zeta}{\int_{\mathbb{R}} |w'|^2} [w''' + f'(w)w'] \equiv 0$$

In addition, observe by definition (5.3) of $\tilde{\phi}$ that is orthogonal in L^2 against w', namely

$$\int_{\mathbb{R}} \tilde{\phi}(s,\zeta) \cdot w'(\zeta) d\zeta = 0 \quad \text{for all } s \in \mathbb{R} \tag{5.4}$$

Now, we claim that $\tilde{\phi}$ *has exponential decay in* t*, uniformly in* $s \in \mathbb{R}$. To prove this, consider θ in $(0,1)$ small so that for a certain $t_0 > 0$ it holds

$$f'(w) < -2\theta^2 \quad \text{for all } |t| > t_0 \tag{5.5}$$

Given that ϕ is bounded, we can easily seen from (5.3) that $\tilde{\phi}$ is also bounded. This motivates to consider for $\varepsilon > 0$ the following barrier function:

$$g_\varepsilon(s,t) = \|\tilde{\phi}\|_\infty e^{-\theta(|t-t_0|)} + \varepsilon \cosh(\theta s) + \varepsilon e^{\theta t} \tag{5.6}$$

First, it can be readily check that

$$\|\tilde{\phi}\|_\infty e^{-\theta|t-t_0|} \geq |\tilde{\phi}(s,t)| \quad \text{for all } \ |t| \leq t_0, \ \forall s \in \mathbb{R}$$

Now given $\varepsilon > 0$ we can always find $s_0 = s_0(\theta, \varepsilon)$ sufficiently large, such that

$$\varepsilon \cosh(\theta s) \geq \|\tilde{\phi}\|_\infty \quad \text{for all } |s| \geq s_0$$

The same argument allow us to find $t_1 = t_1(\theta, \varepsilon)$ with $t_1 >> t_0$, large enough in such way that

$$\varepsilon e^{\theta t} \geq \|\tilde{\phi}\|_\infty \quad \text{for all } |t| > t_1$$

The foregoing justify the fact that $\forall \varepsilon > 0$ the barrier function is above function $\tilde{\phi}$ in $\mathbb{R}^2$ except for a bounded set, namely

$$\tilde{\phi}(s,t) \leq g_\varepsilon(s,t) \quad \text{for } (s,t) \text{ with } |s| \geq s_0, |t| \geq t_1 \tag{5.7}$$

On the other hand, by definition of g_ε we get

$$\begin{aligned} L(g_\varepsilon) &= \partial_{ss} g_\varepsilon + \partial_{tt} g_\varepsilon + f'(w(t)) g_\varepsilon \\ &= \varepsilon\theta^2 \cosh(\theta s) + \theta^2 \|\tilde{\phi}\|_\infty e^{-\theta(|t-t_0|)} + \varepsilon\theta^2 e^{\theta t} + f'(w)(\|\tilde{\phi}\|_\infty e^{-\theta(|t-t_0|)} + \varepsilon\cosh(\theta s) + \varepsilon e^{\theta t}) \\ &< -\theta^2 \cdot g_\varepsilon < 0 \quad \text{for all } \ |t| > t_0, \ \forall s \in \mathbb{R} \end{aligned}$$

where we used the assumption (5.5) and that $g_\varepsilon > 0$ in $\mathbb{R}^2$. Then using maximum principle on the bounded set $\Omega := (-s_0, s_0) \times [(-t_1, t_0) \cup (t_0, t_1)]$, it follows

$$|\tilde{\phi}(s,t)| \leq g_\varepsilon(s,t) \quad \text{for } \ (s,t) \in \Omega \tag{5.8}$$

since $L(\tilde{\phi} - g^*_\varepsilon) \geq 0$ in Ω. Furthermore, from inequalities (5.7)-(5.8) we deduce the global estimate

$$|\tilde{\phi}(s,t)| \leq \|\tilde{\phi}\|_\infty e^{-\theta(|t-t_0|)} + \varepsilon\cosh(\theta s) + \varepsilon e^{\theta t} \ , \ \text{for } \ (s,t) \in \mathbb{R}^2$$

finally, letting $\varepsilon \to 0^+$ we obtain the desired property:

$$|\tilde{\phi}(s,t)| \leq \|\phi\|_\infty \cdot e^{-\theta|t|} \ , \forall (s,t) \in \mathbb{R}^2 \tag{5.9}$$

In view of the above discussion, it turns out that the function

$$\varphi(s) := \int_{\mathbb{R}} \tilde{\phi}^2(s,t)dt \tag{5.10}$$

is well defined, and is bounded. In fact, so are its first and second derivatives by elliptic regularity of ϕ, and differentiation under the integral sign is thus justified. Observe that

$$\frac{d^2}{ds^2}\varphi(s) = \int_{\mathbb{R}} \frac{d}{ds}\left(2\tilde{\phi}(s,t)\partial_s\tilde{\phi}(s,t)\right)dt = 2\left[\int_{\mathbb{R}} |\partial_s\tilde{\phi}(s,t)|^2dt + \int_{\mathbb{R}} \tilde{\phi}(s,t)\partial_{ss}\tilde{\phi}(s,t)dt\right]$$

but as $L(\tilde{\phi}) = 0$ it folllows

$$= 2\left[\int_{\mathbb{R}} |\partial_s\tilde{\phi}(s,t)|^2dt - \int_{\mathbb{R}} (\partial_{tt}\tilde{\phi}(s,t) + f'(w)\tilde{\phi}(s,t))\tilde{\phi}(s,t)dt\right]$$

and integrating by parts in t, for $s \in \mathbb{R}$ fixed

$$\begin{aligned} &= 2\left[\int_{\mathbb{R}} |\partial_s\tilde{\phi}(s,t)|^2dt + \int_{\mathbb{R}} (|\partial_t\tilde{\phi}(s,t)|^2 - f'(w(t))\tilde{\phi}(s,t))\tilde{\phi}(s,t)dt\right] \\ &\geq 2\left[\int_{\mathbb{R}} |\partial_s\tilde{\phi}(s,t)|^2dt + \mu\cdot\int_{\mathbb{R}} [|\tilde{\phi}(s,t)|^2dt\right] \geq 2\vartheta\cdot\int_{\mathbb{R}} \tilde{\phi}^2(s,t)dt \end{aligned}$$

where the last inequality is due to coercivity (5.2), since $\tilde{\phi}(s,t) \in H^1(\mathbb{R})$ in variable t for s fixed, and $\tilde{\phi}$ satisfies (5.4) orthogonality condition. So this estimate can be restated as

$$\varphi''(s) - 2\vartheta\varphi(s) \geq 0 \ , \quad \text{for all } \ s \in \mathbb{R}$$

which is a differential inequality involving a uniformly elliptic operator $L_*(\psi) := \frac{d^2}{ds^2}\psi - 2\vartheta\psi$. Using standards arguments for elliptic equations, it is easy to see that $\varphi(s) \leq \varepsilon\cosh(\sqrt{2\vartheta}s)$ in $\mathbb{R}$, since $g^*_\varepsilon(s) := \varepsilon\cosh(\sqrt{2\vartheta}s)$ is a barrier function for operator L_*. In fact, given that φ is bounded, there exists $s^*_0 > 0$ such that

$$g^*_\varepsilon(s) \geq \|\varphi\|_\infty \quad \text{for all } \ |s| \geq s^*_0$$

Besides, since $L_*(g^*_\varepsilon) = 0$ we can apply maximum principle to justify that

$$\varphi(s) < g^*_\varepsilon(s) \quad \text{in } \ \Omega_* := (-s^*_0, s^*_0)$$

because $L_*(\varphi - g^*_\varepsilon) \geq 0$ in Ω_*. Thus, we obtain the desired estimate in the whole space

$$\varphi(s) \leq \varepsilon\cosh(\sqrt{2\vartheta}s) \ , \quad \forall s \in \mathbb{R}$$

and letting $\varepsilon \to 0^+$ we deduce that

$$\varphi(s) \leq 0 \ , \quad \forall s \in \mathbb{R}$$

Finally, recall from definition (5.10) that $\varphi \geq 0$. In conclusion, we have proved that $\varphi(s) \equiv 0$, which is equivalent to say

$$\forall s \in \mathbb{R} : \quad \tilde{\phi}(s,t) = 0 \quad \text{a.e. for } t \in \mathbb{R}$$

and by definition (5.3) of $\tilde{\phi}$, is the same that

$$\phi(s,t) = C(s) \cdot w'(t) \quad \text{with} \quad C(s) := \frac{\int_{\mathbb{R}} \phi(s,\zeta) w'(\zeta) d\zeta}{\int_{\mathbb{R}} |w'(\zeta)|^2 d\zeta} \in \mathbb{R} \tag{5.11}$$

It only remains to prove that $C(s)$ is actually constant on variable $s \in \mathbb{R}$. Recall that ϕ solves the equation induced by the linearized operator $L(\phi) = \Delta_{(s,t)}\phi + f'(w)\phi = 0$ for all $(s,t) \in \mathbb{R}^2$, so by replacing the expression (5.11) for ϕ in this equation, we obtain

$$C''(s)w'(t) + C(s)w'''(t) + f'(w)C(s)w'(t) = 0 \quad \Leftrightarrow \quad C''(s) + C(s)[w''' + f'(w)w'] = 0$$

that is $C''(s) = 0$, which implies $C(s) = As + B$, $\forall s \in \mathbb{R}$. To conclude, note that $\phi(s,t)$ is bounded solution, we can be restated as $C(s)$ bounded, since (5.11). So it follows that necessarily $A = 0$ and B is any real contact, which finishes Lemma 6. □

5.1.2. A priori estimates for the projected problem

We shall consider a slightly modified version of problem (4.27), with a domain bounded in s-direction. Let us study a slightly different version of the former problem, given by

$$\begin{aligned} \partial_{tt}\phi + \partial_{ss}\phi + f'(w(t))\phi = g(s,t) + c(s)w'(t) \quad &\text{in } \left[-\frac{R}{\varepsilon}, \frac{R}{\varepsilon}\right] \times \mathbb{R} \\ \phi = 0 \quad &\text{on } \left\{-\frac{R}{\varepsilon}, \frac{R}{\varepsilon}\right\} \times \mathbb{R} \\ \int_{\mathbb{R}} \phi(s,t)w'(t)dt = 0 \quad &\text{for all } s \in \left[-\frac{R}{\varepsilon}, \frac{R}{\varepsilon}\right] \end{aligned} \tag{5.12}$$

where we allow $R = +\infty$ and we recall that $c(s)$ is given in (4.28).

Let us first develop a simple tool to compute the size of the errors corresponding to each of the projections. We have the following Lemma

Lemma 8. *Given any $\sigma \in (0, \sqrt{2})$ and $\mu \geq 0$, consider a function $\rho(t)$ such that for every $t \in \mathbb{R}$, $0 < \rho(t) \leq C$. Assume also that there is some function $\Theta = \Theta(s,t)$ in such way that*

$$\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} := \sup_{(s,t)\in\mathbb{R}\times\mathbb{R}} (1 + |\varepsilon s|)^{\mu} e^{\sigma|t|} \|\Theta\|_{C^{0,\lambda}(B_1(s,t))} < +\infty$$

Then the function defined by

$$Z(s) := \int_{\mathbb{R}} \Theta(s,t)\rho(t)dt, \quad \text{in} \quad \mathbb{R}$$

satisfies the following estimate

$$\|Z\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} := \sup_{s\in\mathbb{R}}(1+|\varepsilon s|)^{\mu}\|Z\|_{C^{0,\lambda}(s-1,s+1)} \leq C\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

Proof.-
Take any $s_1, s_2 \in \mathbb{R}$ such that $|s-s_1|, |s-s_1| \leq 1$ and observe that

$$\begin{aligned}|Z(s_1)-Z(s_2)| &\leq \int_{\mathbb{R}}|\Theta(s_1,t)-\Theta(s_2,t)|\rho(t)dt \leq \int_{\mathbb{R}}\|\Theta\|_{C^{0,\sigma}(B_1(s,t))}\rho(t)dt\cdot|s_1-s_2|^{\lambda}\\ &\leq \|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}(1+|\varepsilon s|)^{-\mu}\int_{\mathbb{R}}e^{-\sigma|t|}\rho(t)dt\cdot|s_1-s_2|^{\lambda}\end{aligned}$$

Hence we have that

$$(1+|\varepsilon s|)^{\mu}\|Z\|_{C^{0,\lambda}(s-1,s+1)} \leq C_{\lambda}\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

□

Now we will devote to prove an a priori estimate for the linear projected problem.

Proposition 7. *Let us assume $0<\sigma<\sqrt{2}$ and $\mu \geq 0$. There is a universal constant $C>0$ such that for ε small enough and any given $g(s,t)$ with $\|g\|_{C^{0,\lambda}_{\mu,\sigma}} < \infty$ the following holds. Any solution $\phi = \phi(s,t)$ to problem (5.12) with $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}} < \infty$ satisfies the a priori bound*

$$\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} := \|D^2\phi\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|D\phi\|_{L^{\infty}_{\mu,\sigma}(\mathbb{R}^2)} + \|\phi\|_{L^{\infty}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{5.13}$$

Proof.- For the purpose of this bound, let us study first the case when $c(s) \equiv 0$. We claim that elliptic local estimates reduce the work of proving the entire priori bound, to just the following inequality

$$\|\phi\|_{L^{\infty}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{5.14}$$

Indeed, using the Schauder local elliptic estimates, with the Dirichlet boundary condition $\phi|_{\{-R/\varepsilon,R/\varepsilon\}\times\mathbb{R}} = 0$, we have that for $B_1 := B_1((s,t)),\ B_2 := B_2((s,t))$

$$\|D^2\phi\|_{C^{0,\lambda}(B_1)} + \|D\phi\|_{L^{\infty}(B_1)} + \|\phi\|_{L^{\infty}(B_1)} \leq C(\|\phi\|_{L^{\infty}(B_2)} + \|g\|_{C^{0,\lambda}(B_2)} + \|0\|_{C^{2,\lambda}(B_2)}) \tag{5.15}$$

But notice that from the compactness property of $B_2(s,t)$, can have the existence of an integer number $k \geq 2$, independent of (s,t), and points $(s_1,t_1),\ldots,(s_k,t_k) \in B_2(s,t)$ such that $B_2(s,t) \subset \cup_{i=1}^{k} B_1(s,t)$. This implies that

$$\|g\|_{C^{0,\lambda}(B_2(s,t))} \leq \sum_{j=1}^{k}\|g\|_{C^{0,\lambda}(B_1(s_j,t_j))}$$

Moreover, using the decay of g we have

$$\|g\|_{C^{0,\lambda}(B_1(s_j,t_j))} \leq \frac{e^{-\sigma|t_j|}}{(1+|\varepsilon s_j|)^\mu} \cdot (1+|\varepsilon s_j|)^\mu e^{\sigma|t_j|} \|g\|_{C^{0,\lambda}(B_1(s_j,t_j))} \leq \|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

which implies $\|g\|_{C^{0,\lambda}(B_2(s,t))} \leq k\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$. Then multiplying inequality (5.15) in both sides by $(1+|\varepsilon s|)^\mu e^{\sigma|t|}$, taking supreme in $(s,t) \in [-R/\varepsilon, R/\varepsilon] \times \mathbb{R}$, and using the previous estimate for g and ϕ, we finally get

$$\|D^2\phi\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|D\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} + \|\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} \leq C(\|\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} + \|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)})$$

So in order to prove Proposition 7, we just need to prove the validity of estimate (5.14).

To do so, let us assume by contradiction argument that (5.14) does not hold. Then we have the existence of sequences $\varepsilon_n \to 0$, $R_n \to +\infty$ and g_n so that $\|g_n\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \to 0$, with $\|\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} = 1$ that satisfies

$$\begin{aligned} \partial_{tt}\phi_n(s,t) + \partial_{ss}\phi_n(s,t) + f'(w(t))\phi_n(s,t) = g_n(s,t) \quad &\text{in } I^{R_n}_{\varepsilon_n} \times \mathbb{R} \\ \phi_n(s,t) = 0 \quad &\text{on } \partial I^{R_n}_{\varepsilon_n} \times \mathbb{R} \\ \int_{\mathbb{R}} \phi_n(s,t)w'(t)dt = 0 \quad &\text{for all } s \in I^{R_n}_{\varepsilon_n} \end{aligned} \tag{5.16}$$

where we define $I^{R,\varepsilon}_n$ to be the following compact interval $I^{R_n}_{\varepsilon_n} := \left[-\frac{R_n}{\varepsilon_n}, \frac{R_n}{\varepsilon_n}\right]$

Considering that $\|\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} = 1$, we can find points $(s_n,t_n) \in I^{R,\varepsilon}_n \times \mathbb{R}$ such that

$$(1+|\varepsilon_n s_n|)^\mu e^{\sigma|t_n|}|\phi(s_n,t_n)| \geq \frac{1}{2} \tag{5.17}$$

For our purpose of reaching a contradiction, we will have to study separately the cases in which the first two terms of the left hand side of (5.17) either diverge, or stay bounded. The study of the limit operator in suitable norms lead us to consider different possibilities, in which it holds either $|\varepsilon_n s_n| = O(1)$ or $|\varepsilon_n s_n| \to +\infty$, and the same respective options for $|t_n|$.

1.- Case $|\varepsilon_n s_n|$ bounded

In this case, $\varepsilon_n s_n$ lies in a bounded subregion of $\mathbb{R}$, so we may assume that for a subsequence

$$\bar{s}_n := \varepsilon_n s_n \to \tilde{s}_0$$

Consider now the change of variable given by

$$s := \varepsilon_n^{-1}\bar{s}_n + \mathtt{s} = s_n + \mathtt{s}$$

1.1.- Subcase $|t_n|$ bounded
Let us assume first that $|t_n| \leq C$ for all $n \in \mathbb{N}$, then by subsequence it holds that $t_{n_j} \to \bar{t}$ as $j \to \infty$. Now define

$$\tilde{\phi}_n(\mathtt{s}, t) := \phi_n(s_n + s, t) \quad \tilde{g}_n(\mathtt{s}, t) := g_n(s_n + \mathtt{s}, t)$$

so, from problem (5.16) we realize that $\tilde{\phi}_n$ satisfies the following PDE

$$\partial_{tt}\tilde{\phi}_n(\mathtt{s}, t) + \partial_{ss}\tilde{\phi}_n(\mathtt{s}, t) + f'(w(t))\tilde{\phi}_n(\mathtt{s}, t) = \tilde{g}_n(\mathtt{s}, t) \quad \text{in } I_{\varepsilon_n}^{R_n} \times \mathbb{R} \tag{5.18}$$

Observe that this expression is valid for $\mathtt{s}$ well-inside the domain $I_{\varepsilon_n}^{R_n}$, which is expanding to the entire space $\mathbb{R}$, as $n \to +\infty$. Now, since by hypothesis $\tilde{\phi}_n$ is bounded in $I_{\varepsilon_n}^{R_n} \times \mathbb{R}$ and $\tilde{g}_n \to 0$ in $C^{0,\lambda}_{loc}(\mathbb{R}^2)$, then the Schauder local estimate implies a local uniform bound for gradient of $\tilde{\phi}_n$. Indeed, we have for $B_1 := B_1((\mathtt{s}, t))$, $B_2 := B_2((\mathtt{s}, t))$

$$\|\tilde{\phi}_n\|_{C^1(B_1)} \leq C(\|\tilde{\phi}_n\|_{L^\infty(B_2)} + \|\tilde{g}_n\|_{C^{0,\lambda}(B_2)}) \leq C(1 + |\mathtt{s}|)^\mu e^{-\sigma|t|} \leq \hat{C}, \qquad \forall n \geq n_0$$

We deduce that sequence $\tilde{\phi}_n$ is locally uniformly Lipschitz, and so this family is equicontinuous. By Arzela-Ascoli's compactness criterion, we can extract a subsequence (which will be also denoted by $\tilde{\phi}_n$) converging uniformly over compact subsets of $\mathbb{R}^2$ to $\tilde{\phi}(\mathtt{s}, t)$, which solves the asymptotic PDE (5.18)

$$\partial_{tt}\tilde{\phi}(\mathtt{s}, t) + \partial_{\mathtt{ss}}\tilde{\phi}(\mathtt{s}, t) + f'(w(t))\tilde{\phi}(\mathtt{s}, t) = 0 \quad \text{in } \mathbb{R}^2 \tag{5.19}$$

Due to regularity theory for elliptic PDEs, as coefficients 1 and $f'(w(t))$ of L_0 are $C^\infty(V)$ on any bounded open set, and the right hand side 0 is also $C^\infty(V)$, the infinite differentiability in the interior Theorem asserts that the solution $\tilde{\phi}$ in $C^0(V)$ is actually a $C^\infty(V)$-smooth solution that solves the PDE in the classical sense.

In particular, this means that $\tilde{\phi}$ belongs to the kernel of the linearized operator $L[\phi] = \Delta_{(\mathtt{s},t)}\phi + f'(w)\phi$. Since $\tilde{\phi}$ is bounded and smooth function, Lemma 6 implies that $\tilde{\phi} = Cw'(t)$ for some $C \in \mathbb{R}$. In addition thanks to the exponential decay of $w'(t)$, the uniform convergence $\tilde{\phi}_n \to \tilde{\phi}$ on compacts sets is sufficient to show that

$$0 = \int_{\mathbb{R}} \tilde{\phi}_n(\mathtt{s}, t)w'(t)dt \to \int_{\mathbb{R}} \tilde{\phi}(\mathtt{s}, t)w'(t)dt = C\int_{\mathbb{R}} |w'(t)|^2 dt \quad \text{as } n \to \infty$$

Then $C = 0$ and therefore $\tilde{\psi} = 0$. Finally, the hypothesis (5.17) shows that
$(1 + |\varepsilon_n s_n|)^\mu e^{\sigma|t_n|}|\tilde{\phi}_n(0, t_n)| \geq \frac{1}{2}$, and since $|\varepsilon_n s_n|$ and t_n are both bounded, then the local uniform convergence implies that $\tilde{\phi} \neq 0$. We have reached a contradiction. $\to\leftarrow$

1.2.- Subcase $|t_n| \to +\infty$
The variation is that we now define

$$\tilde{\phi}_n(\mathtt{s}, t) := e^{\sigma(t_n+t)}\phi_n(s_n + \mathtt{s}, t_n + t) \quad g_n(\mathtt{s}, t) := e^{\sigma(t_n+t)}g_n(s_n + \mathtt{s}, t_n + t)$$

Considering that $\|\phi_n\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} = 1$ for all $n \in \mathbb{N}$, it follows that $\tilde{\phi}_n$ is uniformly bounded and $\tilde{g_n} \to 0$ in $C^{0,\lambda}_{loc}(\mathbb{R}^2)$. Furthermore, from problem (5.16) it follows in this case that $\tilde{\phi}_n$ satisfies the equation

$$\partial_{tt}\tilde{\phi}_n(\mathtt{s},t) + \partial_{\mathtt{ss}}\tilde{\phi}_n(\mathtt{s},t) - 2\sigma\partial_t\tilde{\phi}_n(\mathtt{s},t) + (f'(w(t+t_n)) + \sigma^2)\tilde{\phi}_n(\mathtt{s},t) = \tilde{g}_n(\mathtt{s},t) \quad \text{in } I^{R_n}_{\varepsilon_n} \times \mathbb{R} \tag{5.20}$$

Arguing like before, we can find a local uniformly limit $\tilde{\phi}$ that satisfies the limiting situation of (5.20), that is

$$\partial_{tt}\tilde{\phi}(\mathtt{s},t) + \partial_{\mathtt{ss}}\tilde{\phi}(\mathtt{s},t) - 2\sigma\partial_t\tilde{\phi}(\mathtt{s},t) - (2-\sigma^2)\tilde{\phi}(\mathtt{s},t) = 0 \quad \text{in } \mathbb{R}\times\mathbb{R} \tag{5.21}$$

But since $2 - \sigma^2 > 0$, maximum principle applied to equation (5.21) implies that $\tilde{\phi} = 0$. However, from the absurd argument (5.17) follows that $|\tilde{\phi}_n(0,0)| \geq \frac{1}{2}$ and so the punctual convergence implies $|\tilde{\phi}(0,0)| \geq \frac{1}{2}$, reaching a contradiction. $\rightarrow\leftarrow$

2.- Case $|\varepsilon_n s_n| \to +\infty$

In this case we may assume without loss of generality that the sequence $\varepsilon_n s_n$ diverges to $+\infty$, since the case $\varepsilon_n s_n \to -\infty$ is totally analogous. Note, however, that this divergence can happen in two different ways. To see this, let us note first that if $s \in I^{R_n}_{\varepsilon_n}$, where s denotes the variable associate to ϕ, and we also take the this change of variables $s := s_n + \mathtt{s}$, then this implies that $\mathtt{s}$ must belong to the new interval $I_n := [-\frac{R_n}{\varepsilon_n} - s_n, \frac{R_n}{\varepsilon_n} - s_n]$. So given that $R_n \to \infty, \varepsilon_n \to 0^+$ and $s_n \to \infty$, it is direct that left portion of I_n stretches to the semi-space $(-\infty, 0]$ as $n \to \infty$. Nevertheless is it not straightforward the convergence in $\mathbb{R}^+$ of the right portion of I_n as $n \to \infty$. For this reason we must consider the following possibilities:

2.1- Subcase $\frac{R_n}{\varepsilon_n} - s_n \to +\infty$
Here we assume that the growth rate at which s_n diverges at infinity, is such that this sequence does not get close to the boundary point R_n of the interval $I^{R_n}_{\varepsilon_n}$. A direct consequence is that I_n converges to the entire space $\mathbb{R}$, as $n \to \infty$.

2.1.1- Subcase $|t_n|$ *bounded*
In this context, let us set the functions

$$\tilde{\phi}_n(\mathtt{s},t) := (1+|\varepsilon_n(s_n+\mathtt{s})|)^{\mu}\phi_n(s_n+\mathtt{s}, t_n+t), \quad \tilde{g}_n(\mathtt{s},t) := (1+|\varepsilon_n(s_n+\mathtt{s})|)^{\mu}g_n(s_n+\mathtt{s}, t_n+t)$$

Then using these definitions on problem (5.16), we have that $\tilde{\phi}$ solves

$$\partial_{tt}\tilde{\phi}(\mathtt{s},t) + \partial_{\mathtt{ss}}\tilde{\phi}(\mathtt{s},t) + o_n(1)\partial_{\mathtt{s}}\tilde{\phi}_n(\mathtt{s},t) + (f'(w(t)) + o_n(1))\tilde{\phi}_n(\mathtt{s},t) = \tilde{g}_n(\mathtt{s},t), \quad \text{in } I_n \times \mathbb{R}$$

Again, thanks to the hypothesis $\|\phi_n\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} = 1$, $\|g_n\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \to 0$, it follows that $\tilde{\phi}_n$ is uniformly bounded and $\tilde{g}_n \to 0$ in $C^{0,\lambda}_{loc}(\mathbb{R}^2)$. Schauder elliptic estimates give local uniform bounds to $\|\nabla\tilde{\phi}_n\|_{L^\infty}$ and $\|D^2\tilde{\phi}_n\|_{C^{0,\lambda}}$, thus allowing to find a limit function which is bounded. Given the rate at how $|\varepsilon_n s_n| \to +\infty$, this assures that the limit problem solved by $\tilde{\phi}$ is

$$\partial_{tt}\tilde{\phi}(\mathtt{s},t) + \partial_{ss}\tilde{\phi}(\mathtt{s},t) + f'(w(t))\tilde{\phi}_n(\mathtt{s},t) = 0 \text{ in } \mathbb{R}\times\mathbb{R}\ , \qquad \int_{\mathbb{R}} \tilde{\phi}(\mathtt{s},t)w'(t)dt = 0$$

Note also that $\tilde{\phi} \neq 0$, since $\tilde{\phi}_n \to \phi$ uniformly on compact sets of $\mathbb{R}^2$ and as $e^{\sigma|t_n|}|\tilde{\phi}_n(0,0)| \geq \frac{1}{2}$ implied by (5.17). However, as $\tilde{\phi}$ is a bounded and smooth function, Lemma 6 implies that this solution in the kernel of the linearized operator satisfies $\tilde{\phi} = Cw'(t)$ for some $C \in \mathbb{R}$. Moreover, the orthogonality condition on $w'(t)$ allow us to show, using the same arguments as before, that $C = 0$ and therefore $\tilde{\phi} = 0$. $\rightarrow\leftarrow$

2.1.2- Subcase $|t_n| \to +\infty$
The divergence of t_n, motivates this time the setting of

$$\tilde{\phi}_n(\mathtt{s},t) := (1+|\varepsilon_n(s_n+\mathtt{s})|)^\mu e^{\sigma|t_n+t|}\phi_n(s_n+\mathtt{s},t_n+t)$$
$$\tilde{g}_n(\mathtt{s},t) := (1+|\varepsilon_n(s_n+\mathtt{s})|)^\mu e^{\sigma|t_n+t|}g_n(s_n+\mathtt{s},t_n+t)$$

Replacing these expressions on the PDE satisfied by ϕ amounts to the following PDE for $\tilde{\phi}$ in $I_n\times\mathbb{R}$:

$$\partial_{tt}\tilde{\phi}(\mathtt{s},t)+\partial_{\mathtt{ss}}\tilde{\phi}(\mathtt{s},t)-2\sigma\partial_t\tilde{\phi}_n(\mathtt{s},t)+o_n(1)\partial_{\mathtt{s}}\tilde{\phi}_n(\mathtt{s},t)+(f'(w(t))+\sigma^2+o_n(1))\tilde{\phi}_n(\mathtt{s},t) = \tilde{g}_n(\mathtt{s},t)$$

Likewise, the hypothesis on $\|\phi_n\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} = 1$, $\|g_n\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \to 0$, together with local elliptic estimates, justify the convergence of $\tilde{\phi}_n$ to a limit function $\tilde{\phi}$ which is bounded. Once again, the rate at how $|\varepsilon_n s_n| \to +\infty$, assures that the limit problem solved by $\tilde{\phi}$ is

$$\partial_{tt}\tilde{\phi}(\mathtt{s},t) + \partial_{\mathtt{ss}}\tilde{\phi}(\mathtt{s},t) - 2\sigma\partial_t\tilde{\phi}_n(\mathtt{s},t) - (2-\sigma^2)\tilde{\phi}_n(\mathtt{s},t) = 0 \text{ in } \mathbb{R}\times\mathbb{R}\ , \qquad \int_{\mathbb{R}} \tilde{\phi}(\mathtt{s},t)w'(t)dt = 0$$

Note also that $\tilde{\phi} \neq 0$, since (5.17) implies $|\tilde{\phi}_n(0,0)| \geq \frac{1}{2}$. But since $2-\sigma^2 > 0$, the maximum principle applied to this equation leads that $\tilde{\phi} = 0$. $\rightarrow\leftarrow$

2.2- Subcase $\frac{R_n}{\varepsilon_n} - s_n = O(1)$
Now we assume that the growth rate at which s_n diverges at infinity is qualitatively similar to those of the boundary point R_n of the interval $I^{R_n}_{\varepsilon_n}$, in such a way the distance between them is nearly constant. Under this configuration, by subsequence, it holds that I_n converges to the a semi-infinite interval of the form $(-\infty, M]$, for some M.

2.2.1- Subcase $|t_n|$ bounded
We set again, like before, the functions

$$\tilde{\phi}_n(\mathtt{s},t) := (1+|\varepsilon_n(s_n+\mathtt{s})|)^\mu\phi_n(s_n+\mathtt{s},t_n+t), \quad \tilde{g}_n(\mathtt{s},t) := (1+|\varepsilon_n(s_n+\mathtt{s})|)^\mu g_n(s_n+\mathtt{s},t_n+t)$$

which implies that $\tilde{\phi}$ solves

$$\partial_{tt}\tilde{\phi}(\mathtt{s},t)+\partial_{\mathtt{ss}}\tilde{\phi}(\mathtt{s},t)+o_n(1)\partial_{\mathtt{s}}\tilde{\phi}_n(\mathtt{s},t)+(f'(w(t))+o_n(1))\tilde{\phi}_n(\mathtt{s},t)=\tilde{g}_n(\mathtt{s},t), \quad \text{in } I_n\times\mathbb{R}$$

Likewise, the hypothesis on $\|\phi_n\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}=1$, $\|g_n\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}\to 0$, together with local elliptic estimates, justify the convergence of $\tilde{\phi}$ to a limit function $\tilde{\phi}$ which is bounded. Yet this time, the rate at how $|\varepsilon_n s_n|\to+\infty$ only permits that the limit problem solved by $\tilde{\phi}$ be on a half-space, which after constant translation can be assumed to be

$$\partial_{tt}\tilde{\phi}(\mathtt{s},t)+\partial_{ss}\tilde{\phi}(\mathtt{s},t)+f'(w(t))\tilde{\phi}_n(\mathtt{s},t)=0, \text{ in } (-\infty,0]\times\mathbb{R}$$

$$\phi(0,t)=0, \quad \text{for all } t\in\mathbb{R}.$$

$$\int_{\mathbb{R}}\tilde{\phi}(\mathtt{s},t)w'(t)dt=0, \quad \text{for all } \mathtt{s}\in(-\infty,0]$$

By Schwartz's reflection, the odd extension of $\tilde{\phi}$, given for $s>0$ as $\tilde{\phi}(s,t)=-\tilde{\phi}(-s,t)$, satisfies the same equation in $\mathbb{R}\times\mathbb{R}$ and the orthogonality condition in $\mathbb{R}$. Thus proceeding as in *Subcase 2.1.1.* we get the same conclusion, finding again a contradiction. $\rightarrow\leftarrow$

2.2.2- Subcase $|t_n|\to+\infty$

The divergence of t_n, motivates the following

$$\tilde{\phi}_n(\mathtt{s},t):=(1+|\varepsilon_n(s_n+\mathtt{s})|)^{\mu}e^{\sigma|t_n+t|}\phi_n(s_n+\mathtt{s},t_n+t)$$
$$\tilde{g}_n(\mathtt{s},t):=(1+|\varepsilon_n(s_n+\mathtt{s})|)^{\mu}e^{\sigma|t_n+t|}g_n(s_n+\mathtt{s},t_n+t)$$

Replacing these expressions on the PDE satisfied by ϕ amounts to the following PDE for $\tilde{\phi}$ in $I_n\times\mathbb{R}$:

$$\partial_{tt}\tilde{\phi}(\mathtt{s},t)+\partial_{\mathtt{ss}}\tilde{\phi}(\mathtt{s},t)-2\sigma\partial_t\tilde{\phi}_n(\mathtt{s},t)+o_n(1)\partial_{\mathtt{s}}\tilde{\phi}_n(\mathtt{s},t)+(f'(w(t))+\sigma^2+o_n(1))\tilde{\phi}_n(\mathtt{s},t)=\tilde{g}_n(\mathtt{s},t)$$

Proceeding like before, it is possible to prove the convergence of $\tilde{\phi}_n\to\tilde{\phi}$ in $C^0_{loc}(\mathbb{R}^2)$. Yet this time, the rate at how $|\varepsilon_n s_n|\to+\infty$ only permits that $\tilde{\phi}$ solves the limit problem on a half-space, that under translation can be view as

$$\partial_{tt}\tilde{\phi}(\mathtt{s},t)+\partial_{\mathtt{ss}}\tilde{\phi}(\mathtt{s},t)-2\sigma\partial_t\tilde{\phi}_n(\mathtt{s},t)-(2-\sigma^2)\tilde{\phi}_n(\mathtt{s},t)=0, \text{ in } (-\infty,0]\times\mathbb{R}$$

$$\phi(0,t)=0, \quad \text{for all } t\in\mathbb{R}.$$

$$\int_{\mathbb{R}}\tilde{\phi}(\mathtt{s},t)w'(t)dt=0, \quad \text{for all } \mathtt{s}\in(-\infty,0]$$

By Schwartz's reflection, the odd extension of $\tilde{\phi}$, given for $s>0$ as $\tilde{\phi}(s,t)=-\tilde{\phi}(-s,t)$, satisfies the same equation in $\mathbb{R}\times\mathbb{R}$ and the orthogonality condition in $\mathbb{R}$. Proceeding in the same way as in *Subcase 2.1.2.*, we obtain the same a contradiction. $\rightarrow\leftarrow$

In any case, all the study done so far leads us to a contradiction, concluding the proof in the case $c(s) \equiv 0$. Now for the general case of g with $c(s) \neq 0$, observe that thanks to Lemma 8 the function

$$c(s) = c^* \int_{\mathbb{R}} g(s,t) w'(t) dt, \qquad \text{with} \qquad c^* := \|w'\|^{-2}_{L^2(\mathbb{R})}$$

satisfy the estimate

$$\|c\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R}^2)} = \sup_{s\in\mathbb{R}} (1+|\varepsilon s|)^{\mu} \|c\|_{C^{0,\lambda}(s-1,s+1)} \leq C\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

and then for any $0 < \sigma < \sqrt{2}$ it holds

$$\|c(s)w'(t)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq \sup_{t\in\mathbb{R}} e^{\sigma|t|}|w'(t)| \cdot \|c(s)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R}^2)} \leq C\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

In view of the latter fact, plus the hypothesis $\|g\|_{C^{0,\lambda}_{\mu,\sigma}} < \infty$, we can apply the proof already done in the case $\tilde{c}_g(s) = 0$ for the right-hand side $\tilde{g}(s,t) = g(s,t) - c(s)w'(t)$ of equation (5.11). This shows the validity of the estimate of Proposition 7. $\square$

5.1.3. Existence for the projected problem

Let us prove now the existence of a solution of problem (5.16), arguing by approximations. Recall that $I^R_\varepsilon := \left[-\frac{R}{\varepsilon}, \frac{R}{\varepsilon}\right]$. Consider first the right-hand side g to be regular with decay, given by $\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} < \infty$, with compact support in $I^R_\varepsilon \times \mathbb{R}$. For this class of $g(s,t)$, let us propose the following Dirichlet boundary value problem:

$$\begin{aligned} \partial_{tt}\phi + \partial_{ss}\phi + f'(w(t))\phi = g(s,t) + c(s)w'(t) \quad &\text{in} \;\; I^R_\varepsilon \times \mathbb{R}, \\ \phi(s,t) = 0 \;\; \text{on} \;\; \partial I^R_\varepsilon \times \mathbb{R}& \\ \int_{\mathbb{R}} \phi(s,t)w'(t)dt = 0 \;\; \text{for all} \;\; s \in I^R_\varepsilon& \end{aligned} \tag{5.22}$$

where we allow $R/\varepsilon = +\infty$ and we recall that $c(s)$ is given by (4.28).
Step 1.- Let us start first with the study of the adequate functional setting. Define the set

$$H_\perp := \left\{ \phi \in H^1_0(I^R_\varepsilon \times \mathbb{R}) \;/\; \int_{\mathbb{R}} \phi(s,t)w'(t)dt = 0 \;\; \text{a.e. in} \;\; s \in I^R_\varepsilon \right\}$$

endowed with the bilinear form $\mathtt{b} : H_\perp \times H_\perp \to \mathbb{R}$ given by

$$\mathtt{b}(\phi,\psi) := \iint_{I^R_\varepsilon \times \mathbb{R}} \left[\partial_t\phi \cdot \partial_t\psi + \partial_s\phi \cdot \partial_s\psi - f'(w(t))\phi \cdot \psi\right] dsdt \tag{5.23}$$

It can be readily checked that $\mathtt{b}$ defines an inner product in $H_\perp$. Indeed, the symmetry property and the bilinearity satisfied by $\mathtt{b}$ are pretty straightforward. Furthermore, observe that the orthogonality condition for elements in $H_\perp$, validates the following coercivity inequality in H^1: For all $\phi \in H_\perp$,

$$\mathtt{b}(\phi,\phi) = \iint_{I_\varepsilon^R\times\mathbb{R}} |\partial_s\phi|^2 + (|\partial_t\phi|^2 - f'(w(t))\phi^2)dsdt \geq \vartheta \iint_{I_\varepsilon^R\times\mathbb{R}} (|\nabla_{s,t}\phi|^2+\phi^2)dsdt \quad (5.24)$$

thanks to the coercivity Lemma (7) in H^1.

In particular this implies that $\mathtt{b}(\phi,\phi)$ is positive in $H_\perp$, and also proves that the only element vanishing $\mathtt{b}$ is 0. Indeed from (5.24), if $\bar\phi \in H$ were such that $\mathtt{b}(\bar\phi,\bar\phi) = 0$, then it would follow $\int_{I_\varepsilon^R\times\mathbb{R}} |\nabla_{(s,t)}\bar\phi|^2 dsdt = 0$, which guarantees that $\bar\phi = 0$ since $\bar\phi \in H_0^1(I_\varepsilon^R \times \mathbb{R})$.

In this way, the expression $\|\phi\|_{H_\perp} := \mathtt{b}(\phi,\phi)^{1/2}$ defines a norm in $H_\perp$, which makes this vector space to be a closed subspace of $H_0^1(I_\varepsilon^R\times\mathbb{R})$. To see this, note that from (5.24) if $\phi \in H_0^1(I_{ep}^R\times\mathbb{R})$ and $\{\phi_n\}_n \subset H_\perp$ is a sequence such that $\phi_n \to \phi$ in $H_\perp$, then Hölder inequality implies

$$\int_{I_\varepsilon^R}\left(\int_{\mathbb{R}}\phi_n w'(t)dt - \int_{\mathbb{R}}\phi w'(t)dt\right)^2 ds \leq \int_{I_\varepsilon^R}\left(\int_{\mathbb{R}}|\phi_n-\phi|^2 dt\right)\cdot\left(\int_{\mathbb{R}}|w'(t)|^2dt\right)ds$$
$$= c_*\iint_{I_\varepsilon^R\times\mathbb{R}} |\phi_n(s,t)-\phi(s,t)|^2 dt\; ds \leq c_*\vartheta^{-1}\cdot\|\phi_n-\phi\|_H \to 0$$

where $c_* := \int_{\mathbb{R}}|w'|^2dt$. Regularity theory for ϕ and the exponential decay of w' assure that $\int_{\mathbb{R}}\phi(s,t)w'(t)dt = \lim_{n\to+\infty}\int_{\mathbb{R}}\phi_n(s,t)w'(t)dt \equiv 0$ a.e. in $s \in I_\varepsilon^R$, which proves that $\phi \in H_\perp$. All previous analysis show that $H_\perp$ is a Hilbert space when is endowed with its natural norm $\|\phi\|_{H_\perp}$.

Now we turn our attention on the weak formulation of the linear problem, consisting in the following: ϕ is a weak solution of problem (5.22), if and only if, $\phi \in H_\perp$ and also satisfies

$$\mathtt{b}(\phi,\psi) = \iint_{I_\varepsilon^R\times\mathbb{R}}[\nabla_{s,t}\phi\cdot\nabla_{s,t}\psi - f'(w(t))]\phi\psi dsdt = -\iint_{I_\varepsilon^R\times\mathbb{R}} g(s,t)\psi dsdt, \quad \forall\psi\in H_\perp \quad (5.25)$$

Note that since $\|\psi\|_{L^2(I_\varepsilon^R\times\mathbb{R})} \leq C\|\psi\|_{H_\perp}$ and also that $g \in L^2(I_\varepsilon^R\times\mathbb{R})$, which is true under the assumption $\|g\|_{C^{0,\lambda}_{\mu,\sigma}} < \infty$, we have $l_g(\psi) := \iint_{I_\varepsilon^R\times\mathbb{R}} g(s,t)\psi\; dsdt$ defines a dual element in $H_\perp^*$. Therefore, the Riesz's theorem ensures the existence of a unique weak solution $\phi \in H_\perp$ of (5.25).

Step 2.- So far we have that ϕ, the weak solution of equation (5.25), belongs to $H_0^1(I_\varepsilon^R\times\mathbb{R})$. However since ϕ solves $-\Delta\phi + f'(w(t))\phi = g$ within any ball $B_2((s,t))$, the Calderon-Zigmund L^p-elliptic regularity estimate holds:

$$\|\phi\|_{W^{2,2}(B_1(s,t))} \leq C[\|\phi\|_{L^2(B_2(s,t))} + \|g\|_{L^2(B_2(s,t))}] \leq \tilde C[\|\phi\|_{H_\perp} + \|g\|_{C^{0,\lambda}(B_2(s,t))}]$$

Moreover, as $g(s,t)$ decays in both variables, we get a local bound for ϕ uniformly on the point (s,t) chosen: $\|\phi\|_{W^{2,2}(B_1(s,t))} \leq C[\|\phi\|_{H_\perp} + \|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}]$. But, the Sobolev injections in $B_1(s,t)$, for the case $kp > N$ with $k = p = N = 2$ gives that $\|\phi\|_{C^{\theta,\lambda}B_1(s,t)} \leq C\|\phi\|_{W^{2,2}B_1(s,t)}$, where $\theta := k - (\lfloor \frac{N}{p} \rfloor + 1) = 0$. Thus we obtain $\|\phi\|_{C^{0,\lambda}(B_1(s,t))} \leq C[\|\phi\|_{H_\perp} + \|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}]$, and so $\|\phi\|_{L^\infty(I^R_\varepsilon \times \mathbb{R})} < +\infty$. All this analysis allow us to conclude not only ϕ is locally Hölder continuous, but also that is globally bounded.

Additionally, this solution ϕ exhibits an *exponential decay* in variable t. Indeed, as g has compact support, then for some $t_0 > 0$ the equation satisfied by ϕ is

$$\Delta_{s,t}\phi + f'(w(t))\phi = c(s)w'(t), \quad s \in I^R_\varepsilon \text{ and } \quad |t| > t_0 \tag{5.26}$$

with $c(s)$ bounded. So by choosing t_0 large enough, we force $-1 \leq f'(w(t)) \leq -2$ if $|t| > t_0$, and then we see that for $0 < \sigma < \sqrt{2}$ and $\varepsilon > 0$ the function

$$v_\varepsilon(s,t) := (\|\phi\|_{L^\infty(I^R_\varepsilon \times \mathbb{R})} + \|c\|_{L^\infty(I^R_\varepsilon)})e^{-\sigma|t|} + \varepsilon e^{\sigma|t|}$$

is a positive supersolution of equation (5.26). Due to the maximum principle it holds $|\phi| \leq Ce^{-\sigma|t|}$ for $|t| > t_0$, using $v_\varepsilon(s,t)$ as a barrier function. Further, since I^R_ε is a bounded domain, it follows that $(1 + |\varepsilon s|)^\mu$ remains bounded, and thus we deduce $\|\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} < +\infty$.

Finally, as now we have the decay in (s,t)-variables of g and ϕ, using the a priori estimate of Proposition 7 it follows that ϕ is not only locally bounded-with decay, up to its second-order derivatives, but also

$$\|D^2\phi\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|D\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} + \|\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} \leq C\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{5.27}$$

Step 3.- Now consider problem (5.22) allowed above for $R/\varepsilon = \infty$, with $\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} < \infty$. Let us choose any sequence R_n/ε_n such that $R_n/\varepsilon_n \to +\infty$. We can take a suitable sequence $\{g_n\}_n$ of smooth functions compactly supported, in such a way $g_n \to g$ in $C^{0,\lambda}_{loc}(\mathbb{R}^2)$, converging locally and uniformly on $I^{R_n}_{\varepsilon_n} \times \mathbb{R}$, with $\|g_n\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq \|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$. From *steps 1 and 2* we get a sequence $\{\phi_n\}_n$ of solutions to problem (5.22) associated to g_n, which are uniformly $C^{2,\lambda}_{\mu,\sigma}$-bounded thanks to estimate (5.27). This implies in particular that ϕ_R is locally uniformly bounded for n in $C^1_{loc}(\mathbb{R}^2)$, and by using Arzela-Ascoli's compactness criterion we can extract a subsequence $\phi_{n_j} \to \phi$ converging uniformly over compact sets of $\mathbb{R}^2$. Hence ϕ is a limit bounded solution of the full problem on the entire space. Further, ϕ respects estimate (5.27), and therefore ϕ is the solution we are looking for.

This concludes the proof of the existence, and hence that of the Proposition 3. □

5.2. Reducing the gluing system and solving the projected problem

This section is devoted to prove Lemma 5, which reduces the gluing system (4.17)-(4.19) to solving the nonlocal equation (4.25). We also give a proof of Proposition 4 on solving the nonlinear projected problem (4.26), in which the basic ingredient is the linear theory stated in Proposition 3. In what follows, we refer to the notation and to the objects introduced in Sections 4.2-4.3.

5.2.1. Reducing the gluing system

Let us consider equation (4.17) of the gluing system (4.17)-(4.19),

$$\Delta_x\psi - W_\varepsilon(x)\psi + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\psi + (1-\zeta_3)S(\mathtt{w}) + (1-\zeta_4)N_1(\psi+\zeta_3\phi)$$

$$+ 2\nabla_x\zeta_3\nabla_x\phi + \phi\Delta_x\zeta_3 + \varepsilon\phi\frac{\nabla_{\bar{x}}a}{a}\nabla_x\zeta_3 = 0 \quad \text{in} \quad \mathbb{R}^2 \tag{5.28}$$

where

$$-W_\varepsilon(x) := [(1-\zeta_4)f'(u_1) + \zeta_4 f'(H(t))]$$

and the dependence in ε is implicit on the cut-off function ζ_4, given in definition (4.5).

Solving the linear outer problem

We first consider the linear partial differential equation

$$\Delta_x\psi + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\psi - W_\varepsilon(x)\psi = g(x), \quad \text{in} \quad \mathbb{R}^2 \tag{5.29}$$

Let us observe that for any $\varepsilon > 0$ small enough, the term W_ε satisfies the global estimate $0 < \beta_1 < W_\varepsilon(x) < \beta_2$ for a certain positive constants β_1, β_2. In fact, we can chose $\beta_1 := \sqrt{2} - \tau$ for any arbitrary small $\tau > 0$. To address the study of this equation, recall the definition of the weighted norms:

$$\|g\|_{L^\infty_K(\mathbb{R}^2)} := \sup_{x\in\mathbb{R}^2} K(x)\|g\|_{L^\infty(B_1(x))}, \quad \|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} := \sup_{x\in\mathbb{R}^2} K(x)\|g\|_{C^{0,\lambda}(B_1(x))}$$

with K is given by (4.6). In addition, we agree $\|g\|_{C^{0,\lambda}_1(\mathbb{R}^2)} := \sup\left\{\|g\|_{C^{0,\lambda}(B_1(x))} :\ x \in \mathbb{R}^2\right\}$.

Lemma 9. *For any $\lambda \in (0,1)$, there are numbers $C > 0$, and $\varepsilon_0 > 0$ small enough, such that for $0 < \varepsilon < \varepsilon_0$ and any given continuous function $g = g(x)$ with $\|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} < +\infty$, the equation (5.29) has a unique solution $\psi = \Psi(\phi)$ satisfying the a priori estimate:*

$$\|\psi\|_X := \|D^2\psi\|_{C^{0,\lambda}_K(\mathbb{R}^2)} + \|D\psi\|_{L^\infty_K(\mathbb{R}^2)} + \|\psi\|_{L^\infty_K(\mathbb{R}^2)} \le C\|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \tag{5.30}$$

Proof.-
Step 1: In order to prove the desired estimate, we will first justify a weaker inequality

$$\|\psi\|_{L^\infty(\mathbb{R}^2)} \leq C\|g\|_{C_1^{0,\lambda}(\mathbb{R}^2)} \tag{5.31}$$

for any bounded solution of (5.29), with a right-hand side satisfying $\|g\|_{C_1^{0,\lambda}(\mathbb{R}^2)} < \infty$. Let us suppose by absurd the existence of $\varepsilon_n \to 0^+$ and solutions ψ_n to equation (5.29), with $\|\psi_n\|_{L^\infty(\mathbb{R}^2)} = 1$, $\|g_n\|_{C_1^{0,\lambda}(\mathbb{R}^2)} \to 0$. From this condition, we can consider a sequence of points $\{x_n\}_n$ in $\mathbb{R}^2$, such that

$$\psi_n(x_n) \geq \frac{1}{2} \tag{5.32}$$

Then we define

$$\tilde{\psi}_n(x) := \psi_n(x_n + x), \quad \tilde{W}_n(x) := W_{\varepsilon_n}(x_n + x), \quad \tilde{g}_n(x) := g_n(x_n + x)$$

Similarly to what was done in the previous section, we readily check using (5.29) that the equation solved by $\tilde{\psi}_n$ has the form Let us define

$$\Delta_x\tilde{\psi}_n + \varepsilon_n\frac{\nabla_{\bar{x}}a}{a}\nabla_x\tilde{\psi}_n - \tilde{W}_n(x)\psi_n + \tilde{g}_n = 0, \quad \text{in} \quad \mathbb{R}^2$$

We have that $\tilde{\psi}_n$ is uniformly bounded, since $\|\psi_n\|_{L^\infty(\mathbb{R}^2)} = 1$. Moreover, it follows $\tilde{g}_n \to 0$ in $C_{loc}^{0,\lambda}(\mathbb{R}^2)$. Thanks to Schauder local elliptic estimates, the latter implies a L^∞-bound for the sequence $\nabla\tilde{\psi}_n$, because

$$\|\tilde{\psi}_n\|_{C^{1,\lambda}(B_1(x))} \leq C(\|\tilde{\psi}_n\|_{L^\infty(B_1(x))} + \|\tilde{g}_n\|_{C^{0,\lambda}(B_2(x))}) \leq \hat{C}$$

Therefore the Arzela-Ascoli compactness criterion provides the existence of a subsequence converging uniformly on compacts subsets of $\mathbb{R}^2$, to a limit function $\tilde{\psi} \neq 0$ which turns out to be a bounded solution to an asymptotic equation of the form

$$\Delta_x\tilde{\psi} - W_*(x)\tilde{\psi} = 0, \quad \text{in} \quad \mathbb{R}^2$$

with $0 < \beta_1 \leq W_*(x) \leq \beta_2$. Hence, independent of the behavior of sequence $\{x_n\}_n$, a direct consequence of (5.32) and the convergence $\tilde{\psi}_n \to \tilde{\psi}$ is $|\tilde{\psi}(0)| \geq 1/2$. Applying the maximum principle to the limit equation, using a barrier function of the form $\psi_0(x) = \theta\cosh(\sqrt{\beta_1}x)$ for $\theta > 0$ small converging to 0^+, gives that $\tilde{\psi} = 0$, which makes this situation impossible and finishes the proof of estimate (5.31). $\rightarrow\leftarrow$

Step 2: Now we study the existence of a bounded solution to problem (5.29). Given any g with $\|g\|_{C_1^{0,\lambda}(\mathbb{R}^2)} < \infty$, consider a collection of approximations g_n such that $\|g_n\|_{C_1^{0,\lambda}(\mathbb{R}^2)} < \infty$, $g_n \to g$ in $C_{loc}^{0,\lambda}(\mathbb{R}^2)$, and also $\|g_n\|_{C_1^{0,\lambda}(\mathbb{R}^2)} \leq C\|g\|_{C_1^{0,\lambda}(\mathbb{R}^2)}$. For any $n \geq 1$ there exists a unique bounded function ψ_n solving the next problem, provided by Lax-Milgram theorem

$$\Delta_x\psi_n + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\psi_n - W_\varepsilon(x)\psi_n = g_n \quad \text{in} \quad B_n(\vec{0}), \qquad \psi_n\Big|_{\partial B_n(\vec{0})} = 0$$

if $\varepsilon > 0$ is taken very small in order to guarantee $0 < \beta_1 < W_\varepsilon < \beta_2$.

As the coefficients involved in the PDE are smooth, Schauder local elliptic estimates assure that ψ_n belongs to $C^{2,\lambda}_{loc}(\mathbb{R}^2)$. Further, the previous a priori estimate plus a compactness argument allow us to find a subsequence that converges uniformly over compact sets of $\mathbb{R}^2$, to a limit bounded function ψ solving (5.29) in the entire space. We have that ψ satisfies (5.31), due to the convergence of ψ_n and $\|g_n\|_{C^{0,\lambda}_1(\mathbb{R}^2)} \leq C\|g\|_{C^{0,\lambda}_1(\mathbb{R}^2)}$. Thus we have proved the existence of a unique bounded solution of (5.29).

Step 3: We claim now that for $\varepsilon > 0$ small enough, the following a priori estimate

$$\|\psi\|_{L^\infty_K(\mathbb{R}^2)} \leq C\|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \tag{5.33}$$

holds for any bounded solution ψ of (5.29), provided that the right-hand side satisfies $\|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} < \infty$. Since $\sqrt{2} - \tau = \beta_1 \leq W_\varepsilon(x) \leq \beta_2$, we can readily check that choosing $\varepsilon > 0$ sufficiently small, and using that $b_1^2 + b_2^2 < (\sqrt{2} - \tau)/2$, it turns out that the function $\psi_0(x) := e^{R_0}\|\psi\|_\infty \cdot \left\{\zeta_3(x)[e^{-\sigma|t|/2}(1 + |\varepsilon s|)^{-\mu}] + (1 - \zeta_3(x))e^{-b_1|x_1| - b_2|x_2|}\right\}$ is a positive super-solution of (5.29). We can estimate this PDE by diving the analysis in two regions, one near the curve using Fermi coordinates, and on the outer region using Euclidean coordinates. In all cases, we obtain

$$\Delta_x\psi_0 + \varepsilon\frac{\nabla_{\bar{x}}a}{a}\nabla_x\psi_0 - W_\varepsilon(x)\psi_0 \leq -\frac{\beta_1}{2}\psi_0, \quad \text{in} \quad \mathbb{R}^2$$

Adjusting the radii $R_1 > R_0 > 0$ sufficiently large, we can use the maximum principle within the annulus $B_{R_1}(\vec{0}) \setminus B_{R_0}(\vec{0})$ with a barrier function of the form $\psi_0 + \theta e^{\sqrt{\beta_1/2}(|x_1| + |x_2|)}$ for $\theta > 0$ small, where is essential the fact that $g(x)$ decays at a rate $K(x)$. Further, by taking $\theta \to 0^+$ in the latter inequality we get that any bounded solution ψ of (5.29) satisfies

$$|\psi(x)| \leq M\psi_0(x) \quad \Leftrightarrow \quad K(x)|\psi(x)| \leq M\|\psi\|_{L^\infty(\mathbb{R}^2)}$$

However *Step 2* assures the validity of $\|\psi\|_{L^\infty(\mathbb{R}^2)} \leq C\|g\|_{C^{0,\lambda}_1(\mathbb{R}^2)}$. Noting in addition from (4.6) that $K(x) \geq 1$ for all $x \in \mathbb{R}^2$, it follows that

$$K(x)|\psi(x)| \leq \tilde{M}\|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)}$$

where the constant $\tilde{M}$ does not depend on ε. This directly implies the a priori estimate (5.33).

Step 4: Finally we deduce the validity of (5.30) from the previous inequality. Indeed, by multiplying the Schauder local elliptic regularity estimate by the weight function $K(x)$, follows

$$K(x)(\|D^2\psi\|_{C^{0,\lambda}(B_1(x))} + \|D\psi\|_{L^\infty(B_1(x))} + \|\psi\|_{L^\infty(B_1(x))}) \\ \leq CK(x)(\|\psi\|_{L^\infty(B_2(x))} + \|g\|_{C^{0,\lambda}(B_2(x))})$$

and taking $\sup\limits_{x\in\mathbb{R}^2}$ in both sides, we obtain

$$\|D^2\psi\|_{C^{0,\lambda}_K(\mathbb{R}^2)} + \|D\psi\|_{L^\infty_K(\mathbb{R}^2)} + \|\psi\|_{L^\infty_K(\mathbb{R}^2)} \leq C(\|\psi\|_{L^\infty_K(\mathbb{R}^2)} + \|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)})$$

which finishes the proof of Lemma 9. $\square$

The proof of Lemma 5

Let us call $\psi := \Upsilon(g)$ the solution of equation (5.29) predicted by Lemma 9. We can write problem (5.28) as a fixed point problem in the space $X := \{\psi \in C^{2,\lambda}_{loc}(\mathbb{R}^2)/\ \|\psi\|_X < \infty\}$, trough

$$\psi = \Upsilon(g_1 + G(\psi)), \quad \psi \in X \tag{5.34}$$

where

$$g_1 := (1-\zeta_3)S(\mathtt{w}) + 2\nabla_x\zeta_3\nabla_x\phi + \phi\Delta_x\zeta_3 + \varepsilon\phi\frac{\nabla_{\bar{x}}a}{a}\nabla_x\zeta_3, \quad G(\psi) := (1-\zeta_4)N_1(\psi + \zeta_3\phi) \tag{5.35}$$

For what comes, consider numbers $\mu \in (0, 2+\alpha)$, $\sigma \in (0, \sqrt{2})$ and $\alpha > 0$ to be such that h satisfies (4.12). Regard a function $\phi = \phi(s,t)$ defined in $\mathbb{R}\times\mathbb{R}$, satisfying $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq 1$.

Note that the derivatives of ζ_3 are nontrivial only within the region $\rho_\varepsilon - 2 < |t + h(\varepsilon s)| < \rho_\varepsilon - 1$, with ρ_ε defined in (3.53). Therefore, taking into account the weight $K(x)$ (4.6),

$$\begin{aligned} K(x)\left|2\nabla_x\zeta_3\nabla_x\phi + \phi\Delta_x\zeta_3 + \varepsilon\phi\frac{\nabla_{\bar{x}}a}{a}\nabla_x\zeta_3\right| &\leq C_a K(x) e^{-\sigma|t|}(1+|\varepsilon s|)^{-\mu}\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\ &\leq C_a\ e^{-\sigma\delta/2\varepsilon}e^{\sigma/2(-c_0|s|+2+|h|)}\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \end{aligned}$$

Expressions (3.56)-(3.57) for $S(\mathtt{w})$ imply that $\|S(\mathtt{w})\|_{C^{0,\lambda}_{\mu,\sqrt{2}}(\mathbb{R}^2)} \leq C\varepsilon^3$. In particular, the exponential decay exhibited by w', w'', ψ_0, ψ_1 in $t-$variable imply

$$|(1-\zeta_3)S(\mathtt{w})| = |(1-\zeta_3)\zeta_3 S(u_1) + (1-\zeta_3)E| \leq C_a\ e^{-\sqrt{2}|t|}(1+|\varepsilon s|)^{-2-\alpha}$$

Now since this error term is vanishing everywhere but on the region $\rho_\varepsilon - 2 < |t + h(\varepsilon s)| < \rho_\varepsilon - 1$, we can use the definition (4.6) of the weight function $K(x)$ to prove that

$$\begin{aligned} K(x)|(1-\zeta_3)S(\mathtt{w})(x)| &\leq e^{\sigma|t|/2}(1+|\varepsilon s|)^{\mu-2-\alpha}\ C_a e^{-\sigma|t|/2}e^{-(\sqrt{2}-\sigma/2)|t|} \\ &\leq C_a e^{-(\sqrt{2}-\sigma/2)(\delta/\varepsilon + c_0|s| - |h| - 2)} \leq Ce^{-\sigma\tilde{\delta}/\varepsilon} \end{aligned}$$

where we have used the expression (3.53) for ρ_ε, and we set $\tilde{\delta} := (\sqrt{2}/\sigma - 1/2)\delta >> \delta/2$. Further, the regularity in the $s-$variable of the functions involved in g_1, imply that

$$\|g_1\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \leq Ce^{-\sigma\delta/2\varepsilon}$$

On the other hand, consider the set for $A > 0$ large

$$\Lambda = \{\psi \in X: \ \|\psi\|_X \leq A \cdot e^{-\sigma\delta/2\varepsilon}\} \tag{5.36}$$

The definitions of G and N_1 given in (5.35) (4.4) yield

$$\begin{aligned} |G(\psi_1) - G(\psi_2)| &\leq (1-\zeta_4) \sup_{\xi\in(0,1)} |DN_1(\xi\psi_1 + (1-\xi)\psi_2 + \zeta_3\phi)[\psi_1 - \psi_2]| \\ &\leq C\|f''(\mathtt{w})\|_\infty (1-\zeta_4) \sup_{\xi\in(0,1)} |\xi\psi_1 + (1-\xi)\psi_2 + \zeta_3\phi| \cdot |\psi_1 - \psi_2| \end{aligned}$$

The latter, plus the regularity in the $s-$variable leads the Lipschitz character of G:

$$\|G(\psi_1) - G(\psi_2)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \leq C_A e^{-\sigma\delta/\varepsilon} \|\psi_1 - \psi_2\|_{C^{0,\lambda}_K(\mathbb{R}^2)}$$

while

$$\|G(0)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \leq C_{\mathtt{w}} \|(1-\zeta_4)\zeta_3^2\phi^2\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \leq Ce^{-\sigma\delta/\varepsilon}$$

In order to use the fixed point theorem, we need to estimate the size of the nonlinear operator

$$\begin{aligned} \|\Upsilon(g_1 + G(\psi))\|_X &\leq \|\Upsilon(g_1 + G(\psi) - G(0))\|_X + \|\Upsilon(G(0))\|_X \\ &\leq \|\Upsilon\|(\|g_1\|_{C^{0,\lambda}_K(\mathbb{R}^2)} + \|G(\psi) - G(0)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} + \|G(0)\|_{C^{0,\lambda}_K(\mathbb{R}^2)}) \\ &\leq \|\Upsilon\|(C_a\ e^{-\sigma\delta/2\varepsilon} + Ce^{-\sigma\delta/\varepsilon}\|\psi\|_{C^{0,\lambda}_K(\mathbb{R}^2)} + Ce^{-\sigma\delta/\varepsilon}) \\ &\leq \|\Upsilon\| e^{-\sigma\delta/2\varepsilon}(C + \|\psi\|_X) \end{aligned}$$

additionally, we also have

$$\begin{aligned} \|\Upsilon(g_1 + G(\psi_1)) - \Upsilon(g_1 + G(\psi_2))\|_X &= \|\Upsilon\|\|G(\psi_1) - G(\psi_2)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \\ &\leq Ce^{-\sigma\delta/\varepsilon}\|\Upsilon\|\|\psi_1 - \psi_2\|_X \end{aligned}$$

where in both inequalities we used that Υ is a linear and bounded operator.

This means that the right hand side of equation (5.34) defines a contraction mapping on Λ into itself, provided that the number A in definition (5.36) is taken large enough and $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}} \leq 1$. Hence applying Banach fixed point theorem follows the existence of a unique solution $\psi = \Psi(\phi) \in \Lambda$.

In addition, it is direct to check the Lipschitz dependence (4.23) of Ψ on $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}} \leq 1$. Let us make more explicit the dependance on ϕ of the nonlinearity (5.35), by denoting

$g_1 = g_1(\phi),\ G(\psi) = G_\phi(\Psi(\phi))$, then

$$\begin{aligned}\|\Psi(\phi_1) - \Psi(\phi_2)\|_X &= \|\Upsilon[g_1(\phi_1) + G_{\phi_1}(\Psi(\phi_1))] - \Upsilon[g_1(\phi_2) + G_{\phi_2}(\Psi(\phi_2))]\|_X \\ &\leq \|\Upsilon\|(\|g_1(\phi_1) - g_1(\phi_2)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} + \|G_{\phi_1}(\Psi(\phi_1)) - G_{\phi_2}(\Psi(\phi_2))\|_{C^{0,\lambda}_K(\mathbb{R}^2)}) \\ &\leq \|\Upsilon\|\ \|2\nabla_x\zeta_3\nabla_x(\phi_1 - \phi_2) + [\Delta_x\zeta_3 + \nabla_{\bar{x}}a/a \cdot \nabla_x\zeta_3](\phi_1 - \phi_2)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \\ &+ \|\Upsilon\|\ \|(1-\zeta_4)[N_1(\Psi(\phi_1) + \zeta_3\phi_1) - N_1(\Psi(\phi_2) + \zeta_3\phi_2)]\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \\ &\leq C_a\ e^{-\sigma\delta/2\varepsilon}\|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R})} + C\ e^{-\sigma\delta/\varepsilon}\|\Psi(\phi_1) - \Psi(\phi_2)\|_X \end{aligned} \tag{5.37}$$

where we have used that for $\Psi(\phi_i) \in \Lambda$ the next estimate for the difference holds:

$$\begin{aligned}&(1-\zeta_4)|N_1(\Psi(\phi_1) + \zeta_3\phi_1) - N_1(\Psi(\phi_2) + \zeta_3\phi_2)| \leq \\ &C_{\mathtt{w}}(1-\zeta_4) \sup_{t\in(0,1)} |t\Psi(\psi_1) + (1-t)\Psi(\psi_2) + \zeta_3(t\phi_1 + (1-t)\phi_2)| \cdot |\Psi(\psi_1) - \Psi(\psi_2)|\end{aligned}$$

Therefore, choosing $\varepsilon > 0$ sufficiently small in (5.37) follows the desired inequality, which concludes the proof of Lemma 5. □

Now that we have the validity of Lemma 5, we can replace $\psi = \Psi(\phi)$ into the equation (4.19) of the gluing system, thus obtaining the nonlocal problem

$$\partial_{tt}\phi + \partial_{ss}\phi + f'(w)\phi = -\tilde{S}(u_1) - \mathtt{N}(\phi) \quad \text{in } \mathbb{R}\times\mathbb{R} \tag{5.38}$$

where we redefine

$$\mathtt{N}(\phi) := \underbrace{\mathtt{B}(\phi) + [f'(u_1) - f'(w)]\phi}_{\mathtt{N}_1(\phi)} + \underbrace{\varepsilon\nabla_{\bar{x}}a/a \cdot \nabla_x\phi}_{\mathtt{N}_2(\phi)} + \underbrace{\zeta_4[f'(u_1) - f'(H(t))]\Psi(\phi)}_{\mathtt{N}_3(\phi)} + \underbrace{\zeta_4 N_1(\Psi(\phi) + \phi)}_{\mathtt{N}_4(\phi)} \tag{5.39}$$

considering that the operators N_1 and $\mathtt{B}$ are given in (4.4)-(4.18).

For what comes next, we will concentrate our efforts in solving this problem.

5.2.2. Proof of Proposition 4

Recall from Section 4.3, that Proposition 4 refers to the solvability of the projected problem

$$\begin{aligned}&\partial_{tt}\phi + \partial_{ss}\phi + f'(w)\phi = -\tilde{S}(u_1) - \mathtt{N}(\phi) + c(s)w'(t) \quad \text{in} \quad \mathbb{R}\times\mathbb{R} \\ &\int_{\mathbb{R}} \phi(s,t)w'(t)dt = 0, \quad \text{for all } s \in \mathbb{R}\end{aligned} \tag{5.40}$$

and to the adjustment of h so that the projection of the right-hand side onto w' vanishes, namely $c(y) \equiv 0$. Let us define $\phi := T(g)$ as the operator providing the solution of the linearized operator L in Proposition 3, result that was proved in Section 5.1. Then problem (5.40) can be reformulated as the fixed point problem

$$\phi = T(-\tilde{S}(u_1) - \mathtt{N}(\phi)) =: \mathcal{T}(\phi), \quad \|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le K\varepsilon^4 \tag{5.41}$$

which is equivalent to

$$\phi = T(-\tilde{S}(u_1) - \varepsilon^2 \mathcal{J}_a[h] w(t)' - \mathtt{N}(\phi)) =: \mathcal{T}(\phi), \quad \|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le K\varepsilon^4 \tag{5.42}$$

since the term added has the form $\rho(s)w'$ which adds up to $c(s)w'$. The reason to absorb this term is that because of assumption (4.12) then $\|\varepsilon^2 \mathcal{J}_a[h] \cdot w'\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} = O(\varepsilon^3)$, while the remainder has a priori size slightly smaller, $O(\varepsilon^4)$.

The Lipschitz character of $\mathtt{N}$

We will solve problem (5.41) using contraction mapping principle, which motivates to give account of a suitable Lipschitz property for the operator $\mathcal{T}$. This fact is justified in the next result.

Claim 3. *Given $\alpha > 0$, $0 < \mu < 2 + \alpha$ and $0 < \sigma < \sqrt{2}$, there is some constant $C > 0$, possibly depending on the constant $\mathcal{K}$ of* (4.12) *but independent of ε, such that for $M > 0$ and ϕ_1, ϕ_2 satisfying*

$$\|\phi_i\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le M\varepsilon^4, \quad i = 1, 2.$$

then the nonlinearity $\mathtt{N}$ behaves locally Lipschitz, as

$$\|\mathtt{N}(\phi_1) - \mathtt{N}(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le C\varepsilon\ \|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{5.43}$$

where the operator $\mathtt{N}$ is given in (5.39).

Proof.-
We study the Lipschitz character of the operator $\mathtt{N}$ through analyzing each of its components $\mathtt{N}_i$ for $i = 1, 2, 3, 4$, as made explicit in page 85. Let us start with $\mathtt{N}_1$. Note that its first term corresponds to a second order linear operator with coefficients of order ε plus a decay of order at least $O((1 + |\varepsilon s|)^{-1-\alpha})$. In particular, recall from (4.18) that $\mathtt{B} = \zeta_0 \tilde{\mathtt{B}}_0$, where in coordinates $[\Delta_x - \partial_{tt} - \partial_{ss}]$ amounts

$$\begin{aligned}\tilde{B}_0 = &- 2\varepsilon h \partial_{st} - \varepsilon[k(\varepsilon s) + \varepsilon(t+h)k^2(\varepsilon s)]\partial_t + \varepsilon(t+h)A_0(\varepsilon s, \varepsilon(t+h)) \\ &\cdot [\partial_{ss} - 2h_1'\partial_t + \varepsilon^2 |h'|^2 \partial_{tt}] + \varepsilon^2(t+h)B_0(\varepsilon s, \varepsilon(t+h))[\partial_s - \varepsilon h' \partial_t] - \varepsilon^2 h'' \partial_t + \varepsilon^2 |h'|^2 \partial_{tt} \\ &+ \varepsilon^3 (t+h)^2 C_0(\varepsilon s, \varepsilon(t+h))\partial_t\end{aligned} \tag{5.44}$$

in which the following decay holds

- $h(\varepsilon s) = O(\varepsilon)$
- $h'(\varepsilon s) = O(\varepsilon(1+|\varepsilon s|)^{-1-\alpha})$
- $h''(\varepsilon s) = O(\varepsilon(1+|\varepsilon s|)^{-2-\alpha})$
- $k(\varepsilon s) = O((1+|\varepsilon s|)^{-1-\alpha/2})$
- $A_0(\varepsilon s, \varepsilon(t+h)) = O((1+|\varepsilon s|)^{-1-\alpha/2})$
- $B_0(\varepsilon s, \varepsilon(t+h)) = O((1+|\varepsilon s|)^{-2-\alpha/2})$
- $C_0(\varepsilon s, \varepsilon(t+h)) = O((1+|\varepsilon s|)^{-3-3\alpha/2})$

Analyzing each term gives that

$$\|\mathtt{B}(\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

So the linearity of $\mathtt{B}$ shows that this inequality is sufficient to get the estimate.
For the second part of $\mathtt{N}_1$, note that the definition (3.41) of u_1 implies for any $(s,t) \in \mathbb{R}^2$

$$f'(u_1(s,t)) - f'(w(t)) = f''(\xi w(t) + (1-\xi)u_1(s,t))[u_1(s,t) - w(t)] \leq 6(|w(t)| + |u_1(s,t)|)\varphi_1(s,t)$$

So from the definition of φ_1 given (3.42), follows

$$\|(|w|+|u_1|)\varphi_1\|_{C^{0,\lambda}_{0,0}(\mathbb{R}^2)} \leq C\varepsilon^2\|w+u_1\|_{C^{0,\lambda}_{0,0}(\mathbb{R}^2)}(\|Q(\varepsilon s)\psi_0(t)\|_{C^{0,\lambda}_{0,0}(\mathbb{R}^2)} + \varepsilon\|U(\varepsilon s)\psi_1(t)\|_{C^{0,\lambda}_{0,0}(\mathbb{R}^2)})$$

and this implies

$$\|[f'(u_1) - f'(w)](\phi_1 - \phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon^2\|\phi_1 - \phi_2\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

Thus, all the previous analysis leads that $\mathtt{N}_1$ satisfies

$$\|\mathtt{N}_1(\phi_1) - \mathtt{N}_1(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon\|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{5.45}$$

On the other hand, consider functions ϕ_i with

$$\|\phi_i\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq M\varepsilon^3, \quad i = 1,2$$

To analyze $\mathtt{N}_2$, recall that this product was computed for any $(s,t) \in \mathbb{R}^2$ in (A.56)

$$\begin{aligned}
|\mathtt{N}_2(\phi_1) - \mathtt{N}_2(\phi_2)| &= \varepsilon\left|\frac{\nabla_{\bar{x},\bar{y}}a}{a}(\varepsilon x, \varepsilon y)[\nabla_{x,y}\phi_1 - \nabla_{x,y}\phi_2]\right| \\
&\leq \varepsilon\frac{\partial_{\boldsymbol{s}}a}{a}(\varepsilon s, 0)(|\partial_s\phi_1(s,t) - \partial_s\phi_2(s,t)| + \varepsilon|h'(\varepsilon s)|\ |\partial_t\phi_1(s,t) - \partial_t\phi_2(s,t)|) \\
&+ \varepsilon\left[\frac{\partial_{\boldsymbol{t}}a}{a}(\varepsilon s, 0) + \varepsilon(t + h(\varepsilon s))\left(\frac{\partial_{\boldsymbol{tt}}a}{a}(\varepsilon s, 0) - k(\varepsilon s)^2\right)\right]|\partial_t\phi_1(s,t) - \partial_t\phi_2(s,t)| + O(\varepsilon^2)
\end{aligned}$$

so in view of the behavior of $\phi_1(s,t), \phi_2(s,t), a(s,t)$ and $k(\varepsilon s)$, we deduce that

$$\begin{aligned}
\|\mathtt{N}_2(\phi_1) - \mathtt{N}_2(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} &\leq \varepsilon\left\|\frac{\partial_{\boldsymbol{s}}a}{a}(\varepsilon\ \cdot, 0)\right\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}(1 + \varepsilon\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})})\|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\leq \varepsilon\left(\left\|\frac{\partial_{\boldsymbol{s}}a}{a}(\varepsilon s, 0)\right\|_{C^{0,\lambda}_{0,*}(\mathbb{R})} + \varepsilon\|h\|_{C^{2,\lambda}_{0,*}(\mathbb{R})}\|Q\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}\right)\|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + O(\varepsilon^2) \\
&\leq C_{a,h,Q}\cdot\varepsilon\ \|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}
\end{aligned} \tag{5.46}$$

Now, let us analyze $\mathtt{N}_3$ by noting that for any $(s,t) \in \mathbb{R}^2$ the definition (3.42) implies

$$\begin{aligned} f'(u_1(s,t)) - f'(H(t)) &= f''(\xi H(t) + (1-\xi)u_1(s,t)])[u_1(s,t) - H(t)] \\ &\leq 6(|H(t)| + |u_1(s,t)|) \cdot ([w(t) - H(t)] + \varphi_1(s,t)) = O(e^{-\sqrt{2}|t|}) \end{aligned}$$

Then

$$\begin{aligned} &\|\mathtt{N}_3(\phi_1) - \mathtt{N}_3(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\ &\quad = \sup_{(s,t)\in\mathbb{R}^2} e^{\sigma|t|}(1+|\varepsilon s|)^{\mu} \|\zeta_4(x)[f'(u_1) - f'(H)](\Psi(\phi_1) - \Psi(\phi_2))\|_{C^{0,\lambda}(B_1(s,t))} \\ &\quad \leq C \sup_{(s,t)\in\mathbb{R}^2} e^{(\sigma/2-\sqrt{2})|t|}\zeta_4(x) e^{\sigma|t|/2}(1+|\varepsilon s|)^{\mu} \|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))} \\ &\quad \leq C \sup_{(s,t)\in\mathbb{R}^2} e^{(\sigma/2-\sqrt{2})|t|} \sup_{x\in\mathbb{R}^2} K(x) \|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))} \\ &\quad \leq C\|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{0,\lambda}_{K}(\mathbb{R}^2)} = Ce^{-\sigma\delta/2\varepsilon}\|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \end{aligned} \tag{5.47}$$

In order to analyze $\mathtt{N}_4$, note that the definition (4.4) of N_1 also implies

$$\begin{aligned} |\mathtt{N}_4(\phi_1) - \mathtt{N}_4(\phi_2)| &\leq |\zeta_4 N_1(\Psi(\phi_1) + \phi_1) - \zeta_4 N_1(\Psi(\phi_2) + \phi_2)| \\ &\leq C\zeta_4 \sup_{\xi\in(0,1)} |\xi(\Psi(\phi_1) + \phi_1) + (1-\xi)(\Psi(\phi_2) + \phi_2)| \cdot (|\phi_1 - \phi_2| + |\Psi(\psi_1) - \Psi(\psi_2)|) \end{aligned}$$

taking into account the region of $\mathbb{R}^2$ we are considering, it is possible to make appear de weight $K(x)$ in (4.6). Therefore thanks to the hypothesis on ϕ_i and Lemma 5, we obtain

$$\begin{aligned} &\|\mathtt{N}_4(\phi_1) - \mathtt{N}_4(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\ &\quad \leq C \sup_{(s,t)\in\mathbb{R}^2} \Big\{ e^{\sigma|t|/2}[\|\phi_1\|_{C^{0,\lambda}(B_1(s,t))} + \|\phi_2\|_{C^{0,\lambda}(B_1(s,t))} + \|\Psi(\phi_1)\|_{C^{0,\lambda}(B_1(x))} + \|\Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))}] \\ &\qquad \cdot e^{\sigma|t|/2}(1+|\varepsilon s|)^{\mu}[\|\phi_1 - \phi_2\|_{C^{0,\lambda}(B_1(s,t))} + \|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))}] \Big\} \\ &\leq C \sup_{(s,t)\in\mathbb{R}^2} \Big\{ \Big[\|\phi_1\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + K(x)(\|\Psi(\phi_1)\|_{C^{0,\lambda}(B_1(x))} + \|\Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))})\Big] \\ &\qquad \cdot (e^{-\sigma|t|/2}\|\phi_1 - \phi_2\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + K(x)\|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))}) \Big\} \\ &\leq C(\|\phi_1\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\Psi(\phi_1)\|_X + \|\Psi(\phi_2)\|_X [\|\phi_1 - \phi_2\|_{C^{0,\lambda}_{\mu,\sigma}} + \|\Psi(\phi_1) - \Psi(\phi_1)\|_X] \\ &\leq 2C(\varepsilon^3 + e^{-\sigma\delta/2\varepsilon}) \Big\{ \|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + e^{-\sigma\delta/2\varepsilon}\|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \Big\} \end{aligned} \tag{5.48}$$

To reach a conclusion, we note from (5.45)-(5.46)-(5.47) and (5.48) that choosing $\varepsilon > 0$ small enough we obtain the validity of inequality (5.43). The proof of Claim 3 is concluded. □

Conclusion of the proof of Proposition 4

The first observation we make is that formula (4.20) and estimate (4.21) ensure that, for any $0 < \mu < 2 + \alpha$, $\sigma \in (0, \sqrt{2})$ and $\lambda \in (0, 1)$ it holds

$$\|\tilde{S}(u_1) - \varepsilon^2 \mathcal{J}_a[h] \cdot w'\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon^4 \tag{5.49}$$

Let us assume now that $\phi_1, \phi_2 \in B_\varepsilon$, where

$$B_\varepsilon := \{\phi \in C^{2,\lambda}_{loc}(\mathbb{R}^2) \ / \ \|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq K\varepsilon^4\}$$

for a constant K to be chosen. Note that using Claim 3, we are able to bound the size of $\mathtt{N}(\phi)$ for any $\varepsilon > 0$ sufficiently small, as follows

$$\begin{aligned}
\|\mathtt{N}(\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} &\leq C\|\mathtt{N}(0)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + C\varepsilon\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&= C\|\zeta_4[f'(u_1) - f'(H)]\Psi(0) + \zeta_4 N_1(\Psi(0))\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + C\varepsilon\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\leq C \sup_{t\in\mathbb{R}} e^{(\sigma/2-\sqrt{2})|t|} \cdot \|\Psi(0)\|_X + \|\Psi(0)\|^2_X + C\varepsilon \cdot K\varepsilon^4 \\
&\leq Ce^{-\sigma\delta/2\varepsilon} + K\varepsilon^5 \leq \tilde{C}\varepsilon^5 \quad \text{ for all } \quad \phi \in B_\varepsilon
\end{aligned} \tag{5.50}$$

for some constant $\tilde{C}$ that now is independent of K.

Then from the estimates (5.49)-(5.50) follows that the right hand side of the projected problem (5.40) defines an operator $\mathcal{T}$ applying the ball B_ε into itself, provided K is fixed sufficiently large and independent of $\varepsilon > 0$. Indeed using the alternative definition (5.42) of $\mathcal{T}$, and Proposition 3, we can easily find an estimate for the size of ϕ, through

$$\begin{aligned}
\|\mathcal{T}(\phi)\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} &= \|T(-\tilde{S}(u_1) - \varepsilon^2 \mathcal{J}_a[h]w' - \mathtt{N}(\phi))\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\leq \|T\|(\|\tilde{S}(u_1) + \varepsilon^2 \mathcal{J}_a[h]w'\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\mathtt{N}(\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}) \leq C\varepsilon^4
\end{aligned}$$

Further, $\mathcal{T}$ is also a contraction mapping of B_ε in norm $C^{2,\lambda}_{\mu,\sigma}$ provided that $\mu \leq 2 + \alpha$, since Claim (3) asserts that $\mathtt{N}$ has Lipschitz dependence in ϕ:

$$\begin{aligned}
\|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} &= \| - T(\mathtt{N}(\phi_1) - \mathtt{N}(\phi_2))\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\leq C\|\mathtt{N}(\phi_1) - \mathtt{N}(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon \ \|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}
\end{aligned}$$

So by taking $\varepsilon > 0$ small, we can use the contraction mapping principle to deduce the existence of a unique fixed point ϕ to equation (5.41), and thus ϕ turns out to be the only solution of problem (5.40). This justify the existence of ϕ, as required.

On the other hand, the Lipschitz dependence (4.33) of Φ in h, is a consequence of a series of lengthy but straightforward considerations on the operator defining the right hand side of equation (5.40), for the norm $\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})}$ given in (4.12). In particular, this implies that we have to study the operators $\mathtt{N}(h,\phi)$ and $\tilde{S}(u_1)[h]$. For the first one, let us recall formula (5.44) for the operator $\mathtt{B}=\zeta_0\tilde{B}_0$, and consider for example, the terms depending linearly in h:

$$\mathcal{A}(h,\phi) := -2\varepsilon h\partial_{st}\phi - \varepsilon^2 h''\partial_t\phi - \varepsilon(k+\varepsilon(t+h)k^2)\partial_t\phi$$

We have

$$\begin{aligned}
\|\mathcal{A}(h_1,\phi)-\mathcal{A}(h_2,\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} &= \|2\varepsilon(h_1-h_2)\partial_{st}\phi - \varepsilon^2(h_1''-h_2'')\partial_t\phi - \varepsilon^2(h_1-h_2)k^2\partial_t\phi\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\le 2\varepsilon\|h_1-h_2\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}\|\partial_{st}\phi\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \varepsilon^2\|h_1''-h_2''\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}\|\partial_t\phi\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&+ \varepsilon^2\|h_1-h_2\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}\|k^2\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}\|\partial_t\phi\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\le 2(\varepsilon+\varepsilon^2+\varepsilon^2\|k^2\|_{C^{0,\lambda}_{0,*}(\mathbb{R})})\|h_1-h_2\|_{C^{2,\lambda}_{0,*}(\mathbb{R})}\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\le C\varepsilon\|h_1-h_2\|_{C^{2,\lambda}_{0,*}(\mathbb{R})}\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}
\end{aligned}$$

We should also take into account some terms in $\mathtt{B}$ involving nonlinear, yet mild dependence, in h. Recall for instance $\mathcal{B}(h,\phi) := \varepsilon(t+h)A_0(\varepsilon s,\varepsilon(t+h))[\partial_{ss}\phi - 2h'\partial_t\phi + \varepsilon^2|h'|^2\partial_{tt}\phi]$. The difference can be estimated as follows

$$\begin{aligned}
&\|\mathcal{B}(h_1,\phi)-\mathcal{B}(h_2,\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} = \varepsilon\|(\mathcal{A}_0(h_1)-\mathcal{A}_0(h_2))\partial_{ss}\phi\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&+ 2\varepsilon\|(\mathcal{A}_0(h_1)h_1'-\mathcal{A}_0(h_2)h_2')\partial_t\phi\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \varepsilon^3\|(\mathcal{A}_0(h_1)|h_1'|^2-\mathcal{A}_0(h_2)|h_2'|^2)\partial_{tt}\phi\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\le \varepsilon\Big(\|\mathcal{A}_0(h_1)-\mathcal{A}_0(h_2)\|_{C^{0,\lambda}_{0,\sigma}(\mathbb{R}^2)} + 2\|\mathcal{A}_0(h_1)h_1'-\mathcal{A}_0(h_2)h_2'\|_{C^{0,\lambda}_{0,\sigma}(\mathbb{R}^2)} \\
&\quad +\varepsilon^2\|\mathcal{A}_0(h_1)|h_1'|^2-\mathcal{A}_0(h_2)|h_2'|^2\|_{C^{0,\lambda}_{0,\sigma}(\mathbb{R}^2)}\Big)\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}
\end{aligned}$$

where $\mathcal{A}_0(h) := (t+h)A_0(\varepsilon s,\varepsilon(t+h))$.
Nevertheless, for $n=1,2$, the following terms can be bounded like

$$\begin{aligned}
\|\mathcal{A}_0(h_1)|h_1'|^n&-\mathcal{A}_0(h_2)|h_2'|^n\|_{C^{0,\lambda}_{0,\sigma}(\mathbb{R}^2)} \\
&= \|(t+h_1)A_0(\varepsilon s,\varepsilon(t+h_1))|h_1'|^n - (t+h_2)A_0(\varepsilon s,\varepsilon(t+h_2))|h_2'|^n\|_{C^{0,\lambda}_{0,\sigma}(\mathbb{R}^2)} \\
&\le \|A_0\|_{C^{0,\lambda}_{0,0}(\mathbb{R}^2)}\Big(\|t|h_1'|^n - t|h_2'|^n\|_{C^{0,\lambda}_{0,\sigma}(\mathbb{R}^2)} + \|h_1|h_1'|^n - h_2|h_2'|^n\|_{C^{2,\lambda}_{0,*}(\mathbb{R})}\Big) \\
&\le \|A_0\|_{C^{0,\lambda}_{0,0}(\mathbb{R}^2)}\left(\|h_2\|^{n-1}_{C^{2,\lambda}_{0,*}(\mathbb{R})} + 2\|h_1\|^{n-1}_{C^{2,\lambda}_{0,*}(\mathbb{R})} + \|h_2\|^{n+1}_{C^{2,\lambda}_{0,*}(\mathbb{R})}\right)\|h_1-h_2\|_{C^{2,\lambda}_{0,*}(\mathbb{R})} \\
&\le (\mathcal{K}\varepsilon)^{n-1}\|A_0\|_{C^{0,\lambda}_{0,0}(\mathbb{R}^2)}\|h_1-h_2\|_{C^{2,\lambda}_{0,*}(\mathbb{R})}
\end{aligned}$$

We can clearly obtain the same type of bound for $\|\mathcal{A}_0(h_1)-\mathcal{A}_0(h_2)\|_{C^{0,\lambda}_{0,\sigma}(\mathbb{R}^2)}$. So for any $\varepsilon>0$ chosen small enough we get

$$\|\mathcal{B}(h_1,\phi)-\mathcal{B}(h_2,\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}\le C\varepsilon\|h_1-h_2\|_{C^{2,\lambda}_{0,*}(\mathbb{R})}\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

Likewise, the remaining terms of $\mathtt{B}$ having a nonlinear dependence on h

$$\varepsilon^2(t+h)B(\varepsilon s,\varepsilon(t+h))[\partial_s-\varepsilon h'\partial_t],\quad \varepsilon^3(t+h)^2C_0(\varepsilon s,\varepsilon(t+h))\partial_t$$

can be checked to have a similar bound in terms of $\|h_1-h_2\|_{C^{2,\lambda}_{0,*}(\mathbb{R})}$.
Moreover, examining the rest of $\mathtt{N}$ in formula (5.39), just like before, allow us to deduce that there is a Lipschitz dependence on h for the whole operator $\mathtt{N}$, with small constant. Further,

$$\|\mathtt{N}(h_1,\phi)-\mathtt{N}(h_2,\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}\le C\varepsilon\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{5.51}$$

On the other hand, for the error term

$$\mathcal{R}(h)=-\tilde{S}(u_1)[h]-\varepsilon^2\mathcal{J}_a[h]w'(t)$$

we claim using formula (4.20) for $\tilde{S}(u_1)$, that for $\mu\in(0,2+\alpha)$

$$\|\mathcal{R}(h_1)-\mathcal{R}(h_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}\le C\varepsilon^3\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})} \tag{5.52}$$

To see this, let us analyze term by term of $\tilde{S}(u_1)$ in expansion (4.20). The linear part of $\mathcal{R}$ in h complies

$$\|\varepsilon^4Q(\varepsilon s)\psi_0'(h_1''-h_2'')\|_{C^{0,\lambda}(B_1(s,t))}\le C\varepsilon^4(1+|\varepsilon s|)^{-\mu}e^{-\sigma|t|}\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}$$

Now, for the nonlinear dependence of $\mathcal{R}$ in h, note that for suitable μ and σ

$$\|\varepsilon^3(t+h_1-h_2)A_0(\varepsilon s,\varepsilon(t+h_1-h_2))(h_1''-h_2'')w''(t)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}\le\varepsilon^3C_{k,a}\|h_1-h_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R})}$$

In addition, thanks to estimate (4.21)

$$\begin{aligned}|R_1(\varepsilon s,t,h_1,h_1')-&R_1(\varepsilon s,t,h_2,h_2')|\\ &\le\sup_{\xi\in(0,1)}|\nabla_{\imath,\jmath}R_1(\varepsilon s,t,\xi h_1+(1-\xi)h_2,\xi h_1'+(1-\xi)h_2')|\cdot|(h_1-h_2,h_1'-h_2')|\\ &\le C\varepsilon^4(1+|\varepsilon s|)^{-2-2\alpha}e^{-\sqrt{2}|t|}\|h_1-h_2\|_{C^{1,\lambda}_{0,*}(\mathbb{R})}\end{aligned}$$

Note also that there is a Hölder character of $R_1(\varepsilon s,t,h_1,h_1')-R_1(\varepsilon s,t,h_2,h_2')$ in (s,t), since the expansion (3.50) of R_1 shows that all the terms involved are basically linear or quadratic in h and its derivatives. Besides, given the bound (4.12) for h_i, $i=1,2$ and the smoothness plus the decay of $k(\varepsilon s)$, $a(\varepsilon s,0)$, $w(t)$, it follows that the Hölder seminorm $[\cdot]_{0,\lambda,B_1(s,t)}$ of this subtraction can be easily bounded by some constant times $\varepsilon^3\|h_1-h_2\|_{C^{2,\lambda}(B_1(s,t))}$. Let us take a look, as an example ,to the term: $\varepsilon^2(|h_1'(\varepsilon s)|^2 - |h_2'(\varepsilon s)|^2)w''(t)$. We just need to use the difference of squares formula, to get directly the desired bound

$$\|\varepsilon^2(|h_1'|^2 - |h_2'|^2)w''\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq \varepsilon^2\|(h_1+h_2)w''\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}\|h_1-h_2\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\varepsilon^3\|h_1-h_2\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})}$$

This argument justifies inequality (5.52).

Now we may combine both Lipschitz character of $\mathtt{N}$, in h for ϕ fixed (5.51), as in ϕ with h fixed (5.43). In addition, considering the estimate (5.52) for $\mathcal{R}$ we finally obtain the desired Lipschitz dependence (4.33) of Φ in h. In fact, using the fixed point characterization (5.42):

$$\begin{aligned}
\|\Phi(h_1)-\Phi(h_2)\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} &= \|\mathcal{T}(\Phi(h_1)) - \mathcal{T}(\Phi(h_2))\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\leq \|T\|(\|\mathcal{R}(h_1)-\mathcal{R}(h_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\mathtt{N}(h_1,\Phi(h_1)) - \mathtt{N}(h_2,\Phi(h_2))\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}) \\
&\leq \|T\|(C\varepsilon^3\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})} \\
&+ \|\mathtt{N}(h_1,\Phi(h_1)) - \mathtt{N}(h_2,\Phi(h_1))\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\mathtt{N}(h_2,\Phi(h_1)) - \mathtt{N}(h_2,\Phi(h_2))\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}) \\
&\leq C(\varepsilon^3 + \varepsilon\|\Phi(h_1)\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)})\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})} + \tilde{C}\varepsilon\|\Phi(h_1)-\Phi(h_2)\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}
\end{aligned}$$

So by recalling that $\|\Phi(h_1)\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon^4$, we deduce

$$(1-\tilde{C}\varepsilon)\|\Phi(h_1)-\Phi(h_2)\|_{C^{2,\lambda}_{\mu,\lambda}(\mathbb{R}^2)} \leq C\varepsilon^3\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}$$

Finally, by choosing $\varepsilon > 0$ small enough, it follows the proof of Proposition 4. □

5.3. Nonlinear Jacobi problem: Proposition 6

In this section we will prove Proposition 6, based on the linear theory provided by Proposition 5. The reduced problem that must be solved is

$$\mathcal{J}_a[h](\varepsilon s) := h''(\varepsilon s) + \frac{\partial_{\boldsymbol{s}} a(\varepsilon s,0)}{a(\varepsilon s,0)}h'(\varepsilon s) - Q(s)h(\varepsilon s) = \mathtt{G}(h)(\varepsilon s) \quad \text{in} \quad \mathbb{R} \tag{5.53}$$

where $Q(s)$ was defined in (2.16), and the operator $\mathtt{G} = G_1 + G_2$ was given in (4.36)-(4.37). The idea is to use contraction mapping principle to determine the existence of a unique solution $h = h(\bar{s})$ with $\bar{s} := \varepsilon s$, for which constraint (4.12) is satisfied, namely

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} := \|h\|_{L^\infty(\mathbb{R})} + \|(1+|\bar{s}|^{1+\alpha})h'\|_{L^\infty(\mathbb{R})} + \sup_{s\in\mathbb{R}}(1+|\bar{s}|^{2+\alpha})\|h''\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)} \leq \mathcal{K}\varepsilon \tag{5.54}$$

after fixing $\mathcal{K}$ sufficiently large. A crucial step for this purpose, is to analyze the size of the operator $\mathtt{G}$, for which we have the following estimate.

Lemma 10. *Let $\Theta = \Theta(s,t)$ be a function defined in $\mathbb{R}\times\mathbb{R}$, such that, for any $\lambda \in (0,1)$, $\mu \in (1, 2+\alpha]$ and $\sigma \in (0,\sqrt{2})$*

$$\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} := \sup_{(s,t)\in\mathbb{R}\times\mathbb{R}} e^{\sigma|t|}(1+|\varepsilon s|)^{\mu}\|\Theta\|_{C^{0,\lambda}(B_1(s,t))} < +\infty$$

Then the function defined in $\mathbb{R}$ as

$$Z(\varepsilon s) := \int_{\mathbb{R}} \Theta(s,t)w'(t)dt$$

satisfies for some constant $C = C(w,\mu,\sigma) > 0$ the following estimate:

$$\|Z\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \le C\varepsilon^{-1}\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{5.55}$$

Proof.-
Recall the definition of the norm (4.11). By denoting $\bar{s} := \varepsilon s$, we need to prove

$$\|Z\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} = \sup_{\bar{s}\in\mathbb{R}}(1+|\bar{s}|)^{\mu}\left\|\int_{\mathbb{R}}\Theta(\bar{s}/\varepsilon,t)w'(t)dt\right\|_{C^{0,\lambda}(\bar{s}-1,\bar{s}+1)}$$

First, in order to bound the L^{∞} norm, is easy to see that for any $\bar{s}\in\mathbb{R}$

$$|Z(\bar{s})| = \left|\int_{\mathbb{R}}\Theta(\bar{s}/\varepsilon,t)w'(t)dt\right| \le \int_{\mathbb{R}}\|\Theta\|_{L^{\infty}(B_1(\bar{s}/\varepsilon,t))}w'(t)dt \le \frac{\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}}{(1+|\bar{s}|)^{\mu}}\int_{\mathbb{R}}e^{-\sigma|t|}w'(t)dt$$

In particular, as $\mu > 1$, this implies that for any $\bar{s}_1 \in (\bar{s}-1,\bar{s}+1)$

$$|Z(\bar{s}_1)| \le C_w\frac{1}{(1+|\bar{s}_1|)^{\mu}}\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le C_w\frac{2^{\mu}}{(1+|\bar{s}|)^{\mu}}\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

and thus we deduce

$$(1+|\bar{s}|)^{\mu}\|Z\|_{L^{\infty}(\bar{s}-1,\bar{s}+1)} \le \tilde{C}_w\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{5.56}$$

Further, we need to estimate the Hölder seminorm. Take any $\bar{s}_1,\bar{s}_2 \in (\bar{s}-1,\bar{s}+1)$ with $\bar{s}_1 < \bar{s}_2$. Let us consider a increasing sequence, of N_ε points between $s_1 := \bar{s}_1/\varepsilon$ and $s_2 := \bar{s}_2/\varepsilon$, given by the formula

$$s^i = \frac{\bar{s}_1}{\varepsilon} + \frac{i}{N_\varepsilon}\left(\frac{\bar{s}_2}{\varepsilon} - \frac{\bar{s}_1}{\varepsilon}\right) \qquad \text{for} \qquad i = 1,\dots,N_\varepsilon$$

so that their consecutive mutual distance is $|s^{i+1}-s^i| = |\bar{s}_2-\bar{s}_1|/(\varepsilon N_\varepsilon)$. In addition, the number of points must be chosen large enough to make $|s^{i+1}-s^i| < 1$, this implies $N_\varepsilon = O(\varepsilon^{-1})$. In view of all this discussion, we can estimate the difference as follows

$$|Z(\bar{s}_2)-Z(\bar{s}_1)| \le \int_{\mathbb{R}}\left|\Theta\left(\frac{\bar{s}_2}{\varepsilon},t\right) - \Theta\left(\frac{\bar{s}_1}{\varepsilon},t\right)\right|w'(t)dt \le \sum_{i=0}^{N_\varepsilon-1}\int_{\mathbb{R}}\left|\Theta\left(s^{i+1},t\right)-\Theta\left(s^i,t\right)\right|w'(t)dt$$

$$\le \sum_{i=0}^{N_\varepsilon-1}\int_{\mathbb{R}}[\Theta]_{0,\lambda,B_1(s^i,t)}|s^{i+1}-s^i|^{\lambda}w'(t)dt \le \frac{\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}}{(1+|\varepsilon s^i|)^{\mu}}\int_{\mathbb{R}}e^{-\sigma|t|}w'(t)dt\cdot\underbrace{\sum_{i=0}^{N_\varepsilon-1}|s^{i+1}-s^i|^{\lambda}}_{I}$$

Nevertheless, given $\lambda \in (0,1)$ it follows that $|\cdot|^\lambda$ is a concave function, and so

$$I = N_\varepsilon \sum_{i=0}^{N_\varepsilon - 1} \frac{1}{N_\varepsilon}|s^{i+1} - s^i|^\lambda \leq N_\varepsilon \left| \sum_{i=0}^{N_\varepsilon - 1} \frac{1}{N_\varepsilon}(s^{i+1} - s^i) \right|^\lambda \leq N_\varepsilon^{1-\lambda} \left| \sum_{i=0}^{N_\varepsilon - 1} (s^{i+1} - s^i) \right|^\lambda$$

moreover, applying telescopic summation, we get

$$I \leq N_\varepsilon^{1-\lambda}|s_2 - s_1|^\lambda = N_\varepsilon^{1-\lambda}\frac{|\bar{s}_2 - \bar{s}_1|^\lambda}{\varepsilon^\lambda} \leq \left(\frac{|\bar{s}_2 - \bar{s}_1|}{\varepsilon}\right)^{1-\lambda} \frac{|\bar{s}_2 - \bar{s}_1|^\lambda}{\varepsilon^\lambda} = \frac{2^{1-\lambda}|\bar{s}_2 - \bar{s}_1|^\lambda}{\varepsilon}$$

and then, as $\mu > 1$, we obtain

$$|Z(\bar{s}_2) - Z(\bar{s}_1)| \leq 2^{1-\lambda+\mu}\varepsilon^{-1}\frac{\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}}{(1+|\bar{s}|)^\mu} \int_{\mathbb{R}} e^{-\sigma|t|}w'(t)dt \cdot |\bar{s}_2 - \bar{s}_1|^\lambda$$

Therefore for any $\bar{s} \in \mathbb{R}$

$$(1+|\bar{s}|)^{\mu-1}[Z]_{0,\lambda,(\bar{s}-1,\bar{s}+1)} \leq \varepsilon^{-1}\tilde{C}\int_{\mathbb{R}} e^{-\sigma|t|}w'(t)dt \; \|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{5.57}$$

In conclusion, the previous estimates (5.56)-(5.57) justify the desired bound:

$$\sup_{\bar{s}\in\mathbb{R}}(1+|\bar{s}|)^\mu(\|Z\|_{L^\infty(\bar{s}-1,\bar{s}+1)} + [Z]_{0,\lambda,(\bar{s}-1,\bar{s}+1)}) \leq C_{w,\mu,\sigma}\varepsilon^{-1} \; \|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

where we set $C_{w,\mu,\sigma} := 2^{1+\mu} \cdot \int_{\mathbb{R}} e^{-\sigma|t|}w'(t)dt$, thus finishing the proof of Lemma (10). $\square$

Let us apply Lemma (10) to the function $\Theta(s,t) := \mathtt{N}(\Phi(h))(s,t)$, to estimate the size of the operator G_2 in (4.37). Recall that

$$G_2(h)(\varepsilon s) := c_*^{-1}\varepsilon^{-2}\int_{\mathbb{R}} \mathtt{N}(h,\Phi(h))(s,t)w'(t)dt$$

We can estimate the size of the projection of $\mathtt{N}$ using the previous estimate (5.55), and the bound (5.50) for the size of $\mathtt{N}$:

$$\|G_2(h)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\varepsilon^{-2}\cdot\varepsilon^{-1}\|\mathtt{N}(h,\Phi(h))\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon^{-3}\cdot\varepsilon^5 = \varepsilon^2 \tag{5.58}$$

Likewise, for $\phi_i = \Phi(h_i)$, $i = 1,2$ it holds similarly that

$$\|G_2(h_1) - G_2(h_2)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\varepsilon^{-2}\cdot\varepsilon^{-1}\|\mathtt{N}(h_1,\phi_1) - \mathtt{N}(h_2,\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

Nonetheless, the Lipschitz character (5.51) of $\mathtt{N}$ and the bound (4.32) for the size of ϕ imply

$$\|\mathtt{N}(h_1,\phi_1) - \mathtt{N}(h_2,\phi_1)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon\|h_1 - h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}\|\phi_1\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\varepsilon^5\|h_1 - h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}$$

Additionally, using the Lipschitz dependence (5.43) of $\mathtt{N}$ in ϕ, and also the Lipschitz dependence (4.33) of Φ in h, the next term can be bounded as:

$$\|\mathtt{N}(h_2,\phi_1)-\mathtt{N}(h_2,\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le C\varepsilon\|\phi_1-\phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le C\varepsilon^4\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}$$

The previous estimates allow us to deduce

$$\begin{aligned}\|G_2(h_1)-G_2(h_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} &\le C\varepsilon^{-3}\|\mathtt{N}(h_1,\phi_1)-\mathtt{N}(h_2,\phi_1)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}\\ &\quad + C\varepsilon^{-3}\|\mathtt{N}(h_2,\phi_1)-\mathtt{N}(h_2,\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}\\ &\le C\varepsilon\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}\end{aligned}$$

Furthermore, from (5.58) we also have that

$$\|G_2(0)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \le C\varepsilon^2 \tag{5.59}$$

for some $C>0$ possibly depending on $\mathcal{K}$.
We can be similarly bound the difference of the remaining small operator $G_1(h_1)-G_1(h_2)$, that is given in (4.36)

$$\begin{aligned}c_*G_1(h_1) &= \varepsilon h_1''(\varepsilon s)\int_{\mathbb{R}}\zeta_0(t+h_1)A_0(\varepsilon s,\varepsilon(t+h_1))w''(t)w'(t)dt\\ &\quad + \varepsilon^2 Q(\varepsilon s)h_1''(\varepsilon s)\int_{\mathbb{R}}\psi_0'(t)w'(t)dt + \varepsilon^{-2}\int_{\mathbb{R}}\zeta_0\; R_1(\varepsilon s,t,h_1,h_1')w'(t)dt\end{aligned}$$

This is a consequence of an direct bound for the integral in Hölder norm, without the need of using estimate (5.55), since the integrand is also evaluate in εs. In fact, denoting by $\hat{A}_0(h) := (t+h)A_0(\varepsilon s,\varepsilon(t+h))$, it follows that

$$\begin{aligned}&\left\|h_1''\int_{\mathbb{R}}\zeta_0\hat{A}_0(h_1)w''w'dt - h_2''\int_{\mathbb{R}}\zeta_0\hat{A}_0(h_2)w''w'dt\right\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})}\\ &\le \|h_1''-h_2''\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})}\left\|\int_{\mathbb{R}}\hat{A}_0(h_1)w''w'dt\right\|_{C^{0,\lambda}_{0,*}(\mathbb{R})} + \|h_2''\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})}\left\|\int_{\mathbb{R}}\left(\hat{A}_0(h_1)-\hat{A}_0(h_2)\right)w''w'dt\right\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}\\ &\le \|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}\int_{\mathbb{R}}\left\|\hat{A}_0(h_1)(\cdot,t)\right\|_{C^{0,\lambda}_{0,*}(\mathbb{R})} w''w'dt\\ &\qquad + \|h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}\int_{\mathbb{R}}\left\|\hat{A}_0(h_1)-\hat{A}_0(h_2)(\cdot,t)\right\|_{C^{0,\lambda}_{0,*}(\mathbb{R})} w''w'dt\end{aligned} \tag{5.60}$$

However, from the definition of $\hat{A}_0$ and the expansion (3.7) of A_0, we have

$$\left\|\hat{A}_0(h_1)(\cdot,t)\right\|_{C^{0,\lambda}_{0,*}(\mathbb{R})} \le C(t+\underbrace{\|h_1\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}}_{O(\varepsilon)})(\|k\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}+\varepsilon t\|k^2\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}) \le Ct\|k\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}$$

furthermore,

$$\left\|\hat{A}_0(h_1)-\hat{A}_0(h_2)(\cdot,t)\right\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}=\|(t+h_1)A_0(\cdot,\varepsilon(t+h_1))-(t+h_2)A_0(\cdot,\varepsilon(t+h_2))\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}$$

$$\leq \|(t+h_1)[A_0(\cdot,\varepsilon(t+h_1))-A_0(\cdot,\varepsilon(t+h_2))]\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}+\|(h_1-h_2)A_0(\cdot,\varepsilon(t+h_2))\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}$$

$$\leq C\left(\varepsilon(t+\|h_1\|_{C^{0,\lambda}_{0,*}(\mathbb{R})})\|k^2\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}+\|A_0(\cdot,\varepsilon(t+h_2))\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}\right)\|h_1-h_2\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})}$$

$$\leq C\|k\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}\|h_1-h_2\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})}$$

Gathering these estimates in equation (5.60) we obtain

$$\left\|h_1''\int_{\mathbb{R}}\zeta_0\hat{A}_0(h_1)w''w'dt-h_2''\int_{\mathbb{R}}\zeta_0\hat{A}_0(h_2)w''w'dt\right\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})}$$

$$\leq C\left(\int_{\mathbb{R}}tw''(t)w'(t)dt+\mathcal{K}\varepsilon\int_{\mathbb{R}}w''(t)w'(t)dt\right)\|k\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})} \tag{5.61}$$

Similarly, there is a direct bound for the difference of the projection of R_1:

$$\left\|\int_{\mathbb{R}}\zeta_0(R_1(\varepsilon s,t,h_1,h_1')-R_1(\varepsilon s,t,h_2,h_2'))w'dt\right\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}$$

$$\leq \int_{\mathbb{R}}\|R_1(\cdot,t,h_1,h_1')-R_1(\cdot,t,h_2,h_2')\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}w'(t)dt$$

$$\leq \varepsilon^3\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}\int_{\mathbb{R}}e^{-\sqrt{2}|t|}w'(t)dt \tag{5.62}$$

where we used the decomposition (3.50) of R_1 in terms of linear and quadratic powers of h. In sum, thanks to (5.61) and (5.62), we are now able to establish a global estimate for the difference of the entire operator $G_1(h)$ as follows

$$\|G_1(h_1)-G_1(h_2)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})}\leq C\varepsilon\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}+C_w\varepsilon^2\|Q\|_{C^{0,\lambda}_{0,*}(\mathbb{R})}\|h_1''-h_2''\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})}$$

$$+\varepsilon^{-2}\|R_1(\varepsilon s,t,h_1,h_1')-R_1(\varepsilon s,t,h_2,h_2')\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})}$$

$$\leq (C\varepsilon+C_{w,Q}\varepsilon^2+\varepsilon^{-2}\cdot\varepsilon^3)\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}$$

therefore, it satisfies

$$\|G_1(h_1)-G_1(h_2)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})}\leq C\varepsilon\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}$$

Now, a simple but crucial observation we make is that

$$c_*G_1(0)=\varepsilon^{-2}\int_{\mathbb{R}}\zeta_0\ R_1(\varepsilon s,t,0,0)w'(t)dt$$

so given the size (4.21) of R_1, it follows that for some constant C_2 independent of $\mathcal{K}$ in (5.54)

$$\|G_1(0)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\varepsilon^{-2} \cdot \varepsilon^{-1}\|R_1\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R})} \leq C_2\varepsilon \tag{5.63}$$

Therefore, the entire operator $\mathtt{G}(h)$ inherits a Lipschitz character in h, from those of G_1, G_2:

$$\|\mathtt{G}(h_1) - \mathtt{G}(h_2)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\varepsilon\|h_1 - h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})} \tag{5.64}$$

Further, estimates (5.59)-(5.63) imply that $\mathtt{G}$ is such

$$\|\mathtt{G}(0)\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})} \leq 2C_2\varepsilon \tag{5.65}$$

Now let $h = T(f)$ be the linear operator defined in Proposition 5, and let $\mathtt{G}$ be the nonlinear operator given in (4.37). Consider the Jacobi nonlinear equation (5.53), but this time written as a fixed point problem: Find some h such that

$$h = T(\mathtt{G}(h)), \quad \|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq \mathcal{K}\varepsilon \tag{5.66}$$

We need first to estimate the size of $T \circ \mathtt{G}$, in order to solve (5.66) using contraction mapping principle.

$$\begin{aligned}\|T(\mathtt{G}(h))\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} &\leq \|T\|(\|\mathtt{G}(h) - \mathtt{G}(0)\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} + \|\mathtt{G}(0)\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})}) \\ &\leq \varepsilon(\|T\|\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} + 2C_2\|T\|)\end{aligned}$$

where we made use of (5.64)-(5.65), and that T is the bounded linear operator. Hence choosing $\mathcal{K} > 3C_2\|T\|$, we find that for all ε sufficiently small, the operator $T \circ \mathtt{G}$ defines an endomorphism on the ball $\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq \mathcal{K}\varepsilon$.

Moreover, the linearity of T and the Lipschitz character of $\mathtt{G}$, directly imply that for $\varepsilon > 0$ small enough the operator $T \circ \mathtt{G}$ is a contraction mapping

$$\|T(\mathtt{G}(h_1)) - T(\mathtt{G}(h_2))\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq \|T\|\|\mathtt{G}(h_1) - \mathtt{G}(h_2)\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq C\varepsilon\|h_1 - h_2\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})}$$

In conclusion, as a consequence of the Banach's fixed point theorem, we have proved the existence of a unique fixed point of the problem (5.66), that is, a unique solution h to equation (5.53) satisfying (5.54). This finishes the proof of Proposition 6. ☐

Chapter 6

Conclusions and future work

In the Introduction we mentioned the relation between the solutions of the inhomogeneous Allen-Cahn equation (1.6), and the properties of the potential $a(x,y)$ involved in this PDE. This opened a question about the existence of a smooth bounded solution u with transition near a given noncompact curve $\Gamma \subset \mathbb{R}^2$. More specifically, in determining sufficient conditions on $a(x,y)$ and Γ in order to build such solutions. In this direction, Theorem 1 provides some specific conditions on both, the potential $a(x,y)$ and the curve Γ, of which we point out the following: *I*) The smoothness and the uniform positiveness of the potential $a(x,y)$, *II*) The polynomial decay along the curve of the potential, and the decay of curvature k_Γ, *III*) The stationarity of Γ relative to $l_{a,\Gamma}$ plus a nondegeneracy, in relation to the existence of bounded kernel of the Jacobi operator $\mathcal{J}_a[h]$. Furthermore as expected, it turns out that the solution u depends strongly on the potential $a(x,y)$. Indeed, the construction method forces $u(\vec{x})$ to depend on some perturbation h, that is ultimately determined by $a(\vec{x})$; h needs to solve the nonlinear Jacobi equation (4.38).

It is of interest to discuss some technical similarities and differences between this work and what has been developed in the literature so far. To begin with, we should mention that the inhomogeneity term $a(x,y)$ makes profound changes in the formulation for the classic Jacobi operator of a manifold $M \subset \mathbb{R}^N$, which in the presence of the Euclidean metric $a \equiv 1$ simply reduces to

$$\mathcal{J}_M[h] := \Delta_x h + [|A_M| + \mathrm{Ric}_g(\nu_M, \nu_M)]h$$

In our context, given that $N = 2$, the Jacobi operator around Γ turns out to be

$$\mathcal{J}_{a,\Gamma}[h] := h'' + \frac{\partial_{\boldsymbol{s}} a}{a} h' + \left(2k_\Gamma^2 - \frac{\partial_{\boldsymbol{tt}} a}{a} \right) h$$

Therefore, the study of the invertibility of the Jacobi operator needs to take into account the properties of the potential $a(x,y)$. In particular, we discuss the method employed to get an inverse of $\mathcal{J}_{a,\Gamma}$ in the space of bounded functions. The *lack of compactness* is one of the most important aspects of this work, regarding the functional setting of the Jacobi operator where perturbations $h : \Gamma \to \mathbb{R}$ are now defined on an unbounded domain. In contrast, compact or unbounded Manifolds are usually considered in the literature, whose geometrical properties allow some *pseudo-compact* contexts through the use of techniques of geometrical inversion. To cope this difficulty, we benefit from the fact that $\mathcal{J}_{a,\Gamma}$ is a differential operator acting on functions of only one variable, thus allowing the study of the Jacobi equation using theory of ODEs. A precise qualitative description is presented in Proposition 1, for the asymptotic behavior of a solution to $\mathcal{J}_{a,\Gamma}[h] = 0$, provided some conditions on the coefficients of the equation. The study of the kernel of the Jacobi operator is the key aspect from which we obtain the desired invertibility. Proposition 2 assures the sufficiency for the variation of parameters formula, to provide a smooth bounded solution of $\mathcal{J}_{a,\Gamma}[h] = f$, for a locally Hölder right-hand side decaying polynomially. Further, the polynomial decay is inherited to h' and h'' as stated in (2.49). In addition, the last Proposition also shows a high regularity for the solution here provided, unlike what presented on classical contexts of invertibility, where the inverse of $\mathcal{J}_M$ is usually defined in functional spaces of weaker regularity.

Another crucial aspect that deserves our attention, is the difference in the linear invertibility theory developed in Proposition 2 with respect to what it has done before. In this work, both the potential $a(x)$ and the curve Γ are such that Γ presents no *symmetries*, in the sense that rigid motions cannot induce some bounded Jacobi fields $\mathcal{J}_{a,\Gamma}(z) = 0$. The nondegeneracy condition of Γ supposed in Theorem 1, amounts to the latter property, because under this assumption not only the bounded kernel space of $\mathcal{J}_{a,\Gamma}$ has finite dimension, but also is spanned by $\{0\}$. This simplifies the study of the invertibility theory of the Jacobi operator in our case. In contrast, the authors in [8] considered complete, embedded minimal surfaces M in $\mathbb{R}^3$, which they do present *symmetries*. Translations along the coordinate axis x_1, x_2, x_3, and the rotation around the x_3- axis induce functions $\{z_i\}_{i=1}^4$ that form a basis of the bounded kernel:

$$\{z \in L^\infty(\Gamma) : \ \mathcal{J}_M(z) = 0\} = \text{span}\{z_1, z_2, z_3, z_4\}$$

In that context, the solvability for the Jacobi equation $\mathcal{J}_M[h] = f$ is more complex, because it not only depends on the properties of M, but also makes necessary the study of an associate problem known as the *projected Jacobi problem*

$$\mathcal{J}_M[h] = f + \sum_{i\in J} \frac{c_i}{1+r^4}\hat{z}_i, \qquad \int_M \frac{h(y)\cdot \hat{z}_i(y)}{1+r^4(y)} dV = 0, \quad \text{for} \quad i = 1, \dots, J$$

where $\{\hat{z}_i\}_{i\in J}$ represent the bounded Jacobi fields. A consequence of this essential difference in the solvability for the Jacobi equation is the fact that, in our proof we did not have to justify that the right-hand side can be chosen so that it satisfies an orthogonality condition against the bounded Jacobi fields. In particular, we did not have to make use of the invariance of the Allen-Cahn equation nor to use balancing formulas for the geodesic Γ to justify this.

In another topic, it is worth mentioning a previous stage of this study, where we dealt with a slight simplification of the context in this thesis work. It was studied the existence of a solution u to the inhomogeneous Allen-Cahn equation (1.12), in the case where the potential $a : \mathbb{R}^2 \to \mathbb{R}$ has the form $a(\vec{x}) = 1 + \chi(\vec{x})$, where function χ has compact support. An interesting result arose from this analysis, characterizes the *nondegeneracy condition* of the unbounded curve Γ in terms of the solvability of an related ODE in a compact domain. More explicitly, we proved

Proposition 8. *Let Γ be an unbounded curve, intersecting the set $\Omega := \text{supp}(\chi) \subset \mathbb{R}^2$. Assume that the portion of Γ contained in Ω is parametrized as $\Gamma \cap \Omega := \gamma([s_1, s_2])$. Then Γ is a nondegenerate curve with respect to the arclength $\int_\Gamma a(\vec{x})$, if and only if, the following Neumann boundary value problem*

$$\begin{cases} \mathcal{J}_{a,\Gamma}[h](s) = 0, & in \quad (s_1, s_2) \\ h'(s_1) = h'(s_2) = 0 \end{cases} \tag{6.1}$$

does not have the eigenvalue $\lambda = 0$.

From this fact we can easily describe the kernel of $\mathcal{J}_{a,\Gamma}$, since the nontrivial behavior of a bounded basis h_1, h_2 it only could arise on the compact portion Ω, depending on the existence of the eigenfunction associated to $\lambda = 0$. Another appealing geometrical property arisen in this context is related to the stationarity for geodesics. It can be shown, using a similar argument than carried out in [7] for the analysis of the nondegeneracy in $\mathbb{R}^2$ on a bounded domain, that a necessary condition for a curve Γ to be geodesic related to the length $\int_\Gamma a(\vec{x})$, is that Γ must *cross perpendicularly the boundary* $\partial\Omega$, which requires that on each point of intersection $P \in \Gamma \cap \partial\Omega$ the tangent vector $\hat{t}$ of the curve must be perpendicular to the normal of the boundary ν_Ω.

On the other hand, we must say that the two examples 2.3.3-2.3.4 exhibited in Chapter 2 constitutes a major contribution to the understanding of Differential Geometry in relation to Partial Differential Equations. There are only a few examples of this kind in the literature, because of the difficulty in finding nontrivial geometrical configurations in which geodesic curves which are *non-degenerate* with respect to some arclength $\int_\Gamma a(\vec{x})$. Regarding this matter, we can provide another interesting case: Fix any smooth bounded potential of the form $a^*(x, y) := i(y)$ with function $i(y)$ being uniformly positive, and achieving a local minimum around zero, so that $i'(0) = 0, i''(0) > 0$. Under these circumstances, it follows easily that $\Gamma = \overrightarrow{OX}$ is a nondegenerate geodesic relative to the length induced by a^*. Indeed the stationarity condition in Euclidean coordinates trivially holds for $\Gamma = \{(\boldsymbol{x}, 0)\}_{\boldsymbol{x}\in\mathbb{R}}$ and moreover, the Jacobi operator is reduced to $\mathcal{J}_{a^*,\Gamma}[h] = h'' - \lambda_i h$ for $\lambda_i = i''(0)/i(0) > 0$. Hence the kernel of $\mathcal{J}_{a^*,\Gamma}$ comprises linear combinations of $\exp(\sqrt{\lambda_i}\boldsymbol{x})$ and $\exp(-\sqrt{\lambda_i}\boldsymbol{x})$ in the entire $\mathbb{R}$, and thus this operator satisfies the nondegeneracy condition. Note however that this potential does not decay as assumed in (1.15), nonetheless we can repeat the same scheme as performed in this work to build a solution u^* to the inhomogeneous Allen-Cahn equation with a single transition on the curve $\overrightarrow{OX}$, but this time using a $L^\infty-$functional context in the solvability of the nonlinear Jacobi equation and the gluing system. This work would allow to find more examples of this phenomena, in more complex configurations, and in geometrical settings on higher dimensions.

There are natural extensions of this thesis, that can lead to future works. The first one deals with same context that this work does, and concerns the search of more criteria that imply the nondegeneracy condition for geodesic in $\mathbb{R}^2$, related to the length $\int_\Gamma a(\vec{x})$, and different from Corollary 1.

Another open problem consists in a variant of this thesis work, on the existence of smooth bounded solutions u to the inhomogeneous Allen-Cahn equation (1.12) with multiple transitions near an noncompact curve Γ, whose positions are expected to be governed by a Toda-type system. Some other cases consist in the study of the same equation in a variety of settings, where the potential $a(\vec{x})$ is less smooth or has some singularities, or where the uniform positiveness does not hold.

Appendix A

Annexe

The two successive sections are mainly oriented in finding expressions for each one of the terms in the Allen-Cahn equation (1.12) written in Fermi coordinates, that are suitable for the geometrical study of this equation. Since they define a local change of variables in a neighborhood of Γ, our effort will focus on finding an equivalent form of (1.12) in these coordinates.

A.1. Calculation of the Laplacian: Proof of Lemma 3

Hereinafter we regard $u = u(\bar{x}, \bar{y})$ as a function where $(\bar{x}, \bar{y})$ denotes non-dilated Euclidean coordinates. In order to characterize the Euclidean Laplacian $\Delta_{x,y}$ in dilated and translated Fermi coordinates, we will follow a scheme that includes the construction of the operator in 3 steps, for which the analysis is simplified.

For any $\delta > 0$ small but fixed, and a curve $\Gamma \subset \mathbb{R}^2$ parameterized by $\gamma \in C^2(\mathbb{R}, \mathbb{R}^2)$, let us consider first the local Fermi coordinates induced by Γ

$$X : \mathbb{R} \times (-\delta, \delta) \to \mathcal{N}_\delta \ , \quad X(\boldsymbol{s}, \boldsymbol{t}) = \gamma(\boldsymbol{s}) + \boldsymbol{t} \cdot \nu(\boldsymbol{s}) \tag{A.1}$$

where $\nu(\boldsymbol{s})$ denotes the normal vector to the curve Γ at the point $\gamma(\boldsymbol{s})$.

It can be seen that X defines a local change of variables on the tubular open neighborhood

$$\mathcal{N}_\delta := \{(\bar{x}, \bar{y}) = \gamma(\boldsymbol{s}) + \boldsymbol{t} \cdot \nu(\boldsymbol{s}) \ / \ \boldsymbol{s} \in \mathbb{R}, \ |\boldsymbol{t}| < \delta + \varepsilon \cdot 2c_0|\boldsymbol{s}|\}$$

of Γ, where $c_0 > 0$ is a fixed number, and $|\boldsymbol{t}| = \text{dist}((\bar{x}, \bar{y}), \Gamma)$ for every $(\bar{x}, \bar{y}) = X(\boldsymbol{s}, \boldsymbol{t})$. The picture below depicts this geometrical setting

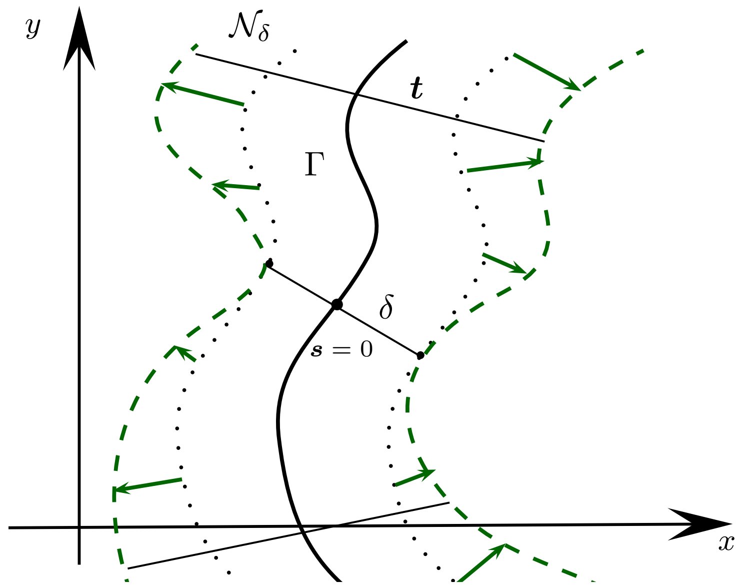

Figure 6: Neighborhood $\mathcal{N}_\delta$ in Fermi coordinates $X(s,t)$

Laplacian in Fermi coordinates

Given that $X(\mathcal{N}_\delta) \subset \mathbb{R}^2$ is a 2-dimensional manifold, we can employ a formula from Differential Geometry that allow us to compute the Euclidean Laplacian in terms of Fermi coordinates, for points $(\bar{x},\bar{y}) = X(\boldsymbol{s},\boldsymbol{t}) \in \mathcal{N}_\delta$ as follows

$$\Delta_X = \frac{1}{\sqrt{\det(g(\boldsymbol{s},\boldsymbol{t}))}}\,\partial_i\left(\sqrt{\det(g(\boldsymbol{s},\boldsymbol{t}))}\cdot g^{ij}(\boldsymbol{s},\boldsymbol{t})\;\partial_j\right), \quad i,j = \boldsymbol{s},\boldsymbol{t} \tag{A.2}$$

where $g_{ij}(\boldsymbol{s},\boldsymbol{t}) = <\partial_{\boldsymbol{i}} X(\boldsymbol{s},\boldsymbol{t}), \partial_{\boldsymbol{j}} X(\boldsymbol{s},\boldsymbol{t})>$ corresponds to the ijth entry of metric g of Γ, and we regard $g^{ij} = (g^{-1})_{i,j}$ as the respective entry for the inverse of the metric. Performing explicit calculations, and using the relations between the tangent and the normal to the curve Γ, it follows

$$\partial_{\boldsymbol{s}} X(\boldsymbol{s},\boldsymbol{t}) = \dot{\gamma}(\boldsymbol{s}) + \boldsymbol{t}\cdot\dot{\nu}(\boldsymbol{s}), \quad \partial_{\boldsymbol{t}} X(\boldsymbol{s},\boldsymbol{t}) = \nu(\boldsymbol{s}) \tag{A.3}$$

And so by (A.3), the metric g can be computed as

$$g_{ss}(\boldsymbol{s},\boldsymbol{t}) = |\dot{\gamma}(\boldsymbol{s})|^2 + 2\boldsymbol{t} <\dot{\gamma}(\boldsymbol{s}),\dot{\nu}(\boldsymbol{s})> + \boldsymbol{t}^2|\dot{\nu}(\boldsymbol{s})|^2 = (1-\boldsymbol{t}k(\boldsymbol{s}))^2 \tag{A.4}$$

$$g_{st}(\boldsymbol{s},\boldsymbol{t}) = g_{ts}(\boldsymbol{s},\boldsymbol{t}) = 0, \quad g_{tt}(\boldsymbol{s},\boldsymbol{t}) = 1 \tag{A.5}$$

Hence, the components of the g^{-1} are

$$g^{ss}(\boldsymbol{s},\boldsymbol{t}) = \frac{1}{(1-\boldsymbol{t}k(\boldsymbol{s}))^2}, \quad g^{st}(\boldsymbol{s},\boldsymbol{t}) = g^{ts}(\boldsymbol{s},\boldsymbol{t}) = 0, \quad g^{tt}(\boldsymbol{s},\boldsymbol{t}) = 1 \tag{A.6}$$

Replacing formula (A.2) and using values obtained in (A.6), we get

$$\Delta_x = \partial_{\boldsymbol{tt}} + g^{ss}\cdot\partial_{\boldsymbol{ss}} + \frac{1}{\sqrt{\det g}}\partial_{\boldsymbol{t}}(\sqrt{\det g})\cdot\partial_{\boldsymbol{t}} + \frac{1}{\sqrt{\det g}}\partial_{\boldsymbol{s}}(\sqrt{\det g}\cdot g^{ss})\cdot\partial_{\boldsymbol{s}} \tag{A.7}$$

where $\sqrt{\det g} = 1 - \boldsymbol{t}k(\boldsymbol{s})$, and with

$$\frac{1}{\sqrt{\det g}}\partial_{\boldsymbol{t}}(\sqrt{\det g}) = \frac{-k(\boldsymbol{s})}{1-\boldsymbol{t}k(\boldsymbol{s})}, \quad \frac{1}{\sqrt{\det g}}\partial_{\boldsymbol{s}}(\sqrt{\det g}\cdot g^{ss}) = \frac{\boldsymbol{t}\dot{k}(\boldsymbol{s})}{(1-\boldsymbol{t}k(\boldsymbol{s}))^3}$$

Further, we can make an approximation of this operator at main order, if k is bounded:

$$\frac{1}{(1-\boldsymbol{t}k)^2} = \underbrace{\left(\sum_{m=0}^{+\infty}(\boldsymbol{t}k)^m\right)^2}_{\boldsymbol{t}\text{ small}} = [1+\boldsymbol{t}k+\boldsymbol{t}^2O(k^2)]^2 = [1+2\boldsymbol{t}k+\boldsymbol{t}^2O(k^2)]$$

$$\frac{1}{(1-\boldsymbol{t}k)^3} = \underbrace{\left(\sum_{m=0}^{+\infty}(\boldsymbol{t}k)^m\right)^3}_{\boldsymbol{t}\text{ small}} = [1+\boldsymbol{t}k+\boldsymbol{t}^2O(k^2)]^3 = [1+3\boldsymbol{t}k+\boldsymbol{t}^2O(k^2)]$$

In this way, we deduce the expansion

$$g^{ss} = 1 + \boldsymbol{t}A_0(\boldsymbol{s},\boldsymbol{t}) \tag{A.8}$$

$$\frac{1}{\sqrt{\det g}}\partial_{\boldsymbol{t}}(\sqrt{\det g}) = -k(\boldsymbol{s}) - \boldsymbol{t}k^2(\boldsymbol{s}) + t^2C_0(\boldsymbol{s},\boldsymbol{t}) \tag{A.9}$$

$$\frac{1}{\sqrt{\det g}}\partial_{\boldsymbol{s}}(\sqrt{\det g}\cdot g^{ss}) = \boldsymbol{t}B_0(\boldsymbol{s},\boldsymbol{t}) \tag{A.10}$$

where

$$A_0(\boldsymbol{s},\boldsymbol{t}) = 2k(\boldsymbol{s}) + O(\boldsymbol{t}\ k^2(\boldsymbol{s})) \tag{A.11}$$

$$B_0(\boldsymbol{s},\boldsymbol{t}) = \dot{k}(\boldsymbol{s}) + O(\boldsymbol{t}\ \dot{k}(\boldsymbol{s})k(\boldsymbol{s})) \tag{A.12}$$

$$C_0(\boldsymbol{s},\boldsymbol{t}) = k^3(\boldsymbol{s}) + O(\boldsymbol{t}\ k^4(\boldsymbol{s})) \tag{A.13}$$

are smooth functions, and these relations can be derived.

So using relations (A.7) to (A.10), we get the Euclidean Laplacian in Fermi coordinates

$$\Delta_X = \partial_{\boldsymbol{tt}} + \partial_{\boldsymbol{ss}} - [k(\boldsymbol{s}) + \boldsymbol{t}k^2(\boldsymbol{s})]\cdot\partial_{\boldsymbol{t}} + \boldsymbol{t}\cdot A_0(\boldsymbol{s},\boldsymbol{t})\cdot\partial_{\boldsymbol{ss}} + \boldsymbol{t}\cdot B_0(\boldsymbol{s},\boldsymbol{t})\cdot\partial_{\boldsymbol{s}} + \boldsymbol{t}^2\cdot C_0(\boldsymbol{s},\boldsymbol{t})\cdot\partial_{\boldsymbol{t}} \tag{A.14}$$

For the remainder of this section, let us denote by $v : \mathbb{R}\times]-\delta,\delta[\to\mathbb{R}$ the function that represents u in Fermi coordinates

$$v(\boldsymbol{s},\boldsymbol{t}) := u\circ X(\boldsymbol{s},\boldsymbol{t})$$

Laplacian in dilated Fermi coordinates

We now write the Euclidean Laplacian $\Delta_{x,y}$ in terms of dilated Fermi variables (s,t), making use of the expression obtained for Δ_X in "shrink" Fermi coordinates $(\boldsymbol{s},\boldsymbol{t})$. This time we consider the dilated curve $\Gamma_\varepsilon := \varepsilon^{-1}\Gamma$ by $\gamma_\varepsilon : s \mapsto \varepsilon^{-1}\gamma(\varepsilon s)$, and we define associated local **dilated Fermi coordinates** in $\mathbb{R}^2$ by

$$X_\varepsilon : \mathbb{R} \times (-\delta/\varepsilon,\ \delta/\varepsilon) \to \mathcal{N}_\varepsilon\ , \quad X_\varepsilon(s,t) := \frac{1}{\varepsilon}X(\varepsilon s, \varepsilon t) = \frac{1}{\varepsilon}\gamma(\varepsilon s) + t \cdot \nu(\varepsilon s) \tag{A.15}$$

on a dilated tubular neighborhood $\varepsilon^{-1}\mathcal{N}_\delta$ of the curve Γ_ε

$$\mathcal{N}_\varepsilon = \left\{(x,y) = X_\varepsilon(s,t) \in \mathbb{R}^2 /\ s \in \mathbb{R},\ |t| < \frac{\delta}{\varepsilon} + 2c_0|s|)\right\} \tag{A.16}$$

where $c_0 > 0$ is a fixed number, and in such way that X_ε defines a local change of variables. The following picture depicts the geometrical setting previously described:

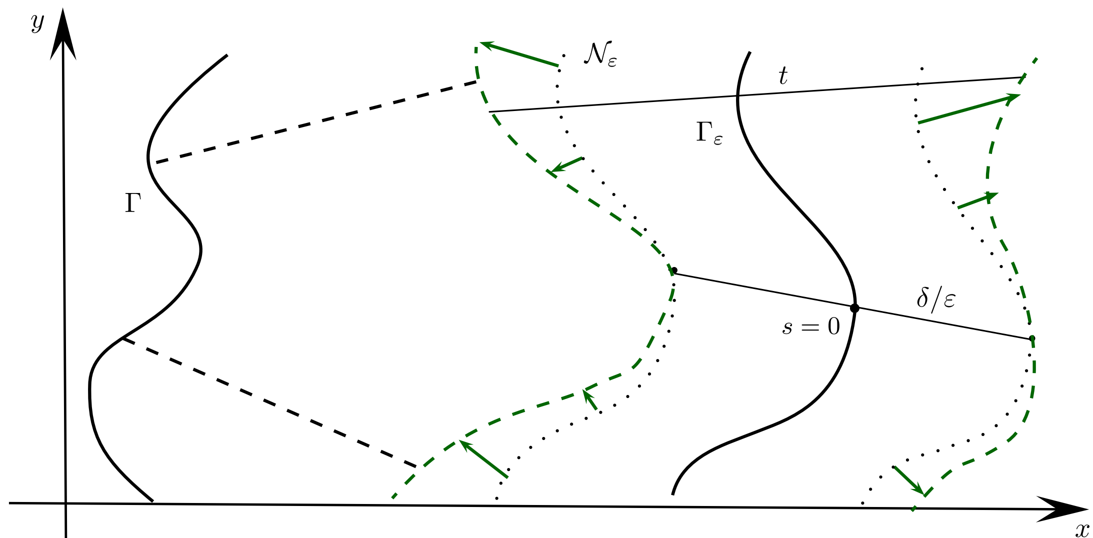

Figure 7: Neighborhood $\mathcal{N}_\varepsilon$ in Fermi coordinates $X_\varepsilon(s,t)$

Hereinafter, regard $\tilde{u} = \tilde{u}(x,y)$ as a smooth function representing $u = u(\bar{x},\bar{y})$ in dilated coordinates of the space, where the pair (x,y) represents the dilated size of Γ in $\mathcal{N}_\varepsilon$. So if $T_\varepsilon(x,y) = (\varepsilon x, \varepsilon y)$ is a shrinking map in $\mathbb{R}^2$ for small $\varepsilon > 0$, then $\tilde{u}$ complies

$$\tilde{u}(x,y) = u \circ T_\varepsilon(x,y) = u(\bar{x},\bar{y})$$

where we will adopting the following convention $(\bar{x},\bar{y}) := (\varepsilon x, \varepsilon y)$, $(\boldsymbol{s},\boldsymbol{t}) := (\varepsilon s, \varepsilon t)$.

Note that we can compute the Euclidean Laplacian of $\tilde{u}$ in terms of the same operator in shrink variables $(\bar{x},\bar{y})$, thus obtaining the first term of the Allen-Cahn equation (2.65). Indeed, a simple calculation shows that

$$\Delta_{x,y}\tilde{u}(x,y) = \Delta_{x,y}u(\varepsilon x, \varepsilon y) = \varepsilon^2\Delta_{\bar{x},\bar{y}}u(\varepsilon x, \varepsilon y) = \varepsilon^2\Delta_{\bar{x},\bar{y}}u(\bar{x},\bar{y}) \tag{A.17}$$

Just like the preceding part, let us define $\tilde{v} : \mathbb{R} \times (-\delta/\varepsilon,\ \delta/\varepsilon) \to \mathcal{N}$ as the function that represents $\tilde{u}$ in dilated Fermi coordinates, through

$$\tilde{v}(s,t) := \tilde{u} \circ X_\varepsilon(s,t)$$

It it easy to see that if $\boldsymbol{t} = \varepsilon t,\ \boldsymbol{s} = \varepsilon s$ are shrink Fermi variables, then it follows

$$\tilde{v}(s,t) = v(\varepsilon s, \varepsilon t) = u \circ X(\boldsymbol{s}, \boldsymbol{t}) \tag{A.18}$$

To check this, note that the following propositions are equivalents

$$(x,y) \in \mathcal{N}_\varepsilon \Leftrightarrow (\bar{x}, \bar{y}) = (\varepsilon x, \varepsilon y) \in \mathcal{N}_\delta$$

because

$$(x,y) = X_\varepsilon(s,t) \Leftrightarrow (x,y) = \frac{1}{\varepsilon}(\varepsilon x, \varepsilon y) = \frac{1}{\varepsilon}\gamma(\varepsilon s) + t\nu(\varepsilon s) \Leftrightarrow$$

$$(\bar{x}, \bar{y}) = (\varepsilon x, \varepsilon y) = \gamma(\varepsilon s) + \varepsilon t \nu(\varepsilon s) \Leftrightarrow (\bar{x}, \bar{y}) = X(\boldsymbol{s}, \boldsymbol{t}) = X(\varepsilon s, \varepsilon t)$$

and then it follows formula (A.18)

$$\tilde{v}(s,t) = \tilde{u}(X_\varepsilon(s,t)) = u(\varepsilon\ X_\varepsilon(s,t)) = u(X(\varepsilon s, \varepsilon t)) = v(\boldsymbol{s}, \boldsymbol{t})$$

Consequently, the latter fact together with formula (A.17) allow us to write the Euclidean Laplacian in shrink Fermi variables $(\boldsymbol{s}, \boldsymbol{t})$ using expression (A.11) for Δ_X

$$\Delta_{x,y} = \varepsilon^2 \left\{ \partial_{\boldsymbol{tt}} + \partial_{\boldsymbol{ss}} - [k(\boldsymbol{s}) + \boldsymbol{t}k^2(\boldsymbol{s})] \cdot \partial_{\boldsymbol{t}} + \boldsymbol{t}A_0(\boldsymbol{s}, \boldsymbol{t}) \cdot \partial_{\boldsymbol{ss}} + \boldsymbol{t}B_0(\boldsymbol{s}, \boldsymbol{t}) \cdot \partial_{\boldsymbol{s}} + \boldsymbol{t}^2 C_0(\boldsymbol{s}, \boldsymbol{t}) \cdot \partial_{\boldsymbol{t}} \right\} \tag{A.19}$$

On the other hand, from relation (A.18) we can compute derivatives of $\tilde{v}$ from those of v

$$\partial_s \tilde{v}(s,t) = \varepsilon \partial_{\boldsymbol{s}} v(\varepsilon s, \varepsilon t) = \varepsilon \partial_{\boldsymbol{s}} v(\boldsymbol{s}, \boldsymbol{t}) \tag{A.20}$$

$$\partial_t \tilde{v}(s,t) = \varepsilon \partial_{\boldsymbol{t}} v(\varepsilon s, \varepsilon t) = \varepsilon \partial_{\boldsymbol{t}} v(\boldsymbol{s}, \boldsymbol{t}) \tag{A.21}$$

$$\partial_{ss} \tilde{v}(s,t) = \varepsilon^2 \partial_{\boldsymbol{ss}} v(\varepsilon s, \varepsilon t) = \varepsilon^2 \partial_{\boldsymbol{ss}} v(\boldsymbol{s}, \boldsymbol{t}) \tag{A.22}$$

$$\partial_{tt} \tilde{v}(s,t) = \varepsilon^2 \partial_{\boldsymbol{tt}} v(\varepsilon s, \varepsilon t) = \varepsilon^2 \partial_{\boldsymbol{tt}} v(\boldsymbol{s}, \boldsymbol{t}) \tag{A.23}$$

Finally using relations (A.18), and (A.20) to (A.23), we get the expression for the Euclidean Laplacian in dilated Fermi coordinates (s,t)

$$\begin{aligned} \Delta_{X_\varepsilon} := \partial_{tt} + \partial_{ss} - \varepsilon[k(\varepsilon s) + \varepsilon t k^2(\varepsilon s)] \cdot\ \partial_t +\ \varepsilon t A_0(\varepsilon s, \varepsilon t) \cdot \partial_{ss} \\ +\ \varepsilon^2 t B_0(\varepsilon s, \varepsilon t) \cdot \partial_s +\ \varepsilon^3 t^2 C_0(\varepsilon s, \varepsilon t) \cdot \partial_t \end{aligned} \tag{A.24}$$

where

$$A_0(\varepsilon s, \varepsilon t) = 2k(\varepsilon s) + \varepsilon O(|t\ k^2(\varepsilon s)|) \tag{A.25}$$

$$B_0(\varepsilon s, \varepsilon t) = \dot{k}(\varepsilon s) + \varepsilon O(|t\ \dot{k}(\varepsilon s) \cdot k^2(\varepsilon s)|) \tag{A.26}$$

$$C_0(\varepsilon s, \varepsilon t) = k^3(\varepsilon s) + \varepsilon O(|t\ k^4(\varepsilon s)|) \tag{A.27}$$

are smooth function, and these relations can be derived.

Therefore for points $(x,y) = X_\varepsilon(s,t) \in \mathcal{N}_\varepsilon$, formula (A.24) allow us to compute the operator $\Delta_{x,y}\tilde{u}(x,y)$ in terms of derivatives in variables (s,t) of the function $\tilde{v} = \tilde{v}(s,t)$, where we adopt the convention that $(x,y) \in \mathbb{R}^2$ are Euclidean dilated coordinated and $(s,t) \in \mathbb{R} \times [-\delta/\varepsilon,\ \delta/\varepsilon]$ are dilated Fermi coordinates.

Laplacian in dilated and translated Fermi coordinates

Like before, define $v^* : \mathbb{R}^2 \to \mathbb{R}$ as a function representing $\tilde{u} = \tilde{u}(x,y)$ in these new Fermi coordinates $v^*(s,t) := \tilde{u} \circ X_{\varepsilon,h}(s,t)$. Let us regard $\tilde{v}(s,z) := \tilde{u} \circ X_\varepsilon(s,z)$, hence relation (2.66) implies that

$$v^*(s,t) = \tilde{v}(s, t + h(\varepsilon s)), \quad \tilde{v}(s,z) = v^*(s, z - h(\varepsilon s)) = \tilde{u} \circ X_\varepsilon(s,z) \tag{A.28}$$

Recall that the Euclidean Laplacian can be computed in dilated Fermi coordinates (s,z) using (A.24)

$$\begin{aligned}\Delta_{X_\varepsilon} \tilde{u}(x,y) &= \partial_{zz}\tilde{v}(s,z) + \partial_{ss}\tilde{v}(s,z) - \varepsilon[k(\varepsilon s) + \varepsilon z k^2(\varepsilon s)]\partial_z \tilde{v}(s,z) \\ &+ \varepsilon z A_0(\varepsilon s, \varepsilon z)\partial_{ss}\tilde{v}(s,z) + \varepsilon^2 z B_0(\varepsilon s, \varepsilon z)\partial_s \tilde{v}(s,z) + \varepsilon^3 z^2 C_0(\varepsilon s, \varepsilon z)\partial_z \tilde{v}(s,z)\end{aligned} \tag{A.29}$$

However, equation (A.28) allow us to compute derivatives of v^* in term of those of $\tilde{v}$ as

$$\partial_s \tilde{v}(s,z) = \partial_s v^*(s, z-h) - \partial_t v^*(s, z-h) \cdot (\varepsilon h'(\varepsilon s)) \tag{A.30}$$

$$\begin{aligned}\partial_{ss}\tilde{v}(s,z) = \partial_{ss}v^*(s,z-h) &- 2\varepsilon h'(\varepsilon s)\partial_{st}v^*(s,z-h) \\ &+ \varepsilon^2 (h'(\varepsilon s))^2 \partial_{tt}v^*(s,z-h) - \varepsilon^2 h''(\varepsilon s)\partial_t v^*(s,z-h)\end{aligned} \tag{A.31}$$

$$\partial_z \tilde{v}(s,z) = \partial_t v^*(s,z-h), \quad \partial_{zz}\tilde{v}(s,t) = \partial_{tt}v^*(s,z-h) \tag{A.32}$$

So replacing in equation (A.29) the expressions (A.30) to (A.32), and making use of relation $\tilde{v}(s,z) = v^*(s,t)$ with $z = t - h(\varepsilon s)$, it is possible to compute $\Delta_{x,y}$ in terms of dilated and translated Fermi coordinates (s,t), as

$$\begin{aligned}\Delta_{X_{\varepsilon,h}}\tilde{u}(x,y) &= \partial_{tt}v^*(s,t) + \partial_{ss}v^*(s,t) - 2\varepsilon h'(\varepsilon s)\partial_{st}v^*(s,t) - \varepsilon^2 h''(\varepsilon s)\partial_t v^*(s,t) \\ &+ \varepsilon^2 |h'(\varepsilon s)|^2 \partial_{tt}v^* - \varepsilon[k(\varepsilon s) + \varepsilon(t + h(\varepsilon s))k^2(\varepsilon s)] \cdot \partial_t v^* + D_{\varepsilon,h}(s,t)\end{aligned} \tag{A.33}$$

where $D_{\varepsilon,h}$ is a small operator with ε, of the form

$$\begin{aligned}D_{\varepsilon,h}(s,t) := \;&\varepsilon(t + h(\varepsilon s))A_0(\varepsilon s, \varepsilon(t+h))[\partial_{ss}v^* - 2\varepsilon h'(\varepsilon s)\partial_{ts}v^* - \varepsilon^2 h''(\varepsilon s)\partial_t v^* + \varepsilon^2 |h'(\varepsilon s)|^2 \partial_{tt}v^*] \\ &+ \varepsilon^2 (t + h(\varepsilon s))B_0(\varepsilon s, \varepsilon(t+h))[\partial_s v^*(s,t) - \varepsilon h'(\varepsilon s)\partial_t v^*(s,t)] \\ &+ \varepsilon^3 (t + h(\varepsilon s))^2 C_0(\varepsilon s, \varepsilon(t+h))\partial_t v^*(s,t)\end{aligned} \tag{A.34}$$

such that

$$A_0(\varepsilon s, \varepsilon[t + h(\varepsilon s)]) = 2k(\varepsilon s) + \varepsilon O(|[t + h(\varepsilon s)]k^2(\varepsilon s)|) \tag{A.35}$$

$$B_0(\varepsilon s, \varepsilon[t + h(\varepsilon s)]) = \dot{k}(\varepsilon s) + \varepsilon O(|(t + h(\varepsilon s))\dot{k}(\varepsilon s) \cdot k^2(\varepsilon s)|) \tag{A.36}$$

$$C_0(\varepsilon s, \varepsilon[t + h(\varepsilon s)]) = k^3(\varepsilon s) + \varepsilon O(|(t + h(\varepsilon s))k^4(\varepsilon s)|) \tag{A.37}$$

are smooth functions and these relations can be derived.
In this way, for points $(x,y) = X_{\varepsilon,h}(s,t) \in \mathcal{N}_{\varepsilon,h}$ formula (A.33) allow us to compute the operator $\Delta_{x,y}\tilde{u}(x,y)$ in terms of derivatives in variables (s,t) of the function $v^* = v^*(s,t)$, thus finishing the proof of Lemma 3. □

A.2. Calculation of Gradients: Proof of Lemma 4

Analogously to what performed in the last section, our main goal is finding a characterization for the product $\varepsilon\nabla_{\bar{x}}a/a \cdot \nabla_x u$ in Fermi coordinates. To achieve this we will follow a scheme in 3 steps, for which the analysis is simplified.

Hereinafter, once again we adopt the convention $a = a(\bar{x}, \bar{y})$ and $u = u(\bar{x}, \bar{y})$, where $(\bar{x}, \bar{y})$ denotes the non-dilated Euclidean coordinates of the space.

Gradients in Fermi coordinates

Recall that $X(\boldsymbol{s}, \boldsymbol{t}) = \gamma(\boldsymbol{s}) + \boldsymbol{t}\nu(\boldsymbol{s})$ provides a local change of variables, so that $X^{-1} : \mathcal{N} \to \mathbb{R} \times (-\delta, \delta)$, $X^{-1}(\bar{x}, \bar{y}) = (\boldsymbol{s}, \boldsymbol{t})$ is well defined in some neighborhood of the curve Γ. Thus the following relation holds

$$X(X^{-1}(\bar{x}, \bar{y})) = I_d(\bar{x}, \bar{y}), \quad \forall (\bar{x}, \bar{y}) \in \mathcal{N} \tag{A.38}$$

Deriving (A.38) in variables $(\bar{x}, \bar{y})$, we get

$$D_{\boldsymbol{s},\boldsymbol{t}}X(X^{-1}(\bar{x}, \bar{y})) \cdot D_{\bar{x},\bar{y}}(X^{-1})(\bar{x}, \bar{y}) = I_d(\bar{x}, \bar{y}) \tag{A.39}$$

so in particular

$$(D_{\boldsymbol{s},\boldsymbol{t}}X(\boldsymbol{s}, \boldsymbol{t}))^{-1} = D_{\bar{x},\bar{y}}(X^{-1})(\bar{x}, \bar{y}) \quad \text{where} \;\; (\boldsymbol{s}, \boldsymbol{t}) = X^{-1}(\bar{x}, \bar{y}) \tag{A.40}$$

Let us set functions $a : \mathbb{R} \times (-\delta, \delta) \to \mathbb{R}$, $v_u : \mathbb{R} \times (-\delta, \delta) \to \mathbb{R}$ as representations of a, u in Fermi coordinates, through

$$a(\boldsymbol{s}, \boldsymbol{t}) := a \circ X(\boldsymbol{s}, \boldsymbol{t}) \;, \quad v_u(\boldsymbol{s}, \boldsymbol{t}) := u \circ X(\boldsymbol{s}, \boldsymbol{t})$$

Given that $(\boldsymbol{s}, \boldsymbol{t}) = X^{-1}(\bar{x}, \bar{y})$, we can set both a, u as implicit functions of variables $(\boldsymbol{s}, \boldsymbol{t})$ by

$$a(\bar{x}, \bar{y}) = a(\boldsymbol{s}(\bar{x}, \bar{y}), \boldsymbol{t}(\bar{x}, \bar{y})) \;, \quad u(\bar{x}, \bar{y}) = v_u(\boldsymbol{s}(\bar{x}, \bar{y}), \boldsymbol{t}(\bar{x}, \bar{y})) \tag{A.41}$$

Deriving function a with respect to $(\bar{x}, \bar{y})$ in (A.41), and making use of the chain rule, we get

$$\begin{aligned} \nabla_{\bar{x},\bar{y}}a(\bar{x}, \bar{y}) &= \nabla_{\boldsymbol{s},\boldsymbol{t}}a(s(\bar{x}, \bar{y}), t(\bar{x}, \bar{y})) \cdot D_{\bar{x},\bar{y}}(\boldsymbol{s}, \boldsymbol{t})(\bar{x}, \bar{y}) \\ &= \nabla_{\boldsymbol{s},\boldsymbol{t}}a(X^{-1}(\bar{x}, \bar{y})) \cdot D_{\bar{x},\bar{y}}(X^{-1})(\bar{x}, \bar{y}) \end{aligned} \tag{A.42}$$

Now from (A.40) it follows that

$$\nabla_{\bar{x},\bar{y}}a(\bar{x}, \bar{y}) = \nabla_{\boldsymbol{s},\boldsymbol{t}}a(X^{-1}(\bar{x}, \bar{y})) \cdot \left[D_{\boldsymbol{s},\boldsymbol{t}}X(X^{-1}(\bar{x}, \bar{y}))\right]^{-1} \tag{A.43}$$

Using the same argument, we readily see that u satisfies

$$\nabla_{\bar{x},\bar{y}}u(\bar{x}, \bar{y}) = \nabla_{\boldsymbol{s},\boldsymbol{t}}v_u(X^{-1}(\bar{x}, \bar{y})) \cdot \left[D_{\boldsymbol{s},\boldsymbol{t}}X(X^{-1}(\bar{x}, \bar{y}))\right]^{-1} \tag{A.44}$$

By these means, expressions (A.43)-(A.44) allow us to write the product of Euclidean gradients in terms of the Fermi coordinates, as follows

$$\begin{aligned}\nabla_{\bar{x},\bar{y}}a\nabla_{\bar{x},\bar{y}}u(\bar{x},\bar{y}) &= \left(\nabla_{\boldsymbol{s},\boldsymbol{t}}a\,[D_{\boldsymbol{s},\boldsymbol{t}}X]^{-1}\right)\cdot\left(\nabla_{\boldsymbol{s},\boldsymbol{t}}v_u\,[D_{\boldsymbol{s},\boldsymbol{t}}X]^{-1}\right)^T\\ &= \left(\nabla_{\boldsymbol{s},\boldsymbol{t}}a\,[D_{\boldsymbol{s},\boldsymbol{t}}X]^{-1}\right)\cdot\left([D_{\boldsymbol{s},\boldsymbol{t}}X]^{-1}\right)^T(\nabla_{\boldsymbol{s},\boldsymbol{t}}v_u)^T\\ &= \nabla_{\boldsymbol{s},\boldsymbol{t}}a\left([D_{\boldsymbol{s},\boldsymbol{t}}X]^{-1}\cdot\left([D_{\boldsymbol{s},\boldsymbol{t}}X]^T\right)^{-1}\right)(\nabla_{\boldsymbol{s},\boldsymbol{t}}v_u)^T\\ &= \nabla_{\boldsymbol{s},\boldsymbol{t}}a\left([D_{\boldsymbol{s},\boldsymbol{t}}X]^T\cdot D_{\boldsymbol{s},\boldsymbol{t}}X\right)^{-1}(\nabla_{\boldsymbol{s},\boldsymbol{t}}v_u)^T \qquad \text{(A.45)}\end{aligned}$$

all the above evaluated in $(\boldsymbol{s},\boldsymbol{t}) = X^{-1}(\bar{x},\bar{y})$. Nonetheless, note that the product term in between corresponds to the inverse g^{-1} of the metric of Γ, since

$$D_{\boldsymbol{s},\boldsymbol{t}}X(\boldsymbol{s},\boldsymbol{t}) = \left[\ \partial_s X(s,t)\mid\partial_t X(s,t)\ \right]_{2\times 2}\ ,\quad [D_{\boldsymbol{s},\boldsymbol{t}}X(\boldsymbol{s},\boldsymbol{t})]^T = \begin{bmatrix}(\partial_s X(s,t))^T\\(\partial_t X(s,t))^T\end{bmatrix}_{2\times 2}$$

$$\Rightarrow\quad [D_{\boldsymbol{s},\boldsymbol{t}}X]^T\cdot D_{\boldsymbol{s},\boldsymbol{t}}X(\boldsymbol{s},\boldsymbol{t}) = \begin{bmatrix}<\partial_s X,\partial_s X> & <\partial_s X,\partial_t X>\\ <\partial_t X,\partial_s X> & <\partial_t X,\partial_t X>\end{bmatrix}(\boldsymbol{s},\boldsymbol{t}) \equiv g(\boldsymbol{s},\boldsymbol{t})$$

Then using (A.6) we have

$$\left([D_{\boldsymbol{s},\boldsymbol{t}}X(\boldsymbol{s},\boldsymbol{t})]^T\cdot D_{\boldsymbol{s},\boldsymbol{t}}X(\boldsymbol{s},\boldsymbol{t})\right)^{-1} = \begin{bmatrix}g^{ss}(\boldsymbol{s},\boldsymbol{t}) & 0\\ 0 & 1\end{bmatrix} \qquad \text{(A.46)}$$

Therefore, using (A.46) we obtain that product (A.45) can be computed as

$$\begin{aligned}\frac{\nabla_{\bar{x},\bar{y}}a}{a}\nabla_{\bar{x},\bar{y}}u(\bar{x},\bar{y}) &= \frac{1}{(1-\boldsymbol{t}k(\boldsymbol{s}))^2}\left(\frac{\partial_{\boldsymbol{s}}a}{a}\cdot\partial_{\boldsymbol{s}}v_u\right)+\frac{\partial_{\boldsymbol{t}}a}{a}\cdot\partial_{\boldsymbol{t}}v_u\\ &= [1+\boldsymbol{t}A_0(\boldsymbol{s},\boldsymbol{t})]\left(\frac{\partial_{\boldsymbol{s}}a}{a}\cdot\partial_{\boldsymbol{s}}v_u\right)+\frac{\partial_{\boldsymbol{t}}a}{a}\cdot\partial_{\boldsymbol{t}}v_u\\ &= \frac{\partial_{\boldsymbol{s}}a}{a}\cdot\partial_{\boldsymbol{s}}v_u+\frac{\partial_{\boldsymbol{t}}a}{a}\cdot\partial_{\boldsymbol{t}}v_u+\boldsymbol{t}A_0(\boldsymbol{s},\boldsymbol{t})\frac{\partial_{\boldsymbol{s}}a}{a}\cdot\partial_{\boldsymbol{s}}v_u\end{aligned}$$

where we have made use of the expansion (A.8) of entry $g^{\boldsymbol{ss}}$ of the metric.
Further, on the above expression we can make a Taylor expansion of the term $\nabla a/a$ in variable t around 0, that is, around the curve Γ:

$$\frac{\partial_{\boldsymbol{s}}a}{a}(\boldsymbol{s},\boldsymbol{t}) = \frac{\partial_{\boldsymbol{s}}a}{a}(\boldsymbol{s},0)+\boldsymbol{t}\ \partial_{\boldsymbol{t}}\left(\frac{\partial_{\boldsymbol{s}}a}{a}\right)(\boldsymbol{s},0)+O\left(\boldsymbol{t}^2\partial_{\boldsymbol{tt}}\left(\frac{\partial_t a}{a}\right)\right)$$

$$\frac{\partial_{\boldsymbol{t}}a}{a}(\boldsymbol{s},\boldsymbol{t}) = \frac{\partial_{\boldsymbol{t}}a}{a}(\boldsymbol{s},0)+\boldsymbol{t}\left(\frac{\partial_{\boldsymbol{tt}}a}{a}(\boldsymbol{s},0)-\left|\frac{\partial_{\boldsymbol{t}}a}{a}(\boldsymbol{s},0)\right|^2\right)+\frac{\boldsymbol{t}^2}{2}\partial_{\boldsymbol{tt}}\left(\frac{\partial_{\boldsymbol{t}}a}{a}(\boldsymbol{s},0)\right)+O\left(\boldsymbol{t}^3\partial_{\boldsymbol{ttt}}\left(\frac{\partial_t a}{a}\right)\right)$$

Hence, if we replace these expansions in the above equation, then we obtain the product of Euclidean gradients in terms of Fermi coordinates:

$$\frac{\nabla_X a}{a}\nabla_X u(\bar{x},\bar{y}) = \frac{\partial_{\boldsymbol{s}} a}{a}(\boldsymbol{s},0)\cdot\partial_{\boldsymbol{s}} v_u + \left[\frac{\partial_{\boldsymbol{t}} a}{a}(\boldsymbol{s},0) + \boldsymbol{t}\left(\frac{\partial_{\boldsymbol{tt}} a}{a}(\boldsymbol{s},0) - \left|\frac{\partial_{\boldsymbol{t}} a}{a}(\boldsymbol{s},0)\right|^2\right)\right]\cdot\partial_{\boldsymbol{t}} v_u$$
$$+\boldsymbol{t}D_0(\boldsymbol{s},\boldsymbol{t})\cdot\partial_{\boldsymbol{s}} v_u + \boldsymbol{t}^2 F_0(\boldsymbol{s},\boldsymbol{t})\cdot\partial_{\boldsymbol{t}} v_u \tag{A.47}$$

for which

$$D_0(\boldsymbol{s},\boldsymbol{t}) = \partial_{\boldsymbol{t}}\left(\frac{\partial_{\boldsymbol{s}} a}{a}\right)(\boldsymbol{s},0) + O\left(\boldsymbol{t}\partial_{\boldsymbol{tt}}\left(\frac{\partial_t a}{a}\right)\right) + A_0(\boldsymbol{s},\boldsymbol{t})\cdot\frac{\partial_{\boldsymbol{s}} a}{a}(\boldsymbol{s},\boldsymbol{t})$$
$$F_0(\boldsymbol{s},\boldsymbol{t}) = \frac{1}{2}\partial_{\boldsymbol{tt}}\left(\frac{\partial_t a}{a}(\boldsymbol{s},0)\right) + O\left(\boldsymbol{t}\partial_{\boldsymbol{ttt}}\left(\frac{\partial_t a}{a}\right)\right)$$

and $A_0(\boldsymbol{s},\boldsymbol{t})$ given by (A.11).

Gradients in dilated Fermi coordinates

As before, let us denote by $\tilde{a}:\mathbb{R}^2\to\mathbb{R},\ \tilde{u}:\mathbb{R}^2\to\mathbb{R}$ the smooth functions representing $a=a(\bar{x},\bar{y})$ and $u=u(\bar{x},\bar{y})$ in dilated Euclidean coordinates of space. For $\tilde{u}=\tilde{u}(x,y)$ we suppose that (x,y) represents dilated size of Γ in $\mathcal{N}_\varepsilon$. This means that $\tilde{u}$ complies

$$\tilde{u}(x,y) = u\circ T_\varepsilon(x,y) = u(\bar{x},\bar{y})$$

where we recall the convention $(\bar{x},\bar{y}) := (\varepsilon x,\varepsilon y),\ (\boldsymbol{s},\boldsymbol{t}) := (\varepsilon s,\varepsilon t)$.

Gradient of $\tilde{u}$ in dilated variables (x,y) can be computed from those of u in shrink variables $(\bar{x},\bar{y})$, trough the scaling

$$\nabla_{x,y}\tilde{u}(x,y) = \nabla_{x,y}u(\varepsilon x,\varepsilon y) = \varepsilon\nabla_{\bar{x},\bar{y}}u(\bar{x},\bar{y}) \tag{A.48}$$

We want to write the product of Euclidean gradients, using the expression written in shrink variables $(\boldsymbol{s},\boldsymbol{t})$ obtained in (A.47). Let us define the function $\tilde{v}_u:\mathbb{R}\times(-\delta/\varepsilon,\ \delta/\varepsilon)\to\mathcal{N}$ that represents the $\tilde{u}$ in dilated Fermi coordinates, by

$$\tilde{v}_u(s,t) := \tilde{u}\circ X_\varepsilon(s,t)$$

So that, if $\boldsymbol{t}=\varepsilon t,\ \boldsymbol{s}=\varepsilon s$ are shrink Fermi variables, it holds

$$\tilde{v}_u(s,t) = v_u(\varepsilon s,\varepsilon t) = u\circ X(\boldsymbol{s},\boldsymbol{t}) \tag{A.49}$$

Remark 8. *Just for convenience, it is not necessary to define the corresponding function $\tilde{a}$ that represents a in dilated Fermi coordinates $X_\varepsilon(s,t)$. It suffices to consider just a in shrink Fermi coordinates $X(\boldsymbol{s},\boldsymbol{t})$, and the result writing it in terms of contractions in ε of dilated variables (s,t).*

Note that (A.49) provides a relation between derivatives of $\tilde{v}_u$ and those of v_u, as follows

$$\partial_s \tilde{v}_u(s,t) = \varepsilon \partial_{\boldsymbol{s}} v_u(\varepsilon s, \varepsilon t) = \varepsilon \partial_{\boldsymbol{s}} v_u(\boldsymbol{s}, \boldsymbol{t}), \quad \partial_t \tilde{v}_u(s,t) = \varepsilon \partial_{\boldsymbol{t}} v_u(\varepsilon s, \varepsilon t) = \varepsilon \partial_{\boldsymbol{t}} v_u(\boldsymbol{s}, \boldsymbol{t}) \tag{A.50}$$

Accordingly, to obtain the second term of the Allen-Cahn equation (2.65), we just need to compute the product of Euclidean gradients of $\tilde{a}, \tilde{u}$ in dilated variables (x,y), and then relate them to those of a, u in shrink variables $(\bar{x}, \bar{y})$. In fact, from (A.48) we have

$$\varepsilon \frac{\nabla_{\bar{x},\bar{y}} \tilde{a}(\bar{x},\bar{y})}{\tilde{a}(\bar{x},\bar{y})} \cdot \nabla_{x,y} \tilde{u}(x,y) = \varepsilon^2 \, \frac{\nabla_{\bar{x},\bar{y}} a(\bar{x},\bar{y})}{a(\bar{x},\bar{y})} \cdot \nabla_{\bar{x},\bar{y}} u(\bar{x},\bar{y}) \tag{A.51}$$

The equation (A.47) provides a way to write the product of gradients in shrink Euclidean coordinates $(\bar{x}, \bar{y})$ of the above expression (A.51), in terms of shrink Fermi variables $(\boldsymbol{s}, \boldsymbol{t})$:

$$\varepsilon \frac{\nabla_X a}{a}(\bar{x},\bar{y}) \nabla_{X_\varepsilon} u(x,y) = \varepsilon^2 \Bigg\{ \frac{\partial_{\boldsymbol{s}} a}{a}(\boldsymbol{s},0) \partial_{\boldsymbol{s}} v_u + \left[\frac{\partial_{\boldsymbol{t}} a}{a}(\boldsymbol{s},0) + \boldsymbol{t} \left(\frac{\partial_{\boldsymbol{tt}} a}{a}(\boldsymbol{s},0) - \left| \frac{\partial_{\boldsymbol{t}} a}{a}(\boldsymbol{s},0) \right|^2 \right) \right] \partial_{\boldsymbol{t}} v_u$$
$$+ \boldsymbol{t} D_0(\boldsymbol{s},\boldsymbol{t}) \cdot \partial_{\boldsymbol{s}} v_u + \boldsymbol{t}^2 F_0(\boldsymbol{s},\boldsymbol{t}) \cdot \partial_{\boldsymbol{t}} \tilde{v}_u \Bigg\}$$

Finally, thanks to (A.50) is possible to find the product of the Euclidean gradients in terms of dilated Fermi coordinates (s,t)

$$\begin{aligned} \varepsilon \frac{\nabla_X a}{a}(\bar{x},\bar{y}) \nabla_{X_\varepsilon} u(x,y) &= \varepsilon \frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s, 0) \cdot \partial_s \tilde{v}_u \\ &+ \varepsilon \left[\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s,0) + \varepsilon t \left(\frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s,0) - \left| \frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s,0) \right|^2 \right) \right] \partial_t \tilde{v}_u + E_\varepsilon(s,t) \end{aligned} \tag{A.52}$$

where the derivatives of function a are with respect to shrink Fermi variables $(\boldsymbol{s}, \boldsymbol{t})$, and with

$$E_\varepsilon(s,t) := \varepsilon^2 t D_0(\varepsilon s, \varepsilon t) \cdot \partial_s \tilde{v}_u + \varepsilon^3 t^2 F_0(\varepsilon s, \varepsilon t) \cdot \partial_t \tilde{v}_u$$

for which

$$\begin{aligned} D_0(\varepsilon s, \varepsilon t) &= \partial_{\boldsymbol{t}} \left(\frac{\partial_{\boldsymbol{s}} a}{a} \right)(\varepsilon s, 0) + \varepsilon O \left(t \partial_{\boldsymbol{tt}} \left[\frac{\partial_t a}{a} \right] \right) + A_0(\varepsilon s, \varepsilon t) \cdot \frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s, \varepsilon t) \\ F_0(\varepsilon s, \varepsilon t) &= \frac{1}{2} \partial_{\boldsymbol{tt}} \left[\frac{\partial_t a}{a}(\varepsilon s, 0) \right] + \varepsilon O \left(t \partial_{\boldsymbol{ttt}} \left(\frac{\partial_t a}{a} \right) \right) \end{aligned}$$

and the function $A_0(\varepsilon s, \varepsilon t)$ given by (A.25).

Gradients in dilated and translated Fermi coordinates

Given any bounded and smooth function $h : \mathbb{R} \to \mathbb{R}$, let us denote by $v_u^* : \mathbb{R}^2 \to \mathbb{R}$ the function that represents $\tilde{u} = \tilde{u}(x, y)$ in the coordinates given in (2.66), through

$$v_u^*(s,t) := \tilde{u} \circ X_{\varepsilon,h}(s,t)$$

By renaming dilated Fermi variables as $\tilde{v}_u(s,z) := \tilde{u} \circ X_\varepsilon(s,z)$, we deduce that

$$v_u^*(s,t) = \tilde{v}_u(s, t + h(\varepsilon s)) \ , \quad \tilde{v}_u(s,z) = v_u^*(s, z - h(\varepsilon s)) = \tilde{u} \circ X_\varepsilon(s,z) \tag{A.53}$$

Recall, using (A.52), that the product of Euclidean gradients in dilated Fermi coordinates is

$$\begin{aligned} \varepsilon \frac{\nabla_{\bar{x},\bar{y}} a}{a}(\bar{x},\bar{y}) \nabla_{x,y} \tilde{u}(x,y) &= \varepsilon \frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s, 0) \cdot \partial_s \tilde{v}_u(s,z) \\ &+ \varepsilon \left[\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s, 0) + \varepsilon z \left(\frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s, 0) - \left| \frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s, 0) \right|^2 \right) \right] \partial_z \tilde{v}_u(s,z) + E_\varepsilon(s,z) \end{aligned} \tag{A.54}$$

Nonetheless, (A.53) allow us to compute derivatives of $\tilde{v}_u$ from those of v_u^*, as

$$\partial_s \tilde{v}_u(s,z) = \partial_s v_u^*(s, z-h) - \partial_t v_u^*(s, z-h) \cdot (\varepsilon h'(\varepsilon s)) \ , \quad \partial_z \tilde{v}_u(s,z) = \partial_t v_u^*(s, z-h) \tag{A.55}$$

Replacing the expressions (A.55) in the equation (A.54), and making use of relation $\tilde{v}_u(s,z) = v_u^*(s,t)$ with $z = t - h(\varepsilon s)$, follows that this product in dilated and translated Fermi coordinates amounts to

$$\begin{aligned} \varepsilon \frac{\nabla_X a}{a}(\varepsilon x, \varepsilon y) \nabla_{X_{\varepsilon,h}} u(x,y) &= \varepsilon \frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s, 0)[\partial_s v_u^*(s,t) - \varepsilon h'(\varepsilon s) \cdot \partial_t v_u^*(s,t)] \\ &+ \varepsilon \left[\frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s, 0) + \varepsilon(t + h(\varepsilon s)) \left(\frac{\partial_{\boldsymbol{tt}} a}{a}(\varepsilon s, 0) - \left| \frac{\partial_{\boldsymbol{t}} a}{a}(\varepsilon s, 0) \right|^2 \right) \right] \partial_t v_u^*(s,t) \\ &+ E_{\varepsilon,h}(\varepsilon s, t) \end{aligned} \tag{A.56}$$

where $E_{\varepsilon,h}$ is a small operator for $\varepsilon > 0$ small enough, and of the form

$$\begin{aligned} E_{\varepsilon,h}(\varepsilon s, t) := \ & \varepsilon^2 (t + h(\varepsilon s)) D_0(\varepsilon s, \varepsilon(t + h(\varepsilon s)))[\partial_s v_u^*(s,t) - \varepsilon h'(\varepsilon s) \cdot \partial_t v_u^*(s,t)] \\ &+ \varepsilon^3 (t + h(\varepsilon s))^2 F_0(\varepsilon s, \varepsilon(t + h(\varepsilon s))) \partial_t v_u^*(s,t) \end{aligned} \tag{A.57}$$

such that the following functions are smooth, and these relations can be derived.

$$\begin{aligned} D_0(\varepsilon s, \varepsilon(t+h)) &= \partial_{\boldsymbol{t}} \left[\frac{\partial_{\boldsymbol{s}} a}{a} \right] (\varepsilon s, 0) + \varepsilon O \left((t + h(\varepsilon s)) \partial_{\boldsymbol{tt}} \left[\frac{\partial_t a}{a} \right] \right) \\ &+ A_0(\varepsilon s, \varepsilon(t+h)) \frac{\partial_{\boldsymbol{s}} a}{a}(\varepsilon s, \varepsilon(t+h)) \end{aligned} \tag{A.58}$$

$$F_0(\varepsilon s, \varepsilon(t+h)) = \frac{1}{2} \partial_{\boldsymbol{tt}} \left[\frac{\partial_t a}{a} \right] (\varepsilon s, 0) + \varepsilon O \left((t + h(\varepsilon s)) \partial_{\boldsymbol{ttt}} \left[\frac{\partial_t a}{a} \right] \right) \tag{A.59}$$

and $A_0(\varepsilon s, \varepsilon(t+h))$ given by (A.35), thus concluding the proof of Lemma 4. □

A.3. Coercivity Property: Proof of Lemma 7

Assume first that $\psi(t) \in C_c^\infty(\mathbb{R})$, so this function and its derivatives decay sufficiently fast as $|t| \to +\infty$, and then we can write it as $\psi(t) = \rho(t) \cdot w'(t)$. Computing ψ into the quadratic form, and by integrating by parts, we obtain

$$\begin{aligned} B(\psi) &= -\int_{\mathbb{R}} [|\psi''(t) + f'(w)\psi(t)]\psi(t)dt \\ &= -\int_{\mathbb{R}} [\rho''(t)w' + 2\rho'(t)w''(t) + \rho(t)w''' + f'(w)w'\rho(t)]\rho(t)w'(t)dt \\ &= -\int_{\mathbb{R}} [\rho''(t)|w'|^2 + 2\rho'(t)w''(t)w'(t) + \rho(t)\underbrace{(w''' + f'(w)w')}_{\equiv 0}]\rho(t)dt \\ &= -\int_{\mathbb{R}} \left(|w'(t)|^2\rho'(t)\right)' \rho(t)dt = \int_{\mathbb{R}} |w'(t)|^2|\rho'(t)|^2 dt \geq 0 \end{aligned} \tag{A.60}$$

This implies that B is positive unless ψ is constant multiple of w'. Besides, using density arguments, we deduce that $B(\psi) \geq 0$, $\forall \psi \in H^1(\mathbb{R})$.

In order to prove Lemma 7, we need first to find a weaker coercivity inequality. We claim that there exists $\vartheta > 0$ such that for any $\psi \in H^1(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \psi(s,\zeta)w'(\zeta)d\zeta = 0$ it holds

$$B(\psi) \geq \vartheta \int_{\mathbb{R}} \psi^2(t)dt \tag{A.61}$$

Arguing by absurd, suppose that property (A.61) does not hold, so for all $\vartheta_n = \frac{1}{n} > 0$ there exists $\hat{\psi}_n \in H^1(\mathbb{R})$ with $\int_{\mathbb{R}} \hat{\psi}_n(s,\zeta)w'(\zeta)d\zeta = 0$, and such that

$$B(\hat{\psi}_n) < \frac{1}{n} \int_{\mathbb{R}} \hat{\psi}_n^2(t)dt$$

In particular, by normalizing in L^2 each term $\hat{\psi}_n$, this construction gives us a sequence of functions $\{\psi_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R})$, orthogonal to w' in $L^2(\mathbb{R})$, such that

$$\forall n \in \mathbb{N} : \int_{\mathbb{R}} \psi_n^2(t)dt = 1 \quad \text{and} \quad B(\psi_n) \xrightarrow[n\to+\infty]{} 0 \tag{A.62}$$

But this implies that

$$\forall n \in \mathbb{N} : \int_{\mathbb{R}} |\psi_n'(t)|^2 dt = B(\psi_n) + \int_{\mathbb{R}} f'(w(t))\psi_n^2(t)dt \leq o_n(1) + \|f'\|_{L^\infty[-1,1]} \cdot 1 \leq C$$

So it follows $\|\psi_n\|_{H^1(\mathbb{R})} \leq C$ is bounded, so there exists a subsequence which will be denoted again by $\{\psi_n\}$, that converges to a weak limit $\bar{\psi}$ in $H^1(\mathbb{R})$. In particular this implies that $\psi_n \rightharpoonup \bar{\psi}$ weak in $L^2(\mathbb{R})$ and so the orthogonality condition for $\bar{\psi}$ against w' holds too:

$$0 \equiv \int_{\mathbb{R}} \psi_n(t) \cdot w'(t)dt \xrightarrow[n\to+\infty]{} \int_{\mathbb{R}} \bar{\psi}(t) \cdot w'(t)dt \tag{A.63}$$

Additionally, it is possible to check that the weak limit $\bar{\psi}$ is not zero. To see this compute

$$B(\psi_n) = \int_{\mathbb{R}} |\psi_n'(t)|^2 dt + 2\int_{\mathbb{R}} \psi_n^2(t)dt - \int_{\mathbb{R}} (2 + f'(w))\psi_n^2(t)dt$$

Using that $B(\psi_n) \to 0$ and the fact that $2 + f'(u) = 3(1 - u^2)$ we obtain the lower bound

$$2 \leq \int_{\mathbb{R}} |\psi_n'(t)|^2 dt + 2 = 3\int_{\mathbb{R}} (1 - w^2(t))\psi_n^2(t)dt + o_n(1)$$

in particular this implies for a subsequence that $\exists n_0 \in \mathbb{N}$, $\forall n_j \geq n_0$

$$\frac{1}{2} \leq \int_{\mathbb{R}} (1 - w^2(t))\psi_{n_j}^2(t)dt \xrightarrow[j\to+\infty]{} \int_{\mathbb{R}} (1 - w^2(t))\bar{\psi}^2(t)dt$$

where the converge of the last integral is justified by both the exponential decay of $(1 - w^2(t)) = O(e^{-2|t|})$, and the compact injection $W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ for any bounded open set $\Omega \subset \mathbb{R}$ applied to the sequence $\psi_n \to \bar{\psi}$. The last condition is due to Rellich-Kondrakov theorem in the case $p = 2 > N = 1$ and the fact that the sequence is bounded $\|\psi_n\|_{H^1(\mathbb{R})} \leq C$. It follows

$$\bar{\psi} \neq 0 \tag{A.64}$$

For the rest of the argument, we will focus our attention on the properties of function $\bar{\psi}$. To begin with, note that this element minimizes the quadratic form Q, as a consequence of the following estimate

$$\begin{aligned}
B(\bar{\psi}) &= \int_{\mathbb{R}} |\bar{\psi}'(t)|^2 dt - \int_{\mathbb{R}} f'(w)\bar{\psi}^2(t)dt \\
&= \|\bar{\psi}'\|_{L^2} + 2\|\bar{\psi}\|_{L^2} - \int_{\mathbb{R}} (2 + f'(w))\bar{\psi}^2(t)dt \\
&\leq \liminf_{n\to\infty} \left[\int_{\mathbb{R}} |\psi_n'(t)|^2 dt + 2\int_{\mathbb{R}} \psi_n^2(t)dt - 3\int_{\mathbb{R}} (1 - w^2(t))\psi_n^2(t)\right] \\
&= \liminf_{n\to\infty} B(\psi_n) = 0
\end{aligned}$$

where we have used that $\bar{\psi}$ is the weak limit of ψ_n in $H^1(\mathbb{R})$, and the converge of the last integral explained in the previous paragraph.

This tells us that $0 \leq B(\bar{\psi}) \leq 0$, where the first inequality is due to estimate (A.60). Then it holds that $B(\bar{\psi}) \equiv 0$, and in particular the same estimate (A.60) implies that B achieves a minimum in $H^1(\mathbb{R})$ at function $\bar{\psi}$. But, as in this case B is Frechet-differentiable in $H^1(\mathbb{R})$, this fact can be characterized by $DB(\bar{\psi})[\xi] \equiv 0$ and consequently

$$\int_{\mathbb{R}} \bar{\psi}'(t)\xi(t)dt - \int_{\mathbb{R}} f'(w(t))\bar{\psi}(t)\xi(t)dt = 0\ ,\quad \forall \xi \in H^1(\mathbb{R})$$

which is exactly to say that $\bar{\psi}$ is a weak solution in $H^1(\mathbb{R})$ to the ODE

$$L_0(\bar{\psi}) := \bar{\psi}''(t) + f'(w(t))\bar{\psi}(t) = 0 \quad \text{in } H^{-1}(\mathbb{R})$$

Thanks to elliptic regularity theory for PDEs, given that coefficients 1 and $f'(w(t))$ of L_0 are $C^\infty(V)$ on any bounded open set, and the right hand side 0 is also $C^\infty(V)$, we can apply the Infinite differentiability in the interior Theorem which asserts that the weak solution $\bar{\psi}$ in $H^1(V)$ is actually a $C^\infty(V)$-smooth solution to $L_0(\bar{\psi}) = 0$, and therefore it satisfies the ODE in the classical sense. Moreover, by Sobolev injections we get that $\bar{\psi}$ is bounded in the whole space $\mathbb{R}$ of dimension one, because it holds $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ for $2 = p > N = 1$.

Furthermore, the Schauder estimates allow us to establish a global control on the size of $\bar{\psi}$ up to its second-order derivable in norm L^∞, since for any $t \in \mathbb{R}$ we have that 1, $f'(w)$, and 0 are $C^{\,0,1}(B(t,2))$ functions. This result states that the classical solution $\tilde{\psi} \in C^2(B(t,2))$ of $L_0(\bar{\psi}) = 0$ satisfies for any $\alpha \in (0,1)$.

$$\|\bar{\psi}\|_{C^{2,\alpha}(B(t,1))} \leq C[\|\bar{\psi}\|_{L^\infty(B(t,1))} + \|0\|_{C^\alpha(B(t,1))}] \leq \tilde{C}\|\bar{\psi}\|_{H^1(\mathbb{R})} \tag{A.65}$$

where the constant depends only on L_0, α, and the radius $R = 1$ of $B(t,1) \subset\subset \mathbb{R}$. In particular, estimate (A.65) allow us to control the global size of $\bar{\psi}'$, as the bound does not depend on the ball $B(t,1)$ chosen, but instead depends on its constant radius.
As a result, we have

$$\|\bar{\psi}'\|_{L^\infty(\mathbb{R})} \leq \sup_{t\in\mathbb{R}} \|\bar{\psi}'\|_{L^\infty(B(t,1))} \leq \tilde{C} \cdot \|\bar{\psi}\|_{H^1(\mathbb{R})}$$

so we deduce that $\bar{\psi}'$ is uniformly bounded in the entire space.

Nonetheless, recall that $w'(t)$ is another solution of this homogeneous equation, since taking the derivative with respect to t in the ODE (3.17) satisfied by w', we obtain $w'''(t) + f'(w(t))w'(t) = 0$ in $\mathbb{R}$. Analyzing the Wronskian of $\bar{\psi}$ and w', we readily see that $W(\bar{\psi}, w')(t) \equiv M$ is constant due to Abel's formula, as there is no term of first-order derivative in the ODE. In particular, this means

$$M \equiv \lim_{s\to\infty} W(\bar{\psi}, w')(t) = \lim_{s\to\infty} (\bar{\psi}'(t)w'(t) - w''(t)\bar{\psi}(t))$$

but in view of the decay behavior of w' and w'' at infinity, and at the same time the boundedness of $\bar{\psi}$ and $\bar{\psi}'$ argued before, we deduce

$$\lim_{s\to\infty} \bar{\psi}'(s)w'(s) - \lim_{s\to\infty} w''(t)\bar{\psi}(t) = 0 \quad \Rightarrow \quad M = 0$$

In other words, $\bar{\psi}$ and w' solutions of this homogeneous second-order ODE are linearly dependent in $\mathbb{R}$, so accordingly

$$\bar{\psi}(t) = C \cdot w'(t)\ , \qquad \text{for all } t \in \mathbb{R}$$

Finally from the orthogonality condition (A.63) satisfied by $\bar{\psi}$ we get

$$\int_{\mathbb{R}} (Cw'(t)) \cdot w'(t)dt = C \cdot \int_{\mathbb{R}} |w'(t)|^2 dt = 0 \quad \Rightarrow \quad C = 0$$

therefore, we conclude $\bar{\psi}(t) = 0 \cdot w' \equiv 0$ which contradicts (A.64), that is, $\bar{\psi}(t) \neq 0$. $\rightarrow\leftarrow$

To conclude Lemma 2, it remains to prove that coercivity inequality (A.61) not only holds for norm L^2 of ψ, but also holds in norm H^1. To see this, take any $\varepsilon \in (0,1)$ and $\psi \in H^1(\mathbb{R})$, then

$$\begin{aligned}(1+\varepsilon)B(\psi) &= \int_{\mathbb{R}} [|\psi'(t)|^2 - f'(w)\psi^2(t)]dt + \varepsilon \int_{\mathbb{R}} |\psi'(t)|^2 dt - \varepsilon \int_{\mathbb{R}} f'(w(t))\psi^2(t)dt \\ &\geq \vartheta \cdot \int_{\mathbb{R}} \psi^2(t)dt + \varepsilon \int_{\mathbb{R}} |\psi'(t)|^2 dt - \varepsilon \|f'\|_{L^\infty[-1,1]} \int_{\mathbb{R}} \psi^2(t)dt \\ &\geq (\vartheta - \varepsilon \|f'\|_{L^\infty[-1,1]}) \cdot \int_{\mathbb{R}} \psi^2(t)dt + \varepsilon \int_{\mathbb{R}} |\psi'(t)|^2 dt \end{aligned} \tag{A.66}$$

where we have used the known coercivity (A.61) in L^2.

Now choosing ε sufficiently small in such way $\vartheta - \varepsilon\|f'\|_{L^\infty[-1,1]} > \vartheta/2$, and using that B is nonnegative in $H^1(\mathbb{R})$, we get the estimate

$$2B(\psi) \geq \frac{\vartheta}{2} \int_{\mathbb{R}} \psi^2(t)dt + \varepsilon \int_{\mathbb{R}} |\psi'(t)|^2 dt$$

Thus we deduce the existence of a constant $\tilde{\vartheta} > 0$ such that

$$B(\psi) \geq \tilde{\vartheta} \int_{\mathbb{R}} [|\psi'(t)|^2 + \psi^2(t)]dt$$

taking $\tilde{\vartheta} := \text{mín}\{\vartheta/4, \varepsilon/2\}$, which concludes the proof of Lemma 7. $\square$

References

[1] S. Allen and J. W. Cahn. A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta. Metall.*, 27:pp 1084–1095, 1979.

[2] L. Caffarelli and A. Cordoba. Uniform convergence of a singular perturbation problem. *Comm. Pure and Appl. Math.*, XLVII:pp 1–12, 1995.

[3] L. Caffarelli and A. Cordoba. Phase transition: uniform regularity of the intermediate layers. *J. Reine Angew. Math.*, 593:pp 209–235, 2006.

[4] M. del Pino. New entire solutions to some classical semilinear elliptic problems. In *Proceedings of the International Congress of Mathematicians*, pages 1–6, Hyderabad, India, 2010.

[5] M. del Pino. New entire solutions of semilinear elliptic equations. In *IMS-CUHK*, pages 17–27, Hong Kong, China, 2011.

[6] M. del Pino, M. Kowalczyk, F. Pacard, and J. C. Wei. Multiple end solutions to the Allen-Cahn equation in $\mathbb{R}^2$, November 2008.

[7] M. del Pino, M. Kowalczyk, and J. C. Wei. The Toda System and Clustering Interfaces in the Allen Cahn equation. *Arch. Rational Mech.Anal.*, 190(DOI 10.1007):141–187, 2008. s00205-008-0083-0143-3.

[8] M. del Pino, M. Kowalczyk, and J. C. Wei. Entire solutions of the Allen-Cahn equation and Complete Embedded Minimal Surfaces of finite total curvature in $\mathbb{R}^3$, 2009. arXiv:0902.2047v1.

[9] M. del Pino, M. Kowalczyk, J. C. Wei, and Jun Yang. Interface foliation near minimal submanifolds in Riemmanian Manifolds with positive Ricci curvature. *GAFA Geometric and Functional Analysis*, 20(DOI 10.1007):918–957, August 2010. s00039-010-0083-6.

[10] A. Floer and A. Weinstein. Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. *J.Funct. Anal.*, 69:397–408, 1986.

[11] J. E. Hutchinson and Y.Tonegawa. Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory. *Calc.Var.*, 10:pp 49–84, 2000.

[12] L. Modica. Convergence to minimal surfaces problem and global solutions of $\triangle u = 2(u^3 - u)$. In *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis*, pages 223—-244, Rome 1978 Pitagora and Bologna 1979, 1979.

[13] F. Pacard and M. Ritore. From the constant mean curvature hypersurfaces to the gradient theory of phase transtition. *J. Differencial Geom. Math.*, 64:pp 359–423, 2003.

[14] Z.Du and B.Lai. Transition layers for an inhomogeneus Allen-Cahn equation in Riemannian Manifolds. *submitted*.

[15] Z.Du and C.Gui. Interior layers for an inhomogeneous Allen-Cahn equation. *J.Differential Equations*, 249:pp 215–239, July 2010.

[16] Z.Du and L.Wang. Interface foliation for an inhomogeneous Allen-Cahn equation in Riemmanian Manifolds. *Calc.Var*, (DOI 10.1007), March 2012. s00526-012-0521-4.