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ANÁLISIS TEÓRICO Y NUMÉRICO DE UN PROBLEMA INVERSO DE RECUPERACIÓN DE FUENTE Y ATENUACIÓN PARA LA ECUACIÓN DE TRANSFERENCIA RADIATIVA CON APLICACIONES A TOMOGRAFÍA SPECT

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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Resumen

En este trabajo buscamos obtener imágenes de las distribuciones internas de fuentes radioactivas y mapa de atenuación para el procedimiento médico usado en tomografía SPECT, usando mediciones balísticas y de primer orden de scattering.

Con este objetivo, modelamos matemáticamente el problema tridimensionalmente utilizando la ecuación de transferencia radiativa, logrando explicitar el operador no-lineal que entrega las mediciones en función de la distribución de fuentes radioactivas y mapa de atenuación. Derivando direccionalmente el operador no-lineal, obtuvimos un operador lineal que define el problema inverso linealizado. Bajo hipótesis de regularidad sobre la distribución de fuentes radioactivas y mapa de atenuación y, considerando baja atenuación, se demostró rigurosamente que el operador lineal es invertible y se calculó explícitamente su inversa.

La demostración de la invertibilidad del operador linealizado consta de varias etapas. En una primera etapa se descompone el operador en una parte L invertible y una perturbación Q que sea pequeña para pequeñas atenuaciones en el espacio funcional adecuado. En una segunda etapa, se estudian las propiedades de regularidad de L y Q mediante métodos que incluyen estimaciones sobre la inversa de la transformada de Radon atenuada y de la transformada de Radon con pesos como operadores integrales en espacios de Sobolev con exponente fraccionario. Finalmente se concluye la invertibilidad de $L+Q$ acotando la norma de $L^{-1}Q$ y usando series de Neumann.

Usando el resultado previo de inversión para el operador lineal, se plantearon en este trabajo nuevos tipos de algoritmos iterativos de recuperación de fuentes y atenuación para la tomografía SPECT. Estos algoritmos incluyen un algoritmo para el problema inverso linealizado usando series de Neumann, un algoritmo para el problema inverso no-lineal usando el método de Newton-Raphson y un algoritmo heurístico para el no-lineal el cual fue implementado numéricamente.

El análisis teórico del problema linealizado provisto por este estudio representa un paso previo fundamental para el estudio de la convergencia de los algoritmos numéricos antes propuestos.

Al comparar el algoritmo heurístico implementado en este trabajo con la metodología tradicional de SPECT, tanto en experimentos con datos reales como sintéticos, se observa una mejora en la recuperación de fuentes, además de contar con la reconstrucción adicional

del mapa de atenuación del medio.

Abstract

In this work we seek to obtain images of the internal distributions of radioactive sources and attenuation map for the medical procedure used in SPECT tomography, using ballistic and first order scattering external measurements.

With this objective, we modeled mathematically the problem three-dimensionally using the radiative transfer equation, obtaining explicitly the non-linear operator that gives the measurements as a function of the radioactive source distribution and attenuation map. Directionally deriving the non-linear operator we obtained a linear operator that defines the linearized inverse problem. Under hypothesis of regularity for the radioactive source distribution and attenuation map and, considering low attenuation, we rigorously proved that the linear operator is invertible and we computed explicitly its inverse.

The proof of the invertibility of the linearized operator consists of several steps. In the first step we decomposed the operator in an invertible part L and a perturbation Q to be small for small attenuation in the appropiate functional space. In the second step, we study the regularity properties of L and Q with methods that includes estimates over the inverse of the attenuated Radon transform and the weighted Radon transform as integral operators in Sobolev spaces with fractional exponent. Finally we conclude the invertibility of $L+Q$ bounding the norm of $L^{-1}Q$ and using Neumann series.

Using the previous inversion result for the linear operator, we propose in this work a new type of iterative algorithms for radioactive source and attenuation map recovery for SPECT tomography. These algorithms include an algorithm for the linearized inverse problem using Neumann series, an algorithm for the non-linear inverse problem using the Newton-Raphson method and an heuristic algorithm using the Banach fixed-point for the non-linear case which was implemented numerically.

The theoretical analysis for the linearized problem provided in this study represents a fundamental previous step for the study of the convergence of the proposed numerical algorithms.

Comparing the implemented heuristic algorithm in this work with the standard method used in SPECT, on experiments with real and synthetic data, we observe and improvement in the source recovery, in addition to recovery of the attenuation map of the medium.

To my family, my girlfriend and my friends.

Contents

Introduction

0.1 Inverse problems and medical imaging

An inverse problem is a general framework that is used to obtain, through observed indirect measurements, information about a physical object or system that we are interested in. In medicine, such problems appear when we want to create an image of internal organs or tissues of a patient. Since that information can be difficult to obtain directly, we must rely on external measurements that allow us to reconstruct an image of the object in study.

To do this, we first need a mathematical model that explains the available measurements as a function of the physical properties that we are interested in, and then seek the best reconstruction that fits the observed data. For example, if we have a model that explains the decay of X-ray photons intensity traversing an object as a function of the object's density, we can expose a patient to X-rays and measure the intensity of exiting X-ray beams to acquire information about the density of internal tissues.

There are many types of medical imaging techniques such as Computerized Tomography (CT), Magnetic Resonance Imaging (MRI), Positron Emition Tomography (PET), among others (see [21]). In this work we deal with a specific method called Single-Photon Emission Computed Tomography (SPECT).

0.2 Single-Photon Emission Computed Tomography (SPECT)

0.2.1 General description

The SPECT method is a nuclear medicine tomographic imaging technique using gamma rays, the idea is to deliver into a patient a gamma-emitting radioisotope (typically technetium-99m) that is designed to get attached to certain types of tissues, thus, after the patient has been inyected, the especific tissue will start to emit gamma rays (for reference see [11], Chapter 2). This radiation can be measured, outside the patient, by a device called gamma camera which can identify the direction and energy level of the gamma rays. With the information gathered, the goal is to reconstruct the distribution of the radioisotope inside the patient, hence obtaining an image of the desired specific tissue in study.

The mathematical model commonly used to describe the external measured photons require two physical parameters, the radioactive source map f and the attenuation map a. The attenuation map represents the capacity of the medium to absorb photons and is, given the medical procedure, an unknown value. The radioactive source map represents the capacity of the medium to radiate photons, and is the function to be obtained.

We consider that generated photons travel in straight lines, thus to obtain the information of a cross-sectional slice of f, say $f|_{\mathcal{P}} = \{f(x),\ x \in \mathcal{P}\}$, it is enough to measure the photons exiting the patient body and travelling in the $\mathcal{P}$ plane. This information can be represented as the operator $R_{a|_{\mathcal{P}}}[f|_{\mathcal{P}}]$ called the attenuated Radon transform of $f|_{\mathcal{P}}$ with attenuation $a|_{\mathcal{P}}$.

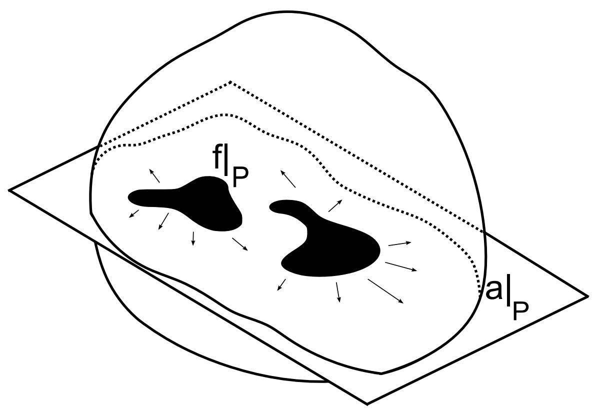

Figure 1: A representation of cross-sections $a|_{\mathcal{P}}$ and $f|_{\mathcal{P}}$, the arrows symbolize photons moving in the $\mathcal{P}$ plane

0.2.2 SPECT Inverse Problem

Since the data that can be measured in a given plane $\mathcal{P}$ can be represented as an explicit operator, the inverse problem that arises is to reconstruct the value of $f|_{\mathcal{P}}$ with the information of the attenuated Radon transform $R_{a|_{\mathcal{P}}}[f|_{\mathcal{P}}]$. Given that the values of $a|_{\mathcal{P}}$ are mixed in the operator with the information of $f|_{\mathcal{P}}$, we have 2 cases.

- $a|_{\mathcal{P}}$ is known
  This case was solved by Novikov in 2000 [23] and obtained an explicit formula for the inverse of the attenuated Radon transform.
- $a|_{\mathcal{P}}$ is unknown
  Up to this date there are no explicit reconstruction methods when $a|_{\mathcal{P}}$ is unknown, although there are some interesting results, for instance see [29]. A commonly used procedure to approximate the values of $f|_{\mathcal{P}}$ is to assume a known value of $a|_{\mathcal{P}}$ and reconstruct using

the inverse of the attenuated Radon transform, as is comented in page 42 of book [18].

There is another inverse problem that contains the case in which $a|_{\mathcal{P}}$ is unknown, this is to reconstruct both the attenuation and source map $a|_{\mathcal{P}}$ and $f|_{\mathcal{P}}$ respectively using the information given by the attenuated Radon transform $R_{a|_{\mathcal{P}}}[f|_{\mathcal{P}}]$. This problem is known in the literature as the *identification problem* for SPECT.

0.3 Our inverse problem

Our main goal is to reconstruct both the attenuation and source maps of an unknown object using the SPECT setting, although this is the same objective as in the identification problem, we tackle a different inverse problem by using aditional measurements by considering assumptions and extensions of the underlying mathematical model.

The model is extended by considering scattering effects, this introduces an unknown coefficient that describes the scattering behavior of photons inside the object in study, the made asumption relates this scattering coefficient to the attenuation coefficient and the additional information is gathered by measuring scattered photons outside the object in study.

When a photon scatters it reduces its energy level, and since gamma cameras can discriminate the energy level of photons, we can measure separately the gamma rays exiting the patient that have not scattered (ballistic photons) and the gamma rays exiting the patient that have scattered, particularly we are interested in measuring photons that just have scattered once (first order scattering photons), hence, we can gather more information using the same medical procedure described for the SPECT method and without the addition of new parameters.

The new information given by first order scattering photons travelling in a $\mathcal{P}$ plane contains information of the whole three dimentional body, because unlike ballistic photons, scattered photons do not, by definition, travel in straight line from the source to the gamma camera, see figure 2. Hence, to use the information there is a need of measuring along a family $(\mathcal{P}_z)_{z\in\mathbb{R}}$ of planes such that their information allows us to reconstruct both the attenuation map and source map of the whole three dimensional body.

Explicitly, consider $\mathcal{P}_z = \{(y,z) \in \mathbb{R}^3,\ y \in \mathbb{R}^2\}$ an horizontal plane at height $z \in \mathbb{R}$, and for a function f defined on $\mathbb{R}^3$ let $f_z(y) = f(y,z), y \in \mathbb{R}^2$ be its restriction to the $\mathcal{P}_z$ plane. If we have an attenuation map a and a source map f (both functions defined on $\mathbb{R}^3$), then the measurements of ballistic photons in the $\mathcal{P}_z$ plane gives us the information of

$$R_{a_z}[f_z](s,\theta) = \int_{\mathbb{R}} f_z(t\theta + s\theta^\perp)\mathrm{e}^{-\int_0^\infty a_z(t\theta+\tau\theta+s\theta^\perp)\mathrm{d}\tau}\mathrm{d}t \quad \forall s \in \mathbb{R}, \theta \in S^1,$$

where $S^1 = \{\theta \in \mathbb{R}^2,\ |\theta| = 1\}$ is the unit sphere in $\mathbb{R}^2$, $\theta^\perp$ is a counterclockwise rotation of

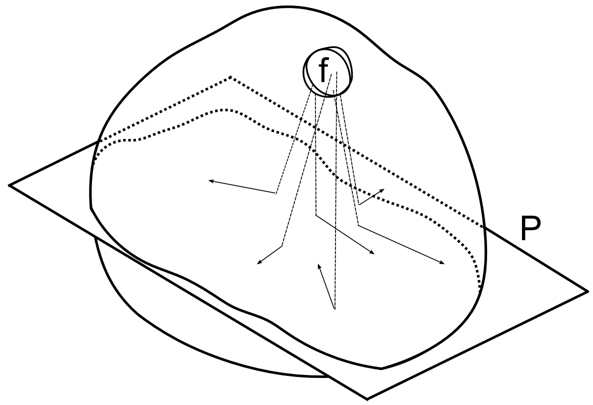

Figure 2: A representation of how measuring the scattering photons in a $\mathcal{P}$ plane, contains information of a the whole three dimentional object.

vector θ (i.e. $\theta^{\perp} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \theta$) and every pair $(s, \theta) \in \mathbb{R} \times S^1$ represents a directed line in $\mathbb{R}^2$, so $R_{a_z}[f_z]$ is obtained by measuring at every position and planar direction at height z outside a patient. Observe that this is the same operator for the SPECT inverse problem and, to our knowledge, there is no inversion formula to obtain both a_z and f_z from it.

Let $M[a, f](x)$ be a function that gives the total ammount of ballistic photons passing by the spatial point $x \in \mathbb{R}^3$

$$M[a,f](x) = \int_{S^2} \int_0^{\infty} f(x + t\phi) \mathrm{e}^{-\int_0^t a(x+\tau\phi)\mathrm{d}\tau} \mathrm{d}t\mathrm{d}\phi, \qquad S^2 = \{\phi \in \mathbb{R}^3,\ |\phi| = 1\}.$$

Then measuring the first order scattering photons in the $\mathcal{P}_z$ plane gives us information of

$$R_{a_z}[a_z \cdot M_z[a,f]](s,\theta) = \int_0^{\infty} a_z(t\theta + s\theta^{\perp}) M_z[a,f](t\theta + s\theta^{\perp}) \mathrm{e}^{-\int_0^{\infty} a_z(t\theta+\tau\theta+s\theta^{\perp})\mathrm{d}\tau} \mathrm{d}t \quad \forall s \in \mathbb{R}, \theta \in S^1.$$

In this case, although we are just using the information of $M[a, f]$ restricted to the $\mathcal{P}_z$ plane, we have that at each point this function contains data of the whole three dimentional object, thus to use the first order scattering measurements we need to reconstruct the whole three dimentional object attenuation and source map.

Taking all this into account, our inverse problem can be formulated as obtaining both the attenuation and radioactive source map a and f defined on $\mathbb{R}^3$, from the knowledge of

$$\begin{aligned} R_{a_z}[f_z](s,\theta) &\quad \forall s, z \in \mathbb{R}, \theta \in S^1 \\ R_{a_z}[a_z \cdot M_z[a,f]](s,\theta) &\quad \forall s, z \in \mathbb{R}, \theta \in S^1. \end{aligned}$$

0.3.1 Main Objectives

There are three main objectives in this work, the first one is to derive an inverse problem that consider scattering effects in the standard mathematical model of SPECT that describes the behavior of photons in a medium, this by means of suitable assumptions that allows us to deal with the gathered information from the ballistic and first order scattering photons.

The second goal is to reconstruct both the attenuation and source map from the available data. To achieve this goal we study the operator that describes the external measurements by means of a linearization process.

The third objective is to tackle numerically the deduced inverse problem, this is, to develop algorithms that use the ballistic and first order scattering photon infomation to reconstruct both the attenuation and source map of an unknown object.

0.4 Previous results

The Attenuated Radon transform (AtRT) plays a central role in the SPECT inverse problem, and particularly in the extension made in this work, hence all significant previous results are about this transform.

The inversion formula with known attenuation was obtained by Novikov in [23] deriving an explicit inverse operator, although from a slightly previous work [2] it can be deduced. There are several generalization for this result, for instance if we integrate along geodesics [27], if the coefficients are complex valued [33] or using more general weight functions [8, 7]. There are also invertibility and stability results for partial measurements, for instance in [19] inyectivity is obtained measuring in an arbitrarily small open set of angles, stability for the direct and inverse problem can be found at [26] and inversion of data in [3].

Regarding the identification problem, first approximations were obtained by assuming a constant attenuation map, this process reduces the operator to one called the Exponential Radon transform which arises a different problem [16, 15, 28], in [28] there are non uniqueness results for specific attenuation and source pairs. Similarly if axial symetry is assumed for the attenuation map, there is also an inversion result [24].

Another approach to study the identification problem is by characterizing the range of the AtRT. In [22] there is a necessary and sufficient condition for the range of the AtRT, restrictions over the attenuation as a function of the range can be found in [18]. Recently in [29, 5], these compatibility conditions and their linearization are used to obtain more information regarding the attenuation map and the identification problem.

Numerical algorithms to compute the inverse of the AtRT can be found in [20, 17, 13]. In [6] there is a fast implementation for partial measurements. Regarding the identification problem, some numerical algorithms focuses on getting a good approximation of the attenuation map first instead of treating (a, f) as a pair, see for example [14, 32, 25], these methods are often called attenuation correction algorithms.

For the mathematical model, the equation that describes the scattering behavior has a general form called the Radiative Transfer Equation, an extensive survey by Bal of the subject can be found in [4].

Chapter 1

Preliminaries

1.1 Basic Concepts

1.1.1 Fourier Trasform

Named after Joseph Fourier, this transform is of great importance in engineering and mathematics, it will be used through the whole text because is highly related to the Radon Transform, and its usefull properties to analyze stability, invertibility and more. For references see [30].

Definition 1.1 **(Fourier transform)** *Let f be integrable (i.e. $\int_{\mathbb{R}^n} |f(x)|\mathrm{d}x < \infty$) or square integrable (i.e. $\int_{\mathbb{R}^n} |f(x)|^2\mathrm{d}x < \infty$) then we can define the Fourier transform of f as*

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\mathrm{d}x.$$

If f is a Tempered Distribution (see definition 1.12), its Fourier transform is defined by

$$\langle \mathcal{F}f, \phi \rangle = \langle f, \mathcal{F}\phi \rangle \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n),$$

and if f is angularly dependant (say $f : \mathbb{R}^n \times S^m \to \mathbb{R}$ with $S^m = \{x \in \mathbb{R}^{m+1},\ |x| = 1\}$), we define its Fourier transform as the Fourier transform taken on the spatial variable (as long as it is a.e. integrable or square integrable)

$$\mathcal{F}f(\xi, \phi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x, \phi)\mathrm{d}x \quad a.\mathrm{e.}\ \phi \in S^m.$$

There are various standard definitions of the Fourier transform which differ in constants added for different purposes, this definition is generally called the "non unitary Fourier transform". Recall some properties of general knowledge whose demonstration will be omitted.

Theorem 1.2 *(***Inverse of the Fourier transform** *) Let f be a function that is integrable and such that its Fourier transform $\hat{f}$ is also integrable, then we have the inversion formula*

$$\mathcal{F}^{-1}\hat{f}(x) = f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} f(x)\mathrm{d}x$$

if f is square integrable, this formula holds for almost every $x \in \mathbb{R}^n$.

Proposition 1.3 *(***Partial derivative formula** *) Let f be a function such that $\partial^n_{x_i} f(x)$ has a well defined Fourier transform, then*

$$\mathcal{F}(\frac{\partial}{\partial x_j} f)(\xi) = \mathrm{i}\xi_j \hat{f}(\xi) \quad j \in \{1, 2, ..n\}.$$

Proposition 1.4 *(***Convolution formula** *) Let f and g be integrable, then their convolution $f * g(y) = \int_{\mathbb{R}^n} f(y-x)g(x)\mathrm{d}x$ is well defined, is integrable and satisfies*

$$\mathcal{F}(f * g)(\xi) = \mathcal{F}(f) * \mathcal{F}(g)(\xi).$$

Theorem 1.5 *(***Plancherel's theorem** *) Let f be integrable and square integrable, then*

$$\int_{\mathbb{R}^n} |f(x)|^2 \mathrm{d}x = (2\pi)^n \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \mathrm{d}\xi.$$

1.1.2 Function spaces and norms

Now we will establish certain spaces and norms that will be used along the text.

Definition 1.6 *(***Space of continuous functions** *) Let $\Omega \subset \mathbb{R}^n$ be any set, then*

$$C^0(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R},\ f \text{ is continuous }\},$$
$$C^0(\Omega) = \{f \in C^0(\mathbb{R}^n),\ \mathrm{supp}(f) = \overline{\{x \in \mathbb{R}^n,\ f(x) \neq 0\}} \subset \Omega\},$$

for angularly dependant functions we define

$$C^0(\Omega \times S^m) = \{f : \mathbb{R}^n \times S^m \to \mathbb{R},\ f \text{ is continuous and } \mathrm{supp}(f) \subset \Omega \times S^m\} \quad m \in \mathbb{N}.$$

Observation There are no needed properties for the set Ω since it just defines a restriction over the support of the functions and not over the domain of the spatial variable.

Definition 1.7 *(***Space of infinitely differentiable functions** *)*

$$C^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R},\ \text{is } f \text{ infinitely differentiable }\}$$
$$C^\infty(\mathbb{R}^n \times S^m) = \{f : \mathbb{R}^n \times S^m \to \mathbb{R},\ \text{is } f \text{ infinitely differentiable }\}$$
$$C^\infty_0(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n),\ f \text{ has compact support}\}$$
$$C^\infty_0(\mathbb{R}^n \times S^m) = \{f \in C^\infty(\mathbb{R}^n \times S^m),\ f \text{ has compact support}\}$$

for $\Omega \subset \mathbb{R}^n$ *we define*

$$\begin{aligned} C^\infty(\Omega) &= \{f \in C^\infty(\mathbb{R}^n),\ \mathrm{supp}(f) \subset \Omega\} \\ C^\infty(\Omega \times S^m) &= \{f \in C^\infty(\mathbb{R}^n \times S^m),\ \mathrm{supp}(f) \subset \Omega\} \\ C_0^\infty(\Omega) &= \{f \in C^\infty(\Omega),\ f \text{ has compact support}\} \\ C_0^\infty(\Omega \times S^m) &= \{f \in C^\infty(\Omega \times S^m),\ f \text{ has compact support}\} \end{aligned}$$

Definition 1.8 *(*L^p **spaces** *) We define* $L^p(\mathbb{R}^n)$ *and* $\|\cdot\|_{L^p(\mathbb{R}^n)}$ *with* $p \in [1,\infty]$ *by the clasical way*

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x\right)^{1/p} \quad p \in [1,\infty), \quad \|f\|_\infty = \text{ess sup}_{x\in\mathbb{R}^n}|f(x)|.$$

For $\Omega \subset \mathbb{R}^n, p \in [1,\infty]$ *we define*

$$L^p(\Omega) = \{f \in L^p(\mathbb{R}^n),\ f(x) = 0 \quad x \notin \Omega \text{ a.e. }\}.$$

Observation Since Ω is a restriction over the support of f and not over the domain, we have that if $K \subset \tilde{K}$ then $L^2(K) \subset L^2(\tilde{K})$.

Proposition 1.9 *(***Density in** L^p **spaces** *) Let* $\Omega \subset \mathbb{R}^n$ *be any set and* $p \in [1,\infty)$*, then*

$$\overline{C_0^\infty(\Omega)}^{\|\cdot\|_{L^p(\mathbb{R}^n)}} \subset L^p(\Omega) \quad p \in [1,\infty).$$

Definition 1.10 *(***Schwartz space** *)*

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n),\ \sup|x^\beta \partial^\alpha f| < \infty \quad \forall \textit{multi-indexes } \alpha, \beta \in \mathbb{N}^m, m \in \mathbb{N}\}.$$

Definition 1.11 *(***Schwartz space topology** *) We say that* $\phi_j \to 0$ *in* $S(\mathbb{R}^n)$ *if for all multi-indices* α *and* β *we have* $x^\beta \partial^\alpha \phi_j \to 0$ *uniformly on* $\mathbb{R}^n$

Definition 1.12 *(***Tempered distribution space** *)* $\mathcal{S}'(\mathbb{R}^n)$ *is the set of sequentially continuous linear functionals on the space* $\mathcal{S}(\mathbb{R}^n)$

Definition 1.13 *(***Sobolev space** H^s *) For* $s \in \mathbb{R}$ *and* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that* $\hat{f}$ *can be represented as a function, we define the Sobolev norm*

$$\|f\|_{H^s(\mathbb{R}^n)} = \|f\|_s = \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s \mathrm{d}\xi\right)^{1/2}.$$

Then for $\Omega \subset \mathbb{R}^n$

$$\begin{aligned} H^s(\mathbb{R}^n) &= \{f \in \mathcal{S}'(\mathbb{R}^n),\ \|f\|_{H^s(\mathbb{R}^n)} < \infty\}, \\ H^s(\Omega) &= \{f \in H^s(\mathbb{R}^n),\ \mathrm{supp}(f) \subset \Omega\}, \end{aligned}$$

if we have a angularly dependant function, such that $f(\cdot,\theta)\in H^s(\mathbb{R}^n)\quad$ a.e. $\theta\in S^m$ *we define its* $H^s(\mathbb{R}^n\times S^m)$ *norm as*

$$\|f\|_{H^s(\mathbb{R}^n\times S^m)}=\left(\int_{S^m}\|f(\cdot,\phi)\|_s^2\mathrm{d}\phi\right)^{1/2},$$

and for $\Omega\subset\mathbb{R}^n$ *the spaces*

$$H^s(\mathbb{R}^n\times S^m)=\{f\in\mathcal{S}'(\mathbb{R}^n\times S^m),\ \|f\|_{H^s(\mathbb{R}^n\times S^m)}<\infty\},$$
$$H^s(\Omega\times S^m)=\{f\in H^s(\mathbb{R}^n\times S^m),\ \mathrm{supp}(f)\subset\Omega\times S^m\}.$$

Note that all these spaces with their respective norms are Banach spaces.

Proposition 1.14 (**Inclusion of** $H^s(\mathbb{R}^n)$ **spaces**) *If* $s<l$ *in* $\mathbb{R}$ *then*

$$H^l(\mathbb{R}^n)\subset H^s(\mathbb{R}^n)$$

Proposition 1.15 (**Regularity loss of derivatives in** $H^s(\mathbb{R})$ **spaces**) *For* $s\in\mathbb{R}$, $f\in H^s(\mathbb{R}^n)$ *(Resp.* $f\in H^s(\mathbb{R}^n\times S^m)$*) and* α *a multi-index such that* $|\alpha|=l$ *then we have that* $\partial^\alpha f\in H^{s-l}(\mathbb{R}^n)$ *(Resp.* $\partial^\alpha f\in H^{s-l}(\mathbb{R}^n\times S^m)$*, with this derivative taken in the spatial variable) and the following inequalities*

$$\|\partial^\alpha f\|_{H^{s-l}(\mathbb{R}^n)}\leq\|f\|_{H^s(\mathbb{R})}\qquad\forall\text{ multi-index }\alpha\text{ such that }|\alpha|=l,$$
$$\|\partial^\alpha f\|_{H^{s-l}(\mathbb{R}^n\times S^m)}\leq\|f\|_{H^s(\mathbb{R}\times S^m)}\qquad\forall\text{ multi-index }\alpha\text{ such that }|\alpha|=l.$$

Proposition 1.16 (**Density of** $C_0^\infty(\mathbb{R}^n)$ **on** $H^s(\mathbb{R}^n)$) *For any* $s\in\mathbb{R}$*, and* $\Omega\subset\mathbb{R}^n$ *then*

$$\overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^s(\mathbb{R}^n)}}=H^s(\Omega).$$

Proposition 1.17 (**Duality inequality of** H^s **spaces**) *Let* $s>0$ *be a positive real, then* $H^{-s}(\mathbb{R}^n)$ *(resp.* $H^{-s}(\mathbb{R}\times S^m)$*) can be identified with the dual space of* $H^s(\mathbb{R}^n)$ *(resp.* $H^s(\mathbb{R}\times S^m)$*) and we have the following inequalities for the duality product* $\langle\cdot,\cdot\rangle$ *(in* $H^s(\mathbb{R}\times S^m)$ *the duality product is* $\langle f,g\rangle=\int_{S^m}\langle f(\cdot,\theta),g(\cdot,\theta)\rangle\,\mathrm{d}\theta$ *with* $f(\cdot,\theta)\in H^s(\mathbb{R}^n),g(\cdot,\theta)\in H^{-s}(\mathbb{R}^n)$ a.e. $\theta\in S^m$*).*

$$|\langle f,g\rangle|\leq C\|f\|_{H^{-s}(\mathbb{R}^n)}\|g\|_{H^s(\mathbb{R}^n)}\quad\forall f\in H^{-s}(\mathbb{R}^n),g\in H^s(\mathbb{R}^n),$$
$$|\langle f,g\rangle|\leq C\|f\|_{H^{-s}(\mathbb{R}^n\times S^m)}\|g\|_{H^s(\mathbb{R}^n\times S^m)}\quad\forall f\in H^{-s}(\mathbb{R}^n\times S^m),g\in H^s(\mathbb{R}^n\times S^m).$$

Proof. The first inequality is classical, the second one follows directly from the definition and the first inequality

$$\begin{aligned}|\langle f,g\rangle|&\leq\int_{S^m}|\langle f(\cdot,\theta),g(\cdot,\theta)\rangle|\,\mathrm{d}\theta\\&\leq C\int_{S^m}\|f(\cdot,\theta)\|_{H^{-s}(\mathbb{R}^n)}\|g(\cdot,\theta)\|_{H^s(\mathbb{R}^n)}\mathrm{d}\theta\\&\leq C\sqrt{\int_{S^m}\|f(\cdot,\theta)\|^2_{H^{-s}(\mathbb{R}^n)}\mathrm{d}\theta}\sqrt{\int_{S^m}\|g(\cdot,\theta)\|^2_{H^s(\mathbb{R}^n)}\mathrm{d}\theta}\\&=C\|f\|_{H^{-s}(\mathbb{R}^n\times S^m)}\|g\|_{H^s(\mathbb{R}^n\times S^m)}.\end{aligned}$$

□

Definition 1.18 (**Hölder Space**) *For $0 < \alpha \leq 1$ and a function f on $\mathbb{R}^n$ we define its Hölder norm as*

$$\|f\|_{C^\alpha(\mathbb{R}^n)} = \sup_x |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha},$$

its Hölder seminorm

$$|f|_{C^\alpha(\mathbb{R}^n)} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha},$$

and for $\Omega \subset \mathbb{R}^n$ the Hölder space

$$C^\alpha(\mathbb{R}^n) = \{f \in C^0(\mathbb{R}^n),\ \|f\|_{C^\alpha(\mathbb{R}^n)} < \infty\},$$
$$C^\alpha(\Omega) = \{f \in C^\alpha(\mathbb{R}^n),\ \operatorname{supp}(f) \subset \Omega\}.$$

If we have f defined on $\mathbb{R}^n \times S^m$ then we define its Hölder norm as

$$\|f\|_{C^\alpha(\mathbb{R}^n \times S^m)} = \sup_\theta \|f(\cdot, \theta)\|_{C^\alpha(\mathbb{R}^n)},$$

and for $\Omega \subset \mathbb{R}^n$ we define the space

$$C^\alpha(\mathbb{R}^n \times S^m) = \{f \in C^0(\mathbb{R}^n \times S^m),\ \|f\|_{C^\alpha(\mathbb{R}^n) \times S^m} < \infty\},$$
$$C^\alpha(\Omega \times S^m) = \{f \in C^\alpha(\mathbb{R}^n \times S^m),\ \operatorname{supp}(f) \subset \Omega \times S^m\}.$$

And finally, a theorem that connects these spaces.

Theorem 1.19 (**Sobolev Embedding Theorem**) *Let s, n be integers, $\alpha \in \mathbb{R}^2$, if $(s - \alpha)/n < 1/2$ then*

$$H^s(\mathbb{R}^n) \subset C^\alpha(\mathbb{R}^n),$$

and the inclusion is continuous.

1.1.3 Hilbert Transform

This transform is widely used in signal analysis, one of its most useful properties is its behavior under the Fourier transform, for further references see [31].

Definition 1.20 (**Hilbert Transform**) *For $f \in L^p(\mathbb{R}), p > 1$, denoting P.V. $1/x$ a distribution that satisfies $\langle \mathrm{P.V.}1/x, f \rangle = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} f(x)/x \, \mathrm{d}x$ then we define the Hilbert transform as*

$$Hf(x) = \quad f * \left(P.V. \frac{1}{\pi x} \right),$$

where $*$ stands for the convolution operator.

Observation The Hilbert transform can be rewritten as

$$\begin{aligned} Hf(x) &= f * \left(P.V.\frac{1}{\pi x}\right) \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{x-\varepsilon}\frac{f(t)}{x-t}\mathrm{d}t + \int_{x+\varepsilon}^{\infty}\frac{f(t)}{x-t}\mathrm{d}t\right) \\ &= \frac{-1}{\pi}\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{\infty}\frac{f(x+t)-f(x-t)}{t}\mathrm{d}t. \end{aligned}$$

Proposition 1.21 (**Hilbert transform in $L^p(\mathbb{R})$ space**) *If $f \in L^p(\mathbb{R}), p > 1$ the Hilbert transform is well defined a.e. and*

$$||Hf||_{L^p(\mathbb{R})} \leq C_p ||f||_{L^p(\mathbb{R})}$$

The demostration will be omitted since is quite technical, it can be seen at [31] chapter 3.

Proposition 1.22 (**Hilbert transform under Fourier transform**) *For $f \in L^2(\mathbb{R})$ the Hilbert transform satisfies*

$$\mathcal{F}(Hf)(\xi) = -\mathrm{i}\,\mathrm{sgn}(\xi)\hat{f},$$

where

$$\mathrm{sgn}(\xi) = \begin{cases} 1 & \text{if} \quad \xi > 0 \\ 0 & \text{if} \quad \xi = 0 \\ -1 & \text{if} \quad \xi < 0 \end{cases}$$

.

Proof. Formally

$$\begin{aligned} \mathcal{F}(Hf)(\xi) &= \mathcal{F}(P.V.\frac{1}{\pi x} * f)(\xi) \\ &= \frac{1}{\pi}\mathcal{F}(\frac{1}{x})(\xi)\hat{f}(\xi) \\ &= -\mathrm{i}\,\mathrm{sgn}(\xi)\hat{f}(\xi). \end{aligned}$$

One of the details in this proof is the use of the convolution property in a convolution with principal value, this is basically an exchange of limit with an integration, which is a valid exchange since we can use the dominated convergence theorem.

The other detail is in the calculation of the Fourier transform of $1/x$, to this we need to know that $\int_{\mathbb{R}} \sin(x)/x \; \mathrm{d}x = \lim_{R\to\infty}\int_{(-R,R)} \sin(x)/x \; \mathrm{d}x = \pi$, since it follows straight forward that

$$\begin{aligned}\mathcal{F}\left(\frac{1}{x}\right)(\xi) &= \int_{\mathbb{R}} e^{ix\xi}\frac{1}{x}dx \\ &= \int_{\mathbb{R}} \cos(x\xi)/x dx - i\int_{\mathbb{R}} \sin(x\xi)/x dx \\ &= 0 - i\int_{\mathbb{R}} \sin(x\xi)/(x\xi)(\xi dx) \\ &= -i\pi\,\mathrm{sgn}(\xi).\end{aligned}$$

□

Corollary 1.23 (**Hilbert transform in $H^s(\mathbb{R})$ spaces**) *Consider $f \in H^s(\mathbb{R})$ with $s \in \mathbb{R}$ then $Hf \in H^s(\mathbb{R})$ and*

$$\|f\|_{H^s(\mathbb{R})} = \|Hf\|_{H^s(\mathbb{R})}.$$

Proof. Recalling the definition of the $H^s(\mathbb{R})$ norm,

$$\begin{aligned}\|Hf\|_{H^s(\mathbb{R})} &= \int_{\mathbb{R}} (1+|\xi|^2)^s|\hat{H}f(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}} (1+|\xi|^2)^s|-i\,\mathrm{sgn}(\xi)\hat{f}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}} (1+|\xi|^2)^s|\hat{f}(\xi)|^2 d\xi = \|f\|_{H^s(\mathbb{R})}.\end{aligned}$$

□

Proposition 1.24 (**Conmutativity of the Hilbert trasnform with derivatives**) *Consider $f \in H^1(\mathbb{R})$ then*

$$\frac{d}{dx}Hf = H\frac{d}{dx}f \qquad \text{a.e. in } \mathbb{R}.$$

Proof. It is just necessary to take the Fourier transform and to use its inyectivity in L^2 to conclude,

$$\begin{aligned}\mathcal{F}(\frac{d}{dx}Hf)(\xi) &= i\xi(-i)\,\mathrm{sgn}(\xi)\hat{f}(\xi) \\ &= -i\,\mathrm{sgn}(\xi)i\xi\hat{f}(\xi) \\ &= \mathcal{F}(H\frac{d}{dx}f)(\xi).\end{aligned}$$

□

1.2 The Radon Transform

1.2.1 Radon Transform

This transform is the most important mathematical object in this thesis. It is the simplest form of the attenuated Radon Transform, which we will present in detail in the next chapters. This transform and its invertibility was first studied by the Austrian mathematician Johann Radon in 1917, for further references see [19].

Definition 1.25 (**Radon transform**) *Given* $f \in \mathcal{S}(\mathbb{R}^n)$*, we define*

$$Rf[\rho,\theta] = \int_{x\cdot\theta=\rho} f(x)\mathrm{d}l(x), \qquad \theta \in S^{n-1}, \rho \in \mathbb{R},$$

where $x\cdot\theta = \sum_{\mathrm{i}=1}^{n} x_{\mathrm{i}}\theta_{\mathrm{i}}$ *is the classical product in* $\mathbb{R}^n$ *and* $\mathrm{d}l(x)$ *is the Lebesgue measure in the* $n-1$ *dimensional hyperplane* $\{x \in \mathbb{R}^n,\ x\cdot\theta = \rho\}$.

This transform corresponds to integrate over hiperplanes, orthogonal to θ and at a distance ρ from the origin. The main interest in developing a theory of this transformation is that it has many applications in tomography, some of these applications will be detailed in the next chapter.

Notation: since the Radon transform has 2 variables, whenever we take a convolution, a Hilbert transform or a Fourier transform, this will be done with respecto to the one dimentional spatial variable ρ for θ fixed.

Theorem 1.26 (**Fourier slice theorem**) *Consider* $f \in \mathcal{S}(\mathbb{R}^n)$ *then*

$$\mathcal{F}(Rf)(\rho,\theta) = \hat{f}(\rho\theta), \qquad \theta \in S^{n-1}, \rho \in \mathbb{R}.$$

PROOF.

$$\begin{aligned}\mathcal{F}(Rf)(\rho,\theta) &= \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}s\rho}[Rf](s,\theta)\mathrm{d}s \\ &= \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}s\rho} \int_{x\cdot\theta=\rho} f(x)\mathrm{d}l(x)\mathrm{d}s \\ &= \int_{\mathbb{R}}\int_{x\cdot\theta=\rho} \mathrm{e}^{-\mathrm{i}(x\cdot\theta)\rho} f(x)\mathrm{d}l(x)\mathrm{d}s \\ &= \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}(x\cdot\theta)\rho} f(x)\mathrm{d}x \qquad (1.1) \\ &= \mathcal{F}(f)(\rho\theta).\end{aligned}$$

□

Proposition 1.27 (**Partial derivative of the Radon transform**) *Let $f \in \mathcal{S}(\mathbb{R}^n)$ be a smooth function, then*

$$\theta_j \partial_\rho Rf(\rho,\theta) = R[\partial_{x_j} f](\rho,\theta) \quad j \in \{1,2,...n\}.$$

Proof. Applicating the Fourier transform to the left hand term and using the Fourier slice theorem 1.26 yields

$$\begin{aligned}\mathcal{F}(\theta_j \partial_\rho Rf(\rho,\theta)) &= \theta_j \mathrm{i}\xi \mathcal{F}(Rf(\xi,\theta)) \\ &= \theta_j \mathrm{i}\xi \hat{f}(\xi\theta), \quad \forall \xi \in \mathbb{R}, \theta \in S^{n-1}.\end{aligned}$$

And doing the same to the right hand term

$$\begin{aligned}\mathcal{F}(R[\partial_{x_j} f](\xi,\theta)) &= \widehat{\partial_{x_j} f}(\xi\theta) \\ &= \mathrm{i}(\xi\theta)_j \hat{f}(\xi\theta) \\ &= \mathrm{i}\xi\theta_j \hat{f}(\xi\theta), \quad \forall \xi \in \mathbb{R}, \theta \in S^{n-1}.\end{aligned}$$

We conclude using the inyectivity of the Fourier transform. □

Corollary 1.28 *For $f \in \mathcal{S}(\mathbb{R}^n)$ we have that*

$$\partial_\rho^2 Rf(\rho,\theta) = R[\Delta f](\rho,\theta).$$

This corollary follows directly from the previous proposition. One of its principal advantages is that it can transform n-dimensional PDE's to 1 dimensional PDE's.

Definition 1.29 (**Formal Transpose of R, R^***) *For $g(\rho,\theta) \in C_0^\infty(\mathbb{R} \times S^{n-1})$, we define*

$$R^* g(x) = \int_{S^{n-1}} g(x \cdot \theta, \theta) \mathrm{d}\theta.$$

The formal adjoint of the Radon transform that satisfies

$$\langle Rf, g \rangle_{L^2(\mathbb{R}\times S^{n-1})} = \langle f, R^* g \rangle_{L^2(\mathbb{R}^n)}.$$

Proof. Let $f \in C_0^\infty(\mathbb{R}^n)$ and $g \in C_0^\infty(\mathbb{R} \times S^{n-1})$, then

$$\begin{aligned}\langle Rf, g \rangle_{L^2(\mathbb{R}\times S^{n-1})} &= \int_{S^{n-1}} \int_{\mathbb{R}} Rf(\rho,\theta) g(\rho,\theta) \mathrm{d}\rho \mathrm{d}\theta \\ &= \int_{S^{n-1}} \int_{\mathbb{R}} \int_{x\cdot\theta=\rho} f(x) g(\rho,\theta) \mathrm{d}l(x) \mathrm{d}\rho \mathrm{d}\theta \\ &= \int_{S^{n-1}} \int_{\mathbb{R}^n} f(x) g(x\cdot\theta,\theta) \mathrm{d}x \mathrm{d}\theta \\ &= \int_{\mathbb{R}^n} f(x) \int_{S^{n-1}} g(x\cdot\theta,\theta) \mathrm{d}\theta \mathrm{d}x \\ &= \langle f, R^* g \rangle_{L^2(\mathbb{R}^n)}.\end{aligned}$$

□

Proposition 1.30 *Let $K \subset \mathbb{R}^n$ be a compact set and $f \in L^2(K)$, then $Rf(\cdot,\theta)$ is well defined as a $L^2(\mathbb{R})$ function for all θ and satisfies*

$$||Rf(\cdot,\theta)||_{L_2(\mathbb{R})} \le C(K)||f||_{L_2(R^n)} \quad \forall \theta \in S^{n-1},$$

with $C(K)$ a constant depending only in the compact set K.

PROOF. Let $g \in C_0^\infty(\mathbb{R}^n)$ with $\mathrm{supp}(g) \subset K$ and lets define the indicatrix function

$$\mathbb{1}_K(x) = \begin{cases} 1 & \text{if} \quad x \in K \\ 0 & \text{if} \quad x \notin K \end{cases},$$

then

$$\begin{aligned} ||Rg(s,\theta)||^2_{L_2(\mathbb{R})} &= \int_{\mathbb{R}} \left(\int_{x\cdot\theta=s} g(x)\mathrm{d}l(x) \right)^2 \mathrm{d}s \\ &= \int_{\mathbb{R}} \left(\int_{x\cdot\theta=s} g(x)\mathbb{1}_K(x)\mathrm{d}l(x) \right)^2 \mathrm{d}s \\ &\le \int_{\mathbb{R}} \left(\int_{x\cdot\theta=s} (g(x))^2\mathrm{d}l(x) \right)\left(\int_{x\cdot\theta=s} \mathbb{1}_K^2(x)\mathrm{d}l(x) \right) \mathrm{d}s \\ &\le \mathrm{diam}(K)^{n-1} \int_{\mathbb{R}} \int_{x\cdot\theta=s} (g(x))^2\mathrm{d}l(x)\mathrm{d}s \\ &= \mathrm{diam}(K)^{n-1} \int_{\mathbb{R}^2} (g(x))^2\mathrm{d}x \\ &= \mathrm{diam}(K)^{n-1} ||g||^2_{L_2(\mathbb{R}^2)}, \end{aligned}$$

and thus by density of $C_0^\infty(K)$ in $L^2(K)$ we can conclude by extending the definition of the Radon transform for $f \in L^2(K)$. □

Corollary 1.31 *Let $K \subset \mathbb{R}^n$ be a compact set and $f \in H^k(K), k \ge 0$, then*

$$||Rf(\cdot,\theta)||_{H^k(\mathbb{R})} \le C(K)||f||_{H^k(\mathbb{R}^n)} \quad \forall \theta \in S^{n-1},$$

with $C(K)$ a constant depending only in the compact K.

PROOF. Lets us prove it for k=1. We already have the inequality for the $L^2(\mathbb{R})$ norm, hence let us prove it for the seminorm. Let $g \in C_0^\infty(K)$, using proposition 1.27 we have that

$$\begin{aligned} ||\theta_i\partial_s Rg(\cdot,\theta)||_{L^2(\mathbb{R})} &= ||R[\partial_{x_i} g](\cdot,\theta)||_{L^2(\mathbb{R})} \\ &\le C(K)||\partial_{x_i} g||_{L^2(\mathbb{R}^n)} \\ &\le C(K)||g||_{H^1(\mathbb{R}^n)}. \end{aligned}$$

Thus we have that

$$\begin{aligned}
\|\partial_s Rg(\cdot,\theta)\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |\partial_s Rg(s,\theta)|^2 \mathrm{d}s \\
&= \int_{\mathbb{R}} \sum_{\mathrm{i}=1}^{n} \theta_{\mathrm{i}}^2 |\partial_s Rg(s,\theta)|^2 \mathrm{d}s \\
&= \sum_{\mathrm{i}=1}^{n} \int_{\mathbb{R}} |\theta_{\mathrm{i}} \partial_s Rg(s,\theta)|^2 \mathrm{d}s \\
&= \sum_{\mathrm{i}=1}^{n} \|\theta_{\mathrm{i}} \partial_s Rg(\cdot,\theta)\|^2_{L^2(\mathbb{R})} \\
&\leq \sum_{\mathrm{i}=1}^{n} C(K)\|g\|^2_{H^1(\mathbb{R}^n)} \\
&\leq nC(K)\|g\|^2_{H^1(\mathbb{R}^n)}.
\end{aligned}$$

□

Thus by density we conclude the result for $k = 1$. Now using induction assume we have the property for all $f \in H^k(K)$, then if we take $g \in H^{k+1}(K)$ we have that $\partial_{x_{\mathrm{i}}} g \in H^k(K)$ and thus

$$\|R[\partial_{x_{\mathrm{i}}} g](\cdot,\theta)\|_{H^k(\mathbb{R})} \leq C(K)\|\partial_{x_{\mathrm{i}}} g\|_{H^k(\mathbb{R}^n)} \leq C(K)\|g\|_{H^{k+1}(\mathbb{R}^n)}$$

We conclude the result for all $k \in \mathbb{N}$.

Theorem 1.32 (**Inverse of the Radon Transform**) *Let $f \in C_0^\infty(\mathbb{R}^n)$ be a smooth function, then*

$$f(x) = c_n \begin{cases} R^* H \partial_\rho^{n-1} Rf(x) & \text{if } n \text{ is even} \\ R^* \partial_\rho^{n-1} Rf(x) & \text{if } n \text{ is odd} \end{cases}$$

$$\text{with } c_n = \frac{1}{2}(2\pi)^{1-n} \begin{cases} (-1)^{(n-2)/2} & \text{if } n \text{ is even} \\ (-1)^{(n-1)/2} & \text{if } n \text{ is odd} \end{cases},$$

where ∂_ρ is the derivative taken in the spatial variable of Rf.

Proof. Consider $f \in C_0^\infty$, then

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\xi} \hat{f}(\xi) \mathrm{d}\xi$$

changing to polar coordinates, choosing $\rho = |\xi|$ and $\theta = \xi/|\xi|$

$$f(x) = \frac{1}{(2\pi)^n} \int_0^\infty \int_{S^{n-1}} \mathrm{e}^{\rho\theta\cdot x} \hat{f}(\rho\theta) \rho^{n-1} \mathrm{d}\theta \mathrm{d}\rho$$

Now we want to extend the integral from $[0,\infty]$ to $\mathbb{R}$, to do this we need to separate in cases depending on the dimension of the space.

- n **odd case:**
  We can extend the integral without problems, because ρ^{n-1} does not change the sign, and the $\rho\theta$ product is not an issue since we can just flip θ without changing the integration set (since it is symmetric).

$$\begin{aligned} f(x) &= \frac{1}{2(2\pi)^n} \int_{\mathbb{R}} \int_{S^{n-1}} \mathrm{e}^{\mathrm{i}\rho\theta\cdot x} \hat{f}(\rho\theta)\rho^{n-1} \mathrm{d}\theta \mathrm{d}\rho \\ &= \frac{1}{2(2\pi)^n} \int_{\mathbb{R}} \int_{S^{n-1}} \mathrm{e}^{\mathrm{i}\rho\theta\cdot x} \frac{1}{\mathrm{i}^{n-1}} \mathcal{F}_\rho[\partial_\rho^{n-1} Rf](\rho,\theta) \mathrm{d}\rho \mathrm{d}\theta \\ &= \frac{\mathrm{i}^{n-1}}{2(2\pi)^{n-1}} \int_{S^{n-1}} \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\rho\theta\cdot x} \mathcal{F}_\rho[\partial_\rho^{n-1} Rf](\rho,\theta) \\ &= \frac{\mathrm{i}^{n-1}}{2(2\pi)^{n-1}} \int_{S^{n-1}} \partial_\rho^{n-1} Rf(\theta\cdot x,\theta) \mathrm{d}\theta \\ &= c_n R^* \partial_\rho^{n-1} Rf(x). \end{aligned}$$

- n **even case:**
  Now we have a change of sign to extend the integral, to fix this we use the $\mathrm{sgn}(x)$ function, and we replicate the last procedure, the only change is the appearance of the Hilbert transform when we introduce the sign function into the Fourier transform.

$$\begin{aligned} f(x) &= \frac{1}{2(2\pi)^n} \int_{\mathbb{R}} \int_{S^{n-1}} \mathrm{e}^{\mathrm{i}\rho\theta\cdot x} \hat{f}(\rho\theta)\,\mathrm{sgn}(\rho)\rho^{n-1} \mathrm{d}\theta \mathrm{d}\rho \\ &= \frac{1}{2(2\pi)^n} \int_{\mathbb{R}} \int_{S^{n-1}} \mathrm{e}^{\mathrm{i}\rho\theta\cdot x} \mathrm{i}\mathcal{F}_\rho[H_\rho(Rf)](\rho,\theta)\rho^{n-1} \mathrm{d}\theta \mathrm{d}\rho \\ &= \frac{1}{2(2\pi)^n} \int_{\mathbb{R}} \int_{S^{n-1}} \mathrm{e}^{\mathrm{i}\rho\theta\cdot x} \frac{\mathrm{i}}{\mathrm{i}^{n-1}} \mathcal{F}_\rho[\partial_\rho^{n-1} H_\rho(Rf)](\rho,\theta) \mathrm{d}\theta \mathrm{d}\rho \\ &= \frac{\mathrm{i}^{n-2}}{2(2\pi)^{n-1}} R^* H_\rho \partial_\rho^{n-1} Rf(x) \end{aligned}$$

□

1.2.2 Beam Transform

This transform is useful to define the attenuated Radon transform and has many properties that will be used in the next chapters

Definition 1.33 (**Beam Transform**) *Let $a \in \mathcal{C}^0(\mathbb{R}^n)$ be a continuous function with compact support, then the beam transform is the integral from one point $x \in \mathbb{R}^n$ in a straight line in the $\theta \in S^{n-1}$ direction.*

$$(Ba)(x,\theta) = \int_0^\infty a(x+t\theta)\mathrm{d}t \quad x \in \mathbb{R}^n, \theta \in S^{n-1}.$$

Proposition 1.34 *Let $R > 0$ be a positive number such that $\forall x \in \mathrm{supp}(a),\ |x| \le R$, then :*

1. $\theta \cdot \partial_x Ba = -a$,

2. $Ba = 0$ *for* $x \cdot \theta > R$,

3. $Ba = R_0[a]$ *for* $x \cdot \theta < -R$.

Proof. The first one is obtained by cosidering $g(t) = a(x + t\theta)$, then $\frac{\mathrm{d}}{\mathrm{d}t} g(t) = \theta \cdot \partial_x a(x + t\theta)$ thus the result is obtained since a has compact support so we can exchange the derivative with the integral. We conclude by using the fundamental theorem of calculus and that $a(x + t\theta)|_{t=\infty} = 0$.

The second one is obtained because a has compact support and because the Beam Transform takes into account the mass from a point and moving forward (θ direction) if $x \cdot \theta > R$ the integral will be out of the support of a. For the third one the explanation is similar. □

Proposition 1.35 *Let $a \in C^\alpha(\mathbb{R}^n)$ with compact support K, then $Ba(x,\theta) \in C^\alpha(\mathbb{R}^n \times S^{n-1})$ with $|Ba|_{C^\alpha} \le C(K)|a|_{C^\alpha}$, where $C(K)$ stands for a constant depending only in the support.*

Proof.

$$\begin{aligned} |Ba(x,\theta) - Ba(y,\theta)| &\le \int_0^\infty |a(x + t\theta) - a(y + t\theta)|\mathrm{d}t \\ &\le \int_0^\infty |a(x + t\theta) - a(y + t\theta)|(\mathbb{1}_K(x + t\theta) + \mathbb{1}_K(y + t\theta))\mathrm{d}t \\ &\le \int_0^\infty |a|_{C^\alpha}|x + \cancel{t\theta} - y - \cancel{t\theta}|^\alpha(\mathbb{1}_K(x + t\theta) + \mathbb{1}_K(y + t\theta))\mathrm{d}t \\ &= |a|_{C^\alpha}|x - y|^\alpha \int_0^\infty (\mathbb{1}_K(x + t\theta) + \mathbb{1}_K(y + t\theta))\mathrm{d}t \\ &\le 2|a|_{C^\alpha} \cdot \mathrm{diam}(K)|x - y|^\alpha. \end{aligned}$$

□

Proposition 1.36 *Let $a \in \mathcal{C}^0(\mathbb{R}^n)$, with compact support K then*

$$|Ba(x,\theta)| \le C_K\|a\|_\infty \qquad \forall x \in \mathbb{R}^n, \forall \theta \in S^{n-1}.$$

Proof.

$$\begin{aligned} |Ba(x,\theta)| &= |\int_0^\infty a(x + t\theta)\mathrm{d}t| \\ &\le \|a\|_\infty |\int_0^\infty \mathbb{1}_K(x + t\theta)\mathrm{d}t| \\ &\le \mathrm{diam}(K)\|a\|_\infty. \end{aligned}$$

□

Proposition 1.37 *Let $a \in H^1(\mathbb{R}^2) \cap \mathcal{C}^0(\mathbb{R}^2)$ with compact support K, then*

$$|Ba(x,\theta)| \leq C\|a\|_{H^1(\mathbb{R})} \quad \forall x \in \mathbb{R}^2, \theta \in S^1.$$

Proof. Set $x = s\theta + \tau\theta^\perp$ then

$$\begin{aligned}|Ba(x,\theta^\perp)| &= \left|\int_0^\infty a(s\theta + (\tau+t)\theta^\perp)\mathrm{d}t\right| \\ &\leq \int_{\mathbb{R}} |a(s\theta + t\theta^\perp)|\mathrm{d}t \\ &= R[\ |a|\ ](s,\theta).\end{aligned}$$

Using the Sobolev embbeding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ we obtain

$$|Ba(x,\theta^\perp)| \leq C\|R[\ |a|\ ](\cdot,\theta)\|_{H^1(\mathbb{R})} \quad \forall x \in \mathbb{R}^2, \theta \in S^1$$

And using Corollary 1.31 we get

$$|Ba(x,\theta^\perp)| \leq C(K)\|\ |a|\ \|_{H^1(\mathbb{R}^2)} \quad \forall x \in \mathbb{R}^2, \theta \in S^1$$

And finally, we use that $\|\ |a|\ \|_{H^1(\mathbb{R}^n)} \leq \|a\|_{H^1(\mathbb{R}^2)}$, this property can be found in [12] in page 152, thus

$$|Ba(x,\theta^\perp)| \leq C(K)\|a\|_{H^1(\mathbb{R}^2)}, \quad \forall x \in \mathbb{R}^2, \forall\theta \in S^1.$$

□

1.2.3 Attenuated Radon transform

To define the attenuated Radon transform we will just consider the 2 dimensional case.

Definition 1.38 (**Attenuated Radon Transform (AtRT)**) *Let $a, f \in \mathcal{C}^0(\mathbb{R}^2)$ be continuous functions, then*

$$R_a f(s,\theta) = \int_{\mathbb{R}} f(s\theta^\perp + t\theta)\mathrm{e}^{-(Ba)(s\theta^\perp + t\theta,\theta)}\mathrm{d}t, \quad s \in \mathbb{R}, \theta \in S^1.$$

The AtRT can be understood as an integral of $f(x)$ in the line parallel to θ and at distance s from the origin, weighted with the exponential term $\mathrm{e}^{-(Ba)(s\theta^\perp + t\theta,\theta)}$ that basically integrates along the same line, but starting from $s\theta^\perp + t\theta$.

Observation Some authors define the AtRT as

$$\tilde{R}_a f(s,\theta) = \int_{x\cdot\theta = s} f(x)\mathrm{e}^{-(Ba)(x,\theta^\perp)}\mathrm{d}l(x), \quad s \in \mathbb{R}, \theta \in S^1$$

that is quite the same functional, related by $R_a f(s,\theta) = \tilde{R}_a f(-s, -\theta^\perp)$ or conversely $\tilde{R}_a f(s,\theta) = R_a f(-s,\theta^\perp)$

Theorem 1.39 (**Inverse of the attenuated Radon transform**) *Let $a, f \in \mathcal{C}^1(\mathbb{R}^2)$ be differentiable functions, then the next formula holds pointwise*

$$f(x) = \frac{1}{4\pi} \text{ Re div} \int_{S^1} \theta \mathrm{e}^{Ba(x,\theta^\perp)} (\mathrm{e}^{-h} H \mathrm{e}^{h} \tilde{R}_a f)(x \cdot \theta, \theta) \mathrm{d}\theta,$$

where $h(s,\theta) = \frac{1}{2}(I + \mathrm{i}H)Ra(s,\theta)$ and, as in the inverse of the Radon Transform, the Hilbert transform is taken in the spatial variable.

The deduction of this inversion formula can be found in [23].

Observation 1 This last inversion formula is for the $\tilde{R}_a f$ version of the AtRT, to obtain an inversion formula for $R_a f$ it is enough to define $\mathcal{G}g(s,\theta) = g(-s,\theta^\perp)$ and then the formula becomes

$$f(x) = \frac{1}{4\pi} \text{Re div} \int_{S^1} \theta \mathrm{e}^{Ba(x,\theta^\perp)} (\mathrm{e}^{-h} H \mathrm{e}^{h} \mathcal{G} R_a f)(x \cdot \theta, \theta) \mathrm{d}\theta.$$

Observation 2 This formula also holds in a distributional sense.

1.2.4 Weighted Radon Transform

This transform is a more general form of the Radon Transform that integrates in hyperplanes with an arbitrary weight function, we will only consider the two dimensional case here.

Definition 1.40 (**Weighted Radon transform**) *Let $f \in L^2(\mathbb{R}^2)$ be a function with compact support and $w \in \mathcal{C}^0(\mathbb{R}^2 \times S^1)$ a weight function, then*

$$I_w f(s,\theta) = \int_{\mathbb{R}} w(s\theta^\perp + t\theta, \theta) f(s\theta^\perp + t\theta) \mathrm{d}t.$$

Observation Given the required regularity for f, $I_w(\cdot,\theta)$ is well defined as an $L^2(\mathbb{R})$ function for all θ (the proof of this is analogous to the proof of Proposition 1.30).

Lemma 1.41 *Let $K \subset \mathbb{R}^2$ be a compact set, $f \in L^2(K)$ and $w \in \mathcal{C}^0(\mathbb{R}^n \times S^{n-1})$ a weight function, then there exists a compact $\tilde{K} \subset \mathbb{R}^2$ independent of θ such that*

$$I_w f(s,\theta) = \mathbb{1}_{\tilde{K}}(s) I_w f(s,\theta) \quad \forall s \in \mathbb{R}, \theta \in S^{n-1}.$$

Proof. Let $R > 0$ such that $K \subset B(0,R)$, then we have that $f(x) = \mathbb{1}_{B(0,R)}(x) f(x)$ thus

$$I_w f(s,\theta) = \int_{\mathbb{R}} w(s\theta^\perp + t\theta, \theta) f(s\theta^\perp + t\theta) \mathbb{1}_{B(0,R)}(s\theta^\perp + t\theta) \mathrm{d}t,$$

we can see that if $s > R$ then $\forall t \in \mathbb{R}$ $s\theta^\perp + t\theta \notin B(0,R)$, so

$$I_w f(s,\theta) = \mathbb{1}_{[-R,R]}(s) I_w f(s,\theta).$$

□

Theorem 1.42 *If* $0 \leq k < 1/2$, $k + 1/2 < \alpha \leq 1$, $f \in H^k(\mathbb{R}^2)$ *with compact support and* $w(x,\theta) \in C^\alpha(\mathbb{R}^2 \times S^1)$ *then*

$$||I_w f||_{H^{k+1/2}(\mathbb{R}\times S^1)} \leq C||w||_{C^\alpha(\mathbb{R}^2\times S^1)}||f||_{H^k(\mathbb{R}^2)}.$$

This is Theorem 1 in [26], we will omit the proof.

Chapter 2

Mathematical Models

2.1 Mathematical Models

The mathematical models that will be shown are stationary three dimentional models of photon propagation in space, considering attenuation, radiation and scattering properties of the medium. We will consider

$$u : \mathbb{R}^3 \times S^2 \to \mathbb{R}$$

as the intensity of photons in the spatial point $x \in \mathbb{R}^3$ travelling in the direction $\phi \in S^2$.

2.1.1 Free Transport Equation

In vacuum, photons propagate in a striaght line by conserving its intensity, which can be written as

$$\phi \cdot \nabla_x u(x, \phi) = 0 \quad \forall x \in \mathbb{R}^3, \phi \in S^2$$

where $\phi \cdot \nabla_x u(x, \phi)$ is the spatial directional derivative of the intensity of photons. All solutions of this equation satisfy

$$u(x + s\phi, \phi) = u(x, \phi), \qquad \forall s \in \mathbb{R}, x \in \mathbb{R}^3, \phi \in S^2$$

i.e. the intensity of photons travelling in direction ϕ remain constant along the line of direction ϕ.

2.1.2 Linear Transport Equation with attenuation and without source

Adding attenuation to the model, if photons pass through some medium, we expect the intensity of photons to decrease according to its attenuation properties. This can be written as

$$\phi \cdot \nabla_x u(x,\phi) = -a(x)u(x,\phi), \tag{2.1}$$

where $a(x) : \mathbb{R}^3 \to \mathbb{R}$ stands for the attenuation coefficient of the modeled element. We assume an isotropic medium (i.e. the attenuation map does not depend on the direction).

Proposition 2.1 $u(x,\phi) = C\mathrm{e}^{-\int_{-\infty}^{0} a(x+t\phi)\mathrm{d}t}$ *satisfies the equation 2.1 for all* $C \in \mathbb{R}$.

Proof. Given $x \in \mathbb{R}^3$, let $\{\phi, \phi_1^\perp, \phi_2^\perp\}$ be an orthonormal basis of $\mathbb{R}^3$, we can write x for unique $s, \tau_1, \tau_2 \in \mathbb{R}$ as $x = s\phi + \tau_1\phi_1^\perp + \tau_2\phi_2^\perp$, thus

$$\begin{aligned}
\phi \cdot \nabla_x u(x,\phi) &= \partial_s u(s\phi + \tau_1\phi_1^\perp + \tau_2\phi_2^\perp, \phi) \\
&= \partial_s \left(C\mathrm{e}^{-\int_{-\infty}^{0} a((s+t)\phi + \tau_1\phi_1^\perp + \tau_2\phi_2^\perp)\mathrm{d}t} \right) \\
&= \partial_s \left(C\mathrm{e}^{-\int_{-\infty}^{s} a(t\phi + \tau_1\phi_1^\perp + \tau_2\phi_2^\perp)\mathrm{d}t} \right) \\
&= \left(C\mathrm{e}^{-\int_{-\infty}^{s} a(t\phi + \tau_1\phi_1^\perp + \tau_2\phi_2^\perp)\mathrm{d}t} \right) \left(-a(s\phi + \tau_1\phi_1^\perp + \tau_2\phi_2^\perp) \right) \\
&= -\left(C\mathrm{e}^{-\int_{-\infty}^{0} a(x+t\phi)\mathrm{d}t} \right) a(x) \\
&= -u(x,\phi)a(x).
\end{aligned}$$

□

2.1.3 Linear Transport Equation with attenuation and source

To model the behavior of photons in presence of attenuation and photon-emiting sources, we consider the following equation

$$\begin{aligned}
\phi \cdot \nabla_x u(x,\phi) + a(x)u(x,\phi) &= f(x) \qquad \forall x \in \mathbb{R}^3, \phi \in S^2, \\
\lim_{t\to\infty} u(x - t\phi, \phi) &= 0 \qquad \forall x \in \mathbb{R}^3, \phi \in S^2,
\end{aligned} \tag{2.2}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ stands for the source map distribution, there is no angular dependence in this function since we assume it emits uniformly in all directions. The boundary condition states that there are no external radiation sources.

Proposition 2.2 *If the source map f and the attenuation map a are integrable in each line in $\mathbb{R}^3$ (i.e. $\int_{\mathbb{R}} |f(x + t\phi)|\mathrm{d}t < \infty \quad \forall x \in \mathbb{R}^3, \phi \in S^2$) the unique solution for the equation* (2.2) *is*

$$u(x,\phi) = \int_{-\infty}^{0} f(x + t\phi)\mathrm{e}^{-\int_t^0 a(x+s\phi)\mathrm{d}s}\mathrm{d}t.$$

PROOF.
Set $x = \eta\phi + \varphi$, with $\varphi \in \{w \in \mathbb{R}^3,\ w \cdot \theta = 0\}, \eta \in \mathbb{R}$, then $\phi \cdot \nabla_x u(x,\phi) = \partial_\eta u(\eta\phi + \varphi, \phi)$, and

$$\begin{aligned}
\phi \cdot \nabla_x u(x,\phi) &= \partial_\eta \left(\int_{-\infty}^{0} f((\eta + t)\phi + \varphi) \mathrm{e}^{-\int_t^0 a((\eta+s)\phi+\varphi)\mathrm{d}s} \mathrm{d}t \right) \\
&= \partial_\eta \left(\int_{-\infty}^{\eta} f(t\phi + \varphi) \mathrm{e}^{-\int_{t-\eta}^0 a((\eta+s)\phi+\varphi)\mathrm{d}s} \mathrm{d}t \right) \\
&= \partial_\eta \left(\int_{-\infty}^{\eta} f(t\phi + \varphi) \mathrm{e}^{-\int_t^\eta a(s\phi+\varphi)\mathrm{d}s} \mathrm{d}t \right)
\end{aligned}$$

Since

$$\partial_\eta \int_0^\eta g(t,\eta)\mathrm{d}t = g(\eta,\eta) + \int_0^\eta \partial_\eta g(t,\eta)\mathrm{d}t$$

we have that

$$\begin{aligned}
\phi \cdot \nabla_x u(x,\phi) &= f(\eta\phi + \varphi)\cancel{\mathrm{e}^{-\int_\eta^\eta a(s\phi+\varphi)\mathrm{d}s}} + \int_{-\infty}^{\eta} f(t\phi + \varphi) \mathrm{e}^{-\int_t^\eta a(s\phi+\varphi)\mathrm{d}s} (-a(\eta\phi + \varphi))\mathrm{d}t \\
&= f(x) - a(x)u(x,\phi).
\end{aligned}$$

There is uniqueness because taking $x = \eta\phi + \varphi$ and $\Phi(\eta) = u(\eta\phi + \varphi, \phi)$ reduces the equation to

$$\begin{gathered}
\partial_\eta \Phi(\eta) + a(\eta\phi + \varphi)\Phi(\eta) = f(\eta\phi + \varphi) \\
\lim_{\eta \to -\infty} \Phi(\eta) = 0,
\end{gathered}$$

which for each ϕ and φ is an ordinary differential equation and has unique solution. □

2.1.4 Radiative Transfer Equation (including Scattering)

Let $s(x,\phi,\phi')$ be a scattering kernel that gives us the distribution according to wich photons at the spatial point $x \in \mathbb{R}^3$, coming from direction $\phi \in S^2$ are scattered in the direction $\phi' \in S^2$. The equation that we use to model the propagation of photons with attenuation, source and scattering is, for all $x \in \mathbb{R}^3$ and $\phi \in S^2$

$$\begin{gathered}
\phi \cdot \nabla_x u(x,\phi) + a(x)u(x,\phi) + \int_{S^2} u(x,\phi)s(x,\phi,\phi')\mathrm{d}\phi' = f(x) + \int_{S^2} u(x,\phi'')s(x,\phi'',\phi)\mathrm{d}\phi'' \\
\lim_{t \to \infty} u(x - t\phi, \phi) = 0.
\end{gathered} \tag{2.3}$$

The term $\int_{S^2} u(x,\phi)s(x,\phi,\phi')\mathrm{d}\phi'$ corresponds to the effect of photons that are scattered away from the path defined by (x,ϕ), term $\int_{S^2} u(x,\phi'')s(x,\phi'',\phi)\mathrm{d}\phi''$ is the opposite, gamma

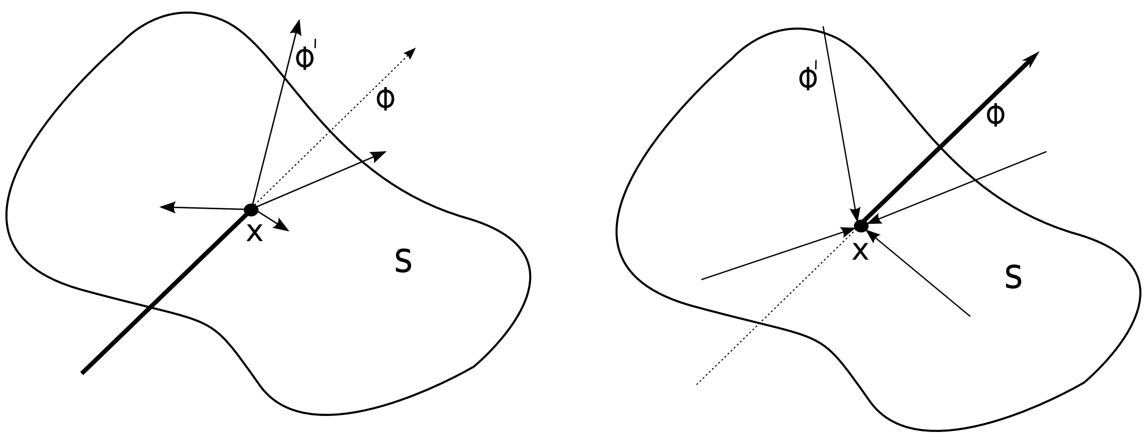

Figure 2.1: In both images the lines represent photons, the dotted line is the line defined by the point x and direction ϕ, the left handed image corresponds to a scattering effect in wich photons are scattered away from the path defined by x and ϕ and the right hand image corresponds to the oposite, photons that by scattering efects in x take the direction defined by the dotted line.

rays travelling in the spatial point $x \in \mathbb{R}^3$ but coming from any direction that by a scattering process take the path defined by (x, ϕ), see Figure 2.1

Assuming isotropy of the scattering kernel, we can write $s(x, \phi, \phi') = s(x, \phi \cdot \phi')$ i.e. the scattering process just depend on the angle at which photons are scattered, hence we can reduce the equation (2.3).

$$\phi \cdot \nabla_x u(x,\phi) + a(x)u(x,\phi) + \int_{S^2} u(x,\phi)s(x,\phi\cdot\phi')\mathrm{d}\phi' = f(x) + \int_{S^2} u(x,\phi'')s(x,\phi''\cdot\phi)\mathrm{d}\phi''$$

$$\Leftrightarrow \phi \cdot \nabla_x u(x,\phi) + a(x)u(x,\phi) + u(x,\phi)\int_{S^2} s(x,\phi\cdot\phi')\mathrm{d}\phi' = f(x) + \int_{S^2} u(x,\phi'')s(x,\phi''\cdot\phi)\mathrm{d}\phi''$$

$$\Leftrightarrow \phi \cdot \nabla_x u(x,\phi) + a_T(x)u(x,\phi) = f(x) + \int_{S^2} u(x,\phi'')s(x,\phi''\cdot\phi)\mathrm{d}\phi'' \qquad x \in \mathbb{R}^3, \phi \in S^2 \quad (2.4)$$

with $a_T(x) = a(x) + \int_{S^2} s(x,\phi\cdot\phi')\mathrm{d}\phi'$.

Let us define $u_{\mathrm{i}}(x,\phi)$ a function denoting the intensity of photons that have scattered i times, thus we can decompose the total intensity as

$$u(x,\phi) = \sum_{\mathrm{i}=0}^{\infty} u_{\mathrm{i}}(x,\phi),$$

(for further reference in this decomposition see [4] page 9) and equation (2.4) becomes system

$$\begin{aligned} \phi\cdot\nabla_x u_0(x,\phi)+a_T(x)u_0(x,\phi) &= f(x) && \forall x\in\mathbb{R}^3,\phi\in S^2\\ \phi\cdot\nabla_x u_{\mathrm{i}}(x,\phi)+a_T(x)u_{\mathrm{i}}(x,\phi) &= \int_{S^2} s(x,\phi\cdot\phi')u_{\mathrm{i}-1}(x,\phi')\mathrm{d}\phi' && \forall \mathrm{i}\geq 1, x\in\mathbb{R}^3,\phi\in S^2\\ \lim_{t\to\infty} u_{\mathrm{i}}(x-t\phi,\phi) &= 0, && \forall \mathrm{i}\geq 0, x\in\mathbb{R}^3,\phi\in S^2 \end{aligned} \tag{2.5}$$

and lastly we separate variables for the scattering kernel (i.e. $s(x,\theta\cdot\theta')=s(x)k(\theta\cdot\theta')$), we asume that the function $s(x)$ is proportional to the attenuation map (i.e. $\exists\tilde{C}$ such that $\tilde{C}s(x)=a(x)$) and for the angular variable we assume the scattering kernel is independent of the scattering angle (i.e. $k(\theta\cdot\theta')=1/4\pi \quad \forall\theta\cdot\theta'\in[0,1]$). With these asumptions we have that

$$a_T(x)=a(x)+\int_{S^2} s(x)k(\theta\cdot\theta')\mathrm{d}\phi' = s(x)(\tilde{C}+1)$$

thus defining $C=(4\pi(1+\tilde{C}))^{-1}$ the system (2.5) becomes

$$\begin{aligned} \phi\cdot\nabla_x u_0(x,\phi)+a_T(x)u_0(x,\phi) &= f(x) && \forall x\in\mathbb{R}^3,\phi\in S^2\\ \phi\cdot\nabla_x u_{\mathrm{i}}(x,\phi)+a_T(x)u_{\mathrm{i}}(x,\phi) &= Ca_T(x)\int_{S^2} u_{\mathrm{i}-1}(x,\phi')\mathrm{d}\phi' && \forall \mathrm{i}\geq 1, x\in\mathbb{R}^3,\phi\in S^2\\ \lim_{t\to\infty} u_{\mathrm{i}}(x-t\phi,\phi) &= 0. && \forall \mathrm{i}\geq 0, x\in\mathbb{R}^3,\phi\in S^2 \end{aligned} \tag{2.6}$$

Proposition 2.3 *If f and a are uniformly line integrable (i.e. $\exists D>0, \int_{\mathbb{R}}|f(x+t\phi)|\mathrm{d}t < D, \quad \forall x\in\mathbb{R}^3,\phi\in S^2$) The system (2.6) has as unique solution*

$$\begin{aligned} u_0(x,\phi) &= \int_{-\infty}^0 f(x+t\phi)\mathrm{e}^{-\int_t^0 a_T(x+s\phi)\mathrm{d}s}\mathrm{d}t\\ u_{\mathrm{i}}(x,\phi) &= C\int_{-\infty}^0 a_T(x+t\phi)\int_{S^2} u_{\mathrm{i}-1}(x+t\phi,\phi')\mathrm{d}\phi'\mathrm{e}^{-\int_t^0 a_T(x+s\phi)\mathrm{d}s}\mathrm{d}t. \end{aligned}$$

Observation The uniform line integrable condition can be obtained, for example, by considering $f,a\in C^0(K)$ with $K\subset\mathbb{R}^3$ some compact set.

Proof. To solve the system we use proposition 2.2 in each equation, it is just needed to prove that the functions $Ca(x)\int_{S^2} u_{\mathrm{i}}(x,\phi')\mathrm{d}\phi'$ are line integrable.

It is direct to see that the equation for u_0 satisfy the hypothesis since a and f are line integrable, hence, the solution formula is valid for u_0 and

$$\begin{aligned} |u_0(x,\phi)| &\leq \int_{-\infty}^0 \left|f(x+t\phi)\mathrm{e}^{-\int_t^0 a_T(x+s\phi)\mathrm{d}s}\right|\mathrm{d}t\\ &\leq \mathrm{e}^{\int_{\mathbb{R}}|a_T(x+s\phi)|\mathrm{d}s}\int_{\mathbb{R}}|f(x+t\phi)|\mathrm{d}t\\ &\leq D\mathrm{e}^D \qquad \forall x\in\mathbb{R}^3,\phi\in S^2, \end{aligned}$$

then

$$\begin{aligned}\int_{\mathbb{R}} |Ca(x+t\phi)\int_{S^2} u_0(x+t\phi,\phi')\mathrm{d}\phi'|\mathrm{d}t &\leq D\mathrm{e}^D \int_{\mathbb{R}} |a(x+t\phi)|\mathrm{d}t \\ &\leq D\mathrm{e}^D \\ &< \infty \quad \forall x \in \mathbb{R}^3, \phi \in S^2.\end{aligned}$$

Thus we have the uniform line integrability for the right hand term of the equation for u_1 and the solution formula is valid. We can iterate this steps by induction to prove that $\forall \mathrm{i} \geq 0$ the equations satisfy the hypothesis of Proposition 2.2. □

2.2 Inverse Problem Measurements

To establish the measurements that will be used for the inverse problem, we will assume compact support for the attenuation a and the source f, and for simplicity the T subscript in the a_T function will be omitted.

We will measure the information given by u_0 the ballistic photons, and u_1 the first order scattering photons at points outside of the support of a and f. Recalling proposition 2.3, assuming that we have the hypothesis to use it and defining $M[a,f](x) = \int_{S^2} u_0(x,\phi')\mathrm{d}\phi'$, we have the expressions

$$\begin{aligned}u_0(x,\phi) &= \int_{-\infty}^{0} f(x+t\phi)\mathrm{e}^{-\int_t^0 a(x+s\phi)\mathrm{d}s}\mathrm{d}t && \forall x \in \mathbb{R}^3, \phi \in S^2, \\ u_1(x,\phi) &= C\int_{-\infty}^{0} a(x+t\phi)M[a,f](x+t\phi)\mathrm{e}^{-\int_t^0 a(x+s\phi)\mathrm{d}s}\mathrm{d}t && \forall x \in \mathbb{R}^3, \phi \in S^2.\end{aligned}$$

The information of the photons corresponds to the measurement of u_0 and u_1 outside the support of the attenuation a and source map f, for each oriented line defined by $(x,\phi) \in \mathbb{R}^3 \times S^2$ (i.e. the set $\{x+t\phi,\ t \in \mathbb{R}\}$ but with orientation). Thus defining $\mathcal{A}_0$ and $\mathcal{A}_1$ as the measured information, they can be written as

$$\begin{aligned}\mathcal{A}_0(x,\phi) &= \int_{-\infty}^{\infty} f(x+t\phi)\mathrm{e}^{-\int_t^\infty a(x+s\phi)\mathrm{d}s}\mathrm{d}t && \forall x \in \mathbb{R}^3, \phi \in S^2, \\ \mathcal{A}_1(x,\phi) &= C\int_{-\infty}^{\infty} a(x+t\phi)M[a,f](x+t\phi)\mathrm{e}^{-\int_t^\infty a(x+s\phi)\mathrm{d}s}\mathrm{d}t && \forall x \in \mathbb{R}^3, \phi \in S^2.\end{aligned}$$

Observation 1 These function represents the measured information on each oriented line (i.e. $(x,\phi) \in \mathbb{R}^3 \times S^2$ define an oriented line), and thus there is no restriction to choose x and phi, because it is assumed that the measurements are taken outside the support of the attenuation and source maps.

Observation 2 Note that the oriented line defined by (x,ϕ) is the same as the one defined by $(x+\tau\phi,\phi)\quad\forall\tau\in\mathbb{R}$, hence, the measures are obtained by evaluating u_0 and u_1 in $(x+\tau\phi,\phi)$ and taking the limit $\tau\to\infty$.

Now we consider only horizontal measurements, this is the same as setting a height $z\in\mathbb{R}$ and taking data with planar angles $\phi=(\theta_1,\theta_2,0)^t$. Defining

$$f_z(y)=f(y,z),\quad a_z(y)=a(y,z),\quad M_z(x)=M(y,z)\quad\forall z\in\mathbb{R},y\in\mathbb{R}^2$$

we have that the observed data at height $z\in\mathbb{R}$ is

$$\begin{aligned}\mathcal{A}_0^z(y,\theta)&=\int_{-\infty}^{\infty}f_z(y+t\theta)\mathrm{e}^{-\int_t^\infty a_z(y+s\theta)\mathrm{d}s}\mathrm{d}t && \forall y\in\mathbb{R}^2,\theta\in S^1\\ \mathcal{A}_1^z(y,\theta)&=C\int_{-\infty}^{\infty}a_z(y+t\theta)M_z[a,f](y+t\theta)\mathrm{e}^{-\int_t^\infty a_z(y+s\theta)\mathrm{d}s}\mathrm{d}t && \forall y\in\mathbb{R}^2,\theta\in S^1\end{aligned}$$

Note that there is a redundancy in the data because each planar oriented line defined by $(y,\theta)\in\mathbb{R}^2\times S^1$ is equal to $(y+\tau\theta,\theta)$, thus to get rid of this redundancy we consider the oriented lines defined by $(s\theta^\perp,\theta),\quad s\in\mathbb{R},\theta\in S^1$ with $\theta^\perp$ a counterclockwise rotation of θ (i.e. if $\theta=(\theta_1,\theta_2)^t$ then $\theta^\perp=(\theta_2,-\theta_1)^t$, see figure 2.2). Thus, abusing the notation, we write the measurements at height z as

$$\begin{aligned}\mathcal{A}_0^z(s,\theta)&=\int_{-\infty}^{\infty}f_z(t\theta+s\theta^\perp)\mathrm{e}^{-\int_t^\infty a_z(\tau\theta+s\theta^\perp)\mathrm{d}\tau}\mathrm{d}t && \forall s\in\mathbb{R},\theta\in S^1,\\ \mathcal{A}_1^z(s,\theta)&=C\int_{-\infty}^{\infty}a_z(t\theta+s\theta^\perp)M_z[a,f](t\theta+s\theta^\perp)\mathrm{e}^{-\int_t^\infty a_z(\tau\theta+s\theta^\perp)\mathrm{d}\tau}\mathrm{d}t && \forall s\in\mathbb{R},\theta\in S^1.\end{aligned}$$

Recalling the definition of the Attenuated Radon transform in 1.38 and changing variables

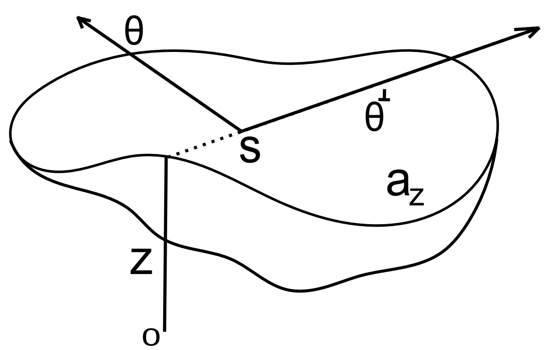

Figure 2.2: A representation of how the variables are choosed.

$\tau\to\tau+t$ we write the total information, at all heighs, for the inverse problems as

$$\begin{aligned}\mathcal{A}_0^z(s,\theta)&=R_{a_z}[f_z](s,\theta) && \forall z,s\in\mathbb{R},\theta\in S^1\\ \mathcal{A}_1^z(s,\theta)&=R_{a_z}[C\cdot a_z\cdot M_z[a,f]](s,\theta) && \forall z,s\in\mathbb{R},\theta\in S^1\end{aligned}\tag{2.7}$$

2.2.1 The $M[a, f](x)$ function

Recall the definition of $M[a, f](x)$

$$M[a, f](x) = \int_{S^2} u_0(x, \phi')\mathrm{d}\phi'.$$

This function is the sum of all the intensity of the ballistic photons with any direction in the spatial point x, since we have an explicit formula for u_0 we can express this function in terms of a and f

$$\begin{aligned} M[a, f](x) &= \int_{S^2} u_0(x, \phi')\mathrm{d}\phi' \\ &= \int_{S^2} \int_{-\infty}^{0} f(x + t\phi)\mathrm{e}^{-\int_t^0 a(x+s\phi)\mathrm{d}s}\mathrm{d}t\mathrm{d}\phi \\ &= \int_{S^2} \int_{0}^{\infty} f(x + t\phi)\mathrm{e}^{-\int_0^t a(x+s\phi)\mathrm{d}s}\mathrm{d}t\mathrm{d}\phi. \end{aligned}$$

Observation Note that $M[a, f](x)$ uses all the three dimentional information of a and f, even if we consider $M_z[a, f](y) = M[a, f](y, z), \quad y \in \mathbb{R}^2, z \in \mathbb{R}$.

Chapter 3

Theoretical analysis of the linearized inverse problem

The main objective in this chapter is to present a linear approximation of the measurements given by equations (2.7) and using that information, reconstruct the attenuation map a and source map f. More precisely, given the equations:

$$\begin{aligned} \mathcal{A}_0^z(s,\theta) &= R_{a_z}[f_z](s,\theta) & s,z \in \mathbb{R}, \theta \in S^1 \\ \mathcal{A}_1^z(s,\theta) &= R_{a_z}[C \cdot a_z \cdot M_z[a,f]](s,\theta) & s,z \in \mathbb{R}, \theta \in S^1 \end{aligned}$$

with

- $a, f : \mathbb{R}^3 \to \mathbb{R}$ the attenuation map and source map respectively, both with compact support.
- $a_z(y) = a(y,z),\ f_z(y) = f(y,z)$ with $y \in \mathbb{R}^2, z \in \mathbb{R}$, these are horizontal slices of a and f.
- $M[a,f](x) = \int_{S^2} \int_0^\infty f(x + t\phi) \mathrm{e}^{-\int_0^t a(x+s\phi)\mathrm{d}s} \mathrm{d}t\mathrm{d}\phi$ with $x \in \mathbb{R}^3$ and $M_z[a,f](y) = M[a,f](y,z) \quad \forall y \in \mathbb{R}^2, z \in \mathbb{R}$
- $R_{a_z}[f_z]$ denotes the AtRT of f_z with attenuation a_z.
- $\mathcal{A}_0^z$ and $\mathcal{A}_1^z$ are the information that can be obtained by measuring in the $\mathcal{P}_z = \{(y,z)^t \in \mathbb{R}^3,\ y \in \mathbb{R}^2\}$ horizontal plane, thus are known values.

The main objective is to linearize the measurements $(\mathcal{A}_0^z)_{z\in\mathbb{R}}$ and $(\mathcal{A}_1^z)_{z\in\mathbb{R}}$ around known attenuation and source, $\breve{a}$ and $\breve{f}$ respectively, and we will obtain an inversion formula for the resulting linear operator.

To simplify, we will consider $C = 1$, this is not an issue since the attenuated Radon transform is linear in that variable.

3.1 Linearization of the inverse problem

The information $\mathcal{A}_0^z$ and $\mathcal{A}_1^z$ will be linearly approximated for every $z \in \mathbb{R}$. This assumes the unknown values a and f are close in some sense to some given known source and att enuation $\breve{f}$ and $\breve{a}$, i.e.

$$f = \breve{f} + \delta f,$$
$$a = \breve{a} + \delta a,$$

with $\delta f, \delta a$ small in some sense that we will precise later, and

$$\mathcal{A}_0^z \approx R_{\breve{a}_z}[\breve{f}_z] + DR_{\breve{a}_z}[\breve{f}_z](\delta f_z, \delta a_z),$$
$$\mathcal{A}_1^z \approx R_{\breve{a}_z}[a_z \cdot M_z[\breve{a}, \breve{f}]] + D\left(R_{\breve{a}_z}[a_z \cdot M_z[\breve{a}, \breve{f}]]\right)(\delta f_z, \delta a_z).$$

Proposition 3.1 *The directional derivative of the attenuated Radon transform is given by*

$$DR_{\breve{a}_z}[\breve{f}_z](\delta a_z, \delta f_z) = I_{w[\breve{a}_z, \breve{f}_z]}[\delta a_z] + R_{\breve{a}_z}[\delta f_z],$$

with

$$w[\breve{a}_z, \breve{f}_z](y, \theta) = -\int_{-\infty}^{0} \mathrm{e}^{-B\breve{a}_z(y+\tau\theta,\theta)} \breve{f}_z(y + \tau\theta)\mathrm{d}\tau \qquad y \in \mathbb{R}^2, \theta \in S^1.$$

Proposition 3.2 *The directional derivative of the operator $R_{\breve{a}_z}[\breve{a}_z \cdot M_z[\breve{a}, \breve{f}]]$ that gives the first order scattering measures is given by*

$$DR_{\breve{a}_z}(\breve{a}_z \cdot M_z[\breve{a}, \breve{f}])(\delta a, \delta f) = I_{w[\breve{a}_z, \breve{a}_z \cdot M_z[\breve{a}, \breve{f}]]}\delta a_z + R_{\breve{a}_z}(\delta a_z \cdot M_z[\breve{a}, \breve{f}]) + R_{\breve{a}_z}(\breve{a}_z \cdot \partial_{\breve{a}} M_z[\breve{a}, \breve{f}]\delta a) + R_{\breve{a}_z}(\breve{a}_z \cdot M_z[\breve{a}, \delta f]),$$

with

$$\partial_{\breve{a}} M_z[\breve{a}, \breve{f}]\delta a(y) = -\int_{S^2} \int_0^{\infty} \breve{f}((y,z)^t + t\phi)\mathrm{e}^{-\int_0^t \breve{a}((y,z)^t + \tau\phi)\mathrm{d}\tau} \int_0^t \delta a((y,z)^t + s\phi)\mathrm{d}s\mathrm{d}t\mathrm{d}\phi \quad y \in \mathbb{R}^2.$$

Proof. Of Proposition 3.1
This proof was adapted from [29], let us consider the Beam transform

$$B\breve{a}_z(y, \theta) = \int_0^{\infty} \breve{a}_z(y + t\theta)\mathrm{d}t \quad y \in \mathbb{R}^2, \theta \in S^1.$$

Using Proposition 1.34 for $y = t\theta + s\theta^{\perp}$ and $\breve{a}_z$ with compact support contained in $B(0, R)$, with $R > 0$, gives:

1. $\theta \cdot \partial_y B\breve{a}_z(y, \theta) = \partial_t B\breve{a}_z(t\theta + s\theta^{\perp}, \theta) = -\breve{a}_z(t\theta + s\theta^{\perp})$,

2. $B\breve{a}_z(y,\theta) = 0$ for $y \cdot \theta > R$ then $B\breve{a}_z(t\theta + s\theta^\perp, \theta) = 0$ for $t > R$,

3. $B\breve{a}_z(y,\theta) = R_0[\breve{a}_z](y \cdot \theta^\perp, \theta)$ for $y \cdot \theta < -R$
then $B\breve{a}_z(t\theta + s\theta^\perp, \theta) = R_0[\breve{a}_z](s,\theta) = \int_{\mathbb{R}} \breve{a}_z(\tau\theta + s\theta^\perp)\mathrm{d}\tau$ for $t < -R$.

Consider

$$R_{\breve{a}_z+\eta\delta a_z}\breve{f}_z(s,\theta) = \int_{\mathbb{R}} \mathrm{e}^{-B[\breve{a}_z+\eta\delta a_z](t\theta+s\theta^\perp,\theta)}\breve{f}_z(t\theta + s\theta^\perp)\mathrm{d}t,$$

thus

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\Big|_{\eta=0} R_{\breve{a}_z+\eta\delta a_z}\breve{f}_z(s,\theta) = \int_{\mathbb{R}} \mathrm{e}^{-B\breve{a}_z(t\theta+s\theta^\perp,\theta)}\breve{f}_z(t\theta + s\theta^\perp)\left(-B[\delta a](t\theta + s\theta^\perp, \theta)\right)\mathrm{d}t.$$

Using Property 1.34.1, we write $\mathrm{e}^{-B\breve{a}_z(t\theta+s\theta^\perp)}\breve{f}_z(t\theta + s\theta^\perp) = -\partial_t B[\mathrm{e}^{-B\breve{a}_z(\cdot,\theta)}\breve{f}_z(\cdot)](t\theta + s\theta^\perp, \theta)$, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\Big|_{\eta=0} R_{\breve{a}_z+\eta\delta a_z}\breve{f}_z(s,\theta) = \int_{\mathbb{R}} \partial_t B[\mathrm{e}^{-B\breve{a}_z(\cdot,\theta)}\breve{f}_z(\cdot)](t\theta + s\theta^\perp, \theta)B[\delta a_z](t\theta + s\theta^\perp, \theta)\mathrm{d}t,$$

integrating by part we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\eta}\Big|_{\eta=0} R_{\breve{a}_z+\eta\delta a_z}\breve{f}_z(s,\theta) = &-\int_{\mathbb{R}} B[\mathrm{e}^{-B\breve{a}_z(\cdot,\theta)}\breve{f}_z(\cdot)](t\theta + s\theta^\perp, \theta)\partial_t B[\delta a_z](t\theta + s\theta^\perp, \theta)\mathrm{d}t \\ &+ B[\mathrm{e}^{-B\breve{a}_z(\cdot,\theta)}\breve{f}_z(\cdot)](t\theta + s\theta^\perp, \theta)B[\delta a_z](t\theta + s\theta^\perp, \theta)\Big|_{t=-\infty}^{t=\infty}\end{aligned}$$

and using all properties of Proposition 1.34

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\eta}\Big|_{\eta=0} R_{\breve{a}_z+\eta\delta a_z}\breve{f}_z(s,\theta) = &\int_{\mathbb{R}} B[\mathrm{e}^{-B\breve{a}_z(\cdot,\theta)}\breve{f}_z(\cdot)](t\theta + s\theta^\perp, \theta)\delta a_z(t\theta + s\theta^\perp)\mathrm{d}t \\ &- \int_{\mathbb{R}} \mathrm{e}^{-B\breve{a}_z(\tau\theta+s\theta^\perp,\theta)}\breve{f}_z(\tau\theta + s\theta^\perp)\mathrm{d}\tau \int_{\mathbb{R}} \delta a_z(t\theta + s\theta^\perp)\mathrm{d}t \\ = \int_{\mathbb{R}} \bigg(B[\mathrm{e}^{-B\breve{a}_z(\cdot,\theta)}\breve{f}_z(\cdot)](t\theta + s\theta^\perp, \theta) &- \int_{\mathbb{R}} \mathrm{e}^{-B\breve{a}_z(\tau\theta+s\theta^\perp,\theta)}\breve{f}_z(\tau\theta + s\theta^\perp)\mathrm{d}\tau \bigg) \delta a_z(t\theta + s\theta^\perp)\mathrm{d}t.\end{aligned}$$

Notice that

$$\begin{aligned}&B[\mathrm{e}^{-B\breve{a}_z(\cdot,\theta)}\breve{f}_z(\cdot)](t\theta + s\theta^\perp, \theta) - \int_{\mathbb{R}} \mathrm{e}^{-B\breve{a}_z(\tau\theta+s\theta^\perp,\theta)}\breve{f}_z(\tau\theta + s\theta^\perp)\mathrm{d}\tau \\ &= \int_0^\infty \mathrm{e}^{-B\breve{a}_z(t\theta+\tau\theta+s\theta^\perp,\theta)}\breve{f}_z(t\theta + \tau\theta + s\theta^\perp)\mathrm{d}\tau - \int_{\mathbb{R}} \mathrm{e}^{-B\breve{a}_z(\tau\theta+s\theta^\perp,\theta)}\breve{f}_z(\tau\theta + s\theta^\perp)\mathrm{d}\tau \\ &= -\int_{-\infty}^0 \mathrm{e}^{-B\breve{a}_z(t\theta+\tau\theta+s\theta^\perp,\theta)}\breve{f}_z(t\theta + \tau\theta + s\theta^\perp)\mathrm{d}\tau,\end{aligned}$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\Big|_{\eta=0} R_{\breve{a}_z+\eta\delta a_z}\breve{f}_z(s,\theta) = \int_{\mathbb{R}}\left(-\int_{-\infty}^{0} \mathrm{e}^{-B\breve{a}_z(t\theta+\tau\theta+s\theta^{\perp},\theta)}\breve{f}_z(t\theta+\tau\theta+s\theta^{\perp})\mathrm{d}\tau\right)\delta a_z(t\theta+s\theta^{\perp})\mathrm{d}t,$$

and defining $w[\breve{a}_z,\breve{f}_z](y,\theta) = -\int_{-\infty}^{0}\mathrm{e}^{-B\breve{a}_z(y+\tau\theta,\theta)}\breve{f}_z(y+\tau\theta)\mathrm{d}\tau$, we can write the directional derivative as a weighted Radon transform

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\Big|_{\eta=0} R_{\breve{a}_z+\eta\delta a_z}\breve{f}_z(s,\theta) = I_{w[\breve{a}_z,\breve{f}_z]}\delta a_z(s,\theta).$$

Finally, using the linearity of the operator in the other coordinate we obtain the directional derivative of the attenuated Radon transform at the point $(\breve{a},\ \breve{f})$ in the direction $(\delta a,\delta f)$:.

$$DR_{\breve{a}_z}[\breve{f}_z](\delta a_z,\delta f_z) = I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z] + R_{\breve{a}_z}[\delta f_z].$$

□

Proof. Of Proposition 3.2

We want to compute the derivate of the operator $R_{\breve{a}_z}[\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]$, thus we proceed by using proposition 3.1 and the chain rule after, denoting $\partial_{\breve{a}}(\cdot)\delta a$ as the directional derivative of the attenuation with direction δa, we obtain

$$\partial_{\breve{a}}(R_{\breve{a}_z}(\breve{a}_z\cdot M_z[\breve{a},\breve{f}]))\delta a = I_{w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]}\delta a_z + R_{\breve{a}_z}(\delta a_z\cdot M_z[\breve{a},\breve{f}]) + R_{\breve{a}_z}(\breve{a}_z\cdot\partial_{\breve{a}}M_z[\breve{a},\breve{f}]\delta a).$$

So we conclude that the directional derivative of the operator at the pair $\breve{a}$ and $\breve{f}$ in the direction $(\delta a,\delta f)$ is

$$\begin{aligned} DR_{\breve{a}_z}(\breve{a}_z\cdot M_z[\breve{a},\breve{f}])(\delta a,\delta f) =& I_{w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]}\delta a_z + R_{\breve{a}_z}(\delta a_z\cdot M_z[\breve{a},\breve{f}]) \\ &+ R_{\breve{a}_z}(\breve{a}_z\cdot\partial_{\breve{a}}M_z[\breve{a},\breve{f}]\delta a) + R_{\breve{a}_z}(\breve{a}_z\cdot M[\breve{a},\delta f]). \end{aligned}$$

The value of $\partial_{\breve{a}}M[\breve{a},\breve{f}]\delta a$ is obtained in a similar way. □

3.2 Analysis of the linear problem. Main results

Recapitulating, $\breve{a},\breve{f}$ are known functions and the goal is to obtain a and f by solving system (3.1) with $\delta a = \breve{a}-a$ and $\delta f = \breve{f}-f$ as unknowns (all these functions have compact support), denoting $\breve{M} = M_z[\breve{a},\breve{f}]$ the linearized system can be written as

$$\begin{aligned} \mathcal{A}_0^z - R_{\breve{a}_z}(\breve{f}_z) &\approx I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z] + R_{\breve{a}_z}[\delta f_z], \\ \mathcal{A}_1^z - R_{\breve{a}_z}(\breve{a}_z\cdot\breve{M}) &\approx I_{w[\breve{a}_z,\breve{a}_z\cdot\breve{M}_z]}\delta a_z + R_{\breve{a}_z}(\delta a_z\cdot\breve{M}_z) + R_{\breve{a}_z}(\breve{a}_z\cdot\partial_a\breve{M}_z\delta a) + R_{\breve{a}_z}(\breve{a}_z\cdot M_z[\breve{a},\delta f]). \end{aligned} \tag{3.1}$$

The first one is the information given by the ballistic measures, and the second one is the information given by the first order scattering measures, both measured at the horizontal plane P_z.

By composing with $R^{-1}_{\breve{a}_z}$ the terms at the system (3.1), yields

$$\begin{aligned} R^{-1}_{\breve{a}_z}\left(\mathcal{A}^z_0 - R_{\breve{a}_z}(\breve{f}_z)\right) &\approx R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z] + \delta f_z, \\ R^{-1}_{\breve{a}_z}\left(\mathcal{A}^z_1 - R_{\breve{a}_z}(\breve{a}_z \cdot \breve{M})\right) &\approx R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z,\breve{a}_z\cdot\breve{M}_z]}\delta a_z + \delta a_z \cdot \breve{M}_z + \breve{a}_z \cdot \partial_a \breve{M}_z \delta a + \breve{a}_z \cdot M_z[\breve{a},\delta f]. \end{aligned} \tag{3.2}$$

Let $K \subset \mathbb{R}$ be a compact set such that $\operatorname{supp}(\breve{f}) \cup \operatorname{supp}(\breve{a}) \cup \operatorname{supp}(\delta a) \cup \operatorname{supp}(\delta f) \subset K^3 = K \times K \times K$, notice this compact also satisfies $\operatorname{supp}(\breve{f}_z) \cup \operatorname{supp}(\breve{a}_z) \cup \operatorname{supp}(\delta a_z) \cup \operatorname{supp}(\delta f_z) \subset K^2 = K \times K \;\; \forall z \in \mathbb{R}$, and for every $z \in K^c$, $\breve{f}_z, \breve{a}_z, \delta a_z, \delta f_z = 0$.

Since we have compact support for all the terms in the system (3.2) except for the ones involving the $R^{-1}_{\breve{a}_z}$ operator, we can multiply both equations by a smooth cut-off function χ (independent of z) with compact support contained in $\tilde{K} \times \tilde{K}$ (with $\tilde{K}$ dependending on K) such that $\chi(y) = 1, \forall y \in K^2$ and $|\chi| \leq 1$, obtaining

$$\begin{aligned} \tilde{\mathcal{A}}^z_0 &= \chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z] + \delta f_z, \\ \tilde{\mathcal{A}}^z_1 &= \chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z,\breve{a}_z\cdot\breve{M}_z]}\delta a_z + \delta a_z \cdot \breve{M}_z + \breve{a}_z \cdot \partial_a \breve{M}_z \delta a + \breve{a}_z \cdot M_z[\breve{a},\delta f]. \end{aligned} \tag{3.3}$$

where $\tilde{\mathcal{A}}^z_0$ and $\tilde{\mathcal{A}}^z_1$ are defined in terms of the original data $\mathcal{A}^z_0$, $\mathcal{A}^z_1$, as

$$\begin{aligned} \chi R^{-1}_{\breve{a}_z}\mathcal{A}^z_0 - \breve{f}_z &\approx \tilde{\mathcal{A}}^z_0 \\ \chi R^{-1}_{\breve{a}_z}\mathcal{A}^z_1 - \breve{a}_z \cdot \breve{M}_z &\approx \tilde{\mathcal{A}}^z_1. \end{aligned}$$

Notice that this cut-off function satisfies $K^2 \subset \operatorname{supp}(\chi)$. Let us formally define for $z \in \mathbb{R}$

$$\begin{aligned} &L_z[\breve{a},\breve{f}], Q_z[\breve{a},\breve{f}] : L^2(\tilde{K}^3) \times L^2(\tilde{K}^3) \to L^2(\tilde{K}^2) \times L^2(\tilde{K}^2), \\ &L_z[\breve{a},\breve{f}](\delta a, \delta f) = \begin{pmatrix} \chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z] + \delta f_z \\ \delta a_z \cdot \breve{M}_z \end{pmatrix}, \\ &Q_z[\breve{a},\breve{f}](\delta a, \delta f) = \begin{pmatrix} 0 \\ \chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z,\breve{a}_z\cdot\breve{M}_z]}[\delta a_z] + (\breve{a}_z \cdot \partial_a \breve{M}_z \delta a) + (\breve{a}_z \cdot M_z[\breve{a},\delta f]) \end{pmatrix}. \end{aligned}$$

Observation Since $\delta a, \delta f \in L^2(K^3),\; K^3 \subset \mathbb{R}^3$, we have that a_z, f_z are defined for almost every $z \in \mathbb{R}$ and $a_z, f_z \in L^2(K^2)$, thus these operators L_z, Q_z are defined for almost every $z \in \mathbb{R}$.

With this notation, system (3.3), for $\delta a, \delta f \in L^2(\tilde{K}^3)$ can be written as

$$(L_z + Q_z)[\breve{a},\breve{f}](\delta a, \delta f) = \begin{pmatrix} \tilde{\mathcal{A}}^z_0 \\ \tilde{\mathcal{A}}^z_1 \end{pmatrix} \quad \text{a.e. } z \in \mathbb{R}.$$

Observation We are extending the posible values for $\delta a, \delta f$ by considering a bigger support, i.e. $\delta a, \delta f \in L^2(\tilde{K}^3)$, this is not an issue since $L^2(K) \subset L^2(\tilde{K})$.

Now defining the operators that gives all the measurements

$$\begin{aligned} L[\breve{a},\breve{f}], Q[\breve{a},\breve{f}] : L^2(\tilde{K}^3) \times L^2(\tilde{K}^3) \to L^2(\tilde{K}; L^2(\tilde{K}^2) \times L^2(\tilde{K}^2)), \\ L[\breve{a},\breve{f}](\delta a, \delta f) = \left(L_z[\breve{a},\breve{f}](\delta a, \delta f)\right)_{z\in\mathbb{R}}, \\ Q[\breve{a},\breve{f}](\delta a, \delta f) = \left(Q_z[\breve{a},\breve{f}](\delta a, \delta f)\right)_{z\in\mathbb{R}}. \end{aligned}$$

System (3.3), describing the model for the measured data, becomes

$$(L+Q)[\breve{a},\breve{f}](\delta a, \delta f) = \begin{pmatrix} \tilde{\mathcal{A}}_0^z \\ \tilde{\mathcal{A}}_1^z \end{pmatrix}_{z\in\mathbb{R}}. \tag{3.4}$$

Observation The image of L and Q will be justified later in Theorem 3.5.

Thus the linearization of our inverse problem, consisting in the reconstruction of both the attenuation and source maps, reduces to the invertibility of the operator $(L+Q)[\breve{a},\breve{f}]$.

Lemma 3.3 *Consider $\breve{a},\breve{f} \in C^0(K^3)$ if $\exists C > 0$ such that $M[\breve{a},\breve{f}](x) \geq C$ $\forall x \in \tilde{K}^3$ then the operator $L[\breve{a},\breve{f}]$ is left.invertible, with left inverse*

$$L^{-1}[\breve{a},\breve{f}] : L^2(\tilde{K}; L^2(\tilde{K}^2) \times L^2(\tilde{K}^2)) \to L^2(\tilde{K}; L^2(\tilde{K}^2) \times L^2(\tilde{K}^2)) \cong L^2(\tilde{K}^3) \times L^2(\tilde{K}^3) \tag{3.5}$$

$$L^{-1}[\breve{a},\breve{f}]\begin{pmatrix} g \\ h \end{pmatrix} = \left(h_z/\breve{M}_z,\ g_z - \chi R_{\breve{a}_z}^{-1} I_{w[\breve{a}_z,\breve{f}_z]}[h_z/\breve{M}_z]\right)_{z\in K} \tag{3.6}$$

Observation 1 A simple way to ensure that $\exists C > 0$ such that $M[\breve{a},\breve{f}](x) \geq C$ $\forall x \in \tilde{K}^3$ is considering $\breve{f} \geq 0$, $f \not\equiv 0$ (notice that since $\breve{a}$ and $\breve{f}$ are continuous, $M[\breve{a},\breve{f}]$ is also continuous and is defined everywhere).

Observation 2 The image and domain of this operator is justified later in Theorem 3.5.

Proof. Since $\breve{a},\breve{f} \in C^0(K^3)$, then $\breve{M}_z(y), \breve{a}_z(y)$ and $\breve{f}_z(y)$ are well defined for all $z \in K, y \in K^2$, now lets define $L_z^{-1}[\breve{a},\breve{f}]$ such that

$$\begin{aligned} &L_z^{-1}[\breve{a},\breve{f}] : L^2(\tilde{K}^2) \times L^2(\tilde{K}^2) \to L^2(\tilde{K}^2) \times L^2(\tilde{K}^2) \\ &L_z^{-1}[\breve{a},\breve{f}]\begin{pmatrix} g_z \\ h_z \end{pmatrix} = \left(h_z/\breve{M}_z, g_z - \chi R_{\breve{a}_z}^{-1} I_{w[\breve{a}_z,\breve{f}_z]}[h_z/\breve{M}_z]\right) \quad g_z, h_z \in L^2(\tilde{K}^2) \end{aligned}$$

Notice that since $M[\breve{a},\breve{f}](x) \geq C > 0$ $\forall x \in \tilde{K}^3$, there are no problems in dividing by $\breve{M}$. Set $z \in K$ such that $\delta a_z, \delta f_z$ are well defined (and thus $\delta a_z, \delta f_z \in L^2(K^2)$), then

$$
\begin{aligned}
L_z^{-1}[\breve{a},\breve{f}]L_z[\breve{a},\breve{f}](\delta a,\delta f) &= L_z^{-1}[\breve{a},\breve{f}]\begin{pmatrix} \chi R_{\breve{a}_z}^{-1} I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z] + \delta f_z \\ \delta a_z \cdot \breve{M}_z \end{pmatrix} \\
&= \left(\delta a_z \cdot \cancel{\breve{M}_z}/\breve{M}_z,\ \delta f_z + \cancel{R_{\breve{a}_z}^{-1} I_{w[\breve{a}_Z,\breve{f}_z]}[\delta a_Z]} - \cancel{R_{\breve{a}_z}^{-1} I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z \cdot \breve{M}_z/\breve{M}_z]} \right) \\
&= (\delta a_z, \delta f_z)_{z\in\mathbb{R}} \\
&= (\delta a, \delta f)
\end{aligned}
$$

thus, by defining

$$
L^{-1}[\breve{a},\breve{f}]\begin{pmatrix} g \\ h \end{pmatrix} = \left(L_z^{-1}\begin{pmatrix} g_z \\ h_z \end{pmatrix} \right)_{z\in\tilde{K}},
$$

we get that

$$
L^{-1}[\breve{a},\breve{f}]L[\breve{a},\breve{f}](\delta a,\delta f) = (\delta a_z,\delta f_z)_{z\in\tilde{K}} = (\delta a,\delta f).
$$

Where the last equality is a.e. $z \in \mathbb{R}$. □

Proof. **of the first observation of Lemma 3.3** If $\breve{f} \geq 0$, $\breve{f} \not\equiv 0$ then $\|\breve{f}\|_\infty > 0$ and there exists a set A with positive measure such that

$$
\breve{f}(x) \geq \mathbb{1}_A(x)\frac{\|\breve{f}\|_\infty}{2} \quad \forall x \in \tilde{K}^3
$$

$$
\begin{aligned}
\Rightarrow M[\breve{a},\breve{f}](x) &= \int_{S^2}\int_0^\infty \breve{f}(x+t\phi)\mathrm{e}^{-\int_0^t \breve{a}(x+s\phi)\mathrm{d}s}\mathrm{d}t\mathrm{d}\phi \\
&\geq \mathrm{e}^{-\mathrm{di}am(K^3)\|\breve{a}\|_\infty}\int_{S^2}\int_0^\infty \breve{f}(x+t\phi)\mathrm{d}t\mathrm{d}\phi \\
&\geq \mathrm{e}^{-\mathrm{di}am(K^3)\|\breve{a}\|_\infty}\frac{\|\breve{f}\|_\infty}{2}\int_{S^2}\int_0^\infty \mathbb{1}_A(x+t\phi)\mathrm{d}t\mathrm{d}\phi \\
&\geq \mathrm{e}^{-\mathrm{di}am(K^3)\|\breve{a}\|_\infty}\frac{\|\breve{f}\|_\infty}{2}\min_{x\in\tilde{K}^3}\int_{S^2}\int_0^\infty \mathbb{1}_A(x+t\phi)\mathrm{d}t\mathrm{d}\phi \\
&> 0.
\end{aligned}
$$

Where the minimum in x is attained because we are minimizing a continuous function over a compact set. □

Proposition 3.4 ***(zero attenuation case)***
If $\breve{a} = 0$, $\breve{f} \in C^0(K^3)$ and $\exists C > 0$ such that $M(x) \geq C \ \ \forall x \in \tilde{K}^3$, then the operator $Q[\breve{a},\breve{f}] = 0$ and thus $(L+Q)[\breve{a},\breve{f}]$ is invertible.

Proof. We already have the hypothesis to use Lemma 3.3 to invert $L[\breve{a},\breve{f}]$, to conclude we need to check that $Q[\breve{a},\breve{f}] = 0$, this is directly since $\breve{a} = 0 \Rightarrow \breve{a}_z = 0$, hence

$$
\breve{a}_z \cdot \partial_a \breve{M}_z \delta a = 0,
$$
$$
\breve{a}_z \cdot M_z[\breve{a},\delta f] = 0,
$$

and

$$w[\breve{a}_z, \breve{a}_z \cdot \breve{M}_z] = w[0,0] = 0,$$

$$\Rightarrow I_{w[0,0]}\delta a_z(s,\theta) = \int_{\mathbb{R}} w[0,0](s\theta^\perp + t\theta, \theta)\delta a_z(s\theta^\perp + t\theta)\mathrm{d}t = 0.$$

Since $Q_z[\breve{a}, \breve{f}]$ is the addition of all these vanishing terms, $Q_z[\breve{a}, \breve{f}] = 0$ and thus $Q[\breve{a}, \breve{f}] = \left(Q_z[\breve{a}, \breve{f}]\right)_{z\in K} = 0$. □

Theorem 3.5 *Let $K, \tilde{K} \subset \mathbb{R}$ be compact set such that $K \subset \tilde{K}$, $\breve{a} \in H^{5/2}(K^3)$, $\breve{f} \in C^\alpha(K^3)$ with $\alpha > 1/2$, $\breve{f} \geq 0$, $\breve{f} \not\equiv 0$ and $\chi \in C_0^\infty(\tilde{K}^2)$ whose norms depend only on K, then L^{-1} and Q are well defined linear operators in the Banach spaces*

$$L^{-1}[\breve{a}, \breve{f}] : L^2(\tilde{K}; L^2(\tilde{K}^2) \times L^2(\tilde{K}^2)) \to L^2(\tilde{K}; L^2(\tilde{K}^2) \times L^2(\tilde{K}^2))$$
$$Q[\breve{a}, \breve{f}] : L^2(\tilde{K}^3) \times L^2(\tilde{K}^3) \to L^2(\tilde{K}; L^2(\tilde{K}^2) \times L^2(\tilde{K}^2))$$

and there exists $D > 0$ such that the operator $(L+Q)[\breve{a}, \breve{f}]$ defined on $L^2(\tilde{K}^3) \times L^2(\tilde{K}^3)$ is invertible for all $\breve{a} \in H^{5/2}(K^3)$ satisfying $\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)} < D$. The inverse is given by

$$(L+Q)^{-1}[\breve{a}, \breve{f}] = L^{-1}[\breve{a}, \breve{f}] \sum_{k=0}^{\infty} \left(-(QL^{-1})[\breve{a}, \breve{f}]\right)^k.$$

3.3 Proof of the inversion theorem

The idea of this proof is a standar argument for a perturbation of an invertible linear operator. From now on will consider $\breve{a}$ and $\breve{f}$ with support contained in $\tilde{K}$, this does not change the Theorem since with our definition of the spaces $H^{5/2}(K^3) \subset H^{5/2}(\tilde{K}^3)$ and $C^\alpha(K^3) \subset C^\alpha(\tilde{K}^3)$.

3.3.1 Part 1: estimates for the operators $I_{w[\breve{a},\breve{f}]}$ and $I_{w[\breve{a},\breve{a}\cdot\breve{M}]}$

Lemma 3.6 *Let $\Omega \subset \mathbb{R}^n$ be any set and $f \in C^\alpha(\Omega)$. For each fixed $z \in \mathbb{R}$ define $f_z(y) = f(y,z), y \in \mathbb{R}^{n-1}$ and $\Omega_z = \{y \in \mathbb{R}^{n-1},\ (y,z) \in \Omega\}$ then $f_z \in C^\alpha(\Omega_z)$ and*

$$|f_z|_{C^\alpha(\mathbb{R}^{n-1})} \leq |f|_{C^\alpha(\mathbb{R}^n)},$$
$$\|f_z\|_{C^\alpha(\mathbb{R}^{n-1})} \leq \|f\|_{C^\alpha(\mathbb{R}^n)}.$$

Proof. It follows from the fact that $\|f_z\|_\infty \leq \|f\|_\infty$ and

$$\begin{aligned}|f_z(y_1) - f_z(y_2)| &= |f(y_1,z) - f(y_2,z)| \\ &\leq |f|_{C^\alpha}|(y_1,z) - (y_2,z)|^\alpha \\ &\leq |f|_{C^\alpha}|y_1 - y_2|^\alpha.\end{aligned}$$

□

Lemma 3.7 *Let $\Omega \subset \mathbb{R}^N$ be any set and $f_1, f_2 \in C^\alpha(\Omega)$, then $f_1 \cdot f_2 \in C^\alpha(\Omega)$ and*

$$|f_1 \cdot f_2|_{C^\alpha(\mathbb{R}^n)} \leq 2\|f_1\|_{C^\alpha(\mathbb{R}^n)}\|f_2\|_{C^\alpha(\mathbb{R}^n)},$$
$$\|f_1 \cdot f_2\|_{C^\alpha(\mathbb{R}^n)} \leq 3\|f_1\|_{C^\alpha(\mathbb{R}^n)}\|f_2\|_{C^\alpha(\mathbb{R}^n)}.$$

Proof.

$$\begin{aligned}|f_1f_2(x) - f_1f_2(y)| &= |f_1(x)(f_2(x) - f_2(y)) - f_2(y)(f_1(y) - f_1(x))| \\ &\leq \|f_1\|_\infty |f_2|_{C^\alpha}|x-y|^\alpha + \|f_2\|_\infty |f_1|_{C^\alpha}|x-y|^\alpha \\ &\leq 2\|f_1\|_{C^\alpha}\|f_2\|_{C^\alpha}|x-y|^\alpha,\end{aligned}$$

and since

$$\|f_1f_2\|_\infty \leq \|f_1\|_\infty\|f_2\|_\infty \leq \|f_1\|_{C^\alpha}\|f_2\|_{C^\alpha},$$

then

$$\|f_1f_2\|_{C^\alpha(\mathbb{R}^n)} \leq 3\|f_1\|_{C^\alpha(\mathbb{R}^n)}\|f_2\|_{C^\alpha(\mathbb{R}^n)}.$$

□

Lemma 3.8 *Let $\tilde{K} \subset \mathbb{R}$ be a compact set and $\breve{a}, \breve{f} \in C^\alpha(\tilde{K}^3)$, then $M[\breve{a}, \breve{f}] \in C^\alpha(\mathbb{R}^3)$ and*

$$\|M[\breve{a}, \breve{f}]\|_{C^\alpha(\mathbb{R}^3)} \leq C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}(1 + |\breve{a}|_{C^\alpha(\mathbb{R}^3)})\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)},$$

with $C(\tilde{K})$ a constant depending only in the compact $\tilde{K}$.

Proof. Recall the definition of M[$\breve{a}$,$\breve{f}$]

$$M[\breve{a}, \breve{f}](x) = \int_{S^2}\int_0^\infty \breve{f}(x + t\phi)\mathrm{e}^{-\int_0^t \breve{a}(x+s\phi)\mathrm{d}s}\mathrm{d}t\mathrm{d}\phi.$$

Let us define $\breve{a}_{t,\phi}(x) = \mathrm{e}^{-\int_0^t \breve{a}(x+s\phi)\mathrm{d}s}$, using the mean value theorem we obtain

$$\begin{aligned}|\breve{a}_{t,\phi}(x) - \breve{a}_{t,\phi}(y)| &\leq \mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left|\int_0^t -\breve{a}(x+s\phi) + \breve{a}(y+s\phi)\mathrm{d}s\right| \\ &\leq \mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\int_0^t |\breve{a}|_{C^\alpha}|x-y|^\alpha|\mathbb{1}_{\tilde{K}^3}(x+s\phi) + \mathbb{1}_{\tilde{K}^3}(y+s\phi)|\mathrm{d}s \\ &\leq C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}|\breve{a}|_{C^\alpha}|x-y|^\alpha,\end{aligned}$$

it is easy to verify that $|\breve{a}_{t,\phi}(x)| \leq \mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}$, thus

$$\|\breve{a}_{t,\phi}\|_{C^\alpha(\mathbb{R}^3)} \leq C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}(1 + |\breve{a}|_{C^\alpha(\mathbb{R})}).$$

By defining $\breve{f}_{t,\phi}(x) = \breve{f}(x+t\phi)$ we know it is a Hölder continuous function the with same Hölder norm as $\breve{f}$, hence

$$
\begin{aligned}
&|M[\breve{a},\breve{f}](x) - M[\breve{a},\breve{f}](y)| \\
&\qquad \le \int_{S^2}\int_0^\infty |\breve{f}_{t,\phi}\breve{a}_{t,\phi}(x) - \breve{f}_{t,\phi}\breve{a}_{t,\phi}(y)|\left(\mathbb{1}_{\tilde{K}^3}(x+t\phi) + \mathbb{1}_{\tilde{K}^3}(y+t\phi)\right)\mathrm{d}t\mathrm{d}\phi \\
&\qquad \le 2\int_{S^2}\int_0^\infty \|\breve{f}\|_{C^\alpha}\|\breve{a}_{t,\phi}\|_{C^\alpha}\left(\mathbb{1}_{\tilde{K}^3}(x+t\phi) + \mathbb{1}_{\tilde{K}^3}(y+t\phi)\right)\mathrm{d}t\mathrm{d}\phi|x-y|^\alpha \\
&\qquad \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}(1+|\breve{a}|_{C^\alpha})\|\breve{f}\|_{C^\alpha}\int_{S^2}\int_0^\infty \left(\mathbb{1}_{\tilde{K}^3}(x+t\phi) + \mathbb{1}_{\tilde{K}^3}(y+t\phi)\right)\mathrm{d}t\mathrm{d}\phi|x-y|^\alpha \\
&\qquad \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}(1+|\breve{a}|_{C^\alpha})\|\breve{f}\|_{C^\alpha}|x-y|^\alpha,
\end{aligned}
$$

since we have a uniform bound over M

$$
|M[\breve{a},\breve{f}](x)| \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\|\breve{f}\|_\infty \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\|\breve{f}\|_{C^\alpha},
$$

we conclude the result

$$
\|M[\breve{a},\breve{f}]\|_{C^\alpha(\mathbb{R}^3)} \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}(1+|\breve{a}|_{C^\alpha})\|\breve{f}\|_{C^\alpha}.
$$

□

Lemma 3.9 *Let $\tilde{K}\subset\mathbb{R}$ be a compact set and $\breve{a},\breve{f}\in C^\alpha(\tilde{K}^3)$ then $w[\breve{a}_z,\breve{a}_z\cdot\breve{M}_z], w[\breve{a}_z,\breve{f}_z]\in C^\alpha(\mathbb{R}^2\times S^1)$ and $\forall z\in\tilde{K}$*

$$
\begin{aligned}
\|w[\breve{a}_z,\breve{f}_z]\|_{C^\alpha(\mathbb{R}^2\times S^1)} &\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}(1+|\breve{a}|_{C^\alpha(\mathbb{R}^3)})\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}, \\
\|w[\breve{a}_z,\breve{a}_z\cdot\breve{M}_z]\|_{C^\alpha(\mathbb{R}^2\times S^1)} &\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}(1+|\breve{a}|_{C^\alpha(\mathbb{R}^3)})^2\|\breve{a}\|_{C^\alpha(\mathbb{R}^3)}\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)},
\end{aligned}
$$

with $C(\tilde{K})$ a constant depending only on the compact $\tilde{K}$.

Proof. Recall the definition of $w[\breve{a}_z,\breve{f}_z]$

$$
w[\breve{a}_z,\breve{f}_z](y,\theta) = -\int_{-\infty}^0 \mathrm{e}^{-\int_0^\infty \breve{a}_z(y+t\theta+\tau\theta)\mathrm{d}\tau} f_z(y+t\theta)\mathrm{d}t \quad y\in\mathbb{R}^2, \theta\in S^1.
$$

In Lemma 3.8 we had in shape a similar operator, the inequality for the Hölder norm is the same obtained with the same process, hence

$$
\|w[\breve{a},\breve{f}]\|_{C^\alpha(\mathbb{R}^2\times S^1)} \le 2\pi C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}_z\|_\infty}(1+|\breve{a}_z|_{C^\alpha})\|\breve{f}_z\|_{C^\alpha},
$$

using Lemma 3.6 we obtain

$$
\|w[\breve{a},\breve{f}]\|_{C^\alpha(\mathbb{R}^2\times S^1)} \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}(1+|\breve{a}|_{C^\alpha})\|\breve{f}\|_{C^\alpha}.
$$

For $w[\breve{a}_z,\breve{a}_z\cdot\breve{M}_z] = w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]$ we use the above result to get

$$
\|w[\breve{a},\breve{a}\cdot\breve{M}]\|_{C^\alpha(\mathbb{R}\times S^1)} \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}(1+|\breve{a}|_{C^\alpha})\|\breve{a}\cdot\breve{M}\|_{C^\alpha(\mathbb{R}^3)},
$$

and using Lemmas 3.8 and 3.7, we obtain

$$\begin{aligned}\|w[\breve{a},\breve{a}\cdot\breve{M}]\|_{C^\alpha(\mathbb{R}\times S^1)} &\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}(1+|\breve{a}|_{C^\alpha})\|\breve{a}\|_{C^\alpha}\|\breve{M}\|_{C^\alpha}\\ &\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}(1+|\breve{a}|_{C^\alpha})^2\|\breve{a}\|_{C^\alpha}\|\breve{f}\|_{C^\alpha}.\end{aligned}$$

□

Proposition 3.10 *Let $\tilde{K}\subset\mathbb{R}$ be a compact set, $\breve{a},\breve{f}\in C^\alpha(\tilde{K}^3)$ with $\alpha>1/2$ then for all $z\in\tilde{K}$*

$$I_{w[\breve{a}_z,\breve{f}_z]}, I_{w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]}: L^2(\tilde{K}^2)\to H^{1/2}(\mathbb{R}\times S^1),$$

and $\forall\delta a_z\in L^2(\tilde{K}^2)$

$$\|I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z]\|_{H^{1/2}(\mathbb{R}\times S^1)}\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+|\breve{a}|_{C^\alpha(\mathbb{R}^3)}\right)\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}\|\delta a_z\|_{L^2(\mathbb{R}^2)},$$

$$\|I_{w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]}[\delta a_z]\|_{H^{1/2}(\mathbb{R}\times S^1)}\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+|\breve{a}|_{C^\alpha(\mathbb{R}^3)}\right)^2\|\breve{a}\|_{C^\alpha(\mathbb{R}^3)}\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}\|\delta a_z\|_{L^2(\mathbb{R}^2)}.$$

Proof. By Lemma 3.9 we have that both weights $w[\breve{a}_z,\breve{f}_z], w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]\in C^\alpha(\mathbb{R}^2\times S^1)$ and we have estimates for their Hölder norms, also since δa_z has compact support then we can use Theorem 1.42 to obtain inequalities for both operators

$$\begin{aligned}\|I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z]\|_{H^{1/2}(\mathbb{R}\times S^1)} &\le \|w[\breve{a}_z,\breve{f}_z]\|_{C^\alpha(\mathbb{R}^2\times S^1)}\|\delta a_z\|_{L^2(\mathbb{R}^2)}\\ &\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+|\breve{a}|_{C^\alpha(\mathbb{R}^3)}\right)\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}\|\delta a_z\|_{L^2(\mathbb{R}^2)},\end{aligned}$$

$$\begin{aligned}\|I_{w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]}[\delta a_z]\|_{H^{1/2}(\mathbb{R}\times S^1)} &\le \|w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]\|_{C^\alpha(\mathbb{R}^2\times S^1)}\|\delta a_z\|_{L^2(\mathbb{R}^2)}\\ &\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}(1+|\breve{a}|_{C^\alpha(\mathbb{R}^3)})^2\|\breve{a}\|_{C^\alpha(\mathbb{R}^3)}\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}\|\delta a_z\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

□

3.3.2 Part 2: Estimates for the operators $\chi R^{-1}_{\breve{a}_z}I_{w[\breve{a}_z,\breve{f}_z]}$ and $\chi R^{-1}_{\breve{a}_z}I_{w[\breve{a}_z,\breve{a}_z\cdot\breve{M}_z]}$

Lemma 3.11 *Let $\tilde{K}\subset\mathbb{R}$ be a compact set and $\breve{a}\in C^\alpha(\tilde{K}^3)$, then for each $z\in\tilde{K}$ the function $k_z(y,\theta)=\mathrm{e}^{B\breve{a}_z(y,\theta^\perp)}$, $y\in\mathbb{R}^2,\theta\in S^1$ is an α Hölder continuous function with Hölder norm $\|k_z\|_{C^\alpha(\mathbb{R}^2\times S^1)}\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+|\breve{a}|_{C^\alpha(\mathbb{R}^3)}\right)$ $\forall z\in\tilde{K}$, where $C(\tilde{K})$ is a constant depending only in the compact set $\tilde{K}$.*

Proof. This proof is analogous to the proof of Lemma 3.8, we obtain

$$|k_z(x,\theta)-k_z(y,\theta)|\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}_z\|_\infty}|\breve{a}_z|_{C^\alpha(\mathbb{R}^2)}|x-y|^\alpha,$$

and since $\|k_z(\cdot,\cdot)\|_\infty\le\mathrm{e}^{C(\tilde{K})\|\breve{a}_z\|_\infty}$, $|\breve{a}_z|_\infty\le|\breve{a}|_\infty$, then

$$\|k_z\|_{C^\alpha(\mathbb{R}^2\times S^1)}\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}(1+|\breve{a}|_{C^\alpha(\mathbb{R}^3)})\quad\forall z\in\tilde{K}.$$

□

Lemma 3.12 *If $f \in H^{1/2}(\mathbb{R})$ and $g \in H^s(\mathbb{R}), s > 1$, then $f \cdot g \in H^{1/2}(\mathbb{R})$ and we have that*

$$||fg||_{H^{1/2}(\mathbb{R})} \leq C||g||_{H^s(\mathbb{R})}||f||_{H^{1/2}(\mathbb{R})}.$$

Proof. First notice that

$$||f \cdot g||_{L^2(\mathbb{R})} \leq ||f||_{L^2(\mathbb{R})}||g||_{L^\infty} \leq C||f||_{H^{1/2}(\mathbb{R})}||g||_{H^s(\mathbb{R})},$$

where the constant C comes from the Sobolev embedding Theorem $H^s(\mathbb{R}) \subset L^\infty(\mathbb{R})$.

Now we compute the value of the semi-norm, we need to prove that $|\xi|^{1/2}\widehat{f \cdot g}(\xi) \in L^2(\mathbb{R})$ thus

$$\begin{aligned}
\left||\xi|^{\frac{1}{2}}\widehat{fg}(\xi)\right|^2 &= \left||\xi|^{\frac{1}{2}}[\hat{f} * \hat{g}](\xi)\right|^2 \\
&= |\xi|\left|\int_{\mathbb{R}} \hat{f}(\xi - t)\hat{g}(t)\mathrm{d}t\right|^2 \\
&= |\xi|\left|\int_{\mathbb{R}} \hat{f}(\xi - t)\frac{1}{(1+|t|^s)}(1+|t|^s)\hat{g}(t)\mathrm{d}t\right|^2 \\
&\leq |\xi|\int_{\mathbb{R}}\left|\hat{f}(\xi - t)\frac{1}{(1+|t|^s)}\right|^2 \mathrm{d}t \int_{\mathbb{R}}|(1+|t|^s)\hat{g}(t)|^2\,\mathrm{d}t \\
&= ||g||^2_{H^s(\mathbb{R})}\int_{\mathbb{R}}|\hat{f}(\xi - t)|^2\frac{|\xi|}{(1+|t|^s)^2}.
\end{aligned}$$

Thus, taking the L^2 norm, we obtain

$$\begin{aligned}
||\ |\xi|^{\frac{1}{2}}\widehat{fg}||^2_{L(\mathbb{R})} &= \int_{\mathbb{R}}\left||\xi|^{\frac{1}{2}}\widehat{fg}(\xi)\right|^2\mathrm{d}\xi \\
&\leq ||g||^2_{H^s(\mathbb{R})}\int_{\mathbb{R}}\int_{\mathbb{R}}|\hat{f}(\xi - t)|^2\frac{|\xi|}{(1+|t|^s)^2}\mathrm{d}t\mathrm{d}\xi \quad (\text{ c.v.}\xi \to y + t) \\
&= ||g||^2_{H^s(\mathbb{R})}\int_{\mathbb{R}}\int_{\mathbb{R}}|\hat{f}(y)|^2\frac{|y+t|}{(1+|t|^s)^2}\mathrm{d}t\mathrm{d}y \\
&\leq ||g||^2_{H^s(\mathbb{R})}\int_{\mathbb{R}}|\hat{f}(y)|^2\left(\int_{\mathbb{R}}\frac{|y|+|t|}{(1+|t|^s)^2}\mathrm{d}t\right)\mathrm{d}y \\
&\leq ||g||^2_{H^s(\mathbb{R})}C\int_{\mathbb{R}}|\hat{f}(y)|^2(|y|+1)\mathrm{d}y \\
&\leq C||g||^2_{H^s(\mathbb{R})}\int_{\mathbb{R}}|\hat{f}(y)|^2(|y|^{\frac{1}{2}}+1)^2\mathrm{d}y \\
&\leq C||g||^2_{H^s(\mathbb{R})}||f||^2_{H^{1/2}}(\mathbb{R}),
\end{aligned}$$

where we used that $(1+|t|^s)^{-2}$ and $|t|(1+|t|^s)^{-2}$ are integrable, which is true for $s > 1$. □

Lemma 3.13 *Let $\tilde{K} \subset \mathbb{R}$ be a compact set and $\breve{a} \in H^{5/2}(\tilde{K}^3)$ then defining $h_z(s,\theta) = \frac{1}{2}(I + \mathrm{i}H)R\breve{a}_z(s,\theta)$ and taking χ a smooth cut-off function depending on the compact $\tilde{K}$ (say $\|\chi(s)\|_{W^{2,\infty}} + \|\chi(s)\|_{W^{2,2}} \le C(\tilde{K})$, with $C(\tilde{K})$ a constant depending only on $\tilde{K}$), we have that $\chi(\cdot)\mathrm{e}^{h_z(\cdot,\theta)} \in H^2(\mathbb{R})$ $\forall z \in \tilde{K}$ and*

$$\|\chi(\cdot)\mathrm{e}^{\pm h_z(\cdot,\theta)}\|_{H^2(\mathbb{R})} \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1 + \|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\right)^2 \quad \forall\theta \in S^1, z \in \tilde{K}.$$

Proof. Since $\breve{a} \in H^{5/2}(\tilde{K}^3)$, using the trace theorem we have that $\breve{a}_z \in H^2(\tilde{K}^2)$ and that $\|\breve{a}_z\|_{H^2(\mathbb{R}^2)} \le C\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}$, also using Property 1.31 we know that $R\breve{a}_z(\cdot,\theta) \in H^2(\mathbb{R})$ and

$$\|R\breve{a}_z(\cdot,\theta)\|_{H^2(\mathbb{R})} \le C(\tilde{K})\|\breve{a}_z\|_{H^2(\mathbb{R}^2)} \le C(\tilde{K})\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)} \quad \forall\theta \in S^1, z \in \tilde{K},$$

since the Hilbert transform is a unitary isometry from $H^k(\mathbb{R}^n)$ onto itself (Corollary 1.23), we have that $h_z(s,\theta) = \frac{1}{2}(I + \mathrm{i}H)R\breve{a}_z(s,\theta)$ is a complex valued function that satisfies

$$\|h_z(\cdot,\theta)\|_{H^2(\mathbb{R})} \le C(\tilde{K})\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)} \quad \forall\theta \in S^1, z \in \tilde{K}.$$

Hence $\mathrm{e}^{h_z(s,\theta)}$ has the desired regularity, and by multipliying by $\chi(s)$ it is integrable after imposing a compact support, hence $\chi(\cdot)\mathrm{e}^{h_z(\cdot,\theta)} \in H^2(\mathbb{R})$. To derive a bound for the norm we need some inequalities:

- $\chi(s), \partial_s\chi(s), \partial_s^2\chi(s) \le C(\tilde{K}) \qquad \forall s \in \mathbb{R}$,
- $\|\chi\|_{L^2(\mathbb{R})}, \|\partial_s\chi\|_{L^2(\mathbb{R})}, \|\partial_s^2\chi\|_{L^2(\mathbb{R})} \le C(\tilde{K})$,
- $|\mathrm{e}^{h_z(s,\theta)}| = \mathrm{e}^{1/2R\breve{a}_z(s,\theta)} \le \mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty} \quad \forall s \in \mathbb{R}, \theta \in S^1$,
- $|\partial_s h_z(s,\theta)| \le \|\partial_s h_z(\cdot,\theta)\|_{H^1(\mathbb{R})} \le \|h_z(\cdot,\theta)\|_{H^2(\mathbb{R})} \le C(\tilde{K})\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)} \quad \forall s \in \mathbb{R}, \theta \in S^1$.

These bounds in the L^∞ norms allows us to take out these terms from inside the $L^2(\mathbb{R})$ norms, thus computing the derivatives of $\chi(s)\mathrm{e}^{h_z(s,\theta)}$ we obtain the following estimates

$$\begin{aligned}
\|\chi(\cdot)\mathrm{e}^{h_z(\cdot,\theta)}\|_{L^2(\mathbb{R})} &\le \mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\|\chi\|_{L^2(\mathbb{R})} \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}, \\
\|\partial_s\left(\chi(\cdot)\mathrm{e}^{h_z(\cdot,\theta)}\right)\|_{L^2(\mathbb{R})} &\le \|\partial_s\chi(\cdot)\mathrm{e}^{h_z(\cdot,\theta)}\|_{L^2(\mathbb{R})} + \|\chi(\cdot)\mathrm{e}^{h_z(\cdot,\theta)}\partial_s h_z(\cdot,\theta)\|_{L^2(\mathbb{R})} \\
&\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(\|\partial_s\chi\|_{L^2(\mathbb{R})} + \|\partial_s h_z(\cdot,\theta)\|_{L^2(\mathbb{R})}\right) \\
&\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1 + \|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\right), \\
\|\partial_s^2\left(\chi(\cdot)\mathrm{e}^{h_z(\cdot,\theta)}\right)\|_{L^2(\mathbb{R})} &\le \|\partial_s^2\chi(\cdot)\mathrm{e}^{h_z(\cdot,\theta)}\|_{L^2(\mathbb{R})} + 2\|\partial_s\chi(\cdot)\mathrm{e}^{h_z(\cdot,\theta)}\partial_s h_z(\cdot,\theta)\|_{L^2(\mathbb{R})} \\
&+ \|\chi(\cdot)\mathrm{e}^{h_z(\cdot,\theta)}\left(\partial_s h_z(\cdot,\theta)\right)^2\|_{L^2(\mathbb{R})} + \|\chi(\cdot)\mathrm{e}^{h_z(\cdot,\theta)}\partial_s^2 h_z(\cdot,\theta)\|_{L^2(\mathbb{R})} \\
&\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(\|\partial_s^2\chi\|_{L^2} + \|\partial_s h_z(\cdot,\theta)\|_{L^2}\right. \\
&\left.+\|(\partial_s h_z(\cdot,\theta))^2\|_{L^2} + \|\partial_s^2 h_z(\cdot,\theta)\|_{L^2}\right) \\
&\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1 + \|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\right)^2.
\end{aligned}$$

We have repeatedly used that $\|fg\|_{L^2} \le \|f\|_{L^\infty}\|g\|_{L^2}$, in particular $\|f^2\|_{L^2} \le \|f\|_{L^\infty}\|f\|_{L^2}$. We conclude

$$\|\chi(\cdot)\mathrm{e}^{h_z(\cdot,\theta)}\|_{H^2(\mathbb{R})} \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\right)^2 \quad \forall\theta\in S^1, z\in\tilde{K}.$$

The case in which we take $-h$ is the same process and yields the same estimate. □

Proposition 3.14 *Let* $\tilde{K}\subset\mathbb{R}$ *be a compact set,* $\breve{a}\in H^{5/2}(\tilde{K}^3)$*,* $J\in H^{1/2}(\mathbb{R}\times S^1)$ *and* $\chi\in C_0^\infty(\mathbb{R}^2), \eta\in C_0^\infty(\mathbb{R})$ *smooths functions whose norms depend only on* $\tilde{K}$ *then*

$$\|\chi R_{\breve{a}_z}^{-1}[\eta J]\|_{L^2(\mathbb{R}^2)} \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\right)^5\|J\|_{H^{1/2}(\mathbb{R}\times S^1)} \quad \forall z\in\tilde{K}.$$

Proof. Define $\tilde{J}=\eta J$ and take $z\in\tilde{K}$ arbitrary, we know an expression for operator $R_{\breve{a}_z}^{-1}$ given by Theorem 1.39, (we will omit the $\mathcal{G}$ operator given in the Observation 1 of Theorem 1.39 since the result is the same) consider $h_z(s,\theta)=1/2(I+\mathrm{i}H)R\breve{a}_z(s,\theta)$, with I the indentity operator and H the Hilbert transform applied in the space variable s, we will compute the $L^2(\mathbb{R}^2)$ norm of the operator $\chi R_{\breve{a}_z}^{-1}[\tilde{J}]$ by duality. Applying it to $g\in C_0^\infty(\mathbb{R}^2)$ yields

$$\begin{aligned}&\langle\chi R_{\breve{a}_z}^{-1}[\tilde{J}],g\rangle_{L^2(\mathbb{R}^2)}\\ =&\frac{1}{4\pi}\langle\chi\ \mathrm{Re}\ \mathrm{div}\int_S^1\theta\mathrm{e}^{(B\breve{a}_z)(y,\theta^\perp)}\left(\mathrm{e}^{-h_z}H\mathrm{e}^{h_z}\tilde{J}\right)(y\cdot\theta,\theta)\mathrm{d}\theta,g\rangle_{L^2(\mathbb{R}^2)}\\ =&\frac{1}{4\pi}\langle\ \mathrm{Re}\ \mathrm{div}\int_S^1\theta\mathrm{e}^{(B\breve{a}_z)(y,\theta^\perp)}\left(\mathrm{e}^{-h_z}H\mathrm{e}^{h_z}\tilde{J}\right)(y\cdot\theta,\theta)\mathrm{d}\theta,\chi g\rangle_{L^2(\mathbb{R}^2)},\end{aligned}\tag{3.7}$$

define $\tilde{g}=\chi g$ (observe that $\|\tilde{g}\|_{L^2}\le\|g\|_{L^2}$ and that $\mathrm{supp}(\tilde{g})$ depends only on $\tilde{K}$), since $\tilde{g}$ has a compact support, we can write $\tilde{g}(y)=\chi_{B(0,R)}(y)\tilde{g}(y)$ with $R>0$ such that $\tilde{K}^2\cup\mathrm{supp}(\tilde{g})\subset B(0,R)$ and $\chi_{B(0,R)}\in C_0^\infty(\mathbb{R}^2)$ a radial cut-off function such that $\chi_{B(0,R)}(y)=1\forall y\in B(0,R)$. In this way, we have that

$$\nabla_y(\tilde{g}(y)\chi_{B(0,R)}(y))=\nabla_y\tilde{g}(y)\chi_{B(0,R)}(y)+\underbrace{\tilde{g}(y)\nabla_y\chi_{B(0,R)}(y)}_{\to 0}.$$

By integrating by parts in (3.7) and passing the cut-off function back to the other side we obtain

$$\langle\chi R_{\breve{a}_z}^{-1}[\tilde{J}],g\rangle_{L^2(\mathbb{R}^2)}=\frac{1}{4\pi}\mathrm{Re}\langle\int_{S^1}\chi_{B(0,R)}(y)\theta\mathrm{e}^{(B\breve{a}_z)(y,\theta^\perp)}\left(\mathrm{e}^{-h_z}H\mathrm{e}^{h_z}\tilde{J}\right)(y\cdot\theta,\theta)\mathrm{d}\theta,\nabla\tilde{g}(y)\rangle_{L^2(\mathbb{R}^2)},$$

since $\chi_{B(0,R)}$ is radial, we can define a cut-off function $\tilde{\chi}\in C_0^\infty(\mathbb{R})$ that satisfies

$$\chi_{B(0,R)}(y)=\tilde{\chi}(y\cdot\theta)\quad\forall y\in\mathbb{R}^2,\theta\in S^1,$$

hence replacing it and defining $F_z(s,\theta)=\tilde{\chi}(s)\mathrm{e}^{-h_z}H\mathrm{e}^{h_z}\tilde{J}(s,\theta)$ yields

$$\langle\chi R_{\breve{a}_z}^{-1}[\tilde{J}],g\rangle_{L^2(\mathbb{R}^2)}=\frac{1}{4\pi}\mathrm{Re}\int_{\mathbb{R}^2}\int_{S^1}\theta\mathrm{e}^{B\breve{a}_z(y,\theta^\perp)}F_z(y\cdot\theta,\theta)\mathrm{d}\theta\cdot\nabla\tilde{g}(y)\mathrm{d}y.$$

Now, by interchanging integrals and parametrizing the integration in $\mathbb{R}^2$ we get

$$\langle \chi R_{\breve{a}_z}^{-1}[\tilde{J}], g\rangle_{L^2(\mathbb{R}^2)} = \frac{1}{4\pi}\mathrm{Re}\int_{S^1}\int_{\mathbb{R}}\int_{y\cdot\theta=s}\theta\cdot\nabla_y\tilde{g}(y)\mathrm{e}^{(B\breve{a}_z)(y,\theta^\perp)}\mathrm{d}l(y)F_z(s,\theta)\mathrm{d}s\mathrm{d}\theta$$
$$= \frac{\mathrm{Re}}{4\pi}\Big\langle\int_{y\cdot\theta=s}\theta\cdot\nabla(\tilde{g}(y)\mathrm{e}^{(B\breve{a}_z)(y,\theta^\perp)})\mathrm{d}l(y) - \int_{y\cdot\theta=s}\tilde{g}(y)\theta\cdot\nabla_y\mathrm{e}^{(B\breve{a}_z)(y,\theta^\perp)}\mathrm{d}l(y), F_z\Big\rangle_{L^2(\mathbb{R}\times S^1)},$$

and taking the absolute value gives the inequality

$$\left|< \chi R_{\breve{a}_z}^{-1}[\tilde{J}], g >_{L^2(\mathbb{R}^2)}\right| \le C\left|\Big\langle\int_{y\cdot\theta=s}\theta\cdot\nabla_y(\tilde{g}(y)\mathrm{e}^{(B\breve{a}_z)(y,\theta^\perp)})\mathrm{d}l(y), F_z\Big\rangle_{L^2(\mathbb{R}\times S^1)}\right| \tag{3.8}$$
$$+ C\left|\Big\langle\int_{y\cdot\theta=s}\tilde{g}(y)\theta\cdot\nabla_y\mathrm{e}^{(B\breve{a}_z)(y,\theta^\perp)}\mathrm{d}l(y), F_z\Big\rangle_{L^2(\mathbb{R}\times S^1)}\right|. \tag{3.9}$$

In order to estimate the term (3.9), we have that

$$\theta\cdot\nabla_y\mathrm{e}^{B\breve{a}_z(y,\theta^\perp)} = \mathrm{e}^{B\breve{a}_z(y,\theta^\perp)}B[\theta\cdot\nabla a_z](y,\theta).$$

Using the Beam transform Properties 1.36 and 1.37, where $C(\tilde{K})$ stands for a constant depending only on compact $\tilde{K}$, we obtain

$$\begin{aligned}|\theta\cdot\nabla_y\mathrm{e}^{B\breve{a}_z(y,\theta^\perp)}| &\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}_z\|_\infty}|B[\theta\cdot\nabla a_z](y,\theta)|\\ &\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}_z\|_\infty}\|\nabla\breve{a}_z\|_{H^1(\mathbb{R}^2)}\\ &\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}_z\|_\infty}\|\breve{a}_z\|_{H^2(\mathbb{R}^2)} \quad \forall y\in\mathbb{R}^2, \theta\in S^1.\end{aligned}$$

Hence we can deduce an estimate for (3.9) as follows

$$\begin{aligned}&\left|\int_{S^1}\int_R\int_{y\cdot\theta=s}\tilde{g}(y)\theta\cdot\nabla_y\mathrm{e}^{B\breve{a}_z(y,\theta^\perp)}\mathrm{d}l(y)F_z(s,\theta)\mathrm{d}s\mathrm{d}\theta\right|\\
&\le \int_{S^1}\int_R\int_{y\cdot\theta=s}|\tilde{g}(y)|\ \left|\theta\cdot\nabla_y\mathrm{e}^{B\breve{a}_z(y,\theta^\perp)}\right|\mathrm{d}l(y)|F_z(s,\theta)|\mathrm{d}s\mathrm{d}\theta\\
&\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}_z\|_\infty}\|\breve{a}_z\|_{H^2(\mathbb{R}^2)}\int_{S^1}\int_R\int_{y\cdot\theta=s}|\tilde{g}(y)|\mathrm{d}l(y)|F_z(s,\theta)|\mathrm{d}s\mathrm{d}\theta\\
&\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}_z\|_\infty}\|\breve{a}_z\|_{H^2(\mathbb{R}^2)}\langle R|\tilde{g}|(s,\theta), |F_z(s,\theta)|\rangle_{L^2(\mathbb{R}\times S^1)}\\
&\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}_z\|_\infty}\|\breve{a}_z\|_{H^2(\mathbb{R}^2)}\|R|\tilde{g}|(s,\theta)\|_{L^2(\mathbb{R}\times S^1)}\|F_z(s,\theta)\|_{L^2(\mathbb{R}\times S^1)}\\
&\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}_z\|_\infty}\|\breve{a}_z\|_{H^2(\mathbb{R}^2)}\|\tilde{g}\|_{L^2(\mathbb{R}^2)}\|F_z(s,\theta)\|_{L^2(\mathbb{R}\times S^1)},\end{aligned}$$

where we used Proposition 1.30 in the last inequality.

For the right hand term in (3.8), the directional derivative $\theta\cdot\nabla_y$ can be taken out of the integral as a partial derivative using Property 1.27, thus

$$\Big\langle\int_{y\cdot\theta=s}\theta\cdot\nabla(\tilde{g}(y)\mathrm{e}^{(B\breve{a}_z)(y,\theta^\perp)})\mathrm{d}l(y), F_z\Big\rangle_{L^2(\mathbb{R}\times S^1)} = \Big\langle\partial_s\int_{y\cdot\theta=s}\tilde{g}(y)\mathrm{e}^{(B\breve{a}_z)(y,\theta^\perp)}\mathrm{d}l(y), F_z\Big\rangle_{L^2(\mathbb{R}\times S^1)}. \tag{3.10}$$

By defining $k_z(y,\theta) = \mathrm{e}^{(B\breve{a}_z)(y,\theta^\perp)}$ we can rewrite (3.10) using the weighted Radon transform defined on 1.40 and obtain an estimate using the duality inequality (1.17) and the Property 1.15 for H^s spaces

$$\begin{aligned}\left|\langle\partial_s \int_{y\cdot\theta=s}\tilde{g}(y)\mathrm{e}^{(B\breve{a}_z)(y,\theta^\perp)}\mathrm{d}l(y), F_z\rangle_{L^2(\mathbb{R}\times S^1)}\right| &= |\langle\partial_s I_{k_z}\tilde{g}(s,\theta), F(s,\theta)\rangle|_{L^2(\mathbb{R}\times S^1)}\\ &\le \|\partial_s I_{k_z}\tilde{g}\|_{H^{-1/2}(\mathbb{R}\times S^1)}\|F\|_{H^{1/2}(\mathbb{R}\times S^1)}\\ &\le \|I_{k_z}\tilde{g}\|_{H^{1/2}(\mathbb{R}\times S^1)}\|F\|_{H^{1/2}(\mathbb{R}\times S^1)}.\end{aligned}$$

Since $\breve{a} \in H^{5/2}(\mathbb{R})$ using the Sobolev embedding theorem we have that $\breve{a} \in C^\alpha(\tilde{K}^3)$ with $\alpha > 1/2$, thus using Lemma 3.11 we deduce $k_z \in C^\alpha(\mathbb{R}^2 \times S^1)$ and an estimate for its Hölder norm, then with the Theorem 1.42 we obtain

$$\begin{aligned}\|I_{k_z}\tilde{g}\|_{H^{1/2}(\mathbb{R}\times S^1)} &\le C(\tilde{K})\|k_z\|_{C^\alpha(\mathbb{R}^2\times S^1)}\|\tilde{g}\|_{L^2(\mathbb{R}^2)}\\ &\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+|\breve{a}|_{C^\alpha(\mathbb{R}^3)}\right)\|\tilde{g}\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

At this point, let us make an important observation: the constant is independent of the support of g since we control its support with the cut-off function χ. The Sobolev embedding theorem gives the inequality $|\breve{a}|_{C^\alpha(\mathbb{R}^3)} \le \|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}$, with which we obtain

$$\langle\int_{y\cdot\theta=s}\theta\cdot\nabla(\tilde{g}(y)\mathrm{e}^{(B\breve{a}_z)(y,\theta^\perp)})\mathrm{d}l(y)\rangle \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\right)\|\tilde{g}\|_{L^2(\mathbb{R}^2)}.$$

Using the inequalities derived for both term (3.9) and the right hand term in (3.8), we deduce the following estimate

$$|\langle\chi R_{\breve{a}_z}^{-1}\tilde{J}, g\rangle| \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^2)}\right)\|\tilde{g}\|_{L^2(\mathbb{R}^2)}\|F_z\|_{H^{1/2}(\mathbb{R}\times S^1)}.$$

Recall $F_z(s,\theta) = \tilde{\chi}(s)\mathrm{e}^{-h_z(s,\theta)}H\mathrm{e}^{h_z(s,\theta)}\eta(s)J(s,\theta)$, using Lemma 3.13 we have that $\tilde{\chi}(\cdot)\mathrm{e}^{\pm h_z(\cdot,\theta)} \in H^2(\mathbb{R})$, with the aid of Lemma 3.12 and the Hilbert transform Property 1.23 we proceed;

$$\begin{aligned}\|F_z(s,\theta)\|^2_{H^{1/2}(\mathbb{R}\times S^1)} &= \int_{S^1}\|\tilde{\chi}(\cdot)\mathrm{e}^{-h_z(\cdot,\theta)}H\mathrm{e}^{h_z(\cdot,\theta)}\eta(\cdot)J(\cdot,\theta)\|^2_{H^{1/2}(\mathbb{R})}\mathrm{d}\theta\\ &\le \int_{S^1}\|\tilde{\chi}(\cdot)\mathrm{e}^{-h_z(\cdot,\theta)}\|^2_{H^2(\mathbb{R})}\|H\mathrm{e}^{h_z(\cdot,\theta)}\eta(\cdot)J(\cdot,\theta)\|^2_{H^{1/2}(\mathbb{R})}\mathrm{d}\theta\\ &= \int_{S^1}\|\tilde{\chi}(\cdot)\mathrm{e}^{-h_z(\cdot,\theta)}\|^2_{H^2(\mathbb{R})}\|\mathrm{e}^{h_z(\cdot,\theta)}\eta(\cdot)J(\cdot,\theta)\|^2_{H^{1/2}(\mathbb{R})}\mathrm{d}\theta\\ &\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^2)}\right)^4\int_{S^1}\|\mathrm{e}^{h_z(\cdot,\theta)}\eta(s)J(\cdot,\theta)\|^2_{H^{1/2}(\mathbb{R})}\mathrm{d}\theta\\ &\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^2)}\right)^8\int_{S^1}\|J(\cdot,\theta)\|^2_{H^{1/2}(\mathbb{R})}\mathrm{d}\theta\\ &= C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^2)}\right)^8\|J\|^2_{H^{1/2}(\mathbb{R}\times S^1)},\end{aligned}$$

hence

$$|\langle \chi R^{-1}_{\breve{a}_z}\eta J, g\rangle_{L^2(\mathbb{R}^2)}| \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\right)^5 \|J\|_{H^{1/2}(\mathbb{R}\times S^1)}\|\tilde{g}\|_{L^2(\mathbb{R}^2)}$$
$$\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\right)^5 \|J\|_{H^{1/2}(\mathbb{R}\times S^1)}\|g\|_{L^2(\mathbb{R}^2)},$$

where the constant $C(\tilde{K})$ depends only on the compact $\tilde{K}$ (i.e. independent of z and g), since this inequality holds $\forall g \in C_0^\infty(\mathbb{R}^2)$ a density argument implies the same inequality holds $g \in L^2(\mathbb{R}^2)$. Also since we took $z \in \tilde{K}$ arbitrary, we conclude the bound for all $z \in \tilde{K}$

$$\|\chi R^{-1}_{\breve{a}_z}\eta J\|_{L^2(\mathbb{R}^2)} \le \mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\right)^5 \|J\|_{H^{1/2}(\mathbb{R}\times S^1)} \quad \forall z \in \tilde{K}.$$

□

Proposition 3.15 *Let $\tilde{K} \subset \mathbb{R}$ be a compact set, $\breve{a} \in H^{5/2}(\tilde{K}^3)$ with $\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)} < D$, $\breve{f} \in C^\alpha(\tilde{K}^3)$ with $\alpha > 1/2$ and $\chi \in C_0^\infty(\tilde{K}^2)$ whose norms depend only in $\tilde{K}$, then*

$$\left\|\chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z,\breve{f}_z]}\right\|_{\mathcal{L}(L^2(\tilde{\tilde{K}}^2),L^2(\tilde{\tilde{K}}^2))} \le C(\tilde{K},D)\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)} \quad \forall z \in \tilde{K}$$
$$\left\|\chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]}\right\|_{\mathcal{L}(L^2(\tilde{\tilde{K}}^2),L^2(\tilde{\tilde{K}}^2))} \le C(\tilde{K},D)\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)} \quad \forall z \in \tilde{K},$$

with $C(\tilde{K},D)$ a constant only depending on $\tilde{K}$ and D, and non-decreasing with respect to D.

Proof. Since $\delta a_z \in L^2(\tilde{K}^2)$, then $I_{w[\breve{a}_z,\breve{f}_z]}, I_{w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]} \in H^{1/2}(\mathbb{R}\times S^1)$ (by Proposition 3.10) and has compact support (by Lemma 1.41), thus there exists $B \subset \mathbb{R}$, a compact set, such that

$$I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z](s,\theta) = \mathbb{1}_B(s) I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z](s,\theta) \quad \forall s,z \in \mathbb{R}, \theta \in S^1$$
$$, I_{w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]}[\delta a_z](s,\theta) = \mathbb{1}_B(s) I_{w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]}[\delta a_z](s,\theta) \quad \forall s,z \in \tilde{K}, \theta \in S^1,$$

using Proposition 3.14 we obtain

$$\|\chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z]\|_{L^2(\mathbb{R}^2)} \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\right)^5 \|I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z]\|_{H^{1/2}(\mathbb{R}\times S^1)},$$
$$\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\right)^5 \left(1+|\breve{a}|_{C^\alpha(\mathbb{R}^3)}\right)\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}\|\delta a_z\|_{L^2(\mathbb{R}^2)},$$

$$\|\chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]}[\delta a_z]\|_{L^2(\mathbb{R}^2)} \le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\right)^5 \|I_{w[\breve{a}_z,\breve{a}_z\cdot \breve{M}_z]}[\delta a_z]\|_{H^{1/2}(\mathbb{R}\times S^1)},$$
$$\le C(\tilde{K})\mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\left(1+\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\right)^5 \left(1+|\breve{a}|_{C^\alpha(\mathbb{R}^3)}\right)^2\|\breve{a}\|_{C^\alpha(\mathbb{R}^3)}\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}\|\delta a_z\|_{L^2(\mathbb{R}^2)}.$$

Using the Sobolev embedding Theorem 1.19 we have that $\|\breve{a}\|_{C^\alpha(\mathbb{R}^3)} \le C\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}$ for $\alpha > 1/2$, and that $\|\breve{a}\|_\infty \le \|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}$, thus if $\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3} \le D$ then

$$\|\chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z,\breve{f}_z]}[\delta a_z]\|_{L^2(\mathbb{R}^2)} \le C(\tilde{K},D)\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}\|\delta a_z\|_{L^2(\mathbb{R}^2)} \quad \forall z \in \tilde{K},$$
$$\|\chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z,\breve{a}_z\cdot M_z[\breve{a},\breve{f}]]}[\delta a_z]\|_{L^2(\mathbb{R}^2)} \le C(\tilde{K},D)\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\|\delta a_z\|_{L^2(\mathbb{R}^2)} \quad \forall z \in \tilde{K}.$$

It is clear that the image of this operators are functions with support contained in $\tilde{K}^2$ since $\mathrm{supp}(\chi) \subset \tilde{K}^2$, also given the dependence of constant $C(\tilde{K}, D)$ it is straightforward to notice its monotone behavior with respect to D. $\square$

3.3.3 Part 3: Estimates for the operators $L^{-1}[\breve{a}, \breve{f}]$, $Q[\breve{a}, \breve{f}]$ and proof of the inversion theorem

Proposition 3.16 *Let $\tilde{K} \subset \mathbb{R}$ be a compact set and $\breve{a} \in H^{5/2}(\tilde{K}^3)$, $\breve{f} \in C^\alpha(\tilde{K}^3)$ with $\alpha > 1/2$, $\chi \in C_0^\infty(\tilde{K}^2)$ whose norm depends only in $\tilde{K}$ and $\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)} < D$, then*

$$Q[\breve{a}, \breve{f}] : L^2(\tilde{K}^3) \times L^2(\tilde{K}^3) \to L^2(\tilde{K}; L^2(\tilde{K}^2) \times L^2(\tilde{K}^2)),$$

defined by

$$Q[\breve{a}, \breve{f}](\delta a, \delta f) = \left(Q_z[\breve{a}, \breve{f}](\delta a, \delta f)\right)_{z\in\mathbb{R}},$$

with Q_z defined almost everywhere by

$$Q_z[\breve{a}, \breve{f}] : L^2(\tilde{K}^3) \times L^2(\tilde{K}^3) \to L^2(\tilde{K}^2) \times L^2(\tilde{K}^2),$$

$$Q_z[\breve{a}, \breve{f}](\delta a, \delta f) = \left(\begin{array}{c} 0 \\ \chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z, \breve{a}_z \cdot \breve{M}_z]} \delta a_z + (\breve{a}_z \cdot \partial_a \breve{M}_z \delta a) + (\breve{a}_z \cdot M_z[\breve{a}, \delta f]) \end{array} \right),$$

satifies

$$\|Q[\breve{a}, \breve{f}]\| \le C(\tilde{K}, D)(1 + \|\breve{f}\|_{C^\alpha(\mathbb{R}^3)})\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)},$$

with $\|\cdot\|$ the operator norm for linear operators between Banach spaces.

Proof. Fix $\delta a, \delta f \in L^2(\tilde{K}^3)$ set $z \in \tilde{K}$ such that $\delta a_z \in L^2(\tilde{K}^2)$ (possible almost everywhere on z), hence Q_z is well defined and

$$\begin{aligned} \|Q_z[\breve{a}, \breve{f}](\delta a, \delta f)\|_{(L^2(\mathbb{R}^2))^2} \le{}& \|\chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z, \breve{a}_z \cdot \breve{M}_z]}[\delta a_z]\|_{L^2(\mathbb{R}^2)} \\ &+ \|\breve{a}_z \cdot \partial_a \breve{M}_z[\delta a]\|_{L^2(\mathbb{R}^2)} + \|\breve{a}_z \cdot M_z[\breve{a}, \delta f]\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

We already know from Proposition 3.15 that

$$\|\chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z, \breve{a}_z \cdot \breve{M}_z]}[\delta a_z]\|_{L^2(\mathbb{R}^2)} \le C(\tilde{K}, D)\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}\|\delta a_z\|_{L^2(\mathbb{R}^2)} \quad \forall z \in \tilde{K}.$$

Also, if $z \notin \tilde{K}$ both terms in the inequality vanish, thus the inequality holds for all $z \in \mathbb{R}$, i.e.

$$\|\chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z, \breve{a}_z \cdot \breve{M}_z]}[\delta a_z]\|_{L^2(\mathbb{R}^2)} \le C(\tilde{K}, D)\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}\|\delta a_z\|_{L^2(\mathbb{R}^2)} \quad \forall z \in \mathbb{R},$$

and if we square it and integrate over $z \in \mathbb{R}$

$$\begin{aligned} \int_{\mathbb{R}} \|\chi R^{-1}_{\breve{a}_z} I_{w[\breve{a}_z, \breve{a}_z \cdot \breve{M}_z]}[\delta a_z]\|^2_{L^2(\mathbb{R}^2)} \mathrm{d}z \le{}& C(\tilde{K}, D)\|\breve{a}\|^2_{H^{5/2}(\mathbb{R}^3)}\|\breve{f}\|^2_{C^\alpha(\mathbb{R}^3)} \int_{\mathbb{R}} \|\delta a_z\|^2_{L^2(\mathbb{R}^2)} \mathrm{d}z \\ \le{}& C(\tilde{K}, D)\|\breve{a}\|^2_{H^{5/2}(\mathbb{R}^3)}\|\breve{f}\|^2_{C^\alpha(\mathbb{R}^3)} \int_{\mathbb{R}} \int_{\mathbb{R}^2} (\delta a(y, z))^2 \,\mathrm{d}y\mathrm{d}z \\ \le{}& C(\tilde{K}, D)\|\breve{a}\|^2_{H^{5/2}(\mathbb{R}^3)}\|\breve{f}\|^2_{C^\alpha(\mathbb{R}^3)}\|\delta a\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Now we proceed to prove the inequalities for the other terms:

- The operator $\breve{a}_z \cdot \partial_a \breve{M}_z[\cdot]$

$$\begin{aligned}|\breve{a}_z \cdot \partial_a \breve{M}_z \delta a(y)| &= |\breve{a}_z(y) \int_{S^2}\int_0^\infty \breve{f}((y,z)+t\phi)\mathrm{e}^{-\int_0^t \breve{a}((y,z)+\tau\phi)\mathrm{d}\tau} \int_0^t \delta a((y,z)+s\phi)\mathrm{d}s\mathrm{d}t\mathrm{d}\phi| \\ &\leq \|\breve{a}\|_\infty \mathbb{1}_{\tilde{K}^3}(y,z) \int_{S^2}\int_0^\infty |\breve{f}((y,z)+t\phi)|\mathrm{e}^{-\int_0^t \breve{a}((y,z)+\tau\phi)\mathrm{d}\tau}\mathrm{d}t \int_{\mathbb{R}} |\delta a((y,z)+s\phi)|\mathrm{d}s\mathrm{d}\phi.\end{aligned}$$

Since we have the indicatrix over $\tilde{K}^3$ for (y,z) then the integration variables τ, t, s can be taken in the $[-\mathrm{diam}(\tilde{K}^3), \mathrm{diam}(\tilde{K}^3)] = J(\tilde{K})$ interval, hence

$$|\breve{a}_z \cdot \partial_a \breve{M}_z \delta a(y)| \leq C(\tilde{K})\|\breve{a}\|_\infty \mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\|\breve{f}\|_\infty \mathbb{1}_{\tilde{K}^3}(y,z)\int_{S^2}\int_{J(\tilde{K})} |\delta a((y,z)+s\phi)|\mathrm{d}s\mathrm{d}\phi,$$

using the Sobolev embedding Theorem we obtain that $\|\breve{a}\|_\infty \leq C\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)} \leq C\cdot D$, and by definition $\|\breve{f}\|_\infty \leq \|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}$, hence

$$|\breve{a}_z \cdot \partial_a \breve{M}_z \delta a(y)| \leq C(\tilde{K}, D)\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)} \mathbb{1}_{\tilde{K}^3}(y,z)\int_{S^2}\int_{J(\tilde{K})} |\delta a((y,z)+s\phi)|\mathrm{d}s\mathrm{d}\phi,$$

so taking the square of the L^2 norm yields

$$\begin{aligned}\|\breve{a}_z \cdot \partial_a \breve{M}_z \delta a\|^2_{L^2(\mathbb{R}^2)} &\leq C(\tilde{K}, D)\|\breve{a}\|^2_{5/2}\|\breve{f}\|^2_{C^\alpha}\mathbb{1}_{\tilde{K}}(z)\int_{\tilde{K}^2}\left(\int_{S^2}\int_{J(\tilde{K})}|\delta a((y,z)+s\phi)|\mathrm{d}s\mathrm{d}\phi\right)^2\mathrm{d}y \\ &\leq C(\tilde{K}, D)\|\breve{a}\|^2_{5/2}\|\breve{f}\|^2_{C^\alpha}\mathbb{1}_{\tilde{K}}(z)\int_{\tilde{K}^2}\int_{S^2}\int_{J(\tilde{K})}(\delta a((y,z)+s\phi))^2\mathrm{d}s\mathrm{d}\phi\mathrm{d}y.\end{aligned}$$

Now if we integrate over z we obtain

$$\begin{aligned}\int_{\mathbb{R}}\|\breve{a}_z \cdot \partial_a \breve{M}_z \delta a\|^2_{L^2(\mathbb{R}^2)} &\leq C(\tilde{K}, D)\|\breve{a}\|^2_{5/2}\|\breve{f}\|^2_{C^\alpha}\int_{\tilde{K}}\int_{\tilde{K}^2}\int_{S^2}\int_{J(\tilde{K})}(\delta a((y,z)+s\phi))^2\mathrm{d}s\mathrm{d}\phi\mathrm{d}y\mathrm{d}z \\ &= C(\tilde{K}, D)\|\breve{a}\|^2_{5/2}\|\breve{f}\|^2_{C^\alpha}\int_{S^2}\int_{J(\tilde{K})}\int_{\tilde{K}}\int_{\tilde{K}^2}(\delta a((y,z)+s\phi))^2\mathrm{d}y\mathrm{d}z\mathrm{d}s\mathrm{d}\phi \\ &= C(\tilde{K}, D)\|\breve{a}\|^2_{5/2}\|\breve{f}\|^2_{C^\alpha}\int_{S^2}\int_{J(\tilde{K})}\|\delta a\|^2_{L^2(\mathbb{R}^2)}\mathrm{d}s\mathrm{d}\phi \\ &\leq C(\tilde{K}, D)\|\breve{a}\|^2_{H^{5/2}(\mathbb{R}^3)}\|\breve{f}\|^2_{C^\alpha(\mathbb{R}^3)}\|\delta a\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

- The operator $\breve{a}_z \cdot M_z[\breve{a}, \cdot]$:
The proof of this inequality is nearly the same as the previous one, therefore some steps will be omitted

$$\begin{aligned}|\breve{a}_z \cdot M_z[\breve{a}, \delta f](y)| &= \left|\breve{a}(y,z)\int_{S^2}\int_0^\infty \delta f((y,z)+t\phi)\mathrm{e}^{-\int_0^\infty \breve{a}((y,z)+\tau\phi)\mathrm{d}\tau}\mathrm{d}t\mathrm{d}\phi\right| \\ &\leq \mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty}\|\breve{a}\|_\infty \mathbb{1}_{\tilde{K}^3}(y,z)\int_{S^2}\int_{J(\tilde{K})}|\delta f((y,z)+t\phi)|\mathrm{d}t\mathrm{d}\phi \\ &\leq C(\tilde{K}, D)\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}\mathbb{1}_{\tilde{K}^3}(y,z)\int_{S^2}\int_{J(\tilde{K})}|\delta f((y,z)+t\phi)|\mathrm{d}t\mathrm{d}\phi.\end{aligned}$$

So, taking the squared L^2 norm as a function of y

$$\begin{aligned}\|\breve{a}_z\cdot M_z[\breve{a},\delta f]\|^2_{L^2(\mathbb{R}^2)} &\le C(\tilde{K},D)\|\breve{a}\|^2_{5/2}\mathbb{1}_{\tilde{K}}(z)\int_{\tilde{K}^2}\left(\int_{S^2}\int_{J(\tilde{K})}|\delta f((y,z)+t\phi)|\mathrm{d}t\mathrm{d}\phi\right)^2\mathrm{d}y\\ &\le C(\tilde{K},D)\|\breve{a}\|^2_{5/2}\mathbb{1}_{\tilde{K}}(z)\int_{\tilde{K}^2}\int_{S^2}\int_{J(\tilde{K})}\left(\delta f((y,z)+t\phi)\right)^2\mathrm{d}t\mathrm{d}\phi\mathrm{d}y,\end{aligned}$$

and integrating $z\in\mathbb{R}$ we obtain

$$\begin{aligned}\int_{\mathbb{R}}\|\breve{a}_z\cdot M_z[\breve{a},\delta f]\|^2_{L^2(\mathbb{R}^2)}\mathrm{d}z &\le C(\tilde{K},D)\|\breve{a}\|^2_{5/2}\int_{\tilde{K}}\int_{\tilde{K}^2}\int_{S^2}\int_{J(\tilde{K})}\left(\delta f((y,z)+t\phi)\right)^2\mathrm{d}t\mathrm{d}\phi\mathrm{d}y\mathrm{d}z\\ &= C(\tilde{K},D)\|\breve{a}\|^2_{5/2}\int_{S^2}\int_{J(\tilde{K})}\int_{\tilde{K}}\int_{\tilde{K}^2}\left(\delta f((y,z)+t\phi)\right)^2\mathrm{d}y\mathrm{d}z\mathrm{d}t\mathrm{d}\phi\\ &\le C(\tilde{K},D)\|\breve{a}\|^2_{H^{5/2}(\mathbb{R}^3)}\|\delta f\|_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Finally let us obtain the estimates for the norm of Q

$$\begin{aligned}\|Q[\breve{a},\breve{f}](\delta a,\delta f)\|_{L^2(\mathbb{R};L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2))} &= \int_{\mathbb{R}}\|Q_z[\breve{a},\breve{f}](\delta a,\delta f)\|^2_{L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)}\mathrm{d}z\\ &\le \int_{\mathbb{R}}\|\chi R^{-1}_{\breve{a}_z}I_{w[\breve{a}_z,\breve{a}_z\cdot\breve{M}_z]}[\delta a_z]\|^2_{L^2(\mathbb{R}^2)}+\|\breve{a}_z\cdot\partial_a\breve{M}_z\delta a\|^2_{L^2(\mathbb{R}^2)}+\|\breve{a}_z\cdot M_z[\breve{a},\delta f]\|^2_{L^2(\mathbb{R}^2)}\mathrm{d}z\\ &\le C(\tilde{K},D)\|\breve{a}\|^2_{H^{5/2}(\mathbb{R}^3)}\left(\|\breve{f}\|^2_{C^\alpha(\mathbb{R}^3)}\|\delta a\|^2_{L^2(\mathbb{R}^3)}+\|\delta f\|^2_{L^2(\mathbb{R}^3)}\right)\quad\forall\delta a,\delta f\in L^2(\tilde{K}^3),\end{aligned}$$

hence

$$\|Q[\breve{a},\breve{f}]\|\le C(\tilde{K},D)\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}(1+\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}).$$

Also,notice that the image of Q is contained in $L^2(\tilde{K},L^2(\tilde{K}^2)\times L^2(\tilde{K}^2))$, because if $z\notin\tilde{K}$ then $\|Q_z[\breve{a},\breve{f}](\delta a,\delta f)\|=0\ \ \forall\delta a,\delta f\in L^2(\tilde{K}^2)$ and it is trivial to notice that $Q_z[\breve{a},\breve{f}](\delta a,\delta f)\in\left(L^2(\tilde{K}^2)\right)^2$ almost everywhere on z.

□

Lemma 3.17 *Let $\tilde{K}\subset\mathbb{R}$ be a compact set, $\breve{a}\in C^0(\tilde{K}^3)$, $\breve{f}\in C^\alpha(\tilde{K}^3)$ with $\breve{f}\ge 0$ and $\breve{f}\not\equiv 0$, then*

$$M[\breve{a},\breve{f}](x)\ge C(\tilde{K})\mathrm{e}^{-C(\tilde{K})\|\breve{a}\|_\infty}\left(\frac{\|\breve{f}\|_\infty}{|\breve{f}|_{C^\alpha(\mathbb{R}^3)}}\right)^{3/\alpha}\|\breve{f}\|_\infty\quad\forall x\in\tilde{K}^3.$$

Proof. Since $\breve{f}$ is continuous and non-negative with compact support, then there exists $\overline{x}\in\tilde{K}^3$ such that $\breve{f}(\overline{x})=\|\breve{f}\|_\infty>0$. Let us define

$$A=\{x\in\tilde{K}^3,\ \breve{f}(x)\ge\frac{\|\breve{f}\|_\infty}{2}\},$$

then, using that $\breve{f}$ is Hölder continuous, we have that

$$\begin{aligned}
&|\breve{f}(\overline{x}) - \breve{f}(y)| \leq |\breve{f}|_{C^\alpha}|\overline{x} - y|^\alpha \quad \forall y \in A^c \\
\Rightarrow &\frac{\|\breve{f}\|_\infty}{2} \leq |\breve{f}|_{C^\alpha}\|\overline{x} - y|^\alpha \quad \forall y \in A^c \\
\Rightarrow &R := \left(\frac{\|\breve{f}\|_\infty}{2|\breve{f}|_{C^\alpha}}\right)^{1/\alpha} \leq \mathrm{dist}(\overline{x}, A^c).
\end{aligned}$$

(notice that $|\breve{f}|_{C^\alpha} \neq 0$ because $\breve{f}$ has compact support and $\breve{f} \not\equiv 0$) Then $B(\overline{x}, R) \subset A$ thus

$$\begin{aligned}
&\breve{f}(x) \geq \frac{\|\breve{f}\|_\infty}{2} \quad \forall x \in B(\overline{x}, R) \\
\Rightarrow &\breve{f}(x) \geq \frac{\|\breve{f}\|_\infty}{2}\mathbb{1}_{B(\overline{x},R)}(x) \quad \forall x \in \tilde{K}^3.
\end{aligned}$$

Hence for $\breve{M}$ and $x \in \tilde{K}^3$

$$\begin{aligned}
M[\breve{a}, \breve{f}](x) &= \int_{S^2}\int_0^\infty f(x + t\phi)\mathrm{e}^{-\int_0^t a(x+s\phi)\mathrm{d}s}\mathrm{d}t\mathrm{d}\phi \\
&\geq \mathrm{e}^{-C(\tilde{K})\|\breve{a}\|_\infty}\int_{S^2}\int_0^\infty f(x + t\phi)\mathrm{d}t\mathrm{d}\phi \\
&\geq \mathrm{e}^{-C(\tilde{K})\|\breve{a}\|_\infty}\frac{\|\breve{f}\|_\infty}{2}\int_{S^2}\int_0^\infty \mathbb{1}_{B(\overline{x},R)}(x + t\phi)\mathrm{d}t\mathrm{d}\phi,
\end{aligned}$$

notice that if $t > \mathrm{diam}(\tilde{K}^3)$ then $(x + t\phi) \notin B(\overline{x}, R)\forall x \in \tilde{K}^3, \phi \in S^2$ because $B(\overline{x}, R) \subset \tilde{K}^3$, hence

$$\begin{aligned}
M[\breve{a}, \breve{f}](x) &\geq \mathrm{e}^{-C(\tilde{K})\|\breve{a}\|_\infty}\frac{\|\breve{f}\|_\infty}{2}\int_{S^2}\int_0^{\mathrm{diam}(\tilde{K}^3)} \mathbb{1}_{B(\overline{x},R)}(x + t\phi)\mathrm{d}t\mathrm{d}\phi \\
&= \mathrm{e}^{-C(\tilde{K})\|\breve{a}\|_\infty}\frac{\|\breve{f}\|_\infty}{2}\int_{S^2}\int_0^{\mathrm{diam}(\tilde{K}^3)} \mathbb{1}_{B(\overline{x},R)}(x + t\phi)\frac{t^2}{t^2}\mathrm{d}t\mathrm{d}\phi \\
&\geq \mathrm{e}^{-C(\tilde{K})\|\breve{a}\|_\infty}\frac{\|\breve{f}\|_\infty}{2\mathrm{diam}(\tilde{K}^3)^2}\int_{S^2}\int_0^{\mathrm{diam}(\tilde{K}^3)} \mathbb{1}_{B(\overline{x},R)}(x + t\phi)t^2\mathrm{d}t\mathrm{d}\phi \\
&= \mathrm{e}^{-C(\tilde{K})\|\breve{a}\|_\infty}\frac{\|\breve{f}\|_\infty}{2\mathrm{diam}(\tilde{K}^3)^2}\int_{\mathbb{R}^3} \mathbb{1}_{B(\overline{x},R)}(x)\mathrm{d}x \\
&= \mathrm{e}^{-C(\tilde{K})\|\breve{a}\|_\infty}\frac{\|\breve{f}\|_\infty}{2\mathrm{diam}(\tilde{K}^3)^2}4\pi R^3 \\
&\geq C(\tilde{K})\mathrm{e}^{-C(\tilde{K})\|\breve{a}\|_\infty}\left(\frac{\|\breve{f}\|_\infty}{|\breve{f}|_{C^\alpha(\mathbb{R}^3)}}\right)^{3/\alpha}\|\breve{f}\|_\infty \quad \forall x \in \tilde{K}^3.
\end{aligned}$$

□

Proposition 3.18 *Let $\tilde{K} \subset \mathbb{R}$ be a compact set and let $\breve{a} \in H^{5/2}(\tilde{K}^3)$, $\breve{f} \in C^\alpha(\tilde{K}^3)$ with $\alpha > 1/2$ and $\breve{f} \not\equiv 0, \breve{f} \geq 0$. Let $\chi \in C_0^\infty(\tilde{K}^2)$ be a smooth cut-off function whose norm depends only in $\tilde{K}$ and $\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)} < D$. Then*

$$L^{-1}[\breve{a}, \breve{f}] : L^2(\tilde{K}^2 \times \tilde{K}) \times L^2(\tilde{K}^2 \times \tilde{K}) \to L^2(\tilde{K}; L^2(\tilde{K}^2) \times L^2(\tilde{K}^2)),$$

defined by

$$L^{-1}[\breve{a}, \breve{f}] \begin{pmatrix} g \\ h \end{pmatrix} = \left(L_z^{-1}[\breve{a}, \breve{f}] \begin{pmatrix} g \\ h \end{pmatrix} \right)_{z \in \mathbb{R}},$$

where L_z^{-1} is defined almost everywhere as

$$L_z^{-1}[\breve{a}, \breve{f}] : L^2(\tilde{K}^2 \times \tilde{K}) \times L^2(\tilde{K}^2 \times \tilde{K}) \to L^2(\tilde{K}^2) \times L^2(\tilde{K}^2)$$

$$L_z^{-1}[\breve{a}, \breve{f}] \begin{pmatrix} g \\ h \end{pmatrix} = \left(h_z / \breve{M}_z,\ g_z - \chi R_{\breve{a}_z}^{-1} I_{w[\breve{a}_z, \breve{f}_z]}[h_z / \breve{M}_z] \right),$$

satifies

$$\|L^{-1}[\breve{a}, \breve{f}]\| \leq 1 + C(\tilde{K}, D)(1 + \|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}) \frac{|\breve{f}|_{C^\alpha(\mathbb{R}^3)}^{3/\alpha}}{\|\breve{f}\|_\infty^{3/\alpha+1}},$$

where $\|\cdot\|$ is the operator norm for linear operators between Banach spaces.

Proof. Given $g, h \in L^2(\tilde{K}^3)$ $g_z, h_z \in L^2(\tilde{K}^3)$ a.e. $z \in \mathbb{R}$, then

$$\|L_z^{-1}[\breve{a}, \breve{f}] \begin{pmatrix} g \\ h \end{pmatrix}\|_{(L^2(\mathbb{R}^2))^2}^2 \leq \|h_z / \breve{M}_z\|_{L^2(\mathbb{R}^2)}^2 + \|g_z\|_{L^2(\mathbb{R}^2)}^2 + \|\chi R_{\breve{a}}^{-1} I_{w[\breve{a}, \breve{f}]}[h_z / \breve{M}_z]\|_{L^2(\mathbb{R}^2)}^2.$$

Then, using Proposition 3.15, we have that

$$\|\chi R_{\breve{a}_z}^{-1} I_{w[\breve{a}_z, \breve{f}_z]}[h_z / \breve{M}_z]\|_{L^2(\mathbb{R}^2)} \leq C(\tilde{K}, D) \|\breve{f}\|_{C^\alpha(\mathbb{R}^3)} \|h_z / \breve{M}_z\|_{L^2(\mathbb{R}^2)} \quad \forall z \in \tilde{K},$$

and both sides vanish for $z \notin \tilde{K}$, hence the inequality remains valid for all $z \in \mathbb{R}$, i.e.

$$\|\chi R_{\breve{a}_z}^{-1} I_{w[\breve{a}_z, \breve{f}_z]}[h_z / \breve{M}_z]\|_{L^2(\mathbb{R}^2)} \leq C(\tilde{K}, D) \|\breve{f}\|_{C^\alpha(\mathbb{R}^3)} \|h_z / \breve{M}_z\|_{L^2(\mathbb{R}^2)} \quad \forall z \in \mathbb{R}.$$

By Lemma 3.17 we have that

$$\begin{aligned} \|h_z / \breve{M}_z\|_{L^2(\mathbb{R}^2)} &\leq C(\tilde{K}) \mathrm{e}^{C(\tilde{K})\|\breve{a}\|_\infty} \left(\frac{|\breve{f}|_{C^\alpha(\mathbb{R}^3)}}{\|\breve{f}\|_\infty} \right)^{3/\alpha} \frac{1}{\|\breve{f}\|_\infty} \|h_z\|_{L^2(\mathbb{R}^2)} \\ &\leq C(\tilde{K}, D) \left(\frac{|\breve{f}|_{C^\alpha(\mathbb{R}^3)}}{\|\breve{f}\|_\infty} \right)^{3/\alpha} \frac{1}{\|\breve{f}\|_\infty} \|h_z\|_{L^2(\mathbb{R}^2)}, \end{aligned}$$

hence a.e. for $z \in \mathbb{R}$, we have that

$$\|L_z^{-1}[\breve{a},\breve{f}]\begin{pmatrix} g \\ h \end{pmatrix}\|^2_{(L^2(\mathbb{R}^2))^2} \leq \|g_z\|^2_{L^2(\mathbb{R}^2)} + \left(1 + C(\tilde{K},D)\|\breve{f}\|_{C^\alpha}\right)\|h_z/\breve{M}_z\|_{L^2(\mathbb{R}^2)}$$
$$\leq \|g_z\|^2_{L^2(\mathbb{R}^2)} + C(\tilde{K},D)(1+\|\breve{f}\|_{C^\alpha})\frac{|\breve{f}|^{3/\alpha}_{C^\alpha}}{\|\breve{f}\|^{3/\alpha+1}_\infty}\|h_z\|_{L^2(\mathbb{R}^2)}.$$

Since $\int_{\mathbb{R}}\|g_z\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}}\int_{\mathbb{R}^2} g(y,z)^2 \mathrm{d}y\mathrm{d}z = \|g\|_{L^2(\mathbb{R}^3)}$ (and the same for h_z), then we can obtain the estimate for the norm of $L^{-1}[\breve{a},\breve{f}]$ by integrating over z:

$$\|L^{-1}[\breve{a},\breve{f}]\begin{pmatrix} g \\ h \end{pmatrix}\|_{L^2(\mathbb{R};L^2(\mathbb{R}^2))} = \int_{\mathbb{R}}\|L_z^{-1}[\breve{a},\breve{f}]\begin{pmatrix} g \\ h \end{pmatrix}\|^2_{(L^2(\mathbb{R}^2))^2}\mathrm{d}z$$
$$\leq \int_{\mathbb{R}}\|g_z\|^2_{L^2(\mathbb{R}^2)} + C(\tilde{K},D)(1+\|\breve{f}\|_{C^\alpha})\frac{|\breve{f}|^{3/\alpha}_{C^\alpha}}{\|\breve{f}\|^{3/\alpha+1}_\infty}\|h_z\|_{L^2(\mathbb{R}^2)}\mathrm{d}z$$
$$\leq \|g\|_{L^2(\mathbb{R}^3)} + C(\tilde{K},D)(1+\|\breve{f}\|_{C^\alpha})\frac{|\breve{f}|^{3/\alpha}_{C^\alpha}}{\|\breve{f}\|^{3/\alpha+1}_\infty}\|h\|_{L^2(\mathbb{R}^3)},$$

thus

$$\|L^{-1}[\breve{a},\breve{f}]\| \leq 1 + C(\tilde{K},D)(1+\|\breve{f}\|_{C^\alpha(\mathbb{R}^3)})\frac{|\breve{f}|^{3/\alpha}_{C^\alpha(\mathbb{R}^3)}}{\|\breve{f}\|^{3/\alpha+1}_\infty}.$$

It is easy to see that the image of the operator $L^{-1}[\breve{a},\breve{f}]$ is contained in $L^2(\tilde{K};L^2(\tilde{K}^2)\times L^2(\tilde{K}^2))$, since we are considering the domain $L^2(\tilde{K}^3)$ then for $z \notin \tilde{K}$ $g_z, h_z, f_z \equiv 0$ almost everywhere, and thus all the terms in $L_z^{-1}[\breve{a},\breve{f}]$ are equal to 0, while for $z \in \tilde{K}$ we have that $g_z, h_z, \in L^2(\tilde{K}^2)$ and the last term in $L_z^{-1}[\breve{a},\breve{f}]$ is square integrable and it is multiplied by a smooth function χ with support in $\tilde{K}^2$, hence the support of all terms is contained in $\tilde{K}^2$.

□

Proof of Theorem 3.5 Let $\tilde{K} \subset \mathbb{R}$ be a compact set, let $\breve{f} \in C^\alpha(\tilde{K}^3)$ with $\alpha > 1/2$ and $\breve{f} \not\equiv 0, \breve{f} \geq 0$, $\chi \in C_0^\infty(\tilde{K}^2)$ whose norms depend only in $\tilde{K}$. Then exists $\tilde{D} > 0$ such that the operator $(L+Q)[\breve{a},\breve{f}]$ defined on $L^2(\tilde{K}^3)\times L^2(\tilde{K}^3)$ is invertible for all $\breve{a} \in H^{5/2}(\tilde{K}^3)$, $\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)} < \tilde{D}$, and its inverse is

$$(L+Q)^{-1}[\breve{a},\breve{f}] = L^{-1}[\breve{a},\breve{f}]\sum_{k=0}^{\infty}\left(-(QL^{-1})[\breve{a},\breve{f}]\right)^k.$$

Proof. To achieve this we will use Neumann series. Since L is invertible by Lemma 3.3, we have that

$$(L+Q)[\breve{a},\breve{f}] = (I+QL^{-1})L[\breve{a},\breve{f}].$$

If $\|QL^{-1}[\breve{a},\breve{f}]\| < 1$, then we can invert the operator $(I + QL^{-1})L$ with inverse

$$\left((I + QL^{-1})L\right)^{-1} = L^{-1}\left(I + QL^{-1}\right)^{-1} = L^{-1}\left(\sum_{k=0}^{\infty}(-1)^k\left(QL^{-1}\right)^k\right).$$

(This can be done because these operators are in a Banach space, the norm condition ensures the convergence of the sum).

First of all, the composition $(QL^{-1})[\breve{a},\breve{f}]$ is well defined as an operator from $L^2(\tilde{K}^2 \times \tilde{K}) \times L^2(\tilde{K}^2 \times \tilde{K})$ to $L^2(\tilde{K}; L^2(\tilde{K}^2) \times L^2(\tilde{K}^2))$ because the image of $L^{-1}[\breve{a},\breve{f}]$ is contained in $L^2(\tilde{K}, L^2(\tilde{K}^2) \times L^2(\tilde{K}^2))$ but this space is isometric to $L^2(\tilde{K}^3) \times L^2(\tilde{K}^3)$. So, we need just to verify that exists $\tilde{D} > 0$ such that if $\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)} \le \tilde{D}$ then $\|(QL^{-1})[\breve{a},\breve{f}]\| < 1$.

Using the Propositions 3.16 and 3.18 we have that for $\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)} \le D$

$$\begin{aligned}
\|Q[\breve{a},\breve{f}]\| &\le C(\tilde{K},D)\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}(1 + \|\breve{f}\|_{C^\alpha(\mathbb{R}^3)}),\\
\|L^{-1}[\breve{a},\breve{f}]\| &\le 1 + C(\tilde{K},D)(1 + \|\breve{f}\|_{C^\alpha(\mathbb{R}^3)})\frac{|\breve{f}|^{3/\alpha}_{C^\alpha(\mathbb{R}^3)}}{\|\breve{f}\|^{3/\alpha+1}_{\infty}}\\
&\le C(\tilde{K},D)(1 + \|\breve{f}\|_{C^\alpha(\mathbb{R}^3)})\left(1 + \frac{|\breve{f}|^{3/\alpha}_{C^\alpha(\mathbb{R}^3)}}{\|\breve{f}\|^{3/\alpha+1}_{\infty}}\right),
\end{aligned}$$

with $C(\tilde{K},D) > 0$ a constant depending only in $\tilde{K}$, D. Thus we have that

$$\begin{aligned}
\|(QL^{-1})[\breve{a},\breve{f}]\| &\le \|Q[\breve{a},\breve{f}]\|\|L^{-1}[\breve{a},\breve{f}]\|\\
&\le C(\tilde{K},D)\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}(1 + \|\breve{f}\|_{C^\alpha(\mathbb{R}^3)})^2\left(1 + \frac{|\breve{f}|^{3/\alpha)}_{C^\alpha(\mathbb{R}^3)}}{\|\breve{f}\|^{3/\alpha+1}_{\infty}}\right).
\end{aligned}$$

Since $C(\tilde{K},D) > 0$ is non-decreasing with respect to D we can choose $\tilde{D} > 0$ such that

$$\tilde{D} < \left(C(\tilde{K},\tilde{D})(1 + \|\breve{f}\|_{C^\alpha(\mathbb{R}^3)})^2\left(1 + \frac{|\breve{f}|^{3/\alpha}_{C^\alpha(\mathbb{R}^3)}}{\|\breve{f}\|^{3/\alpha+1}_{\infty}}\right)\right)^{-1},$$

hence

$$\|(QL^{-1})[\breve{a},\breve{f}]\| < 1 \quad \forall \breve{a} \text{ such that } \|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)} \le \tilde{D},$$

concluding the invertibility by Neumann series. □

Chapter 4

Application to SPECT

4.1 Numerical algorithms in the linearized and non-linear cases

Let us review some numerical algorithms that could be applied in our setting.

4.1.1 Banach fixed-point method

This is a method to find fixed-points of operators (i.e. $\overline{x}$ such that $T(\overline{x}) = \overline{x}$). Let T be an operator such that $T : X \to X$ with X a banach space and T a contraction i.e. $\exists C \in (0,1)$ such that

$$\|T(x) - T(y)\| \leq C\|x - y\| \quad \forall x, y \in X,$$

then there exists $\overline{x} \in X$ such that $T(\overline{x}) = \overline{x}$ and an algorithm to find $\overline{x}$ is given by the steps

$$\begin{aligned} &\text{choose any } x_0 \in X; \\ &\text{let } x_{n+1} = T(x_n) \quad n \in \mathbb{N}. \end{aligned} \tag{4.1}$$

The existence of solution and convergence of the algorithm is given by the following theorem

Theorem 4.1 (**Banach fixed-point theorem**) *Let X be a Banach space and let $T : X \to X$ be a contraction, then T admits a unique fixed-point $\overline{x} \in X$. Furthermore, $\overline{x}$ can be found with Algorithm* (4.1) *and we have an estimate for the error*

$$\|\overline{x} - x_n\| \leq C^n \|\overline{x} - x_0\|,$$

with $C \in (0,1)$ the Lipschitz constant of the operator T.

4.1.2 Neumann series

A Neumann series is a mathematical series of the form $\sum_{k=0}^{\infty} S^k$ where S is a linear operator and S^k are k compositions of the operator. A sufficient condition for the convergence of the series is considering $S : X \to X$ with X a Banach space and such that $\|S\| < 1$.

These series are useful to invert perturbations of invertible linear operators. Let $T : X \to X$ be a linear operator with X a Banach space such that $T = I - S$ with I the identity and S a linear operator, then if $\|S\| < 1$, T is invertible and

$$T^{-1} = \sum_{k=0}^{\infty} S^k.$$

An algorithm derived from this formula is an iteration to approximate T^{-1} with the partial sums $\sum_{k=0}^{m} S^k$. For fixed x, the steps of the algorithm to invert $T(x)$ (i.e. to obtain a sequence $x_n \to x$) are

$$\begin{aligned} x_0 &= T(x), \\ x_{n+1} &= Sx_n + x_n. \end{aligned}$$

From the previous algorithm steps it is easy to see that $x_n = (\sum_{k=0}^{n} S^k)\, T(x)$ and we have an estimate of the error

$$\begin{aligned} \|x_n - x\| &= \|\left(\sum_{k=0}^{n} S^k\right) T(x) - x\| \\ &= \|\left(\sum_{k=0}^{\infty} S^k\right) T(x) - x - \left(\sum_{k=n+1}^{\infty} S^k\right) T(x)\| \\ &= \|\left(\sum_{k=n+1}^{\infty} S^k\right) T(x)\| \\ &\le \sum_{k=n+1}^{\infty} \|S\|^k \|T\|\ \|x\| \\ &= \frac{\|S\|^{m+1}}{1 - \|S\|} \|T\|\ \|x\|. \end{aligned}$$

4.1.3 Newton-Raphson method

The Newton-Raphson method is a technique for finding successively better approximations to the zeroes of a function (or operator), Let $F : X \to Y$ be an operator with X and Y Banach spaces, the iteration algorithm consists in taking

$$\begin{aligned} x_0 &\text{ such that } F(x_0) \approx 0, \\ x_{n+1} &= x_n - [F'(x_n)]^{-1} F(x_n), \end{aligned}$$

where $F'(x_n)$ is the Fréchet derivative computed at x_n. The Newton-Kantorovich theorem gives a sufficient condition for the convergence of the Newton-Raphson method.

Theorem 4.2 (**Newton-Kantorovich theorem**) *Let X and Y be Banach spaces and $F : D \subset X \to Y$. Suppose that on an open convex set $D_0 \subset D$, F is Fréchet differentiable and*

$$\|F'(x) - F'(y)\| \le K\|x - y\| \quad \forall x, y \in D_0.$$

For some $x_0 \in D_0$, assume that $\Gamma_0 = [F'(x_0)]^{-1}$ is defined on all Y and that $h = \beta K \eta \le 1/2$ where $\|\Gamma_0\| \le \beta$ and $\|\Gamma_0 F x_0\| \le \eta$. Set

$$t^* = \frac{1}{\beta K}\left(1 - \sqrt{1 - 2h}\right), \quad t^{**} = \frac{1}{\beta K}\left(1 + \sqrt{1 - 2h}\right),$$

and supppose that $S = \{x \in X,\ \|x - x_0\| \le t^*\} \subset D_0$. Then the Newton-Raphson iterates x_n are well defined, lie in S and converge to a solution x^* of $Fx = 0$ wich is unique in $D_0 \cap \{x \in X,\ \|x - x_0\| \le t^{**}\}$. Moreover, if $h < 1/2$ the order of convergence is at least quadratic.

4.2 SPECT: recovering of source and attenuation

4.2.1 Discretization of the measurements

Given the model derived in Chapter 2, for an attenuation map a and source map f both defined on $\mathbb{R}^3$ and with compact support, the operator that gives the external measurements is

$$F[a, f](s, \theta, z) = \begin{pmatrix} R_{a_z}[f_z](s, \theta) \\ \tilde{C} R_{a_z}[a_z M_z[a, f]](s, \theta) \end{pmatrix} \quad s, z \in \mathbb{R}, \theta \in S^1, \tag{4.2}$$

where $\tilde{C}$ is a fixed constant,

$$M[a, f](x) = \int_{S^2} \int_0^\infty f(x + t\phi) \mathrm{e}^{-\int_0^t a(x+\tau\phi)\mathrm{d}\tau} \mathrm{d}t\mathrm{d}\phi, \qquad x \in \mathbb{R}^3$$
$$a_z(y) = a(y, z) \quad f_z(y) = f(y, z) \quad M_z[a, f](y) = M[a, f](y, z), \qquad \forall y \in \mathbb{R}^2, z \in \mathbb{R}$$

and the variables θ and s are the angular variable and the spatial translation variable respectively, z is the height variable.

Let $\Theta \subset [0, 2\pi]$ be the set of finite angular measurements, $\Sigma \subset \mathbb{R}$ the set of finite spatial translation measurements and $H \subset \mathbb{R}$ the set of finite height measurements, then the discretization of $F[a, f]$ is given by

$$F[a, f](s, \theta, z) = \begin{pmatrix} R_{a_z}[f_z](s, \theta) \\ \tilde{C} R_{a_z}[a_z M_z[a, f]](s, \theta) \end{pmatrix} \quad s \in \Sigma, z \in H, \theta \in \Theta.$$

Observation The validity of the representation of this operator by finite values (in the angular and spatial translation variables) is studied in [19] Chapter 3, but roughly speaking it is given by the Shannon sampling theorem (described in [19]) which for finite measurements allows a reconstruction of a regularized version of the function in study, this also holds for the height variable.

It is also of fundamental importance to have an algorithm to compute accurately the operator R_a^{-1}, there are several algorithms to achieve this, see for example [13, 17, 20]. In our heuristic method we used the algorithm given by Natterer in [20] and implemented by Francois Monard (extracted from his webpage [1]).

4.2.2 Guideline for an algorithm for the linearized inverse problem

We will proceed to describe how to obtain an algorithm to solve our inverse problem in the linearized setting, this has not been implement and for a convergence result it requires many unproven assumptions.

For the linearized inverse problem (recall Chapter 3), we assume known reference attenuation and source map $\breve{a} \in H^{5/2}(K^3), \breve{f} \in C^\alpha(K^3), \alpha > 1/2$ with $K \subset \mathbb{R}$ a compact set and unknown perturbations $\delta a, \delta f \in L^2(K^3)$, the measurements that the algorithm can use are expressed as $F[\breve{a} + \delta a, \breve{f} + \delta f]$ with F defined in (4.2).

Assume the operator $F : H^{5/2}(K^3) \times C^\alpha(K^3) \to Y = (Y_1 \times Y_2)_{z\in\mathbb{R}}$ is Fréchet differentiable and for some Banach space $(Y_1 \times Y_2)_{z\in\mathbb{R}}$. Hence we have that

$$F[\breve{a} + \delta a, \breve{f} + \delta f] = F[\breve{a}, \breve{f}] + DF[\breve{a}, \breve{f}](\delta a, \delta f) + o(\delta a, \delta f).$$

Let $\tilde{K} \subset \mathbb{R}$ be a compact set such that $K \subset \tilde{K}$ and consider a cut-off function $\chi \in C^\infty(\tilde{K}^2)$ whose norms depend only on the compact K, then taking the inverse on the attenuated Radon transform and multiplying by χ at each height and component of $(Y_1 \times Y_2)_{z\in\mathbb{R}}$ (the notation for this will be $\chi R_{\breve{a}}^{-1}[(f,g)_z] = \left(\chi R_{\breve{a}_z}^{-1} f_z, R_{\breve{a}_z}^{-1} g_z\right)_{z\in\mathbb{R}}$), yields

$$\begin{aligned} \chi R_{\breve{a}}^{-1}\left[F[\breve{a} + \delta a, \breve{f} + \delta f] - F[\breve{a}, \breve{f}]\right] &= \chi R_{\breve{a}}^{-1}\left[DF[\breve{a}, \breve{f}](\delta a, \delta f)\right] + R_{\breve{a}}^{-1}[o(\delta a, \delta f)] \\ &= (L + Q)[\breve{a}, \breve{f}](\delta a, \delta f) + R_{\breve{a}}^{-1}[o(\delta a, \delta f)], \end{aligned} \tag{4.3}$$

where L and Q are the operators studied in Theorem 3.5, hence assuming $\breve{f} \geq 0, \breve{f} \not\equiv 0$ and $\|\breve{a}\|_{H^{5/2}(\mathbb{R}^3)}$ sufficient small, we have an inversion formula. Applying it to equation (4.3) and denoting $\varepsilon = (L + Q)^{-1}[\breve{a}, \breve{f}]\left(R_{\breve{a}}^{-1}[o(\delta a, \delta f)]\right)$ we obtain

$$\begin{aligned} & (L + Q)^{-1}[\breve{a}, \breve{f}]\left(\chi R_{\breve{a}}^{-1}\left[F[\breve{a} + \delta a, \breve{f} + \delta f] - F[\breve{a}, \breve{f}]\right]\right) = (\delta a_z, \delta f_z)_{z\in\mathbb{R}} + \varepsilon \\ \Leftrightarrow \quad & L^{-1}[\breve{a}, \breve{f}] \sum_{k=0}^{\infty} \left(-(QL^{-1})[\breve{a}, \breve{f}]\right)^k \left(\chi R_{\breve{a}}^{-1}\left[F[\breve{a} + \delta a, \breve{f} + \delta f] - F[\breve{a}, \breve{f}]\right]\right) = (\delta a_z, \delta f_z)_{z\in\mathbb{R}} + \varepsilon. \end{aligned}$$

Finally, assuming that if $\|(\delta a,\delta f)\|_{L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)}\to 0$ then $\|R_{\breve{a}}^{-1}o(\|\delta a\|+\|\delta f\|)\|_{L^2(\mathbb{R};L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2))}\to 0$, we obtain an algorithm to approximate the unknown small perturbations $\delta a,\delta f\in L^2(K^3)$ by approximating the series by its partial sum and computing each term on the left hand side of the equation.

Observation Under assumptions of the Theorem 3.5, we have an inverse of operator $DF[\breve{a},\breve{f}]$ given by

$$\left(DF[\breve{a},\breve{f}]\right)^{-1}=L^{-1}[\breve{a},\breve{f}]\sum_{k=0}^{\infty}\left(-(QL^{-1})[\breve{a},\breve{f}]\right)^{k}\chi R_{\breve{a}}^{-1}.$$

4.2.3 Formal guideline to solve the non-lineal inverse problem with the Newton-Raphson method

We will describe how to use the Newton-Raphson method to obtain an algorithm to approximate the values of $\breve{a},\breve{f}$ for the non-linear setting. This section is formal since we do not give conditions for the algorithm to converge.

For the non-linear inverse problem the avaliable information is expressed as $F[\breve{a},\breve{f}]$ with F determined in (4.2). Consider for $K\subset\mathbb{R}$ a compact set the unknown values $\breve{a},\breve{f}\in H^{5/2}(K^3)$ and $G_K:L^2(K^3)\to H^{5/2}(K^3)$ an invertible bounded linear operator with bounded inverse (for notational simplicity, if $a,f\in L^2(\mathbb{R}^3)$ we define $G_K(a,f):=(G_K(a),G_K(f))$), consider

$$\begin{aligned}\tilde{F}&:L^2(K^3)\times L^2(K^3)\to Y=(Y_1,Y_2)_{z\in\mathbb{R}}\\ \tilde{F}(a,f)&=F[G_K(a,f)]-F[\breve{a},\breve{f}].\end{aligned}$$

Finding a zero of this functional is the same as finding $a,f\in L^2(K^3)$ such that $F[G_K(a,f)]=F[\breve{a},\breve{f}]$, thus assuming inyectivity of the operator F, we get that $G_K(a,f)=(\breve{a},\breve{f})$ which solves our inverse problem. To achieve this we use the Newton-Raphson method, as presented in Section 4.2, we need (among other requirements) the operator $\tilde{F}$ to be Fréchet differentiable and invertible.

Under the hypothesis and definitions for the linearized inverse problem in Subsection 4.2.2, we have an inverse for operator $DF[G_K(a,f)]$ given by

$$(DF[G_K(a,f)])^{-1}=L^{-1}\left(\sum_{k=0}^{\infty}(-QL^{-1})^k[G_K(a,f)]\right)\chi R_{G_K(a)}^{-1}.$$

Thus, for the operator $\tilde{F}$, we deduce

$$\begin{aligned}D\tilde{F}(a,f)&=DF[G_K(a,f)]\circ G_K\\ \Rightarrow\ \left(D\tilde{F}(a,f)\right)^{-1}&=G_K^{-1}\circ(DF[G_K(a,f)])^{-1}\\ &=G_K^{-1}L^{-1}\left(\sum_{k=0}^{\infty}(-QL^{-1})^k[G_K(a,f)]\right)\chi R_{G_K(a)}^{-1},\end{aligned}$$

and we can implement the Newton-Raphson method

$$(a^{n+1}, f^{n+1}) = (a^n, f^n) - G_K^{-1} L^{-1} \left(\sum_{k=0}^{\infty} (-QL^{-1})^k [G_K(a,f)] \right) \chi R^{-1}_{G_K(a)} F(G_K(a^n, f^n)).$$

There are several conditions that must hold to ensure the success of this method, particularly, the hypothesis over $G_K(a^n)$ and $G_K(f^n)$ needed for Theorem 3.5 must be controled for each n for the convergence of the series.

Observation An example of operator G_K is taking $G_K(a) = \tilde{a}$ where $\tilde{a}$ is the solution of the problem

$$\begin{aligned} \left(I + \varepsilon\, (-\Delta)^{5/4} \right) \tilde{a} &= a \quad \forall x \in K^3 \\ \tilde{a} &= 0 \quad \forall x \in \partial K^3, \end{aligned}$$

for sufficiently small $\varepsilon > 0$ (the operator $(-\Delta)^p$ is the p-laplacian). This is a computable invertible bounded linear operator that regularizes functions and thus can be used for the algorithm.

4.2.4 Heuristical Banach fixed point method results

We developed and programed a fixed-point algorithm to obtain both the attenuation and source for our inverse problem, but we did not demonstrated a convergence condition for the algorithm, thus is purely heuristical.

For a fixed attenuation map a and source map f both defined on $\mathbb{R}^3$ and with compact support. Recall that the ballistic measurements at height $z \in \mathbb{R}$ are given by

$$\mathcal{A}_0^z = R_{a_z}[f_z],$$

and that the first order scattering measurements at height $z \in \mathbb{R}$ are given by

$$\mathcal{A}_1^z = \tilde{C} R_{a_z}[a_z \cdot M_z[a,f]].$$

The algorithm consists in an iterative update of the attenuation and source maps using the information given by the measurements. We start with a known attenuation a^0 defined on $\mathbb{R}^3$, then using the ballistic measurements at every height we can reconstruct a source map f^0, and using both functions and the first order scattering measurements we update the value of the attenuation map to a^1. Following this process we iterate to obtain a^{i} and f^{i} for $\mathrm{i} \in \mathbb{N}$.

Mathematically, for $\mathrm{i} \in \mathbb{N}$, to obtain the source map f^{i} from the attenuation map a^{i} and the ballistic measurements $\mathcal{A}_0$ we use the inverse of the attenuated Radon transform at each height

$$f^{\mathrm{i}}(y,z) = R^{-1}_{a^{\mathrm{i}}_z}[\mathcal{A}^z_0](y) \quad y \in \mathbb{R}^2, z \in \mathbb{R}.$$

Observation Note that if $a^{\mathrm{i}} = a$ then, recalling the definition of $\mathcal{A}_0$, we obtain directly the source function f.

For $\mathrm{i} \in \mathbb{N}$, to update the attenuation map from a^{i} to $a^{\mathrm{i}+1}$, using the information of the first order scattering measurements $\mathcal{A}_1$ and the source function f^{i}, we first compute $M[a^{\mathrm{i}}, f^{\mathrm{i}}]$ and then obtain $a^{\mathrm{i}+1}$ using this formula

$$a^{\mathrm{i}+1}(y,z) = \frac{R^{-1}_{a^{\mathrm{i}}_z}[\frac{1}{\tilde{C}}\mathcal{A}^z_1](y)}{M[a^{\mathrm{i}}, f^{\mathrm{i}}](y,z)} \quad y \in \mathbb{R}^2, z \in \mathbb{R}.$$

Observation If $a^{\mathrm{i}} = a$ and $f^{\mathrm{i}} = f$ then, recalling the definition of $\mathcal{A}_1$, we have that $a^{\mathrm{i}+1} = a$. Also note that if $a^{\mathrm{i}}, f^{\mathrm{i}} \geq 0$ and $f > 0$ in a set with positive measure, then $M[a^{\mathrm{i}}, f^{\mathrm{i}}](y,z) \geq C > 0 \quad \forall y \in \mathbb{R}^2, z \in \mathbb{R}$ for some constant depending on a^{i} and f^{i}.

Thus the operator that given a^{i} returns $a^{\mathrm{i}+1}$ is

$$T[a^{\mathrm{i}}](y,z) = \frac{R^{-1}_{a^{\mathrm{i}}_z}[\frac{1}{\tilde{C}}\mathcal{A}^z_1](y)}{M\left[a^{\mathrm{i}}, \left(R^{-1}_{a^{\mathrm{i}}_{z'}}[\mathcal{A}^{z'}_0]\right)_{z' \in \mathbb{R}}\right](y,z)} \quad y \in \mathbb{R}^2, z \in \mathbb{R}. \tag{4.4}$$

To obtain a convergence result, from the Banach fixed point method described in Subsection 4.1.1, it is needed to define this operator in a suitable Banach space such that T is a contraction.

Examples from synthetic SPECT data

This heuristical algorithm was implemented and tested in synthetic data corresponding to three dimentional objects with attenuation and source distributions. The simulated photons followed the radiative transfer equation derived in Chapter 2 (just ballistic and first order scattering photons were simulated), the constant $\tilde{C}$ was considered equal to 1 in both the algorithm and the synthetic data.

Regarding to the synthetic data, consider the cube $[-1,1] \times [-1,1] \times [-1,1]$ discretized with $128 \times 128 \times 128$ pixels (the size of each pixel is $2/128 \times 2/128 \times 2/128$), the synthetic objects are contained the cube, particularly inside the cylinder of diameter 2 and height 2 centered in the origin. The measuring variables (angle, spatial translation and height) are all taken 128 times, i.e. $|\Theta| = |\Sigma| = |H| = 128$, and uniformly distributed in a reasonable domain, that is $H = \Sigma = \{(2j-1)/128 - 1,\ \mathrm{i} \in \{1,2,..128\}\}, \Theta = \{2\pi j/128,\ j \in \{0,1,2,..127\}\}$ (See Figure 4.1).

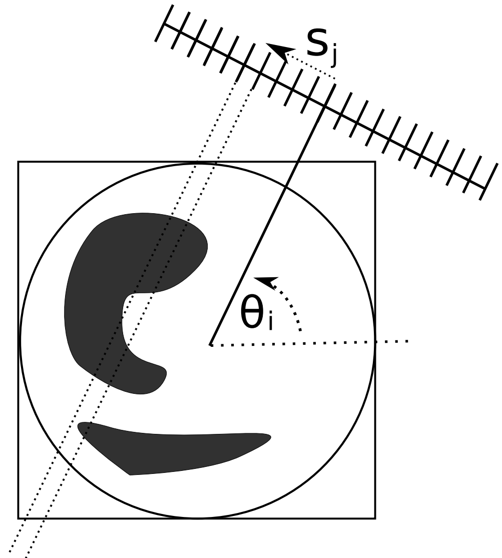

Figure 4.1: Sketch of the measuring variables for fixed height. The grey area represents the radioactive distribution and the parallel dotted lines represent the sector where the photons are being measured.

Example 1: Simulated Thorax

Following the examples used in [9, 10], we constructed a three dimentional synthetic thorax composed of simplified lungs, heart, 5 ribs, vertebral column and a torso, with high radiactive source located in the surface of the heart and low radiactive source in the rest of the object.

In Figure 4.2 we can see cross-sectional images of the example's attenuation and source map. In Figure 4.3 we can see, for fixed angle, the simulated ballistic and first order scattering measurements. In Figure 4.4 we present the obtained reconstructions and their respective errors with three iterations of the algorithm.

To compare the results of our heuristic reconstruction with a standard source reconstruction method (that consists in applying the inverse of the Radon transform to the ballistic measurements asuming a known constant attenuation), we test the algorithms in the next cases:

- **First test:** The algorithm is tested over the displayed data in Figures 4.2, 4.3.
- **Higher attenuation:** The algorithm is tested over the same source map, but the attenuation map is amplified 10 times.
- **10% noise:** The algorithm is tested over the same data of the first test, but we add white noise to the measurements, with values from 0 to 0.1.

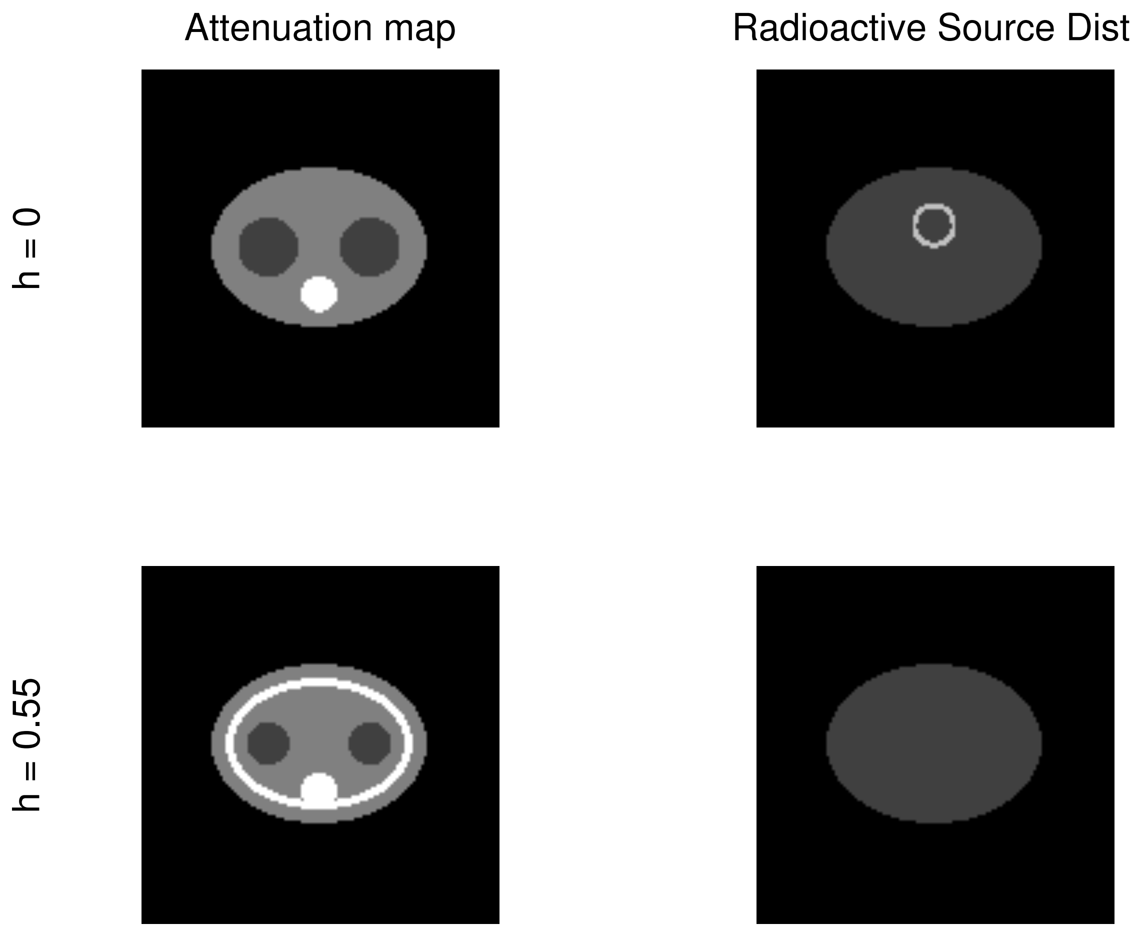

Figure 4.2: Cross-sectional views of the synthetic thorax. At the left hand side there are two views of the attenuation map. At the right hand side there are two views of the radioactive source map. The top images are at height 0 (the circle in the right hand image corresponds to the surface of the heart). The bottom images are at height 0.55.

- **20% noise:** The setting is the same than the previous one, but the noise has values from 0 to 0.2.

The results are sumarized in Table 4.1.

Regarding the convergence of the algorithm, in Table 4.2 we can see the relative error of the algorithm at each iteration in the first test.

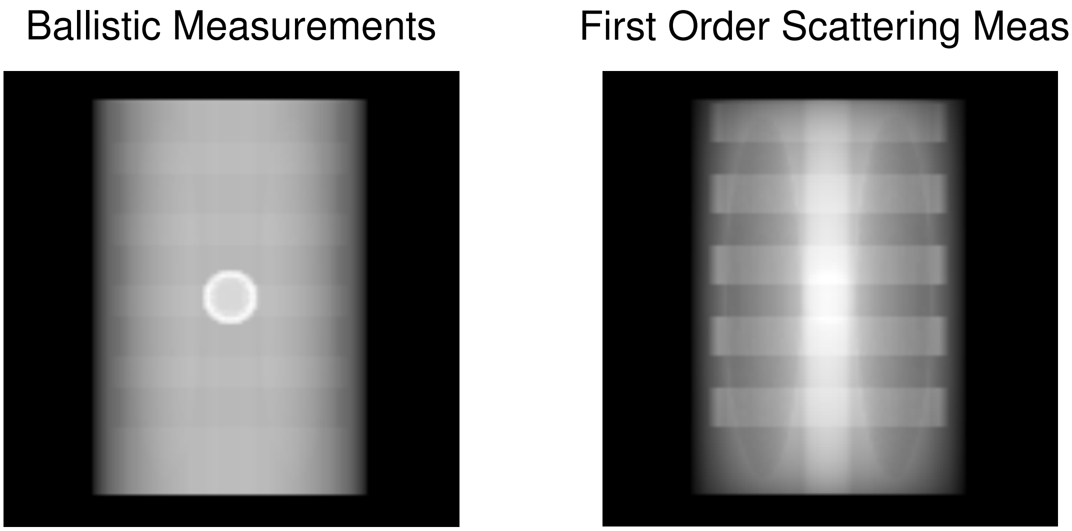

Figure 4.3: For fixed angle (i.e. a frontal view of the thorax), the ballistic and first order scattering measurements. The images are normalized for better view.

Table 4.1: Relative L^2 error of tested reconstructions

	Obtained source	Standard source	Obtained attenuation
First test	0.110	1.029	0.171
Higher attenuation	0.568	0.932	1.000
10% noise	0.237	1.164	0.766
20% noise	0.430	1.429	1.314

Table 4.2: Relative L^2 error for iterations of the algorithm

Number of iterations	1	2	3	4	5
Source reconstruction relat. errors	1.029	0.113	0.110	0.110	0.110
Attenuation reconstruction relat. errors	0.289	0.180	0.171	0.170	0.171

Example 2: Less regular object (LRO)

This example was constructed to test the algorithm with a less regular object compared to the thorax example. The discretization and measuring setting is the same as the previous one.

In the LRO, both the attenuation and source maps at each height z corresponds to shepp-logan phantoms as it can be seen in Figure 4.5, the whole three dimentional object is constructed by rotating constinuously these phantoms at each height, with both maps not

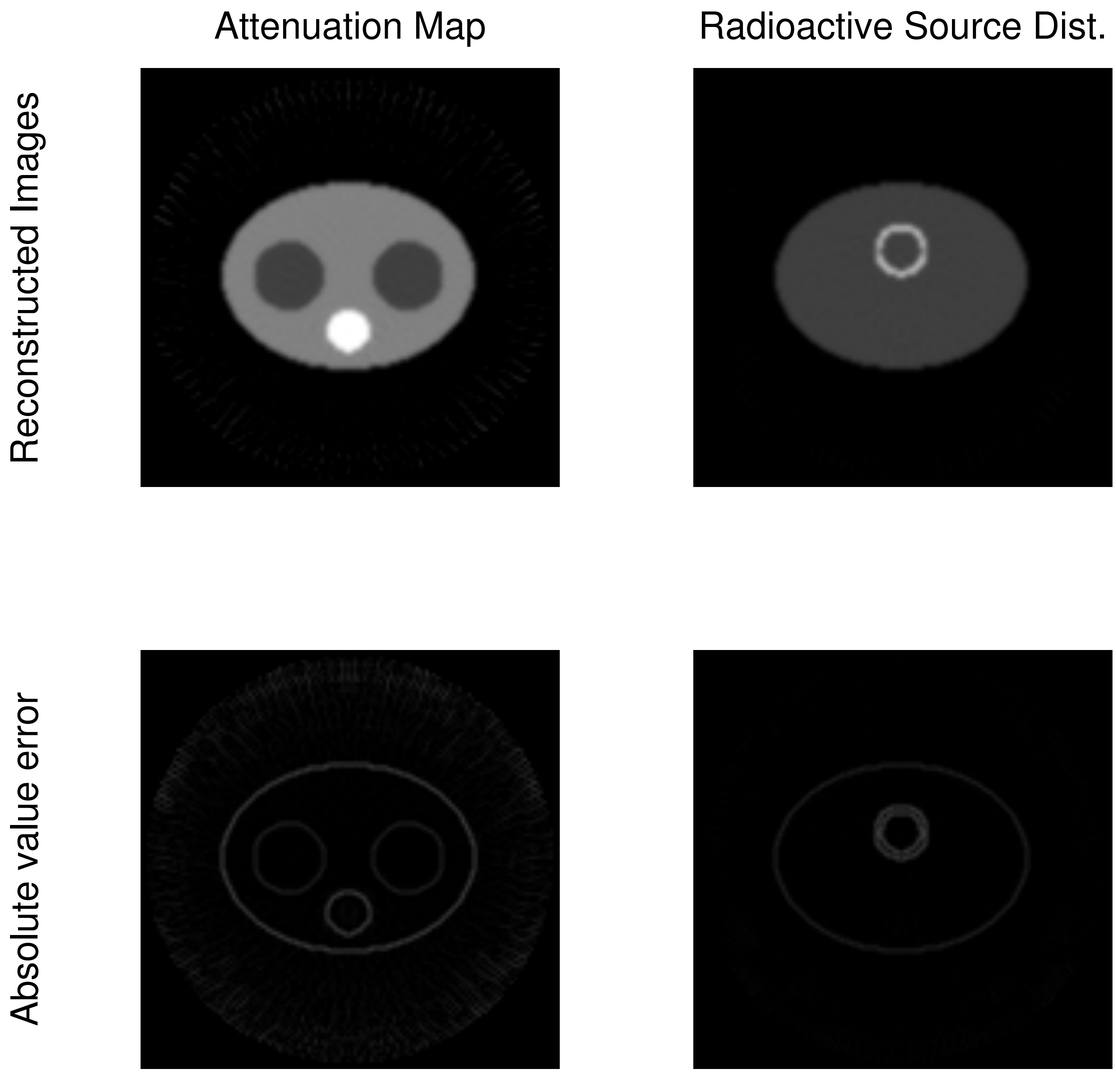

Figure 4.4: For height h=0, the thorax reconstructions. At the top, both images correspond to the reconstructed attenuation map and radioactive source distribution, at the bottom the abolute value error of the reconstructions. As it can be seen, there is a smoothing effect in the reconstruction.

rotating in the same direction. The measurements of the ballistic and first order scattering photons can be seen at Figure 4.6.

In Figure 4.7 we present the obtained reconstructions and their respective errors with three iterations of the algorithm, at the left hand side is the reconstructed attenuation map and at the right hand side the source map reconstruction, the bottom images represent the

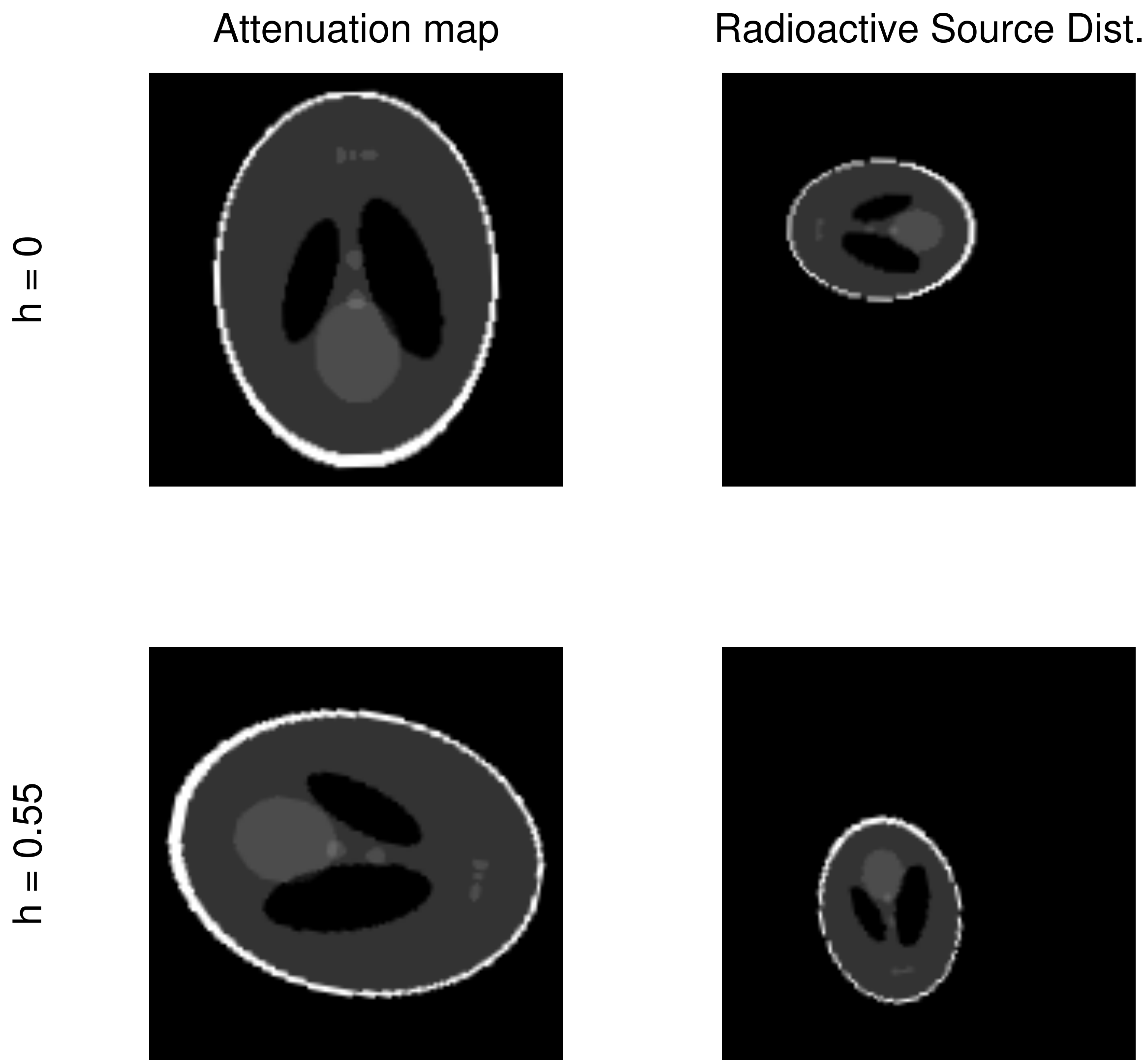

Figure 4.5: A cross-section of the LRO, to the left hand side the attenuation map and to the right hand side the radioactive source map.

respective absolute value error.

We tested the algorithm over the same tests done to the thorax example, the results can be seen in Table 4.3

Regarding the convergence of the algorithm, in Table 4.4 we can see the relative L^2 error of the algorithm at each iteration.

Taking into account the results for synthetic data, the heuristic fixed-point algorithm gives

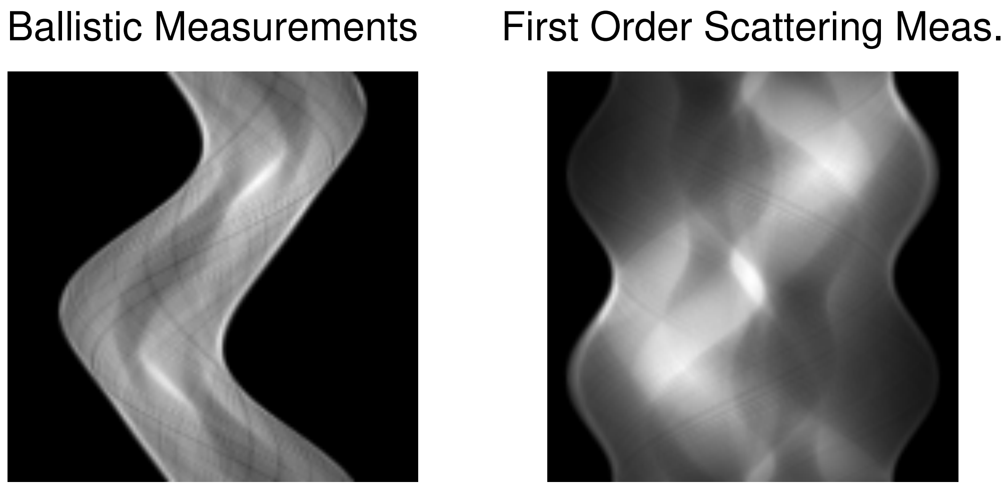

Figure 4.6: External measurements of the synthetic 3-D object at a fixed angle, to the left hand side the ballistic measurements and to the right hand side the first order scattering measurements

Table 4.3: Relative L^2 error of tested reconstructions

	Obtained source	Standard source	Obtained attenuation
First test	0.234	0.352	0.329
Higher attenuation	0.549	0.759	1
10% noise	0.351	0.366	0.38
20% noise	0.407	0.411	0.58

Table 4.4: Relative L^2 error for iterations of the algorithm

Number of iterations	1	2	3	4	5
Source reconstruction relat. errors	0.3519	0.3308	0.3289	0.3287	0.3286
Attenuation reconstruction relat. errors	0.305	0.2389	0.2344	0.2345	0.2347

better approximations to the source map in contrast to a standard method of considering constant attenuation. As it can be seen from Tables 4.4 and 4.2 it stabilizes at the third iteration, but just can achieve a regularized version of the functions to reconstruct. As a comment, although 3 iterations is fast, each iteration requires a lot of computing power, since it must handle the entire three dimentional object and not just a 2 dimensional slice of it. On average the standard method requires less than a minute to compute, our algorithm takes 20 minutes per iteration (These times measured in a personal computer).

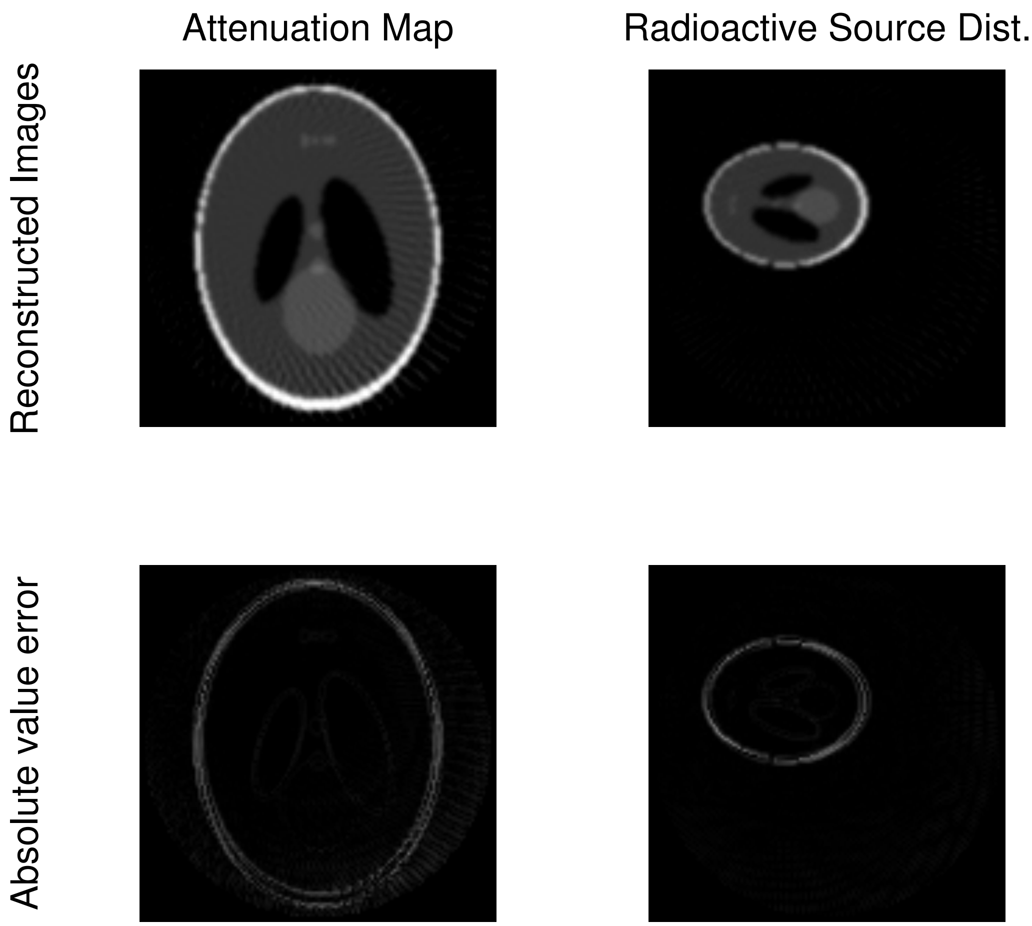

Figure 4.7: The obtained reconstructions for a specific height, the images at the top are the reconstructed maps a_z to the left hand side and f_z to the right hand side, at the bottom there are the respective error.

Example from real SPECT data

We also tried the heuristic algorithm with real data, the measures can be seen in Figure 4.8, these are lateral measurements of the ballistic and first order scattering photons of an unknown object.

With regards to the data, it can be seen that has low resolution and high noise levels, the total amount of angles measured is 128, the height and spatial measures are 128 each one, thus $|\Theta| = |\Sigma| = |H| = 128$. These measurements were provided by the "Biomedical Imaging Center of the Pontificia Universidad Católica de Chile".

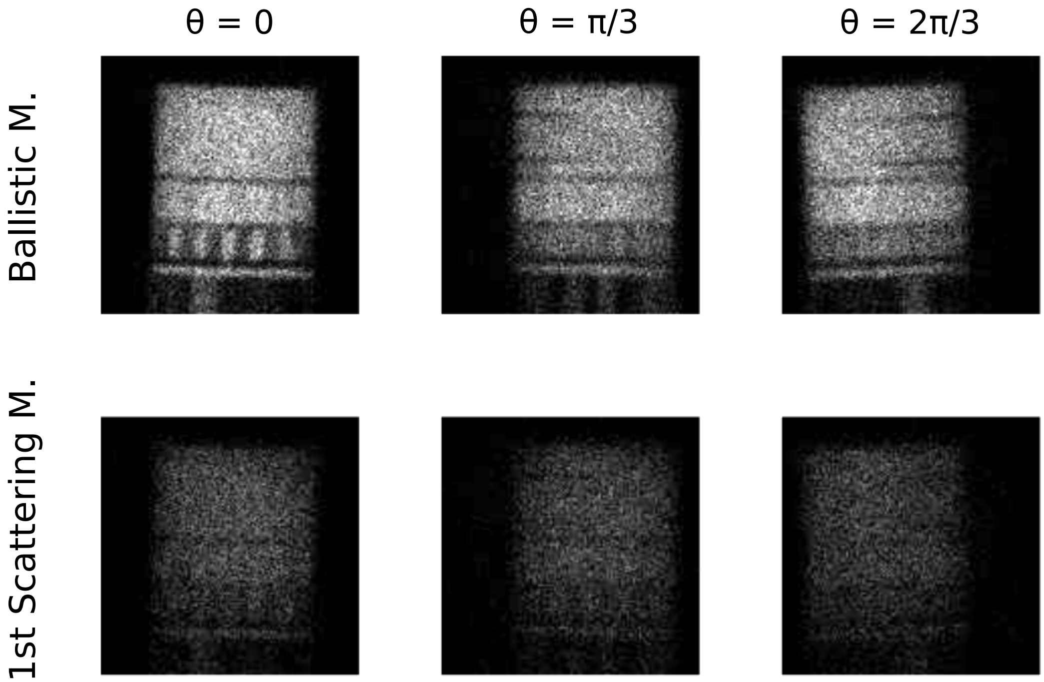

Figure 4.8: The above row are ballistic measurements of the object, the below row are first order scattering measurements of the object

Since the object is unknown we dont know the accuracy of the obtained reconstructions, thus we will only compare the results with the standard method of reconstruction.

The reconstructions of the source map can be seen in Figure 4.9, these are three different heights at which the reconstruction was made. The above images are the result of our fixed-point algorithm and the below images are the reconstructions obtained using a constant attenuation map. All measurements were smoothed for better visualization, also all reconstructions were normalized, the constant $\tilde{C}$ from operator (4.4) was assumed equal to 1 and the starting value of the fixed-point algorithm was a constant attenuation map.

We observe that the apparent quality of the source reconstruction with the fixed-point method is better than the standard reconstruction, the images have less noise and the details are more clear.

Lastly, the reconstructed attenuation map obtained by our algorithm can be seen in Figure 4.10, these reconstructions seem to contain a large amount of noise and are very similar in structure to the reconstructed source map, since we dont know exactly the attenuation map and do not have a alternative method to compare with, we can not judge properly the results.

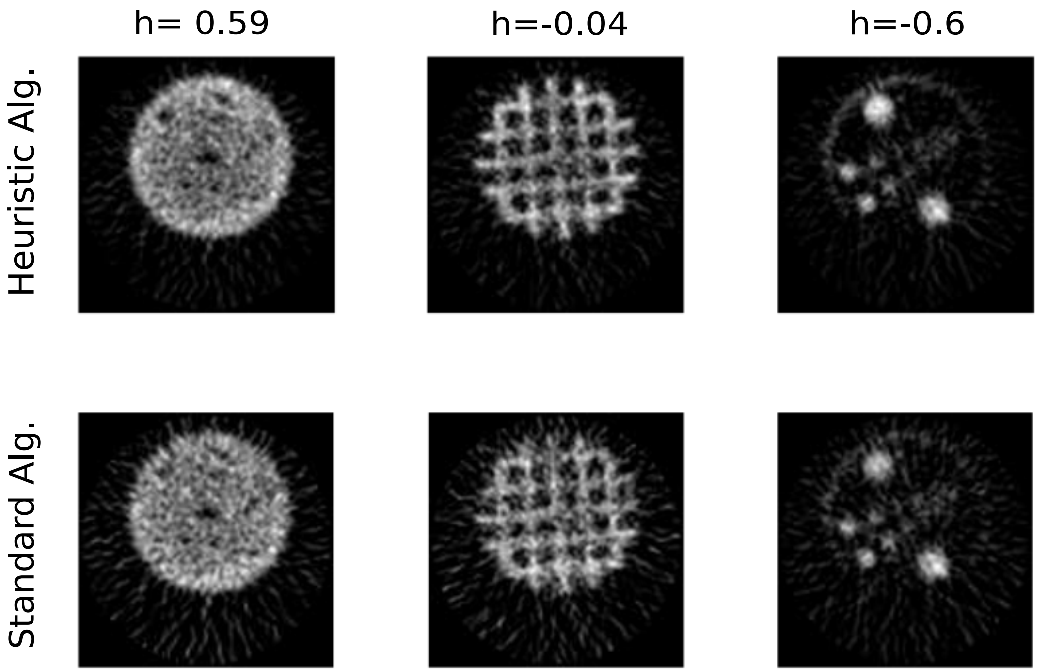

Figure 4.9: These are source map reconstructions at different heights, the above reconstructions corresponds to the fixed-point method, the below reconstructions are those obtained with the standard reconstruction method.

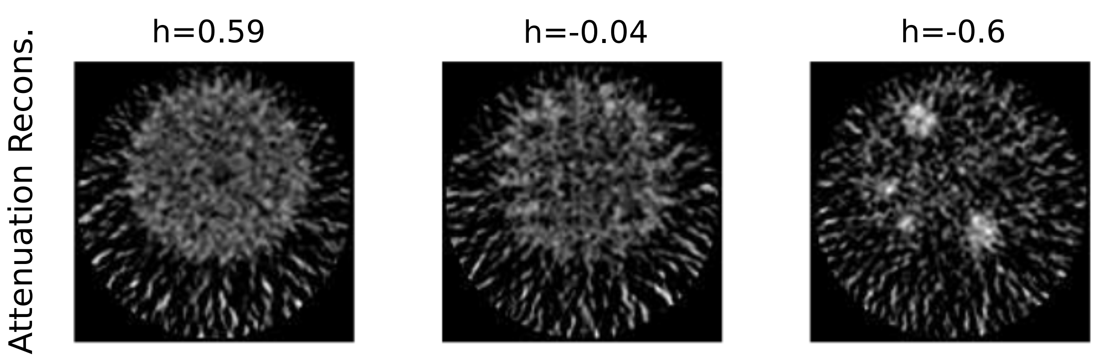

Figure 4.10: Reconstructed attenuation map obtained with the implemented fixed-point algorithm.

Conclusion

We have extended the standard mathematical model that explains the measurements used for the inverse problem regarding the SPECT medical method. This extension is based on considering scattering effects in the photon dynamics and assuming the possibility of measuring scattered photons. To avoid the incorporation of more unknown variables to the inverse problem it is also assumed a strong relation between the scattering properties of a medium and its attenuation properties. As a result of this extension we obtained an inverse problem that has more information available to reconstruct the same unknown variables as the original problem (the identification problem).

We have formally linearized the obtained equations and we have deduced an inversion theorem for the resulting linear operator under small attenuation assumptions and sufficient regularity of the reference attenuation and source map, this linearization was done by means of taking the directional derivative of the operator that represents the available data for our inverse problem

$$F[a,f](s,\theta,z) = \begin{pmatrix} R_{a_z}[f_z](s,\theta) \\ \tilde{C}R_{a_z}[a_z M_z[a,f]](s,\theta) \end{pmatrix} \quad s,z \in \mathbb{R}, \theta \in S^1$$

With the deduced theorem we have provided guidelines to implement algorithms to solve the derived inverse problem for the linear and non-linear cases, elthogh with no convergence results. For the full non-linear case, we have implemented an heuristic algorithm that tested over synthetic and real data works better than the standard inversion procedure, consisting in assuming a constant attenuation map.

Here are some open problems that have arisen from the present work.

- Prove Fréchet differentiability of the operator F that represents the available information for our inverse problem.
- Demonstrate a convergence result for the presented algorithms.

These problems are needed for a deeper understanding of the proposed algorithms.

Chapter 5

Bibliography

[1] Francois Monard's webpage. http://www.math.washington.edu/ fmonard/. Accessed: 2011-06-01.

[2] E.V. Arbuzov, A.L. Bukhgeim, and S.G. Kazantsev. Two-dimensional tomography problems and the theory of A-analytic functions.

[3] Guillaume Bal. On the attenuated radon transform with full and partial measurements. *Inverse Problems*, 20(2):399, 2004.

[4] Guillaume Bal. Inverse transport theory and applications. *Inverse Problems*, 25(5):053001, 2009.

[5] Guillaume Bal and Alexandre Jollivet. Combined source and attenuation reconstructions in spect. *Tomography and Inverse Transport Theory. Contemp. Math*, 559:13–28, 2011.

[6] Guillaume Bal and Philippe Moireau. Fast numerical inversion of the attenuated radon transform with full and partial measurements. *Inverse Problems*, 20(4):1137, 2004.

[7] Jan Boman. Local injectivity for weighted radon transforms. 2012.

[8] Jan Boman and Jan-Olov Strömberg. Novikov's inversion formula for the attenuated radon transform a new approach. *The Journal of Geometric Analysis*, 14(2):185–198, 2004.

[9] A V Bronnikov. Numerical solution of the identification problem for the attenuated radon transform. *Inverse Problems*, 15(5):1315–1324, Oct 1999.

[10] Volker Dicken. A new approach towards simultaneous activity and attenuation reconstruction in emission tomography. *Inverse Problems*, 15(4):931–960, Aug 1999.

[11] Stefano Fanti, Mohsen Farsad, and Luigi Mansi. *Atlas of SPECT-CT*. Springer-Verlag, 2011.

[12] David Gilbarg and Neil S Trudinger. *Elliptic partial differential equations of second order*, volume 224. Springer, 2001.

[13] Daniel Gourion and Dominikus Noll. The inverse problem of emission tomography. *Inverse Problems*, 18(5):1435, 2002.

[14] Daniel Gourion, Dominikus Noll, Pierre Gantet, Anna Celler, and Jean-Paul Esquerré. Attenuation correction using spect emission data only. *Nuclear Science, IEEE Transactions on*, 49(5):2172–2179, 2002.

[15] Alexander Hertle. The identification problem for the constantly attenuated radon transform. *Mathematische Zeitschrift*, 197(1):13–19, 1988.

[16] Peter Kuchment and Eric Todd Quinto. Some problems of integral geometry arising in tomography. *The Universality of the Radon Transform. Oxford Univ. Press, London*, 2003.

[17] Leonid A Kunyansky. A new spect reconstruction algorithm based on the novikov explicit inversion formula. *Inverse Problems*, 17(2):293, 2001.

[18] Frank Natterer. Computerized tomography with unknown sources. *SIAM Journal on Applied Mathematics*, 43(5):1201–1212, 1983.

[19] Frank Natterer. *The mathematics of computerized tomography.* Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1986.

[20] Frank Natterer. Inversion of the attenuated radon transform. *Inverse problems*, 17(1):113, 2001.

[21] Frank Natterer and Frank Wubbeling. *Mathematical Methods in Image Reconstruction.* Society for Industrial and Applied Mathematics, Jan 2001.

[22] RG Novikov. On the range characterization for the two-dimensional attenuated x-ray transformation. *Inverse problems*, 18(3):677, 2002.

[23] Roman Novikov. An inversion formula for the attenuated X-ray transformation. *Arkiv för Matematik*, 40(1):145–167, April 2002.

[24] A Puro and A Garin. Cormack-type inversion of attenuated radon transform. *Inverse Problems*, 29(6):065004, 2013.

[25] Ronny Ramlau and Rolf Clackdoyle. Accurate attenuation correction in spect imaging using optimization of bilinear functions and assuming an unknown spatially-varying attenuation distribution. In *Nuclear Science Symposium, 1998. Conference Record. 1998 IEEE*, volume 3, pages 1684–1688. IEEE, 1998.

[26] Hans Rullgård. An explicit inversion formula for the exponential radon transform using data from 180. *Arkiv för matematik*, 42(2):353–362, 2004.

[27] Mikko Salo and Gunther Uhlmann. The attenuated ray transform on simple surfaces.

J. Diff. Geom, 88(1):161–187, 2011.

[28] Donald C Solmon. The identification problem for the exponential radon transform. *Mathematical methods in the applied sciences*, 18(9):687–695, 1995.

[29] Plamen Stefanov. The identification problem for the attenuated x-ray transform. *preprint*, 2013.

[30] Elias M. Stein and Rami Shakarchi. *Fourier Analysis: An Introduction (Princeton Lectures in Analysis)*. Princeton University Press, 2003.

[31] Edward Charles Titchmarsh. *Introduction to the theory of Fourier integrals*, volume 337. Clarendon Press Oxford, 1937.

[32] Andy Welch, Rolf Clack, Frank Natterer, and Grant T Gullberg. Toward accurate attenuation correction in spect without transmission measurements. *Medical Imaging, IEEE Transactions on*, 16(5):532–541, 1997.

[33] Jiangsheng You. The attenuated radon transform with complex coefficients. *Inverse Problems*, 23(5):1963, 2007.