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REPRESENTATION RESULTS FOR CONTINUOUS-STATE BRANCHING PROCESSES AND LOGISTIC BRANCHING PROCESSES

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

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"Un día bien, otro mal, no hay mal que por bien no venga,
el que quiere andar ya sabe que llevar la sombra cuesta. "
Armando Tejada Gómez

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Resumen

El objetivo de este trabajo es explorar el comportamiento de los procesos de ramificación evolucionando a tiempo y estados continuos, y encontrar representaciones para su trayectoria y su genealogía.

En el primer capítulo se muestra que un proceso de ramificación condicionado a no extinguirse es la única solución fuerte de una ecuación diferencial estocástica conducida por un movimiento Browniano y una medida puntual de Poisson, más un subordinador que representa la inmigración, dónde estos procesos son mutuamente independientes. Para esto se usa el hecho de que es posible obtener la ley del proceso condicionado a partir del proceso original, a través de su h-transformada, y se da una manera trayectorial de construir la inmigración a partir de los saltos del proceso.

En el segundo capítulo se encuentra una representación para los procesos de ramificación con crecimiento logístico, usando ecuaciones estocásticas. En particular, usando la definición general dada por A. Lambert, se prueba que un proceso logístico es la única solución fuerte de una ecuación estocástica conducida por un movimiento Browniano y una medida puntual de Poisson, pero con un drift negativo fruto de la competencia entre individuos. En este capítulo se encuentra además una ecuación diferencial estocástica asociada con un proceso logístico condicionado a no extinguirse, suponiendo que éste existe y que puede ser definido a través de una h-transformada. Esta representación muestra que nuevamente el condicionamiento da origen a un término correspondiente a la inmigración, pero en este caso dependiente de la población.

Por último, en el tercer capítulo se obtiene una representación de tipo Ray-Knight para los procesos de ramificación logísticos, lo que da una descripción de su genealogía continua. Para esto, se utiliza la construcción de árboles aleatorios continuos asociados con procesos de Lévy generales dada por J.-F. Le Gall e Y. Le Jan, y una generalización del procedimiento de "poda" desarrollado por R. Abraham, J.-F. Delmas. Este resultado extiende la representación de Ray-Knight para procesos de difusión logísticos dada por V. Le, E. Pardoux y A. Wakolbinger.

Abstract

The aim of this Ph.D. thesis is to explore the behaviour of continuous-state population processes that evolve over continuous time, and to find some pathwise and genealogical representations for such processes.

In the first chapter, it is shown that a (sub-)critical continuous-state branching process conditioned to be never extinct is the unique strong solution to a stochastic differential equation driven by a Brownian motion and a Poisson point measure, plus an independent subordinator accounting for immigration, and these objects are mutually independent.To this end, the fact that the law of the conditioned CSBP is obtained from the one of the non conditioned process by means of an explicit h-transform is used, and an explicit mechanism to build the immigration term appearing in the conditioned process is given, by randomly selecting jumps of the original one.

In the second chapter, a representation for logistic branching process is found using stochastic differential equations. In particular, Lambert's general definition [36] is used to characterize them as the unique strong solution of a stochastic differential equation driven by a Brownian motion and a Poisson point measure with a negative drift, resulting from negative interactions between each pair of individuals in the population. Also, under the assumption that a logistic branching process conditioned to be never extinct exists and can be defined trough a h-transform, a stochastic differential equation associated is found. Such representation shows again an immigration term, that it is density-dependent in this case.

In the third chapter, a Ray-Knight representation for logistic branching processes is established, giving a description of its continuous genealogy. To this end, the construction of continuum random trees associated with general Lévy processes given by J.-F. Le Gall and Y. Le Jan and a generalization of the pruning procedure developed by R. Abraham and J.-F. Delmas are used. The main result presented in this last chapter extends the Ray-Knight representation for logistic Feller difussion given by V. Le, E. Pardoux and A. Wakolbinger.

Contents

Introduction

The aim of this Ph.D. thesis is to explore the behaviour of continuous-state population processes that evolve over continuous time. In nature, competition for limited resources regulates the growth of these populations, and their behaviour varies according to whether they are isolated or not. Therefore, study their genealogy and possible extinction is an issue of great interest for various sciences, especially biology (see [29, 37, 16, 39, 7]).

In population biology, the most simple process modeling the dynamics of a population is the Malthusian process. If $Y_t \in [0, \infty)$ denotes the total number of individuals at time t, then the Malthusian process satisfies $dY_t = bY_t dt$, where b is the mean birth-death balance per individual and per time unit. The solutions are straightforward exponential functions and when $b > 0$, they rapidly go to ∞, proving useless for long-term models. Moreover, this model does not allow populations with positive growth to become extinct. This elementary model has a random counterpart, called the branching process, where populations may have positive (expected) growth and become extinct. In their discrete time and discrete state-space form, branching processes were introduced by Lord Francis Galton and Irénée-Jules Bienaymé in 1873. The so-called Bienaymé-Galton-Watson (BGW) process is a Markov chain, where time steps are the nonoverlapping generations, with individuals behaving independently from one another and each giving birth to a (random) number of offspring (belonging to the next generation). These (random) offspring all have the same probability distribution. Here, the mean growth is geometric, but the process evolves randomly through time, eventually dying out or tending to ∞, with probability 1. In 1958, M. Jirina [32] defined continuous-state branching processes (CSBP). Later, J. Lamperti [36] showed that they can be obtained as scaling limits of a sequence of Galton-Watson processes, and established a one-to-one correspondence between CSBPs and spectrally positive Lévy processes via a random time change. The definition of branching processes in this context was generalized by K. Kawazu and S. Watanabe [33] to model populations with immigration (CBI).

From an ecological standpoint, the BGW-process shares with the Malthusian process the shortcoming of being able to go to ∞. In the deterministic case, a celebrated improvement of the Malthusian process is the logistic process, characterized by the ordinary differential equation $dY_t = bY_t dt - cY_t^2 dt$, for $t > 0$, where $c > 0$. It is an elementary combination of geometric growth for small population sizes and a quadratic density-dependent regulatory mechanism. The main advantage of this model is that Y_t converges to a finite limit as $t \to \infty$, namely, b/c (if $b > 0$) or 0 (if $b \leq 0$). On the other hand, this model does not allow the population to evolve once it has reached its stable state. A natural continuation will then be for us to replace geometric growth in the logistic equation by random branching (random growth with geometric mean). Alternatively, this can be seen as improving the branching process by, loosely speaking, adding a quadratic regulatory term to it (and thus prevent it from going to ∞). This kind of processes were introduced by R.B. Campbell [14] in the context of continuous-state and continuous-time processes. In this case, branching processes with logistic growth (LBPs) were defined in a general form by A. Lambert [36] by means of a Lamperti transformation on Ornstein-Uhlenbeck processes driven by general spectrally positive Lévy process. The global behavior of the population can be intuitively understood as the result of standard branching behavior, plus a pairwise competition among individuals, resulting in an individual death rate increased by an amount that is proportional to the total instantaneous population descending from the original one.

The study of this kind of population processes has revealed deep connexions between the stochastic differential equations that describe the evolution of such branching populations, and continuous time processes of a different nature that can be used to code their genealogies.

In the first chapter of this work, SDEs are used as a tool to describe the paths of (sub)critic continuous-state branching processes conditioned to never extinct. CSBPs conditioned to stay positive were first studied in the continuous-state framework by S. Roelly and A. Rouault [54], who proved that there is a well defined probability measure for this case and, under this measure, a CSBP has the same law as a CBI, where the immigration corresponds to an independent subordinator (i.e. a Lévy process with no negative jumps). In the particular case of a (sub)-critical CSBP, it is well known that such conditioned CSBP corresponds to a CBI with particular immigration mechanisms (see [54]). Thus, using general results and techniques developed in some of the aforementioned works (see [17, 25]), was possible to obtain such representation in a direct way, by using the fact that the law of the conditioned CSBP is obtained from the one of the non conditioned process by means of an explicit h-transform. It is shown that under the law of a (sub-)critical continuous-state branching process conditioned to be never extinct, the process is the unique strong solution to a stochastic differential equation driven by a Brownian motion and two Poisson point measures and these objects are mutually independent. The relation between the original law and the conditioned law, together with the spine or immortal particle picture of the conditioned process ([40, 23]), suggest that one should be able to identify, after a measure change, copies of the original driving random processes and an independent subordinator accounting for immigration. The stochastic differential equation describes an explicit mechanism to build the immigration term appearing in the conditioned process, by randomly selecting jumps of the original one. This work has already been published [24].

In the second chapter some SDE representations for LBP are studied. In particular, Lambert's definition [36] for general logistic branching processes is used to characterize them as the unique solution of a stochastic differential equation.

Also, the LBP conditioned to non extinction is studied. Unlike the CSBP case, the branching property is not longer true and it is unknown, in general, if there exists such conditioned process (and if it can be defined through an h-transform, as in the CSBP case -see [48]-). Results in that sense are only known for the logistic Feller diffusion case, thanks to the renowned work of Cattiaux *et. al.* [15]. In the general case, the study of this problem requires the use of spectral theory of jump processes, and will not be addressed in this dissertation. However, it is still of interest to describe the dynamics of such conditioned process. Under the assumption that a function h exists that allows one to obtain the law of the conditioned LBP from the one of the non conditioned process by means of an h-transform, the same arguments as in the CSBP case can be applied. Some results are obtained, which shed some light on the pathwise properties of this kind of processes.

In the third chapter of this dissertation, a Ray-Knight representation for LBP is established. At first sight, the continuous-state branching process with logistic growth does not lend itself to a Ray-Knight representation, because the competition between individuals destroys the "branching property", i.e. the independence in the reproduction. The lack of independence between the individuals of the populations modeled by such processes prevents the application of standard tools of excursion theory and of continuous random trees to suitably define the genealogy of such processes, and new points of view must be developed. The use of excursion theory to describe the genealogy of the process start with Kiyosi Itô [30, 31]. He introduced the Poisson point process of excursions of a Markov process from a regular point, inspired by the ideas of P. Lévy [44] in the case of linear Brownian motion. Itô excursion theory has many important applications, since it is a fundamental tool in the analysis of Lévy processes and for studying the asymptotic properties of large random trees, which are deeply connected with branching processes. Around 1990, Davis Aldous constructed a continuum random tree (CRT) as the tree coded by a normalized Brownian excursion [5, 6]. This relationship between exploration and mass excursion had appeared earlier in the classical second Ray-Knight theorem ([52], [34], see [53]) as a connection between Brownian excursions (described by Itô's excursion measure) and excursions of Feller's branching diffusion. In informal terms, this theorem says that: *The time which a (suitably stopped) reflected Brownian motion spends near level* t (and which is formally captured by its *local time* at t), *viewed as a process in* t*, is a Feller branching diffusion.* So, the excursions of reflected Brownian motion can be understood as exploration paths of the trees of descendants of the ancestors of the population at time $t = 0$, with the local time at height t measuring the population size at time t.

This same idea was generalized by T. Duquesne and J.-F. Le Gall [43] for general Lévy processes. In their work, given a Lévy process X with characteristic exponent ψ, they defined a suitable height process H and proved that the process of local times of H at a fixed time is a CSBP. The definition of the corresponding excursions, their heights and their local times at each level, which are needed in order to state a Ray-Knight theorem in that setting, is much more involved than in the diffusion case, with no simple (say, finite dimensional or SDE-like) representation of a Markov process coding the genealogy. Their result therefore required to use of the so-called exploration process, introduced by J.-F. Le Gall and Y. Le Jan [43], which codes the continuum random trees embedded in a spectrally positive Lévy processes X, or Lévy-CRT. Extensions of the Ray-Knight theorem and related genealogical descriptions have since then been obtained for more complex models with branching type behavior, such as super processes and branching processes with immigration, [2, 1, 9], and have been used in the study of several properties of these processes.

Later, V. Le, E. Pardoux and A. Wakolbinger ([42], see also [49]) give another generalization of Ray-Knight result, for the Feller branching diffusion with logistic growth. As in the classical Ray-Knight representation, the excursions of such process are understood as the exploration paths of the trees of descendants of the ancestors at time t $=0$, and the local time of the process at height s measures the population size at time t. Their key idea to define a genealogy is to think of the individuals as being arranged "from left to right" (as given by the exploration time), and decree that interaction between them takes place through "pairwise fights" that are always won by the individual "to the left", hence lethal for the individual "to the right". Deaths following pairwise fights lead in the exploration process of the genealogical forest to a downward drift, proportional to the amount of mass (or local time units) seen to the left of the individual encountered at each exploration time. In this way, excursions which come later in the exploration time tend to be smaller (trees to the right are "under attack from those to the left").

The main purpose of the this last chapter is to extend the previous Ray-Knight representations and genealogical descriptions to LBP associated with general spectrally positive Lévy process with (sub)critical mechanism. To this end, a family of generalized marked exploration processes and correspondent pruned local time processes are constructed, using a Poisson Lévy snake in two dimensions, in a more general way that was done by Abraham and Delmas to prune a Lévy-CRT at constant rate [3]. Through a fix point argument, a limiting progressively marked exploration process (ρ, m^*) and an associated local times process $(L_t^a(m^*) : a \geq 0, t \geq 0)$ are obtained. This new local time process correspond to the local times of the original Lévy-CRT, coded by the exploration process, erased at a rate depending on the population "on the left", at each instant and height. While this idea was inspired by the work of Pardoux and Wakolbinger [42, 50] for the Feller diffusion with logistic growth, the general case presents other challenges, since the height process in this case is not Markovian. Furthermore, the result obtained is not only in law but also allows to obtain the desired representation from a Lévy tree coded by a exploration process in a pathwise manner. The finite-dimensional laws of the associated pruned local times read at increase times of the local time at level 0 are identified as the finite-dimensional laws of a LBP. For this identification, a discretization of this process is used, whose law coincides with the law of a stochastic flow studied by Dawson-Li [18] pruned piecewise in a path-dependent way. This last chapter is a joint work with Professor Julien Berestycki, as a result of two stays in the Laboratoire of Probabilités et Modèles Aléatoires of Université Pierre et Marie Curie (Paris VI).

Chapter 1

On SDE associated with CSBP conditioned to never be extinct

This chapter is largely based on the paper *On SDE associated with continuous-state branching processes conditioned to never be extinct* , with J. Fontbona, published in the volume **17–49** of *ELECTRONIC COMMUNICATIONS in PROBABILITY* in 2012 [24].

1.1 Introduction and preliminaries

Stochastic differential equations (SDE) representing continuous-state branching processes (CSBP) or CSBP with immigration (CBI) have attracted increasing attention in the last years, as powerful tools for studying pathwise and distributional properties of these processes as well as some scaling limits, see e.g. Dawson and Li [17, 18], Lambert [38], Fu and Li [25] and Caballero *et al.* [13]. In this chapter, we are interested in SDE representations for (sub)-critical CSBP conditioned to never be extinct. It is well known that such conditioned CSBP correspond to CBIs with particular immigration mechanisms (see [54]). Thus, it is possible to obtain SDE representations for them by using general results and techniques developed in some of the aforementioned works, see [25, 17]. However, our goal is to directly obtain such representation by rather using the fact that the law of the conditioned CSBP is obtained from the one of the non conditioned process, by means of an explicit h-transform. This relation between the two laws, together with the "spine" or immortal particle picture of the conditioned process [54, 23], suggest that one should be able to identify, after measure change, copies of the original driving random processes and an independent subordinator accounting for immigration. Our proof will show how to obtain these processes by using Girsanov theorem and an enlargement of the probability space in order to select by a suitable marking procedure those jumps of the original (non conditioned) process that will constitute (or will not) the immigrants. The enlargement of the probability space and the marking procedure are both inspired in a construction of Lambert [38] on stable Lévy processes. They are also reminiscent of the sized biased tree representation of measure changes for Galton-Watson trees (Lyons *et al.* [47]) or for branching Brownian motions (see e.g. Kyprianou [35] and Englänger and Kyprianou [22]), but we do not aim at fully developing those ideas in the present framework. In a related direction, using the look-down particle representation of CSBP of Donnelly and Kurtz [20], Hénard obtains in a recently posted article [28] the same

SDE description of the conditioned CSBP. Our proof of the SDE representation contains less information about the process, but in turn is much simpler. The reader is also referred to [19, 41, 46] for further recent developments on representations of CSBP and their conditioned versions.

We start by recalling some definitions and results about CSBP and Lévy processes, following Kyprianou's book [35, Ch. 1,2 and 10]. Then, we remind their relationship through the Lamperti transform, following Kyprianou [35, Ch. 10] and Caballero *et al.* [13].

1.1.1 Continuous-state branching processes

A *continuous-state branching process* (*CSBP*) with probability laws given the initial state $\{\mathbb{P}_x : x \geq 0\}$ is a càdlàg $[0,\infty)$-valued strong Markov processes $Y = (Y_t : t \geq 0)$ satisfying the branching property. That is, for any $t \geq 0$ and $y_1, y_2 \in [0,\infty)$, Y_t under $\mathbb{P}_{y_1+y_2}$ has the same law as the independent sum $Y_t^{(1)} + Y_t^{(2)}$, where the distribution of $Y_t^{(i)}$ is equal to that of Y_t under $\mathbb{P}_{y_i}$ for $i = 1, 2$. Usually, Y_t represents the population at time t descending from an initial population x. The law of Y is completely characterized by its Laplace transform

$$\mathbb{E}_x(e^{-\theta Y_t}) = e^{-x u_t(\theta)}, \quad \forall x > 0, t \geq 0,$$

where u is a differentiable function in t satisfying

$$\begin{cases} \dfrac{\partial u_t}{\partial t}(\theta) + \psi(u_t(\theta)) = 0 \\ \qquad\qquad u_0(\theta) = \theta, \end{cases} \tag{1.1}$$

and ψ is called the *branching mechanism* of Y, which has the form

$$\psi(\lambda) = -q - \alpha\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}})\Pi(dx) \quad \lambda \geq 0, \tag{1.2}$$

for some $q \geq 0$, $\alpha \in \mathbb{R}$, $\sigma \geq 0$ and Π a measure supported in $(0,\infty)$ such that

$$\int_{(0,\infty)} (1 \wedge x^2)\Pi(dx) \; < \; \infty.$$

In particular, ψ is the Laplace exponent of a *spectrally positive* Lévy process, i.e. one with no negative jumps. Since clearly, $\mathbb{E}_x(Y_t) = xe^{-\psi'(0+)t}$, defining $\rho := \psi'(0+)$ one has the following classification of CSBPs :

- *subcritical* if $\rho > 0$,
- *critical* if $\rho = 0$, and
- *supercritical* if $\rho < 0$,

according to whether the process will, on average, decrease, remain constant or increase.

In the following, we will assume that Y is *conservative*, i.e. $\forall\, t > 0$, $\mathbb{P}_x(Y_t < \infty) = 1$. By Grey [26], this is true if and only if

$$\int_{0+} \frac{d\xi}{|\psi(\xi)|} = \infty,$$

so it is sufficient to asume

$$\psi(0) = 0 \quad \text{and} \quad |\psi'(0+)| < \infty.$$

From now on, we also assume that Z is a (sub-)critical CSBP with branching mechanism ψ which satisfies

$$\psi(\infty) = \infty \quad \text{and} \quad \int^{\infty} \frac{d\xi}{\psi(\xi)} < \infty. \tag{1.3}$$

Under these previous conditions, the process does not explode and there is almost surely extinction in finite time.

1.1.2 Lévy Processes and their connection with continuous-state branching processes

A *Lévy process* $X = (X_t : t \geq 0)$ is a process which possesses the following properties:

i). The paths of X are $\mathbb{P}$-a.s. right continuous with left limits.

ii). $\mathbb{P}(X_0 = 0) = 1$.

iii). For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}. (*Stationary Increments*)

iv). For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$. (*Independent Increments*)

A Lévy process X with *characteristic exponent* ψ can be identificated as the independent sum of three processes:

Theorem (Lévy-Itô descomposition)**.** *Given any $a \in \mathbb{R}$, $\sigma \geq 0$ and a measure Π concentrated on $\mathbb{R}\backslash\{0\}$, satisfying*

$$\int_{\mathbb{R}} (1 \wedge x^2)\Pi(dx) < \infty,$$

there exists a probability space on which three independent Lévy processes exist, $X^{(1)}$, $X^{(2)}$ y $X^{(3)}$, where $X^{(1)}$ is a linear Brownian motion with drift given by

$$X_t^{(1)} = \sigma B_t + \alpha t;$$

$X^{(2)}$ is a compound Poisson process given by

$$X_t^{(2)} = \sum_{i=1}^{N_t} \xi_i,$$

where $(N_t : t \geq 0)$ is Poisson process with rate $\Pi(\mathbb{R}\backslash(-1,1))$, and $(\xi_i : i \geq 1)$ are i.i.d. r.v. with distribution $\Pi(dx)/\Pi(\mathbb{R}\backslash(-1,1))$ concentrated on $\{x : |x| \geq 1\}$; and $X^{(3)}$ is a square integrable martingale with an almost surely countable number of jumps on each finite time interval which are of magnitude less than unity and with characteristic exponent given by

$$\psi^{(3)}(\theta) = \int_0^1 (1 - e^{-\theta x} - \theta x)\Pi(dx).$$

The measure Π is called the Lévy (characteristic) measure.

Then, given X a spectrally positive Lévy process with initial position $x \geq 0$, using the Lévy-Itô descomposition one can write

$$X_t = x + \alpha t + \sigma B_t^X + \int_0^t \int_1^\infty r N^X(ds, dr) + \int_0^t \int_0^1 r \tilde{N}^X(ds, dr), \tag{1.4}$$

where α is a real number, $\sigma \geq 0$, B^X is a un Brownian motion and N^X is an independent Poisson measure on $[0, \infty) \times (0, \infty]$ with intensity measure $dt \times \Pi(dr)$, with $\tilde{N}^X$ the associated compensated measure, satisfying

$$\int_0^t \int_0^1 r \tilde{N}^X(ds, dr) := \lim_{\varepsilon \to 0} \left[\int_0^t \int_\varepsilon^1 N^X(ds, dr) - \int_0^t \int_\varepsilon^1 ds \Pi(dr) \right]$$

1.1.3 Lamperti representation of continuous-state branching processes

In [40], Lamperti established a one-to-one correspondence between CSBPs and spectrally positive Lévy processes via a random time change. The correspondence at the level of laws was also proved by Silverstein [55] by analytic methods, and a proof in the conservative case by discrete (probabilistic) approximation was given in [27]. We refer the reader to [13] for self-contained modern proofs of this result in the general case. Given a Lévy process X as above, Lamperti's construction states that the process

$$Y := (Y_t = X_{\theta_t \wedge T_0} : t \geq 0),$$

where $T_0 = \inf\{t > 0 : X_t = 0\}$ and $\theta_t = \inf\left\{s > 0 : \int_0^s \frac{du}{X_u} > t\right\}$, is a continuous-state branching process with branching mechanism ψ and initial value $Y_0 = x$. Conversely, given $Y = (Y_t : t \geq 0)$ a CSBP with branching mechanism ψ, such that $Y_0 = x > 0$, we have that

$$X := (X_t = Y_{\varphi_t \wedge T} : t \geq 0),$$

where $T = \inf\{t > 0 : Y_t = 0\}$ and $\varphi_t = \inf\left\{s > 0 : \int_0^s Y_u du > t\right\}$, is a Lévy process with no negative jumps, stopped at T_0 and satisfying $\psi(\lambda) = \log \mathbb{E}(e^{-\lambda X_1})$, with initial position $X_0 = x$. Relying on this relationship, Caballero *et al.* [13, Prop 4] provide a pathwise description of the dynamics of a CSBP: given a version of the process $(Y_t, t \geq 0)$ on some probability space, there exist in an enlarged probability space a standard Brownian motion B^Y and an independent Poisson measure N^Y on $[0, \infty) \times (0, \infty) \times (0, \infty)$ with intensity measure $dt \times d\nu \times \Pi(dr)$ such that

$$\begin{aligned} Y_t = x + \alpha \int_0^t Y_s ds + \sigma \int_0^t \sqrt{Y_s} dB_s^Y + \int_0^t \int_0^{Y_{s-}} \int_1^\infty r N^Y(ds, d\nu, dr) \\ + \int_0^t \int_0^{Y_{s-}} \int_0^1 r \tilde{N}^Y(ds, d\nu, dr), \end{aligned} \tag{1.5}$$

where $\tilde{N}^Y$ is the compensated Poisson measure associated with N^Y.

1.2 CSBPs conditioned to be never extinct as solutions of SDEs

1.2.1 CSBP conditioned to be never extinct

Branching processes conditioned to stay positive were first studied in the continuous-state framework by Roelly and Rouault [54], who proved that for Z satisfying the above conditions,

$$\mathbb{P}_x^{\uparrow}(A) := \lim_{s\uparrow\infty} \mathbb{P}_x(A|T > t+s), \quad A \in \sigma(Y_s : s \leq t) \tag{1.6}$$

is a well defined probability measure which satisfies

$$\mathbb{P}_x^{\uparrow}(A) = \mathbb{E}(\mathbf{1}_A e^{\rho t}\frac{Y_t}{x}).$$

In particular, $\mathbb{P}_x^{\uparrow}(T < \infty) = 0$, and $(e^{\rho t}Y_t : t \geq 0)$ is a martingale under $\mathbb{P}_x$. Note that $\mathbb{P}_x^{\uparrow}$ is the law of the so-called *Q-process* (for an in-depth look at these processes, we refer the reader to [38], [48] and references therein). They also proved that $(Y, \mathbb{P}^{\uparrow})$ has the same law as a CBI with branching mechanism ψ and immigration mechanism $\phi(\theta) = \psi'(\theta) - \rho$, $\theta \geq 0$. This means that $(Y, \mathbb{P}^{\uparrow})$ is a càdlàg $[0, \infty)$-valued process, such that for all $x, t > 0$ and $\theta \geq 0$

$$\mathbb{E}_x^{\uparrow}(e^{-\theta Y_t}) = \exp\{-xu_t(\theta) - \int_0^t \phi(u_{t-s}(\theta))ds\},$$

where $u_t(\theta)$ is the unique solution to (1.1). Note also that ϕ is the Laplace exponent of a subordinator. We call *subordinators* to Lévy processes whose paths are almost surely non-decreasing. For $\theta \geq 0$,

$$\phi(\theta) = d\theta + \int_{(0,\infty)} (1 - e^{-\theta x})\Lambda(dx),$$

where Λ is a measure concentrated on $(0, \infty)$, satisfying $\int_{(0,\infty)}(1 \wedge x)\Lambda(dx) < \infty$.

1.3 Main Result

The work of Roelly and Rouault is the key for the study of CSBP conditioned on non-extinction, but we seek a more explicit description for the paths of Y under $\mathbb{P}^{\uparrow}$. To this end, we shall prove that $(Y, \mathbb{P}^{\uparrow})$ has a SDE representation, which agrees with the interpretation of a CSBP conditioned on non-extinction as a CBI, but also gives us a pathwise description for the conditioned process. In particular, this result extends Lambert's results for the stable case [38, Theorem 5.2] (see below for details) as well as equation (1.5).

Theorem 1.1. *Under $\mathbb{P}^{\uparrow}$, the process Y is the unique strong solution of the following stochastic differential equation:*

$$\begin{aligned} Y_t = x + \alpha \int_0^t Z_s ds + \sigma \int_0^t \sqrt{Y_s} dB_s^{\uparrow} + \int_0^t \int_0^{Y_{s-}} \int_1^{\infty} rN^{\uparrow}(ds, d\nu, dr) \\ + \int_0^t \int_0^{Y_{s-}} \int_0^1 r\tilde{N}^{\uparrow}(ds, d\nu, dr) + \int_0^t \int_0^{\infty} rN^{\star}(ds, dr) + \sigma^2 t, \end{aligned} \tag{1.7}$$

where $(B_t^\uparrow : t \geq 0)$ is a Brownian motion, $N^\uparrow$ and $N^\star$ are Poisson measures on $[0,\infty)\times(0,\infty)^2$ and $[0,\infty)\times(0,\infty)$ with intensities measures $ds \times d\nu \times \Pi(dr)$ and $ds \times r\Pi(dr)$, respectively, and these objects are mutually independent (as usual, $\tilde{N}^\uparrow$ stands for the compensated measure associated with $N^\uparrow$). Moreover, given a solution to (1.5) in some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, the processes $B^\uparrow$, $N^\uparrow$ and $N^\star$ can be explicitly constructed by a change of measure in an enlargement of $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ by an independent i.i.d. sequence of uniform random variables in $[0,1]$.

This result implies that we can recover Y conditioned on non-extinction as the solution of a SDE driven by a copy of B^Y, a copy of N^Y, and a Poisson random measure with intensity $ds \times r\Pi(dr)$, plus a drift. (Notice that taking out the last two terms, corresponding to a subordinator with drift, one again obtains equation (1.5).)

1.4 Relations to previous results

1.4.1 Stable processes

As pointed out before, the result above is based in the work of Lambert, and we can recover his result using equation (1.7). We consider the case when X is a spectrally positive α-stable process, $1 < \alpha \leq 2$, that is a Lévy process with Laplace exponent ψ proportional to $\lambda \to \lambda\alpha$. In particular, $\rho = 0$ (critical case). In this case, Lambert showed that the associated Q-process is the solution of a certain SDE, which enlightens the immigration mechanism.

Theorem 1.2 (Theorem 5.2 in [38])**.** *The branching process with branching mechanism ψ is the unique solution in law to the following SDE*

$$dY_t = Y_{t-}^{1/\alpha} - dX_t, \tag{1.8}$$

where X is a spectrally positive α-stable Lévy process with Laplace exponent ψ. Moreover, the branching process conditioned to be never extinct is solution to

$$dY_t = Y_{t-}dX_t + d\sigma_t, \tag{1.9}$$

where σ is an $(\alpha-1)$-stable subordinator with Laplace exponent ψ', independent of X.

We show that Lambert's SDE representation of stable branching processes can be seen as a special case of Theorem 1.1.

Let X be a spectrally positive α-stable process with characteristic exponent ψ and characteristic measure $\Pi(dr) = kr^{-(\alpha+1)}dr$, where k is some positive constant and $1 < \alpha \leq 2$. Let Y be the branching process with branching mechanism ψ. Thanks to Theorem 1.1 we know that, under $\mathbb{P}^\uparrow$, Y satisfies the following stochastic differential equation:

$$\begin{aligned} Y_t = x &+ \int_0^t \int_0^{Y_{s-}} \int_1^\infty r N^\uparrow(ds, d\nu, dr) + \int_0^t \int_0^{Y_{s-}} \int_0^1 r\tilde{N}^\uparrow(ds, d\nu, dr) \\ &+ \int_0^t \int_0^\infty r N^\star(ds, dr), \end{aligned} \tag{1.10}$$

where $N^\uparrow$ is a Poisson random measure with intensity $ds \times d\nu \times \Pi(dr)$ and $N^\star$ is an independent Poisson random measure with intensity $ds \times r\Pi(dr)$.

Now, we define

$$\theta_n := \frac{r_n^\uparrow \mathbf{1}_{\{\nu_n^\uparrow \le Y_{t_n-}\}}}{Y_{t_n-}^{1/\alpha}},$$

where $((t_n, \nu_n^\uparrow, r_n^\uparrow) : n \in \mathbb{N})$ are the atoms of $N^\uparrow$. We claim that, under $\mathbb{P}^\uparrow$, $((t_n, \theta_n) : n \in \mathbb{N})$ are atoms of a Poisson random measure N' with intensity $ds \times \Pi(du)$. Indeed, for any bounded non-negative predictable process H, and any positive bounded function f vanishing at zero,

$$M_t := \sum_{t_n \le t} H_{t_n} f(\theta_n) - \int_0^t H_s ds \int_0^\infty \int_0^\infty f\left(r/Y_s^{1/\alpha}\right) \mathbf{1}_{\{\nu \le Y_s\}} \Pi(dr) d\nu$$

is a martingale. If we change variables, the particular form of Π implies that

$$M_t = \sum_{t_n \le t} H_{t_n} f(\theta_n) - \int_0^t H_s ds \int_0^\infty f(u)\Pi(du).$$

Taking expectations, our claim follows thanks to Lemma 1.4 below. Since

$$\sum_{t_n \le t} r_n^\uparrow \mathbf{1}_{\{\nu_n^\uparrow \le Y_{t_n-}\}} = \sum_{t_n \le t} Y_{t_n-}^{1/\alpha} \theta_n,$$

we can rewrite (1.10) as

$$Y_t = x + \int_0^t \int_1^\infty Y_{s-}^{1/\alpha} u N'(ds, du) + \int_0^t \int_0^1 Y_{s-}^{1/\alpha} u \tilde{N}'(ds, du) + \int_0^t \int_0^\infty r N^\star(ds, dr).$$

Defining

$$X_t := \int_0^t \int_1^\infty u N'(ds, du) + \int_0^t \int_0^1 u \tilde{N}'(ds, du),$$

by the Lévy-Ito decomposition it is easy to see that X is an α-stable Lévy process with characteristic exponent ψ. Similarly,

$$S_t := \int_0^t \int_0^\infty r N^\star(ds, dr)$$

is seen to be an $(\alpha-1)$-stable subordinator. Independence of X and S is granted by construction, because the two processes do not have simultaneous jumps. Thus, we have

$$dY_t = Y_t^{1/\alpha} dX_t + dS_t,$$

which corresponds to equation (1.9) in Lambert's result.

1.4.2 CSBP flows as SDE solutions

A family of CSBP processes $Y = (Y_t(a) : t \ge 0, a \ge 0)$ allowing the initial population size $Y_0(a) = a$ to vary, can be constructed simultaneously as a two parameter process or *stochastic flow* satisfying the branching property. This was done by Bertoin and Le-Gall [10] by using families of subordinators. In [11, 12] they later used Poisson measure driven SDE to formulate such type of flows in related contexts, including equations close to (1.5). In the same line, Dawson and Li [18] proved the existence of strong solutions for stochastic flows of continuous-state branching processes with immigration, as SDE families driven by white noise processes and Poisson random measures with joint regularity properties.

In particular, suppose $\sigma \geq 0$ and a real constants and $v \to \gamma(v)$ is a non-negative and non-decreasing continuous function on $[0, \infty)$. Let $W(ds, du)$ be a white noise process on $(0, \infty)^2$ based on the Lebesgue measure $ds \times du$. Let $N(ds, d\nu, dr)$ be a Poisson random measure on $(0, \infty)^3$ with intensity $ds \times d\nu \times \Pi(dr)$ independent of $W(ds, du)$. It is shown in [18] that for any $v \geq 0$ there is a pathwise unique non-negative solution of the stochastic equation

$$Z_t(v) = \ v + \sigma \int_0^t \int_0^{Z_{s-}(v)} W(ds, du) + \int_0^t [\gamma(v) - \alpha Z_s(v)]\, ds + \int_0^t \int_0^{Z_{s-}(v)} \int_0^\infty r \tilde{N}(ds, d\nu, dr),$$

and each solution $Z(v) = (Z_t(v) : t \geq 0)$ is a continuous-state branching process with immigration (CBI-process), so it is natural to call the two-parameter process $(Z_t(v) : t \geq 0, v \geq 0)$ a *flow of CBI-processes.* Moreover, the family of two-parameter processes $(Y_t(v) : t \geq s, v \geq 0)$ has a version with the following properties:

(i) for each $v \geq 0$, $t \to Z_t(v)$ is a càdlàg process on $[0, \infty)$ and solves (3.67);

(ii) for each $t \geq 0$, $v \to Z_t(v)$ is a non-negative and non-decreasing càdlàg process on $[0, \infty)$.

The stochastic equation above is close to equation (1.7), the main difference being the immigration behavior which in their case only covers linear drifts. For simplicity reasons our result is presented in the case of a Brownian motion and Poisson measure driven SDE, but our arguments can be extended to the white-noise and Poisson measure driven stochastic flow considered (in absence of immigration) in [18].

1.5 Proof of the main theorem

This result was inspired for the work of Lambert [38]. In his work, a suitable marking of Poisson point processes was used to firstly construct a stable Lévy process, conditioned to stay positive, out of the realization of the unconditioned one. After time-changing the author takes advantage of the scaling property of α-stable processes to derive an SDE for the branching process. Our proof is inspired in his marking argument but in turn it is carried out directly in the time scale of the CSBP. For the proof, we will need the following version of Girsanov's theorem (c.f. Theorem 37 in Chapter III.8 of [51]):

Theorem 1.3. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, and let M be a $\mathbb{P}$-local martingale with $M_0 = 0$. Let $\mathbb{P}^\star$ be another probability measure absolutely continuous with respect to $\mathbb{P}$, and let $D_t = \mathbb{E}(\frac{d\mathbb{P}^\star}{d\mathbb{P}}\big|_{\mathcal{F}_t})$. Assume that $\langle M, D\rangle$ exists for $\mathbb{P}$. Then $A_t = \int_0^t \frac{1}{D_{s^-}} d\langle M, D\rangle_s$ exists a.s. for the probability $\mathbb{P}^\star$, and $M_t - A_t$ is a $\mathbb{P}^\star$-local martingale.*

The following well-known characterization of Poisson point processes will also be useful:

Lemma 1.4. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, $(S, \mathcal{S}, \eta)$ an arbitrary σ-finite measure space, and $\{(t_n, \delta_n) \in \mathbb{R}_+ \times S\}$ a countable family of random variables such that*

$\{t_n \le t, \delta_n \in A\} \in \mathcal{F}_t$ for all $n \in \mathbb{N}$, $t \ge 0$ and $A \in \mathcal{S}$ and, moreover,

$$\mathbb{E}\left[\sum_{n:t_n \le t} F_{t_n} g(\delta_n)\right] = \mathbb{E}\left[\int_0^t F_s ds \int_S g(x) m(dx)\right] \tag{1.11}$$

for any nonnegative predictable process F_s and any nonnegative measurable function $g : S \to \mathbb{R}$. Then, $(t_n, \delta_n)_{n\in\mathbb{N}}$ are the atoms of a Poisson random measure N on $\mathbb{R}_+ \times S$ with intensity $dt \times m(dx)$.

Proof. Writing

$$\begin{aligned} e^{\sum\limits_{t_n \le t} f(\delta_n))} &= \sum_{n:t_n \le t} \left[\prod_{k:t_k < t_n} e^{f(\delta_k)}\right] (e^{f(\delta_n)} - 1) \\ &= \sum_{n:t_n \le t} \left[e^{\sum\limits_{k:t_k < t_n} f(\delta_k)}\right] (e^{f(\delta_n)} - 1), \end{aligned}$$

we get from (1.11) that

$$\mathbb{E}\left[e^{\sum\limits_{n:t_n \le t} f(\delta_n)}\right] = \int_0^t \mathbb{E}\left[e^{\sum\limits_{k:t_k < s} f(\delta_k)}\right] ds \int_S (e^{f(x)} - 1) m(dx),$$

since $F_s := \prod\limits_{t_k < s} e^{f(\delta_k)}$ is a predictable process. Solving this differential equation yields

$$\mathbb{E}\left[e^{\sum\limits_{t_n \le t} f(\delta_n)}\right] = \exp\left(-t \int_S (1 - e^{f(x)}) m(dx)\right),$$

and the statement follows by Campbell's formula (see for example Theorem 2.7 in [35]). ■

Proof of Theorem 1.1. We will prove that under the laws $\mathbb{P}_x^{\uparrow}$ the process Y in equation (1.5) is a weak solution of (1.7). Pathwise uniqueness, which classically implies also strong existence, will then be shown as in [25].

We write $B = B^Y$ and $N = N^Y$ for the processes in (1.5), and we denote by $(\mathcal{F}_t)$ the filtration

$$\mathcal{F}_t := \sigma(B_s, (r_n, \nu_n)\mathbf{1}_{(t_n \le s)}; n \in \mathbb{N}, s \le t),$$

where $((t_n, r_n, \nu_n) \in [0, \infty) \times (0, \infty) \times (0, \infty))_{n\in\mathbb{N}}$ are the atoms of the Poisson point process N. We will use the absolute continuity of $\mathbb{P}^{\uparrow}$ w.r.t. $\mathbb{P}$ with Radon-Nikodym density $D_t = \frac{e^{\rho t} Y_t}{x}$, applying Theorem 1.3 to the process $(B_t : t \ge 0)$ and, indirectly, to the Poisson random measure N and its compensated measure.

Dealing with the diffusion part is standard since $d\langle D, B\rangle_t = \frac{e^{\rho t}}{x} \sigma \sqrt{Y_t} dt$, so that

$$B_t^{\uparrow} := B_t - \int_0^t \frac{d\langle D, B\rangle_s}{D_s} = B_t - \sigma \int_0^t Y_s^{-\frac{1}{2}} ds$$

is a Brownian motion under $\mathbb{P}^{\uparrow}$ by Theorem 1.3.

We next study the way the Poisson random measure N is affected by the change of probability, which is the main part of the proof. Enlarging the probability space and filtration if needed, we may and shall assume that there is a sequence $(u_n)_{n\geq 1}$ of independent random variables uniformly distributed on $[0,1]$, independent of B and N and such that $u_n \mathbf{1}_{\{t_n \leq t\}}$ is $\mathcal{F}_t$-measurable. Define random variables $(\Delta_n, \delta_n) \in [0,\infty)^2 \times [0,\infty)$ by

$$(\Delta_n, \delta_n) := \begin{cases} ((0,0), r_n \mathbf{1}_{\{\nu_n \leq Z_{t_n^-}\}}) & \text{if} \quad u_n > \dfrac{D_{t_n-}}{D_{t_n}} = \dfrac{Y_{t_n-}}{Y_{t_n}} \text{ and } Y_{t_n} > 0, \\ ((r_n, \nu_n), 0) & \text{if} \quad u_n \leq \dfrac{D_{t_n-}}{D_{t_n}} \text{ and } Y_{t_n} > 0, \\ ((0,0), 0) & \text{if} \quad Y_{t_n} = 0. \end{cases}$$

Let $f_{R,\epsilon}$ be a nonnegative measurable function such that for fixed $R \geq 0$ and $0 < \epsilon \leq 1$, and all (r, ν, s),

- $f_{R,\epsilon}((r,\nu), s) = 0$ when $\nu \geq R$,
- $f_{R,\epsilon}((r,\nu), s) = 0$ when $r < \epsilon$, and
- $f_{R,\epsilon}((0,0), 0) = 0$.

For any non-negative predictable process F we then have (using the third property of $f_{R,\epsilon}$ to pass to the second line)

$$\begin{aligned} &\sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}(\Delta_n, \delta_n) \\ &\quad = \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}\left((0,0), r_n \mathbf{1}_{\left\{\nu_n \leq Z_{t_n^-}\right\}}\right) \mathbf{1}_{\left\{u_n > \frac{Y_{t_n-}}{Y_{t_n}}\right\}} + \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}((r_n, \nu_n), 0) \mathbf{1}_{\left\{u_n \leq \frac{Y_{t_n-}}{Y_{t_n}}\right\}} \\ &\quad = \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}((0,0), r_n) \mathbf{1}_{\{\nu_n \leq Y_{t_n-}\}} \mathbf{1}_{\left\{u_n > \frac{Y_{t_n-}}{Y_{t_n}}\right\}} + \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}((r_n, \nu_n), 0) \mathbf{1}_{\left\{u_n \leq \frac{Y_{t_n-}}{Y_{t_n}}\right\}}. \end{aligned}$$

Therefore, since $1 - \frac{Y_{t_n-}}{Y_{t_n}} = \frac{r_n \mathbf{1}_{\{\nu_n \leq Y_{t_n-}\}}}{Z_{t_n}}$, by the compensation formula the process

$$\begin{aligned} S_t := \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}(\Delta_n, \delta_n) - \int_0^t ds F_s \int_0^\infty \int_0^\infty f_{R,\epsilon}((0,0), r) \frac{r \mathbf{1}_{\{\nu \leq Y_s\}}}{Y_s + r \mathbf{1}_{\{\nu \leq Y_s\}}} x \Pi(dr) d\nu \\ - \int_0^t ds F_s \int_0^\infty \int_0^\infty f_{R,\epsilon}((r,\nu), 0) \frac{Y_s}{Y_s + r \mathbf{1}_{\{\nu \leq Y_s\}}} \Pi(dr) d\nu \end{aligned}$$

is a pure jump martingale under $\mathbb{P}$. The quadratic covariation of S and D is thus given by

$$\begin{aligned}[S,D]_t &= \sum_{t_n\le t}(S_{t_n}-S_{t_n-})\left(\frac{e^{\rho t_n}}{x}Y_{t_n}-\frac{e^{\rho t_n-}}{x}Y_{t_n-}\right)\\ &= \sum_{t_n\le t}F_{t_n}f_{R,\epsilon}(\Delta_n,\delta_n)\frac{e^{\rho t_n}}{x}r_n\mathbf{1}_{\{\nu_n\le Y_{t_n-}\}}\\ &= \sum_{t_n\le t}F_{t_n}f_{R,\epsilon}((0,0),r_n)\frac{e^{\rho t_n}}{x}r_n\mathbf{1}_{\{\nu_n\le Y_{t_n-}\}}\mathbf{1}_{\left\{u_n>\frac{Y_{t_n-}}{Y_{t_n}}\right\}}\\ &\qquad+\sum_{t_n\le t}F_{t_n}f_{R,\epsilon}((r_n,\nu_n),0)\frac{e^{\rho t_n}}{x}r_n\mathbf{1}_{\{\nu_n\le Y_{t_n-}\}}\mathbf{1}_{\left\{u_n\le\frac{Y_{t_n-}}{Y_{t_n}}\right\}}.\end{aligned}$$

By the compensation formula, the conditional quadratic covariation of S and D is then given by

$$\begin{aligned}\langle D,S\rangle_t &= \int_0^t ds\frac{e^{\rho s}}{x}F_s\int_0^\infty\int_0^\infty f_{R,\epsilon}((0,0),r)\frac{r\mathbf{1}_{\{\nu\le Y_s\}}}{Y_s+r\mathbf{1}_{\{\nu\le Y_s\}}}r\Pi(dr)d\nu\\ &\quad+\int_0^t ds\frac{e^{\rho s}}{x}F_s\int_0^\infty\int_0^\infty f_{R,\epsilon}((r,\nu),0)\frac{Y_s}{Y_s+r\mathbf{1}_{\{\nu\le Y_s\}}}r\mathbf{1}_{\{\nu\le Y_s\}}\Pi(dr)d\nu.\end{aligned}$$

Using Theorem 1.3 we see that the process

$$\begin{aligned}S_t^\uparrow := S_t &- \int_0^t\int_0^\infty\int_0^\infty F_sf_{R,\epsilon}((0,0),r)\frac{r\mathbf{1}_{\{\nu\le Y_s\}}}{Y_s+r\mathbf{1}_{\{\nu\le Y_s\}}}\frac{r}{Y_s}\Pi(dr)d\nu ds\\ &-\int_0^t\int_0^\infty\int_0^\infty F_sf_{R,\epsilon}((r,\nu),0)\frac{Y_s}{Y_s+r\mathbf{1}_{\{\nu\le Y_s\}}}\frac{r\mathbf{1}_{\{\nu\le Y_s\}}}{Y_s}\Pi(dr)d\nu ds\end{aligned}$$

is a $(\mathcal{F}_t)$-martingale under $\mathbb{P}^\uparrow$. By definition of S and noting that $\int_0^\infty \frac{r}{Y_s}\mathbf{1}_{\{\nu\le Y_s\}}d\nu = r$, we get

$$\begin{aligned}S_t^\uparrow &= \sum_{t_n\le t}F_{t_n}f_{R,\epsilon}(\Delta_n,\delta_n)-\int_0^t dsF_s\int_0^\infty\int_0^\infty\left[f_{R,\epsilon}((0,0),r)\frac{r}{Y_s}\mathbf{1}_{\{\nu\le Y_s\}}+f_{R,\epsilon}((r,\nu),0)\right]\Pi(dr)d\nu\\ &=\sum_{t_n\le t}F_{t_n}f_{R,\epsilon}(\Delta_n,\delta_n)-\int_0^t dsF_s\left[\int_0^\infty f_{R,\epsilon}((0,0),r)r\Pi(dr)+\int_0^\infty\int_0^\infty f_{R,\epsilon}((r,\nu),0)\right]d\Pi(dr)\nu.\end{aligned}$$

Since $\mathbb{E}^\uparrow(S_t^\uparrow)=\mathbb{E}^\uparrow(S_0^\uparrow)=0$, this implies

$$\begin{aligned}\mathbb{E}^\uparrow\left[\sum_{t_n\le t}F_{t_n}f_{R,\epsilon}(\Delta_n,\delta_n)\right] &= \mathbb{E}^\uparrow\left[\int_0^t dsF_s\int_0^\infty f_{R,\epsilon}((0,0),r)r\Pi(dr)\right]\\ &\qquad+\mathbb{E}^\uparrow\left[\int_0^t dsF_s\int_0^\infty\int_0^\infty f_{R,\epsilon}((r,\nu),0)\Pi(dr)d\nu\right].\end{aligned}$$

By standard arguments, this formula is also true for any nonnegative function f such that $f((0,0),0)=0$. Using Lemma 1.4 we then see that $(t_n,\Delta_n)_{n\ge0}$ and $(t_n,\delta_n)_{n\ge0}$ are under $\mathbb{P}^\uparrow$

the atoms of two Poisson point processes $N^\uparrow$ and $N^\star$, with intensity measures $dt\times d\nu\times\Pi(dr)$ and $dt\times r\Pi(dr)$ on $[0,\infty)\times(0,\infty)\times(0,\infty)$ and $[0,\infty)\times(0,\infty)$, respectively. By construction, $N^\uparrow$ and $N^\star$ are independent because they never jump simultaneously. Now set

$$J_t := \int_0^t\int_0^{Z_{s-}}\int_1^\infty rN(ds,d\nu,dr) = \sum_{t_n\le t} r_n\mathbf{1}_{\{\nu_n\le Y_{t_n-}\}}\mathbf{1}_{\{r_n\ge 1\}}.$$

From the definition of $(\Delta_n,\delta_n)_{n\in\mathbb{N}}$, and writing $\Delta_n^{(i)}$ for the $i-$th coordinate of Δ_n, $i=1,2$, we have

$$\begin{aligned} J_t &= \sum_{t_n\le t}\Delta_n^{(1)}\mathbf{1}_{\{\Delta_n^{(2)}\le Y_{t_n-}\}}\mathbf{1}_{\{\Delta_n^{(1)}\ge 1\}} + \sum_{t_n\le t}\delta_n\mathbf{1}_{\{\delta_n\ge 1\}} \\ &= \int_0^t\int_0^{Y_{s-}}\int_1^\infty rN^\uparrow(ds,d\nu,dr) + \int_0^t\int_1^\infty rN^\star(ds,dr). \end{aligned}$$

Finally, we observe that for given $0<\varepsilon<1$, the process

$$\begin{aligned} \tilde{M}_t^{(\varepsilon)} &:= \int_0^t\int_0^{Y_{s-}}\int_\varepsilon^1 rN(ds,d\nu,dr) - \int_0^t\int_0^{Y_{s-}}\int_\varepsilon^1 r\Pi(dr)d\nu ds \\ &= \sum_{t_n\le t} r_n\mathbf{1}_{\{\nu_n\le Y_{t_n-}\}}\mathbf{1}_{\{\varepsilon<r_n<1\}} - \int_0^t\int_0^{Y_s}\int_\varepsilon^1 r\,\Pi(dr)d\nu ds \end{aligned}$$

is a $\mathbb{P}$-martingale which converges in the $L^2(\mathbb{P})$ sense when $\varepsilon\to 0$ to $\tilde{M}_t := \int_0^t\int_0^{Y_s-}\int_0^1 r\tilde{N}(ds,d\nu,dr)$. In terms of (Δ_n) and (δ_n), we can write

$$\begin{aligned} \tilde{M}^{(\varepsilon)} &= \left(\sum_{t_n\le t}\Delta_n^{(1)}\mathbf{1}_{\{\Delta_n^{(2)}\le Y_{t_n-}\}}\mathbf{1}_{\{\varepsilon<\Delta_n^{(1)}<1\}} - \int_0^t\int_0^{Y_s}\int_\varepsilon^1 r\Pi(dr)d\nu ds\right) + \sum_{t_n\le t}\delta_n\mathbf{1}_{\{\varepsilon<\delta_n<1\}} \\ &= \left(\int_0^t\int_0^{Y_{s-}}\int_\varepsilon^1 rN^\uparrow(ds,d\nu,dr) - \int_0^t\int_0^{Y_s}\int_\varepsilon^1 r\Pi(dr)d\nu ds\right) + \int_0^t\int_\varepsilon^1 rN^\star(ds,dr). \end{aligned} \tag{1.12}$$

Thanks to [35, Theorem 2.10], the $L^2(\mathbb{P}^\uparrow)$ limit as $\varepsilon\to 0$ of the $\mathbb{P}^\uparrow$-martingale given by the expression in the third line of (1.12) exists, and equals the $\mathbb{P}^\uparrow$-martingale

$$\int_0^t\int_0^{Y_{s-}}\int_0^1 r\tilde{N}^\uparrow(ds,d\nu,dr),$$

where $\tilde{N}^\uparrow$ is the compensated measure associated with $N^\uparrow$. Also, as $\int_0^\infty(1\wedge x^2)\Pi(dx)<\infty$, by [35, Theorem 2.9] the last term of (1.12) converges $\mathbb{P}^\uparrow$-a.s. as $\varepsilon\to 0$, and so we have

$$\tilde{M}_t = \int_0^t\int_0^{Y_{s-}}\int_0^1 r\tilde{N}^\uparrow(ds,d\nu,dr) + \int_0^t\int_0^1 rN^\star(ds,dr)\quad \mathbb{P}^\uparrow - a.s.$$

Bringing all parts together, we have shown that Y satisfies under $\mathbb{P}^\uparrow$ the desired SDE, except for the independence of the processes $B^\uparrow$ and $(N^\uparrow,N^\star)$, which we establish in what follows.

Let $\zeta \in \mathbb{R}$, $\lambda_k, \gamma_k \in \mathbb{R}_+$, $m \in \mathbb{N}$ and $k \in \{1, ..., m\}$, and consider $(W_k)_{k=1}^m$ and $(V_k)_{k=1}^m$ disjoint subsets of $(0,\infty)\times(0,\infty)$ and $(0,\infty)$ respectively, such that $\int_{W_k} \Pi(dr)d\nu$ and $\int_{V_K} r\Pi(dr)$ are finite. Set

$$F(x, y_1, .., y_m, z_1, .., z_m) := e^{\zeta x} e^{-\sum_{k=1}^m \lambda_k y_k} e^{-\sum_{k=1}^m \gamma_k z_k}.$$

Applying Itô's formula to the semimartingale

$$X_t = \left(B_t^{\uparrow}, N^{\uparrow}((0,t]\times W_1), .., N^{\uparrow}((0,t]\times W_m), N^{\star}((0,t]\times V_1), .., N^{\star}((0,t]\times V_m)\right),$$

we obtain for $0 \le s \le t$ that

$$\begin{aligned} F(X_t) - F(X_s) = & \int_s^t \zeta F(X_u) dB_u^{\uparrow} + \frac{\zeta^2}{2}\int_s^t F(X_u)du + \sum_{s<t_n\le t} F(X_{t_n}) - F(X_{t_n-}) \\ & + \sum_{s<t_n\le t}\sum_{j=1}^m \left[\lambda_j F(X_{t_n-})\mathbf{1}_{\{\Delta_n\in W_j\}} + \gamma_j F(X_{t_n-})\mathbf{1}_{\{\delta_n \in V_j\}}\right] \\ & - \sum_{j=1}^m \int_s^t \int_{W_j} \lambda_j F(X_{u-}) N^{\uparrow}(du, d\nu, dr) - \sum_{j=1}^m \int_s^t \int_{V_j} \gamma_j F(X_{u-}) N^{\star}(du, dr) \\ = & \int_s^t \zeta F(X_u) dB_u^{\uparrow} + \frac{\zeta^2}{2}\int_s^t F(X_u)du + \sum_{s<t_n\le t} F(X_{t_n^-}) f(\Delta_n, \delta_n), \end{aligned} \tag{1.13}$$

where the second and third lines canceled out by definition of the integrals with respect to $N^\star$ and $N^\uparrow$, and where the notation

$$f((r,\nu), s) := e^{-\sum\limits_{k=1}^m \lambda_k \mathbf{1}_{\{(r,\nu)\in W_k\}} - \sum\limits_{k=1}^m \lambda_k \mathbf{1}_{\{s\in V_k\}}} - 1$$

was used in the last term of the fourth line. Using the fact that $f((0,0),0) = 0$ and previous arguments, we can show that the process

$$\sum_{t_n\le t} F(X_{t_n-}) f(\Delta_n, \delta_n) - \int_0^t F(X_u)du \left[\int_0^\infty \int_0^\infty f((r,\nu),0)\Pi(dr)d\nu + \int_0^\infty f((0,0),r) r\Pi(dr)\right]$$

is a $\mathbb{P}^\uparrow$-martingale with respect to $\mathcal{F}_t$. Since the sum of the two integrals in square braquets is equal to

$$\sum_{k=1}^m \left[\int_{W_k} (e^{-\lambda_k} - 1)\Pi(dr)d\nu + \int_{V_k} (e^{-\gamma_k} - 1) r\Pi(dr)\right],$$

we deduce from the latter and (1.13) that

$$F(X_t) - F(X_s) - \int_s^t F(X_u)du \left(\frac{\zeta^2}{2} + \sum_{k=1}^m \left[\int_{W_k} (e^{-\lambda_k} - 1)\Pi(dr)d\nu + \int_{V_k} (e^{-\gamma_k} - 1) r\Pi(dr)\right]\right)$$

is a martingale increment. Multiplying it by $F((X_s))^{-1}\mathbf{1}_A$ for $A \in \mathcal{F}_s$, taking expectation, and using then Gronwall's lemma, we conclude that

$$\mathbb{E}^\uparrow\left[F(X_t - X_s)\mathbf{1}_A\right] = \mathbb{P}^\uparrow(A) e^{(t-s)\left[\frac{\zeta^2}{2} + \sum\limits_{k=1}^m \int_{W_k} (e^{-\lambda_k}-1)\Pi(dr)d\nu + \sum\limits_{k=1}^m \int_{V_k} (e^{-\gamma_k}-1) r\Pi(dr)\right]}.$$

This means that under $\mathbb{P}^{\uparrow}$, X_t is a multidimensional Lévy process with respect to $\mathcal{F}_t$ with independent coordinates and implies the independence of $B^{\uparrow}$ and $(N^{\uparrow}, N^{\star})$.

We now establish the pathwise uniqueness of solutions following the ideas of Fu and Li [25]. Let $B^{\uparrow}$, $N^{\uparrow}$ and $N^{\star}$ be independent processes as before. Let $(Y_t^{(1)})$ and $(Y_t^{(2)})$ be two solutions of (1.7) with deterministic initial values, and set $\zeta_t := Y_t^{(1)} - Y_t^{(2)}$ for $t \geq 0$. Then, we have

$$\begin{aligned}\zeta_t = \zeta_0 &+ \int_0^t \alpha\left(Y_s^{(1)} - Y_s^{(2)}\right) ds + \int_0^t \sigma\left(\sqrt{Y_s^{(1)}} - \sqrt{Y_s^{(2)}}\right) dB_s^{\uparrow} \\ &+ \int_0^t \int_{U_0} r\left(\mathbf{1}_{\{\nu < Y_s^{(1)}\}} - \mathbf{1}_{\{\nu < Y_s^{(2)}\}}\right) N^{\uparrow}(ds, d\nu, dr) \\ &+ \int_0^t \int_{U_1} r\left(\mathbf{1}_{\{\nu < Y_s^{(1)}\}} - \mathbf{1}_{\{\nu < Y_s^{(2)}\}}\right) \tilde{N}^{\uparrow}(ds, d\nu, dr),\end{aligned} \tag{1.14}$$

where $U_0 = [0,\infty) \times [1,\infty)$ and $U_1 = [0,\infty) \times (0,1)$.

The idea is to construct now a suitable sequence of $\mathcal{C}^2$ functions $\{\phi_k\}$ that approximate the function $|x|$, and to prove then that $\mathbb{E}\left[\phi_k(\zeta_t)\right] = 0$ for each $t \geq 0$ with help of Itô's formula.

First, we establish some notation that will be needed in the sequel:

- Let us define the constant $K := |\alpha| + M$, where $\int_1^{\infty} r\Pi(dr) = M < \infty$. Observe that
$$|\alpha x| + \int_0^{\infty}\int_1^{\infty} r\mathbf{1}_{\{\nu < x\}}\Pi(dr)d\nu = Kx.$$
- Set $L(x) := (\sigma^2 + I)(x)$, where $I = \int_0^1 r^2\Pi(dr)$. Then, the function L satisfies
$$v\sigma^2 x + \int_0^{\infty}\int_0^1 r^2\mathbf{1}_{\{\nu < x\}}d\nu\Pi(dr) = \sigma^2 x + x\int_0^1 r^2\Pi(dr) = L(x).$$
- Let $\beta(z) := (|\alpha| + M)z$. This function satisfies $\int_{0_+}\beta(z)^{-1}dz = \infty$ and, if we suppose without losing generality that $y \leq x$, we have
$$|\alpha(x-y)| + \int_0^{\infty}\int_1^{\infty} r\mathbf{1}_{\{y<\nu<x\}}\Pi(dr)d\nu = \beta(x-y). \tag{1.15}$$
- We define the function $\varrho(x) := [\sigma^2 + I]\sqrt{x}$, where $I = \int_0^1 r^2\Pi(dr)$. Note that, if $y \leq x$, then
$$\begin{aligned}\sigma^2(\sqrt{x} - \sqrt{y})^2 + \int_0^{\infty}\int_0^1 r^2\mathbf{1}_{\{y<\nu<x\}}\Pi(dr)d\nu &= \sigma^2(\sqrt{x} - \sqrt{y})^2 + (x-y)I \\ &\leq \varrho(x-y).\end{aligned} \tag{1.16}$$

Now, fix a sequence $(a_k)_{k\geq 1}$ such that $a_k = a_{k-1}e^{-k[\sigma^2+I]^2}$ and $a_0 = 1$. Note that $a_k \to 0_+$ decreasingly and $\int_{a_k}^{a_{k-1}} \varrho(z)^{-2}dz = k$ for $k \geq 1$. Let $z \mapsto \psi_k(z)$ be a non-negative continuous function on $\mathbb{R}$ which has support in (a_k, a_{k-1}), satisfies $0 \leq \psi_k(z) \leq 2k^{-1}\varrho(z)^{-2}$ for $a_k <$

$z < a_{k-1}$, and $\int_{a_k}^{a_{k-1}} \psi_k(z)dz = 1$. For each $k \geq 1$, we define the non-negative and twice continuously differentiable function

$$\phi_k(x) = \int_0^{|x|} dy \int_0^y \psi_k(z)dz, \quad x \in \mathbb{R}.$$

The sequence (ϕ_k) has the following properties:

i). $\phi_k(x) \to |x|$ non-decreasingly as $k \to \infty$, since for all $y \geq 0$, $\int_0^y \psi_k(z)dz \nearrow 1$;

ii). $0 \leq \phi_k'(x) \leq 1$ for $x \geq 0$ and $-1 \leq \phi_k'(x) \leq 0$ for $x \leq 0$;

iii). $\phi''(x) \geq 0$ for $x \in \mathbb{R}$, and $\phi_k''(x-y)[\sigma\sqrt{x} - \sigma\sqrt{y}]^2 \to 0 \quad (k \to \infty)$, uniformly in x, y;

iv).

$$0 \leq \int_{U_1} D_{l(r,\nu;x,y)}\phi_k(x-y)\Pi(dr)d\nu \leq \frac{1}{k[\sigma^2 + I]} \int_0^1 r^2\Pi(dr) \to_{k\to\infty} 0,$$

uniformly in $x, y \geq 0$, where $l(r, \nu; , x, y) = r\left[\mathbf{1}_{\{\nu<x\}} - \mathbf{1}_{\{\nu<y\}}\right]$.

Property iii.) is true by (3.84). Indeed,

$$\phi_k''(x-y)[\sigma\sqrt{x} - \sigma\sqrt{y}]^2 \leq \psi_k(|x-y|)\varrho(|x-y|)^2 \leq 2/k.$$

Also, by Taylor's expansion,

$$\begin{aligned} D_h\phi_k(\varsigma) &:= \phi_k(\varsigma + h) - \phi_k(\varsigma) - \phi_k'(\varsigma)h \\ &= h^2 \int_0^1 \phi_k''(\varsigma + th)(1-t)dt \\ &= h^2 \int_0^1 \psi_k(|\varsigma + th|)(1-t)dt; \end{aligned}$$

and the monotonicity of $z \mapsto \varrho(z)$ implies

$$0 \leq D_h\phi_k(\varsigma) \leq 2k^{-1}h^2 \int_0^1 \varrho(|\varsigma + th|)^{-2}(1-t)dt \leq k^{-1}h^2\varrho(|\varsigma|)^{-2}$$

for $\varsigma h \geq 0$. Since $x \mapsto r\mathbf{1}_{\{\nu<x\}}$ is non-decreasing, for $x, y \geq 0$ we can use the previous inequalities and (3.84) to prove property iv.).

We now deduce the pathwise uniqueness for equation (1.7). Let $\tau_m = \inf\{t \geq 0 : Y_t^{(1)} \geq m \text{ or } Y_t^{(2)} \geq m\}$, $m \geq 1$. By (1.14) and Itô's formula,

$$\begin{aligned} \phi_k(\zeta_{t\wedge\tau_m}) = \phi_k(\zeta_0) &+ \int_0^{t\wedge\tau_m} \phi_k'(\zeta_s)\alpha\left(Y_s^{(1)} - Y_s^{(2)}\right)ds + \frac{1}{2}\int_0^{t\wedge\tau_m} \sigma^2\phi_k''(\zeta_s)\left[\sqrt{Y_s^{(1)}} - \sqrt{Y_s^{(2)}}\right]^2 ds \\ &+ \int_0^{t\wedge\tau_m} \sigma\phi_k'(\zeta_s)\left(\sqrt{Y_s^{(1)}} - \sqrt{Y_s^{(2)}}\right)dB_s^{\uparrow} \\ &+ \int_0^{t\wedge\tau_m}\int_{U_0} \phi_k'(\zeta_{s-})l(Y_s^{(1)}, Y_s^{(2)})N^{\uparrow}(ds, d\nu, dr) \\ &+ \int_0^{t\wedge\tau_m}\int_{U_1} \phi_k'(\zeta_{s-})l(Y_s^{(1)}, Y_s^{(2)})\tilde{N}^{\uparrow}(ds, d\nu, dr) \\ &+ \sum_{s\leq t}\left[\phi_k(\zeta_s) - \phi_k(\zeta_{s-}) - \phi_k'(\zeta_{s-})\Delta\zeta_s\right], \end{aligned}$$

and so

$$\begin{aligned}\phi_k(\zeta_{t\wedge\tau_m}) = \phi_k(\zeta_0) &+ \int_0^{t\wedge\tau_m} \phi_k'(\zeta_s)\alpha\left(Y_s^{(1)} - Y_s^{(2)}\right) ds \\ &+ \frac{1}{2}\int_0^{t\wedge\tau_m} \phi_k''(\zeta_s)\sigma^2\left[\sqrt{Y_s^{(1)}} - \sqrt{Y_s^{(2)}}\right]^2 ds \\ &+ \int_0^{t\wedge\tau_m} ds \int_{U_0} \triangle_{l(r,\nu;Y_{s-}^{(1)},Y_{s-}^{(2)})}\phi_k(\zeta_{s-})\Pi(dr)d\nu \\ &+ \int_0^{t\wedge\tau_m} ds \int_{U_1} D_{l(r,\nu;Y_{s-}^{(1)},Y_{s-}^{(2)})}\phi_k(\zeta_{s-})\Pi(dr)d\nu + \check{M}_{t\wedge\tau_m},\end{aligned} \tag{1.17}$$

where $\triangle_h f(z) := f(z+h) - f(z)$ and $(\check{M}_{t\wedge\tau_m})$ is $(\mathcal{F}_t)$ - martingale. By property (ii), we see that

$$\phi_k'(\zeta_{s-})a(Y_s^{(1)} - Y_s^{(2)}) \leq |a||Y_s^{(1)} - Y_s^{(2)}|.$$

Observe also that

$$\int_{U_0} \triangle_{l(r,\nu;Y_{s-}^{(1)},Y_{s-}^{(2)})}\phi_k(\zeta_{s-}))\Pi(dr)d\nu \leq \int_{U_0} r|\mathbf{1}_{\{\nu<Y_s^{(1)}\}} - \mathbf{1}_{\{\nu<Y_s^{(2)}\}}|\Pi(dr)d\nu.$$

By (3.83), for any $s \leq \tau_m$ the sum of the right hand sides of the above two inequalities is not larger than $\beta(|\zeta_{s-}|)$. Due to properties (iii) and (iv) we have

$$\phi_k''(\zeta_s)\sigma^2\left[\sqrt{Y_s^{(1)}} - \sqrt{Y_s^{(2)}}\right]^2 \to 0 \quad \text{and} \quad \int_{U_1} D_{l(r,\nu;Y_{s-}^{(1)},Y_{s-}^{(2)})}\phi_k(\zeta_{s-})\Pi(dr)d\nu \to 0,$$

uniformly on the event $\{s \leq \tau_m\}$. Taking expectation in (1.17) and letting $k \to \infty$, we see that

$$\mathbb{E}^{\uparrow}|\zeta_{t\wedge\tau_m}| \leq \mathbb{E}^{\uparrow}|\zeta_0| + \mathbb{E}^{\uparrow}\int_0^{t\wedge\tau_m} \beta(|\zeta_{s-}|)ds.$$

Since $\zeta_{s-} < m$ for $0 < s \leq \tau_m$, we deduce that $t \mapsto \mathbb{E}^{\uparrow}|\zeta_{t\wedge\tau_m}|$ is locally bounded. Note also that $\zeta_{s-} \neq \zeta_s$ for at most countably many $s \geq 0$. Then

$$\begin{aligned}\mathbb{E}^{\uparrow}|\zeta_{t\wedge\tau_m}| &\leq \mathbb{E}^{\uparrow}|\zeta_0| + \int_0^t \mathbb{E}^{\uparrow}(|\alpha| + M)|\zeta_{s\wedge\tau_m}|ds \\ &\leq \mathbb{E}^{\uparrow}|\zeta_0| + \int_0^t (|\alpha| + M)\mathbb{E}^{\uparrow}|\zeta_{s\wedge\tau_m}|ds.\end{aligned}$$

Since $Y_0^{(1)} = Y_0^{(2)}$, we can use Gronwall's lemma to show that $\mathbb{E}^{\uparrow}|\zeta_{t\wedge\tau_m}| = 0$ for all $t \geq 0$, which implies $\mathbb{P}^{\uparrow}\{\zeta_t = 0 \text{ for all } t \leq \tau_m\} = 1$. Since $\tau_m \to \infty$ as $m \to \infty$ a.s., the prove is completed. ■

Chapter 2

SDE representations of logistic branching process

2.1 Continuous-state branching processes with logistic growth

Branching processes with logistic growth or logistic branching processes, abbreviated as LBPs, are density-dependent continuous time branching processes. In the continuous state-space setting, the LBP is a Markov process with (nonnegative) real values and a.s. càdlàg paths. The definition of these processes, given in a general form by Lambert [36], is inspired by Lamperti transform linking continuous-state branching processes and Lévy processes, but using an Ornstein-Uhlenbeck process instead: let R be the unique strong solution, starting from x, of the SDE

$$dR_t = dX_t - cR_t dt, \tag{2.1}$$

where X a Lévy process with Laplace exponent ψ. Then, the continuous-state branching process with logistic growth Z with branching mechanism ψ and rate c is the process defined by

$$Z_t := \begin{cases} R(C_t) & \text{if } 0 \leq t < \eta_\infty, \\ 0 & \text{if } \eta_\infty < \infty \wedge t \geq \eta_\infty, \end{cases} \tag{2.2}$$

with $T_0 := \inf\{t > 0 : R_t = 0\}$ and C is the right inverse of η, where

$$\eta_t = \int_0^{t \wedge T_0} \frac{ds}{R_s}, \quad t > 0.$$

This definition give as a close relationship between logistic and branching processes: given $Y = (Y_t : t \geq 0)$ a CSBP with branching mechanism ψ, we can see $Z = (Z_t : t \geq 0)$ as the analogous of process Y with negative interactions between each pair of individuals in the population.

Next, we give an easier characterization for this kind of processes, as the unique solution of an SDE. This result is a generalization of the CSBP case, found in [13, Prop. 4].

Theorem 2.1 (LB-process as a SDE solution)**.** *There is a standard Brownian motion B and an independent Poisson measure N on $[0, \infty) \times (0, \infty) \times (0, \infty]$ with intensity measure*

$dt \times d\nu \times \Pi(dr)$, such that the LB-process Z is the unique strong solution of the following equation:

$$Z_t = v + \alpha \int_0^t Z_s ds + \sigma \int_0^t \sqrt{Z_s} dB_s + \int_0^t \int_0^{Z_{s-}} \int_1^\infty r N(ds, d\nu, dr) \\ + \int_0^t \int_0^{Z_{s-}} \int_0^1 r \tilde{N}(ds, d\nu, dr) - c \int_0^t {Z_s}^2 ds, \tag{2.3}$$

where $\tilde{N}$ is the compensated Poisson measure associated with N.

Proof of Theorem 2.1. To prove this result, we use similar arguments as in [13, Prop. 4]. Given a Lévy process X with characteristic exponent ψ, there exists a standard Brownian motion B^X and a Poisson random measure N^X on $[0,\infty) \times (0,\infty]$ with intensity $ds \times \Pi(dr)$ such X satisfies

$$dX_t = \alpha dt + \sigma dB_t^X + \int_1^\infty r N^X(dt, dr) + \int_0^1 r \widetilde{N}^X(dt, dr). \tag{2.4}$$

Using this fact along with (2.2), we can deduce that R satisfies

$$dR_t = adt + \sigma dB_t^X + \int_1^\infty r N^X(dt, dr) + \int_0^1 r \widetilde{N}^X(dt, dr) - cR_t dt.$$

Now, we set

$$T := \inf\{t > 0 : Z_t = 0\} = \inf\{t > 0 : R(C_t) = 0\} \wedge \eta_\infty.$$

As C is right-continuous, we have that $R(C_T) = 0$, and from here we can deduce that $C_T = T_0$, using the fact that

$$\eta_l = \int_0^{l \wedge T_0} \frac{ds}{R_s} = \eta_\infty$$

for all $l \geq T_0$.

In the same way that Caballero *et al.* [13], we define a standard Brownian motion B satisfying

$$\int_0^t \sqrt{Z_s} dB_s = B^X(C_t \wedge T_0) \tag{2.5}$$

and a Poisson random measure N with intensity $ds \times d\nu \times \Pi(dr)$ such that

$$\sum_{\{n: t_n^X < C_t\}} r_n^X \mathbf{1}_{\{r_n^X \geq 1\}} = \sum_{\{n: t_n < t\}} \Delta_n \mathbf{1}_{\{\Delta_n \geq 1\}} = \int_0^t \int_0^{Z_{s-}} \int_0^\infty r \mathbf{1}_{\{r \geq 1\}} N(ds, dv, dr),$$

where $((\Delta_n, t_n) : n \in \mathbb{N})$ is an arbitrary labelling of the pairs associating jump times and jump sizes of Z and $((r_n^X, t_n^X) : n \in \mathbb{N})$ are the atoms of N^X, and from here we have that

$$\int_0^t \int_0^{Z_{s-}} \int_1^\infty r N(ds, d\nu, dr) = \int_0^{C_t} \int_1^\infty r N^X(ds, dr). \tag{2.6}$$

In the same way, we have that

$$\lim_{\varepsilon \searrow 0} \left[\sum_{\{n: t_n < t\}} \Delta_n \mathbf{1}_{\{\varepsilon < \Delta_n < 1\}} - \int_0^t Z_s ds \int_\varepsilon^1 r \Pi(dr) \right] = \lim_{\varepsilon \searrow 0} \left[\sum_{\{n: t_n^X < C_t\}} r_n^X \mathbf{1}_{\{\varepsilon < r_n^X < 1\}} - \int_0^{C_t} ds \int_\varepsilon^1 r \Pi(dr) \right],$$

so the compensated measures satisfies

$$\int_0^t \int_0^{Z_{s-}} \int_0^1 r\widetilde{N}(ds, d\nu, dr) = \int_0^{C_t} \int_0^1 r\widetilde{N}^X(ds, dr). \tag{2.7}$$

Putting together expressions (2.5), (2.6) and (2.7) into equation (2.4), we obtain that

$$dR_{C_t} = \alpha dC_t + \sigma\sqrt{Z_t}dB_t + \int_0^{Z_{t-}} \int_1^\infty rN(dt, d\nu, dr) + \int_0^{Z_{t-}} \int_0^1 r\widetilde{N}(dt, d\nu, dr) - cR_{C_t}dC_t.$$

By (2.2), we see that $adC_t = aZ_tdt$ and $cR_{C_t}dC_t = c(Z_t)^2dt$, so the logistic branching process $Z = (Z_t : t \geq 0)$ is a solution of (2.3). Finally, defining the parameters (b, σ, g_0, g_1) by

$$b(x) := (ax - cx^2)\mathbf{1}_{\{x\geq 0\}}, \quad \sigma(x) := \sigma x, \quad \text{and} \quad g_0(x, (\nu, r)) = g_1(x, (\nu, r)) := r\mathbf{1}_{\{\nu\leq x\}}$$

we see that equation (2.3) is included in the jump-type stochastic equations studied by Fu and Li in [25], so the existence of an unique non-negative strong solution to (2.3) follows directly from [25, Thm 2.5]. ■

2.2 Logistic diffusion process conditioned to be never extinct

Now, we want to study the LBP conditioned to non extinction. To this end, we apply the main arguments in the proof of Theorem 1.1 to equation (2.3), assuming that such process exists. Notice that in the LBP case, the branching property is not longer true and it is unknown, in general, if there exists such a Q-process defined through an h-transform, as in the CSBP case (see [48]). Results in that sense are only known for the logistic Feller diffusion case, thanks to the renowned work of Cattiaux *et. al.* [15].

In the particular case when the underlying Lévy process is a Brownian motion with drift, equation (2.3) reduces to

$$Z_t = z + \alpha\int_0^t Z_sds + \sigma\int_0^t \sqrt{Z_s}dB_s - c\int_0^t {Z_s}^2ds, \tag{2.8}$$

In [15], Cattiaux *et. al.* established existence of the Q-process, through the study of quasi-stationary distributions for drifted Brownian motion on $(0, \infty)$ of the form

$$dX_t = dB_t - q(X_t)dt, \quad X_0 = x > 0,$$

where q is a given function C^1 on $(0, \infty)$ and $(B_t : t \geq 0)$ is a standard one-dimensional Brownian motion. Defining $X_t = 2\sqrt{Z_t}/\sigma$, (2.8) turns into

$$dX_t = dB_t - q(X_t)dt, \quad X_0 = x = 2\sqrt{z}/\sigma > 0, \tag{2.9}$$

where $q(x) = \frac{1}{2x} - \frac{2}{\sigma^2}\left(\frac{\alpha\sigma^2 x}{4} - \frac{c\sigma^4x^3}{16}\right)$. For this kind of processes, the authors in [15] defined a measure μ on $(0, \infty)$, given by

$$\mu(dy) := e^{-Q(y)}dy,$$

where $Q(y) := 2\int_1^\infty q(x)dx$. Hence, they have established the existence, under certain conditions (see Remark 2.2 below), of a non-positive self adjoint operator L on $\mathbb{L}^2(\mu)$ with domain $D(L) \supseteq C_0^\infty((0,\infty))$ such that for $g \in C_0^\infty((0,\infty))$,

$$Lg = \frac{1}{2}g'' - qg'.$$

Using spectral theory, they showed also that $-L$ has a purely discrete spectrum $0 \leq \lambda_1 < \lambda_2 < \dots$, and furthermore, each λ_i ($i \in \mathbb{N}$) is associated to a unique (up to a multiplicative constant) eigenfunction η_i of class $C^2((0,\infty))$, which also satisfies the ordinary differential equation

$$\frac{1}{2}\eta_i'' - q\eta_i' = -\lambda_i\eta_i.$$

The sequence $(\eta_i)_{i\geq 1}$ is an orthonormal basis of $\mathbb{L}^2(\mu)$, and η_1 can be chosen to be strictly positive in $(0,\infty)$. Moreover, $\eta_1 \in \mathbb{L}^1(\mu)$ and it is an increasing function.

Remark 2.2. We say that hypothesis (H) is satisfied if

(H1) for all $x > 0$, $\mathbb{P}_x(\tau = T_0 < T_\infty) = 1$;

(H2) $C = -\inf_{y\in(0,\infty)} (q^2(y) - q'(y)) < \infty$ and $\lim_{y\to\infty} (q^2(y) - q'(y)) = +\infty$; and

(H3) $\displaystyle\int_0^1 \frac{e^{-Q(y)}dy}{q^2(y) - q'(y) + C + 2} < \infty$ or $\displaystyle\left(\int_1^\infty e^{-Q(x)}dx < \infty \quad \wedge \quad \int_0^1 xe^{-Q(x)/2}dx < \infty\right)$

hold.

Under (H), Cattiaux *et al* were able to describe the law of the process X conditioned to be never extinct.

Lemma 2.3 (Corollary 6.1 in [15]). *For all $x > 0$ and $t \geq 0$, we have*

$$\lim_{s\to\infty} \mathbb{P}_x(X \in B | T_0 > t + s) = \mathbb{Q}_x(B),$$

for all B Borel measurable subsets of $C([0,t])$, where $\mathbb{Q}_x$ is the law of a diffusion process on $(0,\infty)$, with transition probability densities (w.r.t. the Lebesgue measure) given by

$$q(t,x,y) = e^{\lambda_1 t}\frac{\eta_1(y)}{\eta_1(x)}r(s,x,y)e^{-Q(y)}.$$

That is, $\mathbb{Q}_x$ is locally absolutely continuous w.r.t. $\mathbb{P}_x$ and

$$\mathbb{Q}_x(X \in B) = \mathbb{E}_x\left(\mathbf{1}_B(X)\mathbf{1}_{\{t<T_0\}}e^{\lambda_1 t}\frac{\eta_1(X_t)}{\eta_1(x)}\right).$$

In the result above, r correspond to the density of X under the measure μ, i.e. $r(t,x,\cdot)$ satisfies

$$\mathbb{E}_x\left[f(X_t)\mathbf{1}_{\{t<T_0\}}\right] = \int_0^\infty f(y)r(t,x,y)\mu(dy), \quad \text{for all } x > 0,\ t > 0$$

for all bounded Borel f. In particular, we can deduce from this result an SDE representation for the logistic diffusion under $\mathbb{Q}$.

Proposition 2.4. *Under $\mathbb{Q}$, the process Z satisfies*

$$Z_t = z + \alpha \int_0^t Z_s ds + \sigma \int_0^t \sqrt{Z_s} dB_s^{\mathbb{Q}} - c \int_0^t {Z_s}^2 ds + \sigma^2 \int_0^t \frac{\phi'(Z_s)}{\phi(Z_s)} Z_s ds, \tag{2.10}$$

where $\{B_t^{\mathbb{Q}} : t \geq 0\}$ is an standard Brownian motion on $\mathbb{Q}$ and $\phi(x) = \eta\left(\frac{2\sqrt{x}}{\sigma}\right)$.

Proof. We start by writing the Radom-Nykodim derivative between $\mathbb{Q}$ and $\mathbb{P}$ as a function of Z

$$D_t := \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{e^{\lambda_1 t}}{\eta_1\left(\frac{2\sqrt{z}}{\sigma}\right)} \eta_1\left(\frac{2\sqrt{Z_t}}{\sigma}\right) = \frac{e^{\lambda_1 t}}{\phi(z)} \phi(Z_t)$$

Applying Itô's Formula, we obtain that

$$\begin{aligned}
dD_t &= \frac{e^{\lambda_1 t}}{\eta_1\left(\frac{2\sqrt{z}}{\sigma}\right)} \left\{ \lambda_1 \eta_1\left(\frac{2\sqrt{Z_t}}{\sigma}\right) dt + \frac{\eta_1'\left(\frac{2\sqrt{Z_t}}{\sigma}\right)}{\sigma\sqrt{Z_t}} dZ_t + \frac{1}{2}\left[\frac{\eta_1''\left(\frac{2\sqrt{Z_t}}{\sigma}\right)}{\sigma^2 Z_t} - \frac{\eta_1'\left(\frac{2\sqrt{Z_t}}{\sigma}\right)}{2\sigma Z_t\sqrt{Z_t}}\right] d[Z,Z]_t \right\} \\
&= \frac{e^{\lambda_1 t}}{\eta_1\left(\frac{2\sqrt{z}}{\sigma}\right)} \left\{ \lambda_1\eta_1\left(\frac{2\sqrt{Z_t}}{\sigma}\right) + \left[\frac{(\alpha - cZ_t)\sqrt{Z_t}}{\sigma} - \frac{\sigma}{4\sqrt{Z_t}}\right] \eta_1'\left(\frac{2\sqrt{Z_t}}{\sigma}\right) \right. \\
&\qquad \left. - \frac{\sigma}{2}\sqrt{Z_t}\eta_1''\left(\frac{2\sqrt{Z_t}}{\sigma}\right) \right\} dt + \frac{e^{\lambda_1 t}}{\eta_1\left(\frac{2\sqrt{z}}{\sigma}\right)} \eta_1'\left(\frac{2\sqrt{Z_t}}{\sigma}\right) dB_t,
\end{aligned}$$

and thus

$$d\langle D, B\rangle_t = \frac{e^{\lambda_1 t}}{\eta_1\left(\frac{2\sqrt{z}}{\sigma}\right)} \eta_1'\left(\frac{2\sqrt{Z_t}}{\sigma}\right) dt = \frac{\sigma e^{\lambda_1 t}\sqrt{Z_t}\phi'(Z_t)dt}{\phi(z)}.$$

Applying then the Girsanov's Theorem 1.3, we can define the $\mathbb{Q}$-martingale

$$\begin{aligned}
B_t^{\mathbb{Q}} &:= B_t - \int_0^t \frac{\eta'\left(\frac{2\sqrt{Z_s}}{\sigma}\right)}{\eta\left(\frac{2\sqrt{Z_s}}{\sigma}\right)} ds \\
&= B_t - \int_0^t \sigma\sqrt{Z_s}\frac{\phi'(Z_s)}{\phi(Z_s)} ds.
\end{aligned}$$

In particular, $B^{\mathbb{Q}}$ is a standard Brownian Motion in $\mathbb{Q}$, and we can deduce equation (2.10) from (2.8). ■

2.3 General LBP conditioned to be never extinct

For the general logistic case, we can not assure existence of a conditioned process defined through a h-transform. Nevertheless, it is still of interest to describe the dynamics of such processes under the assumption that such h-transform exists and is well-defined. Thus, given $T = \inf\{t > 0 : Z_t = 0\}$, we assume that

$(H_1^\dagger)$ The probability measure

$$\mathbb{P}_x^\dagger(A) := \lim_{s\uparrow\infty} \mathbb{P}_x(A|T > t+s), \quad A \in \sigma(Z_s : s \leq t),$$

is well defined. Moreover, there exist a positive real number γ and an increasing function $\phi \in C^2((0,\infty))$ such that $\mathbb{P}_x^\dagger(A) = \mathbb{E}(\mathbf{1}_A h(t, Z_t)) = \mathbb{E}(\mathbf{1}_A e^{-\gamma t}\phi(Z_t))$.

Theorem 2.5. *Under hypotesis $(H_1^\dagger)$, $(Z, \mathbb{P}^\dagger)$ satisfies*

$$\begin{aligned} Z_t = \alpha \int_0^t Z_s ds + \sigma \int_0^t \sqrt{Z_s} dB_s^\dagger + \int_0^t \int_0^{Z_{s-}} \int_1^\infty r N^\dagger(ds, d\nu, dr) \\ + \int_0^t \int_0^{Z_{s-}} \int_0^1 r \tilde{N}^\dagger(ds, d\nu, dr) - c \int_0^t {Z_s}^2 ds \\ + \sigma^2 \int_0^t \frac{Z_s \phi'(Z_s)}{\phi(Z_s)} ds + \int_0^t \int_{Z_{s-}}^{\frac{Z_{s-}\phi(Z_{s-}+r)}{\phi(Z_{s-})}} \int_1^\infty r N^\ddagger(ds, d\nu, dr) \\ + \int_0^t \int_{Z_{s-}}^{\frac{Z_{s-}\phi(Z_{s-}+r)}{\phi(Z_{s-})}} \int_0^1 r \tilde{N}^\ddagger(ds, d\nu, dr). \end{aligned} \tag{2.11}$$

where $\{B_t^\dagger : t \geq 0\}$ is a Brownian motion, $N^\dagger$ and $N^\ddagger$ are independent Poisson measures on $[0,\infty) \times (0,\infty)^2$, both with intensity measure $ds \times d\nu \times \Pi(dr)$.

Proof. As in the proof of Lemma 2.3, we start by defining the Radon-Nikodym derivative $D^\dagger$ by

$$D_t^\dagger := \left.\frac{d\mathbb{P}^\dagger}{d\mathbb{P}}\right|_{F_t} = h(t, Z_t).$$

We apply first Ito's theorem to process $D^\dagger$ to obtain

$$\begin{aligned} D_t^\dagger = \phi(x) + \int_0^t e^{-\gamma s}\phi'(Z_s)[\alpha Z_s - c Z_s^2] ds + \frac{\sigma}{2} \int_0^t e^{-\gamma s}\phi''(Z_s) Z_s ds \\ + \sigma \int_0^t e^{-\gamma s}\phi'(Z_s)\sqrt{Z_s} dB_s + \int_0^\infty \int_0^{Z_{s-}} \int_1^\infty e^{-\gamma s}\phi'(Z_s) r N(ds, d\nu, dr) \\ + \int_0^\infty \int_0^{Z_{s-}} \int_0^1 e^{-\gamma s}\phi'(Z_s) r \tilde{N}(ds, d\nu, dr) \\ + \sum_{t_n \leq t} e^{-\gamma t_n} \left[\phi(Z_{t_n-} + r_n \mathbf{1}_{\{\nu_n \leq Z_{t_n-}\}}) - \phi(Z_{t_n-}) - \phi'(Z_{t_n-}) r_n \mathbf{1}_{\{\nu_n \leq Z_{t_n^-}\}}\right]. \end{aligned}$$

Applying Girsanov's theorem (Thm. 1.3) to the Brownian motion B, we have that

$$B_t = B_t^\dagger + \sigma \int_0^t \frac{e^{-\gamma s}\phi'(Z_s)\sqrt{Z_s}}{e^{-\gamma s}\phi(Z_s)},$$

where $B^\dagger$ is a $\mathbb{P}^\dagger$-BM. Thus,

$$\sigma \int_0^t \sqrt{Z_s} dB_s^\dagger = \sigma \int_0^t \sqrt{Z_s} dB_s - \sigma^2 \int_0^t \frac{\phi'(Z_s)}{\phi(Z_s)} Y_s.$$

On the other hand, we want to study the terms driven by the Poisson random measure N. As in the CSBP case, we know that the jumps of $(Z_t : t \geq 0)$ are given by $r_n \mathbf{1}_{\{\nu_n \leq Z_{t_n-}\}}$, where $((t_n, \nu_n, r_n) : n \in \mathbb{N})$ are the atoms of the Poisson measure N. Hence, enlarging the probability space and filtration if needed, we may and shall assume that there is a sequence $(u^\dagger_{t_n})_{n\geq 1}$ of independent random variables uniformly distributed on $[0,1]$, independent of B and N and such that $u^\dagger_{t_n}\mathbf{1}_{\{t_n \leq t\}}$ is $\mathcal{F}^\dagger_t$-measurable, with $(\mathcal{F}^\dagger_t)$ the natural filtration of the conditioned process. As in the proof of Theorem 1.1, we define $\Delta^\dagger_n$ and $\eta^\dagger_n$ by

$$(\Delta^\dagger_n, \eta^\dagger_n) := \begin{cases} (0,0), r^\dagger_n \mathbf{1}_{\{\nu^\dagger_n \leq Z_{t_n-}\}}) & \text{if} \quad u^\dagger_n < \dfrac{\phi(Z_{t_n}) - \phi(Z_{t_n-})}{\phi(Z_{t_n})} \text{ and } Z_{t_n} > 0, \\ ((r^\dagger_n, \nu^\dagger_n), 0) & \text{if} \quad u^\dagger_n \geq \dfrac{\phi(Z_{t_n}) - \phi(Z_{t_n-})}{\phi(Z_{t_n})} \text{ and } Z_{t_n} > 0, \\ ((0,0),0) & \text{if} \quad \phi(Z_{t_n}) = 0. \end{cases} \tag{2.12}$$

For any nonnegative $(\mathcal{F}^\dagger_t)$-predictable F; nonnegative f vanishing on the diagonal, such that $f((r,\nu), s) = 0$ when $\nu \leq R$ for some $R \geq 0$; and $x \geq 0$; we have the martingale

$$\begin{aligned} S_t := & \sum_{t_n \leq t} F_{t_n} f(\Delta^\dagger_n, \eta^\dagger_n) \\ & - \int_0^t ds F_s \int_0^\infty \int_0^\infty f((0,0), r\mathbf{1}_{\{\nu \leq Z_s\}}) \frac{\phi(Z_s + r\mathbf{1}_{\{\nu \leq Z_s\}}) - \phi(Z_s)}{\phi(Z_s + r\mathbf{1}_{\{\nu \leq Z_s\}})} \Pi(dr) d\nu \\ & - \int_0^t ds F_s \int_0^\infty \int_0^\infty f((r,\nu),0) \frac{\phi(Z_s)}{\phi(Z_s + r\mathbf{1}_{\{\nu \leq Z_s\}})} \Pi(dr) d\nu. \end{aligned}$$

and by similar arguments as in the proof of Thm. 1.1, the process $S^\dagger$, given by

$$\begin{aligned} S^\dagger_t &= S_t - \int_0^t \frac{d\langle S, D^\dagger\rangle_s}{D^\dagger_s} \\ &= \sum_{t_n \leq t} F_{t_n} f(\Delta^\dagger_n, \eta^\dagger_n) - \int_0^t ds F_s \int_0^\infty \int_0^\infty f((0,0), r) \frac{[\phi(Z_s + r) - \phi(Z_s)]\, \mathbf{1}_{\{\nu \leq Z_s\}}}{\phi(Z_s)} \Pi(dr) d\nu \\ & \quad - \int_0^t ds F_s \int_0^\infty \int_0^\infty f((r,\nu),0) \Pi(dr) d\nu, \end{aligned}$$

is a $\mathcal{F}^\dagger$-martingale under $\mathbb{P}^\dagger$ with mean zero. Therefore,

$$\begin{aligned} \mathbb{E}^\dagger \left[\sum_{t_n \leq t} F_{t_n} f(\Delta^\dagger_n, \eta^\dagger_n) \right] = & \ \mathbb{E}^\dagger \left[\int_0^t ds F_s Z_s \int_0^\infty f((0,0), r) \frac{\phi(Z_s + r) - \phi(Z_s)}{\phi(Z_s)} \Pi(dr) \right] \\ & + \mathbb{E}^\dagger \left[\int_0^t ds F_s \int_0^\infty \int_0^\infty f((r,\nu),0) \Pi(dr) d\nu \right]. \end{aligned} \tag{2.13}$$

Thanks to Lemma 1.4, the second term on the r.h.s. of previous equation allow us to deduce that $(\Delta^\dagger_n)$ are in fact the atoms of a $\mathcal{F}^\dagger$-Poisson random measure $N^\dagger$ with intensity $ds \times d\nu \times \Pi(dr)$ w.r.t. $\mathbb{P}^\dagger$. Also, we can assume by a standard enlarging procedure that there exist another $\mathcal{F}^\dagger$-Poisson random measure N^* with intensity $ds \times d\nu \times \Pi(dr)$ w.r.t. $\mathbb{P}^\dagger$ and independent of $N^\dagger$, and a sequence $(v_n)_{n\geq 1}$ of independent random variables uniformly distributed on $[0,1]$, such that $v_n\mathbf{1}_{\{t_n\le t\}}$ is $\mathcal{F}^\dagger_t$-measurable and v_n is independent of $\mathcal{F}^\dagger_{t_n-}$. Using these tools, we define the process $N^\ddagger$ as

$$N^\ddagger(ds,d\nu,dr)=\mathbf{1}_{\{\nu\le Z_{s-}\}}N^*(ds,d\nu,dr)+\sum_n\delta_{\left\{s_n,\frac{\phi(Z_{s_n-}+r_n)-\phi(Z_{s_n-})}{\phi(Z_{s_n-})}Y_{s_n}-U_n+Z_{s_n-},\eta^\dagger_n\right\}}(ds,d\nu,dr)$$
$$+\mathbf{1}_{\left\{\nu>\frac{\phi(Z_{s-})Z_s}{\phi(Z_{s-})}\right\}}N^*(ds,d\nu,dr),$$

where δ denotes Dirac measure. For this process, let F be again a non-negative $\mathcal{F}^\dagger$-predictable process, and f be a two-variable non-negative Borel function. Thanks to formula (2.13), we have that

$$\mathbb{E}^\dagger\left[\sum_{t_n\le t}F_{t_n}f(\eta^\dagger_n,\frac{\phi(Z_{t_n-}+r_s)-\phi(Z_{t_n-})}{\phi(Z_{t_n-})}Z_{t_n-}U_n+Z_{t_n-})\right]$$
$$=\mathbb{E}^\dagger\left[\int_0^t\int_0^\infty\int_0^1F_sf(r,\frac{\phi(Z_s+r)-\phi(Z_s)}{\phi(Z_s)}Z_su+Z_s)\frac{\phi(Z_s+r)-\phi(Z_s)}{\phi(Z_s)}Z_s\Pi(dr)duds\right],$$

and by a change of variables we deduce that

$$\mathbb{E}^\dagger\left[\sum_{t_n\le t}F_{t_n}f(\eta^\dagger_n,\ \frac{\phi(Z_{t_n-}+r_n)-\phi(Z_{t_n-})}{\phi(Z_{t_n-})}Z_{t_n-}U_n+Z_{t_n-})\right]$$
$$=\mathbb{E}^\dagger\left[\int_0^t dsF_s\int_{Z_{s-}}^{\frac{\phi(Z_{s-}+r)}{\phi(Z_{s-})}Z_{s-}}\int_0^\infty f(r,\nu)\Pi(dr)d\nu\right].$$

Moreover, since

$$\mathbb{E}^\dagger\left[\int_0^t\int_0^\infty\int_0^\infty F_sf(\nu,r)\mathbf{1}_{\{\nu\le Z_{s-}\}}N^*(ds,d\nu,dr)\right]=\mathbb{E}^\dagger\left[\int_0^t dsF_s\int_0^{Z_{s-}}\int_0^\infty f(\nu,r)\Pi(dr)d\nu\right]$$

and

$$\mathbb{E}^\dagger\left[\int_0^t\int_0^\infty\int_0^\infty F_sf(\nu,r)\ \mathbf{1}_{\left\{\nu>\frac{\phi(Z_{s-})Z_{s-}+r}{\phi(Z_{s-})}\right\}}N^*(ds,d\nu,dr)\right]$$
$$=\mathbb{E}^\dagger\left[\int_0^t dsF_s\int_0^\infty\int_{\frac{\phi(Z_{s-})Z_{s-}+r}{\phi(Z_{s-})}}^\infty f(\nu,r)\Pi(dr)d\nu\right],$$

we deduce that

$$\mathbb{E}^\dagger\left[\int_0^t\int_0^\infty\int_0^\infty F_sf(\nu,r)N^\ddagger(ds,d\nu,dr)\right]=\mathbb{E}^\dagger\left[\int_0^t F_sds\int_0^\infty\int_0^\infty f(\nu,r)d\nu\Pi(dr)\right],\quad(2.14)$$

which shows that $N^\ddagger$ is a $\mathcal{F}^\dagger$-Poisson random measure with intensity $ds\times d\nu\times\Pi(dr)$ w.r.t. $\mathbb{P}^\dagger$.

These previous equalities (2.13) and (2.14) along with the fact that

$$\sum_{t_n\le t}r_n\mathbf{1}_{\{\nu_n\le Z_{t_n-}\}}=\sum_{t_n\le t}p_1(\Delta^\dagger_n)\mathbf{1}_{\{p_2(\Delta^\dagger_n)\le Z_{t_n-}\}}+\sum_{t_n\le t}\eta^\dagger_n$$
$$=\sum_{t_n\le t}r^\dagger_n\mathbf{1}_{\{\nu^\dagger_n\le Z_{t_n-}\}}+\sum_{t_n\le t}r^\ddagger_n\mathbf{1}_{\left\{Z_{t_n-}<\nu^\ddagger_n\le\frac{Z_{t_n-}\phi(Z_{t_n-}+r^\ddagger_n)}{\phi(Z_{t_n-})}\right\}},$$

where p_i are the respective projection of $\{\Delta_n\}$ and $\{(t_n, r_n^\dagger, \nu_n^\dagger)\}_{n\in\mathbb{N}}$ and $\{(t_n, r_n^\ddagger, \nu_n^\ddagger)\}_{n\in\mathbb{N}}$ are the atoms of $N^\dagger$ and $\mathcal{N}^\ddagger$ respectively; imply that the process Z under $\mathbb{P}^\dagger$ satisfies the desired SDE. ■

In particular, this construction allow us to recover the stochastic differential equation for the CSBP case. We know that a continuous-state branching process Y can be seen as a LB-process with rate $c = 0$, and in this case

$$h(t, Y_t) = e^{\rho t}\phi(Y_t) = \frac{e^{\rho t} Y_t}{x}.$$

Here, by Lemma 1.4 together with equation (2.13), we see that

$$\begin{aligned} \mathbb{E}^\dagger\left[\sum_{t_n\le t} F_{t_n} f((0,0),\eta_n^\dagger)\right] &= \mathbb{E}^\dagger\left[\int_0^t F_s ds \int_{Y_s}^{Y_s+r}\int_0^\infty f((0,0),r)d\nu\Pi(dr)\right] \\ &= \mathbb{E}^\dagger\left[\int_0^t F_s ds \int_0^\infty f((0,0),r)r\Pi(dr)\right], \end{aligned} \tag{2.15}$$

from where $\{\eta_n^Y\}$ are the atoms of a Poisson random measure $N^\star$ with intensity $ds \times r\Pi(dr)$ w.r.t. $\mathbb{P}^\dagger$. In particular, this measure is independent of $N^\dagger$ and, as $\int_0^1 (1\wedge r) r\Pi(dr) < \infty$, it not need to be compensated. Therefore, equation (2.11) take the form

$$\begin{aligned} Y_t = a\int_0^t Y_s ds + \sigma\int_0^t \sqrt{Y_s} dB_s^\dagger + \int_0^t\int_0^{Y_{s-}}\int_1^\infty rN^\dagger(ds,d\nu,dr) \\ + \int_0^t\int_0^{Y_{s-}}\int_0^1 r\tilde{N}^\dagger(ds,d\nu,dr) + \sigma^2 t + \int_0^t\int_{Z_{s-}}^{Z_{s-}+r}\int_0^\infty rN^\star(ds,dr), \end{aligned} \tag{2.16}$$

that is in fact our original equation (1.7).

Chapter 3

Ray-Knight representation of Lévy-driven LBPs

This chapter is based on the paper *Ray-Knight representation of Lévy-driven continuous-state branching processes with logistic growth* , with J. Berestycki and J. Fontbona (in preparation) .

3.1 Introduction and preliminaries

The understanding and the description of the genealogical structure of stochastic population models with branching-type behavior have been active fields of research over the last decades, giving rise to powerful mathematical tools for studying such models and the asymptotic behavior of large random tree-like structures. In the framework of *continuous-state branching process*, or CSBP for short, the study of these questions has revealed deep connexions between the Markov processes that describe the evolution of such branching populations, and continuous time processes of a different nature that can be used to code their genealogies. The theorem of Ray and Knight [52], [34] is historically the first result in that direction. It states that the accumulated time which a suitably stopped reflected Brownian motion spends near level s (rigorously understood as its local time at level s) is a Feller branching diffusion when viewed as a process in s. Thanks to the excursion theory introduced by Itô [30, 31], it is well known that the excursions away from 0 of such reflected Brownian motion define a Poisson point process indexed by the local time at level 0. In the nineties, Aldous constructed the (Brownian) continuum random tree (CRT) as the tree coded by the normalized Brownian excursion [5, 6]. Brought together, these objects and results give a precise mathematical meaning to the genealogy of a population governed by the Feller diffusion: the excursions of reflected Brownian motion can be understood as exploration paths of the trees of descendants of the ancestors of the population at time $t = 0$, with the local time at height t measuring the population size at that time.

In the framework of general CSBP, which can be defined from a spectrally positive Lévy process X by means of Lamperti's transform, a Ray-Knight theorem was established by Duquesne and Le Gall [21]. The definition of the corresponding excursions, their heights and their local times at each level, which are needed in order to state a Ray-Knight theorem in that setting, is much more involved than in the diffusion case, with no simple (say, finite dimensional or SDE-like) representation of a Markov process coding the genealogy. Their result therefore required the use of the so-called exploration process, introduced by Le Gall and Le Jan [43], which codes the continuum random trees embedded in a spectrally positive Lévy processes X, or Lévy-CRT.

Extensions of the Ray-Knight theorem and related genealogical descriptions have since then been obtained for more complex models with branching type behavior, such as super processes, branching processes with immigration and generalized Fleming-Viot Processes with mutations [1, 2, 9, 8, 45], and have been used in the study of several properties of these processes.

During the last decade, density-dependent stochastic population models have considerably enlarged the scope of mathematically tractable population models and therefore have attracted increasing attention both in the mathematical and theoretical biology communities. In the present work, we are interested in the genealogical description of continuous state branching processes with logistic growth. *Branching processes with logistic growth* or logistic branching process, abbreviated as LBP, are population-dependent continuous time branching process where moreover, informally speaking, the total population instantaneously decreases at a rate proportional to the squared population size at each instant. In the continuous state-space framework, LBP are Markov process taking nonnegative values and with a.s. càdlàg paths, and were introduced in a general form by Lambert [36], by means of a Lamperti transformation on Ornstein-Ulhenbeck processes driven by general spectrally positive Lévy process. The global behavior of the population can be intuitively understood as the result of standard branching behavior, plus a pairwise competition among individuals, resulting in an individual death rate increased by an amount that is proportional to the total instantaneous population descending from the original one.

The lack of independence between the individuals of the populations modeled by such processes prevents the application of standard tools of excursion theory and of continuous random trees to suitably define the genealogy of such processes, and new points of view must be developed. In that setting, a Ray-Knight theorem was recently obtained in the case of the logistic Feller diffusion by Le, Pardoux and Wakolbinger [42], see also [50], in terms of the local times of a reflected Brownian motion with local time drift. As in the classical Ray-Knight representation, the excursions of such process are understood as the exploration paths of the trees of descendants of the ancestors at time $t = 0$, and the local time of the process at height s measures the population size at time t. Their key idea to define a genealogy is to think of the individuals as being arranged "from left to right " (as given by the exploration time), and decree that interaction between them takes place through "pairwise fights" that are always won by the individual "to the left ", hence lethal for the individual "to the right". Deaths following pairwise fights lead in the exploration process of the genealogical forest to a downward drift, proportional to the amount of mass (or local time units) seen to the left of the individual encountered at each exploration time. In this way, excursions which come later in the exploration time tend to be smaller (trees to the right are "under attack from those to the left") .

The main purpose of the present paper is to extend the previous Ray-Knight representations and genealogical descriptions to LBP associated with general spectrally positive Lévy process with (sub)critical mechanism.

We next briefly recall the basic needed facts on CSBP and LBP and relations between these processes, together with SDE representations that will be useful for our purposes. We discuss then the Ray-Knight theorems of Duquesne and Le Gall and the theorem of Le, Pardoux and Wakolbinger for the logistic Feller diffusion. In order to introduce the tools we will require to formulate the problem in the general setting, we then recall some ideas on pruning of Lévy trees, following Abraham, Delmas and Voisin [3] and recall some of the results therein that will be useful in our approach. Then, in the following section, our main results are stated.

3.1.1 Continuous state branching processes

A CSPB is a càdlàg $[0,\infty)$-valued strong Markov processes $Y=(Y_t : t\geq 0)$ with laws given the initial states $(\mathbb{P}_x : x\geq 0)$ which satisfy the branching property. That is, for any $t\geq 0$ and $y_1, y_2 \in [0,\infty)$, Y_t under $\mathbb{P}_{y_1+y_2}$ has the same law as the independent sum $Y_t^{(1)}+Y_t^{(2)}$, where the distribution of $Y_t^{(i)}$ is equal to that of Y_t under $\mathbb{P}_{y_i}$ for $i=1,2$. The law of Y is completely characterized by its Laplace transform: For $\theta>0$, $x>0$ and $t\geq 0$, one has

$$\mathbb{E}_x(e^{-\theta Y_t}) = e^{-xu_t(\theta)},$$

where u is the unique nonnegative solution of the differential equation

$$\frac{\partial u_t(\theta)}{\partial t} = -\psi(u_t(\theta)), \quad u_0(\theta)=\theta,$$

and the function ψ called *branching mechanism* of Z is of the form

$$\psi(\lambda) = -q-\alpha\lambda+\frac{1}{2}\sigma^2\lambda^2+\int_{(0,\infty)}(e^{-\lambda x}-1+\lambda x\mathbf{1}_{\{x<1\}})\Pi(dx) \quad \lambda\geq 0, \tag{3.1}$$

for some $q\geq 0$, $\alpha\in\mathbb{R}$, $\sigma\geq 0$ and Π a measure supported in $(0,\infty)$ such that

$$\int_{(0,\infty)}(1\wedge x^2)\Pi(dx) \;<\; \infty.$$

Defining $\rho := \psi'(0+)$ and since $\mathbb{E}_x(Y_t)=xe^{-\rho t}$, CSBPs are classified as *subcritical* ($\rho>0$), *critical* ($\rho=0$) and *supercritical* ($\rho<0$), according to whether the process will, on average, decrease, remain constant or increase. In the following, we will assume that Y satisfies the following conditions:

$$\begin{array}{lll}(A_1) & c\psi(0)=q=0 \quad \text{and} \quad |\psi'(0+)|<\infty; & \\ (A_2) & \alpha\leq -\int_1^\infty r\Pi(dr) \quad \text{and} \quad \int_0^\infty (r\wedge r^2)\Pi(dr)<\infty; & \\ (A_3) & \sigma>0 \quad \text{or} \quad \int_0^\infty r\Pi(dr)=\infty; \quad \text{and} & \\ (A_4) & \displaystyle\int_1^\infty \frac{d\lambda}{\psi(\lambda)}<\infty. & \end{array} \tag{A}$$

Assumption (A_1) ensures that Y is *conservative* (i.e. $\forall\, t>0$, $\mathbb{P}_x(Y_t<\infty)=1$), assumption (A_2) when $q=0$ restricts our work to the (sub)critical case and assumption (A_4) implies that there is a.s. extinction for Y. Assumption (A_3) is imposed in order to deal with the more interesting case when the process has infinite variation. The nonnegative function ψ is the *characteristic exponent* of a spectrally positive Lévy process $X=(X_t : t\geq 0)$, i.e. a process with càdlàg paths, stationary and independent increments, and no negative jumps, characterized by its Laplace exponent

$$\mathbb{E}\left(e^{-\theta X_t}\right) = e^{-t\psi(\theta)},$$

and under our assumptions, it has no negative jumps, does not drift to $+\infty$ and its paths are of infinite variation.

Lamperti [40] established his celebrated one-to-one trajectorial correspondence between a CSBP Y as above and the spectrally positive Lévy processes X, killed upon hitting 0, via a random time change. We refer to Caballero *et al.* [13] for modern proofs of that result. The following alternative SDE representation of the dynamics of a CSBP established in [13],[25] and [18] will be practical for our purposes. Given a realization of the process $(Y_t : t \geq 0)$ starting form $y > 0$ in some probability space, one can construct in an enlarged one a standard Brownian motion B and an independent Poisson measure N on $[0,\infty)\times(0,\infty)\times(0,\infty)$ with intensity measure $dt \times d\nu \times \Pi(dr)$ such that

$$\begin{aligned} Y_t = x + \alpha \int_0^t Y_s ds + \sigma \int_0^t \sqrt{Y_s} dB_s + \int_0^t \int_0^{Y_{s-}} \int_1^{\infty} rN(ds, d\nu, dr) \\ + \int_0^t \int_0^{Y_{s-}} \int_0^1 r\tilde{N}(ds, d\nu, dr), \end{aligned} \tag{3.2}$$

where $\tilde{N}$ denotes the compensated Poisson measure associated with N. When N and α are identically null, or equivalently, when the underlying Lévy process is a Brownian motion, the CSBP $(Y_t : t \geq 0)$ is the celebrated Feller diffusion. Consider the reflected Brownian motion

$$H_s = \frac{2}{\sigma} B_s^H + \frac{1}{2} L_s^0(H), \quad s \geq 0, \tag{3.3}$$

where B^H is a standard Brownian motion and $L_s^a(H)$ is the local time accumulated by β at level $a \geq 0$ up to time $s \geq 0$. Then, one has

Theorem 3.1 (Ray-Knight theorem)**.** *Set $T_x = \inf\{t \geq 0 : L_t^0(H) = x\}$. Then, the process $(L_{T_x}^a(H) : a \geq 0)$ has a continuous modification which is a Feller difussion.*

3.1.2 Lévy exploration processes and the Ray-Knight theorem of Duquesne and Le Gall

We next recall Duquesne and Le Gall's extension of the previous result to the case of general CSBP with (sub)critical branching mechanism. To that end the introduction of the exploration process associated with a spectrally positive Lévy process X, as well as its height and local times processes, is needed. The reader is referred to [21] for details and further background.

Under assumption (A), point 0 is regular for the process X reflected both at its running infimum and at its running supremum. The running infimum process of X is denoted by $I_t := \inf_{0\leq s\leq t} X_s$, and it is a local time at 0 for the strong Makov process $X - I$. The *future infimum* of X is the (two parameter) process defined for $0 \leq s \leq t$ by $I_t^s = \inf_{s\leq r\leq t} X_r$. The *height process* $H^0 = (H_t^0 : t \geq 0)$ which roughly speaking measures for each $t \geq 0$ the size of the set $\{s \leq t : X_s = \inf_{[s,t]} X_r\}$, can be firstly defined as

$$H_t^0 = \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{X_s < I_t^s + \varepsilon\}} ds,$$

which is equal, by time reversal at time t, to the local time at the supremum of the dual Lévy process. The process $(H_t^0 : t \geq 0)$ is Markov only when X has no jumps (in which case it is a reflected Brownian motion with drift). But it always has a version which is a measurable function of a measure-valued strong Markov process, called *exploration process*. The exploration process $\rho = (\rho_t : t \geq 0)$ takes values in the space $\mathcal{M}_f(\mathbb{R}_+)$ of finite measures in $\mathbb{R}_+$ and for each $t \geq 0$ it is defined on nonnegative measurable functions f by

$$\langle \rho_t, f \rangle = \int_0^t d_s I_t^s f(H_s^0),$$

where $d_s I_t^s$ denotes the Lebesgue-Stieljes integral with respect to the nondecreasing map $s \mapsto I_t^s$. Equivalently

$$\rho_t(dr) = \beta \mathbf{1}_{[0,H_t^0]}(r) dr \ + \sum_{0 < s \leq t, X_{s-} < I_t^s} (I_t^s - X_{s^-}) \delta_{H_s^0}(dr).$$

In particular, the total mass of ρ_t is $\langle \rho_t, 1 \rangle = X_t - I_t$, The process $H_t := H(\rho_t)$ defined as the supremum of the closed support of the measure ρ_t and with $H(0) := 0$ by convention, is a modification of the height process H_t^0, such that the mapping $t \to H(\rho_t) = H_t$ is lower semicontinuous a.s.

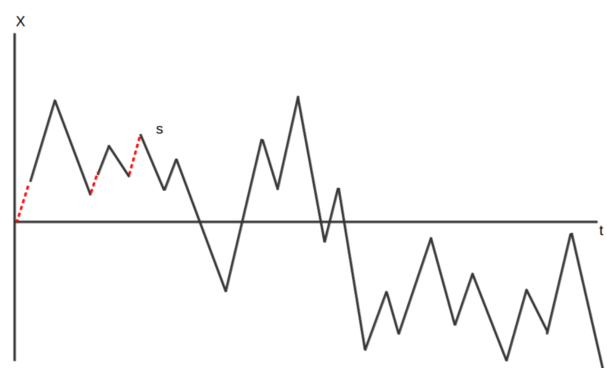


Figure 3.1: Set $\{r \leq s : X_r = \inf_{[r,s]} X_u\}$

The exploration process ρ, or equivalently the process $X - I$, codes a tree structure defined in the following way. Under the excursion measure $\mathbf{N}$ induced on excursions of $X - I$ away from 0, the "law" of the height process H_t puts weight on nonnegative functions $e : [0, \sigma] \to \mathbb{R}_+$ with compact support and such that $e_0 = 0 = e_s$ for all $s \geq \sigma > 0$, where σ denotes the length of the excursion. The random function d_e defined on $[0, \sigma]^2$ by

$$d_e(s,t) = e_s + e_t - 2m_e(s,t),$$

with $m_e(s,t) = \inf_{s\wedge t \le r \le s\vee t} e_r$, defines an equivalence relation in $[0,\sigma]$ through $d_e(s,t) = 0$. This induces an ultrametric distance in the quotient set $\mathcal{T}_e$ which results to be a compact metric space (a "real tree" actually). Informally, each real number $s \in [0,\sigma]$ corresponds to a vertex at level H_s in the tree, and $d_e(s,t)$ is the distance between vertices corresponding to s and t (in particular, s and t correspond to the same vertex if and only if $d_e(s,t) = 0$). The quantity $m_e(s,t)$ is interpreted as the height (or the generation) of the most recent ancestor common to s and t. Thus, ρ_t can be seen as a measure on the ancestral line of the individual labeled by t, which gives the intensity of the sub-trees that are grafted on the right of this ancestral line.

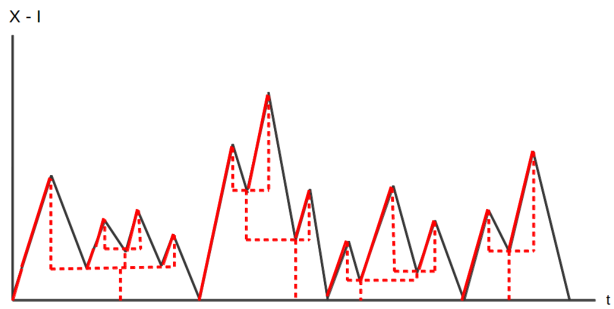


Figure 3.2: Tree structure

The Ray-Knight theorem for CSBP of Duquesne and Le Gall is stated in terms of the local time of the height process. The latter is in general not Markovian nor a semimartingale, and so its local times must be defined in terms of the exploration process ρ. Since $H_t = 0$ iff $\rho_t = 0$, or equivalently $X_t - I_t = 0$, the natural definition for the local time at level 0 of H is the process $L_t^0 := -I_t$. In order to define the local time at a given level $a > 0$ one has to consider the exploration process "above level a", defined as follows. Set for each $t \ge 0$,

$$\tau_t^a = \inf\{s \ge 0 : \int_0^s \mathbf{1}_{\{H_r > a\}} dr > t\} = \inf\{s \ge 0 : \int_0^s \mathbf{1}_{\{\rho_r((a,\infty))>0\}} dr > t\}$$

which is a.s. finite since $\int_0^\infty \mathbf{1}_{\{H_r > a\}} dr = \infty$ a.s., and

$$\tilde{\tau}_t^a = \inf\{s \ge 0 : \int_0^s \mathbf{1}_{\{H_r \le a\}} dr > t\}.$$

For every $t \ge 0$, one then defines a random measure ρ_t^a on $\mathbb{R}_+$ by

$$\langle \rho_t^a, f\rangle = \int_{(a,\infty)} \rho_{\tau_t^a}(dr) f(r-a). \tag{3.4}$$

Then, the process $(\rho_t^a : t \ge 0)$ has the same distribution as $(\rho_t : t \ge 0)$ and is independent of the sigma field $\mathcal{H}_a$ generated by the càdlàg process $((X_{\tilde{\tau}_t^a}, \rho_{\tilde{\tau}_t^a}) : t \ge 0)$ and the class of negligible sets of the canonical filtration of the Lévy process X. Denoting by $l^a = (l^a(s) :$

$s \geq 0$) the local time at 0 of $(\langle \rho^a, 1\rangle : t \geq 0)$, the local time at level a and at time s of the height process H is defined by

$$L_s^a = l^a \left(\int_0^s \mathbf{1}_{\{H_r > a\}} dr \right). \tag{3.5}$$

With these elements, Duquesne and Le Gall proved in [21, Theorem 1.4.1] (see also [43, Theorem 4.2]) the following generalization of the classic Ray-Knight theorem:

Theorem 3.2 (Ray-Knight representation for CSBP)**.** *Set $T_x = \inf\{t \geq 0, L_t^0 = x\}$. Then, the process $(L_{T_x}^a : a \geq 0)$ has a càdlàg modification $(Y_a : a \geq 0)$ which is a CSBP of branching mechanism ψ starting from x.*

3.1.3 Logistic branching processes and genealogy of the logistic Feller diffusion

As we see in Section 2.1, continuous-state branching processes with logistic growth (LBP) were introduced in [36] by means of a similar Lamperti transform as the one linking continuous-state branching processes and Lévy processes, but using a Lévy driven Ornstein-Uhlenbeck (OU) process instead. An LBP $Z = (Z_t : t \geq 0)$ can be seen as an analogue of a CSBP $Y = (Y_t : t \geq 0)$ with additional negative interactions (competition) between pairs of individuals in the population alive at each time instant. As we proved in Theorem 2.1, an LBP corresponding to a time-changed OU process driven by a Lévy process of Laplace exponent ψ given by (3.1) can actually be constructed as a (strong) solution of the SDE

$$\begin{aligned} Z_t = v + \alpha \int_0^t Z_s ds + \sigma \int_0^t \sqrt{Z_s} dB_s + \int_0^t \int_0^{Z_{s-}} \int_1^{\infty} r N(ds, d\nu, dr) \\ + \int_0^t \int_0^{Z_{s-}} \int_0^1 r \tilde{N} ds, d\nu, dr) - c \int_0^t Z_s^2 ds, \end{aligned} \tag{3.6}$$

with a Brownian motion B and an independent Poisson point process N on $[0,\infty) \times (0,\infty) \times (0,\infty]$ of intensity $dt \times d\nu \times \Pi(dr)$ similar as in (3.2), and $c > 0$ a positive constant referred to as *competition intensity*. The above SDE representation of LBP can be deduced from the Lamperti representation (in a similar way as in [13] for CSBP). In the particular case when the underlying Lévy process is a Brownian motion with drift, the previous equation reduces to

$$dZ_t = \left(\alpha Z_t - c Z_t^2\right) dt + \sigma \sqrt{Z_t} dB_t, \quad Z_0 = x, \tag{3.7}$$

the solution of which is known as the "Logistic Feller diffusion".

In [42, 50] the authors established a generalization of the classical Ray-Knight theorem for the process (3.7), in terms of the local times of a reflected Brownian motion H with a local time drift, in the case $\alpha \geq 0$. This is defined as the solution of the SDE

$$H_s = \frac{2}{\sigma} B_s^H + \frac{1}{2} L_s^0(H) + \frac{2\alpha}{\sigma} s - c \int_0^s L_r^{H_r}(H) dr, \quad s \geq 0, \tag{3.8}$$

where B^H is a standard Brownian motion and $L_s^a(H)$ is the local time accumulated by H at level $a \geq 0$ up to time $s \geq 0$. They proved in [42] that SDE (3.8) has a weak solution, unique in law, and moreover

Theorem 3.3 (Ray-Knight theorem for logistic Feller diffusion)**.** *Define for each $x > 0$ the stopping time*

$$T_x = \inf\{s > 0, L_s^0 > x\},$$

Then $((\sigma^2/4)L^a_{T_x} : a \geq 0)$ is a weak solution of (3.7).

The result can be interpreted as follows. The death rate due to the pairwise fights leads in the exploration process of the genealogical forest to a downward drift which is proportional to $L_s^{H_s}$, that is, proportional to the amount of mass seen to the left of the individual encountered at exploration time s (and living at real time H_s). In this way, those excursions of H which come later in the exploration time tend to be smaller (the trees to the right are "under attack from those to the left").

In a similar way as in [42], the key issue in order to define a genealogy of LBP in the Lévy case is to first give a sense to competition between pairs of individuals in the same generation or height of the Lévy tree, entailing the disappearance of the defeated individual and of its whole (potential) descendent line. Keeping in mind the picture in [42] of individuals arranged from left to right, together with pairwise "fights" that are always won by the left-most individual, in the Lévy case this amounts to define a consistent way of randomly "erasing " local time units at a given level, together with the corresponding excursions of the exploration process above that level, at a rate given by the total population on the left of the individual indexed by the erased local time unit.

The idea of "erasing" consistently the local time and the corresponding excursions of the exploration process is now standardly formulated by a means of a "pruning" procedure performed on the Lévy-CRT (see [3, 4, 56]). This procedure is defined in terms of a Poisson Lévy-snake (a particular instance of the powerful Lévy-snake device introduced in [21]) which provides a mechanism to put Poissonian marks on the path of the exploration process, in a way that is consistent with coded tree structure. To a large extent, our formulation of the genealogy of a Lévy driven LBP will be inspired by the ideas of [3], and by a Ray-Knight interpretation of their main result. However, we will need to define the pruning mechanism in a more general way, allowing for some past exploration-path dependence of the marking rates.

In the next subsection we recall the Lévy-tree pruning procedure developed in [3] as well as the main results therein, and state some consequences that will be relevant for the sequel. We will then be ready to state our main results.

3.1.4 Poisson Lévy-snake and Lévy tree pruning

Duquesne and Le Gall [21] introduced the Lévy snake process which combines the continuous genealogical structure coded by the height process H of the exploration process ρ, with the spatial motion of a càdlàg Markov process ξ in a Polish state space E. Recall that the space $\mathscr{W}$ of killed càdlàg paths in E can be equipped with a metric making it a Polish space.

Definition 3.4. Given a fixed starting point $x \in E$, and a realization of the process $(\rho_s : s \geq 0)$, the Lévy snake is the time homogeneous strong Markov process $((\rho_s, \mathcal{W}_s) : s \geq 0)$ with values in the product space $\mathcal{M}_f(\mathbb{R}_+) \times \mathscr{W}$ (and defined in an enlarged probability space) such that, conditionally on $(\rho_s : s \geq 0)$,

- for every $s \geq 0$, $\mathcal{W}_s = (\mathcal{W}_s(t) : 0 \leq t < H_s)$ is a path of ξ started at x killed at time H_s, and
- for each pair of time instants s and s', the paths $\mathcal{W}_s$ and $\mathcal{W}_{s'}$ are the same up to time $t = H_{s,s'} := \inf_{s,s'} H_r$ and then behave independently conditionally on their (common) past up to time $H_{s,s'}$.

(we refer to [21] Ch. 4 for details). The second property above is referred to as the *snake property*.

Definition 3.5. In the case that $E = \mathbb{R}_+$, $x = 0$ and ξ is a Poisson process of rate $\theta > 0$, the process $((\rho_s, \mathcal{W}_s) : s \geq 0)$ is called Poisson Lévy-snake, or simply *Poisson snake*.

For each $s \geq 0$, a Poisson snake $\mathcal{W}_s$ is rather described in terms of its derivative m_s^θ, which is (conditionally on ρ) a Poisson point measure in $[0, H_s)$ of intensity θ times the Lebesque measure. In these terms, the snake property is equivalent to the fact that for $s < s'$ (conditionally on ρ) one has $m_{s'}^\theta(dr)\mathbf{1}_{\{r \leq H_{s,s'}\}} = m_s^\theta(dr)\mathbf{1}_{\{r \leq H_{s,s'}\}}$, and $m_{s'}^\theta(dr)\mathbf{1}_{\{r > H_{s,s'}\}}$ and $m_s^\theta(dr)\mathbf{1}_{\{r > H_{s,s'}\}}$ are independent. The atoms of m_s^θ can be seen as unit mass marks on the ancestral line of the individual labeled s. Thus, atoms of $(m_t^\theta, t \geq 0)$ can be interpreted as marks " on the skeleton" of the tree coded by ρ, which are distributed according to a Poisson point measure with intensity θ times the Hausdorff measure on the tree.

In [3], Abraham et al. study the measure-pair valued process $\mathscr{S}^\theta := ((\rho_t, m_t^\theta) : t \leq 0)$, called the *marked exploration process*. (Actually, they also consider marks on the nodes of infinite degree of the tree, but these will not be needed here; our process $(m_t^\theta : t \leq 0)$ corresponds to the process $(m_t^{\text{ske}} : t \geq 0)$ in [3]). Then, they show that if the underlying Lévy tree is pruned by removing from the original CRT all the individuals who have a marked ancestor, the resulting tree is the Lévy tree associated with the branching mechanism

$$\psi_\theta(\lambda) := \psi(\lambda) + \theta\lambda. \tag{3.9}$$

To be more precise, denoting by $\bar{A}_t$ the Lebesgue measure of the set of the individuals prior (in exploration time) to t, whose lineage does not contain any mark, *i.e.*

$$\bar{A}_t = \int_0^t \mathbf{1}_{\{m_s^\theta = 0\}} ds = \int_0^t \mathbf{1}_{\{m_s^\theta([0,H_s))=0\}} ds,$$

and considering its right-continuous inverse $\bar{C}_t := \inf\{r \geq 0, \bar{A}_r > t\}$, they define the pruned exploration process as

$$\forall t \geq 0, \quad \bar{\rho}_t = \rho_{\bar{C}_t}. \tag{3.10}$$

The main result in [3] then is:

Theorem 3.6. *The pruned exploration process (3.10) is distributed as the exploration process associated with a branching mechanism ψ_θ given in (3.9).*

Using classic approximation results on local times, one can moreover check that for each $a \geq 0$, the local times of the pruned exploration process (3.10) at level a is given by the process

$$\left(\int_0^{\bar{C}_t} \mathbf{1}_{\{m_s^\theta = 0\}} dL_s^a : t \geq 0 \right). \tag{3.11}$$

It is then possible to deduce a "Ray-Knight interpretation" of Theorem 3.6 and of the pruning procedure. The following result, proved in Section 3.4, is the starting point for the ideas developed in the present paper (and also a key technical fact for our results):

Corollary 3.7. *For each $x \geq 0$, the process $\left(\int_0^{T_x} \mathbf{1}_{\{m_s^\theta = 0\}} dL_s^a : a \geq 0 \right)$ has a right continuous version which is a CSBP with branching mechanisms $\psi_\theta(\lambda)$, starting at x.*

3.2 Main statements

Our first goal is to give a sense to the idea of pruning at height $r > 0$ the CRT coded by the exploration process ρ at, roughly speaking, a rate proportional to the local time accumulated by the pruned tree "on the left" of each individual at level r. But rather than defining the pruning of the exploration process itself, this idea will be more easily and naturally formalized in terms of "pruned local times", inspired by the relation (3.11). Indeed, Corollary 3.7 suggests that, in order to state a Ray-Knight representation of a LBP it should be enough to define the "marks" inducing the pruning procedure we are interested in. But to do so, the introduction of a generalized notion of marked exploration process is needed.

We denote in the sequel by $\mathscr{M}(\mathbb{R}_+)$ the space of Borel measures in $\mathbb{R}_+$ endowed with the vague topology and by $\mathscr{M}_a(\mathbb{R}_+)$ the subspace of $\mathscr{M}(\mathbb{R}_+)$ of atomic measures with unit mass atoms. We write $(\mathbb{S}, \hat{d})$ for the (Polish) state-space of the marked exploration process used in [3] (the metric will be recalled below).

Definition 3.8. Let ψ be a branching mechanism satisfying assumption (A). A càdlàg $\mathbb{S}$-valued process $((\rho_t, m_t) : t \geq 0)$ defined in some probability space, where ρ is an exploration processes associated with ψ, will be called **generalized marked exploration process** if, conditionally on ρ,

- for each $s \geq 0$, m_s is an element of $\mathscr{M}_a(\mathbb{R}_+)$ supported in $[0, H_s)$ and
- for each pair of time instants s and s' one has $m_{s'}(dr)\mathbf{1}_{\{r \leq H_{s,s'}\}} = m_s(dr)\mathbf{1}_{\{r \leq H_{s,s'}\}}$.

Moreover, it will be called **progressively marked exploration process** if in addition, conditionally on the sigma field $\sigma((\rho_s, m_s \mathbf{1}_{\{L_s^{H_s} < L_t^{H_s}\}}) : s \leq t)$, m_t is a (non-homogeneous) Poisson point process in $[0, H_t)$.

Notice that $\left((\rho_s, m_s \mathbf{1}_{\{L_s^{H_s} < L_t^{H_s}\}}) : s \leq t \right)$ corresponds to the exploration process coding the trees on the left of (and including) the lineage of the individual labeled t, together with marks put strictly on its left (excluding the marks on its lineage). Also,

$$\left(\sigma((\rho_s, \mathbf{1}_{\{L_s^{H_s} < L_t^{H_s}\}} m_s) : s \leq t) \right)_{t \geq 0}$$

is subfiltration of $(\sigma((\rho_s, m_s) : s \leq t))_{t \geq 0}$.

Definition 3.9. Let $((\rho_t, m_t) : t \geq 0)$ be a progressively marked exploration process. For each $a \geq 0$, we will call **local time at level** a **progressively pruned by** m, or simply m-**pruned local time at level** a, the process defined by

$$L_t^a(m) := \int_0^t \mathbf{1}_{\{m_s = 0\}} dL_s^a, \quad t \geq 0.$$

Our first relevant result is the existence of a progressively marked exploration process $((\rho_t, m^*_t) : t \geq 0)$ that puts marks at each level of a given ancestral line, at a rate that is proportional to the m^*-pruned local time accumulated on the forest on its the left:

Theorem 3.10. *Let $(\rho_t : t \geq 0)$ be the exploration process associated with a branching mechanism ψ satisfying (A) and let $c > 0$. There exists in some extended probability space a progressively marked exploration process $((\rho_t, m^*_t) : t \geq 0)$ such that, for each $t \geq 0$, conditionally on the sigma field $\sigma((\rho_s, m^*_s \mathbf{1}_{\{L^{H_s}_s < L^{H_s}_t\}}) : s \leq t)$, the point process m^*_t is (non-homogeneous) Poisson of intensity measure*

$$c\, L^a_t(m^*) da \mathbf{1}_{\{a < H_t\}},$$

where for each $a \geq 0$, $(L^a_t(m^*) : t \geq 0)$ is the m^*-pruned local time process at level a.

Process $((\rho_t, m^*_t) : t \geq 0)$ will be called the **logistically marked exploration process**.

The main result of the present paper is the identification of the law of the process of cumulated m^*-pruned local times at each level, at increase instants of the local time at level 0.

Theorem 3.11 (Ray-Knight theorem for Lévy-driven logistic branching processes)**.** *Under the assumptions of Theorem 3.10, the process*

$$(L^a_{T_x}(m^*) : a \geq 0)$$

is a continuous-state logistic branching process with branching mechanism ψ and competition rate $c/2$, starting from x.

The fact that the obtained competition rate is $c/2$ instead of the constant c appearing in Theorem 3.10 accounts for the non-symmetric competition between individuals: the ones on the left are kept forever (with respect to exploration time) as part of the population, whereas those further right (or newly arrived in the exploration time sense) are susceptible of being removed (as in [50].)

The techniques we introduce in order to prove Theorem 3.11 actually allow us to state a stronger result, namely the identification of the law of the two-parameter process

$$(L^a_{T_x}(m^*) : x \geq 0, a \geq 0).$$

In particular, we are able to provide a more complete description of the above picture of competition, when competing individuals descend from different ancestors (or initial populations $x \geq 0$) at generation $a = 0$. The key tool to do this, and also a crucial element in the proof of Theorem 3.11, is an extension to the LBP setting of stochastic flows of CSBP introduced by Dawson an Li [18]. We next briefly recall the flow of CSBP and its connection with the Ray-Knight theorem 3.2 and then describe the analogous connection in the present setting.

3.2.1 Stochastic flow and tree interpretation

The flow of CSBPs introduced in [18] is a two-parameter process $(Y_t(v) : t \geq 0, v \geq 0)$, where for every $v \geq 0$ the process $Y(v) = (Y_t(v) : t \geq 0)$ is the unique strong solution of the stochastic differential equation:

$$Y_t(v) = v + \alpha\int_0^t Y_s(v)ds + \sigma\int_0^t\int_0^{Y_{s-}(v)} W(ds,du) + \int_0^t\int_0^{Y_{s-}(v)}\int_1^\infty rN(ds,d\nu,dr) \\ + \int_0^t\int_0^{Y_{s-}(v)}\int_0^1 r\tilde{N}(ds,d\nu,dr), \tag{3.12}$$

where $\Pi(dr)$, $\sigma \geq 0$ and α are the same objects as in (3.1), $W(ds,du)$ is a white noise process on $(0,\infty)^2$ based on the Lebesgue measure $ds \times du$ and N is a Poisson random measure on $(0,\infty)^3$ with intensity $ds \times d\nu \times \Pi(dr)$ as in (3.2). It is easily seen using the properties of stochastic integrals with respect to white noise, that for each $v \geq 0$, the above process satisfies equation (3.2) and hence it is CSBP with branching mechanism ψ given in (3.1) starting with initial population v. The authors in [18] proved that $(Y_t(v) : t \geq 0, v \geq 0)$ has a version with the following properties:

i. for each $v \geq 0$, $t \mapsto Y_t(v)$ is a càdlàg process on $[0,\infty)$;

ii. for each $t \geq 0$, $v \mapsto Y_t(v)$ is a non-negative and non-decreasing càdlàg process on $[0,\infty)$.

iii. For each $0 \leq v_1 \leq v_2 \leq \cdots \leq v_n$, the processes $(Y_t(v_j) - Y_t(v_{j-1}) : t \geq 0)$, $j = 1,\dots n$ are independent CSBP with branching mechanism ψ issued from $v_j - v_{j-1}$,

i.e. $(Y_t(v) : v \geq 0)$ is a subordinator. The stochastic flow of CBSP thus provides a simultaneous construction of a family of CSBP featuring the branching property as a function of the initial population v (in particular it provides a SDE construction of Bertoin and Le Gall's flow of subordinators defined in [10] for similar purposes). Moreover, since a similar additive property is shared by the exploration local times $x \mapsto (L^a_{T_x} : a \geq 0)$ of the Lévy CRT with branching mechanism ψ thanks to the strong Markov property of the exploration process $(\rho_t : t \geq 0)$, the process $(L^a_{T_x} : a \geq 0, x \geq 0)$ and the family $(Y_a(x) : a \geq 0, x \geq 0)$ have the same law. The random "forest" $\mathcal{T}$ associated with height process H, and coded by the exploration process $(\rho_t : t \geq 0)$, can thus be viewed as the genealogical tree $\mathcal{T}$ of the flow of CSBP.

In the case of the logistic branching, adapting techniques of [18], it is not hard to establish

Proposition 3.12. *Let the parameters $\Pi(dr)$, $\sigma \geq 0$ and α and the processes $W(ds,du)$ and $N(ds,d\nu,dr)$ be as above. For every $v \geq 0$ there is a unique strong solution of the stochastic differential equation:*

$$Z_t(v) = v + \alpha\int_0^t Z_s(v)ds + \sigma\int_0^t\int_0^{Z_{s-}(v)} W(ds,du) + \int_0^t\int_0^{Z_{s-}(v)}\int_1^\infty rN(ds,d\nu,dr) \\ + \int_0^t\int_0^{Z_{s-}(v)}\int_0^1 r\tilde{N}(ds,d\nu,dr) - \frac{c}{2}\int_0^t {Z_s}^2(v)ds, \tag{3.13}$$

Moreover, the process $(Z_t(v) : t \geq 0, v \geq 0)$ admits (bi-measurble) version such that

i. for each $v \geq 0$, $t \mapsto Z_t(v)$ is a càdlàg process on $[0,\infty)$ which is a LBP of branching mechanism ψ and competition rate $c/2$ started from v;

ii. *for each* $t \geq 0$*,* $v \mapsto Z_t(v)$ *is a non-negative and non-decreasing càdlàg process on* $[0, \infty)$*.*

iii. *For each* $0 \leq v$*, the conditional law of* $(Z_t(v) - Z_t(u) : t \geq 0, u \leq v)$ *given* $(Z_t(x) : t \geq 0, 0 \leq x \leq u)$ *depends only on* $(Z_t(u) : t \geq 0)$*.*

Notice that this construction generalizes a similar one of [50] in the case of the logistic Feller diffusion. Our proofs of Theorems 3.10 and 3.11 will provide a simultaneous realization of the logistically pruned local times for different initial populations, and can be easily adapted in order to establish the following stronger result:

Corollary 3.13. *The processes* $(L^a_{T_x}(m^*) : a \geq 0, x \geq 0)$ *and* $(Z_a(x) : a \geq 0, x \geq 0)$ *have the same law.*

To show that for every $x \geq 0$ the local times process $(L^a_{T_x}(m^*) : a \geq 0)$ can be indeed interpreted as the Ray-Knight representation for a LBP starting from x, we will construct an approximation $(L^a_t(\varepsilon, \delta) : a \geq 0, t \geq 0)$ in a suitable tree-like height/local time discrete grid, using the pruning procedure employed by Abraham *et a.* in [3] in an iterative way. To identify the law of this approximation, we will define a flow $(Z^{\varepsilon,\delta}_a(v) : a \geq 0, v \geq 0)$ of suitable pruned CSBP that will prove to be an embedding of the local time process $(L^a_{T_v}(\varepsilon, \delta) : a \geq 0, v \geq 0)$. Finally, we prove that the law of the pruned flow $Z^{\varepsilon,\delta}$ (and therefore that of $L(\varepsilon, \delta)$) converges strong enough to the logistic flow.

3.3 The logistic Poisson Lévy-snake

In order to give a meaning to the idea of pruning the Lévy tree or the associated exploration processes in a logistic way, we next introduce a Poisson Lévy-snake $\mathcal{N}$ with values in the space of Poisson point process in $[0, \infty) \times (0, \infty)$, and we will use it to mark the tree at variable random rates, generalizing the main ideas of Abraham *et al.*[3]. In doing so, we will also extend ideas developed [4], where a two dimensional Poisson snake was used to prune a Brownian excursion process simultaneously at different (but constant) rates.

3.3.1 A 2d Poisson Lévy-snake

Let $\mathcal{V}$ denote the set of pairs $(\mu, \eta) \in \mathscr{M}_f(\mathbb{R}_+) \times \mathscr{M}(\mathbb{R}^2_+)$ such that $\operatorname{supp} \eta \subseteq [0, H(\mu)) \times \mathbb{R}_+$. For each $u \in [0, H(\mu))$, we denote by $\eta_{(u)} \in \mathscr{M}(\mathbb{R}_+)$ the measure given by

$$\eta_{(u)}(A) = \eta([0, u] \times A), \quad A \in \mathcal{B}(\mathbb{R}_+) \tag{3.14}$$

and notice that $u \mapsto \eta_{(u)} \in \mathscr{M}(\mathbb{R}_+)$ is vaguely càdlàg (by dominated convergence). Moreover, $\eta, \eta' \in \mathscr{M}(\mathbb{R}^2)$ supported in $[0, H(\mu))$ are equal if and only if $\eta_{(u)} = \eta'_{(u)}$ for all $u \in [0, H(\mu))$.

We endow $\mathcal{V}$ with the distance d given for $(\mu, \eta), (\mu', \eta') \in \mathcal{V}$ by,

$$d\left((\mu, \eta), (\mu', \eta')\right) = D(\mu, \mu') + \int_0^{H(\mu) \wedge H(\mu')} \left(d_u(\eta_{(u)}, \eta'_{(u)}) \wedge 1\right) du + |H(\mu) - H(\mu')|, \tag{3.15}$$

where D is a distance inducing the topology of weak convergence such that the metric space $(\mathscr{M}_f(\mathbb{R}_+), D)$ is complete, and d_u is the Skorohod metric on $\mathbb{D}([0, u], \mathscr{M}(\mathbb{R}_+))$. One can check that $(\mathcal{V}, d)$ is a Polish space.

In an analogous way as in Abraham and Serlet [4, Theorem 5] for the Brownian excursion, one can construct a càdlàg strong Markov process $((\rho_s, \mathcal{N}_s) : s \leq 0)$ with values in $\mathcal{V}$ such that

1. $(\rho_s : s \geq 0)$ is the exploration process associated with the Lévy process X.

2. Conditionally on $(\rho_s : s \geq 0)$, for each $s \geq 0$, $\mathcal{N}_s$ is a Poisson point measure on $[0, H(\rho_s)) \times \mathbb{R}_+$ with intensity the Lebesgue measure and, for all $0 \leq s \leq s'$,

   - $\mathcal{N}_{s'}(dr, d\nu)\mathbf{1}_{\{r \leq H_{s,s'}\}} = \mathcal{N}_s(dr, d\nu)\mathbf{1}_{\{r \leq H_{s,s'}\}}$, where $H_{s,s'} := \inf\{H_u, s \leq u \leq s'\}$, and

   - $\mathcal{N}_{s'}(dr, d\nu)\mathbf{1}_{\{r > H_{s,s'}\}}$ and $\mathcal{N}_s(dr, d\nu)\mathbf{1}_{\{r > H_{s,s'}\}}$ are independent point processes.

Remark 3.14. We stress the fact that in the standard snake terminology of [21] (see Definition 3.4), the above process actually is the Lévy-snake with underlying spatial-Markov process ξ corresponding to the "primitive in the r variable" of a Poisson point measure $\eta(dr, d\nu)$ in $\mathscr{M}(\mathbb{R}_+^2)$. More precisely, in a similar way as for the one-dimensional Poisson snake, a path of ξ here is an increasing càdlàg path $r \mapsto \eta([0, r], d\nu)$ taking values in $\mathscr{M}(\mathbb{R}_+)$ instead of in $\mathbb{R}_+$ (see (3.14)), and we describe it in terms of its "derivative", which is the point measure $\eta(dr, d\nu)$.

We denote by $(\mathcal{F}_t)_{t \geq 0}$ the right continuous completion of the filtration $\sigma((\rho_s, \mathcal{N}_s) : s \leq t))$, $t \geq 0$ and by $(\mathcal{F}_t^\rho)_{t \geq 0}$ the one associated with $\sigma(\rho_s : s \leq t)$.

For all $t \geq 0$, we introduce the "vertical" filtration $(\mathcal{G}_r^{(t)})_{r \geq 0}$ given by the right continuous completion of the filtration generated by

$$\mathcal{G}_r^{(t)} = \sigma\left(\mathcal{F}_t^\rho, \left\{(r_n^{(s)}\mathbf{1}_{\{r_n^{(s)} \leq r\}}, \nu_n^{(s)}\mathbf{1}_{\{r_n^{(s)} \leq r\}})_{n \in \mathbb{N}}, s \leq t\right\}\right), \tag{3.16}$$

where $\{(r_n^{(s)}, \nu_n^{(s)})\}$ are the atoms of the process $\mathcal{N}_s$. Notice that also, for each $r \geq 0$, $(\mathcal{G}_r^{(t)})_{t \geq 0}$ is a sub-filtration of $(\mathcal{F}_t)_{t \geq 0}$ containing $(\mathcal{F}_t^\rho)_{t \geq 0}$.

Thanks to the snake property of $((\rho_s, \mathcal{N}_s) : s \geq 0)$, is is not hard to check

Lemma 3.15. *Conditionally on $\mathcal{G}_0^{(t)}$, the process $\mathcal{N}_t$ is a $(\mathcal{G}_r^{(t)})_{r \geq 0}$-Poisson point process in $\mathbb{R}_+^2$ of intensity $\mathbf{1}_{[0, H(\rho_t))}(r)dr \times d\nu$.*

In particular, if $\mathcal{P}red(\mathcal{G}^{(t)})$ denotes the predictable sigma-field associated with $(\mathcal{G}_r^{(t)})_{r \geq 0}$, one can define integrals of $\mathcal{P}red(\mathcal{G}^{(t)}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable processes $h((r, \omega), \nu)$ with respect to $\mathcal{N}_t$, and they have, conditionally on $\mathcal{F}_t^\rho$, the standard properties of Poisson type integrals, relative to the filtration $(\mathcal{G}_r^{(t)})_{r \geq 0}$.

3.3.2 A operator on generalized marked exploration processes

Our goal now is to use the Poisson Lévy-snake $((\rho_s, \mathcal{N}_s) : s \geq 0)$ to construct a logistic marked exploration process. We will do this by means of an iterative scheme. In order to define its generic step, we need to introduce an operator in the set of generalized marked exploration process and study some of its properties.

Recall that $(\mathbb{S}, \hat{d})$ denotes the state-space of the marked exploration process used in [3], which also contains the trajectories of the generalizes marked exploration processes previously introduced. This is the Polish space of pairs (μ, w) with $\mu \in \mathscr{M}_f(\mathbb{R}_+)$ and $w \in \mathscr{M}_{at}(\mathbb{R}_+)$ such that $supp(w) \subset [0, H(\mu))$, endowed with the distance

$$\hat{d}((\mu, w), (\mu', w')) := D(\mu, \mu') + \int_0^{H(\mu)\wedge H(\mu')} \big(d_u(w_{(u)}, w'_{(u)}) \wedge 1\big)\, du + |H(\mu) - H(\mu')|, \tag{3.17}$$

where $w_{(u)}$ (resp. $w'_{(u)}$) is the cumulative distribution function of the measure w (resp. w') restricted to $[0, u]$ and d_u is the Skorohod metric on the space $\mathbb{D}([0, u], \mathbb{R}_+)$.

Consider $((\rho_t, m_t) : t \geq 0)$ an $(\mathcal{F}_t)$-adapted generalized marked exploration process such that, for each $t \geq 0$, the process $a \mapsto m_t([0, a])$ is $(\mathcal{G}_a^{(t)})$-adapted. Recall that the parameter $c/2 \geq 0$ stands for competition intensity. Define for each $t \geq 0$ and $h \geq 0$:

$$L_t^h(m) := \int_0^t \mathbf{1}_{\{m_s = 0\}} dL_s^h \quad \text{and} \quad m'_t([0, h]) := \int_0^{H_t} \mathbf{1}_{[0,h]}(r) \int_0^\infty \mathbf{1}_{\{\nu < c\ {}^pL_t^r(m)\}} \mathcal{N}_t(dr, d\nu), \tag{3.18}$$

where $({}^pL_t^r(m), r \geq 0)$ is the predictable projection of $(L_t^r(m) : r \geq 0)$ with respect to the filtration $(\mathcal{G}_r^{(t)})_{r\geq 0}$. That is, ${}^pL_t^{\cdot}$ is the unique (up to indistinguishability) $(\mathcal{G}_r^{(t)})_{r\geq 0}$-predictable process such that:

$$\mathbb{E}\left[L_t^R \mathbf{1}_{\{R<\infty\}} \big| \mathcal{G}_{R-}^{(t)}\right] = {}^pL_t^R \mathbf{1}_{\{R<\infty\}} \quad a.s. \tag{3.19}$$

for every predictable $\mathcal{G}^{(t)}$-stopping time R [1]. By properties of exploration local times and generalized marked exploration process, the two parameter process $(L_t^r(m) : r \geq 0, t \geq 0)$ has a bi-measurable version which is continuous in t for each $r \geq 0$. We always work with such a version.

Lemma 3.16 (Basic properties of the mapping $m \mapsto m'$)**.** *The process $\mathcal{S}' = (\rho, m')$ is a generalized marked exploration process. Moreover, the càdlàg process $t \mapsto m'_t$ is $(\mathcal{F}_t)$-adapted and for every $t > 0$, the càdlàg process $a \mapsto m'_t([0, a])$ is $(\mathcal{G}_a^{(t)})$-adapted. Finally, the integer valued process $t \mapsto m'_t([0, H_t))$ of the total number of marks in each lineage is càdlàg.*

Proof. Measurability follows directly from the definition of (3.18). The fact that (ρ, m') is a generalized marked exploration processes, apart from càdlàg paths, follow from (3.18) and the snake property of the process $(\rho, \mathcal{N})$, together with the fact that the processes $r \mapsto L_t^r$ and $r \mapsto L_s^r$ are equal on $[0, H_{t,s})$. As for the path regularity, from (3.17) we have for $s, t \geq 0$ that

$$\hat{d}((\rho_t, m'_t), (\rho_s, m'_s)) = D(\rho_t, \rho_s) + \int_{H_{t,s}}^{H_t \wedge H_s} \big(d_u((m'_t)_{(u)}, (m'_s)_{(u)}) \wedge 1\big)\, du + |H_t - H_s|.$$

Since $(\rho_s : s \geq 0)$ is a càdlàg process and, under our assumptions, $(H_s : s \geq 0)$ is a continuous process, the right-hand side goes to zero when $s \to t$. We deduce that the marked exploration process $((\rho_t, m'_t) : t \geq 0)$ is right-continuous and has left limits (for the latter property one easily checks that (ρ_{s_n}, m'_{s_n}) is Cauchy when $s_n \nearrow t$, with a limit not depending on the sequence). Finally, the facts that for each $t \geq 0$, $m'_t([u, H_t)) = 0$ for some $u \in (0, H_t)$ and that $m'_s(dr)\mathbf{1}_{\{r \leq H_{s,t}\}} = m'_t(dr)\mathbf{1}_{\{r \leq H_{s,t}\}}$ for $s \neq t$ imply, together with the convergence $H_{s,t} \to H_t$ when $s \to t$, the asserted regularity property of the total number of marks. ■

[1] Notice that $(L_t^r : r \geq 0)$ is $\mathcal{G}_0^{(t)}$ measurable but we cannot ensure the existence of a version that is right continuous in r for all t; we circumvent this problem using the predictable projection.

We prove now some estimates that will be crucial for the sequel. To this end, we are going to use the following generalized occupation time formula: a.s. for any nonnegative measurable function $\varphi(s,a)$ and every $t \geq 0$,

$$\int_0^t \varphi(r, H_r)dr = \int_0^\infty \int_0^t \varphi(s,a)dL_s^a da.^2 \tag{3.20}$$

Proposition 3.17. *Let $(m_t : t \geq 0)$ and $(\tilde{m}_t : t \geq 0)$ be two adapted right-continuous processes taking values in the space of finite point measures on $\mathbb{R}_+$, having for each $t \geq 0$ a support contained in $[0, H_t)$ and atoms with mass equal to 1. Let the processes $(m'_t : t \geq 0)$ and $(\tilde{m}'_t : t \geq 0)$ be defined respectively in terms of $(m_t : t \geq 0)$ and $(\tilde{m}_t : t \geq 0)$ by the formulae (3.18).*

i). For $t \geq 0$, set $\Delta_t = |m_t([0,H_t)) - \tilde{m}_t([0,H_t))|$ and $\Delta'_t = |m'_t([0,H_t)) - \tilde{m}'_t([0,H_t))|$. Then, for each $A \geq 0$ and $T \geq 0$, we have

$$\mathbb{E}\left[\mathbf{1}_{\{H_t \leq A\}}\Delta'_t\right] \leq c\int_0^t \mathbb{E}\left[\mathbf{1}_{\{H_s \leq A\}}\Delta_t\right] ds. \tag{3.21}$$

ii). For each $x \geq 0$ and $a \geq 0$, let $T^{a,x}$ denote the $(\mathcal{F}_t)$-stopping time

$$T^{a,x} = \inf\{t \geq 0 : \exists\, b \leq a \text{ s.t. } L_t^b \geq x\}.$$

Define $L_t^r(m') := \int_0^t \mathbf{1}_{\{m'_s = 0\}} dL_s^r$ and $L_t^r(\tilde{m}') := \int_0^t \mathbf{1}_{\{\tilde{m}'_s = 0\}} dL_s^r$. Then, for any $(\mathcal{F}_t)$-stopping τ we have

$$\begin{aligned}\mathbb{E}\left(|L^a_{\tau\wedge T^{a,x}}(m') - L^a_{\tau\wedge T^{a,x}}(\tilde{m}')|\right) &\leq \mathbb{E}\left(\int_0^{\tau\wedge T^{a,x}} |m'_t([0,a)) - \tilde{m}'_t([0,a))|\, dL_t^a\right)\\ &\leq cx\int_0^a \mathbb{E}\left(\int_0^{\tau\wedge T^{a,x}} |m_t([0,r)) - \tilde{m}_t([0,r))|\, dL_t^r\right) dr\\ &= cx\mathbb{E}\left(\int_0^{\tau\wedge T^{a,x}} |m_t([0,H_t)) - \tilde{m}_t([0,H_t))|\, dt\right).\end{aligned} \tag{3.22}$$

Proof.

i). Observe that for all $t \geq 0$, by the assumptions on m_t and $\tilde{m}_t$,

$$\left|\mathbf{1}_{\{m_t=0\}} - \mathbf{1}_{\{\tilde{m}_t=0\}}\right| = \left|\mathbf{1}_{\{m_t([0,H_t))=0\}} - \mathbf{1}_{\{\tilde{m}_t([0,H_t))=0\}}\right| \leq |m_t([0,H_t)) - \tilde{m}_t([0,H_t))|\,,$$

Since $dL_s^a = \mathbf{1}_{\{H_s=a\}}dL_s^a$, integrating this inequality between 0 and $t > 0$ against dL_t^a we deduce that

$$|L_t^a(m) - L_t^a(\tilde{m})| \leq \int_0^t |m_s([0,a)) - \tilde{m}_s([0,a))|\, dL_s^a. \tag{3.23}$$

[2] It is easy to deduce this formula from the occupation time formula in [18, Prop. 1.3.3]

For every $t \geq 0, A \geq 0$, the snake property and the definition of the Poisson random measure $(\mathcal{N}_t)$, together with (3.23) imply that

$$
\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{H_t \leq A\}} \Delta_t'\right] &\leq \mathbb{E}\left[\mathbf{1}_{\{H_t \leq A\}} \mathbb{E}\left(\int_0^{H_t} \int_0^{\infty} \left|\mathbf{1}_{\{\nu < c\ {}^pL_t^r(m)\}} - \mathbf{1}_{\{\nu < c\ {}^pL_t^r(\tilde{m})\}}\right| \mathcal{N}_t(dr, d\nu) \middle| \mathcal{F}_t^\rho\right)\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{H_t \leq A\}} \mathbb{E}\left(\int_0^{H_t} c \left|L_t^r(m) - L_t^r(\tilde{m})\right| dr \middle| \mathcal{F}_t^\rho\right)\right] \\
&\leq c\mathbb{E}\left[\mathbf{1}_{\{H_t \leq A\}} \int_0^{H_t} \left(\int_0^t |m_s([0,r)) - \tilde{m}_s([0,r))| \, dL_s^r\right) dr\right] \\
&\leq c\mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{r \leq A\}} \int_0^t |m_s([0,r)) - \tilde{m}_s([0,r))| \, dL_s^r dr\right].
\end{aligned}
$$

Using the space-time occupation-times formula (3.20) and Fubini's Theorem, we deduce (3.21).

(*ii.*) Since by Lemma 3.16, m'_t and $\tilde{m}'_t$ are for each $t \geq 0$ finite point measures with unit mass atoms and support contained in $[0, H_t)$, in a similar way as (3.23) we now get

$$
|L_t^a(m') - L_t^a(\tilde{m}')| \leq \int_0^t |m'_s([0,a)) - \tilde{m}'_s([0,a))| \, dL_s^a
$$

for all $t \geq 0$, which gives us the first inequality. Let us prove the second inequality. For any $(\mathcal{F}_t^\rho)$-stopping time $\tau \geq 0$ one has

$$
\begin{aligned}
&\int_0^\tau |m'_t([0,a)) - \tilde{m}'_t([0,a))| \, dL_t^a \\
&\quad \leq \int_0^\tau dL_t^a \ \mathbf{1}_{\{H_t = a\}} \int_0^{H_t} \int_0^\infty \left|\mathbf{1}_{\{\nu < c\ {}^pL_t^r(m)\}} - \mathbf{1}_{\{\nu < c\ {}^pL_t^r(\tilde{m})\}}\right| \mathcal{N}_t(dr, d\nu) \\
&\quad \leq \int_0^\tau dL_t^a \ \mathbf{1}_{\{H_t = a\}} g_a(t),
\end{aligned}
$$

where

$$
g_a(t) = \int_0^a \int_0^\infty \mathbf{1}_{\{c[L_s^r(\tilde{m}) \wedge L_s^r(m)] < \nu \leq c[L_s^r(\tilde{m}) \vee L_s^r(m)]\}} \mathcal{N}_t(dr, d\nu).
$$

Thus,

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^\tau |m'_t([0,a)) - \tilde{m}'_t([0,a))| \, dL_t^a\right) \\
&\qquad = \mathbb{E}\left(\mathbb{E}\left(\int_0^\infty |m'_t([0,a)) - \tilde{m}'_t([0,a))| \, \mathbf{1}_{\{t \leq \tau\}} dL_t^a \middle| \mathcal{F}_\tau^\rho\right)\right) \\
&\qquad \leq \mathbb{E}\left(\int_0^\infty dL_t^a \mathbf{1}_{\{t \leq \tau\}} \mathbb{E}\left(\mathbf{1}_{\{H_t = a\}} g_a(t) \middle| \mathcal{F}_{t \wedge \tau}^\rho\right)\right).
\end{aligned}
$$

The snake property and the definition of the Poisson random measure $(\mathcal{N}_t)$, together with (3.23) imply that

$$
\begin{aligned}
&\mathbb{E}\left(\mathbf{1}_{\{H_t=a\}}g_a(t)\middle|\mathcal{F}^\rho_\tau\right)\mathbf{1}_{\{t\le\tau\}}\\
&\qquad\le \int_0^a \mathbb{E}\left(\int_0^\infty \left|\mathbf{1}_{\{\nu<c\ ^pL^r_t(m)\}}-\mathbf{1}_{\{\nu<c\ ^pL^r_t(\tilde m)\}}\right|\mathcal{N}_t(dr,d\nu)\middle|\mathcal{F}^\rho_{t\wedge\tau}\right)\mathbf{1}_{\{t\le\tau\}}\\
&\qquad\le \int_0^a \mathbb{E}\left(\left|cL^r_t(m)-cL^r_t(\tilde m)\right|\middle|\mathcal{F}^\rho_{t\wedge\tau}\right)\mathbf{1}_{\{t\le\tau\}}dr\\
&\qquad\le c\int_0^a \mathbb{E}\left(\int_0^t \left|m_s([0,r))-\tilde m_s([0,r))\right|dL^r_s\middle|\mathcal{F}^\rho_{t\wedge\tau}\right)\mathbf{1}_{\{t\le\tau\}}dr.
\end{aligned}
$$

Taking $\tau\wedge T^{a,x}$ instead of τ, the desired inequality follows with help of Fubini's theorem and definition of $T^{x,a}$. More precisely, from the previous we get that

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^{\tau\wedge T^{a,x}} \left|m'_t([0,a))-\tilde m'_t([0,a))\right|dL^a_t\right)\\
&=c\int_0^a \mathbb{E}\left(\int_0^{\tau\wedge T^{a,x}} dL^a_t\mathbb{E}\left(\int_0^t \left|m_s([0,r))-\tilde m_s([0,r))\right|dL^r_s\middle|\mathcal{F}^\rho_{t\wedge\tau\wedge T^{a,x}}\right)\right)dr\\
&\le c\int_0^a \mathbb{E}\left(\int_0^{T^{a,x}} dL^a_t\,\mathbb{E}\left(\int_0^{\tau\wedge T^{x,a}} \left|m_s((0,r])-\tilde m_s((0,r])\right|dL^r_s\middle|\mathcal{F}^\rho_{t\wedge\tau\wedge T^{a,x}}\right)\right)dr\\
&\le cx\int_0^a \mathbb{E}\left(\int_0^{\tau\wedge T^{a,x}} \left|m_s([0,r))-\tilde m_s([0,r))\right|dL^r_s\right)dr.
\end{aligned}
$$

The last asserted identity is readily obtained with the occupation times formula. ■

3.3.3 Construction of the logistically marked exploration process

In this subsection we shall prove the following result, which is easily seen to imply Theorem 3.10.

Theorem 3.18. *There exists an $(\mathcal{F}_t)$-adapted progressively marked exploration process $(\mathcal{S}^*_t : t\ge 0)=((\rho_t,m^*_t):t\ge 0)$ with associated m^*-pruned local time process*

$$
L^a_t(m^*)=\int_0^t \mathbf{1}_{\{m^*_s=0\}}dL^a_s, \tag{3.24}
$$

such that, for each $t\ge 0$ the càdlàg process $a\mapsto m^*_t([0,a])$ is $(\mathcal{G}^{(t)}_a)$-adapted, and the relation

$$
m^*_t([0,h])=\int_0^{H_t}\mathbf{1}_{[0,h]}(r)\int_0^\infty \mathbf{1}_{\{\nu<c\ ^pL^r_t(m^*)\}}\mathcal{N}_t(d\nu,dr),\quad h\in[0,H_t). \tag{3.25}
$$

hold a.s. for all $t\ge 0$ and $h\ge 0$. Moreover, the pair of processes $((L^a_t(m^*))_{a\ge 0},m^*_t):t\ge 0)$ is the unique solution of the system of equations (3.24)-(3.25) satisfying the previous properties.

We start the construction of the pair $((L^a_t(m^*))_{a\ge 0},m^*_t)$ by an iterative procedure.

For each $t \geq 0$, define m_t^0 as the null measure on $[0, H_t)$, and for each $a \geq 0$ set $L_t^a(0) := L_t^a$ for all $t \geq 0$. We define a marking measure process (m_t^1) such that for each $t \geq 0$, m_t^1 is a (finite) point measure supported in $[0, H_t)$, as follows. Set

$$m_t^1([0,h]) := \int_0^{H_t} \mathbf{1}_{[0,h]}(r) \int_0^\infty \mathbf{1}_{\{\nu < c\ ^pL_t^r(0)\}} \mathcal{N}_t(dr, d\nu), \quad h \geq 0, \tag{3.26}$$

For every $a \geq 0$, we also introduce the local time process $(L_t^a(1) : t \geq 0)$ at level a marked by the measures (m_t^1), defined as

$$L_t^a(1) := \int_0^t \mathbf{1}_{\{m_s^1([0,a))=0\}} dL_s^a = \int_0^t \mathbf{1}_{\{m_s^1=0\}} dL_s^a.$$

Notice that the second equality comes from the fact that $dL_s^a = \mathbf{1}_{\{H_s=a\}} dL_s^a$. This process corresponds to the individuals in the population at each height a having no mark in its ancestral line.[3] For each $a \geq 0$, it is easy to see that

$$L_t^a(0) \geq L_t^a(1) \tag{3.27}$$

a.s. for all $t \geq 0$ (by continuity).

Next, we prune the original local times at each level a at rate proportional to the m^1-pruned local time $L_\cdot^a(1)$ accumulated on its left. More precisely, for every $t \geq 0$ we define a new measure given by

$$m_t^2([0,h]) := \int_0^{H_t} \mathbf{1}_{[0,h]}(r) \int_0^\infty \mathbf{1}_{\{\nu < c\ ^pL_t^r(1)\}} \mathcal{N}_t(dr, d\nu), \quad h \geq 0. \tag{3.28}$$

From (3.27), we deduce that for each $t \geq 0$,

$$m_t^2 \leq m_t^1 \tag{3.29}$$

almost surely, and actually a.s. for all $t \geq 0$ by right continuity. We then associate with $(m_t^2 : t \geq 0)$ a pruned local time process, corresponding to the population at each height a having no mark in its ancestral line:

$$L_t^a(2) := \int_0^t \mathbf{1}_{\{m_s^2([0,a))=0\}} dL_s^a = \int_0^t \mathbf{1}_{\{m_s^2=0\}} dL_s^a.$$

We see from (3.27) and (3.29) that, for all $a \geq 0$,

$$L_t^a(0) \geq L_t^a(2) \geq L_t^a(1)$$

a.s. simultaneously for all $t \geq 0$ and that , for all t,

$$m_t^0 \leq m_t^2 \leq m_t^1$$

as measures. We continue to define inductively for each $n \in \mathbb{N}$, a family of measures $(m_t^n : t \geq 0)$ by

$$m_t^{n+1}([0,h]) := \int_0^{H_t} \mathbf{1}_{[0,h]}(r) \int_0^\infty \mathbf{1}_{\{\nu < c\ ^pL_t^r(n)\}} \mathcal{N}_t(dr, d\nu), \quad h \geq 0 \tag{3.30}$$

[3] Loosely speaking, the population $L_t^a(1)$ is obtained by pruning each individual at height a at rate equal to the size of the (original) population lying on its left.

for each $t \geq 0$, and for each $a \geq 0$ a family of processes

$$L_t^a(n+1) := \int_0^t \mathbf{1}_{\{m_s^{n+1}([0,a))=0\}} dL_s^a = \int_0^t \mathbf{1}_{\{m_s^{n+1}=0\}} dL_s^a. \tag{3.31}$$

Notice that the processes $(m_t^n : t \geq 0)$ have the properties stated in Lemma 3.16 for all $n \in \mathbb{N}$. Also, for each $n \in \mathbb{N}$, the process $(L_t^a : a \geq 0, t \geq 0)$ is bi-measurable, continuous in t for each $a \geq 0$ and $(\mathcal{F}_t)$-adapted.

It is easily checked by induction in n that, for all $a \geq 0$, a.s.

$$L_t^a(0) \geq L_t^a(2) \geq \cdots \geq L_t^a(2n) \geq L_t^a(2n+1) \geq \cdots \geq L_t^a(3) \geq L_t^a(1), \tag{3.32}$$

for all $t \geq 0$ simultaneously. Also, a.s. for all t

$$m_t^0 \leq m_t^2 \leq \cdots \leq m_t^{2n} \leq m_t^{2n+1} \leq \cdots \leq m_t^3 \leq m_t^1 \tag{3.33}$$

as measures. Some relevant consequences of the previous inequalities are next established:

Lemma 3.19 (Convergence of odd and even marking measures and local times)**.**

i). Almost surely for every $t \geq 0$, there exists two finite atomic measures m_t^e and m_t^o such that for all but finitely many $n \in \mathbb{N}$,

$$m_t^e = m_t^{2n} \quad \text{and} \quad m_t^o = m_t^{2n+1} \tag{3.34}$$

and

$$m_t^e \leq m_t^o.$$

ii). For each $a \geq 0$, define two $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $(L_t^a(e) : t \geq 0)$ and $(L_t^a(o) : t \geq 0)$ by

$$L_t^a(e) := \inf_{n\in\mathbb{N}} L_t^a(2n) \geq L_t^a(o) := \sup_{n\in\mathbb{N}} L_t^a(2n+1). \tag{3.35}$$

Then, the processes $(a,t) \mapsto \int_0^t \mathbf{1}_{\{m_s^e=0\}} dL_s^a$ and $(a,t) \mapsto \int_0^t \mathbf{1}_{\{m_s^o=0\}} dL_s^a$ are bi-measurable and continuous in t versions of respectively $L_t^a(e)$ and $L_t^a(o)$.

iii). For all $t \geq 0$ one has a.s. for all $h \geq 0$

$$\begin{aligned} m_t^e([0,h]) &= \int_0^{H_t} \mathbf{1}_{[0,h]}(r) \int_0^\infty \mathbf{1}_{\{\nu < c^{\;p} L_t^r(o)\}} \mathcal{N}_t(dr, d\nu), \\ m_t^o([0,h]) &= \int_0^{H_t} \mathbf{1}_{[0,h]}(r) \int_0^\infty \mathbf{1}_{\{\nu < c^{\;p} L_t^r(e)\}} \mathcal{N}_t(dr, d\nu). \end{aligned}$$

iv) The processes $\mathcal{S}^e = (\rho, m^e)$ and $\mathcal{S}^o = (\rho, m^o)$ are $\mathbb{S}$-progressively marked exploration process.

Proof. We prove each of the statements for the "even" limiting objects, the corresponding proofs for the "odd" ones being similar.

i). For each $t \geq 0$, every measure m_t^n is finite since bounded by m_t^1 and atomic with unit mass atoms. The increasing sequence of integers $m_t^{2n}([0, H_t))$ is convergent and thus $m_t^{2n}([0, H_t)) = m_t^{2n_t}([0, H_t))$ for all n larger or equal than certain integer $n_t > 0$. From such an index on, the sequence of atomic measures m_t^{2n} must be constant since for all $x \in [0, H_t)$, $m_t^{2n}(\{x\}) \in \{0, 1\}$ is a non decreasing sequence and the total mass is constant.

ii). For fixed $a \geq 0$ and $t \geq 0$, we have

$$L_t^a(e) = \inf_{n\in\mathbb{N}} \int_0^t \mathbf{1}_{\{m_s^{2n}([0,H_t))=0\}} dL_s^a = \int_0^t \inf_{n\in\mathbb{N}} \mathbf{1}_{\{m_s^{2n}=0\}} dL_s^a = \int_0^t \mathbf{1}_{\{m_s^e=0\}} dL_s^a,$$

using (3.34) in the third equality. This and the continuity of $t \mapsto L_t^a$ implies the desired statement.

iii). Given $t \geq 0$ and $h \geq 0$, by (3.34) we have for some integer n_t and all $n \geq n_t$ that

$$\begin{aligned} m_t^e([0,h]) = m_t^{2n+2}([0,h]) &= \int_0^{H_t} \mathbf{1}_{[0,h]}(r) \int_0^\infty \mathbf{1}_{\{\nu < c\ {}^pL_t^r(2n+1)\}} \mathcal{N}_t(dr, d\nu) \\ &= \lim_{k\to\infty} \int_0^{H_t} \mathbf{1}_{[0,h]}(r) \int_0^\infty \mathbf{1}_{\{\nu < c\ {}^pL_t^r(2k+1)\}} \mathcal{N}_t(dr, d\nu) \\ &= \int_0^{H_t} \mathbf{1}_{[0,h]}(r) \int_0^\infty \sup_{k\in\mathbb{N}} \mathbf{1}_{\{\nu < c\ {}^pL_t^r(2k+1)\}} \mathcal{N}_t(dr, d\nu) \\ &= \int_0^{H_t} \mathbf{1}_{[0,h]}(r) \int_0^\infty \mathbf{1}_{\{\nu < c\ {}^pL_t^r(o)\}} \mathcal{N}_t(dr, d\nu), \end{aligned}$$

where we used the fact that ${}^pL_t^{\cdot}(o) = \sup_{k\in\mathbb{N}} {}^pL_t^{\cdot}(2k+1)$ by (3.35) and the characterizations of the predictable projection analogue to that in (3.19). Both extremes in the above equalities being right continuous functions of h, the conclusion on m_t^e follows.

Part iv). follows from iii). and Lemma 3.16. ■

We are now ready for the proof of Theorem 3.18.

Proof of Theorem 3.18. The existence part will consist in proving that the processes $((L_t^a(e))_{a\geq 0}, m_t^e) : t \geq 0)$ and $(((L_t^a(o))_{a\geq 0}, m_t^o) : t \geq 0)$ are equal.

By letting $A \nearrow +\infty$ in part i) of Proposition 3.17 applied to $m = m^e$ and $\tilde{m} = m^o$ (and taking into account the relations in part iii) of Lemma 3.19), we see by Gronwall's lemma that

$$m_t^o([0, H_t)) = m_t^e([0, H_t))$$

holds $\mathbb{P}(d\omega)dt$ a.e. Using the right-continuity of the two processes $\mathcal{S}^e = (\rho, m^e)$ and $\mathcal{S}^o = (\rho, m^o)$ in $\mathbb{S}$, the previous identity is seen to hold a.s. for all $t \geq 0$. Since for each $t \geq 0$, $m_t^o \geq m_t^e$ as measures by (3.33), and they both are atomic with unit mass atoms and equal total masses, we deduce that they must be equal.

By Lemma 3.19 ii) and since $dL_s^a = \mathbf{1}_{\{H_s=a\}} dL_s^a$ for each $a \geq 0$, we deduce from the previous that time right continuous versions of $L_t^a(e)$ and $L_t^a(o)$ are indistinguishable. The asserted properties of (m_t^*) follow from Lemma 3.16.

For the uniqueness statement, consider $m = m^*$ and $\tilde{m} = \hat{m}^*$ two different solutions of (3.25) and the associated marked local times $L(m^*)$ and $L(\hat{m}^*)$ given by (3.24). We obtain by similar arguments as in the existence part that a.s. for all $t \geq 0$,

$$m_t^*([0, H_t)) = \hat{m}_t^*([0, H_t)).$$

Using this equality in (3.24) both for m^* and $\hat{m}^*$ (and again the fact that $dL_s^a = \mathbf{1}_{\{H_s=a\}} dL_s^a$), we deduce that for each $a \geq 0$ the processes $L_t^a(m^*)$ and $L_t^a(\hat{m}^*)$ are indistinguishable. The fact that $m^* = \hat{m}^*$ then follows by using the latter in equation (3.25) both for m^* and $\tilde{m}^*$ ■

3.4 Proof of the Ray-Knight Theorem

Our next goal is to prove Theorem 3.11. The proof consists in two main steps. First, we will construct in terms of the same Lévy tree as before an approximation of the logistically pruned local time $(L_t^a(m^*) : a \geq 0, t \geq 0)$, by local time processes pruned at constant rate in the rectangles of some tree-like discrete grid, defined in height and local time units. The results of [3] will be crucial to identify the law of such approximation. The second step will consist in embedding this grid approximation into a white-noise/ Poisson-noise driven stochastic flow, which will correspond to a suitable approximation of the logistic stochastic flow process by an SDE flow with frozen coefficients, and proving then that such SDEs pointwise converges to the desired limit.

3.4.1 Grid approximation of the logistically marked local times

We denote by $(T_y)_{y \geq 0}$ the inverse local time at level 0 of the exploration process ρ. Given fixed parameters $\varepsilon, \delta > 0$ and a fixed amount $x \geq 0$ of cumulated local time at 0 (interpreted as initial population), we next introduce an approximation

$$L(\varepsilon, \delta) = (L_t^a(\varepsilon, \delta) : t \leq T_x, a \geq 0)$$

of $(L^a(m^*) : t \leq T_x, a \geq 0)$, consisting in local time process pruned at rates that are constant on the rectangles of a suitably defined tree-like height/local time discrete grid. The construction of this grid will be done in a lexicographical way. The index $k \in \mathbb{N}$ represents in what follows a discrete height level in the tree-like grid. We denote by $\mathbb{N}^*$ the set $\bigcup_{k \in \mathbb{N}} \mathbb{N}^k$. Notice that the dependance in the initial population x will be implicit, in order to enlighten the notation.

Step 0 : For all $t, a \geq 0$, we set

$$L_t^0(\varepsilon, \delta) := L_t^0 = L_t^0(m^*) \quad \text{and} \quad L_0^a(\varepsilon, \delta) := L_0^a = L_0^a(m^*) = 0.$$

Moreover, for every $n \in \mathbb{N}$ we set $T_{n\delta}^0 := T_{n\delta \wedge x}$.

Step 1 : Let $k = 0$ and $n_k = n_0 = 0$. For every $t \in (0, T_\delta^0]$ and $a \in (0, \varepsilon]$, we set

$$L_t^a(\varepsilon, \delta) := L_{T_0^0}^a(\varepsilon, \delta) + \int_{T_0^0}^t \mathbf{1}_{\{m_s^{\varepsilon,\delta}=0\}} dL_s^a,$$

where the measure $m^{\varepsilon,\delta}$ for $s \in (0, T_\delta^0]$ is defined by

$$m_s^{\varepsilon,\delta}([0,h]) := \int_0^{H_s} \mathbf{1}_{\{r \le h\}} \int_0^\infty \mathbf{1}_{\{\nu < cL_0^0(\varepsilon,\delta)\}} \mathcal{N}_s(dr, d\nu), \quad \forall h \in (0, \varepsilon].$$

If $\exists t \in (0, T_\delta]$ such that

$$L_t^\varepsilon(\varepsilon,\delta) - L_0^\varepsilon(\varepsilon,\delta) > 0,$$

we set $k = k+1 = 1, (n_0, n_1) = (0,0)$, and define

$$T_{n_1\delta}^{\varepsilon,(n_0)} := \sup\{t \ge 0 : L_t^\varepsilon(\varepsilon,\delta) - L_0^\varepsilon(\varepsilon,\delta) = n_1\delta\} \wedge T_\delta^0.$$

Otherwise, if $\exists t \in (T_\delta^0, T_x]$ such that

$$L_t^0(\varepsilon,\delta) - L_{T_\delta^0}(\varepsilon,\delta) > 0,$$

we set $k = k = 0, n_0 = n_0 + 1 = 1$.

Step g : In general, assuming that we have already constructed the processes

$$(L_t^a(\varepsilon,\delta) : \ 0 < t \le T', \varepsilon' < a \le \varepsilon'') \quad \text{and} \quad (L_t^a(\varepsilon,\delta) : \ 0 \le t < T'' \ a \le \varepsilon')$$

respectively on the left of and below the rectangle $[T', T''] \times [\varepsilon', \varepsilon'']$, the process

$$(L_t^a(\varepsilon,\delta) : \ T' < t \le T'', \varepsilon' < a \le \varepsilon'')$$

will be constructed as

$$L_t^a(\varepsilon,\delta) := L_{T'}^a(\varepsilon,\delta) + \int_{T'}^t \mathbf{1}_{\{m_s^{\varepsilon,\delta}=0\}} dL_s^a, \tag{3.36}$$

where the measure $m^{\varepsilon,\delta}$ for $s \in (T', T'']$ is defined by

$$m_s^{\varepsilon,\delta}([0,h]) := m_s^{\varepsilon,\delta}([0,\varepsilon']) + \int_0^{H_s} \mathbf{1}_{\{\varepsilon' < r \le h\}} \int_0^\infty \mathbf{1}_{\{\nu < cL_{T'}^{\varepsilon'}(\varepsilon,\delta)\}} \mathcal{N}_s(dr, d\nu), \quad \forall h \in (\varepsilon', \varepsilon''].$$
$$\tag{3.37}$$

We describe then the general step of the algorithm: Given $z \in \mathbb{N}$ and $(n_0, n_1, ..n_z) \in \mathbb{N}^*$, we use formula (3.36) to construct

$$\left(L_t^a(\varepsilon,\delta) : \ T_{n_z\delta}^{z\varepsilon,(n_0,n_1,..n_{(z-1)})} < t \le T_{(n_z+1)\delta}^{z\varepsilon,(n_0,n_1,..n_{(z-1)})}, z\varepsilon < a \le (z+1)\varepsilon\right),$$

where

$$T_{n_z\delta}^{z\varepsilon,(n_0,n_1,..n_{(z-1)})}$$
$$= \inf\{t \ge T_{n_{(z-1)}\delta}^{(z-1)\varepsilon,(n_0,n_1,..n_{(z-2)})} : L_t^{z\varepsilon}(\varepsilon,\delta) - L^{z\varepsilon}_{T_{n_{(z-1)}\delta}^{(z-1)\varepsilon,(n_0,n_1,..n_{(z-2)})}}(\varepsilon,\delta) = n_z\delta\}$$
$$\wedge\, T_{n_{(z-1)}+1}^{(z-1)\varepsilon,(n_0,n_1,..(n_{(z-2)}))},$$

with the convention that $n_i = 0$ if $i < 0$.

If $\exists t \in (T_{n_z\delta}^{z\varepsilon,(n_0,n_1,..n_z)}, T_{(n_z+1)\delta}^{z\varepsilon,(n_0,n_1,..n_z)}]$ such that

$$L_t^{(z+1)\varepsilon}(\varepsilon,\delta) - L^{(z+1)\varepsilon}_{T_{n_z\delta}^{z\varepsilon,(n_0,n_1,..n_z)}}(\varepsilon,\delta) > 0,$$

we set $k = z+1$ and $(n_0, n_1, ..n_k) = (n_0, n_1, ..n_z, 0)$ and we return to step **g**.

Otherwise,

Step g.1 : If $\exists t \in (T^{z\varepsilon,(n_0,n_1,..n_z+1)}_{n_z\delta}, T^{(z-1)\varepsilon,(n_0,n_1,..n_{z-1})}_{(n_{z-1}+1)\delta}]$ such that

$$L^{z\varepsilon}_t(\varepsilon,\delta) - L^{z\varepsilon}_{T^{z\varepsilon,(n_0,n_1,..n_z+1)}_{n_z\delta}}(\varepsilon,\delta) > 0,$$

we set $k = z$ and $(n_0,..,n_k) = (n_0,..,n_z+1)$, and we return to step **g**.

Otherwise, we set $k = z-1$ and $(n_0,..,n_k) = (n_0,..,n_z+1)$ and return to step **g.1** if $k \geq 0$, or the algorithm stops if $k = -1$.

In words, on each rectangle $(T^{k\varepsilon,(n_0,..,n_{k-1})}_{n_k\delta}, T^{k\varepsilon,(n_0,..,n_{k-1})}_{(n_k+1)\delta}] \times (k\varepsilon,(k+1)\varepsilon]$, marks are produced at each height at constant rate equal to the pruned local time $L^{k\varepsilon}_{T^{k\varepsilon,(n_0,..,n_{k-1})}_{n_k\delta}}(\varepsilon,\delta)$ accumulated at the time and level of the immediately lower-left grid point $(T^{k\varepsilon,(n_0,..,n_{k-1})}_{n_k\delta}, k\varepsilon)$. Then, for each $(t,a) \in (T^{k\varepsilon,(n_0,..,n_{k-1})}_{n_k\delta}, T^{k\varepsilon,(n_0,..,n_{k-1})}_{(n_k+1)\delta}] \times (k\varepsilon,(k+1)\varepsilon]$ inside the rectangle, the local time measure dL^a_t is pruned according to that marks, if the local times below dL^b_t, $b \leq a$ are not yet pruned (or equivalently, it the ancestors of t are not marked). Notice that the algorithms stops at step g if an only if at the end of that step $T^0_{n_0\delta} = T^0_{(n_0+1)\delta} = T_x$.

We therefore have a tree-like partition of the populations (represented by accumulated local times) at each level $k\varepsilon$, $k \in \mathbb{N}$, into subpopulations of size at most δ, in such a way that the partition of the population at height $(k+1)\varepsilon$ is a refinement of the partition induced by its ancestors at height $k\varepsilon$.

In this fashion, the population block at level $k\varepsilon$ indexed by $(n_0,\ldots,n_k)$ corresponds to the $(n_k+1)-$th block of descendants of the population block at level $(k-1)\varepsilon$ indexed by $(n_0,\ldots,n_{k-1})$. Notice that the size of the block $(n_0,\ldots,n_k)$ is

$$L^{k\varepsilon}_{T^{k\varepsilon,(n_0,..,n_{k-1})}_{(n_k+1)\delta}} - L^{k\varepsilon}_{T^{k\varepsilon,(n_0,..,n_{k-1})}_{n_k\delta}},$$

and that the size of a such a block is zero for n_k sufficiently large.

Given $h \geq 0$, we set

$$k_h = k_h(\varepsilon) := \sup\{k \in \mathbb{N} : k\varepsilon \leq h\}$$

and for each $s \geq 0$ and $k \in \mathbb{N}$ such that $k\varepsilon \leq H_s$, we define

$$T^{k,(*)}_s = T^{k,(*)}_s(\varepsilon,\delta) := \sup\{T^{k\varepsilon,(i_0,..i_{k-1})}_{i_k\delta} : \, j \in \{0,...k\}, \, i_j \in \mathbb{N}, T^{k\varepsilon,(i_0,..i_{k-1})}_{i_k\delta} \leq s\},$$

that is, the (exploration) time indexing the first individual in the block of ancestors at level $k\varepsilon$, of the individual indexed by s.

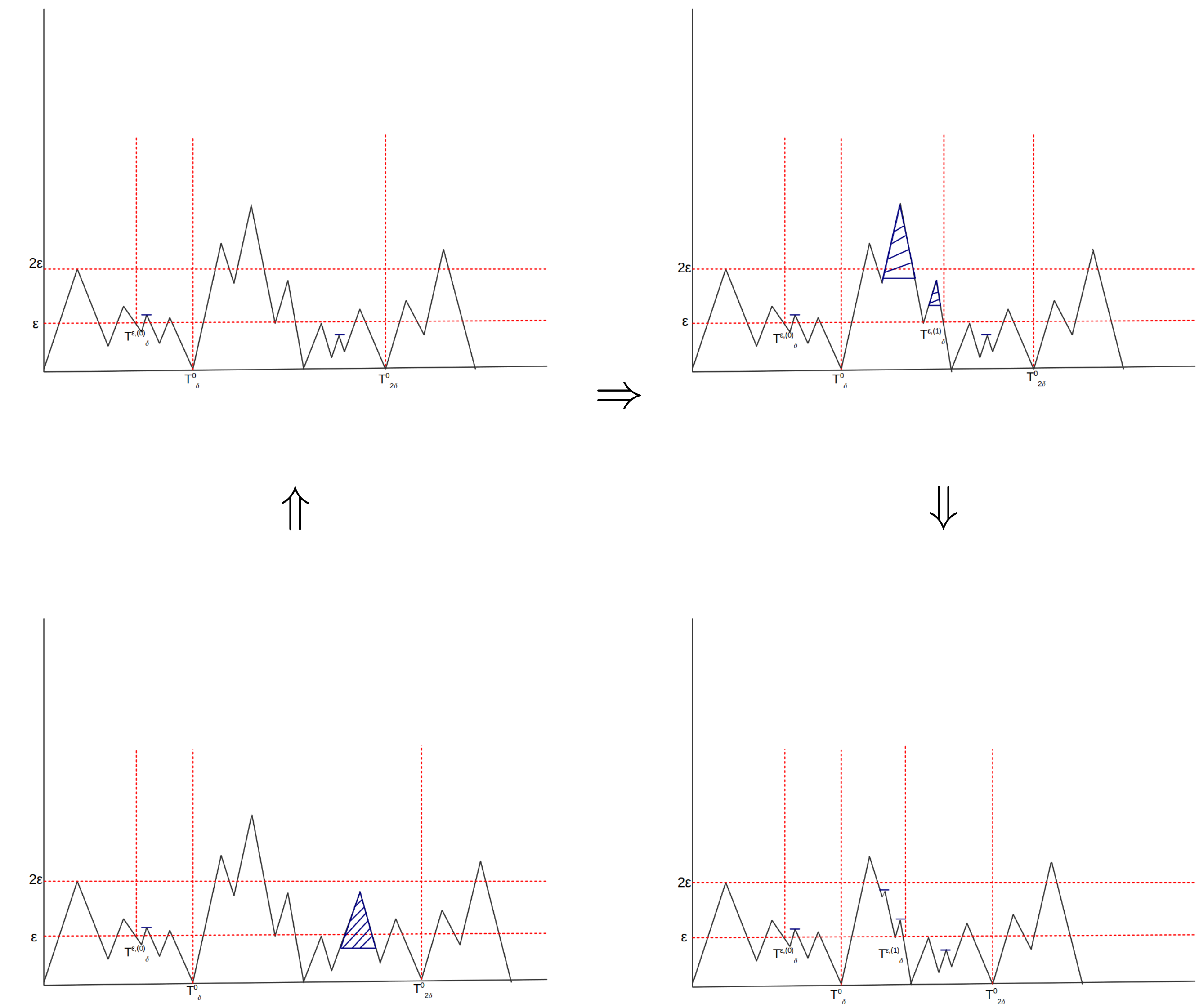
By construction, we have a.s. for all $a \geq 0$ that

$$L^a_t(\varepsilon,\delta) = \int_0^t \mathbf{1}_{\{m^{\varepsilon,\delta}_s = 0\}} dL^a_s = L^a_t(m^{\varepsilon,\delta}), \tag{3.38}$$

for all $t \geq 0$ and, for each $t \geq 0$, a.s. it holds for all $h \geq 0$ that

$$m^{\varepsilon,\delta}_t([0,h]) = \int_0^{H_t} \mathbf{1}_{\{r \leq h\}} \int_0^\infty \mathbf{1}_{\{\nu < cL^{\varepsilon k_r -}_{T^{k_r,(*)}_t}(\varepsilon,\delta)\}} \mathcal{N}_t(d\nu, dr), \quad h \in [0, H_t) \tag{3.39}$$

To check this, we use the fact that $L^{\varepsilon k_r}_{T^{k_r,(*)}_t}(\varepsilon,\delta)$ coincides with $L^{\varepsilon k_r -}_{T^{k_r,(*)}_t}(\varepsilon,\delta)$ except at heights in the grid $r \in \{0,\varepsilon,\ldots,\varepsilon k_{H_t}\}$, where the values of the measures (3.37) and (3.39) coincide by definition of the former.



Construction of the grid approximation.

Observe that, since we are pruning the original local times processes, for each $a \geq 0$ it a.s. holds that

$$L_t^a(\varepsilon, \delta) \leq L_t^a \tag{3.40}$$

for all $t \leq T_x$, but there is not clear pointwise relation between $L_t^a(\varepsilon, \delta)$ and $L_t^a(\varepsilon', \delta')$ for $(\varepsilon, \delta) \neq (\varepsilon', \delta')$. Finally, observe that the constructions are consistent for different initial populations $x \geq 0$.

For each fixed height $a > 0$, we denote in what follows by $\mathcal{E}_a$ the sigma field

$$\mathcal{E}_a := \big((\rho_{\tilde{\tau}_t^a}, \mathcal{N}_{\tilde{\tau}_t^a}) \,:\, t \geq 0\big) \tag{3.41}$$

where $\tilde{\tau}_t^a$ is the right continuous inverse of the process

$$\tilde{A}_t^a := \int_0^t \mathbf{1}_{\{H_s \leq a\}} ds.$$

In the remainder of this section, our goal is to prove the following two results:

Proposition 3.20 (Law of the grid approximation). *Let $\varepsilon, \delta > 0$ and $x \geq 0$ be fixed. For each $(k, (n_0, ..n_k)) \in \mathbb{N} \times \mathbb{N}^*$, conditionally on $\mathcal{F}_{T_{n_k\delta}^{k\varepsilon,(n_0,..,n_{k-1})}} \bigvee \mathcal{E}_{k\varepsilon}$ the process*

$$\left(L^{k\varepsilon+h}_{T_{(n_k+1)\delta}^{k\varepsilon,(n_0,..,n_{k-1})}}(\varepsilon, \delta) - L^{k\varepsilon+h}_{T_{n_k\delta}^{k\varepsilon,(n_0,..,n_{k-1})}}(\varepsilon, \delta) : h \in [0, \varepsilon]\right) \tag{3.42}$$

has the law of a CSBP of branching mechanism

$$\psi_{n_k\delta}^{k\varepsilon,n_0,..,n_{k-1}}(\lambda) := \psi(\lambda) + \lambda L^{k\varepsilon}_{T_{n_k\delta}^{k\varepsilon,(n_0,..,n_{k-1})}}(\varepsilon, \delta)\,; \tag{3.43}$$

with initial population

$$L^{k\varepsilon}_{T_{(n_k+1)\delta}^{k\varepsilon,(n_0,..,n_{k-1})}}(\varepsilon, \delta) - L^{k\varepsilon}_{T_{n_k\delta}^{k\varepsilon,(n_0,..,n_{k-1})}}(\varepsilon, \delta),$$

observed in the time interval $[0, \varepsilon]$.

Proposition 3.21 (Convergence of the grid approximation). *For each $x \geq 0$ and $a \geq 0$, the r.v. $L_{T_x}^a(\varepsilon, \delta)$ converges in probability to $L_{T_x}^a(m^*)$ when both ε and δ go (in an arbitrary way) to 0. In particular, the process $(L_{T_x}^a(\varepsilon, \delta) : x \geq 0, a \geq 0)$ converges to $(L_{T_x}^a(m^*) : x \geq 0, a \geq 0)$ in the sense of finite dimensional distributions.*

Their proofs are based on a series of technical lemmas, relying on the main result on Lévy tree pruning of [3] and on the excursion theory for snake process developed in [21]. We refer to Ch. 1 and 4 for details concerning the forthcoming discussion.

Recall that the exploration process ρ_t starting from 0 at time 0 can be defined in terms of the excursions of the underlying reflected Lévy process $(X_s - I_s \,:\, s \geq 0) = (\langle \rho_s, 1\rangle \,:\, s \geq 0)$, with both processes sharing the same excursion intervals $(\alpha_j, \beta_j)_{j\in J}$ away from their respective 0 elements, and with ρ_t being a function of the excursion of $(X_s - I_s \,:\, s \geq 0)$ straddling the time instant t, for each $t \geq 0$.

The snake process $(\rho_t, \mathcal{N}_t : t \geq 0)$ can also be described in terms of the above excursions of ρ_t and the excursions of the snake component $(\mathcal{N}_t : t \geq 0)$ away from 0, occurring in the same excursion intervals $(\alpha_j, \beta_j)_{j\in J}$. Indeed, one has

$$(\rho_t, \mathcal{N}_t) = \sum_{j\in J} \mathbf{1}_{\{\alpha^j < t < \beta^j\}} (\rho^j_{t-\alpha^j}, \mathcal{N}^j_{t-\alpha^j}). \tag{3.44}$$

where the pair $(\rho^j, \mathcal{N}^j)$ defined by

$$\begin{cases} \rho^j_s = \rho_{\alpha_j + s} & 0 < s < \beta_j - \alpha_j \\ \rho^j_s = 0 & s \geq \beta_j - \alpha_j \end{cases}, \quad \text{and} \quad \begin{cases} \mathcal{N}^j_s = \mathcal{N}_{\alpha_j + s} & 0 < s < \beta_j - \alpha_j \\ \mathcal{N}^j_s = 0 & s \geq \beta_j - \alpha_j \end{cases}$$

is the excursion away from $(0,0)$ of $(\rho_t, \mathcal{N}_t : t \geq 0)$ in the interval (α_j, β_j).

Moreover, the point process in $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, \mathscr{M}_f(\mathbb{R}_+) \times \mathscr{M}(\mathbb{R}^2_+))$ given by

$$\mathbf{M} := \sum_{j\in J} \delta_{(\ell^j, \rho^j, \mathcal{N}^j)}, \tag{3.45}$$

where $\ell^j = L^0_{\alpha_j}$, is Poisson with intensity $dx \times \mathbb{N}(d\rho\,, d\mathcal{N})$, where $\mathbb{N}(d\rho\,, d\mathcal{N}) = \mathbf{N}(d\rho) Q^{H(\rho)}(d\mathcal{N})$, $\mathbf{N}(d\rho)$ is the excursion measure of the exploration process and $Q^{H(\rho)}$ the conditional (probability) law of the snake component $(\mathcal{N}_t)$ of the snake process $(\rho_t, \mathcal{N}_t)$, given ρ. These facts follow from standard excursion theory, or are established in Section 4.1.4 in [21] in what concerns the description of $\mathbb{N}$. (Notice that in the standard snake terminology, they are stated in terms of the excursions of the process $(\rho_t, \mathcal{W}_t)$ where $\mathcal{W}_t = (s \mapsto \mathcal{N}_t([0,s], d\nu))$, see Remark 3.14.)

Reciprocally, given a Poisson point process $\mathbf{M}$ of intensity $dx \times \mathbb{N}(d\rho\,, d\mathcal{N})$ and atoms $(\ell^j, \rho^j, \mathcal{N}^j)_{j\in J}$, a snake process $(\rho_t, \mathcal{N}_t : t \geq 0)$ is uniquely determined through the relation (3.44), with (α_j, β_j) defined in terms of $\mathbf{M}$ by

$$\beta_j := \sum_{k\in J: \ell^k \leq \ell^j} \zeta_k \quad \text{and } \alpha_j := \sum_{k\in J: \ell^k < \ell^j} \zeta_k\,,$$

where for each $j \in J$, $\zeta_j := \inf\{s \geq 0 : \rho^j_s = 0\}$ is the length of excursion j. This follows from the fact that the local time at level 0 of an exploration process, when seen as a measure supported $\{t \geq 0 : \rho_t = 0\}$, is singular with respect to Lebesgue measure.

Let us now fix a height $a \geq 0$. Recall the notation τ^a_t used for the right continuous inverse of the process

$$A^a_t := \int_0^t \mathbf{1}_{\{H_s > a\}} ds.$$

Consider the process $(\rho^a_t, \mathcal{N}^a_t)$ defined by

$$\langle \rho^a_t, f\rangle = \int_{(a,\infty)} \rho_{\tau^a_t}(dr) f(r-a)$$

for measurable $f \geq 0$, and associated snake component $(\mathcal{N}^a_t : t \geq 0)$ defined by

$$\mathcal{N}^a_t(dr, d\nu) := \mathcal{N}_{\tau^a_t}(a + dr, d\nu).$$

We denote by $(\rho^{(i)})_{i\in I}$ the excursions of the process ρ above from height a and by $(\alpha^{(i)},\beta^{(i)})_{i\in I}$ the corresponding excursion intervals. More precisely, for each $i\in I$, we set

$$\begin{cases} \rho^{(i)}_s = \int_{(a,\infty)} \rho_{\alpha^{(i)}+s}(dr) f(r-a) & 0 < s < \beta^{(i)} - \alpha^{(i)} \\ \rho^{(i)}_s = 0 & s \geq \beta^{(i)} - \alpha^{(i)}. \end{cases}$$

These excursions are in one-to-one correspondence with the excursions away from 0 of ρ^a occurring at cumulated local times $L^a_{\alpha^{(i)}} = L^a_{\beta^{(i)}}$ at level a. We also introduce the excursions of $\mathcal{N}$ above level a, but relative to their value at height a. Namely,

$$\begin{cases} \mathcal{N}^{(i)}_s(dr,d\nu) = \mathcal{N}_{\alpha^{(i)}+s}(a+dr,d\nu) & 0 < s < \beta^{(i)} - \alpha^{(i)} \\ \mathcal{N}^{(i)}_s = 0 & s \geq \beta^{(i)} - \alpha^{(i)} \end{cases}.$$

Remark 3.22. Notice that each of these excursions $\mathcal{N}^{(i)}_s$ is issued from 0, instead of from $x_i = \mathcal{N}_{\alpha^{(i)}}\big|_{[0,a)\times\mathbb{R}_+}$, which would be the usual definition of the snake excursion above level a of the snake process $(\rho_t, \mathcal{W}_t)$, with $\mathcal{W}_t = (s \mapsto \mathcal{N}_t([0,s],d\nu))$.

Thus, $(\rho^{(i)},\mathcal{N}^{(i)})_{i\in I}$ are exactly the excursions of the process $((\rho^a_t,\mathcal{N}^a_t)\,:t\geq 0)$ away from $(0,0)$. Moreover, by arguments of snake excursion theory (close to those of the proof of Proposition 4.2.3 in [21]) it is not hard to establish

Lemma 3.23 (Snake excursion process above a given level)**.** *For each $a\geq 0$, the process $((\rho^a_t,\mathcal{N}^a_t)\,:t\geq 0)$ has the same law as $((\rho_t,\mathcal{N}_t)\,:t\geq 0)$ and it is independent of the sigma field $\mathcal{E}_a$ defined in (3.41). Moreover, conditionally on $\mathcal{E}_a$, the point process in $\mathbb{R}_+\times\mathbb{D}(\mathbb{R}_+,\mathcal{V})$ given by*

$$\sum_{i\in I} \delta_{(\ell^{(i)},\rho^{(i)},\mathcal{N}^{(i)})}, \tag{3.46}$$

where $\ell^{(i)} = L_{\alpha^{(i)}}$ for all $i\in I$, has the same law as (3.45) and in particular it is independent from $\mathcal{E}_a$.

Proof. Like for the process $((\rho_t,\mathcal{N}_t)\,:t\geq 0)$, the trajectories of the process $((\rho^a_t,\mathcal{N}^a_t)\,:t\geq 0)$ are determined in a unique (measurable) way from the atoms of (3.46). It is therefore enough to establish the second claim.

To do so, one easily adapts first the arguments of the proof of Proposition 4.2.3 in [21] in order to prove that, under the excursion measure $\mathbb{N}$, the process

$$\sum_{i\in I^j} \delta_{(\ell^{(i)},\rho^{(i)},\mathcal{N}^{(i)})}, \tag{3.47}$$

with $I_j := \{i\in I : (\alpha^{(i)},\beta^{(i)}) \subset (\alpha^j,\beta^j)\}$ the sub excursions above level a of the excursion away from 0 labeled j, is conditionally on $\mathcal{E}_a$ a Poisson point process of intensity $dx\mathbf{1}_{[L^a_{\alpha_j},L^a_{\beta_j}]}\times\mathbb{N}(d\rho\,,d\mathcal{N})$. (We notice that our superscripts (i) correspond to superscripts i therein.) The only difference is that, in the computation analogous to the one in end of that proof, one must consider test functions depending also on the components $\ell^{(i)}$ of the atoms, and depending on the excursions of the spatial component above level a only though their increments respect to their values at that a (recall Remark 3.22). Since I is equal to the disjoint union $\bigcup_{j\in J} I_j$, one then concludes using conditionally on $\mathcal{E}_a$ the additivity of Poisson point processes. ■

Given $\theta > 0$, we next consider $((\rho_t, m_t^\theta) : \theta \geq 0)$ the marked exploration processes (in the sense of [3]) with snake component m_t^θ given conditionally on ρ by

$$m_t^\theta([0,h]) := \mathcal{N}_t([0,h] \times [0,\theta]), \quad h \geq 0,$$

that is, a Poisson process on $[0, H_t)$ of parameter θ. The next result is central for the proof of Proposition 3.20.

Lemma 3.24 (Pruning at constant rate below a given level)**.** *Let $(L_t^a(m^\theta) : t \geq 0)$ denote the m^θ-pruned local time at level a and denote by*

$$I^\theta := \{i \in I \,:\, m^\theta_{\alpha^{(i)}} = 0\}$$

the set of excursions above level a whose lineage below that level does not have any mark. Then, conditionally on $\mathcal{E}_a$, the point process in $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, \mathcal{V})$ given by

$$\sum_{i \in I^\theta} \delta_{(L^a_{\alpha^{(i)}}(m^\theta), \rho^{(i)}, \mathcal{N}^{(i)})} \tag{3.48}$$

has the same law as the point process (3.45) and is independent of $\mathcal{E}_a$.

Notice that $I^\theta = \{i \in I \,:\, m^\theta_{\alpha^{(i)}}([0,a)) = 0\}$ since $H_{\alpha^{(i)}} = a$ for all $i \in I$.

Remark 3.25. Lemma 3.24 can be restated by saying that the removal of local time units corresponding to all individuals at level a with marked ancestors, and of all the excursions starting at the removed local time positions, leaves us a tree and marks above level a which behave (when described in terms of the right time units) exactly as the original exploration and snake processes. (Notice that this is not the situation studied in [3], where the non removed excursions are again pruned above level a.) This is a consequence of Lemma 3.23, and of an elementary fact about Poisson processes in $\mathbb{R}_+$ stated in Lemma 3.26 and proved below for completeness.

Lemma 3.26. *Let $(N_x : x \geq 0)$ be a Poisson process of parameter λ in $\mathbb{R}_+$ with respect to a given filtration, and let $F \subset \mathbb{R}_+$ be a predictable set such that a.s.,*

$$\varphi_x := \int_0^x \mathbf{1}_F(\ell) d\ell \to \infty$$

when $x \to \infty$. Let $(\vartheta_y)_{y \geq 0}$ be the right-continuous inverse of $(\varphi_x)_{x\geq 0}$. Then, the process

$$\left(N_x^\varphi = \int_0^{\vartheta_x} \mathbf{1}_F(\ell) N(d\ell), \quad x \geq 0 \right)$$

is a Poisson process in $\mathbb{R}_+$ with parameter λ.

Proof of Lemma 3.24. As in the proof of Proposition 4.2.3 in [21] we introduce $\tilde{L}_t^a := L^a_{\tilde{\tau}_t^a}$ with $\tilde{\tau}_t^a$ defined above after (3.41) and its left-continuous inverse

$$\gamma^a(r) := \inf\{s \geq 0 : \, \tilde{L}_s^a > r\}.$$

We next rewrite the m^θ-pruned local time at level a in terms of local time units $x \geq 0$. Using the snake property of m^θ in the second equality, we have

$$\begin{aligned} L^a_{T^a_x}(m^\theta) &= \int_0^{T^a_x} \mathbf{1}_{\{m^\theta_v = 0, H_v = a\}} dL^a_v = \int_0^{\tilde{A}^a_{T^a_x}} \mathbf{1}_{\{m^\theta_{\tilde{\tau}^a_v} = 0\}} d\tilde{L}^a_v \\ &= \int_0^{\tilde{L}^a_{\tilde{A}^a_{T^a_x}}} \mathbf{1}_{\{m^\theta_{\tilde{\tau}^a_{\gamma^a(\ell)}} = 0\}} d\ell = \int_0^x \mathbf{1}_{\{\mathbf{m}^\theta_\ell = 0\}} d\ell \end{aligned}$$

where for all $\ell \geq 0$ we have set $\mathbf{m}^\theta_\ell := m^\theta_{\tilde{\tau}^a_{\gamma^a(\ell)}}$. The last equality above stems from the fact that, by definition of $\tilde{A}^a$ and its right inverse,

$$\tilde{L}^a_{\tilde{A}^a_{T^a_x}} = L^a_{\tilde{\tau}^a_{\tilde{A}^a_{T^a_x}}} = L^a_{T^a_x}.$$

Notice that the process $(\mathbf{m}^\theta_\ell : \ell \geq 0)$ is $\mathcal{E}_a$-measurable and thus also $(L^a_{T^a_x}(m^\theta) : x \geq 0)$ is so. Moreover, the function $\ell \mapsto \mathbf{m}^\theta_\ell$ is right continuous since the composition of the right continuous functions $\ell \mapsto \tilde{\tau}^a_{\gamma^a(\ell)}$ and $s \mapsto m^\theta_s([0, H_s))$ (cf. Lemma 3.16). We can thus rewrite the process $(L^a_{T^a_x}(m^\theta) : x \geq 0)$ as

$$L^a_{T^a_x}(m^\theta) = \int_0^x \mathbf{1}_{\{\mathbf{m}^\theta_{\ell-} = 0\}} d\ell.$$

Let us denote by $\vartheta_y := \inf\{x \geq 0 : L^a_{T^a_x}(m^\theta) > y\}$ its right continuous inverse and by $\mathbf{M}^a$ the point process defined in (3.46). For each Borel set $S \subset \mathbb{D}(\mathbb{R}_+, \mathcal{V})$ with $\mathbb{N}(S) < \infty$, we define a Poisson process in $\mathbb{R}_+$ by

$$N^{a,S}([0,\ell]) := \mathbf{M}^a([0,\ell] \times S), \quad \ell \geq 0.$$

Notice that it is a $(\mathcal{Q}^a_\ell)_{\ell \geq 0}$ -Poisson point process, where $(\mathcal{Q}^a_\ell)_{\ell \geq 0}$ is the right continuous completion of the filtration $(\sigma(\mathbf{M}^a([0,x] \times d\rho, d\mathcal{N}) : 0 \leq x \leq \ell))_{\ell \geq 0}$. Setting $F^\theta = \{\ell \in \mathbb{R}_+ : \mathbf{m}^\theta_{\ell-} = 0\}$, we then see by Lemma 3.26 that

$$N^\varphi(S) := \left(\int_0^{\vartheta_x} \mathbf{1}_{F^\theta}(\ell) N^{a,S}(d\ell) : x \geq 0 \right)$$

is a Poisson process in $\mathbb{R}_+$ of parameter $\lambda = \mathbb{N}(S)$, with respect to the time changed filtration $(\mathcal{Q}^a_{\vartheta_\ell})_{\ell \geq 0}$ (the required divergence $\int^{+\infty} \mathbf{1}_{F^\theta}(\ell) d\ell$ is checked e.g. using the additivity of the process $x \mapsto L^a_{T^a_x}(m^\theta)$ following from the strong Markov property of $((\rho_t, m^\theta_t) : t \geq 0))$. Moreover for mutually disjoint sets $S_1, ..., S_n$, the real processes $N^\varphi(S_1), .., N^\varphi(S_n)$ are Poisson with respect to the same filtration $(\mathcal{Q}^a_{\vartheta_\ell})_{\ell \geq 0}$, and hence are independent from each other. We conclude that the point process $\mathbf{M}^{a,\theta}$ defined in $\mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+, \mathcal{V})$ by

$$\mathbf{M}^{a,\theta}([0,x] \times S) = \int_{[0,\vartheta_x] \times S} \mathbf{1}_{F^\theta}(\ell) \mathbf{M}^a(d\ell, d\rho, d\mathcal{N})$$

is Poisson with intensity $dx \times \mathbb{N}(d\rho, d\mathcal{N})$. Finally, it is not hard to see that this is exactly the point process (3.48). ■

Proof of Lemma 3.26. By standard properties of Poisson processes we know that, for any nonnegative predictable process f and stopping time τ in the given filtration,

$$\mathbb{E}\left[e^{-u\int_0^{\tau\wedge t} f(\ell)N(d\ell)+\lambda\int_0^{\tau\wedge t}(1-e^{-uf(\ell)})d\ell}\right]=1$$

for all $u \geq 0$ and $t \geq 0$. If moreover τ is such that $\mathbb{E}\left[e^{\lambda\int_0^{\tau}(1-e^{-uf(\ell)})d\ell}\right]<\infty$, by dominated convergence we deduce letting $t\to\infty$ that

$$\mathbb{E}\left[e^{-u\int_0^{\tau} f(\ell)N(d\ell)+\lambda\int_0^{\tau}(1-e^{-uf(\ell)})d\ell}\right]=1.$$

Since, $e^{\lambda\int_0^{\vartheta_x}(1-e^{-u\mathbf{1}_F(\ell)})d\ell}=e^{\lambda\int_0^{\vartheta_x}(1-e^{-u})\mathbf{1}_F(\ell)d\ell}$ and this, by a change of variable, is equal to $e^{\lambda\int_0^{x}(1-e^{-u})d\varphi}=e^{\lambda x(1-e^{-u})}$, we obtain from the previous that

$$\mathbb{E}\left[e^{-u\int_0^{\vartheta_x}\mathbf{1}_F(\ell)N(d\ell)}\right]=e^{-\lambda x(1-e^{-u})}.$$

We conclude the result by Campbell's formula. ∎

Remark 3.27. Let $\Theta:\mathbb{R}_+\to\mathbb{R}_+$ be a càglàd (left continuous) piecewise constant function, bounded by a constant $\bar{\theta}\geq 0$ and for each $t\geq 0$, define $\theta_t:=\Theta(L_t^0)$ and a progressively marked exploration process $((\rho_t,m_t):t\geq 0)$ by

$$m_t([0,h]):=\mathcal{N}_t([0,h]\times[0,\theta_t]),\quad h\geq 0.$$

A simple variation of the arguments of Lemma 3.24 considering m instead of m^θ allows us to obtain the same result for the Poisson (snake) excursion process pruned below level ε according to m, more precisely taking m instead of m^θ in (3.48). The divergence condition required for the time change therein to work is ensured in this (variable rate) case by comparison with the constant case of rate $\bar{\theta}$. This fact will be used in the proof of Propositions 3.20 and 3.21.

Remark 3.28. It is not hard to check that the snake process $(\rho'_t,\mathcal{N}'_t\ :t\geq 0)$ associated with the Poisson excursion process (3.48) which is equal in law to $(\rho_t,\mathcal{N}_t\ :t\geq 0)$ and $(\rho_t^a,\mathcal{N}_t^a\ :t\geq 0)$, can be described in terms of the latter and the marks below level a, via the time change

$$C'_t:=\inf\left\{s>0\ :\ \int_0^s \mathbf{1}_{\{m^\theta_{\tau_r^a}([0,a))=0\}}dr=\int_0^{\tau_s^a}\mathbf{1}_{m^\theta_u([0,a))=0\}}dA_u^a>t\right\}.$$

More precisely, it is given by $(\rho'_t,\mathcal{N}'_t\ :t\geq 0):=\left(\rho^a_{C'_t},\mathcal{N}^a_{C'_t}\ :t\geq 0\right)$. Notice that although A_u^a varies on intervals where the height of the process $(\rho_t,\mathcal{N}_t\ :t\geq 0)$ is above a, by the snake property the function $\mathbf{1}_{\{m^\theta_u([0,a))=0\}}$ does not. Thus, $(\rho'_t,\mathcal{N}'_t\ :t\geq 0)$ is function of the process $(\rho_t^a,\mathcal{N}_t^a\ :t\geq 0)$ and of an independent ($\mathcal{E}_a$-measurable) removal of some of its excursions.

In the proof of Proposition 3.20 we will also need Corollary 3.7. We therefore provide now its proof, which relies on the following well known approximation of exploration local times (see [21, Prop. 1.3.3]):

$$\lim_{\varepsilon\to 0}\sup_{a\geq 0}\mathbb{E}\left[\sup_{s\leq t}\left|\varepsilon^{-1}\int_0^s\mathbf{1}_{\{a<H_r\leq a+\varepsilon\}}dr-L_s^a\right|\right]=0.\qquad(3.49)$$

Proof of Corollary 3.7. Recall that the height process is a function of the exploration process at each time instant. Recall also the fact that $\bar{C}_t < \infty$ a.s. for all $t \geq 0$ by [3]. Applying (3.49) to the pruned exploration process (3.10), and performing the change of variable $\bar{C}_r \mapsto u$, we deduce that its local times process $\left(\bar{L}_s^a : t \geq 0\right)$ at level a satisfies a.s.,

$$\bar{L}_t^a = \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^{\bar{C}_t} \mathbf{1}_{\{a < H_u \leq a+\varepsilon, m_u^\theta = 0\}} du \quad \text{for all } t \geq 0$$

(the limit being an $L^1(\mathbb{P})$ limit). We therefore just need to check that the above limit is equal to $L_t^a(m^\theta)$.

The approximation (3.49) applied to the original exploration process $(\rho_t : t \geq 0)$ implies, for a subsequence $\varepsilon_n \to 0$ obtained by a diagonal argument, the almost sure convergence in each interval $[0, k]$, $k \in \mathbb{N}$, of the finite measures $\varepsilon_n^{-1} \mathbf{1}_{\{a < H_s \leq a+\varepsilon_n\}} ds$ towards the measure dL_s^a, with respect to the weak topology. Since a.s. for each $t \geq 0$ the function $s \mapsto \mathbf{1}_{\{m_s^\theta([0,H_s))=0, s<\bar{C}_t\}}$ is càdlàg and supported in some interval $[0, k]$, it is bounded and continuous almost everywhere with respect to the continuous measure $\mathbf{1}_{[0,k]}(s) dL_s^a$. In particular, for such subsequence ε_n we a.s. have that

$$L_{\bar{C}_t}^a(m^\theta) = \lim_{\varepsilon_n \to 0} \varepsilon_n^{-1} \int_0^{\bar{C}_t} \mathbf{1}_{\{a < H_u \leq a+\varepsilon_n, m_u^\theta = 0\}} du.$$

Since for each $a \geq 0$, $\bar{L}_t^a$ and $L_{\bar{C}_t}^a(m^\theta)$ are both a.s. continuous functions of $t \geq 0$, we conclude that $\left(\bar{L}_t^a : t \geq 0\right)$ and $\left(\int_0^{\bar{C}_t} \mathbf{1}_{\{m_s^\theta = 0\}} dL_s^a : t \geq 0\right)$ are indistinguishable. In particular, if we denote $\bar{T}_x = \inf\{s > 0 : \bar{L}_t^0 > x\}$, for each $a \geq 0$ we a.s. have

$$\bar{L}_{\bar{T}_x}^a = L_{\bar{C}_{\bar{T}_x}}^a(m^\theta) = L_{T_x}^a(m^\theta)$$

since $\bar{L}^0 = L_{\bar{C}.}^0(m^\theta) = L_{\bar{C}.}^0$. The conclusion follows by combining the above identities, Theorem 3.2 and Theorem 3.6. ∎

We are ready to proceed to the

Proof of Proposition 3.20. We consider first the case $k = 0$ and write $n_0^x = \sup\{n \in \mathbb{N} : n\delta < x\}$. We consecutively apply the strong Markov property of $(\rho, \mathcal{N})$ with respect to $\mathcal{F}_{T_{n_0\delta}^0}$ for each $n_0 \in \{0, \dots, n_0^x + 1\}$. In each step, we deduce with Corollary 3.7 that the process

$$\left(L_{T_{(n_0+1)\delta}^0}^h(\varepsilon, \delta) - L_{T_{n_0\delta}^0}^h(\varepsilon, \delta) : h \in [0, \varepsilon]\right) \tag{3.50}$$

has the required conditional laws. Furthermore, by Remarks 3.27 and 3.28, the process

$$\sum_{i \in I^{\varepsilon,\delta}} \delta_{(L_{\alpha^{(i)}}^a(\varepsilon,\delta), \rho^{(i)}, \mathcal{N}^{(i)})}, \tag{3.51}$$

with $I^{\varepsilon,\delta} := \{i \in I : m_{\alpha^{(i)}}^{\varepsilon,\delta} = 0, L_{\alpha^{(i)}}^0 \leq x\}$ is a Poisson point process of intensity $\mathbf{1}_{(0, L_{T_x}^\varepsilon(\varepsilon,\delta)]} d\ell \times \mathbb{N}(d\rho, d\mathcal{N})$ conditionally on $\mathcal{E}_\varepsilon$, associated with a snake process

$$(\hat{\rho}, \hat{\mathcal{N}}) := \left((\rho_{\hat{C}_t^\varepsilon}^\varepsilon, \mathcal{N}_{\hat{C}_t^\varepsilon}^\varepsilon) \ : t \leq T\right)$$

defined in terms of the original one $(\rho, \mathcal{N})$ via the time change

$$\hat{C}_t^{\varepsilon} := \inf\left\{ s > 0 : \int_0^s \mathbf{1}_{\{m^{\varepsilon,\delta}_{\tau_r^a}([0,\varepsilon))=0\}} dr = \int_0^{\tau_s^a} \mathbf{1}_{\{m^{\varepsilon,\delta}_u([0,\varepsilon))=0, H_u > \varepsilon\}} du > t \right\}.$$

Denoting by $\hat{L}_t^a$, $\hat{T}_x$ and $\hat{H}_t$ the corresponding local time processes, inverse local time process at level 0 and height process, for every $a \geq 0$ and $y \geq 0$ we can write,

$$L^{a+\varepsilon}_{T_y^{\varepsilon}}(\varepsilon,\delta) = \int_0^{T_y^{\varepsilon}} \mathbf{1}_{\{m_s^{\varepsilon,\delta}([0,a+\varepsilon))=0\}} dL_s^{a+\varepsilon} = \int_0^{\hat{T}_{L^{\varepsilon}_{T_y^{\varepsilon}}(\varepsilon,\delta)}} \mathbf{1}_{\{\hat{m}_u^{\varepsilon,\delta}([0,a))=0\}} d\hat{L}_u^a,$$

where

$$\hat{m}_u^{\varepsilon,\delta}([0,h)) := m^{\varepsilon,\delta}_{\hat{C}_u^{\varepsilon}}([\varepsilon, \varepsilon + h)).$$

Therefore, the construction of the approximated pruned local time and marks $(L_t^{a+\varepsilon}(m^{\varepsilon,\delta}) : a \geq 0, t \geq 0)$ and $(m_t^{\varepsilon,\delta}(\varepsilon + dh) : h \geq 0, t \geq 0)$ can be achieved conditionally on $\mathcal{E}_a$ in terms of the process (3.51) in the same way as the processes $(L_t^a(m^{\varepsilon,\delta}) : a \geq 0, t \geq 0)$ and $(m_t^{\varepsilon,\delta} : h \geq 0, t \geq 0)$ were constructed from the process (3.45). This allows us to iterate this argument in order to conclude the desired result by induction in k.

■

The remainder of this subsection is devoted to the proof of Proposition 3.21. Two further technical results are needed. This first one is an approximation result similar to the classic one (3.49) or to the extension we dealt with in the proof of Corollary 3.7 for local times pruned at constant rate, which will allow us to control the accumulated local times at heights that are not in the grid, with respect to those which are in it. We thus need to deal with local times randomly pruned at piecewise constant rates, as they appear when describing the grid construction above level $(k+1)\varepsilon$ in terms of the construction between that level and level $k\varepsilon$. Since the amount of local time accumulated at different levels $k\varepsilon$ of the grid is unbounded even when a bound is known at level 0, the convergence of the grid approximation needs to be established under a suitable localization of those local times (which is why the convergence in Proposition 3.21 is obtained in probability). But in order to remove correctly the localizing parameter, we need to know to dependence on it of the approximation of the pruned local times. We thus need to state some quantitative version of (pruned) local times approximations .

Following [21], for each $K > 0$ we denote by τ^K the stopping time

$$\tau^K := \inf\{s > 0 : \langle \rho_s, 1 \rangle \geq K\} = \inf\{s > 0 : X_s - I_s \geq K\}.$$

Lemma 3.29 (**Approximation of variably pruned local times at level** 0)**.** *Consider as in Remark 3.27 a càglàd (i.e. left continuous) piecewise constant function $\Theta : \mathbb{R}_+ \to [0, \bar{\theta}]$ with $\bar{\theta} \geq 0$ and the progressively marked exploration process $((\rho_t, m_t) : t \geq 0)$ defined by*

$$m_t([0,h]) := \mathcal{N}_t([0,h] \times [0, \theta_t]), \quad h \geq 0$$

with

$$\theta_t := \Theta(L_t^0)$$

a) There exists an explicit nonnegative function $(\varepsilon, \bar{\theta}) \mapsto \hat{\mathcal{C}}(\bar{\theta}, K, \varepsilon)$ *going to* 0 *when* $\varepsilon \to 0$ *and increasing both in* ε *and* $\bar{\theta}$*, such that for all* $x \geq 0$*:*

$$\mathbb{E}\left[\sup_{y\in[0,x]}\left|y - \frac{1}{\varepsilon}\int_0^{T_y} \mathbf{1}_{\{0<H_s\leq\varepsilon,\, m_s=0\}}ds\right|\mathbf{1}_{\{\tau^K>T_x\}}\right] \leq \hat{\mathcal{C}}(\bar{\theta}, K, \varepsilon)(x+\sqrt{x}).$$

b) For all $x \geq 0$ *we have*

$$\mathbb{E}\left[\sup_{t\in[0,T_x]}\left|L_t^{\varepsilon}(m) - L_t^0\right|\mathbf{1}_{\{\tau^K>T_x\}}\right] \leq \mathcal{C}(\bar{\theta}, K, \varepsilon)(x+\sqrt{x})$$

for an explicit nonnegative function $(\varepsilon, \bar{\theta}) \mapsto \mathcal{C}(\bar{\theta}, K, \varepsilon)$ *with similar properties as* $\hat{\mathcal{C}}(\bar{\theta}, K, \varepsilon)$*.*

Proof. a) The proof is inspired by that of Lemma 1.3.2 in [21]. We have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{y\in[0,x]}\left|y - \frac{1}{\varepsilon}\int_0^{T_y} \mathbf{1}_{\{0<H_s\leq\varepsilon,\, m_s=0\}}ds\right|\mathbf{1}_{\{\tau^K>T_x\}}\right)\\
&\leq \mathbb{E}\left(\sup_{y\in[0,x]}\left|\frac{1}{\varepsilon}\int_0^{T_y} \mathbf{1}_{\{0<H_s\leq\varepsilon, m_s=0, \langle\rho_s,1\rangle\leq K\}}ds - \frac{1}{\varepsilon}\mathbb{E}\left(\int_0^{T_y} \mathbf{1}_{\{0<H_s\leq\varepsilon, m_s=0, \langle\rho_s,1\rangle\leq K\}}ds\middle|\mathcal{Q}_y\right)\right|\right)\\
&\qquad + \mathbb{E}\left(\sup_{y\in[0,x]}\left|\frac{1}{\varepsilon}\mathbb{E}\left(\int_0^{T_y} \mathbf{1}_{\{0<H_s\leq\varepsilon, m_s=0, \langle\rho_s,1\rangle\leq K\}}ds\middle|\mathcal{Q}_y\right) - y\right|\right).
\end{aligned} \tag{3.52}$$

The time integral in the above expressions can be written in terms of the excursion point process (3.45). More precisely,

$$\int_0^{T_y} \mathbf{1}_{\{0<H_s\leq\varepsilon, m_s=0,\, \langle\rho_s,1\rangle\leq K\}}ds = \sum_{j\in J_y}\int_0^{\zeta^j} \mathbf{1}_{\{0<H_s^j\leq\varepsilon, m_s^j=0,\, \langle\rho_s^j,1\rangle\leq K\}}ds \tag{3.53}$$

where $J_y := \{j \in J : \ell^j \leq y\}$, $H_s^j = H_s(\rho^j) = H_{\alpha^j+s}$, $m_s^j = \mathcal{N}_s^j(\,\cdot \times [0, \Theta(\ell^j)))$ and ζ^j the length of the excursion j. By compensation, the desintegration $\mathbb{N}(d\rho\,, d\mathcal{N}) = \mathbf{N}(d\rho)Q^{H(\rho)}(d\mathcal{N})$ and the very definition of the snake $(\rho, \mathcal{N})$, we then get

$$\begin{aligned}
\mathbb{E}\left(\int_0^{T_y} \mathbf{1}_{\{0<H_s\leq\varepsilon, m_s=0,\, \langle\rho_s,1\rangle\leq K\}}ds\middle|\mathcal{Q}_y\right) &= \int_0^y d\ell\, \mathbb{N}\left(\int_0^{\zeta} \mathbf{1}_{\{0<H_s(\rho)\leq\varepsilon, \mathcal{N}_s(\,\cdot\times[0,\Theta(\ell)))=0\,, \langle\rho_s,1\rangle\leq K\}}ds\right)\\
&= \int_0^y d\ell\, \mathbf{N}\left(\int_0^{\zeta} e^{-\Theta(\ell)H_s(\rho)}\mathbf{1}_{\{0<H_s(\rho)\leq\varepsilon\,, \langle\rho_s,1\rangle\leq K\}}ds\right).
\end{aligned}$$

Thus, the second term in the r.h.s. of (3.52) is bounded by

$$\int_0^x d\ell\, \mathbb{E}\left[\left|\varepsilon^{-1}\mathbf{N}\left(\int_0^{\zeta} e^{-\Theta(\ell)H_s(\rho)}\mathbf{1}_{\{0<H_s(\rho)\leq\varepsilon\,, \langle\rho_s,1\rangle\leq K\}}ds\right) - 1\right|\right].$$

Using Proposition 1.2.5 in [21] to compute the integral with respect to $\mathbf{N}$ for each $\ell \in [0, y]$, the latter expression is seen to be equal to

$$\int_0^x d\ell\left[1 - \frac{1}{\varepsilon}\int_0^{\varepsilon} e^{\alpha b}e^{-\Theta(\ell)b}\mathbb{P}(S_b \leq K)db\right] \leq x\left[1 - \frac{1 - e^{(\alpha-\bar{\theta})\varepsilon}}{(\alpha-\bar{\theta})\varepsilon}\mathbb{P}(S_{\varepsilon} \leq K)\right] \tag{3.54}$$

where $(S_b)_{b\geq 0}$ is a subordinator of Laplace exponent $\exp\left(-t\widehat{\psi}(\lambda)\right) = \mathbb{E}(\exp -\lambda S_t)$ not depending on the drift coefficient α of the underlying Lévy process X. In particular, the expression on the r.h.s. of (3.54) goes to 0 with ε, and its supremum over $\varepsilon' \in [0,\varepsilon]$ is an increasing function of ε which does so too. As concerns the first term in the r.h.s. of (3.52), which corresponds to the expected supremum of a $(\mathcal{Q}_\ell)_{\ell\geq 0}$-martingale, we can use BDG inequality to bound it by some universal constant C_1 times

$$\sqrt{Var\left[\frac{1}{\varepsilon}\int_0^{T_x}\mathbf{1}_{\{0<H_s\leq\varepsilon, m_s=0, \langle\rho_s,1\rangle\leq K\}}ds\right]}.$$

Written in terms of the excursion Poisson point process (3.45), the above quantity reads

$$\sqrt{Var\left[\frac{1}{\varepsilon}\sum_{j\in J_x}\int_0^{\zeta_j}\mathbf{1}_{\{0<H^j_s\leq\varepsilon, m^j_s=0,\ \langle\rho^j_s,1\rangle\leq K\}}ds\right]}$$

and can be estimated by the same arguments as in the proof of Lemma 1.3.2 of [21] (see also the proof of Lemma 1.1.3 for details on related arguments):

$$\begin{aligned}
Var\left[\frac{1}{\varepsilon}\int_0^{T_x}\mathbf{1}_{\{0<H_s\leq\varepsilon, m_s=0, \langle\rho_s,1\rangle\leq K\}}ds\right] &= \frac{x}{\varepsilon^2}\mathbb{N}\left(\left(\int_0^{\zeta}\mathbf{1}_{\{0<H_s\leq\varepsilon, m_s=0, \langle\rho_s,1\rangle\leq K\}}ds\right)^2\right) \\
&\leq \frac{x}{\varepsilon^2}\mathbf{N}\left(\left(\int_0^{\zeta}\mathbf{1}_{\{0<H_s\leq\varepsilon, \langle\rho_s,1\rangle\leq K\}}ds\right)^2\right) \\
&\leq 2x\mathbb{E}(X_{L^{-1}(\varepsilon)}\wedge K),
\end{aligned} \tag{3.55}$$

where $\varepsilon\mapsto X_{L^{-1}(\varepsilon)}$ is the subordinator of Laplace exponent $\exp\left(-t\left(\widehat{\psi}(\lambda)-\alpha\right)\right)$. That is, the same subordinator S as above, but killed at an independent exponential time of parameter α. Thus, we have $\mathbb{E}(X_{L^{-1}(\varepsilon)}\wedge K)\leq\mathbb{E}(S_\varepsilon\wedge K)+K(1-e^{\alpha\varepsilon})\to 0$ as $\varepsilon\to 0$. The statement now follows by bringing together (3.52), (3.54) and (3.55).

b) We deduce the estimate from the one in part a). Observe first that for all $t\leq T_x$ which is not an increase time of L^0, either one has $L^\varepsilon_t(m)<L^0_t$ in which case for some $y\geq 0$ such that $T_y<t$ one has by continuity of local times that $|L^0_t-L^\varepsilon_t(m)|\leq L^0_{T_y}-L^\varepsilon_{T_y}(m)=y-L^\varepsilon_{T_y}(m)$, or $L^\varepsilon_t(m)\geq L^0_t$ and then $|L^0_t-L^\varepsilon_t(m)|\leq L^\varepsilon_{T_y}(m)-L^0_{T_y}=L^\varepsilon_{T_y}(m)-y$ for some $y\geq 0$ such that $T_y>t$. Therefore, is is enough to establish the required upper bound, for the quantity

$$\mathbb{E}\left[\sup_{y\in[0,x]}\left|L^\varepsilon_{T_y}(m)-y\right|\mathbf{1}_{\{\tau^K>T_x\}}\right]. \tag{3.56}$$

We have

$$\begin{aligned}
L^\varepsilon_{T_y}(m)-y &= \left[L^\varepsilon_{T_y}(m)-\frac{1}{\varepsilon}\int_0^{T_y}\mathbf{1}_{\{\varepsilon<H_s\leq 2\varepsilon, m_s((0,\varepsilon))=0\}}ds\right] \\
&\quad+2\left[\frac{1}{2\varepsilon}\int_0^{T_y}\mathbf{1}_{\{0<H_s\leq 2\varepsilon, m_s((0,\varepsilon))=0\}}ds-y\right] \\
&\quad+\left[y-\frac{1}{\varepsilon}\int_0^{T_y}\mathbf{1}_{\{0<H_s\leq\varepsilon, m_s((0,\varepsilon))=0\}}ds\right],
\end{aligned}$$

the absolute value of the second term on the right hand side being bounded by

$$2\left|\frac{1}{2\varepsilon}\int_0^{T_y}\mathbf{1}_{\{0<H_s\le 2\varepsilon, m_s((0,2\varepsilon))=0\}}ds-y\right|+2\left|\frac{1}{2\varepsilon}\int_0^{T_y}\mathbf{1}_{\{0<H_s\le 2\varepsilon\}}ds-y\right|.$$

It follows from part a) that the expression in (3.56) is upper bounded by

$$\begin{aligned}&\Big(2\hat{\mathcal{C}}(\bar\theta,K,2\varepsilon)+2\hat{\mathcal{C}}(0,K,\varepsilon)+\hat{\mathcal{C}}(\bar\theta,K,\varepsilon)\Big)(x+\sqrt{x})\\&\qquad+\mathbb{E}\left[\sup_{y\in[0,x]}\left|L^\varepsilon_{T_y}(m)-\frac{1}{\varepsilon}\int_0^{T_y}\mathbf{1}_{\{\varepsilon<H_s\le 2\varepsilon, m_s((0,\varepsilon))=0\}}ds\right|\mathbf{1}_{\{\tau^K>T_x\}}\right]\end{aligned}\tag{3.57}$$

and it only remains us to obtain a bound as the required one, for the above expectation. Notice to that end that the inner supremum can be written in terms of the Poisson excursions point process living above level ε we described in Remark 3.27, that is

$$\sum_{i\in I^m}\delta_{(L^\varepsilon_{\alpha^{(i)}}(m),\rho^{(i)},\mathcal{N}^{(i)})}\tag{3.58}$$

where $I^m=\{i\in I: m_{\alpha^{(i)}}([0,\varepsilon))=0\}$. More precisely, denote by $(\hat\rho,\hat{\mathcal{N}})$ the associated (standard) snake process and respectively by $\hat L^0_t$, $\hat T_x$, $\hat H_t$ and $\hat\tau^K$ the corresponding local time at 0, inverse local time at 0, height process, and the stopping time $\hat\tau^K:=\inf\{s>0:\langle\hat\rho_s,1\rangle\ge K\}$. Then, writing in a similar way as in (3.53) the time integral in (3.57) as a sum of integrals over (now) non marked excursion intervals above level ε, we get

$$L^\varepsilon_{T_y}(m)-\frac{1}{\varepsilon}\int_0^{T_y}\mathbf{1}_{\{\varepsilon<H_s\le 2\varepsilon, m_s((0,\varepsilon))=0\}}ds=L^\varepsilon_{T_y}(m)-\frac{1}{\varepsilon}\int_0^{\hat T_{L^\varepsilon_{T_y}(m)}}\mathbf{1}_{\{0<\hat H_s\le\varepsilon\}}ds$$

Since $T_x\ge\hat T_{L^\varepsilon_{T_x}(m)}$ and $\sup_{t\le T_x}\langle\rho_t,1\rangle\ge\sup_{t\le\hat T_{L^a_{T_x}(m)}}\langle\hat\rho_t,1\rangle$, the expectation in (3.57) is bounded from above by

$$\mathbb{E}\left[\sup_{z\in[0,L^\varepsilon_{T_x}(m)]}\left|z-\frac{1}{\varepsilon}\int_0^{\hat T_z}\mathbf{1}_{\{0<\hat H_s\le\varepsilon\}}ds\right|\mathbf{1}_{\{\hat\tau^K>\hat T_{L^\varepsilon_{T_x}(m)}\}}\right]\le\hat{\mathcal{C}}(0,K,\varepsilon)\mathbb{E}\left[L^\varepsilon_{T_x}(m)+\sqrt{L^\varepsilon_{T_x}(m)}\right],$$

where the inequality is obtained by applying part a) (with $m=0$ or equivalently $\bar\theta=0$) conditionally on $\mathcal{E}_\varepsilon$. With the obvious bounds $L^\varepsilon_{T_x}(m)\le L^\varepsilon_{T_x}$ a.s., $\mathbb{E}\left[\sqrt{L^\varepsilon_{T_x}}\right]\le\sqrt{\mathbb{E}\left[L^\varepsilon_{T_x}\right]}$ and the identities $\mathbb{E}(L^\varepsilon_{T_x})=x\mathbf{N}\left(L^\varepsilon_\zeta\right)=xe^{-\alpha\varepsilon}$ following from Corollary 1.3.4 in [21], we conclude that the result holds with $\mathcal{C}(\bar\theta,K,\varepsilon)=\Big(2\hat{\mathcal{C}}(\bar\theta,K,2\varepsilon)+3\hat{\mathcal{C}}(0,K,\varepsilon)+\hat{\mathcal{C}}(\bar\theta,K,\varepsilon)\Big)$. ■

Recall that for $M\ge 0$ and $a\ge 0$ we defined in Proposition 3.17 the stopping time $T^{a,M}=\inf\{t\ge 0:\exists\, r\le a \text{ s.t. } L^r_t\ge M\}$, which obviously satisfies

$$T^{a,M}\le T^r_M \text{ a.s. for all } r\in[0,a].$$

Lemma 3.30. *Let us fix real numbers $a,\varepsilon,\delta,K,M>0$.*

a) *For all* $a>0$,

$$\mathbb{E}\left[\sup_{t\le T^{a,M}\wedge\tau^K}|L_t^{k_a\varepsilon}(\varepsilon,\delta)-L_t^a(\varepsilon,\delta)|\right]\le\mathcal{C}(M,K,\varepsilon)(M+\sqrt{M})+\Gamma(M,K)$$

where $\Gamma(M,K)=2M\mathbb{P}(\tau^K\le T_M)$

b) *For each* $t\ge 0$,

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{t<T^{a,M}\wedge\tau^K\}}\mathbf{1}_{\{H_t\le a\}}\ |m_t^{\varepsilon,\delta}([0,H_t))-m_t^*([0,H_t))|\right]&\\ \le ca\left(\delta+\Gamma(M,K)+\mathcal{C}(M,K,\varepsilon)(M+\sqrt{M})\right)e^{ct}.&\end{aligned}$$

Proof. a) We start noting that if $t\le T^{a,M}$, we have $L_t^{k_a\varepsilon}(\varepsilon,\delta)\le L_t^{k_a\varepsilon}\le M$ and $L_t^a(\varepsilon,\delta)\le L_t^a\le M$. Thus, if $k_a=0$, we have

$$\sup_{t\le T^{a,M}\wedge\tau^K}|L_t^{k_a\varepsilon}(\varepsilon,\delta)-L_t^a(\varepsilon,\delta)|\le\sup_{t\le T_M}|L_t^0-L_t^a(\varepsilon,\delta)|\mathbf{1}_{\{\tau^K>T_M\}}+2M\mathbf{1}_{\{\tau^K\le T_M\}}$$

and the inequality follows from part b) of Lemma 3.29. To prove our claim for any level $k_a=k$, we first observe that

$$\sup_{t\le T^{a,M}\wedge\tau^K}|L_t^{k\varepsilon}(\varepsilon,\delta)-L_t^a(\varepsilon,\delta)|\le\sup_{t\le T_M^{k\varepsilon}}|L_t^{k\varepsilon}(\varepsilon,\delta)-L_t^a(\varepsilon,\delta)|\mathbf{1}_{\{\tau^K>T_M\}}+2M\mathbf{1}_{\{\tau^K\le T_M\}},\quad(3.59)$$

so it enough to bound the first term on the right hand side by $\mathcal{C}(M,K,\varepsilon)(M+\sqrt{M})$ to obtain the desired inequality. In the case $k=1$, we consider again the processes $\hat{L}_t^r$, $\hat{T}_x$ and $\hat{H}_t$ associated with the snake process $(\hat{\rho},\hat{\mathcal{N}})$ already used in the proof of Proposition 3.20. By arguments given in the proof of part b) of Lemma 3.29, we know that $\mathbf{1}_{\{\tau^K>T_M^\varepsilon\}}\le\mathbf{1}_{\{\hat{\tau}^K>\hat{T}_{L^\varepsilon_{T^\varepsilon_M}(\varepsilon,\delta)}\}}$. Moreover, one can check that

$$\sup_{t\le T_M^\varepsilon}|L_t^\varepsilon(\varepsilon,\delta)-L_t^a(\varepsilon,\delta)|\le\sup_{s\le\hat{T}_{L^\varepsilon_{T^\varepsilon_M}(\varepsilon,\delta)}}|\hat{L}_s^0-\hat{L}_s^{\varepsilon'}(\hat{m}_s^{\varepsilon,\delta})|,$$

where $\hat{m}_s^{\varepsilon,\delta}$ was also defined in the proof of Proposition 3.20 and $\varepsilon'=a-k\varepsilon\in[0,\varepsilon]$. By conditioning first on $\mathcal{E}_\varepsilon$ when taking expectation to the first term on the r.h.s. of (3.59), and applying part b) of Lemma (3.29) conditionally on $\mathcal{E}_\varepsilon$, the result follows since $L^\varepsilon_{T^\varepsilon_M}(\varepsilon,\delta)\le M$ and $M\mapsto\mathcal{C}(M,K,\varepsilon)(M+\sqrt{M})$ is increasing.

The result for general $k_a=k+1$ is obtained in a similar way by using the same recursive description of the $m^{\varepsilon,\delta}$-pruned local times above level $(k+1)\varepsilon$ in terms of the non marked excursions above level $k\varepsilon$.

b) For each $t\ge0$ we write $\Delta_t:=|m_t^{\varepsilon,\delta}([0,H_t))-m_t^*([0,H_t))|$. By a similar argument as in the first part of the proof of Proposition 3.17, we have that

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}_{\{t<T^{a,M}\wedge\tau^K\}}\mathbf{1}_{\{H_t\le a\}}\Delta_t\right]\\ &\le\mathbb{E}\left[\mathbf{1}_{\{t<T^{a,M}\wedge\tau^K\}}\mathbf{1}_{\{H_t\le a\}}\int_0^{H_t}\int_0^\infty|\mathbf{1}_{\{\nu<cL^{k_r\varepsilon}_{T_t^{k_r,(*)}}(\varepsilon,\delta)\}}-\mathbf{1}_{\{\nu<c\ {}^pL_t^r(*)\}}|\mathcal{N}_t(dr,d\nu)\right]\\ &\le\mathbb{E}\left[\mathbf{1}_{\{t<T^{a,M}\wedge\tau^K\}}\mathbf{1}_{\{H_t\le a\}}\mathbb{E}\left(\int_0^{H_t}\int_0^\infty|\mathbf{1}_{\{\nu<cL^{k_r\varepsilon}_{T_t^{k_r,(*)}}(\varepsilon,\delta)\}}-\mathbf{1}_{\{\nu<c\ {}^pL_t^r(*)\}}|\mathcal{N}_t(dr,d\nu)\middle|\mathcal{F}_t^\rho\right)\right]\\ &\le c\mathbb{E}\left[\mathbf{1}_{\{t<T^{a,M}\wedge\tau^K\}}\mathbf{1}_{\{H_t\le a\}}\int_0^{H_t}\left|L^{k_r\varepsilon}_{T_t^{k_r,(*)}}(\varepsilon,\delta)-L_t^r(*)\right|dr\right].\end{aligned}$$

We deduce that

$$\mathbb{E}\left[\mathbf{1}_{\{t<T^{a,M}\wedge\tau^K\}}\mathbf{1}_{\{H_t\le a\}}\Delta_t\right] \le c\left\{\mathbb{E}\left[\mathbf{1}_{\{t<T^{a,M}\wedge\tau^K\}}\mathbf{1}_{\{H_t\le a\}}\int_0^{H_t}|L_t^r(\varepsilon,\delta)-L_t^r(*)|\,dr\right]\right. \tag{3.60}$$

$$+\,c\,\mathbb{E}\left[\mathbf{1}_{\{t<T^{a,M}\wedge\tau^K\}}\mathbf{1}_{\{H_t\le a\}}\int_0^{H_t}|L^{k_r\varepsilon}_{T_t^{k_r,(*)}}(\varepsilon,\delta)-L_t^{k_r\varepsilon}(\varepsilon,\delta)|dr\right] \tag{3.61}$$

$$\left.+\,c\,\mathbb{E}\left[\mathbf{1}_{\{t<T^{a,M}\wedge\tau^K\}}\mathbf{1}_{\{H_t\le a\}}\int_0^{H_t}|L_t^{k_r\varepsilon}(\varepsilon,\delta)-L_t^r(\varepsilon,\delta)|dr\right]\right\}. \tag{3.62}$$

By formula (3.38) and the occupation times, the term on the right-hand side of (3.60) satisfies

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{\{t<T^{a,M}\wedge\tau^K\}}\mathbf{1}_{\{H_t\le a\}}\int_0^{H_t}|L_t^r(\varepsilon,\delta)-L_t^r(m^*)|\,dr\right]\\
&\qquad\le \mathbb{E}\left[\mathbf{1}_{\{t<T^{a,M}\wedge\tau^K\}}\int_0^\infty \mathbf{1}_{\{r\le a\}}\int_0^t\left|m_s^{\varepsilon,\delta}([0,r))-m_s^*([0,r))\right|dL_s^r dr\right]\\
&\qquad\le \mathbb{E}\left[\mathbf{1}_{\{t<T^{a,M}\wedge\tau^K\}}\int_0^t \mathbf{1}_{\{H_s\le a\}}\left|m_s^{\varepsilon,\delta}([0,H_s))-m_s^*([0,H_s))\right|ds\right]\\
&\qquad= \int_0^t \mathbb{E}\left[\mathbf{1}_{\{s<T^{a,M}\wedge\tau^K\}}\mathbf{1}_{\{H_s\le a\}}\Delta_s\right]ds.
\end{aligned}$$

From the definition of the random times $T_t^{*k_r}$ the expression (3.61) is bounded by $a\delta$. Finally, by part a) and since $T^{a,M}\le T^{r,M}$ for $r\in[0,a]$, the expression in (3.62) is bounded by

$$\mathbb{E}\left[\int_0^a \mathbf{1}_{\{t<T^{r,M}\wedge\tau^K\}}\sup_{s\le T^{r,M}\wedge\tau^K}|L_s^{k_r\varepsilon}(\varepsilon,\delta)-L_s^r(\varepsilon,\delta)|dr\right]\le a\left(\mathcal{C}(M,K,\varepsilon)(M+\sqrt{M})+2M\Gamma(M,K)\right),$$

and the statement follows by Gronwall's lemma. ■

Proof of Proposition 3.21. By formula (3.38) and the definition of the logistically pruned local times, in an analogous way as in second part of Lemma 3.17, for all $0\le h\le a$ and every stopping time τ with respect to $\mathcal{F}_t^\rho$ we get

$$\begin{aligned}
\mathbb{E}\left[|L^h_{\tau\wedge T^{a,M}}(\varepsilon,\delta)-L^h_{\tau\wedge T^{a,M}}(m^*)|\right] &\le \mathbb{E}\left[\int_0^{\tau\wedge T^{a,M}}|m_s^{\varepsilon,\delta}([0,h))-m_s^*([0,h))|dL_s^h\right]\\
&\le c\mathbb{E}\left[\int_0^{\tau\wedge T^{a,M}}dL_s^h\int_0^{H_s}|L^{k_r\varepsilon}_{T_s^{k_r,(*)}}(\varepsilon,\delta)-L_s^r(m^*)|dr\right],
\end{aligned}$$

from where

$$\mathbb{E}\left[|L^h_{\tau\wedge T^{a,M}}(\varepsilon,\delta)-L^h_{\tau\wedge T^{a,M}}(m^*)|\right]\le c\mathbb{E}\left[\int_0^{\tau\wedge T^{a,M}}dL_s^h\int_0^{H_s}|L^{k_r\varepsilon}_{T_s^{k_r,(*)}}(\varepsilon,\delta)-L_s^{k_r\varepsilon}(\varepsilon,\delta)|dr\right] \tag{3.63}$$

$$+\,c\mathbb{E}\left[\int_0^{\tau\wedge T^{a,M}}dL_s^h\int_0^{H_s}|L_s^{k_r\varepsilon}(\varepsilon,\delta)-L_s^r(\varepsilon,\delta)|dr\right] \tag{3.64}$$

$$+\,c\mathbb{E}\left[\int_0^{\tau\wedge T^{a,M}}dL_s^h\int_0^{H_s}|L_s^r(\varepsilon,\delta)-L_s^r(m^*)|dr\right]. \tag{3.65}$$

Dealing with terms (3.63) and (3.64) as in the proof of Lemma 3.30 b), we get that

$$\begin{aligned}\mathbb{E}\left[|L^h_{\tau\wedge T^{a,M}}(\varepsilon,\delta)\right. & \left. -L^h_{\tau\wedge T^{a,M}}(m^*)|\right] \\ & \leq c\delta a + c\,\mathbb{E}\left[\mathbf{1}_{\{h\leq a\}}\int_0^{\tau\wedge T^{a,M}} dL^h_s \int_0^{H_s} |L^{k_r\varepsilon}_s(\varepsilon,\delta) - L^r_s(\varepsilon,\delta)|dr\right] \\ & \quad + c\,\mathbb{E}\left[\int_0^{\tau\wedge T^{a,M}} dL^h_s \int_0^{H_s} |L^r_s(\varepsilon,\delta) - L^r_s(m^*)|dr\right] \\ & \leq c\delta a + c\,\mathbb{E}\left[\int_0^{\tau\wedge T^{a,M}} dL^h_s \int_0^{h} |L^{k_r\varepsilon}_s(\varepsilon,\delta) - L^r_s(\varepsilon,\delta)|dr\right] \\ & \quad + c\,\mathbb{E}\left[\int_0^{\tau\wedge T^{a,M}} dL^h_s \int_0^{h}\int_0^{s} |m^{\varepsilon,\delta}_u([0,r)) - m^*_u([0,r))|dL^r_u dr\right],\end{aligned}$$

since $dL^h_s = dL^h_s \mathbf{1}_{\{H_s=h\}}$. As $h\leq a$, it follows by using the inequality $T^{a,M}\leq T^{h,M}$ and the occupation times formula in the last expression that

$$\begin{aligned}\mathbb{E} & \left[|L^h_{\tau\wedge T^{a,M}}(\varepsilon,\delta) - L^h_{\tau\wedge T^{a,M}}(m^*)|\right] \\ & \leq c\delta a + c\,\mathbb{E}\left[\int_0^{\tau\wedge T^{a,M}} dL^h_s \int_0^{h} \mathbf{1}_{\{s\leq\tau\wedge T^{h,M}\}}|L^{k_r\varepsilon}_s(\varepsilon,\delta) - L^r_s(\varepsilon,\delta)|dr\right] \\ & + c\,\mathbb{E}\left[\int_0^{\tau\wedge T^{a,M}} dL^h_s \int_0^{s} \mathbf{1}_{\{H_u<h,u\leq\tau\wedge T^{h,M}\}}|m^{\varepsilon,\delta}_u([0,H_u)) - m^*_u([0,H_u))|du\right].\end{aligned} \tag{3.66}$$

We now take $\tau = t\wedge\tau^K\wedge T_x$. We then can bound the integral with respect du in the last term on inequality (3.66) by the corresponding integral between 0 and t no longer depending on s. This trivializes the local time integral therein, yielding a quantity bounded by M. Applying part b) of Lemma 3.30 to the remaining time integral shows that the last term in (3.66) is then bounded by

$$aM\left(\delta + \Gamma(M,K) + \mathcal{C}(M,K,\varepsilon)(M+\sqrt{M})\right)(e^{ct}-1).$$

For the same choice of τ, taking supremum over $s\leq\tau^K\wedge T^{h,M}$ inside the integral with respect to dr in the second term on the right hand side of (3.66), we now deduce with help of part a) of Lemma 3.30 the following upper bound for that term:

$$caM\left(\Gamma(M,K) + \mathcal{C}(M,K,\varepsilon)(M+\sqrt{M})\right).$$

We therefore have shown that

$$\begin{aligned}\mathbb{E}\left[|L^a_{T_x\wedge(\tau^K\wedge t\wedge T^{a,M})}(\varepsilon,\delta)\right. & \left. -L^a_{T_x\wedge(\tau^K\wedge t\wedge T^{a,M})}(m^*)|\right] \\ & \leq a(c+1)M\left(\delta + \Gamma(M,K) + \mathcal{C}(M,K,\varepsilon)(M+\sqrt{M})\right)e^{ct}.\end{aligned}$$

Now, since $\tau^K\to\infty$ when $K\to\infty$, for each $M>0$ there is some $K=K(M)>0$ sufficiently large so that $\mathbb{P}(\tau^K(M)\leq T_M)\leq\frac{1}{2M^3}$ and hence $\Gamma(M,K(K))\leq 1/M^2$ for all $M>0$. We then choose for each $M>0$, $t=t(M):=\ln(M)/2c$. With these choices, we have

$$M\,\Gamma(M,K(K))e^{ct(M)} = 2M^{-1}\to 0$$

when $M \to \infty$, whereas the sequence of stopping times $\mathcal{T}_M := t(M) \wedge \tau^{K(M)} \wedge T^{a,M}$ goes a.s. to ∞. Thus, for each $\eta > 0$,

$$\begin{aligned}\mathbb{P}\left[|L^a_{T_x}(\varepsilon,\delta) \right. & \left. -L^a_{T_x}(m^*)| > \eta\right] \\ &\leq \mathbb{P}\left[|L^a_{T_x\wedge\mathcal{T}_M}(\varepsilon,\delta) - L^a_{T_x\wedge\mathcal{T}_M}(m^*)| > \eta\right] + \mathbb{P}(T_x > \mathcal{T}_M) \\ &\leq \frac{a(c+1)}{\eta} M\left(\delta + \Gamma(M,K) + \mathcal{C}(M,K,\varepsilon)(M+\sqrt{M})\right)e^{ct} + \mathbb{P}(T_x > \mathcal{T}_M)\end{aligned}$$

and hence, for all $M \geq 0$,

$$\limsup_{\varepsilon,\delta\to(0,0)} \mathbb{P}\left[|L^a_{T_x}(\varepsilon,\delta) - L^a_{T_x}(m^*)| > \eta\right] \leq \frac{a(c+1)}{\eta} M\,\Gamma(M,K(K))e^{ct(M)} + \mathbb{P}(T_x > \mathcal{T}_M).$$

Letting $M \to \infty$, we have established that

$$\lim_{\varepsilon,\delta\to(0,0)} \mathbb{P}\left[|L^a_{T_x}(\varepsilon,\delta) - L^a_{T_x}(m^*)| > \eta\right] = 0.$$

■

3.4.2 Stochastic flow embedding of the grid-aproximation

We will now show that the approximating process $(L^a_{T_x}(\varepsilon,\delta) : a \geq 0)$ coincides (in law) with a flow of CSBPs of branching mechanism ψ as studied by Dawson and Li [18], but with additional "frozen" negative drift terms, on rectangles of a suitable time-space grid, accounting for the "pruning" of the original population.

Recall that the flow of CSBPs introduced in [18] is the two-parameter process $(Y_t(v) : t \geq 0, v \geq 0)$, where for every $v \geq 0$ the process $Y(v) = (Y_t(v) : t \geq 0)$ is the unique strong solution of the stochastic differential equation:

$$\begin{aligned}Y_t(v) = v + \alpha\int_0^t Y_s(v)ds + \sigma\int_0^t\int_0^{Y_{s-}(v)} W(ds,du) + \int_0^t\int_0^{Y_{s-}(v)}\int_1^\infty rN(ds,d\nu,dr) \\ + \int_0^t\int_0^{Y_{s-}(v)}\int_0^1 r\tilde{N}(ds,d\nu,dr),\end{aligned} \tag{3.67}$$

where $\Pi(dr)$, $\sigma \geq 0$ and α are the same objects as in (3.1), $W(ds,du)$ is a white noise process on $(0,\infty)^2$ based on the Lebesgue measure $ds \times du$ and N is a Poisson random measure on $(0,\infty)^3$ with intensity $ds \times d\nu \times \Pi(dr)$. Further properties of the two parameter process were recalled in Section 3.2.1.

In a similar way, as we state in Proposition 3.12, we can define also a flow of LBPs as a two-parameter process $(Z_t(v) : t \geq 0, v \geq 0)$, where for every $v \geq 0$ the process $Z(v) = (Z_t(v) : t \geq 0)$ satisfies

$$\begin{aligned}Z_t(v) = v + \alpha\int_0^t Y_s(v)ds + \sigma\int_0^t\int_0^{Z_{s-}(v)} W(ds,du) + \int_0^t\int_0^{Z_{s-}(v)}\int_1^\infty rN(ds,d\nu,dr) \\ + \int_0^t\int_0^{Z_{s-}(v)}\int_0^1 r\tilde{N}(ds,d\nu,dr) - c\int_0^t Z_s^2 ds.\end{aligned} \tag{3.68}$$

Proof of Proposition 3.12 . We first show statements i) and iii) and use them to prove ii).

i) Given the parameters (b,σ,g_0,g_1) defined by

- $x \mapsto b(x) := \alpha x - cx^2$;
- $(x, u) \mapsto \sigma(x, u) := \sigma \mathbf{1}_{u \leq x}$;
- $(x, \nu, r) \mapsto g_0(x, \nu, r) = g_1(x, \nu, r) := r\mathbf{1}_{\nu \leq x}$,

we can check that $(b, \sigma, g0, g1)$ are admissible parameters satisfying conditions $(2.a, b, c, e)$ in [18, Section 2]. Thus, for each $v \leq 0$, we can deduce from [18, Thm. 2.5] that there is a unique strong solution to (3.68). Moreover, for each $v \geq 0$, the solution $(Z_t(v) : t \geq 0)$ satisfies also equation (3.6) with the Brownian motion given by $dB_t := (Z_{s^-}(v))^{-\frac{1}{2}} \int_0^{Z_{s^-}(v)} W(ds, du)$, and it follows from Theorem 2.1 that any solution of equation (3.6) is a LBP with the required parameters ψ and $c > 0$.

iii) Given $v \geq u \geq 0$, $t \geq 0$, we set $\Upsilon_t := Z_t(v) - Z_t(u)$. From (3.68) we deduce that Υ_t satisfies

$$\begin{aligned}\Upsilon_t = \alpha \int_0^t \Upsilon_s ds + \sigma \int_0^t \int_0^{\Upsilon_{s-}} W_1(ds, dw) + \int_0^t \int_0^{\Upsilon_{s-}} \int_1^{\infty} rN_1(ds, d\nu, dr) \\ + \int_0^t \int_0^{\Upsilon_{s-}} \int_0^1 r\tilde{N}_1(ds, d\nu, dr) - c\int_0^t \left[Z_s^2(v) - Z_s^2(u)\right] ds,\end{aligned}$$

where

$$W_1(ds, dw) = W(ds, dw + Z_{s-}(u))$$

is a white noise with intensity $ds \times dw$, and

$$N_1(ds, d\nu, dr) = N(ds, d\nu + Z_{s-}(u), dr)$$

is a Poisson random measure with intensity $ds \times d\nu \times \Pi(dr)$. Thus, $(\Upsilon_t : t \geq 0)$ satisfies

$$\begin{aligned}\Upsilon_t = \alpha \int_0^t \Upsilon_s ds + \sigma \int_0^t \int_0^{\Upsilon_{s-}} W_1(ds, dw) + \int_0^t \int_0^{\Upsilon_{s-}} \int_1^{\infty} rN_1(ds, d\nu, dr) \\ + \int_0^t \int_0^{\Upsilon_{s-}} \int_0^1 r\tilde{N}_1(ds, d\nu, dr) - c\int_0^t \Upsilon_s^2 ds - c\int_0^t Z_s(u)\Upsilon_s ds,\end{aligned} \tag{3.69}$$

and we can deduce that statement iii) is true.

ii) Given $t \geq 0$, the càdlàg property for $v \mapsto Z_t(v)$ can be deduced from the comparison property stated in [18, Thm. 2.3]. Moreover, it is easy to show using similar arguments as in the proof of [18, Thm. 3.4] that there is a locally bounded non-negative function $t \mapsto C(t)$ on $[0, \infty)$ so that

$$\mathbb{E}\left\{\sup_{0 \leq s \leq t} |Z_s(v) - Z_s(u)|\right\} \leq C(t)\left\{(v - u) + \sqrt{v - u}\right\}$$

for $v \geq u \geq 0$. Therefore, using the previous bound and [18, Lemma 3.5] (along with the Markov property stated in iii)), we can deduce that the path-valued process $(Z(v) : v \geq 0)$ has a càdlàg modification, following the arguments in the proof of Theorem 3.6 in [18]. ■

Given fixed parameters $\varepsilon, \delta > 0$ and a fixed initial population $x \geq 0$, we now construct a new flow $(Z_t^{\varepsilon,\delta}(w) : w \leq x, t \geq 0)$ of CSBP with "frozen drifts", driven by the same noise processes W and N as the process Z in (3.68), by means of the following iterative procedure:

Step 0 : For all $t, v \geq 0$, we set

$$Z_t^{\varepsilon,\delta}(0) := Z_t(0) = 0 \quad \text{and} \quad Z_0^{\varepsilon,\delta}(v) := Z_0(v) = v;$$

Step 1 : For every $x \in \mathbb{R}_+$, we set $n_0^x := \sup\{n \in \mathbb{N} : x > n\delta\}$. We define then a new flow $(Z_t^{\varepsilon,\delta}(w) : w \leq x, 0 \leq \varepsilon)$ by

$$Z_t^{\varepsilon,\delta}(w) := \sum_{i_0=0}^{n_0^x} Z_t^{0,(i_0)}\left((w \wedge (i_0+1)\delta) - (w \wedge i_0\delta)\right), \quad t \in (0, \varepsilon],$$

where, for each $i_0 \in \{0, ..n_0^x\}$, the process $(Z_t^{0,(i_0)}(v) : t \geq 0)$ is a CSBP with branching mechanism $\psi^{0,(i_0)}(\lambda) = \psi(\lambda) + c\lambda i_0\delta$, starting from v. More precisely, we consider $Z^{0,(i_0)}(v)$ as the unique strong solution of the SDE

$$\begin{aligned} Z_t^{0,(i_0)}(v) =& v + \alpha \int_0^t Z_s^{0,(i_0)}(v) ds + \sigma \int_0^t \int_0^{Z_{s-}^{0,(i_0)}(v)} W^{0,(n_0)}(ds, du) \\ &+ \int_0^t \int_0^{Z_{s-}^{0,(i_0)}(v)} \int_1^\infty r N^{0,(i_0)}(ds, d\nu, dr) + \int_0^t \int_0^{Z_{s-}^{0,(i_0)}(v)} \int_0^1 r \tilde{N}^{0,(i_0)}(ds, d\nu, dr) \\ &- ci_0\delta \int_0^t Z_s^{0,(i_0)}(v) ds. \end{aligned}$$

Here

$$W^{0,(i_0)}(ds, du) := W(ds, du + Z_s^{\varepsilon,\delta}(i_0\delta))$$

is a white noise with intensity $ds \times du$, and

$$N^{0,(i_0)}(ds, d\nu, dr) := N(ds, d\nu + Z_s^{\varepsilon,\delta}(i_0\delta), dr).$$

is a Poisson random measure with intensity $ds \times d\nu \times \Pi(dr)$.

Step 2 : Now, for every $i_0 \in \{0, .., n_0\}$, we set

$$n_\varepsilon^{(i_0)} := \sup\{n \in \mathbb{N} : Z_\varepsilon^{0,(i_0)}(\delta) > n\delta\},$$

and for every $i_1 \in \{0, .., n_\varepsilon^{(i_0)} + 1\}$, we define

$$x_{i_1\delta}^{\varepsilon,(i_0)} := \left(i_1\delta + Z_\varepsilon^{\varepsilon,\delta}(i_0\delta)\right).$$

We extend then each process $(Z_t^{\varepsilon,\delta}(w) : 0 \leq t \leq \varepsilon)$, with $w \leq x$ to a process $(Z_t^{\varepsilon,\delta}(w) : 0 < t \leq 2\varepsilon)$ by setting

$$Z_t^{\varepsilon,\delta}(w) := Z_{\varepsilon,t}(Z_\varepsilon^{\varepsilon,\delta}(w)), \quad t \in (\varepsilon, 2\varepsilon], \tag{3.70}$$

where the process $(Z_{\varepsilon,t}(y) : t \geq \varepsilon)$ is given by

$$Z_{\varepsilon,t}(y) := \sum_{i_0=0}^{n_0^x} \sum_{i_1=0}^{n_\varepsilon^{(i_0)}} Z_{t-\varepsilon}^{\varepsilon,(i_0,i_1)}\left((x_{(i_1+1)\delta}^{\varepsilon,(i_0)} \wedge y) - (x_{i_1\delta}^{\varepsilon,(i_0)} \wedge y)\right). \tag{3.71}$$

As before, each process $(Z_t^{\varepsilon,(i_0,i_1)}(v):t\geq 0)$ is a CSBP with branching mechanism

$$\psi^{\varepsilon,(i_0,i_1)}(\lambda)=\psi(\lambda)+c\lambda x_{i_1\delta}^{\varepsilon,(i_0)},$$

starting from v, and in fact it is defined as the unique strong solution of the SDE

$$\begin{aligned}Z_t^{\varepsilon,(i_0,i_1)}(v)&=v+\alpha\int_0^t Z_s^{\varepsilon,(i_0,i_1)}(v)ds+\sigma\int_0^t\int_0^{Z_{s-}^{\varepsilon,(i_0,i_1)}(v)}W^{\varepsilon,(i_0,i_1)}(ds,du)\\&+\int_0^t\int_0^{Z_{s-}^{\varepsilon,(i_0,i_1)}(v)}\int_1^\infty rN^{\varepsilon,(i_0,i_1)}(ds,d\nu,dr)\\&+\int_0^t\int_0^{Z_{s-}^{\varepsilon,(i_0,i_1)}(v)}\int_0^1 r\tilde{N}^{\varepsilon,(i_0,i_1)}(ds,d\nu,dr)-cx_{i_1}^{\varepsilon,(i_0)}\int_0^t Z_s^{\varepsilon,(i_0,i_1)}(v)ds,\end{aligned}$$

where

$$W^{\varepsilon,(i_0,i_1)}(ds,du)=W(ds+\varepsilon,du+Z_{\varepsilon,s}(x_{i_1}^{\varepsilon,(i_0)}))$$

is a white noise with intensity $ds\times du$ and

$$N^{\varepsilon,(i_0,i_1)}(ds,d\nu,dr)=N(ds+\varepsilon,d\nu+Z_{\varepsilon,s}(x_{i_1}^{\varepsilon,(i_0)}),dr)$$

is a Poisson random measure with intensity $ds\times d\nu\times\Pi(dr)$.

Step g : If we assume that the flow $(Z_t^{\varepsilon,\delta}(w):t\leq k\delta)$ is already constructed, we define then inductively the extension of the process $Z^{\varepsilon,\delta}(w)$ to $(k\delta,(k+1)\delta]$ as

$$Z_t^{\varepsilon,\delta}(w):=Z_{k\varepsilon,t}(Z_{k\varepsilon}^{\varepsilon,\delta}(w)),\quad t\in(k\varepsilon,(k+1)\varepsilon] \tag{3.72}$$

where

$$Z_{k\varepsilon,t}(y):=\sum_{i_1=0}^{n_0^x}\cdots\sum_{i_k=0}^{n_{k\varepsilon}^{(i_0,..i_{k-1})}}Z_{t-k\varepsilon}^{k\varepsilon,(i_0,i_1,..,i_k)}\left((x_{(i_k+1)\delta}^{k\varepsilon,(i_0,..,i_{k-1})}\wedge y)-(x_{(i_k)\delta}^{k\varepsilon,(i_0,..,i_{k-1})}\wedge y)\right), \tag{3.73}$$

with

$$n_{k\varepsilon}^{(i_0,i_1,..i_{k-1})}=\sup\{n\in\mathbb{N}:Z_\varepsilon^{k-1,(i_0,i_1,..,i_{k-2})}(\delta)>n\delta\}$$

and

$$x_{i_k}^{k\varepsilon,(i_0,..,i_{k-1})}=Z_{(k-1)\varepsilon,k\varepsilon}(i_{k-1}\delta)+i_k\delta,\quad i_k\in\{0,1,..n_{k\varepsilon}^{(i_0,i_1,..i_{k-1})}\}. \tag{3.74}$$

Also, each process $(Z_t^{k\varepsilon,(i_0,i_1,..i_k)}(v):t\geq 0)$ is a CSBP with branching mechanism

$$\psi^{k\varepsilon,(i_0,i_1,..,i_k)}(\lambda)=\psi(\lambda)+c\lambda x_{i_k\delta}^{k\varepsilon,(i_0,..,i_{k-1})},$$

starting from v, given as the unique strong solution of the SDE

$$\begin{aligned}Z_t^{k\varepsilon,(i_0,..,i_k)}(v)&=v+\alpha\int_0^t Z_s^{k\varepsilon,(i_0,..,i_k)}(v)ds+\sigma\int_0^t\int_0^{Z_{s-}^{k\varepsilon,(i_0,..,i_k)}(v)}W^{k\varepsilon,(i_0,..,i_k)}(ds,du)\\&+\int_0^t\int_0^{Z_{s-}^{k\varepsilon,(i_0,..,i_k)}(v)}\int_1^\infty rN^{k\varepsilon,(i_0,..,i_k)}(ds,d\nu,dr)\\&+\int_0^t\int_0^{Z_{s-}^{k\varepsilon,(i_0,..,i_k)}(v)}\int_0^1 r\tilde{N}^{k\varepsilon,(i_0,..,i_k)}(ds,d\nu,dr)\\&-cx_{i_k\delta}^{k\varepsilon,(i_0,..i_{k-1})}\int_0^t Z_s^{k\varepsilon,(i_0,..,i_k)}(v)ds,\end{aligned} \tag{3.75}$$

where

$$W^{k\varepsilon,(i_0,..,i_k)}(ds,du)=W(ds+k\varepsilon,du+Z_{k\varepsilon,s}(x^{k\varepsilon,(i_0,..,i_{k-1})}_{i_k\delta}))$$

is a white noise with intensity $ds\times du$ and

$$N^{k\varepsilon,(i_0,..,i_k)}(ds,d\nu,dr)=N(ds+k\varepsilon,d\nu+Z_{k\varepsilon,s}(x^{k\varepsilon,(i_0,..,i_{k-1})}_{i_k\delta}),dr)$$

is a Poisson random measure with intensity $ds\times d\nu\times\Pi(dr)$.

Remark 3.31. Using the Lévy characterization of Brownian motions, one can check that the white noise processes $W^{k\varepsilon,(i_0,..,i_k)}(ds,du)$ are independent when the indexes $(i_0,..,i_k)$ vary. Indeed, using the quadratic variations of local martingales given by stochastic integrals with respect to Gaussian white noise, each $W^{k\varepsilon,(i_0,..,i_k)}(ds\times[a,b])$ with $a\le b$ is seen to be a Brownian motion (of variance $b-a$) with respect to filtration generated by both W and N, and for different multi-indexes the covariation processes vanish since the integrals are disjointly supported processes. A similar argument can be used for the Poisson integrals. The independence between white and Poisson noise integrals can be checked by an extension of such arguments using Itô calculus to identify the joint characteristic functions of the stochastic integrals (see e.g. [24, Thm. 2.1]).

We will roughly refer to the above processes $(Z^{\varepsilon,\delta}_t(w):t\ge0)$,$w\ge0$ as the "grid approximation of the LBP". We can easily deduce an SDE for each of them

Lemma 3.32. *For every $w\le x$, the process $Z^{\varepsilon,\delta}(w)=(Z^{\varepsilon,\delta}_t(w):t\ge0)$ solves the following stochastic differential equation:*

$$\begin{aligned}Z^{\varepsilon,\delta}_t(w)&=w+\alpha\int_0^t Z^{\varepsilon,\delta}_s(w)ds+\sigma\int_0^t\int_0^{Z^{\varepsilon,\delta}_{s-}(w)}W(ds,du)+\int_0^t\int_0^{Z^{\varepsilon,\delta}_{s-}(w)}\int_1^\infty rN(ds,d\nu,dr)\\&+\int_0^t\int_0^{Z^{\varepsilon,\delta}_{s-}(w)}\int_0^1 r\tilde N(ds,d\nu,dr)-c\sum_{i_0=0}^{n_0^x}\cdots\sum_{i_{k_t}=0}^{n^{(..)}_{k_t\varepsilon}}\int_0^t x^{k_s\varepsilon,(i_0,..,i_{k_s-1})}_{i_{k_s}\delta}Z^{k_s\varepsilon,(i_0,..,i_{k_s})}_{s-k_s\varepsilon}\\&\Big((x^{k_s\varepsilon,(i_0,..i_{k_s-1})}_{(i_{k_s}+1)\delta}\wedge Z^{\varepsilon,\delta}_{k_s\varepsilon}(w))-(x^{k_s\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta}\wedge Z^{\varepsilon,\delta}_{k_s\varepsilon}(w))\Big)\,ds,\end{aligned}\tag{3.76}$$

where $\sigma\ge0$ and α are the same objects as in (3.1), $W(ds,du)$ is a white noise process on $(0,\infty)^2$ based on the Lebesgue measure $ds\times du$ and N is a Poisson random measure on $(0,\infty)^3$ with intensity $ds\times d\nu\times\Pi(dr)$. Moreover, by construction and properties of flows of CSBP, for each $t\ge0$, $w\mapsto Z^{\varepsilon,\delta}_t(w)$ is a non-negative and non-decreasing càdlàg process on $[0,\infty)$.

Proposition 3.33. *For each $x\ge0$, the process $(Z^{\epsilon,\delta}_a(x):a\ge0)$ and the process $(L^a_{T_x}(\epsilon,\delta):a\ge0)$ has the same law.*

Proof. For every $k\in\mathbb{N}$ and $(i_0,i_1,..i_k)\in\mathbb{N}^*$, it is immediate from Proposition 3.20 to see that the process

$$\Big(Z_{k\varepsilon,a}(x^{k\varepsilon,(i_0,..,i_{k-1})}_{(i_k+1)\delta})-Z_{k\varepsilon,a}(x^{k\varepsilon,(i_0,..,i_{k-1})}_{i_k\delta}):k\varepsilon<a\le(k+1)\varepsilon\Big)$$

behaves as the approximated local time process

$$(L^a_{T^{k\varepsilon,(i_0,..,i_{k-1})}_{(i_k+1)\delta}}(\varepsilon,\delta) - L^a_{T^{k\varepsilon,(i_0,..,i_{k-1})}_{(i_k)\delta}}(\varepsilon,\delta) : k\varepsilon < a \leq (k+1)\varepsilon).$$

Furthermore, that result also tells us that, at each height a, the process $L^a_{T_x}(\varepsilon,\delta)$ is obtained as a sum of branching processes defined rectangle by rectangle in the tree-like height/local time discrete grid, whose branching mechanisms are determined by adding to ψ the constant c times the cumulative population in the lower left corner of the rectangle. Thus, it becomes clear form the independence of the noises driving the CSBP stated in Remark 3.31 in each block of the grid and the construction of the flow $Z^{\epsilon,\delta}$, that the latter is " an embedding" of $L(\varepsilon,\delta)$ in the flow framework, so that the law of the process $(Z^{\epsilon,\delta}_a(x) : a \geq 0)$ is the same as the process $(L^a_{T_x} : a \geq 0)$. ■

The following comparison property will be useful in the sequel.

Lemma 3.34 (Comparison property)**.** *For all $\varepsilon,\delta \geq 0$ and $0 \leq v \leq w$, the solution $(Z^{\varepsilon,\delta}_t(v) : t \geq 0)$ of equation (3.76) and the solution $(Z_t(v) : t \geq 0)$ of equation (3.68) satisfy*

$$\mathbb{P}\{Z^{\varepsilon,\delta}_t(v) \leq Y_t(w)\ \forall t \geq 0\} = 1 \quad \textit{and} \quad \mathbb{P}\{Z_t(v) \leq Y_t(w)\ \forall t \geq 0\} = 1, \tag{3.77}$$

where $(Y_t(w) : t \geq 0)$ is the solution of equation (3.67). In both cases we say that the "comparison property" holds.

Proof. The comparison property for the logistic process Z follows directly from [18, Theorem 2.2]. For the grid approximation $Z^{\varepsilon,\delta}$ of the LBP, the same result implies the comparison property for each CSBP $(Z^{k\varepsilon,(i_0,i_1,..i_k)}_t(v) : t \geq 0)$ with branching mechanism

$$\psi^{k\varepsilon,(i_0,i_1,..,i_k)}(\lambda) = \psi(\lambda) + c\lambda x^{k\varepsilon,(i_0,..,i_{k-1})}_{i_k\delta}$$

and initial condition v, with respect to a flow of CSBPs with mechanism $\psi(\lambda)$ driven by the same noise processes, and starting from initial conditions $w \geq v$. Since $Z^{\varepsilon,\delta}_t$ is defined in each band $k\varepsilon \leq t(k+1)\varepsilon$ as a sum over indexes $(i_0,i_1,..,i_k)$ of the above processes, by an inductive argument in k one gets the desired comparison property. ■

We prove know that the process $(Z^{\varepsilon,\delta}_t(x) : t \geq 0)$ is actually a grid approximation of the LBP $(Z_t(x) : t \geq 0)$, in the sense of the following proposition.

Proposition 3.35 (Convergence of the grid approximation of the LBP)**.** *For each $x \geq 0$ and $t \geq 0$, the r.v. $Z^{\varepsilon,\delta}_t(x)$ converges in probability to $Z_t(x)$. In particular, the process $(Z^{\varepsilon,\delta}_a(x) : x \geq 0, a \geq 0)$ converges to $(Z_a(x) : x \geq 0, a \geq 0)$ in the sense of finite dimensional distributions.*

We introduce the filtration

$$\left(\mathcal{S}^{(x)}_t\right)_{t\geq 0} = \left(\bigvee_{k=0}^{k_t} \sigma\left(Z_{k\varepsilon,s}(h), h \leq Z^{\varepsilon,\delta}_{k\varepsilon}(x), k\varepsilon < s \leq (k+1)\varepsilon \wedge t\right)\right)_{t\geq 0},$$

where $(Z_{k\varepsilon,s}(h) : s \geq k\varepsilon)$ is the process defined in equation (3.73), and we define $(\mathcal{S}_t)_{t\geq 0}$ by

$$\mathcal{S}_t := \bigcup_{x\geq 0} \mathcal{S}^{(x)}_t.$$

To unburden the proof of the above proposition, we prove first a technical lemma.

Lemma 3.36. *Set $\tau_m = \inf\{t \geq 0 : Y_t(x) > m\}$, $m \geq 1$. We have then that*

$$\mathbb{E}\left[\mathbf{1}_{\{s\leq\tau_m\}} \sup_{0\leq v\leq Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} |Z_{k_s\varepsilon,s}(v) - v|\right] \leq m\varepsilon\left(|\alpha| + \int_1^\infty r\Pi(dr)\right) + cm^2\varepsilon$$
$$+ \sqrt{m\varepsilon}C_1\left(\sigma + \sqrt{\int_0^1 r^2\Pi(dr)}\right),$$

where $C_1 > 0$ is an universal constant.

Proof. Using (3.73), we have that

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{s\leq\tau_m\}} \sup_{0\leq v\leq Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} |Z_{k_s\varepsilon,s}(v) - v|\right] \leq &|\alpha|\mathbb{E}\left[\mathbf{1}_{\{s\leq\tau_m\}}\int_{k_s\varepsilon}^s Z^{\varepsilon,\delta}_\theta(x)d\theta\right] + \sigma\mathbb{E}\left[\sup_{0\leq v\leq Z^{\varepsilon,\delta}_{k_{s\wedge\tau_m}\varepsilon}(x)} |M^W_v(s\wedge\tau_m)|\right]\\
&+ \mathbb{E}\left[\mathbf{1}_{\{s\leq\tau_m\}}\int_{k_s\varepsilon}^s\int_0^{Z^{\varepsilon,\delta}_{\theta-}(x)}\int_1^\infty rN(d\theta,d\nu,dr)\right]\\
&+ \mathbb{E}\left[\sup_{0\leq v\leq Z^{\varepsilon,\delta}_{k_{s\wedge\tau_m}\varepsilon}(x)} |M^N_v(s\wedge\tau_m)|\right]\\
&+ c\mathbb{E}\left[\mathbf{1}_{\{s\leq\tau_m\}}Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)\int_{k_s\varepsilon}^s Z^{\varepsilon,\delta}_\theta(x)d\theta\right],\end{aligned} \tag{3.78}$$

where the processes $(M^W_v(s\wedge\tau_m))_{v\geq0}$ and $(M^N_v(s\wedge\tau_m))_{v\geq0}$, defined as

$$M^W_v(s) = \int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m}\int_0^{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)} W(d\theta,du) \quad\text{and}\quad M^N_v(s) = \int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m}\int_0^{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)}\int_0^1 r\tilde{N}(d\theta,d\nu,dr),$$

respectively are ("vertical") martingales issued from 0. Indeed, we have

$$\begin{aligned}&\mathbb{E}\left[\int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m}\int_0^{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v+h)} W(d\theta,du)\middle|\mathcal{S}^{k_s\varepsilon,(v)}_s\right]\\
&\quad= \mathbb{E}\left[\int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m}\int_0^{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)} W(d\theta,du)\middle|\mathcal{S}^{k_s\varepsilon,(v)}_s\right] + \mathbb{E}\left[\int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m}\int_{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)}^{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v+h)} W(d\theta,du)\middle|\mathcal{S}^{k_s\varepsilon,(v)}_s\right]\\
&\quad= \int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m}\int_0^{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)} W(d\theta,du) + \mathbb{E}\left[\int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m}\int_{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)}^{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v+h)} W(d\theta,du)\right]\\
&\quad= \int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m}\int_0^{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)} W(d\theta,du),\end{aligned}$$

where the second equality holds by Remark 3.31. By Ito's formula in the time variable s, we see also that

$$\begin{aligned}\mathbb{E}\left[M^W_v(s\wedge\tau_m)^2\right] &= 2\mathbb{E}\left[\int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m}\int_0^{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)} M^W_u(\theta)W(d\theta,du)\right] + \mathbb{E}\left[\int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m} Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)d\theta\right]\\
&= \mathbb{E}\left[\int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m} Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)d\theta\right],\end{aligned}$$

and we deduce that

$$\mathbb{E}\left[\left[M^W_\cdot(s\wedge\tau_m), M^W_\cdot(s\wedge\tau_m)\right]_v\right] = \mathbb{E}\left[\int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m} Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)d\theta\right]$$

by definition of the quadratic variation of $(M^W_v(s\wedge\tau_m))_{v\geq 0}$. Thus, we bound the supremum of M^W using Burkholder-Davis-Gundy's inequality (again in the "vertical" sense)

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq v\leq Z^{\varepsilon,\delta}_{k_{s\wedge\tau_m}\varepsilon}(x)} |M^W_v(s\wedge\tau_m)|\right] &\leq C_1\mathbb{E}\left[\sqrt{[M^W_\cdot(s\wedge\tau_m), M^W_\cdot(s\wedge\tau_m)]_{Z^{\varepsilon,\delta}_{k_{s\wedge\tau_m}\varepsilon}(x)}}\right]\\ &\leq C_1\sqrt{\mathbb{E}\left[\int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m} Z^{\varepsilon,\delta}_\theta(x)d\theta\right]},\end{aligned} \tag{3.79}$$

with $C_1 > 0$ a universal constant. In a similar way, we check that

$$\mathbb{E}\left[\int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m}\int_0^{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v+h)}\int_0^1 r\tilde{N}(d\theta,d\nu,dr)\middle|\mathcal{S}^{k_s\varepsilon,(v)}_s\right] = \int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m}\int_0^{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)}\int_0^1 r\tilde{N}(d\theta,d\nu,dr),$$

and

$$\begin{aligned}\mathbb{E}\left[M^{\tilde{N}}_v(s\wedge\tau_m)^2\right] &= 2\mathbb{E}\left[\int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m}\int_0^{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)}\int_0^1 M^{\tilde{N}}_u(\theta)\tilde{N}(d\theta,d\nu,dr)\right]\\ &\quad+\mathbb{E}\left[\sum_{k_{s\wedge\tau_m}\varepsilon<s_n\leq s\wedge\tau_m} r_n^2\mathbf{1}_{\{0\leq r_n\leq 1\}}\mathbf{1}_{\{\nu_n\leq Z_{k_{s\wedge\tau_m}\varepsilon,s_n}(v)\}}\right]\\ &=\mathbb{E}\left[\int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m}\int_0^{Z_{k_{s\wedge\tau_m}\varepsilon,\theta}(v)}\int_0^1 r^2\Pi(dr)d\theta d\nu\right].\end{aligned}$$

by Itô's formula. Thus, applying Burkholder-Davis-Gundy's inequality we have that

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq v\leq Z^{\varepsilon,\delta}_{k_{s\wedge\tau_m}\varepsilon}(x)} |M^{\tilde{N}}_v(s\wedge\tau_m)|\right] &\leq C_1\mathbb{E}\left[\sqrt{\left[M^{\tilde{N}}_\cdot(s\wedge\tau_m), M^{\tilde{N}}_\cdot(s\wedge\tau_m)\right]_{Z^{\varepsilon,\delta}_{k_{s\wedge\tau_m}\varepsilon}(x)}}\right]\\ &\leq C_1\sqrt{\mathbb{E}\left[\int_{k_{s\wedge\tau_m}\varepsilon}^{s\wedge\tau_m} Z^{\varepsilon,\delta}_\theta(x)d\theta\int_0^1 r^2\Pi(dr)\right]}.\end{aligned} \tag{3.80}$$

Using expressions (3.79),(3.80) to bound equation (3.78) we obtain then that

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{s\leq\tau_m\}}\sup_{0\leq v\leq Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)}|Z_{k_s\varepsilon,s}(v)-v|\right]\leq&|\alpha|\mathbb{E}\left[\mathbf{1}_{\{s\leq\tau_m\}}\int_{k_s\varepsilon}^{s}Z^{\varepsilon,\delta}_{\theta}(x)d\theta\right]\\&+\sigma C_1\sqrt{\mathbb{E}\left[\mathbf{1}_{\{s\leq\tau_m\}}\int_{k_s\varepsilon}^{s}Z^{\varepsilon,\delta}_{\theta}(x)d\theta\right]}\\&+\mathbb{E}\left[\mathbf{1}_{\{s\leq\tau_m\}}\int_{k_s\varepsilon}^{s}Z^{\varepsilon,\delta}_{\theta-}(x)d\theta\int_1^\infty r\Pi(dr)\right]\\&+C_1\sqrt{\mathbb{E}\left[\mathbf{1}_{\{s\leq\tau_m\}}\int_{k_s\varepsilon}^{s}Z^{\varepsilon,\delta}_{\theta}(x)d\theta\int_0^1 r^2\Pi(dr)\right]}\\&+c\mathbb{E}\left[\mathbf{1}_{\{s\leq\tau_m\}}Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)\int_{k_s\varepsilon}^{s}Z^{\varepsilon,\delta}_{\theta}(x)d\theta\right],\end{aligned}$$

and we get the desired bound using the comparison property 3.34. ■

Proof of Proposition 3.35. To establish the desired convergence we adapt computations in [25]. Given $\varepsilon,\delta\geq 0$, and $t,x\geq 0$, we set $\zeta^{\varepsilon,\delta}_t(x):=Z_t(x)-Z^{\varepsilon,\delta}_t(x)$. We have then $\zeta^{\varepsilon,\delta}_t(x)$ satisfies the SDE

$$\begin{aligned}\zeta^{\varepsilon,\delta}_t(x)=&\alpha\int_0^t\left(Z_s(x)-Z^{\varepsilon,\delta}_s(x)\right)ds+\sigma\int_0^t\int_0^\infty\left(\mathbf{1}_{\{\nu<Z_{s-}(x)\}}-\mathbf{1}_{\{\nu<Z^{\varepsilon,\delta}_{s-}(x)\}}\right)W(ds,du)\\&+\int_0^t\int_0^\infty\int_1^\infty\left(\mathbf{1}_{\{\nu<Z_{s-}(x)\}}-\mathbf{1}_{\{\nu<Z^{\varepsilon,\delta}_{s-}(x)\}}\right)rN(ds,d\nu,dr)\\&+\int_0^t\int_0^\infty\int_0^1\left(\mathbf{1}_{\{\nu<Z_{s-}(x)\}}-\mathbf{1}_{\{\nu<Z^{\varepsilon,\delta}_{s-}(x)\}}\right)r\tilde{N}(ds,d\nu,dr)\\&-\frac{c}{2}\int_0^t\left[{Z_s}^2(x)-Z^{\varepsilon,\delta 2}_s(x)\right]ds-\frac{c}{2}\int_0^t Z^{\varepsilon,\delta 2}_s(x)ds\\&+c\sum_{i_0=0}^{n_0^x}..\sum_{i_{k_t}=0}^{n^{(i_0,..i_{k-1})}_{k_t\varepsilon}}\int_0^t x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta}Z^{\varepsilon,(i_0,..,i_{k_s})}_{s-k_s\varepsilon}(x^{\varepsilon,(i_0,..i_{k_s-1})}_{(i_{k_s}+1)\delta}\wedge Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)-x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta})ds\end{aligned}\tag{3.81}$$

We first notice that

$$\int_0^t\left[Z^{\varepsilon,\delta}_s(x)\right]^2ds=2\int_0^t\int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)}Z_{k_s\varepsilon,s}(v)d_vZ_{k\varepsilon,s}(v)ds,$$

and

$$\begin{aligned}Z^{\varepsilon,(i_0,..,i_{k_s})}_{s-k_s\varepsilon}(x^{\varepsilon,(i_0,..i_{k_s-1})}_{(i_{k_s}+1)\delta}\wedge Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)-x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta})&=Z_{k_s\varepsilon,s}(x^{\varepsilon,(i_0,..i_{k_s-1})}_{(i_{k_s}+1)\delta}\wedge Z^{\varepsilon,\delta}_{k_s\varepsilon}(x))-Z_{k_s\varepsilon,s}(x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta})\\&=\int_{x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta}}^{x^{\varepsilon,(i_0,..i_{k_s-1})}_{(i_{k_s}+1)\delta}\wedge Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)}d_vZ_{k_s\varepsilon,s}(v).\end{aligned}$$

Replacing these expressions in the two last terms in the right-hand side of the equation (3.81), we obtain that

$$\begin{aligned}
\zeta_t^{\varepsilon,\delta}(x) &= \alpha \int_0^t \left(Z_s(x) - Z_s^{\varepsilon,\delta}(x)\right) ds + \sigma \int_0^t \int_0^\infty \left(\mathbf{1}_{\{\nu < Z_{s-}(x)\}} - \mathbf{1}_{\{\nu < Z_{s-}^{\varepsilon,\delta}(x)\}}\right) W(ds, du) \\
&+ \int_0^t \int_0^\infty \int_1^\infty \left(\mathbf{1}_{\{\nu < Z_{s-}(x)\}} - \mathbf{1}_{\{\nu < Z_{s-}^{\varepsilon,\delta}(x)\}}\right) r N(ds, d\nu, dr) \\
&+ \int_0^t \int_0^\infty \int_0^1 \left(\mathbf{1}_{\{\nu < Z_{s-}(x)\}} - \mathbf{1}_{\{\nu < Z_{s-}^{\varepsilon,\delta}(x)\}}\right) r \tilde{N}(ds, d\nu, dr) \\
&- \frac{c}{2} \int_0^t \left[Z_s(x) - Z_s^{\varepsilon,\delta}(x)\right] \left[Z_s(x) + Z_s^{\varepsilon,\delta}(x)\right] ds \\
&- c \sum_{i_0=0}^{n_0^x} .. \sum_{i_{k_t}=0}^{n_{k_t\varepsilon}^{(i_0,..i_{k-1})}} \int_0^t \int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} \mathbf{1}_{\{x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta} < v \leq x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta}\}} \\
&\left[Z_{k_s\varepsilon,s}(v) - x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta}\right] d_v Z_{k_s\varepsilon,s}(v) ds.
\end{aligned} \tag{3.82}$$

For what follows, we use some notation developed in the proof of [25, Thm 5.4]. Let us define

- the constant $K := |\alpha| + M$, where $\int_1^\infty r\Pi(dr) = M < \infty$. Observe that
$$|\alpha x| + \int_0^\infty \int_1^\infty r\mathbf{1}_{\{\nu<x\}} d\nu \Pi(dr) \leq K(x+1).$$
- the function $U(x) := (\sigma^2 + I)(x)$, where $I = \int_0^1 r^2\Pi(dr)$. Then, U satisfies
$$\sigma^2 x + \int_0^\infty \int_0^1 r^2 \mathbf{1}_{\{\nu<x\}} d\nu \Pi(dr) \leq U(x).$$
- the function $\beta(z) := (|\alpha+1| + M)z$, which satisfies $\int_{0_+} \beta(z)^{-1}dz = \infty$. If we suppose also without losing generality that $y \leq x$, we have
$$|(\alpha+1)(x-y)| + \int_0^\infty \int_1^\infty r\mathbf{1}_{\{y<\nu<x\}} d\nu \Pi(dr) \leq \beta(x-y). \tag{3.83}$$
- the function $\varrho(x) := [\sigma^2 + I]\sqrt{x}$, where $I = \int_0^1 r^2\Pi(dr)$. Note that, if $y \leq x$, then
$$\sigma^2(\sqrt{x} - \sqrt{y})^2 + \int_0^\infty \int_0^1 r^2 \mathbf{1}_{\{y<\nu<x\}} d\nu \Pi(dr) \leq \varrho(x-y). \tag{3.84}$$

We fix then a sequence $\{a_j\}_{j\geq 1}$ such that $a_j = a_{j-1}\, e^{-j[\sigma^2+I]^2}$ and $a_0 = 1$. Note that $a_j \to 0_+$ decreasingly and $\int_{a_j}^{a_{j-1}} \varrho(z)^{-2}dz = j$ for $j \geq 1$.Thus, let $z \mapsto \psi_j(z)$ be a non-negative continuous function on $\mathbb{R}$ which has support in (a_j, a_{j-1}), satisfies $0 \leq \psi_j(z) \leq 2k^{-1}\varrho(z)^{-2}$ for $a_j < z < a_{j-1}$, and $\int_{a_j}^{a_{j-1}} \psi_j(z)dz = 1$. For each $j \geq 1$, we define the non-negative and twice continuously differentiable function

$$\phi_j(x) = \int_0^{|x|} dy \int_0^y \psi_j(z)dz, \quad x \in \mathbb{R},$$

such that $\phi_j(x) \to |x|$ non-decreasingly as $j \to \infty$, and

$$\begin{cases} 0 \le \phi_j'(x) \le 1 & \text{if } x \ge 0 \\ -1 \le \phi_j'(x) \le 0 & \text{if } x < 0 \end{cases}. \tag{3.85}$$

We have also that $\phi_j''(x) \ge 0$ for $x \in \mathbb{R}$, and

$$\phi_j''(x-y)[\sigma\sqrt{x} - \sigma\sqrt{y}]^2 \to 0$$

when $j \to \infty$, uniformly in x, y. Furthermore,

$$0 \le \int_0^\infty \int_1^\infty D_{l(r,\nu;x,y)}\phi_j(x-y) d\nu \Pi(dr) \le \frac{1}{j[\sigma^2 + I]} \int_0^1 r^2 \Pi(dr) \to_{j\to\infty} 0,$$

uniformly in $x, y \ge 0$, where $l(r,\nu;,x,y) = r\left[\mathbf{1}_{\{\nu<x\}} - \mathbf{1}_{\{\nu<y\}}\right]$.

Notice that $\zeta^{\varepsilon,\delta}_{t\wedge\tau_m}(x) \le 2m$ for each $m \in \mathbb{N}$, by Lemma 3.34. By (3.82) and Itô's formula, we have that

$$\begin{aligned}
\phi_j(\zeta^{\varepsilon,\delta}_{t\wedge\tau_m}(x)) = a &\int_0^{t\wedge\tau_m} \phi_j'(\zeta^{\varepsilon,\delta}_s(x)) \left(Z_s(x) - Z^{\varepsilon,\delta}_s(x)\right) ds \\
&+ \frac{1}{2}\int_0^{t\wedge\tau_m} \sigma^2 \phi_j''(\zeta^{\varepsilon,\delta}_s(x)) \left[\sqrt{Z_s(x)} - \sqrt{Z^{\varepsilon,\delta}_s(x)}\right]^2 ds \\
&+ \int_0^{t\wedge\tau_m} \int_0^\infty \sigma\phi_j'(\zeta^{\varepsilon,\delta}_s(x)) l(Z_s(x), Z^{\varepsilon,\delta}_s(x)) W(ds,du) \\
&+ \int_0^{t\wedge\tau_m} \int_0^\infty \int_1^\infty \phi_j'(\zeta^{\varepsilon,\delta}_s(x)) l(Z_s(x), Z^{\varepsilon,\delta}_s(x)) N(ds,d\nu,dr) \\
&+ \int_0^{t\wedge\tau_m} \int_0^\infty \int_0^1 \phi_j'(\zeta^{\epsilon,\delta}_s(x)) l(Z_s(x), Z^{\varepsilon,\delta}_s(x)) \tilde{N}(ds,d\nu,dr) \\
&+ \sum_{s\le t\wedge\tau_m} \left[\phi_k(\zeta^{\varepsilon,\delta}_s(x)) - \phi_k(\zeta^{\varepsilon,\delta}_{s^-}(x)) - \phi_j'(\zeta^{\varepsilon,\delta}_{s^-}(x))\Delta\zeta^{\varepsilon,\delta}_s(x)\right] \\
&- \frac{c}{2}\int_0^{t\wedge\tau_m} \phi_j'(\zeta^{\varepsilon,\delta}_s(x)) \left[Z_s(x) - Z^{\varepsilon,\delta}_s(x)\right] \left[Z_s(x) + Z^{\varepsilon,\delta}_s(x)\right] ds \\
&- c\sum_{i_0=0}^{n_0^x} .. \sum_{i_{k_t}=0}^{n_{k_t\varepsilon}^{(i_0,..i_{k_t-1})}} \int_0^{t\wedge\tau_m} \int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} \phi_j'(\zeta^{\varepsilon,\delta}_s(x)) \\
&\mathbf{1}_{\left\{x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta} < v \le x^{\varepsilon,(i_0,..i_{k_s-1})}_{(i_{k_s}+1)\delta}\right\}} \left[Z_{k_s\varepsilon,s}(v) - x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta}\right] d_v Z_{k_s\varepsilon,s}(v) ds
\end{aligned}$$

and so

$$
\begin{aligned}
\phi_j(\zeta^{\varepsilon,\delta}_{t\wedge\tau_m}(x)) = a\int_0^{t\wedge\tau_m} & \phi_j'(\zeta^{\varepsilon,\delta}_s(x))\left(Z_s(x)-Z^{\epsilon,\delta}_s(x)\right)ds \\
&+\frac{1}{2}\int_0^{t\wedge\tau_m}\sigma^2\phi_j''(\zeta^{\varepsilon,\delta}_s(x))\left[\sqrt{Z_s(x)}-\sqrt{Z^{\varepsilon,\delta}_s(x)}\right]^2 ds \\
&+\sum_{s\le t}\left[\phi_k(\zeta^{\varepsilon,\delta}_s(x))-\phi_k(\zeta^{\varepsilon,\delta}_{s^-}(x))-\phi_j'(\zeta^{\varepsilon,\delta}_{s^-}(x))\Delta\zeta^{\epsilon,\delta}_s(x)\right] \\
&+\int_0^{t\wedge\tau_m}ds\int_0^\infty\int_0^1 \triangle_{l(r,\nu;Z_{s^-}(x),Z^{\varepsilon,\delta}_{s^-}(x)}\phi_k(\zeta^{\varepsilon,\delta}_{s^-}(x))d\nu\Pi(dr) \\
&+\int_0^{t\wedge\tau_m}ds\int_0^\infty D_{l(r,\nu;Z_{s^-}(x),Z^{\varepsilon,\delta}_{s^-}(x))}\phi_k(\zeta^{\varepsilon,\delta}_{s^-})d\nu\Pi(dr)+\hat{M}_{t\wedge\tau_m} \\
&-\frac{c}{2}\int_0^{t\wedge\tau_m}\phi_j'(\zeta^{\varepsilon,\delta}_s(x))\left[Z_s(x)-Z^{\varepsilon,\delta}_s(x)\right]\left[Z_s(x)+Z^{\varepsilon,\delta}_s(x)\right]ds \\
&-c\sum_{i_0=0}^{n_0^x}\cdots\sum_{i_{k_t}=0}^{n_{k_t\varepsilon}^{(i_0,..i_{k_t-1})}}\int_0^{t\wedge\tau_m}\int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)}\phi_j'(\zeta^{\varepsilon,\delta}_s(x)) \\
&\mathbf{1}_{\{x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta}<v\le x^{\varepsilon,(i_0,..i_{k_s-1})}_{(i_{k_s}+1)\delta}\}}\left[Z_{k_s\varepsilon,s}(v)-x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta}\right]d_vZ_{k_s\varepsilon,s}(v)ds,
\end{aligned}
\tag{3.86}
$$

where $\triangle_h f(z):=f(z+h)-f(z)$ and $(\hat{M}_{t\wedge\tau_m})$ is a $(\mathcal{S}_t)$ - martingale. By the properties of ϕ_j, we see that

$$\phi_j'(\zeta^{\varepsilon,\delta}_{s^-}(x))\alpha|Z_s(x)-Z^{\varepsilon,\delta}_s(x)|\le|\alpha||Z_s(x)-Z^{\varepsilon,\delta}_s(x)|,$$

and

$$\int_0^\infty\int_1^\infty\triangle_{l(r,\nu;Y_{s^-}(x),Z^{\varepsilon,\delta}_{s^-}(x))}\phi_j(\zeta^{\varepsilon,\delta}_{s^-}(x))d\nu\Pi(dr)\le\int_0^\infty\int_1^\infty r|\mathbf{1}_{\{\nu<Z_s(x)\}}-\mathbf{1}_{\{\nu<Z^{\varepsilon,\delta}_s(x)\}}|\Pi(dr)d\nu,$$

so we can deduce that

$$\phi_j''(\zeta^{\varepsilon,\delta}_s(x))\sigma^2\left[\sqrt{Z_s(x)}-\sqrt{Z^{\varepsilon,\delta}_s(x)}\right]^2\to_{j\to 0}0,$$

and

$$\int_0^\infty\int_0^1 D_{l(r,\nu;Y_{s^-}(x),Z^{\varepsilon,\delta}_{s^-}(x))}\phi_j(\zeta^{\varepsilon,\delta}_{s^-}(x))d\nu\Pi(dr)\to_{j\to 0}0,$$

uniformly on the event $\{s\le\tau_m\}$. Taking expectation in (3.86) and letting $j\to\infty$, we see that

$$
\begin{aligned}
\mathbb{E}|\zeta^{\varepsilon,\delta}_{t\wedge\tau_m}(x)|\le\mathbb{E}\left[\int_0^t(|\alpha|+M)|\zeta^{\varepsilon,\delta}_{s\wedge\tau_m}(x)|ds\right]&+\frac{c}{2}\mathbb{E}\left[\int_0^{t\wedge\tau_m}|\zeta^{\varepsilon,\delta}_{s\wedge\tau_m}(x)|\left[Z_s(x)+Z^{\varepsilon,\delta}_s(x)\right]ds\right] \\
+c\mathbb{E}\Bigg[\Bigg|\sum_{i_0=0}^{n_0^x}\cdots\sum_{i_{k_t}=0}^{n_{k_t\varepsilon}^{(i_0,..i_{k_t-1})}}&\int_0^{t\wedge\tau_m}\int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)}\mathbf{1}_{\{x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta}<v\le x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta}\}} \\
\left(Z_{k_s\varepsilon,s}(v)-x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta}\right)&d_vZ_{k_s\varepsilon,s}(v)ds\Bigg|\Bigg]
\end{aligned}
$$

from where

$$\begin{aligned}\mathbb{E}|\zeta^{\varepsilon,\delta}_{t\wedge\tau_m}(x)| \leq \mathbb{E}\left[\int_0^t (|\alpha|+M+cm)|\zeta^{\varepsilon,\delta}_{s\wedge\tau_m}(x)|ds\right] + c\mathbb{E}\Bigg[\sum_{i_0=0}^{n_0^x}\cdot\cdot\sum_{i_{k_t}=0}^{n^{(i_0,..i_{k_t-1})}_{k_t\varepsilon}}\int_0^{t\wedge\tau_m} \\ \int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} \mathbf{1}_{\{x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta}<v\leq x^{\varepsilon,(i_0,..i_{k_s-1})}_{(i_{k_s}+1)\delta}\}}\left|v - x^{\varepsilon,(i_0,..i_{k_s-1})}_{i_{k_s}\delta}\right| d_vZ_{k_s\varepsilon,s}(v)ds\Bigg] \\ + \mathbb{E}\left[\left|\int_0^{t\wedge\tau_m}\int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} (Z_{k_s\varepsilon,s}(v)-v)\, d_vZ_{k_s\varepsilon,s}(v)ds\right|\right].\end{aligned} \tag{3.87}$$

By (3.74), we see that $\left|v - x^{\varepsilon,(i_0,..i_{k-1})}_{i_{k_s}\delta}\right| \leq \delta$. Thus,

$$\begin{aligned}\mathbb{E}|\zeta^{\varepsilon,\delta}_{t\wedge\tau_m}(x)| &\leq \mathbb{E}\left[\int_0^t (|\alpha|+M+cm)|\zeta^{\varepsilon,\delta}_{s\wedge\tau_m}(x)|ds\right] + c\delta\mathbb{E}\left[\int_0^{t\wedge\tau_m} Z^{\varepsilon,\delta}_s(x)ds\right] \\ &\quad + \mathbb{E}\left|c\int_0^{t\wedge\tau_m}\int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} (Z_{k_s\varepsilon,s}(v)-v)d_vZ_{k_s\varepsilon,s}(v)ds\right| \\ &\leq \mathbb{E}\left[\int_0^{t\wedge\tau_m}(|\alpha|+M+cm)|\zeta^{\varepsilon,\delta}_{s\wedge\tau_m}(x)|ds\right] + cm\delta t \\ &\quad + c\mathbb{E}\left|\int_0^{t\wedge\tau_m}\int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} (Z_{k_s\varepsilon,s}(v)-v)d_vZ_{k_s\varepsilon,s}(v)ds\right|.\end{aligned} \tag{3.88}$$

By integration by parts, we have that

$$\begin{aligned}\int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} vd_vZ_{k_s\varepsilon,s}(v) &= vZ_{k_s\varepsilon,s}(v)\Big|_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} - \int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} Z_{k_s\varepsilon,s}(v)dv \\ &= Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)Z^{\varepsilon,\delta}_s(x) - \int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} Z_{k_s\varepsilon,s}(v)dv.\end{aligned}$$

Thus, the last term in the right-hand side of (3.88) can be rewritten as

$$\begin{aligned}c\mathbb{E}\left|\int_0^{t\wedge\tau_m}\int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} (Z_{k_s\varepsilon,s}(v)-v)d_vZ_{k_s\varepsilon,s}(v)ds\right| &= c\mathbb{E}\Bigg|\frac{1}{2}\int_0^{t\wedge\tau_m} Z^{\varepsilon,\delta}_s(x)^2ds - \int_0^{t\wedge\tau_m} Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)Z^{\varepsilon,\delta}_s(x)ds \\ &\quad + \int_0^{t\wedge\tau_m}\int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} Z_{k_s\varepsilon,s}(v)dvds\Bigg| \\ &\leq \frac{c}{2}\mathbb{E}\left|\int_0^{t\wedge\tau_m}\left(Z^{\varepsilon,\delta}_s(x) - Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)\right)^2 ds\right| \\ &\quad + c\mathbb{E}\left|\int_0^{t\wedge\tau_m}\int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} (Z_{k_s\varepsilon,s}(v)-v)\, dvds\right|,\end{aligned}$$

where in the last inequality we used the fact that

$$\frac{1}{2}\int_0^{t\wedge\tau_m} Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)^2ds = \int_0^{t\wedge\tau_m}\int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} vdvds.$$

Then, equation (3.88) yields

$$\begin{aligned}
\mathbb{E}|\zeta^{\varepsilon,\delta}_{t\wedge\tau_m}(x)| \le& \int_0^t (|\alpha|+M+cm)\mathbb{E}\left[|\zeta^{\varepsilon,\delta}_{s\wedge\tau_m}(x)|\right] ds + cm\delta t \\
&+ \frac{c}{2}\int_0^t \mathbb{E}\left[\mathbf{1}_{\{s\le\tau_m\}}\left|Z^{\varepsilon,\delta}_s(x) - Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)\right|\left(Z^{\varepsilon,\delta}_s(x) + Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)\right)\right] ds \\
&+ c\int_0^t \mathbb{E}\left[\mathbf{1}_{\{s\le\tau_m\}}\int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} |Z_{k_s\varepsilon,s}(v)-v|\,ds\right] ds \\
\le \int_0^t (|\alpha|+M+cm)&\mathbb{E}\left[|\zeta^{\varepsilon,\delta}_{s\wedge\tau_m}(x)|\right] ds + cm\delta t \\
&+ cm\int_0^t \mathbb{E}\left[\mathbf{1}_{\{s\le\tau_m\}}\left|Z^{\varepsilon,\delta}_s(x) - Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)\right|\right] ds \\
&+ c\int_0^t \mathbb{E}\left[\mathbf{1}_{\{s\le\tau_m\}}\int_0^{Z^{\varepsilon,\delta}_{k_s\varepsilon}(x)} |Z_{k_s\varepsilon,s}(v)-v|\,dv\right] ds
\end{aligned} \tag{3.89}$$

We use Lemma 3.36 to bound the last two terms in the right-hand side of equation above to obtain that

$$\begin{aligned}
\mathbb{E}|\zeta^{\varepsilon,\delta}_{t\wedge\tau_m}(x)| \le (|\alpha|+M+cm)\mathbb{E}\left[\int_0^t |\zeta^{\varepsilon,\delta}_{s\wedge\tau_m}(x)| ds\right] + cm\delta t + 2cm^2 t\varepsilon\left(|\alpha| + \int_1^\infty r\Pi(dr)\right) \\
+ 2cmt\left(cm^2\varepsilon + \sqrt{m\varepsilon}C_1\left(\sigma + \sqrt{\int_0^1 r^2\Pi(dr)}\right)\right).
\end{aligned}$$

Since $\zeta_s < 2m$ for $0 < s \le \tau_m$, we deduce that $t \mapsto \mathbb{E}|\zeta_{t\wedge\tau_m}|$ is locally bounded. Thus, we have that

$$\begin{aligned}
\mathbb{E}|\zeta^{\epsilon,\delta}_{t\wedge\tau_m}(x)| \le cmt&\left[\delta + 2m\varepsilon\left(|\alpha| + \int_1^\infty r\Pi(dr)\right) + 2cm^2\varepsilon\right. \\
&\left. + 2\sqrt{m\varepsilon}C_1\left(\sigma + \sqrt{\int_0^1 r^2\Pi(dr)}\right)\right] e^{(|\alpha|+M+cm)t}
\end{aligned}$$

by Gronwall's lemma, from where $\mathbb{E}|\zeta^{\epsilon,\delta}_{t\wedge\tau_m}(x)|$ goes to zero when $(\delta,\varepsilon)\to(0,0)$. Since $\tau_m\to\infty$ as $m\to\infty$ a.s., we have the desired result. ■

Finally, we can easily deduce the

Proof of Theorem 3.11. Given $x \ge 0$, by Proposition 3.21 we have the convergence in the sense of finite-dimensional laws of the process $(L^a_{T_x}(\varepsilon,\delta) : a \ge 0)$ to the process $(L^a_{T_x}(*) : a \ge 0)$. Analogously, by Proposition 3.35 we have convergence, in the sense of finite-dimensional distributions, of the process $(Z^{\varepsilon,\delta}_a(x) : a \ge 0)$ to the process $(Z_a(x) : a \ge 0)$. Then, thanks to Proposition 3.33, we obtain the desired result. ■

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