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# LARGE CONFORMAL METRICS WITH PRESCRIBED SIGN-CHANGING GAUSS CURVATURE AND A CRITICAL NEUMANN PROBLEM

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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**MÉTRICAS CONFORMES CON CURVATURA GAUSSIANA PRESCRITA CON CAMBIO DE SIGNO Y UN PROBLEMA DE NEUMANN CRÍTICO**

En esta memoria se estudian dos problemas semilineales elípticos clásicos en la literatura: el problema de la curvatura Gaussiana prescrita en dimensión 2, y el problema de Lin-Ni-Takagi con exponente crítico en dimensión 3. En ambos se encuentran soluciones con reviente cuando el valor de un parámetro involucrado se aproxima a cierto valor crítico.

En el primer capítulo se estudia el siguiente problema: Dada una función escalar $\kappa(x)$, suficientemente regular, definida en una variedad Riemanniana compacta $(M, g)$ de dimensión 2, se desea saber si $\kappa$ puede corresponder a la curvatura Gaussiana de $M$ para una métrica $g_1$, que es adicionalmente conforme a la métrica inicial $g$, es decir, $g_1 = e^u g$ para alguna función escalar $u$ en $M$. Sea $f$ una función regular en $M$ tal que

$$f \geq 0, \quad f \not\equiv 0, \quad \min_M f = 0.$$

Sean $p_1, \ldots, p_n$ una colección de puntos cualesquiera en los que $f(p_i) = 0$ y $D^2 f(p_i)$ es no singular. Se demuestra que para todo $\lambda > 0$ suficientemente pequeño, existe una familia de metricas conformes de tipo burbuja $g_\lambda = e^{u_\lambda} g$ tal que su curvatura Gaussiana está dada por la función que cambia de signo $K_{g_\lambda} = -f + \lambda^2$. Más aún, la familia $u_\lambda$ satisface

$$u_\lambda(p_j) = -4 \log \lambda - 2 \log \left( \frac{1}{\sqrt{2}} \log \frac{1}{\lambda} \right) + O(1), \quad \lambda^2 e^{u_\lambda} \rightharpoonup 8\pi \sum_{i=1}^{n} \delta_{p_i},$$

donde $\delta_p$ corresponde a la masa de Dirac en el punto $p$.

En el segundo capítulo se considera el problema

$$-\Delta u + \lambda u - u^5 = 0, \quad u > 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

donde $\Omega \subset \mathbb{R}^3$ es un dominio acotado con frontera regular $\partial\Omega$, $\lambda > 0$ and $\nu$ denota la normal unitaria exterior a $\partial\Omega$. Se demuestra que cuando $\lambda$ se apoxima por arriba a cierto valor explícitamente caracterizado en términos de funciones de Green, una familia de soluciones con reviente en un cierto punto interior del dominio existe.

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## LARGE CONFORMAL METRICS WITH PRESCRIBED SIGN-CHANGING GAUSS CURVATURE AND A CRITICAL NEUMANN PROBLEM


In this thesis we present a study of two semi-linear elliptic problems classical in the literature: the prescribed Gaussian curvature problem in dimension 2, and the Lin-Ni-Takagi problem with critical exponent in dimension 3. In both problems we find solutions with "bubbling" as a certain parameter involved in the problem approaches a critical value.

In the first chapter we study the following problem: Given a real-valued function $\kappa(x)$, sufficiently smooth, defined on a two dimensional compact Riemannian manifold $(M,g)$, we want to know if $\kappa$ can be realized as the Gaussian curvature of $M$ for a metric $g_1$, which is in addition conformal to $g$, namely, $g_1 = \mathrm{e}^u g$ for some scalar function $u$ on $M$. Let $f$ be a smooth function on $M$ such that

$$f \geq 0, \quad f \not\equiv 0, \quad \min_M f = 0.$$

Let $p_1, \dots, p_n$ be any set of points at which $f(p_\mathrm{i}) = 0$ and $D^2 f(p_\mathrm{i})$ is non-singular. We prove that for all sufficiently small $\lambda > 0$ there exists a family of "bubbling" conformal metrics $g_\lambda = \mathrm{e}^{u_\lambda} g$ such that their Gauss curvature is given by the sign-changing function $K_{g_\lambda} = -f + \lambda^2$. Moreover, the family $u_\lambda$ satisfies

$$u_\lambda(p_j) = -4\log\lambda - 2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{\lambda}\right) + O(1), \quad \lambda^2 \mathrm{e}^{u_\lambda} \rightharpoonup 8\pi \sum_{\mathrm{i}=1}^{n} \delta_{p_\mathrm{i}},$$

where $\delta_p$ designates Dirac mass at the point $p$.

In the second chapter we consider the problem

$$-\Delta u + \lambda u - u^5 = 0, \quad u > 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ and $\nu$ denotes the unit normal to $\partial\Omega$. We prove that when $\lambda$ approaches from a above a certain value, explicitly characterized in terms of Green's functions, a family of solutions with blow-up around an interior point of the domain exists.

*A mis padres.*

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# Contents

# Chapter 1

# Large conformal metrics with prescribed sign-changing Gauss curvature

## 1.1 Introduction

Let $(M, g)$ be a two-dimensional compact Riemannian manifold. We consider in this work the classical *prescribed Gaussian curvature problem*: Given a real-valued, sufficiently smooth funtion $\kappa(x)$ defined on $M$, we want to know if $\kappa$ can be realized as the Gaussian curvature $K_{g_1}$ of $M$ for a metric $g_1$, which is in addition conformal to $g$, namely $g_1 = \mathrm{e}^u g$ for some scalar function $u$ on $M$.

It is well known, by the uniformization theorem, that without loss of generality we may assume that $M$ has constant Gaussian curvature for $g$, $K_g =: -\alpha$. Besides, the relation $K_{g_1} = \kappa$ is equivalent to the following nonlinear partial differential equation

$$\Delta_g u + \kappa\,\mathrm{e}^u + \alpha = 0, \quad \text{in } M, \tag{1.1}$$

where $\Delta_g$ is the Laplace Beltrami operator on $M$. There is a considerable literature on necessary and sufficient conditions on the function $\kappa$ for the solvability of the PDE (1.1). We refer the reader in particular to the classical references [5, 9, 18, 19, 20, 25] and to [7] for a recent review of the state of the art for this problem.

Integrating equation (1.1), assuming that $M$ has surface area equal to one, and using the Gauss-Bonet formula we obtain

$$\int_M \kappa \mathrm{e}^u \mathrm{d}\mu_g = \int_M K_g \mathrm{d}\mu_g = -\alpha = 2\pi\chi(M), \tag{1.2}$$

where $\chi(M)$ is the Euler characteristic of the manifold $M$.

In what follows we shall assume that the surface $M$ has **genus** $g(M)$ **greater than one**, so that $\chi(M) = 2(1 - g(M)) < 0$ and hence

$$-K_g = \alpha > 0.$$

Then (1.2) tells us that a necessary condition for existence is that $\kappa(x)$ be negative somewhere on $M$. More than this, we must have that

$$\int_M \kappa \mathrm{d}\mu_g < 0.$$

Indeed testing equation (1.1) against $\mathrm{e}^{-u}$ we get

$$\int_M \kappa \mathrm{d}\mu_g = -\int_M (|\nabla_g u|^2 + \alpha)\mathrm{e}^{-u}\mathrm{d}\mu_g < 0. \tag{1.3}$$

Solutions $u$ to equation (1.2) correspond to critical points in the Sobolev space $H^1(M,g)$ of the energy functional

$$E_\kappa(u) = \frac{1}{2}\int_M |\nabla_g u|^2 \mathrm{d}\mu_g - \alpha\int_M u\mathrm{d}\mu_g - \int_M \kappa\mathrm{e}^u\mathrm{d}\mu_g.$$

As observed in [5], since $\alpha > 0$, we have that If $\kappa \le 0$ and $\kappa \not\equiv 0$, then this functional is strictly convex and coercive in $H^1(M,g)$. It thus have a unique critical point which is a global minimizer of $E_\kappa$.

A natural question to ask is what happens when $f$ changes sign. A drastic change in fact occurs. If $\sup_M \kappa > 0$, then the functional $E_\kappa$ is no longer bounded below, hence a global minimizer cannot exist. On the other hand, intuition would tell us that if $\kappa$ is “not too positive" on a set “not too big", then the global minimizer should persist in the form of a local minimizer. This is in fact true, and quantitative forms of this statement can be found in [3, 6].

We shall focus in what follows in a special class of functions $\kappa(x)$ which change sign being nearly everywhere negative. Let $f$ be a function of class $C^3(M)$ such that

$$f \ge 0, \quad f \not\equiv 0, \quad \min_M f = 0.$$

For $\lambda > 0$ we let

$$\kappa_\lambda(x) = -f(x) + \lambda^2.$$

so that our problem now reads

$$\Delta_g u - f\mathrm{e}^u + \lambda^2\mathrm{e}^u + \alpha = 0, \quad \text{in } M. \tag{1.4}$$

In [15], Ding and Liu proved that the global minimizer of $E_{\kappa_0}$ persists as a local minimizer $\underline{u}_\lambda$ of $E_{\kappa_\lambda}$ for any $0 < \lambda < \lambda_0$. From (1.3) we see that

$$\lambda_0 < \left(\int_M f\right)^{1/2}.$$

Moreover, they established the existence of a second, non-minimizing solution $u_\lambda$ in this range. Uniqueness of the solution $u_0$ for $\lambda = 0$, and its minimizing character, tell us that we

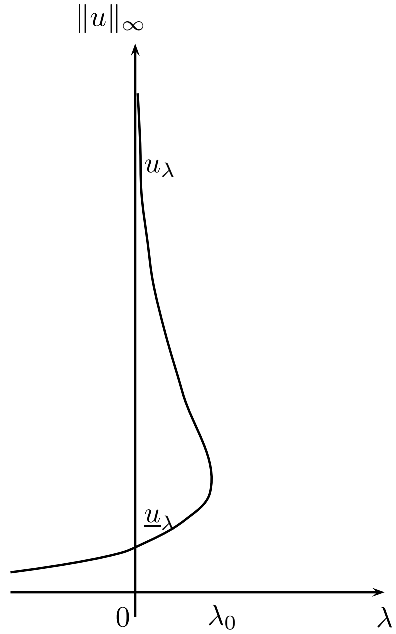


Figure 1.1: Bifurcation diagram for solutions of Problem (1.4)

must have $\underline{u}_\lambda \to u_0$ as $\lambda \to 0$ while $u_\lambda$ must become unbounded. The situation is depicted as a bifurcation diagram in Figure 1.1.

The proof in [15] does not provide information on its asymptotic blowing-up behavior or about the number of such "large" solutions. Borer, Galimberti and Struwe [7] have recently provided a new construction of the mountain pass solution for small $\lambda$, which allowed them to identify further properties of it under the following generic assumption: points of global minima of $f$ are non-degenerate. This means that if $f(p) = 0$ then $D^2 f(p)$ is positive definite. In [7] it is established that blowing-up of the family of large solutions $u_\lambda$ occurs only near zeros of $f$, and the associated metric exhibits "bubbling behavior", namely Euclidean spheres emerge around some of the zero-points of $f$. In fact, the mountain-pass characterization let them estimate the number of bubbling points as no larger than four. More precisely, they find that along any sequence $\lambda = \lambda_k \to 0$, there exist points $p_1^k, \dots, p_n^k$, $1 \le n \le 4$, converging to $p_1, \dots, p_n$ points of global minima of $f$ such that one of the following holds

(i) There exist $\varepsilon_\lambda^1, \dots, \varepsilon_\lambda^k$, such that $\varepsilon_\lambda^{\mathrm{i}}/\lambda \to 0$, $\mathrm{i} = 1, \dots, k$, and in local conformal coordinates around $p_{\mathrm{i}}$ there holds

$$u_\lambda(\varepsilon_\lambda^{\mathrm{i}} x) - u_\lambda(0) + \log 8 \to w(x) := \log \frac{8}{(1 + |x|^2)^2}, \tag{1.5}$$

smoothly locally in $\mathbb{R}^2$. We note that

$$\Delta w + \mathrm{e}^w = 0.$$

(ii) In local conformal coordinates around $p_i$, with a constant $c_i$ there holds

$$u_n(\lambda x) + 4\log(\lambda) + c_i \to w_\infty(x),$$

smoothly locally in $\mathbb{R}^2$, where $w_\infty$ satisfies

$$\Delta_g w_\infty + [1 - (Ax, x)]e^{w_\infty} + \alpha = 0$$

where $A = \frac{1}{2}D^2 f(p_i)$.

In this work we will substantially clarify the structure of the set of large solutions of problem (1.4) with a method that yields both multiplicity and accurate estimates of their blowing-up behavior. Roughly speaking we establish that for any given collection of non-degenerate global minima of $f$, $p_1, \ldots, p_k$, there exist a solution $u_\lambda$ blowing-up in the form (1.5) *exactly* at those points. Moreover

$$\varepsilon_\lambda^i \sim \frac{\lambda}{|\log \lambda|}, \quad u_\lambda(p_i) = -4\log\lambda - 2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{\lambda}\right) + O(1).$$

In particular if $f$ has exactly $m$ non-degenerate global minimum points, then $2^m$ distinct large solutions exist for all sufficiently small $\lambda$.

In order to state our main result, we consider the singular problem

$$\Delta_g G - f e^G + 8\pi \sum_{i=1}^n \delta_{p_i} + \alpha = 0, \quad \text{in } M, \tag{1.6}$$

where $\delta_{p_i}$ designates the Dirac mass at the point $p_i$. We have the following result.

**Lemma 1.1** *Problem (2.7) has a unique solution $G$ which is smooth away from the singularities and in local conformal coordinates around $p_i$ it has the form*

$$G(x) = -4\log|x| - 2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{|x|}\right) + \mathcal{H}(x), \tag{1.7}$$

*where $\mathcal{H}(x) \in C(M)$.*

Our main result is the following.

**Theorem 1.2** *Let $p_1, \ldots, p_n$ be points such that $f(p_i) = 0$ and $D^2 f(p_i)$ is positive definite for each* i. *Then, there exists a family of solutions $u_\lambda$ to (1.4) with*

$$\lambda^2 e^{u_\lambda} \rightharpoonup 8\pi \sum_{i=1}^n \delta_{p_i}, \quad \text{as } \lambda \to 0,$$

*and $u_\lambda \to G$ uniformly in compacts subsets of $M \setminus \{p_1, \ldots, p_k\}$. We define*

$$c_i = \frac{1}{2}e^{\mathcal{H}(p_i)/2}, \quad \delta_\lambda^i = \frac{c_i}{|\log\lambda|}, \quad \varepsilon_\lambda^i = \lambda\delta_\lambda^i$$

*where $\mathcal{H}$ is defined near $p_{\mathrm{i}}$ by relation (1.10). In local conformal coordinates around $p_{\mathrm{i}}$, there holds*

$$u_\lambda(\varepsilon_\lambda^{\mathrm{i}} x) + 4\log\lambda + 2\log\delta_\lambda^{\mathrm{i}} \to \log\frac{8}{(1+|x|^2)^2},$$

*uniformly on compact sets of $\mathbb{R}^2$ as $\lambda \to 0$.*

Our proof consists of the construction of a suitable first approximation of a solution as required, and then solving by linearization and a suitable Lyapunov-type reduction There is a large literature in Liouville type equation in two-dimensional domains or compact manifold, in particular concerning construction and classification of blowing-up families of solutions. See for instance [8, 14, 16, 21, 24, 30] and their references.

We shall present the detailed proof of our main result in the case of one bubbling point $n = 1$. In the last section we explain the necessary (minor, essentially notational) changes for general $n$. Thus, we consider the problem

$$\Delta_g u - f\mathrm{e}^u + \lambda^2 \mathrm{e}^u + \alpha = 0, \quad \text{in } M, \tag{1.8}$$

under the following hypothesis: there exists a point $p \in M$ such that $f(p) = 0$ and $D^2 f(p)$ is positive definite.

## 1.2 A nonlinear Green's function

We consider the singular problem

$$\Delta_g G - f\mathrm{e}^G + 8\pi\delta_p + \alpha = 0, \quad \text{in } M \tag{1.9}$$

where $\delta_p$ is the Dirac mass supported at $p$, which is assume to be a point of global non-degenerate minimum of $f$. In this section we will establish the following result, which corresponds to the case $n = 1$ in Lemma 2.6

**Lemma 1.3** *Problem (1.9) has a unique solution $G$ which is smooth away from the singularities and in local conformal coordinates around $p$ it has the form*

$$G(x) = -4\log|x| - 2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{|x|}\right) + \mathcal{H}(x), \tag{1.10}$$

*where $\mathcal{H}(x) \in C(M)$.*

Proof. In order to construct a solution to this problem, we first consider the equation, in local conformal coordinates around $p$, for $\gamma \ll 1$

$$\Delta\Gamma - f\mathrm{e}^\Gamma + 8\pi\delta_0 = 0, \quad \text{in } B(0,\gamma). \tag{1.11}$$

Since

$$-\Delta \log \frac{1}{|x|^4} = 8\pi\delta_0,$$

we look for a solution of (1.11) of the form $\Gamma = -4\log|x| + h(x)$, where $h$ satisfies

$$\Delta h - f(x)\frac{1}{|x|^4}\mathrm{e}^h = 0, \quad \text{in } B(0,\gamma). \tag{1.12}$$

Since $p$ is a non-degenerate point of minimum of $f$, we may assume that, in local conformal coordinates around $p$, there exist positive numbers $\beta_1, \beta_2, \gamma$ such that

$$\beta_1|x|^2 \leq f(x) \leq \beta_2|x|^2, \tag{1.13}$$

for all $x \in B(0,\gamma)$. Letting $r = |x|$, it is thus important to consider the equation

$$\Delta V - \frac{1}{r^2}\mathrm{e}^V = 0, \quad \text{in } B(0,\gamma). \tag{1.14}$$

For a radial function $V = V(r)$, this equation becomes

$$V''(r) + \frac{1}{r}V'(r) - \frac{1}{r^2}\mathrm{e}^{V(r)} = 0, \quad 0 < r < \gamma. \tag{1.15}$$

We make the change of variables $r = \mathrm{e}^t$, $v(t) = V(r)$, so that equation (1.15) transforms into

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}v(t) = \mathrm{e}^{v(t)}, \quad -\infty < t < \log\gamma.$$

from where it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v'(t)^2}{2} - \mathrm{e}^{v(t)}\right) = 0,$$

or $v'(t)^2 = 2(\mathrm{e}^v + C)$, for some constant $C$. Choosing $C = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-v(t)/2}\right) = -\frac{1}{\sqrt{2}}.$$

Integrating and coming back to the original variable, we deduce that

$$V(r) = -2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{r}\right)$$

is a radial solution of equation (1.14). From condition (1.13) we readily find that $h_1(x) = V(|x|) - \log\beta_1$ is a supersolution of (1.12), while $h_2(x) = V(|x|) - \log\beta_2$ is a subsolution of (1.12). This suggest us to look a solution to (1.12) of the form $V(|x|) + O(1)$.

Now we deal with existence of a solution of problem (1.9). The previous analysis suggest that the singular part of the Green's function, in local conformal coordinates around $p$, is

$$\Gamma(x) := -4\log|x| + V(|x|),$$

so we look for a solution of (1.9) of the form $u = \eta\Gamma + H$, where $\eta$ is a smooth cut-off function such that $\eta \equiv 1$ in $B(0, \frac{\gamma}{2})$ and $\eta \equiv 0$ in $\mathbb{R}^2 \setminus B(0,\gamma)$. Therefore, $H$ satisfies the equation

$$\Delta_g H - f\mathrm{e}^{\eta\Gamma}\mathrm{e}^H + \alpha = -\eta f \mathrm{e}^\Gamma - 2\nabla_g\eta\nabla_g\Gamma - \Gamma\Delta_g\eta =: \Theta, \quad \text{in } M. \tag{1.16}$$

Observe that $f\mathrm{e}^{\eta\Gamma} \in L^1(B(0,\gamma))$. Next we find ordered global sub and supersolutions for (1.16). Let us consider the problem

$$-\Delta_g h_0 + f h_0 = 1, \quad \text{in } M,$$

which has a unique non-negative solution of class $C^{2,\beta}, 0 < \beta < 1$. Observe that

$$\Delta_g \beta h_0 - f\mathrm{e}^{\eta\Gamma}\mathrm{e}^{\beta h_0} + \alpha - \Theta = -\beta + f\beta h_0 - f\mathrm{e}^{\eta\Gamma}\mathrm{e}^{\beta h_0} + \alpha - \Theta,$$

so if we choose $\beta = \beta_1 < 0$ small enough, then $\underline{H} := \beta_1 h_0$ is a subsolution of (1.16), while if we choose $\beta = \beta_2 > 0$ large enough, then $\overline{H} := \beta_2 h_0$ is a supersolution of (1.16).

We consider the space

$$X = \left\{ H \in H^1(M,g) \;\middle|\; \int_M f\mathrm{e}^{\eta\Gamma}\mathrm{e}^H < \infty \right\},$$

and the energy functional

$$E(H) = \frac{1}{2}\int_M |\nabla_g H|^2 + \int_M f\mathrm{e}^{\eta\Gamma}F(H) + \int_M (-\alpha + \Theta)H, \tag{1.17}$$

where

$$F(H(x)) = \begin{cases} \mathrm{e}^{\underline{H}(x)}(H - \underline{H}(x)) & H < \underline{H}(x), \\ \mathrm{e}^H - \mathrm{e}^{\underline{H}(x)} & H \in [\underline{H}(x), \overline{H}(x)], \\ \mathrm{e}^{\overline{H}(x)}(H - \underline{H}(x)) & H > \overline{H}(x). \end{cases}$$

Observe that since $h_0 \in L^\infty(M,g)$ and $f\mathrm{e}^{\eta\Gamma} \in L^1(B(p,\gamma))$, then $\overline{H}, \underline{H} \in X$, which means that the functional $E$ is well defined in $X$. Since

$$\int_M -\Delta_g(\eta\Gamma) = -\lim_{a\to 0}\int_{\partial B(p,a)} \frac{\partial\Gamma}{\partial r} = 8\pi,$$

we conclude that

$$\int_M \Theta = \int_M (-\Delta_g(\eta\Gamma) - 8\pi\delta_p) = 0.$$

Besides $\alpha > 0$, so the functional $E$ is coercive in $X$. We claim that $E$ attains a minimum in $X$. In fact, taking $H_n \in X$ such that

$$\lim_{n\to\infty} E(H_n) = \inf_{H\in X} E(H) > -\infty,$$

and passing to a subsequence if necessary, we obtain

$$H_n \to \mathcal{H} \in X \text{ (in } L^2), \; \nabla_g H_n \rightharpoonup \nabla_g \mathcal{H} \text{ (weakly in } L^2), \; E(\mathcal{H}) = \inf_{H\in X} E(H).$$

Observe that if we take $\varphi \in C^\infty(M)$ then $\mathcal{H} + \varphi \in X$, we can differentiate and obtain

$$\frac{\partial}{\partial t} E(\mathcal{H} + t\varphi)\Big|_{t=0} = 0, \quad \text{for all } \varphi \in C^\infty(M, g)$$

or

$$\int_M \nabla_g \mathcal{H} \cdot \nabla_g \varphi + \int_M f e^{\eta\Gamma} G(\mathcal{H})\varphi + \int_M (-\alpha + \Theta)\varphi = 0, \tag{1.18}$$

where

$$G(H) = \begin{cases} e^{\underline{H}(x)} & H < \underline{H}(x), \\ e^H & H \in [\underline{H}(x), \overline{H}(x)], \\ e^{\overline{H}(x)} & H > \overline{H}(x). \end{cases}$$

By suitably approximating $H_1 = (\underline{H} - \mathcal{H})_+$, we can use it as a test function in (1.18) and obtain

$$\int_M \nabla_g \mathcal{H} \cdot \nabla_g H_1 + \int_M f e^{\eta\Gamma} G(\mathcal{H}) H_1 + \int_M (-\alpha + \Theta) H_1 = 0.$$

Since $\underline{H}$ is a subsolution for Equation (1.16), we have

$$\int_M \nabla_g \underline{H} \cdot \nabla_g H_1 + \int_M f e^{\eta\Gamma} e^{\underline{H}} H_1 + \int_M (-\alpha + \Theta) H_1 \leq 0.$$

Observe that

$$\int_M f e^{\eta\Gamma} G(\mathcal{H}) H_1 = \int_M f e^{\eta\Gamma} e^{\underline{H}} H_1.$$

From the above calculations we deduce

$$\int_M |\nabla_g H_1|^2 \leq 0,$$

hence $H_1 \equiv C$ for some constant $C$. If $C > 0$, necessarily $C \equiv H_1 \equiv \underline{H} - \mathcal{H}$ almost everywhere. Thus, $\underline{H} = \mathcal{H} + C$, and (1.18) traduces into

$$\int_M \nabla_g \underline{H} \cdot \nabla_g \varphi + \int_M f e^{\eta\Gamma} e^{\underline{H}} \varphi + \int_M (-\alpha + \Theta)\varphi = 0,$$

for all $\varphi \in C^\infty(M)$, which contradicts the fact that $\underline{H}$ solves

$$-\Delta_g \underline{H} + f\underline{H} = 1,$$

or in other words, the fact that $\underline{H}$ is not a solution of problem (1.16). Hence $H_1 \equiv 0$, which implies $\underline{H} \leq \mathcal{H}$. In a similar way, we find $\mathcal{H} \leq \overline{H}$ and hence

$$\underline{H}(x) \leq \mathcal{H}(x) \leq \overline{H}(x), \quad \text{a.e. } x \in M.$$

Note that

$$\int_M \nabla_g \mathcal{H} \cdot \nabla_g \varphi + \int_M f e^{\eta\Gamma} e^{\mathcal{H}} \varphi + \int_M (-\alpha + \Theta)\varphi = 0, \tag{1.19}$$

for all $\varphi \in C^\infty(M, g)$. Besides, since the functional $E$ is strictly convex and coercive, we conclude that $\mathcal{H}$ is the unique minimizer in $X$.

So far we have proven that Problem (1.9) has a unique solution $G$ which is smooth away from the singularity point $p$ and in local conformal coordinates around $p$ it has the form

$$G(x) = \eta\left[-4\log|x| - 2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{|x|}\right)\right] + \mathcal{H}(x),$$

where $\mathcal{H} \in X \cap L^\infty(M,g)$, is the unique minimizer of the functional $E$ defined in $X$ by (1.17).

Next we will further study the form of $\mathcal{H}$ near $p$, which in particular yields its continuity at $p$. For this purpose we use local conformal coordinates around $p$.

Let us consider the problem

$$\begin{cases} -\Delta_g \mathcal{J} &=& \alpha & \text{in } B(0,\frac{\gamma}{2}), \\ \mathcal{J} &=& \mathcal{H} & \text{on } \partial B(0,\frac{\gamma}{2}). \end{cases}$$

This problem has a unique solution $\mathcal{J}$, which is smooth in $B(0,\frac{\gamma}{2})$. So we can expand $\mathcal{J}$ as

$$\mathcal{J} = \sum_{k=0}^{\infty} b_k r^k = b_0 + O(r).$$

We write $\mathcal{H} = \mathcal{J} + \mathcal{F}$, therefore $\mathcal{F}$ solves

$$\begin{cases} -\Delta_g \mathcal{F} + \dfrac{f}{r^4}\dfrac{2}{\log^2 r}\mathrm{e}^{\mathcal{J}}\mathrm{e}^{\mathcal{F}} - \dfrac{1}{r^2}\dfrac{2}{\log^2 r} &=& 0 & \text{in } B(0,\frac{\gamma}{2}), \\ \mathcal{F} &=& 0 & \text{on } \partial B(0,\frac{\gamma}{2}), \end{cases}$$

because $\eta\Gamma \equiv \Gamma$ in $B(0,\frac{\gamma}{2})$. Since $\mathcal{F} \in L^2(B(0,\frac{\gamma}{2}))$ we can expand it as

$$\mathcal{F}(r,\theta) = \sum_{k=0}^{\infty} a_k(r)\mathrm{e}^{\mathrm{i}k\theta}.$$

Observe that

$$\frac{f(x)}{r^2} = \frac{\kappa_1 r^2\cos^2(\theta) + \kappa_2 r^2 \sin^2(\theta) + \kappa_3 r^2 \sin\theta\cos\theta}{r^2} + O(r) = a(\theta) + O(r),$$

for $r \neq 0$. Besides, $\beta_1 \leq a(\theta) \leq \beta_2$. Thus

$$\frac{f(x)}{r^4}\frac{2}{\log^2 r}\mathrm{e}^{\mathcal{J}}\mathrm{e}^{\mathcal{F}} - \frac{1}{r^2}\frac{2}{\log^2 r} = \frac{1}{r^2}\frac{2}{\log^2 r}\left[(a(\theta) + O(r))\mathrm{e}^{\mathcal{J}+\mathcal{F}} - 1\right].$$

Moreover, since $\mathcal{H} \in L^\infty(B(0,\frac{\gamma}{2}))$ we have $\mathrm{e}^{\mathcal{J}+\mathcal{F}} \in L^2(B(0,\frac{\gamma}{2}))$, so

$$\frac{1}{r^2}\frac{2}{\log^2 r}\left[(a(\theta) + O(r))\mathrm{e}^{\mathcal{J}+\mathcal{F}} - 1\right] = \sum_{k=0}^{\infty} m_k(r)\mathrm{e}^{\mathrm{i}k\theta},$$

where

$$|m_k(r)| \leq \frac{C}{r^2}\frac{1}{\log^2 r}, \quad \forall k \geq 0,$$

for a constant $C$ independent of $k$. Now, we study the behavior of the coefficients $a_k(r)$. For this purpose let us remember that

$$\Delta u(r,\theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

For $k \geq 1$, we see that $a_k(r)$ satisfies the ordinary differential equation

$$-\frac{\partial^2 a_k}{\partial r^2}(r) - \frac{1}{r}\frac{\partial a_k}{\partial r}(r) + \frac{k^2}{r^2}a_k(r) = m_k(r), \quad 0 < r < \frac{\gamma}{2}, \tag{1.20}$$

under the conditions

$$a_k\left(\frac{\gamma}{2}\right) = 0, \quad a_k(r) \in L^\infty\left(\left[0,\frac{\gamma}{2}\right]\right). \tag{1.21}$$

We recall that the $L^\infty$-condition comes from the fact that $\mathcal{F} \in L^\infty(B(0,\frac{\gamma}{2}))$. Let us make the change of variables $r = \mathrm{e}^t$, $A_k(t) = a_k(\mathrm{e}^t)$, $M_k(t) = m_k(\mathrm{e}^t)$, so the previous problem transform into

$$-\frac{\mathrm{d}^2 A_k}{\mathrm{d}t^2}(t) + k^2 A_k(t) = M_k(t), \quad -\infty < t < \log\frac{\gamma}{2}, \tag{1.22}$$

under the conditions

$$A_k\left(\log\frac{\gamma}{2}\right) = 0, \quad A_k \in L^\infty\left(\left(-\infty,\log\frac{\gamma}{2}\right]\right). \tag{1.23}$$

Besides, $|M_k(t)| \leq Ct^{-2}$ for all $k \geq 1$. All the solutions of the homogeneous equation are given by linear combinations of $\mathrm{e}^{kt}$ and $\mathrm{e}^{-kt}$ and a particular solution $A_k^{part}$ of the non-homogeneous equation (1.22) is given by the variation of parameter formula. We conclude that this problem has a solution of the form

$$C_1\mathrm{e}^{kt} + C_2\mathrm{e}^{-kt} + A_k^{part}.$$

By the $L^\infty$-condition we conclude that $C_2 = 0$ and by the boundary condition in (1.23) we deduce $C_1 = 0$. This implies that the null function is the only solution of the homogeneous equation under condition (1.23). Hence, this problem has a unique solution $A_k(t)$. We claim that for a constant $C$ independent of $k$ we have

$$|A_k(t)| \leq C\frac{1}{k^2t^2}. \tag{1.24}$$

The proof of this fact is based on maximum principle: Observe that since $k^2 > 0$, the operator

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + k^2$$

satisfies the weak maximum principle on bounded subsets of $(-\infty, \log\frac{\gamma}{2}]$. Let us prove that $\phi = \frac{C_1}{k^2t^2} + \rho\mathrm{e}^{-kt}$ is a non-negative supersolution for this problem. Observe first that since $A_k(t)$ is bounded, there exist $\tau_\rho$ such that

$$A_k(t) \leq \phi(t), \quad \text{for all } t \in (-\infty,\tau_\rho].$$

Besides,

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+k^2\right)\phi = -6C_1\frac{1}{k^2t^4}+C_1\frac{1}{t^2}\geq M_k(t),\quad \forall t\in\left(\tau_\rho,\log\frac{\gamma}{2}\right),$$

where the last inequality is valid if we choose $C_1$ large enough. Observe also that $\phi(t)\geq A_k(t)$ for $t=\tau_\rho,\log\frac{\gamma}{2}$. Hence, by weak maximum principle we conclude that for all $\rho>0$

$$A_k(t)\leq\frac{C_1}{k^2t^2}+\rho\mathrm{e}^{-kt},\quad \forall t\in\left(-\infty,\log\frac{\gamma}{2}\right].$$

Taking the limit $\rho\to 0$ in the last expression, we conclude that $A_k(t)\leq C\frac{1}{k^2t^2}$. Analogously, we now prove that $\varphi=-\frac{C_2}{k^2t^2}-\rho\mathrm{e}^{-kt}$ is a non-positive subsolution for this problem. Since $A_k(t)$ is bounded, there exist $\tau_\rho$ such that

$$\varphi(t)\leq A_k(t),\quad \forall t\in(-\infty,\tau_\rho].$$

Besides,

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+k^2\right)\varphi = 6C_2\frac{1}{k^2t^4}-C_2\frac{1}{k^2t^2}\leq M_k(t),\quad \forall t\in\left(\tau_\rho,\log\frac{\gamma}{2}\right),$$

where the last inequality is valid if we choose $C_2$ large enough. Observe also that $\varphi(t)\leq A_k(t)$ for $t=\tau_\rho,\log\frac{\gamma}{2}$. Hence, by weak maximum principle we conclude that for all $\rho>0$

$$-\frac{C_2}{k^2t^2}-\rho\mathrm{e}^{-kt}\leq A_k(t),\quad \forall t\in\left(-\infty,\log\frac{\gamma}{2}\right].$$

Taking the limit $\rho\to 0$ in the last expression, we conclude (1.24). Finally, coming back to the variable $r$ we conclude that there exist a unique solution $a_k(r)$ of problem (1.20)-(1.21), and for a constant $C$ independent of $k$ we have

$$|a_k(r)|\leq C\frac{1}{k^2\log^2 r},\quad 0<r<\frac{\gamma}{2}.$$

Now we deal with $a_0(r)$. Observe that

$$\mathrm{e}^{\mathcal{F}}=\mathrm{e}^{a_0(r)}\left(1+O\left(\frac{1}{\log^2 r}\right)\right),\quad \mathrm{e}^{\mathcal{J}}=\mathrm{e}^{b_0}(1+O(r)),$$

and

$$a(\theta)=\alpha_0+\sum_{k=1}^{\infty}\alpha_k\mathrm{e}^{\mathrm{i}k\theta},\quad \text{with } \alpha_0>0,$$

so we conclude that $a_0(r)$ satisfies the ordinary differential equation

$$-\frac{\partial^2 a_0(r)}{\partial r^2}-\frac{1}{r}\frac{\partial a_0(r)}{\partial r}+2\frac{\alpha_0\mathrm{e}^{b_0}\mathrm{e}^{a_0(r)}-1}{r^2\log^2 r}=O\left(\frac{1}{r^2\log^4 r}\right),$$

under the following conditions

$$a_0\left(\frac{\gamma}{2}\right)=0,\quad a_0\in L^\infty\left(\left[0,\frac{\gamma}{2}\right]\right).$$

We make the change of variables $r = \mathrm{e}^t$, $\tilde{a}_0(t) = a_0(\mathrm{e}^t)$, so the previous problem transform into

$$-\frac{\mathrm{d}^2\tilde{a}_0}{\mathrm{d}t^2} + 2\frac{\alpha_0 \mathrm{e}^{b_0}\mathrm{e}^{\tilde{a}_0} - 1}{t^2} = O\left(\frac{1}{t^4}\right), \tag{1.25}$$

under the conditions

$$\tilde{a}_0\left(\log\frac{\gamma}{2}\right) = 0, \quad \tilde{a}_0 \in L^\infty\left(\left(-\infty, \log\frac{\gamma}{2}\right]\right). \tag{1.26}$$

The $L^\infty$-condition implies that there exist a sequence $t_n \to -\infty$ such that

$$\tilde{a}_0(t_n) \to L, \quad \text{as } n \to \infty,$$

where $L = -\log(\alpha_0 \mathrm{e}^{b_0})$. If not there exist $M < 0$ such that

$$|\alpha_0 \mathrm{e}^{b_0}\mathrm{e}^{\tilde{a}_0} - 1| \geq \varepsilon > 0, \quad \forall t < M,$$

which means that

$$\left|\frac{\mathrm{d}^2\tilde{a}_0}{\mathrm{d}t^2}\right| \geq C\frac{\varepsilon}{t^2}, \quad \forall t < M.$$

Thus

$$|\tilde{a}_0| \geq C\varepsilon \log|t|, \quad \forall t < M,$$

so $\tilde{a}_0$ is unbounded, a contradiction.

We claim that the problem (1.25), (1.26) has at most one solution. In fact, let us suppose by contradiction that $u_1$ and $u_2$ are two diferent solutions. We define $u = u_1 - u_2$, which satisfies the problem

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + 2\alpha_0 \mathrm{e}^{b_0} c(t) u = 0,$$

under the conditions,

$$u\left(\log\frac{\gamma}{2}\right) = 0, \quad u \in L^\infty\left(\left(-\infty, \log\frac{\gamma}{2}\right]\right),$$

and where

$$c(t) = \begin{cases} 0 & \text{if } u_1(t) = u_2(t), \\ \frac{1}{t^2}\frac{\mathrm{e}^{u_1(t)-u_2(t)}}{u_1(t)-u_2(t)} & \text{if } u_1(t) \neq u_2(t). \end{cases}$$

Observe that $c(t) \geq 0$, so we can apply the strong maximum principle in bounded domains for this problem. Moreover, from the $L^\infty$ condition we deduce that there exists a sequence $t_n$ such that $u(t_n) \to 0$ as $n \to \infty$ (the proof of this fact is the same that we gave before). From this two facts, we deduce easily that $u_1 \equiv u_2$.

Let us make the change of variables $-t = \mathrm{e}^s$, $A_0(s) = \tilde{a}_0(-\mathrm{e}^s)$, so the previous problem transform into

$$-\frac{\mathrm{d}^2 A_0}{\mathrm{d}s^2} + \frac{\mathrm{d}A_0}{\mathrm{d}s} + 2(\alpha_0 \mathrm{e}^{b_0}\mathrm{e}^{A_0} - 1) = O(\mathrm{e}^{-2s}), \tag{1.27}$$

under the conditions

$$A_0\left(\log\left(-\log\frac{\gamma}{2}\right)\right) = 0, \quad A_0 \in L^\infty\left(\left[\log\left(-\log\frac{\gamma}{2}\right), \infty\right)\right).$$

We look for a solution of this problem of the form $A_0(s) = L+\phi(s)$, so $\phi$ solves the differential equation

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \frac{\mathrm{d}\phi}{\mathrm{d}s} + 2\phi = N(\phi) + O(\mathrm{e}^{-2s}),$$

where

$$N(\phi) = -2(\mathrm{e}^{\phi} - \phi - 1).$$

Observe that $\phi_+ = \mathrm{e}^{2s}$, $\phi_- = \mathrm{e}^{-s}$ are two linear independent solutions of the homogeneous equation.

From the previous analysis, we deduce that there exists a sequence $s_n \to \infty$ such that $\phi(s_n) = \delta_n \to 0$, as $n \to \infty$. We make the change of variables $\tilde{\phi}_n(\tau_n) = \phi(s) - \delta_n\phi_-(\tau_n)$, where $\tau_n = s - s_n$, so $\tilde{\phi}_n \in L^\infty$ solves the problem

$$\begin{cases} -\tilde{\phi}_n'' + \tilde{\phi}_n' + 2\tilde{\phi}_n &= N(\tilde{\phi}_n + \delta\mathrm{e}^{-\tau_n}) + \mathrm{e}^{-2s_n}O(\mathrm{e}^{-2\tau_n}) \quad \tau_n \in (0,\infty), \\ \tilde{\phi}_n(0) &= 0. \end{cases} \tag{1.28}$$

Let us study the linear problem

$$\begin{cases} -\varphi'' + \varphi' + 2\varphi &= \omega \quad \text{in } (0,\infty), \\ \varphi(0) &= 0, \quad \varphi \in L^\infty(0,\infty) \end{cases}$$

for $\omega \in C([0,\infty))$ given. This problem has an explicit and unique solution $\varphi = T[g]$, in fact

$$\varphi(t) = C_1\mathrm{e}^{\lambda_+ t} + C_2\mathrm{e}^{\lambda_- t} + \mathrm{e}^{\lambda_+ t}\int_0^t \frac{\mathrm{e}^{\lambda_- s}\omega(s)}{3\mathrm{e}^{2s}}\mathrm{d}s - \mathrm{e}^{\lambda_- t}\int_0^t \frac{\mathrm{e}^{\lambda_+ s}\omega(s)}{3\mathrm{e}^{2s}}\mathrm{d}s$$

and we deduce that $C_1 = 0$ and $C_2 = 0$ due to the $L^\infty$ condition and the value at 0 of $\varphi$, respectively. Problem (1.28) can be written as

$$\tilde{\phi}_n = T[N(\tilde{\phi}_n + \delta\mathrm{e}^{-\tau_n}) + \mathrm{e}^{-2s_n}O(\mathrm{e}^{-2\tau_n})] := A[\tilde{\phi}_n]. \tag{1.29}$$

We consider the set

$$B_\varepsilon = \{\varphi \in C([0,\infty)) \,:\, \|\varphi\|_\infty \le \varepsilon\}.$$

It is easy to see that if $s_n$ is large enough and $\delta_n$ small enough we have

$$\|A[\tilde{\phi}_n^1] - A[\tilde{\phi}_n^2]\|_\infty \le C\varepsilon\|\tilde{\phi}_n^1 - \tilde{\phi}_n^2\|,$$

$$\|A[\tilde{\phi}_n]\| \le C\varepsilon,$$

and where $C$ is independent of $n$. It follows that for all sufficiently small $\varepsilon$ we get that $A$ is a contraction mapping of $B_\varepsilon$ (provided $n$ large enough), and therefore a unique fixed point of $A$ exists in this region. We deduce that there exists a unique solution $A_0$ of problem (1.27), and it has the form $A_0(s) = L + \phi(s)$, where $L$ is a fixed constant, and $\phi(s) \to 0$ as $s \to \infty$. This concludes the proof of Lemma 2.8. $\square$

## 1.3 Construction of a first approximation

In this section we will build a suitable approximation for a solution of Problem (1.8) which is large exactly near the point $p$. The "basic cells" for the construction of an approximate solution of problem (1.8) are the radially symmetric solutions of the problem

$$\begin{cases} \Delta w + \lambda^2 e^w &= 0 \quad \text{in } \mathbb{R}^2, \\ w(x) &\to 0 \quad \text{as } |x| \to \infty. \end{cases} \tag{1.30}$$

which are given by the one-parameter family of functions

$$w_\delta(|x|) = \log \frac{8\delta^2}{(\lambda^2\delta^2 + |x|^2)^2},$$

where $\delta$ is any positive number. We define $\varepsilon = \lambda\delta$. In order to construct the approximate solution we consider the equation

$$\Delta F - \frac{\delta^2}{r^2} e^F = 0, \tag{1.31}$$

in the variable $r = |x|/\varepsilon$ and we look for a radial solution $F = F(r)$, away from $r = 0$. For this purpose we solve Problem (1.31) under the following initial conditions

$$F(1/\delta) = 0, \quad F'(1/\delta) = 0.$$

We make the change of variables $r = e^t$, $V(t) = F(r)$, so that equation (1.31) transforms into

$$V'' - \delta^2 e^V = 0.$$

We consider the transformation $V(s) = \tilde{V}(\delta s)$, so $\tilde{V}$ solves problem

$$\tilde{V}'' - e^{\tilde{V}} = 0, \quad \tilde{V}(\delta|\log\delta|) = 0, \quad \tilde{V}'(\delta|\log\delta|) = 0.$$

This problem has a unique regular solution, which blows-up at some finite radius $\gamma > 0$. Coming back to the variable $r = |x|/\varepsilon$, we conclude that the solution $F(r)$ is defined for all $1/\delta \le r \le C e^{1/\delta} = C/\lambda$, for some constant $C$. Besides, we extend by 0 the function $F$ for $r \in [0, 1/\delta)$, which means $F(r) = 0$, for all $r \in [0, 1/\delta)$ and we denote by $\tilde{F}(|x|) = F(|x|/\varepsilon)$. A first local approximation of the solution, in local conformal coordinates around $p$, is given by the radial function $u_\varepsilon(x) = w_\delta(|x|) + \tilde{F}(|x|)$.

In order to build a global approximation, let us consider $\eta$ a smooth radial cutoff function such that $\eta(r) = 1$ if $r \le C_1\delta$ and $\eta(r) = 0$ if $r \ge C_2\delta$, for constants $0 < C_1 < C_2$. We consider as initial approximation $U_\varepsilon = \eta u_\varepsilon + (1-\eta)G$, where $G$ is the Green function that we built in the previous section. In order to have a good approximation around $p$ we have to adjust the parameter $\delta$. The good choice of this number is

$$\log 8\delta^2 = -2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{\lambda}\right) + \mathcal{H}(p),$$

where $\mathcal{H}$ is defined in Section 1.2. With this choice of the parameter $\delta$, the function $u_\varepsilon$ is approaching the Green function $G$ around $p$.

A useful observation is that $u$ satisfies problem (1.8) if and only if

$$v(y) = u(\varepsilon y) + 4\log\lambda + 2\log\delta$$

satisfies

$$\Delta_g v - \lambda^{-2} f(\varepsilon y)\mathrm{e}^v + \mathrm{e}^v + \varepsilon^2\alpha = 0, \quad y \in M_\varepsilon, \tag{1.32}$$

where $M_\varepsilon = \varepsilon^{-1}M$.

We denote in what follows $p' = \varepsilon^{-1}p$ and

$$\tilde{U}_\varepsilon(y) = U_\varepsilon(\varepsilon y) + 4\log\lambda + 2\log\delta,$$

for $y \in M_\varepsilon$. This means precisely in local conformal coordinates around $p$ that

$$\begin{aligned}\tilde{U}_\varepsilon(y) =& \eta(\varepsilon|y|)\left(\log\frac{1}{(1+|y|^2)^2} + \tilde{F}(\varepsilon|y|)\right)\\ &+ (1-\eta(\varepsilon|y|))\left(G(\varepsilon y) + 4\log\lambda + 2\log\delta\right).\end{aligned}$$

Let us consider a vector $k \in \mathbb{R}^2$. We recall that $w_\delta(|x-k|)$ is also a solution of problem (1.30). In order to solve problem (1.32), we need to modify the first approximation of the solution, in order to have a new parameter related to translations. More precisely, we consider for $|k| \ll 1$ the new first approximation of the solution (in the expanded variable)

$$\begin{aligned}V_\varepsilon(y) =& \eta(\varepsilon|y|)\left(\log\frac{1}{(1+|y-k|^2)^2} + \tilde{F}(\varepsilon|y|)\right)\\ &+ (1-\eta(\varepsilon|y|))\left(G(\varepsilon y) + 4\log\lambda + 2\log\delta\right).\end{aligned}$$

We will denote by $v_\varepsilon$ the first approximation of the solution in the original variable, which means

$$v_\varepsilon(x) = \eta(|x|)\left(\log\frac{8\delta^2}{(\varepsilon^2+|x-\varepsilon k|^2)^2} + \tilde{F}(|x|)\right) + (1-\eta(|x|))G(x).$$

Hereafter we look for a solution of problem (1.32) of the form $v(y) = V_\varepsilon(y) + \phi(y)$, where $\phi$ represent a lower order correction. In terms of $\phi$, problem (1.32) now reads

$$L(\phi) = N(\phi) + E, \quad \text{in } M_\varepsilon, \tag{1.33}$$

where

$$\begin{aligned}L(\phi) :=& \Delta_g\phi - \lambda^{-2} f(\varepsilon y)\mathrm{e}^{V_\varepsilon}\phi + \mathrm{e}^{V_\varepsilon}\phi,\\ N(\phi) :=& \lambda^{-2} f(\varepsilon y)\mathrm{e}^{V_\varepsilon}(\mathrm{e}^\phi - 1 - \phi) - \mathrm{e}^{V_\varepsilon}(\mathrm{e}^\phi - 1 - \phi),\\ E :=& -\left(\Delta_g V_\varepsilon - \lambda^{-2} f(\varepsilon y)\mathrm{e}^{V_\varepsilon} + \mathrm{e}^{V_\varepsilon} + \varepsilon^2\alpha\right).\end{aligned}$$

## 1.4 The linearized operator around the first approximation

In this section we will develop a solvability theory for the second-order linear operator $L$ defined in (2.21) under suitable orthogonality conditions. Using local conformal coordinates around $p'$, then formally the operator $L$ approaches, as $\varepsilon, |k| \to 0$, the operator in $\mathbb{R}^2$

$$\mathcal{L}(\phi) = \Delta\phi + \frac{8}{(1+|z|^2)^2}\phi,$$

namely, equation $\Delta w + \mathrm{e}^w = 0$ linearized around the radial solution $w(z) = \log \frac{8}{(1+|z|^2)^2}$. An important fact to develop a satisfactory solvability theory for the operator $L$ is the non-degeneracy of $w$ modulo the natural invariance of the equation under dilations and translations. Thus we set

$$z_0(z) = \frac{\partial}{\partial s}[w(sz) + 2\log s]|_{s=1}, \tag{1.34}$$

$$z_{\mathrm{i}}(z) = \frac{\partial}{\partial \zeta_{\mathrm{i}}} w(z+\zeta)|_{\zeta=0}, \quad \mathrm{i} = 1, 2. \tag{1.35}$$

It turns out that the only bounded solutions of $\mathcal{L}(\phi) = 0$ in $\mathbb{R}^2$ are precisely the linear combinations of the $z_{\mathrm{i}}$, $\mathrm{i} = 0, 1, 2$, see [4] for a proof. We define for $\mathrm{i} = 0, 1, 2$,

$$Z_{\mathrm{i}}(y) = z_{\mathrm{i}}(y-k).$$

Additionally, let us consider $R_0$ a large but fixed number $R_0 > 0$ and $\chi$ a radial and smooth cut-off function such that $\chi \equiv 1$ in $B(k, R_0)$ and $\chi \equiv 0$ in $B(k, R_0+1)^c$.

Given $h$ of class $C^{0,\beta}(M_\varepsilon)$, we consider the linear problem of finding a function $\phi$ such that for certain scalars $c_{\mathrm{i}}$, $\mathrm{i} = 1, 2$, one has

$$\begin{cases} L(\phi) = h + \sum_{\mathrm{i}=1}^{2} c_{\mathrm{i}} \chi Z_{\mathrm{i}} & \text{in } M_\varepsilon, \\ \displaystyle\int_{M_\varepsilon} \chi Z_{\mathrm{i}} \phi = 0 & \text{for } \mathrm{i} = 1, 2. \end{cases} \tag{1.36}$$

We will establish a priori estimates for this problem. To this end we define, given a fixed number $0 < \sigma < 1$, the norm

$$\|h\|_* = \|h\|_{*,p} := \sup_{M_\varepsilon} (\max\{\varepsilon^2, |y|^{-2-\sigma}\})^{-1} |h|. \tag{1.37}$$

Here the expression $\max\{\varepsilon^2, |y|^{-2-\sigma}\}$ is regarded in local conformal coordinates around $p' = \varepsilon^{-1}p$. Since local coordinates are defined up to distance $\sim \frac{1}{\varepsilon}$ that expression makes sense globally in $M_\varepsilon$.

Our purpose in this section is to prove the following result.

**Proposition 1.4** *There exist positive numbers $\varepsilon_0, C$ such that for any $h \in C^{0,\beta}(M_\varepsilon)$, with $\|h\|_* < \infty$ and for all $k$ such that $|k| \le C\lambda/\delta$, there is a unique solution $\phi = T(h) \in C^{2,\beta}(M_\varepsilon)$ of problem (2.22) for all $\varepsilon < \varepsilon_0$, which defines a linear operator of $h$. Besides,*

$$\|T(h)\|_\infty \le C \log\left(\frac{1}{\varepsilon}\right) \|h\|_*. \tag{1.38}$$

Observe that the orthogonality conditions in problem (2.22) are only taken respect to the elements of the approximate kernel due to translations.

The next Lemma will be used for the proof of Proposition 2.5. We obtain an a priori estimate for the problem

$$\begin{cases} L(\phi) &= h \quad \text{in } M_\varepsilon, \\ \int_{M_\varepsilon} \chi Z_{\mathrm{i}} \phi &= 0 \quad \text{for i} = 1,2. \end{cases} \tag{1.39}$$

We have the following estimate.

**Lemma 1.5** *There exist positive constants $\varepsilon_0, C$ such that for any $\phi$ solution of problem (1.39) with $h \in C^{0,\beta}(M_\varepsilon)$, $\|h\|_* < \infty$ and any $k$, $|k| \le C\lambda/\delta$*

$$\|\phi\|_\infty \le C \log\left(\frac{1}{\varepsilon}\right) \|h\|_*,$$

*for all $\varepsilon < \varepsilon_0$.*

Proof. We carry out the proof by a contradiction argument. If the above fact were false, there would exist sequences $(\varepsilon_n)_{n\in\mathbb{N}}$, $(k_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$, $|k_n| \to 0$ and functions $\phi_n$, $h_n$ with $\|\phi_n\|_\infty = 1$,

$$\log(\varepsilon_n^{-1})\|h_n\|_* \to 0,$$

such that

$$\begin{cases} L(\phi_n) &= h_n \quad \text{in } M_{\varepsilon_n}, \\ \int_{M_{\varepsilon_n}} \chi Z_{\mathrm{i}} \phi_n &= 0 \quad \text{for i} = 1,2. \end{cases} \tag{1.40}$$

A key step in the proof is the fact that the operator $L$ satisfies a weak maximum principle in regions, in local conformal coordinates around $p$, of the form $A_\varepsilon = B(p', \varepsilon^{-1}\gamma/2) \setminus B(p', R)$, with $R$ a large but fixed number. Consider the function $z_0(r) = \frac{r^2-1}{r^2+1}$, radial solution in $\mathbb{R}^2$ of

$$\Delta z_0 + \frac{8}{(1+r^2)^2} z_0 = 0.$$

We define a comparison function

$$Z(y) = z_0(a|y-p'|), \quad y \in M_\varepsilon.$$

Let us observe that

$$-\Delta Z(y) = \frac{8a^2(a^2|y-p'|^2 - 1)}{(1+a^2|y-p'|^2)^3}.$$

So, for $100a^{-2} < |y - p'| < \varepsilon^{-1}\gamma/2$, we have

$$-\Delta Z(y) \geq 2\frac{a^2}{(1 + a^2|y - p'|^2)^2} \geq \frac{a^{-2}}{|y - p'|^4}.$$

On the other hand, in the same region,

$$\mathrm{e}^{V_\varepsilon(y)} Z(y) \leq C\frac{1}{|y - p'|^4}.$$

Hence if $a$ is taken small and fixed, and $R > 0$ is chosen sufficiently large depending on this $a$, then

$$\Delta Z + \mathrm{e}^{V_\varepsilon} Z < 0, \quad \text{in } A_\varepsilon.$$

Since $Z > 0$ in $A_\varepsilon$, we have

$$L(Z) < 0, \quad \text{in } A_\varepsilon$$

We conclude that $L$ satisfies weak maximum principle in $A_\varepsilon$, namely if $L(\phi) \leq 0$ in $A_\varepsilon$ and $\phi \geq 0$ on $\partial A_\varepsilon$, then $\phi \geq 0$ in $A_\varepsilon$.

We now give the proof of the Lemma in several steps.

STEP 1. We claim that

$$\sup_{y \in M_{\varepsilon_n} \setminus B(p/\varepsilon_n, \rho/\varepsilon_n)} |\phi_n(y)| = o(1),$$

where $\rho$ is a fixed number. In fact, coming back to the original variable by the transformation

$$\hat{\phi}_n(x) = \phi_n\left(\frac{x}{\varepsilon_n}\right), \quad x \in M.$$

We can see that $\hat{\phi}_n$ satisfies the equation

$$\Delta_g \hat{\phi}_n - f\mathrm{e}^{v_{\varepsilon_n}} \hat{\phi}_n + \lambda_n^2 \mathrm{e}^{v_{\varepsilon_n}} \hat{\phi}_n = \frac{1}{\varepsilon_n^2} h_n\left(\frac{x}{\varepsilon_n}\right), \tag{1.41}$$

where

$$v_{\varepsilon_n}(x) = V_{\varepsilon_n}\left(\frac{x}{\varepsilon_n}\right) - 4\log\lambda_n - 2\log\delta,$$

is the approximation of the solution in the original variable. Taking $n \to \infty$, we can see that $\hat{\phi}_n$ converges uniformly over compacts of $M \setminus \{p\}$ to a function $\hat{\phi} \in H^1(M) \cap L^\infty(M)$ solution of the problem

$$\Delta_g \hat{\phi} - f\mathrm{e}^J \hat{\phi} = 0, \quad \text{in } M \setminus \{p\} \tag{1.42}$$

where $J$ is the limit of $v_{\varepsilon_n}$. We claim that $\hat{\phi} \equiv 0$, in fact, we consider the unique solution $\Phi$ of the problem

$$\Delta_g \Phi - \min\{f\mathrm{e}^J, 1\}\Phi = -\delta_p, \quad \text{in } M.$$

Using local conformal coordinates around $p$ we expand

$$\Phi(x) = -\frac{1}{2\pi}\log(|x|) + H(x)$$

for $H$ bounded. Since $\hat{\phi} \in L^\infty(M)$, we conclude that for all sufficiently small $\varepsilon$ and $\tau$ we have

$$|\hat{\phi}(x)| \leq \varepsilon\Phi(x), \quad x \in \partial B(0,\tau).$$

Multiplying (1.42) by $\varphi = (\hat{\phi} - \varepsilon\Phi)_+$, and integrating by parts over $M_\tau = M \setminus U_\tau$, where $U_\tau$ is the neighborhood around $p$ under the local conformal coordinates that we used, we have

$$\int_{M_\tau} |\nabla_g \varphi|^2 + \int_{M_\tau} f e^J \varphi^2 + \varepsilon \int_{M_\tau} e^J \varphi \Phi = 0.$$

Since $\Phi \geq 0$, we have

$$\int_{M_\tau} |\nabla_g \varphi|^2 + \int_{M_\tau} f e^J \varphi^2 \leq 0.$$

Hence $\varphi = (\hat{\phi} - \varepsilon\Phi)_+ = 0$ in $M_\tau$, so $\hat{\phi} \leq \varepsilon\Phi$ in $M_\tau$. Multiplying by $\varphi = (\hat{\phi} + \varepsilon\Phi)_-$ and integrating by parts, we have $(\hat{\phi} + \varepsilon\Phi)_- = 0$, thus

$$|\hat{\phi}(x)| \leq \varepsilon\Phi(x), \quad x \in M_\tau.$$

Taking $\varepsilon \to 0$ and $\tau \to 0$, we conclude that $\hat{\phi} \equiv 0$.

STEP 2. Let us consider the transformation

$$\tilde{\phi}_n(y) = \phi_n(y + p_n').$$

Thus $\tilde{\phi}_n$ satisfies the equation

$$\Delta_g \tilde{\phi}_n - \lambda_n^{-2} f(\varepsilon_n y + p_n) e^{V_{\varepsilon_n}(y+p_n')} \tilde{\phi}_n + e^{V_{\varepsilon_n}(y'+p_n')} = h_n(y + p_n'),$$

in $M_{\varepsilon_n} - \{p_n'\}$. Taking the limit $n \to \infty$ in the last equation (and also in problem (1.40)), we see that $\tilde{\phi}_n$ converges uniformly over compacts of $M_{\varepsilon_n} - \{p_n'\}$ to a bounded solution $\tilde{\phi}$ of the problem

$$\mathcal{L}(\tilde{\phi}) = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} \chi Z_{\rm i} \tilde{\phi} = 0, \quad {\rm i} = 1, 2.$$

Hence $\tilde{\phi}(x) = C_0 Z_0(x)$.

In what follows we assume without loss of generality that $C_0 \geq 0$. If $C_0 < 0$, we work with $-\phi_n$ instead of $\phi_n$ and the following analysis is also valid.

STEP 3. In this step we will construct a non-negative supersolution in the region, in local conformal coordinates around $p_n'$, $B_n = B(k_n, \rho) \setminus B(k_n, \varepsilon_n^{-1}\gamma/2), \rho > 0$, where the weak maximum principle is valid. We work first in the case $C_0 > 0$. Let us consider the problem

$$\begin{cases} -\Delta \psi_n - e^{V_\varepsilon} \psi_n &= 1 & \text{in } B_n, \\ \psi_n(y) &= C_0 & \text{on } \partial B(k_n, \rho), \\ \psi_n(y) &= o(1) & \text{on } \partial B(k_n, \varepsilon_n^{-1}\gamma/2). \end{cases} \tag{1.43}$$

We define $r = |y - k_n|$. A direct computation shows that

$$\psi_n(y) = C_0 Z_0(r) + C_\varepsilon Y(r) + W(r),$$

where

$$Y(r) = Z_0 \int_\rho^r \frac{1}{s Z_0^2(s)} \mathrm{d}s, \quad W(r) = -Z_0(r) \int_\rho^r sY(s)\mathrm{d}s + Y(r) \int_\rho^r s Z_0(s)\mathrm{d}s,$$

and

$$C_\varepsilon = \frac{o(1) - C_0 Z_0(\varepsilon_n^{-1}\gamma/2) - W(\varepsilon_n^{-1}\gamma/2)}{Y(\varepsilon_n^{-1}\gamma/2)}.$$

We choose $\rho > R$, where $R$ is the fixed minimal radio for which the weak maximum principle is valid in the region $B_n$. Observe that

$$L(\psi_n) = -1 - \lambda^{-2} f(\varepsilon y) \mathrm{e}^{V_\varepsilon} \psi_n \leq h_n = L(\phi_n).$$

Moreover, from steps 1 and 2, we deduce that

$$\psi_n \geq \phi_n, \quad \text{on } \partial B_n, \tag{1.44}$$

which means that $\psi_n$ is a supersolution for the problem

$$L(\phi_n) = h_n, \quad \text{in } B_n.$$

Since $\rho > R$, we can apply the weak maximum principle and we deduce that $\Psi_n \geq \phi_n$ in $B_n$. Observe that

$$\left| \frac{\mathrm{d}\psi_n(\rho)}{\mathrm{d}r} \right| \geq \varepsilon_n^{-1}. \tag{1.45}$$

In the other hand

$$\frac{\mathrm{d}Z_0}{\mathrm{d}r} = -C \frac{r}{(r^2-1)^2}, \tag{1.46}$$

where $C > 0$ is a constant independent of $n$. Since $\phi_n$ converges over compacts of the expanded variable to the function $C_0 Z_0$, we deduce from (1.44), (1.45) and (1.46) that the partial derivative of $\phi_n$ respect to $r$ is discontinuous at $|y - k_n| = \rho$, for large values of $n$, which is a contradiction.

In the case $C_0 = 0$, $\phi_n$ converges to 0 over compacts of the expanded variable. Let us consider the problem

$$\begin{cases} -\Delta\psi_n - \mathrm{e}^{V_\varepsilon}\psi_n &= 1 \quad \text{in } B_n, \\ \psi_n(y) &= 1/2 \quad \text{on } \partial B(k_n, \rho), \\ \psi_n(y) &= o(1) \quad \text{on } \partial B(k_n, \varepsilon_n^{-1}\gamma/2). \end{cases}$$

It is easy to see that $\psi_n \leq 1/2$ in $\overline{B}_n$. Using the previous maximum principle argument we deduce that $\phi_n \leq \psi_n \leq 1/2$ Applying the same argument for the problem that $-\phi_n$ satisfies, we conclude $-\phi_n \leq 1/2$. Thus,

$$\|\phi_n\|_\infty \leq 1/2,$$

which is a contradiction with the fact $\|\phi_n\|_\infty = 1$. This finishes the proof of the a priori estimate. $\square$

We are now ready to prove the main result of this section.

Proof of Proposition 2.5. We begin by establishing the validity of the a priori estimate (2.24). The previous lemma yields

$$\|\phi\|_\infty \le C \log\left(\frac{1}{\varepsilon}\right)\left[\|h\|_* + \sum_{\mathrm{i}=1}^{2} |c_\mathrm{i}|\right], \tag{1.47}$$

hence it suffices to estimate the values of the constants $|c_\mathrm{i}|$, $\mathrm{i} = 1, 2$. We use local conformal coordinates around $p$, and we define again $r = |y|$ and we consider a smooth cut-off function $\eta(r)$ such that $\eta(r) = 1$ for $r < \frac{1}{\sqrt{\varepsilon}}$, $\eta(r) = 0$ for $r > \frac{2}{\sqrt{\varepsilon}}$, $|\eta'(r)| \le C\sqrt{\varepsilon}$, $|\eta''(r)| \le C\varepsilon$. We test the first equation of problem (2.22) against $\eta Z_\mathrm{i}$, $\mathrm{i} = 1, 2$ to find

$$\langle L(\phi), \eta Z_\mathrm{i}\rangle = \langle h, \eta Z_\mathrm{i}\rangle + c_\mathrm{i} \int_{M_\varepsilon} \chi |Z_\mathrm{i}|^2. \tag{1.48}$$

Observe that

$$\langle L(\phi), \eta Z_\mathrm{i}\rangle = \langle \phi, L(\eta Z_\mathrm{i})\rangle,$$

and

$$L(\eta Z_\mathrm{i}) = Z_\mathrm{i}\Delta\eta + 2\nabla\eta\cdot\nabla Z_\mathrm{i} + \eta(\Delta Z_\mathrm{i} + \mathrm{e}^{V_\varepsilon} Z_\mathrm{i}) - \eta\lambda^{-2} f(\varepsilon y)\mathrm{e}^{V_\varepsilon} Z_\mathrm{i}.$$

We have

$$\eta(\Delta Z_\mathrm{i} + \mathrm{e}^{V_\varepsilon} Z_\mathrm{i}) = \varepsilon O((1+r)^{-3}).$$

Observe that

$$\lambda^{-2} f(\varepsilon y)\mathrm{e}^{V_\varepsilon(y)} = \lambda^2\delta^2 f(x)\mathrm{e}^{v_\varepsilon(x)}, \quad \text{where } y = \frac{x}{\varepsilon}, x \in M,$$

thus

$$\eta\lambda^{-2} f(\varepsilon y)\mathrm{e}^{V_\varepsilon} Z_\mathrm{i} = O(\varepsilon^2).$$

Since $\Delta\eta = O(\varepsilon)$, $\nabla\eta = O(\sqrt{\varepsilon})$, and besides $Z_\mathrm{i} = O(r^{-1})$, $\nabla Z_\mathrm{i} = O(r^{-2})$, we find

$$Z_\mathrm{i}\Delta\eta + 2\nabla\eta\cdot\nabla Z_\mathrm{i} = O(\varepsilon\sqrt{\varepsilon}).$$

From the previous estimates we conclude that

$$|\langle \phi, L(\eta Z_\mathrm{i})\rangle| \le C\sqrt{\varepsilon}\|\phi\|_\infty.$$

Combining this estimate with (1.47) and (1.48) we obtain

$$|c_\mathrm{i}| \le C\left[\|h\|_* + \sqrt{\varepsilon}\log\frac{1}{\varepsilon}\right],$$

which implies

$$|c_\mathrm{i}| \le C\|h\|_* \quad \mathrm{i} = 1, 2.$$

It follows from (1.47) that

$$\|\phi\|_\infty \le C\log\left(\frac{1}{\varepsilon}\right)\|h\|_*,$$

and the a priori estimate (2.24) has been thus proven. It only remains to prove the solvability assertion. For this purpose let us consider the space

$$H = \left\{ \phi \in H^1(M_\varepsilon) \, : \, \int_{M_\varepsilon} \chi Z_i \phi = 0, \, i = 1, 2. \right\}$$

endowed with the inner product,

$$\langle \phi, \psi \rangle = \int_{M_\varepsilon} \nabla_g \phi \nabla_g \psi + \int_{M_\varepsilon} \lambda^{-2} f(\varepsilon y) e^{V_\varepsilon} \phi \psi.$$

Problem (2.22) expressed in weak form is equivalent to that of finding $\phi \in H$ such that

$$\langle \phi, \psi \rangle = \int_{M_\varepsilon} \left[ e^{V_\varepsilon} \phi + h + \sum_{i=1}^{2} c_i \chi Z_i \right] \psi, \quad \text{for all } \psi \in H.$$

With the aid of Riesz's representation theorem, this equation gets rewritten in $H$ in the operator form $\phi = K(\phi) + \tilde{h}$, for certain $\tilde{h} \in H$, where $K$ is a compact operator in $H$. Fredholm's alternative guarantees unique solvability of this problem for any $h$ provided that the homogeneous equation $\phi = K(\phi)$ has only zero as solution in $H$. This last equation is equivalent to problem (2.22) with $h \equiv 0$. Thus, existence of a unique solution follows from the a priori estimate (2.24). The proof is complete. □

## 1.5 The nonlinear problem

We recall that our goal is to solve problem (2.21). Rather than doing so directly, we shall solve fist the intermediate problem

$$\begin{cases} L(\phi) = N(\phi) + E + \sum_{i=1}^{2} c_i \chi Z_i & \text{in } M_\varepsilon, \\ \int_{M_\varepsilon} \chi Z_i \phi = 0 & \text{for } i = 1, 2, \end{cases} \tag{1.49}$$

using the theory developed in the previous section. We assume that the conditions in Proposition (2.5) hold. We have the following result

**Lemma 1.6** *Under the assumptions of Proposition (2.5) there exist positive number $C, \varepsilon_0$ such that problem (1.49) has a unique solution $\phi$ which satisfies*

$$\|\phi\|_\infty \le C \varepsilon \log \frac{1}{\varepsilon},$$

*for all $\varepsilon < \varepsilon_0$.*

Proof. In terms of the operator $T$ defined in Proposition (2.5), problem (1.49) becomes

$$\phi = T(N(\phi) + E) =: A(\phi). \tag{1.50}$$

For a given number $\vartheta > 0$, let us consider the space

$$H_\vartheta = \left\{ \phi \in C(M_\varepsilon) \,:\, \|\phi\|_\infty \le \vartheta \varepsilon \log \frac{1}{\varepsilon} \right\}.$$

From Proposition (2.5), we get

$$\|A(\phi)\|_\infty \le C \log\left(\frac{1}{\varepsilon}\right) (\|N(\phi)\|_* + \|E\|_*).$$

Let us first measure how well $V_\varepsilon$ solves problem (1.32). Observe that

$$\mathrm{e}^{V_\varepsilon(y)} = \lambda^4 \delta^2 \mathrm{e}^{v_\varepsilon(x)}, \quad y = \frac{x}{\varepsilon}, x \in M, \tag{1.51}$$

so

$$\|\mathrm{e}^{V_\varepsilon(y)}\|_* \le C\varepsilon.$$

As a consequence of the construction of the first approximation, the choice of the parameter $\delta$, the expansion of the Green function $G$ around $p$, and (1.51), a direct computation yields

$$\|E\|_* \le C\varepsilon.$$

Now we estimate

$$N(\phi) = \lambda^{-2} f(\varepsilon y) \mathrm{e}^{V_\varepsilon} (\mathrm{e}^\phi - 1 - \phi) - \mathrm{e}^{V_\varepsilon} (\mathrm{e}^\phi - 1 - \phi).$$

In one hand, from (1.51) we deduce

$$\|\mathrm{e}^{V_\varepsilon} (\mathrm{e}^\phi - 1 - \phi)\|_* \le C\varepsilon \|\phi\|_\infty^2.$$

In the other hand

$$\lambda^{-2} f(\varepsilon y) \mathrm{e}^{V_\varepsilon(y)} = \lambda^2 \delta^2 \mathrm{e}^{v_\varepsilon(x)}, \quad y = \frac{x}{\varepsilon}, x \in M,$$

so

$$\|\lambda^{-2} f(\varepsilon y) \mathrm{e}^{V_\varepsilon} (\mathrm{e}^\phi - 1 - \phi)\|_* \le C\varepsilon^{-\sigma} \|\phi\|_\infty^2.$$

We conclude,

$$\|N(\phi)\|_* \le C\varepsilon^{-\sigma} \|\phi\|_\infty^2.$$

Observe that for $\phi_1, \phi_2 \in H_\vartheta$,

$$\|N(\phi_1) - N(\phi_2)\|_* \le C\vartheta \varepsilon^{1-\sigma} \log\left(\frac{1}{\varepsilon}\right) \|\phi_1 - \phi_2\|_\infty,$$

where $C$ is independent of $\vartheta$. Hence, we have

$$\|A(\phi)\|_\infty \le C\varepsilon \log\left(\frac{1}{\varepsilon}\right) [\vartheta^2 \varepsilon^{1-\sigma} \log\left(\frac{1}{\varepsilon}\right) + 1],$$

$$\|A(\phi_1) - A(\phi_2)\|_\infty \le C\varepsilon^{1-\sigma} \log\left(\frac{1}{\varepsilon}\right) \|\phi_1 - \phi_2\|_\infty.$$

It follows that there exist $\varepsilon_0$, such that for all $\varepsilon < \varepsilon_0$ the operator $A$ is a contraction mapping from $H_\vartheta$ into itself, and therefore $A$ has a unique fixed point in $H_\vartheta$. This concludes the proof. □

With these ingredients we are now ready for the proof of our main result.

## 1.6 Proof of Theorem 1.2 for $n = 1$

After problem (1.49) has been solved, we find a solution to problem (2.21), and hence to the original problem, if $k = k(\varepsilon)$ is such that

$$c_{\mathrm{i}}(k) = 0, \quad \mathrm{i} = 1, 2. \tag{1.52}$$

Let us consider local conformal coordinates around $p$ and define $r = |y|$. We consider a smooth cut-off function $\eta(r)$ such that $\eta(r) = 1$ for $r < \frac{1}{\sqrt{\varepsilon}}$, $\eta(r) = 0$ for $r > \frac{2}{\sqrt{\varepsilon}}$, $|\eta'(r)| \le C\sqrt{\varepsilon}$, $|\eta''(r)| \le C\varepsilon$. Testing the equation

$$L(\phi) = N(\phi) + E + \sum_{\mathrm{i}=1}^{2} c_{\mathrm{i}} \chi Z_{\mathrm{i}},$$

against $\eta Z_{\mathrm{i}}$, $\mathrm{i} = 1, 2$, we find

$$\langle L(\phi), \eta Z_{\mathrm{i}} \rangle = \int_{M_\varepsilon} [N(\phi) + E] \eta Z_{\mathrm{i}} + c_{\mathrm{i}} \int_{M_\varepsilon} \chi Z_{\mathrm{i}}^2, \quad \mathrm{i} = 1, 2.$$

Therefore, we have the validity of (1.52) if and only if

$$\langle L(\phi), \eta Z_{\mathrm{i}} \rangle - \int_{M_\varepsilon} [N(\phi) + E] \eta Z_{\mathrm{i}} = 0, \quad \mathrm{i} = 1, 2.$$

We recall that in the proof of Proposition (2.5) we obtained

$$|\langle \phi, L(\eta Z_{\mathrm{i}}) \rangle| \le C\sqrt{\varepsilon} \|\phi\|_\infty,$$

thus

$$|\langle \phi, L(\eta Z_{\mathrm{i}}) \rangle| \le C\varepsilon^{3/2} \log \frac{1}{\varepsilon}.$$

Observe that

$$\|N(\phi)\|_\infty \le C\varepsilon^2 \|\phi\|_\infty^2,$$

so

$$\left| \int_{M_\varepsilon} N(\phi) \eta Z_{\mathrm{i}} \right| \le C\varepsilon \|\phi\|_\infty^2 \le C\varepsilon^3 \log^2 \frac{1}{\varepsilon}$$

Let us remember that

$$E = -\Delta V_\varepsilon + \lambda^{-2} f(\varepsilon y) \mathrm{e}^{V_\varepsilon} - \mathrm{e}^{V_\varepsilon} - \varepsilon^2 \alpha.$$

Using (1.51), we have

$$\int_{M_\varepsilon} \mathrm{e}^{V_\varepsilon} \eta Z_{\mathrm{i}} = O(\varepsilon^4).$$

We also have,

$$\int_{M_\varepsilon} \varepsilon^2 \alpha \eta Z_{\mathrm{i}} = O(\varepsilon).$$

Observe that

$$\Delta_g V_\varepsilon(y) = \varepsilon^2 \Delta_g v_\varepsilon(x), \quad y = \frac{x}{\varepsilon}, x \in M,$$

thus

$$\int_{M_\varepsilon} \Delta V_\varepsilon \eta Z_{\rm i} = O(\varepsilon^2).$$

Also, by change of variables we have

$$\int_{M_\varepsilon} f(\varepsilon y) \mathrm{e}^{V_\varepsilon} \eta Z_{\rm i} = \int_{\tilde{M}_\varepsilon} f(p + \varepsilon(y+k)) \mathrm{e}^{V_\varepsilon(y+k+p')} \eta(|y+k|) Z_{\rm i}(y+k+p'),$$

where $\tilde{M}_\varepsilon = M_\varepsilon - k + p'$. Using the fact that $p$ is a local maximum of $f$ of value 0, we have

$$f(p + \varepsilon(y+k)) = \varepsilon^2 \langle (y+k), D^2 f(p)(y+k) \rangle + O(\varepsilon^3),$$

where we used the fact that $f \in C^3(M)$. Thus

$$\lambda^{-2} \int_{M_\varepsilon} f(\varepsilon y) \mathrm{e}^{V_\varepsilon} \eta Z_{\rm i} = I_{\rm i} + II_{\rm i},$$

where

$$I_{\rm i} = \delta^2 \int_{\tilde{M}_\varepsilon} \langle (y+k), H_f(p)(y+k) \rangle \mathrm{e}^{V_\varepsilon(y+k+p')} \eta(|y+k|) Z_{\rm i}(y+k+p')$$
$$II_{\rm i} = \int_{\tilde{M}_\varepsilon} O(\varepsilon) \mathrm{e}^{V_\varepsilon(y+k+p')} \eta(|y+k|) Z_{\rm i}(y+k+p').$$

Observe that $\mathrm{e}^{V_\varepsilon(y+k+p')} \eta(|y+k|) Z_{\rm i}(y+k+p') = O((1+|y|)^{-4})$, so

$$II_{\rm i} = O(\varepsilon).$$

Finally, let us compute $I_{\rm i}$. In the first place, observe that $0 \in \tilde{M}_\varepsilon$. Let us consider a fixed number $A_0$, such that $\mathcal{B}_1 = B(0, A_0/\sqrt{\varepsilon}) \subset \tilde{M}_\varepsilon \cap \operatorname{supp}(\eta(\cdot + k)) := \mathcal{B}$ and $\eta(\cdot + k) = 1$ in $\mathcal{B}_1$. We have the decomposition $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$, where $\mathcal{B}_2 = \tilde{\Omega}_\varepsilon \cap \operatorname{supp}(\eta(\cdot + k)) \setminus \mathcal{B}_1$. Also, observe that

$$Z_{\rm i}(y+k+p') = C_0 \frac{y_{\rm i}}{1+|y|}, \quad {\rm i} = 1, 2,$$

where $C_0$ is a fixed constant independent of $\varepsilon$. We have the following computation

$$\langle (y+k), D^2 f(p)(y+k) \rangle = f_{11}(p)(y_1+k_1)^2 + 2 f_{12}(p)(y_1+k_1)(y_2+k_2) + f_{22}(p)(y_2+k_2)^2,$$

where $f_{11}(p) = \frac{\partial^2 f}{\partial y_1^2}(p)$, $f_{22}(p) = \frac{\partial^2 f}{\partial y_2^2}(p)$ and $f_{12}(p) = f_{21}(p) = \frac{\partial^2 f}{\partial y_1 \partial y_2}(p)$. We recall that

$$\mathrm{e}^{V_\varepsilon(y+k+p')} = \frac{H_0}{(1+|y|^2)^2} (1 + C\sqrt{\varepsilon} + O(\varepsilon)), \tag{1.53}$$

in the region $\tilde{\Omega}_\varepsilon \cap \operatorname{supp}(\eta(\cdot + k))$. We define $t(y) = \mathrm{e}^{V_\varepsilon(y+k+p')}\eta(|y+k|)Z_{\mathrm{i}}(y+k+p')$. We have

$$\begin{aligned}\int_{\mathcal{B}} f_{11}(p)(y_1+k_1)^2 t(y) &= \int_{\mathcal{B}_1} f_{11}(p)(y_1+k_1)^2 t(y) + \int_{\mathcal{B}_2} f_{11}(p)(y_1+k_1)^2 t(y) \\ &= 2k_1 f_{11}(p) \int_{\mathcal{B}_1} C_0 \frac{y_1^2}{1+|y|} \frac{H_0}{(1+|y|^2)^2} + O(\varepsilon).\end{aligned}$$

In order to get the previous result, we used the fact that

$$\int_{\mathcal{B}_1} \frac{y_1}{1+|y|} \frac{\mathrm{d}y}{(1+|y|^2)^2} = \int_{\mathcal{B}_1} \frac{y_1^3}{1+|y|} \frac{\mathrm{d}y}{(1+|y|^2)^2} = 0,$$

and the expansion (1.53). We also have

$$\int_{\mathcal{B}_1+\mathcal{B}_2} 2f_{12}(p)(y_1+k_1)(y_2+k_2)t(y) = 2k_2 f_{12}(p) \int_{\mathcal{B}_1} C_0 \frac{y_1^2}{1+|y|} \frac{H_0}{(1+|y|^2)^2} + O(\varepsilon),$$

where we used the fact that

$$\int_{\mathcal{B}_1} \frac{y_1 y_2}{1+|y|} \frac{1}{(1+|y|^2)^2} = 0,$$

and also the expansion (1.53). Finally, we have

$$\int_{\mathcal{B}_1+\mathcal{B}_2} f_{22}(p)(y_2+k_2)^2 t(y) = O(\varepsilon),$$

where we used the fact that

$$\int_{\mathcal{B}_1} \frac{y_1 y_2^2}{1+|y|} \frac{1}{(1+|y|^2)^2} = 0,$$

and also the expansion (1.53). From the above computations we conclude that

$$I_1 = 2\delta^2 I k_1 f_{11}(p) + 2\delta^2 I k_2 f_{12}(p) + O(\varepsilon),$$

where

$$I = \int_{\mathcal{B}_1} C_0 \frac{y_1^2}{1+|y|} \frac{H_0}{(1+|y|^2)^2} > 0.$$

Similar computations yield

$$I_2 = 2\delta^2 I k_1 f_{12}(p) + 2\delta^2 I k_2 f_{22}(p) + O(\varepsilon).$$

Summarizing, we have the system

$$\delta^2 D^2 f(p) k = \varepsilon b(k), \tag{1.54}$$

where $b$ is a continuous function of $k$ of size $O(1)$. Since $p$ is a non-degenerate critical point of $f$, we know that $D^2 f(p)$ is invertible. A simple degree theoretical argument, yields that system (1.54) has a solution $k = O(\lambda\delta^{-1})$. We thus obtain $c_1(k) = c_2(k) = 0$, and we have found a solution of the original problem. The proof for the case $k = 1$ is thus concluded. □

## 1.7 Proof of Theorem 1.2 for general $n$

In this section we will detail the main changes in the proof of our main result, in the case of multiple bubbling.

Let $p_1, \dots, p_n$ be points such that $f(p_j) = 0$ and $D^2 f(p_j)$ is positive definite for each $j$. We consider the singular problem

$$\Delta_g G - f e^G + 8\pi \sum_{j=1}^{k} \delta_{p_j} + \alpha = 0, \quad \text{in } M, \tag{1.55}$$

where $\delta_p$ designates the Dirac mass at the point $p$. A first remark we make is that the proof of Lemma 2.8 applies with no changes (except some additional notation) to find the result of Lemma 2.6. Indeed, the core of the proof is the local asymptotic analysis around each point $p_j$.

We define the first approximation in the original variable as

$$U_\varepsilon = \sum_{j=1}^{n} \eta_j u_\varepsilon^j + \left(1 - \sum_{j=1}^{n} \eta_j\right) G,$$

where $\eta_j$ is defined around $p_j$ as in Section 1.3 and, in local conformal coordinates around $p_j$, $u_\varepsilon^j(x) = w_{\delta_j}(|x - k_j|) + \tilde{F}_j(|x|)$, for parameters $k_j \in \mathbb{R}^2$. We make the following choice of the parameters $\delta_j$

$$\log 8\delta_{\mathrm{i}}^2 = -2 \log\left(\frac{1}{\sqrt{2}} \log \frac{1}{\lambda}\right) + \mathcal{H}(p_{\mathrm{i}}).$$

We also define the first approximation in the expanded variable around each $p_j$ by

$$V_{\varepsilon_j}(y) = U_\varepsilon(\varepsilon_j y) + 4 \log \lambda + 2 \log \delta_j, \quad y \in M_{\varepsilon_j}$$

where $\varepsilon_j = \lambda \delta_j$ and $M_{\varepsilon_j} = \varepsilon_j^{-1} M$.

We look for a solution of problem (1.8) of the form $u(y) = U_\varepsilon(x) + \phi(x)$, where $\phi$ represent a lower order correction. By simplicity, we denote also by $\phi$ the small correction in the expanded variable around each $p_j$. In terms of $\phi$, the expanded problem around $p_j$

$$\Delta_g v - \lambda^{-2} f(\varepsilon_j y) e^v + e^v + \varepsilon_j^2 \alpha = 0, \quad y \in M_{\varepsilon_j}$$

reads

$$L_j(\phi) = N_j(\phi) + E_j, \quad \text{in } M_{\varepsilon_j},$$

where

$$\begin{aligned} L_j(\phi) :=& \Delta_g \phi - \lambda^{-2} f(\varepsilon_j y) e^{V_{\varepsilon_j}} \phi + e^{V_{\varepsilon_j}} \phi, \\ N_j(\phi) :=& \lambda^{-2} f(\varepsilon_j y) e^{V_{\varepsilon_j}} (e^\phi - 1 - \phi) - e^{V_{\varepsilon_j}} (e^\phi - 1 - \phi), \\ E_j :=& -\left(\Delta_g V_{\varepsilon_j} - \lambda^{-2} f(\varepsilon_j y) e^{V_{\varepsilon_j}} + e^{V_{\varepsilon_j}} + {\varepsilon_j}^2 \alpha\right). \end{aligned}$$

Next we consider the linearized problem around our first approximation $U_\varepsilon$. Given $h$ of class $C^{0,\beta}(M)$, which by simplicity we still denote by $h$ in the expanded variable around each $p_j$, we consider the linear problem of finding a function $\phi$ such that for certain scalars $c_{\rm i}^j$, $\mathrm{i} = 1,2;\ j = 1,\dots,n$, one has

$$\left\{ \begin{array}{rcll} L_j(\phi) & = & h + \sum_{\rm i=1}^2 \sum_{j=1}^n c_{\rm i}^j \chi_j Z_{{\rm i}j} & \text{in } M_{\varepsilon_j}, \\ \displaystyle\int_{M_{\varepsilon_j}} \chi_j Z_{{\rm i}j}\phi & = & 0 & \text{for all } {\rm i}, j. \end{array} \right. \tag{1.56}$$

Here the definitions of $Z_{{\rm i}j}$ and $\chi_j$ are the same as before for $Z_{\rm i}$ and $\chi$, with the dependence of the point $p_j$ emphasized.

To solve this problem we consider now the norm

$$\|h\|_* = \sum_{j=1}^n \|h\|_{*,p_j}. \tag{1.57}$$

where $\|h\|_{*,p_j}$ is defined accordingly with (1.37). With exactly the same proof as in the case $n = 1$, we find the unique bounded solvability of Problem 1.56 for all small $\varepsilon = \max \varepsilon_{\rm i}$ by $\phi = T(h)$, so that

$$\|T(h)\|_\infty \le C \log\left(\frac{1}{\varepsilon}\right) \|h\|_*. \tag{1.58}$$

Then we argue as in the proof of Lemma 1.6 to obtain existence and uniqueness of a small solution $\phi$ of the projected nonlinear problem

$$\left\{ \begin{array}{rcll} L_j(\phi) & = & N_j(\phi) + E_j + \sum_{\rm i=1}^2 \sum_{j=1}^n c_{\rm i}^j \chi_j Z_{{\rm i}j} & \text{in } M_{\varepsilon_j}, \\ \displaystyle\int_{M_{\varepsilon_j}} \chi_j Z_{{\rm i}j}\phi & = & 0 & \text{for all } {\rm i}, j. \end{array} \right.$$

with

$$\|\phi\|_\infty \le C\varepsilon \log \frac{1}{\varepsilon}.$$

After this, we proceed as in Section 1.6 to choose the parameters $k_j$ in such a way that $c_{\rm i}^j = 0$ for all ${\rm i}, j$. Summarizing, we have the system

$$D^2 f(p_j) k_j = \varepsilon_{\rm i} \delta_{\rm i}^{-2} b_j(k_1, \dots, k_n), \tag{1.59}$$

which can be solved by the same degree-theoretical argument employed before. The proof is concluded. $\square$

# Chapter 2

# Critical interior bubbling in a semilinear Neumann problem in dimension 3

## 2.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $\lambda > 0$. Let us consider the problem

$$\begin{cases} -\Delta u + \lambda u - u^q &= 0 \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\nu$ denotes the unit outgoing normal and $q > 1$. This problem has been widely considered in the literature for more than 20 years. Lin Ni and Takagi initiated the study of this problem [22, 26, 27]. Integrating the equation in $\Omega$ yields that a necessary condition for solvability of (2.1) is $\lambda > 0$. This boundary value problem represents a model of different phenomena which exhibit concentrating behavior of families of their solutions. For instance, this equation arises as the so-called shadow system associated to the Gierer-Meinhardt activator-inhibitor model in mathematical theory of biological pattern formation. It also appears in certain models of chemotaxis. In those problems, it is particularly meaningful the presence of solutions exhibiting peaks of concentration, namely one or several local maxima around which the solution remains strictly positive, while being very small away from them.

Problem (2.1) has a variational structure, since its solutions correspond to critical points of the energy functional

$$E_q(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{\lambda}{2}\int_\Omega u^2 \mathrm{d}x - \frac{1}{q+1}\int_\Omega |u|^{q+1}\mathrm{d}x,$$

defined for all $u \in H^1(\Omega) \bigcap L^{q+1}$. It follows from Sobolev's embedding theorem that $H^1(\Omega) \hookrightarrow L^{q+1}$ continuously for $2 < q+1 \le 2^* = \frac{n+2}{n-2}$, thus the functional $E_q$ is well defined on $H^1(\Omega)$, for all $2 < q+1 \le 2^*$. Moreover, the previous embedding is compact if $1 < q < 2^* - 1$.

Observe that Problem 2.1 admits the constant solution $u_\lambda = \lambda^{\frac{1}{q-1}}$. In the case $1 < q < 2^*-1$, the compactness of the embedding and variational techniques imply that if $\lambda$ is sufficiently small, then $u_\lambda$ is the unique solution to Problem 2.1. This result inspired Lin and Ni to conjecture an extension of this result to the critical case $q+1 = 2^*$: For $\lambda$ sufficiently small, the constant solution $u_\lambda$ is the unique solution to Problem 2.1. In the radial case, that is when $\Omega$ is a ball and when $u$ is radially symmetrical, Adimurthi-Yadava solved the problem in [1, 2]. The result depends strongly in the dimension: when $n = 3$ or $n \geq 7$, the answer to Lin-Ni's question is affirmative, and it is negative for $n = 4, 5, 6$. In the asymmetric case, the complete answer is not known yet. When $n = 3$, it was proved by Zhu [31] and Wei-Xu [29] that the answer to Lin-Ni's question is positive when $\Omega$ is convex and $\lambda$ sufficiently small. When $n = 5$, Rey-Wei [28] constructed solutions as a sum of interior peaks for $\lambda \to 0$.

Construction of single and multiple spike-layer patterns for this problem in the subcritical case $q < \frac{n+2}{n-2}$ has been the object of many studies, see for instance [11, 17, 23, 12], and also [13] in the supercritical case $q > \frac{n+2}{n-2}$.

In what follows, we work on the case $n = 3$, $q = 5$. Even when the answer to the Lin-Ni's question is positive in dimension 3 for some special type of domains, we will show that this situation is different when the parameter $\lambda$ converges to some strictly positive critical value. In fact, it turns out that the object driving the location of blowing-up in single-bubble solutions of (2.1) is the *Robin's function* $g_\lambda$ defined as follows. Let $0 < \lambda$ and consider Green's function $G_\lambda(x, y)$, solution for any given $x \in \Omega$ of

$$\begin{aligned} -\Delta_y G_\lambda + \lambda G_\lambda &= \delta_x \qquad y \in \Omega\,, \\ \frac{\partial G_\lambda}{\partial \nu}(x,y) &= 0 \qquad y \in \partial\Omega\,. \end{aligned}$$

Let $H_\lambda(x, y) = \Gamma(y - x) - G_\lambda(x, y)$ with $\Gamma(z) = \frac{1}{4\pi|z|}$, be its regular part. In other words, $H_\lambda(x, y)$ can be defined as the unique solution of the problem

$$\begin{aligned} -\Delta_y H_\lambda + H_\lambda &= \lambda\,\Gamma(x-y) \qquad y \in \Omega\,, \\ \frac{\partial H_\lambda}{\partial \nu} &= \frac{\partial \Gamma(x-y)}{\partial \nu} \qquad y \in \partial\Omega\,. \end{aligned}$$

Let us consider Robin's function of $G_\lambda$, defined as

$$g_\lambda(x) \equiv H_\lambda(x, x)\,.$$

It turns out that $g_\lambda(x)$ is a smooth function (we provide a proof of this fact in the appendix).

We consider here the role of *non-trivial critical values* of $g_\lambda$ in existence of solutions of (2.1). The following is our principal result

**Theorem 2.1** *Suppose that for a number $\lambda = \lambda_0 > 0$, one of the two situations holds* *(a) Either there is an open, bounded set $\mathbb{D}$ of $\Omega$ such that*

$$0 = \sup_{\mathbb{D}} g_{\lambda_0} > \sup_{\partial\mathbb{D}} g_{\lambda_0} .$$

*(b) Or there is a $\zeta_0 \in \Omega$ such that*

$$g_{\lambda_0}(\zeta_0) = 0, \quad \nabla g_{\lambda_0}(\zeta_0) = 0 ,$$

*and $D^2 g_{\lambda_0}(\zeta_0)$ is non-singular.*

*Then for all $\lambda > \lambda_0$ sufficiently close to $\lambda_0$ there exists a solution $u_\lambda$ of Problem (2.1) of the form*

$$u_\lambda(x) \;=\; \frac{3^{1/4}\, M_\lambda}{\sqrt{1 \,+\, M_\lambda^4\, |x - \zeta_\lambda|^2}}\,(1 + o(1)) \tag{2.2}$$

*where $o(1) \to 0$ uniformly in $\overline{\Omega}$ as $\lambda \downarrow \lambda_0$, and the number $M_\lambda$ depends on the Robin's function and $\lambda_0$. Here $\zeta_\lambda \in \mathbb{D}$ in case (a) and $\zeta_\lambda \to \zeta_0$ in case (b).*

The rest of this work will be devoted to the proof of Theorem 2.1.

## 2.2 Energy expansion

For $\varepsilon > 0$, we consider the transformation

$$u(x) = \frac{1}{\varepsilon^{1/2}} v\left(\frac{x}{\varepsilon}\right)$$

therefore $v$ solves the problem

$$\begin{cases} -\Delta v + \varepsilon^2 \lambda v - v^5 &= 0 \quad \text{in } \Omega_\varepsilon, \\ v &> 0 \quad \text{in } \Omega_\varepsilon, \\ \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.3}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$.

We fix a point $\zeta \in \Omega$ and a positive number $\mu$. We denote in what follows

$$w_{\zeta,\mu}(x) = 3^{1/4} \frac{\mu^{1/2}}{\sqrt{\mu^2 + |x - \zeta|^2}}$$

which correspond to all positive solutions of the problem

$$-\Delta w - w^5 = 0, \quad \text{in } \mathbb{R}^3.$$

We define $\pi_{\zeta,\mu}(x)$ to be the unique solution of the problem

$$\begin{cases} -\Delta\pi_{\zeta,\mu} + \lambda\pi_{\zeta,\mu} &= -\lambda w_{\zeta,\mu} \quad \text{in } \Omega, \\ \frac{\partial \pi_{\zeta,\mu}}{\partial\nu} &= -\frac{\partial w_{\zeta,\mu}}{\partial \nu} \quad \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

We consider as a first approximation of the solution of (2.1) one of the form

$$U_{\zeta,\mu} = w_{\zeta,\mu} + \pi_{\zeta,\mu}. \tag{2.5}$$

Observe that $U_{\zeta,\mu}$ satisfies the problem

$$\begin{cases} -\Delta U_{\zeta,\mu} + \lambda U_{\zeta,\mu} &= w^5_{\zeta,\mu} \quad \text{in } \Omega, \\ \frac{\partial U_{\zeta,\mu}}{\partial\nu} &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Let us also observe that

$$\int_\Omega w^5_{\zeta,\mu} = C\mu^{1/2}\left(1+o(1)\right), \quad \text{as } \mu\to 0,$$

which implies that

$$\frac{w^5_{\zeta,\mu}}{\int_\Omega w^5_{\zeta,\mu}} \to 0, \quad \text{as } \mu \to 0,$$

uniformly on compacts subsets of $\overline{\Omega}\setminus\{\zeta\}$. It follows that on each of this subsets

$$U_{\zeta,\mu}(x) = \left(\int_\Omega w^5_{\zeta,\mu}\right) G(x,\zeta) = C\mu^{1/2}\left(1+o(1)\right) G(x,\zeta) \tag{2.7}$$

where $G(x,\zeta)$ denotes the Green's function, solution of the problem

$$\begin{cases} -\Delta G(x,\zeta) + \lambda G(x,\zeta) &= \delta_\zeta \quad \text{in } \Omega, \\ \frac{\partial G(x,\zeta)}{\partial\nu} &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\delta_\zeta$ denotes the Dirac mass centered at the point $\zeta$.

Using the transformation $U_{\zeta,\mu}(x) = \frac{1}{\varepsilon^{1/2}} V\left(\frac{x}{\varepsilon}\right)$ we see that $V$ solves the problem

$$\begin{cases} -\Delta V + \varepsilon^2\lambda V - w^5_{\zeta',\mu'} &= 0 \quad \text{in } \Omega_\varepsilon, \\ \frac{\partial V}{\partial\nu} &= 0 \quad \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $w_{\zeta',\mu'}(x) = 3^{1/4}\frac{\mu'^{1/2}}{\sqrt{\mu'^2+|x-\zeta'|^2}}$ and $\zeta' = \varepsilon^{-1}\zeta$, $\mu' = \varepsilon^{-1}\mu$.

The following lemma establishes the relationship between the functions $\pi_{\zeta,\mu}(x)$ and the regular part of the Green's function $(\zeta, x)$. Let us consider the (unique) radial solution $\mathbb{D}_0(z)$ of the problem in entire space,

$$\begin{cases} -\Delta\mathbb{D}_0 &= \lambda 3^{1/4}\left[\frac{1}{\sqrt{1+|z|^2}} - \frac{1}{|z|}\right] \quad \text{in } \mathbb{R}^3, \\ \mathbb{D}_0 &\to 0 \quad \text{as } |z|\to\infty. \end{cases}$$

$\mathbb{D}_0(z)$ is a $C^{0,1}$ function with $\mathbb{D}_0(z) \sim |z|^{-1}\log|z|$, as $|z|\to\infty$.

**Lemma 2.2** *For any $\sigma > 0$ we have the validity of the following expansion as $\mu \to 0$*

$$\mu^{-1/2}\pi_{\mu,\zeta}(x) = -4\pi 3^{1/4} H_\lambda(\zeta, x) - \mu \mathbb{D}_0\left(\frac{x-\zeta}{\mu}\right) + \mu^{2-\sigma}\theta(\zeta,\mu,x).$$

*where for $j = 0,1,2$, $\mathrm{i} = 0,1$ $\mathrm{i}+j \le 2$, the function $\mu^j \frac{\partial^{\mathrm{i}+j}}{\partial\zeta^{\mathrm{i}}\partial\mu^j}\theta(\zeta,\mu,x)$ is bounded uniformly on $x \in \Omega$, all small $\mu$ and $\zeta$, in compacts subsets of $\Omega$.*

Proof. We recall that $H_\lambda(\zeta,x)$ satisfies the equation

$$\begin{cases} -\Delta_x H_\lambda + \lambda H_\lambda &= \lambda\Gamma(x-\zeta) \quad x \in \Omega, \\ \frac{\partial H_\lambda(\zeta,x)}{\partial\nu} &= \frac{\partial\Gamma(x-\zeta)}{\partial\nu} \quad x \in \partial\Omega, \end{cases}$$

where $\Gamma(z) = \frac{1}{4\pi|z|}$.

Let us set $\mathbb{D}_1(x) = \mu\mathbb{D}_0(\mu^{-1}(x-\zeta))$, so that $\mathbb{D}_1$ satisfies

$$\begin{cases} -\Delta\mathbb{D}_1 &= \lambda\left[\mu^{-1/2}w_{\zeta,\mu}(x) - 4\pi 3^{1/4}\Gamma(x-\zeta)\right] \quad x \in \Omega, \\ \frac{\partial\mathbb{D}_1}{\partial\nu} &\sim \mu^3\log\mu \quad \text{on } \partial\Omega, \text{ as } \mu \to 0. \end{cases}$$

Let us write

$$S_1(x) = \mu^{-1/2}\pi_{\zeta,\mu}(x) + 4\pi 3^{1/4} H_\lambda(\zeta,x) + \mathbb{D}_1(x).$$

With the notation of Lemma 2.2, this means

$$S_1(x) = \mu^{2-\sigma}\theta(\mu,\zeta,x).$$

Observe that for $x \in \partial\Omega$, as $\mu \to 0$,

$$\nabla(\mu^{-1/2}w_{\zeta,\mu}(x) + 4\pi 3^{1/4}\Gamma(x-\zeta))\cdot\nu \sim \mu^2|x-\zeta|^{-5}.$$

Using the above equations we find that $S_1$ satisfies

$$\begin{cases} -\Delta S_1 + \lambda S_1 &= -\lambda\mathbb{D}_1 \quad x \in \Omega, \\ \frac{\partial S_1}{\partial\nu} &= O(\mu^3\log\mu) \quad \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Observe that, for any $p > 3$,

$$\int_\Omega |\mathbb{D}_1(x)|^p \mathrm{d}x \le \mu^{p+3}\int_{\mathbb{R}^3}|\mathbb{D}_0(x)|^p\mathrm{d}x,$$

so that $\|\mathbb{D}_1\|_{L^p} \le C_p\mu^{1+3/p}$. Elliptic estimates applied to problem (2.8) yield that, for any $\sigma > 0$, $\|S_1\|_\infty = O(\mu^{2-\sigma})$ uniformly on $\zeta$ in compacts subsets of $\Omega$. This yields the assertion of the lemma for $\mathrm{i}, j = 0$.

We consider now the quantity $S_2 = \partial_\zeta S_1$. Observe that $S_2$ satisfies

$$\begin{cases} -\Delta S_2 + \lambda S_2 &= -\lambda\partial_\zeta\mathbb{D}_1 \quad x \in \Omega, \\ \frac{\partial S_2}{\partial\nu} &= O(\mu^3\log\mu) \quad \text{on } \partial\Omega. \end{cases}$$

Observe that $\partial_\zeta \mathbb{D}_1(x) = -\nabla D_0\left(\frac{x-\zeta}{\mu}\right)$, so that for any $p > 3$,

$$\int_\Omega |\partial_\zeta \mathbb{D}_1(x)|^p \mathrm{d}x \le \mu^{3+p} \int_{\mathbb{R}^3} |\nabla \mathbb{D}_0(x)|^p \mathrm{d}x$$

We conclude that $\|S_2\|_\infty = O(\mu^{2-\sigma})$, for any $\sigma > 0$. This gives the proof of the lemma for $\mathrm{i} = 1$, $j = 0$. Now we consider $S_3 = \mu\partial_\mu S_1$. Then

$$\begin{cases} -\Delta S_3 + \lambda S_3 &= -\lambda\mu\partial_\mu \mathbb{D}_1 \quad x \in \Omega, \\ \frac{\partial S_3}{\partial \nu} &= O(\mu^3 \log \mu) \quad \text{on } \partial\Omega. \end{cases}$$

Observe that

$$\mu\partial_\mu D_1(x) = \mu(\mathbb{D}_0 - \overline{\mathbb{D}}_0)\left(\frac{x-\zeta}{\mu}\right),$$

where $\overline{\mathbb{D}}_0(z) = \nabla\mathbb{D}_0(z)\cdot z$. Thus, similarly as the estimate for $S_1$ itself we obtain $\|S_3\|_\infty = O(\mu^{2-\sigma})$, for any $\sigma > 0$. This yields the assertion of the lemma for $\mathrm{i} = 0$, $j = 1$. The proof of the remaining estimates comes after applying again $\mu\partial_\mu$ to the equations obtained for $S_2$ and $S_3$ above, and the desired result comes after exactly the same arguments. This concludes the proof. □

Classical solutions to (2.1) correspond to critical points of the energy functional

$$E_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{6}\int_\Omega |u|^6. \tag{2.9}$$

If there was a solution very close to $U_{\zeta^*,\mu^*}$ for a certain pair $(\zeta^*, \mu^*)$, then we would formally expect $E_\lambda$ to be nearly stationary with respect to variations of $(\zeta,\mu)$ on $U_{\zeta,\mu}$ around this point. It seems important to understand critical points of the functional $(\zeta,\mu) \to E_\lambda(U_{\zeta,\mu})$. In the following lemma we find explicit asymptotic expressions for this functional.

**Lemma 2.3** *For any $\sigma > 0$, as $\mu \to 0$, the following expansion holds*

$$E_\lambda(U_{\zeta,\mu}) = a_0 + a_1\mu g_\lambda(\zeta) + a_2\mu^2\lambda - a_3\mu^2 g_\lambda^2(\zeta) + \mu^{3-\sigma}\theta(\zeta,\mu) \tag{2.10}$$

*where for $j = 0,1,2$, $\mathrm{i} = 0,1$, $\mathrm{i} + j \le 2$, the function $\mu^j \frac{\partial^{\mathrm{i}+j}}{\partial\zeta^{\mathrm{i}}\partial\mu^j}\theta(\zeta,\mu)$ is bounded uniformly on all small $\mu$ and $\zeta$ in compact subsets of $\Omega$.*

Proof. Observe that

$$E_\lambda(U_{\zeta,\mu}) = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI},$$

where

$$\begin{aligned}
\mathrm{I} &= \int_\Omega \left(\frac{1}{2}|\nabla w_{\zeta,\mu}|^2 - \frac{1}{6} w_{\zeta,\mu}^6\right),\\
\mathrm{II} &= \int_\Omega \left(\nabla w_{\zeta,\mu}\cdot\nabla\pi_{\zeta,\mu} - w_{\zeta,\mu}^5\pi_{\zeta,\mu}\right),\\
\mathrm{III} &= \frac{1}{2}\int_\Omega \left[|\nabla\pi_{\zeta,\mu}|^2 + \lambda(w_{\zeta,\mu}+\pi_{\zeta,\mu})\pi_{\zeta,\mu}\right],\\
\mathrm{IV} &= \frac{\lambda}{2}\int_\Omega (w_{\zeta,\mu}+\pi_{\zeta,\mu})w_{\zeta,\mu},\\
\mathrm{V} &= -\frac{5}{2}\int_\Omega w_{\zeta,\mu}^4\pi_{\zeta,\mu}^2,\\
\mathrm{VI} &= -\frac{1}{6}\int_\Omega \left[(w_{\zeta,\mu}+\pi_{\zeta,\mu})^6 - w_{\zeta,\mu}^6 - 6w_{\zeta,\mu}^5\pi_{\zeta,\mu} - 15w_{\zeta,\mu}^4\pi_{\zeta,\mu}^2\right].
\end{aligned}$$

Multiplying equation $-\Delta w_{\zeta,\mu} = w_{\zeta,\mu}^5$ by $w_{\zeta,\mu}$ and integrating by parts in $\Omega$ we obtain

$$\begin{aligned}
\mathrm{I} &= \frac{1}{2}\int_{\partial\Omega}\frac{\partial w_{\zeta,\mu}}{\partial\nu}w_{\zeta,\mu} + \frac{1}{3}\int_\Omega w_{\zeta,\mu}^6\\
&= \frac{1}{2}\int_{\partial\Omega}\frac{\partial w_{\zeta,\mu}}{\partial\nu}w_{\zeta,\mu} + \frac{1}{3}\int_{\mathbb{R}^3} w_{\zeta,\mu}^6 - \frac{1}{3}\int_{\mathbb{R}^3\setminus\Omega} w_{\zeta,\mu}^6.
\end{aligned}$$

Now, testing the same equation against $\pi_{\zeta,\mu}$, we find

$$\mathrm{II} = \int_{\partial\Omega}\frac{\partial w_{\zeta,\mu}}{\partial\nu}\pi_{\zeta,\mu} = -\int_{\partial\Omega}\frac{\partial\pi_{\zeta,\mu}}{\partial\nu}\pi_{\zeta,\mu},$$

where we have used the fact that $\pi_{\zeta,\mu}$ solves problem (2.4). Testing the equation $-\Delta\pi_{\zeta,\mu} + \lambda\pi_{\zeta,\mu} = -\lambda w_{\zeta,\mu}$ against $\pi_{\zeta,\mu}$ and integrating by parts in $\Omega$, we get

$$\mathrm{III} = \frac{1}{2}\int_{\partial\Omega}\frac{\partial\pi_{\zeta,\mu}}{\partial\nu}\pi_{\zeta,\mu}.$$

Testing equation $-\Delta w_{\zeta,\mu} = w_{\zeta,\mu}^5$ against $U_{\zeta,\mu} = w_{\zeta,\mu} + \pi_{\zeta,\mu}$ and integrating by parts twice, we obtain

$$\mathrm{IV} = \frac{1}{2}\int_{\partial\Omega}\frac{\partial\pi_{\zeta,\mu}}{\partial\nu}w_{\zeta,\mu} - \frac{1}{2}\int_{\partial\Omega}\frac{\partial w_{\zeta,\mu}}{\partial\nu}w_{\zeta,\mu} - \frac{1}{2}\int_\Omega w_{\zeta,\mu}^5\pi_{\zeta,\mu}.$$

From the mean value formula, we get

$$\mathrm{VI} = -10\int_0^1 \mathrm{d}s(1-s)^2\int_\Omega (w_{\zeta,\mu}+s\pi_{\zeta,\mu})^3\pi_{\zeta,\mu}^3.$$

Adding up the previous expressions we get so far

$$E_\lambda(U_{\zeta,\mu}) = \frac{1}{3}\int_{\mathbb{R}^3} w_{\zeta,\mu}^6 - \frac{1}{2}\int_\Omega w_{\zeta,\mu}^5\pi_{\zeta,\mu} - \frac{5}{2}\int_\Omega w_{\zeta,\mu}^4\pi_{\zeta,\mu}^2 + \mathcal{R}_1, \tag{2.11}$$

where

$$\mathcal{R}_1 = -\frac{1}{3}\int_{\mathbb{R}^3\setminus\Omega} w_{\zeta,\mu}^6 - 10\int_0^1 \mathrm{d}s(1-s)^2\int_\Omega (w_{\zeta,\mu}+s\pi_{\zeta,\mu})^3\pi_{\zeta,\mu}^3. \tag{2.12}$$

We will expand the second integral term of expression (2.11). Using the change of variable $x = \zeta + \mu z$ and calling $\Omega_\mu = \mu^{-1}(\Omega - \zeta)$, we find that

$$A_1 = \int_\Omega w_{\zeta,\mu}^5 \pi_{\zeta,\mu} \mathrm{d}x = \mu \int_{\Omega_\mu} w_{0,1}^5(z) \mu^{-1/2} \pi_{\zeta,\mu}(\zeta + \mu z) \mathrm{d}z.$$

From Lemma 2.2, we have the expansion

$$\mu^{-1/2}\pi_{\zeta,\mu}(\zeta + \mu z) = -4\pi 3^{1/4} H_\lambda(\zeta + \mu z, \zeta) - \mu \mathbb{D}_0(z) + \mu^{2-\sigma}\theta(\zeta, \mu, \zeta + \mu z).$$

According to Appendix A,

$$H_\lambda(\zeta + \mu z, \zeta) = g_\lambda(\zeta) + \frac{\lambda}{8\pi}\mu|z| + \Theta(\zeta, \zeta + \mu z),$$

where $\Theta$ is a function of class $C^2$ with $\Theta(\zeta, \zeta) = 0$. Using this fact , we obtain

$$A_1 = -4\pi 3^{1/4} \mu g_\lambda(\zeta) \int_{\mathbb{R}^3} w_{0,1}^5(z) \mathrm{d}z - \mu^2 \int_{\mathbb{R}^3} w_{0,1}^5(z) \left[\mathbb{D}_0(z) + \frac{3^{1/4}}{2}\lambda|z|\right] \mathrm{d}z + \mathcal{R}_2$$

with

$$\begin{aligned} \mathcal{R}_2 =& \mu \int_{\Omega_\mu} w_{0,1}^5(z)[\Theta(\zeta, \zeta + \mu z) + \mu^{2-\sigma}\theta(\zeta, \mu, \zeta + \mu z)] \mathrm{d}z \\ &+ \mu^2 \int_{\mathbb{R}^3 \setminus \Omega_\mu} w_{0,1}^5(z) \left[\mathbb{D}_0(z) + \frac{3^{1/4}}{2}\lambda|z|\right] \mathrm{d}z \\ &+ 4\pi 3^{1/4} \mu g_\lambda(\zeta) \int_{\mathbb{R}^3 \setminus \Omega_\mu} w_{0,1}^5(z) \mathrm{d}z. \end{aligned} \tag{2.13}$$

Let us recall that

$$-\Delta \mathbb{D}_0 = 3^{1/4}\lambda \left[\frac{1}{\sqrt{1 + |z|^2}} - \frac{1}{|z|}\right],$$

so that,

$$\begin{aligned} -\int_{\mathbb{R}^3} w_{0,1}^5 \mathbb{D}_0(z) &= \int_{\mathbb{R}^3} \Delta w_{0,1} \mathbb{D}_0(z) \\ &= \int_{\mathbb{R}^3} w_{0,1} \Delta \mathbb{D}_0(z) = 3^{1/4}\lambda \int_{\mathbb{R}^3} w_{0,1} \left[\frac{1}{|z|} - \frac{1}{\sqrt{1 + |z|^2}}\right]. \end{aligned}$$

Combining the above relations we get

$$\begin{aligned} A_1 =& -4\pi 3^{1/4} \mu g_\lambda(\zeta) \int_{\mathbb{R}^3} w_{0,1}^5(z) \mathrm{d}z \\ &- \mu^2 \lambda 3^{1/4} \int_{\mathbb{R}^3} \left[w_{0,1}(z) \left(\frac{1}{\sqrt{1 + |z|^2}} - \frac{1}{|z|}\right) + \frac{1}{2} w_{0,1}^5 |z|\right] \mathrm{d}z + \mathcal{R}_2. \end{aligned}$$

Let us consider now $A_2 = \int_\Omega w^4_{\zeta,\mu}\pi^2_{\zeta,\mu}$. We have

$$\begin{aligned} A_2 =&\mu \int_{\Omega_\mu} w^4_{0,1}(z)\pi^2_{\zeta,\mu}(\zeta+\mu z)\mathrm{d}z \\ =&\mu^2 \int_{\Omega_\mu} w^4_{0,1}(z)\left[-4\pi 3^{1/4} H_\lambda(\zeta+\mu z,\zeta) - \mu\mathbb{D}_0(z) + \mu^{2-\sigma}\theta(\zeta,\mu,\zeta+\mu z)\right]^2 \mathrm{d}z, \end{aligned}$$

which we expand as

$$A_2 = \mu^2 g^2_\lambda(\zeta) 16\pi^2 3^{1/2}\int_{\mathbb{R}^3} w^4_{0,1} + \mathcal{R}_3.$$

Combining relation (2.11) with the above expressions, we get so far

$$E_\lambda(U_{\zeta,\mu}) = a_0 + a_1\mu g_\lambda(\zeta) + a_2\lambda\mu^2 - a_3\mu^2 g^2_\lambda(\zeta) + \mathcal{R}_1 - \frac{1}{2}\mathcal{R}_2 - \frac{5}{2}\mathcal{R}_3,$$

where

$$\begin{aligned} a_0 =&\frac{1}{3}\int_{\mathbb{R}^3} w^6_{0,1}, \\ a_1 =&2\pi 3^{1/4}\int_{\mathbb{R}^3} w^5_{0,1}, \\ a_2 =&\frac{3^{1/4}}{2}\int_{\mathbb{R}^3}\left[w_{0,1}(z)\left(\frac{1}{\sqrt{1+|z|^2}} - \frac{1}{|z|}\right) + \frac{1}{2}w^5_{0,1}|z|\right]\mathrm{d}z, \\ a_3 =&40\pi^2 3^{1/2}\int_{\mathbb{R}^3} w^4_{0,1}. \end{aligned}$$

We want to establish the estimate

$$\mu^j \frac{\partial^{\mathrm{i}+j}}{\partial\zeta^{\mathrm{i}}\partial\mu^j}\mathcal{R}_l = O(\mu^{3-\sigma}),$$

for each $j = 0,1,2$, $\mathrm{i} = 0,1$, $\mathrm{i}+j \le 2$, $l = 1,2,3$, uniformly on all small $\mu$ and $\zeta$ in compact subsets of $\Omega$. This needs a corresponding analysis for each of the individual terms arising in the expressions for $\mathcal{R}_l$.

Since several of these computations are similar, we shall only carry in detail those that appear as most representative.

In (2.12) let us consider for instance the integral

$$\int_{\mathbb{R}^3\setminus\Omega} w^6_{\mu,\zeta} = 3^{\frac{3}{2}}\mu^3\int_{\mathbb{R}^3\setminus\Omega}\frac{1}{(\mu^2+|y-\zeta|^2)^3}.$$

From this expression it easily follows that

$$\mu^j \frac{\partial^{\mathrm{i}+j}}{\partial\zeta^{\mathrm{i}}\partial\mu^j}\int_{\mathbb{R}^3\setminus\Omega} w^6_{\mu,\zeta} = O(\mu^3),$$

uniformly in $\zeta$ in compact subsets of $\Omega$.

In (2.13), let us consider the term

$$B \equiv \mu \int_{\Omega_\mu} w_{1,0}^5(z) \left[\theta_1(\zeta, \zeta + \mu z) + \mu^{2-\sigma}\theta(\mu, \zeta, \zeta + \mu z)\right] \mathrm{d}z = B_1 + B_2 \,.$$

Let us observe that

$$B_2 \equiv \mu \int_{\Omega_\mu} w_{1,0}^5(z)\, \mu^{2-\sigma}\, \theta(\mu, \zeta, \zeta + \mu z)\, \mathrm{d}z = \mu^{-\sigma} \int_\Omega w_{1,0}^5 \left(\frac{y-\zeta}{\mu}\right) \theta(\mu, \zeta, y)\, \mathrm{d}y \,.$$

The size of this quantity in absolute value is clearly $O(\mu^{3-\sigma})$. We have then that

$$\partial_\zeta B_2 = I_{21} + I_{22} \,,$$

$$I_{21} = -\mu^{-\sigma} \int_\Omega \mu^{-1} D(w_{1,0}^5) \left(\frac{y-\zeta}{\mu}\right) \theta(\mu, \zeta, y)\, \mathrm{d}y \,,$$

$$I_{22} = \mu^{-\sigma} \int_\Omega w_{1,0}^5 \left(\frac{y-\zeta}{\mu}\right) \partial_\zeta \theta(\mu, \zeta, y)\, \mathrm{d}y \,.$$

Since $\partial_\zeta \theta(\mu, \zeta, y)$ is uniformly bounded for $\zeta$ ranging on compact subsets of $\Omega$, $B_{22}$ is of size $O(\mu^{3-\sigma})$. Now, using symmetry,

$$\begin{aligned} I_{22} &= \mu^{2-\sigma} \int_{\Omega_\mu} D(w_{1,0}^5) \left[\theta(\mu, \zeta, \zeta + \mu z) - \theta(\mu, \zeta, \zeta)\right] \\ &\quad - \mu^{2-\sigma} \theta(\mu, \zeta, \zeta) \int_{\mathbb{R}^3 \setminus \Omega_\mu} D(w_{1,0}^5) \\ &= \mu^{2-\sigma} \int_{\Omega_\mu} D(w_{1,0}^5) \left[\theta(\mu, \zeta, \zeta + \mu z) - \theta(\mu, \zeta, \zeta)\right] + o(\mu^3) \,. \end{aligned}$$

Now, $\theta$ is symmetric in $\zeta$ and $y$, hence has bounded derivative over compacts with respect to each of its arguments. Thus

$$\begin{aligned} &\left| \mu^{2-\sigma} \int_{\Omega_\mu} D(w_{1,0}^5)(z) \Big[\theta(\mu, \zeta, \zeta + \mu z) - \theta(\mu, \zeta, \zeta)\Big] \mathrm{d}z \right| \\ &\quad \le C\, \mu^{2-\sigma} \int_{\mu|z| \le \delta} \mu \left|D(w_{1,0}^5)(z)\right| |z|\, \mathrm{d}z + C\, \mu^{2-\sigma} \int_{\mu|z| > \delta} |z|^{-6}\, \mathrm{d}z = O(\mu^{3-\sigma}) \,. \end{aligned}$$

Let us consider now $B_1$. We can expand

$$\theta_1(\zeta, \zeta + \mu z) = \mu\, \mathbf{c} \cdot z + \theta_2(\zeta, \zeta + \mu z)$$

for a constant vector $\mathbf{c}$, where $\theta_2$ is a $C^2$ function with $|\theta_2(\zeta, y)| \le C\, |\zeta - y|^2$. Observe that by symmetry,

$$\mu^2 \int_{\Omega_\mu} w_{1,0}^5(z)\, \mathbf{c} \cdot z\, \mathrm{d}z = -\mu^2 \int_{\mathbb{R}^3 \setminus \Omega_\mu} w_{1,0}^5(z)\, \mathbf{c} \cdot z\, \mathrm{d}z = O(\mu^3) \,.$$

From here it easily follows that $B_1 = O(\mu^3 \log \mu)$. Let us decompose it as

$$B_1 = B_{11} + B_{12}\,,$$

$$B_{11} \equiv 3^{\frac{5}{2}}\,\mu^{-2} \int_\Omega (1 + \mu^{-2}|y-\zeta|^2)^{-\frac{5}{2}}\,\theta_2(\zeta, y)\,\mathrm{d}y\,,$$
$$B_{12} \equiv -3^{\frac{5}{2}}\,\mu^3 \int_{\mathbb{R}^3 \setminus \Omega} (\mu^2 + |y-\zeta|^2)^{-\frac{5}{2}}\,(y-\zeta)\cdot \mathbf{c}\,\mathrm{d}y\,.$$

$B_{12}$ has derivatives with respect to $\zeta$ uniformly bounded by $O(\mu^3)$. As for the first integral,

$$B_{11} = \mu^{-2} \int_\Omega w_{1,0}^5 \left(\frac{y-\zeta}{\mu}\right) \theta_2(\zeta, y)\,\mathrm{d}y\,,$$

we obtain that $\partial_\zeta B_{11}$ can be written as $I_{111} + I_{112}$ with

$$I_{111} = \mu^{-3} \int_\Omega D(w_{1,0}^5) \left(\frac{y-\zeta}{\mu}\right) \theta_2(\zeta, y)\,\mathrm{d}y\,,$$
$$I_{112} = \mu^{-2} \int_\Omega w_{1,0}^5 \left(\frac{y-\zeta}{\mu}\right) \partial_\zeta \theta_2(\zeta, y)\,\mathrm{d}y\,.$$

Let us estimate the second integral

$$I_{112} = \mu^{-2} \int_\Omega w_{1,0}^5 \left(\frac{y-\zeta}{\mu}\right) \partial_\zeta \theta_2(\zeta, y)\,\mathrm{d}y = \mu \int_\Omega w_{1,0}^5(z)\,\partial_\zeta \theta_2(\zeta, \zeta + \mu\, z)\,\mathrm{d}z\,.$$

We have that

$$\partial_\zeta \theta_2(\zeta, \zeta + \mu\, z) = \mu\,\mathbf{A}\,z + O(|\mu\, z|^2)$$

where $\mathbf{A} = D_2^2 \theta_2(\zeta, \zeta)$, where we have used the expansion for $H_\lambda$ made in the appendix. Replacing the above expression and making use of symmetry we get that $I_{112} = O(\mu^3 \log \mu)$. As for the integral $B_{11}$, we observe that after an integration by parts,

$$I_{111} = O(\mu^3) - \mu^{-2} \int_\Omega w_{1,0}^5 \left(\frac{y-\zeta}{\mu}\right) \partial_y \theta_2(\zeta, y)\,\mathrm{d}y\,.$$

The integral in the above expression can be treated in exactly the same way as $B_{12}$, and we thus find $\partial_\zeta B = O(\mu^{3-\sigma})$ uniformly over compacts of $\Omega$ in the variable $\zeta$ variable. In analogous way, we find similar bounds for $\mu\,\partial_\mu B$. The same type of estimate does hold for second derivatives $\mu^2 \partial_\mu^2 B$ and $\mu^2 \partial_{\mu\zeta}^2 B$. As an example, let us estimate, as a part of the latter, the quantity $\mu\,\partial_\mu I_{21}$. We have

$$\begin{aligned} \mu\,\partial_\mu I_{21} = {} & -\mu\,\partial_\mu \left[\mu^{-1-\sigma} \textstyle\int_\Omega D(w_{1,0}^5) \left(\frac{y-\zeta}{\mu}\right) \theta(\mu, \zeta, y)\,\mathrm{d}y\right] \\ = {} & (1+\sigma) I_{21} + \mu^{-\sigma} \textstyle\int_\Omega \mu^{-1} D^2(w_{1,0}^5) \left(\frac{y-\zeta}{\mu}\right) \cdot \left(\frac{y-\zeta}{\mu}\right) \theta(\mu, \zeta, y)\,\mathrm{d}y \\ & - \mu^{-1-\sigma} \textstyle\int_\Omega D(w_{1,0}^5) \left(\frac{y-\zeta}{\mu}\right) \mu\,\partial_\mu \theta(\mu, \zeta, y)\,\mathrm{d}y\,. \end{aligned}$$

Let us consider the term

$$\mu^{-\sigma}\int_\Omega \mu^{-1}D^2(w_{1,0}^5)\left(\frac{y-\zeta}{\mu}\right)\cdot\left(\frac{y-\zeta}{\mu}\right)\theta(\mu,\zeta,y)\,\mathrm{d}y\,,$$

the others being estimated in exactly the same way as before. The observation is that the estimate of this integral by $O(\mu^{3-\sigma})$ goes over exactly as that one before for $I_{21}$, where we simply need to replace the function $D(w_{1,0}^5)(z)$ by $D^2(w_{1,0}^5)z\cdot z$ which enjoys the same properties used in the former computation. Corresponding estimates for the remaining terms in $\mathcal{R}_2$ and $\mathcal{R}_3$ are obtained with similar computations, so that we omit them.

Summarizing, we have the validity of the desired expansion (2.10), which with the aid of the formula

$$\int_0^\infty \left(\frac{r}{1+r^2}\right)^q \frac{\mathrm{d}r}{r^{\alpha+1}} = \frac{\Gamma(\frac{q-\alpha}{2})\Gamma(\frac{q+\alpha}{2})}{2\Gamma(q)},$$

has constant $a_{\mathrm{i}}$ given by

$$a_0=\frac{1}{4}\sqrt{3}\pi^2,\quad a_1=8\sqrt{3}\pi^2,\quad a_2=2\sqrt{3}\pi(4-\pi),\quad a_3=120\sqrt{3}\pi^4.$$

The proof is completed. $\square$

## 2.3 Critical single-bubbling

The purpose of this section is to establish that in the situation of Theorems 2.1 there are critical points of $E_\lambda(U_{\mu,\zeta})$ which persist under properly small perturbations of the functional. As we shall rigorously establish later, this analysis does provide critical points of the full functional $E_\lambda$, namely solutions of (2.1), close to a single bubble of the form $U_{\mu,\zeta}$.

Let us suppose the situation (a) of local maximizer. In this section, and in the following lemma we work with $-g$ instead of $g$ in the situation (a). Thus we have the assumption of a local minimizer

$$0=\inf_{x\in\mathbb{D}} g_{\lambda_0}(x) < \inf_{x\in\partial\mathbb{D}} g_{\lambda_0}(x)\,.$$

Then for $\lambda$ close to $\lambda_0$, $\lambda>\lambda_0$, we will have

$$\inf_{x\in\mathbb{D}} g_\lambda(x) < -A\,(\lambda-\lambda_0)\,.$$

Let us consider the shrinking set

$$\mathbb{D}_\lambda=\left\{y\in\mathbb{D}\,:\,g_\lambda(x)<-\frac{A}{2}(\lambda-\lambda_0)\right\}$$

Assume $\lambda>\lambda_0$ is sufficiently close to $\lambda_0$ so that $g_\lambda=\frac{A}{2}(\lambda-\lambda_0)$ on $\partial\mathbb{D}_\lambda$.

Now, let us consider the situation of Part (b). Since $g_\lambda(\zeta)$ has a non-degenerate critical point at $\lambda = \lambda_0$ and $\zeta = \zeta_0$, this is also the case at a certain critical point $\zeta_\lambda$ for all $\lambda$ close to $\lambda_0$ where $|\zeta_\lambda - \zeta_0| = O(\lambda - \lambda_0)$.

Besides, for some intermediate point $\tilde{\zeta}_\lambda$,

$$g_\lambda(\zeta_\lambda) = g_\lambda(\zeta_0) + Dg_\lambda(\tilde{\zeta}_\lambda)(\zeta_\lambda - \zeta_0) \geq A(\lambda - \lambda_0) + o(\lambda - \lambda_0)$$

for a certain $A > 0$. Let us consider the ball $B_\rho^\lambda$ with center $\zeta_\lambda$ and radius $\rho\,(\lambda - \lambda_0)$ for fixed and small $\rho > 0$. Then we have that $g_\lambda(\zeta) > \frac{A}{2}(\lambda - \lambda_0)$ for all $\zeta \in B_\rho^\lambda$. In this situation we set $\mathbb{D}_\lambda = B_\rho^\lambda$.

It is convenient to make the following relabeling of the parameter $\mu$. Let us set

$$\mu \equiv -\frac{a_1}{2\,a_2}\,\frac{g_\lambda(\zeta)}{\lambda}\,\Lambda\,, \tag{2.14}$$

where $\zeta \in \mathbb{D}_\lambda$. We have the following result

**Lemma 2.4** *Assume the validity of one of the conditions (a) or (b) of Theorem 2.1, and consider a functional of the form*

$$\psi_\lambda(\Lambda, \zeta) = E_\lambda(U_{\mu,\zeta}) + g_\lambda(\zeta)^2\,\theta_\lambda(\Lambda, \zeta) \tag{2.15}$$

*where $\mu$ is given by (2.14) and*

$$|\theta_\lambda| + |\nabla\theta_\lambda| + |\nabla\partial_\Lambda\theta_\lambda| \to 0 \tag{2.16}$$

*uniformly on $\zeta \in \mathbb{D}_\lambda$ and $\Lambda \in (\delta, \delta^{-1})$. Then $\psi_\lambda$ has a critical point $(\Lambda_\lambda, \zeta_\lambda)$ with $\zeta_\lambda \in \mathbb{D}_\lambda$, $\Lambda_\lambda \to 1$.*

Proof. Using the expansion for the energy with $\mu$ given by (2.14) we find now that

$$\psi_\lambda(\Lambda, \zeta) \equiv E_\lambda(U_{\zeta,\mu}) = a_0 + \frac{a_1^2}{4\,a_2}\,\frac{g_\lambda(\zeta)^2}{\lambda}\,\left[-2\,\Lambda + \Lambda^2\right] + g_\lambda(\zeta)^2\,\theta_\lambda(\Lambda, \zeta) \tag{2.17}$$

where $\theta_\lambda$ satisfies property (2.16). Observe then that $\partial_\Lambda\psi_\lambda = 0$ if and only if

$$\Lambda = 1 + o(1)\,\theta_\lambda(\Lambda, \zeta)\,, \tag{2.18}$$

where $\theta_\lambda$ is bounded in $C^1$-sense. This implies the existence of a unique solution close to 1 of this equation, $\Lambda = \Lambda_\lambda(\zeta) = 1 + o(1)$ with $o(1)$ small in $C^1$ sense. Thus we get a critical point of $\psi_\lambda$ if we have one of

$$p_\lambda(\zeta) \equiv \psi_\lambda(\Lambda_\lambda(\zeta), \zeta) = a_0 + c\,g_\lambda(\zeta)^2\,[1 + o(1)] \tag{2.19}$$

with $o(1)$ uniformly small in $C^1$-sense and $c < 0$. In the case of Part (a), *i.e.* of the minimizer, it is clear that we get a local maximum in the region $\mathbb{D}_\lambda$ and therefore a critical point.

Let us consider the case (b). With the same definition for $p_\lambda$ as above, we have

$$\nabla p_\lambda(\zeta) = g_\lambda(\zeta)\Big[\nabla g_\lambda + o(1)\, g_\lambda\Big]. \tag{2.20}$$

Consider a point $\zeta \in \partial\mathbb{D}_\lambda = \partial B_\rho^\lambda$. Then $|\nabla g_\lambda(\zeta)| = |D^2 g_\lambda(\tilde{x})(\zeta - \zeta_\lambda)| \geq \alpha\rho(\lambda - \lambda_0)$, for some $\alpha > 0$. We also have $g_\lambda(\zeta) = O(\lambda - \lambda_0)$. We conclude that for all $t \in (0,1)$, the function $\nabla g_\lambda + t\, o(1)\, g_\lambda$ does not have zeros on the boundary of this ball, provided that $\lambda - \lambda_0$ is small. In conclusion, its degree on the ball is constant along $t$. Since for $t = 0$ is not zero, thanks to non-degeneracy of the critical point $\zeta_\lambda$, we conclude the existence of a zero of $\nabla p_\lambda(\zeta)$ inside $\mathbb{D}_\lambda$. This concludes the proof. □

## 2.4 The method

Hereafter we will look for a solution of (2.3) of the form $v = V + \phi$, so that $\phi$ solves the problem

$$\begin{cases} L(\phi) & = & N(\phi) + E & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} & = & 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.21}$$

where

$$\begin{aligned} L(\phi) &:= -\Delta\phi + \varepsilon^2\lambda\phi - 5V^4\phi, \\ N(\phi) &:= (V+\phi)^5 - V^5 - 5V^4\phi, \\ E &:= V^5 - w^5_{\zeta',\mu'}. \end{aligned}$$

Let us remember that the only bounded solutions of the linear problem

$$\Delta z + 5w^4_{\zeta',\mu'} z = 0, \quad \text{in } \mathbb{R}^3$$

are given by linear combinations of the functions

$$\begin{aligned} z_{\mathrm{i}}(x) &= \frac{\partial w_{\zeta',\mu'}}{\partial \zeta'_{\mathrm{i}}}(x), \quad \mathrm{i} = 1,2,3, \\ z_4(x) &= \frac{\partial w_{\zeta',\mu'}}{\partial \mu'}(x). \end{aligned}$$

In fact, the functions $z_{\mathrm{i}}$, $\mathrm{i} = 1,2,3,4$ span the space of all bounded functions of the kernel of $L$ in the case $\varepsilon = 0$. Observe also that

$$\int_{\mathbb{R}^3} z_j z_k = 0, \text{if } j \neq k.$$

Rather than solving (2.21) directly, we will look for a solution of the following problem first: Find a function $\phi$ such that for certain numbers $c_{\mathrm{i}}$,

$$\begin{cases} L(\phi) & = & N(\phi) + E + \sum_{\mathrm{i}=1}^4 c_{\mathrm{i}} w^4_{\zeta',\mu'} z_{\mathrm{i}} & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} & = & 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} w^4_{\zeta',\mu'} z_{\mathrm{i}}\phi & = & 0 & \text{for } \mathrm{i} = 1,2,3,4. \end{cases} \tag{2.22}$$

## 2.5 The linear problem

In this section we will study the linear part of the problem (2.22). Given a function $h$, we consider the linear problem of finding $\phi$ and numbers $c_{\rm i}$, i $= 1, 2, 3, 4$ such that

$$\begin{cases} L(\phi) &= h + \sum_{\rm i=1}^4 c_{\rm i} w^4_{\zeta',\mu'} z_{\rm i} & \text{in } \Omega_\varepsilon, \\ \frac{\partial \phi}{\partial \nu} &= 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} w^4_{\zeta',\mu'} z_{\rm i}\phi &= 0 & \text{for i} = 1,2,3,4. \end{cases} \tag{2.23}$$

Given a fixed number $0 < \sigma < 1$ we define the following norms

$$\|f\|_* := \sup_{x\in\Omega_\varepsilon}(1 + |x-\zeta'|^\sigma)|f(x)|, \quad \|f\|_{**} := \sup_{x\in\Omega_\varepsilon}(1 + |x-\zeta'|^{2+\sigma})|f(x)|.$$

**Proposition 2.5** *There exist positive numbers $\delta_0$, $\varepsilon_0$, $\alpha_0$, $\beta_0$ and a constant $C > 0$ such that if*

$$\text{dist}(\zeta', \partial\Omega_\varepsilon) > \frac{\delta_0}{\varepsilon} \quad \textit{and} \quad \alpha_0 < \mu' < \beta_0,$$

*then for any $h \in C^{0,\alpha}(\Omega_\varepsilon)$ with $\|h\|_{**} < \infty$ and for all $\varepsilon < \varepsilon_0$, problem (2.23) admits a unique solution $\phi = T(h) \in C^{2,\alpha}(\Omega_\varepsilon)$. Besides,*

$$\|T(h)\|_* \le C\|h\|_{**} \quad \textit{and} \quad |c_{\rm i}| \le C\|h\|_{**}, \text{ i} = 1,2,3,4. \tag{2.24}$$

For the proof of Proposition (2.5) we will need the next

**Lemma 2.6** *Assume the existence of a sequences $(\mu'_n)_{n\in\mathbb{N}}$, $(\zeta'_n)_{n\in\mathbb{N}}$, $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\alpha_1 < \mu'_n < \beta_1$, $\text{dist}(\zeta'_n, \partial\Omega_\varepsilon) > \delta_1$, $\varepsilon_n \to 0$ and for certain functions $\phi_n$ and $h_n$ with $\|h_n\|_{**} \to 0$ and scalars $c^n_{\rm i}$, i $= 1, 2, 3, 4$, one has*

$$\begin{cases} L(\phi_n) &= h_n + \sum_{\rm i=1}^4 c^n_{\rm i} w^4_{\zeta'_n,\mu'_n} z^n_{\rm i} & \textit{in } \Omega_{\varepsilon_n}, \\ \frac{\partial \phi_n}{\partial \nu} &= 0 & \textit{on } \partial\Omega_{\varepsilon_n}, \\ \int_{\Omega_{\varepsilon_n}} w^4_{\zeta'_n,\mu'_n} z^n_{\rm i}\phi_n &= 0 & \textit{for } \text{i} = 1,2,3,4 \end{cases}$$

*where*

$$z^n_{\rm i} = \partial_{(\zeta'_n)_{\rm i}} w_{\zeta'_n,\mu'_n}, \text{ i} = 1,2,3, \quad z^n_4 = \partial_{\mu_n} w_{\zeta'_n,\mu'_n}$$

*then*

$$\lim_{n\to\infty} \|\phi_n\|_* = 0$$

Proof. By contradiction, we may assume that $\|\phi_n\|_* = 1$. We will proof first the weaker assertion that

$$\lim_{n\to\infty} \|\phi_n\|_\infty = 0.$$

Also, by contradiction, we may assume up to a subsequence that $\lim_{n\to\infty}\|\phi_n\|_\infty = \gamma$, where $0 < \gamma \le 1$. Let us see that

$$\lim_{n\to\infty} c^n_{\rm i} = 0, \text{ i} = 1,2,3,4.$$

Up to subsequence, we can suppose that $\mu_n' \to \mu'$, where $\alpha_1 \le \mu' \le \beta_1$. Testing the above equation against $z_j^n(x)$ and integrating by parts twice we get the relation

$$\int_{\Omega_{\varepsilon_n}} L(z_j^n)\phi_n + \int_{\partial\Omega_{\varepsilon_n}} \frac{\partial z_j^n}{\partial \nu}\phi_n = \int_{\Omega_{\varepsilon_n}} h_n z_j^n + \sum_{\mathrm{i}=1}^{4} c_{\mathrm{i}}^n \int_{\Omega_{\varepsilon_n}} w_{\zeta_n',\mu_n'}^4 z_{\mathrm{i}}^n z_j^n.$$

Observe that

$$\left|\int_{\Omega_{\varepsilon_n}} L(z_j^n)\phi_n + \int_{\partial\Omega_{\varepsilon_n}} \frac{\partial z_j^n}{\partial \nu}\phi_n - \int_{\Omega_{\varepsilon_n}} h_n z_j^n\right| \le C\|h_n\|_* + o(1)\|\phi_n\|_*,$$

$$\int_{\Omega_{\varepsilon_n}} w_{\zeta_n',\mu_n'}^4 z_{\mathrm{i}}^n z_j^n = C\delta_{\mathrm{i},j} + o(1).$$

Hence as $n \to \infty$, $c_{\mathrm{i}}^n \to 0$, $\mathrm{i} = 1,2,3,4$.

Let $x_n \in \Omega_{\varepsilon_n}$ be such that $\sup_{x\in\Omega_{\varepsilon_n}} \phi_n(x) = \phi_n(x_n)$, so that $\phi_n$ maximizes at this point. We claim that there exists $R > 0$ such that

$$|x_n - \zeta_n'| \le R,\ \forall n \in \mathbb{N}.$$

This fact follows immediately from the assumption $\|\phi_n\|_* = 1$. We define $\tilde{\phi}_n(x) = \phi(x + \zeta_n')$ Hence, up to subsequence, $\tilde{\phi}_n$ converges uniformly over compacts of $\mathbb{R}^3$ to a nontrivial bounded solution of

$$\begin{cases} -\Delta\tilde{\phi} - 5w_{0,\mu'}^4\tilde{\phi} &= 0 \quad \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} w_{0,\mu'}^4 z_{\mathrm{i}}\tilde{\phi} &= 0 \quad \text{for i} = 1,2,3,4 \end{cases}$$

where $z_{\mathrm{i}}$ is defined in terms of $\mu'$ and $\zeta' = 0$. Then $\tilde{\phi} = \sum_{\mathrm{i}=1}^4 \alpha_{\mathrm{i}} z_{\mathrm{i}}(x)$. From the orthogonality conditions $\int_{\mathbb{R}^3} w_{0,\mu'}^4 z_{\mathrm{i}}\tilde{\phi} = 0$, $\mathrm{i} = 1,2,3,4$, we deduce that $\alpha_{\mathrm{i}} = 0$, $\mathrm{i} = 1,2,3,4$. This implies that $\tilde{\phi} = 0$, which is a contradiction with the hypothesis $\lim_{n\to\infty}\|\phi_n\|_\infty = \gamma > 0$.

Now we prove the stronger result: $\lim_{n\to\infty}\|\phi_n\|_* = 0$. Let us observe that $\zeta_n$ is a bounded sequence, so $\zeta_n \to \zeta$, as $n \to \infty$, up to subsequence. Let $R > 0$ be a fixed number. Without loss of generality we can assume that $|\zeta_n - \zeta| \le R/2$, for all $n \in \mathbb{N}$ and $B(\zeta, R) \subseteq \Omega$. We define $\psi_n(x) = \frac{1}{\varepsilon_n^\sigma}\phi_n\left(\frac{x}{\varepsilon_n}\right)$, $x \in \Omega$ (here we suppose without loss of generality that $\mu_n > 0$, $\forall n \in \mathbb{N}$). From the assumption $\lim_{n\to\infty}\|\phi_n\|_* = 1$ we deduce that

$$|\psi_n(x)| \le \frac{1}{|x - \zeta_n|^\sigma},\ \text{for } x \in B(\zeta, R).$$

Also, $\psi_n(x)$ solves the problem

$$\begin{cases} -\Delta\psi_n + \lambda\psi_n &= \varepsilon_n^{-(2+\sigma)}\{5(\varepsilon_n^{1/2}U_{\zeta_n,\mu_n})^4\psi + g_n + \sum_{\mathrm{i}=1}^4 c_{\mathrm{i}}^n \varepsilon_n^2 w_{\zeta_n,\mu_n}^4 Z_{\mathrm{i}}^n\} & \text{in } \Omega, \\ \frac{\partial\psi_n}{\partial\nu} &= 0 & \text{on } \partial\Omega, \end{cases}$$

where $g_n(x) = h_n\left(\frac{x}{\varepsilon_n}\right)$ and $Z_{\mathrm{i}}^n(x) = z_{\mathrm{i}}^n\left(\frac{x}{\varepsilon_n}\right)$. Since $\lim_{n\to\infty}\|h_n\|_{**} = 0$, we know that

$$|g_n(x)| \le o(1)\frac{\varepsilon_n^{2+\sigma}}{\varepsilon_n^{2+\sigma} + |x - \zeta_n|^{2+\sigma}},\ \text{for } x \in \Omega.$$

Also, by (2.7), we see that

$$(\varepsilon_n^{1/2} U_{\zeta_n,\mu_n}(x))^4 = C\varepsilon_n^4(1+o(1))G(x,\zeta_n) \tag{2.25}$$

away from $\zeta_n$. It's easy to see that $\varepsilon^{-\sigma}\sum_{\mathrm{i}=1}^4 c_{\mathrm{i}}^n w_{\zeta_n,\mu_n}^4 Z_{\mathrm{i}} = o(1)$ as $\varepsilon_n \to 0$, away from $\zeta_n$. We conclude (by a diagonal convergence method) that $\psi_n(x)$ converges uniformly over compacts of $\overline{\Omega}\setminus\{\zeta\}$ to $\psi(x)$, a bounded solution of

$$\begin{cases} -\Delta\psi + \lambda\psi &= 0 \quad \text{in } \Omega\setminus\{\zeta\}, \\ \frac{\partial\psi}{\partial\nu} &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

such that $|\psi(x)| \le \frac{1}{|x-\zeta|^\sigma}$ in $B(\zeta,R)$. So $\psi$ has a removable singularity at $\zeta$, and we conclude that $\psi(x)=0$. This implies that over compacts of $\overline{\Omega}\setminus\{\zeta\}$, we have

$$|\psi_n(x)| = o(1)\varepsilon_n^\sigma.$$

In particular, we conclude that for all $x\in\Omega\setminus B(\zeta_n,R/2)$ we have $|\psi_n(x)|\le o(1)\varepsilon_n^\sigma$, which traduces into the following for $\phi_n$

$$|\phi_n(x)| \le o(1)\varepsilon_n^\sigma, \text{ for all } x\in\Omega_{\varepsilon_n}\setminus B(\zeta_n', R/2\varepsilon_n). \tag{2.26}$$

Consider a fixed number $M$, such that $M < R/2\varepsilon_n$, for all $n$. Observe that $\|\phi_n\|_\infty = o(1)$, so

$$(1+|x|^\sigma)|\phi_n(x)| \le o(1), \text{ for all } x\in\overline{B(\zeta_n',M)}. \tag{2.27}$$

We claim that

$$(1+|x|^\sigma)|\phi_n(x)| \le o(1), \text{ for all } x\in A_{\varepsilon_n,M}, \tag{2.28}$$

where $A_{\varepsilon_n,M} = B(\zeta_n',R/2\varepsilon_n)\setminus\overline{B(\zeta_n',M)}$. This assertions follows from the fact that the operator $L$ satisfies the weak maximum principle in $A_{\varepsilon_n,M}$ (choosing a larger M and a subsequence if necessary): If $u$ satisfies $L(u)\le 0$ in $A_{\varepsilon_n,M}$ and $u\le 0$ in $\partial A_{\varepsilon_n,M}$, then $u\le 0$ in $A_{\varepsilon_n,M}$. This result in just a consequence of the fact that $L(|x-\zeta_n'|^{-\sigma})\ge 0$ in $A_{\varepsilon_n,M}$, if $M$ is larger enough but independent of $n$.

We now prove (2.28) with the use of a suitable barrier. Observe that from (2.26) we deduce the existence of $\eta_n^1\to 0$, as $n\to 0$ such that $\varepsilon_n^{-\sigma}|\phi_n(x)|\le\eta_n^1$, for all $x$ such that $|x| = R/2\varepsilon_n$. From (2.27) we deduce the existence of $\eta_n^2\to 0$, as $n\to\infty$ such that $M^\sigma|\phi_n(x)|\le\eta_n^2$, for all $x$ such that $|x| = M$. Also, there exists $\eta_n^3\to 0$, as $n\to\infty$ such that

$$|x+\zeta_n'|^{2+\sigma}|L(\phi_n)| \le \eta_n^3, \text{in } A_{\varepsilon_n,M}.$$

We define the barrier function $\varphi_n(x) = \eta_n\frac{1}{|x-\zeta_n'|^\sigma}$, with $\eta_n = \max\{\eta_n^1,\eta_n^2,\eta_n^3\}$. Observe that $L(\varphi_n) = \sigma(1-\sigma)\eta_n\frac{1}{|x-\zeta_n'|^{2+\sigma}} + (\varepsilon_n^2\lambda - 5V^4)\eta_n\frac{1}{|x-\zeta_n'|^\sigma}$. It's not hard to see that $|L(\phi_n)|\le CL(\varphi_n)$ in $A_{\mu_n,M}$ and $|\phi_n(x)|\le C\varphi_n$ in $\partial A_{\varepsilon_n,M}$, where $C$ is a constant independent of $n$. From the weak maximum principle we deduce (2.28) and the fact $\|\phi_n\|_\infty = o(1)$. From (2.26), (2.27), (2.28), and $\|\phi_n\|_\infty = o(1)$ we conclude that

$$\|\phi_n\|_* = o(1)$$

which is a contradiction with the assumption $\|\phi_n\|_* = 1$. The proof of Lemma (2.6) is completed. □

Proof of proposition 2.5. Let us consider the space

$$H = \left\{ \phi \in H^1(\Omega) \middle| \int_{\Omega_\varepsilon} w^4_{\zeta',\mu'} z_{\mathrm{i}} \phi = 0,\ \mathrm{i} = 1,2,3,4. \right\}$$

endowed with the inner product,

$$[\phi, \psi] = \int_{\Omega_\varepsilon} \nabla\phi\nabla\psi + \varepsilon^2 \lambda \int_{\Omega_\varepsilon} \phi\psi.$$

Problem (2.23) expressed in the weak form is equivalent to that of finding $\phi \in H$ such that

$$[\phi, \psi] = \int_{\Omega_\varepsilon} \left[ 5V^4\phi + h + \sum_{\mathrm{i}=1}^{4} c_{\mathrm{i}} w^4_{\zeta',\mu'} z_{\mathrm{i}} \right] \psi, \quad \text{for all } \psi \in H.$$

The a priori estimate $\|T(h)\|_* \le C\|h\|_{**}$ implies that for $h \equiv 0$ the only solution is 0. With the aid of Riesz's representation theorem, this equation gets rewritten in $H$ in operational form as one in which Fredholm's alternative is applicable, and its unique solvability thus follows. Besides, its easy to conclude (2.24) from an application of Lemma (2.6). □

It is important, for later purposes, to understand the differentiability of the operator $T : h \to \phi$, with respect to the variables $\mu'$ and $\zeta'$, for a fixed $\varepsilon$ (we only let $\zeta$ and $\mu$ be variables). We have the following result

**Proposition 2.7** *Under the conditions of Proposition (2.24), the map $T$ is of class $C^1$. Besides, we have*

$$\|\nabla_{\zeta',\mu'} T(h)\|_* \le C\|h\|_{**}.$$

Proof. Let us consider differentiation with respect to the variable $\zeta'_k$, $k = 1,2,3$. For notational simplicity we write $\frac{\partial}{\partial \zeta'_k} = \partial_{\zeta'_k}$. Let us set, still formally, $X_k = \partial_{\zeta'_k}\phi$. Observe that $X_k$ satisfies the following equation

$$L(X_k) = 5\partial_{\zeta'_k}(V^4)\phi + \sum_{\mathrm{i}=1}^{4} \mathrm{d}^k_{\mathrm{i}} w^4_{\zeta',\mu'} z_{\mathrm{i}} + \sum_{\mathrm{i}=1}^{4} c_{\mathrm{i}} \partial_{\zeta'_k}(w^4_{\zeta',\mu'} z_{\mathrm{i}}), \quad \text{in } \Omega_\varepsilon.$$

Here $\mathrm{d}^k_{\mathrm{i}} = \partial_{\zeta'_k} c_{\mathrm{i}}$, $\mathrm{i} = 1,2,3$. Besides, from differentiating the orthogonality conditions $\int_{\Omega_\varepsilon} w^4_{\zeta',\mu'} z_{\mathrm{i}} = 0$, $\mathrm{i} = 1,2,3,4$, we further obtain the relations

$$\int_{\Omega_\varepsilon} X_k w^4_{\zeta',\mu'} z_{\mathrm{i}} = -\int_{\Omega_\varepsilon} \phi \partial_{\zeta'_k}(w^4_{\zeta',\mu'} z_{\mathrm{i}}), \quad \mathrm{i} = 1,2,3,4.$$

Let us consider constants $b_{\mathrm{i}}$, $\mathrm{i} = 1,2,3,4$, such that

$$\int_{\Omega_\varepsilon} \left( X_k - \sum_{\mathrm{i}=1}^{4} b_{\mathrm{i}} z_{\mathrm{i}} \right) w^4_{\zeta',\mu'} z_j = 0, \quad j = 1,2,3,4.$$

These relations amount to

$$\sum_{i=1}^{4} b_i \int_{\Omega_\varepsilon} w_{\zeta',\mu'} z_i z_j = \int_{\Omega_\varepsilon} \phi \partial_{\zeta'_k} (w^4_{\zeta',\mu'} z_j), \quad j = 1,2,3,4.$$

Since this system is diagonal dominant with uniformly bounded coefficients, we see that it is uniquely solvable and that

$$b_i = O(\|\phi\|_*)$$

uniformly on $\zeta'$, $\mu'$ in the considered region. Also, it is not hard to see that

$$\|\phi \partial_{\zeta'_k}(V^4)\|_{**} \le C\|\phi\|_*.$$

From Proposition (2.24), we conclude

$$\left\| \sum_{i=1}^{4} c_i \partial_{\zeta'_k} (w^4_{\zeta',\mu'} z_i) \right\|_{**} \le C\|h\|_{**}.$$

We set $X = X_k - \sum_{i=1}^4 b_i z_i$, so $X$ satisfies

$$L(X) = f + \sum_{i=1}^{4} b_i^k w^4_{\zeta',\mu'} z_i, \quad \text{in } \Omega_\varepsilon,$$

where

$$f = 5\partial_{\zeta'_k}(V^4)\phi \sum_{i=1}^{4} b_i L(z_i) + \sum_{i=1}^{4} c_i \partial_{\zeta',\mu'} (w^4_{\zeta',\mu'} z_i)$$

Observe that also,

$$\int_{\Omega_\varepsilon} X w^4_{\zeta',\mu'} z_i = 0, \quad i = 1,2,3,4.$$

This computation is not just formal. Indeed, one gets, as arguing directly by definition shows,

$$\partial_{\xi'_k} \phi = \sum_{i=1}^{4} b_i z_i + T(f)$$

and

$$\|\partial_{\xi'_k} \phi\|_* \le C\|h\|_{**}.$$

The corresponding result for differentiation with respect to $\mu'$ follows similarly. This concludes the proof. □

## 2.6 The nonlinear problem

We recall that our goal is to solve the problem (2.21). Rather than doing so directly, we shall solve first the intermediate nonlinear problem (2.22) using the theory developed in the previous section. We have the next result

**Lemma 2.8** *Under the assumptions of Proposition (2.5), there exist numbers $\varepsilon_1 > 0$, $C_1 > 0$, such that if $\mu$ and $\zeta$ are additionally such that $\|E\|_{**} < \varepsilon_1$, then problem (2.22) has a unique solution $\phi$ which satisfies*

$$\|\phi\|_* \le C_1 \|E\|_{**}.$$

Proof. In terms of the operator $T$ defined in Proposition (2.5), problem (2.22) becomes

$$\phi = T(N(\phi) + E) \equiv A(\phi).$$

For a given $\gamma > 0$, let us consider the region

$$\mathcal{F}_\gamma := \{\phi \in C(\overline{\Omega}_\varepsilon) \big| \ \|\phi\|_* \le \gamma \|E\|_{**}\}.$$

From Proposition (2.5), we get

$$\|A(\phi)\|_* \le C\left[\|N(\phi)\|_{**} + \|E\|_{**}\right].$$

The definition of $N$ immediately yields $\|N(\phi)\|_{**} \le C_0 \|\phi\|_*^2$. It is also easily checked that $N$ satisfies, for $\phi_1, \phi_2 \in \mathcal{F}_\gamma$,

$$\|N(\phi_1) - N(\phi_2)\|_{**} \le C_0 \gamma \|E\|_{**} \|\phi_1 - \phi_2\|_*.$$

Hence for a constant $C_1$ depending on $C_0$, $C$, we get

$$\begin{aligned} \|A(\phi)\|_* &\le C_1 \left[\gamma^2 \|E\|_{**} + 1\right] \|E\|_{**} \\ \|A(\phi_1) - A(\phi_2)\|_* &\le C_1 \gamma \|E\|_{**} \|\phi_1 - \phi_2\|_*. \end{aligned}$$

Choosing

$$\gamma = C_1, \quad \varepsilon_1 = \frac{1}{2C_1^2},$$

we conclude that $A$ is a contraction mapping of $\mathcal{F}_\gamma$, and therefore a unique fixed point of $A$ exists in this region. □

We shall next analyze the differentiability of the map $(\zeta', \mu') \to \phi$. Concerning the differentiability of the function $\phi(\zeta')$, let us write

$$A(x, \varphi) = \varphi - T(N(\varphi) + E).$$

Observe that $A(\zeta', \phi) = 0$ and

$$\partial_\phi A(\zeta', \phi) = I + O(\varepsilon).$$

It follows that for small $\varepsilon$, the linear operator $\partial_\phi A(\zeta', \phi)$ is invertible, with uniformly bounded inverse. It also depends continuously on its parameters. Differentiating respect to $\zeta'$ we obtain

$$\partial_{\zeta'} A(\zeta', \phi) = -(\partial_{\zeta'} T)(N(\phi) + E) - T(\partial_{\zeta'} N(\phi) + \partial_{\zeta'} R).$$

where the previous expression depend continuously on their parameters. Hence the implicit function theorem yields that $\phi(\zeta')$ is a $C^1$ function. Moreover, we have

$$\partial_{\zeta'}\phi = -(\partial_\phi A(\zeta',\phi))^{-1}[\partial_{\zeta'}A(\zeta',\phi)].$$

By Taylor expansion we conclude that

$$\|\partial_{\zeta'}N(\phi)\|_{**} \le C(\|\phi\|_* + \|\partial_{\zeta'}\phi\|_*)\|\phi\|_* \le C(\|E\|_{**} + \|\partial_{\zeta'}\|_*)\|E\|_{**}.$$

Using Proposition (2.7), we have

$$\|\partial_{\zeta'}\phi\|_* \le C(\|N(\phi)+E\|_{**} + \|\partial_{\zeta'}N(\phi)\|_{**} + \|\partial_{\zeta'}E\|_{**}),$$

for some constant $C > 0$. Hence, we conclude that

$$\|\partial_{\zeta'}\phi\|_* \le C(\|E\|_{**} + \|\partial_{\zeta'}E\|_{**}).$$

A similar argument shows that, as well

$$\|\partial_{\mu'}\phi\|_* \le C(\|E\|_{**} + \|\partial_{\mu'}E\|_{**}).$$

This can be summarized as follows

**Lemma 2.9** *Under the assumptions of Proposition (2.5) and (2.8) consider the map $(\zeta',\mu') \to \phi$. The partial derivatives $\nabla_{\zeta'}\phi$ and $\nabla_{\mu'}\phi$ exist and define continuous functions of $(\zeta',\mu')$. Besides, there exist a constant $C_2 > 0$, such that*

$$\|\nabla_{\zeta'}\phi\|_* + \|\nabla_{\mu'}\phi\|_* \le C_2(\|E\|_{**} + \|\nabla_{\zeta'}E\|_{**} + \|\nabla_{\mu'}E\|_{**}).$$

After Problem (2.21) has been solved, we will find solutions to the full problem (2.22) if we manage to adjust the pair $(\zeta',\mu')$ in such a way that $c_i(\zeta',\mu') = 0$, i $= 1,2,3,4$. This is the *reduced problem*. A nice feature of this system of equations is that it turns out to be equivalent to finding critical points of a functional of the pair $(\zeta',\mu')$ which is close, in appropriate sense, to the energy of the single bubble $U$.

## 2.7 Variational formulation of the reduced problem.

In order to obtain a solution of (2.1) we need to solve the system of equations

$$c_j(\zeta',\mu') = 0 \quad \text{for all } j = 1,\dots,4. \tag{2.29}$$

If (2.29) holds, then $v = V + \phi$ will be a solution to (2.21). This system turns out to be equivalent to a variational problem. We define

$$F(\zeta',\mu') = E_\lambda(V+\phi),$$

where $\phi = \phi(\zeta', \mu')$ is the unique solution of (2.22) that we found in the previous section, and $E_\lambda$ is the energy functional defined in the previous sections. Critical points of $F$ correspond to solutions of (2.29), under the assumption that the error $E$ is small enough. The proof of this fact is similar to the one of Lemma 7.2 in [12].

Additionally, the following expansion holds

$$F(\zeta', \mu') = E_\lambda(V) + (\|E\|_{**}^2 + \|\nabla_{\zeta'} E\|_{**}^2 + \|\nabla_{\mu'} E\|_{**}^2)\theta(\zeta', \mu'),$$

where for a certain constant $C > 0$ the function $\theta$ satisfies $|\theta| + |\nabla_{\zeta'}\theta| + |\nabla_{\mu'}\theta| \le C$. Using this expansion and the hypothesis of Theorem 2.1, we conclude the proof using a similar argument to the one given for the proof of Theorem 3 part (b) in [12]. This concludes the proof of our main theorem. □

## 2.8 Appendix – Robin's function

In this appendix we prove two facts we have used in the course of the proofs about *Robin's function* $g_\lambda$. Recall that $g_\lambda(x) \equiv H_\lambda(x,x)$ where the function $y \mapsto H_\lambda(x,y)$ satisfies the boundary value problem

$$\begin{cases} -\Delta_y H_\lambda + \lambda H_\lambda &= \lambda\Gamma(x-y) \quad y \in \Omega, \\ \frac{\partial H_\lambda(x,y)}{\partial\nu} &= \frac{\partial\Gamma(x-y)}{\partial\nu} \quad x \in \partial\Omega, \end{cases}$$

where $\Gamma(z) = \frac{1}{4\pi|z|}$.

**Lemma 2.10** *The function $g_\lambda$ is of class $C^\infty(\Omega)$.*

Proof. We will show that $g_\lambda \in C^k$, for any $k$. Fix $x \in \Omega$. Let $h_{1,\lambda}$ be the function defined in $\Omega \times \Omega$ by the relation

$$H_\lambda(x,y) = \beta_1 |x-y| + h_{1,\lambda}(x,y)\,,$$

where $\beta_1 = -\frac{\lambda}{8\pi}$. Then $h_{1,\lambda}$ satisfies the boundary value problem

$$\begin{cases} -\Delta_y h_{1,\lambda} + \lambda h_{1,\lambda} &= -\lambda\beta_1|x-y| \quad x\text{in } \Omega, \\ \frac{\partial h_{1,\lambda}(x,y)}{\partial\nu} &= \frac{\partial\Gamma(x-y)}{\partial\nu} - \beta_1\frac{\partial|x-y|}{\partial\nu} \quad \text{on } \partial\Omega. \end{cases}$$

Elliptic regularity then yields that $h_{1,\lambda}(x,\cdot) \in C^2(\Omega)$. Its derivatives are clearly continuous as functions of the joint variable. Let us observe that the function $H_\lambda(x,y)$ is symmetric, thus so is $h_1$, and then $h_{1,\lambda}(\cdot,y)$ is also of class $C^2$ with derivatives jointly continuous. It follows that $h_1(x,y)$ is a function of class $C^2(\Omega\times\Omega)$. Iterating this procedure, we get that, for any $k$

$$H_\lambda(x,y) = \sum_{j=1}^{k} \beta_j|x-y|^{2j-1} + h_{k,\lambda}(x,y)$$

with $\beta_{j+1} = -\lambda\,\beta_j/((2j+1)(2j+2))$ and $h_{k,\lambda}$ solution of the boundary value problem

$$\left\{\begin{array}{rcll} -\Delta_y h_{k,\lambda} + \lambda h_{k,\lambda} & = & -\lambda\beta_k|x-y|^{2k-1} & \text{in } \Omega, \\ \frac{\partial h_{k,\lambda}(x,y)}{\partial\nu} & = & \frac{\partial\Gamma(x-y)}{\partial\nu} - \sum_{j=1}^{k}\beta_j\frac{\partial|x-y|^{2j-1}}{\partial\nu} & \text{on } \partial\Omega. \end{array}\right.$$

We may remark that

$$-\Delta_y h_{k+1,\lambda} + \lambda\, h_{k,\lambda} = 0 \quad \text{in } \Omega\,.$$

Elliptic regularity then yields that $h_{k,\lambda}$, is a function of class $C^{k+1}(\Omega\times\Omega)$. Let us observe now that by definition of $g_\lambda$ we have $g_\lambda(x) = h_{k,\lambda}(x,x)$, and the conclusion of the Lemma follows.

□

**Lemma 2.11** *The function $\frac{\partial g_\lambda}{\partial\lambda}$ is well defined, smooth and strictly positive in $\Omega$. Its derivatives depend continuously on $\lambda$.*

Proof. For a fixed given $x\in\Omega$, consider the unique solution $F(y)$ of

$$\left\{\begin{array}{rcll} -\Delta_y F + \lambda\,F & = & G(x,y) & y\in\Omega, \\ \frac{\partial F}{\partial\nu} & = & 0 & y\in\partial\Omega. \end{array}\right.$$

Using elliptic regularity, $F$ is at least of class $C^{0,\alpha}$. Besides a convergence argument using elliptic estimates shows that actually

$$F(y) = \frac{\partial H_\lambda}{\partial\lambda}(x,y)\,.$$

Since $\lambda>0$ and $G$ is positive in $\Omega$, using $F_-$ as a test function we get that $F_- = 0$ in $\Omega$, thus $F>0$. Hence, in particular

$$\frac{\partial g_\lambda}{\partial\lambda}(x) = F(x) > 0\,.$$

Arguing as in the previous lemma, this function turns out to be smooth in $x$. The resulting expansions easily provide the continuous dependence in $\lambda$ of its derivatives in $x$-variable. □

# Chapter 3

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