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FENÓMENOS DE CONCENTRACIÓN EN GEOMETRÍA Y ANÁLISIS NO LINEAL

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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Resumen

El trabajo presentado en esta memoria se sitúa en la interfaz entre el análisis y la geometría. El interés recae en el estudio de fenómenos de concentración para dos problemas "geométricos" no lineales: la existencia de hipersuperficies con r-curvatura constante en variedades Riemannianas, y una ecuación de Schrödinger no lineal. Esta memoria se puede dividir en dos partes principales. La primera está dedicada a explorar algunos resultados sobre concentración de familias de hipersuperficies de curvatura media constante (o en general curvatura r-media constante) con topología no trivial en variedades Riemannianas compactas. Se recuerda que la curvatura r-media de una hipersuperficie se define como la r-ésima función simétrica elemental de las curvaturas principales de la hipersuperficie. Se prueba que las técnicas desarrolladas en el trabajo de Mahmoudi, Mazzeo y Pacard [28] se pueden extender para manejar el caso de curvatura r-media con $r \geq 1$. Este fenómeno de concentración se relaciona en general con un fenómeno de resonancia, que hace el análisis particularmente delicado y que también se encuentra en el estudio de una clase de ecuaciones elápticas no lineales que presentan concentración sobre conjuntos de dimensión mayor.

En la segunda parte, correspondiente al paper [29], se prueba un nuevo resultado sobre concentración en subvariedades para una ecuación de Schrödinger no lineal con potencial definido en una variedad Riemanniana suave y compacta M o el espacio Euclídeo $\mathbb{R}^n$, resolviendo en completa generalidad una conjetura planteada por Ambrosetti, Malchiodi y Ni, ver [1]. Precisamente, se estudian soluciones positivas de la siguiente ecuación semilineal:

$$\varepsilon^2 \Delta_{\bar{g}} u - V(z)u + u^p = 0 \ \text{ en } \ M,$$

donde $(M, \bar{g})$ es una variedad Riemanniana n-dimensional suave, compacta y sin borde o el espacio Euclídeo $\mathbb{R}^n$, ε es un parámetro positivo pequeño, $p > 1$ y V es un potencial uniformemente positivo. Se prueba que dado $k = 1, \dots, n-1$ y $1 < p < \frac{n+2-k}{n-2-k}$, y suponiendo que K es una subvariedad k-dimensional suave y encajada de M, que es estacionaria y no degenerada con respecto al funcional $\int_K V^{\frac{p+1}{p-1} - \frac{n-k}{2}} dvol$, entonces existe una secuencia $\varepsilon = \varepsilon_j \to 0$ y soluciones positivas asociadas $u = u_\varepsilon$ que concentran sobre K en el sentido de que decaen exponencialmente a cualquier distancia positiva a K. En particular este enfoque explora una conexión entre soluciones de esta ecuación de Schrödinger no lineal y subvariedades f-minimales en variedades con densidad.

Abstract

The work presented in this thesis is located in the interface of analysis and geometry. The interest is the study of concentration phenomena for two nonlinear "geometric" problems: the existence of hypersurfaces with constant r-curvature in Riemannian manifolds, and a nonlinear Schrödinger equation. This thesis can be divided in two main parts. The first one is devoted to explore some results about concentration of families of constant mean curvature (or in general constant r-mean curvature) hypersurfaces with nontrivial topology in compact Riemannian manifolds. Recall that the r-mean curvature of a hypersurface is defined to be the r-th elementary symmetric function of the principal curvatures of the hypersurface. It is shown that the techniques developed in the paper by Mahmoudi, Mazzeo and Pacard [28] can be extended to handle the case of r-mean curvature with $r \geq 1$. This concentration phenomenon is in general related to a resonance phenomenon which makes the analysis particularly delicate and which one also meets in the study of a class of nonlinear elliptic equations presenting concentration on higher dimensional sets.

In the second part, corresponding to the paper [29] (submitted), a new result about concentration on submanifolds for a nonlinear Shrödinger equation with potential defined on a smooth compact Riemannian manifold M or the Euclidean space $\mathbb{R}^n$ is proved, solving in full generality a conjecture stated by Ambrosetti, Malchiodi and Ni, see [1]. Precisely, positive solutions of the following semilinear equation are studied:

$$\varepsilon^2 \Delta_{\bar{g}} u - V(z)u + u^p = 0 \ \text{ on } \ M,$$

where $(M, \bar{g})$ is a compact smooth n-dimensional Riemannian manifold without boundary or the Euclidean space $\mathbb{R}^n$, ε is a small positive parameter, $p > 1$ and V is a uniformly positive smooth potential. It is proved that given $k = 1, \dots, n-1$ and $1 < p < \frac{n+2-k}{n-2-k}$, and assuming that K is a k-dimensional smooth, embedded compact submanifold of M, which is stationary and non-degenerate with respect to the functional $\int_K V^{\frac{p+1}{p-1} - \frac{n-k}{2}} dvol$, then there exist a sequence $\varepsilon = \varepsilon_j \to 0$ and associated positive solutions u_ε that concentrate along K in the sense that they decay exponentially at each distance to K. In particular this approach explores a connection between solutions of this nonlinear Schrödinger equation and f-minimal submanifolds in manifolds with density.

Contents

Chapter 1

Hypersurfaces of constant r-mean curvature condensing on a submanifold

1.1 Introduction

Let Σ^m be an oriented embedded (or possibly immersed) hypersurface in a compact Riemannian manifold (M^{m+1}, g). The shape operator A_Σ is the symmetric endomorphism of the tangent bundle of K associated with the second fundamental form of Σ, b_Σ, by

$$b_\Sigma(X, Y) = g_\Sigma(A_\Sigma X, Y), \quad \forall X, Y \in T\Sigma; \qquad \text{here} \qquad g_\Sigma = g|_{T\Sigma}\,.$$

The eigenvalues κ_i of the shape operator A_Σ are the principal curvatures of the hypersurface Σ. The r-curvature of Σ is defined to be the k-th symmetric function of the principal curvatures of Σ, i.e.

$$\sigma_r(\Sigma) := \sum_{i_1 < \ldots < i_r} \kappa_{i_1} \ldots \kappa_{i_r}.$$

In particular σ_1 equals m times the mean curvature of Σ and σ_m equals the Gauss-Kronecker curvature of Σ.

Hypersurfaces of constant mean curvature and Gauss-Kronecker curvature constitute a very important class of submanifolds in a compact Riemannian manifold (M^{m+1}, g) and have been studied extensively. In this work, degenerating families of submanifolds with constant r-curvature which 'condense' to the submanifold $K^k \subset M^{m+1}$ of codimension greater than 1 are studied. Under fairly reasonable geometric assumptions, cf. [34], the existence of such a family for $r = 1$ implies that K is minimal. Some cases have been studied previously: Ye [47], [48] proved the existence of a local foliation by constant mean curvature hypersurfaces when K is a point (which is required to be a nondegenerate critical point of the scalar curvature function); more recently, Mazzeo and Pacard [34] proved existence of a partial foliation in the mean curvature case when K is a nondegenerate geodesic and Mahmoudi [24] proved existence of a local foliation in the case when K is a point and general r. Finally, Mahmoudi, Mazzeo and Pacard [28] studied the case when K is an arbitrary nondegenerate minimal submanifold (no extra curvature hypotheses are required) and $r = 1$. The aim of the first part of this work is to give a overview of the above mentionned results and to show that the methods used in the paper [28] can be extanded to handle the general case, i.e. for arbitrary k and r.

This result is described in more detail. Let K^k be a closed (embedded or immersed) submanifold in M^{m+1}, $1 \leq k \leq m-1$, and define the geodesic tube of radius ρ about K by

$$\bar{S}_\rho := \{q \in M^{m+1} : \quad \mathrm{dist}_g(q, K) = \rho\}.$$

This is a smooth (immersed) hypersurface provided ρ is smaller than the radius of curvature of K, and henceforth it is always assumed that this is the case. The r-curvature of this tube satisfies

$$\sigma_r(\bar{S}_\rho) = C(m,k,r)\,\rho^{-r} + \mathcal{O}(\rho^{-r}), \qquad \text{as} \qquad \rho \searrow 0,$$

with $n = m+1-k$ and $C(m,k,r) = C^r_{n-1}$, a quantity depending only on m, k and r, and hence it is plausible that this tube might be able to be perturbed to a constant r-curvature hypersurface with $\sigma_r \equiv C(m,k,r)\,\rho^{-r}$. This is not quite true since the r-mean curvature of $\bar{S}_\rho$ is not sufficiently close to being constant, however when K is minimal there is a better estimate

$$\sigma_r(\bar{S}_\rho) = C(m,k,r)\,\rho^{-r} + \mathcal{O}(\rho),$$

cf. Section 1.4 for more details. Even in this case, there are other more subtle obstructions to carrying out this procedure at certain radii ρ related to eigenvalues of the linearized mean curvature operator on $\bar{S}_\rho$, which in turn are related to a genuine bifurcation phenomenon, at least when $k = r = 1$, cf. [34]. Thus the existence of the constant mean curvature perturbation is not obtained for every small radius. The precise statement of the obtained result is the following:

Theorem 1.1 *Suppose that K^k is a nondegenerate closed minimal submanifold $1 \leq k \leq m-1$ and $r \leq m-k$. Then there exists a sequence of disjoint nonempty intervals $I_i = (\rho_i^-, \rho_i^+)$, $\rho_i^\pm \to 0$, such that for all $\rho \in I := \cup_i I_i$, the geodesic tube $\bar{S}_\rho$ may be perturbed to a constant mean curvature hypersurface S_ρ with $\sigma_r(S_\rho) = C^r_{m-k}\,\rho^{-r}$.*

The nondegeneracy condition on K is simply that the linearized mean curvature operator, also called the Jacobi operator, is invertible; this restriction is quite mild and holds generically [45]. As mentioned above, this result was already known when $k = 0, 1$, but the case $k > 1$ requires a more complicated analysis. This approach was inspired by the works of Malchiodi and Montenegro in different context, see [32, 33].

The hypersurface S_ρ is a small perturbation of $\bar{S}_\rho$ in the sense that it is the normal graph of some function (with L^∞ norm bounded by a constant times ρ^3) over a submanifold obtained by 'translating' K by a section of its normal bundle (with L^∞ norm bounded by a constant times ρ^2); the reader is referred to 1.3.1 for the precise formulation of the construction of S_ρ. When K is embedded, then so are the hypersurfaces S_ρ for ρ sufficiently small. In addition, the hypersurfaces in each of the families $\{S_\rho\}_{\rho \in I_i}$ are leaves of a local foliation of some annular neighborhood of K.

The fact that the construction fails for certain values of ρ is related to a bifurcation phenomenon. When $k = 1$ the families of surfaces which bifurcate off are (perturbations of) Delaunay unduloids [23]; however, when $k \geq 2$, this bifurcation is only known to exist in special cases, and the geometry of the surfaces in the putative bifurcating branches is less clear. In any case, such bifurcations are inherent to the problem and occur also in [32] and in many other situations. Furthermore, the index of the hypersurfaces S_ρ, $\rho \in I_i$, tends to $+\infty$ as $i \to \infty$. On the other hand, it is proved that the set $I = \cup_i I_i$ is quite dense near 0 in the sense that for any $q \geq 2$ there exists a $c_q > 0$ such that

$$|\mathcal{H}^1((0,\rho) \cap I) - \rho| \leq c_q \, \rho^q,$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure.

In the next section the asymptotic expansion of the metric on M in Fermi coordinates around K is calculated; this is applied in the (quite technical) Section 1.3 to derive the expansions of various geometric quantities for the tubes $\bar{S}_\rho$ and their perturbations. This is used in Section 1.4 to obtain the expression for the mean curvature of the perturbed tubes, which gives the equation to be solved. An iteration scheme is introduced in Section 1.5 which allows to find a preliminary perturbation for which the error term is much better, and estimates for the gaps in the spectrum of the linearization are obtained in 1.6; finally, the existence of the constant mean curvature hypersurfaces S_ρ is obtained in 1.7.

1.2 Fermi coordinates near K and expansion of the metric

1.2.1 Fermi coordinates

The construction of Fermi coordinates in a neighborhood of K is recalled. For a given $p \in K$, there is a natural splitting

$$T_p M = T_p K \oplus N_p K.$$

Orthonormal bases E_a, $a = n+1, \ldots, m+1$, for $T_p K$, and E_i, $i = 1, \ldots, n$, of $N_p K$, are chosen.

Notation: The convention that indices $a, b, c, d, \ldots \in \{n+1, \ldots, m+1\}$, indices $i, j, k, \ell, \ldots \in \{1, \ldots, n\}$ and indices $\alpha, \beta, \gamma, \ldots \in \{1, \ldots, m+1\}$, is adopted.

Consider, in a neighborhood of p in K, normal geodesic coordinates

$$f(y) := \exp_p^K(y^a \, E_a), \qquad y := (y^{n+1}, \ldots, y^{m+1}),$$

where $\exp^K$ is the exponential map on K and summation over repeated indices is understood. This yields the coordinate vector fields $X_a := f_*(\partial_{y^a})$. For any $E \in T_p K$, the curve

$$s \longrightarrow \gamma_E(s) := \exp_p^K(sE),$$

is a geodesic in K, so that

$$\nabla_{X_a} X_b|_p \in N_p K.$$

As usual, the Christoffel symbols Γ_{ab}^i are defined by

$$\nabla_{X_a} X_b|_p = \Gamma_{ab}^i \, E_i.$$

The E_i are extended along each $\gamma_E(s)$ so that they are parallel with respect to the induced connection on the normal bundle NK. This yields an orthonormal frame field X_i for NK in a neighborhood of p in K which satisfies

$$\nabla_{X_a} X_i\big|_p \in T_p K,$$

and hence defines coefficients $\Gamma_{ai}^{\,b}$ by

$$\nabla_{X_a} X_i\big|_p = \Gamma_{ai}^{\,b}\, E_b.$$

A coordinate system in a neighborhood of p in M is now defined by

$$F(x, y) := \exp^M_{f(y)}(x^i\, X_i), \qquad (x, y) := (x^1, \dots, x^n, y^{n+1}, \dots, y^{m+1}),$$

with corresponding coordinate vector fields

$$X_i := F_*(\partial_{x^i}) \qquad \text{and} \qquad X_a := F_*(\partial_{y^a}).$$

By construction, $X_\alpha|_p = E_\alpha$.

1.2.2 Taylor expansion of the metric

As usual, the definition of the Fermi coordinates above make the metric coefficients

$$g_{\alpha\beta} = g(X_\alpha, X_\beta)$$

equal $\delta_{\alpha\beta}$ at p; furthermore, $g(X_a, X_i) = 0$ in some neighborhood of p in K. This implies that

$$X_b\, g(X_a, X_i) = g(\nabla_{X_b} X_a, X_i) + g(X_a, \nabla_{X_b} X_i) = 0$$

on K, which yields the identity

$$\Gamma_{ai}^{\,b} = -\Gamma_{ab}^{\,i} \tag{1.1}$$

at p.

Denote by $\Gamma_a^{\,b} : N_pK \longrightarrow \mathbb{R}$ the linear form with coefficients

$$\Gamma_a^{\,b}(\cdot) := g(\nabla_{E_a} E_b, \cdot) = -g(\nabla_{E_a} \cdot, E_b)$$

The higher terms in the Taylor expansions of the functions $g_{\alpha\beta}$ are now computed. The metric coefficients at $q := F(x, 0)$ are given in terms of geometric data at $p := F(0, 0)$ and $|x| = \mathrm{dist}_g(p, q)$.

Notation: The symbol $\mathcal{O}(|x|^r)$ indicates a function such that it and its partial derivatives of any order, with respect to the vector fields X_a and $x^i\, X_j$, are bounded by $c\,|x|^r$ in some fixed neighborhood of 0.

Start with the expansion of the covariant derivative:

Lemma 1.2 *At the point of $q = F(x,0)$, the following expansions hold:*

$$\begin{aligned} \nabla_{X_i} X_j &= \mathcal{O}(|x|) X_\gamma, \\ \nabla_{X_a} X_b &= -\Gamma_a^b(E_i)\, X_i + \mathcal{O}(|x|) X_\gamma, \\ \nabla_{X_a} X_i &= \nabla_{X_i} X_a = \Gamma_a^b(E_i)\, X_b + \mathcal{O}(|x|) X_\gamma, \end{aligned} \tag{1.2}$$

Proof. Observe that, because coordinate vector fields are used, $\nabla_{X_\alpha} X_\beta = \nabla_{X_\beta} X_\alpha$ for any α, β. It also holds that $\nabla_X X|_p = 0$ since any $X \in N_p K$ is tangent to the geodesic $s \mapsto \exp_p^M(sX)$, and hence

$$\nabla_{X_i + X_j}(X_i + X_j)\big|_p = 0.$$

Therefore

$$(\nabla_{X_i} X_j + \nabla_{X_j} X_i)\big|_p = 0,$$

which proves the first estimate.

Also, by construction

$$\nabla_{X_a} X_b = \Gamma_{ab}^i\, X_i + \mathcal{O}(|x|)\, X_\gamma$$

and

$$\nabla_{X_a} X_i = \nabla_{X_i} X_a = \Gamma_{ai}^b\, X_b + \mathcal{O}(|x|)\, X_\gamma.$$

From this, the definition of Γ_a^b and (1.1) the next two estimates follow. $\square$

The expansion of the metric coefficients can now be given. The expansion of the g_{ij}, $i, j = 1, \ldots, n$, agrees with the well known expansion for the metric in normal coordinates [41], [21], [46], but the proof is recalled here for completeness.

Proposition 1.3 *At the point $q = F(x,0)$, the following expansions hold*

$$\begin{aligned} g_{ij} &= \delta_{ij} + \tfrac{1}{3}\, g(R(E_k, E_i)\, E_\ell, E_j)\, x^k\, x^\ell + \mathcal{O}(|x|^3) \\ g_{ai} &= \mathcal{O}(|x|^2) \\ g_{ab} &= \delta_{ab} - 2\,\Gamma_a^b(E_i)\, x^i + \left(g(R(E_k, E_a)\, E_\ell, E_b) + \Gamma_a^c(E_k)\, \Gamma_c^b(E_\ell)\right) x^k\, x^\ell + \mathcal{O}(|x|^3). \end{aligned} \tag{1.3}$$

Proof. By construction, $g_{\alpha\beta} = \delta_{\alpha\beta}$ at p, and so

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(|x|).$$

Now, from

$$X_i\, g_{\alpha\beta} = g(\nabla_{X_i} X_\alpha, X_\beta) + g(X_\alpha, \nabla_{X_i} X_\beta),$$

Lemma 1.2 and (1.1), it holds that

$$X_i\, g_{aj}\big|_p = 0, \qquad X_i\, g_{jk}\big|_p = 0 \qquad \text{and} \qquad X_i\, g_{ab}\big|_p = \Gamma_{ai}^b + \Gamma_{ib}^a = 2\Gamma_{ai}^b.$$

This yields the first order Taylor expansion

$$g_{aj} = \mathcal{O}(|x|^2), \qquad g_{ij} = \delta_{ij} + \mathcal{O}(|x|^2) \qquad \text{and} \qquad g_{ab} = \delta_{ab} + 2\,\Gamma_{ai}^b\, x^i + \mathcal{O}(|x|^2).$$

To compute the second order terms, it suffices to compute $X_k\, X_k\, g_{\alpha\beta}$ at p and polarize (i.e. replace X_k by $X_i + X_j$, etc.). It holds that

$$X_k\, X_k\, g_{\alpha\beta} = g(\nabla^2_{X_k} X_\alpha, X_\beta) + g(X_\alpha, \nabla^2_{X_k} X_\beta) + 2\, g(\nabla_{X_k} X_\alpha, \nabla_{X_k} X_\beta) \tag{1.4}$$

To proceed, first observe that

$$\nabla_X X\big|_{p'} = \nabla^2_X X\Big|_{p'} = 0$$

at $p' \in K$, for any $X \in N_{p'} K$. Indeed, for all $p' \in K$, $X \in N_{p'} K$ is tangent to the geodesic $s \mapsto \exp^M_{p'}(sX)$, and so $\nabla_X X = \nabla^2_X X = 0$ at the point p'.

In particular, taking $X = X_k + \varepsilon\, X_j$, it holds that

$$0 = \nabla_{X_k + \varepsilon X_j} \nabla_{X_k + \varepsilon X_j} (X_k + \varepsilon X_j)|_p.$$

Equating the coefficient of ε to 0 gives $\nabla_{X_j} \nabla_{X_k} X_k|_p = -2 \nabla_{X_k} \nabla_{X_k} X_j|_p$, and hence

$$3\, \nabla^2_{X_k} X_j\Big|_p = R(E_k, E_j)\, E_k,$$

So finally, using (1.4) together with the result of Lemma 1.2, it is obtained that

$$X_k\, X_k\, g_{ij}\big|_p = \frac{2}{3}\, g(R(E_k, E_i)\, E_k, E_j).$$

The formula for the second order Taylor coefficient for g_{ij} now follows at once.

Recall that, since X_γ are coordinate vector fields, it follows from (1.4) that

$$\nabla^2_{X_k} X_\gamma = \nabla_{X_k} \nabla_{X_\gamma} X_k = \nabla_{X_\gamma} \nabla_{X_k} X_k + R(X_k, X_\gamma)\, X_k.$$

Using (1.4), this yields

$$\begin{aligned} X_k\, X_k\, g_{ab} &= 2\, g(R(X_k, X_a) X_k, X_b) + 2\, g(\nabla_{X_k} X_a, \nabla_{X_k} X_b) \\ &+ g(\nabla_{X_a} \nabla_{X_k} X_k, X_b) + g(X_a, \nabla_{X_b} \nabla_{X_k} X_k). \end{aligned}$$

Using the result of Lemma 1.2 together with the fact that $\nabla_X X = 0$ at $p' \in K$ for any $X \in N_{p'} K$, it is concluded that

$$X_k\, X_k\, g_{ab}\big|_p = 2\, g(R(E_k, E_a) E_k, E_b) + 2\, \Gamma^c_{ak}\, \Gamma^c_{bk}$$

and this gives the formula for the second order Taylor expansion for g_{ab}. □

Later on, an expansion of some covariant derivatives which is more accurate than the one given in Lemma 1.2 is needed. These are given in the following:

Lemma 1.4 *At the point $q = F(x, 0)$, the following expansion holds*

$$\begin{aligned} \nabla_{X_a} X_b &= \Gamma^b_a(E_j)\, X_j - g(R(E_i, E_a)\, E_j, E_b)\, x^i\, X_j \\ &+ \tfrac{1}{2} \left(g(R(E_a, E_b)\, E_i, E_j) - \Gamma^c_a(E_i)\, \Gamma^b_c(E_j) - \Gamma^c_a(E_j)\, \Gamma^b_c(E_i) \right) x^i\, X_j \\ &+ \mathcal{O}(|x|)\, X_c + \mathcal{O}(|x|^2)\, X_j. \end{aligned} \tag{1.5}$$

Proof.

$$
\begin{aligned}
X_i\, g(\nabla_{X_a} X_b, X_j) &= g(\nabla_{X_i} \nabla_{X_a} X_b, X_j) + g(\nabla_{X_a} X_b, \nabla_{X_i} X_j) \\
&= g(R(X_i, X_a)\, X_b, X_j) + g(\nabla_{X_a} \nabla_{X_b} X_i, X_j) + g(\nabla_{X_a} X_b, \nabla_{X_i} X_j).
\end{aligned}
$$

Observe that, by construction, it follows that

$$
\nabla_{X_a + \varepsilon X_b} X_i = (\Gamma^c_{ai} + \varepsilon\, \Gamma^c_{bi})\, X_c
$$

along the geodesic $s \mapsto \exp^K_p(s(E_a + \varepsilon E_b))$. Hence

$$
\nabla^2_{X_a + \varepsilon X_b} X_i = ((X_a + \varepsilon\, X_b)(\Gamma^c_{ai} + \varepsilon\, \Gamma^c_{bi}))\; X_c + (\Gamma^c_{ai} + \varepsilon\, \Gamma^c_{bi})\, \nabla_{X_a + \varepsilon\, X_b} X_c. \tag{1.6}
$$

Evaluating this at the point p and considering the coefficient of ε, it follows that

$$
(\nabla_{X_a} \nabla_{X_b} X_i + \nabla_{X_b} \nabla_{X_a} X_i)\big|_p - (\Gamma^c_{ai}\, \nabla_{X_b} X_c + \Gamma^c_{bi}\, \nabla_{X_a} X_c)\big|_p \in T_p K
$$

and therefore

$$
\begin{aligned}
g(\nabla_{X_a} \nabla_{X_b} X_i, X_j)\big|_p + g(\nabla_{X_b} \nabla_{X_a} X_i, X_j)\big|_p &= \Gamma^c_{ai}\; g(\nabla_{X_b} X_c, X_j)\big|_p \\
&+ \Gamma^c_{bi}\, g(\nabla_{X_a} X_c, X_j)\big|_p \\
&= \Gamma^c_{ai}\, \Gamma^j_{bc} + \Gamma^c_{bi}\, \Gamma^j_{ac}.
\end{aligned}
$$

Finally, using the fact that

$$
g(\nabla_{X_b} \nabla_{X_a} X_i, X_j) = g(R(X_b, X_a)\, X_i, X_j) + g(\nabla_{X_a} \nabla_{X_b} X_i, X_j)
$$

it is concluded that, at the point p

$$
2\, g(\nabla_{E_a} \nabla_{E_b} E_i, E_j)\big|_p = g(R(E_a, E_b)\, E_i, E_j) + \Gamma^c_{ai}\, \Gamma^j_{bc} + \Gamma^c_{bi}\, \Gamma^j_{ac}
$$

Collecting these estimates together with the fact that $\nabla_{E_i} E_j\big|_p = 0$ it is concluded that

$$
2\, X_i\, g(\nabla_{X_a} X_b, X_j)\big|_p = -2g(R(E_i, E_a)\, E_j, E_b) + g(R(E_a, E_b)\, E_i, E_j) + \Gamma^c_{ai}\, \Gamma^j_{bc} + \Gamma^c_{bi}\, \Gamma^j_{ac}
$$

And this implies (1.5). □

1.3 Geometry of tubes

Expansions as ρ tends to 0 for the metric, second fundamental form and the r-mean curvature of the tubes $\bar{S}_\rho$ and suitable perturbations are derived. This is an extension of the computation in [34].

1.3.1 Perturbed tubes

A suitable class of deformations of the geodesic tubes $\bar{S}_\rho$, depending on a section Φ of NK and a scalar function w on the spherical normal bundle SNK is now described.

Fix $\rho > 0$. It is convenient to introduce the scaled variable $\bar{y} = y/\rho$; also a local parametrization $z \mapsto \Theta(z)$ of S^{n-1} is used. Define the map

$$G(z, \bar{y}) := F\left(\rho\,(1 + w(z, \bar{y}))\,\Theta(z) + \Phi(\rho\,\bar{y}), \rho\,\bar{y}\right),$$

and denote its image by $S_\rho(w, \Phi)$, so in particular

$$S_\rho(0, 0) = \bar{S}_\rho.$$

Notation: Because of the definition of these hypersurfaces using the exponential map, various vector fields used may be regarded either as fields along K or along $S_\rho(w, \Phi)$. To help with this confusion, the following notation is used:

$$\Phi := \Phi^j\, E_j \qquad \Phi_a := \partial_{y^a}\, \Phi^j\, E_j \qquad \Phi_{ab} := \partial_{y^a}\partial_{y^b}\, \Phi^j\, E_j$$

$$\Theta := \Theta^j\, E_j \qquad \Theta_i := \partial_{z^i}\Theta^j\, E_j.$$

These are all vectors in the tangent space T_pM at the fixed point $p \in K$. On the other hand, the vectors

$$\Psi := \Phi^j\, X_j \qquad \Psi_a := \partial_{y^a}\, \Phi^j\, X_j,$$

$$\Upsilon := \Theta^j\, X_j \qquad \Upsilon_i := \partial_{z^i}\Theta^j\, X_j$$

lie in the tangent space T_qM, $q = F(z, y)$.

For brevity, it is also written

$$w_j := \partial_{z^j} w, \quad w_{\bar{a}} := \partial_{\bar{y}^a} w, \quad w_{ij} := \partial_{z^i}\,\partial_{z^j} w, \quad w_{\bar{a}\bar{b}} := \partial_{\bar{y}^a}\,\partial_{\bar{y}^b} w, \quad w_{\bar{a}j} := \partial_{\bar{y}^a}\,\partial_{z^j} w.$$

In terms of this notation, the tangent space to $S_\rho(w, \Phi)$ at any point is spanned by the vectors

$$\begin{aligned} Z_{\bar{a}} &= G_*(\partial_{\bar{y}^a}) = \rho\,(X_a + w_{\bar{a}}\,\Upsilon + \Psi_a), \qquad a = n+1, \ldots, m+1 \\ Z_j &= G_*(\partial_{z^j}) = \rho\,((1 + w)\,\Upsilon_j + w_j\,\Upsilon), \qquad j = 1, \ldots, n-1. \end{aligned} \tag{1.7}$$

1.3.2 Notation for error terms

The formulas for the various geometric quantities of $S_\rho(\Phi, w)$ are potentially very complicated, and so it is important to condense notation as much as possible.

Any expression of the form $L(w, \Phi)$ denotes a linear combination of the functions w together with its derivatives with respect to the vector fields $\rho\, X_a$ and X_i up to order 2, and Φ^j together with their derivatives with respect to the vector fields X_a up to order 2. The coefficients are assumed to be smooth functions on SNK which are bounded by a constant independent of ρ in the $\mathcal{C}^\infty$ topology (i.e. derivatives taken with respect to X_a and X_i).

Similarly, an expression of the form $Q(w, \Phi)$ denotes a nonlinear operator in the functions w together with its derivatives with respect to the vector fields ρX_a and X_i up to order 2, and Φ^j together with their derivatives with respect to the vector fields X_a up to order 2. Again, the coefficients of the Taylor expansion of the corresponding differential operator are smooth on SNK, and Q vanishes quadratically at $(w, \Phi) = (0, 0)$.

Finally, any term denoted $\mathcal{O}(\rho^d)$ is a smooth function on SNK which is bounded in $\mathcal{C}^\infty(SNK)$ by a constant times ρ^d.

1.3.3 The first fundamental form

The aim of this section is to compute the coefficients of the first fundamental form of $S_\rho(w, \Phi)$. Setting $p := G(0, 0)$ and

$$q := G(z, 0) = F(\rho(1 + w(z, 0))\,\Theta(z) + \Phi(\rho z), 0)$$

formula (1.3) yileds

$$\begin{aligned}
g(X_a, X_b) &= \delta_{ab} - 2\,\rho\,\Gamma_a^b(\Theta) + \mathcal{O}(\rho^2) - 2\,\Gamma_a^b\,(\Phi) + \rho\,L(w, \Phi) + Q(w, \Phi) \\
g(X_i, X_j) &= \delta_{ij} + \tfrac{\rho^2}{3}\,g(R(\Theta, E_i)\,\Theta, E_j) + \mathcal{O}(\rho^3) \\
&+ \tfrac{\rho}{3}\,(g(R(\Theta, E_i)\,\Phi, E_j) + g(R(\Phi, E_i)\,\Theta, E_j)) + \rho^2\,L(w, \Phi) + Q(w, \Phi) \\
g(X_i, X_a) &= \mathcal{O}(\rho^2) + \rho\,L(w, \Phi) + Q(w, \Phi).
\end{aligned} \tag{1.8}$$

Using the previous expansions, the following computation follows:

$$\begin{aligned}
g(\Upsilon, \Upsilon_j) &= g(\Theta, \Theta_j) + \tfrac{\rho^2}{3}\,g(R(\Theta, \Theta)\,\Theta, \Theta_j) + \mathcal{O}(\rho^3) \\
&+ \tfrac{\rho}{3}\,(g(R(\Theta, \Theta)\,\Phi, \Theta_j) + g(R(\Phi, \Theta)\,\Theta, \Theta_j)) + \rho^2\,L(w, \Phi) + Q(w, \Phi).
\end{aligned}$$

Evaluating this expression with $w = 0$ and $\Phi = 0$, it follows that $g(\Upsilon, \Upsilon_j) = 0$ since Υ is normal and Υ_j is tangent to $S_\rho(0, 0)$. Since the first three terms on the right are independent of w and Φ, they must vanish and therefore

$$g(\Upsilon, \Upsilon_j) = \frac{\rho}{3}\,g(R(\Phi, \Theta)\,\Theta, \Theta_j) + \rho^2\,L(w, \Phi) + Q(w, \Phi) \tag{1.9}$$

Using a similar argument, it follows that

$$\begin{aligned}
g(\Upsilon, \Upsilon) &= g(\Theta, \Theta) + \tfrac{\rho^2}{3}\,g(R(\Theta, \Theta)\,\Theta, \Theta_j) + \mathcal{O}(\rho^3) \\
&+ \tfrac{\rho}{3}\,(g(R(\Theta, \Theta)\,\Phi, \Theta) + g(R(\Phi, \Theta)\,\Theta, \Theta)) + \rho^2\,L(w, \Phi) + Q(w, \Phi),
\end{aligned}$$

and when $w = 0$ and $\Phi = 0$ this gives $g(\Upsilon, \Upsilon) = 1$, yielding

$$g(\Upsilon, \Upsilon) = 1 + \rho^2\,L(w, \Phi) + Q(w, \Phi) \tag{1.10}$$

Using these expansions, the expansion of the first fundamental form of $S_\rho(\Phi, w)$ is obtained:

Proposition 1.5 *It holds that*

$$
\begin{aligned}
\rho^{-2}\, g(Z_{\bar a}, Z_{\bar b}) &= \delta_{ab} - 2\,\rho\,\Gamma_a^b(\Theta) + \mathcal{O}(\rho^2) - 2\,\Gamma_a^b(\Phi) + \rho\, L(w,\Phi) + Q(w,\Phi)\\
\rho^{-2}\, g(Z_{\bar a}, Z_j) &= \mathcal{O}(\rho^2) + L(w,\Phi) + Q(w,\Phi)\\
\rho^{-2}\, g(Z_i, Z_j) &= g(\Theta_i,\Theta_j) + \tfrac{\rho^2}{3}\, g(R(\Theta,\Theta_i)\,\Theta,\Theta_j) + \mathcal{O}(\rho^3) + 2\, g(\Theta_i,\Theta_j)\, w\\
&+ \tfrac{\rho}{3}\,(g(R(\Theta,\Theta_i)\Phi,\Theta_j) + g(R(\Theta,\Theta_j)\Phi,\Theta_i)) + \rho^2 L(w,\Phi) + Q(w,\Phi),
\end{aligned}
\tag{1.11}
$$

where summation over repeated indices is understood.

1.3.4 The normal vector field

The task of this section is to give the expansion of the unit normal N to $S_\rho(w,\Phi)$ in terms of (w,Φ). The following proposition is proved:

Proposition 1.6

$$
N = -\Upsilon + \alpha^j \Upsilon_j + \beta^a X_a + (\rho L(w,\Phi) + Q(w,\Phi))\, X_a + (\rho^2 L(w,\Phi) + Q(w,\Phi))\, X_j \tag{1.12}
$$

where the coefficients α^j are solutions of the system

$$
\alpha^j\, g(\Theta_j,\Theta_i) = w_i + \frac{\rho}{3}\, g(R(\Phi,\Theta)\,\Theta,\Theta_i) + \rho^2 L(w,\Phi) + Q(w,\Phi), \qquad i = 1,\dots,n-1,
$$

where summation again over j is understodd, and the coefficients β^a are given by

$$
\beta^a = w_{\bar a} + g(\Phi_a,\Theta) + \rho L(w,\Phi) + Q(w,\Phi).
$$

Proof. Define the normal (not necessarily unitary) vector field

$$
\tilde N := -\,\Upsilon + A^j\, Z_j + B^a\, Z_{\bar a},
$$

and choose the coefficients A^j and B^a so that that $\tilde N$ is orthogonal to all of the $Z_{\bar b}$ and Z_i. This leads to a linear system for A^j and B^a.

From (1.8) together with the fact that $g(\Upsilon, Z_{\bar a}) = 0$ and $g(\Upsilon, Z_j) = 0$ when $w = 0$ and $\Phi = 0$, it follows that

$$
\begin{aligned}
g(\Upsilon, Z_{\bar a}) &= \rho\, w_{\bar a} + \rho\, g(\Phi_a,\Theta) + \rho^2\, L(w,\Phi) + \rho\, Q(w,\Phi)\\
g(\Upsilon, Z_j) &= \rho\, w_j + \tfrac{\rho^2}{3}\, g(R(\Phi,\Theta)\,\Theta,\Theta_j) + \rho^3\, L(w,\Phi) + \rho\, Q(w,\Phi).
\end{aligned}
\tag{1.13}
$$

Using Proposition 1.5, it follows that

$$
B^a = w_{\bar a} + g(\Theta,\Phi_a) + \rho\, L(w,\Phi) + \frac{1}{\rho}\, Q(w,\Phi)
$$

and

$$
A^j\, g(\Theta_j,\Theta_i) = \frac{1}{\rho}\, w_i + \frac{1}{3}\, g(R(\Phi,\Theta)\,\Theta,\Theta_i) + \rho\, L(w,\Phi) + \frac{1}{\rho}\, Q(w,\Phi).
$$

Recall also that $Z_j = \rho\,\Upsilon_j + \rho\,L(w,\Phi)$ and $Z_{\bar a} = \rho\,X_a + \rho\,L(w,\Phi)$. Collecting these, together with the fact that, at q,

$$|\tilde N| = 1 + \rho^2\,L(w,\Phi) + Q(w,\Phi),$$

it follows that

$$\begin{array}{rcl} N & := & -\,\Upsilon + \frac{1}{\rho}\,(\alpha^j\,Z_j + \beta^a\,Z_{\bar a}) + \Big(L(w,\Phi) + \frac{1}{\rho}Q(w,\Phi)\Big)\,Z_{\bar a} \\ & + & \Big(\rho\,L(w,\Phi) + \frac{1}{\rho}Q(w,\Phi)\Big)\,Z_j + (\rho^2\,L(w,\Phi) + Q(w,\Phi))\,\Upsilon. \end{array} \tag{1.14}$$

The result then follows. □

1.3.5 The second fundamental form

The second fundamental form of $S_\rho(w,\Phi)$ can now be computed. To simplify the computations it can be henceforth assumed that, at the point $\Theta(z)\in S^{n-1}$,

$$g(\Theta_i,\Theta_j) = \delta_{ij} \qquad \text{and} \qquad \overline\nabla_{\Theta_i}\Theta_j = 0, \quad i,j = 1,\ldots,n-1 \tag{1.15}$$

(where $\overline\nabla$ is the connection on TS^{n-1}).

Proposition 1.7 *The following expansions hold:*

$$\begin{array}{rcl} \rho^{-2}\,g(N,\nabla_{Z_{\bar a}}Z_{\bar a}) & = & -\Gamma_a^a(\Theta) + \rho\,g(R(\Theta,E_a)\,\Theta,E_a) + \rho\,\Gamma_a^c(\Theta)\,\Gamma_c^a(\Theta) + \mathcal{O}(\rho^2) \\ & - & \frac{1}{\rho}\,w_{\bar a\bar a} - g(\Phi_{aa},\Theta) + g(R(\Phi,E_a)\Theta,E_a) + \Gamma_a^c(\Theta)\,\Gamma_c^a(\Phi) + w_j\,\Gamma_a^a(\Theta_j) \\ & + & \rho\,L(w,\Phi) + \frac{1}{\rho}\,Q(w,\Phi), \\ \rho^{-2}\,g(N,\nabla_{Z_j}Z_j) & = & \frac{1}{\rho} + \frac{2}{3}\,\rho\,g(R(\Theta,\Theta_j)\,\Theta,\Theta_j) + \mathcal{O}(\rho^2) \\ & - & \frac{1}{\rho}\,w_{jj} + \frac{1}{\rho}\,w + \frac{2}{3}\,g(R(\Phi,\Theta_j)\,\Theta,\Theta_j) \\ & + & \rho\,L(w,\Phi) + \frac{1}{\rho}\;Q(w,\Phi) \\ \rho^{-2}\,g(N,\nabla_{Z_{\bar a}}Z_{\bar b}) & = & -\Gamma_a^b(\Theta) - \frac{1}{\rho}\,w_{\bar a\bar b} + \mathcal{O}(\rho) + L(w,\Phi) + \frac{1}{\rho}\,Q(w,\Phi) \quad a\neq b \\ \rho^{-2}\,g(N,\nabla_{Z_{\bar a}}Z_j) & = & \mathcal{O}(\rho) + \frac{1}{\rho}\,L(w,\Phi) + \frac{1}{\rho}\,Q(w,\Phi) \\ \rho^{-2}\,g(N,\nabla_{Z_i}Z_j) & = & \mathcal{O}(\rho) + \frac{1}{\rho}\,L(w,\Phi) + \frac{1}{\rho}\,Q(w,\Phi), \quad i\neq j. \end{array} \tag{1.16}$$

Because of the extensive computations involved, the proof of this proposition is postponed to section 1.8.

1.3.6 The shape operator of the perturbed tubes

Given the above expansion of the first and the second fundamental forms, and using the fact that in terms of matrix representations it holds that $\rho^{-2}g = I + M$ implies that $\rho^2 g^{-1} = I - M + \mathcal{O}(|M|^2)$ and $A = g^{-1}B$, the following expansion of the shape operator follows:

Proposition 1.8 *Under the previous hypothesis, the shape operator is given by*

$$
\begin{aligned}
\rho\, A_{aa}(w,\Phi) &= \rho^2\, g(R(\Theta,E_a)\Theta,E_a) - \rho^2\, \Gamma_a^c(\Theta)\,\Gamma_c^a(\Theta) + \mathcal{O}(\rho^3) \\
&- w_{aa} - \rho\, g(\Phi_{aa} + R(\Phi,E_a)E_a\,,\Theta) + \rho\,\Gamma_a^c(\Phi)\,\Gamma_c^a(\Theta) \\
&- 2\,\rho\,\Gamma_a^c(\Theta)\, w_{ac} + \rho^2\, L(w,\Phi) + Q(w,\Phi) \\
\rho\, A_{ii}(w,\Phi) &= 1 + \tfrac{1}{3}\,\rho^2\, g(R(\Theta,\Theta_i)\Theta,\Theta_i) - w_{ii} - w + \mathcal{O}(\rho^3) \\
&+ \rho^2\, L(w,\Phi) + Q(w,\Phi) \\
\rho\, A_{aj}(w,\Phi) &= \mathcal{O}(\rho^2) + L(w,\Phi) + Q(w,\Phi) \\
\rho\, A_{ij}(w,\Phi) &= \mathcal{O}(\rho^2) + L(w,\Phi) + Q(w,\Phi) \quad i \neq j \\
\rho\, A_{ab}(w,\Phi) &= -\rho\,\Gamma_a^b(\Theta) - w_{ac} + \mathcal{O}(\rho^2) + L(w,\Phi) + Q(w,\Phi) \quad a \neq b
\end{aligned}
$$

where all curvature terms are computed at the point p.

1.4 The r-mean curvature of perturbed tubes

Given any symmetric matrix A, and any $r = 1, \ldots, m$, define

$$\sigma_r(A) := \sum_{i_1 < \ldots < i_r} \lambda_{i_1} \ldots \lambda_{i_r},$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of A. Recall that the r-th Newton transform of A is defined by

$$T_r(A) := \sigma_r(A) I - \sigma_{r-1}(A) A + \cdots + (-1)^r\, A^r$$

with $T_m(A) = 0$ by the Cayley–Hamilton theorem. It is proved in [39] that if $A = A(t)$ is a one parameter family of symmetric matrices which depends smoothly on t then

$$\frac{d}{dt}\sigma_r(A) = \operatorname{tr}\left(T_{r-1}(A) \frac{d}{dt} A \right), \tag{1.17}$$

from which it follows that, given any $m \times m$ symmetric matrices A and M,

$$\sigma_r(A + M) = \sigma_r(A) + \operatorname{tr}(T_{r-1}(A) M) + \mathcal{O}(|M|^2).$$

Let $\tilde{I}$ be the matrix

$$\tilde{I} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$$

where $n = m + 1 - k$. Observe that $\sigma_r(\tilde{I}) = C^r_{n-1}$ *if* $r \leq n-1$ and $\sigma_r(\tilde{I}) = 0$ *if* $r \geq n$. Using this together with the previous expansion of the shape operator, it is not hard to check that the r-mean curvature of the hypersurface $S_\rho(w,\Phi)$ can be expanded as

$$\begin{array}{rcl}\rho^r\,\sigma_r(S_\rho(w,\Phi)) & = & C^r_{n-1}+C^{r-1}_{n-1}\,\rho^2\left(\frac{1}{3}\,\frac{n-r}{n-1}\,g(R(\Theta,\Theta_i)\Theta,\Theta_i)+g(R(\Theta,E_a)\Theta,E_a)\right)\\ & - & C^{r-1}_{n-1}\,\rho^2\,\Gamma^c_a(\Theta)\,\Gamma^a_c(\Theta)+\mathcal{O}(\rho^3)\\ & - & C^{r-1}_{n-1}\,\left(\rho^2\,\Delta_K w+\frac{n-r}{n-1}\,(\Delta_{S^{n-1}}w+(n-1)\,w)\right)-2\,\rho\,C^{r-1}_{n-1}\,\Gamma^c_a(\Theta)\,w_{ac}\\ & - & \rho\,C^{r-1}_{n-1}\,(\,g(\Delta_K\Phi+R(\Phi,E_a)E_a\,,\Theta)-\Gamma^c_a(\Phi)\,\Gamma^a_c(\Theta))\\ & + & \rho^2\,L(w,\Phi)+Q(w,\Phi),\end{array}$$

where summation over repeated indices is understood and the linear and nonlinear operators appearing on the expression are different from the ones before, but enjoy similar properties.

This expression can be simplified as follows: First, note that

$$K \text{ minimal} \Longleftrightarrow \Gamma^a_a=0.$$

Define

$$\mathcal{L}_\rho := -C^{r-1}_{n-1}\left(\rho^2\,\Delta_K+\frac{n-r}{n-1}\,(\Delta_{S^{n-1}}+(n-1))\right), \tag{1.18}$$

as an operator on the spherical normal bundle SNK with the expression (1.18) in any local coordinates.

Introduce the quadratic form

$$\Omega\,(\cdot,\cdot) := -C^{r-1}_{n-1}\left(\left(1-\frac{1}{3}\,\frac{n-r}{n-1}\right)g(\mathcal{R}^N\cdot,\cdot)-\frac{1}{3}\,\frac{n-r}{n-1}\,\mathrm{Ric}(\cdot,\cdot)-g(\mathcal{B}^N\cdot,\cdot)\right)$$

acting on N_pK, cf. [20]. Also, the Jacobi operator, for K is defined by

$$\mathfrak{J} := C^{r-1}_{n-1}\left(-\Delta^N-\mathcal{B}^N+\mathcal{R}^N\right). \tag{1.19}$$

To explain the terms here, recall that the Levi-Civita connection for g induces not only the Levi-Civita connection on K, but also a connection ∇^N on the normal bundle NK. The first term here is simply the rough Laplacian for this connection, i.e.

$$\Delta^N=(\nabla^N)^*\nabla^N=\nabla^N_{E_a}\nabla^N_{E_a}-\nabla^N_{(\nabla_{E_a}E_a)^T}.$$

In the coordinates chosen. The third term is the contraction (in normal directions) of the curvature operator for this connection:

$$\mathcal{R}^N := \left(R(E_i,\cdot)\,E_i\right)^N,$$

where E_i is any orthonormal frame for N_pK. Finally, the second fundamental form

$$B: T_pK\times T_pK \longrightarrow N_pK, \qquad B(X,Y):=\left(\nabla_X Y\right)^N, \quad X,Y\in T_pK,$$

defines a symmetric operator

$$\mathcal{B}^N := B^t\circ B;$$

in terms of the coefficients $\Gamma_a^b := B(E_a, E_b)$,

$$g(\mathcal{B}^N X, Y) = \Gamma_a^b(X)\, \Gamma_b^a(Y).$$

Also define the Ricci tensor

$$\mathrm{Ric}(X, Y) = g(R(X, E_\gamma)\, Y, E_\gamma), \qquad X, Y \in T_pM.$$

Finally, introduce the operator

$$g(\cdot, B) \circ \nabla_K^2 = g(\cdot, B(E_a, E_b))(\nabla_{E_a} \nabla_{E_b} - \nabla_{(\nabla_{E_a} E_b)}),$$

in the coordinates chosen.

In terms of this notation, the following holds:

Proposition 1.9 *Let K be a minimal submanifold. Then the r-mean curvature of $\mathcal{S}_\rho(w, \Phi)$ can be expanded as*

$$\begin{aligned}
\rho^r\, \sigma_r(\mathcal{S}_\rho(w, \Phi)) &= C_{n-1}^r - \Omega(\Theta, \Theta)\, \rho^2 + \mathcal{O}(\rho^3) \\
&+ \mathcal{L}_\rho\, w + \rho\, g(\mathfrak{J}\, \Phi, \Theta) - 2\, C_{n-1}^{r-1}\, \rho^3\, g(\Theta, B) \circ \nabla_K^2 w \\
&+ \rho^2\, L(w, \Phi) + Q(w, \Phi).
\end{aligned}$$

The equation $\rho^r\, \sigma_r(\mathcal{S}_\rho(w, \Phi)) = C_{n-1}^r$ can now be written as

$$\begin{aligned}
\mathcal{L}_\rho\, w + \rho\, g(\mathfrak{J}\, \Phi, \Theta) &= \Omega(\Theta, \Theta)\, \rho^2 + \mathcal{O}(\rho^3) + 2\, C_{n-1}^{r-1}\, \rho^3\, g(\Theta, B) \circ \nabla_K^2 w \\
&+ \rho^2\, L(w, \Phi) + Q(w, \Phi).
\end{aligned} \tag{1.20}$$

1.4.1 Decomposition of functions on SNK

Let (φ_j, λ_j) be the eigendata of $\Delta_{S^{n-1}}$, with eigenfunctions orthonormal and counted with multiplicity. Define the subspace $\mathcal{S} \subset L^2(SNK)$ as the set of functions $v : SNK \to \mathbb{R}$ such that the restriction of v to each fibre of SNK is spanned by $\{\varphi_1, \ldots, \varphi_n\}$. Denote by Π and $\Pi^\perp$ the L^2 orthogonal projections of $L^2(SNK)$ onto $\mathcal{S}$ and $\mathcal{S}^\perp$, respectively.

Now, given any function $v \in L^2(SNK)$, write

$$\Pi v = g(\Phi, \Theta), \qquad \Pi^\perp v = \rho w,$$

so $v = \rho w + g(\Phi, \Theta)$; here Φ is a section of the normal bundle NK, and the somewhat elaborate notation in the second summand here reflects the fact that any element of $\mathcal{S}$ can be written (locally) as the inner product of a section of NK and the vector Θ, whose components are the linear coordinate functions on each S^{n-1}. This summand is identified with Φ, and thus, in the following, w and Φ always represent the components of v in $\mathcal{S}^\perp$ and $\mathcal{S}$, respectively. Thus

$$w = \frac{1}{\rho} \Pi^\perp v, \qquad g(\Phi, \Theta) = \Pi v.$$

Later on the decomposition

$$w = w_0 + w_1 \tag{1.21}$$

is used, where w_0 is a function on K and the integral of w_1 over each fibre of SNK vanishes.

Note that the operator

$$J : v = g(\Phi, \Theta) \mapsto g(\mathfrak{J}\Phi, \Theta)$$

preserves $\mathcal{S}$ and is invertible since K is a nondegerate minimal submanifold.

1.5 Improving the approximate solution

The first step in solving (1.20) is to use an iteration scheme of Picard's type to find a sequence of approximate solutions $(w^{(i)}, \Phi^{(i)})$ for which the estimates for the error term are increasingly small, namely

$$\rho^r \sigma_r(S_\rho(w^{(i)}, \Phi^{(i)})) = C(m, k, r) + \mathcal{O}(\rho^{i+3}).$$

Letting $(w^{(0)}, \Phi^{(0)}) = (0, 0)$, define the sequence $(w^{(i+1)}, \Phi^{(i+1)}) \in \mathcal{S}^\perp \oplus \mathcal{S}$ inductively as the unique solution to

$$\begin{aligned}\mathcal{L}_0\, w^{(i+1)} + \rho\, g(\mathfrak{J}\, \Phi^{(i+1)}, \Theta) &= \Omega(\Theta, \Theta)\rho^2 + \mathcal{O}(\rho^3) - \rho^2\, \Delta_K w^{(i)} \\ &+ 2C^{r-1}_{n-1}\rho^3 g(\Theta, B) \circ \nabla^2_K w^{(i)} + \rho^2\, L(w^{(i)}, \Phi^{(i)}) + Q(w^{(i)}, \Phi^{(i)}).\end{aligned} \tag{1.22}$$

Here

$$\mathcal{L}_0 := -C^{r-1}_{n-1} \cdot \frac{n-r}{n-1} \left(\Delta_{S^{n-1}} + (n-1)\right).$$

Observe that the operator Δ_K has been moved to the right hand side and hence the operator on the left hand side is not elliptic anymore. This equation becomes simpler when divided into its $\mathcal{S}^\perp$ and $\mathcal{S}$ components. Thus using that $\mathcal{L}_0$ annihilates $\mathcal{S}$ and

$$\Omega(\Theta, \Theta) \in \mathcal{S}^\perp,$$

since it is quadratic in Θ, (1.22) can be rewritten as the two separate equations:

$$\begin{aligned}\mathcal{L}_0\, w^{(i+1)} \;&=\; \Pi^\perp \big(\Omega(\Theta, \Theta)\rho^2 + \mathcal{O}(\rho^3) - \rho^2\, \Delta_K w^{(i)} \\ &+\; 2C^{r-1}_{n-1}\rho^3 g(\Theta, B) \circ \nabla^2_K w^{(i)} + \rho^2\, L(w^{(i)}, \Phi^{(i)}) + Q(w^{(i)}, \Phi^{(i)})\big)\,,\end{aligned} \tag{1.23}$$

and

$$\mathfrak{J}\, \Phi^{(i+1)} \;=\; \Pi \left(\mathcal{O}(\rho^2) + 2C^{r-1}_{n-1}\rho^2 g(\Theta, B) \circ \nabla^2_K w^{(i)} + \rho\, L(w^{(i)}, \Phi^{(i)}) + \rho^{-1} Q(w^{(i)}, \Phi^{(i)})\right),$$

since $\Pi(\Delta_K w) = 0$ for all $w \in \mathcal{S}$.

That there is a unique solution now follows from the invertibility of the operators J on $\mathcal{S}$ and $\mathcal{L}_0$ on $\mathcal{S}^\perp$, so the only issue is to obtain estimates.

Lemma 1.10 *For this sequence $(w^{(i)}, \Phi^{(i)})$, the estimates*

$$w^{(i)} = \mathcal{O}(\rho^2) \qquad\qquad \Phi^{(i)} = \mathcal{O}(\rho^2),$$

$$w^{(i+1)} - w^{(i)} = \mathcal{O}(\rho^{i+3}) \qquad\qquad \Phi^{(i+1)} - \Phi^{(i)} = \mathcal{O}(\rho^{i+2})$$

hold, for all $i \geq 1$.

PROOF. The estimates for $(w^{(1)}, \Phi^{(1)})$ are immediate, and the result for $i > 1$ is proved by a standard induction using the general structure of the operators L and Q. $\square$

As already mentioned, the operator in the right hand side of (1.23) is not elliptic since $\mathcal{L}_0$ acts on functions defined on SNK and does not involve any derivatives with respect to y^a. Nevertheless, since the pertinent functions are in $\mathcal{S}$, the equation

$$\mathcal{L}_0 w = f$$

can always be solved for any $f \in \mathcal{S}$ (this equation is solved on each fiber of NK with the base point as a parameter), but without any gain of regularity in the y^a variables and in fact there is a "loss" of two derivatives in the y^a variables at each iteration. At first glance, it would have been more natural to work with the operator $\mathcal{L}_\rho$, which is elliptic, and solve the equation

$$\mathcal{L}_\rho w = f,$$

but the operator $\mathcal{L}_\rho$ has the disadvantage to have a nontrivial kernel in S each time $\frac{C(m,k,r)}{\rho^{r+1}}$ belongs to the spectrum of $-\Delta_K$. This implies that the corresponding iteration scheme, using the operator $\mathcal{L}_\rho$ instead of $\mathcal{L}_0$, does not work for any value of ρ. In addition, even if $\frac{C(m,k,r)}{\rho^{r+1}}$ is chosen not to belong to the spectrum of $-\Delta_K$, the norm of the inverse of $\mathcal{L}_\rho$ blows up as ρ tends to 0 and hence the estimates for $w^{(i)}$ and $\Phi^{(i)}$ are not as good as the ones stated in Lemma 1.10.

To conclude, the use of the iteration scheme (1.22) allows one to improve the approximate solution to any finite order. Observe that the error $\Omega(\Theta,\Theta)\rho^2 + \mathcal{O}(\rho^3)$ in (1.22) is smooth in the y^a variables and hence losing finite regularity in these variables is not a real issue.

Finally, replacing (w, Φ) by $(w^{(i)} + w, \Phi^{(i)} + \Phi)$ in (1.20), the expansion of the equation

$$\sigma_r(S_\rho(w^{(i)} + w, \Phi^{(i)} + \Phi)) = \frac{C(m,k,r)}{\rho^r}.$$

becomes

$$\tfrac{1}{\rho}\mathcal{L}_\rho\, w + g(\mathfrak{J}\,\Phi, \Theta) - 2C_{n-1}^{r-1}\rho^2 g(\Theta, B) \circ \nabla_K^2 w^{(i)} + \rho\, L_i(w, \Phi) = \mathcal{O}_i(\rho^{i+2})\tfrac{1}{\rho}Q_i(w, \Phi). \quad (1.24)$$

The linear and nonlinear operators L_i and Q_i appearing on this equation are different from the ones before and depend on i, but enjoy similar properties uniformly in i.

1.6 Estimates of the spectrum of linearized operators

The mapping properties of the linear operator

$$(w, \Phi) \mapsto \frac{1}{\rho}\mathcal{L}_\rho\, w + g(\mathfrak{J}\,\Phi, \Theta) - 2C_{n-1}^{r-1}\rho^2 g(\Theta, B) \circ \nabla_K^2 w^{(i)} + \rho\, L_i(w, \Phi), \quad (1.25)$$

which appears in (1.24), are examined. This is not precisely the usual Jacobi operator (applied to the function $\rho\, w + g(\Phi, \Theta)$), because this hypersurface is parametrized as a graph over $S_\rho(w^{(i)}, \Phi^{(i)})$ using the vector field $-\Upsilon$ rather than the unit normal.

To understand the difference between (1.25) and the Jacobi operator, recall that if N is the unit normal to a hypersurface Σ and $\tilde{N}$ is any other transverse vector field, then hypersurfaces which are $\mathcal{C}^2$-close to Σ can be parameterized as either

$$\Sigma \ni q \mapsto \exp_q^M(wN) \qquad \text{or} \qquad \Sigma \ni q \mapsto \exp_q^M(\tilde{w}\tilde{N}).$$

The corresponding linearized r-mean curvature operators $\mathbb{L}_{\Sigma,N}$ and $\mathbb{L}_{\Sigma,\tilde{N}}$ are related by

$$\mathbb{L}_{\Sigma,N}(g(N,\tilde{N})\,w) + m\,(\tilde{N}^T H_\Sigma)\,w = \mathbb{L}_{\Sigma,\tilde{N}}w,$$

here $\tilde{N}^T$ is the orthogonal projection of $\tilde{N}$ onto $T\Sigma$. Since $\mathbb{L}_{\Sigma,N}$ is self-adjoint with respect to the usual inner product, it follows that $\mathbb{L}_{\Sigma,\tilde{N}}$ is self-adjoint with respect to the inner product

$$\langle v, v'\rangle := \int_\Sigma v\,v'\,g(N,\tilde{N})\,dvol_\Sigma.$$

Now suppose that $\Sigma = S_\rho(w^{(i)},\Phi^{(i)})$ and $\tilde{N} = \Upsilon$. From Lemma 1.10 and Proposition 1.6 it follows that

$$g(N,-\Upsilon) = 1 + \mathcal{O}(\rho^2).$$

Furthermore, from Proposition 1.5 and Lemma 1.10, and the fact that K is minimal, the volume forms of the tubes $S_\rho(w^{(i)},\Phi^{(i)})$ and SNK are related by

$$\sqrt{\det(g_{S_\rho(w^{(i)},\Phi^{(i)})})} = \rho^{k/2}\,(1+\mathcal{O}(\rho^2))\,\sqrt{\det(g_{SNK})};$$

hence

$$A_\rho := g(N,-\Upsilon)\frac{\sqrt{\det(g_{S_\rho(w^{(i)},\Phi^{(i)})})}}{\rho^{k/2}\,\sqrt{\det(g_{SNK})}} = 1 + \mathcal{O}(\rho^2). \tag{1.26}$$

Now define

$$\begin{aligned}\mathbb{L}_\rho v = \mathbb{L}_\rho(\rho\,w + g(\Phi,\Theta)) &:= A_\rho\,\left(\frac{1}{\rho}\,\mathcal{L}_\rho\,w + g(\mathfrak{J}\,\Phi,\Theta) + \mathcal{O}(\rho^2)\,\nabla_K^2 w + \rho\,\bar{L}(w,\Phi)\right)\\ &= \left(\frac{1}{\rho}\,\mathcal{L}_\rho\,w + g(\mathfrak{J}\,\Phi,\Theta) + \mathcal{O}(\rho^2)\,\nabla_K^2 w + \rho\,\bar{L}(w,\Phi)\right),\end{aligned} \tag{1.27}$$

where the last equality follows from (1.26).

Finally, multiplying (1.24) by A_ρ gives one further equivalent form of this equation,

$$\mathbb{L}_\rho\,v = \mathcal{O}(\rho^{2+i}) + \frac{1}{\rho}\,\tilde{Q}\left(\frac{1}{\rho}\Pi^\perp v, \Pi v\right), \tag{1.28}$$

where the nonlinear operator on the right has the same properties as before.

Associated to $\mathbb{L}_\rho$ is the quadratic form

$$\mathcal{Q}_\rho(w,\Phi) := \int_{SNK}(\rho\,w + g(\Phi,\Theta))\,\mathbb{L}_\rho(\rho\,w + g(\Phi,\Theta)),$$

and its corresponding polarization, the bilinear form $\mathcal{C}_\rho$. These forms are studied as perturbations of the model forms

$$\mathcal{Q}_0(w,\Phi) := \int_{SNK} (\rho^2\,|\nabla_K w|^2 + |\nabla_{S^{n-1}} w|^2 - (n-1)\,|w|^2) + \frac{\omega_{n-1}}{n}\int_K g(\mathfrak{J}\Phi,\Phi)$$

and associated polarization $\mathcal{C}_0$.

To make precise the sense in which $\mathcal{Q}_0$ and $\mathcal{Q}_\rho$ are close, define the weighted norm

$$\|(w,\Phi)\|^2_{H^1_\rho} := \int_{SNK} (\rho^2\,|\nabla_K w|^2 + |\nabla_{S^{n-1}} w|^2 + |w|^2) + \omega_n \int_K (|\nabla_K \Phi|^2 + |\Phi|^2)$$

and also

$$\|(w,\Phi)\|^2_{L^2} := \int_{SNK} |w|^2 + \omega_n \int_K |\Phi|^2.$$

Using (1.26) and the properties of $\bar{L}$, it follows that

$$|\mathcal{C}_\rho((w,\Phi),(w',\Phi')) - \mathcal{C}_0((w,\Phi),(w',\Phi'))| \le c\,\rho\,\|(w,\Phi)\|_{H^1_\rho}\,\|(w',\Phi')\|_{H^1_\rho}. \tag{1.29}$$

1.6.1 Estimates for eigenfunctions with small eigenvalues

Lemma 1.11 *Let σ be an eigenvalue of $\mathbb{L}_\rho$ and (w,Φ) a corresponding eigenfunction. There exist constants $c, c_0 > 0$ such that if $|\sigma| \le c_0$, then using the decomposition $w = w_0 + w_1$ from (1.21),*

$$\|(w - w_0, \Phi)\|_{H^1_\rho} \le c\,\rho\,\|(w,\Phi)\|_{H^1_\rho}.$$

Proof. For any (w',Φ'),

$$\begin{aligned}\mathcal{C}_\rho((w,\Phi),(w',\Phi')) &= \sigma \int_{SNK} (\rho^2 w\,w' + g(\Phi,\Theta)g(\Phi',\Theta)) \\ &= \sigma \int_{SNK} \rho^2 w\,w' + \sigma\,\frac{\omega_n}{n}\int_K g(\Phi,\Phi').\end{aligned}$$

In addition, (1.29) gives

$$\begin{aligned}\Big|\int_{SNK} &(\rho^2\,\nabla_K w\,\nabla_K w' + \nabla_{S^{n-1}} w \nabla_{S^{n-1}} w' - (n-1+\sigma)\,w\,w') \\ &+ \frac{\omega_{n-1}}{n}\int_K (g(\mathfrak{J}\Phi,\Phi') - \sigma\,g(\Phi,\Phi'))\Big| \le c\,\rho\,\|(w,\Phi)\|_{H^1_\rho}\,\|(w',\Phi')\|_{H^1_\rho}.\end{aligned} \tag{1.30}$$

Step 1: First take $w' = 0$ and $\Phi' = \Phi$ in (1.30); this yields

$$\Big|\int_K (g(\mathfrak{J}\Phi,\Phi) + \sigma\,g(\Phi,\Phi))\Big| \le c\,\rho\,\|(w,\Phi)\|_{H^1_\rho}\,\|(0,\Phi)\|_{H^1_\rho}.$$

Since $\mathfrak{J}$ is invertible, there exists $c_1 > 0$ such that

$$2\, c_1 \, \|(0,\Phi)\|^2_{H^1_\rho} \leq \left| \int_K g(\mathfrak{J}\Phi, \Phi) \right|,$$

hence

$$(2\, c_1 - |\sigma|) \, \|(0,\Phi)\|_{H^1_\rho} \leq c\, \rho \, \|(w,\Phi)\|_{H^1_\rho}.$$

Assuming $c_1 \geq |\sigma|$, it follows that

$$\|(0,\Phi)\|_{H^1_\rho} \leq c\, \rho \, \|(w,\Phi)\|_{H^1_\rho}.$$

Step 2: Using now (1.30) with $\Phi' = 0$ and $w = w_1$ it follows that

$$\left| \int_{SNK} (\rho^2 \, |\nabla_K w_1|^2 + |\nabla_{S^{n-1}} w_1|^2 - (n-1-\sigma)\, |w_1|^2) \right| \leq c\, \rho \, \|(w,\Phi)\|_{H^1_\rho} \, \|(w_1, 0)\|_{H^1_\rho}.$$

However, since $\Pi w_1 = 0$ and $\int_{S^{n-1}} w_1 = 0$, this implies

$$\int_{S^{n-1}} |\nabla_{S^{n-1}} w_1|^2 \geq 2\, n \int_{S^{n-1}} |w_1|^2,$$

hence

$$\left| \int_{SNK} (\rho^2 \, |\nabla_K w_1|^2 + \frac{1}{2} \, |\nabla_{S^{n-1}} w_1|^2 + (1-|\sigma|)\, |w_1|^2) \right| \leq c\, \rho \, \|(w,\Phi)\|_{H^1_\rho} \, \|(w_1, 0)\|_{H^1_\rho}.$$

This implies that

$$\|(w_1, 0)\|_{H^1_\rho} \leq c\, \rho \, \|(w,\Phi)\|_{H^1_\rho}$$

provided $|\sigma| \leq 1/4$. This completes the proof taking $c_0 = \min\{c_1, 1/4\}$. □

1.6.2 Variation of small eigenvalues with respect to ρ

In order to estimate the spectral gaps of $\mathbb{L}_\rho$ when the parameter ρ is small, it is necessary to understand the rate of variation of the small eigenvalues of this operator. This is stated in the following lemma

Lemma 1.12 *There exist constants $c_0, c > 0$ such that, if σ is an eigenvalue of $\mathbb{L}_\rho$ with $|\sigma| < c_0$, then*

$$\rho \, \partial_\rho \sigma \geq 2\,(n-1) - c\rho$$

provided ρ is small enough.

Proof. It is relevant to note that for simple eigenvalues, there is a well-known formula which allows to compute its variation with respect to the parameter. This formula is given by

$$\partial_\rho \sigma = \int_{SNK} v\, (\partial_\rho \mathbb{L}_\rho) v \, dvol_{SNK}.$$

There is in general complications in the presence of multiplicities, but a result of Kato [17] shows that if one considers the derivative of the eigenvalue as a multi-valued function, then an analogue of this same formula holds for self-adjoint operators. In fact it holds that in this case

$$\partial_\rho \sigma \in \left\{ \int_{SNK} v\,(\partial_\rho \mathbb{L}_\rho) v : \quad v = \rho\, w + g(\Phi, \Theta), \quad \mathbb{L}_\rho v = \sigma\, v, \quad \|v\|_{L^2} = 1 \right\}.$$

Hence bounds should be provided for the set on the right. This is done by comparing to the model case and using the bounds for eigenfunctions obtained in the last subsection.

Let v be an eigenfunction of $\mathbb{L}_\rho$ corresponding to a small eigenvalue σ, namely $\mathbb{L}_\rho v = \sigma\, v$. Rather than normalizing by $\|v\|_{L^2} = 1$, it is assumed instead that $\|(w, \Phi)\|_{L^2} = 1$. Recall that $w = \rho^{-1} \Pi^\perp v$ and hence it holds that

$$\mathbb{L}_\rho v = \frac{1}{\rho^2} \mathcal{L}_\rho \Pi^\perp v + g(\mathfrak{J}\Phi, \Theta) + \mathcal{O}(\rho) \nabla_K^2 \Pi^\perp v + \rho\, \bar{L}(\rho^{-1}\, \Pi^\perp v, \Pi v).$$

Then, since Π and $\Pi^\perp$ are independent of ρ, it follows that

$$\begin{aligned} \partial_\rho \mathbb{L}_\rho v &= -\frac{2}{\rho^3} \mathcal{L}_\rho(\Pi^\perp v) + \frac{1}{\rho^2}(-2\rho \Delta_K \Pi^\perp v) + \mathcal{O}(1)\, \nabla_K^2 \Pi^\perp v + \bar{L}(\rho^{-1}\, \Pi^\perp v, \Pi v) \\ &= -\frac{2}{\rho^2} \mathcal{L}_0 w + \mathcal{O}(\rho) \nabla_K^2 w + \bar{L}(w, \Phi) \end{aligned}$$

where the operator $\bar{L}$ varies from line to line but satisfies the usual assumptions. This now gives

$$\left| \int_{SNK} v\,(\partial_\rho \mathbb{L}_\rho) v + \frac{2}{\rho} \int_{SNK} (|\nabla_{S^{n-1}} w|^2 - (n-1)\, |w|^2) \right| \le c\, \|(w, \Phi)\|^2_{H^1_\rho}. \tag{1.31}$$

Now, for this eigenfunction v, $Q_\rho(v, v) = \sigma \int \rho^2 |w|^2 + g(\Phi, \Phi)$, and hence by (1.29),

$$\left| \int_{SNK} (\rho^2\, |\nabla_K w|^2 + |\nabla_{S^{n-1}} w|^2 - (n-1+\sigma)\, |w|^2) + \frac{\omega_{n-1}}{n} \int_K (g(\mathfrak{J}\Phi, \Phi) - \sigma\, g(\Phi, \Phi)) \right| \le c\, \rho\, \|(w, \Phi)\|^2_{H^1_\rho}, \tag{1.32}$$

By Lemma 1.11,

$$\int_{SNK} |\nabla_{S^{n-1}} w|^2 + \int_K (|\nabla_K \Phi|^2 + |\Phi|^2) \le c\, \rho\, \|(w, \Phi)\|^2_{H^1_\rho}, \tag{1.33}$$

and inserting this in (1.32) gives

$$\left| \int_{SNK} (\rho^2\, |\nabla_K w|^2 - (n-1+\sigma)\, |w|^2) \right| \le c\, \rho\, \|(w, \Phi)\|^2_{H^1_\rho}. \tag{1.34}$$

Adding these last two estimates now implies that

$$\|(w, \Phi)\|^2_{H^1_\rho} \le c\, \rho\, \|(w, \Phi)\|^2_{H^1_\rho} + c \int_{SNK} |w|^2.$$

Thus, when ρ is small enough,

$$\|(w,\Phi)\|^2_{H^1_\rho} \leq c\|(w,\Phi)\|_{L^2} \leq c$$

by the choice of normalization. From (1.33) again

$$\int_{SNK} |\nabla_{S^{n-1}} w|^2 + \int_K (|\nabla_K \Phi|^2 + |\Phi|^2) \leq c\,\rho.$$

Inserting this into (1.31), and using again that $\|(w,\Phi)\|_{L^2} = 1$, it follows that

$$\left| \int_{SNK} v\,(\partial_\rho \mathbb{L}_\rho) v - \frac{2}{\rho}\,(n-1) \right| \leq c \tag{1.35}$$

for all v such that $\mathbb{L}_\rho v = \sigma\, v$ and $\|(w,\Phi)\|_{L^2} = 1$.

This already implies that $\partial_\rho \sigma > 0$ for ρ small enough. But observing that $\|v\|_{L^2} \leq \|(w,\Phi)\|_{L^2}$ always holds, it follows that

$$\inf_{\substack{v:\mathbb{L}_\rho v = \sigma \\ \|v\|_{L^2}=1}} \int_{SNK} v\,(\partial_\rho \mathbb{L}_\rho) v \quad \geq \quad \inf_{\substack{v:\mathbb{L}_\rho v = \sigma\, v \\ \|(w,\Phi)\|_{L^2}=1}} \int_{SNK} v\,(\partial_\rho \mathbb{L}_\rho) v$$

and (1.35) implies that

$$\partial_\rho \sigma \geq \frac{2}{\rho}\,(n-1) - c.$$

This completes the proof of the result. □

1.6.3 The spectral gap at 0 of $\mathbb{L}_\rho$

A quantitative statement about the clustering of the spectrum at 0 of $\mathbb{L}_\rho$ as $\rho \searrow 0$ can now be proven. The ultimate goal is to estimate the norm of the inverse of this operator, but by self-adjointness, this is equivalent to an estimate on the size of the spectral gap at 0.

Lemma 1.13 *Fix any $q \geq 2$. Then there exists a sequence of disjoint nonempty intervals $I_i = (\rho_i^-, \rho_i^+)$, $\rho_i^\pm \to 0$ and a constant $c_q > 0$ such that when $\rho \in I := \cup_i I_i$, the operator $\mathbb{L}_\rho$ is invertible and*

$$\mathbb{L}_\rho^{-1} : L^2(SNK) \longrightarrow L^2(SNK)$$

has norm bounded by $c_q\,\rho^{-k-q+1}$, uniformly in $\rho \in I$. Furthermore, $I := \cup_i I_i$ satisfies

$$\left|\mathcal{H}^1((0,\rho) \cap I) - \rho\right| \leq c\,\rho^q, \qquad \rho \searrow 0.$$

Proof. An estimate for the size of the spectral gap at 0 is related to the spectral flow of $\mathbb{L}_\rho$, and so it suffices to find an asymptotic estimate for the number of negative eigenvalues of $\mathbb{L}_\rho$. Define the two quadratic forms

$$\mathcal{Q}^\pm(w,\Phi) := \mathcal{Q}_0(w,\Phi) \pm \gamma\,\rho\,\|(w,\Phi)\|^2_{H^1_\rho}$$

From (1.29), if $\gamma > 0$ is sufficiently large, then

$$\mathcal{Q}^- \leq \mathcal{Q}_\rho \leq \mathcal{Q}^+,$$

and this gives a two-sided bound for the index of $\mathcal{Q}_\rho$.

Decomposing $w = w_0 + w_1$ with w_0 depending only on $y \in K$, write

$$D_0^\pm(w_0) = (1 \pm \gamma\, \rho) \int_K \rho^2\, |\nabla_K w_0|^2 - (n-1 \mp \gamma\, \rho) \int_K |w_0|^2,$$

$$D_1^\pm(w_1) = (1 \pm \gamma\, \rho) \int_{SNK} (\rho^2\, |\nabla_K w_1|^2 + |\nabla_{S^{n-1}} w_1|^2) - (n-1 \mp \gamma\, \rho) \int_{SNK} |w_1|^2,$$

and finally

$$D^\pm(\Phi) = -(1 \pm \gamma\, \rho) \int_K g(\mathfrak{J}\, \Phi, \Phi),$$

so that

$$\mathcal{Q}^\pm(w, \Phi) = \omega_{n-1}\, D_0^\pm(w_0) + D_1^\pm(w_1) + \frac{\omega_{n-1}}{n}\, D^\pm(\Phi)$$

If $1 - \gamma\, \rho > 0$, then the index of $D^\pm$ equals the index of the minimal submanifold K, and hence does not depend on ρ. Next, if $(1 - \gamma\, \rho)\, 2\, n - (n - 1 + \gamma\, \rho) > 0$, then the index of $D_1^\pm$ equals 0. So it remains only to study the index of $D_0^\pm$. This is equal to the largest $j \in \mathbb{N}$ such that

$$(1 \pm \gamma\, \rho)\, \rho^2\, \mu_j \leq (n - 1 \mp c\, \rho)$$

Weyl's asymptotic formula states that

$$\operatorname{Ind} \mathcal{Q}^\pm \sim c_K\, \rho^{-k},$$

and hence the index of $D_0^\pm$, and finally $\operatorname{Ind} \mathcal{Q}_\rho$ too, is asymptotic to $c_K\, \rho^{-k}$.

Let $\rho_i \searrow 0$ be the decreasing sequence corresponding to the values at which the index of $\mathcal{Q}_\rho$ changes, counted according to the dimension of the nullspace of $\mathbb{L}_{\rho_i}$, i.e.

$$\rho_{i-1} < \rho_i = \ldots = \rho_j < \rho_{j+1}$$

if $\dim \operatorname{Ker} \mathbb{L}_{\rho_i} = j + 1 - i$. This is well-defined since, by Lemma 1.12 the small eigenvalues of $\mathbb{L}_\rho$ are monotone increasing for ρ small enough and hence, the function $\rho \to \mathcal{Q}_\rho$ is monotone decreasing for ρ small.

The estimates for $\operatorname{Ind} Q_{2\rho}$ and $\operatorname{Ind} Q_\rho$ imply that

$$r_\rho := \#\{\rho_i \in (\rho, 2\rho)\} \sim c\, \rho^{-k}.$$

Letting l_ρ denote the sum of lengths of intervals (ρ_{i+1}, ρ_i) for which $\rho_{i+1} \in (\rho, 2\rho)$ and $(\rho_i - \rho_{i+1}) \leq \rho^{k+q}$, then it holds that $l_\rho \leq c\, \rho^q$; from this it is concluded that ℓ_σ, the sum of lengths of all intervals (ρ_{i+1}, ρ_i) where $\rho_{i+1} < \rho$ and $(\rho_i - \rho_{i+1}) \leq \rho^{k+q}$ is also estimated by $c\, \rho^q$.

Define

$$\tilde{I} = \bigcup_{i \in J} (\rho_{i+1}, \rho_i), \qquad \text{where} \qquad i \in J \Leftrightarrow \rho_i - \rho_{i+1} \geq \rho_i^{k+q}.$$

Then by the above, it holds that

$$\left|\mathcal{H}^1((0,\rho)\cap I)-\rho\right|\le c_q\,\rho^q.$$

Finally, consider for any $\rho\in(\rho_{i+1},\rho_i)$, $i\in J$, the eigenvalues of $\mathbb{L}_\rho$ which are closest to 0, say

$$\sigma^-(\rho)<0<\sigma^+(\rho).$$

(Thus for each $\rho\in(\rho_{i+1},\rho_i)$, $\sigma^-(\rho)=\sigma_j$ where $j=\operatorname{Ind}Q_\rho$.) By construction,

$$\lim_{\rho\searrow\rho_{i+1}}\sigma^+(\rho)=\lim_{\rho\nearrow\rho_i}\sigma^-(\rho)=0.$$

By Lemma 1.12,

$$\sigma^-(\rho)\le 2\,(n-1)\,\frac{\rho-\rho_i}{\rho_i}+c\,\rho_i^{k+q},\qquad \rho\in(\rho_{i+1},\rho_i),$$

and

$$\sigma^+(\rho)\ge 2\,(n-1)\,\frac{\rho-\rho_{i+1}}{\rho_{i+1}}-c\,\rho_{i+1}^{k+q},\qquad \rho\in(\rho_{i+1},\rho_i).$$

Hence by the monotonicity of small eigenvalues, if

$$\rho\in I:=\bigcup_i\left(\rho_{i+1}+\frac{1}{4}\,\rho_i^{k+q},\rho_i-\frac{1}{4}\,\rho_i^{k+q}\right)$$

then the infimum of the absolute value of the eigenvalues of $\mathbb{L}_\rho$ is bounded from below by a constant (only depending on n) times ρ_i^{k+q-1}, provided ρ is small enough. The result then follows. $\square$

1.7 Existence of r-CMC hypersurfaces

The results of the previous sections are now used in order to solve the equation (1.28), which reduces to finding a fixed point

$$\rho\,w+g(\Phi,\Theta)=\mathbb{L}_\rho^{-1}\left(\mathcal{O}(\rho^{2+i})+\frac{1}{\rho}\,\tilde{Q}(w,\Phi)\right).$$

Since any function v defined on SNK can be decomposed as $v=\rho\,w+g(\Phi,\Theta)$ where the function w satisfies

$$\int_{S^{n-1}}w\,\varphi_j=0$$

for all $j=1,\dots,n$, this equation can be re-written as

$$v=\mathbb{L}_\rho^{-1}\left(\mathcal{O}(\rho^{2+i})+\frac{1}{\rho}\,\tilde{Q}\left(\frac{1}{\rho}\,\Pi^\perp v,\Pi v\right)\right).$$

Start with the following observation:

Lemma 1.14 *There exists a constant $c > 0$ such that*

$$\rho^{2+\alpha}\,\|v\|_{\mathcal{C}^{2,\alpha}} \leq c\,\rho^2\,\|\mathbb{L}_\rho\, v\|_{\mathcal{C}^{0,\alpha}} + c\,\rho^{-\frac{k}{2}}\,\|v\|_{L^2}.$$

Proof. This is an application of (rescaled) standard elliptic estimates. Set $f := \mathbb{L}_\rho\, v$ and, as in 1.3.1, use local normal coordinates $\bar{y} = y/\rho$ to parameterize a ball of radius $2\rho R$ in K, for some fixed small constant $R > 0$, and local coordinates z to parameterize S^{n-1}. Define the functions

$$\bar{v}(z,\bar{y}) := v(z,\rho\,\bar{y}) \qquad \text{and} \qquad \bar{f}(z,\bar{y}) := \rho^2\,f(z,\rho\,\bar{y}).$$

It is easy to check that $f := \mathbb{L}_\rho\, v$ translates into $\bar{\mathbb{L}}_\rho \bar{v} = \bar{f}$, where $\bar{\mathbb{L}}_\rho$ is a second order elliptic operator whose coefficients are bounded uniformly in ρ as ρ tends to 0. Moreover, the principal part of $\bar{\mathbb{L}}_\rho$ is the Laplace operator on SNK. Standard elliptic estimates yield

$$\|\bar{v}\|_{\bar{\mathcal{C}}^{2,\alpha}(B_R\times S^{n-1})} \leq c\,\|\bar{f}\|_{\bar{\mathcal{C}}^{0,\alpha}(B_R\times S^{n-1})} + c\left(\int_{S^{n-1}}\left(\int_{B_{2R}} |\bar{v}|^2\, d\bar{y}\right)\right)^{1/2}$$

where, to evaluate the Hölder norms in $\bar{\mathcal{C}}^{p,\alpha}$ one takes derivatives with respect to $\bar{y}$ and z. Going back to the functions v and f this yields

$$\rho^{2+\alpha}\,\|v\|_{\mathcal{C}^{2,\alpha}(B_{\rho R}\times S^{n-1})} \leq c\,\|\bar{v}\|_{\bar{\mathcal{C}}^{2,\alpha}(B_R\times S^{n-1})}, \qquad \|\bar{f}\|_{\bar{\mathcal{C}}^{2,\alpha}(B_{\rho R}\times S^{n-1})} \leq c\,\rho^2\,\|f\|_{\mathcal{C}^{2,\alpha}(B_R\times S^{n-1})}$$

and

$$\left(\int_{S^{n-1}}\left(\int_{B_{2R}} |\bar{v}|^2\, d\bar{y}\right)\right)^{1/2} \leq c\,\rho^{-\frac{k}{2}}\left(\int_{S^{n-1}}\left(\int_{B_{2\rho R}} |v|^2\, dy\right)\right)^{1/2}$$

the result then follows at once. □

Fix $q \geq 2$ and $\alpha \in (0,1)$. Collecting the result of Lemma 1.13 and the result of the previous Lemma, it follows that, if $\rho \in I$, then

$$\|v\|_{\mathcal{C}^{2,\alpha}} \leq c\,\rho^{-D}\,\|\mathbb{L}_\rho\, v\|_{\mathcal{C}^{0,\alpha}} \tag{1.36}$$

where the constant $c > 0$ does not depend on ρ and where $D := 3\frac{k}{2} + q + 1 + \alpha$.

Given $R > 0$, set

$$B(R) := \{v \in \mathcal{C}^{2,\alpha}(SNK)\ :\ \|v\|_{\mathcal{C}^{2,\alpha}} \leq R\}.$$

and define the mapping

$$\mathcal{N}_\rho(v) := \mathbb{L}_\rho^{-1}\left(\mathcal{O}(\rho^{2+i}) + \frac{1}{\rho}\,\tilde{Q}\left(\frac{1}{\rho}\,\Pi^\perp v, \Pi v\right)\right).$$

It follows from (1.36) that

$$\|\mathcal{N}_\rho(0)\|_{\mathcal{C}^{2,\alpha}} \leq \frac{c_0}{2}\,\rho^{2+i-D}$$

for some constant $c_0 > 0$, independent of $\rho \in I$.

Choose $i \in \mathbb{N}$ such that $i > 2D+1$. Using the properties of the operator $\tilde{Q}$, it is easy to check that there exists $\rho_0 > 0$ such that, for all $\rho \in (0, \rho_0) \cap I$,

$$\|\mathcal{N}_\rho(v)\|_{\mathcal{C}^{2,\alpha}} \leq c_0\, \rho^{2+i-D}$$

and

$$\|\mathcal{N}_\rho(v) - \mathcal{N}_\rho(v')\|_{\mathcal{C}^{2,\alpha}} \leq c\, \rho^{i-1-2D}\, \|v - v'\|_{\mathcal{C}^{2,\alpha}}$$

for all $v, v' \in B(c_0\, \rho^{2+i-D})$. Therefore the mapping $\mathcal{N}_\rho$ admits a (unique) fixed point v_ρ in $B(c_0\, \rho^{2+i-D})$. This yields the existence of a constant r-mean curvature perturbation of the tube $S_\rho(w^{(i)}, \Phi^{(i)})$ for all $\rho \in (0, \rho_0) \cap I$. The proof of the Theorem is complete.

1.8 Proof of Proposition 1.7

The aim of this section in to prove Proposition 1.7. Note first that by Lemma 1.2, it holds that

$$\begin{aligned}
\nabla_{X_a} X_b &= \Gamma_a^b(E_i)\, X_i + (\mathcal{O}(\rho) + L(w,\Phi) + Q(w,\Phi))\, X_\gamma,\\
\nabla_{X_i} X_j &= (\mathcal{O}(\rho) + L(w,\Phi) + Q(w,\Phi))\, X_\gamma,\\
\nabla_{X_a} X_i &= -\Gamma_a^b(E_i)\, X_b + (\mathcal{O}(\rho) + L(w,\Phi) + Q(w,\Phi))\, X_\gamma.
\end{aligned} \tag{1.37}$$

In particular, this, together with the expression of $Z_{\bar{a}}$ implies that

$$\begin{aligned}
\nabla_{Z_{\bar{a}}} X_i &= \rho\, \Gamma_a^b(E_i)\, X_b + (\mathcal{O}(\rho^2) + \rho\, L(w,\Phi) + \rho\, Q(w,\Phi))\, X_\gamma,\\
\nabla_{Z_{\bar{a}}} X_b &= -\rho\, \Gamma_a^b(E_i)\, X_i + (\mathcal{O}(\rho^2) + \rho\, L(w,\Phi) + \rho\, Q(w,\Phi))\, X_\gamma.
\end{aligned} \tag{1.38}$$

The following expansion which follows from the result of Lemma 1.4 is also be needed:

$$\begin{aligned}
\nabla_{X_a} X_b &= \Gamma_a^b(E_j)\, X_j - g(R(\rho\,\Theta + \Phi, E_a)\, E_j, E_b)\, X_j\\
&+ \tfrac{1}{2}\left(g(R(E_a, E_b)\,\rho\,\Theta + \Phi, E_j) - \Gamma_a^c(\rho\,\Theta + \Phi)\,\Gamma_c^b(E_j) - \Gamma_c^b(\rho\,\Theta + \Phi)\,\Gamma_a^c(E_j)\right)\, X_j\\
&+ (\mathcal{O}(\rho) + L(w,\Phi) + Q(w,\Phi))\, X_c + (\mathcal{O}(\rho^2) + \rho\, L(w,\Phi) + Q(w,\Phi))\, X_j.
\end{aligned} \tag{1.39}$$

Finally, the following expansions is needed:

$$\begin{aligned}
g(\Upsilon, X_a) &= \rho\, L(w,\Phi) + Q(w,\Phi)\\
g(\Upsilon, \Upsilon_j) &= \rho\, L(w,\Phi) + Q(w,\Phi),
\end{aligned} \tag{1.40}$$

the proof of which can be obtained as in 1.3.2, starting from the estimates (1.8).

First estimate: Estimate $g(N, \nabla_{Z_{\bar{a}}} Z_{\bar{b}})$ when $a = b$, since the corresponding estimate, when $a \neq b$ follows from the same proof. The expression to be expanded is

$$\rho^{-2}\, g(N, \nabla_{Z_{\bar{a}}} Z_{\bar{a}}) = \rho^{-1}\, \left(g(N, \nabla_{Z_{\bar{a}}} X_a) + g(N, \nabla_{Z_{\bar{a}}} (w_{\bar{a}}\, \Upsilon)) + g(N, \nabla_{Z_{\bar{a}}} \Psi_a)\right).$$

The estimate is broken into three steps:

Step 1 From (1.12), it follows that

$$\begin{aligned}
g(N,\Upsilon) &= -g(\Upsilon,\Upsilon)+\alpha^j\,g(\Upsilon_j,\Upsilon)+\beta^b\,g(X_b,\Upsilon)+(\rho\,L^1(w,\Phi)+Q^1(w,\Phi))\,g(X_c,\Upsilon)\\
&+ (\rho^2\,L(w,\Phi)+Q(w,\Phi))\,g(X_j,\Upsilon)\\
&= -1+\rho^2\,L(w,\Phi)+Q(w,\Phi).
\end{aligned}$$

Substituting $N=-\Upsilon+N+\Upsilon$ gives

$$g(N,\nabla_{Z_{\bar{a}}}\Upsilon)=-\frac{1}{2}\,\partial_{\bar{y}^a}g(\Upsilon,\Upsilon)+g(N+\Upsilon,\nabla_{Z_{\bar{a}}}\Upsilon).$$

But it follows from (1.10) that

$$\partial_{\bar{y}^a}\,g(\Upsilon,\Upsilon)=\rho^2\,L(w,\Phi)+Q(w,\Phi),$$

and (1.38) together with the expression of N implies that

$$g(N+\Upsilon,\nabla_{Z_{\bar{a}}}\Upsilon)=\rho\,L(w,\Phi)+\rho\,Q(w,\Phi).$$

Collecting these estimates,

$$g(N,\nabla_{Z_{\bar{a}}}\Upsilon)=\rho\,L(w,\Phi)+Q(w,\Phi).$$

Hence it is concluded that

$$g(N,\nabla_{Z_{\bar{a}}}(w_{\bar{a}}\,\Upsilon))=w_{\bar{a}\bar{a}}\,g(N,\Upsilon)+w_{\bar{a}}\,g(N,\nabla_{Z_{\bar{a}}}\Upsilon)=-w_{\bar{a}\bar{a}}+Q(w,\Phi).$$

Step 2 Next,

$$g(N,\nabla_{Z_{\bar{a}}}\Psi_a)=\rho\,g(N,\Psi_{aa})+\Phi^j_a\,g(N,\nabla_{Z_{\bar{a}}}X_j).$$

From (1.38), it follows that

$$\Phi^j_a\,g(N,\nabla_{Z_{\bar{a}}}X_j)=\rho\,L(w,\Phi)+Q(w,\Phi).$$

Also, using the decomposition of N and (1.8), it holds that

$$\begin{aligned}
g(N,\Psi_{aa}) &= -g(\Upsilon,\Psi_{aa})+g(N+\Upsilon,\Psi_{aa})\\
&= -g(\Theta,\Phi_{aa})+\rho\,L(w,\Phi)+Q(w,\Phi)).
\end{aligned}$$

Collecting these gives

$$g(N,\nabla_{Z_{\bar{a}}}\Psi_a)=-\rho\,g(\Phi_{aa},\Theta)+\rho^2\,L(w,\Phi)+\rho\,Q(w,\Phi)).$$

Step 3 Expanding $Z_{\bar{a}}$ gives

$$g(N,\nabla_{Z_{\bar{a}}}X_a)=\rho\,g(N,\nabla_{X_a}X_a)+\rho\,w_{\bar{a}}\,g(N,\nabla_{\Upsilon}X_a)+\rho\,\Phi^j_a\,g(N,\nabla_{X_j}X_b). \tag{1.41}$$

With the help of (1.38), the following can be evaluated:

$$\begin{aligned}
g(N,\nabla_{\Upsilon}X_a) &= \mathcal{O}(\rho)+L(w,\Phi)+Q(w,\Phi)\\
g(N,\nabla_{X_j}X_a) &= \mathcal{O}(\rho)+L(w,\Phi)+Q(w,\Phi)\\
g(N+\Upsilon,\nabla_{X_a}X_a) &= -\alpha^j\,\Gamma^a_a\,(\Theta_j)+\rho\,L(w,\Phi)+Q(w,\Phi),
\end{aligned}$$

and plugging these into (1.41) gives

$$g(N, \nabla_{Z_{\bar a}} X_a) = -\rho\, g(\Upsilon, \nabla_{X_a} X_a) + \rho^2\, L(w, \Phi) + \rho\, Q(w, \Phi)$$

Using (1.39) the following expansion is obtained:

$$\begin{array}{rcl} \nabla_{X_a} X_a & = & -\Gamma^a_a(E_j)\, X_j - g(R(\rho\Theta + \Phi, E_a)\, E_j, E_a)\, X_j + \Gamma^c_a(\rho\,\Theta + \Phi)\, \Gamma^a_c(E_j)\, X_j \\ & + & (\mathcal{O}(\rho) + L(w, \Phi) + Q(w, \Phi))\, X_c + (\mathcal{O}(\rho^2) + \rho\, L(w, \Phi) + Q(w, \Phi))\, X_j, \end{array}$$

Finally, using (1.8) again, it follows that

$$\begin{array}{rcl} g(N, \nabla_{Z_{\bar a}} X_b) & = & \rho\, \Gamma^b_a(\Theta) + \rho^2\, g(R(\Theta, E_a)\, \Theta, E_a) + \mathcal{O}(\rho^3) \\ & + & \rho\, g(R(\Phi, E_a)\, \Theta, E_a) + \rho\, \Gamma^c_a(\rho\, \Theta + \Phi)\, \Gamma^a_c(\Theta) - \rho\, \alpha^j\, \Gamma^b_a\, (\Theta_j) \\ & + & \rho^2\, L(w, \Phi) + \rho\, Q(w, \Phi), \end{array}$$

which, together with the results of Step 1 and Step 2, completes the proof of the first estimate.

Second estimate: Estimate $g(N, \nabla_{Z_i} Z_j)$ when $i = j$ since, just as before, the corresponding estimate when $i \neq j$ follows similarly. This part is taken directly from [34]. Observe that, by Proposition 1.5, it can also be written

$$N = -\Upsilon + \frac{1}{\rho}\, \alpha^j\, Z_j + \hat{N},$$

where

$$\hat{N} = (L(w, \Phi) + Q(w, \Phi))\, X_a + (\rho^2\, L(w, \Phi) + Q(w, \Phi))\, X_j. \tag{1.42}$$

Now write

$$\begin{array}{rcl} g(N, \nabla_{Z_j} Z_j) & = & g(N, \nabla_{Z_j} Z_j) \\ & = & g(\nabla_{Z_j} \Upsilon, Z_j) - g(\nabla_{Z_j} (\alpha^i\, Z_i), Z_j) \\ & + & g(\hat{N}, \nabla_{Z_j} Z_j) - \partial_{z_j}\, g(\hat{N}, Z_j). \end{array}$$

Step 1: By (1.37), it can be estimated

$$\begin{array}{rcl} \nabla_{Z_j} Z_j & = & \rho\, w_j\, Y_j + \rho\, w_{jj}\, \Upsilon + \rho\, (1 + w)\, \nabla_{Z_j} Y_j + \rho\, w_j\, \nabla_{Z_j} \Upsilon \\ & = & (\mathcal{O}(\rho^3) + \rho^2\, L(w, \Phi) + \rho^2\, L(w, \Phi)\, (L(w, \Phi) + Q(w, \Phi)))\, X_a \\ & + & (\mathcal{O}(\rho^3) + \rho\, L(w, \Phi) + \rho^2\, Q(w, \Phi))\, X_k, \end{array}$$

Observe that the coefficient of X_a is slightly better than the coefficient of X_k since the first two terms only involve the X_k. Using this together with (1.42) it is concluded that

$$g(\hat{N}, \nabla_{Z_j} Z_j) = \rho^3\, L(w, \Phi) + \rho\, Q(w, \Phi).$$

Step 2: Next, using (1.42) together with (1.8), it follows that

$$\partial_{z_j}\, g(\hat{N}, Z_j) = \rho^3\, L(w, \Phi) + \rho\, Q(w, \Phi).$$

Step 3: Now estimate

$$C := 2\,g(\nabla_{Z_j}\Upsilon, Z_j).$$

It is convenient to define

$$C' := \frac{2}{1+w}\,g(\nabla_{Z_j}(1+w)\,\Upsilon, Z_j).$$

It follows from (1.13) that

$$C = C' + \rho\,Q(w,\Phi)$$

hence it is enough to focuss on the estimate of C'. To analyze this term, for the moment being regard w and Φ as functions of the coordinates $(z,\bar{y})$ again and also consider ρ as a variable instead of just a parameter. Thus consider

$$\tilde{F}(\rho,z,\bar{y}) = F\big(\rho(1+w(z,\bar{y}))\Upsilon(z) + \Phi(t\,\bar{y}), t\,\bar{y}\big).$$

The coordinate vector fields Z_j are still equal to $\tilde{F}_*(\partial_{z_j})$, but now it also holds that $(1+w)\Upsilon = \tilde{F}_*(\partial_\rho)$, which is the identity to be used below. Now, from (1.11), write

$$C' = \frac{1}{1+w}\,g(\nabla_{\partial_\rho}Z_j, Z_j) = \frac{1}{1+w}\,\partial_\rho g(Z_j,Z_j).$$

Therefore, it follows from (1.11) in Proposition 1.5 that

$$\begin{aligned}
C &= \tfrac{1}{1+w}\,\partial_\rho\,[\rho^2\,g(\Theta_j,\Theta_j) + \tfrac{\rho^4}{3}\,g(R(\Theta,\Theta_j)\,\Theta,\Theta_j) + \mathcal{O}(\rho^5) \\
&+ \; 2\,\rho^2\,w\,g(\Theta_j,\Theta_j) + \tfrac{\rho^3}{3}\,(g(R(\Theta,\Theta_j)\,\Phi,\Theta_j) + g(R(\Theta,\Theta_j)\,\Phi,\Theta_j)) \\
&+ \; \rho^4\,L(w,\Phi) + \rho^2\,Q(w,\Phi)] + \rho\,Q(w,\Phi) \\
&= \tfrac{1}{1+w}\,[2\,\rho\,g(\Theta_j,\Theta_j) + \tfrac{4}{3}\,\rho^3\,g(R(\Theta,\Theta_j)\,\Theta,\Theta_j) + \mathcal{O}(\rho^4) \\
&+ \; 4\,\rho\,w\,g(\Theta_j,\Theta_j) + \rho^2\,\,(g(R(\Theta,\Theta_j)\,\Phi,\Theta_j) + g(R(\Theta,\Theta_j)\,\Phi,\Theta_j)) \\
&+ \; \rho^3\,L(w,\Phi) + \rho\,Q(w,\Phi)] \\
&= 2\,\rho\,g(\Theta_j,\Theta_j) + \tfrac{4}{3}\,\rho^3\,g(R(\Theta,\Theta_j)\,\Theta,\Theta_j) + \mathcal{O}(\rho^4) \\
&+ \; 2\,\rho\,w\,g(\Theta_j,\Theta_j) + \rho^2\,(g(R(\Theta,\Theta_j)\,\Phi,\Theta_j) + g(R(\Theta,\Theta_j)\,\Phi,\Theta_j)) \\
&+ \; \rho^3\,L(w,\Phi) + \rho\,Q(w,\Phi).
\end{aligned}$$

Step 4: Finally, the expression to be computed is

$$\begin{aligned}
D &:= 2\,g(\nabla_{Z_j}(\alpha^i\,Z_i), Z_j) \\
&= 2\,g(Z_i,Z_j)\,\partial_{z^j}\alpha^i + 2\,\alpha^i\,g(\nabla_{Z_i}Z_j, Z_j) \\
&= 2\,g(Z_i,Z_j)\,\partial_{z^j}\alpha^i + \alpha^i\,\partial_{z^i}\,g(Z_j,Z_j).
\end{aligned}$$

Observe that (1.15) implies

$$\partial_{z^j}g(\Theta_i,\Theta_{j'}) = 0$$

at the point p. Using this together with (1.11) and the expression for the α^i given in Proposition 1.6, it follows that

$$\alpha^i\,\partial_{z^i}\,g(Z_j,Z_j) = \rho^4\,L(w,\Phi) + \rho^2\,Q(w,\Phi).$$

It follows from (1.11) and the definition of α^i again that

$$g(Z_i, Z_j)\, \partial_{z^j}\alpha^i = \rho^2\, g(\Theta_i, \Theta_j)\, \partial_{z^j}\alpha^i + \rho^4\, L(w, \Phi) + \rho^2\, Q(w, \Phi).$$

Therefore, it remains to estimate $g(\Upsilon_i, \Upsilon_{j'})\, \partial_{z^j}\alpha^i$. By definition, it holds that

$$\alpha^i\, g(\Theta_i, \Theta_j) = w_j + \frac{\rho}{3}\, g(R(\Phi, \Theta)\, \Theta, \Theta_j).$$

Differentiating with respect to z^j this yields

$$\big(g(\Theta_i, \Theta_j)\, \partial_{z^j}\alpha^i + \alpha^i\, \partial_{z^j} g(\Theta_i, \Theta_j)\big) = w_{jj} + \frac{\rho}{3}\, \partial_{z^j} g(R(\Phi, \Theta)\, \Theta, \Theta_j). \tag{1.43}$$

Again, it follows from (1.15) that $\partial_{z^j} g(\Theta_i, \Theta_j) = 0$. Moreover, using (1.38), it can first be estimated

$$\nabla_{Z_j} \Upsilon = \Upsilon_j + \mathcal{O}(\rho^2) + \rho\, L(w, \Phi) + \rho\, Q(w, \Phi);$$

and, using in addition (1.15), it also holds that

$$\nabla_{Z_j} \Upsilon_j = a\, \Upsilon + \mathcal{O}(\rho^2) + \rho\, L(w, \Phi) + \rho\, Q(w, \Phi)$$

for some $a \in \mathbb{R}$. Reinserting this in (1.43) yields

$$\begin{aligned} g(\Theta_i, \Theta_j)\, \partial_{z^j}\alpha^i \;&=\; w_{jj} + \tfrac{\rho}{3}\, g(R(\Phi, \Theta_j)\, \Theta, \Theta_j) + \tfrac{\rho}{3}\, g(R(\Phi, \Theta)\, \Theta_j, \Theta_j) + \\ &+\; \rho^3\, L(w, \Phi) + \rho^2\, Q(w, \Phi)), \end{aligned}$$

since $R(\Theta, \Theta) = 0$.

Collecting these estimates, it is concluded that

$$D = \rho^2\, w_{jj} + \frac{\rho^3}{3}\, g(R(\Phi, \Theta_j)\, \Theta, \Theta_j) + \rho^4\, L(w, \Phi) + \rho^2\, Q(w, \Phi)$$

since $g(R(\Phi, \Theta)\, \Theta_j, \Theta_j) = 0$. With the estimates of the previous steps, this finishes the proof of the estimate.

Third estimate: Decompose

$$\frac{1}{\rho}\, g(N, \nabla_{Z_{\bar a}} Z_j) = g(N, \Upsilon_j)\, w_{\bar a} + g(N, \Upsilon)\, w_{\bar a j} + (1 + w)\, g(N, \nabla_{Z_{\bar a}} \Upsilon_j) + w_j\, g(N, \nabla_{Z_{\bar a}} \Upsilon).$$

As above the expression of N given in (1.12) is used, to estimate

$$g(N, \Upsilon_j) = -g(\Upsilon, \Upsilon_j) + g(N + \Upsilon, \Upsilon_j) = L(w, \Phi) + Q(w, \Phi).$$

Similarly

$$g(N, \Upsilon) = -1 + L(w, \Phi) + Q(w, \Phi).$$

But now, by (1.38), it holds that

$$g(N, \nabla_{Z_{\bar a}}\, \Upsilon_j) = \mathcal{O}(\rho^2) + \rho\, L(w, \Phi) + \rho\, Q(w, \Phi)$$

and, as already shown in Step 1

$$g(N, \nabla_{Z_{\bar a}}\, \Upsilon) = \rho^2\, L(w, \Phi) + Q(w, \Phi),$$

and the proof of the estimate follows directly. □

Chapter 2

Concentration on a submanifold for a nonlinear Shrödinger equation

The construction of a one-parameter family of constant r-mean curvature hypersurfaces, described in the previous chapter, is comparable to many concentration results that have been highlighted in recent years in the study of semilinear elliptic equations. The first section of this chapter is devoted to recall some results in this direction while in the second section a new result obtained in collaboration with F. Mahmoudi and W. Yao, see [29] is presented.

2.1 Semilinear PDE's presenting concentration phenomena

In this section some previous results are briefly described. The first results in this direction are recalled, referring to Subsection 2.2.1 below for more bibliography.

A. Malchiodi [31] has studied the existence of periodic solutions for the equation

$$\ddot{x} + \frac{1}{\varepsilon^2} V(x) = 0, \qquad x \in \mathbb{R}^n \tag{2.1}$$

for $\varepsilon > 0$ small enough and $V : \mathbb{R}^n \to \mathbb{R}$ is a smooth function whose the set of critical points is a hypersurface $M \subset \mathbb{R}^n$. He distinguished two cases depending on the nature of the potential V: where V is of *repulsive* type with respect to M, i.e.

$$V''(x)\,(n_x, n_x) < 0 \qquad \forall\, x \in\, M, \quad 0 \neq n_x \perp T_xM,$$

he proved that every closed non degenerate geodesic $x_0 : S^1 \to\, M$ is a limit, as $\varepsilon > 0$ tends to 0, of a one-parameter family u_ε of solutions to equation (2.1). In particular, there is no restriction on the the parameter ε. In the *attractive* case

$$V''(x)\,(n_x, n_x) > 0 \qquad \forall\, x \in\, M, \quad 0 \neq n_x \perp T_xM$$

the situation is radically different: given the existence of a resonance phenomenon and under some technical assumptions, A. Malchiodi has demonstrated the existence of a sequence $(\varepsilon_j)_j$

tending to zero and a sequence of solutions of equation (2.1) which converges to x_0 when j tends to $+\infty$.

J. Shatah and C. Zeng [42] considered the equation

$$D_t\dot{p} + \Pi_p\left(w'(p)\right) = 0 \text{ sur } M, \tag{2.2}$$

where M is a submanifold of dimension k embedded in $\mathbb{R}^{m+1}$, Π_p is the orthogonal projection of $T_p\mathbb{R}^{m+1}$ over T_pM and D_t is the covariant derivative on M in the direction of $\dot{p}$. The problem is to show that the periodic solutions of equation (2.2) are limits of a sequence of periodic solutions to the penalized equation

$$\ddot{x} + w'(x) + \frac{1}{\varepsilon^2}\, G'(x) = 0 \tag{2.3}$$

where this time x is a curve in $\mathbb{R}^{m+1}$ and the penalization potential is $G(\cdot) = \operatorname{dist}(\cdot, M)^2$ in a neighborhood of M. They proved in [42] that given p_0 a nondegenerate periodic solution of (2.2), there exists $(\varepsilon_j)_j$ tending to 0 and a sequence of periodic solutions x_j of equation (2.3) with $\varepsilon = \varepsilon_k$ which converges to p_0 in a suitable sence. As in [31], the existence is only proved for a subsequence ε_j and not for all $\varepsilon > 0$ small. This is related to a resonance phenomenon coresponding to values of ε for which the linearisation of (2.3) around p_0 admits a nontrivial kernel. In fact, if one looks formally for solutions of (2.3) as perturbations of solutions to (2.2), namely solutions of the form

$$x = p_0 + y^t + y^n$$

where p_0 is a solutions of (2.2), y^n and y^t are respectively normal and tangential perturbations to M, then the linearized operator of (2.3), projected to normal fibers can be written as

$$L(y^n) = \ddot{y}^n + A(y^n) + \frac{1}{\varepsilon^2}\, y^n$$

and the resonant modes correspond to the values of ε satisfying

$$\frac{1}{\varepsilon^2} = \lambda_j$$

where λ_j are the eigenvalues of the operator $y^n \longmapsto \ddot{y}^n + A(y^n)$.

In [8], M. del Pino, M. Kowalczyk and J. Wei studied existence of solutions to the nonlinear Schrödinger equation

$$-\,i\,\varepsilon\,\frac{\partial\psi}{\partial t} = \varepsilon^2\,\Delta\psi - Q(y)\,\psi + \psi\,|\psi|^p \tag{2.4}$$

in $\mathbb{R}^2$, where $p > 1$. If one looks for a solution ψ of the form $\psi(t, y) = \exp\left(-i\,\lambda\,t/\varepsilon\right)u(y)$, then u is a solution for the nonlinear equation

$$\varepsilon^2\,\Delta u - \left(Q(y) + \lambda\right)u + u^p = 0, \quad u > 0, \tag{2.5}$$

By developing an infinite dimensional version of the Lyapunov-Schmidt reduction method, they proved that given a stationary non-degenerate curve Γ for the potential energy

$$\Gamma \longrightarrow \int_\Gamma V^{\frac{p+1}{p-1}-\frac{1}{2}}\, d\gamma$$

and if $Q + \lambda$ is a uniformly positive function, then for some constant $c > 0$, there exists $\varepsilon_0 > 0$ and $\lambda_* > 0$ such that for all $\varepsilon < \varepsilon_0$ satisfying $|\varepsilon^2 j^2 - \lambda_*| \geq c\varepsilon$, $\forall\, j \in \mathbb{N}$, the equation (2.5) admits a positive solution u_ε which concentrate along Γ. This result generalizes a previous result obtained by A. Ambrosetti, A. Malchiodi and W.M. Ni [1] in the case where the potential V depends only on the distance to the origin. In particular the result of M. del Pino, M. Kowalczyk and J. Wei is the first positive answer in the case $n = 2$ and $k = 1$ to a conjecture stated by A. Ambrosetti, A. Malchiodi and W.M. Ni which says that *concentration on k-dimensional sets for $k = 1, \cdots, n-1$ is expected* under suitable non-degeneracy assumptions and the limit set K should satisfy

$$\theta_k \nabla^N V = V\mathbf{H}, \tag{2.6}$$

where ∇^N is the normal gradient to K and $\mathbf{H}$ is the mean-curvature vector on K.

Later on Mahmoudi, Malchiodi and Montenegro in [27] constructed different type of solutions. Indeed, they studied complex-valued solutions whose phase is highly oscillatory carrying a quantum mechanical momentum along the limit curve. In particular they established the validity of the above conjecture for the case $n \geq 2$ arbitrary and $k = 1$. Recently, by applying the method developed in [8], Wang-Wei-Yang [43] considered the one-codimensional case $n \geq 3$ and $k = n - 1$ in the flat Euclidean space $\mathbb{R}^n$.

The main purpose of this chapter is to prove the validity of the above conjecture for all $k = 1, \ldots, n-1$.

2.2 On the Ambrosetti-Malchiodi-Ni conjecture for general submanifolds *(joint work with F. Mahmoudi and W. Yao), http://arxiv.org/abs/1405.6752*

2.2.1 Introduction and main results

In this chapter concentration phenomena for positive solutions of the nonlinear elliptic problem

$$-\varepsilon^2 \Delta_{\bar{g}} u + V(z)u = |u|^{p-1}u \quad \text{on } M, \tag{2.7}$$

are studied, where M is an n-dimensional compact Riemannian manifold without boundary (or the flat Euclidean space $\mathbb{R}^n$), $\Delta_{\bar{g}}$ stands for the Laplace-Beltrami operator on $(M, \bar{g})$, V is a smooth positive function on M satisfying

$$0 < V_1 \leq V(z) \leq V_2, \quad \text{for all } z \in M \text{ and for some constants } V_1, V_2, \tag{2.8}$$

u is a real-valued function, $\varepsilon > 0$ is a small parameter and p is an exponent greater than one.

The above semilinear elliptic problem arises from the standing waves for the nonlinear Schrödinger equation on M, see [1, 8] and some references therein for more details. An interesting case is the *semiclassical limit* $\varepsilon \to 0$. For results in this direction, when $M = \mathbb{R}$ and $p = 3$, Floer-Weinstein [12] first proved the existence of solutions highly concentrated near critical points of V. Later on this result was extended by Oh [37] to $\mathbb{R}^n$ with $1 < p < \frac{n+2}{n-2}$.

More precisely, the profile of these solutions is given by the *ground state* $U_{V(x_0)}$ of the limit equation

$$-\Delta u + V(x_0)u - u^p = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{2.9}$$

where x_0 is the concentration point. That is, the solutions obtained in [12] and [37] behave qualitatively like

$$u_\varepsilon(x) \sim U_{V(x_0)}\left(\frac{x - x_0}{\varepsilon}\right), \quad \text{as } \varepsilon \text{ tends to zero.}$$

Since $U_{V(x_0)}$ decays exponentially to 0 at infinity, u_ε vanishes rapidly away from x_0. In other words, in the semiclassical limit, solutions constructed in [12, 37] concentrate at points and they are always called *peak solutions* or spike solutions. In recent years, these existence results have been generalized in different directions, including: multiple peaks solutions, degenerate potentials, potentials tending to zero at infinity and for more general nonlinearities. *An important and interesting question is whether solutions exhibiting concentration on higher dimensional sets exist.*

Only recently it has been proven the existence of solutions concentrating at higher dimensional sets, like curves or spheres. In all these results (except for [2]), the profile is given by (real) solutions to (2.9) which are independent of some of the variables. If concentration occurs near a k-dimensional set, then the profile in the directions orthogonal to the limit set (concentration set) will be given by a soliton in $\mathbb{R}^{n-k}$. For example, some first results in the case of radial symmetry were obtained by Badiale-D'Aprile [4] and Benci-D'Aprile [5]. These results were improved by Ambrosetti-Malchiodi-Ni [1], where necessary and sufficient conditions for the location of the concentration set have been given. Unlike the point concentration case, the limit set is not stationary for the potential V : in fact a solution concentrated near a sphere carries a *potential energy* due to V and a *volume energy.* Define

$$E(u) = \frac{\varepsilon^2}{2}\int_M |\nabla_{\bar{g}} u|^2 + V(z)u^2 - \frac{1}{p+1}\int_M |u|^{p+1} \tag{2.10}$$

and let K be a k-dimensional submanifold of M and U_K be a proper approximate solution concentrated along K, see (2.51) below. One has

$$E(U_K) \sim \varepsilon^{n-k}\int_K V^{\theta_k} dvol, \qquad \text{with} \qquad \theta_k = \frac{p+1}{p-1} - \frac{1}{2}(n-k).$$

Based on the above energy considerations, Ambrosetti-Malchiodi-Ni [1] conjectured that *concentration on k-dimensional sets for $k = 1, \cdots, n-1$ is expected* under suitable non-degeneracy assumptions and the limit set K should satisfy

$$\theta_k \nabla^N V = V\mathbf{H}, \tag{2.11}$$

where ∇^N is the normal gradient to K and $\mathbf{H}$ is the mean-curvature vector on K. In particular, they suspected that concentration occurs in general along sequences $\varepsilon_j \to 0$.

By developing an infinite dimensional version of the Lyapunov-Schmidt reduction method, del Pino-Kowalczyk-Wei [8] successfully proved the validity of the above conjecture for $n = 2$ and $k = 1$. Actually they proved that: given a non-degenerate stationary curve K in $\mathbb{R}^2$ (for the weighted length functional $\int_K V^{\frac{p+1}{p-1}-\frac{1}{2}}$), suppose that ε is sufficiently small and satisfies the following *gap condition*:

$$|\varepsilon^2\ell^2 - \mu_0| \geq c\,\varepsilon, \quad \forall\, \ell \in \mathbb{N},$$

where μ_0 is a fixed positive constant, then problem (2.7) has a positive solution u_ε which concentrates on K, in the sense that it is exponentially small away from K. After some time Mahmoudi-Malchiodi-Montenegro in [27] constructed a different type of solutions. Indeed, they studied complex-valued solutions whose phase is highly oscillatory carrying a quantum mechanical momentum along the limit curve. In particular they established the validity of the above conjecture for the case $n \geq 2$ arbitrary and $k = 1$. Recently, by applying the method developed in [8], Wang-Wei-Yang [43] considered the one-codimensional case $n \geq 3$ and $k = n-1$ in the flat Euclidean space $\mathbb{R}^n$. The main purpose of this chapter is to prove the validity of the above conjecture for all $k = 1, \ldots, n-1$.

To prove the validity of the Ambrosetti-Malchiodi-Ni conjecture for all cases, one possible way is to generalize the method developed in [8] and [43]. For this purpose, the key steps in [8] and [43] are first recalled. According to our knowledge, the first key step is the construction of proper approximate solutions, and the second key step is to develop an infinite dimensional Lyapunov-Schmidt reduction method so that the original problem can be reduced to a simpler one that can be handled easily. Actually this kind of infinite dimensional reduction argument has been used in many constructions in PDE and geometric analysis. It has been developed by many authors working on this subject or on closely related problems, see for example [8, 9, 13, 28, 25] and references therein.

Let us now go back to our problem. To construct proper approximate solutions for general submanifolds, first the Laplace-Betrami operator for arbitrary submanifolds is expanded, see Proposition 2.5. Then by an iterative scheme of Picard's type, a family of very accurate approximate solutions can be obtained, see Section 3. Next an infinite dimensional reduction is developed such that the construction of positive solutions of problem (2.7) can be reduced to the solvability of a reduced system (2.62). For more details about the setting-up of the problem, refer to Subsection 4.1. It is slightly different from the arguments in [8] and [43]. Finally, by noticing the recent development on manifolds with density in differential geometry (cf. e.g. [22, 35]), our method explores a connection between solutions of the nonlinear Schrödinger equation and f-minimal submanifolds in Riemannian manifolds with density.

The main result can now be stated.

Theorem 2.1 *Let M be a compact n-dimensional Riemannian manifold (or the Euclidean space $\mathbb{R}^n$) and let $V : M \to \mathbb{R}$ be a smooth positive function satisfying (2.8). Given $k = 1, \ldots, n-1$, and $1 < p < \frac{n+2-k}{n-2-k}$. Suppose that K be a stationary non-degenerate smooth compact submanifold in M for the weighted functional*

$$\int_K V^{\frac{p+1}{p-1}-\frac{n-k}{2}}dvol,$$

then there is a sequence $\varepsilon_j \to 0$ such that problem (2.7) possesses positive solutions u_{ε_j} which concentrate near K. Moreover, for some constants C, $c_0 > 0$, the solutions u_{ε_j} satisfies *globally*

$$|u_{\varepsilon_j}(z)| \leq C \exp\big(- c_0\, dist(z, K)/\varepsilon_j \big).$$

Remark 2.2.1 The assumptions on K are related to the existence of non-degenerate compact minimal submanifold in manifolds M with density $V^{\frac{p+1}{p-1}-\frac{n-k}{2}}\,dvol$. In fact writing $V^{\frac{p+1}{p-1}-\frac{n-k}{2}} = e^{-f}$, then K is called f-minimal submanifold in differential geometry (cf. [22]).

Remark 2.2.2 Actually it can be proved that the same result holds true under a gap condition on ε, which is due to a resonance phenomena. Similar conditions can be found in [8, 43] and some references therein.

Before closing this introduction, notice that problem (2.7) is similar to the following singular perturbation problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{2.12}$$

This latter problem arises in the study of some biological models and as (2.7) it exhibits concentration of solutions at some points of $\overline{\Omega}$. Since this equation is homogeneous, then the location of concentration points is determined by the geometry of the domain. On the other hand, it has been proven that solutions exhibiting concentration on higher dimensional sets exist. For results in this direction refer to [9, 25, 26, 30, 32, 33, 44].

In general, these results can be divided into two types: The first one is the case where the concentration set lies totally on the boundary. The second one is where the concentration set is inside the domain and which intersect the boundary transversally. For this second type of solutions refer to Wei-Yang [44], who proved the existence of layer on the line intersecting with the boundary of a two-dimensional domain orthogonally. See also Ao-Musso-Wei [3], where triple junction solutions have been constructed. In the over-mentioned two results, [3] and [44], only the one dimensional concentration case has been considered. We believe the method developed here to the above problem (2.12) can be used to handle the higher dimensional situation, namely concentration at arbitrary dimensional submanifolds which intersect the boundary transversally. Interestingly, our preliminary result shows that our method explores a connection between solutions of problem (2.12) and minimal submanifolds with free boundary in geometric analysis.

It is worth pointing out that [44] applied an infinite dimensional reduction method while [3] used a finite dimensional one. Interested readers are encouraged to refer to the paper [10] for an intermediate reduction method which can be interpreted as an intermediate procedure between the finite and the infinite dimensional ones. Moreover, it is interesting to consider Open Question 4 in [10], which can be seen as the Ambrosetti-Malchiodi-Ni Conjecture without the small parameter ε.

The chapter is organized as follows. In Section 2 the Fermi coordinates in a tubular neighborhood of K in M are introduced and the Laplace-Beltrami operator is expanded in these Fermi coordinates. In Section 3, a family of very accurate approximate solutions is constructed. Section 4 will be devoted to develop an infinite dimensional Lyapunov-Schmidt reduction and to prove Theorem 2.1.

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2.2.2 Geometric background

In this section some geometric background is given. In particular, the so-called Fermi coordinates, which play important role in the higher dimensional concentrations, are introduced. Before doing this, the auxiliary weighted functional corresponding to problem (2.7) is first introduced.

The auxiliary weighted functional

Let K be a k-dimensional closed (embedded or immersed) submanifold of M^n, $1 \le k \le n-1$. Let $\{K_t\}_t$ be a smooth one-parameter family of submanifolds such that $K_0 = K$. Define

$$\mathcal{E}(t) = \int_{K_t} V^\sigma dvol, \qquad \text{with} \qquad \sigma = \frac{p+1}{p-1} - \frac{n-k}{2}. \tag{2.13}$$

Denote ∇^T and ∇^N to be connections projected to the tangential and normal spaces on K. The following definitions on K which appear in Theorem 2.1 are given:

Definition 2.2.1 (Stationary condition) A submanifold K is said to be stationary relative to the functional $\int_K V^\sigma dvol$ if

$$\sigma \nabla^N V = -VH \text{ on } K, \tag{2.14}$$

where H is the mean curvature vector on K, i.e., $H_j = -\Gamma^a_{aj}$ (here the minus sign depends on the orientation, and Γ^b_a are the 1-forms on the normal bundle of K (see (2.19) below for the definition).

Definition 2.2.2 (Nondegeneracy (ND) condition) It is said that K is non-degenerate if the quadratic form

$$\int_K \Big\{ \Big\langle \Delta_K \Phi + \frac{\sigma}{V} \nabla_K V \cdot \nabla_K \Phi, \Phi \Big\rangle + \sigma^{-1} H(\Phi)^2 - \frac{\sigma}{V} \big(\nabla^N\big)^2 V\,[\Phi, \Phi] - \mathrm{Ric}(\Phi, \Phi) \\ + \Gamma^a_b(\Phi)\Gamma^b_a(\Phi) \Big\} V^\sigma \sqrt{\det(g)}\, dvol \tag{2.15}$$

defined on the normal bundle to K, is non-degenerate.

Remark 2.2.3 As in the first chapter the Einstein summation convention is used, that is, summation over repeated indices is understood.

Setting $V^\sigma = e^{-f}$, i.e., $f = -\sigma \ln V$, then the stationary and ND conditions correspond to the first and second variation formulas of an f-minimal submanifold in [22], i.e.,

$$H = \nabla^N f,$$

where $H = -\sum_a \nabla^N_{e_a} e_a$ is the mean curvature vector, e_a $(1 \le a \le k)$ is an orthonormal frame in an open set of K. And at $t = 0$,

$$\frac{d^2}{dt^2}\Big(\int_{K_t} e^{-f}\Big) = \int_K e^{-f}\Big(-\sum_{a=1}^k R_{avva} - \frac{1}{2}\Delta_K(|v|^2) + |\nabla_K v|^2 - 2|A^v|^2 - f_{vv} + \frac{1}{2}\langle \nabla^T f, \nabla^T(|v|^2)\rangle\Big),$$

where K_t is a smooth family of submanifolds such that $K_0 = K$, the variational normal vector field v is compactly supported on K_t, and $A^v_{ab} = -\langle \nabla_{e_a} e_b, v\rangle$.

Fermi coordinates and expansion of the metric

Let K be a k-dimensional submanifold of $(M, \bar g)$ $(1 \le k \le n-1)$. Define $N = n - k$, and choose along K a local orthonormal frame field $\big((E_a)_{a=1,\cdots,k}, (E_i)_{i=1,\cdots,N}\big)$ which is oriented. At points of K, the natural splitting

$$TM = TK \oplus NK$$

hold, where TK is the tangent space to K and NK represents the normal bundle, which are spanned respectively by $(E_a)_a$ and $(E_i)_i$.

Denote by ∇ the connection induced by the metric $\bar g$ and by ∇^N the corresponding normal connection on the normal bundle. Given $p \in K$, some geodesic coordinates y centered at p are used. It is also assumed that at p the normal vectors $(E_i)_i$, $i = 1, \ldots, N$, are transported parallely (with respect to ∇^N) through geodesics from p, so in particular

$$\bar g\,(\nabla_{E_a} E_j\,, E_i) = 0 \ \text{ at } p, \quad \forall\, i, j = 1, \ldots, N,\ a = 1, \ldots, k. \tag{2.16}$$

In a neighborhood of p in K, consider normal geodesic coordinates

$$f(\bar y) := \exp^K_p(y_a E_a), \quad \forall\, \bar y := (y_1, \ldots, y_k),$$

where $\exp^K$ is the exponential map on K and summation over repeated indices is understood. This yields the coordinate vector fields $X_a := f_*(\partial_{y_a})$. Extend the E_i along each geodesic $\gamma_E(s)$ so that they are parallel with respect to the induced connection on the normal bundle NK. This yields an orthonormal frame field X_i for NK in a neighborhood of p in K which satisfies

$$\nabla_{X_a} X_i\big|_p \in T_pK.$$

A coordinate system in a neighborhood of p in M is now defined by

$$F(\bar y, \bar x) := \exp^M_{f(\bar y)}(x_i\, X_i), \quad \forall\, (\bar y, \bar x) := (y_1, \ldots, y_k, x_1, \ldots, x_N), \tag{2.17}$$

with corresponding coordinate vector fields

$$X_i := F_*(\partial_{x_i}) \quad \text{and} \quad X_a := F_*(\partial_{y_a}).$$

By our choice of coordinates, on K the metric $\bar{g}$ splits in the following way

$$\bar{g}(q) = \bar{g}_{ab}(q)\, dy_a \otimes dy_b + \bar{g}_{ij}(q)\, dx_i \otimes dx_j, \quad \forall q \in K. \tag{2.18}$$

We denote by $\Gamma_a^b(\cdot)$ the 1-forms defined on the normal bundle, NK, of K by the formula

$$\bar{g}_{bc}\Gamma_{ai}^c := \bar{g}_{bc}\Gamma_a^c(X_i) = \bar{g}(\nabla_{X_a}X_b, X_i) \quad \text{at } q = f(\bar{y}). \tag{2.19}$$

Notice that

$$K \text{ is minimal} \quad \Longleftrightarrow \quad \sum_{a=1}^{k} \Gamma_a^a(E_i) = 0 \quad \text{for any } i = 1, \ldots N. \tag{2.20}$$

Define $q = f(\bar{y}) = F(\bar{y}, 0) \in K$ and let $(\widetilde{g}_{ab}(y))$ be the induced metric on K. When we consider the metric coefficients in a neighborhood of K, we obtain a deviation from formula (2.18), which is expressed by the next lemma. We will denote by $R_{\alpha\beta\gamma\delta}$ the components of the curvature tensor with lowered indices, which are obtained by means of the usual ones $R_{\beta\gamma\delta}^{\sigma}$ by

$$R_{\alpha\beta\gamma\delta} = \bar{g}_{\alpha\sigma}\, R_{\beta\gamma\delta}^{\sigma}. \tag{2.21}$$

Lemma 2.2 *At the point $F(\bar{y}, \bar{x})$, the following expansions hold, for any $a = 1, ..., k$ and any $i, j = 1, ..., N$, where $N = n - k$,*

$$\bar{g}_{ij} = \delta_{ij} + \frac{1}{3}\, R_{istj}\, \bar{x}_s\, \bar{x}_t \;+\; \mathcal{O}(|\bar{x}|^3);$$

$$\bar{g}_{aj} = \frac{2}{3}\widetilde{g}_{ab}R_{kjl}^b\bar{x}^k\bar{x}^l + \mathcal{O}(|\bar{x}|^3);$$

$$\bar{g}_{ab} = \widetilde{g}_{ab} - \Big\{\widetilde{g}_{ac}\,\Gamma_{bi}^c + \widetilde{g}_{bc}\,\Gamma_{ai}^c\Big\}\,\bar{x}_i + \Big[R_{sabl} + \widetilde{g}_{cd}\Gamma_{as}^c\,\Gamma_{bl}^d\Big]\bar{x}_s\bar{x}_l + \mathcal{O}(|\bar{x}|^3).$$

Here R_{istj} are computed at the point of K parameterized by $(\bar{y}, 0)$.

Proof. The proof is somewhat standard and is thus omitted, we refer to [9] for details, see also Proposition 2.1 in [28]. $\square$

By the Whitney embedding theorem, $K \subset M \hookrightarrow \mathbb{R}^{2n}$. Thus we can define $K_\varepsilon := K/\varepsilon$ and $M_\varepsilon := M/\varepsilon$ in a natural way. On the other hand since $F(\bar{y}, \bar{x})$ is a Fermi coordinate system on M, then $F_\varepsilon(y, x) := F(\varepsilon y, \varepsilon x)/\varepsilon$ defines a Fermi coordinate system on M/ε. With this notation, here and in the sequel, by slight abuse of notation we denote $V(\varepsilon y, \varepsilon x)$ to actually mean $V(\varepsilon z) = V\big(F(\varepsilon y, \varepsilon x)\big)$ in the Fermi coordinate system. The same way is understood to its derivatives with respect to y and x.

Now we can introduce our first parameter function Φ which is a normal vector field defined on K and define $x = \xi + \Phi(\varepsilon y)$. Then (y, ξ) is the Fermi coordinate system for the submanifold

K_Φ. Adjusting the parameter Φ, it is later shown that there are solutions concentrating on K_Φ for a subsequence of ε.

We denote by $g_{\alpha\beta}$ the metric coefficients in the new coordinates (y, ξ). It follows that

$$g_{\alpha\beta} = \sum_{\gamma,\delta} \bar{g}_{\gamma\delta} \frac{\partial z_\alpha}{\partial \xi_\gamma} \frac{\partial z_\beta}{\partial \xi_\delta}.$$

Which yields

$$g_{ij} = \bar{g}_{ij}|_{\xi+\Phi}, \qquad g_{aj} = \bar{g}_{aj}|_{\xi+\Phi} + \varepsilon\, \partial_{\bar{a}} \Phi^l \bar{g}_{jl}|_{\xi+\Phi},$$

and

$$g_{ab} = \bar{g}_{ab}|_{\xi+\Phi} + \varepsilon \left\{ \bar{g}_{aj}\, \partial_{\bar{b}} \Phi^j + \bar{g}_{bj}\, \partial_{\bar{a}} \Phi^j \right\}|_{\xi+\Phi} + \varepsilon^2\, \partial_{\bar{a}} \Phi^i\, \partial_{\bar{b}} \Phi^j\, \bar{g}_{ij}|_{\xi+\Phi}$$

where summations over repeated indices is understood.

To express the error terms, it is convenient to introduce some notations. For a positive integer q, we denote by $R_q(\xi)$, $R_q(\xi, \Phi)$, $R_q(\xi, \Phi, \nabla\Phi)$, and $R_q(\xi, \Phi, \nabla\Phi, \nabla^2\Phi)$ error terms such that the following bounds hold for some positive constants C and d:

$$|R_q(\xi)| \le C\varepsilon^q (1 + |\xi|^d),$$

$$\begin{aligned} &|R_q(\xi, \Phi)| \le C\varepsilon^q (1 + |\xi|^d), \\ &|R_q(\xi, \Phi) - R_q(\xi, \bar{\Phi})| \le C\varepsilon^q (1 + |\xi|^d)|\Phi - \bar{\Phi}|, \end{aligned}$$

$$\begin{aligned} &|R_q(\xi, \Phi, \nabla\Phi)| \le C\varepsilon^q (1 + |\xi|^d), \\ &|R_q(\xi, \Phi, \nabla\Phi) - R_q(\xi, \bar{\Phi}, \nabla\bar{\Phi})| \le C\varepsilon^q (1 + |\xi|^d)\big(|\Phi - \bar{\Phi}| + |\nabla\Phi - \nabla\bar{\Phi}|\big), \end{aligned}$$

and

$$|R_q(\xi, \Phi, \nabla\Phi, \nabla^2\Phi)| \le C\varepsilon^q (1 + |\xi|^d) + C\varepsilon^{q+1} (1 + |\xi|^d)|\nabla^2\Phi|,$$

$$\begin{aligned} &\big|R_q(\xi, \Phi, \nabla\Phi, \nabla^2\Phi) - R_q(\xi, \bar{\Phi}, \nabla\bar{\Phi}, \nabla^2\bar{\Phi})\big| \\ &\le C\varepsilon^q (1 + |\xi|^d)\big(|\Phi - \bar{\Phi}| + |\nabla\Phi - \nabla\bar{\Phi}|\big)\big(1 + \varepsilon|\nabla^2\Phi| + \varepsilon|\nabla^2\bar{\Phi}|\big) \\ &\quad + C\varepsilon^{q+1} (1 + |\xi|^d)|\nabla^2\Phi - \nabla^2\bar{\Phi}|. \end{aligned}$$

Using the expansion of the previous lemma, one can easily show that the following lemma holds true.

Lemma 2.3 *In the coordinate (y, ξ), the metric coefficients satisfy*

$$\begin{aligned} g_{ab} &= \widetilde{g}_{ab} - \varepsilon \left\{ \widetilde{g}_{bf} \Gamma_{ak}^f + \widetilde{g}_{af} \Gamma_{bk}^f \right\} (\xi^k + \Phi^k) + \varepsilon^2 \big(R_{kabl} + \widetilde{g}_{cd}\, \Gamma_{ak}^c \Gamma_{bl}^d \big)(\xi^k + \Phi^k)(\xi^l + \Phi^l) \\ &\quad + \varepsilon^2 \partial_{\bar{a}} \Phi^j \partial_{\bar{b}} \Phi^j + R_3(\xi, \Phi, \nabla\Phi), \\ g_{aj} &= \varepsilon \partial_{\bar{a}} \Phi^j + \frac{2}{3} \varepsilon^2 R_{kajl} (\xi^k + \Phi^k)(\xi^l + \Phi^l) + R_3(\xi, \Phi, \nabla\Phi), \\ g_{ij} &= \delta_{ij} + \frac{1}{3} \varepsilon^2 R_{kijl} (\xi^k + \Phi^k)(\xi^l + \Phi^l) + R_3(\xi, \Phi, \nabla\Phi). \end{aligned}$$

Denote the inverse metric of $(g_{\alpha\beta})$ by $(g^{\alpha\beta})$. Recall that, given the expansion of a matrix as $M = I + \varepsilon A + \varepsilon^2 B + \mathcal{O}(\varepsilon^3)$, we have

$$M^{-1} = I - \varepsilon A - \varepsilon^2 B + \varepsilon^2 A^2 + \mathcal{O}(\varepsilon^3).$$

Lemma 2.4 *In the coordinate (y, ξ), the metric coefficients $g^{\alpha\beta}$ satisfy*

$$\begin{aligned} g^{ab} &= \tilde{g}^{ab} + \varepsilon \left\{ \tilde{g}^{cb}\, \Gamma^a_{ci} + \tilde{g}^{ca}\, \Gamma^b_{ci} \right\} (\xi^i + \Phi^i) - \varepsilon^2\, \tilde{g}^{cb}\, \tilde{g}^{ad}\, R_{kcdl}\, (\xi^k + \Phi^k)(\xi^l + \Phi^l) \\ &\quad + \varepsilon^2 \left(\tilde{g}^{ac}\, \Gamma^b_{dk}\Gamma^d_{cl} + \tilde{g}^{bc}\, \Gamma^a_{dk}\Gamma^d_{cl} + \tilde{g}^{cd}\, \Gamma^a_{dk}\Gamma^b_{cl} \right) (\xi^k + \Phi^k)(\xi^l + \Phi^l) + R_3(\xi, \Phi, \nabla\Phi), \end{aligned}$$

$$\begin{aligned} g^{aj} &= -\varepsilon\, \tilde{g}^{ab}\, \partial_{\bar{b}}\Phi^j - \frac{2\,\varepsilon^2}{3} R_{kajl}(\xi^k + \Phi^k)(\xi^l + \Phi^l) + \varepsilon^2 \partial_{\bar{b}}\Phi^j \left\{ \tilde{g}^{bc}\, \Gamma^a_{ci} + \tilde{g}^{ac}\, \Gamma^b_{ci} \right\} (\xi^i + \Phi^i) \\ &\quad + R_3(\xi, \Phi, \nabla\Phi), \end{aligned}$$

$$g^{ij} = \delta_{ij} - \frac{\varepsilon^2}{3}\, R_{kijl}(\xi^k + \Phi^k)(\xi^l + \Phi^l) + \varepsilon^2\, \tilde{g}^{ab}\, \partial_{\bar{a}}\Phi^i \partial_{\bar{b}}\Phi^j + R_3(\xi, \Phi, \nabla\Phi).$$

Furthermore, we have the validity of the following expansion for the log of the determinant of g:

$$\begin{aligned} \log\big(\det g\big) &= \log\big(\det \tilde{g}\big) - 2\varepsilon\, \Gamma^b_{bk}\, (\xi^k + \Phi^k) + \frac{1}{3}\, \varepsilon^2\, R_{mssl}\, (\xi^m + \Phi^m)\, (\xi^l + \Phi^l) \\ &\quad + \varepsilon^2 \left(\tilde{g}^{ab}\, R_{mabl} - \Gamma^c_{am}\Gamma^a_{cl} \right) (\xi^m + \Phi^m)\, (\xi^l + \Phi^l) + R_3(\xi, \Phi, \nabla\Phi). \end{aligned}$$

Proof. The expansions of the metric in the above lemma follow from Lemma 2.2 while the expansion of the log of the determinant of g follows from the fact that one can write $g = G + M$ with

$$G = \begin{pmatrix} \tilde{g} & 0 \\ 0 & Id_{\mathbb{R}^N} \end{pmatrix} \quad \text{and} \quad M = \mathcal{O}(\varepsilon),$$

then we have the following expansion

$$\log\big(\det g\big) = \log\big(\det G\big) + \operatorname{tr}(G^{-1}M) - \frac{1}{2}\operatorname{tr}\Big((G^{-1}M)^2\Big) + \mathcal{O}(\|M\|^3).$$

and the lemma follows at once. □

2.2.3 Expansion of the Laplace-Beltrami operator

In terms the above notations, we have the following expansion of the Laplace-Beltrami operator.

Proposition 2.5 *Let u be a smooth function on M_ε. Then in the Fermi coordinate (y, ξ),* *we have that*

$$
\begin{aligned}
\Delta_g u = {} & \partial^2_{ii} u + \Delta_{K_\varepsilon} u - \varepsilon\, \Gamma^b_{bj} \partial_j u - 2\varepsilon\, \widetilde{g}^{ab}\, \partial_{\bar b} \Phi^j\, \partial^2_{aj} u + 2\,\varepsilon\, \widetilde{g}^{cb}\, \Gamma^a_{cs}\, (\xi^s + \Phi^s) \partial^2_{ab} u \\
& + \varepsilon^2\, \nabla_K \Phi^i \cdot \nabla_K \Phi^j\, \partial^2_{ij} u - \frac{1}{3} \varepsilon^2 R_{kijl} (\xi^k + \Phi^k)(\xi^l + \Phi^l) \partial^2_{ij} u - \varepsilon^2\, \Gamma^d_{dk}\, \partial_{\bar b} \Phi^k\, \widetilde{g}^{ab} \partial_a u \\
& - \frac{4}{3} \varepsilon^2 R_{kajl} (\xi^k + \Phi^k)(\xi^l + \Phi^l) \partial^2_{aj} u + 2\varepsilon^2 \partial_{\bar b} \Phi^j \left\{ \widetilde{g}^{bc}\, \Gamma^a_{ci} + \widetilde{g}^{ac}\, \Gamma^b_{ci} \right\} (\xi^i + \Phi^i)\, \partial^2_{aj} u \\
& + \varepsilon^2 \left\{ - \widetilde{g}^{cb}\, \widetilde{g}^{ad}\, R_{kcdl} + \widetilde{g}^{ac}\, \Gamma^b_{dk} \Gamma^d_{cl} + \widetilde{g}^{bc}\, \Gamma^a_{dk} \Gamma^d_{cl} + \widetilde{g}^{cd}\, \Gamma^a_{dk} \Gamma^b_{cl} \right\} (\xi^k + \Phi^k)(\xi^l + \Phi^l)\, \partial^2_{ab} u \\
& + \varepsilon^2 \left(\widetilde{g}^{ab}\, R_{kabj} + \frac{2}{3} R_{kiij} - \Gamma^c_{ak} \Gamma^a_{cj} \right) (\xi^k + \Phi^k) \partial_j u - \varepsilon^2 \Delta_K \Phi^j \partial_j u \\
& + 2\varepsilon^3 \partial^2_{\bar a \bar b} \Phi^j \Gamma^b_{ak} (\xi^k + \Phi^k) \partial_j u \\
& - \varepsilon^2 \left(\widetilde{g}^{ab}\, \partial_{\bar a} \Gamma^d_{dk} - \partial_{\bar a} \left\{ \widetilde{g}^{cb} \Gamma^a_{ck} + \widetilde{g}^{ca} \Gamma^b_{ck} \right\} \right) (\xi^k + \Phi^k) \partial_b u - \frac{2}{3} \varepsilon^2 R_{jajk} (\xi^k + \Phi^k) \partial_a u \\
& + 2\varepsilon^2 \left\{ \widetilde{g}^{cb}\, \Gamma^a_{ci} + \widetilde{g}^{ca}\, \Gamma^b_{ci} \right\} \partial_{\bar b}\, \Phi^i\, \partial_a u + \frac{1}{2}\, \varepsilon^2\, \partial_{\bar a} (\log \det \widetilde{g})\, \left\{ \widetilde{g}^{cb} \Gamma^a_{ci} + \widetilde{g}^{ca} \Gamma^b_{ci} \right\} (\xi^i + \Phi^i) \partial_b u \\
& + R_3(\xi, \Phi, \nabla \Phi, \nabla^2 \Phi)(\partial_j u + \partial_a u) + R_3(\xi, \Phi, \nabla \Phi)(\partial^2_{ij} u + \partial^2_{aj} u + \partial^2_{ab} u).
\end{aligned}
$$

Remark 2.2.4 The proof of Proposition 2.5 will be postponed to section 2.2.6. It is worth mentioning that the coefficients of all the derivatives of u in the above expansion are smooth bounded functions of the variable $\bar y = \varepsilon y$. The slow dependence of theses coefficients of y is important in our construction of some proper approximate solutions.

2.2.4 Construction of approximate solutions

To prove Theorem 2.1, the first key step in our method is to construct some proper approximate solutions. To achieve this goal, we have introduced some geometric background, especially the Fermi coordinates. The main objective of this section is to construct some very accurate local approximate solutions in a tubular neighbourhood of K_ε by an iterative scheme of Picard's type and to define some proper global approximate solutions by the gluing method.

Facts on the limit equation

Recall that by the scaling, equation (2.7) becomes

$$
\Delta_g u - V(\varepsilon z) u + u^p = 0. \tag{2.22}
$$

In the Fermi coordinate (y, x), we can write $V(\varepsilon z) = V(\varepsilon y, \varepsilon x)$. Taking $x = \xi + \Phi(\varepsilon y)$, we have the following expansion of potential:

$$
V(\varepsilon y, \varepsilon x) = V(\varepsilon y, 0) + \varepsilon \langle \nabla^N V(\varepsilon y, 0), \xi + \Phi \rangle + \frac{\varepsilon^2}{2} (\nabla^N)^2 V(\varepsilon y, 0)[\xi + \Phi]^2 + R_3(\xi, \Phi). \tag{2.23}
$$

If the profile of solutions depends only on ξ or varies slower on y, by the expansion of the Laplace-Beltrami operator in Proposition 2.5 and the above expansion of potential, the leading equation is

$$\sum_{i=1}^{N} \partial^2_{\xi_i \xi_i} u - V(\varepsilon y, 0) u + u^p = 0. \tag{2.24}$$

Define

$$\mu(\varepsilon y) = V(\varepsilon y, 0)^{1/2}, \quad h(\varepsilon y) = V(\varepsilon y, 0)^{1/(p-1)}, \quad \forall\, y \in K_\varepsilon. \tag{2.25}$$

For the leading equation (2.24), by the scaling

$$u(y, \xi) = h(\varepsilon y) v\big(\mu(\varepsilon y)\xi\big) = h(\varepsilon y) v(\bar{\xi}),$$

the function v satisfies

$$\Delta_{\mathbb{R}^N} v - v + v^p = 0. \tag{2.26}$$

We call this equation *the limit equation.*

We now turn to the equation (2.22), in the spirit of above argument, we look for a solution u of the form

$$u(y, \xi) = h(\varepsilon y) v\big(y, \bar{\xi}\big) \quad \text{with } \bar{\xi} = \mu(\varepsilon y)\xi \in \mathbb{R}^N. \tag{2.27}$$

An easy computation shows that

$$\begin{aligned}
\partial_a u &= h\, \partial_a v + \varepsilon (\partial_{\bar{a}} h) v + \varepsilon\, h\, \partial_{\bar{a}} \mu\, \xi^j \partial_j v, \\
\partial^2_{ij} u &= h\, \mu^2\, \partial^2_{ij} v, \\
\partial^2_{aj} u &= \varepsilon \left(\mu \partial_{\bar{a}} h + h \partial_{\bar{a}} \mu\right) \partial_j v + \, h\, \mu\, \partial^2_{aj} v + \varepsilon\, h\, \mu\, \xi^i\, \partial_{\bar{a}} \mu\, \partial^2_{ij} v, \\
\partial^2_{ab} u &= h\, \partial^2_{ab} v + \varepsilon \left(\partial_{\bar{b}} h\, \partial_a v + \partial_{\bar{a}} h\, \partial_b v + h \partial_{\bar{b}} \mu\, \xi^j \partial^2_{aj} v + h \partial_{\bar{a}} \mu\, \xi^j \partial^2_{bj} v\right) \\
&\quad + \varepsilon^2 \Big(\partial_{\bar{a}} h \partial_{\bar{b}} \mu \xi^j \partial_j v + \partial_{\bar{b}} h \partial_{\bar{a}} \mu \xi^j \partial_j v + \partial^2_{\bar{a}\bar{b}} h v + h \partial_{\bar{a}} \mu \partial_{\bar{b}} \mu \xi^i \xi^j \partial^2_{ij} v + h \partial^2_{\bar{a}\bar{b}} \mu \xi^j \partial_j v\Big),
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{K_\varepsilon} u &= \varepsilon^2 \Delta_K h\, v + h\, \Delta_{K_\varepsilon} v + 2\varepsilon\, \nabla_K h \cdot \nabla_{K_\varepsilon} v + \varepsilon^2 \left(h\, \Delta_K \mu + 2\, \nabla_K h \cdot \nabla_K \mu\right) \xi^j\, \partial_j v \\
&\quad + \varepsilon^2\, h\, |\nabla_K \mu|^2\, \xi^j \xi^l\, \partial^2_{jl} v + 2\varepsilon\, h\, \xi^j\, \nabla_K \mu \cdot (\nabla_{K_\varepsilon} \partial_j v).
\end{aligned}$$

Therefore, we get the following expansion of the Laplace-Beltrami operator on u:

$$h^{-1} \mu^{-2}\, \Delta_g u = \Delta_{\mathbb{R}^N} v + \mu^{-2}\, \Delta_{K_\varepsilon} v + B(v),$$

with $B(v) = B_1(v) + B_2(v)$. Where B_j's are respectively given by

$$
\begin{aligned}
B_1(v) = &- \varepsilon\, \mu^{-1}\, \Gamma_{bj}^{b}\, \partial_j v + \varepsilon^2\, \mu^{-1} \Big(\widetilde{g}^{ab}\, R_{kabj} + \frac{2}{3} R_{kiij} - \Gamma_{ak}^{c} \Gamma_{cj}^{a} \Big) (\frac{1}{\mu} \bar{\xi}^k + \Phi^k) \partial_j v \\
&+ \varepsilon^2\, h^{-1}\, \mu^{-2}\, \Delta_K h\, v + 2\varepsilon^2\, (h\, \mu^2)^{-1}\, \nabla_K h \cdot \Big(\frac{\bar{\xi}^j}{\mu}\, \nabla_K \mu - \mu\, \nabla_K \Phi^j \Big)\, \partial_j v \\
&+ 2\varepsilon\;\, h^{-1}\, \mu^{-2}\, \nabla_K h \cdot \nabla_{K_\varepsilon} v - \frac{1}{3} \varepsilon^2\, R_{kijl} (\frac{1}{\mu} \bar{\xi}^k + \Phi^k)(\frac{1}{\mu} \bar{\xi}^l + \Phi^l) \partial_{ij}^2 v \\
&+ \varepsilon^2 \Big(\mu^{-2} \bar{\xi}^i\, \nabla_K \mu - \nabla_K \Phi^i \Big) \Big(\mu^{-2} \bar{\xi}^j\, \nabla_K \mu - \nabla_K \Phi^j \Big)\, \partial_{ij}^2 v \\
&+ \varepsilon^2\;\, \mu^{-2} \Big(\frac{\bar{\xi}^j}{\mu}\, \Delta_K \mu - 2\, \nabla_K \mu \cdot \nabla_K \Phi^j - \mu\, \Delta_K \Phi^j \Big)\, \partial_j v \\
&+ 2\varepsilon\, \mu^{-2} \Big(\frac{\bar{\xi}^j}{\mu}\, \nabla_K \mu - \mu\;\, \nabla_K \Phi^j \Big) \cdot \nabla_{K_\varepsilon} \big(\partial_j v \big),
\end{aligned}
$$

and

$$
\begin{aligned}
&h\mu^2 B_2(v) = -\varepsilon^2\, h\, \Gamma_{dj}^{d}\, \nabla_K \Phi^j\; \cdot \nabla_{K_\varepsilon} v \\
&+ 2\, \varepsilon\, \widetilde{g}^{cb}\, \Gamma_{cs}^{a} \Big(\frac{1}{\mu}\, \bar{\xi}^s + \Phi^s \Big) \Big(h\, \partial_{ab}^2 v + \varepsilon \Big\{ \partial_{\bar{b}} h\, \partial_a v + \partial_{\bar{a}} h\, \partial_b v + h \partial_{\bar{b}} \mu\, \frac{\bar{\xi}^j}{\mu} \partial_{aj}^2 v + h \partial_{\bar{a}} \mu\, \frac{\bar{\xi}^j}{\mu}\, \partial_{bj}^2 v \Big\} \Big) \\
&- \frac{4}{3} \varepsilon^2\, h\, \mu\, R_{kajl} \Big(\frac{1}{\mu} \bar{\xi}^k + \Phi^k \Big) \Big(\frac{1}{\mu} \bar{\xi}^l + \Phi^l \Big) \partial_{aj}^2 v + 2\varepsilon^2\, h\, \mu\, \partial_b \Phi^j\, \Big\{ \widetilde{g}^{bc}\, \Gamma_{ci}^{a} + \widetilde{g}^{ac}\, \Gamma_{ci}^{b} \Big\} \Big(\frac{1}{\mu} \bar{\xi}^i + \Phi^i \Big)\, \partial_{aj}^2 v \\
&+ \varepsilon^2\, h \Big\{ - \widetilde{g}^{cb}\, \widetilde{g}^{ad}\, R_{kcdl} + 2 \widetilde{g}^{ac}\, \Gamma_{dk}^{b} \Gamma_{cl}^{d} + \widetilde{g}^{cd}\, \Gamma_{dk}^{a} \Gamma_{cl}^{b} \Big\} \Big(\frac{1}{\mu} \bar{\xi}^k + \Phi^k \Big) \Big(\frac{1}{\mu} \bar{\xi}^l + \Phi^l \Big)\, \partial_{ab}^2 v \\
&+ 2\varepsilon^3\, h\, \mu\, \partial_{\bar{a}\bar{b}}^2 \Phi^j \Gamma_{ak}^{b} \Big(\frac{1}{\mu} \bar{\xi}^k + \Phi^k \Big) \partial_j v \\
&- \varepsilon^2\, h \Big(\widetilde{g}^{ab}\, \partial_{\bar{a}} \Gamma_{dk}^{d} - \partial_{\bar{a}} \Big\{ \widetilde{g}^{cb} \Gamma_{ck}^{a} + \widetilde{g}^{ca} \Gamma_{ck}^{b} \Big\} \Big) \Big(\frac{1}{\mu} \bar{\xi}^k + \Phi^k \Big) \partial_b v - \frac{2}{3} \varepsilon^2\, h\, R_{jajk} \Big(\frac{1}{\mu} \bar{\xi}^k + \Phi^k \Big) \partial_a v \\
&+ 2\varepsilon^2\, h\, \Big\{ \widetilde{g}^{cb}\, \Gamma_{ci}^{a} + \widetilde{g}^{ca}\, \Gamma_{ci}^{b} \Big\}\, \partial_{\bar{b}}\, \Phi^i\, \partial_a v + \frac{1}{2}\, \varepsilon^2\, h\, \partial_{\bar{a}} (\log \det \widetilde{g})\, \Big\{ \widetilde{g}^{cb} \Gamma_{ci}^{a} + \widetilde{g}^{ca} \Gamma_{ci}^{b} \Big\} \Big(\frac{1}{\mu} \bar{\xi}^i + \Phi^i \Big) \partial_b v \\
&+ R_3(\xi, \Phi, \nabla\Phi, \nabla^2\Phi) \Big(\partial_j v + \partial_a v \Big) + R_3(\xi, \Phi, \nabla\Phi) \Big(\partial_{ij}^2 v + \partial_{aj}^2 v + \partial_{ab}^2 v \Big).
\end{aligned}
$$

Setting

$$
S_\varepsilon(u) = -\Delta_g u + V(\varepsilon z) u - u^p,
$$

then by using the above expansion we can write

$$
\begin{aligned}
h^{-1} \mu^{-2}\, S_\varepsilon(u) &= -\Delta_{\mathbb{R}^N} v - \mu^{-2}\, \Delta_{K_\varepsilon} v - B(v) + \mu^{-2}\, V(\varepsilon z) v - h^{p-1} \mu^{-2}\, v^p \\
&= -\Delta_{\mathbb{R}^N} v + v - v^p - \mu^{-2}\, \Delta_{K_\varepsilon} v + \mu^{-2} \Big(V(\varepsilon y, \varepsilon x) - V(\varepsilon y, 0) \Big)\, v - B(v).
\end{aligned}
$$

Now using the following expansion of potential:

$$
V(\varepsilon y, \varepsilon x) = V(\varepsilon y, 0) + \varepsilon \langle \nabla^N V(\varepsilon y, 0), \frac{\bar{\xi}}{\mu} + \Phi \rangle + \frac{\varepsilon^2}{2} (\nabla^N)^2 V(\varepsilon y, 0) [\frac{\bar{\xi}}{\mu} + \Phi]^2 + R_3(\bar{\xi}, \Phi),
$$

we obtain

$$h^{-1}\mu^{-2}\, S_\varepsilon(u) = -\Delta_{\mathbb{R}^N} v + v - v^p - \mu^{-2}\, \Delta_{K_\varepsilon} v - \widetilde{B}(v) =: \widetilde{S}_\varepsilon(v), \tag{2.28}$$

where $\widetilde{B}(v) = \widetilde{B}_1(v) + \widetilde{B}_2(v)$ with

$$\widetilde{B}_1(v) = B_1(v) - \mu^{-2}\Big(\varepsilon\langle \nabla^N V(\varepsilon y, 0), \frac{\bar{\xi}}{\mu} + \Phi\rangle + \frac{\varepsilon^2}{2}(\nabla^N)^2 V(\varepsilon y, 0)[\frac{\bar{\xi}}{\mu} + \Phi]^2\Big)\, v$$

and

$$\widetilde{B}_2(v) = B_2(v) - R_3(\bar{\xi}, \Phi)\, v.$$

At the end of this subsection, let us list some basic and useful properties of positive solutions of the limit equation (2.26).

Proposition 2.6 *If $1 < p < \infty$ for $N = 2$ and $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$, then every solution of problem:*

$$\begin{cases} -\Delta_{\mathbb{R}^N} v + v - v^p = 0 \ \textit{in}\ \mathbb{R}^N, \\ v > 0 \ \textit{in}\ \mathbb{R}^N,\ \ v \in H^1(\mathbb{R}^N), \end{cases} \tag{2.29}$$

has the form $w_0(\cdot - Q)$ for some $Q \in \mathbb{R}^N$, where $w_0(x) = w_0(|x|) \in C^\infty(\mathbb{R}^N)$ is the unique positive radial solution which satisfies

$$\lim_{r\to\infty} r^{\frac{N-1}{2}} e^r w_0(r) = c_{N,p}, \quad \lim_{r\to\infty} \frac{w_0'(r)}{w_0(r)} = -1. \tag{2.30}$$

Here $c_{N,p}$ is a positive constant depending only on N and p. Furthermore, w_0 is non-degenerate in the sense that

$$Ker\left(-\Delta_{\mathbb{R}^N} + 1 - p w_0^{p-1}\right) \cap L^\infty(\mathbb{R}^N) = \mathrm{Span}\Big\{\partial_{x_1} w_0, \cdots, \partial_{x_N} w_0\Big\},$$

and the Morse index of w_0 is one, that is, the linear operator

$$L_0 := -\Delta_{\mathbb{R}^N} + 1 - p w_0^{p-1}$$

has only one negative eigenvalue $\lambda_0 < 0$, and the unique even and positive eigenfunction corresponding to λ_0 can be denoted by Z.

Proof. This result is well known. For the proof we refer the interested reader to [6] for the existence, [14] for the symmetry, [18] for the uniqueness, Appendix C in [36] for the nondegeneracy, and [7] for the Morse index. □

As a corollary, there is a constant $\gamma_0 > 0$ such that

$$\int_{\mathbb{R}^N} \Big\{|\nabla\phi|^2 + \phi^2 - p w_0^{p-1}\phi^2\Big\}\, d\bar{\xi} \geq \gamma_0 \int_{\mathbb{R}^N} \phi^2\, d\bar{\xi}, \tag{2.31}$$

whenever $\phi \in H^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \phi\, \partial_j w_0\, d\bar{\xi} = 0 = \int_{\mathbb{R}^N} \phi Z\, d\bar{\xi}, \quad \forall\, j = 1, \ldots, N.$$

Local approximate solutions

In a tubular neighbourhood of K_ε, (2.28) makes it obvious that $S_\varepsilon(u) = 0$ is equivalent to $\widetilde{S}_\varepsilon(v) = 0$.

By the expression of $\widetilde{S}_\varepsilon(v)$ and Remark 2.2.4, we look for approximate solutions of the form

$$v = v(y, \bar{\xi}) = w_0(\bar{\xi}) + \sum_{\ell=1}^{I} \varepsilon^\ell w_\ell(\varepsilon y, \bar{\xi}) + \varepsilon e(\varepsilon y) Z(\bar{\xi}), \tag{2.32}$$

where $I \in \mathbb{N}_+$, w_0 and Z are given in Proposition 2.6, w_ℓ's and e are smooth bounded functions on their variables.

The idea for introducing eZ in (2.32) comes directly from [8, 43]. The reason is the linear theory in Section 4.2.2, especially Lemma 2.11.

To solve $\widetilde{S}_\varepsilon(v) = 0$ accurately, the normal section Φ is to be chosen in the following form

$$\Phi = \Phi_0 + \sum_{\ell=1}^{I-1} \varepsilon^\ell \, \Phi_\ell,$$

where $\Phi_0, \ldots, \Phi_{I-1}$ are smooth bounded functions on $\bar{y}$.

Expansion at first order in ε :

We first solve the equation $\widetilde{S}_\varepsilon(v) = 0$ up to order ε. Here and in the following we will write $\mathcal{O}(\varepsilon^j)$ for terms that appear at the j-th order in an expansion.

Suppose v has the form (2.32), then

$$\begin{aligned}\widetilde{S}_\varepsilon(v) = {} & \varepsilon\Big(- \Delta_{\mathbb{R}^N} w_1 + w_1 - p w_0^{p-1} w_1\Big) + \varepsilon\big(- \varepsilon^2 \mu^{-2} \Delta_K e + \lambda_0 e\big) Z \\ & + \varepsilon\Big(\mu^{-1} \Gamma_{bj}^b \partial_j w_0 + \mu^{-2} \, \langle \nabla^N V(\varepsilon y, 0), \frac{\bar{\xi}}{\mu} + \Phi_0 \rangle w_0\Big) + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Hence the term of order ε in the right-hand side of above equation vanishes if and only if the function w_1 solves

$$L_0 w_1 = -\mu^{-1} \, \Gamma_{bj}^b \partial_j w_0 - \mu^{-2} \, \langle \nabla^N V(\varepsilon y, 0), \frac{\bar{\xi}}{\mu} + \Phi_0 \rangle w_0. \tag{2.33}$$

Here and in the following, we will keep the term $\varepsilon\big(- \varepsilon^2 \mu^{-2} \Delta_K e + \lambda_0 e\big) Z$ in the error. The reason is simply that it cannot be cancelled without solving an equation of e since $L_0 Z = \lambda_0 Z$.

By Proposition 2.6, equation (2.33) is solvable if and only if for all $i = 1, \ldots, N$,

$$\int_{\mathbb{R}^N} \Big(\mu^{-1} \, \Gamma_{bj}^b \partial_j w_0 + \mu^{-2} \, \langle \nabla^N V(\varepsilon y, 0), \frac{\bar{\xi}}{\mu} + \Phi_0 \rangle \, w_0\Big) \partial_i w_0 \, d\bar{\xi} = 0. \tag{2.34}$$

Since w_0 is radially symmetric, (2.34) is equivalent to

$$\Gamma_{bi}^b \int_{\mathbb{R}^N} |\partial_1 w_0|^2 \, d\bar{\xi} = \frac{1}{2} \, \mu^{-2} \, \partial_i V(\varepsilon y, 0) \int_{\mathbb{R}^N} w_0^2 \, d\bar{\xi}.$$

Recalling the identity

$$\frac{1}{2}\int_{\mathbb{R}^N} w_0^2 \, d\bar{\xi} = \sigma \int_{\mathbb{R}^N} |\partial_1 w_0|^2 \, d\bar{\xi} \quad \text{with } \sigma = \frac{p+1}{p-1} - \frac{N}{2}, \tag{2.35}$$

we get

$$\sigma \nabla^N V(\varepsilon y, 0) = -V(\varepsilon y, 0) H(\varepsilon y), \tag{2.36}$$

where $H = (-\Gamma_{bi}^b)_i$ is the mean curvature vector on K. This is exactly our stationary condition on K.

When (2.36) holds, the equation of w_1 becomes

$$L_0 w_1 = -\mu^{-1}\,\Gamma_{bj}^b\Big(\partial_j w_0 + \sigma^{-1}\bar{\xi}^j w_0\Big) + \sigma^{-1}\langle H, \Phi_0\rangle w_0. \tag{2.37}$$

Hence we can write

$$w_1 = w_{1,1} + w_{1,2}, \tag{2.38}$$

where

$$w_{1,1} = -\mu^{-1}\,\Gamma_{bj}^b U_j \quad \text{and} \quad w_{1,2} = \sigma^{-1}\langle H, \Phi_0\rangle U_0. \tag{2.39}$$

Here U_j is the unique smooth bounded function satisfying

$$L_0 U_j = \partial_j w_0 + \sigma^{-1}\,\bar{\xi}^j\, w_0, \quad \int_{\mathbb{R}^N} U_j \,\partial_i w_0 \, d\bar{\xi} = 0, \ \forall\, i = 1, \dots, N, \tag{2.40}$$

and U_0 is the unique smooth bounded function such that

$$L_0 U_0 = w_0, \qquad \int_{\mathbb{R}^N} U_0 \,\partial_i w_0 \, d\bar{\xi} = 0, \ \forall\, i = 1, \dots, N. \tag{2.41}$$

It follows immediately that $w_1 = w_1(\varepsilon y, \bar{\xi})$ is smooth bounded on its variable. Furthermore, it is easily seen that U_j is odd on variable $\bar{\xi}^j$ and is even on other variables. Moreover, U_0 has an explicit expression

$$U_0 = -\frac{1}{p-1} w_0 - \frac{1}{2}\bar{\xi}\cdot\nabla w_0. \tag{2.42}$$

Expansion at second order in ε

In this subsection we will solve the equation $\widetilde{S}_\varepsilon(v) = 0$ up to order ε^2 by solving w_2 and Φ_0 together.

Suppose v has the form (2.32), then

$$\begin{aligned}\widetilde{S}_\varepsilon(v) = \varepsilon^2\Big(-\Delta_{\mathbb{R}^N} w_2 + w_2 - p w_0^{p-1} w_2\Big) + \varepsilon\big(-\varepsilon^2\mu^{-2}\Delta_K e + \lambda_0 e\big) Z \\ + \varepsilon^2 \mathfrak{F}_2 + \varepsilon^2 \mathfrak{G}_2 + \mathcal{O}(\varepsilon^3),\end{aligned}$$

where

$$
\begin{aligned}
\mathfrak{F}_2 =& \mu^{-1}\,\Gamma^b_{bj}\partial_j w_1 + \mu^{-2}\,\langle \nabla^N V, \Phi_1\rangle\, w_0 + \frac{1}{3}\,R_{kijl}(\frac{1}{\mu}\bar{\xi}^k + \Phi_0^k)(\frac{1}{\mu}\bar{\xi}^l + \Phi_0^l)\partial^2_{ij} w_0 \\
& - \mu^{-1}\Big(\widetilde{g}^{ab}\,R_{kabj} + \frac{2}{3}R_{kiij} - \Gamma^c_{ak}\Gamma^a_{cj}\Big)(\frac{\bar{\xi}^k}{\mu} + \Phi_0^k)\partial_j w_0 \\
& - \mu^{-2}\Big(\frac{\bar{\xi}^j}{\mu}\,\Delta_K\mu - 2\,\nabla_K\mu\cdot\nabla_K\Phi_0^j - \mu\,\Delta_K\Phi_0^j\Big)\,\partial_j w_0 \\
& - h^{-1}\mu^{-2}\,\Delta_K h\, w_0 - 2(h\mu^2)^{-1}\nabla_K h\cdot\Big(\frac{\bar{\xi}^j}{\mu}\,\nabla_K\mu - \mu\,\nabla_K\Phi_0^j\Big)\,\partial_j w_0 \\
& - \Big(\mu^{-2}\bar{\xi}^i\,\nabla_K\mu - \nabla_K\Phi_0^i\Big)\Big(\mu^{-2}\bar{\xi}^j\,\nabla_K\mu - \nabla_K\Phi_0^j\Big)\,\partial^2_{ij} w_0 \\
& + \mu^{-2}\,\langle \nabla^N V, \frac{\bar{\xi}}{\mu} + \Phi_0\rangle w_1 + \frac{1}{2}\,\mu^{-2}\,(\nabla^N)^2 V[\frac{\bar{\xi}}{\mu} + \Phi_0, \frac{\bar{\xi}}{\mu} + \Phi_0]\, w_0 - \frac{1}{2}p(p-1)w_0^{p-2}w_1^2,
\end{aligned}
$$

and

$$
\mathfrak{G}_2 = \mu^{-1}\Gamma^b_{bj}\,e\,\partial_j Z + \mu^{-2}\,\langle \nabla^N V, \frac{\bar{\xi}}{\mu} + \Phi_0\rangle eZ - \frac{1}{2}p(p-1)w_0^{p-2}\Big\{(w_1 + eZ)^2 - w_1^2\Big\}.
$$

Hence the term of order ε^2 vanishes (except the term $\varepsilon\big(-\varepsilon^2\mu^{-2}\Delta_K e + \lambda_0 e\big)Z$) if and only if w_2 satisfies the equation

$$
L_0 w_2 = -\mathfrak{F}_2 - \mathfrak{G}_2.
$$

By Freedholm alternative this equation is solvable if and only if $\mathfrak{F}_2 + \mathfrak{G}_2$ is L^2 orthogonal to the kernel of linearized operator L_0, which is spanned by the functions $\partial_i w_0$, $i = 1, \ldots, N$.

It is convenient to write $\mathfrak{F}_2$ as

$$
\mathfrak{F}_2 = \mu^{-2}\langle \nabla^N V, \Phi_1\rangle w_0 + \widetilde{\mathfrak{F}}_2.
$$

Then $\widetilde{\mathfrak{F}}_2$ does not involve Φ_1. By (2.36), similar to w_1, we can write w_2 as

$$
w_2 = w_{2,1} + w_{2,2},
$$

where $w_{2,2} = \sigma^{-1}\langle H, \Phi_1\rangle U_0$ solves the equation

$$
L_0 w_{2,2} = -\mu^{-2}\langle \nabla^N V, \Phi_1\rangle w_0,
$$

and $w_{2,1}$ will solve the equation

$$
L_0 w_{2,1} = -\widetilde{\mathfrak{F}}_2 - \mathfrak{G}_2.
$$

To solve the equation on $w_{2,1}$ we write

$$
\widetilde{\mathfrak{F}}_2 = \widetilde{\mathfrak{F}}_2(\Phi_0) = S_{2,0} + S_2(\Phi_0) + N_2(\Phi_0),
$$

where $S_{2,0} = \widetilde{\mathfrak{F}}_2(0)$ does not involve Φ_0, $S_2(\Phi_0)$ is the sum of linear terms of Φ_0, and $N_2(\Phi_0)$ is the nonlinear term of Φ_0.

Recall that $w_1 = w_{1,1} + w_{1,2}$ with

$$w_{1,1} = -\mu^{-1}\,\Gamma^b_{bj} U_j \quad \text{and} \quad w_{1,2} = \sigma^{-1}\langle H, \Phi_0\rangle U_0.$$

Then

$$\begin{aligned}
S_{2,0} =& \mu^{-1}\,\Gamma^b_{bj}\,\partial_j w_{1,1} + \frac{1}{3}\mu^{-2} R_{kijl}\,(\bar\xi^k\,\bar\xi^l\,\partial^2_{ij} w_0) - \mu^{-2}\Big(\widetilde{g}^{ab}\,R_{kabj} + \frac{2}{3}R_{kiij} - \Gamma^c_{ak}\Gamma^a_{cj}\Big)(\bar\xi^k\,\partial_j w_0) \\
& - (\mu^{-3}\Delta_K\mu)(\bar\xi^j\,\partial_j w_0) - (h^{-1}\mu^{-2}\Delta_K h)\,w_0 - 2(h\mu^3)^{-1}(\nabla_K h\cdot\nabla_K\mu)(\bar\xi^j\,\partial_j w_0) \\
& - \mu^{-4}|\nabla_K\mu|^2\,(\bar\xi^i\,\bar\xi^j\,\partial^2_{ij} w_0) + \mu^{-3}\langle\nabla^N V, \bar\xi\rangle w_{1,1} + \frac{1}{2}\mu^{-4}(\nabla^N)^2 V[\bar\xi,\bar\xi]\,w_0 \\
& - \frac{1}{2}p(p-1)w_0^{p-2}w_{1,1}^2,
\end{aligned}$$

$$\begin{aligned}
S_2(\Phi_0) =& \mu^{-1}\Gamma^b_{bj}\,\partial_j w_{1,2} + \frac{2}{3}\mu^{-1}R_{kijl}\,\Phi_0^l\,(\bar\xi^k\,\partial^2_{ij}w_0) - \mu^{-1}\Big(\widetilde{g}^{ab}\,R_{kabj} + \frac{2}{3}R_{kiij} - \Gamma^c_{ak}\Gamma^a_{cj}\Big)\Phi_0^k\,\partial_j w_0 \\
& + \mu^{-2}\Big(2\,\nabla_K\mu\cdot\nabla_K\Phi_0^j + \mu\,\Delta_K\Phi_0^j\Big)\partial_j w_0 + 2(h\mu)^{-1}\Big(\nabla_K h\cdot\nabla_K\Phi_0^j\Big)\partial_j w_0 \\
& + 2\mu^{-2}\Big(\nabla_K\mu\cdot\nabla_K\Phi_0^j\Big)\,(\bar\xi^i\,\partial^2_{ij}w_0) + \mu^{-3}\langle\nabla^N V,\bar\xi\rangle w_{1,2} + \mu^{-2}\langle\nabla^N V,\Phi_0\rangle w_{1,1} \\
& + \mu^{-3}(\nabla^N)^2 V[\Phi_0,\bar\xi]\,w_0 - p(p-1)w_0^{p-2}w_{1,1}w_{1,2},
\end{aligned}$$

and

$$\begin{aligned}
N_2(\Phi_0) =& \frac{1}{3}R_{kijl}\,\Phi_0^k\,\Phi_0^l\,\partial^2_{ij}w_0 - (\nabla_K\Phi_0^i\cdot\nabla_K\Phi_0^j)\,\partial^2_{ij}w_0 + \mu^{-2}\langle\nabla^N V,\Phi_0\rangle w_{1,2} \\
& + \frac{1}{2}\,\mu^{-2}\,(\nabla^N)^2 V[\Phi_0,\Phi_0]\,w_0 - \frac{1}{2}p(p-1)w_0^{p-2}w_{1,2}^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^N} S_2(\Phi_0)\,\partial_s w_0 =& \mu^{-1}\Gamma^b_{bj}\int_{\mathbb{R}^N}\partial_j w_{1,2}\,\partial_s w_0 + \frac{2}{3}\mu^{-1}R_{kijl}\,\Phi_0^l\int_{\mathbb{R}^N}\bar\xi^k\,\partial^2_{ij}w_0\,\partial_s w_0 \\
& - \mu^{-1}\Big(\widetilde{g}^{ab}\,R_{kabj} + \frac{2}{3}R_{kiij} - \Gamma^c_{ak}\Gamma^a_{cj}\Big)\,\Phi_0^k\int_{\mathbb{R}^N}\partial_j w_0\,\partial_s w_0 \\
& + \mu^{-2}\Big(2\nabla_K\mu\cdot\nabla_K\Phi_0^j + \mu\,\Delta_K\Phi_0^j\Big)\int_{\mathbb{R}^N}\partial_j w_0\,\partial_s w_0 \\
& + 2(h\mu)^{-1}\Big(\nabla_K h\cdot\nabla_K\Phi_0^j\Big)\int_{\mathbb{R}^N}\partial_j w_0\,\partial_s w_0 \\
& + 2\mu^{-2}\Big(\nabla_K\mu\cdot\nabla_K\Phi_0^j\Big)\int_{\mathbb{R}^N}\bar\xi^i\,\partial^2_{ij}w_0\,\partial_s w_0 \\
& + \mu^{-2}\partial_j V(\varepsilon y,0)\,\Big(\mu^{-1}\int_{\mathbb{R}^N}\bar\xi^j\,w_{1,2}\,\partial_s w_0 + \Phi_0^j\int_{\mathbb{R}^N}w_{1,1}\,\partial_s w_0\Big) \\
& + \mu^{-3}\partial^2_{ij}V(\varepsilon y,0)\,\Phi_0^j\int_{\mathbb{R}^N}\bar\xi^i\,w_0\,\partial_s w_0 \\
& - p(p-1)\int_{\mathbb{R}^N}w_0^{p-2}\,w_{1,1}\,w_{1,2}\,\partial_s w_0.
\end{aligned}$$

Let us denote by A the sum of terms involving $w_{1,1}$ and $w_{1,2}$ in the above formula. Using (2.36) and (2.39) we can write

$$A = \mu^{-1}\,\sigma^{-1}\,\langle H, \Phi_0\rangle \Gamma^a_{aj} \int_{\mathbb{R}^N} \Big(\partial_j U_0 + U_j + \sigma^{-1}\,\bar{\xi}^j\, U_0 + p(p-1) w_0^{p-2}\, U_j\, U_0\Big\}\partial_s w_0.$$

To compute this term we differentiate the equation (2.40) on U_j with respect to the variable $\bar{\xi}^j$ to obtain

$$L_0(\partial_j U_j) - p(p-1)w_0^{p-2} U_j \partial_j w_0 = \partial^2_{jj} w_0 + \sigma^{-1} w_0 + \sigma^{-1}\,\bar{\xi}^j\,\partial_j w_0. \tag{2.43}$$

Multiplying the above equation by U_0 and integrating by parts, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big\{\partial_j U_0 + U_j + \sigma^{-1}\,\bar{\xi}^j\, U_0 + p(p-1)w_0^{p-2} U_j\, U_0\Big\}\partial_j w_0 \\
&= -\int_{\mathbb{R}^N} \big(2\partial^2_{jj} w_0 + \sigma^{-1}\, w_0\big) U_0 \\
&= -2\int_{\mathbb{R}^N} \big(-\frac{1}{p-1} w_0 - \frac{1}{2}\,\bar{\xi}^l \partial_l w_0\big)\partial^2_{jj} w_0 - \sigma^{-1}\int_{\mathbb{R}^N} \big(-\frac{1}{p-1} w_0 - \frac{1}{2}\,\bar{\xi}^l \partial_l w_0\big)\, w_0 \\
&= -\big(\frac{2}{p-1} + 1 - \frac{N}{2}\big)\int_{\mathbb{R}^N} |\partial_1 w_0|^2 - \sigma^{-1}\big(\frac{N}{4} - \frac{1}{p-1}\big)\int_{\mathbb{R}^N} w_0^2 \\
&= -\int_{\mathbb{R}^N} |\partial_1 w_0|^2.
\end{aligned}$$

On the other hand, by direct computations we have

$$\int_{\mathbb{R}^N} \partial_j w_0\,\partial_s w_0 = \delta_{js}\int_{\mathbb{R}^N} (\partial_1 w_0)^2,$$

$$\int_{\mathbb{R}^N} \partial^2_{kj} w_0\,\bar{\xi}^k \partial_s w_0 = \frac{1}{2}\,\delta_{js}\int_{\mathbb{R}^N} \bar{\xi}^k \partial_k (\partial_j w_0)^2 = -\frac{N}{2}\,\delta_{js}\int_{\mathbb{R}^N} (\partial_1 w_0)^2,$$

$$R_{kijl}\,\Phi_0^l \int_{\mathbb{R}^N} \bar{\xi}^k\,\partial^2_{ij} w_0\,\partial_s w_0 = R_{sjjl}\,\Phi_0^l\int_{\mathbb{R}^N} (\partial_1 w_0)^2,$$

$$\Big(\widetilde{g}^{ab}\,R_{kabj} + \frac{2}{3} R_{kiij} - \Gamma^c_{ak}\Gamma^a_{cj}\Big)\,\Phi_0^k \int_{\mathbb{R}^N} \partial_j w_0\,\partial_s w_0 = \Big(\widetilde{g}^{ab}\,R_{kabs} + \frac{2}{3} R_{kiis} - \Gamma^c_{ak}\Gamma^a_{cs}\Big)\,\Phi_0^k \int_{\mathbb{R}^N} (\partial_1 w_0)^2.$$

Summarizing, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} S_2(\Phi_0)\,\partial_s w_0 = \mu^{-1}\Big\{ &\Delta_K \Phi_0^s - \Big(\widetilde{g}^{ab}\,R_{kabs} - \Gamma^c_{ak}\Gamma^a_{cs}\Big)\Phi_0^k + (2-N)\mu^{-1}\nabla_K \mu\cdot\nabla_K \Phi_0^s \\
&+ 2h^{-1}\nabla_K h\cdot\nabla_K\Phi_0^s - \sigma\mu^{-2}\,\partial^2_{sj} V(\varepsilon y, 0)\Phi_0^j - \sigma^{-1}\Gamma^a_{as}\langle H, \Phi_0\rangle\Big\}\int_{\mathbb{R}^N} (\partial_1 w_0)^2.
\end{aligned}$$

Now, using the fact that

$$\mu^{-1}\,\nabla_K \mu = \frac{1}{2}\,V^{-1}\,\nabla_K V \quad \text{and} \quad h^{-1}\,\nabla_K h = \frac{1}{p-1}\,V^{-1}\,\nabla_K V,$$

we obtain (recalling the definition of σ) that

$$(2-N)\mu^{-1}\nabla_K\mu\cdot\nabla_K\Phi_0^s+2h^{-1}\nabla_K h\cdot\nabla_K\Phi_0^s=\sigma\,V^{-1}\,\nabla_K V\cdot\nabla_K\Phi_0^s.$$

Hence we summarize

$$\int_{\mathbb{R}^N} S_2(\Phi_0)\,\partial_s w_0=\mu^{-1}\Big\{\Delta_K\Phi_0^s-\Big(\widetilde{g}^{ab}\,R_{kabs}-\Gamma_{ak}^c\Gamma_{cs}^a\Big)\Phi_0^k+\sigma V^{-1}\nabla_K V\cdot\nabla_K\Phi_0^s$$
$$-\sigma\mu^{-2}\partial_{sj}^2 V(\varepsilon y,0)\Phi_0^j+\sigma^{-1}\Gamma_{bj}^b\,\Gamma_{as}^a\,\Phi_0^j\Big\}\int_{\mathbb{R}^N}|\partial_1 w_0|^2.$$

Define $\mathfrak{J}_K: NK\mapsto NK$ is a linear operator from the family of smooth sections of normal bundle to K into itself, whose components are given by

$$\begin{aligned}(\mathfrak{J}_K\Phi_0)^s=\Delta_K\Phi_0^s-\Big(\widetilde{g}^{ab}\,R_{kabs}-\Gamma_{ak}^c\Gamma_{cs}^a\Big)\Phi_0^k+\sigma V^{-1}\nabla_K V\cdot\nabla_K\Phi_0^s\\ -\sigma\mu^{-2}\partial_{sj}^2 V(\bar{y},0)\Phi_0^j+\sigma^{-1}\Gamma_{bj}^b\,\Gamma_{as}^a\,\Phi_0^j.\end{aligned}\tag{2.44}$$

Then

$$\int_{\mathbb{R}^N} S_2(\Phi_0)\,\partial_s w_0=\mu^{-1}\big(\int_{\mathbb{R}^N}|\partial_1 w_0|^2\big)(\mathfrak{J}_K\Phi_0)^s(\varepsilon y).\tag{2.45}$$

On the other hand, it is easy to check that

$$\int_{\mathbb{R}^N} S_{2,0}\,\partial_s w_0=0=\int_{\mathbb{R}^N} N_2(\Phi_0)\,\partial_s w_0\tag{2.46}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N}\mathfrak{G}_2\,\partial_s w_0&=\Big\{\mu^{-1}\Gamma_{bs}^b\int_{\mathbb{R}^N}\partial_s Z\,\partial_s w_0+\mu^{-3}\partial_s V(\varepsilon y,0)\int_{\mathbb{R}^N}\bar{\xi}^s\,Z\,\partial_s w_0\\&\qquad-p(p-1)\int_{\mathbb{R}^N}w_0^{p-2}w_{1,1}Z\,\partial_s w_0\Big\}e\\&=\mu^{-1}\Gamma_{bs}^b e\int_{\mathbb{R}^N}\Big\{\partial_s Z+\sigma^{-1}Z\,\bar{\xi}^s+p(p-1)w_0^{p-2}Z\,U_s\Big\}\partial_s w_0\\&=c_0\mu^{-1}\Gamma_{bs}^b e.\end{aligned}$$

Therefore, the solvability of equation on w_2 is equivalent to the solvability of following equation on Φ_0:

$$\mathfrak{J}_K\Phi_0=\mathfrak{H}_2(\bar{y};e),\tag{2.47}$$

where $\mathfrak{H}_2(\bar{y};e)=c_0He$ is a smooth bounded function.

By the non-degeneracy condition on K, (2.47) is solvable. Moreover, for any given e, it is easy to check that $\Phi_0=\Phi_0(\bar{y};e)$ is a smooth bounded function on $\bar{y}$ and is Lipschitz continuous with respect to e.

Now let us go back to the equation of $w_{2,1}$:

$$L_0 w_{2,1}=-\widetilde{\mathfrak{F}}_2-\mathfrak{G}_2.$$

Since both $\widetilde{\mathfrak{F}}_2$ and $\mathfrak{G}_2$ are smooth bounded functions of $(\varepsilon y, \bar{\xi})$. Hence $w_{2,1} = w_{2,1}(\varepsilon y, \bar{\xi})$ is also a smooth bounded function of $(\varepsilon y, \bar{\xi})$. Moreover, $w_{2,1} = w_{2,1}(\varepsilon y, \bar{\xi}; e)$ is Lipschitz continuous with respect to e.

Higher order approximations

The construction of higher order terms follows exactly from the same calculation. Indeed, to solve the equation up to an error of order ε^{j+1} for some $j \geq 3$, we use an iterative scheme of Picard's type : assuming all the functions w_i's $(1 \leq i \leq j-1)$ constructed, we need to choose a function w_j to solve an equation similar to that of w_2 (with obvious modifications) by solving an equation of Φ_{j-2} similar to that of Φ_0.

When we collect all terms of order $\mathcal{O}(\varepsilon^j)$ in $\widetilde{S}_\varepsilon(v)$, assuming all w_i's for $i = 1, \cdots j-1$ constructed (by the iterative scheme), we have

$$\begin{aligned}\widetilde{S}_\varepsilon(v) = {} & \varepsilon^j\Big(- \Delta_{\mathbb{R}^N} w_j + w_j - p w_0^{p-1} w_j\Big) + \varepsilon\big(- \varepsilon^2 \mu^{-2} \Delta_K e + \lambda_0 e\big) Z \\ & + \varepsilon^j \mathfrak{F}_j + \varepsilon^j \mathfrak{E}_j\, e\, Z + \varepsilon^j \mathcal{A}_j^i(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, e\, \partial_i Z \\ & + \varepsilon^j \mathcal{B}_j^{i\ell}(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, e\, \partial_{i\ell}^2 Z + \varepsilon^j \mathcal{C}_j^i(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3}) \cdot \nabla_K e\, \partial_i Z \\ & + \varepsilon^j \mathcal{D}_j^{ab}(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, \partial_{ab}^2 e\, Z + \mathcal{O}(\varepsilon^{j+1}),\end{aligned}$$

with

$$\begin{aligned}\mathfrak{F}_j = {} & \mu^{-1} \Gamma_{bl}^b\, \partial_l w_{j-1} + \frac{2}{3} \mu^{-1} R_{kisl}\, \bar{\xi}^k\, \Phi_{j-2}^l\, \partial_{is}^2 w_0 - \mu^{-1}\Big(\widetilde{g}^{ab} R_{kabs} + \frac{2}{3} R_{kiis} - \Gamma_{ak}^c \Gamma_{cs}^a\Big) \Phi_{j-2}^k\, \partial_s w_0 \\ & + \mu^{-2}\Big(2\, \nabla_K \mu \cdot \nabla_K \Phi_{j-2}^s + \mu\, \Delta_K \Phi_{j-2}^s\Big) \partial_s w_0 + 2(h\mu)^{-1}\Big(\nabla_K h \cdot \nabla_K \Phi_{j-2}^s\Big) \partial_s w_0 \\ & + 2\mu^{-2}\Big(\nabla_K \mu \cdot \nabla_K \Phi_{j-2}^s\Big)(\bar{\xi}^i\, \partial_{is}^2 w_0) + \mu^{-2} \langle \nabla^N V, \Phi_0 \rangle w_{j-1} + \mu^{-2} \langle \nabla^N V, \Phi_{j-2} \rangle w_1 \\ & + \mu^{-2} \langle \nabla^N V, \Phi_{j-1} \rangle\, w_0 + \mu^{-2} \langle \nabla^N V, \frac{\bar{\xi}}{\mu} \rangle\, w_{j-1} + \mu^{-3} \partial_{kl}^2 V(\varepsilon y, 0)\, \Phi_{j-2}^l\, \bar{\xi}^k\, w_0 \\ & - p(p-1) w_0^{p-2} w_1 w_{j-1} + G_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3}) \\ = {} & \mu^{-2} \langle \nabla^N V, \Phi_{j-1} \rangle\, w_0 + \widetilde{\mathfrak{F}}_j\end{aligned}$$

and

$$\mathfrak{E}_j = -p(p-1) w_0^{p-2} w_{j-1} + \mu^{-2} \langle \nabla^N V, \Phi_{j-2} \rangle + \widetilde{\mathfrak{E}}_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3}),$$

where $\mathcal{A}_j^i$, $\mathcal{B}_j^{i\ell}$, $\mathcal{C}_j^i$, $\mathcal{D}_j^{ab}$ and $\widetilde{\mathfrak{E}}_j$ are smooth bounded functions on their variables.

Except for $\varepsilon\big(- \varepsilon^2 \mu^{-2} \Delta_K e + \lambda_0 e\big) Z$, the term of order ε^j vanishes if and only if w_j satisfies the equation

$$\begin{aligned}L_0 w_j = {} & -\mathfrak{F}_j - \mathfrak{E}_j\, e\, Z - \mathcal{A}_j^i(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, e\, \partial_i Z - \mathcal{B}_j^{i\ell}(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, e\, \partial_{i\ell}^2 Z \\ & - \mathcal{C}_j^i(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3}) \cdot \nabla_K e\, \partial_i Z - \mathcal{D}_j^{ab}(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, \partial_{ab}^2 e\, Z.\end{aligned}$$

By Freedholm alternative this equation is solvable if and only if the right hand side is L^2 orthogonal to the kernel of linearized operator L_0. Before computing the projection against $\partial_s w_0$, let us recall that

$$w_{j-1} = w_{j-1,1} + \sigma^{-1} \langle H, \Phi_{j-2} \rangle U_0,$$

where $w_{j-1,1} \perp \partial_i w_0$ is a function which does not involve Φ_{j-2}.

As before we look for a solution w_j of the form

$$w_j = w_{j,1} + \sigma^{-1}\langle H, \Phi_{j-1}\rangle U_0,$$

where $w_{j,1} \perp \partial_i w_0$ solves

$$\begin{aligned} L_0 w_{j,1} = &-\widetilde{\mathfrak{F}}_j - \mathfrak{E}_j\, e\, Z - \mathcal{A}^i_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, e\, \partial_i Z - \mathcal{B}^{i\ell}_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, e\, \partial^2_{i\ell} Z \\ &- \mathcal{C}^i_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3}) \cdot \nabla_K e\, \partial_i Z - \mathcal{D}^{ab}_j(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})\, \partial^2_{ab} e\, Z. \end{aligned}$$

Since $j \geq 3$, we can write

$$\widetilde{\mathfrak{F}}_j = \widetilde{\mathfrak{F}}_j(\Phi_{j-2}) = S_{j,0} + S_j(\Phi_{j-2}),$$

where $S_{j,0} = S_{j,0}(\varepsilon y, \bar{\xi}; \Phi_0, \cdots, \Phi_{j-3})$ does not involve Φ_{j-2}, and $S_j(\Phi_{j-2})$ is the sum of linear terms of Φ_{j-2}. Since

$$\int_{\mathbb{R}^N} S_j(\Phi_{j-2})\, \partial_s w_0 = \mu^{-1}\Big(\int_{\mathbb{R}^N} |\partial_1 w_0|^2\Big)(\mathcal{J}_K \Phi_{j-2})^s(\varepsilon y), \tag{2.48}$$

the equation on $w_{j,1}$ (and then on w_j) is solvable if and only if Φ_{j-2} satisfies an equation of the form

$$\mathcal{J}_K \Phi_{j-2} = \mathfrak{H}_j(\bar{y}; \Phi_0, \cdots, \Phi_{j-3}, e).$$

This latter equation is solvable by the non-degeneracy condition on K. Moreover, for any given e, by induction method one can get $\Phi_{j-2} = \Phi_{j-2}(\bar{y}; e)$ is a smooth bounded function on $\bar{y}$ and is Lipschitz continuous with respect to e. When this is done, since the right hand side of equation of $w_{j,1}$ is a smooth bounded function of $(\varepsilon y, \bar{\xi})$, we see at once that $w_{j,1} = w_{j,1}(\varepsilon y, \bar{\xi})$ is a smooth bounded function of $(\varepsilon y, \bar{\xi})$. Furthermore, $w_{j,1} = w_{j,1}(\varepsilon y, \bar{\xi}; e)$ is Lipschitz continuous with respect to e.

Remark 2.2.5 To get the higher order approximations, our argument only need the expansion of the Laplace-Beltrami operator up to second order. It is slightly different from the argument used in [43].

Summary

Let v_I be the local approximate solution constructed in the previous section, i.e.,

$$v_I(y, \bar{\xi}) = w_0(\bar{\xi}) + \sum_{\ell=1}^{I} \varepsilon^\ell w_\ell(\varepsilon y, \bar{\xi}) + \varepsilon e(\varepsilon y) Z(\bar{\xi}), \tag{2.49}$$

for $I \in \mathbb{N}_+$ an arbitrary positive integer.

From the analysis in the previous subsections, the stationary and non-degeneracy conditions on K can be seen as conditions such that v_I is very close to a genuine solution and can be reformulated as follows.

Proposition 2.7 *Let K^k be a closed (embedded or immersed) submanifold of M^n. Then the stationary condition on K is* (2.36)*, and the non-degeneracy condition on K is equivalent to the invertibility of operator $\mathfrak{J}_K$ defined in* (2.44).

Summarizing, we have the following proposition by taking $j = I+1$, $w_{I+1} = 0$, and $\Phi_{I+1} = 0$ in Section 3.2.3.

Proposition 2.8 *Let $I \geq 3$ be an arbitrary positive integer, for any given smooth functions Φ_{I-1} and e on K, there are smooth bounded functions*

$$w_\ell = w_{\ell,1}(\varepsilon y, \bar{\xi}; e) + \sigma^{-1}\langle H, \Phi_{\ell-1}\rangle U_0, \quad \ell = 1, \ldots, I,$$

and

$$\Phi_j = \Phi_j(\bar{y}; e), \quad j = 0, \ldots, I-2,$$

such that

$$\begin{aligned}\widetilde{S}_\varepsilon(v_I) = {} & \varepsilon\big(-\varepsilon^2\mu^{-2}\Delta_K e + \lambda_0 e\big)Z + \varepsilon^{I+1}\widetilde{\mathfrak{F}}_{I+1} + \varepsilon^{I+1}\mathfrak{E}_{I+1}\, e\, Z \\ & + \varepsilon^{I+1}\mathcal{A}^i_{I+1}(\varepsilon y, \bar{\xi}; e)\, e\, \partial_i Z + \varepsilon^{I+1}\mathcal{B}^{i\ell}_{I+1}(\varepsilon y, \bar{\xi}; e)\, e\, \partial^2_{i\ell} Z \\ & + \varepsilon^{I+1}\mathcal{C}^i_{I+1}(\varepsilon y, \bar{\xi}; e)\cdot \nabla_K e\, \partial_i Z + \varepsilon^{I+1}\mathcal{D}^{ab}_{I+1}(\varepsilon y, \bar{\xi}; e)\, \partial^2_{ab} e\, Z + \mathcal{O}(\varepsilon^{I+2}),\end{aligned} \tag{2.50}$$

where

$$\begin{aligned}\widetilde{\mathfrak{F}}_{I+1} = {} & \mu^{-1}\Gamma^b_{bl}\,\partial_l w_I + \frac{2}{3}\mu^{-1}R_{kisl}\,\bar{\xi}^k\,\Phi^l_{I-1}\,\partial^2_{is}w_0 - \mu^{-1}\Big(\widetilde{g}^{ab}\,R_{kabs} + \frac{2}{3}R_{kiis} - \Gamma^c_{ak}\Gamma^a_{cs}\Big)\Phi^k_{I-1}\,\partial_s w_0 \\ & + \mu^{-2}\Big(2\,\nabla_K\mu\cdot\nabla_K\Phi^s_{I-1} + \mu\,\Delta_K\Phi^s_{I-1}\Big)\partial_s w_0 + 2(h\mu)^{-1}\Big(\nabla_K h\cdot\nabla_K\Phi^s_{I-1}\Big)\partial_s w_0 \\ & + 2\mu^{-2}\Big(\nabla_K\mu\cdot\nabla_K\Phi^s_{I-1}\Big)(\bar{\xi}^i\,\partial^2_{is}w_0) + \mu^{-2}\langle\nabla^N V, \Phi_0\rangle w_I + \mu^{-2}\langle\nabla^N V, \Phi_{I-1}\rangle w_1 \\ & + \mu^{-2}\langle\nabla^N V, \frac{\bar{\xi}}{\mu}\rangle\, w_I + \mu^{-3}\partial^2_{kl}V(\varepsilon y, 0)\,\Phi^l_{I-1}\,\bar{\xi}^k\,w_0 - p(p-1)w_0^{p-2}w_1 w_I + G_{I+1}(\varepsilon y, \bar{\xi}; e),\end{aligned}$$

$$\mathfrak{E}_{I+1} = -\,p(p-1)w_0^{p-2}w_I + \mu^{-2}\langle\nabla^N V, \Phi_{I-1}\rangle + \widetilde{\mathfrak{E}}_{I+1}(\varepsilon y, \bar{\xi}; e),$$

and $\mathcal{A}^i_{I+1}$, $\mathcal{B}^{i\ell}_{I+1}$, $\mathcal{C}^i_{I+1}$, $\mathcal{D}^{ab}_{I+1}$, $\widetilde{\mathfrak{E}}_{I+1}$ and G_{I+1} are smooth bounded functions on their variables and are Lipschitz continuous with respect to e.

Remark 2.2.6 For example, $\widetilde{\mathfrak{E}}_{I+1}$ involves the term $\mu^{-3}\partial^2_{kl}V(\varepsilon y, 0)\,\Phi^l_{I-2}\,\bar{\xi}^k$.

Global approximation

In the previous sections, some very accurate local approximate solution v_I have been defined.

Denote

$$u_I(y, \xi) = h(\varepsilon y)v_I(y, \bar{\xi}),$$

in the Fermi coordinate. Since K is compact, by the definition of Fermi coordinate, there is a constant $\delta > 0$ such that the normal coordinate x on K_ε is well defined for $|x| < 1000\delta/\varepsilon$.

Now we can simply define our global approximation:

$$W(z) = \eta^{\varepsilon}_{3\delta}(x)u_I(y,\xi) \quad \text{for } z \in M_\varepsilon, \tag{2.51}$$

where $\eta^{\varepsilon}_{\ell\delta}(x) := \eta(\frac{\varepsilon|x|}{\ell\delta})$ and η is a nonnegative smooth cutoff function such that

$$\eta(t) = 1 \quad \text{if } |t| < 1 \qquad \text{and} \quad \eta(t) = 0 \quad \text{if } |t| > 2.$$

It is easy to see that W has the concentration property as required. Note that W depends on the parameter functions Φ_{I-1} and e, thus we can write $W = W(\,\cdot\,; \Phi_{I-1}, e)$ and define the configuration space of (Φ_{I-1}, e) by

$$\Lambda := \left\{ (\Phi_{I-1}, e) \,\middle|\, \begin{array}{l} \|\Phi_{I-1}\|_{C^{0,\alpha}(K)} + \|\nabla\Phi_{I-1}\|_{C^{0,\alpha}(K)} + \|\nabla^2\Phi_{I-1}\|_{C^{0,\alpha}(K)} \le 1, \\ \|e\|_{C^{0,\alpha}(K)} + \varepsilon\|\nabla e\|_{C^{0,\alpha}(K)} + \varepsilon^2\|\nabla^2 e\|_{C^{0,\alpha}(K)} \le 1 \end{array} \right\}. \tag{2.52}$$

Clearly, the configuration space Λ is infinite dimensional.

For $(\Phi_{I-1}, e) \in \Lambda$, it is not difficult to show that for any $0 < \tau < 1$, there is a positive constant C (independent of ε, Φ_{I-1}, e) such that

$$|v_I(y,\bar{\xi})| \le Ce^{-\tau|\bar{\xi}|}, \quad \forall\,(y,\bar{\xi}) \in K_\varepsilon \times \mathbb{R}^N. \tag{2.53}$$

2.2.5 An infinite dimensional reduction and the proof of Theorem 2.1

To construct the solutions stated in Theorem 2.1, we will apply the so-called infinite dimensional reduction which can be seen as a generalization of the classical Lyapunov-Schmidt reduction in an infinite dimensional setting. It has been used in many constructions in PDE and geometric analysis. We present it here in a rather simple and synthetic way since it uses many ideas which have been developed by all the different authors working on this subject or on closely related problems. In particular, we are benefited from the ideas and tricks in [8, 38, 43].

Setting-up of the problem

Given $(\Phi_{I-1}, e) \in \Lambda$, we have defined a global approximate solution W. an infinite dimensional reduction will be applied to claim that there exist Φ_{I-1} and e such that a small perturbation of the global approximation W is a genuine solution.

For this purpose, we denote

$$E := -\Delta_g W + V(\varepsilon z)W - W^p,$$

$$L_\varepsilon[\phi] := -\Delta_g\phi + V(\varepsilon z)\phi - pW^{p-1}\phi,$$

and

$$N(\phi) := -\big[(W+\phi)^p - W^p - pW^{p-1}\phi\big].$$

Obviously, $W + \phi$ is a solution of equation (2.22) is equivalent to

$$L_\varepsilon[\phi] + E + N(\phi) = 0. \tag{2.54}$$

To solve (2.54), we look for a solution ϕ of the form

$$\phi := \eta^\varepsilon_{3\delta}\phi^\sharp + \phi^\flat,$$

where $\phi^\flat : M_\varepsilon \to \mathbb{R}$ and $\phi^\sharp : K_\varepsilon \times \mathbb{R}^N \to \mathbb{R}$. This nice argument has been used in [8, 38, 43] and is called *the gluing technique*. It seems rather counterintuitive, but this strategy will make the linear theory of L_ε clear.

An easy computation shows that

$$-L_\varepsilon[\phi] = \eta^\varepsilon_{3\delta}\left(\Delta_g\phi^\sharp - V\phi^\sharp + pW^{p-1}\phi^\sharp\right) + \Delta_g\phi^\flat - V\phi^\flat + pW^{p-1}\phi^\flat + (\Delta_g\eta^\varepsilon_{3\delta})\phi^\sharp + 2\nabla_g\eta^\varepsilon_{3\delta}\cdot\nabla_g\phi^\sharp.$$

Therefore, ϕ is a solution of (2.54) if the pair $(\phi^\flat, \phi^\sharp)$ satisfies the following coupled system:

$$\begin{cases} \Delta_g\phi^\flat - V\phi^\flat = -(\Delta_g\eta^\varepsilon_{3\delta})\phi^\sharp - 2\nabla_g\eta^\varepsilon_{3\delta}\cdot\nabla_g\phi^\sharp + (1-\eta^\varepsilon_\delta)\Big[E + N(\eta^\varepsilon_{3\delta}\phi^\sharp + \phi^\flat) - pW^{p-1}\phi^\flat\Big], \\ \eta^\varepsilon_{3\delta}\left(\Delta_g\phi^\sharp - V\phi^\sharp + pW^{p-1}\phi^\sharp\right) = \eta^\varepsilon_\delta\Big[E + N(\eta^\varepsilon_{3\delta}\phi^\sharp + \phi^\flat) - pW^{p-1}\phi^\flat\Big]. \end{cases}$$

In order to solve the above system, we first define

$$L^\flat_\varepsilon[\phi^\flat] := \Delta_g\phi^\flat - V\phi^\flat \quad \text{on } M_\varepsilon, \tag{2.55}$$

and note that it is a strongly coercive operator thanks to the conditions on the potential V, see (2.8). Then, in the support of $\eta^\varepsilon_{3\delta}$, we define

$$\phi^\sharp := h(\varepsilon y)\phi^*(y, \bar{\xi}), \qquad \text{with} \quad \phi^* : K_\varepsilon \times \mathbb{R}^N \to \mathbb{R}.$$

A straightforward computation as in Subsection 3.1 yields

$$\eta^\varepsilon_{3\delta}\Big(\Delta_g\phi^\sharp - V\phi^\sharp + pW^{p-1}\phi^\sharp\Big) = \eta^\varepsilon_{3\delta}h^p\Big(\Delta_{\mathbb{R}^N}\phi^* + \mu^{-2}\Delta_{K_\varepsilon}\phi^* - \phi^* + (\eta^\varepsilon_{3\delta})^{p-1}pv_I^{p-1}\phi^* + \widetilde{B}[\phi^*]\Big).$$

where $\widetilde{B} = \mathcal{O}(\varepsilon)$ is a linear operator defined in Subsection 3.1. Now we extend the linear operator $\widetilde{B}$ to $K_\varepsilon \times \mathbb{R}^N$ and we define

$$\mathbb{L}_\varepsilon[\phi^*] := \Delta_{\mathbb{R}^N}\phi^* + \mu^{-2}\Delta_{K_\varepsilon}\phi^* - \phi^* + (\eta^\varepsilon_{3\delta})^{p-1}pv_I^{p-1}\phi^* + \eta^\varepsilon_{6\delta}\widetilde{B}[\phi^*] \quad \text{on } K_\varepsilon \times \mathbb{R}^N,$$

and

$$L^*_\varepsilon[\phi^*] := \Delta_{\mathbb{R}^N}\phi^* + \mu^{-2}\Delta_{K_\varepsilon}\phi^* - \phi^* + pw_0^{p-1}\phi^* = -L_0[\phi^*] + \mu^{-2}\Delta_{K_\varepsilon}\phi^* \quad \text{on } K_\varepsilon \times \mathbb{R}^N.$$

Since $\eta^\varepsilon_{3\delta}\cdot\eta^\varepsilon_\delta = \eta^\varepsilon_\delta$ and $\eta^\varepsilon_{3\delta}\cdot\eta^\varepsilon_{6\delta} = \eta^\varepsilon_{3\delta}$, ϕ is a solution of (2.54) if the pair $(\phi^\flat, \phi^*)$ solves the following coupled system:

$$\begin{cases} L^\flat_\varepsilon[\phi^\flat] = -(\Delta_g\eta^\varepsilon_{3\delta})h\phi^* - 2\nabla_g\eta^\varepsilon_{3\delta}\cdot\nabla_g(h\phi^*) + (1-\eta^\varepsilon_\delta)\Big[E + N(\eta^\varepsilon_{3\delta}\phi^\sharp + \phi^\flat) - pW^{p-1}\phi^\flat\Big], \\ L^*_\varepsilon[\phi^*] = \eta^\varepsilon_\delta\, h^{-p}\Big[E + N(\eta^\varepsilon_{3\delta}h\phi^* + \phi^\flat) - pW^{p-1}\phi^\flat\Big] - (\mathbb{L}_\varepsilon - L^*_\varepsilon)[\phi^*]. \end{cases}$$

It is easy to check that

$$-(\Delta_g\eta^\varepsilon_{3\delta})h\phi^* - 2\nabla_g\eta^\varepsilon_{3\delta}\cdot\nabla_g(h\phi^*) = (1-\eta^\varepsilon_\delta)\Big[-(\Delta_g\eta^\varepsilon_{3\delta})h\phi^* - 2\nabla_g\eta^\varepsilon_{3\delta}\cdot\nabla_g(h\phi^*)\Big]$$

and

$$(1-\eta^\varepsilon_\delta) = (1-\eta^\varepsilon_\delta)(1-\eta^\varepsilon_{\delta/2}).$$

Now, we define

$$\begin{aligned}\mathcal{N}_\varepsilon(\phi^\flat,\phi^*,\Phi_{I-1},e) :=& -(\Delta_g\eta^\varepsilon_{3\delta})h\phi^* - 2\nabla_g\eta^\varepsilon_{3\delta}\cdot\nabla_g(h\phi^*)\\ &+(1-\eta^\varepsilon_{\delta/2})\Big[E+N(\eta^\varepsilon_{3\delta}\phi^\sharp+\phi^\flat)-pW^{p-1}\phi^\flat\Big],\end{aligned}$$

and

$$\mathcal{M}_\varepsilon(\phi^\flat,\phi^*,\Phi_{I-1},e) := \eta^\varepsilon_\delta\, h^{-p}\Big[E+N(\eta^\varepsilon_{3\delta}h\phi^*+\phi^\flat)-pW^{p-1}\phi^\flat\Big]-(\mathbb{L}_\varepsilon-L^*_\varepsilon)[\phi^*].$$

Then $W+\phi$ is a solution of equation (2.22) if $(\phi^\flat,\phi^*,\Phi_{I-1},e)$ solves the following system:

$$\begin{cases} L^\flat_\varepsilon[\phi^\flat] = (1-\eta^\varepsilon_\delta)\,\mathcal{N}_\varepsilon(\phi^\flat,\phi^*,\Phi_{I-1},e),\\ L^*_\varepsilon[\phi^*] = \mathcal{M}_\varepsilon(\phi^\flat,\phi^*,\Phi_{I-1},e).\end{cases} \tag{2.56}$$

To solve the above system (2.56), we first study the linear theory : on one hand, since the operator $L^\flat_\varepsilon$ is strongly coercive, then we have the solvability of equation $L^\flat_\varepsilon[\phi^\flat]=\psi$. On the other hand, one can check at once that L^*_ε has bounded kernels, e.g., $\partial_j w_0$, $j=1,\ldots,N$. Actually, since L_0 has a negative eigenvalue λ_0 with the corresponding eigenfunction Z, there may be more bounded kernels of L^*_ε.

Let ψ be a function defined on $K_\varepsilon\times\mathbb{R}^N$, we define Π to be the $L^2(d\bar\xi)$-orthogonal projection on $\partial_j w_0$'s and Z, namely

$$\Pi[\psi] := \Big(\Pi_1[\psi],\ldots,\Pi_N[\psi],\Pi_{N+1}[\psi]\Big), \tag{2.57}$$

where for $j=1,\ldots,N$,

$$\Pi_j[\psi] := \frac{1}{c_0}\int_{\mathbb{R}^N}\psi(y,\bar\xi)\,\partial_j w_0(\bar\xi)\,d\bar\xi, \quad \text{with } c_0=\int_{\mathbb{R}^N}|\partial_1 w_0|^2\,d\bar\xi,$$

and

$$\Pi_{N+1}[\psi] := \int_{\mathbb{R}^N}\psi(y,\bar\xi)\,Z(\bar\xi)\,d\bar\xi.$$

Let us also denote by $\Pi^\perp$ the orthogonal projection on the orthogonal of $\partial_j w_0$'s and Z, namely

$$\Pi^\perp[\psi] := \psi - \sum_{j=1}^N \Pi_j[\psi]\,\partial_j w_0 - \Pi_{N+1}[\psi]\,Z.$$

With these notations, as in the Lyapunov-Schmidt reduction, solving the system (2.56) amounts to solving the system

$$\begin{cases} L_\varepsilon^\flat[\phi^\flat] = (1-\eta_\delta^\varepsilon)\mathcal{N}_\varepsilon(\phi^\flat,\phi^*,\Phi_{I-1},e), \\ L_\varepsilon^*[\phi^*] = \Pi^\perp\Big[\mathcal{M}_\varepsilon(\phi^\flat,\phi^*,\Phi_{I-1},e)\Big], \\ \Pi\Big[\mathcal{M}_\varepsilon(\phi^\flat,\phi^*,\Phi_{I-1},e)\Big] = 0. \end{cases} \tag{2.58}$$

It is to see that one can write

$$E = \eta_{3\delta}^\varepsilon\, h^p\, \widetilde{S}_\varepsilon(v_I) - (\Delta_g\eta_{3\delta}^\varepsilon)(hv_I) - 2(\nabla_g\eta_{3\delta}^\varepsilon)\cdot\nabla_g(hv_I) - \eta_{3\delta}^\varepsilon\Big[(\eta_{3\delta}^\varepsilon)^{p-1}-1\Big]h^p v_I^p.$$

Hence by Proposition 2.8,

$$\begin{aligned}\mathcal{M}_\varepsilon(\phi^\flat,\phi^*,\Phi_{I-1},e) =& \varepsilon\big(-\varepsilon^2\mu^{-2}\Delta_K e+\lambda_0 e\big)Z+\varepsilon^{I+1}S_{I+1}(\Phi_{I-1})\\ &+\varepsilon^{I+1}G_{I+1}(\varepsilon y,\bar\xi;e)+\varepsilon^{I+2}J_{I+1}(\varepsilon y,\bar\xi;\Phi_{I-1},e)\\ &+\eta_\delta^\varepsilon\, h^{-p}\Big[N(\eta_{3\delta}^\varepsilon h\phi^*+\phi^\flat)-pW^{p-1}\phi^\flat\Big]-(\mathbb{L}_\varepsilon-L_\varepsilon^*)[\phi^*].\end{aligned}$$

On the other hand, since

$$\int_{\mathbb{R}^N} S_{I+1}(\Phi_{I-1})\,\partial_s w_0 = c_0\mu^{-1}(\mathcal{J}_K\Phi_{I-1})^s(\varepsilon y), \tag{2.59}$$

by some rather tedious and technical computations, one can show that

$$\Pi\Big[\mathcal{M}_\varepsilon(\phi^\flat,\phi^*,\Phi_{I-1},e)\Big]=0 \Longleftrightarrow \begin{cases} \varepsilon^{I+1}\mathcal{J}_K[\Phi_{I-1}] = \varepsilon^{I+1}\mathfrak{H}_{I+1}(\bar y;e)+\mathcal{M}_{\varepsilon,1}(\phi^\flat,\phi^*,\Phi_{I-1},e); \\ \varepsilon\,\mathcal{K}_\varepsilon[e] = \mathcal{M}_{\varepsilon,2}(\phi^\flat,\phi^*,\Phi_{I-1},e), \end{cases} \tag{2.60}$$

where $\mathfrak{H}_{I+1}(\bar y;e)$ is a smooth bounded function on $\bar y$ and is Lipschitz continuous with respect to e, $\mathcal{J}_K$ is the Jacobi operator on K, and $\mathcal{K}_\varepsilon$ is a Schrödinger operator defined by

$$\mathcal{K}_\varepsilon[e] := -\varepsilon^2\Delta_K e+\lambda_0\mu^2 e \tag{2.61}$$

where λ_0 is the unique negative eigenvalue of L_0.

We summarize the above discussion by saying that the function

$$u = W(\,\cdot\,;\Phi_{I-1},e)+\eta_{3\delta}^\varepsilon\, h\,\phi^*+\phi^\flat,$$

is a solution of the equation

$$\Delta_g u - V(\varepsilon z)u + u^p = 0,$$

if the functions $\phi^\flat$, ϕ^*, Φ_{I-1} and e satisfy the following system

$$\begin{cases} L_\varepsilon^\flat[\phi^\flat] = (1-\eta_\delta^\varepsilon)\,\mathcal{N}_\varepsilon(\phi^\flat,\phi^*,\Phi_{I-1},e), \\ L_\varepsilon^*[\phi^*] = \Pi^\perp\Big[\mathcal{M}_\varepsilon(\phi^\flat,\phi^*,\Phi_{I-1},e)\Big], \\ \varepsilon^{I+1}\mathcal{J}_K[\Phi_{I-1}] = \varepsilon^{I+1}\mathfrak{H}_{I+1}(\bar y;e)+\mathcal{M}_{\varepsilon,1}(\phi^\flat,\phi^*,\Phi_{I-1},e), \\ \varepsilon\,\mathcal{K}_\varepsilon[e] = \mathcal{M}_{\varepsilon,2}(\phi^\flat,\phi^*,\Phi_{I-1},e). \end{cases} \tag{2.62}$$

Remark 2.2.7 1. In general there are two different approaches to set-up the problem: the first one, as used in [8] and [43], consists in solving first the equations of $\phi^\flat$ and ϕ^* for fixed Φ_{I-1} and e, and then solve the left equations of Φ_{I-1} and e. The second one, as in [26, 30] consists in solving first the linear problem $L_\varepsilon[\phi] + \psi = 0$ under some non-degeneracy and gap conditions; and then solve the nonlinear problem $L_\varepsilon[\phi]+E+N(\phi) = 0$ by using a fixed point arguments.

Our approach is slightly different from those in [8]-[43] and [26]-[30].

2. After solving the system (2.62), one can prove the positivity of u by contradiction since both $\phi^\flat$ and ϕ^* are small.

Analysis of the linear operators

By the above analysis, what is left is to show that (2.62) has a solution. To this end, we will apply a fixed point theorem. Before we do this, a linear theory will be developed.

Analysis of a strongly coercive operator

To deal with the term $-\eta^\varepsilon_\delta h^{-p} pW^{p-1}\phi^\flat$ in $\mathcal{M}_\varepsilon(\phi^\flat, \phi^*, \Phi_{I-1}, e)$ in applying a fixed point theorem, one needs to choose norms with the property that $\mathcal{M}_\varepsilon(\phi^\flat, \phi^*, \Phi_{I-1}, e)$ depends slowly on $\phi^\flat$. To this end, we define

$$\|\phi^\flat\|_{\varepsilon,\infty} = \|(1 - \eta^\varepsilon_{\delta/4})\phi^\flat\|_\infty + \frac{1}{\varepsilon}\|\eta^\varepsilon_{\delta/4}\phi^\flat\|_\infty. \tag{2.63}$$

With this notation, by the exponential decay of W, we have

$$\|\mathcal{M}_\varepsilon(\phi^\flat, \phi^*, \Phi_{I-1}, e)\|_\infty \le C\varepsilon\|\phi^\flat\|_{\varepsilon,\infty}$$

and

$$\|\mathcal{M}_\varepsilon(\phi^\flat_1, \phi^*, \Phi_{I-1}, e) - \mathcal{M}_\varepsilon(\phi^\flat_2, \phi^*, \Phi_{I-1}, e)\|_\infty \le C\varepsilon\|\phi^\flat_1 - \phi^\flat_2\|_{\varepsilon,\infty}.$$

Since (2.8), we have the following lemma.

Lemma 2.9 *For any function $\psi(z) \in L^\infty(M_\varepsilon)$, there is a unique bounded solution ϕ of*

$$L^\flat_\varepsilon[\phi] = (1 - \eta^\varepsilon_\delta)\psi. \tag{2.64}$$

Moreover, there exists a constant $C > 0$ (independent of ε) such that

$$\|\phi\|_{\varepsilon,\infty} \le C\|\psi\|_\infty. \tag{2.65}$$

For $\phi^\flat \in C^{0,\alpha}_0(M_\varepsilon)$, we define

$$\|\phi^\flat\|_{\varepsilon,\alpha} = \|(1 - \eta^\varepsilon_{\delta/4})\phi^\flat\|_{C^{0,\alpha}_0} + \frac{1}{\varepsilon}\|\eta^\varepsilon_{\delta/4}\phi^\flat\|_{C^{0,\alpha}_0}. \tag{2.66}$$

As a consequence of standard elliptic estimates, the following lemma holds.

Lemma 2.10 *For any function $\psi \in C_0^{0,\alpha}(M_\varepsilon)$, there is a unique solution $\phi \in C_0^{2,\alpha}(M_\varepsilon)$ of*

$$L_\varepsilon^\flat[\phi] = (1 - \eta_\delta^\varepsilon)\psi. \tag{2.67}$$

Moreover, there exists a constant $C > 0$ (independent of ε) such that

$$\|\phi\|_{2,\varepsilon,\alpha} := \|\phi\|_{\varepsilon,\alpha} + \|\nabla\phi\|_{\varepsilon,\alpha} + \|\nabla^2\phi\|_{\varepsilon,\alpha} \le C\|\psi\|_{C_0^{2,\alpha}(M_\varepsilon)}. \tag{2.68}$$

Study of the model linear operator L_ε^*

First, we will prove an injectivity result which is the key result. Then, we will use this result to obtain an a priori estimate and the existence result for solutions of $L_\varepsilon^*[\phi] = \psi$ when $\Pi[\phi] = 0 = \Pi[\psi]$.

Lemma 2.11 (The injectivity result) *Suppose that $\phi \in L^\infty(K_\varepsilon \times \mathbb{R}^N)$ satisfies $L_\varepsilon^*[\phi] = 0$ and $\Pi[\phi] = 0$. Then $\phi \equiv 0$.*

Proof. We will prove this lemma by two steps.

Step 1: The function $\phi(y, \bar{\xi})$ decays exponentially in the variables $\bar{\xi}$.

To prove this fact, it suffices to apply the maximum principle since $w_0(\bar{\xi})$ has exponential decay and ϕ is bounded.

Step 2: We next prove that

$$f(y) := \int_{\mathbb{R}^N} \phi^2(y, \bar{\xi})\, d\bar{\xi} = 0, \quad \forall\, y \in K_\varepsilon.$$

Indeed, by Step 1, for all $y \in K_\varepsilon$, $f(y)$ is well defined. Since $L_\varepsilon^*[\phi] = 0$, we have

$$\begin{aligned}
\Delta_{K_\varepsilon} f &= \int_{\mathbb{R}^N} 2\phi\Delta_{K_\varepsilon}\phi\, d\bar{\xi} + \int_{\mathbb{R}^N} 2|\nabla_{K_\varepsilon}\phi|^2\, d\bar{\xi} \\
&= 2\mu^2 \int_{\mathbb{R}^N} \Big\{|\nabla_{\bar{\xi}}\phi|^2 + \phi^2 - pw_0^{p-1}\phi^2\Big\}\, d\bar{\xi} + 2\int_{\mathbb{R}^N} |\nabla_{K_\varepsilon}\phi|^2\, d\bar{\xi} \\
&\ge 2\mu^2\gamma_0 \int_{\mathbb{R}^N} \phi^2(y, \bar{\xi})\, d\bar{\xi},
\end{aligned}$$

where in the last inequality since $\Pi[\phi] = 0$ we use the following inequality

$$\int_{\mathbb{R}^N} \Big\{|\nabla_{\bar{\xi}}\phi|^2 + \phi^2 - pw_0^{p-1}\phi^2\Big\}\, d\bar{\xi} \ge \gamma_0 \int_{\mathbb{R}^N} \phi^2\, d\bar{\xi}. \tag{2.69}$$

Therefore, by the definition of f, the above inequality gives

$$\Delta_{K_\varepsilon} f \ge 2\mu^2\gamma_0 f.$$

Since f is nonnegative and K_ε is compact, we just get $f \equiv 0$ by the integration. If K_ε is non compact, one can first show that f goes to zero at infinity by the comparison theorem and then get $f \equiv 0$ by the maximum principle. □

Remark 2.2.8 Actually, following the argument of proof of Lemma 3.7 in [38], one can show that

$$\phi = \sum_{j=1}^{N} c^j(y)\, \partial_j w_0 + c^{N+1}(y)\, Z, \tag{2.70}$$

if ϕ is a bounded solution of $L^*_\varepsilon[\phi] = 0$, where $c_j(y)$ $(j = 1, \ldots, N)$ can be any bounded function, but $c^{N+1}(y)$ must satisfy the equation

$$\Delta_{K_\varepsilon} c^{N+1} = \lambda_0 \mu^2 c^{N+1}. \tag{2.71}$$

It is worth noting that (2.71) is just another form of $\mathcal{K}_\varepsilon[e] = 0$. When ε satisfies some gap condition (cf. Proposition 2.14 below), equation (2.71) does not have a bounded solution.

Moreover, one can show that under the orthogonal conditions $\Pi[\phi] = 0$, the linear operator L^*_ε has only negative eigenvalues λ^ε_j's and there exists a constant c_0 such that

$$\lambda^\varepsilon_j \le -c_0 < 0.$$

To prove it, since $\mu^2 = V(\bar{y}, 0)$ and (2.8), the inequality (2.69) implies

$$\int_{K_\varepsilon \times \mathbb{R}^N} -L^*_\varepsilon[\phi]\, \phi \ge c \int_{K_\varepsilon \times \mathbb{R}^N} (-L^*_\varepsilon[\phi])(\mu^2 \phi) \ge c\gamma_0 \int_{K_\varepsilon \times \mathbb{R}^N} \phi^2.$$

Before stating the surjectivity result, we define

$$\|\psi\|_{\varepsilon,\alpha,\rho} := \sup_{(y,\bar{\xi}) \in K_\varepsilon \times \mathbb{R}^N} e^{\rho|\bar{\xi}|} \|\psi\|_{C^{0,\alpha}(B_1((y,\bar{\xi})))},$$

where α and ρ are small positive constants.

Proposition 2.12 (The surjectivity result) *For any function ψ with $\|\psi\|_{\alpha,\sigma} < \infty$ and $\Pi[\psi] = 0$, the problem*

$$L^*_\varepsilon[\phi] = \psi \tag{2.72}$$

has a unique solution ϕ with $\Pi[\phi] = 0$. Moreover, the following estimate holds:

$$\|\phi\|_{2,\varepsilon,\alpha,\rho} := \|\phi\|_{\varepsilon,\alpha,\rho} + \|\nabla\phi\|_{\varepsilon,\alpha,\rho} + \|\nabla^2\phi\|_{\varepsilon,\alpha,\rho} \le C\|\psi\|_{\varepsilon,\alpha,\rho}, \tag{2.73}$$

where C is a constant independent of ε.

Remark 2.2.9 Here we choose to use weighted Hölder norms, actually one can also use weighted Sobolev norms.

Non-degeneracy condition and invertibility of $\mathfrak{J}_K$

Proposition 2.13 *Suppose that K is non-degenerate, then for any $\Psi \in (C^{0,\alpha}(K))^N \cap NK$, there exists a unique $\Phi \in (C^{2,\alpha}(K))^N \cap NK$ such that*

$$\mathfrak{J}_K[\Phi] = \Psi \tag{2.74}$$

with the property

$$\|\Phi\|_{2,\alpha} := \|\Phi\|_{C^{0,\alpha}(K)} + \|\nabla\Phi\|_{C^{0,\alpha}(K)} + \|\nabla^2\Phi\|_{C^{0,\alpha}(K)} \le C\|\Psi\|_{C^{0,\alpha}(K)}, \tag{2.75}$$

where C is a positive constant depending only on K.

PROOF. Since the Jacobi operator $\mathcal{J}_K$ is self-adjoint, this result follows from standard elliptic estimates, cf. [15, 19]. $\square$

Gap condition and invertibility of $\mathcal{K}_\varepsilon$

Proposition 2.14 *There is a sequence $\varepsilon = \varepsilon_j \searrow 0$ such that for any $\varphi \in C^{0,\alpha}(K)$, there exists a unique $e \in C^{2,\alpha}(K)$ such that*

$$\mathcal{K}_\varepsilon[e] = \varphi \tag{2.76}$$

with the property

$$\|e\|_* := \|e\|_{C^{0,\alpha}(K)} + \varepsilon\|\nabla e\|_{C^{0,\alpha}(K)} + \varepsilon^2\|\nabla^2 e\|_{C^{0,\alpha}(K)} \le C\varepsilon^{-3k}\|\varphi\|_{C^{0,\alpha}(K)}, \tag{2.77}$$

where C is a positive constant independent of ε_j.

PROOF. This is a semiclassical analysis of a Schrödinger operator. The arguments are similar in spirit as the ones used in the proof of Proposition 8.1 in [43]. We summarize them in the following two steps.

Step 1: There is a sequence $\varepsilon_j \searrow 0$ such that for any $\varphi \in L^2(K)$, there exists a unique solution to (2.76) and satisfies

$$\|e\|_{L^2(K)} \le C\varepsilon_j^{-k}\|\varphi\|_{L^2(K)}. \tag{2.78}$$

This fact follows from the variational characterization of the eigenvalues and Weyl's asymptotic formula.

Step 2: The unique solution satisfies (2.77). This follows from standard elliptic estimates and Sobolev embedding theorem. $\square$

The nonlinear scheme

Now we can develop the nonlinear theory and complete the proof of Theorem 2.1.

Size of the error

Lemma 2.15 *There is a constant C independent of ε such that the following estimates hold:*

$$\big\|\mathcal{N}_\varepsilon(0,0,0,0)\big\|_{C_0^{2,\alpha}(M_\varepsilon)} + \big\|\Pi^\perp\big[\mathcal{M}_\varepsilon(0,0,0,0)\big]\big\|_{\varepsilon,\alpha,\rho} \le C\varepsilon^{I+1}. \tag{2.79}$$

Moreover,

$$\big\|\mathcal{M}_{\varepsilon,1}(0,0,0,0)\big\|_{C^{0,\alpha}(K)} \le C\varepsilon^{I+2}, \quad \big\|\mathcal{M}_{\varepsilon,2}(0,0,0,0)\big\|_{C^{0,\alpha}(K)} \le C\varepsilon^{I+1}. \tag{2.80}$$

PROOF. It follows from the definitions and the estimate (2.53). $\square$

Lipschitz continuity

According to the estimate of error, we define

$$\mathcal{B}_\lambda := \Big\{ (\phi^\flat, \phi^*, \Phi_{I-1}, e) \,\Big|\, \quad \|\phi^\flat\|_{2,\varepsilon,\alpha} \le \lambda \varepsilon^{I+1}, \|\phi^*\|_{2,\varepsilon,\alpha,\rho} \le \lambda \varepsilon^{I+1}, \\ \|\Phi_{I-1}\|_{2,\alpha} \le \lambda\varepsilon, \|e\|_* \le \lambda \varepsilon^{I-3k} \Big\}. \tag{2.81}$$

Lemma 2.16 *Given $(\phi_1^\flat, \phi_1^*, \Phi_{I-1}, e_1), (\phi_2^\flat, \phi_2^*, \widetilde{\Phi}_{I-1}, e_2) \in \mathcal{B}_\lambda$, there is a constant C independent of ε such that the following estimates hold:*

$$\begin{aligned}&\big\|\mathcal{N}_\varepsilon(\phi_1^\flat, \phi_1^*, \Phi_{I-1}, e_1) - \mathcal{N}_\varepsilon(\phi_2^\flat, \phi_2^*, \widetilde{\Phi}_{I-1}, e_2)\big\|_{C_0^{2,\alpha}(M_\varepsilon)} \\ &\le C\varepsilon^{I+1}\Big(\|\phi_1^\flat - \phi_2^\flat\|_{2,\varepsilon,\alpha} + \|\phi_1^* - \phi_2^*\|_{2,\varepsilon,\alpha,\rho} + \|\Phi_{I-1} - \widetilde{\Phi}_{I-1}\|_{2,\alpha} + \|e_1 - e_2\|_* \Big),\end{aligned}$$

$$\begin{aligned}&\big\|\Pi^\perp\big[\mathcal{M}_\varepsilon(\phi_1^\flat, \phi_1^*, \Phi_{I-1}, e_1)\big] - \Pi^\perp\big[\mathcal{M}_\varepsilon(\phi_2^\flat, \phi_2^*, \widetilde{\Phi}_{I-1}, e_2)\big]\big\|_{\varepsilon,\alpha,\rho} \\ &\le C\varepsilon^{I+1}\Big(\|\phi_1^\flat - \phi_2^\flat\|_{2,\varepsilon,\alpha} + \|\phi_1^* - \phi_2^*\|_{2,\varepsilon,\alpha,\rho} + \|\Phi_{I-1} - \widetilde{\Phi}_{I-1}\|_{2,\alpha} + \|e_1 - e_2\|_* \Big),\end{aligned}$$

$$\begin{aligned}&\big\|\mathcal{M}_{\varepsilon,1}(\phi_1^\flat, \phi_1^*, \Phi_{I-1}, e_1) - \mathcal{M}_{\varepsilon,1}(\phi_2^\flat, \phi_2^*, \widetilde{\Phi}_{I-1}, e_2)\big\|_{C^{0,\alpha}(K)} \\ &\le C\varepsilon^{I+2}\Big(\|\phi_1^\flat - \phi_2^\flat\|_{2,\varepsilon,\alpha} + \|\phi_1^* - \phi_2^*\|_{2,\varepsilon,\alpha,\rho} + \|\Phi_{I-1} - \widetilde{\Phi}_{I-1}\|_{2,\alpha} + \|e_1 - e_2\|_* \Big),\end{aligned}$$

and

$$\begin{aligned}&\big\|\mathcal{M}_{\varepsilon,2}(\phi_1^\flat, \phi_1^*, \Phi_{I-1}, e_1) - \mathcal{M}_{\varepsilon,2}(\phi_2^\flat, \phi_2^*, \widetilde{\Phi}_{I-1}, e_2)\big\|_{C^{0,\alpha}(K)} \\ &\le C\varepsilon^{I+1}\Big(\|\phi_1^\flat - \phi_2^\flat\|_{2,\varepsilon,\alpha} + \|\phi_1^* - \phi_2^*\|_{2,\varepsilon,\alpha,\rho} + \|\Phi_{I-1} - \widetilde{\Phi}_{I-1}\|_{2,\alpha} + \|e_1 - e_2\|_* \Big).\end{aligned}$$

Proof. This proof is rather technical but does not offer any real difficulty. It is worth noting that the use of the norm $\|\phi^\flat\|_{2,\varepsilon,\alpha}$ is crucial to estimate the term $-\eta_\delta^\varepsilon\, h^{-p} p W^{p-1} \phi^\flat$ in $\mathcal{M}_\varepsilon(\phi^\flat, \phi^*, \Phi_{I-1}, e)$. □

Proof of Theorem 2.1

By the analysis in Section 4.1, the proof of Theorem 2.1 follows from the solvability of (2.62).

Now we can use the results in the linear theory to rephrase the solvability of (2.62) as a fixed point problem. To do this, let $\Phi_{I-1} = \Phi_{I-1,0} + \widetilde{\Phi}_{I-1}$, where $\Phi_{I-1,0}$ solve the equation

$$\mathfrak{J}_K[\Phi_{I-1,0}] = \mathfrak{H}_{I+1}(\bar{y}; e). \tag{2.82}$$

Thus $\Phi_{I-1,0} = \Phi_{I-1,0}(\bar{y}; e)$. Moreover, the reduced system (2.62) becomes

$$
\begin{cases}
L^\flat_\varepsilon[\phi^\flat] = (1-\eta^\varepsilon_\delta)\,\mathcal{N}_\varepsilon(\phi^\flat, \phi^*, \Phi_{I-1}, e),\\[2mm]
L^*_\varepsilon[\phi^*] = \Pi^\perp\Big[\mathcal{M}_\varepsilon(\phi^\flat, \phi^*, \Phi_{I-1}, e)\Big],\\[2mm]
\varepsilon^{I+1}\mathcal{J}_K[\widetilde{\Phi}_{I-1}] = \widetilde{\mathcal{M}}_{\varepsilon,1}(\phi^\flat, \phi^*, \widetilde{\Phi}_{I-1}, e),\\[2mm]
\varepsilon\,\mathcal{K}_\varepsilon[e] = \widetilde{\mathcal{M}}_{\varepsilon,2}(\phi^\flat, \phi^*, \widetilde{\Phi}_{I-1}, e).
\end{cases}
\tag{2.83}
$$

It is a simple matter to check that both $\widetilde{\mathcal{M}}_{\varepsilon,1}$ and $\widetilde{\mathcal{M}}_{\varepsilon,2}$ satisfy the properties in Lemmas 2.15 and 2.16. Taking $I \geq 3k+1$ and λ sufficiently large, Theorem 2.1 is now a simple consequence of a fixed point theorem for the contraction mapping $\mathcal{B}_\lambda$.

2.2.6 Proof of Proposition 2.5

The proof is based on the Taylor expansion of the metric coefficients. Recall that the Laplace-Beltrami operator is given by

$$
\Delta_g u = \frac{1}{\sqrt{\det g}}\,\partial_\alpha\left(\sqrt{\det g}\, g^{\alpha\beta}\,\partial_\beta u\right)
$$

which can be rewritten as

$$
\Delta_g u = g^{\alpha\beta}\partial^2_{\alpha\beta}u + (\partial_\alpha g^{\alpha\beta})\partial_\beta u + \frac{1}{2}\, g^{\alpha\beta}\partial_\alpha(\log\det g)\partial_\beta u.
$$

Using the expansion of the metric coefficients determined above, it can be easily proved that

$$
\begin{aligned}
g^{\alpha\beta}\,\partial^2_{\alpha\beta}u &= \widetilde{g}^{ab}\,\partial^2_{ab}u + \partial^2_{ii}u + \varepsilon\left\{\widetilde{g}^{cb}\,\Gamma^a_{ci} + \widetilde{g}^{ca}\,\Gamma^b_{ci}\right\}(\xi^i + \Phi^i)\,\widetilde{g}^{ab}\,\partial^2_{ab}u - 2\varepsilon\,\widetilde{g}^{ab}\,\partial_{\bar{b}}\Phi^j\,\partial^2_{aj}u\\
&+ \varepsilon^2\Big(-\widetilde{g}^{cb}\,\widetilde{g}^{ad}\,R_{kcdl} + \widetilde{g}^{ac}\,\Gamma^b_{dk}\Gamma^d_{cl} + \widetilde{g}^{bc}\,\Gamma^a_{dk}\Gamma^d_{cl} + \widetilde{g}^{cd}\,\Gamma^a_{dk}\Gamma^b_{cl}\Big)(\xi^k + \Phi^k)(\xi^l + \Phi^l)\,\partial^2_{ab}u\\
&- \frac{4\,\varepsilon^2}{3}R_{kajl}(\xi^k + \Phi^k)(\xi^l + \Phi^l)\,\partial^2_{aj}u + 2\varepsilon^2\partial_{\bar{b}}\Phi^j\left\{\widetilde{g}^{bc}\,\Gamma^a_{ci} + \widetilde{g}^{ac}\,\Gamma^b_{ci}\right\}(\xi^i + \Phi^i)\,\partial^2_{aj}u\\
&- \frac{\varepsilon^2}{3}\,R_{kijl}(\xi^k + \Phi^k)(\xi^l + \Phi^l)\,\partial^2_{ij}u + \varepsilon^2\,\widetilde{g}^{ab}\,\partial_{\bar{a}}\Phi^i\partial_{\bar{b}}\Phi^j\,\partial^2_{ij}u\\
&+ R_3(\xi, \Phi, \nabla\Phi)(\partial^2_{ij}u + \partial^2_{aj}u + \partial^2_{ab}u).
\end{aligned}
$$

An easy computations yields

$$\partial_b g^{ab} = \partial_b \widetilde{g}^{ab} + \varepsilon^2\, \partial_{\bar b}\Big\{ \widetilde{g}^{cb}\,\Gamma^a_{ci} + \widetilde{g}^{ca}\,\Gamma^b_{ci}\Big\}\,(\xi^i+\Phi^i) + \varepsilon^2\,\Big\{ \widetilde{g}^{cb}\,\Gamma^a_{ci} + \widetilde{g}^{ca}\,\Gamma^b_{ci}\Big\}\,\partial_{\bar b}\,\Phi^i$$
$$+ R_3(\xi,\Phi,\nabla\Phi,\nabla^2\Phi),$$

$$\partial_j g^{ja} = -\frac{2}{3}\varepsilon^2 R_{jajl}(\xi^l+\Phi^l) + \varepsilon^2\partial_{\bar b}\Phi^j\,\Big\{ \widetilde{g}^{bc}\,\Gamma^a_{cj} + \widetilde{g}^{ac}\,\Gamma^b_{cj}\Big\} + R_3(\xi,\Phi,\nabla\Phi),$$

$$\partial_a g^{aj} = -\varepsilon^2\,\partial_{\bar a}\widetilde{g}^{ab}\,\partial_{\bar b}\Phi^j - \varepsilon^2\,\widetilde{g}^{ab}\,\partial^2_{\bar a\bar b}\Phi^j + \varepsilon^3\partial^2_{\bar a\bar b}\Phi^j\,\Big\{ \widetilde{g}^{bc}\,\Gamma^a_{ci} + \widetilde{g}^{ac}\,\Gamma^b_{ci}\Big\}\,(\xi^i+\Phi^i)$$
$$+ R_3(\xi,\Phi,\nabla\Phi,\nabla^2\Phi),$$

$$\partial_i g^{ij} = -\frac{1}{3}\varepsilon^2 R_{kiji}(\xi^k+\Phi^k) + R_3(\xi,\Phi,\nabla\Phi).$$

Then the following expansion holds

$$(\partial_\alpha g^{\alpha\beta})\partial_\beta u =$$
$$\partial_b\widetilde{g}^{ab}\,\partial_a u + \varepsilon^2\,\partial_{\bar b}\Big\{ \widetilde{g}^{cb}\,\Gamma^a_{ci} + \widetilde{g}^{ca}\,\Gamma^b_{ci}\Big\}\,(\xi^i+\Phi^i)\,\partial_a u + \varepsilon^2\,\Big\{ \widetilde{g}^{cb}\,\Gamma^a_{ci} + \widetilde{g}^{ca}\,\Gamma^b_{ci}\Big\}\,\partial_{\bar b}\,\Phi^i\,\partial_a u$$
$$-\frac{2}{3}\varepsilon^2 R_{jajl}(\xi^l+\Phi^l)\,\partial_a u + \varepsilon^2\partial_{\bar b}\Phi^j\,\Big\{ \widetilde{g}^{bc}\,\Gamma^a_{cj} + \widetilde{g}^{ac}\,\Gamma^b_{cj}\Big\}\,\partial_a u$$
$$-\varepsilon^2\,\partial_{\bar a}\widetilde{g}^{ab}\,\partial_{\bar b}\Phi^j\,\partial_j u - \varepsilon^2\,\widetilde{g}^{ab}\,\partial^2_{\bar a\bar b}\Phi^j\,\partial_j u + \varepsilon^3\partial^2_{\bar a\bar b}\Phi^j\,\Big\{ \widetilde{g}^{bc}\,\Gamma^a_{ci} + \widetilde{g}^{ac}\,\Gamma^b_{ci}\Big\}\,(\xi^i+\Phi^i)\,\partial_j u$$
$$-\frac{1}{3}\varepsilon^2 R_{kiji}(\xi^k+\Phi^k)\,\partial_j u + R_3(\xi,\Phi,\nabla\Phi,\nabla^2\Phi)(\partial_j u + \partial_a u).$$

On the other hand using the expansion of the log of determinant of g given in Lemma 2.4, it holds that

$$\partial_b \log\big(\det g\big) = \partial_b \log\big(\det\widetilde{g}\big) - 2\varepsilon^2\,\partial_{\bar b}\big(\Gamma^a_{ak}\big)\,(\xi^k+\Phi^k) - 2\varepsilon^2\,\Gamma^a_{ak}\,\partial_{\bar b}\Phi^k + R_3(\xi,\Phi,\nabla\Phi,\nabla^2\Phi).$$

and

$$\partial_i(\log\det g) = -2\varepsilon\,\Gamma^b_{bi} + 2\varepsilon^2\bigg(\widetilde{g}^{ab}\,R_{kabi} + \frac{1}{3}R_{kjji} - \Gamma^c_{ak}\Gamma^a_{ci}\bigg)(\xi^k+\Phi^k) + R_3(\xi,\Phi,\nabla\Phi),$$

which implies that

$$\frac{1}{2}g^{\alpha\beta}\partial_\alpha(\log\det g)\partial_\beta u =$$
$$\frac{1}{2}\,\partial_a(\log\det\widetilde{g})\,\Big(\widetilde{g}^{ab}\,\partial_b u + \varepsilon\big\{\widetilde{g}^{cb}\Gamma^a_{ci} + \widetilde{g}^{ca}\Gamma^b_{ci}\big\}(\xi^i+\Phi^i)\partial_b u - \varepsilon\,\widetilde{g}^{ab}\partial_{\bar b}\Phi^j\partial_j u\Big)$$
$$-\varepsilon\,\Gamma^b_{bi}\partial_i u + \varepsilon^2\bigg(\widetilde{g}^{ab}R_{kabi} + \frac{1}{3}R_{kjji} - \Gamma^c_{ak}\Gamma^a_{ci}\bigg)(\xi^k+\Phi^k)\,\partial_i u$$
$$-\varepsilon^2\,\bigg(\partial_{\bar b}\big(\Gamma^d_{dk}\big)\,(\xi^k+\Phi^k) + \Gamma^d_{dk}\,\partial_{\bar b}\Phi^k\bigg)\,\widetilde{g}^{ab}\partial_a u + R_3(\xi,\Phi,\nabla\Phi,\nabla^2\Phi)(\partial_j u + \partial_a u).$$

Collecting the above terms and recalling that

$$\Delta_{K_\varepsilon} u = \widetilde{g}^{ab}\partial^2_{ab}u + (\partial_a \widetilde{g}^{ab})\partial_b u + \frac{1}{2}\,\widetilde{g}^{ab}\partial_a(\log\det\widetilde{g})\partial_b u,$$

the desired result then follows at once. $\square$

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