Semi-linear singular elliptic equations with dependence on the gradient

Friedemann Brock[a,*], Leonelo Iturriaga[a], Pedro Ubilla[b]

[a] *Departamento de Matemática, Universidad de Chile, Casilla 653, Chile*
[b] *Departamento de Matemáticas y C. C., Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile*

Abstract

We establish the existence of a positive solution for the following non-variational equation

$$\begin{cases} -\operatorname{div}(|x|^{-2a}\nabla u) = |x|^{-2(a+1)+c} f(x, u, \nabla u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where the non-linearity $f(x, t, \xi)$ belongs to a class of functions that are superlinear in the variable t and sublinear in the variable ξ. For this purpose we used an idea of a recent work by De Figueiredo et al. [D. De Figueiredo, M. Girardi, M. Matzeu, Semilinear elliptic equations with dependence on the gradient via mountain-pass techniques, Diff. Integral Equ. (in press)] and we established a new regularity result for a class of Singular Elliptic Equations.

1. Introduction

We consider the problem

$$\begin{cases} -\operatorname{div}(|x|^{-2a}\nabla u) = |x|^{-2(a+1)+c} f(x, u, \nabla u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $0 \le a < \frac{N-2}{2}$, $c \ge 1$ and Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary such that $0 \in \Omega$. Since the nonlinearity f depends on ∇u, (1.1) cannot be treated directly by variational

* Corresponding author. Tel.: +56 2 978 7307.
E-mail addresses: fbrock@uchile.cl (F. Brock), ozz2112@gmail.com (L. Iturriaga), pubilla@lauca.usach.cl (P. Ubilla).

methods. Our approach is based on an idea of De Figueiredo et al. [4] for an equation involving the Laplacian. This idea consists of analyzing a family of associated elliptic equations without dependence on the gradient (see also [8]). More precisely, given $w \in C^{0,1}(\Omega)$, we consider the following problem

$$\begin{cases} -\mathrm{div}\,(|x|^{-2a}\nabla u) = |x|^{-2(a+1)+c} f(x, u, \nabla w), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Then the result is obtained by a combination of Truncation techniques, the Mountain Pass Theorem and Monotone Iteration. This also requires a proof of Lipschitz regularity of the solutions that occur in the iteration (see Section 2 below). At this stage we would like to point out that if $a < 0$ or if $c < 1$, then we cannot expect a solution of (1.2) to be Lipschitz continuous (for counter examples, see the remark at the end of Section 2). This means that the method of [4] is not applicable in the cases $a < 0$ or $c < 1$. We assign the following hypotheses on the nonlinearity f:

(f_0) $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is measurable, and $f(x, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$.
(f_1) $\lim_{t\to 0} \frac{f(x,t,\xi)}{t} = 0$ uniformly for $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$.
(f_2) $|f(x, t, \xi)| \le a_1(1 + |t|^p)(1 + |\xi|^r)$ $\forall (x, t, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$, for some constants $a_1 > 0$, $1 < p < \min\left\{\frac{N+2}{N-2}, \frac{N-2(a+1)+2c}{N-2(a+1)}\right\}$ and $r \in (0, 1)$.
(f_3) $0 < \theta F(x, t, \xi) \le t f(x, t, \xi)$ $\forall x \in \overline{\Omega}$, $|t| \ge t_0$, $\xi \in \mathbb{R}^N$, for some constants $\theta > 2$ and $t_0 > 0$, where $F(x, t, \xi) = \int_0^t f(x, s, \xi)\,\mathrm{d}s$.

We notice that (f_3) implies that there exist constants $a_2, a_3 > 0$ such that

$$F(x, t, \xi) \ge a_2|t|^\theta - a_3 \quad \forall x \in \overline{\Omega}, t \in \mathbb{R},\ \xi \in \mathbb{R}^N. \tag{1.3}$$

The above hypotheses allow us to apply the Mountain Pass Theorem of Ambrosetti and Rabinowitz (see [2]) on Eq. (1.2). The solvability of problem (1.1) is then ensured if the function f satisfies two local Lipschitz conditions that are given in (f_4) below,

(f_4) $|f(x, t', \xi) - f(x, t'', \xi)| \le L_1|t' - t''|$ $\quad \forall x \in \overline{\Omega}, t', t'' \in [0, \rho_1], |\xi| \le \rho_2$, and $|f(x, t, \xi') - f(x, t, \xi'')| \le L_2|\xi' - \xi''|$ $\quad \forall x \in \overline{\Omega}, t \in [0, \rho_1], |\xi'|, |\xi''| \le \rho_2$, where ρ_1 and ρ_2 depend on $p, N, \theta, a_1, a_2, a_3$ given in (f_2), (f_3) and (1.3).

Let us first recall some basic facts about the weighted Sobolev spaces that we will work with (compare, e.g., [11]). Given $l \ge 1$ and $\alpha \in \mathbb{R}$, we denote by $L^l(\Omega, |x|^{-\alpha})$ the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^l(\Omega,|x|^{-\alpha})} \equiv \int_\Omega |x|^{-\alpha}|u|^l\,\mathrm{d}x < +\infty.$$

If $a \in (-\infty, (N-2)/2)$, then let $W_0^{1,p}(\Omega, |x|^{-2a})$ denote the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| \equiv \int_\Omega |x|^{-2a}|\nabla u|^2\,\mathrm{d}x.$$

Let $l \in (1, 2N/(N-2))$ and $\alpha \le (1+a)l + N(1-(l/2))$. Then there is a constant $C_0 > 0$ such that

$$C_0\left(\int_\Omega |x|^{-\alpha}|u|^l\,\mathrm{d}x\right)^{2/l} \le \int_\Omega |x|^{-2a}|\nabla u|^2\,\mathrm{d}x. \tag{1.4}$$

The case $l = 2$ in (1.4) requires special attention in our analysis. Consider the weighted Rayleigh quotient

$$Q_{a,c}(v) := \frac{\int_\Omega |x|^{-2a} |\nabla v|^2 \, \mathrm{d}x}{\int_\Omega |x|^{-2(a+1)+c} v^2 \, \mathrm{d}x}, \ v \in W_0^{1,2}(\Omega, |x|^{-2a}), \ v \neq 0,$$

where $a \in (-\infty, (N-2)/2)$ and $c \geq 0$ and set

$$S(\Omega, a, c) := \inf\{Q_{a,c}(v) : v \in W_0^{1,2}(\Omega, |x|^{-2a}), \ v \neq 0\}. \tag{1.5}$$

If $c > 0$, then $S(\Omega, a, c)$ is equal to the first eigenvalue of the following problem

$$\begin{cases} -\mathrm{div}(|x|^{-2a} \nabla u) = \lambda |x|^{-2(a+1)+c} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

and $S(\Omega, a, c)$ is attained for any first eigenfunction of (1.6) (see [11]).

If $c = 0$, then $S(\mathbb{R}^N, a, 0) = ((N-2-2a)/2)^2$, but the infimum in (1.5) is not attained (see [3]). It is easy to see that this also implies that $S(\Omega, a, 0) = ((N-2-2a)/2)^2$, and that $S(\Omega, a, 0)$ is not attained. We are now in a position to formulate our main result.

Theorem 1.1. *Let Ω be a C^1-domain, and assume that either $0 < a < (N-2)/2$ and $c \geq 1$, or $a = 0$ and $c > 1$. Furthermore, suppose that f satisfies $(f_0), \ldots, (f_4)$. Then problem (1.1) has a positive and a negative solution in $W_0^{1,2}(\Omega, |x|^{-2a})$ provided that*

$$\frac{L_1}{S(\Omega, a, c)} + \frac{L_2}{\sqrt{S(\Omega, a, 2(c-1))}} < 1. \tag{1.7}$$

Our paper is organized as follows. In Section 2 we obtain regularity properties for the solutions of problem (1.2). The proof of Theorem 1.1 is given in Section 3.

2. A regularity result

In this section we prove boundedness and smoothness for solutions of problem (1.2).

Theorem 2.1. *Let $a \in (-\infty, (N-2)/2)$, $c > 0$, $M > 0$, $\beta := 2(a+1) - c$, $1 < q < \min\{(N+2)/(N-2); (N-2(a+1)+2c)/(N-2(a+1))\}$, and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$, a Caratheodory function such that*

$$|g(x,t)| \leq M(1 + |t|^q) \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{2.8}$$

Furthermore, let $u \in W_0^{1,2}(\Omega, |x|^{-2a})$ satisfy weakly

$$-\mathrm{div}\,(|x|^{-2a} \nabla u) = |x|^{-\beta} g(x,u) \qquad \text{in } \Omega. \tag{2.9}$$

Then there is a constant $C > 0$, depending only on N, Ω, a, c and q, such that

$$|u(x)| \leq MC \qquad \text{in } \Omega. \tag{2.10}$$

Moreover, there is a number $\alpha \in (0,1]$ such that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$. Finally, if $\partial\Omega \in C^{0,1}$ then $u \in C^{0,\alpha}(\overline{\Omega})$.

Proof. We introduce new coordinates

$$x = |y|^{k-1} y, \ x \in \mathbb{R}^N, \qquad \text{where } k = \frac{N-2}{N-2(a+1)},$$

and set $D := \{y : x \in \Omega\}$, $v(y) := u(x)$, $h(y,t) := g(x,t)$, $\forall (x,t) \in \Omega \times \mathbb{R}$. It is then easy to see that $v \in W_0^{1,2}(D)$, and v satisfies weakly

$$-\frac{\partial}{\partial y_i} a_i(y, \nabla v) = k^2 |y|^{-\gamma} h(y, v) \qquad \text{in } D, \tag{2.11}$$

where

$$a_i(y,\xi) = k^2 \xi_i + (1-k^2)\frac{y_i y_j \xi_j}{|y|^2} \quad \forall (y,\xi) \in D \times \mathbb{R}^N,$$

and $\gamma = 2 - c(N-2)/(N-2(a+1))$. Notice that $k > 0$,

$$\xi_i a_i(y,\xi) \geq \begin{cases} |\xi|^2 & \text{if } k \geq 1 \\ k^2|\xi|^2 & \text{if } k < 1 \end{cases} \text{ and} \tag{2.12}$$

$$\sqrt{\sum_{i=1}^N a_i^2(x,\xi)} \leq \begin{cases} |\xi| & \text{if } k < 1 \\ k^2|\xi| & \text{if } k \geq 1. \end{cases} \tag{2.13}$$

Now we write

$$d(y) := \frac{|y|^{-\gamma} h(y, v(y))}{1 + |v(y)|}, \ \forall y \in D.$$

Since $v \in L^{2N/(N-2)}(\Omega)$ by the Sobolev Embedding Theorem, and since $q < 1 + 2c/(N - 2(a+1))$, we find using Hölder's inequality and (2.8),

$$\begin{aligned} \int_D |d|^{N/2}\,\mathrm{d}y &\leq c_2 \int_D |y|^{-\gamma N/2}(1 + |v|^{(q-1)N/2}) \\ &\leq c_2 \left(\int_D |y|^{-\frac{2\gamma N}{4-(N-2)(q-1)}}\right)^{\frac{4-(N-2)(q-1)}{4}} \\ &\quad \times \left(c_3 + \left(\int_D |v|^{2N/(N-2)}\right)^{(N-2)(q-1)/4}\right) \\ &< +\infty, \end{aligned}$$

where c_2, c_3 are some positive constants. In other words, we have that

$$-(\partial/\partial y_i) a_i(y, \nabla v) = d(y)(1 + |v|) \qquad \text{in } D,$$

where $d \in L^{N/2}(D)$. We can now apply Lemma B3 of [10], p. 244 ff., to obtain that $v \in L^r(D)$ for every $r \geq 1$. (Notice that the above mentioned Lemma B3 has been formulated only for the Laplace operator, but its proof carries over without difficulty to the general case, due to the properties (2.12) and (2.13).) Hence we have that $|y|^{-\gamma} g(y, v(y)) \in L^\rho(D)$ for some $\rho > N/2$. The assertions then follow from [9]. □

Next we consider the problem

$$\begin{aligned} &u \in W_0^{1,2}(\Omega, |x|^{-2a}), \\ &-\mathrm{div}(|x|^{-2a}\nabla u) = |x|^{-2a-2+c} f(x) \qquad \text{in } \Omega, \end{aligned} \tag{2.14}$$

where $c \geq 1$ and $f \in L^\infty(\Omega)$. Notice first that if $c > 0$, then a result of [5], Theorem 1.1, tells us that u is bounded and $u \in C^{0,\alpha}(\Omega')$ for some $\alpha \in (0,1)$ and for every $\Omega' \subset\subset \Omega$. Our proof is based on a blow-up argument as used by Gidas and Spruck (see [6]), and requires the following Liouville-type result:

Theorem 2.2. *Let $a \in (-\infty, (N-2)/2)$ and*

$$m_1 = -\frac{N-2}{2} + a + \sqrt{\left(\frac{N-2}{2} - a\right)^2 + N - 1}. \tag{2.15}$$

Then, if $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N, |x|^{-2a})$ satisfies

$$-\mathrm{div}(|x|^{-2a}\nabla u) = 0 \quad \textit{and} \tag{2.16}$$

$$|u(x)| \le C(1 + |x|^{m_1-\varepsilon}) \quad \textit{on } \mathbb{R}^N, \tag{2.17}$$

for some $C > 0$, and $\varepsilon \in (0, m_1)$, it follows that u is constant on $\mathbb{R}^N$.

Proof. Let (r, θ) denote N-dimensional polar coordinates, ($r = |x|, \theta \in \mathcal{S}^{N-1}$), and let $\{v_n\}$ be the sequence of orthonormal eigenfunctions for the Laplace–Beltrami operator on $\mathcal{S}^{N-1}$, that is,

$$-\Delta_\theta\, v_k = \lambda_k v_k \qquad \text{on } \ \mathcal{S}^{N-1}, \ k = 0, 1, 2, \ldots, \tag{2.18}$$

$$\int_{\mathcal{S}^{N-1}} v_i v_j \,\mathrm{d}\theta = \delta_{ij}, \qquad i, j = 0, 1, 2, \ldots, \ \text{and} \tag{2.19}$$

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots. \tag{2.20}$$

Notice that $\lambda_0 = 0$, $v_0 = \text{const.} \neq 0$, $\lambda_1 = \cdots = \lambda_N = N-1$, $v_k = Cx_k/|x|$, $k = 1, \ldots, N$, for some $C > 0$, and the eigenvalues $\lambda (= \lambda_k)$ can be calculated from the relation

$$\lambda = n^2 + n(N-2), \qquad n = 0, 1, 2, \ldots.$$

Let $R > 0$, and let $b_k(R)$, $k = 0, 1, \ldots$, (unique!) numbers such that

$$u(R, \theta) = \sum_{k=0}^{+\infty} b_k(R) v_k(\theta), \quad \forall\, \theta \in \mathcal{S}^{N-1}. \tag{2.21}$$

Then we have the following representation of u (see [1], proof of Theorem 4.4),

$$u(r, \theta) = \sum_{k=0}^{+\infty} b_k(R) r^{m_k} v_k(\theta), \quad \forall\, r \in [0, R], \ \forall\, \theta \in \mathcal{S}^{N-1}, \tag{2.22}$$

$$\text{where} \quad m_k = -\frac{N-2}{2} + a + \sqrt{\left(\frac{N-2}{2} - a\right)^2 + \lambda_k}. \tag{2.23}$$

Since $R > 0$ is arbitrary, we have that

$$b_k(R) = c_k R^{m_k}, \tag{2.24}$$

for some numbers $c_k \in \mathbb{R}$, $k = 0, 1, 2, \ldots$. Using Parseval's identity on ∂B_R and assumption (2.17), we then find that

$$C(1 + R^{2m_1 - 2\varepsilon}) \ge \int_{\mathcal{S}^{N-1}} u^2(R, \theta)\,\mathrm{d}\theta = \sum_{k=0}^{+\infty} c_k^2 R^{2m_k} \quad \forall R > 0, \tag{2.25}$$

for some $C > 0$. Passing to the limit $R \to +\infty$, this gives $c_k = 0$ for $k \ge 1$. Hence u is constant on $\mathbb{R}^N$. $\square$

Lemma 2.1. *Let $a \in (-\infty, (N-2)/2)$, $c > 0$, $f \in L^\infty(\Omega)$, and let u be a solution of (2.14). Then for every $\delta > 0$ satisfying $\delta \le c$ and $\delta < m_1$, there is a number $c_1 > 0$ depending only on δ, c, a, N and Ω such that*

$$|u(x) - u(0)| \le c_1 M |x|^\delta \quad \forall x \in \Omega, \tag{2.26}$$

where $M := \|f\|_{L^\infty(\Omega)}$.

Proof. First assume that $M = 1$. Suppose that (2.26) is wrong. Then there is a number $\delta > 0$ with $\delta \le c$ and $\delta < m_1$ and a sequence $\{x_n\} \subset \Omega \setminus \{0\}$ with $x_n \to 0$ such that

$$\lim_{n\to\infty} |u(x_n) - u(0)||x_n|^{-\delta} = +\infty. \tag{2.27}$$

Define rotations ρ_n of the coordinate system about the origin such that $\rho_n x_n = (\varepsilon_n, 0, \dots, 0) =: y_n$, $(\varepsilon_n > 0)$, and let $\Omega_n := \rho_n \Omega$, $f_n(x) := f(\rho_n x)$, $u_n(x) := u(\rho_n x)$, $n = 1, 2, \dots$. We may assume w.l.o.g. that $\{\varepsilon_n\}$ is decreasing and

$$|u_n(x) - u_n(0)||x|^{-\delta} \le |u_n(y_n) - u_n(0)|\varepsilon_n^{-\delta} \quad \forall x \in \Omega_n \text{ with } |x| \ge \varepsilon_n. \tag{2.28}$$

Setting $D_n := \{(1/\varepsilon_n)x : x \in \Omega_n\}$, $g_n(x) := f_n(\varepsilon_n x)$, and

$$v_n(x) := \frac{u_n(\varepsilon_n x) - u_n(0)}{u_n(y_n) - u_n(0)},$$

we find $v_n(0) = 0$, $v_n(e) = 1$, where e is the unit vector $(1, 0, \dots, 0)$,

$$|v_n(x)| \le |x|^\delta \quad \text{in } D_n \setminus B_1, \tag{2.29}$$

$v_n \in W_0^{1,2}(D_n, |x|^{-2a})$, and

$$-\mathrm{div}(|x|^{-2a}\nabla v_n) = \frac{|x|^{-2a-2+c} g_n(x)\varepsilon_n^\delta}{u_n(\varepsilon_n e) - u_n(0)} =: h_n(x) \quad \text{in } D_n. \tag{2.30}$$

By (2.27) we have that

$$\lim_{n\to\infty} \frac{\varepsilon_n^\delta}{u_n(\varepsilon_n e) - u_n(0)} = 0,$$

so that

$$\lim_{n\to\infty} h_n(x) = 0 \quad \text{uniformly in any compact subset of } \mathbb{R}^N. \tag{2.31}$$

Furthermore, using elliptic estimates separately in B_1 and in $D_n \setminus B_1$, we find – see [5] – that the v_ns are uniformly bounded and $v_n \in C^{0,\alpha}(D')$ for some $\alpha \in (0,1)$, for every $D' \subset\subset D_n$. Hence, in view of (2.29)–(2.31), there is a subsequence $\{v_{n'}\}$ and a function $v \in W_{\mathrm{loc}}^{1,2}(\mathbb{R}^N, |x|^{-2a}) \cap C^{0,\alpha}(\mathbb{R}^N)$ such that

$$v_n' \longrightarrow v \quad \text{in } W^{1,2}(B_R, |x|^{-2a}) \quad \text{and in } C^{0,\alpha}(B_R),\ \forall R > 0, \tag{2.32}$$

$$\mathrm{div}\,(|x|^{-2a}\nabla v) = 0 \quad \text{on } \mathbb{R}^N, \tag{2.33}$$

$$|v(x)| \le |x|^\delta \quad \text{for } |x| \ge 1, \quad \text{and} \tag{2.34}$$

$$v(0) = 0, \quad v(e) = 1. \tag{2.35}$$

Using the previous theorem, conditions (2.33) and (2.34) imply that v must be constant, contradicting (2.35).

In the general case, the result follows from the above analysis, replacing u by $M^{-1}u$. □

We now prove the main results of this section.

Theorem 2.3. *Let $0 < a < (N-2)/2$, $c > 1$, $f \in L^\infty(\Omega)$, and let u be a solution of (2.14). Then $u \in C^{1,\beta}(\Omega)$ for every $\Omega' \subset\subset \Omega$, and for every $\beta \in (0,1)$ with $\beta \le c-1$ and $\beta < m_1 - 1$. Moreover, for every such β, and Ω', there is a constant c_2 depending only on c, β, a, and Ω' such that*

$$\|u\|_{C^{1,\beta}(\Omega')} \le c_2 M, \tag{2.36}$$

where $M := \|f\|_{L^\infty(\Omega)}$. Finally, if Ω is a $C^{1,\beta}$-domain, then $u \in C^{1,\beta}(\overline{\Omega})$ and (2.36) holds with Ω' replaced by $\overline{\Omega}$.

Proof. As in the proof of Lemma 2.1, we may assume that $M = 1$. First observe that standard regularity theory tells us (see, e.g., [7]) that

$$u \in C^{1,\alpha}(\Omega' \setminus \overline{B_\varepsilon}) \quad \text{for every } \Omega' \subset\subset (\Omega \setminus \{0\}) \text{ and } \forall \alpha \in (0,1). \tag{2.37}$$

Let $\delta \in (1, c]$ with $\delta < m_1$, and $\varepsilon_0 > 0$ such that $B_{4\varepsilon_0} \subset \Omega$, and $\varepsilon \in (0, \varepsilon_0)$. Setting $u_\varepsilon(x) := \varepsilon^{-\delta}(u(\varepsilon x) - u(0))$, $f_\varepsilon(x) := f(\varepsilon x)$, and $\Omega_\varepsilon = \{(1/\varepsilon)x : x \in \Omega\}$, we have that $u_\varepsilon \in W_0^{1,2}(\Omega_\varepsilon, |x|^{-2a})$, and

$$-\mathrm{div}(|x|^{-2a}\nabla u_\varepsilon) = |x|^{-2a-2+c} f_\varepsilon(x)\varepsilon^{c-\delta} \quad \text{in } \Omega_\varepsilon. \tag{2.38}$$

By the previous lemma, the u_εs are uniformly bounded. Hence, using elliptic estimates in $B_4 \setminus \overline{B_{1/2}}$, we obtain from (2.38) that for every $\alpha \in (0,1)$ there is a constant $c_2(\alpha)$, independent of ε such that

$$|\nabla u_\varepsilon(x) - \nabla u_\varepsilon(y)| \le c_2(\alpha)|x-y|^\alpha \quad \text{in } B_2 \setminus \overline{B_1},$$

which implies

$$|\nabla u(x) - \nabla u(y)| \le c_2(\alpha)|x-y|^\alpha \varepsilon^{\delta-1-\alpha} \quad \text{in } B_{2\varepsilon} \setminus \overline{B_\varepsilon}.$$

Choosing $\alpha \le \delta - 1$, this shows that

$$|\nabla u(x) - \nabla u(y)| \le c_2(\alpha)|x-y|^\alpha \quad \text{in } B_{2\varepsilon} \setminus \overline{B_\varepsilon}. \tag{2.39}$$

By the previous lemma and by (2.37) we have that $u \in C^1_{\mathrm{loc}}(\Omega)$ and $\nabla u(0) = 0$. Together with (2.37), this proves (2.36).

Finally, if Ω is a $C^{1,\beta}$-domain, then one has

$$u \in C^{1,\beta}(\overline{\Omega} \setminus B_\varepsilon) \quad \forall \varepsilon > 0. \tag{2.40}$$

This implies $u \in C^{1,\beta}(\overline{\Omega})$, by the above considerations. □

A slight modification of the above proof in the case $c = 1$ leads to the following:

Theorem 2.4. *Let $0 < a < (N-2)/2$, $c = 1$, $f \in L^\infty(\Omega)$, and let u be a solution of (2.14). Then $u \in C^{0,1}(\Omega')$ for every $\Omega' \subset\subset \Omega$. Moreover, there is a constant d_2 depending only on a and Ω' such that*

$$\|\nabla u\|_{L^\infty(\Omega')} \le d_2 M, \tag{2.41}$$

where M is as in Theorem 2.3. Finally, if Ω is a C^1-domain, then $u \in C^{0,1}(\overline{\Omega})$ and (2.41) holds with Ω' replaced by Ω.

Proof. We proceed similarly as in the previous proof. Notice first that u satisfies (2.26) with $\delta = 1$, and that (2.37). Moreover, if Ω is a C^1-domain, then one has

$$u \in C^{0,1}(\overline{\Omega} \setminus B_\varepsilon) \quad \forall \varepsilon > 0. \tag{2.42}$$

Choosing ε_0 as before and $\varepsilon \in (0, \varepsilon_0)$, we set $u_\varepsilon(x) := (u(\varepsilon x) - u(0))/\varepsilon$. Then we have that

$$-\mathrm{div}(|x|^{-2a}\nabla u_\varepsilon) = |x|^{-2a-1} f_\varepsilon(x) \quad \text{in } \Omega_\varepsilon. \tag{2.43}$$

Using elliptic estimates in $B_4 \setminus \overline{B_{1/2}}$, we see from (2.43) that there is a constant c_3 independent of ε, such that

$$|\nabla u_\varepsilon(x)| \le c_3 \quad \text{in } B_2 \setminus \overline{B_1}.$$

This implies

$$|\nabla u(x)| \le c_3 \quad \text{in } B_{2\varepsilon} \setminus \overline{B_\varepsilon}.$$

Now the assertion follows from the continuity of u and from (2.26) with $\delta = 1$ and (2.42). □

Remarks. (1) Let us briefly report about the well-known Laplacian case, $a = 0$. Notice that we cannot argue as in the proof of Theorem 2.3, since $m_1 = 1$.

Assume that $c \in (1, 2)$. Since $|x|^{-2a-2+c} f(x) \in L^p(\Omega)$ for every $p > N/(2-c)$, we have that $W^{2,p}_{\mathrm{loc}}(\Omega)$ for these p. By the Embedding Theorem this implies that $u \in C^{1,\beta}(\Omega')$ for every $\Omega' \subset\subset \Omega$, and $\beta \in (0, c-1)$. Moreover, if Ω is a $C^{1,\beta}$-domain then $u \in C^{1,\beta}(\overline{\Omega})$.

(2) We wish to demonstrate that the restrictions on the parameters a and c in the above Theorems 2.3 and 2.4 are optimal.

Choose $R_0 > 0$ such that $B_{R_0} \subset\subset \Omega$, and let $a \in (-\infty, (N-2)/2)$, and $c_i > 0$, $i = 1, 2, 3$, with $c_3 = m_1$, where m_1 is given by (2.15). Setting $u_1(x) = |x|^{c_1}$, $u_2(x) = x_1|x|^{m_1-1}$, and $u_3(x) = x_1|x|^{m_1-1}\log|x|$ in B_{R_0}, we have that

$$-\mathrm{div}(|x|^{-2a}\nabla u_i) = |x|^{-2a-2+c_i} f_i(x) \quad \text{in } B_{R_0}, \quad i = 1, 2, 3, \tag{2.44}$$

where $f_1(x) = -c_1(N + c_1 - 2 - 2a)$, $f_2(x) \equiv 0$, and

$$f_3(x) = -2x_1|x|^{-1}\sqrt{((N-2-2a)/2)^2 + N - 1}.$$

Clearly we may continue u_i to a function in $C^2(\overline{\Omega} \setminus \{0\})$ with compact support in Ω, and such that u_i is a solution of problem (2.14) with right-hand side $|x|^{-2a-2+c_i} f_i(x)$, where $f_i \in L^\infty(\Omega)$, $i = 1, 2, 3$. The examples show that an estimate (2.26) with $\delta \ge m_1$ or with $\delta > c$ does not hold in general. In particular, the first example, u_1, shows that one cannot expect a solution of (2.14) to be Lipschitz continuous if $c < 1$. Moreover, if $a < 0$ then we have that $m_1 < 1$, so that u_2 provides an example of a solution of (2.14) that is not Lipschitz continuous. Finally, a counter example for Lipschitz continuity in the case $a = 0$ and $c = 1$ is given by u_3.

3. Truncation argument

From now on, we assume that Ω, a and c are as in Theorem 1.1. In order to obtain a solution of (1.2) we first consider a truncated problem. Fix some number $R > 0$. Then let

$$f_R(x, t, \xi) = f(x, t, \xi\varphi_R(\xi)), \quad \text{and}$$

$$F_R(x, t, \xi) = \int_0^t f_R(x, \tau, \xi)\,\mathrm{d}\tau \qquad \forall (x, t, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N,$$

where $\varphi_R \in C^1(\mathbb{R}^N)$ and satisfies the following conditions

$$\begin{cases} |\varphi_R(\xi)| \le 1 & \forall \xi \in \mathbb{R}^N, \\ \varphi_R(\xi) = 1 & \forall |\xi| \le R \\ \varphi_R(\xi) = 0 & \forall |\xi| \ge R+1. \end{cases} \tag{3.45}$$

Furthermore, for any fixed $w \in W_0^{1,2}(\Omega, |x|^{-2a})$ we define a functional $I_w^R : W_0^{1,2}(\Omega, |x|^{-2a}) \to \mathbb{R}$ by

$$I_w^R(v) = \frac{1}{2}\int_\Omega |x|^{-2a}|\nabla v|^2 - \int_\Omega |x|^{-2(a+1)+c} F_R(x, v, \nabla w).$$

The critical points u_w^R of I_w^R are weak solutions of the semi-linear elliptic problem

$$\begin{cases} -\mathrm{div}(|x|^{-2a}\nabla u_w^R) = |x|^{-2(a+1)+c} f_R(x, u_w^R, \nabla w) & \text{in } \Omega \\ u_w^R = 0 & \text{on } \partial\Omega. \end{cases}$$

Our aim is to show that the functional I_w^R has a structure of Mountain Pass type for any $w \in W_0^{1,2}(\Omega, |x|^{-2a})$. Indeed one can state the following two lemmata.

Lemma 3.1. *For every $R > 0$ there exist positive numbers $\rho < 1$ and α such that*

$$\begin{aligned} I_w^R(v) \ge \alpha \qquad & \forall\, w \in W_0^{1,2}(\Omega, |x|^{-2a}) \quad \textit{and} \\ & \forall\, v \in W_0^{1,2}(\Omega, |x|^{-2a}) \quad \textit{satisfying } \|v\| = \rho. \end{aligned} \tag{3.46}$$

Lemma 3.2. *There exists some $\overline{v} \in W_0^{1,2}(\Omega, |x|^{-2a})$ with $\overline{v} \ge 0$, $\|\overline{v}\| > 1$ such that*

$$I_w^R(\overline{v}) < 0 \qquad \forall\, R > 0 \textit{ and } \forall\, w \in W_0^{1,2}(\Omega, |x|^{-2a}). \tag{3.47}$$

Proof of Lemma 3.1. It follows from (f_1) and (f_2) that there is a positive constant k_ε that depends only on ε, such that

$$|F_R(x, t, \xi)| \le \frac{\varepsilon t^2}{2} + k_\varepsilon (R+2)^r |t|^{p+1}.$$

In view of (1.4) we have that

$$\begin{aligned} \int_\Omega |x|^{-2(a+1)+c} F_R(x, v, \nabla w) \ &\le\ \frac{\varepsilon}{2}\int_\Omega |x|^{-2(a+1)+c} v^2 \\ &\quad + k_\varepsilon (R+2)^r \int_\Omega |x|^{-2(a+1)+c} |v|^{p+1} \\ &\le\ C\left(\frac{\varepsilon}{2} + k_\varepsilon (R+2)^r \|v\|^{p-1}\right)\|v\|^2, \end{aligned} \tag{3.48}$$

for some constant $C > 0$. Now, choosing

$$\|v\| < \left(\frac{\varepsilon}{2k_\varepsilon (R+2)^r}\right)^{\frac{1}{p-1}}$$

in the above inequality, one gets

$$\int_\Omega |x|^{-2(a+1)+c} F_R(x, v, \nabla w) \le C\varepsilon \|v\|^2,$$

so that (3.46) easily follows by taking $\varepsilon < (2C)^{-1}$, $\rho < \min\{1; (4k_\varepsilon(R+2)^r C)^{-1/(p-1)}\}$ and $\alpha = (\frac{1}{2} - C\varepsilon)\rho^2$. $\square$

Proof of Lemma 3.2. We fix some function $v_0 \in W_0^{1,2}(\Omega, |x|^{-2a})$, with $v_0 \geq 0$, $v_0 \neq 0$. By (1.3) one gets, for any $t > 0$,

$$I_w^R(tv_0) \leq \frac{t^2}{2}\int_\Omega |x|^{-2a}|\nabla v_0|^2 - a_2 \int_\Omega |x|^{-2(a+1)+c} t^\theta |v|^\theta + \widetilde{a_3},$$

where $\widetilde{a_3} = a_3 \int_\Omega |x|^{-2(a+1)+c}$. Then we choose $\overline{v} = \overline{t}v_0$ with $\overline{t}$ sufficiently large such that $\|\overline{v}\| > 1$ and $I_w^R(\overline{v}) < 0$ for all $R > 0$. $\square$

Proposition 3.1. *Let $(f_0), \ldots, (f_3)$ be satisfied and let $w \in W_0^{1,2}(\Omega, |x|^{-2a})$ and $\overline{v}$ be given by Lemma 3.2. Then, for every $R > 0$, there exists some $v = v(w, R)$ such that*

$$\begin{aligned} &D(I_w^R)(v) = 0 \qquad \text{and} \\ &I_w^R(v) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_w^R, \end{aligned} \tag{3.49}$$

where

$$\Gamma = \{\gamma \in C^0([0,1]; W_0^{1,2}(\Omega, |x|^{-2a})) : \gamma(0) = 0,\ \gamma(1) = \overline{v}\}. \tag{3.50}$$

Proof. We have that $I_w^R(0) = 0$. Furthermore, the functional I_w^R satisfies the (PS)-condition in view of $(f_0), \ldots, (f_3)$. Then the existence of an element v such that (3.49) and (3.50) hold is an immediate consequence of the Lemmatas 3.1 and 3.2 and of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz (see [2]). $\square$

Next, we will obtain a positive and a negative solution of (1.1). To this end, we fix an arbitrary element $u_0 \in W_0^{1,2}(\Omega, |x|^{-2a})$ and $R > 0$, and we consider the following iterative scheme:

$$\begin{aligned} &\text{Given } n \in \mathbb{N},\ \text{fix an element } v = u_n^R \text{ satisfying (3.49) and (3.50) with} \\ &w = u_{n-1}^R. \end{aligned} \tag{3.51}$$

Notice that the elements u_n^R above are not unique in general. Now we obtain a uniform estimate from above for the $W_0^{1,2}(\Omega, |x|^{-2a})$-norms of u_n^R. This will finally allow us to get rid of the dependence on R, and to pass to the following iteration scheme:

$$(P)_n \qquad \begin{cases} -\mathrm{div}(|x|^{-2a}\nabla u_n) = |x|^{-2(a+1)+c} f(x, u_n, \nabla u_{n-1}) & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 3.3. *There exists a positive constant c_1 such that*

$$\|u_n^R\| \leq c_1 \tag{3.52}$$

for every $n \in \mathbb{N}$ and $R > 0$.

Proof. Using the definition of u_n^R and choosing the path in Γ given by the line segment joining 0 and $\overline{v}$, one gets from (1.3)

$$I_{u_{n-1}^R}^R(u_n^R) \leq \sup_{t \geq 0} \left\{ \frac{t^2}{2}\int_\Omega |x|^{-2a}|\nabla \overline{v}|^2 - a_2 t^\theta \int_\Omega |x|^{-(a+1)p+c}|\overline{v}|^\theta + \widetilde{a_3} \right\},$$

where $\widetilde{a_3}$ is defined in the proof of Lemma 3.2. Since $\theta > 2$, the function

$$\mathbb{R}_+ \ni t \mapsto \frac{t^2}{2}\int_\Omega |x|^{-2a}|\nabla\overline{v}|^2 - a_2 t^\theta \int_\Omega |x|^{-(a+1)p+c}|\overline{v}|^\theta + \widetilde{a_3}$$

attains a positive maximum. Hence

$$I^R_{u^R_{n-1}}(u^R_n) \le \text{const} \qquad \forall\, n \in \mathbb{N} \text{ and } \forall\, R > 0. \tag{3.53}$$

Now (3.53), (f_3), the fact that $|\varphi_R| \le 1$, and the criticality of u^R_n for $I^R_{u^R_{n-1}}$ imply

$$\begin{aligned} \frac{1}{2}\|u^R_n\|^2 &\le \text{const} + \frac{1}{\theta}\int_\Omega |x|^{-(a+1)p+c} f_R(x, u^R_n, \nabla u^R_{n-1})u^R_n \\ &= \text{const} + \frac{1}{\theta}\|u^R_n\|^2, \end{aligned}$$

and (3.52) follows in view of $\theta > 2$. $\square$

Using the results of Section 2, we now obtain uniform estimates for the L^∞-norms of $\{u^R_n\}$ and $\{\nabla u^R_n\}$, by assuming additionally that

$$u^R_0 \in C^{0,1}(\overline{\Omega}) \quad \text{for every } R > 0. \tag{3.54}$$

Lemma 3.4. *Assume (3.54). Then, for every $n \in \mathbb{N}$ and $R > 0$, $u^R_n \in C^{0,1}(\overline{\Omega})$.*

Proof. We have that u^R_1 is the weak solution of

$$\begin{cases} -\operatorname{div}(|x|^{-2a}\nabla u^R_1) = f_R(x, u^R_1, \nabla u^R_0) & \text{in } \Omega, \\ u^R_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since

$$|f_R(x, u^R_1, \nabla u^R_0)| \le M(1 + |u^R_1|^p)(2 + R)^r,$$

that is, $\|f_R(x, u^R_1, \nabla u^R_0)\|_{L^\infty(\Omega)} \le \widetilde{M}(2 + R)^r$, we may apply Theorem 2.1. Hence $u^R_1 \in C^{0,\alpha}(\overline{\Omega})$. In view of Theorems 2.3 and 2.4 and the remark at the end of Section 2, this means that u^R_1 is Lipschitz continuous on Ω, for any $R > 0$. Our result now follows by induction. $\square$

Lemma 3.5. *Assume (3.54). Then there exist $\mu_0 > 0$ and $\mu_1 > 0$ such that*

$$\|u^R_n\|_{L^\infty(\Omega)} \le k_0 = \mu_0(R+2)^r, \tag{3.55}$$

$$\|\nabla u^R_n\|_{L^\infty(\Omega)} \le k_1 = \mu_1(R+2)^r \qquad \forall\, R > 0 \text{ and } \forall\, n \in \mathbb{N}. \tag{3.56}$$

Proof. Recall that any Lipschitz function is a.e. differentiable with bounded gradient. Then, arguing as in Lemma 3.4, the condition (f_2) and the definition of f_R yield the estimates (3.55) and (3.56). $\square$

Lemma 3.6. *Assume (3.54). Then there exists some $\overline{R} > 0$, such that*

$$\|u^{\overline{R}}_n\|_{L^\infty(\Omega)} \le k_0 = \mu_0(\overline{R}+2)^r \le \overline{R}, \tag{3.57}$$

$$\|\nabla u^{\overline{R}}_n\|_{L^\infty(\Omega)} \le k_1 = \mu_1(\overline{R}+2)^r \le \overline{R}. \tag{3.58}$$

Proof. (3.57) and (3.58) are an obvious consequence of (3.55) and (3.56) and the fact that $r \in (0, 1)$. $\square$

Lemma 3.7. *Assume* (3.54). *Then* $u_n := u_n^{\overline{R}}$ *is a solution of* $(P)_n$ *and the following estimates hold, for any* $n \in \mathbb{N}$,

$$\|u_n\| \le c_1, \tag{3.59}$$

$$\|u_n\|_{L^\infty(\Omega)} \le k_0 = \mu_0(\overline{R}+2)^r \tag{3.60}$$

$$\|\nabla u_n\|_{L^\infty(\Omega)} \le k_1 = \mu_1(\overline{R}+2)^r. \tag{3.61}$$

Proof. The fact that u_n solves $(P)_n$, is a consequence of the definition of f_R and the assumptions (3.45) and (3.57) with $R = \overline{R}$. Moreover, (3.52), (3.57) and (3.58), respectively, imply (3.59)–(3.61) with $R = \overline{R}$. $\square$

The function u_n given in Lemma 3.7 is a *nontrivial* solution of $(P)_n$. More precisely, there holds

Lemma 3.8. *For any* $n \in \mathbb{N}$, *there exists a positive constant* c_2 *such that*

$$\|u_n\| \ge c_2. \tag{3.62}$$

Proof. For any $v \in W_0^{1,2}(\Omega, |x|^{-2a})$ we have that

$$\int_\Omega |x|^{-2a}\nabla u_n \nabla v = \int_\Omega |x|^{-(a+1)p+c} f(x, u_n, \nabla u_{n-1}).$$

Setting $v = u_n$ in the relation above, we obtain that

$$\int_\Omega |x|^{-2a}|\nabla u_n|^2 = \int_\Omega |x|^{-(a+1)p+c} f(x, u_n, u_{n-1})u_n.$$

Hence (f_1) and (f_2) imply that, for any $\delta > 0$, there exists a number $c(\delta) > 0$ such that

$$\begin{aligned}\int_\Omega |x|^{-2a}|\nabla u_n|^2 &\le \delta \int_\Omega |x|^{-(a+1)p+c}|u_n|^2 + c(\delta)\int_\Omega |x|^{-(a+1)p+c}|u_n|^{p+1}\\ &\le C(\delta\|u_n\|^2 + c(\delta)\|u_n\|^{p+1}),\end{aligned}$$

for any $n \in \mathbb{N}$ and for some constant $C > 0$. Now (3.62) follows, by choosing $\delta C < 1$. $\square$

Lemma 3.9. *Let*

$$\overline{k_0} := \min\{k_0 > 0 : \text{(3.60) } \textit{holds}\}$$

$$\overline{k_1} := \min\{k_1 > 0 : \text{(3.61) } \textit{holds}\},$$

and choose $\rho_1 = \overline{k_0}$ *and* $\rho_2 = \overline{k_1}$ *in* (f_4). *Then the sequence* $\{u_n\}$ *converges strongly in* $W_0^{1,2}(\Omega, |x|^{-2a})$.

Proof. By the criticality of u_{n+1} and u_n, one has, for every $n \in \mathbb{N}$,

$$\int_\Omega |x|^{-2a}\nabla u_{n+1}(\nabla(u_{n+1}-u_n)) = \int_\Omega |x|^{-2(a+1)+c} f(x, u_{n+1}, \nabla u_n)(u_{n+1}-u_n), \tag{3.63}$$

$$\int_\Omega |x|^{-2a}\nabla u_n(\nabla(u_{n+1}-u_n)) = \int_\Omega |x|^{-2(a+1)+c} f(x, u_n, \nabla u_{n-1})(u_{n+1}-u_n). \tag{3.64}$$

Subtracting (3.64) from (3.63), we obtain that

$$\begin{aligned}\|u_{n+1}-u_n\|^2 = \int_\Omega |x|^{-2(a+1)+c}\{&[f(x, u_{n+1}, \nabla u_n) - f(x, u_n, \nabla u_n)](u_{n+1}-u_n)\\ &+[f(x, u_n, \nabla u_n) - f(x, u_n, \nabla u_{n-1})](u_{n+1}-u_n)\}.\end{aligned}$$

Using hypothesis (f_4), this leads to the following estimate,

$$\|u_{n+1}-u_n\|^2 \le L_1 \int_\Omega |x|^{-2(a+1)+c} |u_{n+1}-u_n|^2 + L_2 \int_\Omega |x|^{-2(a+1)+c} |\nabla(u_n-u_{n-1})||u_{n+1}-u_n|. \tag{3.65}$$

Using Cauchy–Schwarz and singular Poincaré inequalities, and since $c \ge 1$, we have from (3.65),

$$\|u_{n+1}-u_n\|^2 \le L_1 S(\Omega,a,c)^{-1} \|u_{n+1}-u_n\|^2 + L_2 S(\Omega,a,2(c-1))^{-1/2} \|u_{n+1}-u_n\| \, \|u_n-u_{n-1}\|.$$

This means that

$$\|u_{n+1}-u_n\| \le \frac{L_2 S(\Omega,a,2(c-1))^{-1/2}}{1-L_1 S(\Omega,a,c)^{-1}} \|u_n-u_{n-1}\| =: k\|u_n-u_{n-1}\|.$$

By our assumptions, we have $k < 1$. Hence the sequence $\{u_n\}$ converges in $W_0^{1,2}(\Omega,|x|^{-2a})$ to some function $u \in W_0^{1,2}(\Omega,|x|^{-2a})$. Furthermore, since $\|u_n\| \ge c_2$ by Lemma 3.8, it follows that $u \ne 0$. In this way we obtain a nontrivial solution of (1.1). $\square$

Lemma 3.10. *Problem $(P)_n$ has a positive solution u_n^+ and a negative solution u_n^-. Moreover, the sequences $\{u_n^+\}$ and $\{u_n^-\}$ converge strongly in $W_0^{1,2}(\Omega,|x|^{-2a})$.*

Proof. We consider only the case of the positive solution. The argument leading to a negative solution is analogous. We replace the function $f(x,t,\xi)$ in (1.1) by the function

$$f^+(x,t,\xi) = \begin{cases} 0 & \text{if } f(x,t,\xi) < 0 \\ f(x,t,\xi) & \text{if } f(x,t,\xi) \ge 0. \end{cases}$$

Of course, f^+ satisfies (f_3) only for $t \ge 0$. But this is of no importance if we choose $v_0 > 0$ in the proof of Lemma 3.2. Indeed, proceeding analogously as before, we obtain a solution of the problem

$$\begin{cases} -\operatorname{div}(|x|^{-2a}\nabla u_n) = f^+(x,u_n^+,\nabla u_{n-1}^+) & \text{in } \Omega, \\ u_n^+ = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying the differential equation by the negative part of u_n and integrating by parts, we conclude that u_n is positive, that is, $u_n^+ = u_n$. $\square$

Proof of Theorem 1.1. The proof is a direct consequence of the Lemmatas 3.9 and 3.10. $\square$

Acknowledgements

The first author was supported by FONDECYT 1050412. The second author was supported by a CONICYT grant. The third author author was supported by FONDECYT 1040990.

References

[1] B. Abdellaoui, I. Peral, On quasilinear elliptic equations related to some Caffarelli–Kohn–Nirenberg inequalities, Commun. Pure Appl. Anal. 2 (4) (2003) 539–566.

[2] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.

[3] F. Catrina, Z.-Q. Wang, On the Caffarelli–Kohn–Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions, Commun. Pure Appl. Math. 54 (2001) 229–258.
[4] D. De Figueiredo, M. Girardi, M. Matzeu, Semilinear elliptic equations with dependence on the gradient via mountain-pass techniques, Diff. Integral Equ. (in press).
[5] V. Felli, M. Schneider, A note on regularity of solutions to degenerate elliptic equations of Caffarelli–Kohn–Nirenberg type, Adv. Nonlinear Stud. 3 (4) (2003) 431–443.
[6] B. Gidas, J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, Commun. Partial Differential Equations 6 (8) (1981) 883–901.
[7] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd edition, Springer-Verlag, Berlin, 2001.
[8] M. Girardi, M. Matzeu, Positive and negative solutions of a quasi-linear elliptic equation by a Mountain Pass method and truncature techniques, Nonlinear Anal. 59 (2004) 199–210.
[9] O.A. Ladyshenskaya, N.N. Ural'tseva, Linear and Quasilinear Equations of Elliptic Type, Izdat. Nauka, Moscow 1964; English translation: Academic Press, NY, 1968.
[10] M. Struwe, Variational Methods, 3rd edition, Springer-Verlag, Berlin, 2000.
[11] B.J. Xuan, The eigenvalue problem of a singular quasilinear elliptic equation, Electron. J. Differential Equations 2004 (16) (2004) 1–11.

Further reading

[1] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, in: C.B.M.S. Regional Confer. Ser. in Math, No. 65, Am. Math. Soc., Providence, RI, 1986.