Growth of groups and diffeomorphisms of the interval

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Abstract. We prove that, for all $\alpha > 0$, every finitely generated group of $C^{1+\alpha}$ diffeomorphisms of the interval with sub-exponential growth is almost nilpotent. Consequently, there is no group of $C^{1+\alpha}$ interval diffeomorphisms having intermediate growth. In addition, we show that the $C^{1+\alpha}$ regularity hypothesis for this assertion is essential by giving a C^1 counter-example.

Introduction

A theory for groups of diffeomorphisms of the interval has been extensively developed by many authors (see for example [19, 22, 23, 28, 31, 34, 35, 36, 37, 38, 40]). One of the most interesting topics of this theory is the interplay between the differentiability class of the diffeomorphisms and the algebraic (as well as dynamical) properties of the group (and the action). For instance, as a consequence of the classical Bounded Distortion Principle, groups of C^2 diffeomorphisms appear to have a very rigid behavior. This is no longer true for subgroups of $\mathrm{Diff}^1_+([0,1])$, as it is well illustrated in the literature [6, 26, 36]. Major progress has recently been made in the understanding of the lost of rigidity in intermediate differentiability classes between C^2 and C^1 (see [5]). The object of this work is to study the latter phenomenon for a remarkable class of groups, first introduced by R. Grigorchuk.

Given a finitely generated group (provided with a finite and symmetric system of generators), the growth function assigns to each positive integer n the number of elements of the group that may be written as a product of no more than n generators. One says that the group has polynomial, exponential or intermediate growth, if its growth function has the corresponding asymptotic behaviour. (These notions do not depend on the choice of the finite system of generators.) A celebrated theorem by M. Gromov establishes that a group has polynomial growth if and only if it is almost nilpotent, *i.e.* if it contains a finite index nilpotent subgroup (see [12] and references therein). Typical examples of groups with exponential growth are those that contain free semi-groups on two generators. (However, there exist groups with exponential growth and no free semi-group on two generators; see [25].) The difficult question (raised by J. Milnor [21]) concerning the existence of groups with intermediate growth was positively answered by R. Grigorchuk in [9] (see also [10]). Some years later, one of his examples was realized (by R. Grigorchuk himself and A. Maki [11]) as a subgroup of $\mathrm{Homeo}_+([0,1])$. The problem of improving the regularity for this embedding is at the core of this work. In the first part of this article we prove the following.

Theorem A. *There exists a finitely generated subgroup of* $\mathrm{Diff}^1_+([0,1])$ *with intermediate growth.*

This theorem solves by the negative a conjecture of [11]. In fact, the group we consider turns out to be isomorphic to the group introduced by R. Grigorchuk in [10] and studied in more detail in [11]. In the sequel, we will denote this group by H. We will prove more generally that, for every C^1-neighborhood V of the identity map of $[0,1]$, there exists an embedding $H \hookrightarrow \mathrm{Diff}^1_+([0,1])$ sending some canonical system of generators of H into V. This last issue is interesting because it is known for instance that subgroups of $\mathrm{Diff}^2_+(\mathrm{S}^1)$ generated by elements near the identity (with respect to the C^2-topology) have very restrictive dynamical properties [24].

The proof of Theorem A has two main technical ingredients. One is that, instead of embedding directly H into $\mathrm{Diff}^1_+([0,1])$ (which seems to be very difficult), we construct a coherent sequence of embeddings of some almost nilpotent groups H/H_n which in some sense converge to H. (The group H turns out to be residually almost nilpotent.) An equicontinuity argument allows us to obtain, at the limit, the desired embedding. However, in order to apply this argument, it is necessary to ensure a uniform control for the derivatives of the generators of each group in the afore mentioned sequence with respect to some fixed modulus of continuity. To do this, the other ingredient of the construction is to use a technique inspired by Chapter X of M. Herman's thesis [16]. This is related to the classical construction of $C^{1+\alpha}$ Denjoy counter-examples which, as explained to the author by F. Sergeraert, seems to go back to J. Milnor.

The preceding method of proof is quite natural because the dynamics of (the canonical action of) the group H has *infinitely many levels* (in the sense of [14]; see also [3, Section 8.3]), and a certain amount of regularity is lost when passing from one level to another (*i.e.* from the embedding of H/H_n to that of H/H_{n+1}). In particular, at the limit we do not obtain an inclusion of H into $\mathrm{Diff}_+^{1+\alpha}([0,1])$ for any $\alpha > 0$. (We get an uniform control for the derivatives of the generators only with respect to a logarithmic modulus of continuity.) And indeed, this issue is impossible, because of the following theorem.

Theorem B. *For all $\alpha > 0$ every finitely generated subgroup of $\mathrm{Diff}_+^{1+\alpha}([0,1])$ with sub-exponential growth is almost nilpotent.*

This result is proved in the second part of this article (which is essentially independent of the first part), and holds more generally for (finitely generated) groups without free semi-groups on two generators. The proof relies on the rigidity theory for centralizers of diffeomorphisms of the interval. The foundations of this theory are related to the so-called Kopell Lemma [19] (each Abelian subgroup of $\mathrm{Diff}_+^2([0,1[)$ either acts freely on $]0,1[$ or has a global fixed point therein), and Szekeres' theorem [40] (the centralizer in $\mathrm{Diff}_+^1([0,1[)$ of every element of $\mathrm{Diff}_+^2([0,1[)$ without fixed points in $]0,1[$ is conjugate to the group of translations). For other classes of groups there is Plante-Thurston Theorem [28] (nilpotent subgroups of $\mathrm{Diff}_+^2([0,1[)$ are Abelian), and the classification of solvable subgroups of $\mathrm{Diff}_+^2([0,1[)$ obtained by the author in [23] (see also [22]). Remark, however, that in the afore mentioned results, a C^2 regularity hypothesis is always assumed. (Or at least it is supposed that the maps are C^1 with derivatives having finite total variation.) The possibility of obtaining a result like Theorem B in intermediate regularity class was first suggested in [5]. Let us mention that it is relatively simple to adapt the methods of proof to show that finitely generated subgroups of $\mathrm{Diff}_+^{1+\alpha}(\mathbb{R})$ or $\mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$ with sub-exponential growth (or without free semi-groups on two generators) are also almost nilpotent.

Although no non Abelian nilpotent group can be contained in $\mathrm{Diff}_+^2([0,1[)$, a result of [6] establishes that every finitely generated torsion free nilpotent group can be seen as a subgroup of $\mathrm{Diff}_+^1([0,1])$. Using the methods of [36], it seems that the regularity of these inclusions can be improved up to the class $C^{1+\alpha}$ for every $\alpha < 1/(k-1)$, where k is the nilpotence degree of the corresponding group. However, as a consequence of one of the main results of [5], these actions cannot be made $C^{1+\alpha}$ for any $\alpha > 1/(k-1)$. (And the same should be true for $\alpha = 1/(k-1)$.) Quite surprisingly, the proof of Theorem B is somehow different than the proof of this last statement. On the one hand it does not use the probabilistic techniques introduced in [5] (see also [18]) to get control of distortion estimates in sharp intermediate differentiability classes. However, since the topological dynamics for the action is not prescribed *a priori*, it needs of an accurate study of the combinatorial properties for continuous actions of subexponential growth groups on the interval.

As it was previously recalled, among finitely generated groups the almost nilpotent ones are exactly those whose growth is polynomial [12]. Hence, Theorem B can be restated by saying that finitely generated sub-exponential growth subgroups of $\mathrm{Diff}_+^{1+\alpha}([0,1])$ have polynomial growth. As a direct corollary we obtain the following result, which implies that the conjecture of [11] was true up to some $\alpha > 0$!

Corollary. *For all $\alpha > 0$ there is no finitely generated subgroup of $\mathrm{Diff}_+^{1+\alpha}([0,1])$ having intermediate growth.*

We conclude this Introduction with some remarks. According to the comments after Proposition 5.15 of [5], Grigorchuk-Maki's group H seems to be the first example of a group of C^1 diffeomorphisms of the interval which cannot be seen as a group of $C^{1+\alpha}$ interval diffeomorphisms. On the other hand, it is well known that intermediate growth groups cannot appear as subgroups of Lie groups. These facts show that $\mathrm{Diff}_+^{1+\alpha}([0,1])$ is more appropriate than $\mathrm{Diff}^1([0,1])$ as an infinite dimensional model of a Lie group. Finally, note that the statements of Theorems A and B suggest a certain relationship with the classical Pesin Theory for diffeomorphisms with hyperbolic properties (see for instance [17]). Although these theorems seem to be of a different nature (our growth condition concerns the group and not the dynamics, and the control of distortion in class $C^{1+\alpha}$ is not related to any hyperbolicity of the action), possible further developments of a Pesin like theory for group actions could lead to a nice framework where these results appear as natural pieces.

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1 A group of C^1 interval diffeomorphisms with intermediate growth

The first half of this article is devoted to the proof of Theorem A. In §1.1, we recall the definition of the first Grigorchuk's group G as well as Grigorchuk-Maki's group H mainly as groups acting on spaces of sequences. In §1.2, we introduce several procedures for obtaining natural actions of H by interval homeomorphisms. In §1.3, we study in detail the analytic properties for one of these procedures for getting some technical but quite useful estimates. Finally, in §1.4, we use these analytic estimates to obtain a faithful action of H by C^1 diffeomorphisms of $[0,1]$.

1.1 Continuous actions on the interval and the Cantor set

First Grigorchuk's group G can be seen in many different ways: as the group generated by a finite automaton, as a group acting on the binary rooted tree $\mathcal{T}_2$, and as a group acting isometrically on the Cantor set $\{0,1\}^{\mathbb{N}}$. The last two points of view are essentially the same, since the boundary at infinity of $\mathcal{T}_2$ can be identified with $\{0,1\}^{\mathbb{N}}$. Using the convention $(l_1,(l_2,l_3,\ldots)) = (l_1,l_2,l_3,\ldots)$ for $l_i \in \{0,1\}$, the generators of G are the elements $\bar{\mathtt{a}},\bar{\mathtt{b}},\bar{\mathtt{c}},\bar{\mathtt{d}}$ whose actions on sequences $(l_1,l_2,l_3,\ldots)$ in $\{0,1\}^{\mathbb{N}}$ are defined recursively by

$$\bar{\mathtt{a}}(l_1,l_2,l_3,\ldots) = (1-l_1,l_2,l_3,\ldots),$$

$$\bar{\mathtt{b}}(l_1,l_2,l_3,\ldots) = \begin{cases} (l_1,\bar{\mathtt{a}}(l_2,l_3,\ldots)), & l_1 = 0, \\ (l_1,\bar{\mathtt{c}}(l_2,l_3,\ldots)), & l_1 = 1, \end{cases}$$

$$\bar{\mathtt{c}}(l_1,l_2,l_3,\ldots) = \begin{cases} (l_1,\bar{\mathtt{a}}(l_2,l_3,\ldots)), & l_1 = 0, \\ (l_1,\bar{\mathtt{d}}(l_2,l_3,\ldots)), & l_1 = 1, \end{cases}$$

$$\bar{\mathtt{d}}(l_1,l_2,l_3,\ldots) = \begin{cases} (l_1,l_2,l_3,\ldots), & l_1 = 0, \\ (l_1,\bar{\mathtt{b}}(l_2,l_3,\ldots)), & l_1 = 1. \end{cases}$$

The action on $\mathcal{T}_2$ of the element $\bar{\mathtt{a}} \in G$ consists of permuting the first two edges (and consequently, the trees rooted on the final vertex of each one of those edges). Elements $\bar{\mathtt{b}}$, $\bar{\mathtt{c}}$ and $\bar{\mathtt{d}}$ fix the first two edges of $\mathcal{T}_2$, and their action on higher levels is illustrated in Figure 1 below.

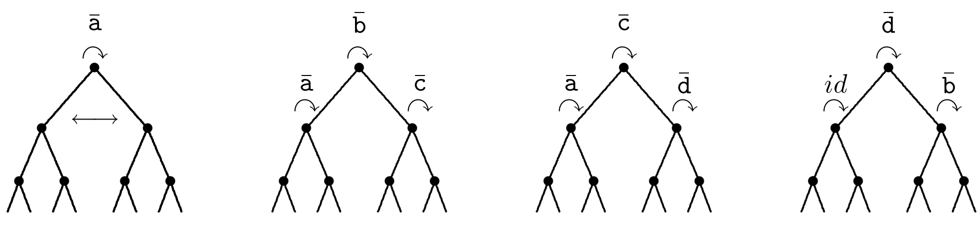

Figure 1

It can be shown that G is a torsion group: each element has order a power of 2 (see [8] or [13]). The first example of a torsion free group with intermediate growth was given in [10]. Geometrically, the idea consists of replacing $\mathcal{T}_2$ by a rooted tree having vertices of infinite (countable) degree. In other terms, we consider the group H acting on the space $\Omega = \mathbb{Z}^{\mathbb{N}}$ which is generated by the elements $\mathtt{a},\mathtt{b},\mathtt{c}$ and $\mathtt{d}$ defined recursively by

$$\mathtt{a}(l_1,l_2,l_3,\ldots) = (1+l_1,l_2,l_3,\ldots),$$

$$\mathtt{b}(l_1,l_2,l_3,\ldots) = \begin{cases} (l_1,\mathtt{a}(l_2,l_3,\ldots)), & l_1 \text{ even}, \\ (l_1,\mathtt{c}(l_2,l_3,\ldots)), & l_1 \text{ odd}, \end{cases}$$

$$\mathtt{c}(l_1,l_2,l_3,\ldots) = \begin{cases} (l_1,\mathtt{a}(l_2,l_3,\ldots)), & l_1 \text{ even}, \\ (l_1,\mathtt{d}(l_2,l_3,\ldots)), & l_1 \text{ odd}, \end{cases}$$

$$\mathtt{d}(l_1,l_2,l_3,\ldots) = \begin{cases} (l_1,l_2,l_3,\ldots), & l_1 \text{ even}, \\ (l_1,\mathtt{b}(l_2,l_3,\ldots)), & l_1 \text{ odd}. \end{cases}$$

The group H preserves the lexicographic order on Ω. It is then a left orderable group [7], and so it can be realized as a group of orientation preserving homeomorphisms of the interval. These facts were first established (by an indirect method) in [11].

We next give an elementary proof of the fact that H can be realized as a group of bi-Lipschitz homeomorphisms of $[0,1]$. Fix a sequence $(\ell_i)_{i \in \mathbb{Z}}$ of positive numbers such that $\sum \ell_i = 1$ and

$$\max\left\{\frac{\ell_{i+1}}{\ell_i}, \frac{\ell_i}{\ell_{i+1}}\right\} \leq M < \infty \quad \text{for all} \quad i \in \mathbb{Z}.$$

We let I_i denote the interval $]\sum_{j<i} \ell_j, \sum_{j \leq i} \ell_j[$. Let $f:[0,1] \to [0,1]$ be the orientation preserving homeomorphism sending each interval I_i onto I_{i+1} affinely. Let g be the orientation preserving affine homeomorphism sending $[0,1]$ onto I_0, and let us denote by $\lambda = 1/\ell_0$ the (constant) value of its derivative. Consider the maps A, B, C and D defined recursively on a dense subset of $[0,1]$ by setting $A(x) = f(x)$ and, for $x \in I_i$,

$$B(x) = \begin{cases} f^i g A g^{-1} f^{-i}(x), & i \text{ even}, \\ f^i g C g^{-1} f^{-i}(x), & i \text{ odd}, \end{cases}$$

$$C(x) = \begin{cases} f^i g A g^{-1} f^{-i}(x), & i \text{ even}, \\ f^i g D g^{-1} f^{-i}(x), & i \text{ odd}, \end{cases}$$

$$D(x) = \begin{cases} x, & i \text{ even}, \\ f^i g B g^{-1} f^{-i}(x), & i \text{ odd}. \end{cases}$$

We claim that A, B, C and D are bi-Lipschitz homeomorphisms with bi-Lipschitz constant bounded above by M. Indeed, this is clear for A. For B, C and D, this fact can be easily verified by induction. For example, if $x \in I_i$ for an even integer i, then

$$B'(x) = \frac{(f^i)'(gAg^{-1}f^{-i}(x))}{(f^i)'(f^{-i}(x))} \cdot \frac{g'(Ag^{-1}f^{-i}(x))}{g'(g^{-1}f^{-i}(x))} \cdot A'(g^{-1}f^{-i}(x)),$$

and since $g'|_{[0,1]} = \lambda$ and $(f^i)'|_{I_0} = \ell_i/\ell_0$, we obtain $B'(x) = A'(g^{-1}f^{-i}(x)) \leq M$. Therefore, the maps A, B, C and D extend to bi-Lipschitz homeomorphisms of the whole interval $[0,1]$, and it is geometrically clear that they generate a group isomorphic to H. Remark finally that the constant M may be chosen so near to 1 as we want.

Remark 1.1. The fact that H can be realized as a group of bi-Lipschitz homeomorphisms is not surprising. Indeed, a simple argument using the harmonic measure allows to show that if Γ is any finitely generated subgroup of $\mathrm{Homeo}_+(\mathrm{S}^1)$ (resp. of $\mathrm{Homeo}_+([0,1])$), then Γ is topologically conjugate to a group of bi-Lipschitz homeomorphisms of the circle (resp. of the interval): see [5, Theorem D]. Nevertheless, the fact that the Lipschitz constant of the generators of H can be taken so near to 1 as desired is a particular property of the group H. This property seems to be shared by any other (finitely generated) group of homeomorphisms of the interval or the circle without free semi-groups on two generators, but it is easy to construct examples showing that it does not hold in general.

The preceding idea is not appropriate for obtaining an embedding of H into $\mathrm{Diff}^1_+([0,1])$. Indeed, the discontinuities for the derivative repeat at each level of the action of H. In the subsequent section, we will give a method of construction to obtain such an embedding. For this we will have to renormalize suitably the geometry at each step. Denoting by H_n the stabilizer of the level n of the tree $\mathcal{T}_\infty$ for the action of H, we will construct embeddings of H/H_n into $\mathrm{Diff}^1_+([0,1])$ in a coherent way and keeping some uniform control for the derivatives of generators; then using Arzelá-Ascoli Theorem, we will pass to the limit and obtain the desired embedding. Unfortunately, this method of construction will involve some technical issues. As a mater of fact, the action we will obtain is only semi-conjugate, but not conjugate, to the bi-Lipschitz action constructed above. (However, this seems to be a necessary condition for C^1-actions of H.)

Remark 1.2. In [9], R. Grigorchuk gives a general procedure for constructing groups of intermediate growth as groups acting on the dyadic rooted tree $\mathcal{T}_2$. It is not very difficult to see that the induced groups acting on the tree $\mathcal{T}_\infty$ by order preserving maps still have intermediate growth. For all of these induced groups, the methods of the first part of this work lead to realizations as subgroups of $\mathrm{Diff}^1_+([0,1])$.

Remark 1.3. A nice example of a group having *non uniform exponential growth* (that is, its exponential rate of growth is positive but becomes arbitrarily small under suitable changes of the system of generators) has been recently given by J. Wilson in [39]. This group acts faithfully by automorphisms of a rooted tree. It would be interesting to know wether there exists an associated group of non uniform exponential growth acting on $\mathcal{T}_\infty$ by order preserving transformations. If this is the case, then the methods of the first part of this work should certainly lead to realizations as a subgroup of $\mathrm{Diff}^1_+([0,1])$.

1.2 Embeddings using equivariant families of homeomorphisms

Henceforth, we will deal only with *orientation preserving* homeomorphisms between intervals. A family $\{\varphi_{u,v} : [0,u] \to [0,v]; u > 0, v > 0\}$ of such homeomorphisms will be called *equivariant* if for all $u > 0, v > 0$ and $w > 0$, one has $\varphi_{v,w} \circ \varphi_{u,v} = \varphi_{u,w}$. Given such a family and two nondegenerate intervals $I = [x_1, x_2]$ and $J = [y_1, y_2]$, we let $\varphi(I,J) : I \to J$ denote the homeomorphism defined by

$$\varphi(I,J)(x) = \varphi_{x_2-x_1,y_2-y_1}(x - x_1) + y_1.$$

Remark that $\varphi(I,I)$ is forced to be the identity map.

The simplest family of equivariant homeomorphisms is the one consisting of the affine maps $\varphi_{u,v}(x) = vx/u$. However, this family is not adequate if we want to fit maps smoothly together. Let us then introduce a general and simple procedure for constructing families of equivariant homeomorphisms as follows. Let $\{\varphi_u : \mathbb{R} \to]0,u[; u > 0\}$ be any family of homeomorphisms. Define $\varphi_{u,v} :]0,u[\to]0,v[$ by $\varphi_{u,v} = \varphi_v \circ \varphi_u^{-1}$. We have

$$\varphi_{v,w} \circ \varphi_{u,v} = (\varphi_w \circ \varphi_v^{-1}) \circ (\varphi_v \circ \varphi_u^{-1}) = \varphi_w \circ \varphi_u^{-1} = \varphi_{u,w}.$$

Thus, extending $\varphi_{u,v}$ continuously to the whole interval $[0,u]$ by setting $\varphi_{u,v}(0) = 0$ and $\varphi_{u,v}(u) = v$, we obtain the desired equivariant family.

Example 1.4. Let $\varphi_u : \mathbb{R} \to]0,u[$ be given by

$$\varphi_u(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{ds}{s^2 + (1/u)^2} = \frac{u}{2} + \frac{u}{\pi}\arctan(ux).$$

The corresponding equivariant family $\{\varphi_{u,v} : [0,u] \to [0,v]; u > 0, v > 0\}$ will be essential in what follows. This family was introduced by J. C. Yoccoz, and it has been already used in [6]. The regularity properties of the maps $\varphi_{u,v}$ will be studied in §1.3.

Now fix any equivariant family of homeomorphisms $\{\varphi_{u,v} : [0,u] \to [0,v]; u > 0, v > 0\}$. For each $n \in \mathbb{N}$ and each $(l_1,\ldots,l_n) \in \mathbb{Z}^n$, let us consider a non degenerate closed interval $I_{l_1,\ldots,l_n} = [a_{l_1,\ldots,l_n}, b_{l_1,\ldots,l_n}]$ and a (perhaps degenerate) closed interval $J_{l_1,\ldots,l_n} = [c_{l_1,\ldots,l_n}, d_{l_1,\ldots,l_n}]$, both contained in some interval $[0,T]$. Let us suppose that the following conditions are satisfied (see Figure 2):
(i) $\sum_{l_1\in\mathbb{Z}} |I_{l_1}| = T$ (where $|\cdot|$ denotes the length of the corresponding interval),
(ii) $a_{l_1,\ldots,l_n} < c_{l_1,\ldots,l_n} \le d_{l_1,\ldots,l_n} = b_{l_1,\ldots,l_n}$, so in particular $J_{l_1,\ldots,l_n} \subset I_{l_1,\ldots,l_n}$,
(iii) $b_{l_1,\ldots,l_{n-1},l_n} = a_{l_1,\ldots,l_{n-1},1+l_n}$,
(iv) $\lim_{l_n\to-\infty} a_{l_1,\ldots,l_{n-1},l_n} = a_{l_1,\ldots,l_{n-1}}$,
(v) $\lim_{l_n\to\infty} a_{l_1,\ldots,l_{n-1},l_n} = c_{l_1,\ldots,l_{n-1}}$,
(vi) $\lim_{n\to\infty} \sup_{(l_1,\ldots,l_n)\in\mathbb{Z}^n} |I_{l_1,\ldots,l_n}| = 0$.

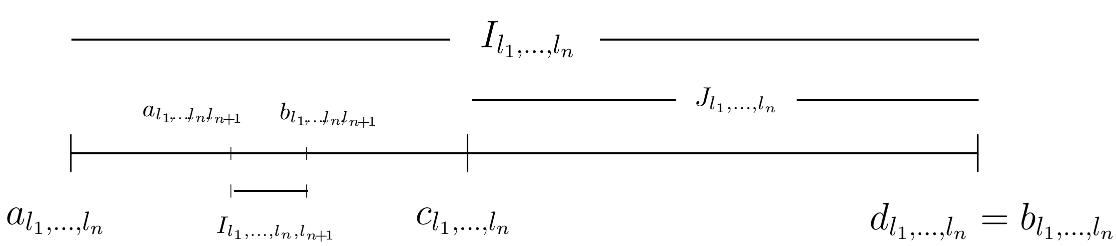

Figure 2

Note that

$$|J_{l_1,\ldots,l_n}| + \sum_{l_{n+1}\in\mathbb{Z}} |I_{l_1,\ldots,l_n,l_{n+1}}| = |I_{l_1,\ldots,l_n}|. \quad (1)$$

For each $n \in \mathbb{N}$, we will define homeomorphisms A_n, B_n, C_n and D_n in such a way that the group generated by them will be isomorphic to H/H_n. For this, let us consider the homomorphisms ϕ_0 and ϕ_1 from the subgroup of H generated by $\mathtt{b}$, $\mathtt{c}$ and $\mathtt{d}$ into H defined by

$$\phi_0(\mathtt{b}) = \mathtt{a}, \quad \phi_0(\mathtt{c}) = \mathtt{a}, \quad \phi_0(\mathtt{d}) = id, \qquad \text{and} \qquad \phi_1(\mathtt{b}) = \mathtt{c}, \quad \phi_1(\mathtt{c}) = \mathtt{d}, \quad \phi_1(\mathtt{d}) = \mathtt{b}.$$

Definition of A_n

– If $p \in J_{l_1,\ldots,l_i}$ for some $i < n$, let $A_n(p) = \varphi(J_{l_1,l_2,\ldots,l_i}, J_{1+l_1,l_2,\ldots,l_i})(p)$.
– If $p \in I_{l_1,\ldots,l_n}$, let $A_n(p) = \varphi(I_{l_1,l_2,\ldots,l_n}, I_{1+l_1,l_2,\ldots,l_n})(p)$.

Definition of B_n

Suppose that $p \in]0,1[$ belongs to $I_{l_1,\ldots,l_n}$, and denote the corresponding sequence reduced modulo 2 by $(\bar{l}_1,\ldots,\bar{l}_n) \in \{0,1\}^n$.

– If $\phi_{\bar{l}_1}(\mathtt{b})$, $\phi_{\bar{l}_2}\phi_{\bar{l}_1}(\mathtt{b})$, $\ldots$, $\phi_{\bar{l}_n}\cdots\phi_{\bar{l}_2}\phi_{\bar{l}_1}(\mathtt{b})$ are well defined, let $B_n(p) = p$.
– Otherwise, we denote the smallest integer $i \leq n$ such that $\phi_{\bar{l}_i}\cdots\phi_{\bar{l}_2}\phi_{\bar{l}_1}(\mathtt{b})$ is not defined by $i(p)$.
  · If $p \in J_{l_1,\ldots,l_j}$ for some $j < i(p)$, let $B_n(p) = p$.
  · If $p \in J_{l_1,\ldots,l_{i(p)},\ldots,l_j}$ for some $i(p) \leq j < n$, let $B_n(p) = \varphi(J_{l_1,\ldots,l_{i(p)},\ldots,l_j}, J_{l_1,\ldots,1+l_{i(p)},\ldots,l_j})(p)$.
  · If $p \in I_{l_1,\ldots,l_n}$, let $B_n(p) = \varphi(I_{l_1,\ldots,l_{i(p)},\ldots,l_n}, I_{l_1,\ldots,1+l_{i(p)},\ldots,l_n})(p)$.

The definitions of C_n and D_n are similar to that of B_n. Clearly, the maps A_n, B_n, C_n and D_n extend to homeomorphisms of $[0,T]$. The fact that they generate a group isomorphic to H/H_n is geometrically clear and follows easily from the equivariant properties of the maps $\varphi_{u,v}$. Moreover, condition (vi) implies that the sequences of maps A_n, B_n, C_n and D_n converge to limit homeomorphisms A, B, C and D respectively, which generate a group isomorphic to H.

Example 1.5. Given a sequence $(\ell_i)_{i\in\mathbb{Z}}$ of positive numbers such that $\sum \ell_i = 1$, define $|I_{l_1,\ldots,l_n}|$ and $|J_{l_1,\ldots,l_n}|$ by $|J_{l_1,\ldots,l_n}| = 0$ and $|I_{l_1,\ldots,l_n}| = \ell_{l_1}\cdots\ell_{l_n}$ respectively. If we carry out the preceding construction (for $T = 1$) using the equivariant family of affine maps $\varphi_{u,v}(x) = ux/v$, then we recover the embedding of H into the group of bi-Lipschitz homeomorphisms of the interval constructed at the end of §1.1 (under the assumptions that $\ell_{i+1}/\ell_i \leq M$ and $\ell_i/\ell_{i+1} \leq M$ for all $i \in \mathbb{Z}$).

1.3 Modulus of continuity for the derivatives

Let $\sigma : [0,1] \to [0,\sigma(1)]$ be an increasing homeomorphism. A continuous map $\psi : [0,1] \to \mathbb{R}$ is σ-continuous if there exists $M < \infty$ such that, for all $x \neq y$ in $[0,1]$,

$$\left|\frac{\psi(x)-\psi(y)}{\sigma(|x-y|)}\right| \leq M.$$

We denote the supremum of the left hand side expression by $\|\psi\|_\sigma$, and we call it the σ-norm of ψ. The interest in the notion of σ-continuity relies on the obvious fact that, if (ψ_n) is a sequence of functions defined on $[0,1]$ such that

$$\sup_{n\in\mathbb{N}} \|\psi_n\|_\sigma < \infty,$$

then (ψ_n) is an equicontinuous sequence.

Example 1.6. For $\sigma(s) = s^\alpha$, with $0<\alpha<1$, the notions of σ-continuity and α-Hölder continuity coincide.

Example 1.7. For $\varepsilon > 0$ suppose that $\sigma = \sigma_\varepsilon$ is such that $\sigma_\varepsilon(s) = s\log(1/s)^{1+\varepsilon}$ for s small. If a map φ is σ_ε-continuous, then it is α-Hölder continuous for all $0 < \alpha < 1$. Indeed, it is easy to verify that

$$s\left(\log\left(\frac{1}{s}\right)\right)^{1+\varepsilon} \leq C_{\varepsilon,\alpha}s^\alpha, \qquad \text{where} \quad C_{\varepsilon,\alpha} = \frac{1}{e^{1+\varepsilon}}\left(\frac{1+\varepsilon}{1-\alpha}\right)^{1+\varepsilon}.$$

We remark that the map $s \mapsto s\log(1/s)^{1+\varepsilon}$ is not Lipschitz. As a consequence, σ_ε-continuity for a function does not imply that the function is Lipschitz.

Example 1.8. A modulus of continuity σ satisfying $\sigma(s) = 1/\log(1/s)$ for s small enough is weaker than any Hölder modulus $s \mapsto s^\alpha$, with $\alpha > 0$. Nevertheless, such a modulus will be essential for our construction.

We will now investigate several upper bounds with respect to some moduli of continuity for the derivatives of maps in Yoccoz's family (see Example 1.4). Letting $y = \varphi_u^{-1}(x)$, we have

$$\varphi_{u,v}'(x) = \varphi_v'(y)(\varphi_u^{-1})'(x) = \frac{\varphi_v'(y)}{\varphi_u'(y)} = \frac{y^2+1/u^2}{y^2+1/v^2}.$$

Note that, when $x \to 0$ (resp. $x \to u$), we have that $y \to -\infty$ (resp. $y \to +\infty$), and $\varphi'_{u,v}(x) \to 1$. Therefore, the map $\varphi_{u,v}$ extends to a C^1 diffeomorphism from $[0,u]$ to $[0,v]$ which is tangent to the identity at the end points of $[0,u]$. Moreover, for $u \geq v$ (resp. $u \leq v$), the function $s \mapsto \frac{s^2+1/u^2}{s^2+1/v^2}$ attains its minimum (resp. maximum) value at $s=0$. Since this value is equal to v^2/u^2, we have

$$\sup_{x\in[0,u]} |\varphi'_{u,v}(x) - 1| = \left| \frac{v^2}{u^2} - 1 \right|.$$

For the second derivative of $\varphi_{u,v}$ we have

$$\varphi''_{u,v}(x) = \frac{d\varphi'_{u,v}(x)}{dy} \cdot \frac{dy}{dx} = \frac{2y(y^2+1/v^2) - 2y(y^2+1/u^2)}{(y^2+1/v^2)^2} \pi(y^2+1/u^2) = \pi \frac{y^2+1/u^2}{(y^2+1/v^2)^2} \left[2y\left(\frac{1}{v^2} - \frac{1}{u^2} \right) \right].$$

Therefore,

$$|\varphi''_{u,v}(x)| = \pi \frac{y^2+1/u^2}{y^2+1/v^2} \cdot \frac{|2y(1/v^2 - 1/u^2)|}{y^2+1/v^2}.$$

It follows from this equality that $\varphi_{u,v}$ is a C^2 diffeomorphism, with $\varphi''_{u,v}(0) = \varphi''_{u,v}(u) = 0$. Moreover, the inequality $\frac{2|y|}{y^2+t^2} \leq \frac{1}{t}$ applied to $t = 1/v$ yields

$$\left|\varphi''_{u,v}(x)\right| \leq \pi \frac{y^2+1/u^2}{y^2+1/v^2} \left| \frac{1}{v^2} - \frac{1}{u^2} \right| v.$$

For $u \leq v$, this implies

$$\left|\varphi''_{u,v}(x)\right| \leq \pi \frac{v^2}{u^2} \left(\frac{v^2-u^2}{u^2v^2} \right) v = \frac{\pi v}{u^2} \left(\frac{v^2}{u^2} - 1 \right).$$

So, if $u \leq v \leq 2u$, then

$$\left|\varphi''_{u,v}(x)\right| \leq 6\pi \left| \frac{v}{u} - 1 \right| \frac{1}{u}.$$

Analogously, if $2v \geq u \geq v$, then

$$\left|\varphi''_{u,v}(x)\right| \leq \frac{\pi}{v} \left(1 - \frac{v^2}{u^2} \right) \leq 2\pi \left| \frac{v}{u} - 1 \right| \frac{1}{v} \leq 4\pi \left| \frac{v}{u} - 1 \right| \frac{1}{u}.$$

Thus, in both cases, we have

$$\left|\varphi''_{u,v}(x)\right| \leq 6\pi \left| \frac{v}{u} - 1 \right| \frac{1}{u}. \tag{2}$$

The last inequality and the next elementary proposition show that the family of maps $\varphi_{u,v}$ is in some sense optimal (at least among families of maps which are tangent to the identity at the end points; see Remark 1.11).

Proposition 1.9. *If $\varphi : [0,u] \to [0,v]$ is a C^2 diffeomorphism such that $\varphi'(0) = \varphi'(u) = 1$, then there exists a point $y \in]0,u[$ such that*

$$|\varphi''(y)| \geq \frac{2}{u} \left| \frac{v}{u} - 1 \right|.$$

Proof. Let us suppose that $v \geq u$ (the case where $v \leq u$ is similar). Since $\varphi(0) = 0$ and $\varphi(u) = v$, there exists some point $x \in]0,u[$ such that $\varphi'(x) \geq v/u$. There are two cases:

(i) $x \geq u/2$: the Mean Value Theorem gives some point $y \in]x,u[$ such that

$$|\varphi''(y)| = \frac{\varphi'(x) - \varphi'(u)}{u - x} \geq \frac{2}{u} |\varphi'(x) - 1| \geq \frac{2}{u} \left| \frac{v}{u} - 1 \right|;$$

(ii) $x \leq u/2$: again, there exists $y \in]0,x[$ such that

$$|\varphi''(y)| = \frac{\varphi'(x) - \varphi'(0)}{x} \geq \frac{2}{u} |\varphi'(x) - 1| \geq \frac{2}{u} \left| \frac{v}{u} - 1 \right|.$$

The next lemma should be compared to §3.17 of Chapter X of [16]. Note that the moduli from Examples 1.6, 1.7 and 1.8 can be taken satisfying the decreasing hypothesis on the function $s \mapsto \sigma(s)/s$.

Lemma 1.10. *Suppose that σ is a modulus of continuity such that the function $s \mapsto \sigma(s)/s$ is decreasing. If $u > 0$ and $v > 0$ satisfy $u/v \leq 2$, $v/u \leq 2$, and*

$$\left|\frac{v}{u} - 1\right| \frac{1}{\sigma(u)} \leq M,$$

then the σ-norm of $\varphi'_{u,v}$ is less than or equal to $6\pi M$.

Proof. By inequality (2), for all $x \in [0, u]$ we have

$$|\varphi''_{u,v}(x)| \leq \frac{6\pi M \sigma(u)}{u}.$$

If $y < z$ are points in $[0, u]$ then there exists $x \in [y, z]$ such that $\varphi'_{u,v}(z) - \varphi'(y) = \varphi''_{u,v}(x)(z - y)$. Since $s \mapsto \sigma(s)/s$ is a decreasing function and $z - y \leq u$, this gives

$$\left|\frac{\varphi'_{u,v}(z) - \varphi'_{u,v}(y)}{\sigma(z-y)}\right| = |\varphi''_{u,v}(x)| \left|\frac{z-y}{\sigma(z-y)}\right| \leq |\varphi''_{u,v}(x)| \left|\frac{u}{\sigma(u)}\right| \leq 6\pi M.$$

This finishes the proof of the lemma.

Remark 1.11. In the case where non tangencies to the identity at the end points are allowed, there are equivariant procedures to construct maps with slightly better regularity properties than those of the maps in Yoccoz's family. This is related to the famous Pixton's actions [26], for which an alternative and precious reference is [36]. Nevertheless, for our construction we do not need a sequence of optimal embeddings of the groups H/H_n, but only a sequence of "good enough" embeddings which allow to preserve the differentiability when passing to the limit. For this reason, we will not use the sharp constructions of [36], for which the computations are much more involved.

We finish this section with an elementary technical lemma which will be useful to control the σ-norm of the derivative of a map obtained by fitting together many diffeomorphisms defined on sub-intervals. The zero Lebesgue measure hypothesis below will be trivially satisfied in our constructions because the corresponding sets will be countable.

Lemma 1.12. *Let $\{I_n : n \in \mathbb{N}\}$ be a family of closed intervals in $[0,1]$ having disjoint interiors and such that the complement of their union has zero Lebesgue measure. Suppose that φ is a homeomorphism of $[0,1]$ such that its restrictions to each interval I_n are $C^{1+\sigma}$ diffeomorphisms which are C^1-tangent to the identity at both end points of I_n and whose derivatives have σ-norms bounded above by a constant M. Then φ is a $C^{1+\sigma}$ diffeomorphism of the whole interval $[0,1]$, and the σ-norm of its derivative is less than or equal to $2M$.*

Proof. Let $x < y$ be two points of $\cup_{n\in\mathbb{N}} I_n$. If they belong to the same interval I_n then, by hypothesis,

$$\left|\frac{\varphi'(y) - \varphi'(x)}{\sigma(y-x)}\right| \leq M.$$

Suppose now that $x \in I_i = [x_i, y_i]$ and $y \in I_j = [x_j, y_j]$, with $y_i \leq x_j$. In this case,

$$\begin{aligned}
\left|\frac{\varphi'(y) - \varphi'(x)}{\sigma(y-x)}\right| &= \left|\frac{(\varphi'(y) - 1) + (1 - \varphi'(x))}{\sigma(y-x)}\right| \\
&\leq \left|\frac{\varphi'(y) - \varphi'(x_j)}{\sigma(y-x)}\right| + \left|\frac{\varphi'(y_i) - \varphi'(x)}{\sigma(y-x)}\right| \\
&\leq M\left[\frac{\sigma(y - x_j)}{\sigma(y-x)} + \frac{\sigma(y_i - x)}{\sigma(y-x)}\right] \\
&\leq 2M.
\end{aligned}$$

The map $x \mapsto \varphi'(x)$ is then uniformly continuous on the dense set $\cup_{n\in\mathbb{N}} I_n$, and so it extends to some continuous function on $[0,1]$ having σ-norm bounded above by $2M$. Finally, since the complementary set of $\cup_{n\in\mathbb{N}} I_n$ has zero Lebesgue measure, the Fundamental Theorem of Calculus shows that this continuous function coincides (everywhere) with the derivative of φ.

1.4 The embedding of H into $\mathrm{Diff}^1_+([0,1])$

In this section, σ will denote a fixed modulus of continuity satisfying $\sigma(s) = 1/\log(1/s)$ for s small enough (namely, for $s \leq 1/e$), and such that the function $s \mapsto \sigma(s)/s$ is decreasing. We will prove that there exist embeddings $H \hookrightarrow \mathrm{Diff}^{1+\sigma}_+([0,1])$ sending the generators $\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}$ of H (and their inverses) to diffeomorphisms so near as we want (in the $C^{1+\sigma}$-topology) to the identity map. To do that, fix any number $M > 0$, and for each $k \in \mathbb{N}$ define $T_k = \sum_{i\in\mathbb{Z}} \frac{1}{(|i|+k)^2} < \infty$. Consider an increasing sequence (k_n) of positive integer numbers such that $k_1 \geq 4$. For $n \in \mathbb{N}$ and $(l_1,\ldots,l_n) \in \mathbb{Z}^n$ let

$$|I_{l_1,\ldots,l_n}| = \frac{1}{(|l_1| + \cdots |l_n| + k_n)^{2n}}.$$

Note that

$$\begin{aligned}\sum_{l_{n+1}\in\mathbb{Z}} |I_{l_1,\ldots,l_n,l_{n+1}}| &= \sum_{l_{n+1}\in\mathbb{Z}} \frac{1}{(|l_1| + \cdots |l_n| + |l_{n+1}| + k_{n+1})^{2n+2}} \\ &\leq 2\int_{|l_1|+\cdots+|l_n|+k_{n+1}-1}^{\infty} \frac{ds}{s^{2n+2}} \\ &= \frac{2}{2n+1}\cdot\frac{1}{(|l_1| + \cdots + |l_n| + k_{n+1} - 1)^{2n+1}}.\end{aligned}$$

Thus

$$\frac{\sum_{l_{n+1}\in\mathbb{Z}} |I_{l_1,\ldots,l_n,l_{n+1}}|}{|I_{l_1,\ldots,l_n}|} \leq \frac{2}{2n+1}\cdot\frac{(|l_1| + \cdots |l_n| + k_n)^{2n}}{(|l_1| + \cdots + |l_n| + k_{n+1} - 1)^{2n+1}} \leq \frac{2}{(2n+1)(|l_1| + \cdots + |l_n| + k_n)}.$$

In particular, we can define $|J_{l_1,\ldots,l_n}|$ by (1), that is

$$|J_{l_1,\ldots,l_n}| = |I_{l_1,\ldots,l_n}| - \sum_{l_{n+1}\in\mathbb{Z}} |I_{l_1,\ldots,l_n,l_{n+1}}|,$$

and for this choice we have

$$|I_{l_1,\ldots,l_n}| \geq |J_{l_1,\ldots,l_n}| \geq \left(1 - \frac{2}{(2n+1)(|l_1| + \cdots + |l_n| + k_n)}\right)|I_{l_1,\ldots,l_n}|. \tag{3}$$

The procedure of §1.2 (using Yoccoz's equivariant family of maps) gives subgroups of $\mathrm{Diff}^1_+([0,T_{k_1}])$ isomorphic to H/H_n and generated by elements A_n, B_n, C_n and D_n. Our next step is to estimate the σ-norm of the derivatives of these maps.

Lemma 1.13. *If the sequence (k_n) satisfy the conditions*

$$\frac{(2n+1)k_n}{(2n+1)k_n - 2}\left(\frac{k_n+1}{k_n}\right)^{2n} \leq 2, \qquad \left(1 - \frac{2}{(2n+1)k_n}\right)\left(\frac{k_n-1}{k_n}\right)^{2n} \geq \frac{1}{2}, \tag{4}$$

$$2n\log(k_n) \geq \log\left(\frac{(2n+1)k_n}{(2n+1)k_n - 2}\right), \tag{5}$$

and

$$\frac{\log(k_n)}{k_n}\left(n2^{2n+3} + \frac{32}{2n+1}\right) \leq \frac{M}{12\pi}, \tag{6}$$

then the σ-norms of the derivatives of A_n, B_n, C_n and D_n are less than or equal to M for all $n \in \mathbb{N}$.

Proof. First of all, it is easy to verify that inequality (3) and hypothesis (4) imply that

$$\frac{1}{2} \leq \frac{|I_{l_1,\ldots,1+l_i,\ldots,l_n}|}{|I_{l_1,\ldots,l_i,\ldots,l_n}|} \leq 2 \quad \text{and} \quad \frac{1}{2} \leq \frac{|J_{l_1,\ldots,1+l_i,\ldots,l_n}|}{|J_{l_1,\ldots,l_i,\ldots,l_n}|} \leq 2. \tag{7}$$

We also have

$$\frac{|l_1| + \cdots + |l_i| \cdots + |l_n| + k_n}{|l_1| + \cdots + |1+l_i| + \cdots + |l_n| + k_n} \leq 2. \tag{8}$$

According to the construction of the maps and inequalities (7), the problem reduces to estimating expressions of the form

$$\left|\frac{|I_{l_1,\dots,1+l_i,\dots,l_n}|}{|I_{l_1,\dots,l_i,\dots,l_n}|}-1\right|\frac{1}{\sigma(|I_{l_1,\dots,l_i,\dots,l_n}|)} \qquad \text{and} \qquad \left|\frac{|J_{l_1,\dots,1+l_i,\dots,l_n}|}{|J_{l_1,\dots,l_i,\dots,l_n}|}-1\right|\frac{1}{\sigma(|J_{l_1,\dots,l_i,\dots,l_n}|)}.$$

Indeed, if we verify that these expressions are bounded above by $M/12\pi$ for all possible choices of sub-indices, then Lemmas 1.10 and 1.12 will imply that the σ-norm of the derivatives of A_n, B_n, C_n and D_n are less than or equal to M.

Using the identity $s^{2n}-1=(s-1)(s^{2n-1}+\cdots+1)$ and inequality (8) we obtain

$$\begin{aligned}\left|\frac{|I_{l_1,\dots,1+l_i,\dots,l_n}|}{|I_{l_1,\dots,l_i,\dots,l_n}|}-1\right| &= \left|\left(\frac{|l_1|+\cdots+|l_i|+\cdots+|l_n|+k_n}{|l_1|+\cdots+|1+l_i|+\cdots+|l_n|+k_n}\right)^{2n}-1\right|\\ &\leq \frac{\big||l_i|-|1+l_i|\big|}{|l_1|+\cdots+|1+l_i|+\cdots+|l_n|+k_n}\cdot(2^{2n-1}+2^{2n-2}+\cdots+1)\\ &\leq \frac{2^{2n}}{|l_1|+\cdots+|1+l_i|+\cdots+|l_n|+k_n}.\end{aligned}$$

Since the function $s\mapsto \log(s)/s$ is decreasing for $s\geq e$, by (6) we conclude

$$\begin{aligned}\left|\frac{|I_{l_1,\dots,1+l_i,\dots,l_n}|}{|I_{l_1,\dots,l_i,\dots,l_n}|}-1\right|\frac{1}{\sigma(|I_{l_1,\dots,l_i,\dots,l_n}|)} &\leq \frac{2^{2n}\log([|l_1|+\cdots+|l_i|+\cdots+|l_n|+k_n]^{2n})}{|l_1|+\cdots+|1+l_i|+\cdots+|l_n|+k_n}\\ &\leq \frac{n2^{2n+1}\log(|l_1|+\cdots+|l_i|+\cdots+|l_n|+k_n)}{|l_1|+\cdots+|1+l_i|+\cdots+|l_n|+k_n}\\ &\leq \frac{n2^{2n+2}\log(k_n)}{k_n}\\ &\leq \frac{M}{12\pi}.\end{aligned}$$

Let us now deal with the case of the intervals $J_{l_1,\dots,l_n}$. First of all, a straightforward computation using (3) and (5) shows that

$$\sigma(|J_{l_1,\dots,l_i,\dots,l_n}|)\geq \sigma(|I_{l_1,\dots,l_i,\dots,l_n}|)/2. \tag{9}$$

Then, using (3), (6), (9), and the triangle inequality

$$\left|\frac{|J_{l_1,\dots,1+l_i,\dots,l_n}|}{|J_{l_1,\dots,l_i,\dots,l_n}|}-1\right|\leq\left|\frac{|I_{l_1,\dots,1+l_i,\dots,l_n}|}{|I_{l_1,\dots,l_i,\dots,l_n}|}-1\right|+\left|\frac{|J_{l_1,\dots,1+l_i,\dots,l_n}|}{|J_{l_1,\dots,l_i,\dots,l_n}|}-\frac{|I_{l_1,\dots,1+l_i,\dots,l_n}|}{|I_{l_1,\dots,l_i,\dots,l_n}|}\right|,$$

it is easy to verify that

$$\left|\frac{|J_{l_1,\dots,1+l_i,\dots,l_n}|}{|J_{l_1,\dots,l_i,\dots,l_n}|}-1\right|\frac{1}{\sigma(|J_{l_1,\dots,l_i,\dots,l_n}|)}$$

is less than or equal to

$$\frac{n2^{2n+3}\log(k_n)}{k_n}+\frac{8}{(2n+1)(|l_1|+\cdots+|l_i|+\cdots+|l_n|+k_n)}\cdot\frac{|I_{l_1,\dots,1+l_i,\dots,l_n}|}{|I_{l_1,\dots,l_i,\dots,l_n}|}\cdot\frac{1}{\sigma(|I_{l_1,\dots,l_i,\dots,l_n}|)},$$

and the value of this expression is bounded above by

$$\frac{n2^{2n+3}\log(k_n)}{k_n}+\frac{32n}{2n+1}\cdot\frac{\log(|l_1|+\cdots+|l_i|+\cdots+|l_n|+k_n)}{|l_1|+\cdots+|l_i|+\cdots+|l_n|+k_n}\leq\frac{\log(k_n)}{k_n}\left(n2^{2n+3}+\frac{32n}{2n+1}\right)\leq\frac{M}{12\pi}.$$

This concludes the proof of the lemma.

Remark that similar computations lead to the same estimate for the σ-norms of $(A_n^{-1})'$, $(B_n^{-1})'$, $(C_n^{-1})'$ and $(D_n^{-1})'$. Thus, A_n', B_n', C_n' and D_n' converge to some σ-continuous functions (having σ-norm bounded above by M), and these functions are the derivatives of $C^{1+\sigma}$ diffeomorphisms A, B, C and D which generate a group isomorphic to H. Note however that this group acts on the interval $[0, T_{k_1}]$. In order to obtain a group acting on $[0,1]$, we can conjugate by the affine map $g:[0,1]\to[0,T_{k_1}]$, *i.e.* $g(x)=T_{k_1}x$. Since $k_1\geq 4$ we have $T_{k_1}\leq 1$, and therefore this procedure does not increase σ-norms of derivatives. Indeed, for instance

$$\begin{aligned}\left|\frac{(g^{-1}Ag)'(x)-(g^{-1}Ag)'(y)}{\sigma(|x-y|)}\right| &= \left|\frac{A'(g(x))-A'(g(y))}{\sigma(|x-y|)}\right|\\ &= \left|\frac{A'(g(x))-A'(g(y))}{\sigma(|g(x)-g(y)|)}\right|\cdot\left|\frac{\sigma(T_{k_1}|x-y|)}{\sigma(|x-y|)}\right|\\ &\leq M.\end{aligned}$$

Since every (orientation preserving) diffeomorphism f of $[0,1]$ has a point for which the derivative is equal to 1, if the σ-norm of f' is bounded above by M then

$$\sup_{x\in[0,1]}|f'(x)-1|\leq M\sigma(1).$$

So, if M is small, then f is near the identity in the $C^{1+\sigma}$-topology. Finally, it is easy to construct sequences (k_n) of integer positive numbers satisfying (4), (5) and (6). This finishes the proof of Theorem A.

Remark 1.14. Note that the action we obtained is given by diffeomorphisms which are tangent to the identity at the end points of $[0,1]$. Therefore, gluing together these two end points, we obtain an action by C^1 diffeomorphisms of the circle. Using the classical procedure of suspension, this allows to construct a codimension-1 foliation (on a 3-dimensional compact manifold) which is transversely of class C^1 and whose leaves have sub-exponential growth. It turns out that a countable number of leaves have polynomial growth and a continuum of leaves have intermediate growth, a fact that should be compared with [15]. In [4] one can find further interesting examples of codimension-1 foliations which are transversely of class C^1 but not C^2.

2 Sub-exponential growth groups of $C^{1+\alpha}$ interval diffeomorphisms

The second part of this article is mainly devoted to the proof of Theorem B. Note that, though this theorem is stated in terms of growth, we will prove an *a priori* stronger result, namely that finitely generated subgroups of $\mathrm{Diff}_+^{1+\alpha}([0,1])$ *without free semi-groups on two generators* are almost nilpotent.[1] For this, first recall that a classical result of J. Rosenblatt ([29]; see also [2]) establishes that every (finitely generated) solvable group without free semi-groups on two generators is almost nilpotent.[2] Hence, to prove (the *a priori* stronger version of) Theorem B, it suffices to show that finitely generated groups of $C^{1+\alpha}$ interval diffeomorphisms without free semi-groups on two generators are solvable.

At this point we would like to make a curious remark: for groups without free semi-groups on two generators, it is no longer necessary to assume finite generation in order to ensure solvability. This is due to the fact that what we will show is that finitely generated groups of $C^{1+\alpha}$ diffeomorphisms of the interval without free semi-groups on two generators are solvable with solvability degree bounded by some constant depending only on α.

Unfortunately, even the proof of the solvability is somehow technical. For this reason we decided to present the ideas in a progressive manner. In §2.1, we recall a simple criterium for existence of free semi-groups arising from the theory of codimension-1 foliations which is closely related to the so-called *resilient leaves*. Furthermore, we discuss a useful tool also intimately related to all of this, namely the translation number homomorphism associated to an invariant Radon measure. In §2.2, we discuss the problem (already treated in [22]) of the embedding of groups with sub-exponential growth into $\mathrm{Diff}_+^2([0,1])$, and in §2.3, we give a relatively short proof of the fact that, for any $\alpha>0$, there is no embedding of H into $\mathrm{Diff}_+^{1+\alpha}([0,1])$ which is semi-conjugate to those of the first part of this work. These last two paragraphs can be skipped by the reader who is pressed to go into the proof of Theorem B. (He or she will only need Lemma 2.7, which is mostly self-contained.) However, they can serve as a useful guide, because it is in these two sections where we introduce (and explain) our main ideas: §2.2 contains a general strategy of proof when it is possible to control distortions individually (*i.e.* in class C^2), whereas §2.3 contains a very simple method to get control of distortion in class $C^{1+\alpha}$. All these ideas are putted together in §2.4, which is divided into three subsections

[1] It seems to be unknown wether there exists a left orderable group without free semi-groups on two generators and having exponential growth.

[2] It may be possible to give a simple proof of this result for left orderable groups or at least for groups of C^1 diffeomorphisms of the interval.

where a complete proof for Theorem B is given. Finally, in §2.5, we prove the extensions of Theorem B for groups acting on the circle and the real line.

We would like to conclude this small introduction to the second part of this article by addressing the question of the uniformity of the exponential growth rate for groups of interval diffeomorphisms (see Remark 1.3):

Question. If Γ is a finitely generated non almost nilpotent subgroup of $\mathrm{Diff}_+^{1+\alpha}([0,1])$, does Γ necessarily have uniform exponential growth? Is this true at least for non-Abelian groups of C^2 diffeomorphisms of the interval?

2.1 Crossed elements, invariant Radon measures, and translation numbers

We say that two homeomorphisms of the interval $[0,1]$ are *crossed* on a sub-interval $[u,v]$ if one of them fixes u and v and no other point in $[u,v]$, while the other sends u or v into $]u,v[$. The following elementary criterion for existence of free semi-groups on two generators is well known.

Lemma 2.1. *If a subgroup* Γ *of* $\mathrm{Homeo}_+([0,1])$ *contains two crossed elements, then* Γ *contains a free semi-group on two generators.*

Proof. Suppose that there exist f, g in Γ and an interval $]u,v[$ which is fixed by f and contains no fixed point of f in its interior, and such that $g(u) \in]u,v[$. (The case where $g(v) \in]u,v[$ is analogous.) Changing f by its inverse if necessary, we may assume that $f(x) < x$ for all $x \in]u,v[$. Let $w = g(u) \in]u,v[$, and let us fix a point $z' \in]w,v[$. Since $gf^n(u) = w$ for all $n \in \mathbb{N}$, and since $gf^n(z')$ converges to w as n tends to infinity, the map gf^n has a fixed point in $]u,z'[$ for $n \in \mathbb{N}$ large enough. Fix such a $n \in \mathbb{N}$ and let $z > w$ be the infimum of the fixed points of gf^n in $]u,v[$. For $m \in \mathbb{N}$ large enough we have $f^m(z) < w$, and so the (positive) Ping-Pong Lemma applied to the restrictions of f^m and fg^n to $[u,v]$ shows that the semi-group generated by these two elements is free (see [13], Chapter VII).

The preceding lemma has the following important consequence: if f and g are interval homeomorphisms which generate a group without free semi-groups, and if f has a fixed point x_0 which is not fixed by g, then the fixed points of g immediately to the left and to the right of x_0 are also fixed by f. This gives a quite clear geometric picture for the action of a group without free semi-groups. However, it is sometimes difficult to use this picture without entering into rather complicated combinatorial discussions. In order to avoid this problem, there is an extremely useful tool for detecting fixed points of elements, namely the translation number associated to an invariant Radon measure. We begin by recalling a result due to J. Plante [27] for groups with sub-exponential growth, and to V. Solodov [32] and L. Beklaryan [1] for groups without free semi-groups. We offer a short proof of our own for the convenience of the reader.

Proposition 2.2. *Let* Γ *be a finitely generated group of homeomorphisms of* $[0,1]$. *If* Γ *has no crossed elements, then* Γ *preserves a (non trivial) Radon measure on* $]0,1[$ *(i.e. a measure on the Borelean sets which is finite on compact subsets of* $]0,1[$*).*

Proof. If Γ has global fixed points inside $]0,1[$ then the claim is obvious: the delta measure on any of such points is invariant by the action. Assume in what follows that the Γ-action on $]0,1[$ has no global fixed point, and take a finite system $\{f_1,\ldots,f_k\}$ of generators for Γ. We first claim that (at least) one of these generators does not have interior fixed points. Indeed, suppose by contradiction that all the maps f_i have interior fixed points, and let $x_1 \in]0,1[$ be any fixed point f_1. If f_2 fixes x_1 then letting $x_2 = x_1$ we have that x_2 is fixed by both f_1 and f_2. If not, choose a fixed point $x_2 \in]0,1[$ for f_2 such that f_2 does not fix any point between x_1 and x_2. Since f_1 and f_2 are non crossed on any interval, x_2 must be fixed by f_1. Now if x_2 is fixed by f_3 let $x_3 = x_2$; if not, take a fixed point $x_3 \in]0,1[$ for f_3 such that f_3 has no fixed point between x_2 and x_3. The same argument as before shows that x_3 is fixed by f_1, f_2, and f_3. Continuing in this way we find a common fixed point for all the generators f_i, and so a global fixed point for Γ, which contradicts our assumption.

Now we claim that there exists a non empty minimal invariant closed set for the action of Γ inside $]0,1[$. To prove this consider a generator $f = f_i$ without fixed points, fix any point $x_0 \in]0,1[$, and let I be the interval $[x_0, f(x_0)]$ if $f(x_0) > x_0$, and $[f(x_0), x_0]$ if $f(x_0) < x_0$. In the family $\mathcal{F}$ of non empty closed invariant subsets of $]0,1[$ let us consider the order relation $\preceq$ given by $K_1 \succeq K_2$ if $K_1 \cap I \subset K_2 \cap I$. Since f has no fixed point, every orbit by Γ must intersect the interval I, and so $K \cap I$ is a non empty compact set for all $K \in \mathcal{F}$. Therefore, we can apply Zorn Lemma to obtain a maximal element for the order $\preceq$, and this element is the intersection with I of a minimal non empty closed subset of $]0,1[$ invariant by the action of Γ.

Now fix the non empty closed invariant minimal set K obtained above. Note that its boundary ∂K as well as the set of its accumulation points K' are also closed sets invariant by Γ. Because of the minimality of K, there are three possibilities:
– $K' = \emptyset$: In this case K is discrete, that is K coincides with the set of points of a sequence $(y_n)_{n\in\mathbb{Z}}$ satisfying $y_n < y_{n+1}$ for all n and without accumulation points inside $]0,1[$. It is then easy to see that the Radon measure $\mu = \sum_{n\in\mathbb{Z}} \delta_{y_n}$ is invariant by Γ.

– $\partial K = \emptyset$: In this case K coincides with the whole interval $]0,1[$. We claim that the action of Γ is free. Indeed, if not let $[u,v]$ be an interval strictly contained in $[0,1]$ and for which there exists an element $g \in \Gamma$ fixing $]u,v[$ and with no fixed point inside it. Since the action is minimal, there must be some $h \in \Gamma$ sending u or v inside $]u,v[$; however, this implies that g and h are crossed on $[u,v]$, contradicting our assumption. Now the action of Γ being free, Hölder Theorem [7] implies that Γ is topologically conjugate to a (in this case dense) group of translations. Pulling back the Lebesgue measure by this conjugacy we obtain an invariant Radon measure for the action of Γ.
– $\partial K = K' = K$: In this case K is "locally" a Cantor set. Collapsing to a point the closure of each connected component of the complementary set of K we obtain a new (topological) open interval, and the original action of Γ induces (by semi-conjugacy) an action by homeomorphisms on this new interval. As in the second case, one easily checks that the induced action is free, and so it preserves a Radon measure. Pulling back this measure by the semi-conjugacy, one obtains a Radon measure on $]0,1[$ which is invariant by the original action.

Remark 2.3. It is easy to see that the finite generation hypothesis is necessary for the preceding proposition. Note however that, during the proof, this hypothesis was only used to ensure the existence of a minimal non empty invariant closed subset of $]0,1[$, and hence it can be replaced by any other hypothesis leading to the same conclusion. For instance, the proposition is still true for non finitely generated groups containing elements without fixed points, or for groups having a system of generators all whose elements send each fixed point $x \in]0,1[$ into some compact subset of $]0,1[$ (which depends on x).

In Proposition 2.2 it is sometimes better to think of Γ as a group of homeomorphisms of the real line. Recall that for (non necessarily finitely generated) groups of homeomorphisms of the real line preserving a Radon measure μ there is an associated *translation number* function $\tau_\mu : \Gamma \to \mathbb{R}$ defined by

$$\tau_\mu(g) = \begin{cases} \mu([x_0, g(x_0)[) & \text{if } g(x_0) > x_0, \\ 0 & \text{if } g(x_0) = x_0, \\ -\mu([g(x_0), x_0[) & \text{if } g(x_0) < x_0, \end{cases}$$

where x_0 is any point of the real line. (One easily checks that this definition is independent of x_0.) The following properties are satisfied (the verification is easy; see for instance [27]):
(i) τ_μ is a group homomorphism,
(ii) $\tau_\mu(g) = 0$ if and only if g has fixed points; in this case the support of μ is contained in the set of these points,
(iii) τ_μ is trivial if and only if Γ has global fixed points (*i.e.* there are points which are fixed by all the elements of Γ).

Note that (iii) is a direct consequence of (ii). Moreover, (i) and (iii) imply the existence of global fixed points for the action of the derived group $\Gamma_1 = [\Gamma, \Gamma]$. If Γ does not already have global fixed points then, according to the proof of Proposition 2.2, there are two possibilities depending on the nature of μ:
– if μ has no atoms then Γ is semi-conjugate to a group of translations,
– if μ has some atom x_0 then there exists an element $f \in \Gamma$ such that the orbit of x_0 by Γ coincides with the orbit of x_0 by f; moreover, for each $g \in \Gamma$ there exists an integer $i = i(g)$ such that $g(f^n(x_0)) = f^{n+i}(x_0)$ for all $n \in \mathbb{Z}$, and the homomorphism τ_μ is given by a scalar multiple of the function $g \mapsto i(g)$.

Invariant Radon measures will be very useful because the functorial properties of the associated translation numbers will allow us to show that, roughly, every continuous action on the interval of a group without free semi-groups on two generators has a level structure which is somehow similar to that of H.

Remark 2.4. For the rest of this article one can forget about invariant Radon measures, and simply retain an elementary fact, already remarked by Salhi in [30] and by Solodov in [32], and that can be checked directly: if Γ is a (non necessarily finitely generated) subgroup of $\mathrm{Homeo}_+([0,1])$ without crossed elements, then the set of elements of Γ having fixed points inside $]0,1[$ is a normal subgroup. Nevertheless, we think that the presentation of our arguments in terms of invariant Radon measures is more transparent.

2.2 On the non embedding of the group H into $\mathrm{Diff}_+^{\mathbf{2}}([\mathbf{0},\mathbf{1}])$

It is possible to give a direct and simple argument to prove that, for any homomorphism $\phi : H \to \mathrm{Diff}_+^2([0,1])$, the image $\phi(H)$ is Abelian. This proof uses only the facts that $\mathtt{a}^2 \in H$ belongs to the center of H and that the elements $\mathtt{b}$, $\mathtt{c}$ and $\mathtt{d}$ commute between them.

Denote by $Fix_\phi(\mathtt{a})$ the set of points in $[0,1]$ which are fixed by $\phi(\mathtt{a}^2)$ (equivalently, by $\phi(\mathtt{a})$). Fix any interval $[x_0,y_0] \subset [0,1]$ such that $Fix_\phi(\mathtt{a}) \cap [x_0,y_0] = \{x_0,y_0\}$. Since $\mathtt{a}^2$ and $\mathtt{b}$ commute, $\phi(\mathtt{b})^n(x_0)$ and $\phi(\mathtt{b})^n(y_0)$ belong to $Fix_\phi(\mathtt{a})$ for all $n \in \mathbb{Z}$, and so they are not in $]x_0,y_0[$. If $\phi(\mathtt{b})(x_0) \leq x_0$ let $x = \lim_{n\to\infty} \phi(\mathtt{b})^n(x_0) \leq x_0$ and $y = \lim_{n\to\infty} \phi(\mathtt{b})^{-n}(y_0) \geq y_0$. If $\phi(\mathtt{b})(x_0) \geq x_0$ let $x = \lim_{n\to\infty} \phi(\mathtt{b})^{-n}(x_0) \leq x_0$ and $y = \lim_{n\to\infty} \phi(\mathtt{b})^n(y_0) \geq y_0$. Note that both x and y belong to $Fix_\phi(\mathtt{a})$. Kopell Lemma applied to the restrictions of $\phi(\mathtt{a}^2)$ and $\phi(\mathtt{b})$ to $[x,y]$ shows that $x = x_0$ and $y = y_0$. Thus the restriction of $\phi(\mathtt{b})$ to $[x_0,y_0]$ is contained in the centralizer (in $\mathrm{Diff}_+^2([x_0,y_0])$) of the restriction of $\phi(\mathtt{a})^2$, which by Szekeres' Theorem is an Abelian group. Similar arguments can be given for $\phi(\mathtt{c})$ and $\phi(\mathtt{d})$, concluding that $\phi(H)$ fixes $[x_0,y_0]$ and the corresponding restriction is an Abelian group. On the other hand, on the set $Fix_\phi(\mathtt{a})$ the action induced by ϕ factors through an action of the Abelian group generated by $\mathtt{b}$, $\mathtt{c}$ and $\mathtt{d}$. This finishes the proof of the commutativity of $\phi(H)$.

The preceding argument cannot be applied to general sub-exponential growth subgroups of $\mathrm{Diff}_+^2([0,1])$ because it uses commutativity in a way that is too strong. However, in §2.1 we saw that, though the center of a sub-exponential growth subgroup of $\mathrm{Homeo}_+([0,1])$ may be trivial, its elements have a lot of "dynamically commuting features". We will see that this property still allows applying some of the techniques of the classical rigidity theory for centralizers (of C^2 maps). Note that this last issue already appears (as the main technical ingredient) in [23]. Indeed, that paper contains the dynamical description for the actions by C^2 interval diffeomorphisms of solvable groups, for which the center is in most cases trivial...

Using the techniques introduced in [23] it is proved in [22] that finitely generated groups of C^2 diffeomorphisms of the interval with sub-exponential growth are Abelian, thus slightly generalizing Plante-Thurston Theorem [28]. Actually, this result is now an easy corollary of our Theorem B. However, in the rest of this section we will give a complete proof of it in order to introduce a second criterium for proving existence of free semi-groups, as well as to illustrate the usefulness of invariant Radon measures and translation numbers to simplify many combinatorial arguments. This will also allow us to clarify some unclear points of the proof given in [22]. But before getting into the proof, let's take a look at a relevant example.

Example 2.5. The wreath product $\mathbb{Z}\wr\mathbb{Z}$ has a natural action on the interval. This action is obtained by identifying one of the two canonical generators of this group with a homeomorphism f of $[0,1]$ satisfying $f(x) < x$ for all $x \in]0,1[$, and the other generator with a homeomorphism g satisfying $g(x) \neq x$ for all $x \in]f(x_0), x_0[$ and $g(x) = x$ for all $x \in [0,1] \setminus [f(x_0), x_0]$, where x_0 is some point in $]0,1[$. (This action can be easily smoothed up to the class C^∞; see [23, Section 1].) Note that $\mathbb{Z}\wr\mathbb{Z}$ is a metabelian non almost nilpotent group, so according to Theorem B it must contain free semi-groups on two generators. Actually, the semi-group generated by f and g is free. The argument bellow is the essence of our second criterium for existence of free semi-groups.

Claim: The semi-group generated by f and g is free.

Proof. Actually, we will prove a much more general statement. Suppose that f and g are two homeomorphisms of $[0,1]$ such that for some $x_0 \in]0,1[$ one has $f(x_0) < x_0$, and such that g fixes all the points of the orbit of x_0 by f. Assume moreover that for some interval $[u,v] \subset]f(x_0), x_0[$ disjoint from $g(]u,v[)$ the following holds: for each $m \in \mathbb{N}$ the intervals $g(f^m(]u,v[))$ and $f^m(]u,v[)$ coincide or are disjoint, and if they coincide then g fixes $f^n g^i([u,v])$ for every $i \in \mathbb{Z}$. In this situation we will show that if the set of integers

$$\mathcal{N} = \{m \geq 0 : \quad gf^m(]u,v[) \text{ and } f^m(]u,v[) \text{ are disjoint}\}$$

is finite, then the semi-group generated by f and g is free. To do this, let's consider two different words in positive powers of f and g, and let's try to prove that they represent distinct homeomorphisms. After conjugacy, we may suppose that these words are of the form $W_1 = f^n g^{m_r} f^{n_r} \cdots g^{m_1} f^{n_1}$ and $W_2 = g^q f^{p_s} g^{q_s} \cdots f^{p_1} g^{q_1}$, where m_j, n_j, p_j, q_j are positive integers, $n \geq 0$, and $q \geq 0$ (with $n > 0$ if $r = 0$, and $q > 0$ if $s = 0$). Let $N_1 = n_1 + \ldots + n_r + n$ and $N_2 = p_1 + \ldots + p_s$, and let m be the maximal value in $\mathcal{N}$. By the choice of m one has $W_1(f^m(u)) = f^{m+N_1}(u)$ and $W_2(f^m(u)) = f^{N_2}(g^{q_1}(f^m(u)))$. However, since $g^{q_1}(f^m(u)) \neq f^m(u)$ (because m belongs to $\mathcal{N}$), it is easy to verify that $f^{N_2}(g^{q_1}(f^m(u)))$ cannot be equal to $f^{m+N_1}(u)$. Hence $W_1(f^m(u)) \neq W_2(f^m(u))$, and so $W_1 \neq W_2$. This finishes the proof of the Claim.

Now let Γ be a finitely generated subgroup of $\mathrm{Diff}_+^2([0,1])$ with sub-exponential growth (or more generally, without free semi-groups on two generators). In order to prove that Γ is Abelian, it is of no loss of generality to assume that Γ has no global fixed point inside $]0,1[$. Then according to the proof of Proposition 2.2, Γ contains an element f such that $f(x) < x$ for all $x \in]0,1[$. Let μ be a Γ-invariant Radon measure on $]0,1[$, and let K be the set of points inside $]0,1[$ which are fixed by all the elements of the first derived group $\Gamma_1 = [\Gamma,\Gamma]$. This set is non empty, since it contains the support of μ. If K coincides with $]0,1[$ then Γ is Abelian. Suppose now that K is strictly contained in $]0,1[$ and that the restriction of Γ_1 to each connected component of $]0,1[\setminus K$ is free. Then by Hölder Theorem [7], the restriction of Γ_1 to every such connected component is Abelian, which implies that Γ is metabelian. By Rosenblatt's theorem [29], Γ is almost nilpotent, and because of Plante-Thurston Theorem, Γ is almost Abelian. Finally, Szekeres' theorem implies easily that almost Abelian groups of C^2 interval diffeomorphisms are in fact Abelian (see [22, Lemme 5.4]), thus finishing the proof in this case.

It remains the case where the action of Γ_1 on some connected component I of the complementary set of K is non free. Fix an element $h \in \Gamma_1$ and an interval $]u,v[$ strictly contained in I such that $]u,v[$ is fixed by h but no point inside $]u,v[$ is fixed by h. There must be some element $g \in \Gamma_1$ sending $]u,v[$ into a disjoint interval (contained in I). Indeed, if this is not the case then, since Γ has no crossed elements, every element of Γ_1 should fix $]u,v[$, and so the points u and v would be contained in K, contradicting the fact that $]u,v[$ was strictly contained in the connected component I of $]0,1[\setminus K$. We will finish the proof by showing that the semi-group generated by f and g is free.

Claim: There exists $N_0 \in \mathbb{N}$ such that g fixes the interval $f^n(]u,v[)$ for every $n \geq N_0$.

Proof. Following the proof of Kopell Lemma given in [3], denote $I =]w,z[$ and fix a constant λ such that

$$1 < \lambda < 1 + \frac{v-u}{e^M(u-w)},$$

where M is the Lipschitz constant of the function $\log(f')$. Since g fixes all the points $f^n(x_0)$, its derivative at the origin must be equal to 1. Let N_0 be such that

$$g'(x) \leq \lambda \quad \text{and} \quad (g^{-1})'(x) \leq \lambda \qquad \text{for all } x \in f^n(I) \text{ and all } n \geq N_0. \tag{10}$$

We will show that this N_0 works for the Claim (see Figure 3). Indeed, since Γ has no crossed elements, if g does not fix $f^n(]u,v[)$ then $f^n(]u,v[)$ and $g(f^n(]u,v[))$ are disjoint. In other words, one has $g(f^n(u)) \geq f^n(v)$ or $g(f^n(v)) \leq f^n(u)$. Fix $n \in \mathbb{N}$ and assume that the first case holds for this n (for the second case just follow the same arguments changing g by g^{-1}). Remark that for some $\bar{u} \in [u,v] \subset [w,z]$ and $\bar{v} \in [w,u] \subset [w,z]$ one has

$$\frac{g(f^n(u)) - f^n(u)}{f^n(u) - f^n(w)} \geq \frac{f^n(v) - f^n(u)}{f^n(u) - f^n(w)} = \frac{(f^n)'(\bar{u})}{(f^n)'(\bar{v})} \cdot \frac{v-u}{u-w}.$$

Since f preserves K and has no interior fixed points, the intervals in $\{f^j(I) : j \in \mathbb{Z}\}$ must be pairwise disjoint. The well known Bounded Distortion Principle then gives

$$\frac{(f^n)'(\bar{u})}{(f^n)'(\bar{v})} \geq \exp(-M),$$

and so

$$\frac{g(f^n(u)) - f^n(u)}{f^n(u) - f^n(w)} \geq \frac{v-u}{e^M(u-w)} > \lambda - 1.$$

This implies that

$$\frac{g(f^n(u)) - f^n(w)}{f^n(u) - f^n(w)} = 1 + \frac{g(f^n(u)) - f^n(u)}{f^n(u) - f^n(w)} > \lambda.$$

Since g fixes $f^n(w) \in K$, the left hand side member of this inequality is equal to $g'(x)$ for some point $x \in f^n([w,u])$. By (10), the integer n must be smaller than N_0, and this finishes the proof of the Claim.

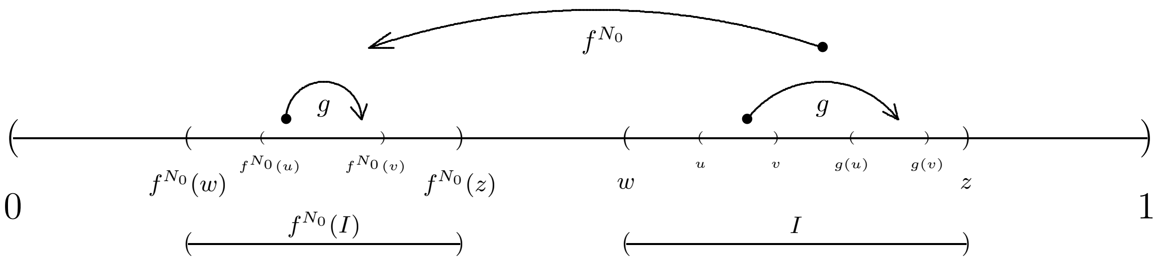

Figure 3

Now in order to prove that the semi-group generated by f and g is free, let us consider two words of the form $W_1 = f^n g^{m_r} f^{n_r} \cdots g^{m_1} f^{n_1}$ and $W_2 = g^q f^{p_s} g^{q_s} \cdots f^{p_1} g^{q_1}$, where m_j, n_j, p_j, q_j are positive integers, $n \geq 0$, and $q \geq 0$ (with $n > 0$ if $r = 0$, and $p > 0$ if $s = 0$). Note that

$$\tau_\mu(W_1) = (n_1 + \ldots + n_r + n)\tau_\mu(f) \quad \text{and} \quad \tau_\mu(W_2) = (p_1 + \ldots + p_s)\tau_\mu(f).$$

So, if the values of $(n_1 + \ldots + n_r + n)$ and $(p_1 + \ldots + p_s)$ are unequal, then $W_1 \neq W_2$. Assuming that these values are equal to some $N \in \mathbb{N}$, the proof is finished by the following Claim.

Claim: The elements $f^{-N}W_1$ and $f^{-N}W_2$ are different.

Proof. Changing $]u,v[$ by its image by some positive iterate of f (if necessary), we may assume that the intervals $]u,v[$ and $g(]u,v[)$ are disjoint, and that g fixes all the intervals $f^n(]u,v[)$ for $n > 0$. Let J be the (open) convex closure of the union $\cup_{j\in\mathbb{Z}} g^j(]u,v[)$. Note that g fixes the interval J and has no fixed point inside it. Now remark that

$$\begin{aligned} f^{-N}W_1 &= f^{-N} f^n g^{m_r} f^{n_r} \cdots g^{m_1} f^{n_1} \\ &= f^{-N} f^n g^{m_r} f^{n_r} \cdots g^{m_2} f^{n_1+n_2} \big(f^{-n_1} g^{m_1} f^{n_1}\big) \\ &= f^{-N} f^n g^{m_r} f^{n_r} \cdots g^{m_3} f^{n_1+n_2+n_3} \big(f^{-(n_1+n_2)} g^{m_2} f^{n_1+n_2}\big)\big(f^{-n_1} g^{m_1} f^{n_1}\big) \\ &\vdots \\ &= \big(f^{-(N-n)} g^{m_r} f^{N-n}\big) \cdots \big(f^{-(n_1+n_2)} g^{m_2} f^{n_1+n_2}\big)\big(f^{-n_1} g^{m_1} f^{n_1}\big) \end{aligned}$$

and

$$\begin{aligned} f^{-N}W_2 &= f^{-N} g^q f^{p_s} g^{q_s} \cdots f^{p_1} g^{q_1} \\ &= f^{-N} g^q f^{p_s} g^{q_s} \cdots f^{p_3} g^{q_3} f^{p_1+p_2} \big(f^{-p_1} g^{q_2} f^{p_1}\big) g^{q_1} \\ &\vdots \\ &= \big(f^{-N} g^q f^N\big) \cdots \big(f^{-p_1} g^{q_2} f^{p_1}\big) g^{q_1}. \end{aligned}$$

Since Γ has no crossed elements, and since all the maps

$$\big(f^{-(N-n)} g^{m_r} f^{N-n}\big), \ldots, \big(f^{-(n_1+n_2)} g^{m_2} f^{n_1+n_2}\big), \big(f^{-n_1} g^{m_1} f^{n_1}\big) \quad \text{and} \quad \big(f^{-N} g^q f^N\big), \ldots, \big(f^{-p_1} g^{q_2} f^{p_1}\big)$$

have fixed points inside J, they must fix the interval J. On the other hand, g^{q_1} fixes J but has no fixed point inside it. Therefore, if ν is any Radon measure on J which is invariant by the group generated by (the restrictions to J of) all those maps (including g^{q_1}), then $\tau_\nu(f^{-N}W_1) = 0$ and $\tau_\nu(f^{-N}W_2) = \tau_\nu(g^{q_1}) \neq 0$, and this shows that $f^{-N}W_1 \neq f^{-N}W_2$.

Remark 2.6. It is not difficult to adapt the arguments of this section to prove a similar result for groups of diffeomorphisms of the interval $[0,1]$ having derivatives with finite total variation.

2.3 On the non embedding of the group H into $\mathrm{Diff}_+^{1+\alpha}([0,1])$

Like in the beginning of §2.2, one can also give a direct proof of the fact that the action of H of §1.1 is not the semi-conjugate of an action by $C^{1+\alpha}$ diffeomorphisms for any $\alpha > 0$. For this, it suffices to note that inside H there are "a lot" of commuting elements, and then apply the Generalized Kopell Lemma (*i.e.* Théorème B of [5]). But again, we would like to give a proof of this fact which does not use commutativity in an essential way. Since we are prescribing the combinatorial structure for the dynamics of the action (up to topological semi-conjugacy), the main difficulty for this will consist in getting good control of distortion estimates in class $C^{1+\alpha}$. What follows is much inspired by §1.2 of [5]; for instance, the lemma below appears as Lemme 1.3 therein. For the reader's convenience, and because of its simplicity, we recall its proof.

Lemma 2.7. *Let h be any $C^{1+\alpha}$ diffeomorphism of a closed interval $[u,v]$. If M denotes the α-Hölder constant for h', then for every $x \in [u,v]$ one has $|h(x)-x| \leq M|v-u|^{1+\alpha}$.*

Proof. By the Mean Value Theorem, there exist some points $y \in [u,x]$ and $z \in [u,v]$ such that

$$h'(y) = \frac{h(x)-u}{x-u} \qquad \text{and} \qquad h'(z) = \frac{h(v)-h(u)}{v-u} = 1.$$

Since $|h'(y)-1| = |h'(y)-h'(z)| \leq M|y-z|^{\alpha} \leq M|v-u|^{\alpha}$, this gives

$$|h(x)-x| = |x-u||h'(y)-1| \leq M|v-u|^{1+\alpha}.$$

Now let us suppose that for some $\alpha > 0$ there exists an embedding $H \hookrightarrow \mathrm{Diff}_+^{1+\alpha}([0,1])$ which is semi-conjugate to that of Example 1.5. Fix the smallest positive integer k such that

$$\alpha(1+\alpha)^{k-2} \geq 1. \tag{11}$$

Note that viewing $k = k(\alpha)$ as a function of α we have that $k(\alpha) \to \infty$ as $\alpha \to 0$. For each $(l_1,\ldots,l_k) \in \mathbb{Z}^k$ denote by $L_{l_1,\ldots,l_k}$ the preimage under the semi-conjugacy of the corresponding interval $I_{l_1,\ldots,l_k}$. We leave to the reader the easy task of verifying the existence of elements $h_1,\ldots,h_k$ in H such that, for each $j \in \{1,\ldots,k\}$ and each $(l_1,\ldots,l_k) \in \mathbb{Z}^k$,

$$h_j(L_{l_1,\ldots,l_{j-1},l_j,l_{j+1},\ldots,l_k}) = L_{l_1,\ldots,l_{j-1},l_j-2,l_{j+1},\ldots,l_k}. \tag{12}$$

For instance, one can take $h_1 = \mathtt{a}^{-2}$, $h_2 = \mathtt{b}^{-2}$, $h_3 = \mathtt{a}^{-1}\mathtt{b}^{-2}\mathtt{a}$, etc. The contradiction is then given by the following general proposition.

Proposition 2.8. *Given an integer $k \geq 3$ let $\{L_{l_1,\ldots,l_k} : (l_1,\ldots,l_k) \in \mathbb{Z}^k\}$ be a family of closed intervals with disjoint interiors and disposed inside $[0,1]$ respecting the (direct) lexicographic order, that is, $L_{l_1,\ldots,l_k}$ is to the left of $L_{l'_1,\ldots,l'_k}$ if and only if $(l_1,\ldots,l_k)$ is lexicographically smaller than $(l'_1,\ldots,l'_k)$. Let $h_1,\ldots,h_k$ be C^1 diffeomorphisms of $[0,1]$ such that for each $j \in \{1,\ldots,k\}$ and each $(l_1,\ldots,l_k) \in \mathbb{Z}^k$ one has*

$$h_j(L_{l_1,\ldots,l_{j-1},l_j,\ldots,l_k}) = L_{l_1,\ldots,l_{j-1},l'_j,\ldots,l'_k} \quad \text{for some} \quad (l'_j,l'_{j+1},\ldots,l'_k) \in \mathbb{Z}^{k-j+1} \text{ satisfying } \quad l'_j \neq l_j. \tag{13}$$

If $\alpha > 0$ is such that $k \geq k(\alpha)$, then $h_1,\ldots,h_{k-1}$ cannot be simultaneously contained in $\mathrm{Diff}_+^{1+\alpha}([0,1])$.

Proof. Suppose by contradiction that $h_1,\ldots,h_{k-1}$ are $C^{1+\alpha}$ diffeomorphisms of $[0,1]$. Let $M \geq 1$ be a simultaneous α-Hölder constant for $h_2,\ldots,h_{k-1}$ as well as for $\log(h'_1)$, and let $\bar{M}$ be defined by

$$\log(\bar{M}) = \big(1+\alpha[1+(1+\alpha)+(1+\alpha)^2+\ldots+(1+\alpha)^{k-3}]\big)\log(M).$$

Denote $[a_k,b_k] = L_{0,\ldots,0}$, and for $i \in \{k-1,\ldots,1\}$ define by induction the interval $[a_i,b_i]$ as being the closed convex closure of the union

$$\bigcup_{n\in\mathbb{Z}} h_{i+1}^n\big([a_{i+1},b_{i+1}]\big).$$

Let λ be a constant such that

$$1 < \lambda < 1 + \frac{b_k - a_k}{e^{\bar{M}}(b_k - a_{k-1})}.$$

The derivative of h_k must be equal to 1 at the accumulation points of the intervals $L_{i_1,0,\ldots,0}$ as $|i_1| \to \infty$. Therefore, we can fix a (large) positive integer n such that $h'_k(x) \leq \lambda$ and $(h_k^{-1})'(x) \leq \lambda$ for all $x \in h_1^n([a_{k-1},b_{k-1}])$. The interval $h_1^n([a_k,b_k])$ is of the form $L_{l'_1,\ldots,l'_k}$ for some $(l'_1,\ldots,l'_k) \in \mathbb{Z}^k$ satisfying $l'_1 = l_1 + nd$ for some fixed $d \neq 0$. Hence, by the hypothesis (13), the intervals $h_1^n(]a_k,b_k[)$ and $h_k(h_1^n(]a_k,b_k[))$ are disjoint. Assume for instance that $h_k(h_1^n(a_k)) \geq h_1^n(b_k)$ (for the case where $h_k(h_1^n(b_k)) \leq h_1^n(a_k)$ just follow the same arguments changing h_k by h_k^{-1}). Then for some $u \in [a_k, b_k] \subset [a_{k-1}, b_{k-1}]$ and $v \in [a_{k-1}, a_k] \subset [a_{k-1}, b_{k-1}]$ one has

$$\frac{h_k(h_1^n(a_k)) - h_1^n(a_k)}{h_1^n(a_k) - h_1^n(a_{k-1})} \geq \frac{h_1^n(b_k) - h_1^n(a_k)}{h_1^n(a_k) - h_1^n(a_{k-1})} = \frac{(h_1^n)'(u)}{(h_1^n)'(v)} \cdot \frac{b_k - a_k}{a_k - a_{k-1}}. \tag{14}$$

Note that

$$\begin{aligned}
\left|\log\Big(\frac{(h_1^n)'(u)}{(h_1^n)'(v)}\Big)\right| &= \left|\log\Big(\prod_{j=0}^{n-1} \frac{h_1'(h_1^j(u))}{h_1'(h_1^j(v))}\Big)\right| \\
&\leq \sum_{j=0}^{n-1} \Big| \log(h_1'(h_1^j(u))) - \log(h_1'(h_1^j(v)))\Big| \\
&\leq M \sum_{j=0}^{n-1} \big|h_1^j(u) - h_1^j(v)\big|^{\alpha} \\
&\leq M \sum_{j=0}^{n-1} \big|h_1^j(b_{k-1}) - h_1^j(a_{k-1})\big|^{\alpha}.
\end{aligned}$$

Now using the combinatorial hypothesis (13) and applying Lemma 2.7 to $h_{k-1}, \ldots, h_2$ one obtains

$$\begin{aligned}
\big|h_1^j(b_{k-1}) - h_1^j(a_{k-1})\big| &\leq M\big|h_1^j(b_{k-2}) - h_1^j(a_{k-2})\big|^{1+\alpha} \\
&\leq M\big(M\big|h_1^j(b_{k-3}) - h_1^j(a_{k-3})\big|^{1+\alpha}\big)^{1+\alpha} = M^{1+(1+\alpha)}\big|h_1^j(b_{k-3}) - h_1^j(a_{k-3})\big|^{(1+\alpha)^2} \\
&\vdots \\
&\leq M^{1+(1+\alpha)+(1+\alpha)^2+\ldots+(1+\alpha)^{k-3}}\big|h_1^j(b_1) - h_1^j(a_1)\big|^{(1+\alpha)^{k-2}}.
\end{aligned}$$

We then deduce

$$\left|\log\Big(\frac{(h_1^n)'(u)}{(h_1^n)'(v)}\Big)\right| \leq \bar{M} \sum_{j=0}^{n-1} \big|h_1^j(b_1) - h_1^j(a_1)\big|^{\alpha(1+\alpha)^{k-2}}. \tag{15}$$

The hypothesis $k \geq k(\alpha)$ implies that $\alpha(1+\alpha)^{k-2} \geq 1$. Therefore, since the intervals $h_1^j(]a_1, b_1[)$ are two by two disjoint, the right hand side expression of (15) is bounded by $\bar{M}$, which implies that

$$\frac{(h_1^n)'(u)}{(h_1^n)'(v)} \geq \frac{1}{e^{\bar{M}}}.$$

Introducing this inequality into (14) we get

$$\frac{h_k(h_1^n(a_k)) - h_1^n(a_k)}{h_1^n(a_k) - h_1^n(a_{k-1})} \geq \frac{b_k - a_k}{e^{\bar{M}}(a_k - a_{k-1})},$$

and summing 1 to both members this gives

$$\frac{h_k(h_1^n(a_k)) - h_1^n(a_{k-1})}{h_1^n(a_k) - h_1^n(a_{k-1})} \geq 1 + \frac{b_k - a_k}{e^{\bar{M}}(a_k - a_{k-1})} > \lambda.$$

But since $h_k(h_1^n(a_{k-1})) = h_1^n(a_{k-1})$, the left hand side member in this inequality is equal to $h_k'(x)$ for some point $x \in h_1^n([a_{k-1}, b_{k-1}])$, and so by our choice of n it is less than or equal to λ. This contradiction finishes the proof.

Remark 2.9. Using the methods of [5], it is possible to prove that the preceding proposition is still true for $k > 1 + 1/\alpha$; moreover, this regularity is sharp, in the sense that for every α such that $k < 1 + 1/\alpha$ there exist $C^{1+\alpha}$ counter-examples.

2.4 Proof of Theorem B

In what follows Γ will be supposed to be a finitely generated subgroup of $\mathrm{Diff}_+^{1+\alpha}([0,1])$ without free semi-groups on two generators. As explained at the beginning of the second part of this article, to prove that Γ is almost nilpotent it suffices to show that Γ is solvable; in fact, we will prove that the corresponding degree of solvability is bounded by $k = 1 + k(\alpha)$, where $k(\alpha)$ is the smallest integer satisfying the inequality (11).[3] Note that, because of this uniform bound in terms of α, we may (and we will) assume that the action of Γ on $]0,1[$ has no global fixed point. By the first part of the proof of Proposition 2.2, every finite system of generators of Γ contains an element without interior fixed points. Let us fix once and for all an element $h_1 \in \Gamma$ such that $h_1(x) < x$ for all $x \in]0,1[$.

2.4.1 Capturing minimal levels

Let us denote by $\Gamma_\star$ the subgroup of Γ consisting of the elements having fixed points inside $]0,1[$. Note that the fact that $\Gamma_\star$ is a subgroup follows from that $\Gamma_\star$ can be identified with the kernel of the translation number homomorphism τ_μ associated to any fixed Γ-invariant Radon measure μ on $]0,1[$. (This can also be directly checked, and holds more generally for groups without crossed elements which are non necessarily finitely generated; see Remark 2.4). In particular, it contains the first derived group Γ_1. For each $h \in \Gamma_\star$ distinct from the identity denote by $\mathcal{I}(h)$ the family of open intervals I such that h fixes I but no point inside I is fixed by h. For $I \in \mathcal{I}(h)$ put

$$\mathcal{N}(h,I) = \{n \geq 0 : h_1^{-n} h h_1^n \text{ has no fixed point inside } I\}.$$

Note that $n=0$ belongs to $\mathcal{N}(h,I)$. Moreover, since Γ does not have crossed elements, if n is contained in $\mathcal{N}(h,I)$ then the intervals $h_1^{-n} h h_1^n(I)$ and I either coincide or are disjoint. For each $n \in \mathcal{N}(h,I)$ define I_n as being the (open) convex closure of the union $\cup_{j \in \mathbb{Z}} h_1^{-n} h^j h_1^n(I)$. Now let us consider the preorder[4] relation $\preceq$ on $\mathcal{N}(h,I)$ defined by $m \preceq n$ if $I_m \subset I_n$. Again the non existence of crossed elements implies that for $m \prec n$ the geometric picture is as in Figure 4. Our first task is to prove that there are integers n for which I_n is maximal. We would like to point out that there is a somehow related issue in the classical level theory for codimension-1 foliations which concerns the existence of local minimal sets (see [3, Theorem 8.1.8]). However, in that context this property is established using C^2 control of distortion estimates which are no longer available in the $C^{1+\alpha}$ case. This is the main reason why our argument is so different: we will show that "there must exist some minimal level" because of the absence of free semi-groups on two generators inside Γ. (This should be compared with Lemma 2.3 of [28].)

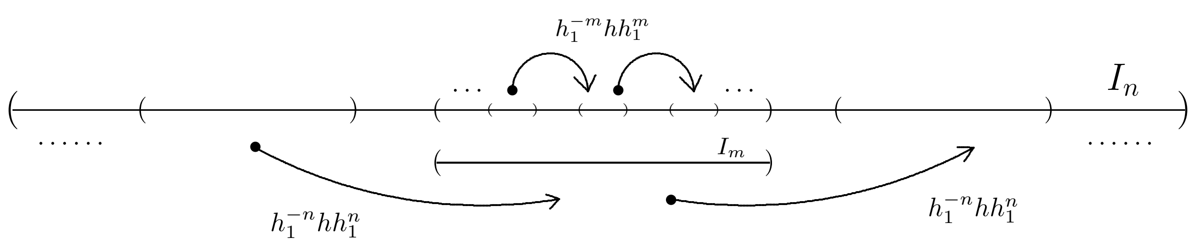

Figure 4

Lemma 2.10. *For every non trivial element h in $\Gamma_\star$ and every interval $I \in \mathcal{I}(h)$, the preorder relation $\preceq$ on $\mathcal{N}(h,I)$ has a maximal element.*

Proof. Suppose by contradiction that there is no maximal element for $\preceq$, and fix an increasing sequence (k_i) of non negative integers such that $k_i \succ n$ holds for every $n < k_i$. We will use this sequence to prove that the semi-group generated by h_1^{-1} and h is free, thus providing a contradiction. So let us consider two different words in positive powers of h_1^{-1} and h, and let's try to prove that they do not represent the same element of Γ. After conjugacy, we may assume that these words are of the form $W_1 = h_1^{-n} h^{m_r} h_1^{-n_r} \cdots h^{m_1} h_1^{-n_1}$ and $W_2 = h^q h_1^{-p_s} h^{q_s} \cdots h_1^{-p_1} h^{q_1}$, where m_j, n_j, p_j, q_j are positive integers, $n \geq 0$, and $q \geq 0$ (with $n > 0$ when $r = 0$, and $q > 0$ when $s = 0$).

[3]According to [5], it is quite possible that the solvability degree is bounded by any integer k satisfying $k > 1 + 1/\alpha$ (compare with Remark 2.9). An analogous remark applies to §2.5.

[4]Recall that a *preoder* is a relation which is reflexive and transitive, but not necessarily antisymmetric.

We first claim that if the integers $N_1 = n_1 + \ldots + n_r + n$ and $N_2 = p_1 + \ldots + p_s$ are different, then W_1 and W_2 are distinct elements. Indeed, since h is in $\Gamma_\star$ one has $\tau_\mu(h) = 0$, and so $\tau_\mu(W_1) = -N_1\tau_\mu(h_1)$ and $\tau_\mu(W_2) = -N_2\tau_\mu(h_1)$. On the other hand, since h_1 has no fixed point inside $]0,1[$ one has $\tau_\mu(h_1) \neq 0$. Therefore, $\tau_\mu(W_1) \neq \tau_\mu(W_2)$ when $N_1 \neq N_2$.

Assume in the rest of the proof that N_1 and N_2 are equal, and denote by N their common value. Fix an integer $i \in \mathbb{N}$ such that $k_i \geq N$. We will prove that $h_1^N W_1$ and $h_1^N W_2$ are different. For this, note that

$$\begin{aligned} h_1^N W_1 &= h_1^N h_1^{-n} h^{m_r} h_1^{-n_r} \cdots h^{m_1} h_1^{-n_1} \\ &= h_1^N h_1^{-n} h^{m_r} h_1^{-n_r} \cdots h^{m_2} h_1^{-(n_1+n_2)} \big(h_1^{n_1} h^{m_1} h_1^{-n_1}\big) \\ &= h_1^N h_1^{-n} h^{m_r} h_1^{-n_r} \cdots h^{m_3} h_1^{-(n_1+n_2+n_3)} \big(h_1^{n_1+n_2} h^{m_2} h_1^{-(n_1+n_2)}\big)\big(h_1^{n_1} h^{m_1} h_1^{-n_1}\big) \\ &\vdots \\ &= \big(h_1^{N-n} h^{m_r} h_1^{-(N-n)}\big) \cdots \big(h_1^{n_1+n_2} h^{m_2} h_1^{-(n_1+n_2)}\big)\big(h_1^{n_1} h^{m_1} h_1^{-n_1}\big), \end{aligned}$$

and

$$\begin{aligned} h_1^N W_2 &= h_1^N h^q h_1^{-p_s} h^{q_s} \cdots h_1^{-p_2} h^{q_2} h_1^{-p_1} h^{q_1} \\ &= h_1^N h^q h_1^{-p_s} h^{q_s} \cdots h^{q_3} h_1^{-(p_1+p_2)} \big(h_1^{p_1} h^{q_2} h_1^{-p_1}\big) h^{q_1} \\ &\vdots \\ &= \big(h_1^N h^q h_1^{-N}\big)\big(h_1^{N-p_s} h^{q_s} h_1^{-(N-p_s)}\big) \cdots \big(h_1^{p_1} h^{q_2} h_1^{-p_1}\big) h^{q_1}. \end{aligned}$$

Now from the facts that $k_i \geq N$ and that $k_i \succ n$ for all $n < k_i$ it follows easily that all the maps

$$\big(h_1^{N-n} h^{m_r} h_1^{-(N-n)}\big), \ldots, \big(h_1^{n_1+n_2} h^{m_2} h_1^{-(n_1+n_2)}\big), \big(h_1^{n_1} h^{m_1} h_1^{-n_1}\big),$$

as well as

$$\big(h_1^N h^q h_1^{-N}\big), \big(h_1^{N-p_s} h^{q_s} h_1^{-(N-p_s)}\big), \ldots, \big(h_1^{p_1} h^{q_2} h_1^{-p_1}\big), h^{q_1},$$

fix the interval $h_1^{k_i}(I_{k_i})$, but only the last one (namely h^{q_1}) has no fixed point inside it. This implies that if ν is a Radon measure on $h_1^{k_i}(I_{k_i})$ which is invariant by the group generated by all of them, then

$$\tau_\nu\big(h_1^N W_1\big) = 0 \qquad \text{and} \qquad \tau_\nu\big(h_1^N W_2\big) = \tau_\nu(h^{q_1}) \neq 0.$$

In particular, $h_1^N W_1 \neq h_1^N W_2$, and so $W_1 \neq W_2$, which finishes the proof of the lemma.

Now let us define the *star operation* of "minimization of levels" as follows: for each non trivial $h \in \Gamma_\star$ and each $I \in \mathcal{I}(h)$, fix an integer $n \in \mathcal{N}(h, I)$ which is maximal for the preorder relation $\preceq$, and denote

$$h^* = h^*_I = h_1^{-n} h h_1^n \qquad \text{and} \qquad I^*(h, I) = I_n.$$

Note that the choice of n (and hence the choice of h^*) is not necessarily unique, but the interval $I^*(h, I)$ does not depend on it. For simplicity, we will assume in addition that $h^* = h$ when $n = 0$ belongs to $\mathcal{N}(h, I)$. Remark finally that h^* is always in $\Gamma_\star$, and $I^*(h, I)$ is an element of $\mathcal{I}(h^*)$. Moreover, it follows easily from the definitions that for all $m \geq 0$ the element h fixes the interval $h_1^m(I^*(h, I))$.

2.4.2 On the structure of minimal levels

The commutator of a finitely generated group is not necessarily finitely generated, even in the case of groups of intermediate growth (it is however the case for nilpotent groups). This is a small algebraic difficulty for the proof of the following lemma.

Lemma 2.11. *If Γ is not solvable with degree of solvability less than or equal to k, then there exist elements $h_2, \ldots, h_k$ in $\Gamma_\star$, and open intervals $I_1, \ldots, I_{k-1}$, such that I_j is strictly contained in I_i for $i < j$, the interval I_{i-1} coincides with $I^*(h_i, I_i)$ for each $i \in \{2, \ldots, k\}$, and one has $h_i = (h_i)^*_{I_i}$ for each $i \in \{2, \ldots, k-1\}$.*

Proof. Let us start by considering a non trivial element f_k in the k-derived group Γ_k of Γ, and let us fix an interval J in $\mathcal{I}(f_k)$. We then let $h_k = f_k^*$ and $I_{k-1} = I^*(f_k, J)$. We claim that there exists an element

$f_{k-1} \in \Gamma_{k-1}$ such that I_{k-1} and $f_{k-1}(I_{k-1})$ are disjoint. Indeed, if this is not true then by Lemma 2.1 one has $h(I_{k-1}) = I_{k-1}$ for every $h \in \Gamma_{k-1}$. Now h_k can be written as a product $[g_1, g_2] \cdots [g_{2n-1}, g_{2n}]$, where the elements $g_1, \ldots, g_{2n}$ belong to Γ_{k-1}. The group $\Gamma_{g_1,\ldots,g_{2n}}$ generated by them fixes the interval I_{k-1}, and since it contains no free semi-group on two generators, it preserves a Radon measure on I_{k-1}. By properties (i) and (iii) of §2.1, the derived group $[\Gamma_{g_1,\ldots,g_{2n}}, \Gamma_{g_1,\ldots,g_{2n}}]$ has global fixed points in I_{k-1}. In particular, the element $h_k \in [\Gamma_{g_1,\ldots,g_{2n}}, \Gamma_{g_1,\ldots,g_{2n}}]$ has fixed points inside I_{k-1}, which contradicts the fact that I_{k-1} belongs to $\mathcal{I}(h_k)$.

With respect to $I_{k-1} \in \mathcal{I}(f_{k-1})$ we can define $h_{k-1} = f_{k-1}^*$ and $I_{k-2} = I^*(f_{k-1}, I_{k-1})$. As above, one can prove the existence of an element $f_{k-2} \in \Gamma_{k-2}$ such that I_{k-2} and $f_{k-2}(I_{k-2})$ are disjoint, and then with respect to $I_{k-2} \in \mathcal{I}(f_{k-2})$ we let $h_{k-2} = f_{k-2}^*$ and $I_{k-3} = I^*(f_{k-2}, I_{k-2})$... Continuing this procedure inductively we finally get the desired elements h_i and intervals I_j, thus concluding the proof of the lemma.

From now on we assume that Γ is not solvable with degree of solvability less than or equal to k. Note that in this case the elements h_i constructed in the preceding lemma do not necessarily satisfy a property of "periodicity" so strong as (12) or (13). Nevertheless, as the next lemma shows, they must be "non lacunary" in a very precise sense.

Lemma 2.12. *For every $i \in \{2, \ldots, k\}$ the set of integers*

$$\mathcal{N}_i = \{m \geq 0: \ \ h_i \text{ has no fixed point inside } h_1^m(I_{i-1})\}$$

is syndetic, i.e. it cannot have arbitrarily large gaps.

Proof. Assuming that $\mathcal{N}_i$ has arbitrarily large gaps, we will prove that h_1 and $h = h_i$ generate a free semi-group on two generators, thus giving a contradiction. For this let us consider two words $W_1 = h_1^n h^{m_r} h_1^{n_r} \cdots h^{m_1} h_1^{n_1}$ and $W_2 = h^q h_1^{p_s} h^{q_s} \cdots h_1^{p_1} h^{q_1}$ in positive powers of h_1 and h, where m_j, n_j, p_j, q_j are positive integers, $n \geq 0$, and $q \geq 0$ (with $n > 0$ when $r = 0$, and $q > 0$ when $s = 0$). We have to prove that these words represent different elements of Γ.

First of all, using a similar argument to that of the beginning of the proof of Lemma 2.10, one easily checks that if the numbers $N_1 = n_1 + \ldots + n_r + n$ and $N_2 = p_1 + \ldots + p_s$ are different, then W_1 and W_2 are distinct elements. Assume in what follows that N_1 and N_2 coincide with some $N \in \mathbb{N}$. We will finish the proof by checking that $h_1^{-N} W_1$ and $h_1^{-N} W_2$ are different. For this note that

$$\begin{aligned}
h_1^{-N} W_1 &= h_1^{-N} h_1^n h_i^{m_r} h_1^{n_r} \cdots h_i^{m_1} h_1^{n_1} \\
&= h_1^{-N} h_1^n h_i^{m_r} h_1^{n_r} \cdots h_i^{m_2} h_1^{n_1+n_2} \big(h_1^{-n_1} h_i^{m_1} h_1^{n_1}\big) \\
&= h_1^{-N} h_1^n h_i^{m_r} h_1^{n_r} \cdots h_i^{m_3} h_1^{n_1+n_2+n_3} \big(h_1^{-(n_1+n_2)} h_i^{m_2} h_1^{n_1+n_2}\big) \big(h_1^{-n_1} h_i^{m_1} h_1^{n_1}\big) \\
&\;\;\vdots \\
&= \big(h_1^{-(N-n)} h_i^{m_r} h_1^{N-n}\big) \cdots \big(h_1^{-(n_1+n_2)} h_i^{m_2} h_1^{n_1+n_2}\big) \big(h_1^{-n_1} h_i^{m_1} h_1^{n_1}\big),
\end{aligned}$$

and

$$\begin{aligned}
h_1^{-N} W_2 &= h_1^{-N} h_i^q h_1^{p_s} h^{q_s} \cdots h_1^{p_2} h_i^{q_2} h_1^{p_1} h_i^{q_1} \\
&= h_1^{-N} h_i^q h_1^{p_s} h_i^{q_s} \cdots h_i^{q_3} h_1^{p_1+p_2} \big(h_1^{-p_1} h_i^{q_2} h_1^{p_1}\big) h_i^{q_1} \\
&\;\;\vdots \\
&= \big(h_1^{-N} h_i^q h_1^N\big) \big(h_1^{-(N-p_s)} h_i^{q_s} h_1^{N-p_s}\big) \cdots \big(h_1^{-p_1} h_i^{q_2} h_1^{p_1}\big) h_i^{q_1}.
\end{aligned}$$

Recall that, by the definition of the star operation, for every $n \geq 0$ the interval $h_1^n(I_{i-1})$ is fixed by h_i. Now take $m \in \mathcal{N}_i$ such that the element next to it in $\mathcal{N}_i$ is bigger than $m + N$. For this choice of m, all the elements

$$\big(h_1^{-(N-n)} h_i^{m_r} h_1^{N-n}\big), \big(h_1^{-(n_1+n_2)} h_i^{m_2} h_1^{n_1+n_2}\big), \ldots, \big(h_1^{-n_1} h_i^{m_1} h_1^{n_1}\big)$$

and

$$\big(h_1^{-N} h_i^q h_1^N\big), \big(h_1^{-(N-p_s)} h_i^{q_s} h_1^{N-p_s}\big), \ldots, \big(h_1^{-p_1} h_i^{q_2} h_1^{p_1}\big), h_i^{q_1}$$

fix the interval $h_1^m(I_{i-1})$, but only the last one (namely $h_i^{q_1}$) has no fixed point inside it. Therefore, if we let ν be any Radon measure on this interval which is invariant by the group generated by (the restrictions) of all these maps, then one has

$$\tau_\nu(h_1^{-N} W_1) = 0 \quad \text{ and } \quad \tau_\nu(h_1^{-N} W_2) = \tau_\nu(h_i^{q_1}) \neq 0.$$

This shows that $h_1^{-N} W_1 \neq h_1^{-N} W_2$ and finishes the proof of the lemma.

2.4.3 End of the proof

We are now ready to finish the proof of Theorem B. For this first note that the preceding lemma shows the existence of a (large) positive integer N satisfying the following property: for every $i \in \{2, \ldots, k\}$ and every integer $m \geq 0$, at least one of the maps

$$h_i, h_1^{-1} h_i h_1^1, \ldots, h_1^{-N} h_i h_1^N$$

has no fixed point on the interval $h_1^m(I_{i-1})$.

Now proceed as in the proof of Proposition 2.8. Let M be a common α-Hölder constant for the function $\log(h_1')$ and for all the maps of the form $h_1^{-j} h_i h_1^j$, where $j \in \{0, 1, \ldots, N\}$ and $i \in \{2, \ldots, k-2\}$. Let $\bar{M}$ now be defined by

$$\log(\bar{M}) = \big(1 + \alpha[1 + (1+\alpha) + (1+\alpha)^2 + \ldots + (1+\alpha)^{k-4}]\big) \log(M).$$

For simplicity let us denote $]a_i, b_i[= I_i$ for every $i \in \{1, \ldots, k-1\}$, and let us fix a constant λ such that

$$1 < \lambda < 1 + \frac{b_{k-1} - a_{k-1}}{e^{\bar{M}}(a_{k-1} - a_{k-2})}.$$

The derivative of h_{k-1} must be equal to 1 at the origin, which is the accumulation point of the intervals $h_1^m([a_{k-2}, b_{k-2}])$ for $m \geq 0$. Therefore, we can fix a (large) positive integer n such that $(h_1^{-j} h_{k-1} h_1^j)'(x) \leq \lambda$ and $(h_1^{-j} h_{k-1}^{-1} h_1^j)'(x) \leq \lambda$ for all $x \in h_1^n([a_{k-2}, b_{k-2}])$ and all $j \in \{0, 1, \ldots, N\}$. Since $]a_{k-1}, b_{k-1}[$ belongs to $\mathcal{I}(h_k)$ and Γ has no crossed elements, one of the maps

$$h_{k-1}, h_1^{-1} h_{k-1} h_1, \ldots, h_1^{-N} h_{k-1} h_1^N$$

must send the interval $h_1^n(]a_{k-1}, b_{k-1}[)$ into a disjoint interval (still contained in $h_1^n(]a_{k-2}, b_{k-2}[)$). Assume for instance that this element is h_{k-1} (all the other cases are treated similarly), and that $h_{k-1}(h_1^n(]a_{k-1}, b_{k-1}[))$ is to the right of $h_1^n(]a_{k-1}, b_{k-1}[)$, *i.e.* that is $h_{k-1}(h_1^n(a_{k-1})) \geq h_1^n(b_{k-1})$ (if not then replace h_{k-1} by its inverse). There must exist points $u \in [a_{k-1}, b_{k-1}] \subset [a_{k-2}, b_{k-2}]$ and $v \in [a_{k-2}, a_{k-1}] \subset [a_{k-2}, b_{k-2}]$ such that

$$\frac{h_{k-1}(h_1^n(a_{k-1})) - h_1^n(a_{k-1})}{h_1^n(a_{k-1}) - h_1^n(a_{k-2})} \geq \frac{h_1^n(b_{k-1}) - h_1^n(a_{k-1})}{h_1^n(a_{k-1}) - h_1^n(a_{k-2})} = \frac{(h_1^n)'(u)}{(h_1^n)'(v)} \cdot \frac{b_{k-1} - a_{k-1}}{a_{k-1} - a_{k-2}}. \tag{16}$$

Note that

$$\begin{aligned}
\left|\log\Big(\frac{(h_1^n)'(u)}{(h_1^n)'(v)}\Big)\right| &= \left|\log\Big(\prod_{m=0}^{n-1} \frac{h_1'(h_1^m(u))}{h_1'(h_1^m(v))}\Big)\right| \\
&\leq \sum_{m=0}^{n-1} \big|\log(h_1'(h_1^m(u))) - \log(h_1'(h_1^m(v)))\big| \\
&\leq M \sum_{m=0}^{n-1} \big|h_1^m(u) - h_1^m(v)\big|^{\alpha} \\
&\leq M \sum_{m=0}^{n-1} \big|h_1^m(b_{k-2}) - h_1^m(a_{k-2})\big|^{\alpha}.
\end{aligned}$$

Now for each $m \in \{0, 1, \ldots, n-1\}$ there exists $j \in \{0, 1, \ldots, N\}$ (depending on m) such that the intervals $h_1^m\big(]a_{k-2}, b_{k-2}[\big)$ and $h_1^{-j} h_{k-2} h_1^j\big(h_1^m(]a_{k-2}, b_{k-2}[)\big)$ are disjoint. Lemma 2.7 applied to $h_1^{-j} h_{k-2} h_1^j$ gives

$$|h_1^m(b_{k-2}) - h_1^m(a_{k-2})| \leq M |h_1^m(b_{k-3}) - h_1^m(a_{k-3})|^{1+\alpha},$$

and so

$$\left|\log\Big(\frac{(h_1^n)'(u)}{(h_1^n)'(v)}\Big)\right| \leq M \sum_{m=0}^{n-1} M^{\alpha} |h_1^m(b_{k-3}) - h_1^m(a_{k-3})|^{\alpha(1+\alpha)}.$$

Repeating this argument several times we deduce

$$\begin{aligned}
\left|\log\Big(\frac{(h_1^n)'(u)}{(h_1^n)'(v)}\Big)\right| &\leq M^{1+\alpha} \sum_{m=0}^{n-1} M^{\alpha(1+\alpha)} \big|h_1^m(b_{k-4}) - h_1^m(a_{k-4})\big|^{\alpha(1+\alpha)^2} \\
&\;\;\vdots \\
&\leq M^{1+\alpha+\alpha[(1+\alpha)+\ldots+(1+\alpha)^{k-4}]} \sum_{m=0}^{n-1} \big|h_1^m(b_1) - h_1^m(a_1)\big|^{\alpha(1+\alpha)^{k-3}},
\end{aligned}$$

that is

$$\left|\log\Big(\frac{(h_1^n)'(u)}{(h_1^n)'(v)}\Big)\right| \leq \bar{M}\sum_{m=0}^{n-1}\big|h_1^m(b_1)-h_1^m(a_1)\big|^{\alpha(1+\alpha)^{k-3}}. \tag{17}$$

By the definition of k (namely $k = 1+k(\alpha)$), one has $\alpha(1+\alpha)^{k-3} \geq 1$. Moreover, the intervals $h_1^m(]a_1, b_1[)$ must be pairwise disjoint. Hence, the right hand side expression of (17) is bounded by $\bar{M}$, which implies that

$$\frac{(h_1^n)'(u)}{(h_1^n)'(v)} \geq \frac{1}{e^{\bar{M}}}.$$

Introducing this inequality into (16) we get

$$\frac{h_k(h_1^n(a_{k-1}))-h_1^n(a_{k-1})}{h_1^n(a_{k-1})-h_1^n(a_{k-2})} \geq \frac{b_{k-1}-a_{k-1}}{e^{\bar{M}}(a_{k-1}-a_{k-2})},$$

and summing 1 to both members this gives

$$\frac{h_{k-1}(h_1^n(a_{k-1}))-h_1^n(a_{k-2})}{h_1^n(a_{k-1})-h_1^n(a_{k-2})} \geq 1+\frac{b_{k-1}-a_{k-1}}{e^{\bar{M}}(a_{k-1}-a_{k-2})} > \lambda.$$

However, since h_{k-1} fixes the point $h_1^n(a_{k-2})$, the left hand side member of this inequality is equal to $h_{k-1}'(x)$ for some point $x \in [h_1^n(a_{k-2}), h_1^n(a_{k-1})]$, and so it is less than or equal to λ by our choice of n. This contradiction finishes the proof of Theorem B.

Remark 2.13. A careful reading of the arguments given along the second part of this article shows that the differentiability of the maps involved is needed only at one of the end points of $[0,1]$. More precisely, Theorem B still holds (with the very same proof) for finitely generated subgroups of $\mathrm{Diff}_+^{1+\alpha}([0,1[)$ or $\mathrm{Diff}_+^{1+\alpha}(]0,1])$ without free semi-groups on two generators. This also applies to Remark 2.6. However, we ignore if the theorem is still true for groups of germs of $C^{1+\alpha}$ diffeomorphisms; this seems to be an interesting problem.

2.5 The cases of the circle and the real line

The aim of this final section is to prove two claims made in the Introduction of this work, namely that finitely generated subgroups of $\mathrm{Diff}_+^{1+\alpha}(\mathbb{R})$ or $\mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$ with sub-exponential growth are also almost nilpotent. Again, we will prove this for subgroups without free semi-groups on two generators by showing that they are solvable with degree of solvability at most $2+k(\alpha)$. (Remark that this issue will still be true for non finitely generated groups without free semi-groups on two generators.) The nilpotence will then be a direct consequence of Rosenblatt's theorem [29].

Let us first consider the (simpler) case of a finitely generated subgroup Γ of $\mathrm{Diff}_+^{1+\alpha}(\mathbb{R})$ without free semi-groups on two generators. By §2.1, the action of the first derived group Γ_1 has global fixed points. Looking at the action of Γ_1 on the closure of each connected component of the complement of the set of its global fixed points, and using (the arguments of the proof of) Theorem B (as well as Remark 2.13), we obtain that Γ_1 is solvable with degree of solvability at most $1+k(\alpha)$. Then one deduces that Γ is solvable itself with degree of solvability smaller than or equal to $2+k(\alpha)$.

Now let Γ be a finitely generated subgroup of $\mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$ without free semi-groups on two generators. Note that, in contrast to the case of sub-exponential growth subgroups, Γ is not *a priori* amenable.[5] This is the reason why the following Claim is not completely trivial.[6]

Claim: The group Γ preserves a probability measure on the circle.

Proof. Let $\tilde{\Gamma}$ be the covering of Γ acting on the real line. This group $\tilde{\Gamma}$ is still finitely generated. Moreover, it cannot contain crossed elements: if it contains two such elements then they project on Γ into two elements for which one can apply the argument of Proposition 2.1 in order to show that Γ contains free semi-groups on two generators, thus giving a contradiction.

By Proposition 2.2, the group $\tilde{\Gamma}$ preserves a Radon measure on the real line. This measure is invariant by the integer translations, and so it projects into a finite measure on the circle which is invariant by Γ. Hence, up to normalization, we have obtained the desired Γ-invariant probability measure on S^1.

[5]There exist non amenable groups without free semi-groups on two generators (see for example [25]), but it seems to be unknown if such a group can act faithfully on the circle.

[6]According to a beautiful result by Margulis [20], the Claim still holds when Γ has no free subgroup on two generators.

For groups of circle homeomorphisms preserving a probability measure it is easy to see that the *rotation number* function is a homomorphism into $\mathbb{T}^1$. (See for instance [7].) Using this fact one easily deduces that in our situation the first derived group Γ_1 has global fixed points. Again, looking at the action of Γ_1 on the closure of each connected component of the complement of the set of its global fixed points, and using (the arguments of the proof of) Theorem B, we obtain that Γ_1 is solvable with degree of solvability at most $1+k(\alpha)$. From this one concludes that Γ is solvable itself with degree of solvability less than or equal to $2+k(\alpha)$, thus finishing the proof.

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